On the Use of Gauss' Principle in Vibration Analysis

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August 2015

A Thesis Submitted to McGill University in Partial Fulfillment of the Requirement of the Degree of Master of Science

DEDICATION

This dissertation is dedicated to my mother, who taught me that even the largest task can be accomplished if done one step at a time. This work is also dedicated to my father, who has been a constant source of support and encouragement during the challenges of graduate school and life.

ACKNOWLEDGEMENTS

Foremost, I would like to extend my sincere gratitude to Dr. Mathias Legrand for guiding me as my advisor throughout this research. I am grateful for his expert advice, patience, and most of all diligent feedback throughout the entire process. I am grateful to him for holding me to a high research standard and enforcing strict validations for each research result, and thus teaching me how to do research.

I would like to give special thanks to Dr. Anders Thorin, for his assistance throughout this research. The completion of this study would not have been possible without his contributions.

I would also like to thank my colleagues in Structural Dynamics and Vibration Laboratory for their technical assistance whenever I needed.

I take this opportunity to thank my friends whose support and care helped me overcome setbacks and stay optimistic during my graduate studies.

Most importantly, I am thankful to my family: my parents and my sister, for supporting me spiritually throughout writing this dissertation. I greatly value their patience and I deeply appreciate their faith in me.

ABSTRACT

In 1829, C.F. Gauss stated his celebrated principle of least constraints which primarily describes the relation between the actual and free accelerations of a system of punctual masses: the actual acceleration satisfies the constraints acting on the system while the free acceleration ignores them. The principle takes the form of a constrained quadratic optimization problem and is based on minimizing the squared difference between the actual and free accelerations. In vibration analysis, the main advantage of this approach is the possibility of obtaining the steady state forced response directly. The availability of numerical minimization tools which were not developed when this principle was first introduced makes this topic appealing again.

In this exploratory study, Gauss' principle of least constraints is illustrated through examples from vibration analysis and its functionality is examined in properly predicting the steady state forced response of various simple mechanical systems such as a single pendulum, a Duffing oscillator, a piecewise linear system and a unilaterally constrained mass-spring system. The vibration analysis of periodically forced systems is investigated and the dynamic behavior of the system obtained from Gauss' approach is compared with those from ordinary differential equations solvers.

ABRÉGÉ

En 1829, C.F. Gauss a énoncé son célèbre principe des moindres contraintes qui décrit principalement la relation entre l'accélération réelle et l'accélération de mouvement libre d'unb système de masses ponctuelles: à l'inverse de l'accélération libre, l'accélération réelle satisfait les contraintes agissant sur le système. Le principe s'écrit sous la forme d'un problème d'optimisation quadratique contraint et est basée sur la minimisation de la valeur résiduelle entre les accélérations réelles et libres. Dans le domaine de l'analyse vibratoire, le principal avantage de cette approche est la possibilité d'obtenir directement la réponse forcée. La disponibilité d'outils de minimisation numériques inexistants lorsque ce principe a été introduit le rend attractif à nouveau.

Dans cette étude exploratoire, le principe de Gauss des moindres contraintes est illustré par des exemples d'analyse vibratoire et sa capacité à correctement prédire la réponse forcée est examinée. Le concept des moindres contraintes est utilisé pour trouver des solutions approximatives des équations différentielles du mouvement de divers systèmes mécaniques tels que le pendule simple, l'oscillateur de Duffing, un système linéaire par morceaux et un système masse-ressort unilatéralement contraint. L'analyse des vibrations des systèmes périodiquement forcés est étudiée et le comportement dynamique du système obtenu à partir de l'approche de Gauss est comparé à celui prédit par des méthodes de résolution d'équations différentielles ordinaires.

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Chapter 1

Introduction

The main concepts used in this document are defined in this chapter. For unfamiliar terms, one can refer to the Glossary at the end of the dissertation.

1.1 History of Classical Dynamics

Classical mechanics is the study of non-relativistic and non-quantum motion (effects) and forces (causes). It was founded in the 17^{th} century by the works of Galileo (1638) and Newton (1687) [1]. The science of mechanics can be divided into *kinematics*, *dynamics*, and *statics*. Kinematics studies the *description* of a flexible or rigid body in motion. While statics deals with details of external and internal forces that hold a body at rest, dynamics explains *causes* of that motion [2, 3].

The developments in mechanics can be classified into two major branches: vectorial mechanics which starts from Newton's laws of motion, and analytical mechanics [3]. Galileo inspired Newtonian mechanics by introducing the concept of acceleration and inertia which became the base for the Newton's laws later in 1687 [3]. Newton's laws explain the dynamics of a single particle. Euler extended these laws to cover dynamics of a system of particles such as rigid bodies [4]. Newton's laws are based on vectors and the concept of forces. Through this approach, equations of motion of dynamically known systems can be easily obtained

considering physical coordinates. However for constrained systems where the forces induced by the constraints are not determined yet, Newtonian approaches do not straightforwardly result in the equations of motion. Furthermore, the restriction of choosing physical coordinates may complicate the formulations [5].

Analytical mechanics was first established by Lagrange a century after Newton formulated his laws to tackle such problems of constrained motion. Analytical dynamics is based on the concept of virtual works. Two widely known techniques of this branch are Lagrangian and Hamiltonian methods which use *generalized coordinates* instead of *physical coordinates* previously used in Newton's laws. Since analytical dynamics' methods use a virtual work approach, they deal with scalar quantities rather than vectors [3]. These approaches are further discussed in the following sections.

1.1.1 Constraints and Generalized Coordinates

In the physical world, three coordinates are needed to fully locate a free particle in space. These coordinates can be in the form of Cartesian, cylindrical or spherical.

To locate a mechanical system of N particles, $3N$ coordinates are needed. Existence of the constraints reduces the minimum number of coordinates necessary for describing the system. If the system is subjected to m equality constraints, the minimum number of coordinates needed to completely describe the motion of the system at any given time is $n = 3N - m$. The chosen coordinates do not have to be necessarily physical and can be dimensionless quantities, length, angle, energy or any other quantity as long as they completely describe the configuration of the system. The instantaneous location of a particle identified by i, $i = 1, \ldots, N$, relative to an origin or reference point is given by its position vector $\mathbf{r}_i(t)$ [1]. For instance, in Cartesian system of coordinates: $\mathbf{r}_i(t) = (x_i(t), y_i(t), z_i(t))$. *Generalized coordinates* are "any set of quantities that completely describe the state or configuration of a system" [6]. In this study, generalized coordinates are expressed as $\mathbf{q} = [q_1, q_2, \ldots, q_n]$. Coordinates might be dependent on each other by constraints. Constraints that affect the

motion only on the configuration level are called *holonomic* constraints. Holonomic constraints for a single particle can be written in the form of algebraic equations among coordinates of displacement:

$$\phi_\ell(\mathbf{r}(t), t) = 0 \qquad \ell = 1, 2, \ldots, h \tag{1.1}$$

Nonholonomic constraints define dependencies on the velocity level and cannot be integrated to equations among displacements. Nonholonomic constraints take the general form:

$$\phi_\ell(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) = 0 \qquad \ell = h+1, h+2, \ldots, m \tag{1.2}$$

It should be noted that only bilateral (equality) constraints are considered in the above-mentioned definitions.

Consider a discrete system of N particles subjected to h holonomic and $m-h$ nonholonomic bilateral constraints. Assuming the constraints are at least twice piecewise continuously differentiable functions of time t, differentiating the holonomic constraint equations twice with respect to time and the nonholonomic constraint equations once yields

$$\mathbf{A}(\mathbf{r}, \dot{\mathbf{r}}, t)\ddot{\mathbf{r}} = \mathbf{b}(\mathbf{r}, \dot{\mathbf{r}}, t), \tag{1.3}$$

where $\mathbf{A}$ is an $m \times 3N$ matrix and $\mathbf{b}$ is an m-vector. It will be shown in next chapters that expressing constraints in the form of (1.3) is useful for incorporating constraints in different forms of equations of motion[1].

1.1.2 Principle of Least Action

There is a general tendency in physical systems to settle down to conditions of minimum energy [8]. Principle of least action, first enunciated by Maupertuis in 1740 is an example of this statement [9]. This variational principle is the base of analytical dynamics and is widely

[1]Equations of motion are equations that describe the behavior of a physical system in terms of its motion as a function of time [7].

used in both classical and quantum mechanics.

Suppose there is a particle moving from point P_1 to point P_2 starting at time t_1 to t_2 as shown in figure 1–1. For one particle of mass m and velocity v, the principle of least action

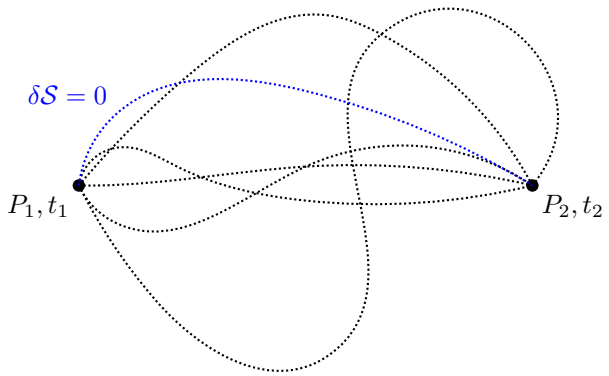


Figure 1–1: Schematic of the movement of a particle from point 1 to point 2

states that among all possible paths between the points P_1 to P_2, the actual path is the one which the integral called the *action*

$$\mathcal{S} = \int_{P_1}^{P_2} (mv) ds \tag{1.4}$$

is an extremum (minimum or maximum) [9]. The quantity s is the curvilinear coordinates abscissa on the trajectory of the particle. This principle can be rephrased as

$$\delta\mathcal{S} = \delta \int_{P_1}^{P_2} (mv) ds = 0 \tag{1.5}$$

This principle introduced a new concept in mechanics and was the initial point for the variational principles. Unlike Newton's approach which suggests that particles follow trajectories as a reason of being pushed by external forces, variational principles propose that natural movement is in a way that minimizes a particular integral called *action* [10].

1.1.3 Principle of Virtual Work

The principle of virtual work for static equilibrium was first formulated by Johann Bernoulli. It was later extended to dynamics by d'Alembert [5].

It is necessary to understand the concept of virtual work before getting into these variational principles. Virtual displacements are infinitesimal arbitrary changes in coordinates, consistent with all the constraints without any change in time [3]. It is useful to distinguish between *virtual displacements* and *actual displacements*. Actual displacements, denoted $d\mathbf{r}$, describe the true infinitesimal motion of a particle subjected to forces and constraints. Actual displacements are consistent with both equations of motion and equations of constraints. On the other hand, virtual displacements denoted $\delta\mathbf{r}$ are possible displacements based on *restrictions* applied on the system in the form of constraints. Hence virtual displacements are required to satisfy equations of constraints and may or may not be consistent with equations of motion [3].

For a particle in static equilibrium, the resultant of all forces acting on it is equal to zero[2]

$$\mathbf{F}_{\text{sum}} = \mathbf{0} \tag{1.6}$$

From equation (1.6), it is obvious that the virtual work done by the resultant of forces vanishes:

$$\delta W = \mathbf{F}_{\text{sum}} \cdot \delta\mathbf{r} = 0 \tag{1.7}$$

The resultant of forces accounts both external (applied) and internal (constraint and reaction) forces:

$$\delta W = (\mathbf{F}_{\text{applied}} + \mathbf{F}_{\text{constraint}}) \cdot \delta\mathbf{r} = 0 \tag{1.8}$$

[2]Basic result from Newton's laws.

Considering ideal constraints[3], the sum of virtual works of applied forces are zero:

$$\mathbf{F}_{\text{applied}} \cdot \delta\mathbf{r} = 0 \tag{1.9}$$

The principle of virtual works for a system of particles is expressed as

$$\sum_{i=1}^{N} \mathbf{F}_{i,\text{applied}} \cdot \delta\mathbf{r}_i = 0 \tag{1.10}$$

It should be noted that $\delta\mathbf{r}_i$ may be constrained and one cannot conclude that $\sum_i \mathbf{F}_{i,\text{applied}}$ is equal to zero. The principle in the form of generalized coordinates is:

$$\sum_{k=1}^{n} Q_k \cdot \delta q_k = 0 \tag{1.11}$$

where Q_k is the generalized force corresponding to the generalized coordinate q_k.

An extension of the principle of virtual works to dynamics is known as d'Alembert's principle which says:

$$\delta W = \sum_{i=1}^{N} \Big(\mathbf{F}_{i,\text{applied}} - \frac{d}{dt}(m_i \dot{\mathbf{r}}_i) \Big) \cdot \delta\mathbf{r}_i = 0 \tag{1.12}$$

which could be specified in terms of generalized coordinates as well.

As mentioned earlier, the principles of virtual works and d'Alembert are applicable to systems with ideal constraints. In [11] and [12], extensions of principles of virtual works and d'Alembert are proposed which are applicable to both ideal and non-ideal systems.

1.1.4 Lagrange's Equations

Lagrange's equations represent a reformulation of Newton's laws which allows using them in a general coordinate system which is not Cartesian. Important examples are

[3] *Ideal constraints* are constraints that do no work under permissible virtual displacements. Rolling without slippage, objects hinged together and a particle restricted to move on a surface are all examples of systems with ideal constraints. On the other hand, Coulomb's friction is an example of a non-ideal constraint force.

polar coordinates in the plane, and spherical or cylindrical coordinates in three dimensions. Lagrange's equations are formulated in terms of scalar quantities of kinetic energy and work.

L is a scalar quantity known as *Lagrangian* and can be obtained by subtracting potential energy U from kinetic energy K of the system:

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = K(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{1.13}$$

The general form of Lagrange's equations of motion for a set of single particles in generalized coordinates is

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_k}\Big) - \frac{\partial L}{\partial q_k} = Q_{\mathrm{nc},k} \qquad k = 1, 2, \ldots, n \tag{1.14}$$

where $Q_{\mathrm{nc},k}$ is the resultant of nonconservative forces or torques associated with the generalized coordinate q_k.

For a particle subjected only to conservative forces[4], $Q_{\mathrm{nc},k} = 0$ in equation (1.14). The great power of the Lagrange's method is that its basic equations take the same form in all coordinate systems. Another advantage of the Lagrangian formulation is the ability to automatically account for constraint forces in the equations of motion by using Lagrange's multipliers [5]. The general form of Lagrange's equations for a system with constraint conditions of the form (1.3) is

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_k}\Big) - \frac{\partial L}{\partial q_k} = Q_{\mathrm{nc},k} + \sum_{j=1}^{m} \lambda_j A_{jk} \qquad k = 1, 2, \ldots, n \tag{1.15}$$

where λ_j is the Lagrange multiplier associated with the j^{th} constraint equation.

This formulation will allow us to treat connected bodies as a single system, rather than individual entities. The primary kinetic quantity for Lagrange's equations of motion is mechanical energy (kinetic and potential), whereas the Newtonian equations of motion are time derivatives of vectorial momentum principles [10].

Note that coordinates can be either independent or depend on each other by constraint

[4]Conservative forces are forces which satisfy the relation $Q_k = -\frac{\partial U}{\partial q_k}$

equations.

1.1.5 Hamilton's Principle

One limitation of Lagrange's method is that equations are restricted to systems composed of rigid bodies with kinetic energy and massless springs and other sources of conservative forces. This is an idealization since no material can sustain stresses without deforming, and all springs have mass [10]. There is a more general formulation which was presented by Sir William Rowan Hamilton in 1834 [13]. Hamilton's principle is an alternative derivation of Lagrange's equations with adjustments in the definition of energy which makes it more general than Lagrange's principle. It also provides the basis for many approximation techniques, including finite element analysis, that are used to derive discrete models of continua [10].

Consider the motion of a dynamical system in the interval of time from t_1 to t_2. Assuming $\delta\mathbf{q}(t_1) = \delta\mathbf{q}(t_2) = 0$, Hamilton's principle of varying action is expressed as:

$$\int_{t_1}^{t_2} (\delta K + \delta W_{\text{c,nc}}) dt = 0 \tag{1.16}$$

where the virtual work of both conservative and nonconservative forces are taken into account:

$$\delta W_{\text{c,nc}} = \delta W_{\text{nc}} - \delta U \tag{1.17}$$

For conservative systems equation (1.16) can be written as

$$\int_{t_1}^{t_2} \delta L \, dt = 0 \tag{1.18}$$

where L is the *Lagrangian* defined previously. When the system is holonomic, integration and variational operations are interchangeable, which yields:

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \tag{1.19}$$

Hamilton's principle for a holonomic conservative system states that among all the possible paths, the actual path followed is the one which gives a stationary value for the time integral

of Lagrangian.

The implementation of Hamilton's principle for finding equations of motion requires the evaluation of the variations of the kinetic and potential energies which makes it more laborious than Lagrange's method. The direct use of Hamilton's principle is useful when investigating motions of deformable bodies, such as vibrations of beams, plates and shells. In such systems, partial differential equations of motion will be resulted from Hamilton's method with accompanying boundary conditions.

Principle of virtual work, d'Alembert's principle, Lagrange's principle and Hamilton's principle are closely related to each other. Hamilton's principle and Lagrange's principle are reformulations of d'Alembert's principle. D'Alembert's principle makes an independent statement at each instant of time, while Hamilton's principle includes a period of time in one single statement.

1.1.6 Gibbs-Appell Method

About half a decade after Hamilton, Gibbs (1879) and Appell (1899) independently devised what is now known as Gibbs-Appell method for obtaining the equations of motions of systems with nonholonomic constraints [14, 15].

Unlike Lagrange's approach which uses kinetic energy to obtain equations of motion, the Gibbs-Appell method makes use of a scalar function in terms of accelerations. In Lagrange's equations, the nonholonomic constraints are dealt with by using Lagrange multipliers. Gibbs-Appell method approaches such systems by the use of quasi coordinates.

For a system of N particles, the Gibbs-Appell function, denoted by S is formed as:

$$S = \frac{1}{2}\sum_{i=1}^{N} m_i \ddot{\mathbf{q}}_i \cdot \ddot{\mathbf{q}}_i \tag{1.20}$$

S is also referred to as the *energy of the acceleration* or the *Gibbs function* [5]. Note that equation (1.20) is similar to the definition of the kinetic energy, except that accelerations are

used instead of velocities. Gibbs-Appell equations are represented as

$$\frac{\partial S}{\partial \ddot{\mathbf{q}}_i} = \mathbf{Q}_i \tag{1.21}$$

where $\mathbf{Q}_i$ are generalized forces corresponding to quasi coordinates [16]. By selecting quasi-velocities to be the same as generalized velocities, Lagrange's equations can be shown to be a special case of Gibbs-Appell equations [5].

Gibbs-Appell equations represent the more general case compared to Lagrange's equations and can handle both holonomic and nonholonomic systems. However, in certain cases, one may be more interested in using Lagrange's equations. From a physical perspective, velocities are easier to visualize than accelerations and kinetic and potential energies are more tangible concepts than the function S. Obtaining equations of motion using Gibbs-Appell method is more cumbersome since it requires the calculation of acceleration terms, as opposed to the velocity terms needed for Lagrange's equations [5].

Udwadia and Kalaba develop an extended form of Gibbs-Appell equation from Gauss' principle [17] and discuss the conceptual and practical differences of these two very similar principles.

1.2 Previous Works On Gauss' Principle and Motivation

Gauss introduced his principle of least constraints in an article in 1829 [18]. He began by stating that the d'Alembert principle reduces all of dynamics to statics and the principle of virtual works reduces all of statics to a mathematical problem. Hence these two principles cover all possible mechanical systems. He then suggested his own new principle which relies on d'Alembert and virtual works principles and reduces all mechanics, dynamics and statics to a single minimization problem [11]. Gauss' principle of least constraints is a minimum principle based on the statement that the actual motion of a kinematically constrained or unconstrained system is in a way that minimizes a parameter called *Gaussian* [19] while

satisfying all the constraints enforced [20].

For a system of N particles of mass m_i with position vector $\mathbf{r}_i = [x_i, y_i, z_i]^T$, true acceleration $\ddot{\mathbf{r}}_i$, unknown to be obtained, and external forces $\mathbf{F}_i \in \mathbb{R}^3$, the Gaussian is:

$$G = \sum_{i=1}^{N} \frac{1}{2m_i}(\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i)^T(\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i) \tag{1.22}$$

In some references G is referred to as *Gauss' constraint* [21].

The physical meaning of Gauss' principle is that the actual motion satisfying all the constraints of the system keeps as near as it is permitted to the unconstrained motion [22].

1.2.1 Discrete Systems

Udwadia and Kalaba have studied the Gauss' Principle of least constraint for discrete systems in several publications [23–27].

Defining $\mathbf{a}$ as a vector of the unconstrained accelerations, that is:

$$\mathbf{a}_i = \mathbf{F}_i/m_i, \quad i = 1, 2, \ldots, N \tag{1.23}$$

the Gaussian (1.22) can also be presented in a matrix form:

$$G = (\ddot{\mathbf{r}} - \mathbf{a})^T\mathbf{M}(\ddot{\mathbf{r}} - \mathbf{a}) \tag{1.24}$$

which has to be minimized respecting the constraints expressed in equation (1.3) for the constrained accelerations to be obtained. The mass matrix $\mathbf{M}$ is symmetric and positive definite. The initial conditions are assumed in a way that satisfy the constraints:

$$\phi_\ell(\mathbf{r}_0, t_0) = 0 \qquad \ell = 1, 2, \ldots, h \tag{1.25}$$

$$\phi_\ell(\mathbf{r}_0, \dot{\mathbf{r}}_0, t_0) = 0 \qquad \ell = h+1, h+2, \ldots, m \tag{1.26}$$

Gaussian in generalized coordinates is formulated the same way, that is:

$$G = (\ddot{\mathbf{q}} - \mathbf{a})^T \mathbf{M} (\ddot{\mathbf{q}} - \mathbf{a}) \tag{1.27}$$

$$\mathbf{A}(\mathbf{q}, \dot{\mathbf{q}}, t)\ddot{\mathbf{q}} = \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{1.28}$$

Note that in equation (1.27), $\mathbf{a}$ is the unconstrained generalized acceleration. It should be mentioned that the constraint equations (1.28) are linear on the acceleration.

It is evident that for an unconstrained system of particles, minimization of Gaussian simply reduces to the second law of Newton $m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i$.

An explicit solution of minimizing equation (1.27) under constraints (1.28), offered by Udwadia and Kalaba for an ideal system described by the mentioned constraints [19] is

$$\ddot{\mathbf{q}} = \mathbf{a} + \mathbf{M}^{-\frac{1}{2}} (\mathbf{A}\mathbf{M}^{-\frac{1}{2}})^{+} (\mathbf{b} - \mathbf{A}\mathbf{a}) \tag{1.29}$$

which consists of the unconstrained acceleration and a perturbation term. In equation (1.29), $^{+}$ denotes the Moore-Penrose inverse and is explained in Appendix 1. The employment of pseudo-inverse matrices was not known to previous investigators and makes this method innovative and unique. For a more general case where non-ideal constraints such as Coulombs friction are involved, the explicit solution for the acceleration is [11]:

$$\ddot{\mathbf{q}} = \mathbf{a} + \mathbf{M}^{-\frac{1}{2}} (\mathbf{A}\mathbf{M}^{-\frac{1}{2}})^{+} (\mathbf{b} - \mathbf{A}\mathbf{a}) + \mathbf{M}^{-\frac{1}{2}} [\mathbf{I} - (\mathbf{A}\mathbf{M}^{-\frac{1}{2}})] \mathbf{M}^{-\frac{1}{2}} \mathbf{c} \tag{1.30}$$

$\mathbf{c}$ is a known $3N$-vector and depends on the nature of the constraint (holonomic or nonholonomic) and is a specification of the constraint forces that do work in virtual displacements[5]. The amount of $\mathbf{c}$ for a particular system can be determined through experimentation or analogy with similar systems [12]. In Lagrangian systems, the nature of constraints is ignored and it is assumed that $\mathbf{c} = \mathbf{0}$ for all systems. The equation of motion given in (1.30) is the

[5]Example of a holonomic non-ideal constraint is a particle bound to move on a surface with friction. The equation of the surface restricting the motion is holonomic but the friction force makes this constraint non-ideal. $\mathbf{c}$ for this system is nonzero.

most general form of equation of motion of a constrained system and is uniquely defined[6].

If the mass matrix is in the form $\mathbf{M} = m\mathbf{I}$, equations (1.29) and (1.30) can be simplified by substituting the following equality [19]:

$$\mathbf{M}^{-\frac{1}{2}}(\mathbf{A}\mathbf{M}^{-\frac{1}{2}})^{+} = \mathbf{A}^{+} \tag{1.31}$$

An extended form of Gaussian for non-ideal discrete systems is [23]:

$$G_{\text{extended}} = (\ddot{\mathbf{q}} - \mathbf{a} - \mathbf{M}^{-1}\mathbf{c})^{T}\mathbf{M}(\ddot{\mathbf{q}} - \mathbf{a} - \mathbf{M}^{-1}\mathbf{c}) \tag{1.32}$$

Computational programs such as MATLAB or WOLFRAM MATHEMATICA have built-in functions for calculating Moore-Penrose inverses, which makes these explicit solutions suitable for numerical studies.

1.2.2 Rigid Bodies

Rigid bodies are investigated as a system of particles attached to each other. For a system of N particles of mass m_i subjected to forces $\mathbf{F}_i$, Gaussian is previously defined in equation (1.22).

For a system of N_b rigid bodies, equation (1.22) is rewritten as [28]

$$G = \sum_{j=1}^{N_b}\sum_{i=1}^{N}\frac{1}{2m_{ji}}(\mathbf{F}_{ji} - m_{ji}\ddot{\mathbf{u}}_{ji})^{T}(\mathbf{F}_{ji} - m_{ji}\ddot{\mathbf{u}}_{ji}) \tag{1.33}$$

where j runs over the number of rigid bodies in the system and i runs over the points of the j^{th} body. Since rigid bodies are investigated here, $\mathbf{u}$ consists of both linear and rotational displacements. m_{ji}, $\ddot{\mathbf{u}}_{ji}$ and $\mathbf{F}_{ji}$ are the mass, acceleration and force acting on the i^{th} point of the j^{th} body:

$$\mathbf{u}_{ji} = \mathbf{c}_j + \mathbf{r}_{ji}, \quad \dot{\mathbf{u}}_{ji} = \mathbf{v}_j + \boldsymbol{\omega}_j \times \mathbf{r}_{ji}, \quad \text{and} \quad \ddot{\mathbf{u}}_{ji} = \dot{\mathbf{v}}_j + \dot{\boldsymbol{\omega}}_j \times \mathbf{r}_{ji} \tag{1.34}$$

[6] Assuming the mass matrix $\mathbf{M}$ is non-singular

where c_j is the linear displacement of the center of mass of the j^{th} body. Let the net force and net torque, respectively, acting on each body be

$$\overline{\mathbf{F}}_j = \sum_{i=1}^{N} \mathbf{F}_{ji} \quad \text{and} \quad \boldsymbol{\tau}_j = \sum_{i=1}^{N} \mathbf{r}_{ji} \times \mathbf{f}_{ji} \tag{1.35}$$

Substituting equation (1.34) in (1.33) and the relations

$$\mathbf{M}_j = \sum_{i=1}^{N} m_{ji} \quad \text{and} \quad \sum m_{ji}\mathbf{r}_{ji} = 0 \tag{1.36}$$

the Gaussian for the system of rigid bodies would be:

$$G = \sum_{j=1}^{N_b}\sum_{i=1}^{N} \frac{\mathbf{F}_{ji}{}^T\mathbf{F}_{ji}}{2m_{ji}} - \sum_{j=1}^{N_b} \dot{\mathbf{v}}_j^T\overline{\mathbf{F}}_j - \sum_{j=1}^{N_b} \dot{\boldsymbol{\omega}}_j^T\boldsymbol{\tau}_j + \frac{1}{2}\sum_{j=1}^{N_b} \dot{\mathbf{v}}_j^T\mathbf{M}_j\dot{\mathbf{v}}_j + \frac{1}{2}\sum_{j=1}^{N_b} \dot{\boldsymbol{\omega}}_j^T\mathbf{I}_j\dot{\boldsymbol{\omega}}_j \tag{1.37}$$

The global mass matrix $\overline{\mathbf{M}}$ and acceleration $\overline{a}$ are defined to simplify equation (1.37) as:

$$\overline{\mathbf{M}} = \begin{bmatrix} \mathbf{M}_1 & 0 & 0 & \dots & 0 \\ 0 & \mathbf{I}_1 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \mathbf{M}_{N_b} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I}_{N_b} \end{bmatrix} \quad \text{and} \quad \overline{\mathbf{a}} = \begin{pmatrix} \dot{\mathbf{v}}_1 \\ \dot{\boldsymbol{\omega}}_1 \\ \vdots \\ \dot{\mathbf{v}}_{N_b} \\ \dot{\boldsymbol{\omega}}_{N_b} \end{pmatrix} \tag{1.38}$$

Assuming the net torque is zero, equation (1.37) is simplified to[7]

$$G = \sum_{j=1}^{N_b}\sum_{i=1}^{N} \frac{\mathbf{F}_{ji}{}^T\mathbf{F}_{ji}}{2m_{ji}} - \overline{\mathbf{a}}^T\mathbf{Q}_F + \frac{1}{2}\overline{\mathbf{a}}^T\overline{\mathbf{M}}\overline{\mathbf{a}} \tag{1.39}$$

where $\mathbf{Q}_F = [\overline{\mathbf{F}}_1\ \mathbf{0}\ \overline{\mathbf{F}}_2\ \mathbf{0}\ \cdots\ \overline{\mathbf{F}}_{N_b}\ \mathbf{0}]^T$.

[7] See [28] for detailed derivations.

1.3 Similar Principles To Gauss' Principle

In this section, two similar principles to Gauss' principle are briefly explained and their differences are highlighted.

1.3.1 Hertz's Principle of Least Curvature

Hertz's principle of least curvature or theorem of the most straight path is a special case of Gauss' principle postulated by Hertz in 1894 [29]. It states that the motion of an unforced discrete mechanical system with a mass matrix of equal generalized components subjected to constraints takes place in a way that the curvature of the path traveled is minimal compared to any other possible path. In other words, the spontaneous motion of a system of identical masses subjected to constraints goes along the straightest path. Hertz defines ds as the element of the arc in space and K as the curvature of the path traveled by the particle [30]:

$$ds^2 = \sum_{k=1}^{N} d\mathbf{r}_k^2 \quad \Longrightarrow \quad K = \sum_{k=1}^{N} \Big(\frac{d^2\mathbf{r}_k}{ds^2} \Big)^2 \tag{1.40}$$

Hertz's principle can be formulated as [31]:

$$\delta K = 0 \tag{1.41}$$

1.3.2 Jordain's Principle

Jourdain's principle is a variational principle established by P. Jourdain in 1908. Jourdain's principle is very similar to Gauss' principle, with the difference that it deals with constraints on the velocity level. Jourdain's principle states that "a constrained mechanical system performs motions such that the total virtual power of the constraint force and torque is zero" [32].

For a system of N particles of mass m_i with position vector $\mathbf{r}_i = [x_i, y_i, z_i]^T$, acceleration

of $\ddot{\mathbf{r}}_i$ and forces $\mathbf{F}_i \in \Re^3$, Jourdain's principle is expressed as:

$$\sum_{i=1}^{N}(\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i) \cdot \delta_J\dot{\mathbf{r}}_i = 0 \tag{1.42}$$

where δ_J is Jourdain's variation $\delta_J t = \delta_J \mathbf{r} = 0$. For comparison purposes, d'Alembert's and Gauss' principles respectively are shown in similar terms

$$\sum_{i=1}^{N}(\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i) \cdot \delta\mathbf{r}_i = 0, \quad \delta t = 0 \tag{1.43}$$

$$\sum_{i=1}^{N}(\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i) \cdot \delta_G\ddot{\mathbf{r}}_i = 0, \quad \delta_G t = \delta_G \mathbf{r} = \delta_G \dot{\mathbf{r}} = 0 \tag{1.44}$$

In [33], authors show the equivalence of d'Alembert's, Jourdain's and Gauss' principles for certain class of systems.

1.4 Outline

In this exploratory investigation, Gauss' principle of least constraints is first used to derive equations of motion of a single pendulum. The obtained equations are compared with Lagrange's equations of motion. A new approach based on Gauss' principle is explained and used to analyze vibrations of various externally periodically forced oscillators. In this method, a trial solution consisting of Fourier series with unknown coefficients is used as an approximation of the steady state response. Making use of the orthogonality of Fourier functions, the principle is adapted to use an integration of the Gaussian over a full period of excitation instead of the original form of Gaussian presented in section 1.2. Minimizing this adjusted Gaussian while respecting the constraints of the system specifies the suggested trial solution. This study mainly targets discrete systems of very small dimension. However, application of Gauss' principle to approximate the response of a continuous systems is illustrated through an example in section 2.3. The effectiveness of this method of approximation based on Gauss' principle is investigated for a Duffing oscillator, a mass-spring system with piecewise linear stiffness, and a unilaterally constrained mass-spring system.

Chapter 2

Methodology

In this chapter and the next ones, the dynamical behavior of various forced mechanical systems are investigated using Gauss' principle. The external force applied on each system is $F\cos(\Omega t)$ where F and Ω are specified. The solution of the initial value problem in the form of second order differential equations obtained from Lagrange's formulations is used as reference. The accuracy of the computational approaches based on Gauss' method is evaluated by comparing the results to those obtained from Lagrange's formulations.

Two different approaches based on Gauss' principle are studied. The first method is based on the works done by Kalaba and Udwadia in several publications [12, 19, 23, 34]. A closed form formulation and its solution introduced in equations (1.27) and (1.29) respectively is used. The second method based on Gauss' principle is a novel approach. In this approach, approximating the steady state response of the system is of interest. In the presence of an external forcing, the system of equations is nonhomogeneous; hence its solution is the sum of the solution of the homogeneous equation and a particular solution of the nonhomogeneous equation (in linear oscillators). The nonhomogeneous part describes the steady-state response which is the response as $t \longrightarrow \infty$ [35]. A slight linear damping is considered in all investigated systems to stabilize the response and cancel out the effect of the initial conditions.

An appropriate trial solution must be assumed. Generally, this trial solution can be a one

term function or a set of Fourier series. A periodic trial solution consisting of Fourier series with the same frequency of the applied force is assumed:

$$q(t) = A_1 + \sum_{i=1}^{N} A_{2i} \cos(i\Omega t) + \sum_{i=1}^{N} A_{2i+1} \sin(i\Omega t) \tag{2.1}$$

where i resembles a counter and must not be mistaken as the imaginary unit.

Due to nonlinearities and other complexities, the suggested trial solution in (2.1) may not be the only possible solution. However the target of this study is to find the best approximation of the Fourier trial solutions with the same frequency as the applied force. The Gaussian of the system is formed over one period of the motion substituting the trial solution as the actual displacement. For a system consists of a single mass m with the coordinate $q(t)$ approximated by (2.1), Gaussian is formed as

$$\begin{aligned} G(t) &= \frac{1}{2m}(F_{\text{sum}} - m\ddot{q}(t))^2 \\ &= \frac{1}{2m}\Big(F_{\text{sum}} - m\frac{d^2}{dt^2}(A_1 + \sum_{i=1}^{N} A_{2i} \cos(i\Omega t) + \sum_{i=1}^{N} A_{2i+1} \sin(i\Omega t))\Big)^2 \end{aligned} \tag{2.2}$$

According to Gauss' principle, the formed Gaussian has to be minimum for the true accelerations of the constrained system (refer to section 1.2). Based on this principle, minimization of the Gaussian results in approximated values for the unknown coefficients of the trial solution. The accuracy of the solution highly depends on suitability of the trial solution in describing the system's dynamics.

By targeting periodic solutions in form of (2.1), an integration of $G(t)$ over a full period of the excitation $T = 2\pi/\Omega$ can be used instead of the original form of Gaussian shown in (2.2) where time t is explicit. An integrated form of Gaussian will advantageously make use of the orthogonality of Fourier functions. The constant term $\frac{1}{2m}$ can be removed from the integration without effecting the minimization with respect to A_i. This integrated form of Gaussian is shown as G_{int} in this study:

$$G_{\text{int}} = \int_0^T \Big(F_{\text{sum}} - m\frac{d^2}{dt^2}(A_1 + \sum_{i=1}^{N} A_{2i} \cos(i\Omega t) + \sum_{i=1}^{N} A_{2i+1} \sin(i\Omega t))\Big)^2 \tag{2.3}$$

To have a different perceptive, G_{int} is formed by substituting the trial solution (2.1) in the equation of motion of the system $F_{\text{sum}} = m\ddot{q}(t)$. This concept is used directly to form the Gaussian.

2.1 Numerical Methods

In the remainder, numerical computations of any kind will be carried out using either MATLAB or WOLFRAM MATHEMATICA. The numerical methods used to obtain the solution of ordinary differential equations and numerical integrations are explained in this section. The optimization method used to minimize the Gaussian varies depending on the system and is discussed for each case individually.

2.1.1 Differential Equation Solving Method

The ordinary differential equations in the this study are solved using two methods depending on the stiffness of the system. There are several definitions of a stiff differential equation. A common feature among all definitions is that the step size for numerically solving these equations needs to be very small to maintain stability [36]. In systems with stiff differential equations, the components of the solutions vary on very different scales [37].

An ordinary differential equation of the form

$$\frac{dy}{dx} = g(x, y) \tag{2.4}$$

for $a \leq x \leq b$ is *stiff* in a neighborhood of a solution y, if there is a component of y whose variation is large compared to $(b - a)^{-1}$ [38].

In this study, nonstiff systems are solved with Adams-Bashforth predictor-corrector method, whereas stiff ones are solved with backward differentiation formula method. The stiffness of a system of equations is measured as the ratio of the highest eigenvalue to the smallest eigenvalue [39].

Predictor-corrector methods proceed by extrapolating a polynomial fit to the derivative from the previous points to the new point (the predictor step), then using this to interpolate the derivative (the corrector step) [40]. Assume a first-order ODE of the form

$$\frac{dx}{dt} = f(t, x) \tag{2.5}$$

and let the step interval be

$$h_n = t_{n+1} - t_n \tag{2.6}$$

where the subscript n denotes the step number. With the assumption that the step size is constant, the Adams-Bashforth formula of the first order is [41]

$$x_{n+1} = x_n + h\Big(\frac{3}{2}f(t_n, x_n) - \frac{1}{2}f(t_{n-1}, x_{n-1})\Big) \tag{2.7}$$

Two initial conditions on x are needed for (2.7). The equations in this study are initial value problems, meaning a pair of initial conditions of displacement and velocity are given for each ODE. Using Euler's method, the second displacement initial condition is obtained:

$$x_n = x_{n-1} + hf(t_{n-1}, x_{n-1}) \tag{2.8}$$

The backward differentiation formula (BDF) of order k approximates the solution $x(t)$ by the polynomial $P(t)$ that interpolates x_{n+1} and the previously computed approximations $x_n, x_{n-1}, \ldots$ The polynomial must satisfy the differential equation at t_{n+1}:

$$P(t_{n+1}) = f(t_{n+1}, P(t_{n+1})) = f(t_{n+1}, x_{n+1}) \tag{2.9}$$

A BDF of order 2 interpolates with a quadratic polynomial at the three points t_{n+1}, t_n and t_{n-1} and is written as

$$\Big(\frac{1+2r}{1+r}\Big)x_{n+1} - (1+r)x_n + \Big(\frac{r^2}{1+r}\Big)x_{n-1} = h_n f(t_{n+1}, x_{n+1}) \tag{2.10}$$

where $r = h_n/h_{n-1}$ and h_n is defined in equation (2.6). If a constant step is assumed,

equation (2.10) is simplified as [41]

$$\frac{3}{2}x_{n+1} - 2x_n + \frac{1}{2}x_{n-1} = hf(t_{n+1}, x_{n+1}) \tag{2.11}$$

which must later be solved in x_{n+1} with an appropriate technique depending on f.

2.1.2 Numerical Integration

An integration rule computes an estimate of an integral over a region $\int_a^b f(t)dt$, typically using a weighted sum $\sum w_i f(t_i)$. An integration rule samples the integrand at a set of points, called sampling points. Corresponding to each sampling point t_i, there is a weight number w_i.

In this study, integrations of G_{int} are performed numerically in WOLFRAM MATHEMATICA using *Gauss-Kronrod quadrature* rule with five Gaussian points and six Kronrod points. The Gauss-Kronrod rule is an adaptive *Gaussian quadrature* method for numerical integration. The Gaussian quadrature uses optimal sampling points to form the weighted sum, whereas the Kronrod extension adds new sampling points in between the Gaussian points and forms a higher-order rule that reuses the Gaussian rule integrand evaluations [42].

2.2 Gauss' Principle in a Single Pendulum

The system of interest consists of a point mass attached to a rigid arm. A periodic horizontal force of $F\cos(\Omega t)$ is applied on the mass. The system is shown in figure 2–1. The arm constrains the motion of the mass and lets it move on a circle with the radius of the length of the arm ℓ around the point O. The system has one degree of freedom. The equation of motion is obtained using both Gauss' and Lagrange's formulations. To investigate the effectiveness of Gauss' approach, two different set of coordinates (x, y) and (r, θ) are considered.

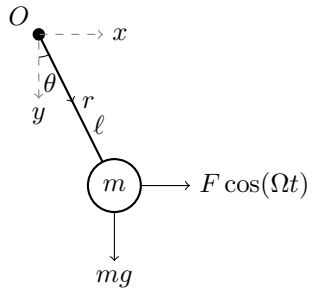


Figure 2–1: Forced single pendulum system

2.2.1 Cartesian Coordinates

Equations of motion of the system in Cartesian coordinates are studied first. The coordinates x and y are used to describe the motion of the system, that is $\mathbf{q} = [x \quad y]^T$.

Gauss' Approach The constraint equation at the configuration level is:

$$x^2 + y^2 = \ell^2 \tag{2.12}$$

Differentiating (2.12) twice with respect to time gives matrix $\mathbf{A}$ and vector $\mathbf{b}$ in (1.3):

$$2x\dot{x} + 2y\dot{y} = 0 \tag{2.13}$$

$$x\ddot{x} + y\ddot{y} = -\dot{x}^2 - \dot{y}^2 \tag{2.14}$$

which can be recast in the matrix form

$$[x \quad y] \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = -\dot{x}^2 - \dot{y}^2 \tag{2.15}$$

that is

$$\mathbf{A} = [x \quad y], \quad \ddot{\mathbf{q}} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix}, \quad b = -\dot{x}^2 - \dot{y}^2 \tag{2.16}$$

The applied force enters the equations through the unconstrained equation of motion of the system:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} F\cos(\Omega t) \\ -mg \end{bmatrix} \tag{2.17}$$

where $\mathbf{a} = [a_1 \ a_2]^T$ stores the free accelerations in the x and y directions. Following the same procedure shown for an unforced single pendulum, the accelerations are obtained. Based on Gauss' Principle, G defined as:

$$G = \left(\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} - \begin{bmatrix} \frac{F\cos(\Omega t)}{m} \\ -g \end{bmatrix} \right)^T \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \left(\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} - \begin{bmatrix} \frac{F\cos(\Omega t)}{m} \\ -g \end{bmatrix} \right) \tag{2.18}$$

must reach a minimum when $\ddot{\mathbf{q}} = [\ddot{x} \ \ddot{y}]^T$ is the actual acceleration of the system. Using equation (1.29), the solution to this minimization is:

$$\ddot{\mathbf{q}} = \begin{bmatrix} \frac{F\cos(\Omega t)}{m} \\ -g \end{bmatrix} - \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}^{-\frac{1}{2}} \left(\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}^{-\frac{1}{2}} \right)^+ \left(\dot{x}^2 + \dot{y}^2 + \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \frac{F\cos(\Omega t)}{m} \\ -g \end{bmatrix} \right) \tag{2.19}$$

Since the mass matrix is a factor of the identity matrix, the simplification offered in equation (1.31) is eligible, that is

$$\ddot{\mathbf{q}} = \begin{bmatrix} \frac{F\cos(\Omega t)}{m} \\ -g \end{bmatrix} + \begin{bmatrix} x & y \end{bmatrix}^+ \left(-\dot{x}^2 - \dot{y}^2 - \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \frac{F\cos(\Omega t)}{m} \\ -g \end{bmatrix} \right) \tag{2.20}$$

Using Appendix 1, MP-inverse of the row vector $\mathbf{A}$ is obtained as

$$\begin{bmatrix} x & y \end{bmatrix}^+ = \frac{1}{x^2 + y^2} \begin{bmatrix} x \\ y \end{bmatrix} \tag{2.21}$$

the final solution is

$$\ddot{\mathbf{q}} = \begin{bmatrix} \frac{F\cos(\Omega t)}{m} \\ -g \end{bmatrix} + \frac{-\dot{x}^2 - \dot{y}^2 - x\frac{F\cos(\Omega t)}{m} + yg}{x^2 + y^2} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} \tag{2.22}$$

which can be rewritten as

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \frac{F\cos(\Omega t)}{m} + \frac{x(-\dot{x}^2-\dot{y}^2-x\frac{F\cos(\Omega t)}{m}+yg)}{\ell^2} \\ -g + \frac{y(-\dot{x}^2-\dot{y}^2-x\frac{F\cos(\Omega t)}{m}+yg)}{\ell^2} \end{bmatrix} = \begin{bmatrix} \frac{-x\dot{x}^2-x\dot{y}^2+y^2\frac{F\cos(\Omega t)}{m}+xyg}{\ell^2} \\ \frac{-y\dot{x}^2-y\dot{y}^2-xy\frac{F\cos(\Omega t)}{m}-gy^2}{\ell^2} \end{bmatrix} \tag{2.23}$$

Equation (2.23) provides the equations of motion of a single pendulum driven by a horizontal periodic force. Knowing the initial displacements and velocities $(x(0), y(0), \dot{x}(0), \dot{y}(0))$, the state of the system at time unfolds.

Lagrange's Approach In this section, the motion of a single pendulum under periodic force is investigated using Lagrange's method and the obtained equations of motion are compared with the ones previously obtained from Gauss' approach in (2.23). The Lagrangian is obtained from kinetic and potential energies:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad \text{Kinetic Energy} \tag{2.24}$$

$$U = mgy \qquad \text{Potential Energy} \tag{2.25}$$

$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy \qquad \text{Lagrangian} \tag{2.26}$$

A periodic non-conservative force is being applied on the mass.

$$Q_{\text{app,nc}} = \delta w = F\cos(\Omega t)\delta x \tag{2.27}$$

Since the system involves one constraint generating one constraint force, one multiplier is considered in Lagrange's equations. The constraint force substituted in the equations of motion is

$$\mathbf{Q}_{\text{cnst}} = \mathbf{A}^T\lambda = \begin{bmatrix} x \\ y \end{bmatrix}\lambda \tag{2.28}$$

Substituting the obtained forces in equation (1.15) gives

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} = \begin{bmatrix} F\cos(\Omega t) \\ 0 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}\lambda \tag{2.29}$$

Equation (2.29) is solved for $q_1 = x$ and $q_2 = y$ respectively, that is:

$$m\ddot{x} = x\lambda \tag{2.30}$$

$$m\ddot{y} + mg = y\lambda \tag{2.31}$$

Solving equations (2.14), (2.30), and (2.31) yields

$$\lambda = m\Big(\frac{-\dot{x}^2 - \dot{y}^2 - \frac{xF\cos(\Omega t)}{m} + gy}{\ell^2}\Big) \tag{2.32}$$

and

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} xm\lambda + \frac{F\cos(\Omega t)}{m} \\ \frac{y\lambda}{m} - g \end{bmatrix} = \begin{bmatrix} \frac{-x\dot{x}^2 - x\dot{y}^2 + y^2\frac{F\cos(\Omega t)}{m} + xyg}{\ell^2} \\ \frac{-y\dot{x}^2 - y\dot{y}^2 - xy\frac{F\cos(\Omega t)}{m} - gy^2}{\ell^2} \end{bmatrix} \tag{2.33}$$

Evidently, the equations of motion (2.33) obtained from Lagrange's formulation are identical to (2.23) obtained from Gauss' method.

2.2.2 Cylindrical Coordinates

The same forced system is studied considering cylindrical coordinates of r and θ. These coordinates are not dependent on each other, but one may treat the constant length of the arm as a constraint on the system.

$$r = \ell \quad \Rightarrow \quad \dot{r} = 0 \quad \Rightarrow \quad \ddot{r} = 0 \tag{2.34}$$

Gauss' Approach Considering $\mathbf{q} = [\theta\ r]^T$, the constraint matrix $\mathbf{A}$ is

$$\begin{bmatrix} 0 & 1 \end{bmatrix}^+ \begin{bmatrix} \ddot{\theta} \\ \ddot{r} \end{bmatrix} = 0 \tag{2.35}$$

Using Appendix 1, the MP-inverse of $\mathbf{A}$ is

$$\mathbf{A}^+ = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.36}$$

Forces in r and θ direction are

$$F_r = mg\cos(\theta) + F\cos(\Omega t)\sin(\theta) = m\ddot{a}_r \tag{2.37}$$

$$F_\theta = F\cos(\Omega t)\cos(\theta) - mg\sin(\theta) = m\ell\ddot{a}_\theta \tag{2.38}$$

Hence, the free motion acceleration is

$$\mathbf{a} = \begin{bmatrix} \frac{F\cos(\Omega t)\cos(\theta)}{\ell m} - \frac{g\sin(\theta)}{\ell} \\ g\cos(\theta) + \frac{F\cos(\Omega t)\sin(\theta)}{m} \end{bmatrix} \tag{2.39}$$

Using (1.31) and substituting in equation (1.29) gives

$$\ddot{\mathbf{q}} = \mathbf{a} + \mathbf{A}^+(\mathbf{b} - \mathbf{A}\mathbf{a}) = \mathbf{a} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}(0 - \ddot{a}_r) = \begin{bmatrix} \frac{F\cos(\Omega t)\cos(\theta)}{\ell m} - \frac{g\sin(\theta)}{\ell} \\ 0 \end{bmatrix} \tag{2.40}$$

Equation (2.40) expresses equations of motion of a single forced pendulum in cylindrical coordinates obtained using Gauss' method.

Lagrange's Approach The kinetic and potential energy and Lagrangian of the system need to be expressed in the cylindrical coordinates:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\ell^2\dot{\theta}^2 \qquad \text{Kinetic Energy} \tag{2.41}$$

$$U = mgy = -mg\ell\cos(\theta) \qquad \text{Potential Energy} \tag{2.42}$$

$$L = T - U = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos(\theta) \qquad \text{Lagrangian} \tag{2.43}$$

The periodic force of $F\cos(\Omega t)$ in the x direction is recast in the cylindrical coordinate system: $x = r\sin\theta$ and $\delta r = 0$ implies $\delta x = r\cos\theta\delta\theta = \ell\cos\theta\delta\theta$. The virtual work associated to the forcing term becomes then $Q_{\text{app,nc}}\delta w = F\cos(\Omega t)\delta x = F\ell\cos(\Omega t)\cos\theta\delta\theta$. Substituting the obtained forces in equation (1.15) and assuming $q_1 = \theta$ and $q_2 = r$ gives

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{r} \end{bmatrix} = \begin{bmatrix} \frac{F\cos(\Omega t)\cos(\theta)}{\ell m} - \frac{g\sin(\theta)}{\ell} \\ 0 \end{bmatrix} \tag{2.44}$$

which is identical to the equations of motion in cylindrical coordinates obtained using Gauss' method in (2.40).

2.2.3 Approximate Periodic Solution Based on Gauss' Principle

In the previous sections, it was shown how to obtain the governing equations of motion using Gauss' principle of least constraints and a method offered in [19]. In this section, the motion of a forced single pendulum system, displayed in Figure 2–2, is numerically studied under periodic tangential force with linear damping.

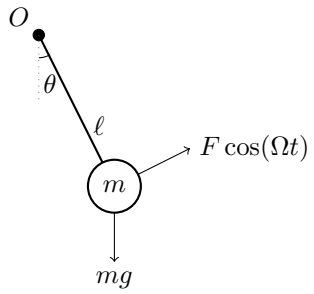


Figure 2–2: Forced single pendulum system

On one hand, the assumed trial solution is of the form of (2.1). On the other hand, the Lagrange's ordinary differential equations of motion are solved using Adams-Bashforth predictor-corrector method in WOLFRAM MATHEMATICA. Then the concept of Gauss' principle is used to approximate the solution of the equation of motion obtained using Lagrange's formulation. The results of this approximation are compared with solving the same equation directly in WOLFRAM MATHEMATICA[1]. The numerical values used in the analysis are shown in table 2–1.

m [kg]	ℓ [m]	F [N]	Ω [rad s^{-1}]	c [N s m^{-1}]	$\theta(0)$ [rad]	$\dot{\theta}(0)$ [rad s^{-1}]	g [m s^{-2}]
1	0.5	3	1	0.1	0	0	9.8

Table 2–1: Forced single pendulum parameters

[1]Using "NDSolve" command

Equation of Motion and Reference Solution The independent equation of motion is obtained using Lagrange's method, assuming $q = \theta$. The kinetic and potential energy have the same values as in section 2.2.2. The damping force is treated as a non conservative applied force:

$$Q_{\text{app,nc}} = F\cos(\Omega t)\delta\theta - c\dot{\theta}\delta\theta \tag{2.45}$$

Substituting the forces and Lagrangian in equation (1.15), the equation of motion is

$$ml^2\ddot{\theta} + mg\ell\sin\theta = F\cos(\Omega t) - c\dot{\theta} \tag{2.46}$$

This ordinary differential equation is solved using the methods explained in section 2.1.1 in WOLFRAM MATHEMATICA and the results for θ are plotted in figure 2–3.

Approximate Periodic Solution The approximation is started by assuming $N = 1$ in equation (2.1):

$$\theta(t) = A_1 + A_2\cos(\Omega t) + A_3\sin(\Omega t) \tag{2.47}$$

and G_{int} is formed as

$$\begin{aligned} G_{\text{int}} &= \int_0^{\frac{2\pi}{\Omega}} (m\ell^2\ddot{\theta} + mg\ell\sin(\theta) + c\dot{\theta} - F\cos(\Omega t))^2 dt \\ &= \int_0^{\frac{2\pi}{\Omega}} \Big[m\ell^2(-A_1\Omega^2\cos(\Omega t) - A_2\Omega^2\sin(\Omega t)) + mg\ell\sin(A_1\cos(\Omega t) + A_2\sin(\Omega t)) + \\ &\qquad c(-A_1\Omega\sin(\Omega t) + A_2\Omega\cos(\Omega t)) - F\cos(\Omega t)\Big]^2 dt \end{aligned} \tag{2.48}$$

Integration is performed in WOLFRAM MATHEMATICA using methods explained in section 2.1.2 and G_{int} is minimized to obtain A_1, A_2 and A_3. Solutions from both approaches are plotted in figure 2–3.

As seen from figure 2–3, a basic trial solution with a single harmonic can describe the steady state of the system approximately. Increasing the number of harmonics in the trial solution results in a smaller minimum of G_{int}, and hence a more accurate approximation.

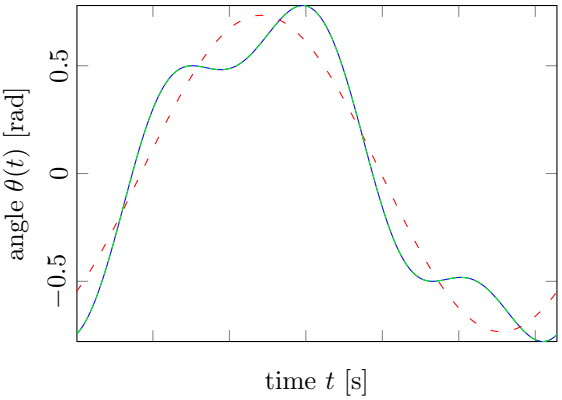


Figure 2–3: Comparison of numerical solution and Gauss' approximation: Reference sol. (——), Gauss' sol. $N = 1$ (- - -), $N = 7$ (----)

2.3 Gauss' Principle for a Uniform Bar

A uniform bar is considered as an example of a continuous system. Bars deform longitudinally along the longitudinal axis. In this section, longitudinal vibrations of a bar is investigated using Gauss' method of approximation. For a bar of length ℓ and cross section

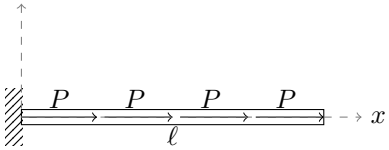


Figure 2–4: Schematic of a clamped-free forced bar

A (shown in figure 2–4) under longitudinal force of P, general equation of motion is [43]

$$EA\frac{\partial^2 u(x,t)}{\partial x^2} + P = \rho A\frac{\partial^2 u(x,t)}{\partial t^2} \tag{2.49}$$

where $u(x,t)$ is the displacement in the x direction and at time t. A periodic force of $P = F\cos(\Omega t)$ is distributed along the length of the bar. The cross section of the bar A, mass per unit volume of the material ρ and Young's modulus E are assumed to be constant. Adding a simple damper c to the system, equation (2.49) becomes:

$$EA\frac{\partial^2 u(x,t)}{\partial x^2} - \rho A\frac{\partial^2 u(x,t)}{\partial t^2} + c\frac{\partial u(x,t)}{\partial t} + F\cos(\Omega t) = 0 \tag{2.50}$$

The chosen clamped-free configuration of the bar is such that the boundary conditions are [43]

$$u(0,t) = 0, \quad \frac{\partial}{\partial x}u(\ell,t) = 0 \tag{2.51}$$

Condition $\frac{\partial}{\partial x}u(\ell,t) = 0$ is deduced from the fact that there is no effective force and hence no tension at the free end point.

The trial solution assumed in approximating the solution consists of both displacement and time dependent functions:

$$u(x,t) = \sin(\alpha x)[B_1 \sin(\Omega t) + B_2 \cos(\Omega t)] \tag{2.52}$$

α is determined in a way that respects the boundary conditions in equation (2.51)

$$\frac{\partial}{\partial x}u(\ell,t) = 0 \quad \Rightarrow \quad \cos(\alpha\ell) = 0 \quad \Rightarrow \quad \alpha = \frac{2n-1}{2}\pi, \quad n \in \mathbb{N} \tag{2.53}$$

where $n = 1$ is chosen. It should be noted that $u(0,t) = 0$ is already satisfied.

Since this is a continuous system varying in space (along the bar) and time, G_{int} is formed by integrating the equation of motion in both space and time [20]:

$$G_{\text{int}} = \int_0^T \int_0^\ell \Big(EA\frac{\partial^2 u(x,t)}{\partial x^2} - \rho A\frac{\partial^2 u(x,t)}{\partial t^2} + c\frac{\partial u(x,t)}{\partial t} + F\cos(\Omega t) \Big)^2 dxdt \tag{2.54}$$

The trial solution presented in equation (2.52) is plugged in (2.54) and global minimization is performed to obtain B_1 and B_2. The assumed values for numerical analysis are shown in table 2–2.

ℓ [m]	ρ [kg m^{-3}]	A [m^2]	E [N m^{-2}]	F [N]	Ω [rad s^{-1}]	c [N s m^{-1}]
1000	4	10^{-4}	2×10^9	100	10	1

Table 2–2: Numerical values for a clamped-free bar

Nelder-Mead method of optimization is used to minimize G_{int} of this system. This method uses the concept of a *simplex* [44]. For example, in $\mathbb{R}^2$, a simplex is a *triangle*, and in $\mathbb{R}^3$, a simplex is a *tetrahedron*.

The optimization is performed to obtain two unknown variables in (2.54), hence the

simplex is a triangle. The method is a pattern search that compares the function values at three vertices of the triangle. The worst vertex (which gives the highest amount of G_{int}) is replaced with a new vertex at each step and the search is continued with the new triangle. The process generates a sequence of triangles for which the value of G_{int} decreases. The size of the triangles is reduced to locate the coordinates of the minimum point [45].

The partial differential equation (2.50) is discretized in space by using the *method of lines* in WOLFRAM MATHEMATICA. Discretization is performed only on the spatial derivatives and the time derivatives are left continuous. This leads to a system of ordinary differential equations which can be solved by numerical methods for initial value ordinary equations explained in section 2.1.1.

The displacement response at the free end of the bar from the Gauss' approximation and numerical approach are plotted in figure 2–5. It is observed that even though only trigonometric functions of the first harmonic are assumed, the Gauss' approach gives an accurate approximation. The assumed trial solution (2.52) gives a satisfactory approximation of the displacement field. We should keep in mind and this trial solution might not describe all possible forms of excitation P and boundary conditions. Experience is needed to select the appropriate trial solution.

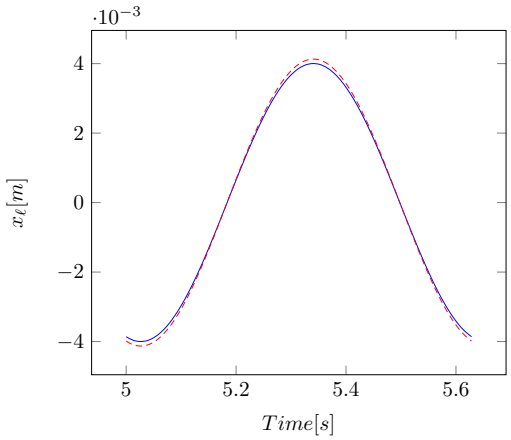


Figure 2–5: Displacement response at the free end: Reference sol. (——), Gauss' sol (----)

Chapter 3

Duffing Oscillator

The *Duffing* oscillator is a prototype system in nonlinear dynamics. As pictured in figure 3–1, it represents a forced single degree of freedom damped oscillator with a cubic stiffness which makes the governing equation nonlinear. It is named after Georg Duffing, who

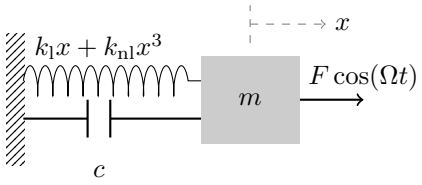


Figure 3–1: Schematic of a Duffing oscillator

studied responses of linear and nonlinear oscillatory systems [46]. The Duffing equation has been successfully used to model a variety of physical systems which include stiffening springs, beam buckling and nonlinear electronic circuits [47]. Although most physical systems cannot be described accurately in this way for a wide range of operating conditions, in many cases it is possible to use this equation as an approximate description so that their behavior can be studied qualitatively. In some situations, quantitative analysis can be conducted for small amplitudes of excitation [35].

3.1 Equation of Motion

The Duffing equation is in the form of:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k_{\mathrm{l}}}{m}x + \frac{k_{\mathrm{nl}}}{m}x^3 = \frac{F}{m}\cos(\Omega t) \tag{3.1}$$

Equation (3.1) describes the motion of a driven nonlinear spring mass system with viscous damping. k_{l}, k_{nl}, m, c and F denote the linear stiffness, nonlinear stiffness, mass, damping factor and magnitude of the external force respectively. For positive damping ($c > 0$), the free oscillation term decays with time and the steady state response consists of only the particular solution of equation (3.1). The steady state has the same frequency as the excitation with a shift in phase [48].

The Duffing oscillator exhibits multi valued responses and chaotic dynamical behavior for certain given conditions. Hysteresis, also called jump phenomenon, is a nonlinear behavior observed in Duffing oscillator and can be induced by either varying the amplitude or the frequency of the excitation [49]. In other words, for a given frequency or amplitude of excitation, the system exhibits more than one possible stable state.

Traditionally a 2-dimensional response curve displaying the root mean square of the amplitude of response as a function of either the amplitude or the frequency of the excitation, has been used to study hysteresis [49]. The same curve is used in this study to investigate peculiar dynamical behavior of nonlinear systems. The frequency-response curve of a Duffing oscillator shows complex dynamical behavior and has been the subject of many studies [49,50]. The responses of a Duffing's oscillator depend on the initial conditions and history of the oscillator, hence there is no exact analytical solution for such systems [49]. In figure 3–2, the frequency-response curve for a Duffing oscillator with positive linear and nonlinear stiffness is shown. It can be observed how the bending of the frequency-response curve leads to multivalued amplitudes and hence a jump phenomenon.

In this chapter, Gauss' method of least constrains is used to find the periodic steady state behavior of a Duffing oscillator with linear damping. The obtained solutions are compared

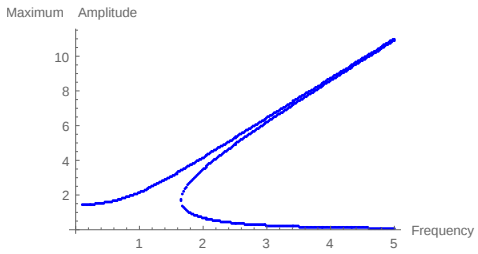


Figure 3–2: Frequency-reference curve in a Duffing oscillator

with the results from directly solving the Duffing equation through numerical methods explained in section 2.1.1.

3.2 Displacement Response

The Gauss' trial solution is in the form of equation (2.1). Substituting the trial solution in (3.1), G_{int} is formed as

$$G_{\text{int}} = \int_0^T \left(\ddot{x} + \frac{c}{m}\dot{x} + \frac{k_{\text{l}}}{m}x + \frac{k_{\text{nl}}}{m}x^3 - \frac{F}{m}\cos(\Omega t) \right)^2 dt \tag{3.2}$$

Computations show that the constant term and even harmonics in the trial solution do not participate in the solution of the Duffing oscillator. This is induced by the cubic nonlinearity in the system and the external excitation. Thus without losing accuracy, the trial solution can be simplified as

$$x(t) = \sum_{i=1}^{N} A_{2i-1}\cos((2i-1)\Omega t) + A_{2i}\sin((2i-1)\Omega t) \tag{3.3}$$

to include only the terms associated to odd harmonics.

It should be noted that N in equation (3.3) is the number of pairs of harmonics used in the approximation.

Local minimization using *Newton's method* is performed on G_{int} to find the unknown coefficients A_i. In one dimension, minimizing $f(x)$ is equivalent to solving $f'(x) = 0$ and checking $f''(x) > 0$. Newton's method uses a guess of x_k as the solution of the optimization.

Linearized f' around x_0 is formed and solved for the point where the linear function vanishes:

$$f'(x_0) + f''(x_0)(x - x_0) = 0 \tag{3.4}$$

This point is used as the next guess for x_1 and the procedure is repeated to obtain the extrema of f [51]. Let x_k be the current guess, the next guess is given by

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)} \tag{3.5}$$

For the data shown in table 3–1, the results of Gauss' approximation along with the solution obtained from direct numerical methods (sections 2.1.1 and 2.1.2) are plotted in figure 3–3. The effectiveness of the method is assessed by comparing the results during one steady cycle. The optimization is started at the origin.

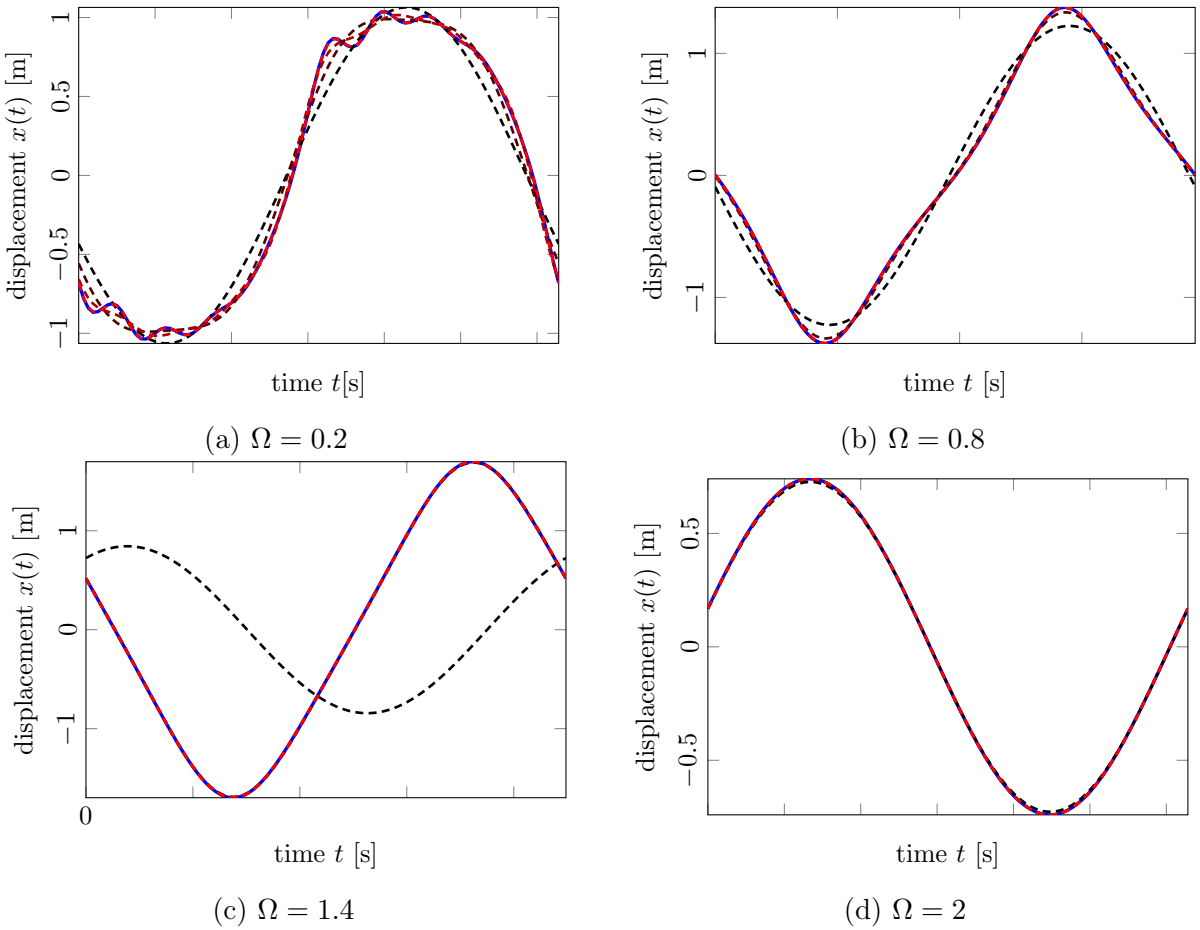
m [kg]	F [N]	k_{l} [N m^{-1}]	k_{nl} [N m^{-1}]	c [N s m^{-1}]
1	2	1	1	0.4

Table 3–1: Numerical values for a Duffing oscillator

For $\Omega = 1.4$, the results show significant discrepancy for $N = 1$. This results from the fact that G_{int} has multiple local minima and the optimization finds a different minimum for Gaussian at $N = 1$. In figure 3–4, the natural logarithm of G_{int} is plotted versus A_1 and A_2 for $N = 1$ at $\Omega = 1.4$ to show existence of multiple minima[1]. Changing the initial guess from $(0, 0)$ to $(2, 2)$ leads to another minimum for G_{int} and results in a curve consistent with the results from the numerical approach.

Figure 3–5 shows three dimensional plots of $\ln(G_{\text{int}})$. For higher Ω, the global minimum is also the locally dominant minimum around the origin, which indicates less dependency of the response on the initial guesses as long as the guesses are in the vicinity of the origin. Selecting initial points far from the origin for these frequencies results in obtaining other local minima and hence responses that do not accord with numerical results.

[1]It is common in science and engineering to use natural logarithm of a set of data to observe growth or decay [52].



(a) $\Omega = 0.2$ (b) $\Omega = 0.8$

(c) $\Omega = 1.4$ (d) $\Omega = 2$

Figure 3–3: Displacement comparison: Reference sol. (——), Gauss' sol. $N = 1$ (----), $N = 2$ (----), $N = 4$ (----), $N = 6$ (----)

3.3 Frequency-Response Curve

The *frequency-response* is defined as the root mean square of the data. The root mean square of the trial solution defined in equation (2.1) is

$$H_{\text{rms,G}} = \sqrt{A_1^2 + \frac{\sum_{i=1}^{N}(A_{2i}^2 + A_{2i+1}^2)}{2}} \tag{3.6}$$

Let n_p be the number of data points in the numerical solution. The root mean square of the reference curve is

$$H_{\text{rms,N}} = \sqrt{\frac{1}{n_p}\sum_{i=1}^{n_p} x_i^2} \tag{3.7}$$

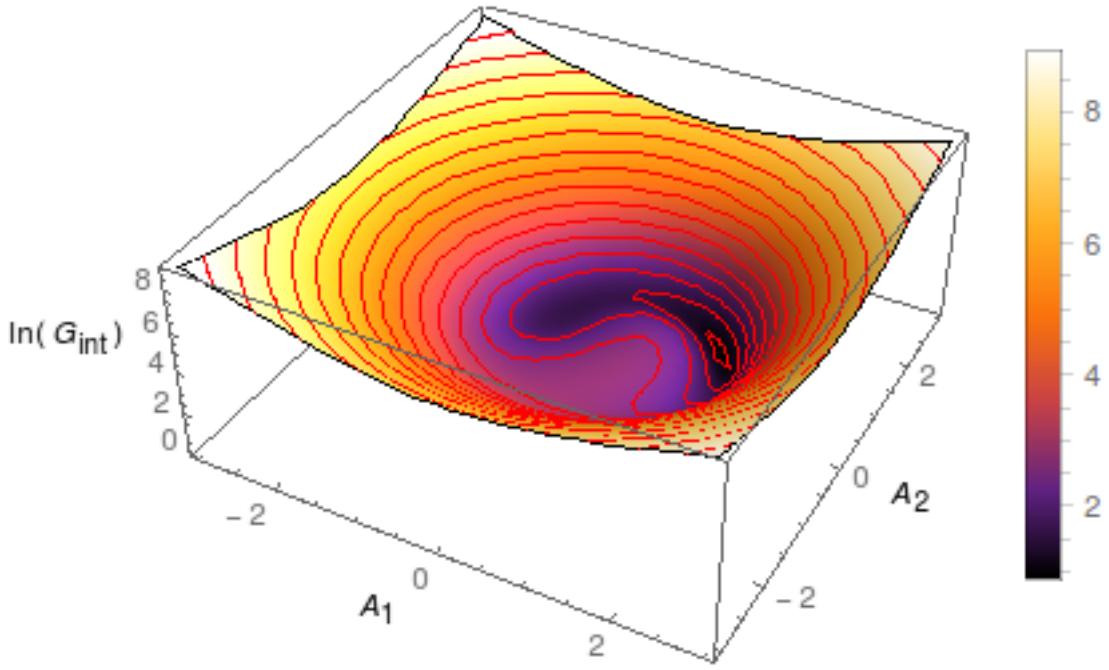


Figure 3–4: Residue of Gaussian G_{int} at $\Omega = 1.4$

To take into account the effect of history of the Duffing oscillator, a frequency marching technique is used in obtaining the reference frequency-response curve. In this approach the initial conditions are updated during each step to match the final condition of the previous step. The initial conditions of the first step are considered as $x(t_0) = 0$ and $\dot{x}(t_0) = 0$. The frequency marching is performed once in the forward direction from $\Omega = 0.1$ to 5 with the frequency step of 0.025[2], and once in the backward direction from $\Omega = 5$ to 0.1 considering the same step. The results are plotted in figure 3–6.

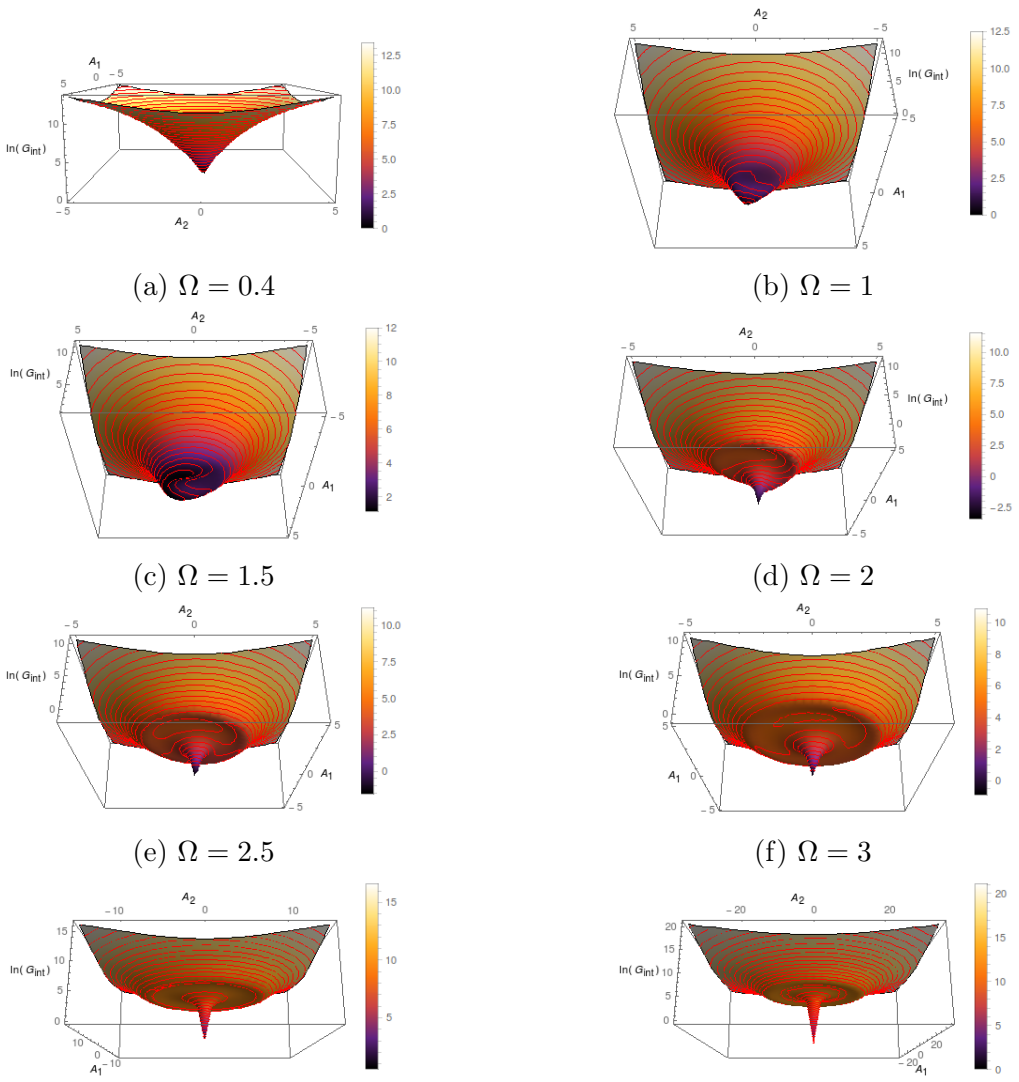
The assumed numerical values are displayed in table 3–2. To investigate the effectiveness of Gauss' method in slight nonlinearity, k_{nl} is kept small compared to k_{l}. The forward and

m [kg]	F [N]	k_{l} [N m^{-1}]	k_{nl} [N m^{-1}]	Ω[rad s^{-1}]	c [N s m^{-1}]
1	2	10	0.1	[0.1,5]	0.1

Table 3–2: Numerical values for a Duffing oscillator

backward curves do not exhibit discrepancies. However, to obtain more frequency response

[2]meaning the initial condition is updated at each step to match the previous state at $\Omega_{\text{current}}+0.025$



(a) $\Omega = 0.4$ (b) $\Omega = 1$

(c) $\Omega = 1.5$ (d) $\Omega = 2$

(e) $\Omega = 2.5$ (f) $\Omega = 3$

(g) $\Omega = 8$ (h) $\Omega = 15$

Figure 3–5: Residue of Gaussian G_{int} at $\Omega = 1.4$

points, both approaches are incorporated in obtaining the reference curve. The used numerical method shows incapability in obtaining the unstable part of the frequency-response.

To obtain the frequency-response curve through Gauss' approach, H_{rms} is defined as (3.6). The minimization is performed locally using Newton's method explained in section 3.2. To increase the possibility of finding multiple responses, a set of initial guesses consisting of integers from -2 to 2 are considered in the optimization. For example, the checked initial

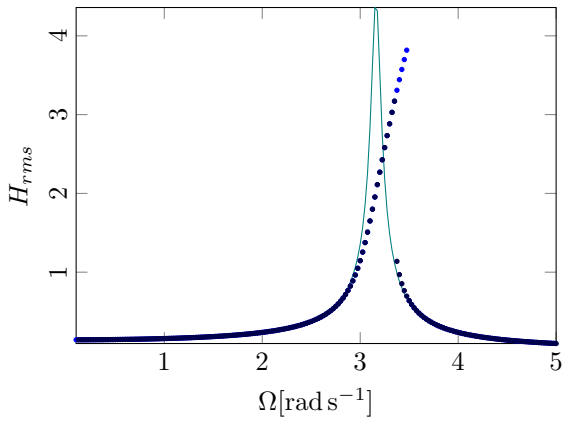


Figure 3–6: Reference frequency-response curve: Linear system (——), Forward marching (·····), Backward marching (·····)

guesses for $N = 1$ are

$$(-2,-2),(-2,-1),(-2,0),(-2,1),(-2,2),(-1,-2),(-1,-1),(-1,0),(-1,1),$$
$$(-1,2),(0,-2),(0,-1),(0,0),(0,1),(0,2),(1,-2),(1,-1),(1,0),(1,1),(1,2),$$
$$(2,-2),(2,-1),(2,0),(2,1),(2,2)$$

In figure 3–7, Gauss' frequency-response of $N = 1$ is plotted along with the reference curve. It is observed that results from the Gauss' approach and the numerical method are a perfect match in the lower branches of the Duffing curve. Gauss' approach has the advantage of predicting parts of the unstable frequency-response curve that the numerical method does not seem to be able to produce. The blank parts of the Gauss' frequency-response curve may be obtained by considering more initial guesses around the primary resonance frequency. This theory has not been verified in this study due to computational costs.

3.3.1 Sensitivity to the Number of Harmonics

The number of terms assumed in the trial solution of Gauss' approximation are increased to $N = 3$ and results are plotted along with the reference curve in figure 3–8. Due to computational costs, only the even integers from -2 to 2 are selected as the initial guesses for $N = 2$ and $N = 3$. The curves corresponding to the Gauss' method show an extra branch

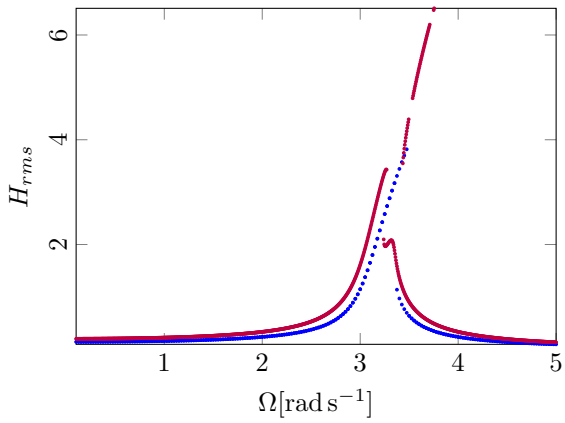


Figure 3–7: Comparison of the frequency-response curves: Reference (·····), Gauss' (·····)

around $\Omega = 1$ for higher number of terms. This branch is associated to the super harmonic resonance frequency at Ω equal to 1/3 of the primary resonance [35]. The Gauss' method of approximation is capable of finding this branch because of assuming multiple initial guesses and performing local minimization. The system being nonlinear, the steady state response depends on the initial conditions. Assuming $x(t_0) = 0$ and $\dot{x}(t_0) = 0$ to obtain the reference plot does not lead to find super harmonic resonance in the frequency-response curve. Trying different initial conditions may show the jump around the super harmonic resonance.

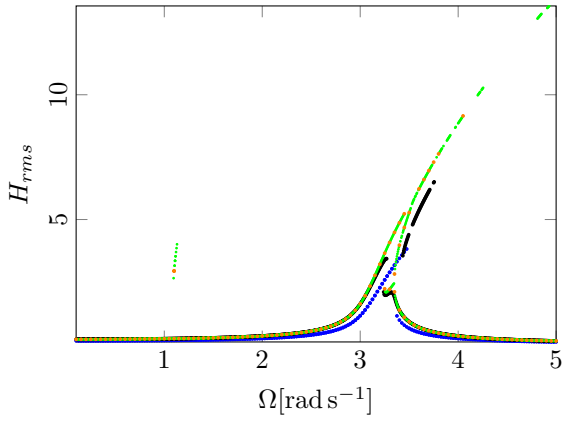
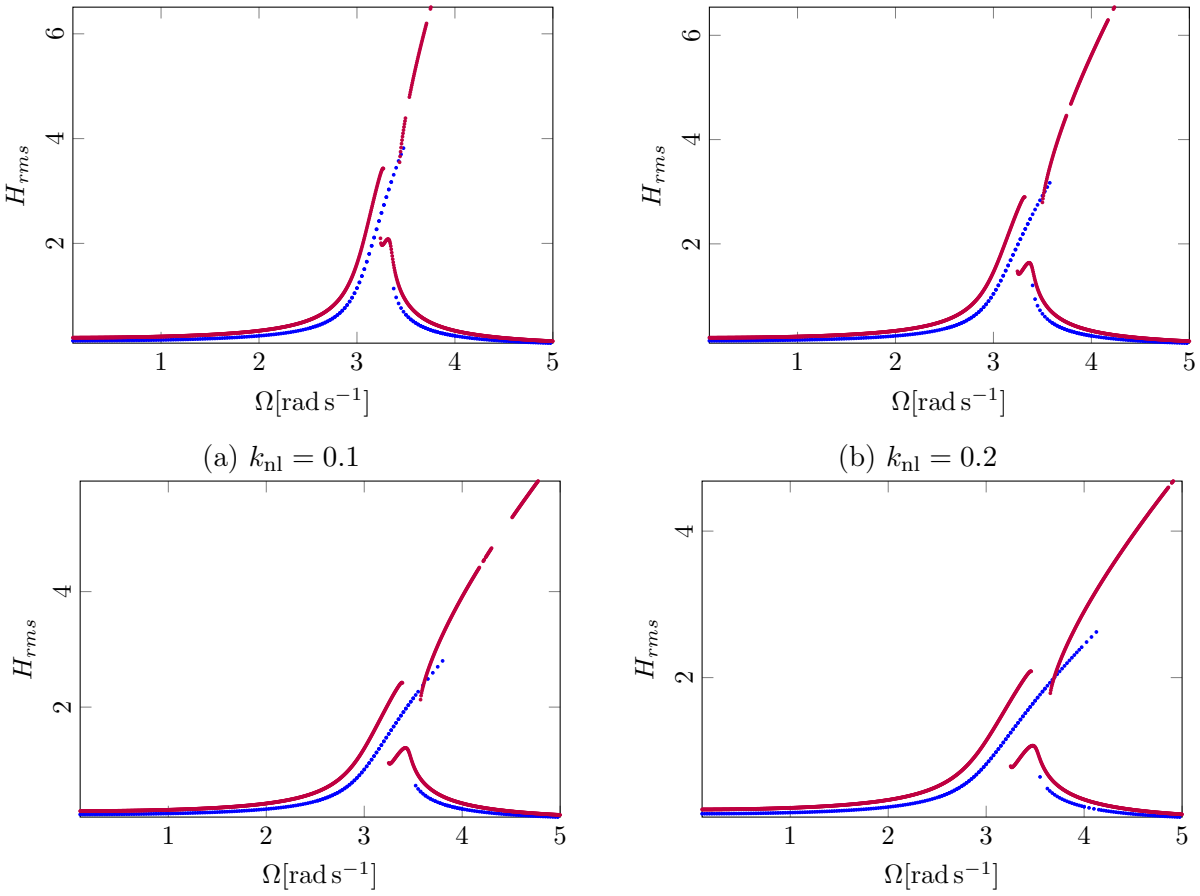


Figure 3–8: Sensitivity of the frequency-response to number of harmonics: Reference response (·····), Gauss' response $N = 1$ (·····), $N = 2$ (·····), $N = 3$ (·····)

3.3.2 Sensitivity to the Nonlinear Stiffness

The nonlinear stiffness of the Duffing oscillator is changed to investigate effectiveness of Gauss' approximation. The frequency-response curves are plotted in figure 3–9. The discrepancy between the two methods slightly grows as k_{nl} is increased. For larger nonlinear stiffnesses, Gauss' frequency-response includes a small protuberance on the lower branch after the primary resonance occurs. The reasons for this peculiarity is to be investigated in future studies.



(a) $k_{\text{nl}} = 0.1$ (b) $k_{\text{nl}} = 0.2$

(c) $k_{\text{nl}} = 0.4$ (d) $k_{\text{nl}} = 0.7$

Figure 3–9: Sensitivity of frequency-response to k_{nl}: Reference (——), Gauss' (——)

Increasing the nonlinear stiffness forces the unstable part of the curve to the right, which results in increasing the range of Ω where multi-valued responses happen. This is known as the *hardening effect* [48]. Gauss' approximation appears to be capable of capturing the

hardening effect as hightlighted in figure 3–10.

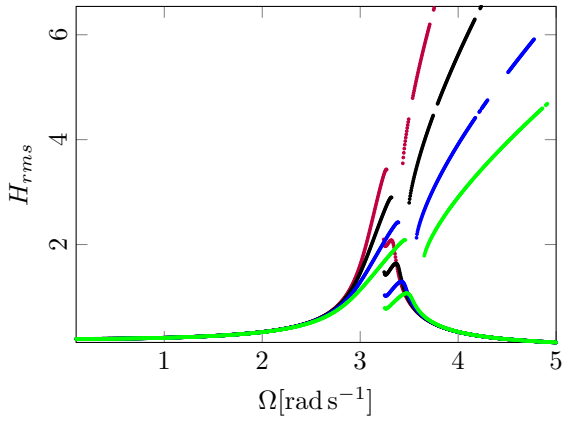


Figure 3–10: Hardening effect in Gauss' approximation: $k_{\text{nl}} = 0.1$ (——), $k_{\text{nl}} = 0.2$ (——), $k_{\text{nl}} = 0.4$ (——), $k_{\text{nl}} = 0.7$ (——)

In figure 3–11, the minimized amount of G_{int} is plotted for different nonlinearities. It is observed that slight variations in nonlinear stiffness does not change the magnitude of the local minimum of Gaussian at the main resonance. At frequencies further from resonance, increasing nonlinearities causes slight increase in the amount of G_{int}. It can be induced that nonlinearities in the system, prevent the minimum of the Gaussian to maintain its lowest amount at 0. The independence minimum of G_{int} at the resonance frequency is distinctive and should be investigated further.

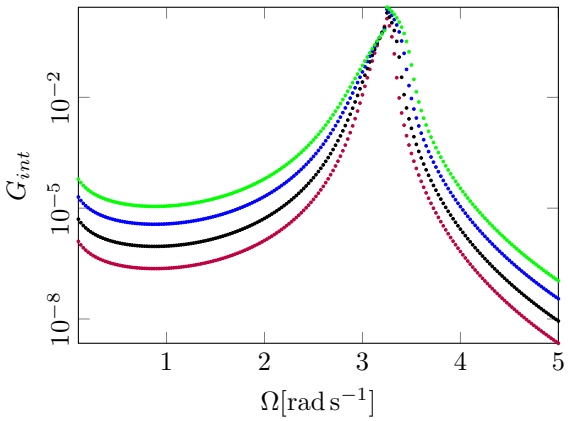


Figure 3–11: Sensitivity of the obtained minimum of G_{int} to k_{nl}: $k_{\text{nl}} = 0.1$ (——), $k_{\text{nl}} = 0.2$ (——), $k_{\text{nl}} = 0.4$ (——), $k_{\text{nl}} = 0.7$ (——)

3.3.3 Sensitivity to the Viscosity

The frequency-response curve obtained from Gauss' approximation is plotted for $c = 0$, 0.01, 0.05, 0.1, and 0.15 in figure 3–12. Although slight variations in viscosity do not change the outline of the curve, assuming different viscosities helps finding the unstable points of the frequency-response curve around the primary resonance frequency. The protuberance on the lower branch which was noticed in section 3.3.2 is more conspicuous for lower magnitudes of the damping coefficient.

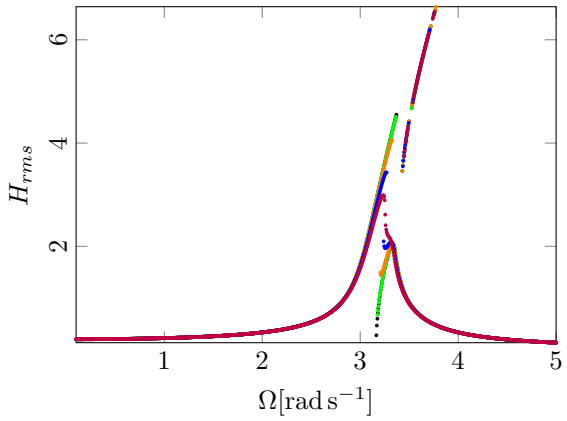
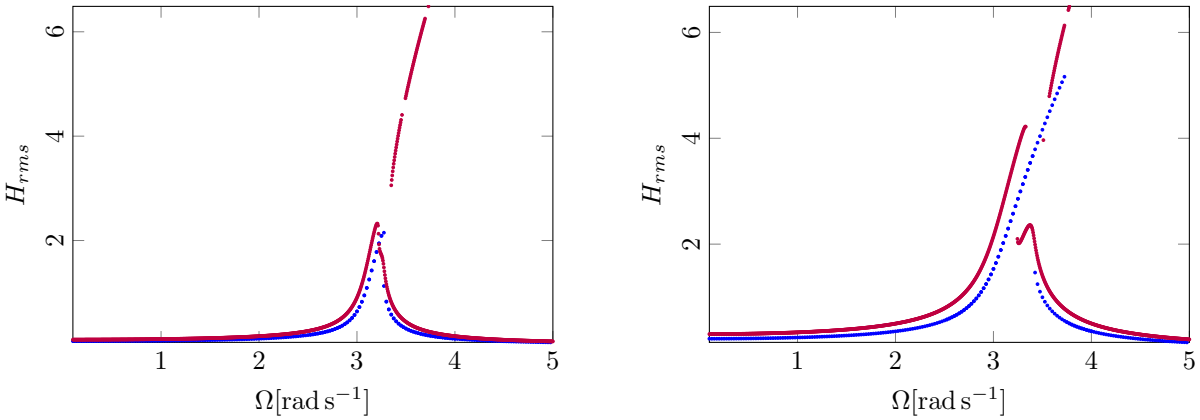


Figure 3–12: Sensitivity of frequency-response to c: $c = 0$ (——), $c = 0.01$ (——), $c = 0.05$ (——), $c = 0.1$ (——), $c = 0.15$ (——)

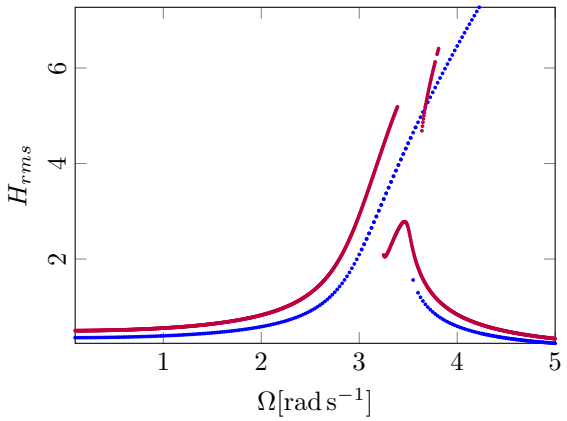
3.3.4 Sensitivity to the Magnitude of Excitation

In figure 3–13, the frequency-response curves are plotted for $F = 1, 3, 5, 7$. It is observed that increasing the magnitude of the driving force significantly decreases the effectiveness of Gauss' method in approximating the frequency-response curve. The protuberance on the lower branch noticed in sections 3.3.2 and 3.3.3 is more conspicuous for larger driving forces.

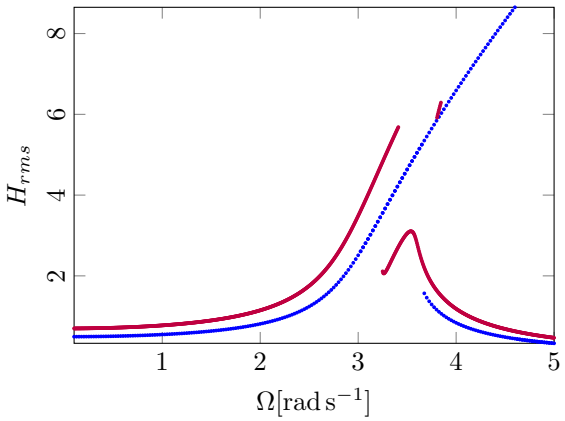


(a) $F = 1$

(b) $F = 3$



(c) $F = 5$



(d) $F = 7$

Figure 3–13: Sensitivity of frequency-response to F: Reference (——), Gauss' (——)

Chapter 4

Piecewise Linear System

Piecewise linear systems are commonly used to model actual engineering systems where unilateral contact naturally arises. Gears, rolling element bearings and clutches are examples of such systems. These systems are inherently nonlinear and their parameters change by experiencing different contact regimes and separations caused by clearances or backlash [53].

In this chapter, a mass-spring system with piecewise stiffness is studied and efficiency of Gauss' approach in predicting the periodic steady state forced response is assessed by comparing the results with those from numerical solvers.

4.1 Modelling

The systems is composed of a mass m attached to a linear spring of stiffness k_1 and a linear damper with damping factor c. When the displacement x exceeds the value d, a second linear spring of stiffness k_2 contacts the mass. The two springs result an overall restoring force which is piecewise linear. The schematic of a mass-spring system with piecewise linear stiffness is shown in figure 4–1.

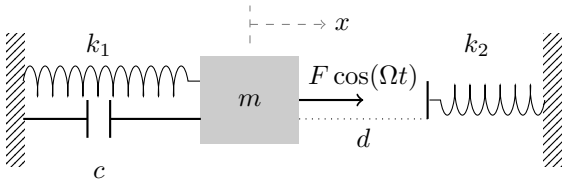


Figure 4–1: Schematic of a piecewise linear system

The dynamics of the system when externally excited by a harmonic force is

$$\ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{f_k}{m} = \frac{F}{m}\cos(\Omega t) \quad \text{with} \quad f_k = \begin{cases} k_1 x & \text{if } x \leq d \\ k_1 x + (x-d)k_2 & \text{else} \end{cases} \tag{4.1}$$

Figure 4–2 plots the restoring force versus displacement for two different clearance stiffnesses. The system is specially interesting when $k_2 \longrightarrow \infty$. In such case, the system is referred to as an *impact oscillator* [54].

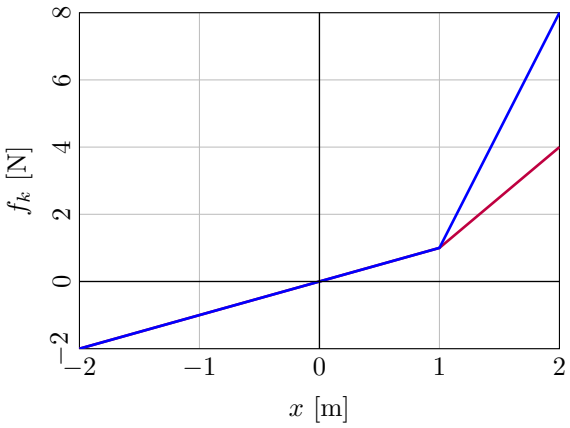


Figure 4–2: Piecewise linear stiffness with $d = 1$, $k_1 = 1$ and $k_2 = 3$ (——), $k_2 = 7$ (——),

The same method of approximation based on Gauss' principle is used to analyze the dynamical behavior of the piecewise linear system[1]. The Gauss' trial solution is in the form of equation (2.1). Substituting the trial solution in (4.1), G_{int} is formed as

$$G_{\text{int}} = \int_0^T \left(\ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{f_k}{m} - \frac{F}{m}\cos(\Omega t)\right)^2 dt \tag{4.2}$$

[1]See chapter 2

The displacement x is constantly monitored and compared to d to select the right f_k defined in equation (4.1) for every step of the integral computation. Value of $x(t_n)$ determines the branch of the f_k that must be used to evaluate $x(t_{n+1})$. The time steps are decreased prior to the discontinuity point $x = d$ to decrease the error and maintain efficiency.

Local minimization on G_{int} is performed by the *Quasi-Newton BFGS* technique. Quasi-Newton BFGS algorithm is an iterative method for solving unconstrained nonlinear optimization problems based on Newton's method previously explained in section 3.2. The major difference between these approaches is that in Quasi-Newton methods, providing second derivatives is not required. Instead, the *Hessian* is built up by updates based on the past steps [55].

The results are compared with those from the numerical differential equation solving method discussed in section 2.1.1.

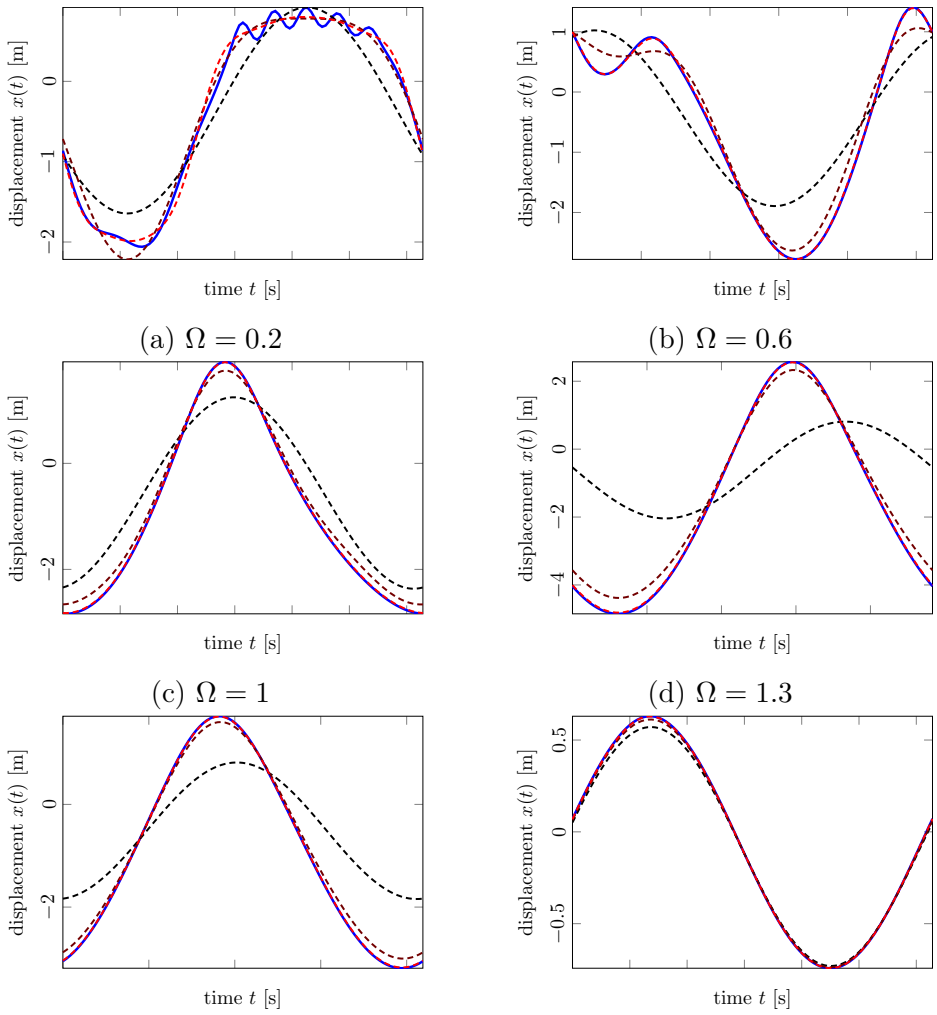
4.2 Displacement Response

In figure 4–3, the approximated displacement from Gauss' method is compared to the numerical solution for different numbers of harmonics and frequencies. The parameters of the system are shown in table 4–1. Local minimization is initiated from the origin (a matrix with all zero elements).

m [kg]	F [N]	k_1 [N m^{-1}]	k_2 [N m^{-1}]	d [m]	c [N s m^{-1}]	$x(0)$ [m]	$\dot{x}(0)$ [m s^{-1}]
1	2	1	4	0.5	0.1	0	0

Table 4–1: Numerical values for a piecewise linear system

It is observed from figure 4–3 that the approximation is close to the reference solution for N as low as 3 at all frequencies. Around $\Omega = 1.3$, the displacement approximation with $N = 1$ shows significant discrepancies when compared to the reference curve. This is the resonance frequency of the nonlinear system. Increasing the number of harmonics used in Gauss' trial solution gives better approximations. For the data shown in table 4–1, results



(a) $\Omega = 0.2$ (b) $\Omega = 0.6$

(c) $\Omega = 1$ (d) $\Omega = 1.3$

(e) $\Omega = 1.5$ (f) $\Omega = 2$

Figure 4–3: Displacement comparison: Reference sol. (——), Gauss' sol. $N = 1$ (- - - -), $N = 3$ (- - - -), $N = 7$ (- - - -)

for higher frequencies ($\Omega = 2$) match well for N as low as 1.

4.3 Frequency-Response Curve

The efficiency of Gauss' approximation in predicting the frequency-response curve is investigated in this section. The *frequency-response* is previously defined in section 3.3. The dependency of the frequency-response curve and efficiency of Gauss' approach is studied for different number of harmonics and varying system parameters such as clearance stiffness k_2

and driving force F.

The reference frequency-response curve for numerical data shown in table 4–1 along with the linear case ($k_2 = 0$) are shown in figure 4–4. For the linear system, resonance at $\Omega = 1$ is observed as expected. Adding nonlinearity to the system shifts the frequency-response curve to right. The primary resonance occurs around $\Omega = 1.3$ when $k_2 = 4$. The curve shows another resonance around $\Omega = 0.7$. The frequency-response of the system between $\Omega = 2.1$ and $\Omega = 2.4$ is distinctive. Since the system of equations is highly nonlinear, this peculiarity may be *chaos* in the response.

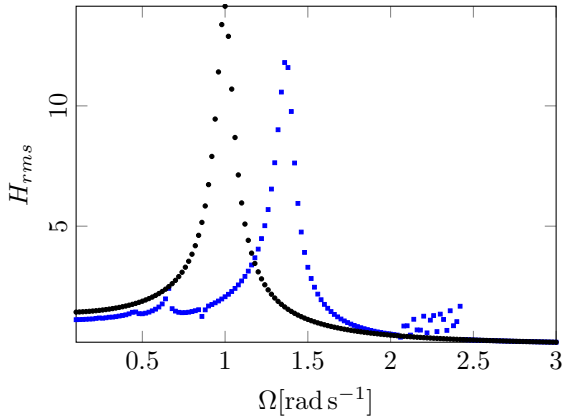


Figure 4–4: Reference frequency-response: $k_2 = 0$ (——), $k_2 = 4$ (——)

4.3.1 Sensitivity to the Initial Guess

First, the sensitivity of frequency-response from Gauss' approximation to the initial guesses provided to the minimization tool is examined. This is of importance since local minimization is performed on G_{int} and results can vary significantly depending on the initial guesses.

The numerical data shown in table 4–1 is used to plot frequency-response curves in figure 4–5. Gauss' approximation is conducted twice, once using constant guess of a matrix of all zero elements for each Ω and then assuming progressive guesses which are updated to the values obtained for each A_i from the previous frequency step. The number of harmonics

is $N = 3$. Progressive selection of initial guesses results in a curve closer to the reference frequency-response obtained from direct numerical equation solving methods. Furthermore, the computation time is considerably less when updating the initial guess at each frequency step. Hence the progressive initial guess approach is used for studying the effect of other parameters on the frequency-response of the piecewise linear system.

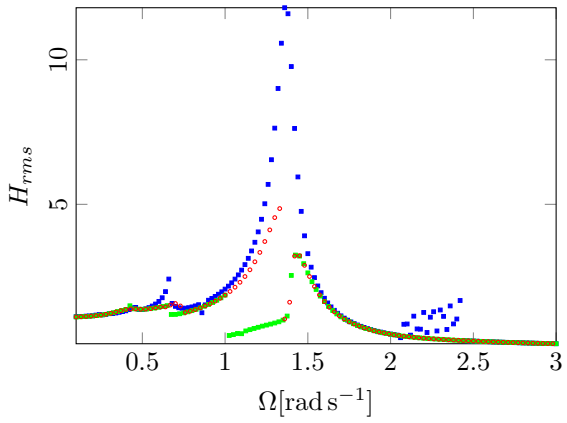


Figure 4–5: Sensitivity of Gauss' approximation to the initial guess: Reference (·····), Gauss' sol. $N = 7$ with constant initial guess (·····), Gauss' sol. with updating initial guess (·····),

4.3.2 Sensitivity to the Number of Harmonics

Figure 4–6 plots the frequency-response curve for the data shown in table 4–1. The initial guesses are updated at each frequency step and the reference curve is obtained by solving the equation numerically. The accuracy of the results highly depend on the number of terms used in Gauss' trial solution (2.1). For $N = 1$, the frequency-response approximation shows significant disparity around the main resonance frequency. The approximation improves significantly by increasing the number of harmonics in the trial solution to $N = 7$.

The values of G_{int} after minimization are plotted in figure 4–7. A logarithmic scale is used for G_{int} to better observe the slight variations. Figure 4–7 shows that assuming more harmonics in the trial solution does not result in converging to different minimum points for G_{int}. Hence the improvement in the response by increasing N is solely a result of better fitting trial solutions.

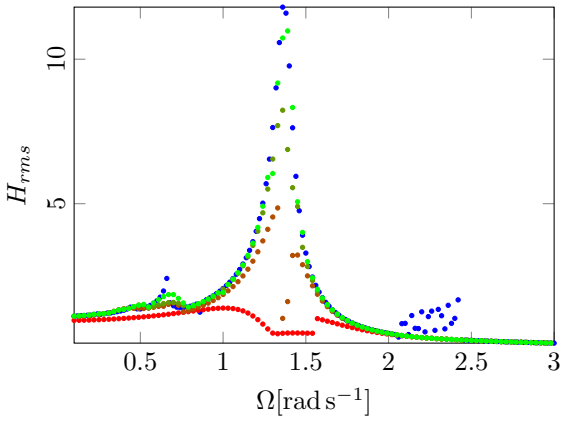


Figure 4–6: Sensitivity of frequency-response to the number of harmonics: Reference (·····), Gauss' sol. $N = 1$ (·····), $N = 3$ (·····), $N = 5$ (·····), and $N = 7$ (·····)

The discontinuity at $\Omega = 2.2$ may be because of a steep variation in G_{int}. However, since the number of harmonics in G_{int} is at least 3, obtaining a three dimensional plot of G_{int} is not feasible. Further investigations are needed to verify the cause of discontinuity in G_{int}.

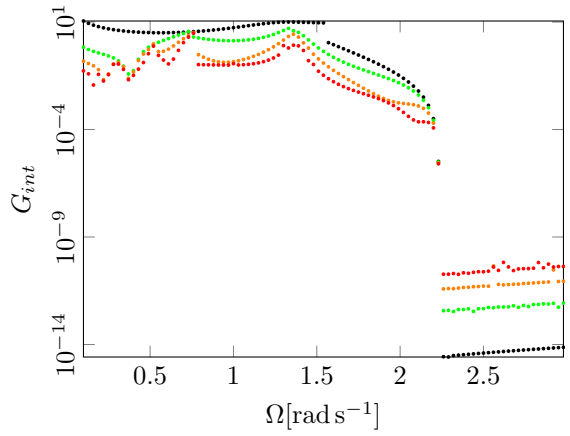


Figure 4–7: Sensitivity of the obtained minimum of G_{int} to number of tems: Reference (·····), Gauss' sol. $N = 1$ (·····), $N = 3$ (·····), $N = 5$ (·····), and $N = 7$ (·····)

4.3.3 Sensitivity to Clearance Stiffness and Excitation

To evaluate the effect of the clearance stiffness k_2 on the effectiveness of Gauss' approximation, frequency-response curves are plotted in figure 4–8 for $k_2 = 0, 4, 10, 20$ while keeping $k_1 = 1$ constant. For the linear system, results are identical. Increasing the nonlinearity of the system by increasing k_2, generates discrepancies in the results. The Gauss' approach does

not adequately approximate the harmonic steady state of the system for large values of k_2. Thus, Gauss' approximation in piecewise linear form is not recommended to obtain solutions of impact oscillators.

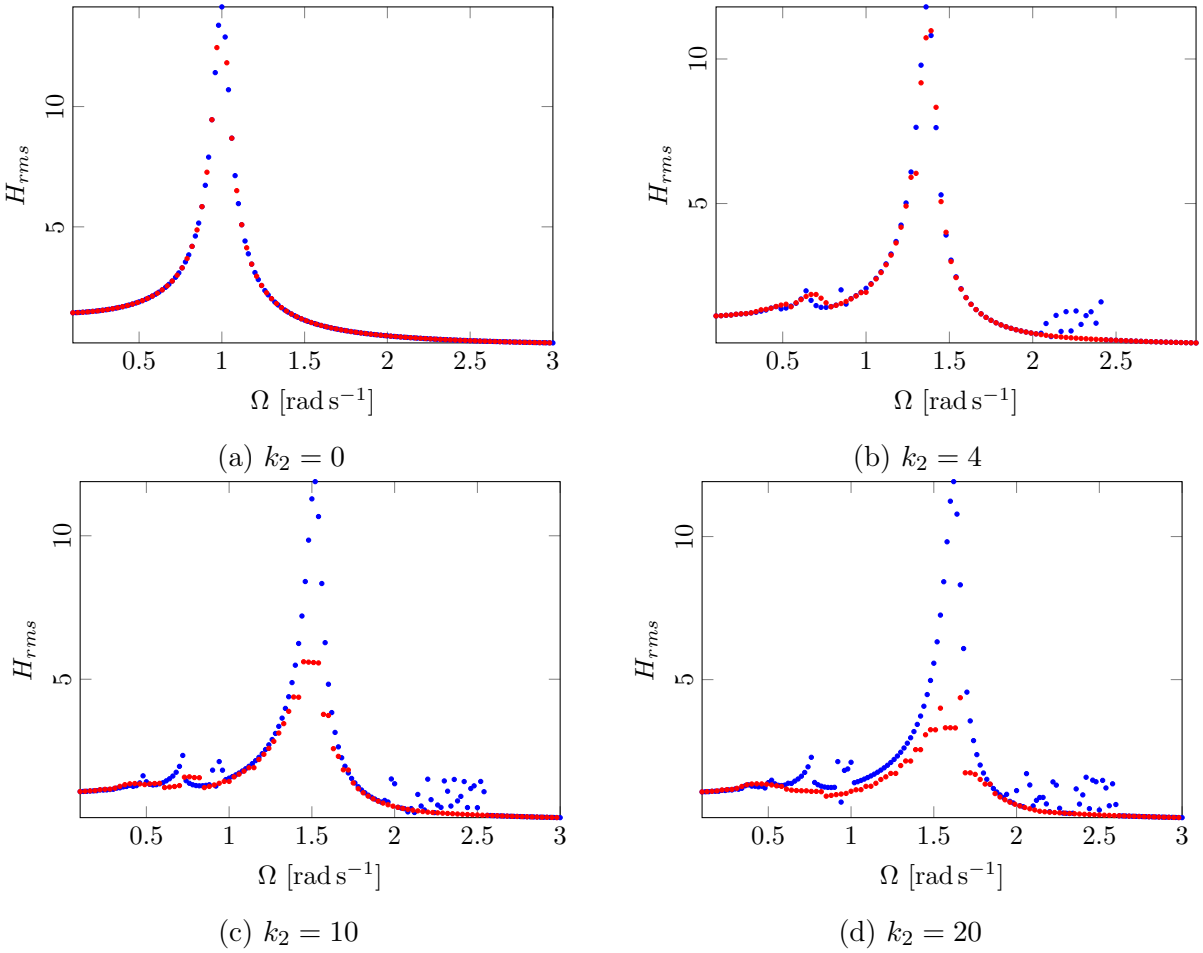
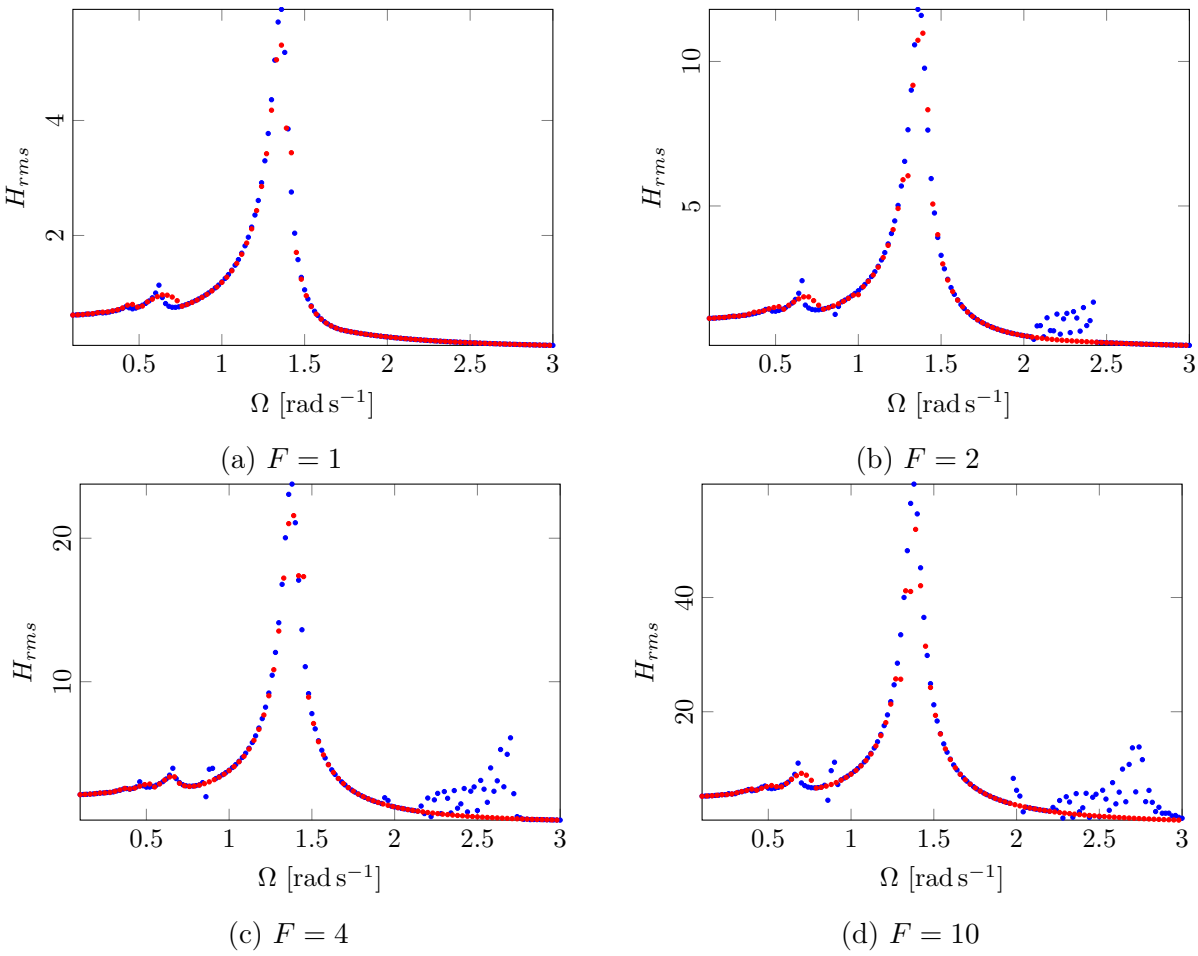


Figure 4–8: Sensitivity of frequency-response to k_2: Reference (·····), Gauss' (·····)

The capability of Gauss' approach in approximating the frequency response curve is studied for different magnitudes of external excitation F. The frequency-response curves are shown in figure 4–9. Although the root mean square of the amplitudes significantly increases for stronger excitations, the approximated frequency-response curve remains accurate.



(a) $F = 1$

(b) $F = 2$

(c) $F = 4$

(d) $F = 10$

Figure 4–9: Sensitivity of frequency-response to F : Reference (·····), Gauss' (·····)

Chapter 5

Unilaterally Constrained System

Unilateral constraints are another example of nonlinear systems. A unilateral constraint is a constraint that prevents penetration between two bodies. Kinematical specification of unilateral constraints results in inequalities in contrast with the bilateral constraints which are described by equalities. A unilateral constraint can be formulated as $\mathbf{f}(\mathbf{q}) \geq 0$ where the normal vector points outwards the constraint surface [56].

5.1 Modelling

Assume a mass-spring system with m and k denoting the mass and the stiffness respectively. A rigid wall is placed at the distance d from the mass as depicted in figure 5–1. Once the point

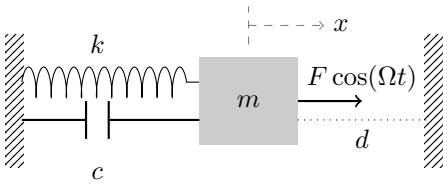


Figure 5–1: Schematic of a unilaterally constrained mass-spring system

mass contacts the rigid surface, a *reaction force* λ appears [57]. The impenetrability feature of the contact is expressed by enforcing the distance between two bodies (mass and wall) to be nonnegative. Also, it is assumed that contacting bodies are not attracting each other,

hence the reaction force λ is nonpositive and vanishes when the contact is not active [58]. These conditions are mathematically expressed as

$$x(t) \leq d \tag{5.1}$$

$$\lambda(t) \leq 0 \tag{5.2}$$

$$\lambda(t)(x(t) - d) = 0 \tag{5.3}$$

and are known as *Hertz-Signorini-Moreau* complementarity conditions in contact mechanics [57]. The *penalty method* and *Lagrange multipliers method* are two popular techniques in solving these equations. Penalty method has been very popular and was adopted in several researches because of its easy implementation [59, 60]. This method, however, only approximately fulfills the contact constraints, and a large penalty parameter has to be used which may give rise to ill-conditioning effective stiffness matrix, leading to instability of equilibrium equation. By contrast, the Lagrange multiplier method can exactly satisfy the contact constraints while not adversely affecting stability, and this method has been successfully combined with explicit time integration in [61]. This will stand as the reference solution in the remainder.

The equation of motion is expressed as

$$\ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{k}{m}x(t) + \lambda = \frac{F}{m}\cos(\Omega t) \tag{5.4}$$

The constraint is expressed as in (5.1). The Lagrange multiplier λ reflects the surface contact force. The Lagrange multiplier method proceeds by treating λ as unknown and solving equations (5.4), (5.1), (5.2), and (5.3) simultaneously [61]. The well-known central finite difference scheme in time is used.

For Gauss' approximation, the trial solution is of the form (2.1) and the Gaussian is formed as before

$$G_{\text{int}} = \int_0^T \left(\ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{k}{m}x(t) - \frac{F}{m}\cos(\Omega t)\right)^2 dt \tag{5.5}$$

where $x(t)$ and its derivatives are substituted by the trial solution. As opposed to other variational principles, Gauss' principle is a *true* minimal principle and thus inequality constraints of the type (5.1) can be readily incorporated in the formulation. Hence, finding the forced periodic steady state response reduces to minimizing G_{int} under only one set of inequality constraints (5.1). An issue remains: time t is a continuous quantity in (5.1). In order to fall in the common framework of Convex Optimization Theory dealing with discrete quantities, the period $[0, T]$ should then be discretized in such a way that the inequalities would be satisfied at the discrete time instants only. Let's consider a set of n_c instants $t_i \in [0, T]$, $i = 1, \ldots, n_c$. The problem to be solved is now:

$$\begin{aligned} &\min_{A_j, j=1,\ldots,2N+1} G_{\text{int}}(A_1, \ldots, A_{2N+1}) \\ &\qquad \text{under constraints } x(t_i) \leq d, \quad \forall i = 1, \ldots, n_c \end{aligned} \tag{5.6}$$

keeping in mind that the displacement x is now a function of the harmonics A_j through (2.1). This is a well-posed conventional optimization problem that can be solved with various numerical techniques. In this work, the minimization is achieved using the *interior point method* through the use of *barrier function*. In this approach, the nonnegative constraints of $d - x(t) \geq 0$ (to be respected), are replaced by adding a barrier term to the objective function G_{int}:

$$B(x, \mu) = G_{\text{int}} - \mu \ln(d - x(t)) \tag{5.7}$$

where μ is a small positive scalar known as the *barrier parameter*. As μ converges to zero the minimum of $B(x, \mu)$ should converge to the minimum of G_{int} while respecting the constraint $x(t_i) \leq d$, $\forall i = 1, \ldots, n_c$ [62]. The number of points the constraint condition is checked for in this study is 30. Minimization of (5.7) is done using Newton's method.

This formulation seems to be the most interesting among all the proposed approaches as it makes use of Gauss' principle with inequality constraints, where most of the other existing variational principles if not all would fail. For instance, it should be noted that such a formulation is not accessible with Hamilton's principle, as the solution is known to be a

stationary point only, and would be a saddle point for the system of interest and not a true minimum. Finding a saddle point satisfying a set of inequalities is not a well-posed problem.

Still, minimizing a function defined in terms of accelerations with unilateral constraints on the displacements does not seem to be trivial.

5.2 Displacement Response

The displacement results of a unilaterally constrained mass-spring system are compared using the numerical data of table 5–1. The length of the clearance d is changed for each Ω so

m [kg]	F [N]	k [N m^{-1}]	c [N s m^{-1}]	$x(0)$ [m]	$\dot{x}(0)$ [m s^{-1}]
1	2	1	0.1	0	0

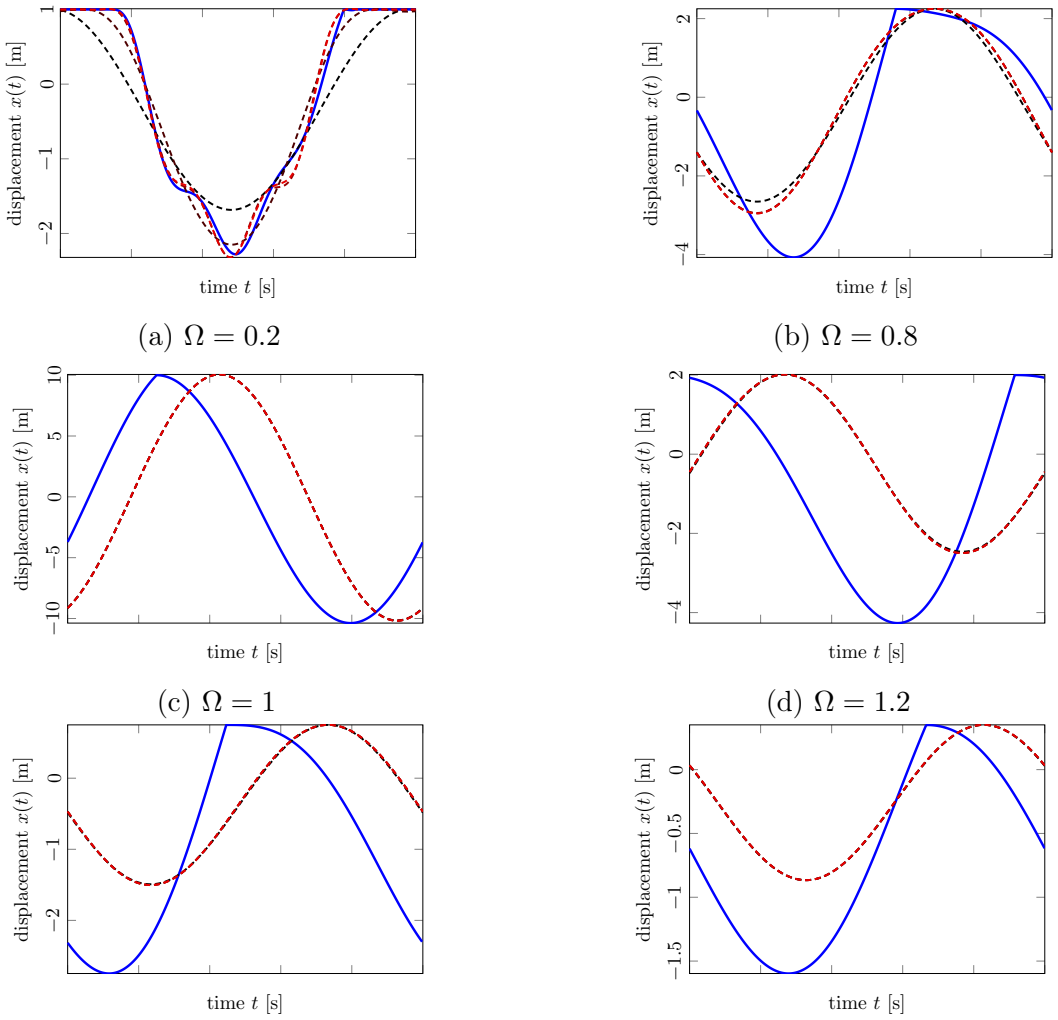
Table 5–1: Numerical values for the unilaterally constrained system

that it is equal to half the amplitude of the steady state displacement response. The effect of changing the number of harmonics used in Gauss' approximation N, the magnitude of the external force F, and the length of the gap d are investigated.

5.2.1 Sensitivity to the Number of Harmonics

Figure 5–2 shows displacement responses. It is observed that Gauss' approximation returns the similar results with the reference curve when the excitation frequency is considerably smaller than the natural resonance frequency of the system. Increasing Ω further from the resonance frequency does not result in an acceptable approximation. It is interesting to have a look at the reaction force obtained from both methods. Figure 5–3 shows that for $\Omega = 0.2$ Gauss' approximation gives similar results for λ compared to the reference method.

In the vicinity of the natural frequency of the system, the latter behaves in such a way that the contact forces are instantaneous impulses and cannot be appropriately captured by the proposed Fourier series. Still, for a specific set of parameters, the comparison is very convincing. It shows that the proposed Gauss' formulation is appropriate but necessitates



(a) $\Omega = 0.2$ (b) $\Omega = 0.8$

(c) $\Omega = 1$ (d) $\Omega = 1.2$

(e) $\Omega = 1.5$ (f) $\Omega = 2$

Figure 5–2: Sensitivity of displacement response to number of harmonics:Reference sol. (——), Gauss' sol. $N = 1$ (- - - -), $N = 3$ (- - - -), $N = 7$ (- - - -), $N = 10$ (- - - -)

numerical tuning that goes beyond this work.

5.2.2 Sensitivity to the Magnitude of Excitation

Effectiveness of Gauss' approximation for different amounts of F is shown in figure 5–4 for $\Omega = 0.2$. It is observed that Gauss' approximation gives similar results to the reference solution with excitation amplitudes ranging from very small $F = 1$ to large amounts $F = 100$.

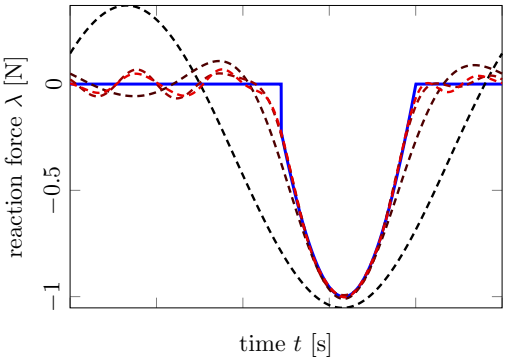
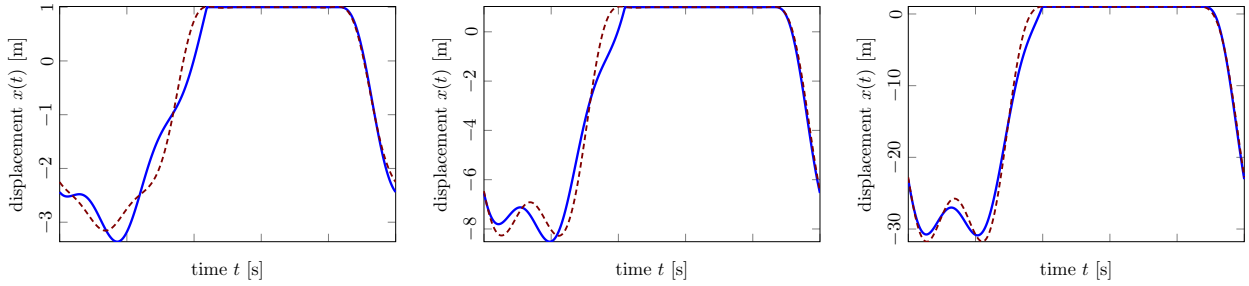


Figure 5–3: λ for $\Omega = 0.2$: Reference sol. (——), Gauss' sol. $N = 1$ (- - - -), $N = 3$ (- - - -), $N = 7$ (- - - -), $N = 10$ (- - - -)

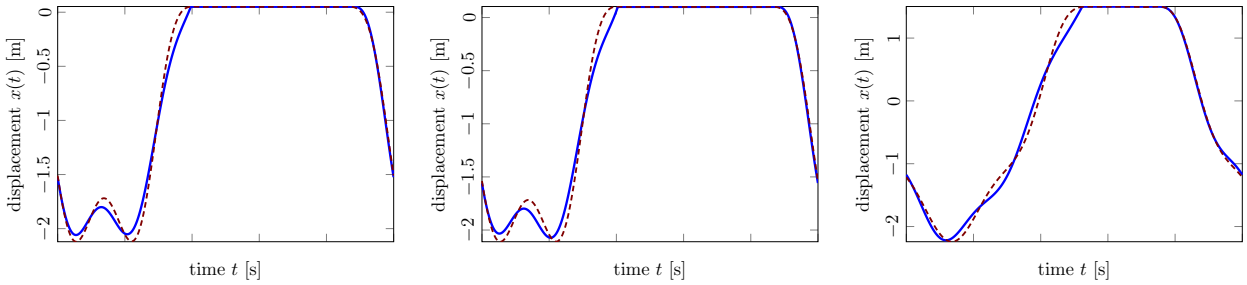


(a) $F = 3$ (b) $F = 8$ (c) $F = 30$

Figure 5–4: Sensitivity of displacement response to F: Reference (——), Gauss' $N = 10$ (- - - -)

5.2.3 Sensitivity to the Clearance

In figure 5–5, the length of the gap is varied for $\Omega = 0.2$ and $F = 2$. It is observed that effectiveness of Gauss' approximation is independent of the length of the gap and the results are similar to the reference curve for d as small as 0.05.



(a) $d = 0.05$ (b) $d = 0.1$ (c) $d = 1.5$

Figure 5–5: Sensitivity to clearance d: Reference (——), Gauss' $N = 10$ (- - - -)

Chapter 6

Conclusion

6.1 Summary

In 1829 Gauss gave an aesthetic and novel reinterpretation of d'Alembert's principle, changing it into a true minimum principle. He formulated the principle of *least constraint* for describing the motion of mechanical systems. Gauss' principle of least constraints is mentioned as a fundamental principle in many treatises e.g. [5, 21, 30]. However, the computational difficulties of directly solving a minimization problem made Gauss's principle unattractive at the time. With more developed numerical methods nowadays, it is worthwhile to examine the effectiveness of this principle in analyzing dynamical behavior of different systems.

In this dissertation, Gauss' principle of least constraints is compared with other variational principles in classical mechanics. Previous works on deriving equations of motion using Gauss' principle is illustrated through examples. An innovative approach based on Gauss' principle is used to approximate the steady state response of slightly damped oscillators under harmonic excitation. In this method, a trial solution consists of Fourier series is used as an approximation of the steady state response. Making use of the orthogonality of Fourier functions, the principle is adapted to use an integration of the Gaussian over a full period of excitation instead of the original form of Gaussian. This adjusted Gaussian is formed for a

single pendulum, clamped-free bar, Duffing oscillator, a mass-spring system with piecewise linear stiffness, and a unilaterally constrained system. By minimizing this adjusted Gaussian, the unknown coefficients for the trial solution are obtained.

In all the systems investigated, it was found that providing a proper initial guess for the minimization is critical for obtaining satisfactory approximations.

In a Duffing oscillator, the approximation is efficient in predicting the displacement responses. Increasing the number of harmonics used in the trial solution gives more accurate approximations. It is observed that Gauss' approximation can detect super harmonics when the number of terms are increased.

In a mass-spring system with piecewise linear stiffness, the approximation is effective in predicting the displacement responses. The accuracy of the approximated frequency-response curve highly depends on the number of harmonics considered in the trial solution. The proposed method of approximation does not render accurate results for increased amounts of clearance stiffness.

In a unilaterally constrained system, the optimization of Gaussian becomes a quadratic minimization problem with a global minimum. Hence only one of the possible solutions of the system is obtained through the approximation. Effectiveness of Gauss' method is examined for a particular frequency of excitation in which the approximated displacement response is similar to the reference response. The approximation gives adequately accurate results for increased magnitudes of excitation and decreased lengths of the gap.

6.2 Suggestions for Future Work

There is clearly more work to be done in determining the effectiveness of Gauss' principle in approximating different systems. One direction for future work would be investigating deeper in the systems studied in this research. Conducting the minimization with more initial guesses would help understanding the Gaussian function better. *Continuation techniques*

could be used to better follow the frequency-response curve and understand the system's behavior in the vicinity of the natural frequencies of the investigated systems.

Another direction for future work in this area would be applying the proposed method of approximation based on Gauss' principle to other nonlinear systems. Investigating the systems presented in this work with more degrees of freedom would be of interest.

Appendix A

Moore-Penrose Inverse

Moore-Penrose pseudo-inverse ($\mathbf{P}^+$) of matrix $\mathbf{P}$ is a matrix that satisfies all these four conditions:

1. $\mathbf{P}\mathbf{P}^+\mathbf{P} = \mathbf{P}$
2. $\mathbf{P}^+\mathbf{P}\mathbf{P}^+ = \mathbf{P}+$
3. $\mathbf{P}\mathbf{P}^+ = (\mathbf{P}\mathbf{P}^+)^T$
4. $\mathbf{P}^+\mathbf{P} = (\mathbf{P}^+\mathbf{P})^T$

The third and forth conditions require $\mathbf{P}^+\mathbf{P}$ and $\mathbf{P}\mathbf{P}^+$ to be symmetric.

MP-inverse-inverse of any given matrix uniquely exists. The following shortcuts can be used to determine the MP-inverse of a matrix:

- The pseudo-inverse of a reversible matrix is its inverse. ($\mathbf{P}_r^+ = \mathbf{P}_r^{-1}$)
- The pseudo-inverse of a zero matrix is its transpose. ($\mathbf{P}_z^+ = \mathbf{P}_z^T$)
- The pseudo-inverse of a nonzero row vector of $\mathbf{p}$ is $\mathbf{p}^+ = \frac{1}{\mathbf{p}\mathbf{p}^T}\mathbf{p}^T$.
- For a nonzero scalar a, $(a\mathbf{P})^+ = \frac{1}{a}\mathbf{P}^+$.

Glossary

abscissa perpendicular distance of a point from the vertical axis. 4

BFGS Broyden Fletcher Goldfarb Shanno. 47

boundary conditions conditions that correspond to the boundaries of a physical system. 9

configuration a set containing the positions of all particles of the body. 2

conservative force a force with the property that the work done in moving a particle between two points is independent of the taken path [63]. 7, 8

constraint a restriction on the freedom of movement of a particle [64]. 1, 2, 3, 5, 6, 5, 9, 10, 11, 12, 15, 16

curvilinear coordinates coordinate system for Euclidean space in which the coordinate lines may be curved. 4

generalized coordinate the generalized coordinate values uniquely define any possible position of the system relative to the intial position [64]. 2, 6, 7, 11

kinetic energy energy that an object possesses due to its motion [63]. 6, 7, 8, 9

motion a change in position of an object with respect to time [65]. 2, 3, 5, 7, 8, 9, 10, 11, 12, 15, 16

MP-inverse Moore-Penrose pseudo-inverse. 64

nonconservative force a force with the property that the work done in moving a particle between two points depends on the path followed [63]. 7, 8

ODE Ordinary Differential Equation. 19, 20

particle a special rigid body whose rotation can be neglected relative to its other motions [1]. ix, 1, 2, 3, 4, 5, 6, 5, 7, 9, 11, 12, 13, 15

path a trajectory that a moving object follows through space as a function of time [66]. 4, 8, 15

potential energy energy stored in an object due to its position in a force field or due to its configuration [63]. 7

quasi coordinates the generalized coordinates that correspond to the angular velocity components . 9, 10

rigid body a special solid whose deformation can be neglected relative to its other motions [1]. 1

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