Local Stability Method for Hypergraph Turán problems

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Preface and Contribution of Authors

The novel parts of this thesis are in Chapters 3, 4, 5, 6 and 7. Chapter 4 and Chapter 5 are based on the following submitted manuscripts, in order.

- *S. Norin and L. Yepremyan. The Turán Number of Generalized Triangle. 2016+. arxiv:1501.01913, submitted to JCTA*
- *S. Norin and L. Yepremyan. The Turán numbers of extensions. 2016+. arxiv:1510.04689, submitted to JCTA*

Chapter 3 includes (but is not limited to) some parts of both of the above-mentioned papers. Chapter 7 is based on the following published article.

- *S. Norin and L. Yepremyan. Sparse halves in dense triangle-free graphs.* J. Comb. Theory Ser. B*, 115(C):1–25, November 2015.*

Finally, Chapter 6 does not appear in any manuscript. All of the presented work is joint with Sergey Norin.

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Abstract

One of the earliest results in Extremal Combinatorics is Mantel's theorem from 1907 which says that the largest triangle-free graph on a given number of vertices is the complete bipartite graph with sizes of partition classes as equal as possible. In 1961 Turán asked the analogous question for 3-uniform hypergraphs - what is the largest 3-uniform hypergraph on a given vertex set with no tetrahedron? To this date, this number is unknown even asymptotically. Since the original question by Turán a new branch in Combinatorics, called hypergraph Turán-type problems, emerged.

A typical Turán-type problem for an r-uniform hypergraph $\mathcal{F}$ asks for the maximum number of edges in an r-uniform hypergraph on given number of vertices without a copy of $\mathcal{F}$; this number is called the Turán number of $\mathcal{F}$. The major part of this thesis is devoted to such problems. In particular, we generalize and extend the classical stability method; a method pioneered by Erdős and Simonovits that is ubiquitous in the study of Turán-type problems. The developed method, referred as local stability method, is generically applicable and is of independent interest. In particular, it allows us to find new Turán numbers of several families of hypergraphs. Furthermore, we solve a conjecture of Frankl and Füredi from 1980's by determining the Turán number of a hypergraph called generalized triangle, for uniformities five and six.

In the final part of the thesis we make some progress on one of the old conjectures of Erdős which states that every triangle-free graph on n vertices contains a subset of $n/2$ vertices that spans at most $n^2/50$ edges. We prove the conjecture under several natural assumptions, improving and generalizing previous results of of Keevash, Krivelevich and Sudakov.

Abrégé

L'un des premiers résultats en Combinatoire Extrémale est le théorème de Mantel de 1907: le plus grand graphe sans triangle sur un certain nombre de sommets est le graphe biparti complet avec des classes de séparation ayant des tailles aussi égales que possible. En 1961 Turán a posé la question analogue sur les hypergraphes 3-uniforme: quel est le plus grand hypergraphe 3-uniforme sur un ensemble de sommets dépourvu de tétraèdre? À ce jour, ce nombre est inconnu, même asymptotiquement. La question initiale de Turán a poussé à la création d'une nouvelle branche de la Combinatoire, portée sur l'étude des problèmes similaires, dits de type Turán, sur les hypergraphes.

Un problème typique de type Turán pour un hypergraphe r-unforme demande le nombre maximum d'arêtes dans un hypergraphe r-uniforme sur le nombre de sommets sans copie de $\mathcal{F}$, nombre appelé nombre de Turán. La majeure partie de cette thèse est consacrée à de tels problèmes. En particulier, nous généralisons et étendons la méthode de stabilité classique initiée par Erdős et Simonovits et qui est omniprésente dans l'étude des problems de type Turán. La méthode développée, appelé méthode de stabilité locale, est génériquement applicable et présente un intérêt indépendant. En particulier, elle nous permet de trouver de nouveaux nombres de Turán pour plusieurs familles de hypergraphes. En outre, nous résolvons une conjecture de Frankl et Füredi de 1980 par la détermination du nombre de Turán d'un hypergraphe appelé triangle généralisé pour des uniformités de cinq ou six.

Enfin, dans la dernière partie de la thèse, nous faisons des progrès sur une ancienne conjecture de Erdős qui postule que tout graphe de n sommets sans triangles contient un sous-ensemble de $n/2$ sommets qui produit au plus $n^2/50$ arêtes. Nous prouvons cette conjecture sous plusieurs hypothèses naturelles, améliorant et généralisant ainsi des résultats antérieurs de Keevash, Krivelevich et Sudakov.

Dedication

To the memory of Stepan Markosyan

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This thesis would have never been possible without my supervisor, Sergey Norin. I cannot imagine a better supervisor for a graduate student than he was for me; I have learned a lot from him in many aspects. He taught me how to do mathematics and how not to. Working with him was a constant process of pushing myself to expand my knowledge and to be better. I am grateful to him for always having time for me, for his patience, guidance and generous support, especially for many travels throughout my Ph.D. I was lucky to be his student.

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Chapter 1

Introduction

In this chapter we introduce basic notation, give some brief historical overview of the subject, describe the problems studied and state our main results.

1.1. Notation

A hypergraph $\mathcal{F}$ is a pair $(V(\mathcal{F}), \mathcal{E}(\mathcal{F}))$, where $V(\mathcal{F})$ is a finite set whose elements are called *vertices* and $\mathcal{E}(\mathcal{F})$, *the edge set*, is a collection of subsets of $V(\mathcal{F})$. We denote $\mathrm{v}(\mathcal{F}) = |V(\mathcal{F})|$ and $\mathrm{e}(\mathcal{F}) = |\mathcal{E}(\mathcal{F})|$. The cardinality of the vertex set is called the *order* of $\mathcal{F}$ and the cardinality of $\mathcal{E}(\mathcal{F})$ is the *size* of $\mathcal{F}$. When there is no confusion we will abbreviate $\mathcal{E}(\mathcal{F})$ by $\mathcal{F}$, that is, we identify the hypergraph $\mathcal{F}$ by its edge set. If we write $F \in \mathcal{F}$, it means F is an edge from $\mathcal{E}(\mathcal{F})$. We call a hypergraph r-uniform or simply, r-graph if all the edges have exactly r elements. As a separate case, 2-graphs are called *graphs*. To make a distinction between 2-graphs and r-graphs, $r \geq 3$, we use different style letters (for example, G and $\mathcal{F}$ correspondingly).

The (edge) *density* of an r-graph $\mathcal{F}$ is defined as follows:

$$\mathrm{d}(\mathcal{F}) = \frac{|\mathcal{F}|}{\binom{n}{r}}.$$

For a family of r-graphs $\mathfrak{F}$, we denote by $\mathfrak{F}_n$ the subfamily of r-graphs on n vertices. We use the notation $m(\mathfrak{F}, n)$ to denote the size of the largest r-graph in $\mathfrak{F}_n$, that is,

$$m(\mathfrak{F}, n) = \max_{\mathcal{F} \in \mathfrak{F}_n} |\mathcal{F}|.$$

We say that an r-graph $\mathcal{F}$ *covers pairs* if every pair of vertices is contained in an edge. For 2-graphs, these graphs are called *cliques*. An r-graph $\mathcal{F}$ is *complete* if every r-subset of the vertex set is an edge. We denote by K_t the complete graph on n vertices and by $\mathcal{K}_t^{(r)}$ the complete r-graph on t vertices, for $r \geq 3$. An r-graph $\mathcal{F}$ is said to be *k-partite* if its vertex set $V(\mathcal{F})$ can be partitioned into k sets so that every edge in the edge set $\mathcal{E}(\mathcal{F})$ of $\mathcal{F}$ consists of a choice of at most one vertex from each partition class. We denote by $K_{s,t}$ the *complete bipartite graph* with partition classes of sizes s and t. For general $r \geq 3$, we denote by $\mathcal{K}^{(r)}_{t_1,t_2,\ldots,t_k}$ *the complete k-partite r-graph* with partition classes of sizes $t_1, t_2, \ldots, t_k$.

For any r-graph $\mathcal{F}$ we define the operation of *blowing up* as follows. Suppose $\mathrm{v}(\mathcal{F}) = k$. Given any natural numbers $m_1, m_2, \ldots, m_k$, we denote by $\mathcal{F}(m_1, m_2, \ldots, m_k)$ *the blowup of* $\mathcal{F}$ on $m = \sum_{i=1}^{k} m_i$ vertices, that is, we replace every vertex v_i of $\mathcal{F}$ by an independent set of size m_i and each edge by the corresponding complete r-partite r-graph. If for every $i, j \in [k]$ $|m_i - m_j| \leq 1$, then the blowup is called *balanced*. When $m_1 = m_2 = \cdots = m_k = m$, the blowup is denoted simply by $\mathcal{F}(m)$.

Given two r-graphs $\mathcal{F}$ and $\mathcal{H}$, we say that they are *isomorphic* if there is an edge-preserving bijective map between the vertex sets, that is a map $f : V(\mathcal{H}) \to V(\mathcal{F})$ such that for any r-tuple $I \subseteq V(\mathcal{H})$, $f(I) \in \mathcal{F}$ if and only if $I \in \mathcal{H}$. For two r-graphs $\mathcal{F}$ and $\mathcal{G}$ on the same vertex set, the *edit distance* between them, denoted by $d(\mathcal{F}, \mathcal{G})$, is defined as

$$d(\mathcal{F}, \mathcal{G}) = |\mathcal{F} \triangle \mathcal{G}|.$$

A hypergraph (graph) $\mathcal{H}$ is a *subhypergraph* (*subgraph*) of another hypergraph (graph) $\mathcal{F}$ if $V(\mathcal{H}) \subseteq V(\mathcal{F})$ and $\mathcal{E}(\mathcal{H}) \subseteq \mathcal{E}(\mathcal{F})$. For simplicity we use the term subgraph both for hypergraphs and graphs. For an r-graph $\mathcal{F}$ and a subset $X \subseteq V(\mathcal{F})$ we denote by $\mathcal{F}[X]$ the *induced subgraph* of $\mathcal{F}$ by X, that is, the r-graph on vertex set X with edge set being those edges of $\mathcal{F}$ that contain only the vertices in X. In an r-graph

$\mathcal{F}$, the *link* of the vertex v is defined as $L_{\mathcal{F}}(v) = \{I | I \cup \{v\} \in \mathcal{F} \text{ and } v \notin I\}$. More generally, the *link* of any tuple I is defined as, $L_{\mathcal{F}}(I) = \{J | J \cap I = \emptyset \text{ and } I \cup J \in \mathcal{F}\}$.

For two hypergraphs $\mathcal{F}$ and $\mathcal{H}$, $\mathcal{F}$ is called $\mathcal{H}$*-free* if it does not contain a subgraph isomorphic to $\mathcal{H}$. For an r-graph $\mathcal{F}$ and a family of r-graphs $\mathfrak{H}$, we say that $\mathcal{F}$ is $\mathfrak{H}$-free if $\mathcal{F}$ is $\mathcal{H}$-free for every $\mathcal{H} \in \mathfrak{H}$. For a family of r-graphs $\mathfrak{F}$, we denote by $\text{Forb}(\mathfrak{F})$ the family of all $\mathfrak{F}$-free r-graphs. When $\mathfrak{F} = \{\mathcal{F}\}$ for some r-graph $\mathcal{F}$, we simply write $\text{Forb}(\mathcal{F})$ instead of $\text{Forb}(\mathfrak{F})$.

1.2. Turán-type Problems For Graphs

In Extremal Combinatorics a typical problem asks to determine or estimate the size of the largest configuration with a given property. One of the earliest results in this area, Mantel's Theorem [Man07] determines the size of the largest graph without a triangle, that is, K_3.

Theorem 1.2.1 (Mantel, [Man07])**.** *If G is a triangle-free graph on n vertices then* $\text{e}(G) \leq \lfloor n^2/4 \rfloor$.

This is clearly the best possible, as one may partition the set of n vertices into two sets of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ and form the complete bipartite graph between them (in other words, the balanced blowup of an edge on n vertices). This graph has no triangles and $\lfloor n^2/4 \rfloor$ edges. Turán's Theorem [Tur61] from 1941 generalizes this result to all complete graphs K_t.

Theorem 1.2.2 (Turán, [Tur61])**.** *If G is a K_{t+1}-free graph on n vertices then* $\text{e}(G) \leq \lfloor \frac{(t-1)n^2}{2t} \rfloor$.

This bound is tightly achieved by the balanced blowup of K_t on n vertices. Since Turán's theorem, many similar problems, often referred as *Turán-type problems*, have emerged.

For a family of r-graphs $\mathfrak{F}$ the *Turán number* $\text{ex}(n, \mathfrak{F})$ is defined to be the largest number of edges in an $\mathfrak{F}$-free r-graph on n vertices. The definition of the Turán number for a single r-graph $\mathcal{F}$ is analogous. A typical Turán-type problem asks to find $\text{ex}(n, \mathcal{F})$ for a given r-graph $\mathcal{F}$ and for all n (or for all sufficiently large n).

For an r-graph $\mathcal{F}$, we define the family of *extremal examples* as the family of all r-graphs achieving the bound $\text{ex}(n, \mathcal{F})$, that is $\mathfrak{G} \subset \text{Forb}(\mathcal{F})$ such that for any n (or for all sufficiently large n), $m(n, \mathfrak{G}) = \text{ex}(n, \mathcal{F})$ and $m(n, \overline{\mathfrak{G}}) < \text{ex}(n, \mathcal{F})$, where $\overline{\mathfrak{G}} = \text{Forb}(\mathcal{F}) \setminus \mathfrak{G}$. Note that typically this family $\mathfrak{G}$ is much smaller and has some precise descriptive structure. For example, it can be derived from Theorem 1.2.2 that for $\mathcal{F} = K_{t+1}$, the family $\mathfrak{G}$ is the family of all balanced blowups of K_t.

A classical result in Extremal Graph Theory, obtained by Erdős, Stone and Simonovits, determines $ex(n, \mathfrak{F})$ asymptotically for any family $\mathfrak{F}$ of 2-graphs, except the ones containing some bipartite graph.

Theorem 1.2.3 (Erdős, Stone, Simonovits, [Erd81, ES46]). *For any finite family of 2-graphs $\mathfrak{F}$, there exists n_0 such that for all $n \geq n_0$,*

$$\text{ex}(n, \mathfrak{F}) = \left(1 - \frac{1}{\min_{F \in \mathfrak{F}} \chi(F) - 1}\right) \frac{n^2}{2} + o(n^2),$$

where $\chi(F)$ is the chromatic number of the graph F.

However, when the family $\mathfrak{F}$ does contain a bipartite graph the theorem only tells us that $\text{ex}(n, \mathfrak{F}) = o(n^2)$. To demonstrate how little we know about these numbers, note that Turán numbers are not known even for as simple graphs as C_8 (the cycle on eight vertices) and $K_{4,4}$ (the complete bipartite graph with four vertices on each side). These numbers have applications also outside of Extremal Graph Theory, such as Additive Combinatorics [EN77, Woo04] and Geometry [Erd46]. The following celebrated Zarankiewicz problem is related to the Turán numbers of complete bipartite graphs, $K_{s,t}$.

Let $Z(m, n, s, t)$ be the largest integer for which there is an $m \times n$ matrix of 0's and 1's containing $Z(m, n, s, t)$ 1's without an $s \times t$ submatrix consisting entirely of 1's. In 1951 Zarankiewicz posed the problem of determining $Z(n, n, 3, 3)$ for $n \leq 6$. While this original problem was solved shortly afterwards, the general problem of determining $Z(m, n, s, t)$ is widely open. It is not hard to see that this problem is equivalent to determining the maximum number of edges in a $K_{s,t}$-free bipartite graph with partition classes having sizes equal to n and m.

The most general bound on the numbers $\text{ex}(n, K_{s,t})$ were obtained by Kővári, Sós and Turán [KST54] who showed that for all $s \leq t$, $\mathrm{ex}(n, K_{s,t}) = O_{s,t}(n^{2-1/s})$. As for lower bounds, constructions matching this upper bound were found by Erdős, Rényi and Sós [ERS66] for $s = t = 2$, for $s = t = 3$ by Brown [Bro66] and for $t \geq s = 2$ by Füredi [F9̈6] (these results determine the right constants as well). For more general s and t optimal constructions matching Kővári–Sós–Turán bound up to a constant are only known when t is sufficiently larger than s [ARS99, KRS96]. These are geometric constructions based on algebraic hypersurfaces of bounded degree with no grids of size $s \times t$. In such finite geometric construction there is a d-dimensional space, where each vertex is joined to some collection of t-dimensional subspaces that contain it. Here the degrees are around $n^{t/d}$ and the construction has around $n^{1+(t/d)}$ edges. This motivated the following conjecture of Erdős and Simonovits which proposes that one can always find such almost extremal constructions.

Conjecture 1.2.1 (Erdős, Simonovits, [Erd81, ES82]). *For any finite family of 2-graphs $\mathfrak{G}$ which contains a bipartite graph there exists a rational $r > 0$ and an absolute constant $c > 0$ such that*

$$\lim_{n \to \infty} \frac{\mathrm{ex}(n, \mathfrak{G})}{n^{1+r}} = c.$$

Erdős and Simonovits also considered the following problem which can be viewed as the inverse of the one above.

Conjecture 1.2.2 (Erdős, Simonovits, [Erd81]). *For every rational number $r \in (1, 2)$ there exists a graph G such that $\mathrm{ex}(n, G) = \theta(n^r)$.*

While the first conjecture seems to be still far from its solution, a recent work of Bukh and Conlon [BC] suggests a positive answer to Conjecture 1.2.2. They showed that Conjecture 1.2.2 holds if in the statement we replace the single graph G by a family of graphs $\mathcal{G}$. Their proof is yet another application of random algebraic method, which has been extensively used in Turán-type problems for bipartite graphs (see [ARS99, BBK11, Buk, KRS96]).

To conclude, determining the asymptotics of Turán numbers for bipartite graphs remains among the hardest problems in Extremal Combinatorics. An interested reader is referred to a comprehensive survey by Füredi and Simonovits on this subject [FS13].

1.3. The Turán Density and Its Properties

Using a simple averaging argument it is easy to see that for any family of r-graphs $\mathfrak{F}$, the ratios $\binom{n}{r}^{-1}\mathrm{ex}(n, \mathfrak{F})$ form a non-increasing sequence of real numbers in $[0, 1]$, therefore their limit

$$\pi(\mathfrak{F}) := \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathfrak{F})}{\binom{n}{r}}$$

exists. Indeed, by averaging over all n-vertex subsets in an r-graph on $n+1$ vertices we obtain:

$$\mathrm{ex}(n+1, \mathfrak{F}) \le \binom{n+1}{n}\frac{\mathrm{ex}(n, \mathfrak{F})}{\binom{n+1-r}{n-r}} = \frac{n+1}{n+1-r}\,\mathrm{ex}(n, \mathfrak{F}),$$

which implies that $\frac{\mathrm{ex}(n+1,\mathfrak{F})}{\binom{n+1}{r}} \le \frac{\mathrm{ex}(n,\mathfrak{F})}{\binom{n}{r}}$. (This argument is originally by Katona, Nemetz and Simonovits [Kat75].) $\pi(\mathfrak{F})$ is called the *Turán density* of $\mathfrak{F}$. The definition of Turán density for a single r-graph $\mathcal{F}$ is analogous. Observe that Turán density of the family $\mathfrak{F}$ captures the asymptotic behaviour of $\mathrm{ex}(n, \mathfrak{F})$ unless $\pi(\mathfrak{F}) = 0$. So for every family of r-graphs $\mathfrak{F}$, the first natural question is to determine $\pi(\mathfrak{F})$. The answer to such question is usually referred as the (Turán) *density result* for the family $\mathfrak{F}$. The second question is to determine $\mathrm{ex}(n, \mathfrak{F})$ for all n (or for all sufficiently large n). Such a result on $\mathrm{ex}(n, \mathfrak{F})$ is called the (Turán) *exact result* for the family $\mathfrak{F}$. Frequently one first obtains the density result, then the exact one. So now let us see when is the density result non-trivial, that is, when does a family $\mathfrak{F}$ of r-graphs have non-zero Turán density?

If for an r-graph $\mathcal{F}$, $\pi(\mathcal{F}) = 0$ both $\mathcal{F}$ and the corresponding problem of determining $\mathrm{ex}(n, \mathcal{F})$ are called *degenerate*. As mentioned in the previous section, note that all bipartite graphs are degenerate. For general $r \ge 2$, it is easy to see that $\pi(\mathcal{F}) > 0$ if $\mathcal{F}$ is not an r-partite r-graph, since then the balanced blowup of an r-edge on n vertices gives a non-zero lower bound on $\pi(\mathcal{F})$. In fact, it shows that for every such $\mathcal{F}$, $\pi(\mathcal{F}) \geq \frac{r!}{r^r}$. Erdős [Erd64] proved that the opposite holds as well.

Theorem 1.3.1 (Erdős, [Erd64]). *For any $r \geq 2$ and any $t \geq 1$, there exists $c = c(r,t) > 0$ and $n_0 = n_0(r,t)$ such that for all $n \geq n_0$,*

$$\mathrm{ex}\left(n, \mathcal{K}^{(r)}_{t,t,\ldots,t}\right) \leq cn^{r-\frac{1}{t^{r-1}}},$$

where $\mathcal{K}^{(r)}_{t,t,\ldots,t}$ is the complete r-partite r-graph with partition classes of size t.

It is easy to see that Theorem 1.3.1 implies that $\pi\left(\mathcal{K}^{(r)}_{t,t,\ldots,t}\right) = 0$ and hence, $\pi(\mathcal{F}) = 0$ for every r-partite r-graph. Now for any finite family of r-graphs $\mathfrak{F}$, $\pi(\mathfrak{F}) \leq \min_{\mathcal{F}\in\mathfrak{F}} \pi(\mathcal{F})$ holds. (This inequality might be strict, we discuss this phenomenon at the end of the section). Hence, if $\mathfrak{F}$ contains an r-partite r-graph then $\pi(\mathfrak{F}) = 0$ by Theorem 1.3.1. It is easy to check that the opposite holds as well. If $\mathfrak{F}$ does not contain any r-partite r-graph then the balanced blowup of an r-edge on n vertices gives a non-zero-lower bound on $\pi(\mathfrak{F})$, just as for a single graph case. So the only degenerate graphs are the r-partite r-graphs. Consequently, a family of r-graphs $\pi(\mathfrak{F})$ has non-zero Turán density if and only if it does not contain any r-partite r-graph.

Theorem 1.3.1 shows that there are no Turán densities in the range $(0, \frac{r!}{r^r})$, the numbers in this interval are called *jumps*. Let the set of all possible Turán densities for $r \geq 2$ be denoted by Γ_r, that is,

$$\Gamma_r = \{\pi(\mathfrak{F}) : \mathfrak{F} \text{ is a family of } r\text{-graphs } \}.$$

Erdős-Stone-Simonovits Theorem stated earlier, completely characterizes Γ_2. It says that

$$\Gamma_2 = \left\{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{m-1}{m}, \ldots\right\}.$$

For general $r \geq 3$, such a characterization is unknown. We say that a real number $\alpha \in [0,1)$ is a *jump* for r, if there exists $c = c(\alpha) > 0$ such that $\Gamma_r \cap (\alpha, \alpha + c) = \emptyset$. Equivalently, $\alpha \in [0,1)$ is a *jump* for r, if there exists $c = c(\alpha) > 0$ such that for any $\varepsilon > 0$ and any integer m, $m \geq r$ there exists $n_0 = n_0(\varepsilon, m)$ such that any r-graph $\mathcal{G}$ on $n \geq n_0$ vertices with $\mathrm{e}(G) \geq (\alpha + \varepsilon)\binom{n}{r}$ contains a subgraph $\mathcal{H}$ on m vertices with $\mathrm{e}(\mathcal{H}) \geq (\alpha + c)\binom{m}{r}$. From Erdős-Stone-Simonovits Theorem it follows that every $\alpha \in [0, 1)$ is a jump for 2-graphs. Note that Theorem 1.3.1 implies that every $\alpha \in [0, \frac{r!}{r^r})$ is a jump for all $r \geq 2$. The famous *jumping constant conjecture* of Erdős says that each $\alpha \in [0, 1)$ is a jump for all $r \geq 2$. This was disproved by Frankl and Rödl [FR84]. They showed that for each $r \geq 3$ an infinite sequence of non-jumps exists. Until recently not much else was known about the structure of Γ_r, for $r \geq 3$. In 2012, Pikhurko [Pik14] obtained crucial results on the structure of these sets. In particular, he showed that they have cardinality of the continuum, thus proving that the sets of non-jumps for each $r \geq 3$ have cardinality of the continuum as well. However, still many questions remain unanswered in this area. For example, the next two questions are credited to Erdős [Erd64]. Is $\frac{r!}{r^r}$ a jump for $r \geq 3$? What are the smallest non-jumps for each $r \geq 3$?

We conclude this section by a small discussion of what is called, the *non-prinicipality* of Turán densities for $r \geq 3$. For any finite family of 2-graphs $\mathfrak{G}$, Erdős-Stone-Simonovits theorem (Theorem 1.2.1) implies that $\pi(\mathfrak{G}) = \min_{G \in \mathfrak{G}} \pi(G)$. So we say that the Turán density for graphs is *principal*. The same is not true for r-graphs, when $r \geq 3$. Balogh [Bal02] showed a family of 3-graphs $\mathfrak{F}$ such that $\pi(\mathfrak{F}) < \min_{\mathcal{F} \in \mathfrak{F}} \pi(\mathcal{F})$, thus proving a conjecture of Mubayi and Rődl [MR02]. In fact, even a family of two hypergraphs may be non-principal. Indeed, Mubayi and Pikhurko [MP08] showed the existence of such families for every $r \geq 3$.

1.4. Turán-type Problems for Hypergraphs

The very first Turán-type result for hypergraphs is the famous Erdős-Ko-Rado theorem (published in 1961 but proved in 1938, see [Erd87]). An r-graph is called *intersecting* if it contains no two disjoint edges.

Theorem 1.4.1 (Erdős, Ko, Rado [EKR61])**.** *If $\mathcal{F}$ is an intersecting r-graph on $n \geq 2r$ vertices then*

$$|\mathcal{F}| \leq \binom{n-1}{r-1}.$$

In contrast to graphs, Turán-type problems turn out to be much harder for hypergraphs. In 1961 Turán [Tur61], as a natural generalization of his result on complete graphs, proposed to determine the Turán numbers of complete r-graphs on t vertices, $\mathcal{K}_t^{(r)}$, for $r \geq 3$. To this date none of these numbers are known even assymptotically for any $t > r \geq 3$. Erdős offered \$500 for the solution of any case and \$1000 for a general solution. The surveys of De Caen [DC91] and Sidorenko [Sid] contain most of the known bounds on these numbers. Some progress on special cases have been achieved in the past two decades but the best general known bound is still from 1980's by De Caen.

Theorem 1.4.2 (De Caen, [Cae83]). *For any $n \geq t \geq r$,*

$$\mathrm{ex}\left(n, \mathcal{K}_t^{(r)}\right) \leq \binom{n}{r} - \frac{\binom{n}{r}}{\binom{t-1}{r-1}} \times \frac{n-t+1}{n-r+1}.$$

As for special cases, the most studied one is the first non-trivial one, that is, $t = 4$, $r = 3$. In [Tur61] Turán conjectured that $ex\left(n, \mathcal{K}_4^{(3)}\right)$ is obtained for the graph $\mathcal{T}_4^{(3)}(n)$ defined as follows. Equipartition the vertex set $[n]$ into V_1, V_2, V_3. A triple is an edge in $\mathcal{T}_4^{(3)}(n)$ if and only if it either intersects all V_i's or contains two vertices of V_i and one from $V_{(i+1) \bmod 3}$. It is easy to check that $\mathcal{T}_4^{(3)}(n)$ has edge density $5/9$ and every four vertices span at most three edges, hence we get a lower bound $\pi\left(\mathcal{K}_4^{(3)}\right) \geq 5/9$. Turán conjectured that this is the exact value of $\pi\left(\mathcal{K}_4^{(3)}\right)$.

Conjecture 1.4.1 (Turán, [Tur61]). $\pi\left(\mathcal{K}_4^{(3)}\right) = \frac{5}{9}$.

Many people worked on this problem, including Katona, Nemetz, Simonovits, de Caen, Giraud, Chung and Lu, Razborov and others, but the problem remains unsolved up to this date. The main difficulty lies in the non-uniqueness of the conjectured extremal examples, as proven by Brown [Bro66] and Todorov [Tod84]. In fact, Kostochka [Kos82] showed that if Conjecture 1.4.1 is true, then there are exponentially many non-isomorphic $\mathcal{K}_4^{(3)}$-free graphs achieving the conjectured bound. The current best known on $\pi\left(\mathcal{K}_4^{(3)}\right)$ is by Razborov [Raz10], who used the Flag Algebra method to obtain the bound $\pi\left(\mathcal{K}_4^{(3)}\right) \leq 0.561666$.

After Turán's original question on $\mathcal{K}_t^{(r)}$, people started to study Turán numbers of other hypergraphs, which on top of being interesting on their own, also helped the community to develop various methods in this area, in hopes of being useful for attacking the original problem. Nevertheless, the Turán numbers are exactly known only for a handful number of hypergraphs. The survey of Keevash [Kee11a] includes all such numbers known up to 2011. It is also worth to mention that almost all of these numbers are only known for sufficiently large n, for smaller n these problems are much harder, at least the methods available in the literature do not apply.

1.5. A Quick Glimpse Into The Methodology

In the course of recent years, many powerful methods have been developed in the theory of Turán-type problems, including but not limited to the stability method, flag algebras, algebraic and probabilistic methods. See the survey of Keevash [Kee11a] for in-depth overview of all these methods. In this thesis we mainly use the stability method. It is a powerful tool for problems when the extremal (conjectured) configuration is unique. For graphs it was introduced in late 1960's by Erdős and Simonovits who obtained the stability result for Turán's theorem.

Theorem 1.5.1 (Erdős, Simonovits, [Sim68])**.** *For any $\varepsilon > 0$ there exists $\delta > 0$ such that if G is a K_{t+1}-free graph with at least $(1-\delta)\,\mathrm{ex}(n, K_{t+1})$ edges then there exists a partition of the vertices of G, say $\{P_1, P_2, \ldots, P_t\}$ such that*

$$\sum_{i=1}^{t} \mathrm{e}(G[P_i]) < \varepsilon n^2.$$

Turán's theorem implies that the largest K_{t+1}-free graph on given number of vertices is the balanced blowup of K_t. So, informally speaking, this stability result says that any K_{t+1}-free graph G on n vertices with approximately $\mathrm{ex}(n, K_{t+1})$ many edges is "close" to the balanced blowup of K_t with respect to the edit distance.

In general, the stability method allows us to obtain the exact result for a family of hypergraphs (graphs) or a single hypergraph (graph) once the corresponding stability result is settled, given that the (conjectured) extremal configuration is unique. For any Turán-type problem the method has two main steps. First, one shows that every graph or hypergraph that has edge density close to the optimal (conjectured) one, has approximately the "correct" structure. The second step is to show that any imperfection in the structure leads to a suboptimal configuration.

For hypergraphs the stability method was first used by Keevash and Mubayi in [KM04]. The same year, Füredi, Simonovits [FS05] and Keevash, Sudakov [KS05a] independently applied stability approach to determine the Turán number of the Fano plane, the finite projective plane of order two, thus proving a conjecture of Sós from 1976. After that, this method became more popular in hypergraph Turán theory([FPS05, FPS06, KS05b, Mub06, Pik08, Pik05]), even for degenerate problems ([KM10, MV07]). We further discuss the method in Chapter 3.

Together with stability approach, we also use the Lagrangian function. For hypergraphs it was introduced independently by Frankl and Rödl [FR84] and Sidorenko [Sid87], generalizing the work of Motzkin and Straus [MS65], who used the Lagrangian function for graphs to give a new proof of Turán's Theorem. The Lagrangian of an r-graph $\mathcal{F}$ is defined as follows:

$$\lambda(\mathcal{F}) = \max_{\mu \in \mathcal{M}(\mathcal{F})} \sum_{F \in \mathcal{F}} \prod_{v \in F} \mu(v),$$

where $\mathcal{M}(\mathcal{F})$ is the set of all probability distributions on the vertex set $V(\mathcal{F})$, that is, the set of functions $\mu : V(\mathcal{F}) \to [0,1]$ such that $\sum_{v \in V(\mathcal{F})} \mu(v) = 1$. One can think of the Lagrangian of an r-graph as the probability of sampling an edge given some distribution on vertices. We describe how Lagrangians are used for Turán-type problems in full detail in Section 2.2. But let us here mention one of its properties to give a glimpse how the Turán density of an r-graph is linked to the Lagrangian of some family of r-graphs.

For any two r-graphs $\mathcal{F}$ and $\mathcal{G}$, any edge-preserving map $\varphi : V(\mathcal{F}) \to V(\mathcal{G})$ is called *homomorphism*, that is, for every $F \in \mathcal{F}$, $\varphi(F) \in \mathcal{G}$. If such a map exists we say that $\mathcal{F}$ *admits a homomorphism to* $\mathcal{G}$. $\mathcal{G}$ is called $\mathcal{F}$*-hom-free* if there is no homomorphism from $\mathcal{F}$ to $\mathcal{G}$. For a family of r-graphs $\mathfrak{F}$, we say that $\mathcal{G}$ is $\mathfrak{F}$*-hom-free* if $\mathcal{G}$ is $\mathcal{F}$-hom-free for every $\mathcal{F} \in \mathfrak{F}$. The following lemma was first established by Frankl, Rödl [FR84] and independently by Sidorenko in [Sid87].

Lemma 1.5.2. *For any family* $\mathfrak{F}$ *of* r*-graphs,*

$$\pi(\mathfrak{F}) = r! \sup_{\mathcal{G} \in \mathrm{Forb}_{\mathrm{hom}}(\mathfrak{F})} \lambda(\mathcal{G}),$$

where $\mathrm{Forb}_{\mathrm{hom}}(\mathfrak{F})$ *is the family of all* r*-graphs that are* $\mathfrak{F}$*-hom-free.*

In this thesis we develop a generalization of the classical stability method by utilizing the Lagrangian function; we call it *local stability method* (see Chapter 3). This method is a tool to obtain the exact Turán result from the corresponding Turán density result. One does so by proving stability around the extremal family and stability in weighted (Lagrangian) setting (this is obtained using the density result). Then further reductions allow to prove the stability only in some local neighbourhood of the extremal configuration. In the method we also use the so-called *symmetrization procedure*, whose pioneers were Zykov [Zyk49] and Sidorenko [Sid87].

1.6. Our Results

We study Turán-type problems for hypergraphs and a conjecture of Erdős on triangle-free graphs. In particular, we develop previously mentioned local stability method and apply it to find new Turán numbers for several hypergraphs. Next we discuss all our results briefly.

1.6.1. The Turán Number Of The Generalized Triangle

The *generalized triangle*, $\mathcal{T}_r$ is an r-graph on vertex set $\{1, 2, \ldots, 2r-1\}$ with three edges, $\{1, 2, \ldots, r\}$, $\{1, 2, \ldots, r-1, r+1\}$ and $\{r, r+1, \ldots, 2r-1\}$. (Alternatively, once could adopt a more symmetric definition, that is, with edges $\{1, 2, \ldots, r\}$, $\{r, r+1, \ldots, 2r-1\}$, $\{1, 2, \ldots, r-1, 2r-1\}$, but we adopt the first version as it is easier to think of as an "extension" of two edges sharing $(r-1)$-vertices.) The origins of the question on the Turán number of the generalized triangle, go back to Katona [Kat75]. As a generalization of Mantel's theorem for graphs, he proposed to study the Turán number of the family of all r-graphs with three edges such that one edge contains the symmetric difference of the other two (denoted by $\mathfrak{T}_r$, $r \geq 3$). For

$r = 3$ this question was answered by Bollobás in [Bol74] who showed that the Turán number of this family is the same as the one for the family where we require two of the edges intersect in exactly $(r-1)$ points. More formally, Σ_r is defined to be the family of all r-graphs with three edges D_1, D_2, D_3 such that $|D_1 \cap D_2| = r-1$ and $|D_1 \triangle D_2| \subseteq D_3$. Bollobás also conjectured that the Turán numbers of the families $\mathfrak{T}_r$ and Σ_r are the same for all $r \geq 3$, which turned out not to be the case, as shown by Shearer in [She96] (the conjecture fails starting $r \geq 10$). However, the analogous question for Σ_r and $\mathcal{T}_r$ remains open. In [FF89], where Frankl and Füredi showed that these numbers are asymptotically the same, also posed the following conjecture.

Conjecture 1.6.1 (Frankl, Füredi, [FF89])**.** *For every $r \geq 3$, there exists some n_0 such that for all $n \geq n_0$, $\mathrm{ex}(n, \mathcal{T}_r) = \mathrm{ex}(n, \Sigma_r)$.*

Frankl and Füredi also showed that the conjecture is true for $r = 3$ by determining the Turán number of $\mathcal{T}_3$. The conjecture for $r = 4$ follows from the results of Pikhurko [Pik08] and Sidorenko [Sid87]. In Chapter 4 we determine the Turán numbers of $\mathcal{T}_5$ and $\mathcal{T}_6$ for large n which together with density results obtained by Frankl and Füredi [FF89] verify the conjecture for $r = 5, 6$. Recall that a *Steiner system* $S(m, r, t)$ is an r-graph on m vertices such that every t-set is contained in a unique r-edge.

Theorem 1.6.1. *There exists n_0 such that for all $n \geq n_0$, $\mathrm{ex}(n, \mathcal{T}_r) = \mathrm{ex}(n, \Sigma_r)$ for $r = 5, 6$. Moreover, there exists some $n_0 := n_0(r)$ such that for all $n \geq n_0$ the extremal graphs are the balanced blowups of unique Steiner systems $S(11, 5, 4)$ and $S(12, 6, 5)$, for $r = 5$ and $r = 6$, respectively.*

1.6.2. Turán Numbers of Some Extensions

The methods developed in this thesis are more generally applicable for determining Turán numbers of so-called *extensions*. A pair of vertices of a hypergraph or a graph is called *uncovered* if it is not contained in any edge. Given an r-graph $\mathcal{G}$, the *extension of* $\mathcal{G}$, denoted by $\mathrm{Ext}(\mathcal{G})$, is an r-graph defined as follows. For every uncovered pair P in $\mathcal{G}$ we add $r-2$ new vertices $v_1^P, v_2^P, \ldots, v_{r-2}^P$ to $V(\mathcal{G})$, and add the edge $P \cup \{v_1^P, v_2^P, \ldots, v_{r-2}^P\}$ to $\mathcal{G}$. For example, the generalized triangle is the extension of the r-graph on $[r+1]$ with two r-edges, sharing $(r-1)$ vertices. In this section we state our results on Turán numbers of extensions of certain graphs. Our first result is related to the famous Erdős-Sós conjecture from 1962.

Conjecture 1.6.2 (Erdős, Sós). *If G is a simple graph of order n with average degree more than $k-2$, then G contains every tree on k vertices as a subgraph.*

This conjecture has been verified for several families of trees, and in early 1990's the proof of the conjecture for large enough k was announced by Ajtai, Komlós, Simonovits and Szemerédi. We say that a tree is *an Erdős-Sós-tree* if it satisfies the conjecture. Given a 2-graph G, define the $(r-2)$*-expansion of* G to be the r-graph obtained by adding $(r-2)$ vertices to G and enlarging each edge of G to contain these vertices. In [Sid89] Sidorenko obtained the Turán density result for the extensions of $(r-2)$-expansions of sufficiently large Erdős-Sós-trees.

Theorem 1.6.2 (Sidorenko, [Sid89]). *For every $r \geq 2$, there exists M_r such that if T is an Erdős-Sós-tree on $t \geq M_r$ vertices then $\pi(\mathrm{Ext}(\mathcal{T})) = r!(t+r-3)^{-r}\binom{t+r-3}{r}$, where $\mathcal{T}$ is the $(r-2)$-expansion of T.*

Note that the quantity $(t+r-3)^{-r}\binom{t+r-3}{r}$ above is the Lagrangian of the complete r-graph on $(t+r-3)$ vertices. In Chapter 5 we prove the following exact version of Theorem 1.6.2.

Theorem 1.6.3. *For every $r \geq 2$, there exists M_r such that the following holds. Let T be an Erdős-Sós-tree on $t \geq M_r$ vertices and let $\mathcal{T}$ be the $(r-2)$ expansion of T. Then there exists n_0 such that the balanced blowup of $\mathcal{K}^{(r)}_{t+r-3}$ on n vertices is the unique $\mathrm{Ext}(\mathcal{T})$-free r-graph on n vertices with the maximum number of edges for all $n \geq n_0$.*

The next result concerns extensions of a different class of sparse hypergraphs. Let $\bar{\mathcal{K}}^{(r)}_t$ denote the edgeless r-graph on t vertices. Mubayi [Mub06] determined the Turán density of $\mathrm{Ext}(\bar{\mathcal{K}}^{(r)}_t)$ and Pikhurko [Pik05] obtained the corresponding exact result.

Theorem 1.6.4 (Pikhurko, [Pik05]). *For every $t > r \geq 3$ there exists n_0 such that the balanced blowup of $\mathcal{K}^{(r)}_t$ on n vertices is the unique $\mathrm{Ext}(\bar{\mathcal{K}}^{(r)}_{t+1})$-free r-graph on $n \geq n_0$ vertices with the maximum number of edges.*

Keevash [Kee11a] considered the following generalization of the above problem. Let $\mathcal{F}$ be any r-graph that covers pairs, and let $\mathcal{F}^{+t}$ be obtained from $\mathcal{F}$ by adding new isolated vertices so that $\mathrm{v}(\mathcal{F}^{+t}) = t$. (We have $\emptyset^{+t} = \bar{\mathcal{K}}_t^{(r)}$, where $\emptyset$ denotes the null r-graph.) Keevash generalized the density argument from [Mub06] as follows.

Theorem 1.6.5 (Keevash, [Kee11a])**.** *Let $\mathcal{F}$ be an r-graph that covers pairs with $\mathrm{v}(\mathcal{F}) \leq t+1$. If $\pi(\mathcal{F}) \leq r!t^{-r}\binom{t}{r}$, then $\pi(\mathrm{Ext}(\mathcal{F}^{+(t+1)})) = r!t^{-r}\binom{t}{r}$.*

We obtain the exact version of a slight weakening of Theorem 1.6.5.

Theorem 1.6.6. *Let $\mathcal{F}$ be an r-graph that covers pairs with $\mathrm{v}(\mathcal{F}) \leq t$. If $\pi(\mathcal{F}) < r!t^{-r}\binom{t}{r}$ then there exists n_0 such that the balanced blowup of $\mathcal{K}_t^{(r)}$ on $n \geq n_0$ vertices is the unique $\mathrm{Ext}(\mathcal{F}^{+(t+1)})$-free r-graph on n vertices with maximum number of edges.*

The proofs of Theorem 1.6.3 and Theorem 1.6.6 share a common part; the local stability around extremal configurations, therefore we prove these results in the same chapter, Chapter 5.

1.6.3. The Turán Number of The Extension of a Two-Matching

Let ${\mathcal{M}_2}^{(r)}$ be the r-graph on $[2r]$ with two disjoint edges, that is, a two-matching. In [HK13] Hefetz and Keevash found the Turán number of $\mathrm{Ext}({\mathcal{M}_2}^{(3)})$ and showed that the unique extremal graphs are the balanced blowups of $\mathcal{K}_5^{(3)}$ on n vertices, for large enough n. In the same paper, Hefetz and Keevash also posed the question of determining the Turán number of $\mathrm{Ext}\left({\mathcal{M}_2}^{(r)}\right)$, for $r \geq 4$ and suggested that the extremal configurations are star-like objects. Let us define them formally.

We say that a partition (A, B) of the vertex set of an r-graph $\mathcal{F}$ is a *star-partition* if for every $F \in \mathcal{F}$, $|F \cap A| = 1$. We say that $\mathcal{F}$ is a *star* if it admits a star-partition. We denote by $\mathcal{S}^{(r)}(n)$ the r-graph on n vertices that is a star and has the maximum possible number of edges. It is easy to check that $|\mathcal{S}^{(r)}(n)| = \left(1 - \frac{1}{r}\right)^{r-1}\binom{n}{r} + o(n^r)$ and, moreover, if (A, B) is a star-partition of $\mathcal{S}^{(r)}(n)$, then $|A| = \frac{n}{r} + o(n)$. Hefetz and Keevash conjectured that for $r \geq 4$ and large enough n, the extremal graph for $\mathrm{Ext}\left({\mathcal{M}_2}^{(r)}\right)$ is $\mathcal{S}^{(r)}(n)$. We settle this conjecture positively.

Theorem 1.6.7. *For every $r \geq 4$, there exists $n_0 := n_0(r)$ such that for all $n \geq n_0$, the largest* $\operatorname{Ext}\left(\mathcal{M}_2{}^{(r)}\right)$*-free r-graph on n vertices is unique and is $\mathcal{S}^{(r)}(n)$.*

1.6.4. Erdős's Conjecture On Sparse Halves

Mantel's theorem implies that every graph on n vertices with more than $n^2/4$ edges contains a triangle. The following generalization of Mantel's theorem was first studied by Erdős, Faudree, Rousseau and Schelp [EFRS94]. Suppose for given $0 < \alpha \leq 1$ every set of αn vertices of a graph spans more than βn^2 edges. What is the smallest $\beta = \beta(\alpha)$ such that every such graph necessarily contains a triangle? In particular, one of the old and favorite conjectures of Erdős is on $\beta(\frac{1}{2})$. We say that a graph has a *sparse half* if there is a set of $\lfloor n/2 \rfloor$ vertices that spans at most $n^2/50$ edges.

Conjecture 1.6.3 (Erdős, [Erd75b])**.** *Every triangle-free graph has a sparse half.*

Here the bound $1/50$ is tight, it is achieved by the balanced blowup of C_5, the cycle on five vertices, and the balanced blowup of the Petersen graph. To the best of our knowledge, no other extremal examples are known.

In [Kri95a] Krivelevich proved that the conjecture holds if $1/50$ is replaced by $1/36$. He also showed that it is true for triangle-free graphs with minimum degree at least $2n/5$. Later, Keevash and Sudakov [KS06a] showed that the conjecture holds for graphs with average degree $2n/5$. We improve on these previous results in three directions.

Theorem 1.6.8. *Every triangle-free graph on n vertices with minimum degree at least $5n/14$ has a sparse half.*

Theorem 1.6.9. *There exists some constant $\alpha > 0$ such that every triangle-free graph on n vertices with average degree $(2/5 - \alpha)n$ has a sparse half.*

Theorem 1.6.10. *There exists $\delta > 0$ such that if a triangle-free graph on n vertices is in the δ-neighbourhood (w.r.t. edit distance) of the balanced blowup of the Petersen graph, then it has a sparse half.*

1.6.5. Summary

Despite substantial technical differences in our methods for graphs and hypergraphs they still have some similarities in fashion, in the spirit of recent results of Lovász [Lov11] and Razborov [Raz13a]. To explain, let us mention that Lovász [Lov11] proved the Sidorenko conjecture locally in the neighborhood of the conjectured extremal example, and Razborov [Raz13a] accomplished a similar goal for the Caccetta-Häggkvist conjecture. And in this sense, some of our results can also be viewed as proving certain conjectures around some local neighbourhoods of extremal examples. But for hypergraphs, in addition, we also establish a framework which under some technical conditions allows one to derive the global result from its local version.

The rest of this thesis is organized as follows. Chapter 2 is a brief review of some results in Extremal Graph Theory which we will be used in the subsequent chapters. In Chapter 3 we develop the local stability method. Afterwards we apply this method to prove Theorems 1.6.1, 1.6.3, 1.6.6, 1.6.7 in Chapters 4, 5 and 6 respectively. Chapter 7 is devoted to Erdős's conjecture on sparse halves. Finally in Chapter 8 we discuss some directions for future research.

Chapter 2

The Lagrangian Function

The main motivation behind this chapter is to show the applications of the Lagrangian function in Turán-type problems. To do so, we discuss a notion in combinatorics called supersaturation, discovered by Erdős and Simonovits [ES83] in 1983. Then we introduce weighted graphs and show how certain parameters are related in weighted and non-weighted setting, in particular for families of graphs which are closed under the operation of taking blowups.

2.1. Supersaturation and Homomorphisms

Suppose we are given two r-graphs $\mathcal{F}$ and $\mathcal{H}$ on n vertices where n is some large number. If we know that $\mathcal{G}$ has density slightly above the Turán density of $\mathcal{F}$ then $\mathcal{G}$ must contain a copy of $\mathcal{F}$ by definition. The supersaturation tells us that in fact $\mathcal{G}$ contains many copies of $\mathcal{F}$, not just one.

Lemma 2.1.1 (Erdős, Simonovits, [ES83])**.** *For any r-graph $\mathcal{F}$ and $\varepsilon > 0$ there exist $\delta > 0$ and n_0 such that if $\mathcal{G}$ is an r-graph on $n \geq n_0$ vertices with $|\mathcal{G}| > (\pi(\mathcal{F})+\varepsilon)\binom{n}{r}$ then $\mathcal{G}$ contains at least $\delta\binom{n}{\mathrm{v}(\mathcal{F})}$ copies of $\mathcal{F}$.*

Supersaturation is quite useful in Turán-type problems, for example, it can be used to show that blowing up does not change the Turán density of a graph. The following lemma is folklore (see, for example, [Kee11b]). Recall that for an r-graph $\mathcal{F}$, $\mathcal{F}(m)$ denotes the balanced blowup of $\mathcal{F}$ on $m\,\mathrm{v}(\mathcal{F})$ vertices where each vertex is being replaced by an independent set of size m.

Lemma 2.1.2. *For any $m \in \mathbb{N}$ and any r-graph $\mathcal{F}$, $\pi(\mathcal{F}) = \pi(\mathcal{F}(m))$.*

Let us recall the definition of homomorphisms from Section 1.1. For any two r-graphs $\mathcal{F}$ and $\mathcal{G}$, any edge-preserving map $\varphi : V(\mathcal{F}) \to V(\mathcal{G})$ is called *homomorphism*, that is, for every $F \in \mathcal{F}$, $\varphi(F) \in \mathcal{G}$. $\mathcal{G}$ is called $\mathcal{F}$*-hom-free* if there is no homomorphism from $\mathcal{F}$ to $\mathcal{G}$. For a family of r-graphs $\mathfrak{F}$, we say that $\mathcal{G}$ is $\mathfrak{F}$*-hom-free* if $\mathcal{G}$ is $\mathcal{F}$-hom-free for every $\mathcal{F} \in \mathfrak{F}$.

Fact 2.1.3. *$\mathcal{G}$ is $\mathcal{F}$-hom-free if and only if for every integer m, $\mathcal{G}(m)$ is $\mathcal{F}$-free.*

It is natural to try to extend the notions of Turán number and the Turán density for homomorphisms. Let $\mathrm{ex}_{\mathrm{hom}}(n, \mathcal{F})$ be the maximum number of edges in an $\mathcal{F}$-hom-free r-graph on n vertices and $\pi_{\mathrm{hom}}(\mathcal{F}) = \lim_{n\to\infty} \mathrm{ex}_{\mathrm{hom}}(n, \mathcal{F})$. However, Lemma 2.1.2 and Fact 2.1.3 together imply that $\pi_{\mathrm{hom}}(\mathcal{F}) = \pi(\mathcal{F})$. Using this, we can actually approximate the Turán density of any r-graph. Let us show how $\mathcal{F}$-hom-free graphs can be used to give a lower bound on $\pi(\mathcal{F})$.

Given an r-graph $\mathcal{F}$, fix some N and suppose $\mathcal{G}$ is a largest $\mathcal{F}$-hom-free r-graph on N vertices with $\alpha\binom{N}{r}$ edges. It follows that $\pi(\mathcal{F}) \le \alpha$. On the other hand, since $\mathcal{G}$ is $\mathcal{F}$-hom-free, for any m the blowup $\mathcal{G}(m)$ is $\mathcal{F}$-free, hence

$$\pi(\mathcal{F}) \ge \frac{\alpha m^r \binom{N}{r}}{\binom{mN}{r}} = \alpha \prod_{i=1}^{r-1} \left(1 - \frac{i}{N}\right).$$

This gives an approximation of $\pi(\mathcal{F})$ with an error-term of cardinality $O(\frac{r^2}{N})$. Of course, the larger N, the better the approximation, but even for small values of N it can be hard to find such a graph $\mathcal{G}$. In this perspective, the theory of flag algebras developed by Razborov [Raz07] gives more sophisticated search techniques.

2.2. Lagrangians

The Lagrangian function for hypergraphs was introduced independently by Frankl and Rödl [FR84] and Sidorenko [Sid87], generalizing the work of Motzkin and Straus [MS65], who used the Lagrangian function for graphs to give a new proof of Turán's Theorem in 1965. In this section we prove some standard properties of the Lagrangian and provide some simple applications of the Lagrangian method for Turán-type problems.

Let $\mathcal{F}$ be an r-graph, recall the definition of $\mathcal{M}(\mathcal{F})$ from the introduction, it is the set of all probability distributions on $V(\mathcal{F})$, that is, the set of functions $\mu : V(\mathcal{F}) \to [0,1]$ such that $\sum_{v \in V(\mathcal{F})} \mu(v) = 1$. Let us also remind the definition of the *Lagrangian* $\lambda(\mathcal{F})$ *of an* r*-graph* $\mathcal{F}$:

$$\lambda(\mathcal{F}) := \max_{\mu \in \mathcal{M}(\mathcal{F})} \lambda(\mathcal{F}, \mu),$$

where $\lambda(\mathcal{F}, \mu) = \sum_{F \in \mathcal{F}} \prod_{v \in F} \mu(v)$. For a family of r-graphs $\mathfrak{F}$, the Lagrangian is defined as $\lambda(\mathfrak{F}) := \sup_{\mathcal{F} \in \mathfrak{F}} \lambda(\mathcal{F})$. Note that we can look at $\lambda(\mathcal{F}, \mu)$ as a multivariable function of variables $\mu_v := \mu(v)$, $v \in V(\mathcal{F})$. For an r-graph $\mathcal{F}$ and any $\mu \in \mathcal{M}(\mathcal{F})$ we define the *support* of μ to be the following subset of vertices:

$$\text{supp}_{\mathcal{F}}(\mu) := \{v \in V(\mathcal{F}) | \mu(v) > 0\}.$$

We skip the index $\mathcal{F}$ whenever the graph $\mathcal{F}$ is clear from the context. The following properties of Lagrangians were first established in [FR84] and [Sid87], we include the proof for completeness. Recall that for $X \subseteq V(\mathcal{F})$, $\mathcal{F}[X]$ stands for the subgraph of $\mathcal{F}$ induced by X.

Lemma 2.2.1. *Let $\mathcal{F}$ be an r-graph. Suppose $\mu^* \in \mathcal{M}(\mathcal{F})$ is such that $\lambda(\mathcal{F}) = \lambda(\mathcal{F}, \mu^*)$ and $\text{supp}(\mu^*)$ is minimal, then*

(i) $\mathcal{F}[\text{supp}(\mu^*)]$ *covers pairs,*

(ii) $\frac{\partial \lambda(\mathcal{F}, \mu)}{\partial \mu_v}\big|_{\mu^*} = r\lambda(\mathcal{F})$, *for every* $v \in \text{supp}(\mu^*)$,

Proof. Suppose $\mathcal{F}[\text{supp}(\mu^*)]$ does not cover pairs. Let v_1 and v_2 be such that they are not contained in any edge together. Then for any $\mu \in \mathcal{M}(\mathcal{F})$, we can express $\lambda(\mathcal{F}, \mu)$ as follows:

$$\lambda(\mathcal{F}, \mu) = a \cdot \mu_{v_1} + b \cdot \mu_{v_2} + c,$$

where a, b and c are polynomials not depending on the variables μ_{v_1} and μ_{v_2}. Let $\mu_i \in \mathcal{M}(\mathcal{F})$ be defined as follows,

$$\mu_i(v) = \begin{cases} \mu^*(v), & \text{if } v \neq v_1, v_2 \\ 0, & \text{if } v = v_i, \\ \mu^*(v_1) + \mu^*(v_2), & \text{otherwise.} \end{cases}$$

Since $a \cdot \mu_{v_1} + b \cdot \mu_{v_2} \leq (\mu_{v_1} + \mu_{v_2}) \max\{a, b\}$, it follows that $\lambda(\mathcal{F}, \mu^*) \leq \max\{\lambda(\mathcal{F}, \mu_1), \lambda(\mathcal{F}, \mu_2)\}$, therefore there exists μ with $\lambda(\mathcal{F}, \mu) = \lambda(\mathcal{F}, \mu^*)$ and $\mu_{v_1} = 0$ or $\mu_{v_2} = 0$, which contradicts to the minimality of $\text{supp}(\mu^*)$. Thus, there is no such pair of vertices v_1, v_2.

By the minimality of $\text{supp}(\mu^*)$, μ^* is an interior extremal point, therefore the partial derivatives $\frac{\partial \lambda(\mathcal{F}, \mu)}{\partial \mu_v}|_{\mu^*}$ are equal for all $v \in \text{supp}(\mu^*)$. (This easily follows using Lagrangian multipliers and by the definition of the feasibility region of the Lagrangian function). Now let $c := \frac{\partial \lambda(\mathcal{F}, \mu)}{\partial \mu_v}|_{\mu^*}$, then clearly

$$r\lambda(\mathcal{F}, \mu^*) = \sum_{v \in \text{supp}(\mu^*)} \frac{\partial \lambda(\mathcal{F}, \mu)}{\partial \mu_v}|_{\mu^*} \mu^*(v) = c \sum_{v \in \text{supp}(\mu^*)} \mu^*(v) = c \cdot 1 = c,$$

as desired. □

Corollary 2.2.2. *For any* 2*-graph* G*,* $\lambda(G) = \frac{1}{2}\left(1 - \frac{1}{r-1}\right)$*, where* $r := \min\{t : K_t \notin G\}$*.*

Proof. A covering 2-graph is complete, hence Lemma 2.2.1 implies that the Lagrangian of a 2-graph is achieved on its maximum clique. It is easy to check that for a clique on r vertices, the Lagrangian is achieved with the uniform measure, that is, $\lambda(K_r) = \binom{r}{2}\frac{1}{r^2} = \frac{1}{2}\left(1 - \frac{1}{r}\right)$. This finishes the proof. □

Observe that the Turán theorem follows easily from Corollary 2.2.2 using the simple fact that for any r-graph $\mathcal{F}$, $|\mathcal{F}| \le \lambda(\mathcal{F})n^r$. (The above argument essentially repeats the proof of Motzkin and Straus [MS65]). Now let us show how the Turán density for any family is related to the Lagrangian function. The following lemma was first established by Frankl, Rödl in [FR84] and independently by Sidorenko in [Sid87].

Lemma 2.2.3. *For any family $\mathfrak{F}$ of r-graphs,*

$$\pi(\mathfrak{F}) = r! \sup_{\mathcal{G} \in \mathrm{Forb}_{\mathrm{hom}}(\mathfrak{F})} \lambda(\mathcal{G}),$$

where $\mathrm{Forb}_{\mathrm{hom}}(\mathfrak{F})$ *is the family of all r-graphs that are $\mathfrak{F}$-hom-free.*

Proof. Let $\mathcal{G} \in \mathrm{Forb}_{\mathrm{hom}}(\mathfrak{F})$. First we show that $\pi(\mathcal{F}) \ge r!\lambda(\mathcal{G})$. Any blowup (not necessarily balanced) of $\mathcal{G}$ is $\mathcal{F}$-free . Let $m_1, m_2, \ldots, m_{\mathrm{v}(\mathcal{G})}$ be any integers with $m = \sum_{i=1}^{\mathrm{v}(\mathcal{G})} m_i$. We have

$$\pi(\mathcal{F}) \ge \lim_{n\to\infty} \frac{|\mathcal{G}(nm_1, nm_2, \ldots, nm_{\mathrm{v}(\mathcal{G})})|}{\binom{mn}{r}} = r!\lambda(\mathcal{G}, \mu_0),$$

where $\mu_0 \in \mathcal{M}(\mathcal{G})$ with $\mu_0(v_i) = \frac{m_i}{m}$. This holds for any choice of integers m_i, that is, for any rational $\mu \in \mathcal{M}(\mathcal{F})$. Since every non-rational $\mu \in \mathcal{M}(\mathcal{F})$ can be approximated by a sequence of rational $\mu_n \in \mathcal{M}(\mathcal{F})$, it follows that $\pi(\mathcal{F}) \ge r!\lambda(\mathcal{G})$.

Now let us show the other direction. If $\xi \in \mathcal{M}(\mathcal{G})$ denotes the uniform measure, then

$$\lambda(\mathcal{G}) \ge \lambda(\mathcal{G}, \xi) = \frac{|\mathcal{G}|}{\mathrm{v}(\mathcal{G})^r} = \frac{|\mathcal{G}|}{r!\binom{\mathrm{v}(\mathcal{G})}{r}} - O\left(\frac{1}{\mathrm{v}(\mathcal{G})}\right) = \frac{1}{r!}d(\mathcal{G}) - O\left(\frac{1}{\mathrm{v}(\mathcal{G})}\right),$$

where $d(\mathcal{G})$ denotes the edge density of the graph $\mathcal{G}$ (recall the definition from the introduction). But since $\pi(\mathcal{F}) = \pi_{\mathrm{hom}}(\mathcal{F})$, then $\pi(\mathcal{F})$ is the limit supremum of $d(\mathcal{G})$ over all $\mathcal{F}$-hom-free $\mathcal{G}$ and hence, $\lambda(\mathcal{G}) \ge \frac{\pi(\mathcal{F})}{r!}$, as desired. □

We can further restrict the search of the graphs achieving the bound $\pi(\mathfrak{F})$ as follows. We say that an r-graph $\mathcal{F}$ is *dense* if for every proper subgraph $\mathcal{F}'$, $\lambda(\mathcal{F}') < \lambda(\mathcal{F})$. Note that dense graphs must cover pairs by Lemma 2.2.1. Thus we obtain the following corollary from Lemma 2.2.3.

Corollary 2.2.4. *For any family* $\mathfrak{F}$ *of* r*-graphs,*

$$\pi(\mathfrak{F}) = r! \sup_{\mathcal{G} \in \mathrm{Forb}_{\mathrm{hom}}^{\mathrm{dense}}(\mathfrak{F})} \lambda(\mathcal{G}),$$

where $\mathrm{Forb}_{\mathrm{hom}}^{\mathrm{dense}}(\mathfrak{F})$ *is the family of all dense* r*-graphs that are* $\mathfrak{F}$*-hom-free.*

Lemma 2.2.3 is very useful for determining Turán densities of graphs or families. For example, classical Erdős-Stone Theorem is easily implied by it.

Corollary 2.2.5 (Erdős, Stone, [ES46])**.** *Let* $\mathfrak{F}$ *be a family of* 2*-graphs, then*

$$\pi(\mathfrak{F}) = \frac{1}{2}\left(1 - \frac{1}{\min_{\mathcal{F}\in\mathfrak{F}} \chi(\mathcal{F}) - 1}\right).$$

Proof. The result easily follows from Lemma 2.2.3 and from the fact that for any r and any graph $\mathcal{F}$, K_r is $\mathcal{F}$-hom-free if and only if $\chi(\mathcal{F}) > r$. □

Now let us show an application of the Lagrangian function to a Turán-type problem for hypergraphs. Let Σ_r be defined as the family of all r-graphs with three edges D_1, D_2, D_3 such that $|D_1 \cap D_2| = r-1$ and $D_1 \triangle D_2 \subseteq D_3$. For demonstration, we show Sidorenko's argument on the Turán number of Σ_4 for all n multiples of four.

Theorem 2.2.6 (Sidorenko, [Sid87])**.** $ex(n, \Sigma_4) = \left(\frac{n}{4}\right)^4$ *for all* $4|n$*.*

Proof. Let $\mathcal{F}$ be a Σ_4-free graph on n vertices. It is enough to show that $\lambda(\mathcal{F}) \le 1/4^4$. Indeed, then from the definition of the Lagrangian it follows that $|\mathcal{F}| \le \left(\frac{n}{4}\right)^4$ and as for the lower bound, the balanced blowup of $K_4^{(4)}$ on n vertices clearly does not contain any graph in Σ_4.

Now let $\mu \in \mathcal{F}$ be such that $\lambda(\mathcal{F}) = \lambda(\mathcal{F}, \mu)$. Let $X = \mathrm{supp}(\mu)$ and $m = |X|$. By Lemma 2.2.1, $\mathcal{F}[X]$ covers pairs. Since $\mathcal{F}$ is Σ_4-free, it follows that for any $F, F' \in \mathcal{F}[X]$, $|F \cap F'| \le 2$. Thus for any two vertices $u, v \in X$, the links $L_{\mathcal{F}}(u), L_{\mathcal{F}}(v)$ are pairwise disjoint. The latter implies for any such pair u, v, the polynomials $\frac{\partial\lambda(\mathcal{F},\mu)}{\partial\mu_u}$, $\frac{\partial\lambda(\mathcal{F},\mu)}{\partial\mu_v}$ have no common term. Thus, by Lemma 2.2.1 (ii),

$$4m\lambda(\mathcal{F}) = \sum_{v\in X} \frac{\partial\lambda(\mathcal{F},\mu)}{\partial\mu_v} \le \sum_{1\le i<j<k<m} \mu_{v_i}\mu_{v_j}\mu_{v_k} \le m^{-3}\binom{m}{3},$$

where the last inequality holds because $\sum_{1\leq i<j<k<m} \mu_{v_i}\mu_{v_j}\mu_{v_k}$ is the third elementary symmetric polynomial with m variables. From here it follows that

$$\lambda(\mathcal{F}) \leq \frac{(m-1)(m-2)}{4!m^3},$$

which is less than or equal to $1/4^4$ unless $m = 5$ and moreover, the equality is achieved only for $m = 4$. Thus it remains to show that $m = 5$ case cannot happen. Indeed, for $\mathcal{F}[X]$ to cover pairs on $m = 5$ vertices it must have at least two edges, hence there exist two edges $F, F' \in \mathcal{F}[X]$ with $|F \cap F'| = 3$. Then, there must be an edge F'' covering the pair of vertices $\{u, v\} = F \triangle F'$. The edges F, F', F'' together induce a copy of a graph in Σ_4. □

From this result it follows that the balanced blowup of $K_4^{(4)}$, is an extremal example for Σ_4-free graphs, when $n|4$. In fact, Sidorenko [Sid87] showed that this holds for all n large enough and even more, the balanced blowup of $K_4^{(4)}$ on n vertices is the unique extremal example. For these results, he used the so-called *symmetrization* trick. The main idea is the following. For any r-graph $\mathcal{F}$ defined by some forbidden configurations (f.e. Σ_4-free) if there exist some two vertices that are not covered by a common edge, we can delete one and clone the other without introducing any forbidden subgraph nor decreasing the size of $\mathcal{F}$. In fact, one can apply this symmetrization operation to sets of vertices. For example, for Σ_4-free graphs, Sidorenko showed that for large enough n, any such graph on n vertices with maximum edge density either can be "symmetrized" to a hypergraph on at most four vertices that covers pairs (hence, it is $\mathcal{K}_4^{(4)}$) or to a hypergraph with small minimum degree. In the latter case one can remove a vertex from this graph and apply induction to the rest of the graph.

The essential idea of the symmetrization is to modify the graph until we reach the optimal configuration (the one obtained by the Lagrangian argument) and then show that the original graph is obtained from it by just blowing up the vertices appropriately. So one can think of the symmetrization as a tool to obtain the Turán number from the Lagrangian result.

However, in many cases the symmetrization alone does not give the desired result. One approach is to combine it with some kind of stability arguments. For example, in [Pik05] Pikhurko used this approach to determine the Turán number of the generalized triangle of uniformity four. In particular, he proved that Σ_4-free graphs are stable (we skip the rigorous definition of his notion of stability for brevity). The problem with the symmetrization argument itself is that it is not clear why it should preserve the property of being close to a 4-partite 4-graph? Could not the deleting and cloning a large set of vertices change the graph structure sharply? It might if we delete-clone a vertex of small degree. To overcome this potential problem, Pikhurko suggested to iteratively and constantly remove vertices whose degree becomes too small at any step of the symmetrization. This ensures that at every deletion-cloning step we have the large minimum degree. Then, when reversing one step, each undeleted vertex must have many incident edges, which forces all of them to fit perfectly into the 4-partition.

In Chapter 3 we utilize the symmetrization further and prove a theorem (Theorem 3.2.4), which provides a general framework for all families closed under the operation of taking blowups to obtain stability from the Lagrangian result. For this to work, we only require the stability to hold in some "local" setting. Note that our argument also allows us to overcome the above discussed potential problem of low degree vertices. For the proof we use the probabilistic method and induction.

2.3. Clonable Families

In this section we assume that all families of hypergraphs are closed under isomorphism. Let us first give an alternative definition of blowups of graphs. We say that an r-graph $\mathcal{G}$ is obtained from an r-graph $\mathcal{F}$ by *cloning a vertex v to a set W* if $\mathcal{F} \subseteq \mathcal{G}$, $V(\mathcal{G}) \setminus V(\mathcal{F}) = W \setminus \{v\}$ and $L_{\mathcal{G}}(w) = L_{\mathcal{F}}(v)$ for every $w \in W$. We say that $\mathcal{G}$ is *a blowup of $\mathcal{F}$* if $\mathcal{G}$ is isomorphic to an r-graph obtained from $\mathcal{F}$ by repeatedly cloning and deleting vertices. We denote the set of all blowups of $\mathcal{F}$ by $\mathfrak{B}(\mathcal{F})$. We say that a family $\mathfrak{F}$ of r-graphs is *clonable* if every blowup of any r-graph in $\mathfrak{F}$, also lies in $\mathfrak{F}$. The Hypergraph Removal Lemma [Gow07, RS06a] (which we state in Chapter 3) allows one to restrict many arguments related to Turán-type problems to clonable families, and some of the more general results of this paper hold for all clonable families.

Let us introduce another class of hypergraph families, which are important for us. For a family of r-graphs $\mathfrak{F}$, let

$$m(\mathfrak{F}, n) := \max_{\substack{\mathcal{F} \in \mathfrak{F} \\ \mathrm{v}(\mathcal{F}) = n}} |\mathcal{F}|.$$

We say that $\mathfrak{F}$ is *smooth* if there exists $\lim_{n\to\infty} m(\mathfrak{F}, n)/n^r$. For a smooth family $\mathfrak{F}$ we denote the above limit by $m(\mathfrak{F})$. Our first lemma establishes a connection between clonable and smooth families.

Lemma 2.3.1. *Every clonable family is smooth.*

Proof. Let $\mathfrak{F}$ be a clonable family of r-graphs. Let

$$d := \limsup_{n\to\infty} \frac{m(\mathfrak{F}, n)}{n^r}.$$

We need to show that for every $0 < \varepsilon < 1$ there exists $N > 0$ such that $m(\mathfrak{F}, n)/n^r \geq d - \varepsilon$ for every $n \geq N$. Let $\mathcal{F} \in \mathfrak{F}$ be chosen so that $|\mathcal{F}| \geq (d - \delta)\,\mathrm{v}(\mathcal{F})^r$ for $\delta := \varepsilon/(d+1)$. Let $s := \mathrm{v}(\mathcal{F})$. For a positive integer k, let $\mathcal{F}^{(k)}$ be an r-graph obtained by cloning every vertex of $\mathcal{F}$ to a set of size k. Then $\mathcal{F}^{(k)} \in \mathfrak{F}$, $\mathrm{v}(\mathcal{F}^{(k)}) = ks$ and $|\mathcal{F}^{(k)}| = k^r|\mathcal{F}| \geq (d - \delta)(ks)^r$. Therefore, for $n \geq (s-1)r/\delta$, we have

$$\begin{aligned} \frac{m(\mathfrak{F}, n)}{n^r} &\geq (d - \delta)\left(\frac{s\lfloor n/s \rfloor}{n}\right)^r \geq (d - \delta)\left(1 - \frac{s-1}{n}\right)^r \\ &\geq (d - \delta)\left(1 - \frac{(s-1)r}{n}\right) \geq (d - \delta)(1 - \delta) \geq d - \varepsilon, \end{aligned}$$

as desired. □

2.4. Weighted Graphs

For any r-graph $\mathcal{F}$ and any $\mu \in \mathcal{M}(\mathcal{F})$, we call the pair $(\mathcal{F}, \mu)$ a *weighted graph.* We define *the density* $\lambda(\mathcal{F}, \mu)$ *of a weighted graph* $(\mathcal{F}, \mu)$, by

$$\lambda(\mathcal{F}, \mu) := \sum_{F \in \mathcal{F}} \prod_{v \in F} \mu(v).$$

Note that the Lagrangian of $\mathcal{F}$ is just the density of the largest weighted graph $(\mathcal{F}, \mu)$, over all $\mu \in \mathcal{M}(\mathcal{F})$.

If an r-graph $\mathcal{F}'$ is obtained from an r-graph $\mathcal{F}$ by cloning a vertex $u \in V(\mathcal{F})$ to a set W, $\mu \in \mathcal{M}(\mathcal{F})$, $\mu' \in \mathcal{M}(\mathcal{F}')$, then we say that $(\mathcal{F}', \mu')$ is *a one vertex blowup of* $(\mathcal{F}, \mu)$, if $\mu(v) = \mu'(v)$ for all $v \in V(\mathcal{F}) \setminus \{u\}$ and $\mu(u) = \sum_{w \in W} \mu'(w)$. We say that $(\mathcal{F}', \mu')$ is *a blowup* of $(\mathcal{F}, \mu)$ if $(\mathcal{F}', \mu')$ is isomorphic to a weighted r-graph which can be obtained from $(\mathcal{F}, \mu)$ by repeatedly taking one vertex blowups. Two weighted graphs $(\mathcal{F}, \mu)$ and $(\mathcal{F}', \mu')$ are *isomorphic* if there exists an isomorphism $\varphi : V(\mathcal{F}) \to V(\mathcal{F}')$ between $\mathcal{F}$ and $\mathcal{F}'$ such that $\mu'(\varphi(v)) = \mu(v)$ for every $v \in V(\mathcal{F})$. As in the case of unweighted graphs, we generally do not distinguish between isomorphic weighted graphs. We denote by $\mathfrak{B}(\mathcal{F}, \mu)$ the family of weighted graphs isomorphic to the blowups of $(\mathcal{F}, \mu)$.

Remark 2.4.1. *An r-graph $\mathcal{F}'$ is a blowup of $\mathcal{F}$ with $V(\mathcal{F}) = [n]$ if and only if there exists a partition $\{P_1, P_2, \dots, P_n\}$ of $V(\mathcal{F}')$ such that $\{v_1, v_2, \dots, v_r\} \in \mathcal{F}'$, $v_j \in P_{i_j}$ for $j \in [r]$ if and only if $\{i_1, i_2, \dots, i_r\} \in \mathcal{F}$. When $\mathcal{F}$ is understood from the context we refer to $\mathcal{P} = \{P_1, P_2, \dots, P_n\}$ as* a blowup partition of $\mathcal{F}'$. *If $\mathcal{F}$* covers pairs*, that is, for every $u, v \in V(\mathcal{F})$, there exists some $F \in \mathcal{F}$ containing u and v, then the blowup partition is unique up to the order of parts and its elements are the maximal independent sets in $\mathcal{F}$.*

Let us also note that a weighted r-graph $(\mathcal{F}', \mu')$ is a blowup of $(\mathcal{F}, \mu)$ if and only if there exists a partition as above with the additional property $\sum_{v \in P_i} \mu'(v) = \mu(i)$, for every $i \in [n]$.

Next we define the notion of distance between graphs both in unweighted and weighted setting. Recall that the edit distance for two unweighted graphs $\mathcal{F}$ and

$\mathcal{G}$ on the same vertex set is defined as $d(\mathcal{F}, \mathcal{G}) = |\mathcal{F} \triangle \mathcal{G}|$. For an r-graph $\mathcal{F}$ and a family of r-graphs $\mathfrak{H}$, let $d_{\mathfrak{H}}(\mathcal{F})$ be the *edit distance* from $\mathcal{F}$ to $\mathfrak{H}$, defined as

$$d_{\mathfrak{H}}(\mathcal{F}) = \min_{\substack{\mathcal{H} \in \mathfrak{H} \\ \mathrm{v}(\mathcal{H}) = \mathrm{v}(\mathcal{F})}} |\mathcal{F} \triangle \mathcal{H}|.$$

To define the distance between weighted graphs, we need couple of steps. If $\mathcal{F}_1, \mathcal{F}_2$ are two r-graphs such that $V(\mathcal{F}_1) = V(\mathcal{F}_2)$ and $\mu \in \mathcal{M}(\mathcal{F}_1) (= \mathcal{M}(\mathcal{F}_2))$, we define

$$d'(\mathcal{F}_1, \mathcal{F}_2, \mu) := \sum_{F \in \mathcal{F}_1 \triangle \mathcal{F}_2} \prod_{v \in F} \mu(v).$$

We define the distance between general weighted r-graphs $(\mathcal{F}_1, \mu_1)$ and $(\mathcal{F}_2, \mu_2)$, as

$$d((\mathcal{F}_1, \mu_1), (\mathcal{F}_2, \mu_2)) := \inf d'(\mathcal{F}'_1, \mathcal{F}'_2, \mu),$$

where the infimum is taken over all r-graphs $\mathcal{F}'_1, \mathcal{F}'_2$, with $V(\mathcal{F}'_1) = V(\mathcal{F}'_2)$ and $\mu \in \mathcal{M}(\mathcal{F}'_1) = \mathcal{M}(\mathcal{F}'_2)$ satisfying $(\mathcal{F}'_i, \mu) \in \mathfrak{B}(\mathcal{F}_i, \mu_i)$ for $i = 1, 2$. If $(\mathcal{F}, \mu)$ is a weighted r-graph and $\mathfrak{F}$ is a family of r-graphs we define *the distance from* $(\mathcal{F}, \mu)$ *to* $\mathfrak{F}$ as

$$d^w_{\mathfrak{F}}(\mathcal{F}, \mu) := \inf_{\mathcal{F}' \in \mathfrak{F}, \mu' \in \mathcal{M}(\mathcal{F}')} d((\mathcal{F}, \mu), (\mathcal{F}', \mu')).$$

Remark 2.4.2. *Note that d^w is not distance unless we identify any two weighted graphs (F_1, μ_0) and (F_2, μ_0), where $\mu_0 \equiv 0$. Indeed, otherwise there would exist distinct graphs with distance zero.*

We write $d_{\mathfrak{F}}(\mathcal{F}, \mu)$ instead of $d^w_{\mathfrak{F}}(\mathcal{F}, \mu)$, except for the cases when we want to emphasize the difference between weighted and unweighted distance. Next we explore how weighted and unweighted distances are related.

For an r-graph $\mathcal{F}$, let $\xi_{\mathcal{F}} \in \mathcal{M}(\mathcal{F})$ denote the uniform distribution on $V(\mathcal{F})$, that is, $\xi_{\mathcal{F}}(v) = 1/\mathrm{v}(F)$ for every $v \in V(\mathcal{F})$. We will omit the index and write ξ instead of $\xi_{\mathcal{F}}$ when $\mathcal{F}$ is understood from the context. It is easy to bound the weighted distance of $(\mathcal{F}, \xi)$ to a family $\mathfrak{H}$ by a function of unweighted distance as $d_{\mathfrak{H}}(\mathcal{F}, \xi) \leq \frac{1}{n^r} d_{\mathfrak{H}}(\mathcal{F})$. The other direction is more involved and we present it below.

Lemma 2.4.1. *For any clonable family $\mathfrak{H}$, if $\mathcal{F}$ is a graph with $\mathrm{v}(\mathcal{F}) = n$ then*

$$d_{\mathfrak{H}}(\mathcal{F}) \leq \frac{r!n}{n-r^2}\binom{n}{r} d^w_{\mathfrak{H}}(\mathcal{F},\xi).$$

Proof. Choose an arbitrary $0 < \varepsilon < 1$ and let $d := d^w_{\mathfrak{H}}(\mathcal{F},\xi)$. Let $(\mathcal{B},\mu)$ be a blowup of $(\mathcal{F},\xi)$ such that there exists $\mathcal{H} \in \mathfrak{H}$ satisfying $d((\mathcal{B},\mu),(\mathcal{H},\mu)) \leq d + \varepsilon$.

Let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ be a blowup partition of $V(\mathcal{B})$. Suppose $v_1, v_2, \ldots, v_r$ are chosen independently at random from $V(\mathcal{H})$ according to the distribution μ. Let A be the event that that $\{v_1, v_2, \ldots, v_r\}$ is a *transversal* of $\mathcal{P}$, that is, $|\{v_1, v_2, \ldots, v_r\} \cap P_j| \leq 1$ for every $P_j \in \mathcal{P}$. Since $\mu(P_i) = 1/n$ for every $1 \leq i \leq n$, we have

$$\mathbb{P}\left[A\right] = \prod_{i=0}^{r-1}\left(1 - \frac{i}{n}\right) \geq \left(1 - \frac{r}{n}\right)^r \geq 1 - \frac{r^2}{n}.$$

Thus, it follows that

$$\mathbb{P}\left[\{v_1, v_2, \ldots, v_r\} \in \mathcal{B} \triangle \mathcal{H} \mid A\right] \leq \frac{r!(d+\varepsilon)n}{n-r^2}. \tag{2.1}$$

Now consider $v_1, v_2, \ldots, v_n$ to be chosen independently at random according to the distribution given by μ, such that $v_i \in P_i$ for every $i \in [n]$. Let $\mathcal{H}'$ and $\mathcal{B}'$ be the random subgraphs induced by $\{v_1, v_2, \ldots, v_n\}$, respectively, in $\mathcal{H}$ and $\mathcal{B}$. It follows from (2.1) and the linearity of expectation that

$$\mathbb{E}\left[|\mathcal{B}' \triangle \mathcal{H}'|\right] \leq \frac{r!(d+\varepsilon)n}{n-r^2}\binom{n}{r}. \tag{2.2}$$

As $\mathcal{B}'$ is isomorphic to $\mathcal{F}$, the inequality (2.2) implies the lemma. □

Lemma 2.4.2. *For every weighted r-graph $(\mathcal{F},\mu)$ there exists a sequence $\{\mathcal{F}_n\}$ of blowups of $\mathcal{F}$, such that $\mathrm{v}(\mathcal{F}_n) \to_{n\to\infty}$ and*

(i) $\lim_{n\to\infty} \frac{|\mathcal{F}_n|}{\mathrm{v}(\mathcal{F}_n)^r} = \lambda(\mathcal{F},\mu)$

(ii) $\lim_{n\to\infty} \frac{d_{\mathfrak{H}}(\mathcal{F}_n)}{\mathrm{v}(\mathcal{F}_n)^r} = d_{\mathfrak{H}}(\mathcal{F},\mu)$ *for every clonable family $\mathfrak{H}$.*

Proof. Let $\mu_n \in \mathcal{M}(\mathcal{F})$ be a sequence of rational measures such that $\mu_n \to \mu$. For every n, consider $(\mathcal{F},\mu_n)$ and choose k_n integer such that $\mu_n(v)k_n$ is an integer for every $v \in V(\mathcal{F})$. Let $\mathcal{F}_n$ be an r-graph obtained by cloning $v \in V(\mathcal{F})$ to a set of size $\mu_n(v)k_n$. Then, clearly, $\mathrm{v}(\mathcal{F}_n) = k_n$ and $|\mathcal{F}_n| = \lambda(\mathcal{F}, \mu_n)k_n^r$. Now both (i) and (ii) follow easily, (in particular, (i) does because $\lambda(\mathcal{F}, \cdot)$ is continuous). $\square$

We can use Lemma 2.4.2 to establish the following connection between the parameters we explore in weighted and unweighted settings.

Lemma 2.4.3. *If $\mathfrak{F}$ is clonable, then $\lambda(\mathfrak{F}) = m(\mathfrak{F})$.*

Proof. By Lemma 2.3.1, $m(\mathfrak{F}) = \lim_{n\to\infty} \frac{m(\mathfrak{F},n)}{n^r}$ is finite. For any n,

$$\lambda(\mathfrak{F}) \geq \sup_{\mathcal{F}\in\mathfrak{F}} \lambda(\mathcal{F}, \xi_{\mathcal{F}}) \geq \frac{m(\mathfrak{F}, n)}{n^r},$$

therefore it follows that $\lambda(\mathfrak{F}) \geq m(\mathfrak{F})$.

For the other direction we use Lemma 2.4.2. We may assume that $\lambda(\mathfrak{F}) = \lambda(\mathcal{F}, \mu)$ for some $\mathcal{F} \in \mathfrak{F}$, $\mu \in \mathcal{M}(\mathcal{F})$. (Otherwise, we select a sequence $\mathcal{F}_m$ such that $\lambda(\mathcal{F}_m) \to \lambda(\mathfrak{F})$ and similar but slightly more technical arguments follow.) By Lemma 2.4.2 we get a sequence $\mathcal{F}_n$ with $\mathrm{v}(\mathcal{F}_n) \to \infty$ such that the conditions (i) and (ii) hold. For any such $\mathcal{F}_n$ with $\mathrm{v}(\mathcal{F}_n) = k_n$ we have

$$\frac{m(\mathfrak{F}, k_n)}{k_n^r} \geq \frac{|\mathcal{F}_n|}{k_n^r}.$$

Taking the limit on both sides as $n \to \infty$ and using the fact that $k_n \to \infty$ as $n \to \infty$, we get $m(\mathfrak{F}) \geq \lambda(\mathcal{F}, \mu) = \lambda(\mathfrak{F})$. $\square$

Chapter 3

Local Stability Method

In this chapter we formalize and extend the notion of classical stability method, which is ubiquitous in the analysis of Turán-type problems. To summarize, we establish a generic method which allows one to obtain exact Turán numbers from Turán density results. One does so by proving stability around the extremal family and stability in weighted setting (the latter is obtained using the density result). Then further reductions allow to prove the stability only in some local neighbourhood of the extremal configuration.

3.1. Classical Stability Method

In general, for any r-graph $\mathcal{F}$ there are two questions we are interested in. The first one is to determine $\pi(\mathcal{F})$, the second is to determine $\mathrm{ex}(n, \mathcal{F})$ for large n. We refer to the corresponding results as *density result* and *exact result*. The stability method is a tool to obtain the exact result from the density one. The idea is to use the density result to prove an approximate structure theorem for graphs with density close to the maximum possible, and then to find the exact structure that has maximum size among the approximate structures. This method was first introduced for graphs by Erdős and Simonovits in 1966 who obtained the stability result for Turán's theorem.

Theorem 3.1.1 (Erdős, Simonovits, [Sim68]). *For every $\varepsilon > 0$ there is $\delta > 0$ such that if G is a K_{t+1} -free graph with at least $(1-\delta)ex(n, K_{t+1})$ edges then there is a* *partition of the vertices of G, say $\{V_1, V_2, \dots, V_t\}$, with $\sum_{i=1}^{t} \mathrm{e}(G[V_i]) < \varepsilon n^2$.*

While such stability results are interesting on their own, they are more commonly used to prove the exact result, meaning, to show that all maximal $\mathcal{F}$-free graphs have the same (conjectured) optimal structure. For example, it is easy to obtain Turán's theorem for sufficiently large n from Theorem 3.1.1 (although note that originally Turán's theorem was proven much earlier). So how would we do it? Suppose we have a graph G on n vertices that is K_{t+1}-free and has $\mathrm{ex}(n, K_{t+1})$ edges. Our goal is to show that in fact this graph is identical to the balanced blowup of K_t, that is, $K_t(n)$. By Theorem 3.1.1 it follows that we can partition the vertex set of G into t parts, $\{V_1, V_2, \dots, V_t\}$, such that $\sum_{i=1}^{t} \mathrm{e}(G[V_i]) < \varepsilon n^2$. Then we show that if any of these possible εn^2 edges are present in G, then we will get a suboptimal configuration since the graph has to be K_{t+1} -free. The idea is that every edge that is contained in a partition class V_i for some $1 \le i \le t$, forces many edges between V_i and other partition classes to be missing, which leads to G having in total less edges than $\mathrm{ex}(n, K_{t+1})$.

The described approach is an instance of the general strategy how the stability result is used to obtain the exact result. For any Turán-type problem, the stability result implies that any graph with maximum edge density has approximately the correct structure and what is left to do is to show that actually there cannot be any perturbations at all. Now let us discuss an example of stability result for hypergraphs. As we mentioned earlier in the introduction, one of the first applications of stability method was to determine the Turán number of the Fano plane [FS05, KS05a]. The *Fano plane*, denoted by $\mathrm{PG}_2(2)$, is the unique 3-graph with 7 vertices and 7 edges, in which every pair of vertices is contained in a unique triple. It can also be described as the projective plane over the field with two elements, $\mathbb{F}_2$. It has 7 vertices, which can be identified with the non-zero vectors of length 3. It has 7 edges, corresponding to the lines of the plane. A triple $\{x, y, z\}$ is an edge if $x + y = z$. In 1976 it was conjectured by Sós [S76́] that

$$\mathrm{ex}(n, \mathrm{PG}_2(2)) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}. \tag{3.1}$$

It is easy to check that the Fano plane is not 2-colorable, and therefore any 2-colorable hypergraph cannot contain the Fano plane. Equipartition $[n]$ into two parts and take all the triples that intersect both of them. Clearly, this is the largest 2-colorable 3-uniform hypergraph on n vertices. Sós [S76] also conjectured that this is the unique extremal graph achieveng the bound in (3.1). In 2005, this conjecture was independently proved by Keevash and Sudakov [KS05a] and Füredi and Simonovits [FS05], for all sufficiently large n. Let us state the stability theorem for graphs that do not contain the Fano plane as stated in [KS05a].

Theorem 3.1.2 (Keevash, Sudakov, [KS05a])**.** *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{H}$ is a 3-graph that does not contain the Fano plane, has at least $(1-\delta)\frac{3}{4}\binom{n}{3}$ edges, there exists a partition of $V(\mathcal{H})$, say $\{A, B\}$, so that $\mathrm{e}(\mathcal{H}[A]) + \mathrm{e}(\mathcal{H}[B]) < \varepsilon n^3$.*

Let us discuss how the exact result for the Fano plane is derived from Theorem 3.1.2. It follows the general template mentioned earlier. Take any graph $\mathcal{H}$ which does not contain the Fano plane and has maximum possible density. By the stability result it follows that $\mathcal{F}$ has approximately the optimal structure so we can consider "the best" partition of $V(\mathcal{H})$. Fix such a partition of the vertex set of $\mathcal{H}$, say $\{A, B\}$. Any edge that is completely contained in A or in B is considered to be "bad" and all the vertices that are in many "bad" edges are also "bad". The last two steps, that actually require most of the work are used to show that there are no "bad" vertices and then, using that, to prove that there cannot be any "bad" edges in $\mathcal{H}$. This outline is the most general pattern (but not the only one) that is followed to derive the exact result from the corresponding stability theorem.

Now that we have discussed how classic stability results look like, let us put this notion of stability in a more general setting. Recall that for a family of r-graphs $\mathfrak{F}$, $m(\mathfrak{F}, n) := \max_{\substack{\mathcal{F}\in\mathfrak{F} \\ \mathrm{v}(\mathcal{F})=n}} |\mathcal{F}|$. Let $\mathfrak{F}$ and $\mathfrak{H}$ be two families of r-graphs. In the following discussion, one should think about $\mathfrak{F}$ being the family of all r-graphs that are $\mathcal{G}$-free, for some r-graph $\mathcal{G}$ and the family $\mathfrak{H}$ being the family of (conjectured) extremal examples. Typically $\mathfrak{H}$ is a substantially more structured subfamily of $\mathfrak{F}$, and our goal is to show that $m(\mathfrak{F}, n) = m(\mathfrak{H}, n)$ for sufficiently large n. Also recall that

$d_{\mathfrak{H}}(\mathcal{F})$ is the edit distance of an r-graph $\mathcal{F}$ from the family $\mathfrak{H}$, that is,

$$d_{\mathfrak{H}}(\mathcal{F}) := \min_{\substack{\mathcal{H}\in\mathfrak{H} \\ \mathrm{v}(\mathcal{F})=\mathrm{v}(\mathcal{H})}} |\mathcal{F}\triangle\mathcal{H}|.$$

Using Theorem 3.1.1 and Theorem 3.1.2 as a guide, we can formulate classical stability as follows in our terminology.

Definition 3.1.3 (Classical Stability). *We say that the family $\mathfrak{F}$ is $\mathfrak{H}$-stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\mathcal{F} \in \mathfrak{F}$ with $\mathrm{v}(\mathcal{F}) = n$ and $|\mathcal{F}| \geq m(\mathfrak{H}, n) - \delta n^r$ one has $d_{\mathfrak{H}}(\mathcal{F}) \leq \varepsilon n^r$.*

So for example, in this language, Theorem 3.1.1 says that $\mathfrak{F}$ is $\mathfrak{H}$-stable with $\mathfrak{F}$ being the family of K_{t+1}-free graphs and $\mathfrak{H}$ being the family of all blowups of K_t.

As we have already shown, if we prove that $\mathfrak{F}$ is $\mathfrak{H}$-stable then it implies that every graph in $\mathfrak{F}$ with maximum density has approximately the same structure and then we still need to get rid of possible small proportion of perturbations to prove the exact structure. In the next section we define a different kind of stability which, in particular, allows us to obtain the exact result directly from the stability result.

3.2. Our Stability

Definition 3.2.1 (Our Stability). *We say that $\mathfrak{F}$ is $\mathfrak{H}$-*stable *if there exists some $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that for all $\mathcal{F} \in \mathfrak{F}$ with $\mathrm{v}(\mathcal{F}) = n \geq n_0$ we have*

$$|\mathcal{F}| \leq m(\mathfrak{H}, n) - \alpha d_{\mathfrak{H}}(\mathcal{F}).$$

Observe that our notion of stability is stronger in two respects:

- It implies linear dependence between δ and ε in Definition 3.1.3.
- It is meaningful in the regime $d_{\mathfrak{H}}(\mathcal{F}) = o(n^r)$, allowing us to compute Turán numbers exactly. Note that if $\mathfrak{F}$ is $\mathfrak{H}$-stable using our definition then $m(\mathfrak{H}, n) \geq m(\mathfrak{F}, n)$ for sufficiently large n.

To make this disinction perhaps a name like "sharp stability" would have been more appropriate, however we simply use the term "stability", for brevity. So from now on whenever we say $\mathfrak{F}$ is $\mathfrak{H}$-stable we mean as in Definition 3.2.1.

Remark 3.2.1. *Note that in the above definition of stability decreasing the parameter α makes the definition weaker, in contrast to more common behavior (the smaller the parameter, the more restrictive the definition). Rewriting (3.2) as*

$$d_{\mathfrak{H}}(\mathcal{F}) \leq \frac{m(\mathfrak{H}, n) - |\mathcal{F}|}{\alpha},$$

one can see more clearly that our notion of stability gives a bound on the edit distance as a constant multiple of the distance between the sizes of $\mathcal{F}$ and the densest graph in $\mathfrak{H}$.

So now let us discuss how do we obtain stability? One of the novel ideas in this thesis is to provide a general framework which can be used to obtain the stability for any Turán-type problem from its local version. We define local stability next.

Definition 3.2.2 (Local Stability)**.** *For $\varepsilon, \alpha > 0$, we say that $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon, \alpha)$*-locally stable *if there exists $n_0 \in \mathbb{N}$ such that for all $\mathcal{F} \in \mathfrak{F}$ with $\mathrm{v}(\mathcal{F}) = n \geq n_0$ and $d_{\mathfrak{H}}(\mathcal{F}) \leq \varepsilon n^r$ we have*

$$|\mathcal{F}| \leq m(\mathfrak{H}, n) - \alpha d_{\mathfrak{H}}(\mathcal{F}). \tag{3.2}$$

We say that $\mathfrak{F}$ is $\mathfrak{H}$-locally stable *if it is $(\mathfrak{H}, \varepsilon, \alpha)$*-locally stable *for some $\varepsilon, \alpha > 0$.*

Note that $\mathfrak{F}$-is $\mathfrak{H}$-stable if it is $(\mathfrak{H}, 1, \alpha)$-*locally stable* for some $\alpha > 0$. In Section 3.4 we show how to obtain that stability from local stability. To state this result we need to introduce the notion of weakly weight-stability which can be viewed as the analogue of classical stability (Definition 3.1.3) for weighted graphs.

Definition 3.2.3 (Weak Weight-Stability)**.** *We say that $\mathfrak{F}$ is $\mathfrak{H}$*-weakly weight-stable *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\mathcal{F} \in \mathfrak{F}$ and $\mu \in \mathcal{M}(\mathcal{F})$ if $\lambda(\mathcal{F}, \mu) \geq \lambda(\mathfrak{H}) - \delta$, then $d_{\mathfrak{H}}(\mathcal{F}, \mu) \leq \varepsilon$.*

Theorem 3.2.4. *Let $\mathfrak{F}, \mathfrak{H}$ be clonable families of r-graphs. Let $\mathfrak{F}^*$ consist of all r-graphs in $\mathfrak{F}$ that cover pairs. If $\mathfrak{F}^*$ is $\mathfrak{H}$-weakly weight-stable and $\mathfrak{F}$ is $\mathfrak{H}$-locally stable then $\mathfrak{F}$ is $\mathfrak{H}$-stable.*

This result can be considered as a generalization of the symmetrization argument by Sidorenko [Sid89] which we discussed in Chapter 2. Note that the symmetrization was modified and employed by Pikhurko [Pik08] and Hefetz and Keevash [HK13] for some special cases. However, Theorem 3.2.4 provides us with a generic tool to obtain stability from local stability for clonable families. Both of the imposed conditions (i.e. local stability of $\mathfrak{F}$ and weak weight-stability of $\mathfrak{F}^*$) depend only on the forbidden subgraph, so the tool is universally applicable. For example, to obtain Mantel's theorem (for large n) using Theorem 3.2.4, we would need to check only one condition. Indeed, in this case $\mathfrak{F}$ is the family of all triangle-free graphs, $\mathfrak{H}$ is the family of all bipartite graphs and $\mathfrak{F}^*$ contains a single graph - $\{K_2\}$. Hence, $\mathfrak{F}^*$ is trivially $\mathfrak{H}$-weakly weight stable as $K_2 \in \mathfrak{H}$. So we only need to check that $\mathfrak{F}$ is $\mathfrak{H}$-locally stable. A demonstration of this part can be found in Section 3.6 where we show how to prove local stability more generally, for K_t-free graphs.

3.3. How Stability and Weight-Stability Are Related?

In this section we define several notions of stability in weighted setting and describe their relations to the corresponding stability notion in unweighted setting.

First let us discuss weakly weight-stability which can be viewed as the analogue of classical stability (Definition 3.1.3) for weighted graphs.

Definition 3.3.1 (Weak Weight-Stability)**.** *We say that* $\mathfrak{F}$ *is* $\mathfrak{H}$-weakly weight-stable *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every* $\mathcal{F} \in \mathfrak{F}$ *and* $\mu \in \mathcal{M}(\mathcal{F})$ *if* $\lambda(\mathcal{F}, \mu) \geq \lambda(\mathfrak{H}) - \delta$, *then* $d_{\mathfrak{H}}(\mathcal{F}, \mu) \leq \varepsilon$.

One can also consider the analogue of the stability from Definition 3.2.1, and we do so.

Definition 3.3.2 (Weight-Stability)**.** *We say that* $\mathfrak{F}$ *is* $\mathfrak{H}$-weight-stable *if there exists* $\alpha > 0$ *such that for every* $\mathcal{F} \in \mathfrak{F}$ *and* $\mu \in M(\mathcal{F})$, $\lambda(\mathcal{F}, \mu) \leq \lambda(\mathfrak{H}) - \alpha d_{\mathfrak{H}}(\mathcal{F}, \mu)$.

Note that the analogous observation to the one in Remark 3.2.1 holds in this setting, in particular, weight-stability is a stronger notion than weak weight-stability (thus, we use the word "weak"). Next we exploit the relations of stabilities in weighted and unweighted settings. First we show that for clonable families stability implies weight-stability.

Lemma 3.3.3. *Let $\mathfrak{F}, \mathfrak{H}$ be two clonable families. If $\mathfrak{F}$ is $\mathfrak{H}$-stable then $\mathfrak{F}$ is $\mathfrak{H}$-weight-stable.*

Proof. Let $\alpha > 0$ be such that $\mathfrak{F}$ is $(\mathfrak{H}, \alpha)$-stable. We show that $\mathfrak{F}$ is $(\mathfrak{H}, \alpha/4)$-weight-stable. Suppose $\mathcal{F} \in \mathfrak{F}$, $\mu \in \mathcal{M}(\mathcal{F})$. We show that

$$\lambda(\mathcal{F}, \mu) \leq \lambda(\mathfrak{H}) - \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F}, \mu). \tag{3.3}$$

By Lemma 2.4.2 there exists a blowup of $\mathcal{F}$, say $\mathcal{B} \in \mathfrak{F}$, with $\mathrm{v}(\mathcal{B}) = n$ sufficiently large such that the following inequalities hold:

$$\frac{3}{4} d_{\mathfrak{H}}(\mathcal{F}, \mu) \leq \frac{d_{\mathfrak{H}}(\mathcal{B})}{n^r} \leq \frac{5}{4} d_{\mathfrak{H}}(\mathcal{F}, \mu), \tag{3.4}$$

$$\lambda(\mathcal{F}, \mu) - \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F}, \mu) \leq \frac{|\mathcal{B}|}{n^r}. \tag{3.5}$$

We can also assume that n is large enough such that

$$\frac{m(\mathfrak{H}, n)}{n^r} \leq m(\mathfrak{H}) + \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F}, \mu). \tag{3.6}$$

By $(\mathfrak{H}, \alpha)$-stability of $\mathfrak{F}$, we have

$$|\mathcal{B}| \leq m(\mathfrak{H}, n) - \alpha d_{\mathfrak{H}}(\mathcal{B}). \tag{3.7}$$

Finally, we have

$$\begin{aligned}
\lambda(\mathcal{F}, \mu) &\overset{(3.5)}{\leq} \frac{|\mathcal{B}|}{n^r} + \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F}, \mu) \\
&\overset{(3.7)}{\leq} \frac{m(\mathfrak{H}, n) - \alpha d_{\mathfrak{H}}(\mathcal{B})}{n^r} + \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F}, \mu) \\
&\overset{(3.6)}{\leq} m(\mathfrak{H}) + \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F}, \mu) - \frac{3\alpha}{4} d_{\mathfrak{H}}(\mathcal{F}, \mu) + \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F}, \mu) \\
&= \lambda(\mathfrak{H}) - \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F}, \mu),
\end{aligned}$$

implying (3.3), as desired. □

Next we show that for clonable $\mathfrak{H}$, local stability together with weight-stability implies stability.

Lemma 3.3.4. *Let $\mathfrak{H}$ be a clonable family. If the family $\mathfrak{F}$ is $\mathfrak{H}$-locally stable and $\mathfrak{H}$-weight-stable, then $\mathfrak{F}$ is $\mathfrak{H}$-stable.*

Proof. Let $\alpha, \varepsilon > 0$ be such that the family $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon, \alpha)$-locally stable and $(\mathfrak{H}, \alpha)$-weight-stable. We will show that $\mathfrak{F}$ is $(\mathfrak{H}, \alpha/4)$-stable, that is, for every $\mathcal{F} \in \mathfrak{F}$ with $n := \mathrm{v}(\mathcal{F})$ sufficiently large,

$$|\mathcal{F}| \leq m(\mathfrak{H}, n) - \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F}). \tag{3.8}$$

We can assume that $d_{\mathfrak{H}}(\mathcal{F}) > \varepsilon n^r$, since otherwise (3.8) holds because $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon, \alpha)$-locally stable.

By Lemma 2.3.1 the family $\mathfrak{H}$ is smooth. We choose n to be sufficiently large so that

$$\left(1 - \frac{r^2}{n}\right) n^r \geq \frac{1}{2} r! \binom{n}{r},$$

and

$$m(\mathfrak{H}, n) \geq \left(m(\mathfrak{H}) - \frac{\alpha\varepsilon}{4}\right) n^r.$$

Using Lemmas 2.4.3, 2.4.1, the inequalities above and the fact that $\mathfrak{F}$ is $(\mathfrak{H}, \alpha)$-weight-stable, we have

$$\begin{aligned}
\frac{|\mathcal{F}|}{n^r} &= \lambda(\mathcal{F}, \xi_{\mathcal{F}}) \leq \lambda(\mathfrak{H}) - \alpha d^w_{\mathfrak{H}}(\mathcal{F}, \xi) \\
&\leq \left(\frac{m(\mathfrak{H}, n)}{n^r} + \frac{\alpha\varepsilon}{4}\right) - \alpha\left(1 - \frac{r^2}{n}\right) \frac{d_{\mathfrak{H}}(\mathcal{F})}{r!\binom{n}{r}} \\
&\leq \left(\frac{m(\mathfrak{H}, n)}{n^r} + \frac{\alpha\varepsilon}{4}\right) - \frac{\alpha d_{\mathfrak{H}}(\mathcal{F})}{2n^r} \\
&= \frac{(m(\mathfrak{H}, n) - \alpha/4 d_{\mathfrak{H}}(\mathcal{F})) + \alpha/4(\varepsilon n^r - d_{\mathfrak{H}}(\mathcal{F}))}{n^r} \\
&\leq \frac{m(\mathfrak{H}, n) - \alpha d_{\mathfrak{H}}(\mathcal{F})/4}{n^r},
\end{aligned}$$

implying (3.8). □

Now let us introduce local stability in weighted setting.

Definition 3.3.5 (Local Weight-Stability)**.** *For $\varepsilon, \alpha > 0$, we say that $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon, \alpha)$-* locally weight-stable *if for every $\mathcal{F} \in \mathfrak{F}, \mu \in \mathcal{M}(\mathcal{F})$ such that $d_{\mathfrak{H}}(\mathcal{F}, \mu) \leq \varepsilon$, we have*

$$\lambda(\mathcal{F}, \mu) \leq \lambda(\mathfrak{H}) - \alpha d_{\mathfrak{H}}(\mathcal{F}, \mu).$$

*We say that $\mathfrak{F}$ is $\mathfrak{H}$-*locally weight-stable *if $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon, \alpha)$-locally weight-stable for some choice of ε and α.*

Note that $\mathfrak{F}$ is $\mathfrak{H}$-*weight-stable* if $\mathfrak{F}$ is $(\mathfrak{H}, 1, \alpha)$-*locally weight-stable* for some choice of α. Note that the analogue of Lemma 3.3.3 holds in local setting, as we show next.

Lemma 3.3.6. *Let $\mathfrak{F}, \mathfrak{H}$ be two clonable families. If $\mathfrak{F}$ is $\mathfrak{H}$-locally stable then $\mathfrak{F}$ is $\mathfrak{H}$-locally weight-stable.*

Proof. Let $\varepsilon, \alpha > 0$ be such that $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon, \alpha)$-locally stable. We show that $\mathfrak{F}$ is $(\mathfrak{H}, \alpha/4, 4\varepsilon/5)$-locally weight-stable. Suppose $\mathcal{F} \in \mathfrak{F}$, $\mu \in \mathcal{M}(\mathcal{F})$ are such that $d_{\mathfrak{H}}(\mathcal{F}, \mu) \leq 4\varepsilon/5$. We show that

$$\lambda(\mathcal{F}, \mu) \leq \lambda(\mathfrak{H}) - \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F}, \mu). \tag{3.9}$$

By Lemma 2.4.2 there exists a blowup of $\mathcal{F}$, say $\mathcal{B} \in \mathfrak{F}$, with $\mathrm{v}(\mathcal{B}) = n$ sufficiently large such that the following inequalities hold:

$$\frac{3}{4} d_{\mathfrak{H}}(\mathcal{F}, \mu) \leq \frac{d_{\mathfrak{H}}(\mathcal{B})}{n^r} \leq \frac{5}{4} d_{\mathfrak{H}}(\mathcal{F}, \mu), \tag{3.10}$$

$$\lambda(\mathcal{F}, \mu) - \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F}, \mu) \leq \frac{|\mathcal{B}|}{n^r}. \tag{3.11}$$

We can also assume that n is large enough such that

$$\frac{m(\mathfrak{H}, n)}{n^r} \leq m(\mathfrak{H}) + \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F}, \mu). \tag{3.12}$$

From (3.10) it follows that $d_{\mathfrak{H}}(\mathcal{B}) \leq \varepsilon n^r$, thus by $(\mathfrak{H}, \varepsilon, \alpha)$-local stability of $\mathfrak{F}$, we have

$$|\mathcal{B}| \leq m(\mathfrak{H}, n) - \alpha d_{\mathfrak{H}}(\mathcal{B}). \tag{3.13}$$

Finally, we have

$$\begin{aligned}
\lambda(\mathcal{F},\mu) &\overset{(3.11)}{\leq} \frac{|\mathcal{B}|}{n^r} + \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F},\mu) \\
&\overset{(3.13)}{\leq} \frac{m(\mathfrak{H},n) - \alpha d_{\mathfrak{H}}(\mathcal{B})}{n^r} + \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F},\mu) \\
&\overset{(3.12)}{\leq} m(\mathfrak{H}) + \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F},\mu) - \frac{3\alpha}{4} d_{\mathfrak{H}}(\mathcal{F},\mu) + \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F},\mu) \\
&= \lambda(\mathfrak{H}) - \frac{\alpha}{4} d_{\mathfrak{H}}(\mathcal{F},\mu),
\end{aligned}$$

implying (3.9), as desired. □

Remark 3.3.1. *By our results it follows that when $\mathfrak{F}$ and $\mathfrak{H}$ are clonable families the stability implies weight-stability, however for the opposite to hold, Lemma 3.3.4 requires local stability. Although we do not have any examples of clonable $\mathfrak{F}$ and $\mathfrak{H}$ such that only weight-stability does not imply stability, but we suspect that local stability is a necessary condition.*

And our final lemma of this section is an auxiliary lemma which we use in the proof of Theorem 3.2.4.

Lemma 3.3.7. *For any two families $\mathfrak{F}$ and $\mathfrak{H}$, if $\mathfrak{F}$ is $\mathfrak{H}$-weakly weight stable and $\mathfrak{H}$-locally weight-stable, then $\mathfrak{F}$ is $\mathfrak{H}$-weight-stable.*

Proof. Let $\varepsilon, \beta > 0$ be such that $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon, \beta)$-locally weight-stable. Since $\mathfrak{F}$ is $\mathfrak{H}$-weakly weight-stable, there exists $\gamma > 0$ such that if $\mathcal{F} \in \mathfrak{F}$ and $\mu \in \mathcal{M}(\mathcal{F})$ are such that $\lambda(\mathcal{F},\mu) \geq \lambda(\mathfrak{H}) - \gamma$, then $d_{\mathfrak{H}}(\mathcal{F},\mu) \leq \varepsilon$. We claim that $\mathfrak{F}$ is (ε, α)-weight-stable with $\alpha = \min(\beta, \gamma)$.

Indeed, for graphs $\mathcal{F} \in \mathfrak{F}$, $\mu \in \mathcal{M}(\mathcal{F})$ with $d_{\mathfrak{H}}(\mathcal{F},\mu) > \varepsilon$ we have

$$\lambda(\mathcal{F},\mu) \leq \lambda(\mathfrak{H}) - \gamma \leq \lambda(\mathfrak{H}) - \alpha d_{\mathfrak{H}}(\mathcal{F},\mu),$$

and, otherwise, we have

$$\lambda(\mathcal{F},\mu) \leq \lambda(\mathfrak{H}) - \beta d_{\mathfrak{H}}(\mathcal{F},\mu) \leq \lambda(\mathfrak{H}) - \alpha d_{\mathfrak{H}}(\mathcal{F},\mu),$$

as desired. □

3.4. Stability From Local Stability (Symmetrization)

In this section we prove Theorem 3.2.4 using the relations between different notions of stabilities proved in the previous section.

Proof of Theorem 3.2.4. By Lemma 3.3.4, it suffices to show that $\mathfrak{F}$ is $\mathfrak{H}$-weight-stable. By Corollary 3.3.6, $\mathfrak{F}$ is $\mathfrak{H}$-locally weight-stable (and hence, also $\mathfrak{F}^*$ is $\mathfrak{H}$-locally stable). It follows from Lemma 3.3.7 that $\mathfrak{F}^*$ is $\mathfrak{H}$-weight-stable. Choose $\varepsilon, \alpha > 0$ such that $\mathfrak{F}^*$ is (α, ε)-weight-stable and $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon, \alpha)$-locally weight-stable. Define $\delta := \alpha\varepsilon/2$. We will prove that for every $\mathcal{F} \in \mathfrak{F}$ and $\mu \in \mathcal{M}(\mathcal{F})$ such that

$$\lambda(\mathcal{F}, \mu) \geq \lambda(\mathfrak{H}) - \delta, \tag{3.14}$$

we have

$$d_{\mathfrak{H}}(\mathcal{F}, \mu) \leq \varepsilon. \tag{3.15}$$

Note that this statement implies that $\mathfrak{F}$ is $(\mathfrak{H}, \delta)$-weight-stable as $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon, \alpha)$-locally weight-stable and $\delta \leq \alpha$.

The proof is by induction on $\mathrm{v}(\mathcal{F})$. The base case of induction is trivial. For the induction step, first suppose that $\mathcal{F} \in \mathfrak{F}^*$, then we have

$$d_{\mathfrak{H}}(\mathcal{F}, \mu) \leq \frac{\lambda(\mathfrak{H}) - \lambda(\mathcal{F}, \mu)}{\alpha} \leq \frac{\delta}{\alpha} \leq \varepsilon,$$

as $\mathfrak{F}^*$ is $(\mathfrak{H}, \alpha)$-weight-stable and $\delta \leq \alpha\varepsilon$.

Thus we assume that $\mathcal{F} \notin \mathfrak{F}^*$ and there exist $v_1, v_2 \in V(\mathcal{F})$, such that the pair $\{v_1, v_2\}$ is not contained in any edge of $\mathcal{F}$. We assume that $\mu(v_1) \neq 0$ and $\mu(v_2) \neq 0$, since otherwise the conclusion follows from the induction hypothesis. We will consider a family of probability distributions on $V(\mathcal{F})$ defined as follows. For $t \in [0, 1]$, let $\mu_t \in \mathcal{M}(\mathcal{F})$ be defined by $\mu_t(v) = \mu(v)$ for all $v \in V(\mathcal{F}) \setminus \{v_1, v_2\}$, $\mu_t(v_1) = t(\mu(v_1) + \mu(v_2))$, and $\mu_t(v_2) = (1-t)(\mu(v_1) + \mu(v_2))$. Note that $\mu = \mu_x$, for $x := \mu(v_1)/(\mu(v_1) + \mu(v_2))$. As $\mu(v_1) \neq 0$ and $\mu(v_2) \neq 0$, it follows that $x \notin \{0, 1\}$.

Note that $(\mathcal{F}, \mu_0)$ and $(\mathcal{F}, \mu_1)$ can be considered as weighted r-graphs on $\mathrm{v}(\mathcal{F})-1$ vertices and, therefore, the induction hypothesis is apllicable to them. Moreover,

$$\lambda(\mathcal{F}, \mu) = x\lambda(\mathcal{F}, \mu_0) + (1-x)\lambda(\mathcal{F}, \mu_1). \tag{3.16}$$

If $\lambda(\mathfrak{F}, \mu_i) < \lambda(\mathfrak{H}) - \delta$ for $i = 1, 2$, then by (3.16), $\lambda(\mathcal{F}, \mu) < \lambda(\mathfrak{H}) - \delta$, in contradiction with (3.14). Thus, without loss of generality, we assume that $\lambda(\mathfrak{F}, \mu_0) \geq \lambda(\mathfrak{H}) - \delta$. By the induction hypothesis we have $d_{\mathfrak{H}}(\mathcal{F}, \mu_0) \leq \varepsilon$.

Now suppose for a contradiction that $d_{\mathfrak{H}}(\mathcal{F}, \mu) > \varepsilon$. Note that $d_{\mathfrak{H}}(\mathcal{F}, \mu_t)$ is a continuous function of t. (The measure μ_t depends continuously on t, and $d_{\mathfrak{H}}(\mathcal{F}, \cdot)$ is a continuous function on $\mathcal{M}(\mathcal{F})$) Thus there exists $y \in [0, x]$ such that $d_{\mathfrak{H}}(\mathcal{F}, \mu_y) = \varepsilon$. Since $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon, \alpha)$-locally weight-stable, we have

$$\lambda(\mathcal{F}, \mu_y) \leq \lambda(\mathfrak{H}) - \alpha\varepsilon. \tag{3.17}$$

On the other hand,

$$\begin{aligned}\lambda(\mathcal{F}, \mu_y) &= \frac{x-y}{x}\lambda(\mathcal{F}, \mu_0) + \frac{y}{x}\lambda(\mathcal{F}, \mu_x)\\ &\geq \frac{x-y}{x}(\lambda(\mathfrak{H}) - \delta) + \frac{y}{x}(\lambda(\mathfrak{H}) - \delta) = \lambda(\mathfrak{H}) - \delta > \lambda(\mathfrak{H}) - \alpha\varepsilon,\end{aligned} \tag{3.18}$$

as $\delta < \alpha\varepsilon$. The contradiction between inequalities (3.17) and (3.18) concludes the proof. □

3.5. Local Stability From Vertex Local Stability

Often in the applications of the stability method, we may assume that the degrees of the vertices of any graph with maximum density are essentially the same. For example, below is a variant of stability result for the graphs without Fano plane, which was proven by Füredi and Simonovits in [FS05].

Theorem 3.5.1 (Füredi, Simonovits, 2005)**.** *There exist some n_0 and $\delta > 0$ such that if $\mathcal{F}$ is a 3-graph on $n \geq n_0$ vertices, does not contain a copy of the Fano plane and $|L_{\mathcal{F}}(v)| > (3/4 - \delta)\binom{n}{2}$ for every vertex $v \in V(\mathcal{F})$, then there exists some*

$X \subseteq [n]$ such that $\mathcal{F} \subseteq \mathcal{F}(X, \overline{X})$, where $\mathcal{F}(X, \overline{X})$ is the 3-graph on $[n]$ such that the edge set is the set of all triples meeting both X and $\overline{X}$.

Note that given this stability result, the approach to obtain the exact result is not very different from the one discussed earlier, in Section 3.1. One still needs to show that if there is any imperfection, then in fact, $\mathcal{F}$ has smaller density, which implies a contradiction with the minimum degree assumption. For this to work we should first show that the minimum degree assumption is valid; this is a fairly standard observation common in the literature. In this section we put this in a general setting and show that in most cases one can indeed assume that the degrees of vertices are large. So we introduce the notion of *vertex local stability* which is a weaker version of local stability since the requirements imposed on the family are stronger. However, for clonable families we show that stability can be derived from this version.

Definition 3.5.2 (Vertex Local Stability)**.** *Let $\mathfrak{H}$ be a smooth family of r-graphs. For $\varepsilon, \alpha > 0$, we say that a family $\mathfrak{F}$ of r-graphs is $(\mathfrak{H}, \varepsilon, \alpha)$-*vertex locally stable *if there exists $n_0 \in \mathbb{N}$ such that for all $\mathcal{F} \in \mathfrak{F}$ with $\mathrm{v}(\mathcal{F}) = n \geq n_0$, $d_{\mathfrak{H}}(\mathcal{F}) \leq \varepsilon n^r$, and $|L_{\mathcal{F}}(v)| \geq (rm(\mathfrak{H}) - \varepsilon)\, n^{r-1}$ for every $v \in V(\mathcal{F})$, we have*

$$|\mathcal{F}| \leq m(\mathfrak{H}, n) - \alpha d_{\mathfrak{H}}(\mathcal{F}).$$

*We say that $\mathfrak{F}$ is $\mathfrak{H}$-*vertex locally stable *if $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon, \alpha)$-vertex locally stable for some ε, α.*

Our main result of the section is the following.

Theorem 3.5.3. *Let $\mathfrak{F}, \mathfrak{H}$ be families of r-graphs such that $\mathfrak{H}$ is clonable. If $\mathfrak{F}$ is $\mathfrak{H}$-vertex locally stable, then $\mathfrak{F}$ is $\mathfrak{H}$-locally stable.*

The proof of Theorem 3.5.3 is based on the following two auxiliary lemmas which we prove first.

Lemma 3.5.4. *Let $\mathfrak{F}$ be a clonable family of r-graphs. Then for every $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ satisfying the following. For every $\mathcal{F} \in \mathfrak{F}$ with $\mathrm{v}(\mathcal{F}) = n \geq n_0$*

and $|\mathcal{F}| \geq (m(\mathfrak{F}) - \delta)\, n^r$ *there exists* $X \subseteq V(\mathcal{F})$ *such that* $|X| \geq (1-\varepsilon)n$ *and*

$$\Big||L_{\mathcal{F}}(v)| - rm(\mathfrak{F})n^{r-1}\Big| \leq \varepsilon n^{r-1}$$

for every $v \in X$*.*

Proof. Clearly, it is enough to prove the lemma for sufficiently small ε. Thus we assume without loss of generality that $\max\{\varepsilon, \varepsilon^2 r^2 m(\mathfrak{F})\} < 1$. We show that $\delta := (\varepsilon^6 - \varepsilon^8 r^2 m(\mathfrak{F}))/(1+r+r^2)$ satisfies the lemma for sufficiently large n_0. Let $X \subseteq V(\mathcal{F})$ be the set of all $v \in V(\mathcal{F})$ satisfying

$$\Big||L_{\mathcal{F}}(v)| - rm(\mathfrak{F})n^{r-1}\Big| \leq \varepsilon n^{r-1}.$$

To prove that $|X| \geq (1-\varepsilon)n$, we first show the following claim.

Claim 3.5.5.

$$|L_{\mathcal{F}}(v)| \leq (rm(\mathfrak{F}) + \varepsilon^2)n^{r-1}$$

for every $v \in V(\mathcal{F})$*.*

Proof. Suppose for a contradiction that

$$|L_{\mathcal{F}}(v)| > (rm(\mathfrak{F}) + \varepsilon^2)n^{r-1}$$

for some $v \in V(\mathcal{F})$. Let $n' := \lceil (1+\varepsilon^4)n \rceil$ and let $\mathcal{F}'$ be obtained from $\mathcal{F}$ by cloning v into a set of size $\lceil \varepsilon^4 n \rceil + 1$. We have $\mathcal{F}' \in \mathfrak{F}$, as $\mathfrak{F}$ is clonable. For sufficiently large n, we have

$$m(\mathfrak{F}, n') \leq (m(\mathfrak{F}) + \delta)n'^r \leq (m(\mathfrak{F}) + \delta)(1 + \varepsilon^4 r + \varepsilon^8 r^2)n^r. \tag{3.19}$$

On the other hand,

$$\begin{aligned} m(\mathfrak{F}, n') \geq |\mathcal{F}'| > |\mathcal{F}| + \varepsilon^4 n(rm(\mathfrak{F}) + \varepsilon^2)n^{r-1} \\ \geq (m(\mathfrak{F}) - \delta)n^r + \varepsilon^4(rm(\mathfrak{F}) + \varepsilon^2)n^r. \end{aligned} \tag{3.20}$$

But now (3.19) and (3.20) together imply that

$$\varepsilon^6 - \delta < \delta(1 + \varepsilon^4 r + \varepsilon^8 r^2) + \varepsilon^8 r^2 m(\mathfrak{F}),$$

which contradicts to our choice of δ. Thus, the claim holds. $\square$

By the preceding claim we have that

$$|L_{\mathcal{F}}(v)| < (rm(\mathfrak{F}) - \varepsilon)\, n^{r-1}$$

for all $v \in V(\mathcal{F}) \setminus X$. Now suppose for a contradicton that $|X| < (1-\varepsilon)n$. Then

$$\begin{aligned} |\mathcal{F}| &= \frac{1}{r}\left(\sum_{v \in V(\mathcal{F})\setminus X} |L_{\mathcal{F}}(v)| + \sum_{v \in X} |L_{\mathcal{F}}(v)|\right) \\ &< \frac{1}{r}\left((n - |X|)\,(rm(\mathfrak{F}) - \varepsilon) + |X|\left(rm(\mathfrak{F}) + \varepsilon^2\right)\right) n^{r-1} \\ &= m(\mathfrak{F})n^r + \frac{\varepsilon}{r}\,((1+\varepsilon)|X| - n)\, n^{r-1} \\ &< m(\mathfrak{F})n^r - \frac{\varepsilon^3}{r} n^r \\ &\le (m(\mathfrak{F}) - \delta)n^r, \end{aligned}$$

a contradiction. $\square$

Lemma 3.5.6. *Let $\mathfrak{F}$ be a clonable family of r-graphs. Then for every $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that for all $n_2 \ge n_1 \ge n_0$, we have*

$$m(\mathfrak{F}, n_2) \ge m(\mathfrak{F}, n_1) + (n_2 - n_1)(rm(\mathfrak{F}) - \varepsilon)n_1^{r-1}$$

Proof. Consider $\mathcal{F}_1 \in \mathfrak{F}$ with $v(\mathcal{F}_1) = n_1$ such that $|\mathcal{F}_1| = m(\mathfrak{F}, n_1)$. For large enough n_1 we have

$$m(\mathfrak{F}, n_1) \ge \left(m(\mathfrak{F}) - \frac{\varepsilon}{r}\right) n_1^r.$$

By averaging, there exists $v \in V(\mathcal{F}_1)$ such that

$$|L_{\mathcal{F}_1}(v)| \ge (rm(\mathfrak{F}) - \varepsilon)\, n_1^{r-1}. \tag{3.21}$$

Let $\mathcal{F}_2$ be obtained from $\mathcal{F}_1$ by cloning v to a set of size $n_2 - n_1 + 1$. As $\mathcal{F}_2 \in \mathfrak{F}$, we have

$$\begin{aligned} m(\mathfrak{F}, n_2) \geq |\mathcal{F}_2| &\geq |\mathcal{F}_1| + (n_2 - n_1)\,(rm(\mathfrak{F}) - \varepsilon)\, n_1^{r-1} \\ &= m(\mathfrak{F}, n_1) + (n_2 - n_1)\,(rm(\mathfrak{F}) - \varepsilon)\, n_1^{r-1}, \end{aligned}$$

as desired. □

Proof of Theorem 3.5.3: Let ε, α be such that $\mathfrak{F}$ is $(\mathfrak{F}', \varepsilon, \alpha)$-vertex locally stable. We choose constants $\varepsilon', \varepsilon''$ such that $0 < \varepsilon' \ll \varepsilon'' \ll \varepsilon$ so that the inequalities throughout the proof are satisfied. Let $\alpha' := \min\{\alpha, 2\varepsilon'' r^2(1 - m(\mathfrak{H}))\}$. We will show that $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon', \alpha')$-locally stable.

Consider $\mathcal{F} \in \mathfrak{F}$ with $V(\mathcal{F}) = [n]$ and $d_{\mathfrak{H}}(\mathcal{F}) \leq \varepsilon' n^r$. We assume that

$$|\mathcal{F}| \geq m(\mathfrak{H}, n) - \varepsilon' n^r,$$

since otherwise the result follows, as $\alpha' < 1$. Let $\mathcal{H} \in \mathfrak{H}$ be such that $|\mathcal{F} \triangle \mathcal{H}| = d_{\mathfrak{H}}(\mathcal{F})$. For large enough n, we have $|\mathcal{H}| \geq (m(\mathfrak{H}) - 3\varepsilon')n^r$. By Lemma 3.5.4 applied to $\mathcal{H}$ with $\varepsilon = \varepsilon''$, there exists $X \subseteq [n]$ with $|X| \geq (1 - \varepsilon'')n$ such that for each $v \in X$,

$$\Big| |L_{\mathcal{H}}(v)| - rm(\mathfrak{H})n^{r-1} \Big| \leq \varepsilon'' n^{r-1}. \tag{3.22}$$

Consider the set

$$J = \{v \in V(\mathcal{F}) : |L_{\mathcal{F}}(v)| < (rm(\mathfrak{H}) - (2r^2 + 1)\varepsilon'')n^{r-1}\}.$$

We will show that J has relatively small size. From the definition of J and X, it follows that for each $v \in J \cap X$, we have $|L_{\mathcal{F}}(v) \triangle L_{\mathcal{H}}(v)| \geq \varepsilon'' n^{r-1}$. Thus,

$$|J \cap X| \varepsilon'' n^{r-1} \leq \sum_{v \in V(\mathcal{F})} |L_{\mathcal{F}}(v) \triangle L_{\mathcal{H}}(v)| = r|\mathcal{F} \triangle \mathcal{H}| \leq \varepsilon' r n^r,$$

and therefore, $|J| \leq |J \cap X| + |J \setminus X| \leq (\frac{\varepsilon' r}{\varepsilon''} + \varepsilon'')n \leq 2\varepsilon'' n$. Let $\mathcal{F}' := \mathcal{F}|_{V(\mathcal{F}) \setminus J}$,

$\mathcal{H}' := \mathcal{H}|_{V(\mathcal{F})\setminus J}$ and $n' := n - |J|$. We have

$$d_{\mathfrak{H}}(\mathcal{F}') \leq |\mathcal{F}' \triangle \mathcal{H}'| \leq |\mathcal{F} \triangle \mathcal{H}| \leq \varepsilon' n^r \leq \varepsilon n'^r. \tag{3.23}$$

Also, for every $v \in V(\mathcal{F}) \setminus J$, we have

$$\begin{aligned} |L_{\mathcal{F}'}(v)| \geq |L_{\mathcal{F}}(v)| - |J| n^{r-2} &\geq \left(rm(\mathfrak{H}) - 2r^2\varepsilon'' - 3\varepsilon''\right) n^{r-1} \\ &\geq (rm(\mathfrak{H}) - \varepsilon) n'^{r-1}. \end{aligned} \tag{3.24}$$

Since $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon, \alpha)$-vertex locally stable, (3.23) and (3.24) imply that

$$|\mathcal{F}'| \leq m(\mathfrak{H}, n') - \alpha d_{\mathfrak{H}}(\mathcal{F}'). \tag{3.25}$$

Let $\mathcal{H}'' \in \mathfrak{H}$ be such that $|\mathcal{H}'' \triangle \mathcal{F}'| = d_{\mathfrak{H}}(\mathcal{F}')$. Let $\mathcal{H}_0$ be obtained from $\mathcal{H}''$ by blowing up a vertex in $V(\mathcal{F}) \setminus J$ to a set of size $n - n' + 1$. We have

$$|\mathcal{F} \triangle \mathcal{H}_0| \leq |\mathcal{F}' \triangle \mathcal{H}''| + |J| n^{r-1}. \tag{3.26}$$

By Lemma 3.5.6, for sufficiently large n, we have

$$\begin{aligned} m(\mathfrak{H}, n) &\geq m(\mathfrak{H}, n') + (n - n') \left(rm(\mathfrak{H}) - \frac{\varepsilon''}{1 - 2r\varepsilon''}\right) n'^{r-1} \\ &\geq m(\mathfrak{H}, n') + |J| \left(rm(\mathfrak{H}) - \frac{\varepsilon''}{1 - 2r\varepsilon''}\right) (1 - 2r\varepsilon'') n^{r-1}. \end{aligned} \tag{3.27}$$

Now we are ready to put all the obtained inequalities together to show that $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon', \alpha')-$ locally stable.

$$
\begin{aligned}
|\mathcal{F}| &\leq |\mathcal{F}'| + |J|(rm(\mathfrak{H}) - (2r^2+1)\varepsilon'')n^{r-1} \\
&\overset{(3.25)}{\leq} m(\mathfrak{H}, n') - \alpha d_{\mathfrak{H}}(\mathcal{F}') + |J|(rm(\mathfrak{H}) - (2r^2+1)\varepsilon'')n^{r-1} \\
&\overset{(3.27)}{\leq} m(\mathfrak{H}, n) - |J|\left(rm(\mathfrak{H}) - \frac{\varepsilon''}{1-2r\varepsilon''}\right)(1-2r\varepsilon'')n^{r-1} \\
&\qquad - \alpha|\mathcal{F}' \triangle \mathcal{H}''| + |J|(rm(\mathfrak{H}) - (2r^2+1)\varepsilon'')n^{r-1} \\
&= m(\mathfrak{H}, n) - \alpha|\mathcal{F}' \triangle \mathcal{H}''| - 2\varepsilon'' r^2(1 - m(\mathfrak{H}))|J|n^{r-1} \\
&\leq m(\mathfrak{H}, n) - \alpha'|\mathcal{F}' \triangle \mathcal{H}''| - \alpha'|J|n^{r-1} \\
&\overset{(3.26)}{\leq} m(\mathfrak{H}, n) - \alpha'|\mathcal{F} \triangle \mathcal{H}_0| \\
&\leq m(\mathfrak{H}, n) - \alpha' d_{\mathfrak{H}}(\mathcal{F}),
\end{aligned}
$$

as desired. □

Corollary 3.5.7. *Let $\mathfrak{F}, \mathfrak{H}$ be clonable families of r-graphs. Let $\mathfrak{F}^*$ consist of all r-graphs in $\mathfrak{F}$ that cover pairs. If $\mathfrak{F}^*$ is $\mathfrak{H}$-weakly weight-stable and $\mathfrak{F}$ is $\mathfrak{H}$-vertex locally stable then $\mathfrak{F}$ is $\mathfrak{H}$-stable.*

Proof. The result follows from Theorem 3.2.4 and Theorem 3.5.3 as a corollary, using the fact that clonable families are smooth (Lemma 2.3.1). □

Our final result of the section allows one to prove vertex local stability with respect to the subfamily $\mathfrak{H}'$ of those graphs who attain the maximum density under some technical condition. Note that this subfamily $\mathfrak{H}'$ may not be clonable at all while we typically require $\mathfrak{H}$ to be so. So it turns out that for vertex local stability, being closed under taking blowups is not a necessary condition. However, note that it is crucial in our proof of Lemma 3.5.3, that is, for deriving local stability from vertex local stability.

Lemma 3.5.8. *Let $\mathfrak{F}$,$\mathfrak{H}$,$\mathfrak{H}'$ be families of r-graphs such that $\mathfrak{H}$ is smooth and $\mathfrak{H}' \subseteq \mathfrak{H}$. If $\mathfrak{H}'$ is such that*

(i) $m(\mathfrak{H}') = m(\mathfrak{H})$,

(ii) for every $\mathcal{H} \in \mathfrak{H}$ there exists $\mathcal{H}' \in \mathfrak{H}'$ such that $V(\mathcal{H}') = V(\mathcal{H})$ and $\mathcal{H}' \subseteq \mathcal{H}$,

then if $\mathfrak{F}$ is $\mathfrak{H}'$-vertex locally stable, then it is also $\mathfrak{H}$-vertex locally stable.

Proof. Let $\varepsilon', \alpha > 0$ be such that $\mathfrak{F}$ is $(\mathfrak{H}', \varepsilon', \alpha)$-vertex locally stable. We show that $\mathfrak{F}$ is $(\mathfrak{H}, \frac{\varepsilon'}{4}, \alpha)$-vertex locally stable. Denote $\varepsilon = \varepsilon'/4$ and assume n is sufficiently large. Suppose $\mathcal{F} \in \mathfrak{F}$ is such that $\mathrm{v}(\mathcal{F}) = n$ and $d_{\mathfrak{H}}(\mathcal{F}) \leq \varepsilon n^r$, and $|L_{\mathcal{F}}(v)| \geq (rm(\mathfrak{H}) - \varepsilon)n^{r-1}$, for every $v \in V(\mathcal{F})$. Then we also have $|L_{\mathcal{F}}(v)| \geq (rm(\mathfrak{H}') - \varepsilon)n^{r-1}$, for every $v \in V(\mathcal{F})$ but to be able to apply $(\mathfrak{H}', \varepsilon', \alpha)$-vertex local stability, we need to bound the distance $d_{\mathfrak{H}'}(\mathcal{F})$ by $\varepsilon' n^r$. That is what we do next.

Let $\mathcal{H} \in \mathfrak{H}$ be such that $V(\mathcal{H}) = V(\mathcal{F})$ and $d_{\mathfrak{H}}(\mathcal{F}) = |\mathcal{H} \triangle \mathcal{F}|$. We have

$$|\mathcal{F}| = \frac{1}{r} \sum_{v \in V(\mathcal{F})} |L_{\mathcal{F}}(v)| \geq \left(m(\mathfrak{H}) - \frac{\varepsilon}{r} \right) n^r.$$

Therefore, $|\mathcal{H}| \geq (m(\mathfrak{H}) - 2\varepsilon)n^r$. Let $\mathcal{H}' \in \mathfrak{H}$ such that $\mathrm{V}(\mathcal{H}') = V(\mathcal{H})$ and $\mathcal{H}' \subseteq \mathcal{H}$. Then by definition of $m(\mathfrak{H}')$,

$$|\mathcal{H}'| \leq m(\mathfrak{H}', n) \leq (m(\mathfrak{H}') + \varepsilon)n^r = (m(\mathfrak{H}) + \varepsilon)n^r.$$

Therefore, $|\mathcal{H} \triangle \mathcal{H}'| \leq 3\varepsilon n^r$. Hence,

$$d_{\mathfrak{H}'}(\mathcal{F}) \leq |\mathcal{H}' \triangle \mathcal{F}| \leq |\mathcal{H}' \triangle \mathcal{H}| + |\mathcal{H} \triangle \mathcal{F}| \leq 4\varepsilon n^r = \varepsilon' n^r.$$

Hence, by $(\mathcal{H}, \varepsilon', \alpha)$-vertex local stability, we get that

$$|\mathcal{F}| \leq m(\mathfrak{H}', n) - \alpha d_{\mathfrak{H}'}(\mathcal{F}) \leq m(\mathfrak{H}, n) - \alpha d_{\mathfrak{H}}(\mathcal{F}),$$

where the last inequality holds because $d_{\mathfrak{H}'}(\mathcal{F}) \geq d_{\mathfrak{H}}(\mathcal{F})$ and $m(\mathfrak{H}', n) \leq m(\mathfrak{H}, n)$. □

We use Lemma 3.5.8 in Chapter 6, in the proof of Theorem 6.2.

3.6. An Application: Erdős-Simonovits Stability Theorem

In this section we give a sample application of the techniques we developed thus far. Recall the classical Erdős-Simonovits stability theorem which can be stated in our language as follows, where here by stability we mean as in Definition 3.1.3.

Theorem 3.6.1 (Erdős, Simonovits, [Sim68])**.** *Let $t \geq 2$ be a fixed positive integer, then* $\mathrm{Forb}(K_t)$ *is* $\mathfrak{B}(K_{t-1})$*-stable.*

To demonstrate how the local stability method can be used to derive the exact Turán result from Turán density result, we prove Theorem 3.6.1 but with the stability notion as in Definition 3.2.1. As discussed in Remark 3.2.1 this will be a stronger statement and moreover, will imply Turán's theorem directly. Note that in the proof we use only the Turán density result for the cliques.

Theorem 3.6.2 (Turán, [Tur61])**.** *For any $t \geq 3$, $\pi(K_{t+1}) = 1 - \frac{1}{t}$.*

Proof. Let $\mathfrak{F} := \mathrm{Forb}(K_t)$, $\mathfrak{H} := \mathfrak{B}(K_{t-1})$ and $\mathfrak{F}^*$ be the family of graphs in $\mathfrak{F}$ that cover pairs. Obviously both $\mathfrak{F}$ and $\mathfrak{H}$ are clonable. By Theorem 3.2.4 we need to show that $\mathfrak{F}$ is $\mathfrak{H}$-vertex locally stable and that the family $\mathfrak{F}^*$ is $\mathfrak{H}$-weakly weight-stable. Now let us look at the graphs in the family $\mathfrak{F}^*$ closer, observe that these are just all the cliques on at most $(t-1)$ vertices. This family is trivially $\mathfrak{H}$-weakly weight-stable. So for the theorem to hold we only need to show that $\mathfrak{F}$ is $\mathfrak{H}$-vertex locally stable.

We will show that $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon, 1)$-vertex locally stable, that is, there exist $\varepsilon > 0$, $n_0 \in \mathbb{N}$ such that if $\mathcal{F} \in \mathfrak{F}$ satisfies $\mathrm{v}(\mathcal{F}) = n \geq n_0$, $d_{\mathfrak{H}}(\mathcal{F}) \leq \varepsilon n^2$ and

$$|L_{\mathcal{F}}(v)| \geq \left(\frac{t-2}{t-1} - \varepsilon\right) n, \tag{3.28}$$

for every $v \in V(\mathcal{F})$, then $|\mathcal{F}| \leq m(\mathfrak{H}, n) - d_{\mathfrak{H}}(\mathcal{F})$. In fact, we prove a stronger statement. We show that if the above conditions hold then there exists $\mathcal{H}_0 \in \mathfrak{H}$ such that $\mathcal{F} \subseteq \mathcal{H}_0$, that is, $\mathcal{F}$ is $(t-1)$-partite.

Remark 3.6.1. *An even stronger result was proved by Andrásfai, Erdős and Sós [AES74]. They show that the condition $d_{\mathfrak{H}}(\mathcal{F}) \leq \varepsilon n^2$ is unnecessary, and (3.28) suffices to deduce that $\mathcal{F}$ is $(t-1)$-partite for $\varepsilon < \frac{1}{(3t-4)(t-1)}$. We, however, include the proof which exploits the bound on the distance from $\mathcal{F}$ to $\mathfrak{H}$ to demonstrate our method.*

Let $0 \ll \varepsilon \ll \delta \ll \gamma \ll 1/t$ be chosen to satisfy the inequalities appearing further in the proof and let n be sufficiently large. Given $\mathcal{F}$ as above, let $\mathcal{H} \in \mathfrak{H}$ be such that $V(\mathcal{H}) = V(\mathcal{F})$ and $|\mathcal{F} \triangle \mathcal{H}| = d_{\mathfrak{H}}(\mathcal{F})$. Since, $d_{\mathfrak{H}}(\mathcal{F}) \leq \varepsilon n^2$, we have

$$|\mathcal{H}| \geq |\mathcal{F}| - \varepsilon n^2 \geq \left(\frac{t-2}{t-1} - 3\varepsilon\right)\frac{n^2}{2}. \tag{3.29}$$

Let $\mathcal{P} = \{P_1, P_2, \ldots, P_{t-1}\}$ be the blowup partition of $V(\mathcal{H})$. It is easy to see that (3.29) implies that

$$\left||P_i| - \frac{n}{t-1}\right| \leq \gamma n,$$

for all $i \in [t-1]$ with an appropriate choice of $\varepsilon \ll \gamma$ (for a similar proof, see Lemma 3.9.1 or Lemma 6.2.2).

Next we show that the neighborhood of every vertex in $\mathcal{F}$ is "close" to the neighborhood of some vertex in $\mathcal{H}$. For $v \in V(\mathcal{F})$, let $I(v) = \{i \mid |N(v) \cap P_i| \geq \gamma n\}$, where $N(v)$ denotes the neighborhood of v. Then (3.28) implies that $|I(v)| \geq t-2$ for every $v \in V(\mathcal{F})$. Suppose that $|I(v)| = t-1$, and choose $Q_i \subseteq N(v) \cap P_i$ so that $|Q_i| = \gamma n$ for $i \in [t-1]$. For simplicity, we assume that γn is an integer. Let $Q = \cup_{i\in[t-1]} Q_i \subseteq N(v)$. Then $\mathcal{F}|_Q$ is K_{t-1}-free and, therefore, Theorem 3.6.2 implies that if n is large enough,

$$|\mathcal{F}|_Q| \leq \frac{(t-3)((t-1)\gamma n)^2}{2(t-2)} + \delta n^2 \tag{3.30}$$

On the other hand, $\mathcal{H}|_Q$ is balanced $(t-1)$-partite, thus,

$$|\mathcal{H}|_Q| \geq \frac{(t-2)((t-1)\gamma n)^2}{2(t-1)}. \tag{3.31}$$

Combining (3.29) and (3.30), we deduce that

$$
\begin{aligned}
|\mathcal{F} \triangle \mathcal{H}| &\geq |\mathcal{F}|_Q \triangle \mathcal{H}|_Q| \\
&\geq \left(\frac{t-2}{t-1} - \frac{t-3}{t-2}\right) \frac{((t-1)\gamma n)^2}{2} - \delta n^2 > \varepsilon n^2.
\end{aligned}
$$

This contradiction implies that $|I(v)| = t-2$ for all $v \in V(\mathcal{F})$.

Finally, we construct a partition $\mathcal{P}' = \{P'_1, P'_2, \ldots, P'_{t-1}\}$ of $V(\mathcal{F})$ so that $\mathcal{F} \subseteq \mathcal{F}''$, where $\mathcal{F}''$ is a blowup of K_{t-1} with the blowup partition $\mathcal{P}'$. Define $P'_i := \{v \in V(\mathcal{F}) \mid i \notin I(v)\}$ for $i \in [t-1]$. Note that (3.28) and the bounds on the size of P_j imply that

$$
|N(v) \cap P_j| \geq n/(t-1) - (t-1)\gamma n
$$

for every $v \in P'_i$, $i \neq j$. It follows that, if $v, v' \in P'_i$, then $\{v, v'\} \notin \mathcal{F}$. (Otherwise, $\mathcal{F}|_{N(v)\cap N(v')}$ is K_{t-2}-free and $|N(v) \cap N(v') \cap P'_j| \geq n/(t-1) - (2t-1)\gamma n$ for every $j \in [t-1] \setminus \{i\}$. This leads to a contradiction using an argument completely analogous to the one used in the preceding paragraph.) Thus, $\mathcal{F} \subseteq \mathcal{F}''$, as desired. □

3.7. Extending Local Stability Method For Non-Clonable $\mathfrak{F}$

Theorem 3.2.4 provides us with a tool to obtain the stability of the family from its local stability when the family under consideration is clonable. However, the family $\mathfrak{F}$ is typically non-clonable. In this section we overcome this obstacle and obtain the analogous tool for general $\mathfrak{F}$.

The trick is to consider the maximal clonable subfamily of $\mathfrak{F}$ instead. We define *the core of* $\mathfrak{F}$ to be

$$
\operatorname{core}(\mathfrak{F}) = \{\mathcal{F} \in \mathfrak{F} \mid \mathfrak{B}(\mathcal{F}) \subseteq \mathfrak{F}\}.
$$

As an application of Hypergraph Removal Lemma we are able to derive the stability of a family from its local stability when its core is stable. For these purposes, we need the following corollary of the Hypergraph Removal Lemma and a classical result of Erdős.

Lemma 3.7.1 (Rődl, Skokan [RS06b]). *For every r-graph $\mathcal{G}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that every r-graph on n vertices which contains at most $\delta n^{\mathrm{v}(\mathcal{G})}$ copies of $\mathcal{G}$ can be made $\mathcal{G}$-free by removing at most εn^r edges.*

Corollary 3.7.2 (Erdős, [Erd64]). *For every r-graph $\mathcal{H}$, a blowup $\mathcal{B} \in \mathfrak{B}(\mathcal{H})$ of $\mathcal{H}$ and $\delta > 0$ there exists n_0 such that any r-graph on $n \geq n_0$ vertices that does not contain $\mathcal{B}$ contains at most $\delta n^{\mathrm{v}(\mathcal{H})}$ many copies of $\mathcal{H}$.*

Lemma 3.7.3. *Let $\mathfrak{G}$ be a finite family of r-graphs, and let $\mathfrak{F} = \mathrm{Forb}(\mathfrak{G})$. Then for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $\mathcal{F} \in \mathfrak{F}$ with $\mathrm{v}(\mathcal{F}) = n \geq n_0$ there exists $\mathcal{F}' \in \mathrm{core}(\mathfrak{F})$ with $V(\mathcal{F}') = V(\mathcal{F})$ and $\mathcal{F}' \subseteq \mathcal{F}$ such that*

$$|\mathcal{F}'| \geq |\mathcal{F}| - \varepsilon n^r.$$

Proof. Let $\mathfrak{H}$ be the family of all minimal graphs $\mathcal{H}$ such that $\mathfrak{B}(\mathcal{H}) \not\subseteq \mathfrak{F}$. It is easy to see that $\mathfrak{H}$ is finite. In particular every element of $\mathfrak{H}$ is a subgraph of some graph in $\mathfrak{G}$: For every $\mathcal{H} \in \mathfrak{H}$ there exists $\mathcal{B}_{\mathcal{H}} \in \mathfrak{G} \cap \mathfrak{B}(\mathcal{H})$.

By Lemma 3.7.1 there exists $\delta > 0$ such that for every $\mathcal{H} \in \mathfrak{H}$ every r-graph on n vertices which contains at most $\delta n^{\mathrm{v}(\mathcal{H})}$ copies of $\mathcal{H}$ can be made $\mathcal{H}$-free by removing at most $\frac{\varepsilon}{|\mathfrak{H}|} n^r$ edges. By Corollary 3.7.2 there exists n_0 such that for every $n \geq n_0$ and every $\mathcal{H} \in \mathfrak{H}$ every $\mathcal{B}_{\mathcal{H}}$-free graph $\mathcal{F}$ on $n \geq n_0$ vertices contains at most $\delta n^{\mathrm{v}(\mathcal{H})}$ copies of $\mathcal{H}$. Hence, by removing at most εn^r edges from any graph $\mathcal{F} \in \mathfrak{F}$ on $n \geq n_0$ vertices, we can obtain a subgraph $\mathcal{F}'$ of $\mathcal{F}$, which is $\mathfrak{H}$-free. We have $\mathcal{F}' \in \mathrm{core}(\mathfrak{F})$, as desired. □

The following result establishes the desired connection between the stability of the family $\mathfrak{F}$ and the stability of the core($\mathfrak{F}$).

Theorem 3.7.4. *Let $\mathfrak{G}, \mathfrak{H}$ be families of r-graphs, such that $\mathfrak{G}$ is finite, and let $\mathfrak{F} = \mathrm{Forb}(\mathfrak{G})$. If $\mathrm{core}(\mathfrak{F})$ is $\mathfrak{H}$-stable and $\mathfrak{F}$ is $\mathfrak{H}$-locally stable, then $\mathfrak{F}$ is $\mathfrak{H}$-stable.*

Proof. Let $\varepsilon, \alpha > 0$ be chosen such that $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon, \alpha)$-locally stable and $\mathrm{core}(\mathfrak{F})$ is $(\mathfrak{H}, \alpha)$-stable. We claim that $\mathfrak{F}$ is $(\mathfrak{H}, \alpha/2)$-stable. Let $\mathcal{F} \in \mathfrak{F}$ with $\mathrm{v}(\mathcal{F}) = n \geq n_0$, where n_0 is chosen sufficiently large. We want to show that

$$|\mathcal{F}| \leq m(\mathfrak{H}, n) - \alpha d_{\mathfrak{H}}(\mathcal{F}). \tag{3.32}$$

If $d_{\mathfrak{H}}(\mathcal{F}) \leq \varepsilon$, then since $\mathfrak{F}$ is $(\mathfrak{H}, \varepsilon, \alpha)$-locally stable, (3.32) holds directly. Now we may assume $d_{\mathfrak{H}}(\mathcal{F}) > \varepsilon$. By Lemma 3.7.3, applied with $\varepsilon' = \frac{\alpha}{2(\alpha+1)}\varepsilon$, there exists $\mathcal{F}' \subseteq \mathcal{F}$ such that $|\mathcal{F}'| \geq |\mathcal{F}| - \varepsilon' n^r$. Hence,

$$d_{\mathfrak{H}}(\mathcal{F}) \leq d_{\mathfrak{H}}(\mathcal{F}') + \varepsilon' n^r.$$

On the other hand, since $\text{core}(\mathfrak{F})$ is $\mathfrak{H}$-stable, we have

$$|\mathcal{F}'| \leq m(\mathfrak{H}, n) - \alpha d_{\mathfrak{H}}(\mathcal{F}') \leq m(\mathfrak{H}, n) - \alpha(d_{\mathfrak{H}}(\mathcal{F}) - \varepsilon' n^r).$$

Putting all this together, we get

$$\begin{aligned}
|\mathcal{F}| &\leq |\mathcal{F}'| + \varepsilon' n^r \\
&\leq m(\mathfrak{H}, n) - \alpha(d_{\mathfrak{H}}(\mathcal{F}) - \varepsilon' n^r) + \varepsilon' n^r \\
&\leq m(\mathfrak{H}, n) - \frac{\alpha}{2} d_{\mathfrak{H}}(\mathcal{F}) - \left(\frac{\alpha}{2} d_{\mathfrak{H}}(\mathcal{F}) - (\alpha+1)\varepsilon' n^r\right) \\
&\leq m(\mathfrak{H}, n) - \frac{\alpha}{2} d_{\mathfrak{H}}(\mathcal{F}),
\end{aligned}$$

where in the last inequality we used that $d_{\mathfrak{H}}(\mathcal{F}) > \varepsilon n^r$. □

We can further simplify the conditions of Theorem 3.7.4 when the family $\mathfrak{H}$ is clonable.

Corollary 3.7.5. *Let $\mathfrak{G}, \mathfrak{H}$ be families of r-graphs, such that $\mathfrak{G}$ is finite, $\mathfrak{H}$ is clonable. Let $\mathfrak{F} = \text{Forb}(\mathfrak{G})$ and $\mathfrak{F}^*$ be the subfamily of all r-graphs in* $\text{core}(\mathfrak{F})$ *that cover pairs. If $\mathfrak{F}$ is $\mathfrak{H}$-vertex locally stable and $\mathfrak{F}^*$ is $\mathfrak{H}$-stable then $\mathfrak{F}$ is $\mathfrak{H}$-stable.*

Proof. Theorem 3.5.3 implies that $\mathfrak{F}$ is $\mathfrak{H}$-locally stable. In particular, it implies the local stability of $\text{core}(\mathfrak{F})$. But the family $\text{core}(\mathfrak{F})$ is clonable, thus Theorem 3.2.4 applies and it follows that $\text{core}(\mathfrak{F})$ is $\mathfrak{H}$-stable. Finally Theorem 3.7.4 implies that $\mathfrak{F}$ is $\mathfrak{H}$-stable. □

3.8. Applications of Local Stability Method for Extensions

Recall the definition of extensions from introduction. Given an r-graph $\mathcal{G}$, the *extension of* $\mathcal{G}$, denoted by $\mathrm{Ext}(\mathcal{G})$, is an r-graph defined as follows. For every uncovered pair P in $\mathcal{G}$ we add $r-2$ new vertices $v_1^P, v_2^P, \ldots, v_{r-2}^P$ to $V(\mathcal{G})$, and add the edge $P \cup \{v_1^P, v_2^P, \ldots, v_{r-2}^P\}$ to $\mathcal{G}$. It turns out that for any $\mathfrak{F} = \mathrm{Forb}(\mathrm{Ext}(\mathcal{G}))$, $\mathrm{core}(\mathrm{Forb}(\mathfrak{F})$ is easy to describe structurally, thus allowing us to further simplify Theorem 3.7.5.

A *weak extension* of an r-graph $\mathcal{G}$ is an r-graph obtained from $\mathcal{G}$ by adding a new edge through every uncovered pair of vertices which could contain up to $(r-2)$ new vertices. Note that in particular, $\mathrm{Ext}(\mathcal{G})$ is a weak extension of $\mathcal{G}$. We denote by $\mathrm{WExt}(\mathcal{G})$ the family of all weak extensions of the graph $\mathcal{G}$.

Lemma 3.8.1. *Let $\mathcal{G}$ be an r-graph and $\mathfrak{F} = \mathrm{Forb}(\mathrm{Ext}(\mathcal{G}))$. Then*

(i) $\mathrm{core}(\mathfrak{F} = \mathrm{Forb}(\mathrm{WExt}(\mathcal{G}))$, *and*

(ii) If $\mathfrak{F}^*$ is a family of all graphs in $\mathrm{Forb}(\mathrm{WExt}(\mathcal{G}))$ which cover pairs, then $\mathfrak{F}^* \subseteq \mathrm{Forb}(\mathcal{G})$.

Proof. (i) We only show that $\mathrm{Forb}(\mathrm{WExt}(\mathcal{G})) \subseteq \mathrm{core}(\mathfrak{F})$, the other direction is similar. Let $\mathcal{F} \in \mathrm{core}(\mathfrak{F})$. We claim that $\mathcal{F}$ is $\mathrm{WExt}(\mathcal{G})$-free. Suppose not and there exists $\mathcal{H} \in \mathrm{WExt}(\mathcal{G})$ such that $\mathcal{F}$ contains a copy of $\mathcal{H}$. $\mathcal{H}$ contains a copy of $\mathcal{G}$ and for every uncovered pair of vertices u, v in this copy, there exists some edge $H \in \mathcal{H}$ such that $u, v \in H$. Iteratively clone all the vertices in the set $H \setminus \{u, v\}$ and do this for every such u, v. The resulting graph will be a blowup of $\mathcal{F}$, containing a copy of $\mathrm{Ext}(\mathcal{G})$, a contradiction.

(ii) Let $\mathcal{F} \in \mathfrak{F}^*$. Suppose it contains a copy of $\mathcal{G}$. Since $\mathcal{F}$ covers pairs, every pair of vertices in this copy is covered by some edge in $\mathcal{F}$, thus creating a copy of some graph in $\mathrm{WExt}(\mathcal{G})$, a contradiction. □

This lemma together with Theorem 3.7.5 implies the desired simpler tool for deriving stability from local stability for all families $\mathfrak{F}$ which are of form $\mathfrak{F} = \mathrm{Forb}(\mathrm{Ext}(\mathcal{G}))$ for some r-graph $\mathcal{G}$.

Theorem 3.8.2. *Let $\mathcal{G}$ be an r-graph, $\mathfrak{F}$ and $\mathfrak{H}$ be two families of r-graphs such that $\mathfrak{F} = \mathrm{Forb}(\mathrm{Ext}(\mathcal{G}))$ and $\mathfrak{H}$ is clonable. Suppose $\mathfrak{F}^*$ is the family of all r-graphs in $\mathrm{Forb}(\mathcal{G})$ that cover pairs and the following conditions hold:*

(C1) *$\mathfrak{F}$ is $\mathfrak{H}$-vertex locally stable,*

(C2) *$\mathfrak{F}^*$ is $\mathfrak{H}$-weakly weight-stable.*

Then $\mathfrak{F}$ is $\mathfrak{H}$-stable. In particular, there exists $n_0 \in \mathbb{N}$ such that if $\mathcal{F} \in \mathfrak{F}$ satisfies $\mathrm{v}(\mathcal{F}) = n$ and $|\mathcal{F}| = m(\mathfrak{F}, n)$ for some $n \geq n_0$ then $\mathcal{F} \in \mathfrak{H}$.

Proof. The result follows directly from Corollary 3.7.5 and Lemma 3.8.1. □

Finally, let us restate our main results from Section 1.6 in the language of extensions. By applying Theorem 3.8.2, we are able to determine the Turán numbers of the following graphs or families of graphs, for large enough n.

(1) The extension of two r-edges sharing $(r-1)$-vertices, for uniformities $r = 5, 6$.

(2) The extension of two disjoint r-edges (that is, a two matching), for all $r \geq 4$.

(3) The extension of the r-graph $\mathcal{F}^{+t}$ for any r-graph $\mathcal{F}$ that covers pairs and $t \geq \mathrm{v}(\mathcal{F})$, where the graph $\mathcal{F}^{+t}$ is obtained from $\mathcal{F}$ by adding new isolated vertices such that $\mathrm{v}(\mathcal{F}^{+t}) = t$.

(4) Recall that the *l-expansion* of a 2-graph G is an $(l+2)$-graph obtained from G by adding l vertices and enlarging each edge of G using these vertices. For $r \geq 3$, we determine the Turán numbers of the extensions of $(r-2)$-expansions of sufficiently large Erdős-Sós-trees (recall that these are the trees that satisfy Conjecture 1.6.2).

3.9. Some Properties of Dense Blowups and Transversal Lemma

In this section we prove some auxiliary lemmas which are common for our proofs of local stabilities of certain families (more specifically, in the proofs of Theorem 4.2.1 and Theorem 5.2.2). In the following discussion, one should think of $\mathcal{F}$ as being the forbidden graph, $\mathcal{H}$ being the graph whose balanced blowups are the (conjectured) extremal examples.

Recall that for an r-graph $\mathcal{H}$, $\xi_{\mathcal{H}} \in \mathcal{M}(\mathcal{H})$ denotes the uniform measure on $V(\mathcal{H})$. Let $\mathfrak{B} := \mathfrak{B}(\mathcal{H})$ denote the family of all blowups of the graph $\mathcal{H}$ and for each $\mathcal{B} \in \mathfrak{B}$, let $\mathcal{P}(\mathcal{B})$ denote the corresponding blowup partition. We say that an r-graph $\mathcal{H}$ is *uniquely dense* if $\lambda(\mathcal{H})$ is uniquely achieved on some $\mu \in \mathcal{M}(\mathcal{H})$. We denote such μ by $\mu^*_{\mathcal{H}}$. Suppose $V(\mathcal{H}) = \{1, 2, \dots, m\}$, for a blowup $\mathcal{B} \in \mathfrak{B}$ we denote by P_i the partition class in $\mathcal{P}(\mathcal{B})$ corresponding to the vertex $i \in V(\mathcal{H})$. We say that $\mathcal{B} \in \mathfrak{B}(\mathcal{H})$ is *(ε, μ)-trimmed* for some $\varepsilon \in (0, 1)$, $\mu \in \mathcal{M}(\mathcal{H})$ if

$$\left| \frac{|P_i|}{\mathrm{v}(\mathcal{B})} - \mu(i) \right| \le \varepsilon,$$

for every $i \in [t]$. Furthermore, we simply say $\mathcal{B}$ is *ε-trimmed* if $\mathcal{B}$ is $(\varepsilon, \xi_{\mathcal{H}})$-trimmed.

Our first lemma ensures that if a blowup of a uniquely dense graph has density close to the maximum possible, then the blowup partition is trimmed with respect to the measure maximizing the Lagrangian of the graph $\mathcal{H}$.

Lemma 3.9.1. *Given $r \ge 2$, let $\mathcal{H}$ be a uniquely dense r-graph. For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{B} \in \mathfrak{B}(\mathcal{H})$ with $\mathrm{v}(\mathcal{B}) = n$ and $|\mathcal{B}| \ge (m(\mathfrak{B}) - \delta)n^r$, then $\mathcal{B}$ is $(\varepsilon, \mu^*_{\mathcal{H}})$-trimmed.*

Proof. Suppose $V(\mathcal{H}) = \{1, 2, \dots, m\}$, we define $\mu(i) = \frac{|P_i|}{n}$, for every $i \in [m]$. Clearly, $\mu \in \mathcal{M}(\mathcal{H})$. For our purposes, it suffices to show that $||\mu - \mu^*||_\infty \le \varepsilon$. We have

$$\lambda(\mathcal{H}, \mu^*) \ge \lambda(\mathcal{H}, \mu) = \frac{|\mathcal{B}|}{n^r} \ge m(\mathfrak{B}) - \delta = \lambda(\mathfrak{B}) - \delta = \lambda(\mathcal{H}, \mu^*) - \delta, \tag{3.33}$$

where in the second equality we used the fact that $\lambda(\mathfrak{B}) = m(\mathfrak{B})$ since $\mathfrak{B}$ is clonable (see Lemma 2.4.3). But $\lambda(\mathcal{H}, \cdot)$ is a continuous function and $\mathcal{H}$ is uniquely dense, therefore for every $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$\lambda(\mathcal{H}, \mu^*) \geq \lambda(\mathcal{H}, \mu) \geq \lambda(\mathcal{H}, \mu^*) - \delta,$$

then $||\mu - \mu^*||_\infty \leq \varepsilon$, as desired. □

We say that an r-graph $\mathcal{H}$ is *balanced* if $\lambda(\mathcal{H}) = \lambda(\mathcal{H}, \xi_{\mathcal{H}})$. In our proofs we only use the following direct corollary of Lemma 3.9.1, which says that if a blowup of a uniquely dense and balanced graph has density close to the maximum possible, then the blowup partition is almost an equipartition.

Corollary 3.9.2. *Given $r \geq 2$, let $\mathcal{H}$ be a uniquely dense and balanced r-graph. For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{B} \in \mathfrak{B}$ with $\mathrm{v}(\mathcal{B}) = n$ and $|\mathcal{B}| \geq (m(\mathfrak{B}) - \delta)n^r$, then $\mathcal{B}$ is ε-trimmed.*

To state our second auxiliary lemma, we need few more definitions. For an r-graph $\mathcal{H}$ with $V(\mathcal{H}) = [m]$, we say that $\mathcal{B} \in \mathfrak{B}(\mathcal{H})$ is *α-dense* for some $\alpha \in (0, 1)$, if $|P_i| \geq \alpha n$, for every $i \in [m]$.

Given a collection of sets $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$ we say that a set F is *$\mathcal{X}$-transversal* if $|X_i \cap F| \leq 1$ for every $1 \leq i \leq k$. We say that an r-graph $\mathcal{F}$ is *$\mathcal{X}$-transversal* if every $F \in \mathcal{F}$ is $\mathcal{X}$-transversal.

Recall that for two r-graphs $\mathcal{F}$ and $\mathcal{H}$, we say that $\mathcal{H}$ is *$\mathcal{F}$-hom-free* if there is no homomorphism from $\mathcal{F}$ to $\mathcal{H}$. We say that $\mathcal{H}$ is *$\mathcal{F}$-hom-critical* if $\mathcal{H}$ is $\mathcal{F}$-hom-free but there exists an edge $F \in \mathcal{F}$ such that there exists a homomorphism from $\mathcal{F} \setminus F$ to $\mathcal{H}$. More specifically, if F is such an edge in $\mathcal{F}$, we say that $\mathcal{H}$ is *$(\mathcal{F}, F)$-hom-critical.*

For an r-graph $\mathcal{F}$ and an edge $F \in \mathcal{F}$, we say that the pair $(\mathcal{F}, F)$ is *loose* if the vertices of F can be partitioned into two sets, F_c and F_f, such that

- $|F_c| = 2$, the vertices of F_c do not share any edge other than F,
- every vertex of F_f is not contained in any edge other than F.

The vertices in F_f are called *free*, the vertices in F_c are called *critical.*

Our next lemma says that if $\mathcal{H}$ is $\mathcal{F}$-hom-free then given any large blowup $\mathcal{B} \in \mathfrak{B}$ and an $\mathcal{F}$-free r-graph $\mathcal{G}$ on the same vertex set such that every vertex has only a small portion of neighbours in $\mathcal{G}$ which are non-neighbours in $\mathcal{B}$, then $\mathcal{F}$ is $\mathcal{P}(\mathcal{B})$-transversal. This lemma is quite useful for us. Why? Suppose we are given such $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{B}$ and also we know that both $\mathcal{B}$ and $\mathcal{G}$ have density very close to the maximum possible one (i.e. $m(n, \mathcal{B})$), then we would like to show that the edit distance between $\mathcal{G}$ and $\mathcal{B}$ is small. In particular, we need to show that the number of *bad* edges is small, that is, those edges which are in $\mathcal{F}$ but not in $\mathcal{B}$. Our next result tells us that there are no non-transversal bad edges, which helps us to bound the number of total bad edges.

Lemma 3.9.3. *Given $r \geq 2$, let $\mathcal{F}$ and $\mathcal{H}$ be two r-graphs such that $(\mathcal{F}, F)$ is loose for some $F \in \mathcal{F}$, $\mathcal{H}$ is vertex transitive, $(\mathcal{F}, F)$-hom-critical and covers pairs. For every $\alpha > 0$ there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that if $\mathcal{G}$ is an $\mathcal{F}$-free r-graph and $\mathcal{B} \in \mathfrak{B}(\mathcal{H})$ with $V(\mathcal{G}) = V(\mathcal{B}) := V$, $\mathrm{v}(\mathcal{G}) = \mathrm{v}(\mathcal{B}) = n \geq n_0$ such that*

- *$\mathcal{B}$ is α-dense,*
- *$|L_{\mathcal{B}}(v) \setminus L_{\mathcal{G}}(v)| \leq \varepsilon n^{r-1}$, for every $v \in V$,*

then $\mathcal{G}$ is $\mathcal{P}(\mathcal{B})$-transversal. Moreover, if $\mathcal{G}'$ is an $\mathcal{F}$-free r-graph such that $\mathcal{G} \subseteq \mathcal{G}'$, then $\mathcal{G}'$ is also $\mathcal{P}(\mathcal{B})$-transversal.

Proof. Assume $V(\mathcal{H}) = [t]$. Since $\mathcal{H}$ is vertex transitive, in particular, it is also regular, let $d := |L_{\mathcal{H}}(i)|$, for some $i \in [t]$. Let $\mathcal{P} := \mathcal{P}(\mathcal{B}) = \{P_1, P_2, \ldots, P_t\}$ and suppose $F = \{c_1, c_2, f_1, \ldots, f_{r-2}\}$, where c_1, c_2 are the critical vertices and $f_1, \ldots, f_{r-2}$ are the free ones. Let m be the smallest integer such that $\mathcal{H}(m)$ (recall that $\mathcal{H}(m)$ is the balanced blowup of $\mathcal{H}$ with m vertices in each partition class) contains a copy of $\mathcal{F} \setminus F$. Such a finite m exists by Fact 2.1.3. Throughout the proof we assume n is sufficiently large, in particular, $m \ll n$. Choose $\varepsilon \ll \min\{\alpha, \frac{1}{m}, \frac{1}{t}\}$.

It suffices to verify only the last conclusion of the lemma. We assume, for a contradiction, that there exists a non-transversal edge $G \in \mathcal{G}'$, with $v_1, v_2 \in G \cap P_j$ for some j. Without loss of generality, assume $j = 1$. We will show that $\mathcal{G}' \setminus G$ contains a copy of $\mathcal{F} \setminus F$ such that v_1 plays the role of c_1 and v_2 plays the role of c_2. Together with G this copy will induce a copy of $\mathcal{F}$, yielding the desired contradiction.

Let φ be a homomorphism from $\mathcal{F} \setminus F$ to $\mathcal{H}$. Since c_1 and c_2 are contained in no edge together in $\mathcal{F} \setminus F$, $\mathcal{H}$ is vertex transitive and covers pairs, we may assume $\varphi(c_1) = \varphi(c_2)$. Furthermore, since $\mathcal{H}$ is vertex-transitive, we may also assume $\varphi(c_1) = \varphi(c_2) = 1$.

We sample m vertices from $P_i \setminus G$ uniformly at random for every $i = 2, \dots, t$ and $(m-2)$ vertices from P_1. Let $\mathcal{R}$ be the subgraph of $\mathcal{G}'$ induced by these vertices, v_1 and v_2. Note that the same vertices together induce $\mathcal{H}(m)$ in $\mathcal{B}$ which contains a copy of $\mathcal{F} \setminus F$. Thus, it suffices to show that with a positive probability every $\mathcal{P}$-transversal r-tuple $I \subseteq V(\mathcal{R})$ is an edge of $\mathcal{G}$ if it is an edge in $\mathcal{B}$. Therefore, it is enough to show that

1. If I is a set of $r-1$ vertices sampled uniformly at random from distinct parts of $\mathcal{P} - \{P_1\}$ then
$$\mathbb{P}\left[I \cup \{v_i\} \in \mathcal{B} \setminus \mathcal{G}\right] < \frac{1}{4t^{r-1}m^{r-1}}$$
for $i = 1, 2$, and

2. If I is a set of r vertices sampled uniformly at random from distinct parts of $\mathcal{P}$ then
$$\mathbb{P}\left[I \in \mathcal{B} \setminus \mathcal{G}\right] < \frac{1}{4t^r m^r}$$
for $i = 1, 2$.

Let us show that both conditions hold. First note that because $\mathcal{B}$ is α-dense, the following statements follow easily.

- $|\mathcal{B}| \geq \mathrm{e}(\mathcal{H})\alpha^r n^r = \frac{\alpha^r dt}{r} n^r$,
- $|L_{\mathcal{B}}(v)| \geq d\alpha^{r-1}n^{r-1}$, for every $v \in V$.

Thus, for any I as in (1), we have

$$\mathbb{P}\left[I \cup \{v_i\} \in \mathcal{B} \setminus \mathcal{G}\right] \leq \frac{|L_{\mathcal{B}}(v_i) \setminus L_{\mathcal{G}}(v_i)|}{|L_{\mathcal{B}}(v_i)|} \leq \frac{\varepsilon n^{r-1}}{d\alpha^{r-1}n^{r-1}} \ll \frac{1}{t^r m^r},$$

and thus the probability that a transversal r-tuple containing v_i is in $\mathcal{B} \setminus \mathcal{F}$ is sufficiently small. Similarly, for I as in (2), we have

$$\mathbb{P}\left[I \in \mathcal{B} \setminus \mathcal{G}\right] \leq \frac{|\mathcal{B} \setminus \mathcal{G}|}{|B|} \leq \frac{\frac{\varepsilon}{r} n^r}{\frac{\alpha^r dt}{r} n^r} = \frac{\varepsilon}{\alpha^r dt} \ll \frac{1}{t^r m^r}.$$

So, with positive probability $\mathcal{R}$ induces a copy of $\mathcal{F} \setminus F$ in $\mathcal{G}$, as desired. □

For our purposes we only need the following corollary from Lemma 3.9.3.

Corollary 3.9.4. *Given $r \geq 2$, let $\mathcal{F}$ and $\mathcal{H}$ be two r-graphs such that $(\mathcal{F}, F)$ is loose with some $F \in \mathcal{F}$, $\mathcal{H}$ is vertex transitive, uniquely dense and balanced, $(\mathcal{F}, F)$-hom-critical and covers pairs. There exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that if $\mathcal{G}$ is an $\mathcal{F}$-free r-graph and $\mathcal{B} \in \mathfrak{B}(\mathcal{H}) := \mathfrak{B}$ with $V(\mathcal{G}) = V(\mathcal{B})$, $\mathrm{v}(\mathcal{G}) = \mathrm{v}(\mathcal{B}) = n \geq n_0$ such that*

- $|\mathcal{B}| \geq (m(\mathfrak{B}) - \varepsilon)\, n^r$,
- $|L_{\mathcal{B}}(v) \setminus L_{\mathcal{G}}(v)| \leq \varepsilon n^{r-1}$, *for every* $v \in V$,

then $\mathcal{G}$ is $\mathcal{P}(\mathcal{B})$-transversal. Moreover, if $\mathcal{G}'$ is an $\mathcal{F}$-free r-graph such that $\mathcal{G} \subseteq \mathcal{G}'$, then $\mathcal{G}'$ is $\mathcal{P}(\mathcal{B})$-transversal.

Proof. Assume $V(\mathcal{H}) = [t]$. Since $\mathcal{H}$ is vertex transitive, in particular, it is also regular, let $d := |L_{\mathcal{H}}(i)|$, for some $i \in [t]$. It is easy to see that $m(\mathfrak{B}) = \frac{d}{rt^{r-1}}$. Let $\varepsilon_{3.9.2} = \frac{1}{2t}$ and $\delta_{3.9.2}$ be derived from Lemma 3.9.2 applied with $\varepsilon = \varepsilon_{3.9.2}$. Let $\varepsilon_{3.9.3}$ be derived from Lemma 3.9.3 applied with $\alpha = \frac{1}{2t}$ and finally choose $\varepsilon = \min\{\delta_{3.9.2}, \varepsilon_{3.9.3}\}$. Then we have

$$|\mathcal{B}| \geq \left(\frac{d}{rt^{r-1}} - \varepsilon\right) n^r \geq \left(\frac{d}{rt^{r-1}} - \delta_{3.9.2}\right) n^r,$$

hence $\mathcal{B}$ is $\varepsilon_{3.9.2}$-trimmed. Thus for every $i \in [t]$ and $P_i \in \mathcal{P}(\mathcal{B})$, $|P_i| \geq \left(\frac{1}{t} - \varepsilon_{3.9.2}\right) n \geq \frac{n}{2t}$. On the other hand, for every $v \in V$,

$$|L_{\mathcal{B}}(v) \setminus L_{\mathcal{G}}(v)| \leq \varepsilon n^{r-1} \leq \varepsilon_{3.9.3} n^{r-1},$$

thus we can apply Lemma 3.9.3 and obtain that $\mathcal{G}$ is $\mathcal{P}(\mathcal{B})$-transversal. □

Chapter 4

The Turán Number of the Generalized Triangle

The *generalized triangle*, is the r-graph on vertex set $[2r-1]$ with edges $\{1, 2, \ldots, r\}$, $\{1, 2, \ldots, r-1, r+1\}$ and $\{r, r+1, \ldots, 2r-1\}$. In this chapter we obtain the Turán number of the generalized triangle for uniformities five and six, thus solving a conjecture of Frankl and Füredi from 1980's. We use the local stability method described in the previous chapter and Turán density results obtained earlier by Frankl and Füredi [FF89].

4.1. The History

Let $\mathfrak{T}_r$ be the family of all r-graphs with three edges such that one edge contains the symmetric difference of the other two. As a generalization of Turán's theorem, Katona suggested to determine $\mathrm{ex}(n, \mathfrak{T}_3)$. This question was answered by Bollobás in early 1970's.

Theorem 4.1.1 (Bollobás, [Bol74]). *For any $n \geq 3$,*

$$\mathrm{ex}(n, \mathfrak{T}_3) = \left\lfloor \frac{n}{3} \right\rfloor \times \left\lfloor \frac{n+1}{3} \right\rfloor \times \left\lfloor \frac{n+2}{3} \right\rfloor .$$

Moreover, the complete balanced 3-partite 3-graph on n vertices, $\mathcal{K}_3^{(3)}(n)$, is the *unique extremal graph.*

In the same paper Bollobás also conjectured that the same result holds for all $r \geq 4$. While the conjecture fails in general (Shearer showed that it does so for all $r \geq 10$, [She96]), Sidorenko proved this conjecture for $r = 4$ and, in fact, he showed something much stronger. Let Σ_r be the family of all r-graphs with three edges D_1, D_2, D_3 such that $|D_1 \cap D_2| = r - 1$ and $D_1 \triangle D_2 \subseteq D_3$. Clearly, $\Sigma_r \subseteq \mathfrak{T}_r$ for all r, and $\Sigma_r = \mathfrak{T}_r$, for $r = 2, 3$. The question on the Turán number of the family Σ_r goes back to De Caen [DC85]. In [FF83] Frankl and Füredi determined $\text{ex}(n, \mathcal{T}_3)$. Then De Caen [DC85], while giving an alternative proof of $\text{ex}(n, \mathcal{T}_3)$, suggested to look at the Turán numbers of the families Σ_r for general r.

Theorem 4.1.2 (De Caen [DC85], Frankl and Füredi [FF83])**.** *For any $n > 3000$,*

$$\text{ex}(n, \Sigma_3) = \text{ex}(n, \mathcal{T}_3) = \left\lfloor \frac{n}{3} \right\rfloor \times \left\lfloor \frac{n+1}{3} \right\rfloor \times \left\lfloor \frac{n+2}{3} \right\rfloor.$$

Morever, the complete balanced 3-partite 3-graph on n vertices, $\mathcal{K}_3^{(3)}(n)$, is the unique extremal graph.

In [Sid87] Sidorenko determined the Turán number of Σ_4 (actually he also reproved the above mentioned result on Σ_3).

Theorem 4.1.3 (Sidorenko, [Sid87])**.** $\text{ex}(n, \mathfrak{T}_4) = \text{ex}(n, \Sigma_4) = \left\lfloor \frac{n}{4} \right\rfloor \times \left\lfloor \frac{n+1}{4} \right\rfloor \times \left\lfloor \frac{n+2}{4} \right\rfloor \times \left\lfloor \frac{n+3}{4} \right\rfloor$ *for all $n \geq 4$. Moreover, the complete balanced 4-partite 4-graph on n vertices, $\mathcal{K}_4^{(4)}(n)$, is the unique extremal graph.*

Much later Keevash and Mubayi [KM04] showed that in Theorem 4.1.3 one can take $n_0 = 33$. Using the supersaturation techinque of Erdős and Simonovits [Erd81] (it also follows from our Lemma 3.7.3), it can be shown that

$$\text{ex}(n, \mathcal{T}_r) - \text{ex}(n, \Sigma_r) = o(n^r).$$

In [FF89] Frankl and Füredi conjectured that these numbers actually are the same for all sufficiently large n.

Conjecture 4.1.1 (Frankl, Füredi, [FF89])**.** *For every* $r \geq 2$*, there exists* $n_0 := n_0(r)$ *such that for all* $n \geq n_0$

$$\mathrm{ex}(n, \mathcal{T}_r) = \mathrm{ex}(n, \Sigma_r).$$

The conjecture is trivially true for $r = 2$, for $r = 3$ it follows from Theorem 4.1.2. Recently it was proven for $r = 4$ by Pikhurko [Pik08]. He determined the Turán number of $\mathcal{T}_4$ which together with Theorem 4.1.3 verifies Conjecture 4.1.1 for $r = 4$.

Theorem 4.1.4 (Pikhurko, [Pik08])**.** *There exists some* n_0 *such that for all* $n \geq n_0$*,* $\mathrm{ex}(n, \mathcal{T}_4) = \left\lfloor \frac{n}{4} \right\rfloor \times \left\lfloor \frac{n+1}{4} \right\rfloor \times \left\lfloor \frac{n+2}{4} \right\rfloor \times \left\lfloor \frac{n+3}{4} \right\rfloor$*. Moreover, the complete balanced* 4*-partite* 4*-graph on* n *vertices,* $\mathcal{K}_4^{(4)}(n)$*, is the unique extremal graph.*

Here we show that Conjecture 4.1.1 is true for $r = 5$ and $r = 6$. Recall that an (m, r, t)-*Steiner system* is an r-graph on m vertices such that every t-tuple is contained in a unique r-tuple. More generally, an (m, r, t)-*partial Steiner system* is an r-graph on m vertices such that every t-tuple is contained in at most one r-tuple. It is easy to see that every $(m, r, r-1)$-partial Steiner system is Σ_r-hom-free. The opposite is also true, if an r-graph is Σ_r-hom-free then it must be an $(m, r, r-1)$-partial Steiner system for some m. Thus, by Lemma 2.2.3,

$$\pi(\Sigma_r) = r! sup_{\mathcal{G}} \lambda(\mathcal{G}),$$

where the supremum is taken over all $(m, r, r-1)$-partial Steiner systems, for all m. Independently, Frankl and Füredi [FF89] and Sidorenko [Sid87] proved that one can reduce the search of the optimal partial Steiner systems to a finite set.

Theorem 4.1.5 (Frankl and Füredi [FF89] and Sidorenko [Sid87])**.** *For any* $r \geq 3$*,*

$$\pi(\Sigma_r) = r! sup_{\mathcal{G}} \lambda(\mathcal{G}),$$

where the supremum is taken over all $(m, r, r-1)$*-partial Steiner systems with* $m \leq \frac{r^r}{r!}$*.*

Using this result, Frankl and Füredi determined the Turán density of Σ_5 and

Σ_6. Before we state their result, note that it was proved by Cameron and Van Lint [CVL75] that the Steiner systems with parameters $(11,5,4)$ amd $(12,6,5)$ are unique. These are alternatively called *Witt designs* in the literature. We will denote them by $\mathcal{S}_5$ and $\mathcal{S}_6$ respectively.

Theorem 4.1.6 (Frankl, Füredi, [FF89]). *$\pi(\Sigma_5) = \frac{720}{11^4}$ and $\pi(\Sigma_6) = \frac{55}{12^3}$. Moreover, for $r = 5,6$ there exists some $n_0 := n_0(r)$ such that for all $n \geq n_0$ the unique extremal graphs are the balanced blowups of $\mathcal{S}_5$ and $\mathcal{S}_6$ if $11|n$ and $12|n$, for $r = 5,6$, respectively.*

In the same paper Frankl and Füredi also gave the following asymptotic result for general $r \geq 7$.

Theorem 4.1.7 (Frankl, Füredi, [FF89]). *For all $r \geq 7$,*

$$\left(1 + O\left(\frac{1}{n}\right)\right) \frac{n^r}{r!\binom{r}{2}e^{1+\frac{1}{r-1}}} < \mathrm{ex}(n, \Sigma_r) < \frac{n^r}{r!e\binom{r-1}{2}}.$$

We prove Conjecture 4.1.1 for $r = 5$ and $r = 6$ using Theorem 4.1.6. In fact, we prove much more general result from which Theorem 4.1.10 follows using another result by Frankl and Füredi on the properties of $\mathcal{S}_5$ and $\mathcal{S}_6$. Recall the definitions of an r-graph being uniquely dense and balanced from Section 3.9. We say that an r-graph $\mathcal{H}$ is *uniquely dense* if $\lambda(\mathcal{H})$ is uniquely achieved on some $\mu \in \mathcal{M}(\mathcal{H})$ and $\mathcal{H}$ is *balanced* if such measure μ is in fact the uniform measure on $V(\mathcal{H})$, that is,$\xi_{\mathcal{H}}$. For family of r-graphs $\mathfrak{F}$ if there exists only one graph $\mathcal{F} \in \mathfrak{F}$ such that $\lambda(\mathfrak{F}) = \lambda(\mathcal{F}) = \lambda(\mathcal{F}, \mu)$ for some $\mu \in \mathfrak{F}$, then we call such a graph $\mathcal{F}$ the *unique Lagrangian maximizer* of the family $\mathfrak{F}$. Now we are ready to state the aforementioned results of ours and Frankl and Füredi. Let $\mathfrak{F}_r^*$ be the subfamily of all r-graphs in $\mathrm{Forb}(\Sigma_r)$ that cover pairs.

Theorem 4.1.8. *Let $m \geq r \geq 3$ and $\mathcal{S}$ be an $(m,r,r-1)$-Steiner system that is uniquely dense and balanced. If $\mathcal{S}$ is the unique Lagrangian maximizer of $\mathfrak{F}_r^*$, then* $\mathrm{Forb}(\mathcal{T}_r)$ *is $\mathfrak{B}(\mathcal{S})$-stable.*

Theorem 4.1.9 (P. Frankl, Z. Füredi, [FF89]). *$\mathfrak{F}_5^*$ and $\mathfrak{F}_6^*$ have unique Lagrangian maximizers; these are $\mathcal{S}_5$ and $\mathcal{S}_6$, respectively. Furthermore, $\mathcal{S}_5$ and $\mathcal{S}_6$ are both balanced and uniquely dense.*

Our main theorem of this section follows.

Theorem 4.1.10. *There exists n_0 such that for all $n \geq n_0$, $\mathrm{ex}(n, \mathcal{T}_r) = \mathrm{ex}(n, \Sigma_r)$ for $r = 5, 6$. Moreover, there exists some $n_0 := n_0(r)$ such that for all $n \geq n_0$ the extremal graphs are the balanced blowups of $\mathcal{S}_5$ and $\mathcal{S}_6$, for $r = 5$ and $r = 6$, respectively.*

Proof. The result follows directly from Theorem 4.1.8 and Theorem 4.1.9. Indeed, together these two results imply that $\mathrm{Forb}(\mathcal{T}_r)$ is $\mathfrak{B}(\mathcal{S}_r)$-stable for $r = 5, 6$. By Remark 3.2.1, this implies that the exact result holds. □

Sketch of the proof of Theorem 4.1.8: To prove the theorem, we use the local stability method and the corresponding tools for extensions developed in Section 3.8. Recall that $\mathcal{T}_r = \mathrm{Ext}(\mathcal{D}_r)$, where $\mathcal{D}_r$ denotes the r-graph with two edges D_1, D_2 such that $|D_1 \cap D_2| = r - 1$. Also note that the family $\mathrm{Forb}(\mathcal{T}_r)$ is not clonable. Thus we follow the framework that Theorem 3.8.2 provides.

We need to prove that given an $(m, r, r-1)$-Steiner system $\mathcal{S}$ which is uniquely dense and balanced and is the unique Lagrangian maximizer of $\mathfrak{F}_r^*$, the following two conditions hold.

(C1) $\mathrm{Forb}(\mathcal{T}_r)$ is $\mathfrak{B}(\mathcal{S})$-vertex locally stable,

(C2) The family $\mathfrak{F}_r^*$ is $\mathfrak{B}(\mathcal{S})$-weakly weight-stable. (Note that here we are using the fact that $\mathfrak{F}_r^*$ is also the subfamily of $\mathrm{Forb}(\mathcal{D}_r)$ of graphs covering pairs.)

In Section 4.2 we prove that (C1) holds and in fact, for this we only need $\mathcal{S}$ to be balanced and uniquely dense (Theorem 4.2.1). The property of $\mathcal{S}$ being the unique Lagrangian maximizer of $\mathfrak{F}_r^*$ is only used for (C2). The latter we prove in Section 4.3 (Theorem 4.3.1, note that here we do not require $\mathcal{S}$ to be balanced). □

4.2. Local Stability of Forb($\mathcal{T}_r$)

The main result of this section, stated below, applies to all balanced and uniquely dense Steiner systems. Recall that we say that a Steiner system $\mathcal{S}$ is *balanced* if $\lambda(\mathcal{S}) = \lambda(\mathcal{S}, \xi_{\mathcal{S}})$, where $\xi_{\mathcal{S}}$ is defined to be the uniform distribution on $V(\mathcal{S})$. We say that $\mathcal{S}$ is *uniquely dense* if $\xi_{\mathcal{S}}$ is the only measure $\mu \in \mathcal{M}(\mathcal{S})$ achieving the equality $\lambda(\mathcal{S}) = \lambda(\mathcal{S}, \mu)$.

Theorem 4.2.1. *If $\mathcal{S}$ is a balanced and uniquely dense $(m, r, r-1)$-Steiner system for some $m \geq r \geq 3$, then Forb($\mathcal{T}_r$) is $\mathfrak{B}(\mathcal{S})$-vertex locally stable.*

For the proof we use the auxillary lemmas developed in Section 3.9. First let us verify that $\mathcal{T}_r$ and $\mathcal{S}$ satisfy the required conditions. Indeed, any $(m, r, r-1)$-Steiner system is clearly vertex transitive and covers pairs. Recall that $\mathcal{T}_r$ has three edges, D_1, D_2, D_3 such that $D_1 = \{1, 2, \ldots, r\}$, $D_2 = \{1, 2, \ldots, r-1, r+1\}$, $D_3 = \{r-1, r+1, \ldots, 2r-1\}$. It is easy to see that $(\mathcal{T}_r, D_3)$ is loose.

Lemma 4.2.2. *For any $m \geq r \geq 3$, every $(m, r, r-1)$-Steiner system is $(\mathcal{T}_r, D_3)$-hom-critical.*

Proof. Let $\mathcal{S}$ be an $(m, r, r-1)$-Steiner system. If there was a homomorphism φ from $\mathcal{T}_r$ to $\mathcal{S}$, then it would have to map the vertices r and $r+1$ to two different points, because of the edge D_3. But then $\varphi(1), \varphi(2), \ldots, \varphi(r-1)$ together with $\varphi(r)$ and $\varphi(r+1)$ would create an $(r-1)$-tuple contained in two different edges, a contradiction. However, one can easily map $\mathcal{T}_r \setminus D_3$ to any edge in $\mathcal{S}$, this creates the desired homomorphism. □

In addition to the mentioned tools developed in Section 3.9, we need the following "embedding lemma" which says that for every $\mathcal{T}_r$-free r-graph $\mathcal{G}$ with sufficiently large minimum degree there exists a blowup $\mathcal{B}_0$ of $\mathcal{S}$ such that every vertex of $\mathcal{G}$ has "similar" neighborhoods in $\mathcal{G}$ and $\mathcal{B}_0$. The proof of this lemma contains the bulk of technical difficulties involved in proving Theorem 4.2.1.

Lemma 4.2.3. *For given $m \geq r \geq 3$, let $\mathcal{S}$ be a uniquely dense, balanced $(m, r, r-1)$-Steiner system. For every $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. If $\mathcal{G}$ is a $\mathcal{T}_r$-free r-graph with $\mathrm{v}(\mathcal{G}) = n \geq n_0$,*

- $d_{\mathfrak{B}(\mathcal{S})}(\mathcal{G}) \leq \delta n^r$,

- $|L_{\mathcal{G}}(v)| \geq \left(\frac{\binom{m-1}{r-2}}{(r-1)m^{r-1}} - \delta\right) n^{r-1}$, *for every* $v \in V(\mathcal{G})$,

then there exists $\mathcal{B}_0 \in \mathfrak{B}$ *with* $V(\mathcal{B}_0) = V(\mathcal{G}) := V$ *such that for every* $v \in V$

$$|L_{\mathcal{G}}(v) \triangle L_{\mathcal{B}_0}(v)| \leq \varepsilon n^{r-1}.$$

Proof: We denote $\mathfrak{B} := \mathfrak{B}(\mathcal{S})$, $\mathrm{d}(m,r) := \frac{\binom{m-1}{r-2}}{(r-1)m^{r-1}}$. It is easy to see that $m(\mathfrak{B}) = \frac{\mathrm{d}(m,r)}{r}$. Let $\varepsilon_{3.9.4}$ be chosen to satisfy Lemma 3.9.4 applied with $\mathcal{H} = \mathcal{S}$ and $\mathcal{F} = \mathcal{T}_r$. We choose constants

$$0 < \delta \ll \varepsilon_{3.9.2} \ll \gamma \ll \min\{\varepsilon_{3.9.4}, \varepsilon\}$$

to satisfy the constraints appearing further in the proof. Let $\delta_{3.9.2}$ be chosen to satisfy Corollary 3.9.2 applied with $\varepsilon = \varepsilon_{3.9.2}$ for $\mathcal{H} = \mathcal{S}$ and $\mathcal{F} = \mathcal{T}_r$. We also impose on δ the condition $\delta \ll \delta_{3.9.2}$.

Let $\mathcal{B} \in \mathfrak{B}$ be such that $|\mathcal{G} \triangle \mathcal{B}| = d_{\mathfrak{B}}(\mathcal{G})$, and let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be the blowup partition of $\mathcal{B}$. First note that

$$|\mathcal{G}| = \frac{1}{r} \sum_{v \in V} |L_{\mathcal{G}}(v)| \geq \left(\frac{\mathrm{d}(m,r)}{r} - \frac{\delta}{r}\right) n^{r-1}.$$

Thus,

$$|\mathcal{B}| \geq |\mathcal{G}| - \delta n^r \geq \left(\frac{\mathrm{d}(m,r)}{r} - \delta_{3.9.2}\right) n^{r-1},$$

therefore, by Corollary 3.9.2, $\mathcal{B}$ is $\varepsilon_{3.9.2}$-trimmed. Now consider the set

$$J := \left\{v \in V \mid |L_{\mathcal{G}}(v) \triangle L_{\mathcal{B}}(v)| > \gamma n^{r-1}\right\},$$

these are the vertices which have potentially very different neihbourhoods in $\mathcal{B}$ and in $\mathcal{G}$. Since the distance between $\mathcal{G}$ and $\mathcal{B}$ is small, it is natural to expect that the set J is also small. Indeed,

$$|J|\gamma n^{r-1} < \sum_{v\in V} |L_{\mathcal{G}}(v)\triangle L_{\mathcal{B}}(v)| = r|\mathcal{G}\triangle\mathcal{B}| \leq \delta r n^r.$$

Let $\delta_1 := \delta r/\gamma$, then $|J| \leq \delta_1 n$, by the above. Let $\mathcal{G}' := \mathcal{G}|_{V\setminus J}$, $n' = \mathrm{v}(\mathcal{G}')$, $\mathcal{B}' := \mathcal{B}|_{V\setminus J}$, $P'_j := P_j \setminus J$ for each $j \in [m]$, and $\mathcal{P}' = \{P'_1, P'_2, \ldots, P'_m\}$. The graph $\mathcal{G}'$ satisfies the assumptions of Lemma 3.9.4. Indeed, for every $v \in V \setminus J$,

$$|L_{\mathcal{G}'}(v)\triangle L_{\mathcal{B}'}(v)| \leq \gamma n^{r-1} \leq \varepsilon_{3.9.4}(1-\delta_1)^{r-1}n^{r-1} \leq \varepsilon_{3.9.4}(n')^{r-1}.$$

Similarly, $|\mathcal{G}|' \geq \left(\frac{\mathrm{d}(m,r)}{r} - \varepsilon_{3.9.4}\right)(n')^{r-1}$. Thus both $\mathcal{G}'$ and $\mathcal{G}$ are $\mathcal{P}'$-transversal by Lemma 3.9.4. Our next goal is to extend $\mathcal{B}'$ to a blowup $\mathcal{B}_0$ of $\mathcal{S}$ with $V(\mathcal{B}_0) = V$, as follows. For each $u \in J$ we will find a unique index $j_u \in [m]$, such that u "behaves" as the vertices in the partition class P'_{j_u}, and add the vertex u to this partition class. By doing so for all vertices of J, we will extend the partition $\mathcal{P}'$, and since J has relatively small size, this operation will not increase the degrees of vertices in $\mathcal{G}'$ drastically. So let us fix some $u \in J$ and show that such an index j_u exists.

For $I \subseteq [m]$, $|I| = r-1$, let

$$E_I(u) := \{G \in \mathcal{G}\,|u \in G,\ |G \cap P'_i| = 1 \text{ for every } i \in I\}.$$

We construct an auxiliary $(r-1)$-graph $\mathcal{L}(u)$ with $V(\mathcal{L}(u)) = [m]$ such that $I \in \mathcal{L}(u)$ if and only if $|E_I(u)| \geq \gamma n^{r-1}$. We aim to show that there exists a unique $j_u \in [m]$ such that $\mathcal{L}(u)$ is isomorphic to the link graph of j_u in $\mathcal{S}$. Note that because of vertex transitivity of $\mathcal{S}$, for all $j \in [m]$, the $(r-1)$-graphs $L_{\mathcal{S}}(j)$ are all isomorphic.

Claim 4.2.4. $|\mathcal{L}(u)| \geq \mathrm{d}(m,r)m^{r-1}$.

Proof. Denote by $E_J(u)$ the set of all the edges in $\mathcal{G}$ that contain u and at least one other vertex from J. Clearly, $|E_J(u)| \leq |J|n^{r-2} \leq \delta_1 n^{r-1}$. Therefore,

$$\begin{aligned}(\mathrm{d}(m,r)-\delta)n^{r-1} \leq |L_{\mathcal{G}}(u)| &\leq |E_J(u)| + \sum_{I\in\mathcal{L}(u)} |E_I(u)| + \sum_{I\notin\mathcal{L}(u)} |E_I(u)| \\ &\leq \delta_1 n^{r-1} + |\mathcal{L}(u)| \left(\frac{1}{m} + \varepsilon_{3.9.2}\right)^{r-1} n^{r-1} + \gamma\binom{m}{r-1} n^{r-1}.\end{aligned}$$

It follows that

$$\begin{aligned}|\mathcal{L}(u)| &\geq \frac{\mathrm{d}(m,r)m^{r-1}}{(1+\varepsilon_{3.9.2}m)^{r-1}} - \frac{(\delta+\delta_1+\gamma/(r-1)!)m^{r-1}}{(1+\varepsilon_{3.9.2}m)^{r-1}} \\ &> \mathrm{d}(m,r)m^{r-1} - 1,\end{aligned}$$

where the last inequality holds, as long as $\varepsilon_{3.9.2}, \delta, \delta_1$ and γ are sufficiently small compared to $1/m^r$. It follows that $|\mathcal{L}(u)| \geq \mathrm{d}(m,r)m^{r-1}$. □

For $j \in [m]$, let

$$L_j(u) = \left\{ v \in P'_j \middle| |L_{\mathcal{G}}(u,v) \geq \frac{\gamma}{2} n^{r-2} \right\},$$

$$K = \left\{ j \middle| |L_j(u)| < \frac{\gamma}{2} n \right\}.$$

Claim 4.2.5. *If $j \in I \in \mathcal{L}(u)$, then $j \notin K$.*

Proof. Suppose the opposite holds, that is, there exists $I \in \mathcal{L}(u)$ and $j \in I$ such that $j \in K$. Then

$$|E_I(u)| \leq |L_j(u)|n^{r-2} + |P'_j \setminus L_j(u)|\beta n^{r-2} < \frac{\gamma}{2} n^{r-1} + \frac{\gamma}{2} n^{r-1} = \gamma n^{r-1},$$

a contradiction. □

By the claim above if we prove that $|K| = 1$, then $j_u \in K$ will be the index of the partition class we were looking for, that is, $\mathcal{L}(u)$ is isomorphic to $L_{\mathcal{S}}(j_u)$ and for every $(r-1)$-tuple $I \subseteq [m]$ with $j_u \in I$, $|E_I(u)| < \gamma n^{r-1}$. More importantly, j_u is unique satisfying all this properties. So let us prove that K is singleton. We do so in two steps, first we show that it is not empty, then we show that if it had to contain more than one index, then we would get a copy of $\mathcal{T}_r$ in $\mathcal{G}$.

Claim 4.2.6. $|K| \neq \emptyset$.

Proof. Fix $I \in \mathcal{L}(u)$. As $\mathcal{S}$ is a Steiner system, there exists unique j such that $I \cup \{j\} \in \mathcal{S}$. We claim that $j \in K$. Assume not, and further assume, without loss of generality, that $I = \{1, 2, \ldots, r-1\}$. Then there exists $\{v_1, v_2, \ldots, v_{r-1}\} \in E_I(u)$ and $v_r \in L_j(u)$, such that $\{v_1, v_2, \ldots, v_{r-1}, v_r\} \in \mathcal{G}$. Otherwise, for every $G \in E_I(u)$ and every $v \in L_j(u)$, $(G \setminus \{u\}) \cup \{v\}$ is a missing edge and that would imply

$$|\mathcal{G} \triangle \mathcal{B}| \geq |E_I(u)||L_j(u)| \geq \gamma n^{r-1} \cdot \gamma n > \delta n^r,$$

a contradiction.

Let $v_1, v_2, \ldots, v_{r-1}, v_r$ be as above. Since $\mathcal{G}$ is $\mathcal{T}_r$-free, every edge in $\mathcal{F}$ that contains both u and v_r, must also contain a vertex among $\{v_1, v_2, \ldots, v_{r-1}\}$. Therefore, we must have $|L(u, v_r)| \leq (r-1)n^{r-3}$, while, by definition of $L_j(u)$, $|L(u, v_r)| \geq \gamma n^{r-2}$, yielding a contradiction when n is large enough. Thus $K \neq \emptyset$. □

Claim 4.2.7. $|K| = 1$.

Proof. Let $k := |K|$, we have already shown that $k \geq 1$. Suppose for a contradiction that $k \geq 2$. Let A be a $\mathcal{P}'$-transversal $(r-2)$-tuple. We want to show that

$$|L(A \cup \{u\})| \leq \left(\frac{1}{m} + \varepsilon_{3.9.2} + \gamma(m-1)\right) n. \tag{4.1}$$

Suppose that there exist $j_1 \neq j_2$ such that $|L(A \cup \{u\}) \cap P_{j_1}| \geq \gamma n$ and $|L(A \cup \{u\}) \cap P_{j_2}| \geq \gamma n$. Since $\mathcal{G}$ is $\mathcal{T}_r$-free, for every $v_1 \in L(A \cup \{u\}) \cap P_{j_1}$ and $v_2 \in L(A \cup \{u\}) \cap P_{j_2}$, we must have

$$|L(v_1, v_2)| \leq (r-1)n^{r-3}.$$

It follows that

$$\begin{aligned}|\mathcal{F} \triangle \mathcal{B}| \geq& \gamma^2 n^2 \left(\left(\frac{1}{m} - \varepsilon_{3.9.2}\right)^{r-2} n^{r-2} - (r-1)\binom{n}{r-3}\right)\\ \geq& \frac{1}{2}\gamma^2 \left(\frac{1}{m} - \varepsilon_{3.9.2}\right)^{r-2} n^r > \delta n^r,\end{aligned}$$

which is a contradiction. Thus, no such j_1 and j_2 exist, and (4.1) follows.

Using (4.1), we obtain an upper bound on $E_I(u)$, for every $(r-2)$-tuple $I \subseteq [m]$. Without loss of generality, suppose $I = \{1, 2, \ldots, r-2\}$. We apply (4.1) to every $A \in [n]^{r-2}$ which is I-transversal in $\mathcal{G}$. As

$$\prod_{j=1}^{r-2} |P_j| \leq \left(\frac{n}{m} + \varepsilon_{3.9.2} n\right)^{r-2},$$

we derive

$$|E_I(u)| \leq \left(\frac{1}{m} + \varepsilon_{3.9.2}\right)^{r-2} \left(\frac{1}{m} + \varepsilon_{3.9.2} + \gamma(m-1)\right) n^{r-1}. \tag{4.2}$$

And finally, we are ready to derive an upper bound on the size of $L_{\mathcal{G}}(u)$, which will contradict the initial assumption $|L_{\mathcal{G}}(u)| \geq (\mathrm{d}(m,r) - \delta)n^{r-1}$:

$$\begin{aligned}
|L_{\mathcal{G}}(u)| &\leq |E_J(u)| + \sum_{\substack{I \subseteq [m], |I| = r-1 \\ I \cap K = \emptyset}} |E_I(u)| + \sum_{\substack{I \subseteq [m], |I| = r-1 \\ I \cap K \neq \emptyset}} |E_I(u)| \\
&\leq |J| n^{r-2} + \frac{1}{r-1} \sum_{\substack{I \subseteq [m], |I| = r-2 \\ I \cap K = \emptyset}} |E_I(u)| \\
&+ \sum_{j \in K} \left(|L_j(u)| n^{r-2} + (n - |L_j(u)|)\gamma n^{r-2}\right) \\
&\overset{(4.2)}{\leq} \frac{1}{r-1} \binom{m-s}{r-2} \left(\frac{1}{m} + \varepsilon_{3.9.2}\right)^{r-2} \left(\frac{1}{m} + \varepsilon_{3.9.2} + \gamma(m-1)\right) n^{r-1} \\
&+ \delta_1 n^{r-1} + 2\gamma m n^{r-1} \\
&\leq \left(\left(\frac{\binom{m-2}{r-1}}{m^{r-1}} + \gamma\right) + 2\gamma m + \delta_1\right) n^{r-1} \\
&< (\mathrm{d}(m,r) - \delta) n^{r-1},
\end{aligned}$$

a contradiction. Thus, $k = 1$. $\square$

As discussed earlier, Claim 4.2.7 implies that for every $u \in J$ there exists unique j_u such that $\mathcal{L}(u) \simeq L_{\mathcal{S}}(j_u)$. We extend the blowup $\mathcal{B}'$ as we discussed earlier. For every $j \in [m]$, define

$$P_j^0 := P_j' \cup \{u \in J \mid j_u = j\}.$$

Let $\mathcal{B}_0 \supseteq \mathcal{B}'$ be the blowup of $\mathcal{S}$ with the blowup partition $\mathcal{P}_0$.

Finally we are ready to show that for every $v \in V$, $|L_{\mathcal{B}_0}(v) \triangle L_{\mathcal{G}}(v)| \leq \varepsilon n^{r-1}$. Indeed, if $v \in V \setminus J$, then

$$\begin{aligned}|L_{\mathcal{B}_0}(v) \triangle L_{\mathcal{G}}(v)| &\leq |L_{\mathcal{B}'}(v) \triangle L_{\mathcal{G}'}(v)| + |J|n^{r-2} \\ &\leq \gamma n^{r-1} + \delta_1 n^{r-1} \leq \varepsilon n^{r-1}.\end{aligned}$$

We now consider $v \in J$. Since $\mathcal{G}$ is $\mathcal{P}'$-transversal and by the choice of j_v, it follows that for every $G \in L_{\mathcal{G} \setminus \mathcal{B}_0}(v)$, either $G \cap J \neq \emptyset$, or there exists $I \notin \mathcal{L}(v)$ such that $G \in L_I(v)$ (recall that for all such I, $j_v \in I$). Thus,

$$|L_{\mathcal{G} \setminus \mathcal{B}_0}(v)| \leq \delta_1 n^{r-1} + \left(\binom{m}{r-1} - |\mathcal{L}(v)| \right) \gamma n^{r-1} < \frac{\varepsilon}{4} n^{r-1}. \tag{4.3}$$

Finally,

$$\begin{aligned}&|L_{\mathcal{G}}(v) \triangle L_{\mathcal{B}_0}(v)| = 2|L_{\mathcal{G} \setminus \mathcal{B}_0}(v)| + |L_{\mathcal{B}_0}(v)| - |L_{\mathcal{G}}(v)| \\ &\overset{(5.1)}{\leq} \frac{\varepsilon}{2} n^{r-1} + \mathrm{d}(m,r) \left(\frac{1}{m} + \varepsilon_{3.9.2} + \delta_1 \right)^{r-1} n^{r-1} - (\mathrm{d}(m,r) - \delta) n^{r-1} \\ &\leq \varepsilon n^{r-1}.\end{aligned}$$

This concudes the proof. □

Proof of Theorem 4.2.1. We denote $\mathfrak{B}(\mathcal{S})$ simply by $\mathfrak{B}$ and $\mathrm{d}(m,r) := \frac{\binom{m-1}{r-2}}{(r-1)m^{r-1}}$. Our goal is to show that there exist $\varepsilon, \alpha, n_0 > 0$ such that the following holds. If $\mathcal{G} \in \mathrm{Forb}(\mathcal{T}_r)$ with $\mathrm{v}(\mathcal{G}) = [n]$, $n \geq n_0$ such that $d_{\mathfrak{B}}(\mathcal{G}) \leq \varepsilon n^r$, and $|L_{\mathcal{G}}(v)| \geq (\mathrm{d}(m,r) - \varepsilon) n^{r-1}$ for every $v \in V(\mathcal{G})$, then

$$|\mathcal{G}| \leq m(\mathfrak{B}, n) - \alpha d_{\mathfrak{B}}(\mathcal{G}). \tag{4.4}$$

In fact, we show that one can take $\alpha = \frac{1}{2}$. Now we specify dependencies between constants used further in the proof. Let $\varepsilon_{3.9.4}$ be taken to satisfy Lemma 3.9.4. Define $\varepsilon_{3.9.2} := \frac{1}{4m}$. Let $\delta_{3.9.2}$ be taken to satisfy Lemma 3.9.2 applied with $\varepsilon = \varepsilon_{3.9.2}$ and $\mathcal{H} = \mathcal{S}$. We choose $0 \ll \varepsilon \ll \varepsilon_{4.2.3} \ll \min\{\delta_{3.9.2}, \varepsilon_{3.9.4}\}$ to satisfy the inequalities appearing in the proof. Furthermore, we will use $\varepsilon < \delta_{4.2.3}/2$, where $\delta_{4.2.3}$ is chosen to satisfy Theorem 4.2.3 applied with $\varepsilon_{4.2.3}$, $\mathcal{H}$ and $\mathcal{T}_r$.

We can assume that

$$|\mathcal{G}| \geq \left(\frac{\mathrm{d}(m,r)}{r} - 2\varepsilon\right) n^r \geq \left(\frac{\mathrm{d}(m,r)}{r} - \delta_{4.2.3}\right) n^r,$$

since otherwise the result follows directly with $\alpha = 1$. By Theorem 4.2.3 there exists $\mathcal{B} \in \mathfrak{B}$ with $V(\mathcal{B}) = V(\mathcal{G}) := V$ such that

$$|L_{\mathcal{F}}(v) \triangle L_{\mathcal{B}}(v)| \leq \varepsilon_{4.2.3} n^{r-1}$$

for every $v \in V$.

We call the edges in $\mathcal{G} \setminus \mathcal{B}$ *bad*, the edges in $\mathcal{B} \setminus \mathcal{G}$ *missing* and, finally, the edges in $\mathcal{G} \cap \mathcal{B}$ *good*. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be the blowup partition of $\mathcal{B}$. By our assumptions,

$$|\mathcal{G}| \geq \left(\frac{\mathrm{d}(m,r)}{r} - \varepsilon_{3.9.4}\right) n^r$$

and $|L_{\mathcal{G}}(v) \triangle L_{\mathcal{B}}(v)| \leq \varepsilon_{4.2.3} n^r \leq \varepsilon_{3.9.4} n^r$ for every $v \in V$. Thus by Lemma 3.9.4 all bad edges in $\mathcal{G}$ are $\mathcal{P}$-transversal.

Also note that since

$$|\mathcal{B}| \geq |\mathcal{G}| - |\mathcal{G} \triangle \mathcal{B}| \geq \left(\frac{\mathrm{d}(m,r)}{r} - 2\varepsilon - \frac{\varepsilon_{4.2.3}}{r}\right) n^r \geq \left(\frac{\mathrm{d}(m,r)}{r} - \delta_{3.9.2}\right) n^r,$$

hence $\mathcal{B}$ is $\varepsilon_{3.9.2}$-balanced.

Generalizing the notions of bad, good and missing edges, we introduce the following notation. For every $I \subset V$ with $0 \leq |I| \leq r$, we denote

$$A(I) := \{G \in \mathcal{B} \setminus \mathcal{G} | I \subseteq G\},$$

$$B(I) := \{G \in \mathcal{G} \setminus \mathcal{B} | I \subseteq G\},$$

$a(I) := |A(I)|$ and $b(I) := |B(I)|$. So $a(I)$ and $b(I)$ respectively denote the number of missing and bad edges that the tuple I is in. We have $\mathcal{G} \triangle \mathcal{B} = A(\emptyset) \cup B(\emptyset)$ and

$|\mathcal{G}\triangle\mathcal{B}| = a(\emptyset) + b(\emptyset)$. It is easy to see that for every I, such that $0 \le |I| \le r-1$, the following inequalities hold

$$\sum_{j\notin I} a(I\cup\{j\}) \ge a(I) \ge \frac{1}{r}\sum_{j\notin I} a(I\cup\{j\}), \tag{4.5}$$

$$\sum_{j\notin I} b(I\cup\{j\}) \ge b(I) \ge \frac{1}{r}\sum_{j\notin I} b(I\cup\{j\}). \tag{4.6}$$

It is not hard to see that to derive the inequality (4.4) it suffices to show that $a(\emptyset) \ge 3b(\emptyset)$. Let us assume for a contradiction that $b(\emptyset) > \frac{1}{3}a(\emptyset)$. Our next claim shows that we can bound the number of bad edges that contain some i-tuple from above by the proportion of the missing edges that contain any of its $(i-1)$-subtuples.

Claim 4.2.8. *There exists $c > 0$ such that for every $I \subseteq V(\mathcal{F}), 1 \le |I| \le r$, and every $I' \subset I$ with $|I'| = |I|-1$, we have $a(I') \ge cb(I)n$.*

Proof. We proceed by induction on $r - |I|$. We prove that for each $1 \le i \le r$, and every $I \subseteq [n]$ with $|I| = i$ there exists $c_i > 0$ such that for all $I' \subset I$ and $|I'| = i-1$, we have $a(I') \ge c_i b(I)n$. This clearly implies the claim.

We start the base case: $|I| = r$ and we assume that I is a bad edge, as otherwise the statement is trivial. Without loss of generality, assume $I = \{v_1, v_2, \ldots, v_r\}$, where $v_j \in P_j$, and $I' = \{v_1, v_2, \ldots, v_{r-1}\}$. Since I is a bad edge, it means that $\{1, 2, \ldots, r\} \notin \mathcal{S}$ which implies that $\{1, 2, \ldots, r-1, k\} \in \mathcal{S}$ for some $k \ne r$. Without loss of generality, we assume $k = r+1$.

Let $N := L(I') \cap P_{r+1}$. For every $u \in N$, we have

$$a(u, v_r) \ge (\min_i |P_i|)^{r-2} - |L(u, v_r)|.$$

However, every edge that covers u and v_r, must have a non-empty intersection with $\{v_1, v_2, \ldots, v_{r-1}\}$, as $\mathcal{F}$ is $\mathcal{T}_r$-free, therefore

$$|L(u, v_r)| \le (r-1)n^{r-3}.$$

On the other hand, since$\mathcal{B}$ is $\varepsilon_{3.9.2}$-balanced, we have

$$a(v_r) \geq |N| \left(\left(\frac{n}{m} - \varepsilon_{3.9.2} n \right)^{r-2} - (r-1)n^{r-3} \right).$$

But $a(\{v_r\}) \leq \varepsilon_{4.2.3} n^{r-1}$ and we have

$$|N| \leq \frac{2\varepsilon_{4.2.3}}{\left(\frac{1}{m} - \varepsilon_{3.9.2}\right)^{r-2}} n = 2\varepsilon_{4.2.3} \left(\frac{4m}{3}\right)^{r-2} n \leq \frac{n}{2m},$$

for sufficiently large n. The latter directly implies that $a(I') \geq |P_{r+1} \setminus N| \geq \frac{n}{4m}$, thus concluding the proof of the base case with $c_r = \frac{1}{4m}$.

We now turn to the induction step. For every $I' \subset I$ with $|I'| = |I| - 1$ we have

$$ra(I') \overset{(4.5)}{\geq} \sum_{\substack{I' \subset J, \\ |J|=i}} a(J) \qquad \geq \sum_{\substack{I' \subset J, J \neq I \\ |J|=i}} c_{i+1} b(J \cup I) n \overset{(4.6)}{\geq} c_{i+1} b(I) n,$$

where the second inequality follows from th induction hypothesis. Thus $a(I') \geq c_i b(I) n$, where $c_i := \frac{c_{i+1}}{r} > 0$, as desired. □

Let c be as in Claim 4.2.8. Then $a(\emptyset) \geq cb(v)n$ for every $v \in V(\mathcal{F})$. Direct averaging shows that for every $I \subseteq V(\mathcal{F})$ with $0 \leq |I| \leq r-1$ and every $c' > 0$ such that $b(I) > c'a(I)$, there exists $v \notin I$ such that $b(I \cup \{v\}) > c'a(I \cup \{v\})$. Therefore, since $b(\emptyset) > \frac{1}{3} a(\emptyset)$, there exists $v_1 \in V(\mathcal{F})$ such that $b(v_1) > \frac{1}{3} a(v_1)$. Similarly, $a(v_1) \geq cb(v_1, v)$ for every $v \in V(\mathcal{F}) \setminus \{v_1\}$, and there exists $v_2 \in V(\mathcal{F}) \setminus \{v_1\}$, such that $b(v_1, v_2) > \frac{1}{3} a(v_1, v_2)$. Applying this argument iteratively, we get the following series of inequalities:

$$\begin{aligned} a(\emptyset) &\geq cb(v_1)n > \frac{c}{3} a(v_1) n \geq \frac{c^2}{3} b(v_1, v_2) n^2 > \frac{c^2}{9} a(v_1, v_2) n^2 \geq \ldots \\ &> \frac{c^{r-1}}{3^{r-1}} a(v_1, v_2, \ldots, v_{r-1}) n^{r-1} \geq \frac{c^r}{3^{r-1}} b(v_1, v_2, \ldots, v_r) n^r \\ &> \frac{c^r}{3^r} a(v_1, v_2, \ldots, v_r) n^r. \end{aligned}$$

In particular, $b(v_1, v_2, \ldots, v_r) > 0$, i.e. $b(v_1, v_2, \ldots, i_r) = 1$. Thus,

$$a(\emptyset) > \frac{c^r}{3^{r-1}} n^r \geq \frac{\varepsilon_{4.2.3}}{r} n^r \geq |\mathcal{F} \triangle \mathcal{B}|,$$

a contradiction. $\square$

4.3. Weak Stability From Lagrangians

In this section we prove that if $\mathcal{S}$ is a uniquely dense $(m, r, r-1)$-Steiner system and is the unique Lagrangian maximizer of the family $\mathfrak{F}_r^*$, the subfamily of $\mathrm{Forb}(\mathcal{D}_r)$ that cover pairs (or equivalently, the subfamily of $\mathrm{Forb}(\Sigma_r)$ covering pairs), then (weak) stability in weighted setting holds. That is, if $\mathcal{F} \in \mathfrak{F}_r^*$ and $\lambda(\mathcal{F}, \mu)$ is "close" to $\lambda(\mathfrak{F}) = \lambda(\mathcal{S}, \mu_{\mathcal{S}}^*)$, for some $\mu \in \mathcal{M}(\mathcal{F})$, then $(\mathcal{F}, \mu)$ is "close" to $\mathfrak{B}(\mathcal{S})$ in weighted setting.

Theorem 4.3.1. *Let $m \geq r \geq 3$, $\mathcal{S}$ be an $(m, r, r-1)$-Steiner system that is uniquely dense. If $\mathcal{S}$ is the unique Lagrangian maximizer of $\mathfrak{F}_r^*$, then $\mathfrak{F}_r^*$ is $\mathfrak{B}(\mathcal{S})$-weakly weight-stable.*

Theorem 4.3.1 together with Theorem 4.1.9, implies the weak weight-stability of the corresponding subfamilies of $\mathrm{Forb}(\mathcal{D}_5)$ and $\mathrm{Forb}(\mathcal{D}_6)$.

Corollary 4.3.2. *$\mathfrak{F}_5^*$, $\mathfrak{F}_6^*$ are respectively $\mathfrak{B}(\mathcal{S}_5)$ and $\mathfrak{B}(\mathcal{S}_6)$-weakly weight-stable.*

So it remains to prove Theorem 4.3.1. The latter is obtained as a corollary from the following more general statement. For simplicity we call the r-graphs not containing $\mathcal{D}_r$ *thin*. A family of r-graphs is called *thin* if every member is thin, or equivalently, if $\mathfrak{F}$ is a subfamily of $\mathrm{Forb}(\mathcal{D}_r)$.

Theorem 4.3.3. *If $\mathfrak{F}^*$ is a thin family such that $\lambda(\mathfrak{F}^*) = \lambda(\mathcal{F}^*)$ for some $\mathcal{F}^* \in \mathfrak{F}$, then it is $\mathfrak{F}^{**}$-weakly weight-stable, where*

$$\mathfrak{F}^{**} = \{\mathcal{F}^*|_{\mathrm{supp}(\mu)} \mid \mathcal{F}^* \in \mathfrak{F}^*,\ \lambda(\mathcal{F}^*, \mu) = \lambda(\mathfrak{F}^*)\ for\ some\ \mu \in \mathcal{M}(\mathcal{F}^*)\}.$$

The proof of this result mainly relies on the continuity of the Lagrangian function and the property of the family $\mathrm{Forb}(\mathcal{D}_r)$, not having an $(r-1)$-tuple contained in more than one edge.

Proof. We will consider infinite r-graphs in the proof of this theorem. Let $\mathfrak{F}_{\mathbb{N}}$ denote the family of r-graphs such that $V(\mathcal{F}) = \mathbb{N}$ for every $\mathcal{F} \in \mathfrak{F}_{\mathbb{N}}$ and every finite subgraph $\mathcal{H}$ of a graph in $\mathfrak{F}_{\mathbb{N}}$ is obtained from a subgraph of a graph in $\mathfrak{F}^*$ by adding isolated vertices. Clearly, $\mathfrak{F}_{\mathbb{N}}$ is thin. We enhance $\mathfrak{F}_{\mathbb{N}}$ with a metric ς defined as follows. For $\mathcal{F}, \mathcal{F}' \in \mathfrak{F}_{\mathbb{N}}$, let $\varsigma(\mathcal{F}, \mathcal{F}') := 1/2^k$, where k is the minimum integer such that $\mathcal{F}|_{[k]} \neq \mathcal{F}'|_{[k]}$. Note that $(\mathfrak{F}_{\mathbb{N}}, \varsigma)$ is compact.

Let

$$\mathcal{M}(\mathbb{N}) := \{\mu : \mathbb{N} \to \mathbb{R}_+ \mid \mu(1) \geq \mu(2) \geq \mu(3) \geq \dots, \sum_{i=1}^{\infty} \mu(i) \leq 1\}.$$

It is not hard to verify that $\mathcal{M}(\mathbb{N})$ is compact with L^1 norm $\|\cdot\|_1$. Let $\mathfrak{X}$ be the product of $(\mathfrak{F}_{\mathbb{N}}, \varsigma)$ and $(\mathcal{M}(\mathbb{N}), \|\cdot\|_1)$.

Note that every pair $(\mathcal{F}, \mu)$ with $\mathcal{F} \in \mathfrak{F}^*, \mu \in \mathcal{M}(\mathcal{F})$ naturally corresponds to an element of $\mathfrak{X}$, as we can assume that $V(\mathcal{F}) = [\mathrm{v}(\mathcal{F})]$ and $\mu(i) \geq \mu(j)$ for all $i \leq j$, $i, j \in V(\mathcal{F})$. For $(\mathcal{F}, \mu) \in \mathfrak{X}$ the density $\lambda(\mathcal{F}, \mu) := \sum_{F \in \mathcal{F}} \prod_{v \in F} \mu(v)$ is defined as before.

Claim 4.3.4. *λ is continuous on $\mathfrak{X}$.*

Proof. It is easy to see that

$$|\lambda(\mathcal{F}, \mu) - \lambda(\mathcal{F}, \mu')| \leq \|\mu - \mu'\|_1$$

for every $\mathcal{F} \in \mathfrak{F}_{\mathbb{N}}$ and all $\mu, \mu' \in \mathcal{M}(\mathbb{N})$. Thus, it suffices to show that for all $\mathcal{F}, \mathcal{F}' \in \mathfrak{F}_{\mathbb{N}}$ and every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $\mathcal{F}'|_{[N]} = \mathcal{F}|_{[N]}$ then $|\lambda(\mathcal{F}, \mu) - \lambda(\mathcal{F}', \mu)| \leq \varepsilon$ for every $\mu \in \mathcal{M}(\mathbb{N})$. We show that $N := \lceil \frac{1}{\varepsilon(r-1)!} \rceil$ satisfies the above. Let $\mathcal{H} := \mathcal{F}'|_{[N]} = \mathcal{F}|_{[N]}$. It suffices to show that $\lambda(\mathcal{F}, \mu) \leq \lambda(\mathcal{H}, \mu) + \varepsilon$. We have

$$\begin{aligned}
\lambda(\mathcal{F}, \mu) - \lambda(\mathcal{H}, \mu) &= \sum_{F \in \mathcal{F}, F \not\subseteq [N]} \prod_{i \in F} \mu(i) \\
&\leq \mu(N+1) \sum_{I \subseteq \mathbb{N}^{(r-1)}} \prod_{i \in I} \mu(i) \\
&\leq \mu(N+1) \frac{1}{(r-1)!} \left(\sum_{i \in N} \mu(i) \right)^{r-1} \leq \frac{1}{N(r-1)!} \leq \varepsilon,
\end{aligned}$$

as desired. Note that in the second inequality above we use the fact that $\mathcal{F}$ is thin. $\square$

It follows from the above claim that

$$\lambda(\mathfrak{F}^*) = \max_{(\mathcal{F},\mu)\in\mathfrak{X}} \lambda(\mathcal{F},\mu), \tag{4.7}$$

as every $(\mathcal{F},\mu) \in \mathfrak{X}$ is a limit of a sequence of weighted graphs in $\mathfrak{F}^*$. Let

$$\mathfrak{X}^{**} = \{(\mathcal{F},\mu) \in \mathfrak{X} \mid \mathcal{F}|_{\mathrm{supp}(\mu)} \in \mathfrak{F}^{**}\},$$

That is, $\mathfrak{X}^{**}$ is a set of weighted graphs in $\mathfrak{X}$ with finite support, coinciding with some graph in $\mathfrak{F}^{**}$ on its support.

Claim 4.3.5. *If $\lambda(\mathcal{F},\mu) = \lambda(\mathfrak{F}^*)$ for some $(\mathcal{F},\mu) \in \mathfrak{X}$, then $(\mathcal{F},\mu) \in \mathfrak{X}^{**}$.*

Proof. Suppose for a contradiction that there exists some $(\mathcal{F},\mu) \in \mathfrak{X} \setminus \mathfrak{X}^{**}$ such that $\lambda(\mathcal{F},\mu) = \lambda(\mathfrak{F}^*)$. By definition of $\mathfrak{F}^{**}$, it follows that $\mathrm{supp}(\mu)$ must be infinite, and hence, $\mathrm{supp}(\mu) = \mathbb{N}$, since μ is non-decreasing. As $\lambda(\mathcal{F},\nu)$ considered as a function of ν is maximized at $\nu = \mu$ we have

$$\left.\frac{\partial\lambda(\mathcal{F},\nu)}{\partial\nu(i)}\right|_{\nu=\mu} = r\lambda(\mathfrak{F}^*),$$

for every $i \in \mathbb{N}$. Thus, we have

$$\sum_{\substack{J\in\mathbb{N}^{(r-1)},|J|=r-1\\ J\cup\{i\}\in\mathcal{F}}} \prod_{j\in J} \mu(j) = r\lambda(\mathfrak{F}^*) \tag{4.8}$$

for every $i \in \mathbb{N}$. To show that (4.8) cannot hold we employ an argument similar to the one used in the proof of the previous claim. Choose an integer N such that $N > \frac{1}{r(r-2)!\lambda(\mathfrak{F}^*)}$, and let i be such that $|F \cap [N]| \le r-2$ for every $F \in \mathcal{F}$ with $i \in F$.

Then

$$\sum_{\substack{J\in\mathbb{N}^{(r-1)},|J|=r-1\\ J\cup\{i\}\in\mathcal{F}}}\prod_{j\in J}\mu(j)\leq\mu(N+1)\sum_{K\in\mathbb{N}^{(r-2)},|K|=r-2}\prod_{j\in K}\mu(j)$$
$$\leq\frac{1}{N(r-2)!}<r\lambda(\mathfrak{F}^*).$$

This contradiction finishes the proof of the claim. □

Now we are ready to finish the proof. We will show that for every $\varepsilon>0$ there exists $\delta>0$ such that for every $\mathcal{F}\in\mathfrak{F}^*$ and $\mu\in\mathcal{M}(\mathcal{F})$, if $\lambda(\mathcal{F}^*,\mu)\geq\lambda(\mathfrak{F}^*)-\delta$, then $d_{\mathfrak{F}^{**}}(\mathcal{F}^*,\mu)\leq\varepsilon$. (Clearly $\lambda(\mathfrak{F}^*)$=$\lambda(\mathfrak{F}^{**})$ so the above implies the theorem.) Abusing notation slightly we consider pairs $(\mathcal{F},\mu)$ as above as elements of $\mathfrak{X}$.

From continuity of λ and Claim 4.3.5 it follows that for every $\varepsilon>0$ there exists $\delta>0$ such that for every $(\mathcal{F},\mu)\in\mathfrak{X}$ satisfying $\lambda(\mathcal{F}^*,\mu)\geq\lambda(\mathfrak{F}^*)-\delta$ there exists $(\mathcal{F}^{**},\mu^{**})\in\mathfrak{X}^{**}$ such that $\mathcal{F}|_{[n]}=\mathcal{F}^{**}|_{[n]}$ for all $n\leq\frac{2}{\varepsilon}(r-1)!+1$.

Following the argument in Claim 4.3.4, let $\mathcal{H}:=\mathcal{F}|_{[N]}(=\mathcal{F}^{**}|_{[N]})$, for $N:=\lceil\frac{2}{\varepsilon}(r-1)!\rceil$. As in Claim 4.3.4 we have

$$\lambda(\mathcal{F},\mu^*)-\lambda(\mathcal{H},\mu^*)\leq\frac{1}{N(r-1)!},$$

$$\lambda(\mathcal{F}^{**},\mu^*)-\lambda(\mathcal{H},\mu^*)\leq\frac{1}{N(r-1)!}.$$

Finally, we have

$$\begin{aligned}d_{\mathfrak{F}^{**}}(\mathcal{F}^*,\mu^*)&\leq d((\mathcal{F}^*,\mu^*),(\mathcal{H},\mu^*))+d((\mathcal{H},\mu^*),(\mathcal{F}^{**},\mu^*))\\&\leq(\lambda(\mathcal{F}^*,\mu^*)-\lambda(\mathcal{H},\mu^*))+(\lambda(\mathcal{F}^{**},\mu^*)-\lambda(\mathcal{H},\mu^*))\\&\leq\frac{2}{N(r-1)!}\leq\varepsilon,\end{aligned}$$

as desired. □

Proof of Theorem 4.3.1. We only need to show that the family $\mathfrak{F}^{**}$ is not empty under the assumptions on $\mathcal{S}$ and $\mathfrak{F}^*_r$. Indeed by assumptions, we know that $\lambda(\mathcal{S},\mu^*_{\mathcal{S}})\geq\lambda(\mathcal{F}^*,\mu)$ for every $\mathcal{F}^*\in\mathfrak{F}^*_r$, $\mu\in\mathcal{M}(\mathcal{F}^*)$ and, further, the equality holds only when

$\mathcal{F}^*$ is isomorphic to $\mathcal{S}$ and $\mu = \mu^*_{\mathcal{S}}$. Therefore $\mathfrak{F}^{**} = \{\mathcal{S}\}$ and $\mathfrak{F}^*_r$ is $\mathcal{S}$-weakly weight stable, which in its turn trivially implies $\mathfrak{B}(\mathcal{S})$-weak weight stability. □

4.4. Proof of Theorem 4.1.8

Proof of Theorem 4.1.8. By Theorem 3.8.2, we need to show that the following conditions hold.

(C1) $\mathrm{Forb}(\mathcal{T}_r)$ is $\mathfrak{B}(\mathcal{S})$-vertex locally stable.

(C2) The subfamily of $\mathrm{Forb}(\mathcal{D}_r)$ of graphs covering pairs is $\mathfrak{B}(\mathcal{S})$-weakly weight stable.

(C1) holds by Theorem 4.2.1. (C2) holds by Theorem 4.3.1. This finishes the proof. □

Chapter 5

Turán numbers of Two Families

In this chapter we study Turán numbers of two families of hypergraph extensions. Keevash [Kee11a] and Sidorenko [Sid89] have previously determined Turán densities of these families. We determine their Turán numbers using the local stability method.

5.1. History

Our first main result is connected to the famous Erdős-Sós conjecture from 1963 (Conjecture 1.6.2), which asserts that if G is a simple graph of order n with average degree more than $k-2$, then G contains every tree on k vertices as a subgraph. This conjecture has been verified for several families of trees, and in early 1990's the proof of the conjecture for large enough k was announced by Ajtai, Komlós, Simonovits and Szemerédi. We say that a tree is *an Erdős-Sós-tree* if it satisfies the conjecture. Recall the definition of expansions from Section 1.6. Given a 2-graph G, the $(r-2)$-*expansion of* G is the r-graph obtained by adding $(r-2)$ vertices to G and enlarging each edge of G to contain these vertices. In [Sid89] Sidorenko proved the following.

Theorem 5.1.1 (Sidorenko, [Sid89]). *For every $r \geq 2$, there exists M_r such that if T is an Erdős-Sós-tree on $t \geq M_r$ vertices then $\pi(\mathrm{Ext}(\mathcal{T})) = r!(t+r-3)^{-r}\binom{t+r-3}{r}$, where $\mathcal{T}$ is the $(r-2)$-expansion of T.*

We prove the following exact version of Theorem 5.1.1.

Theorem 5.1.2. *For every $r \geq 2$, there exists M_r such that the following holds. Let T be an Erdős-Sós-tree on $t \geq M_r$ vertices and let $\mathcal{T}$ be the $(r-2)$-expansion of T. Then there exists n_0 such that for all $n \geq n_0$, $\mathcal{K}^{(r)}_{t+r-3}(n)$ is the unique largest* $\mathrm{Ext}(\mathcal{T})$*-free r-graph on n vertices.*

Our second result concerns extensions of a different class of sparse hypergraphs. Let $\bar{\mathcal{K}}^{(r)}_t$ denote the edgeless r-graph on t vertices. Mubayi [Mub06] determined $\pi(\mathrm{Ext}(\bar{\mathcal{K}}^{(r)}_t))$ and Pikhurko [Pik05] obtained the corresponding exact result.

Theorem 5.1.3 (Pikhurko,[Pik05])**.** *For every $t > r \geq 3$ there exists n_0 such that for all $n \geq n_0$, $\mathcal{K}^{(r)}_t(n)$ is the unique largest* $\mathrm{Ext}(\bar{\mathcal{K}}^{(r)}_{t+1})$*-free r-graph on n vertices.*

Keevash [Kee11a] considered the following generalization of the above problem. Let $\mathcal{F}$ be any r-graph that covers pairs, and let $\mathcal{F}^{+t}$ be obtained from $\mathcal{F}$ by adding new isolated vertices so that $\mathrm{v}(\mathcal{F}^{+t}) = t$. (We have $\emptyset^{+t} = \bar{\mathcal{K}}^{(r)}_t$, where $\emptyset$ denotes the null r-graph.) In [Kee11a] Keevash, generalizing the density argument from [Mub06], proved the following.

Theorem 5.1.4 (Keevash, [Kee11a])**.** *Let $\mathcal{F}$ be an r-graph that covers pairs with $\mathrm{v}(\mathcal{F}) \leq t+1$. If $\pi(\mathcal{F}) \leq r!t^{-r}\binom{t}{r}$, then $\pi(\mathrm{Ext}(\mathcal{F}^{+(t+1)})) = r!t^{-r}\binom{t}{r}$.*

We obtain the exact version of a slight weakening of Theorem 5.1.4.

Theorem 5.1.5. *Let $\mathcal{F}$ be an r-graph that covers pairs with $\mathrm{v}(\mathcal{F}) \leq t$. If $\pi(\mathcal{F}) < r!t^{-r}\binom{t}{r}$ then there exists n_0 such that $\mathcal{K}^{(r)}_t(n)$ is the unique* $\mathrm{Ext}(\mathcal{F}^{+(t+1)})$*-free r-graph on n vertices with maximum number of edges for all $n \geq n_0$.*

Our proofs of Theorems 5.1.2 and 5.1.5 share a common part. Both graphs $\mathrm{Ext}(\mathcal{T})$ and $\mathrm{Ext}(\mathcal{F}^{+(t+1)})$ belong to a general class of graphs, which we call *sharply t-critical*. In the next section we prove a theorem establishing the local stability of $\mathrm{Forb}(\mathcal{F})$ for all such $\mathcal{F}$. Then we prove the stability in weighted setting separately for $\mathrm{Forb}(\mathrm{Ext}(\mathcal{T}))$ and $\mathrm{Forb}(\mathrm{Ext}(\mathcal{F}^{+(t+1)}))$ in Sections 5.3 and 5.4, respectively.

5.2. Local Stability of Sharply Critical Graphs

We say that an r-graph $\mathcal{F}$ is *strongly t-colorable*, if the vertices of $\mathcal{F}$ can be colored in t colors such that every edge contains no two vertices of the same color. Equivalently, $\mathcal{F}$ is strongly t-colorable if and only if $\mathcal{K}_t^{(r)}$ is not $\mathcal{F}$-hom-free. Recall that an r-graph is *t-colorable* if the vertices can be colored in t colors such that no edge is monochromatic. For $r = 2$ the definitions of strong t-colorability and t-colorability coincide, but for $r \geq 3$ they differ.

We say that $\mathcal{F}$ is *t-critical* if $\mathcal{F}$ is not strongly t-colorable, but there exists an edge $F \in \mathcal{F}$ such that $\mathcal{F} \setminus F$ is strongly t-colorable. We are interested in a subfamily of t-critical graphs. Let us remind the reader few definitions from Section 3.9.

Recall that for two r-graphs $\mathcal{F}$ and $\mathcal{H}$, we say that $\mathcal{H}$ is *$\mathcal{F}$-hom-free* if there is no homomorphism from $\mathcal{F}$ to $\mathcal{H}$. We say that $\mathcal{H}$ is *$\mathcal{F}$-hom-critical* if $\mathcal{H}$ is $\mathcal{F}$-hom-free but there exists an edge $F \in \mathcal{F}$ such that there exists a homomorphism from $\mathcal{F} \setminus F$ to $\mathcal{H}$. More specifically, if F is such an edge in $\mathcal{F}$, we say that $\mathcal{H}$ is *$(\mathcal{F}, F)$-hom-critical*.

For an r-graph $\mathcal{F}$ and an edge $F \in \mathcal{F}$, we say that the pair $(\mathcal{F}, F)$ is *loose* if the vertices of F can be partitioned into two sets, F_c and F_f, such that

- $|F_c| = 2$, the vertices of F_c do not share any edge other than F,
- every vertex of F_f is not contained in any edge other than F.

The vertices in F_f are called *free*, the vertices in F_c are called *critical*.

Definition 5.2.1. *For an r-graph $\mathcal{F}$ and $v \in F \in \mathcal{F}$, we say that the triple $(\mathcal{F}, F, v)$ is a t-spike if*

(i) $(\mathcal{F}, F)$ is loose and v is a critical vertex in F,

(ii) $L_{\mathcal{F}}(v)$ is a matching,

(iii) $\mathcal{K}_t^{(r)}$ is $(\mathcal{F}, F)$-hom-critical,

(iv) for every $\mathcal{L} \subseteq [t]^{[r-1]}$ with $|\mathcal{L}| \geq \binom{t-1}{r-1}$ such that $\mathcal{L}$ is not isomorphic to $\mathcal{K}_{t-1}^{(r-1)}$, there exists a mapping

$$\varphi : V(\mathcal{F}) \setminus \{v\} \to [t] \text{ such that}$$

(a) $\varphi|_{\mathcal{F}-v}$ (i.e. φ restricted to the r-graph $\mathcal{F}-v$) is a homomorphism to $\mathcal{K}_t^{(r)}$.

(b) for every $I \in L_{\mathcal{F}}(v)$, $\varphi(I) \in \mathcal{L}$ and $|\varphi(I)| = r-1$.

We say that $\mathcal{F}$ is *sharply* t-critical if there exist $v \in F \in \mathcal{F}$ such that $(\mathcal{F}, F, v)$ is a t-spike. Note that for 2-graphs the technical definition above simplifies considerably. Indeed if $\mathcal{F}$ is a 2-graph which is not t-colorable, and $v \in F \in \mathcal{F}$ are such that $\mathcal{F} \setminus F$ is t-colorable, then $(\mathcal{F}, F, v)$ trivially is a t-spike. We are now ready to state the main result of this section.

Theorem 5.2.2. *For $t \geq r \geq 2$, if an r-graph $\mathcal{F}$ is sharply t-critical then* $\mathrm{Forb}(\mathcal{F})$ *is $\mathfrak{B}\left(\mathcal{K}_t^{(r)}\right)$-vertex locally stable.*

Theorem 5.2.2 and the remark preceding it imply that for every t-critical 2-graph $\mathcal{F}$ the family $\mathrm{Forb}(\mathcal{F})$ is $\mathfrak{B}(\mathcal{K}_t)$-vertex locally stable. Thus, by Corollary 3.5.7, as a consequence of Theorem 5.2.2 we obtain a classical theorem of Simonovits, which using our language can be stated as follows.

Corollary 5.2.3 (Simonovits, [Sim68])**.** *Let $\mathcal{F}$ be a t-critical 2-graph. Then* $\mathrm{Forb}(\mathcal{F})$ *is $\mathfrak{B}(\mathcal{K}_t)$-stable.*

For the proof of Theorem 5.2.2 we use the tools developed in Section 3.9. In particular, if $\mathcal{F}$ is sharply t-critical then clearly Corollary 3.9.2 and Corollary 3.9.4 are applicable to $\mathcal{F}$ and $\mathcal{K}_t^{(r)}$. As in the case of the generalized triangle (Lemma 4.2.3), here also we need an auxiliary lemma which allows us to find a "good embedding" of every large $\mathcal{F}$-free graph, meaning that if $\mathcal{G}$ is an $\mathcal{F}$-free graph with large minimum degree and has a small edit distance to the family $\mathfrak{B}(\mathcal{K}_t^{(r)})$ then we can find a blowup $\mathcal{B} \in \mathfrak{B}$ on the same vertex set as $\mathcal{G}$ such that every vertex has almost the same neighbourhoods in $\mathcal{G}$ and $\mathcal{B}$.

Lemma 5.2.4. *For $t \geq r \geq 2$, let $\mathcal{F}$ be a sharply t-critical r-graph. Then for every $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that if $\mathcal{G}$ is an $\mathcal{F}$-free r-graph with* $\mathrm{v}(\mathcal{G}) = n \geq n_0$ *such that*

- $d_{\mathfrak{B}(\mathcal{K}_t^{(r)})}(\mathcal{G}) \leq \delta n^r$,
- $|L_{\mathcal{G}}(v)| \geq \left(\frac{\binom{t-1}{r-1}}{t^{r-1}} - \delta\right) n^{r-1}$,

then there exists $\mathcal{B}_0 \in \mathfrak{B}(\mathcal{K}_t^{(r)})$ *with* $V(\mathcal{B}_0) = V(\mathcal{G}) := V$ *such that for every* $v \in V$

$$|L_{\mathcal{G}}(v) \triangle L_{\mathcal{B}_0}(v)| \leq \varepsilon n^{r-1}.$$

Proof. Denote $\mathfrak{B} := \mathfrak{B}(\mathcal{K}_t^{(r)})$ and $\mathrm{d}(t,r) = \frac{\binom{t-1}{r-1}}{t^{r-1}}$. It is easy to see that $m(\mathfrak{B}) = \frac{\mathrm{d}(t,r)}{r}$. Let $\varepsilon_{3.9.4}$ be derived from Corollary 3.9.4 applied with $\mathcal{H} = \mathcal{K}_t^{(r)}$ and $\mathcal{F}$. Given ε, we choose the constants $\varepsilon_{3.9.2}$, δ, β, γ as follows

$$0 < \delta \ll \varepsilon_{3.9.2} \ll \gamma \ll \beta \ll \min\left\{\varepsilon_{3.9.4}, \varepsilon, \frac{1}{\mathrm{v}(\mathcal{F})}\right\}.$$

Let $\delta_{3.9.2}$ be derived from Corollary 3.9.2 with respect to $\mathcal{F}$ and $\mathcal{K}_t^{(r)}$ applied with $\varepsilon = \varepsilon_{3.9.2}$, we also impose on δ the condition $\delta \ll \delta_{3.9.2}$. Let $\mathcal{B} \in \mathfrak{B}$ be such that $|\mathcal{G} \triangle \mathcal{B}| = d_{\mathfrak{B}}(\mathcal{B})$. First note that

$$|\mathcal{G}| = \frac{1}{r} \sum_{v \in V(\mathcal{G})} |L_{\mathcal{G}}(v)| \geq \left(\frac{\mathrm{d}(t,r)}{r} - \frac{\delta}{r}\right) n^r.$$

Because of the choice $\delta \ll \delta_{3.9.2}$ we get

$$|\mathcal{B}| \geq |\mathcal{G}| - \delta n^r \geq \left(\frac{\mathrm{d}(t,r)}{r} - \delta_{3.9.2}\right) n^r,$$

therefore $\mathcal{B}$ is $\varepsilon_{3.9.2}$-trimmed. The first part of the proof goes along the same lines as in Lemma 4.2.3. We consider the set of "non-behaving" vertices, that is,

$$J = \left\{v \in V \middle| |L_{\mathcal{G}}(v) \triangle L_{\mathcal{B}}(v)| > \gamma n^{r-1}\right\}$$

and can derive the bound $|J| \leq \frac{\delta r}{\gamma} n$ easily. Then we consider $\mathcal{G}' := \mathcal{G}|_{V \setminus J}$, $n' = \mathrm{v}(\mathcal{G}')$, $\mathcal{B}' := \mathcal{B}|_{V \setminus J}, \mathcal{P}' = \mathcal{P}|_{V \setminus J}$. Just as in Lemma 4.2.3, we can show that the graph $\mathcal{G}'$ satisfies the assumptions of Lemma 3.9.4 and obtain that both $\mathcal{G}'$ and $\mathcal{G}$ are $\mathcal{P}'$-transversal.

Using these properties, we extend $\mathcal{B}'$ to a blowup $\mathcal{B}_0 \in \mathfrak{B}$ with $V(\mathcal{B}_0) = V$, as follows. For each $u \in J$ we find a unique index $j_u \in [m]$, such that u "behaves" as the vertices in the partition class P'_{j_u}, and add the vertex u to this partition class.

For $I \subseteq [t]$, $|I| = r-1$, let

$$E_I(u) := \{G \in \mathcal{G} \,|u \in G,\ |G \cap P'_i| = 1 \text{ for every } i \in I\}.$$

We construct an auxiliary $(r-1)$-graph $\mathcal{L}(u)$ with $V(\mathcal{L}(u)) = [t]$ such that $I \in \mathcal{L}(u)$ if and only if $|E_I(u)| \geq \beta n^{r-1}$. We aim to show that $\mathcal{L}(u)$ is isomorphic to $\mathcal{K}_{t-1}^{(r-1)}$. Just as in Claim 4.2.4, we can show that $|\mathcal{L}(u)| \geq d(t,r)t^{r-1}$. Next we consider the sets

$$L_j(u) = \left\{ v \in P'_j \middle| |L_{\mathcal{G}}(u,v) \geq \frac{\beta}{2} n^{r-2} \right\}, \forall j \in [t]$$

$$K = \left\{ j \middle| |L_j(u)| < \frac{\beta}{2} n \right\}.$$

Just as we did in Claim 4.2.5 we can obtain the following claim.

Claim 5.2.5. *If $j \in I \in \mathcal{L}(u)$ then $j \notin K$.*

So if we prove that $\mathcal{L}(u) \simeq \mathcal{K}_{t-1}^{(r-1)}$ it follows that there exists a unique j_u such that every $I \notin \mathcal{L}(u)$ contains j_u.

Note that in difference to the proof of Lemma 4.2.3, we use a probabilistic argument to show that $\mathcal{L}(u)$ is isomorphic to $\mathcal{K}_{t-1}^{(r-1)}$ rather than a deterministic one. We do so using sharp t-criticality of $\mathcal{F}$, in particular, the existence of a map as in Definition 5.2.1-(iv).

Claim 5.2.6. *$\mathcal{L}(u)$ is isomorphic to $\mathcal{K}_{t-1}^{(r-1)}$.*

Proof. If it is not the case, let $v_c \in F \in \mathcal{F}$ be such that $(\mathcal{F}, F, v_c)$ is a t-spike. Let

$$\varphi : V(\mathcal{F}) \setminus \{v_c\} \to [t]$$

be as in Definition 5.2.1. Let $\rho : V(\mathcal{F}) \to V(\mathcal{G})$ be a random map such that $\rho(v_c) = u$, and let $\rho(w)$ be chosen uniformly at random in $P_{\varphi(w)}$ for every $w \in V(\mathcal{F}) \setminus v_c$. We will show that with probability bounded away from zero as a function of β and independent on n, the map ρ maps all edges of $\mathcal{F}$ to edges of $\mathcal{G}$. It will follow that $\mathcal{G}$ is not $\mathcal{F}$-free yielding the desired contradiction.

If $I \in \mathcal{F}$, $v_c \notin I$ then

$\mathbb{P}[\rho(I) \notin \mathcal{G}] \leq 2\gamma t^r$, as in Lemma 3.9.3. Thus,

$$\mathbb{P}[\rho(I) \notin \mathcal{G} \text{ for some } I \in \mathcal{F} \text{ such that } v \notin I] \leq 2\gamma \, \mathrm{v}(\mathcal{F})^r t^r \ll \beta.$$

If $I \in L_{\mathcal{F}}(v)$ then $\mathbb{P}[\rho(I \cup \{v\}) \in \mathcal{G}] \geq \beta$. As $L_{\mathcal{F}}(v)$ is a matching it follows that the events $\{\rho(I \cup \{v\}) \in \mathcal{G}\}_{I \in L_{\mathcal{F}}(v_c)}$ are independent. Thus

$$\mathbb{P}[\rho(I) \in \mathcal{G} \text{ for every } I \in L_{\mathcal{F}}(v_c)] \geq \beta^{|L_{\mathcal{H}}(v_c)|}.$$

The desired conclusion follows. □

From the claim above it follows that $|K| = 1$. For every u, let j_u be the unique index in the corresponding set K. As discussed above, we extend the blowup $\mathcal{B}'$ as follows. For every $j \in [t]$, define

$$P_j^0 := P_j' \cup \{u \in J \mid j_u = j\}.$$

Let $\mathcal{B}_0 \supseteq \mathcal{B}'$ be the blowup of $\mathcal{H}$ with the blowup partition $\mathcal{P}_0$. Now we are ready to show that for every $v \in V$, $|L_{\mathcal{B}_0}(v) \triangle L_{\mathcal{G}}(v)| \leq \varepsilon n^{r-1}$. If $v \in V \setminus J$, then

$$\begin{aligned} |L_{\mathcal{B}_0}(v) \triangle L_{\mathcal{G}}(v)| &\leq |L_{\mathcal{B}'}(v) \triangle L_{\mathcal{G}'}(v)| + |J| n^{r-2} \\ &\leq \gamma n^{r-1} + \frac{\delta r}{\gamma} n^{r-1} \leq \varepsilon n^{r-1}. \end{aligned}$$

We now consider $v \in J$. Since $\mathcal{G}$ is $\mathcal{P}'$-transversal and by the choice of j_v, it follows that for every $G \in L_{\mathcal{G} \setminus \mathcal{B}_0}(v)$, either $G \cap J \neq \emptyset$, or there exists $I \notin \mathcal{L}(v)$ such that $G \in L_I(v)$. Thus,

$$|L_{\mathcal{G} \setminus \mathcal{B}_0}(v)| \leq \frac{\delta r}{\gamma} n^{r-1} + \left(\binom{m}{r-1} - d \right) \beta n^{r-1} < \frac{\varepsilon}{4} n^{r-1}. \tag{5.1}$$

Finally,

$$\begin{aligned}|L_{\mathcal{G}}(v)\triangle L_{\mathcal{B}_0}(v)| &= 2|L_{\mathcal{G}\setminus\mathcal{B}_0}(v)| + |L_{\mathcal{B}_0}(v)| - |L_{\mathcal{G}}(v)| \\ &\overset{(5.1)}{\leq} \frac{\varepsilon}{2}n^{r-1} + d\left(\frac{1}{m} + \varepsilon_{3.9.2} + \frac{\delta r}{\gamma}\right)^{r-1} n^{r-1} - (d-\delta)n^{r-1} \\ &\leq \varepsilon n^{r-1},\end{aligned}$$

as desired. This finishes the proof of the theorem. □

Proof of Theorem 5.2.2: Suppose $v \in F \in \mathcal{F}$ are such that $(\mathcal{F}, F, v)$ is a t-spike. We denote $\mathfrak{B} = \mathfrak{B}\left(\mathcal{K}_t^{(r)}\right)$ and $\mathrm{d}(t,r) := \frac{\binom{t-1}{r-1}}{t^{r-1}}$. Note that $m(\mathfrak{B}) = \frac{\mathrm{d}(t,r)}{r}$.

Let $\varepsilon_{3.9.4}$ be obtained from Corollary 3.9.4 applied to $\mathcal{F}, F$ and $\mathcal{K}_t^{(r)}$. Let $\delta_{5.2.4}$ be obtained from Theorem 5.2.4 applied with $\varepsilon = \varepsilon_{3.9.4}$ to $\mathcal{F}$, F and $\mathcal{K}_t^{(r)}$.

We want to show that there exist $\varepsilon, \alpha, n_0 > 0$ such that for every $\mathcal{G} \in \mathrm{Forb}(\mathcal{F})$ with $\mathrm{v}(\mathcal{G}) = n \geq n_0$, such that $d_{\mathfrak{B}}(\mathcal{G}) \leq \varepsilon n^r$, and $|L_{\mathcal{G}}(v)| \geq (\mathrm{d}(t,r) - \varepsilon)\, n^{r-1}$ for every $v \in V(\mathcal{F})$, we have

$$|\mathcal{G}| \leq m(\mathfrak{B}, n) - \alpha d_{\mathfrak{B}}(\mathcal{G}).$$

We claim that $\alpha = 1$ and $\varepsilon = \min\{\frac{\varepsilon_{3.9.4}}{4}, \frac{\delta_{5.2.4}}{2}\}$ are as required.

We may assume $|\mathcal{G}| \geq (m(\mathfrak{B}) - 2\varepsilon)n^r$, as otherwise the result follows with $\alpha = 1$. Hence,

$$|\mathcal{G}| \geq (m(\mathfrak{B}) - 2\varepsilon)n^r \geq (m(\mathfrak{B}) - \delta_{5.2.4})n^r$$

and can apply Theorem 5.2.4 and obtain $\mathcal{B} \in \mathfrak{B}$ with $V(\mathcal{B}) = V(\mathcal{G}) := V$ such that for every $v \in V$, $|L_{\mathcal{G}}(v)\triangle L_{\mathcal{B}}(v)| \leq \varepsilon_{3.9.4}n^{r-1}$.

On the other hand,

$$|\mathcal{B}| \geq |\mathcal{G}| - |\mathcal{G}\triangle\mathcal{B}| \geq (m(\mathfrak{B}) - 2\varepsilon)\, n^r - \frac{\varepsilon_{3.9.4}}{r}n^r \geq (m(\mathfrak{B}) - \varepsilon_{3.9.4})n^r$$

Hence now we can apply Corollary 3.9.4 to $\mathcal{G}$ and $\mathcal{B}$ and obtain that $\mathcal{G}$ is $\mathcal{P}(\mathcal{B})$-transversal. And therefore, $\mathcal{G} \subseteq \mathcal{B}$ and thus, $|\mathcal{G}| = |\mathcal{B}| - |\mathcal{B}\setminus\mathcal{G}| \leq m(\mathfrak{B}, n) - d_{\mathfrak{B}}(\mathcal{G})$, as desired. □

5.3. Proof of Theorem 5.1.2

Let $\mathcal{T}$ denote the $(r-2)$-expansion of the tree T. By Theorem 3.8.2, it suffices to prove that

(C1) $\mathrm{Forb}(\mathrm{Ext}(\mathcal{T}))$ is $\mathfrak{B}(\mathcal{K}_{t+r-3}^{(r)})$-vertex locally stable,

(C2) $\mathrm{Forb}(\mathcal{T})$ is $\mathfrak{B}(\mathcal{K}_{t+r-3}^{(r)})$-weakly weight-stable.

By Theorem 5.2.2, the following lemma establishes that (C1) holds.

Lemma 5.3.1. *If T is a tree on $t \geq 3$ vertices, then* $\mathrm{Ext}(\mathcal{T})$ *is sharply* $(t+r-3)$-*critical.*

Proof. Let v be a leaf of T, and let F be an edge of $\mathrm{Ext}(\mathcal{T}) - \mathcal{T}$ containing v. We show that $(\mathrm{Ext}(\mathcal{T}), F, v)$ is a $(t+r-3)$-spike. Let u be the unique vertex in $(F \cap V(T)) - \{v\}$. Note that $\mathrm{v}(\mathcal{T}) = t+r-2$, and every pair of vertices in $V(\mathcal{T})$ is covered in $\mathrm{Ext}(\mathcal{T})$. Condition (i)-(iii) in Definition 5.2.1 are easy to verify. Indeed, $\mathrm{Ext}(\mathcal{T})$ is not strongly $(t+r-3)$-colorable, but $\mathrm{Ext}(\mathcal{T}) - F$ is, since one can use the same color on u and v. Also since v is adjacent to the unique vertex in T, $L_{\mathrm{Ext}(\mathcal{T})}(v)$ is trivially a matching.

It remains to verify (iv). For $\mathcal{L} \subseteq [t+r-3]^{(r-1)}$ such that $|\mathcal{L}| \geq \binom{t+r-4}{r-1}$ and $\mathcal{L}$ is not isomorphic to $\mathcal{K}_{t+r-4}^{(r-1)}$ we define mapping $\varphi : V(\mathrm{Ext}(\mathcal{T})) \setminus \{v\} \to [t+r-3]$ satisfying condition (iv) in the definition of a t-spike as follows.

Consider the subgraph $\mathcal{T}'$ of $\mathrm{Ext}(\mathcal{T})$ induced by the vertex set $V(\mathrm{Ext}(\mathcal{T})) \setminus (V(L_{\mathrm{Ext}(\mathcal{T})}(v)) \setminus V(\mathcal{T}))$. Then $\mathcal{T}'$ is strongly $(t+r-3)$-colorable. Let φ be defined on $V(\mathcal{T}')$ so that φ is a strong $(t+r-3)$-coloring of $\mathcal{T}'$, and moreover, $\varphi(L - \{v\}) \in \mathcal{L}$ for the unique edge $L \in \mathcal{T}'$ such that $v \in L$. It follows that (iv-a) holds for φ.

It remains to extend φ so that it satisfies (iv-b). For every $w \in V(T) - \{u, v\}$ there exists a unique $I \in \mathcal{L}_{\mathrm{Ext}(\mathcal{T})}(v)$ such that $w \in I$. Since $\mathcal{L}$ is not isomorphic to $\mathcal{K}_{t+r-4}^{(r-1)}$, there exists $L \in \mathcal{L}$ such that $\varphi(w) \in L$. We extend φ to $I - \{w\}$ so that $\varphi(I) = L$. Clearly, the resulting map $\varphi : V(\mathrm{Ext}(\mathcal{T})) \to [t+r-3]$ satisfies (iv-b). □

The following theorem shows that (C2) holds, thus completing the proof of Theorem 5.1.2.

Theorem 5.3.2. *For every $r \geq 2$ there exists real M_r such that, if T is an Erdős-Sós-tree on $t \geq M_r$ vertices, then* $\mathrm{Forb}(\mathcal{T})$ *is* $\mathfrak{B}(\mathcal{K}_{t+r-3}^{(r)})$*-weakly weight-stable.*

The proof of Theorem 5.3.2 relies on a result by Sidorenko [Sid89]. Its statement involves a function

$$f_r(x) = \frac{1}{(x+r-3)^r}\binom{x+r-3}{r}\frac{t-2}{x-2}.$$

Let us first note the following useful observations concerning $f_r(x)$ from [Sid89]:

(F1) The function $f_r(x)$ is strictly decreasing for sufficiently large x,

(F2)

$$f_r(x) = \frac{1}{r}\left(\frac{x+r-4}{x+r-3}\right)^{r-1} f_{r-1}(x), \qquad \text{and}$$

(F3) $\lambda(\mathcal{K}_{t+r-3}^{(r)}) = f_r(t)$.

Theorem 5.3.3 ([Sid89, Lemma 3.3])**.** *For $r \geq 2$ let M_r be such that $f_r(x)$ is decreasing for $x \geq M_r$. If T is an Erdős-Sós-tree on $t \geq M_r$ vertices then $\lambda(\mathcal{F}, \mu) \leq f_r(x)$ for every $\mathcal{F} \in \mathrm{Forb}(\mathcal{T}), \mu \in \mathcal{M}(\mathcal{F})$ with $x = \max\{t, \frac{1}{\gamma} - r + 3\}$ and $\gamma = \max_{v \in V(\mathcal{F})} \mu(v)$. In particular, $\lambda(\mathrm{Forb}(\mathcal{T})) = f_r(t)$.*

Given an r-graph $\mathcal{F}$, $u, v \in V(\mathcal{F})$ and $\mu \in \mathcal{M}(\mathcal{F})$, let $\lambda_u(\mathcal{F}, \mu) = \sum_{I \in L_{\mathcal{F}}(u)} \mu(I)$, and let $\lambda_{u,v}(\mathcal{F}, \mu) = \sum_{I \in L_{\mathcal{F}}(u,v)} \mu(I)$. The following technical lemma is useful in the proof of Theorem 5.3.2.

Lemma 5.3.4. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{F}$ is an r-graph, $u \in V(\mathcal{F})$, $\mu \in \mathcal{M}(\mathcal{F})$, $\lambda(\mathcal{F}, \mu) \geq \lambda(\mathcal{F}) - \delta$ and $\mu(u) \geq \varepsilon$, then $\lambda_u(\mathcal{F}, \mu) \geq r\lambda(\mathcal{F}) - \varepsilon$.*

Proof. We assume without loss of generality that $\varepsilon < 1$, and let $\delta = (\varepsilon^3 - \varepsilon^4)/r$. Suppose for a contradiction that $\lambda_u(\mathcal{F}, \mu) < r\lambda(\mathcal{F}) - \varepsilon$. We have $r\lambda(\mathcal{F}, \mu) = \sum_{v \in V(\mathcal{F})} \mu(v)\lambda_v(\mathcal{F}, \mu)$. Thus, there exists $u' \in V(\mathcal{F}) \setminus \{u\}$ such that $\lambda_{u'}(\mathcal{F}, \mu) \geq r\lambda(\mathcal{F})$.

Let $\mu' \in \mathcal{M}(\mathcal{F})$ be defined as follows. Let $\mu'(u') = \mu(u') + \varepsilon^2$, $\mu'(u) = \mu(u) - \varepsilon^2$, and let $\mu'(v) = \mu(v)$ for every $v \in V(\mathcal{F}) - \{u, u'\}$. We have

$$\begin{aligned} r\lambda(\mathcal{F}, \mu') &= \sum_{v \in V(\mathcal{F})} \mu'(v)\lambda_v(\mathcal{F}, \mu') \\ &= r\lambda(\mathcal{F}, \mu) + \varepsilon^2(\lambda_{u'}(\mathcal{F}, \mu) - \lambda_u(\mathcal{F}, \mu)) - \varepsilon^4\lambda_{u,u'}(\mathcal{F}, \mu) \\ &> r\lambda(\mathcal{F}, \mu) + \varepsilon^3 - \varepsilon^4 \\ &\geq r\lambda(\mathcal{F}) - r\delta + \varepsilon^3 - \varepsilon^4 \\ &\geq r\lambda(\mathcal{F}), \end{aligned}$$

a contradiction. □

In the rest of the section it will be convenient for us to occasionally consider subprobabilistic measures on the vertex set of a graph, rather than probabilistic ones. For an r-graph $\mathcal{F}$, let $\mathcal{M}_{\leq 1}(\mathcal{F}) \supset \mathcal{M}(\mathcal{F})$ denote the set of functions $\mu : V(\mathcal{F}) \to \mathrm{R}_+$ such that $\mu(V(\mathcal{F})) \leq 1$. The density $\lambda(\mathcal{F}, \mu)$ and the distance $d((\mathcal{F}, \mu), (\mathcal{F}, \mu'))$ for $\mu, \mu' \in \mathcal{M}_{\leq 1}(\mathcal{F})$ are defined as in Section 2.4. Moreover, it is easy to check that the following result holds.

Fact 5.3.5. *For any r-graph $\mathcal{F}$ and $\mu, \mu' \in \mathcal{M}_{\leq 1}(\mathcal{F})$,*

(i) $|\lambda(\mathcal{F}, \mu) - \lambda(\mathcal{F}, \mu')| \leq ||\mu - \mu'||_1$,

(ii) $d((\mathcal{F}, \mu), (\mathcal{F}, \mu')) \leq ||\mu - \mu'||_1$.

All the technical work in the proof of Theorem 5.3.2 is accomplished in the following lemma.

Lemma 5.3.6. *For every $r \geq 2$ there exists M_r such that if $\mathcal{T}$ is the $(r-2)$-expansion of an Erdős-Sós-tree T on $t \geq M_r$ vertices then the following holds. For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{F} \in \mathrm{Forb}(\mathcal{T})$, $\mu \in \mathcal{M}(\mathcal{F})$ with $\lambda(\mathcal{F}, \mu) \geq \lambda(\mathrm{Forb}(\mathcal{T})) - \delta$, then there exists $S \subseteq V(\mathcal{F})$ such that $\mu(S) \geq 1 - \varepsilon$, and $\mathcal{F}[S]$ is isomorphic to $\mathcal{K}^{(r)}_{t+r-3}$.*

Proof. First note that, by Theorem 5.3.3, $\lambda(\mathrm{Forb}(\mathcal{T})) = \frac{1}{(t+r-3)^r}\binom{t+r-3}{r}$. The proof is by induction on r. The base case is $r = 2$. In this case we have $\mathcal{T} = T$ and

$\lambda(\text{Forb}(T)) = \frac{t-2}{t-1}$. For any 2-graph $\mathcal{F}$, by Corollary 2.2.2, $\lambda(\mathcal{F}) = \frac{\omega-2}{\omega-1}$, where ω is the size of the maximum clique of $\mathcal{F}$. Thus, if $\mathcal{F}$ and $\mu \in \mathcal{M}(\mathcal{F})$ are such that $\lambda(\mathcal{F}, \mu) \geq (1-\delta)\frac{t-2}{t-1}$, it follows that $\omega = t-1$ given that δ is sufficiently small. However, the graph $\mathcal{F}$ is T-free, and therefore every complete subgraph of $\mathcal{F}$ of size $t-1$ is a component of $\mathcal{F}$. Note that if $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ and $V(\mathcal{F}_1) \cap V(\mathcal{F}_2) = \emptyset$ then

$$\lambda(\mathcal{F}, \mu) \leq \mu(V(\mathcal{F}_1))^2\lambda(\mathcal{F}_1) + \mu(V(\mathcal{F}_2))^2\lambda(\mathcal{F}_2).$$

Thus if $\mathcal{C}$ is a clique on $(t-1)$ vertices in $\mathcal{F}$, then

$$\begin{aligned}
(1-\delta)\frac{t-2}{t-1} &\leq \lambda(\mathcal{F}, \mu) \\
&\leq \mu(V(\mathcal{C}))^2\lambda(\mathcal{C}) + (1-\mu(V(\mathcal{C}))^2\lambda(\mathcal{F}-\mathcal{C}) \\
&\leq \frac{t-2}{t-1}\left(\mu(V(\mathcal{C}))^2 + (1-\mu(V(\mathcal{C}))^2\right)
\end{aligned}$$

It follows that $2\mu(V(\mathcal{C}))(1-\mu(V(\mathcal{C}))) \leq \delta$. Therefore if we take $S = V(\mathcal{C})$, it is routine to check that $\delta = \varepsilon - \varepsilon^2$ satisfies the theorem in the base case.

We move on to the induction step. Let M_r be chosen so that $f_k(x)$ is strictly decreasing for $x > M_r$ and $k \leq r$. This choice is possible by (F1). By the induction hypothesis there exists $\delta_{r-1} > 0$ such that the claim holds for the $(r-3)$-expansion of T. The parameters $0 < \delta \ll \delta'' \ll \varepsilon' \ll \delta_{r-1} \ll 1$ will be chosen to satisfy the inequalities (occasionally implicit) appearing further in the proof.

Let $u \in V(\mathcal{F})$ be such that $\mu(u) = \max_{v \in V(\mathcal{F})} \mu(v)$, and let $\gamma = \mu(u)$. Suppose that $\gamma < 1/(t+r-3)$. Then by Theorem 5.3.3 and our assumptions we have $f_r(t) - \delta \leq \lambda(\mathcal{F}, \mu) \leq f_r(1/\gamma - r + 3)$. This inequality and the choice of M_r imply that

$$\gamma \geq \frac{1-\varepsilon'}{t+r-3}, \tag{5.2}$$

as long as δ is sufficiently small compared to ε'. By Lemma 5.3.4 we have

$$\lambda_u(\mathcal{F}, \mu) \geq r\lambda(\mathcal{F}) - \varepsilon'/2 \geq rf_r(t) - \varepsilon', \tag{5.3}$$

once again assuming that δ is sufficiently small compared to ε' for the conditions of

Lemma 5.3.4 to be satisfied.

Let $\mathcal{T}'$ be the $(r-3)$-expansion of the tree T, and let $\mathcal{F}' = L_{\mathcal{F}}(u)$. Then $\mathcal{F}'$ is $\mathcal{T}'$-free. Let $\mu' \in \mathcal{M}(\mathcal{F}')$ be given by $\mu'(v) = \frac{\mu(v)}{1-\gamma}$, for every $v \in V(\mathcal{F}')$. Using (5.2),(5.3) and (F2) we have

$$\begin{aligned}\lambda(\mathcal{F}', \mu') &= \frac{\lambda_u(\mathcal{F}, \mu)}{(1-\gamma)^{r-1}} \geq \left(\frac{t+r-3}{t+r-4+\varepsilon'}\right)^{r-1} (r f_r(t) - \varepsilon') \\ &= \left(\frac{t+r-4}{t+r-4+\varepsilon'}\right)^{r-1} f_{r-1}(t) - \varepsilon' \left(\frac{t+r-3}{t+r-4+\varepsilon'}\right)^{r-1} \geq f_{r-1}(t) - \delta_{r-1},\end{aligned}$$

where the last inequality holds for ε' sufficiently small compared to δ_{r-1}. By the choice of δ_{r-1} there exists $S' \subseteq V(\mathcal{F}) \backslash \{u\}$ with $|S'| = t+r-4$ such that $\mu'(S') \geq 1-\varepsilon$. Let $S = S' \cup \{u\}$ then

$$\mu(S) = \mu(u) + \mu'(S')(1-\mu(u)) \geq 1 - \varepsilon.$$

It remains to show that $\mathcal{F}[S]$ is complete. We assume without loss of generality that ε is sufficiently small. Let $\mathcal{F}^* = \mathcal{F}[S]$, and let $\mu^* = \mu|_S$. Then $\mu^* \in \mathcal{M}_{\leq 1}(\mathcal{F}^*)$. Also define $\mu' \in \mathcal{M}_{\leq 1}(\mathcal{F})$ as $\mu'(v) = \mu(v)$, for any $v \in S$ and zero, otherwise. By Fact 5.3.5,

$$\lambda(\mathcal{F}^*, \mu^*) = \lambda(\mathcal{F}, \mu') \geq \lambda(\mathcal{F}, \mu) - \|\mu - \mu'\| \geq f_r(t) - \delta - \varepsilon = \lambda(\mathcal{K}^{(r)}_{t+r-3}) - (\delta + \varepsilon).$$

It follows that $\mathcal{F}^*$ is isomorphic to $\mathcal{K}^{(r)}_{t+r-3}$, as long as ε and δ are sufficiently small. □

Lemma 5.3.6 directly implies Theorem 5.3.2, as follows.

Proof of Theorem 5.3.2: Let $\mathfrak{B} = \mathfrak{B}(\mathcal{K}^{(r)}_{t+r-3})$. For every $\varepsilon > 0$ we need to show the existence of $\delta > 0$ such that if $\mathcal{F} \in \mathrm{Forb}(\mathcal{T})$, $\mu \in \mathcal{M}(\mathcal{F})$ with $\lambda(\mathcal{F}, \mu) \geq \lambda(\mathrm{Forb}(\mathcal{T})) - \delta$, then $d_{\mathfrak{B}}(\mathcal{F}, \mu) \leq \varepsilon$. Let δ be chosen so that Lemma 5.3.6 holds. Then there exists $S \subseteq V(\mathcal{F})$ such that $\mu(S) \geq 1 - \varepsilon$, and $\mathcal{F}^* := \mathcal{F}[S]$ is isomorphic to $\mathcal{K}^{(r)}_{t+r-3}$. Let $\mu' \in \mathcal{M}_{\leq 1}(\mathcal{F})$ be the measure obtained from μ by setting $\mu'(v) = 0$

for every $v \in V(\mathcal{F}) \setminus S$. Then,

$$d_{\mathfrak{B}}(\mathcal{F}, \mu) \leq d((\mathcal{F}^*, \mu|_S), (\mathcal{F}, \mu)) = d((\mathcal{F}, \mu'), (\mathcal{F}, \mu)) \leq \varepsilon,$$

as desired. □

5.4. The proof of Theorem 5.1.5

Just as we did in the previous section, using Theorem 3.8.2 we only need to verify that for every r-graph $\mathcal{F}$ that covers pairs and $t \geq \mathrm{v}(\mathcal{F})$, the following two conditions hold.

(C1) $\mathrm{Forb}(\mathrm{Ext}(\mathcal{F}^{+(t+1)}))$ is $\mathfrak{B}(\mathcal{K}_t^{(r)})$-vertex locally stable,

(C2) $\mathrm{Forb}(\mathcal{F}^{+(t+1)})$ is $\mathfrak{B}(\mathcal{K}_t^{(r)})$-weakly weight-stable.

By Theorem 5.2.2, the following lemma establishes (C1).

Lemma 5.4.1. *If $\mathcal{F}$ is an r-graph that covers pairs, then for any $t \geq \mathrm{v}(\mathcal{F})$, the r-graph $\mathrm{Ext}(\mathcal{F}^{+(t+1)})$ is sharply t-critical.*

Proof. Let $\mathcal{H} = \mathrm{Ext}(\mathcal{F}^{+(t+1)})$. Consider $v \in V(\mathcal{F}^{+(t+1)}) \setminus V(\mathcal{F})$ and let H be any edge containing v. We will show that $(\mathcal{H}, H, v)$ is a t-spike. Conditions (i)-(iii) in the Definition 5.2.1 are easy to verify. In particular, $\mathcal{H} \setminus H$ is strongly t-colorable. Indeed, let φ be an an arbitrary coloring of $V(\mathcal{F}^{+(t+1)}) \setminus \{v\}$ by t colors. We define the the colors of vertices in $V(\mathcal{H}) \setminus V(\mathcal{F}^{+(t+1)}) \cup \{v\}$ as follows.

Let $w \in V(\mathcal{F}^{+(t+1)})$ be the unique vertex different from v such that $w \in H$. Color v with the same color as w. For every edge $H' \in \mathcal{H} \setminus \mathcal{F}$, which contains exactly two vertices of $V(\mathcal{F}^{+(t+1)})$, say u_1, u_2, color the $(r-2)$-tuple $H \setminus \{u_1, u_2\}$ arbitrarily by the colors in $[t]$ not used on u_1 and u_2. Since all these $(r-2)$-tuples are disjoint, it is easy to see that φ is a strong t-coloring of $\mathcal{H} \setminus H$.

Consider $\mathcal{L} \subseteq [t]$ with $|\mathcal{L}| \geq \binom{t-1}{r-1}$ with $\mathcal{L}$ not isomorphic to $\mathcal{K}_{t-1}^{(r-1)}$. We define the map $\varphi : V(\mathcal{H}) \setminus \{v\} \to [t]$ satisfying the conditions (iv-a) and (iv-b) as follows.

Consider the subgraph $\mathcal{H}'$ of $\mathcal{H}$ induced by the vertex set $V(\mathcal{H}) \setminus V(L_{\mathcal{H}}(v)) \cup V(\mathcal{F}^{+(t+1)})$. Let $\varphi|_{V(\mathcal{H}')}$ be any strong t-coloring of $\mathcal{H}'$. Then (iv-a) holds. For every

$I \in L_{\mathcal{H}}(v)$, let w be the unique vertex in $(I \cap V(\mathcal{F}^{+(t+1)})) \setminus \{v\}$. Since $\mathcal{L}$ is not isomorphic to $\mathcal{K}_{t-1}^{(r-1)}$, there exists $L \in \mathcal{L}$ such that $w \in L$. Extend φ to $I \setminus \{w\}$ so that $\varphi(I) = L$. The resulting map φ satisfies (iv-b). □

Lemma 5.4.2. *Let $\mathcal{F}$ be an r-graph that covers pairs, and let t be such that $t \geq \mathrm{v}(\mathcal{F})$ and $\pi(\mathcal{F}) < r!\lambda(\mathcal{K}_t^{(r)})$. Let $\mathfrak{G}^*$ be the family of all r-graphs in* $\mathrm{Forb}(\mathcal{F}^{+(t+1)})$ *which cover pairs. Then $\mathfrak{G}^*$ is $\mathfrak{B}(\mathcal{K}_t^{(r)})$-weakly weight-stable.*

Proof. If $\mathcal{G} \in \mathfrak{G}^*$, then either $\mathrm{v}(\mathcal{G}) \leq t$ or $\mathcal{G}$ is $\mathcal{F}$-free, as otherwise we get a copy of $\mathcal{F}^{+(t+1)}$ using the property of $\mathcal{G}$ covering pairs. We want to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{G} \in \mathfrak{G}^*$, $\mu \in \mathcal{M}(\mathcal{G})$ and $\lambda(\mathcal{G}, \mu) \geq \lambda(\mathcal{K}_t^{(r)}) - \delta$, then $d_{\mathfrak{B}(\mathcal{K}_t^{(r)})}(\mathcal{G}) \leq \varepsilon$. We prove something even stronger.

Let $\mathfrak{T}$ be the family of all r-graphs on at most t vertices not isomorphic to $\mathcal{K}_t^{(r)}$, and let $\lambda^* = \max\{\lambda(\mathfrak{T}), \pi(\mathcal{F})/r!\}$. Then $\lambda^* < \lambda(\mathcal{K}_t^{(r)})$. For our purposes, it suffices to show that, if $\lambda(\mathcal{G}) > \lambda^*$ for some $\mathcal{G} \in \mathfrak{G}^*$, then $\mathcal{G}$ is isomorphic to $\mathcal{K}_t^{(r)}$.

If $\mathrm{v}(\mathcal{G}) \leq t$ then $\mathcal{G}$ is isomorphic to $\mathcal{K}_t^{(r)}$, as otherwise $\lambda(\mathcal{G}) \leq \lambda(\mathfrak{T})$. Thus we assume $\mathrm{v}(\mathcal{G}) > t$. Then $\mathcal{G}$ is $\mathcal{F}$-free, and, as $\mathcal{F}$ covers pairs, it follows that $\mathfrak{B}(\mathcal{G})$ is $\mathcal{F}$-free. Thus, $\pi(\mathcal{F}) \geq \sup_{\mathcal{B} \in \mathfrak{B}(\mathcal{G})} \frac{|\mathcal{B}|}{\binom{\mathrm{v}(\mathcal{B})}{r}}$. On the other hand, it is easy to see for any graph $\mathcal{G}$,

$$\lambda(\mathcal{G}) = \sup_{\mathcal{B} \in \mathfrak{B}(\mathcal{G})} \frac{|\mathcal{B}|}{\mathrm{v}(\mathcal{B})^r}.$$

Thus, we obtain $\lambda(\mathcal{G}) \leq \frac{\pi(\mathcal{F})}{r!} \leq \lambda^*$, a contradiction. □

Acknowledgements

1. As the referee pointed out, "critical" hypergraphs have been considered prior to our work. In [BBH+16], the authors describe a family of "critical" r-graphs for which they can prove that the balanced blowup of the complete graphs are the extremal examples (see, Theorem 2 in [BBH+16]. Their notion of "criticality", on top of similar conditions as (i) and (ii) in Definition 5.2.1, also requires the stability in classical setting to hold (i.e. as in Definition 3.1.3). In comparison, we exploit sufficient conditions ((iii) and (iv) in Definition 5.2.1) to obtain stability (as in Definition 3.2.2).

2. Brandt, Irwin and Jiang [BIJ] independently proved Theorems 5.1.2 and 5.1.5. Actually their results were initially announced prior to the beginning of our work. While their proofs also mainly use stability techniques, the details are different. For example, a substantial part of [BIJ] involves a symmetrization argument, paralleling [Pik08].

Chapter 6

The Turán number of the extension of a two matching

In [HK13] Hefetz and Keevash determined the the Turán number of a 3-uniform matching of size two, the extremal graphs are the balanced blowups of the complete 3-graph on five vertices. In the same paper they also conjectured what is the Turán number of an r-uniform matching of size two, for $r \geq 4$. In this chapter we affirmatively settle their conjecture.

6.1. History

Recall that ${\mathcal{M}_2}^{(r)}$ is the r-graph with two disjoint edges. In [HK13] Hefetz and Keevash found the Turán number of $\mathrm{Ext}({\mathcal{M}_2}^{(3)})$.

Theorem 6.1.1 (Hefetz, Keevash, [HK13])**.** *There exists n_0 such that for all $n \geq n_9$, the largest $\mathrm{Ext}({\mathcal{M}_2}^{(3)})$-free 3-graph on n vertices is unique and is the balanced blowup of $\mathcal{K}_5^{(3)}$.*

We study the Turán number of $\mathrm{Ext}\left({\mathcal{M}_2}^{(r)}\right)$ for $r \geq 4$. We say that a partition (A, B) of the vertex set of an r-graph $\mathcal{F}$ is a *star-partition* if for every $F \in \mathcal{F}$, $|F \cap A| = 1$. We say that $\mathcal{F}$ is a *star* if it admits a star-partition. For fixed partition (A, B), we denote by $\mathcal{S}^{(r)}(A, B)$ the maximal r-graph with a star-partition (A, B).

We denote by $\mathcal{S}^{(r)}(n)$ the r-graph on n vertices that is a star and has the maximum possible number of edges. It is easy to check that $|\mathcal{S}^{(r)}(n)| = \left(1 - \frac{1}{r}\right)^{r-1} \binom{n}{r} + o(n^r)$ and, moreover, if (A, B) is a star-partition of $\mathcal{S}^{(r)}(n)$, then $|A| = \frac{n}{r} + o(n)$. Our main result of this chapter follows.

Theorem 6.1.2. *For every $r \geq 4$, there exists $n_0 := n_0(r)$ such that for all $n \geq n_0$, the largest* $\mathrm{Ext}\left(\mathcal{M}_2{}^{(r)}\right)$*-free r-graph on n vertices is unique and is $\mathcal{S}^{(r)}(n)$.*

In difference to other graphs or families whose Turán numbers we studied in this thesis, the Turán density of $\mathrm{Ext}\left(\mathcal{M}_2{}^{(r)}\right)$, $r \geq 4$ was not known prior to our research. As for $r = 3$, Hefetz and Keevash determined the Turán density of $\mathrm{Ext}(\mathcal{M}_2{}^{(3)})$. It turns out that the Turán density of the graphs $\mathrm{Ext}\left(\mathcal{M}_2{}^{(r)}\right)$ is related to the Lagrangian of the family of *intersecting* r-graphs, that is, the graphs in which every two edges intersect. A version of the following lemma for $r = 3$ is present in [HK13] [Theorem 4.1].

Lemma 6.1.3. *For all $r \geq 3$, $\pi\left(\mathrm{Ext}\left(\mathcal{M}_2{}^{(r)}\right)\right) = r!\lambda(\mathfrak{H})$, where $\mathfrak{H}$ is the family of all intersecting r-graphs.*

Proof. By Corollary 2.2.4, $\pi\left(\mathrm{Ext}\left(\mathcal{M}_2{}^{(r)}\right)\right) = r! \sup_{\mathcal{H}} \lambda(\mathcal{H})$, over all dense $\mathrm{Ext}\left(\mathcal{M}_2{}^{(r)}\right)$-hom-free r-graphs $\mathcal{H}$. Clearly every intersecting r-graph is $\mathrm{Ext}\left(\mathcal{M}_2{}^{(r)}\right)$-hom-free. Let us show the other direction.

Suppose $\mathcal{H}$ is dense $\mathrm{Ext}\left(\mathcal{M}_2{}^{(r)}\right)$-hom-free r-graph. Suppose there are two disjoint edges F_1 and F_2 in $\mathcal{H}$. Since dense graphs cover pairs, for every pair of vertices $v_1 \in F_1$ and $v_2 \in F_2$, there exists an edge covering them, thus creating a homomorphic copy of $\mathrm{Ext}\left(\mathcal{M}_2{}^{(r)}\right)$, a contradiction. □

Thus, to find the Turán density of $\mathrm{Ext}\left(\mathcal{M}_2{}^{(r)}\right)$, one needs to determine the supremum of Lagrangians over all intersecting r-graphs. For $r = 3$, Hefetz and Keevash did so in [HK13] and showed that the maximum Lagrangian among all intersecting 3-graphs is uniquely achieved by $\mathcal{K}_5^{(3)}$. It is perhaps natural to suspect the analogous result to hold for $r \geq 4$, that is, is it true that the maximum Lagrangian over all intersecting r-graphs is achieved by $\mathcal{K}_{2r-1}^{(r)}$? As observed in [HK13] this is not true. A better candidate is the following. We say that an intersecting

r-graph is *principally intersecting* if there exists a vertex contained in all edges. Let $\mathcal{P}^{(r)}(n)$ be the complete principally intersecting r-graph on $n+1$ vertices. It is easy to see that $\lim_{n\to\infty}\lambda(\mathcal{P}^{(r)}(n)) = \frac{1}{r!}(1-r)^{r-1}$ while $\lambda(\mathcal{K}^{(r)}_{2r-1}) = \frac{1}{(2r-1)^r}\binom{2r-1}{r}$ and $\frac{1}{r!}(1-r)^{r-1} > \frac{1}{(2r-1)^r}\binom{2r-1}{r}$, for $r \geq 4$. Hefetz and Keevash conjectured that the graphs $\mathcal{P}^{(r)}(n)$ assymptotically achieve the supremum of Lagrangians over all intersecting r-graphs. In [NW16], Norin and Watts settled this conjecture in much stronger sense, which we use in the proof of Theorem 6.1.2.

Theorem 6.1.4 (Norin, Watts, [NW16])**.** *For every $r \geq 4$, there exists a constant c_r such that if $\mathcal{H}$ is an intersecting but not principally intersecting r-graphs, then $\lambda(\mathcal{H}) < \frac{1}{r!}\left(1-\frac{1}{r}\right)^{r-1} - c_r$.*

In the rest of the chapter we denote by $\mathfrak{S}$ the the family of all r-uniform stars and by $\mathfrak{F}$ the family $\mathrm{Forb}\left(\mathrm{Ext}\left(\mathcal{M}_2^{(r)}\right)\right)$. Clearly $\mathfrak{S}$ is clonable, hence by Theorem 3.8.2 if we show that the following two conditions hold:

(C1) $\mathfrak{F}$ is $\mathfrak{S}$-vertex locally stable,

(C2) the family $\mathfrak{F}^*$ of all r-graphs in $\mathrm{Forb}(\mathcal{M}_2^{(r)})$ that cover pairs is $\mathfrak{S}$-weakly weight stable,

then it would follow that $\mathfrak{F}$ is $\mathfrak{S}$-stable. Once we established this, it is easy to derive Theorem 6.1.2. In the next section we show that (C1) holds. Section 6.3 contains the final proof of Theorem 6.1.2, including the proof of (C2) which is derived from Theorem 6.1.4.

6.2. Vertex Local Stability of $\mathrm{Forb}\left(\mathrm{Ext}\left({\mathcal{M}_2}^{(r)}\right)\right)$

In this section we use the following notations for the normalized degree and edge density in $\mathcal{S}^{(r)}(n)$, $\mathrm{d}_r := \frac{1}{(r-1)!}\left(1-\frac{1}{r}\right)^{r-1}$, $\mathrm{e}_r := \frac{1}{r!}\left(1-\frac{1}{r}\right)^{r-1}$. Recall that $\mathfrak{S}$ denotes the family of all r-uniform stars and $\mathfrak{F} := \mathrm{Forb}\left(\mathrm{Ext}\left(\mathcal{M}_2^{(r)}\right)\right)$.It is easy to see $\lambda(\mathfrak{S}) = \mathrm{e}_r$. The main result of this section follows.

Theorem 6.2.1. *For every $r \geq 4$, $\mathfrak{F}$ is $\mathfrak{S}$-vertex locally stable.*

In fact, we will prove that $\mathfrak{F}$ is $\mathfrak{S}'$-vertex locally stable, where $\mathfrak{S}' \subseteq \mathfrak{S}$ is the family of all $\mathcal{S}^{(r)}(A,B)$ for all possible partitions (A,B). It is easy to see that by Lemma 3.5.8, this implies Theorem 5.2.2. Our next section contains some auxiliary lemmas which we use in the main proof several times.

6.2.1. Auxiliary Lemmas

Our first lemma is analogous to Lemma 3.9.2; it ensures that if a large $\mathcal{S} \in \mathfrak{S}$ has density close to the maximum possible (i.e. $\mathrm{e}_r\, n^r$), then the star-partition of $\mathcal{S}$ is almost as "balanced" as the one for $\mathcal{S}^{(r)}(n)$. More precisely, recall the definition of ε-trimmed from Section 3.9. We utilize this notion in this case as well. We say that a star with star-partition (A,B) is *ε-trimmed*, if $\left|\frac{|A|}{n} - \frac{1}{r}\right| \le \varepsilon$.

Lemma 6.2.2. *For every $r \ge 4$ and every $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. If $\mathcal{S} \in \mathfrak{S}$ on $n \ge n_0$ vertices such that $|\mathcal{S}| \ge (\mathrm{e}_r - \delta)\, n^r$, then $\mathcal{S}$ is ε-trimmed.*

Proof. Consider the complete principally intersecting r-graph on $|B|+1$ vertices, that is $\mathcal{P}^{(r)}(|B|)$ (note that $|B| = \Omega(n)$, otherwise the size of $\mathcal{S}$ is much smaller than assumed). For simplicity, we denote it by $\mathcal{P}$. Let v be the vertex common for all edges. Assign $\mu(v) = \frac{|A|}{n}$ and for all $u \neq v$, let $\mu(u) = \frac{1}{n}$. Clearly, $\mu \in \mathcal{M}(\mathcal{P})$. Moreover,

$$|\mathcal{S}| \le |A|\binom{|B|}{r-1} = n^r \cdot \binom{|B|}{r-1}\frac{|A|}{n} \cdot \frac{1}{n^{r-1}} = \lambda(\mathcal{P},\mu)n^r.$$

On the other hand, it is easy to see that $\mathrm{e}_r = \lim_{n\to\infty} \mathcal{P}^{(r)}(n)$ and since we assume n is sufficiently large, it follows that $\mathrm{e}_r \ge \lambda(\mathcal{P},\mu^*) - \delta$, where $\mu^* \in \mathcal{M}(\mathcal{P})$ is defined as $\mu^*(v) = \frac{1}{r}$ and $\mu^*(v) = \frac{1-1/r}{|B|}$ for every other $u \neq v$. So we obtain that

$$\lambda(\mathcal{P},\mu^*) \ge \lambda(\mathcal{P},\mu) \ge \lambda(\mathcal{P},\mu^*) - 2\delta, \tag{6.1}$$

but $\lambda(\mathcal{P},\cdot)$ is a continuos function with a unique maximum achieved at μ^*, therefore, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if (6.1) holds then $||\mu - \mu^*||_\infty \le \varepsilon$, from which it follows that $\left|\frac{|A|}{n} - \frac{1}{r}\right| \le \varepsilon$, as desired. □

Our second lemma provides a necessary condition for an r-graph to be $\operatorname{Ext}\left(\mathcal{M}_2{}^{(r)}\right)$-free. A version of this lemma for $r = 3$ can be found in [HK13].

Lemma 6.2.3. *For every $r \geq 3$ there exists $c_{6.2.3} > 0$ and n_0 such that the following holds. Let $\mathcal{F}$ be an r-graph on $n \geq n_0$ vertices, F_1 and F_2 be two disjoint edges of $\mathcal{F}$ and $v_1 \in F_1, v_2 \in F_2$. If there exists an edge F such that $F \cap (F_1 \cup F_2) = \{v_1, v_2\}$ and for every $u_1 \in F_1, u_2 \in F_2$, $\{u_1, u_2\} \neq \{v_1, v_2\}$, $|L_{\mathcal{F}}(u_1, u_2)| \geq c_{6.2.3} n^{r-3}$, then $\mathcal{F}$ contains a copy of $\operatorname{Ext}\left(\mathcal{M}_2{}^{(r)}\right)$.*

Proof. For any such u_1, u_2 the number of edges that contain both u_1 and u_2 and another vertex from $F_1 \cup F_2 \cup F$ is at most $(3r-4)n^{r-3}$. Also, for a fixed $(r-2)$-tuple disjoint from the vertices of $F_1 \cup F_2 \cup F$, the number of edges that contain u_1, u_2 and at least one vertex from this tuple is bounded by $(r-2)n^{r-3}$. Therefore, if every such pair is in at least $\left((3r-4) + (r^2-1)(r-2)\right) n^{r-3}$ edges then we can greedily pick distinct $z_1, z_2, \ldots, z_{r-2}$ such that $\{u_1, u_2, z_1, z_2, \ldots, z_{r-2}\} \in \mathcal{F}$ and moreover, for different pairs u_1, u_2 the sets $\{z_1, z_2, \ldots, z_{r-2}\}$ are disjoint. Thus, together with the edge F, these vertices induce a copy of $\operatorname{Ext}\left(\mathcal{M}_2{}^{(r)}\right)$ in $\mathcal{F}$. □

Corollary 6.2.4. *For every $r \geq 3$ there exists $c_{6.2.4} > 0$ and n_0 such that the following holds. Let $\mathcal{F}$ be an r-graph on $n \geq n_0$ vertices, F_1 and F_2 be two disjoint edges of $\mathcal{F}$. If for every $u_1 \in F_1, u_2 \in F_2$, $|L_{\mathcal{F}}(u_1, u_2)| \geq c_{6.2.4} n^{r-3}$, then $\mathcal{F}$ contains a copy of $\operatorname{Ext}\left(\mathcal{M}_2{}^{(r)}\right)$.*

We use the notation $c_e := c_e(r)$ as the maximum of the two constants derived from Lemma 6.2.3 and Corollary 6.2.4. Our next two lemmas together provide a tool for finding pairs of vertices which are in many "missing" edges.

Lemma 6.2.5. *Given $r \geq 4$ and $0 < \varepsilon < 1$, let $\mathcal{F} \in \mathfrak{F}$ and $\mathcal{S} \in \mathfrak{S}'$ be two r-graphs with $V(\mathcal{F}) = V(\mathcal{S})$ and $|\mathcal{F} \triangle \mathcal{S}| \leq \varepsilon n^r$. Suppose $\mathcal{S}$ has a star-partition (A, B) and $C \subseteq A, D \subseteq B$, $0 < \alpha, \beta < 1$ satisfy the following conditions:*

(i) $|C| \geq \alpha n$,

(ii) $|D| \geq \beta n$,

(iii) $\varepsilon < \frac{\alpha \beta^{r-1}}{2(r-1)^{r-1}}$,

then for large enough n the r-graph $\mathcal{F}' \cap \mathcal{S}'$ has a matching of size at least γn, where $\mathcal{F}' = \mathcal{F}[C \cup D]$, $\mathcal{S}' = \mathcal{S}[C \cup D]$ and $\gamma = \frac{\alpha\beta^{r-1}(r-1)!}{4(r-1)^{r-1}}$.

Proof. Let M be the maximal matching of the graph $\mathcal{F}' \cap \mathcal{S}'$. Note that

$$|\mathcal{F}'| \le |M| + |M|\binom{|C|+|D|}{r-1} \le 2|M|\frac{n^{r-1}}{(r-1)!}. \tag{6.2}$$

On the other hand,

$$|\mathcal{S}'| = |C|\binom{|D|}{r-1} \ge \alpha n\binom{\beta n}{r-1} \ge \frac{\alpha\beta^{r-1}}{(r-1)^{r-1}}n^r. \tag{6.3}$$

By initial assumptions we have $|\mathcal{F}'\triangle\mathcal{S}'| \le |\mathcal{F}\triangle\mathcal{S}| \le \varepsilon n^r$, which together with (6.2) and (6.3) implies

$$|M| \ge \left(\frac{\alpha\beta^{r-1}(r-1)!}{2(r-1)^{r-1}} - \frac{\varepsilon(r-1)!}{2}\right)n \ge \frac{\alpha\beta^{r-1}(r-1)!}{4(r-1)^{r-1}}n,$$

as desired. □

Lemma 6.2.6. *Given $r \ge 4$ and $0 < \varepsilon, \varepsilon', \alpha, \beta < 1$, let $\mathcal{F} \in \mathfrak{F}$ and $\mathcal{S} \in \mathfrak{S}'$ be two r-graphs on n vertices with $|\mathcal{F}\triangle\mathcal{S}| \le \varepsilon n^r$. Suppose $\mathcal{S}$ is ε'-trimmed, L is a set of disjoint $(r-1)$-tuples, $L \subseteq L_{\mathcal{F}}(v)$ for some vertex v, such that $V(L) \subseteq B$ and M is some matching of the r-graph $\mathcal{F} \cap \mathcal{S}$ such that $v \notin M$. If the following conditions hold:*

(i) $|L| \ge \alpha n$,

(ii) $|M| \ge \beta n$,

(iii) L *and* M *are disjoint,*

(iv) $\varepsilon \le \min\left\{\frac{\alpha\beta(1-\frac{1}{r}-\varepsilon')^{r-2}}{4(r-2)^{r-2}}, \frac{\alpha\beta(\frac{1}{r}-\varepsilon')(1-\frac{1}{r}-\varepsilon')^{r-3}}{4(r-2)^{r-2}}\right\}$,

then for large enough n there exists $u \in V(M)$ such that $|L_{\mathcal{F}}(u, v)| < c_e n^{r-3}$.

Proof. Suppose the lemma does not hold and for every $u \in V(M)$, $|L_{\mathcal{F}}(u, v)| \ge c_e n^{r-3}$. Let (A, B) be the star-partition of $\mathcal{S}$. For every $F_1 \in \mathcal{F}$ such that $F_1 = \{v\} \cup I$ for some $I \in L$ and $F_2 \in M$ by Corollary 6.2.4 there exists a pair of vertices $w_1 \in F_1$

and $w_2 \in F_2$ such that w_1 and w_2 together are in less than $c_e n^{r-3}$ edges. Note that by our initial assumption, $w_1 \neq v$. Thus, $w_1 \in B$. On the other hand, $w_2 \in A$ or $w_2 \in B$. By pigeonhole principle we may assume that at least $\frac{|M|}{2}$ many such $w_2 \in F_2$ are from A. Thus,

$$\begin{aligned}|\mathcal{F} \triangle \mathcal{S}| &> |L| \frac{|M|}{2} \left(\frac{(1 - \frac{1}{r} - \varepsilon')^{r-2} n^{r-2}}{(r-2)^{r-2}} - c_e n^{r-3} \right) \\ &\geq \frac{\alpha\beta(1 - \frac{1}{r} - \varepsilon')^{r-2}}{4(r-2)^{r-2}} n^r \\ &\geq \varepsilon n^r,\end{aligned}$$

which is a contradiction. □

Remark 6.2.1. *Note that for Lemma 6.2.5 and Lemma 6.2.6 to hold it is enough to require that $\varepsilon \ll \alpha, \beta \ll 1$. Thus, these are the conditions that we ensure in the applications of the lemmas.*

6.2.2. Proof of Theorem 6.2.1

Recall that $\mathfrak{S}'$ is the subfamily of $\mathfrak{S}$ of all $\mathcal{S}^{(r)}(A, B)$, for all possible partitions (A, B). It is easy to see that $m(\mathfrak{S}') = m(\mathfrak{S})$, because in particular, $\mathfrak{S}'$ contains $\mathcal{S}^{(r)}(n)$ for every n. Moreover, if we have a star $\mathcal{S}$ with partition (A, B), then $\mathcal{S} \subseteq \mathcal{S}^{(r)}(A, B)$, hence we can apply Lemma 3.5.8 to $\mathfrak{F}, \mathfrak{S}$ and $\mathfrak{S}'$ and obtain that if we prove that $\mathfrak{F}$ is $\mathfrak{S}'$-vertex locally-stable, then it would follow that $\mathfrak{F}$ is also $\mathfrak{S}$-vertex locally stable. To escape the notation overload, we will use the notation $\mathfrak{S}$ instead of more precise $\mathfrak{S}'$.

So we want to show that there exist $\varepsilon, \alpha > 0$ and $n_0 \in \mathbb{N}$ such that if $\mathcal{F} \in \mathfrak{F}$ with $\mathrm{v}(\mathcal{F}) = n \geq n_0$, such that

- $d_{\mathfrak{S}}(\mathcal{F}) \leq \varepsilon n^r$,
- $|L_{\mathcal{F}}(v)| \geq (\mathrm{d}_r - \varepsilon) n^{r-1}$ for every $v \in V(\mathcal{F})$,

then

$$|\mathcal{F}| \leq m(\mathfrak{S}, n) - \alpha d_{\mathfrak{S}}(\mathcal{F}).$$

In fact, we show that α can be taken to be one. Consider such an r-graph $\mathcal{F} \in \mathfrak{F}$ and let $\mathcal{S} := \mathcal{S}^{(r)}(A, B) \in \mathfrak{S}$ with $V(\mathcal{S}) = A \cup B$ and $|\mathcal{F} \triangle \mathcal{S}| = d_{\mathfrak{S}}(\mathcal{F})$. We call an edge $F \in \mathcal{F}$ *A-bad* if $|F \cap A| \geq 2$, *B-bad* if $F \subseteq B$ and simply *bad* if it is either A-bad or B-bad. We call an edge $F \in \mathcal{F}$ *good* if it is not bad. We say that an edge F is *missing* if $F \in \mathcal{S}$ but $F \notin \mathcal{F}$. Since the distance between $\mathcal{F}$ and $\mathcal{S}$ is "small" we expect the number of bad edges to be "small" as well. In fact, we show that there are no bad edges in $\mathcal{F}$ at all. It is easy to see that this finishes the proof. Indeed, in that case we have that $\mathcal{F} \setminus \mathcal{S} = \emptyset$, thus

$$|\mathcal{F}| = |\mathcal{S}| - |\mathcal{S} \setminus \mathcal{F}| \leq m(\mathfrak{S}, n) - |\mathcal{F} \triangle \mathcal{S}|,$$

as desired. The main idea behind proving that there cannot be any bad edges is that every bad edge forces "a lot of" edges to be missing to preserve $\mathrm{Ext}\left(\mathcal{M}_2{}^{(r)}\right)$-freeness, thus contributing to the distance between $\mathcal{F}$ and $\mathcal{S}$. However, the formal proof of this statement requires several steps.

First we prove that every vertex is in "many" good edges (Section 6.2.3). This is relatively easy to do; we only use the fact that $|\mathcal{F} \triangle \mathcal{S}|$ is "small". Next we show that there are no bad vertices, where a vertex is *bad* if it is contained in "many" bad edges. This is where the main bulk of our work is concentrated; here we use the first step and the fact that every vertex has very similar degrees in $\mathcal{F}$ and $\mathcal{S}$ (Section 6.2.4). Finally, we derive that there are no bad edges (Section 6.2.5). Now we are ready to start the formal proof.

Let $\varepsilon, \varepsilon_{6.2.2}, \mu > 0$ be real numbers satisfying $\varepsilon \ll \mu \ll \varepsilon_{6.2.2} \ll 1$. Let $\delta_{6.2.2}$ be derived from Lemma 6.2.2 applied with $\varepsilon_{6.2.2}$. In addition to all the constraints imposed on ε, we also require that $\varepsilon \leq \delta_{6.2.2}/3$. Let $\mathcal{F}$ and $\mathcal{S}$ be as discussed in the sketch of the proof. We may assume that $|\mathcal{F}| \geq (\mathrm{e}_r - 2\varepsilon)\, n^r$, since otherwise we are done. It follows that

$$|\mathcal{S}| \geq |\mathcal{F}| - \varepsilon n^r \geq (\mathrm{e}_r - 3\varepsilon)\, n^r \geq (\mathrm{e}_r - \delta_{6.2.2}) n^r.$$

So we can apply Lemma 6.2.2 to $\mathcal{S}$ and obtain that it is $\varepsilon_{6.2.2}$-trimmed. We say that vertex v is *bad* if it is contained in at least $2\mu n^{r-1}$ bad edges. For a vertex v,

$g(v)$ and $b(v)$ will denote the number of good and bad edges that v is contained in, respectively. More generally, we also use the notation $g(I)$ and $b(I)$ for any tuple I to denote the number of good and bad edges that I is in. We say that an edge F is (k, A)-bad for some $2 \le k \le r$, if $|F \cap A| = k$.

6.2.3. Every Vertex Is In Many Good Edges

Lemma 6.2.7. *For every $v \in V$, $g(v) \ge \mu n^{r-1}$.*

Proof. Suppose not and there exists $v \in V$ with $g(v) < \mu n^{r-1}$. We consider two cases.

Case 1: $v \in A$. If v is in at least μn^{r-1} $(2, A)$-bad edges, then consider a new star-partition (A', B') where $A' := A \setminus \{v\}$ and $B' := B \cup \{v\}$. Note that edges that do not contain v preserve their "goodness" or "badness" with respect to the new partition. Among the edges that contain v, the ones that are $(2, A)$-bad become good, edges that are good become B'-bad and all the edges that are (k, A)-bad for some $k \ge 3$ become $(k-1, A')$-bad. Thus, the total number of bad edges with respect to star with the star-partition (A', B') are less than the ones with $\mathcal{S}$. This contradicts to the choice of $\mathcal{S}$, thus v must be contained in less than μn^{r-1} $(2, A)$-bad edges. Now the following is an upper bound for the total number of (k, A)-bad edges for all $k \ge 3$ that v may be contained in.

$$\begin{aligned}\sum_{k=2}^{r-1} \binom{|A|}{k}\binom{|B|}{r-1-k} &= \sum_{k=0}^{r-1} \binom{|A|}{k}\binom{|B|}{r-1-k} - \binom{|B|}{r-1} - \binom{|A|}{1}\binom{|B|}{r-2} \\ &\le \frac{n^{r-1}}{(r-1)!} - \frac{\left(1-\frac{1}{r}-\varepsilon_{6.2.2}\right)^{r-1} n^{r-1}}{(r-1)!} - \frac{\left(\frac{1}{r}-\varepsilon_{6.2.2}\right)\left(1-\frac{1}{r}-\varepsilon_{6.2.2}\right)^{r-2} n^{r-1}}{(r-2)!} \\ &\le (\mathrm{d}_r - \varepsilon - 2\mu) n^{r-1},\end{aligned}$$

for large enough n. (See Section 6.4 for details.) Thus we obtain

$$\begin{aligned}|L_{\mathcal{F}}(v)| &\le g(v) + b(v) < \mu n^{r-1} + (d(r) - \varepsilon - 2\mu) n^{r-1} + \mu n^{r-1} \\ &\le (\mathrm{d}_r - \varepsilon) n^{r-1},\end{aligned}$$

a contradiction.

Case 2: $v \in B$**.** This case is very similar to the previous one. We can assume that v is in less than μn^{r-1} B-bad edges. Otherwise, consider a new star-partition (A', B'), where $A' := A \cup \{v\}$ and $B' := B \setminus \{v\}$; the total number of bad edges is less with respect to (A', B') than with (A, B), which contradicts to the choice of $\mathcal{S}$. Now we can bound the total number of A-bad edges that v is in, as follows.

$$\begin{aligned} b(v, A) = \sum_{k=2}^{r-1} b(v, A, k) &\leq \sum_{k=2}^{r-1} \binom{|A|}{k} \binom{|B|}{r-1-k} \\ &< (\mathrm{d}_r - \varepsilon - 2\mu) n^{r-1}, \end{aligned}$$

just as in the previous case. And similarly, $|L_{\mathcal{F}}(v)| \leq g(v) + b(v) < (\mathrm{d}_r - \varepsilon) n^{r-1}$. This finishes the proof of the lemma. □

Now let us define *the good neighbourhood* of a vertex v, denoted by $N_g(v)$ as follows. If $v \in A$, then let $N_g(v)$ be all the vertices u in B such that $g(u, v) \geq c_e n^{r-3}$. If $v \in B$, then $N_g(v)$ is the set of all vertices in $A \cup B$ with the same property. In comparison, note that in $\mathcal{S}$ for every vertex v in A, $N_g(v) = B$ and for $v \in B$, $N_g(v) = A \cup B \setminus \{v\}$. Because of the small distance between $\mathcal{F}$ and $\mathcal{S}$, it is natural to expect the good neighbourhoods of vertices to have relatively large size in $\mathcal{F}$ as well. We are able to derive this easily from Lemma 6.2.7.

Corollary 6.2.8. *For every* $v \in V$*,* $|N_g(v)| \geq (1 - \frac{1}{r}) \mu n$*.*

Proof. We can use a simple counting argument as follows. For both $v \in A$ or $v \in B$, we can write

$$\mu n^{r-1} \leq g(v) \leq \sum_{u \in N_g(v)} g(v, u) + \sum_{u \notin N_g(v)} g(v, u) \leq |N_g(v)| n^{r-2} + c_e n^{r-2},$$

and therefore,

$$|N_g(v)| \geq \mu n - c_e \geq \left(1 - \frac{1}{r}\right) \mu n.$$

for sufficiently large n. Note that this bound can be easily improved but for our purposes we only need to show that $N_g(v)$ contains positive proportion of vertices, meaning of order n. $\square$

Corollary 6.2.9. *If $v \in A$, then there exists a set of disjoint $(r-1)$-tuples, $L \subseteq L_{\mathcal{F}}(v)$, such that $V(L) \subseteq B$ and $|L| \geq \frac{\mu}{r}n$.*

Proof. Let L be the maximal set of pairwise disjoint $(r-1)$-tuples $I \subseteq B$ such that $\{v\} \cup I$ are good edges. Then, using Lemma 6.2.7 we get

$$\mu n^{r-1} \leq |L| + (r-1)|L|n^{r-2} \leq r|L|n^{r-2},$$

thus $|L| \geq \frac{\mu}{r}n$, as desired. $\square$

6.2.4. No Bad Vertices Exist.

Analogous to the notion of the good neighbourhood, we can consider the *bad neighbourhood* of a vertex v, denoted by $N_b(v)$, as follows. For $v \in A$, let $N_b(v)$ to be all the vertices $u \in A$ such that $b(u,v) \geq c_e n^{r-3}$. For $v \in B$, let $N_b(v)$ to be all the vertices $u \in A \cup B$ such that $b(u,v) \geq c_e n^{r-3}$. Note that in $\mathcal{S}$ no vertex has a bad neighbourhood. So we would like to show that in $\mathcal{F}$ these sets have relatively small sizes. This is easy to do, the proof is analogous to the proof of Corollary 6.2.8.

Lemma 6.2.10. *If v is a bad vertex then $|N_b(v)| \geq 2\left(1-\frac{1}{r}\right)\mu n$.*

However, just Lemma 6.2.10 itself is not enough to show that there are no bad vertices. We need to consider the cases when a vertex is in many A-bad or B-bad edges separately. For these purposes we use the following lemma whose proof goes along the lines of the Corollary 6.2.8 and Corollary 6.2.9 together. For a vertex v, let $b_A(v)$ and $b_B(v)$ denote the number of A-bad and B-bad edges v is contained in, respectively.

Lemma 6.2.11. *If $b_B(v) \geq \mu n^{r-1}$, then $|N_b(v) \cap B| \geq (1-\frac{1}{r})\mu n$ and there exists a set of disjoint $(r-1)$-tuples, $L \subseteq L_{\mathcal{F}}(v)$, such that $V(L) \subseteq B$ and $|L| \geq \frac{\mu}{r}n$.*

And now we are ready to prove the main result of this section.

Lemma 6.2.12. *There are no bad vertices.*

Proof. Suppose there exists a bad verttex v. We consider two cases.

Case 1: $v \in A$ or ($v \in B$ and $b_B(v) \geq \mu n^{r-1}$).

By Corollary 6.2.9 and Lemma 6.2.11 in this case there exists $L \subseteq L_{\mathcal{F}}(v)$ with $V(L) \subseteq B$ such that $|L| \geq \frac{\mu}{r}n$. We may assume $|L| = \frac{\mu}{2r}n$ by choosing a smaller subset of it. Our plan is to apply Lemma 6.2.6 with v, L and some matching M of $\mathcal{F} \cap \mathcal{S}$ that is disjoint from L.

We need to find a large enough matching M such that for every $u \in V(M)$, $|L_{\mathcal{F}}(u,v)| \geq c_e n^{r-3}$. This would contradict to the conclusion of Lemma 6.2.6, thus proving that v cannot be a bad vertex. So in the next steps, we are going to find such a matching M using Lemma 6.2.5.

If $v \in A$ then let $C = N_b(v)$ and choose $D \subseteq N_g(v)$ with $|D| = (1-\frac{1}{r})\frac{\mu}{2}n$ disjoint from L. If $v \in B$, let $C = N_g(v)$ and choose $D \subseteq N_b(v)$ with $|D| = (1-\frac{1}{r})\frac{\mu}{2}n$. In both cases this is possible to do since the number of vertices that are contained in L is bounded by $(1-\frac{1}{r})\frac{\mu}{2}n$ and for $v \in A$, $|N_g(v)| \geq (1-\frac{1}{r})\mu n$ while for $v \in B$, $|N_b(v)| \geq (1-\frac{1}{r})\mu n$.

Now we are ready to apply Lemma 6.2.5 with C, D and a matching M of the subgraph $\mathcal{F}[C \cup D] \cap \mathcal{S}[C \cup D]$ of size at least γn, where we can guarantee $\varepsilon \ll \gamma$ by our choice of $\varepsilon \ll \mu$. Next we apply Lemma 6.2.6 with L and M and derive a contradiction, since for every $u \in V(M)$, $|L_{\mathcal{F}}(u,v)| \geq c_e n^{r-3}$. Note that here we are using the fact that $\varepsilon \ll \mu, \gamma$.

Case 2: $v \in B$ and $b(v,B) < \mu n^{r-1}$.

Our goal is to find a subgraph of $\mathcal{F}$ which contributes to the edit distance with $\mathcal{S}$ more than εn^r. This contradiction would finish the proof. Here is an outline.

(1) Let $\eta := \frac{\mu}{2(\frac{1}{r}+\varepsilon_{6.2.2})^2}$. We construct an auxilary graph G on A where $(a_1, a_2) \in E(G)$ if and only if $b(v, a_1, a_2) \geq \eta n^{r-3}$. We show that G is dense (Claim 6.2.13).

(2) Next we consider the vertices in G of large degree, namely, we let $L = \{a \in A | d_G(a) \geq \eta |A|\}$. It is easy to see that $|L| \geq \eta |A|$.

(3) We consider a maximal matching M of $\mathcal{F}(A, B \setminus \{v\})$, let us denote $A_M = V(M) \cap A$ and $B_M = V(M) \cap B$. We show that M must cover almost all the

vertices of A. (Claim 6.2.14). It follows that M has linear intersection both with L and with the neighbourhood of every vertex in L (Corollary 6.2.15).

(4) Then we show that for some pairs of edges of M we can find a pair of vertices which are in many missing edges (Claim 6.2.16). And finally, using the bounds on L and M we can derive that the number of such pairs is large, thus showing that these missing edges contibute to the edit distance between $\mathcal{F}$ and $\mathcal{S}$ more than we are allowed to have, that is, εn^r.

Claim 6.2.13. $|E(G)| \geq \eta |A|^2$.

Proof. Since $b(v, B) < \mu n^{r-1}$, it follows that $b(v, A) \geq \mu n^{r-1}$. Hence we can bound the number of A-bad edges that v is in from above as follows.

$$\begin{aligned}\mu n^{r-1} \leq b(v, A) &\leq \sum_{u_1, u_2 \in A} b(v, u_1, u_2) \\ &= \sum_{(u_1,u_2) \in E(G)} b(v, u_1, u_2) + \sum_{(u_1,u_2) \notin E(G)} b(v, u_1, u_2) \\ &\leq |E(G)| n^{r-3} + \eta n^{r-2}.\end{aligned}$$

Therefore, for sufficiently large n,

$$|E(G)| \geq \mu n^2 - \eta n \geq \frac{\mu}{2} n^2 = \eta \left(\frac{1}{r} + \varepsilon_{6.2.2} \right)^2 n^2 \geq \eta |A|^2.$$

□

Claim 6.2.14. $|M| \geq (1 - \frac{\eta}{2})|A|$.

Proof. Suppose the claim does not hold, and there exists a set $C \subseteq A$ such that $|C| \geq \frac{\eta}{2}|A|$ and C is uncovered by M. It follows that if we let $D := B \setminus B_M$ then the graph $\mathcal{F}[C \cup D] \cap \mathcal{S}[C \cup D]$ is empty since otherwise we could have extended M.

Since $|B_M| < (r-1)(1-\frac{\eta}{2})|A|$ we have

$$\begin{aligned}|D| &= |B| - |B_M| \\ &> |B| - (r-1)\left(1-\frac{\eta}{2}\right)|A| \\ &\geq \left(1-\frac{1}{r}-\varepsilon_{6.2.2}\right)n - (r-1)\left(1-\frac{\eta}{2}\right)\left(\frac{1}{r}-\varepsilon_{6.2.2}\right)n \\ &= \left(\varepsilon_{6.2.2}(r-2) + \frac{\eta}{2}\left(1-\frac{1}{r}-\varepsilon_{6.2.2}(r-1)\right)\right)n := \xi n.\end{aligned}$$

It follows that

$$\begin{aligned}|\mathcal{F}\triangle\mathcal{S}| &\geq |C|\binom{|D|}{r-1} \\ &\geq \frac{\eta}{2}|A|\frac{|D|^{r-1}}{(r-1)^{r-1}} \\ &\geq \frac{\eta(\frac{1}{r}-\varepsilon_{6.2.2})\xi^{r-1}}{2(r-1)^{r-1}}n^r \\ &> \varepsilon n^r,\end{aligned}$$

a contradiction. □

Corollary 6.2.15. $|L \cap A_M| \geq \frac{\eta}{2}|A|$ *and* $|N_G(a) \cap A_M| \geq \frac{\eta}{2}|A|$ *for every* $a \in L$.

Claim 6.2.16. *For every* $a_1, a_2 \in A_M$ *with* $a_i \in F_i \in M$, $i = 1, 2$, *if* $a_1a_2 \in E(G)$ *then there exists a pair of vertices* $\{u, w\} \neq \{a_1, a_2\}$ *with* $u \in F_1, w \in F_2$ *such that* $|L_{\mathcal{F}}(u, w)| < c_e n^{r-3}$.

Proof. We claim that there exists an edge which intersects F_1 and F_2 only at a_1 and a_2, respectively. This is because $b(v, a_1, a_2) \geq \eta n^{r-3}$ by definition of G. On the other hand, the number of those edges that contain v, a_1, a_2 and some other vertex from $F_{a_1} \cup F_{a_2}$ is bounded by $(2r-2)n^{r-4} < \eta n^{r-3}$, for sufficiently large n. So there exists an edge that contains v, a_1, a_2 and otherwise is disjoint from $F_1 \cup F_2$. Let us denote this edge by F_{a_1,a_2}. Now we can apply Lemma 6.2.2 with F_1, F_2 and F_{a_1,a_2} and obtain such a pair $\{u, w\}$ with the desired property. □

For every $a \in L \cap A_M$ we can apply Claim 6.2.16 with every $a' \in N_G(a) \cap A_M$, thus obtaining at least $\eta|A|/2$ pairs of vertices $\{u, w\} \neq \{a, a'\}$ such that $|L_{\mathcal{F}}(u, w)| <$

$c_e n^{r-3}$. Using Corollary 6.2.15, we know that the number of such pairs is at least $\frac{|L\cap A_M|\times\eta|A|/2}{2}$ (note that here we are using the fact that M is a matching and every pair (a,a') is counted only twice). Note that for every pair (a,a') there are three possible cases for the pair $\{u,w\}$; either both u,w are from B or $u=a$ and $w\in B$ or $u\in B$ and $v=a'$. By pigenhole principle and, without loss of generality, we can assume that in at least one third of the cases the pair is from B. Thus, for at least $\frac{\eta^2|A|^2}{12}$ pairs $\{u,w\}$, we have

$$\begin{aligned}|L_{\mathcal{S}}(u,w)\setminus L_{\mathcal{F}}(u,w)| &\geq |A|\binom{|B|}{r-3}-c_e n^{r-3}\\ &\geq \frac{\left(\frac{1}{r}-\varepsilon_{6.2.2}\right)(1-\frac{1}{r}-\varepsilon_{6.2.2})^{r-3}n^{r-2}}{(r-2)^{r-3}}-c_e n^{r-3}\\ &\geq \frac{\left(\frac{1}{r}-\varepsilon_{6.2.2}\right)(1-\frac{1}{r}-\varepsilon_{6.2.2})^{r-3}n^{r-2}}{2(r-2)^{r-3}},\end{aligned}$$

assuming n is large enough. Therefore,

$$\begin{aligned}|\mathcal{F}\triangle\mathcal{S}| &\geq \eta^2\frac{|A|^2}{12}\cdot\frac{\left(\frac{1}{r}-\varepsilon_{6.2.2}\right)(1-\frac{1}{r}-\varepsilon_{6.2.2})^{r-3}}{2(r-2)^{r-3}}n^{r-2}\\ &\geq \eta^2\frac{\left(\frac{1}{r}-\varepsilon_{6.2.2}\right)^3(1-\frac{1}{r}-\varepsilon_{6.2.2})^{r-3}}{24(r-2)^{r-3}}n^r\\ &> \varepsilon n^r,\end{aligned}$$

a contradiction. This shows that v cannot be a bad vertex, thus concluding the proof. □

Now that we have proved that there are no bad vertices, we can show that for every vertex $v\in V$ there are only small number of vertices with whom v is in many missing edges. For a vertex $v\in A$, we define its *missing neighbourhood*, denoted by $N_m(v)$, to be all the vertices $u\in B$ such that $g(u,v)<c_e n^{r-3}$. For vor $v\in B$, $N_m(v)$ is the set of all vertices $u\in V$ such that $|L_{\mathcal{F}}(u,v)|<c_e n^{r-3}$. Note that we can relate the good and missing neighbourhoods of vertices as follows. For $v\in A$, $N_m(v)=B\setminus N_g(v)$ and for $v\in B$, $N_m(v)=(A\cup B)\setminus\overline{N_g(v)}$, where $\overline{N_g(v)}=N_g(v)\cup\{v\}$. In comparison, note that in the graph $\mathcal{S}$ the missing neighbourhoods are empty for

all vertices since for $v \in A$, $N_g(v) = B$ and for $v \in B$, $\overline{N_g(v)} = A \cup B$. Thus, it is natural to expect that the corresponding sets are small in $\mathcal{F}$. That is what we prove next.

Corollary 6.2.17. *For every $v \in V$, $|N_m(v)| \leq \frac{1}{12r!}\left(1-\frac{1}{r}\right)^r n$.*

Proof. Assume $v \in A$. Let $B' = B \backslash N_m(v)$. Since we have proved that $b(v) < 2\mu n^{r-1}$ we have that

$$g(v) = |L_{\mathcal{F}}(v)| - b(v) \geq \left(\frac{\left(1-\frac{1}{r}\right)^{r-1}}{(r-1)!} - \varepsilon - 2\mu\right) n^{r-1}$$

On the other hand, we can upper bound all the good edges that contain v and some other vertex $u \in N_m(v)$ easily by $|N_m(v)|c_e n^{r-3}$. The number of all the remaining good edges that contain v is bounded by $\binom{|B'|}{r-1}$. Hence we obtain

$$g(v) \leq |N_m(v)|c_e n^{r-3} + \binom{|B'|}{r-1} \leq \varepsilon n + \frac{|B'|^{r-1}}{(r-1)!}$$

So we can put together the upper and lower bounds on $g(v)$ and obtain that

$$\begin{aligned}\frac{|B'|^{r-1}}{(r-1)!} &\geq \left(\frac{\left(1-\frac{1}{r}\right)^{r-1}}{(r-1)!} - 2\varepsilon - 2\mu\right) n^{r-1} \\ &\geq \frac{1}{(r-1)!}\left(\left(1-\frac{1}{r}\right)^{r-1} - 2(r-1)!(\varepsilon+\mu)\right) n^{r-1} \\ &\geq \frac{1}{(r-1)!}\left(1-\frac{1}{r} - 2(r-1)!(\varepsilon+\mu)\right)^{r-1} n^{r-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}|N_m(v)| &= |B| - |B'| \\ &\leq \left(1-\frac{1}{r}+\varepsilon_{6.2.2}\right) n - \left(1-\frac{1}{r} - 2(r-1)!(\varepsilon+\mu)\right) n \\ &= (\varepsilon_{6.2.2} + 2(\varepsilon+\mu)(r-1)!)n \\ &\leq \frac{1}{12r!}\left(1-\frac{1}{r}\right)^r n.\end{aligned}$$

The analysis of $v \in B$ is analogous but slightly more technical since we need to consider the sets $N_m(v) \cap A$ and $N_m(v) \cap B$, however, the same inequality holds by our choice of constants $\varepsilon, \varepsilon_{6.2.2}, \mu \ll 1$. We skip the proof of this case for brevity. □

The non-existence of bad vertices also allows us to give an absolute bound on the number of disjoint $(r-1)$-tuples that are in good edges with a vertex in A, thus improving the bound given by Corollary 6.2.9. We will use this in the next section.

Corollary 6.2.18. *For every vertex $v \in A$ there exists a set of disjoint $(r-1)$-tuples, L, such that every for every $I \in L$, $I \cup \{v\}$ is a good edge and $|L| \geq \frac{1}{r!}\left(1-\frac{1}{r}\right)^r n$.*

Proof. Let L be the maximal set of pairwise disjoint $(r-1)$-tuples $I \subseteq B$ such that $\{v\} \cup I$ are good edges in $\mathcal{F}$. Then, on one hand, we have

$$g(v) = |L_{\mathcal{F}}(v)| - b(v) \geq (d(r) - \varepsilon - 2\mu)n^{r-1}.$$

On the other hand,

$$g(v) \leq |L| + (r-1)|L|n^{r-2} \leq r|L|n^{r-2}.$$

Thus,

$$|L| \geq \frac{1}{r}(\mathrm{d}_r - 2\mu - \varepsilon)n^r \geq \frac{1}{r}\left(1-\frac{1}{r}\right)\mathrm{d}_r\, n = \frac{1}{r!}\left(1-\frac{1}{r}\right)^r n.$$

□

6.2.5. No Bad Edges Exist.

Lemma 6.2.19. *There are no bad edges.*

Proof. Suppose there is one, say F. First suppose F is a B-bad edge, that is, $F \subseteq B$. Let $C = A \setminus \cup_{v\in F}\overline{N_m(v)}, D = B \setminus \cup_{v\in F}\overline{N_m(v)}$. We claim that there exists a good edge F' contained in $\mathcal{F}[C \cup D]$. In particular, note that this edge will be disjoint from F. Indeed, by Lemma 6.2.17, we have $|C| \geq \left(\frac{1}{r} - \varepsilon_{6.2.2} - \theta\right)n$ and $|D| \geq \left(1 - \frac{1}{r} - \varepsilon_{6.2.2} - \theta\right)n$, where $\theta = \frac{1}{12(r-1)!}\left(1-\frac{1}{r}\right)^r$. And hence if there is no such an edge then

$$\begin{aligned}|\mathcal{F}\triangle\mathcal{S}| &\geq |\mathcal{F}[C\cup D]\triangle\mathcal{S}[C\cup D]|\\ &\geq |C|\binom{|D|}{r-1}\\ &\geq \left(\frac{1}{r}-\varepsilon_{6.2.2}-\theta\right)\frac{\left(1-\frac{1}{r}-\varepsilon_{6.2.2}-\theta\right)^{r-1}}{(r-1)^{r-1}}n^r\\ &> \varepsilon n^r,\end{aligned}$$

a contradiction. Now we can apply Corollary 6.2.4 to F and F' and obtain a pair of vertices $u \in F$ and $w \in F'$ with $|L_{\mathcal{F}}(u,w)| < c_e n^{r-3}$. But this pair cannot exist by the definition of C and D.

So we may assume F is an A-bad edge. Hence it contains at least two vertices from A, let them be $a_1, a_2 \in F$. We show that there exist two disjoint good edges F_1 and F_2 such that F_1 and F_2 intersect F only at a_1 and a_2 and for both $i = 1, 2$, every $v \in F_i \setminus \{a_i\}$, the condition $v \notin \cup_{u\in F_{3-i}}\overline{N_m(u)}$ holds. Observe that if such edges exists then we can apply Lemma 6.2.3 to F, F_1 and F_2 and obtain a pair of vertices $v \in F_1, w \in F_2$ such that $\{v,w\} \neq \{a_1,a_2\}$ and $|L_{\mathcal{F}}(u,w)| < c_e n^{r-3}$. But this contradicts to the properties of F_1 and F_2. So it remains to prove the existence of such edges.

Let L_1 and L_2 be the maximal number of disjoint $(r-1)$-tuples that are in a good edge with a_1 and a_2, respectively. By Corollary 6.2.18, $|L_i| \geq \frac{1}{r!}\left(1-\frac{1}{r}\right)^r n$. Now let $D := B \setminus (N_m(a_1) \cup N_m(a_2) \cup F)$. By Lemma 6.2.17,

$$\begin{aligned}|D| &\geq |B| - \frac{1}{6r!}\left(1-\frac{1}{r}\right)^r n - r\\ &\geq \left(1-\frac{1}{r}-\varepsilon_{6.2.2}\right)n - \frac{1}{6r!}\left(1-\frac{1}{r}\right)^r n - \varepsilon_{6.2.2}n\\ &\geq \left(1-\frac{1}{r}+\varepsilon_{6.2.2}\right)n - \frac{1}{2r!}\left(1-\frac{1}{r}\right)^r n\\ &\geq |B| - \frac{1}{2}|L_i|.\end{aligned}$$

Hence, D contains at least half of the $(r-1)$-tuples of each L_i. Thus, for both

$i = 1, 2$ we can find $L'_i \subset L_i$ such that $V(L'_i) \subset D$, $|L'_i| = \frac{|L_i|}{4}$ and L'_1 and L'_2 are disjoint. If we cannot find the desired edges F_1 and F_2, it means that for every pair of $(r-1)$-tuples $I_1 \in L'_1$ and $I_2 \in L'_2$ there exist some pair of vertices $u \in I_1$ and $w \in I_2$, such that $|L_{\mathcal{F}}(u, w)| < c_e n^{r-3}$. On the other hand,

$$|L_{\mathcal{S}}(u,w)| = |A|\binom{|B|}{r-2} \geq \left(\frac{1}{r} - \varepsilon_{6.2.2}\right)\left(1 - \frac{1}{r} - \varepsilon_{6.2.2}\right)^{r-3} \frac{1}{(r-2)^{r-2}} n^{r-2}.$$

Therefore,

$$\begin{aligned} |\mathcal{F} \triangle \mathcal{S}| &\geq |L'_1||L'_2| \left(\left(\frac{1}{r} - \varepsilon_{6.2.2}\right)\left(1 - \frac{1}{r} - \varepsilon_{6.2.2}\right)^{r-3} \frac{n^{r-2}}{(r-2)^{r-2}} - c_e n^{r-3} \right) \\ &\geq \frac{1}{8(r!)^2}\left(1 - \frac{1}{r}\right)^{2r} \left(\frac{1}{r} - \varepsilon_{6.2.2}\right)\left(1 - \frac{1}{r} - \varepsilon_{6.2.2}\right)^{r-3} \frac{n^r}{(r-2)^{r-2}} \\ &> \varepsilon n^r, \end{aligned}$$

a contradiction. So the desired edges F_1 and F_2 exist, and we can apply Lemma 6.2.3 as described earlier. So there are no A-bad edges and hence, no bad edges at all. □

As we have discussed earlier, this finishes the proof. □

6.3. The proof of Theorem 6.1.2

As we discussed earlier, using Theorem 3.8.2, we need to verify that conditions

(C1) $\mathfrak{F}$ is $\mathfrak{S}$-vertex locally stable,

(C2) the family $\mathfrak{F}^*$ of all r-graphs in $\mathrm{Forb}(\mathcal{M}_2^{(r)})$ that cover pairs is $\mathfrak{S}$-weakly weight stable,

hold. The first one does as we showed in the previous section (Theorem 6.2.1). Now let us show that the family $\mathfrak{F}^*$ is $\mathfrak{S}$-weakly weight stable. Note that every graph in $\mathfrak{F}^*$ is intersecting. We need to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{F} \in \mathfrak{F}^*$, $\mu \in \mathcal{M}(\mathcal{F})$ with $\lambda(\mathcal{F}, \mu) \geq \lambda(\mathfrak{S}) - \delta$ then $d_{\mathfrak{S}}(F, \mu) \leq \varepsilon$. Choose $\delta = c_r$, where c_r is derived from Theorem 6.1.4. Then if

$$\lambda(\mathcal{F}, \mu) \geq \lambda(\mathfrak{S}) - \delta = \frac{1}{r!}\left(1 - \frac{1}{r}\right)^{r-1} - c_r,$$

then $\mathcal{F}$ is principally intersecting. Let $v \in V(\mathcal{F})$ be the vertex contained in all the edges of $\mathcal{F}$, then clearly $(\{v\}, V(\mathcal{F}) \setminus \{v\})$ is a star-partition for $\mathcal{F}$, thus $\mathcal{F} \in \mathfrak{S}$, hence, $d_{\mathfrak{S}}(F, \mu) = 0$.

So by Theorem 3.8.2, $\mathfrak{F}$ is $\mathfrak{S}$-stable. In particular, for all sufficiently large n, if $\mathcal{F} \in \mathfrak{F}$ and $|\mathcal{F}| = m(\mathfrak{F}, n)$ then $\mathcal{F} \in \mathfrak{S}$. But $\mathcal{S}^{(r)}(n) \in \mathfrak{S}$ gives a lower bound on $m(\mathfrak{F}, n)$ and moreover, $|\mathcal{S}^{(r)}(n)| = m(\mathfrak{S}, n)$. Thus, $|\mathcal{F}| = m(\mathfrak{F}, n) = m(\mathfrak{S}, n) = \mathcal{S}^{(r)}(n)$ and, furthermore,$\mathcal{F}$ is isomorphic to $\mathcal{S}^{(r)}(n)$. This finishes the proof. □

6.4. Proof of A Numerical Lemma

Lemma 6.4.1. *For every $r \geq 4$, there exist constants $\varepsilon, \beta \in (0,1)$, $\beta \ll \varepsilon$ such that*

$$\frac{1}{(r-1)!} - \frac{\left(1-\frac{1}{r}-\varepsilon\right)^{r-1}}{(r-1)!} - \frac{\left(\frac{1}{r}-\varepsilon\right)\left(1-\frac{1}{r}-\varepsilon\right)^{r-2}}{(r-2)!} \leq \frac{\left(1-\frac{1}{r}\right)^{r-1}}{(r-1)!} - \beta.$$

Proof. Let us rewrite the inequality as follows.

$$1 + \beta(r-1)! \leq 2\left(1-\frac{1}{r}-\varepsilon\right)^{r-1} + \left(1-\frac{1}{r}\right)^{r-1} - \varepsilon(r-2)\left(1-\frac{1}{r}-\varepsilon\right)^{r-2}.$$

We can choose ε small enough such that

$$1 < 2\left(1-\frac{1}{r}-\varepsilon\right)^{r-1} + \left(1-\frac{1}{r}\right)^{r-1}. \tag{6.4}$$

This is true because for fixed r, if we consider the function $f(\varepsilon) = 2\left(1-\frac{1}{r}-\varepsilon\right)^{r-1} + \left(1-\frac{1}{r}\right)^{r-1}$, then $f(0) = 3\left(1-\frac{1}{r}\right)^{r-1} > 1$. (Since as r increases the sequence $\left(1-\frac{1}{r}\right)^{r-1}$ decreases and its limit is $1/e$.) Hence, by continuity we can always choose small enough ε for (6.4) to hold. Then by making ε even smaller we can guarantee

$$1 < 2\left(1-\frac{1}{r}-\varepsilon\right)^{r-1} + \left(1-\frac{1}{r}\right)^{r-1} - \varepsilon(r-2)\left(1-\frac{1}{r}-\varepsilon\right)^{r-2}.$$

After choosing ε, we can always choose β small enough such that the final inequality holds. □

Chapter 7

On a conjecture of Erdős on sparse halves

Erdős [Erd75a] conjectured that every triangle-free graph G on n vertices contains a set of $\lfloor n/2 \rfloor$ vertices that spans at most $n^2/50$ edges. Krivelevich proved the conjecture for graphs with minimum degree at least $\frac{2}{5}n$ [Kri95b]. In [KS06b] Keevash and Sudakov improved this result to graphs with average degree at least $\frac{2}{5}n$. We strengthen these results by showing that the conjecture holds for graphs with minimum degree at least $\frac{5}{14}n$ and for graphs with average degree at least $\left(\frac{2}{5} - \varepsilon\right) n$ for some absolute $\varepsilon > 0$. Moreover, we show that the conjecture is true for graphs which are close to the Petersen graph in edit distance.

7.1. Background

In this chapter we consider the edge distribution in triangle-free graphs. One can consider the following generalization of Mantel's theorem first studied by Erdős, Faudree, Rousseau and Schelp in [EFRS94].

Suppose for given $0 < \alpha \leq 1$ every set of αn vertices of graph G spans more than βn^2 edges. A natural question arises - what is the smallest $\beta = \beta(\alpha)$ such that every such graph G necessarily contains a triangle? In particular, one of the Erdős' old and favorite conjectures says that $\beta(\frac{1}{2}) = \frac{1}{50}$. The bound 1/50 is obtained on the balanced blowup of C_5, the cycle on five vertices, and the balanced blowup of the Petersen graph. We say G contains *a sparse half* if there exists a set of $\lfloor n/2 \rfloor$ vertices in G that spans at most $n^2/50$ edges.

Conjecture 7.1.1 (Erdős, [Erd75b]). *Every triangle-free graph has a sparse half.*

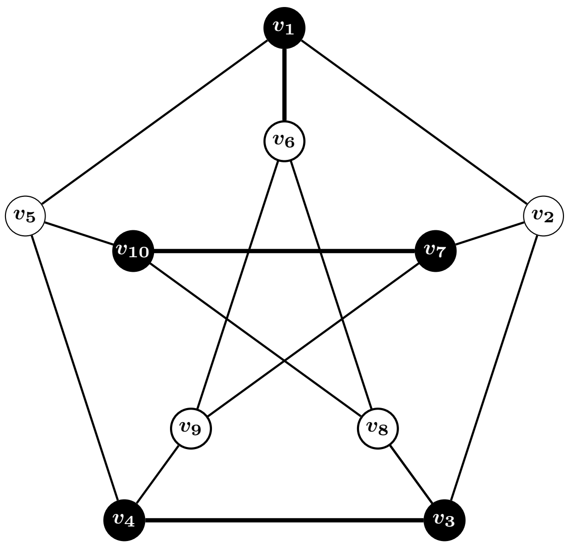

Figure 7.1: A sparse half in the uniform blowup of Petersen graph.

In his paper [Kri95b] Krivelevich proved that the conjecture holds if $n^2/50$ is replaced by $n^2/36$. He also showed that it is true for triangle-free graphs with minimum degree $\frac{2}{5}n$. In Section 7.3 we improve this result by proving the following theorem.

Theorem 7.1.1. *Every triangle-free graph on n vertices with minimum degree $\frac{5}{14}n$ contains a sparse half.*

Our proof of Theorem 7.1.1 is mainly based on the structural characterization of these graphs established by Jin, Chen and Koh in [CJK97, Jin95]. We also use some averaging arguments similar to the ones used in [KS06b, Kri95b].

Sudakov and Keevash [KS06b] improved the result of [Kri95b] showing that the conjecture holds for graphs with average degree $\frac{2}{5}n$. In Section 7.5 we extend their result as follows.

Theorem 7.1.2. *There exists $\gamma > 0$ such that every triangle-free graph on n vertices with at least $(\frac{1}{5} - \gamma)n^2$ edges contains a sparse half.*

Finally, we study the validity of the conjecture in the neighborhood of the known extremal examples, in the following sense. To the best of our knowledge, the uniform blowups of the Petersen graph and C_5 are the only known examples for which the conjecture is tight. In Section 7.4 we develop a set of tools which allow us to prove the it for the classes of graphs which are close to a fixed graph in edit distance. In Section 7.6 we use these tools to verify the conjecture for graphs which are close to the Petersen graph, while Theorem 7.1.2 shows that it also holds for graphs close to the 5-cycle. These results can be considered as a proof of a local version of the conjecture, in the spirit of recent results of Lovász [Lov11], which proves the Sidorenko conjecture locally in the neighborhood of the conjectured extremal example, and Razborov [Raz13b], which accomplishes a similar goal for the Caccetta-Häggkvist conjecture.

7.2. Notation and Preliminary Results

In this section we introduce some notation specifically for 2-graphs. In a graph G, we denote by $N(v)$ the neighborhood of a vertex $v \in V(G)$ and by $d_G(v)$ (or just $d(v)$) the degree of v. The maximum and the minimum degrees of the graph are denoted by $\Delta(G)$ and $\delta(G)$, respectively.

We also consider weighted graphs in a slightly different setting. We say that $\omega : V(G) \to (0,1)$ is a *weight function on G* if $\sum_{v \in V(G)} \omega(v) = 1$. Observe that we do not allow $w(v)$ to be zero or one for any $v \in V(G)$. (This is done for technical reasons and could have been avoided with some extra work in the proofs following.) As before, the pair (G, ω) is called a *weighted graph*. The weight $\omega(e)$ of an edge $e = (u, v)$ in (G, ω) is defined as $\omega(u) \cdot \omega(v)$. For a set X of vertices or edges of G let $\omega(X) := \sum_{x \in X} \omega(x)$. The *degree* of a vertex v in a weighted graph (G, ω) is defined as $\omega(N(v))$. The *minimum degree* of the weighted graph (G, ω) we denote by $\delta(G, \omega)$.

We call a real function $s : V(G) \to \mathbb{R}^+$ *a half* of (G, ω) if $s(v) \leq \omega(v)$ for every

$v \in V(G)$, and $s(V(G)) = 1/2$. If in addition to these conditions also $s(E(G)) \leq \frac{1}{50}$ then s is called a *sparse half*. Note that this is different from the definition of a sparse half for unweighted graphs but Lemma 7.2.1 gives the connection between these two notions. As in earlier chapters of this thesis, we use ξ_G (or simply ξ) to denote the uniform weight on the vertex set, that is $\xi(v) = \frac{1}{\mathrm{v}(G)}$, for every $v \in V(G)$. The weighted graph (G, ξ) we call *uniformly weighted G*.

Lemma 7.2.1. *If (G, ξ) has a sparse half, then so does G.*

Proof. We claim that, if (G, ξ) has a sparse half, then it has a sparse half s such that $s(v) = 0$ or $s(v) = \frac{1}{n}$ for all $v \in V(G)$ except for possible one vertex.

Indeed, let us choose a sparse half s of G such that the number of vertices $v \in V(G)$ such that either $s(v) = 0$ or $s(v) = \frac{1}{n}$ is maximum. We show that there exists at most one vertex u such that $0 < s(u) < \frac{1}{n}$.

Suppose not and there exist $u, v \in V(G)$ such that $0 < s(u), s(v) < \frac{1}{n}$. We define a new half $s' : V(G) \to \mathbb{R}^+$. Let s' be the same as s on all vertices of G, except u and v. Without loss of generality, suppose $s(N(u)) \leq s(N(v))$. Let $\delta = \min\{s(u), \frac{1}{n} - s(v)\}$ and define $s'(u) = s(u) - \delta$ and $s'(v) = s(v) + \delta$. We will show that s' is a sparse half.

Suppose that u and v are adjacent, then

$$\begin{aligned}
s'(E(G)) &= s(E(G)) - \sum_{\substack{x \in N(u) \\ x \neq v}} \delta s(x) + \sum_{\substack{y \in N(v) \\ y \neq u}} \delta s(y) + s'(u)s'(v) - s(u)s(v) \\
&= s(E(G)) - \sum_{\substack{x \in N(u) \\ x \neq v}} \delta s(x) + \sum_{\substack{y \in N(v) \\ y \neq u}} \delta s(y) - \delta s(v) + \delta s(u) - \delta^2 \\
&= s(E(G)) - \delta \left(s(N(u)) - s(N(v)) \right) - \delta^2 < s(E(G)).
\end{aligned}$$

The calculation in the case when u and v are non-adjacent is similar.

It follows that s' contradicts the choice of s. Hence for all vertices v of the graph except maybe one vertex either $s(v) = 0$ or $s(v) = \frac{1}{n}$. Let $S = \{v \in V(G) \mid s(v) = 1/n\}$. It follows from the above that $|S| \geq \lfloor n/2 \rfloor$. It is easy to see that $E(G[S]) \leq n^2 s(E(G)) \leq n^2/50$. It follows that S ia a sparse half in G, as desired. □

Lemma 7.2.1 allows us to work with weighted graphs, which proves convenient.

We prove that every weighted triangle-free graph with minimum degree at least $5/14$ contains a sparse half. Our proof uses structural characterization of these graphs found by Jin, Chen and Koh in [CJK97, Jin95]. To state their result we need a few additional definitions.

Let $\varphi : V(G) \to V(H)$ be a surjective homomorphism and let ω be a weight function on G. We define a weight function ω_φ on H in the following way. For every vertex $v \in V(H)$, let $\omega_\varphi(v) := \omega(\varphi^{-1}(v))$. The next lemma shows that a sparse half in a homomorphic image of the graph G can be lifted to a sparse half in the graph G.

Lemma 7.2.2. *Let G, H be graphs and let $\varphi : V(G) \to V(H)$ be a surjective homomorphism. Then for any weight function ω, if (H, ω_φ) has a sparse half, then so does (G, ω).*

Proof. Let ω be a weight function on G and let s_H be a sparse half on (H, ω_φ). Define

$$s_G(u) := \frac{\omega(u)}{\omega_\varphi(\varphi(u))} s_H(\varphi(u))$$

for every $u \in V(G)$ It is easy to check that s is a sparse half of G. □

Now let us introduce the family of the graphs that plays a key role in our results. For an integer $d \geq 1$, let F_d be the graph with

$$V(F_d) = \{v_1, v_2, ..., v_{3d-1}\},$$

such that the vertex v_j has neighbors $v_{j+d}, \ldots, v_{j+2d-1}$, these values taken modulo $3d-1$. Throughout this paper whenever we deal with F_d graphs, we always take the indices of the vertices modulo $3d-1$. In [Jin95] it is shown that every triangle-free graph G of order n with minimum degree $\delta(G) > 10n/29$ contains a homomorphic copy of F_9 and hence is 3-colorable. In [CJK97] Chen, Jin and Koh proved that every triangle-free graph of order n, with chromatic number $\chi(G) \leq 3$ and minimum degree $\delta > \left\lfloor \frac{(d+1)n}{3d+2} \right\rfloor$ admits a homomorphism to F_d. Therefore, the following theorem holds.

Theorem 7.2.1 (Chen, Jin, Koh [CJK97], [Jin95])**.** *Every triangle-free graphs of order n with minimum degree at least $\frac{5}{14}n$ admits a homomorphism to F_4.*

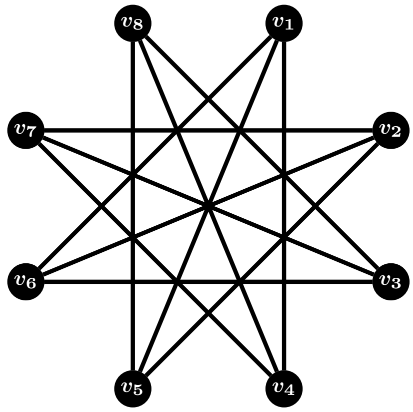

Figure 7.2: The graph F_3

To use Lemma 7.2.1 and Theorem 7.2.1 together we need to make sure that the homomorphism in the statement of Theorem 7.2.1 is surjective. Fortunately, this is not difficult if we allow ourselves to change the target graph.

Lemma 7.2.3. *Let φ be a homomorphism from graph G to F_d, $d \geq 2$. Then either φ is surjective or G admits a homomorphism to F_{d-1}.*

Proof. Suppose φ is a homomorphism from graph G to F_d that is not surjective. Let $V_i = \varphi^{-1}(v_i)$ for all $i = 1, 2, \ldots, 3d-1$. Without loss of generality suppose V_1 is empty. Define a mapping $\varphi' : V(G) \to V(F_{d-1})$ as follows,

$$\varphi'(v) = \begin{cases} v_{i-1}, & \text{if } v \in V_i \text{ } and \text{ } 2 \leq i \leq d-1 \\ v_{d-1}, & \text{if } v \in V_d \cup V_{d+1}, \\ v_{i-2}, & \text{if } v \in V_i \text{ } and \text{ } d+2 \leq i \leq 2d-2 \\ v_{2d-3}, & \text{if } v \in V_{2d-1} \cup V_{2d}, \\ v_{i-3,} & \text{if } v \in V_i \text{ } and \text{ } 2d+1 \leq i \leq 3d-1, \end{cases}$$

It is easy to check that φ' is a homomorphism from G to F_{d-1}. □

In the next section we show that for $1 \leq d \leq 4$ the weighted graph (F_d, ω) with minimum degree at least $5/14$, has a sparse half for any positive weight function ω. In particular, if $\varphi : V(G) \to F_d$ is a surjective homomorphism, (F_d, ξ) has a sparse half. By Lemmas 7.2.1 this implies that graph G has a sparse half. Hence the Theorem 7.1.1 will follow from the results of the next section and the results we have introduced.

7.3. Weighted triangle-free graphs with minimum degree at least $\frac{5}{14}$

Theorem 7.3.1. *Let $d \leq 4$ be a positive integer and let (F_d, ω) be a weighted graph with the minimum degree at least $5/14$. Then (F_d, ω) has a sparse half.*

Proof. The argument is separated into cases based on the value of d.

d=1: Suppose $V(F_1) = \{v_1, v_2\}$ then since $\omega(v_1) + \omega(v_2) = 1$, either $\omega(v_1) \geq \frac{1}{2}$ or $\omega(v_2) \geq \frac{1}{2}$, therefore v_1 or v_2 supports a sparse half in (F_1, ω).

d=2: Let $V(F_2) = \{v_1, v_2, \ldots, v_5\}$. If any two consecutive vertices together have total weight at least $1/2$, then they induce an independent set, which means that they support a sparse half.

Now suppose that no two consecutive vertices have total weight at least $1/2$, then any three consecutive vertices have total weight at least $1/2$. We define the following halves s_i for each $i = 1, 2, \ldots, 5$ on the vertices of the graph and prove that there is at least one sparse half among them.

$$s_i(v) = \begin{cases} \omega(v), & \text{if } v = v_i \textit{ or } v = v_{i+1}, \\ \frac{1}{2} - \left(\omega(v_i) + \omega(v_{i+1})\right), & \text{if } v = v_{i+2}, \\ 0, & \text{otherwise.} \end{cases}$$

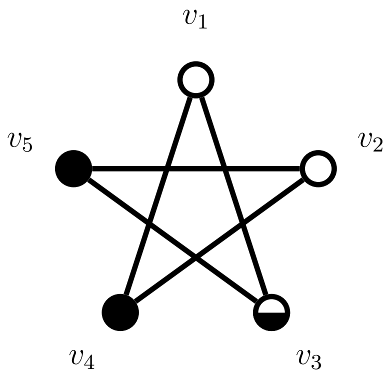

Figure 7.3: A sparse half in uniformly weighted C_5

Note that

$$s_i(E(G)) = \omega(v_i)\left(1/2 - (\omega(v_i) + \omega(v_{i+1}))\right). \tag{7.1}$$

Summing up the equations (7.1) over all $i = 1, \ldots, 5$, we get

$$\begin{aligned}\sum_{i=1}^{5} s_i(E(G)) &= 1/2 - \sum_{i=1}^{5} \omega(v_i)\left(\omega(v_i) + \omega(v_{i+1})\right) \\ &= \frac{1}{2} - \frac{1}{2}\sum_{i=1}^{5}\left(\omega(v_i) + \omega(v_{i+1})\right)^2 \\ &\le \frac{1}{2} - \frac{1}{2}\cdot 5 \cdot \frac{4}{25} = \frac{1}{10},\end{aligned}$$

using Jensen's inequality for the function x^2. Thus one of the functions s_i is a sparse half.

Note that the proof above did not use the minimum degree condition, while we use it in the other two cases.

d=3: Let $F_3 = \{v_1, v_2, \ldots, v_8\}$. As the minimum degree of (F_3, ω) is at least $5/14$, we have

$$\omega(v_j) + \omega(v_{j+1}) + \omega(v_{j+2}) \ge \delta(F_3, \omega) \ge 5/14, \tag{7.2}$$

for all $j = 1, 2, \ldots, 8$. Summing these inequalities for $j = i, i+1$ and $j = i+2$, we obtain $\omega(v_j) \ge 1/14$ for $j = 1, 2, \ldots, 8$.

As in the case of $d = 2$, if there exist three consecutive vertices of total weight at least $1/2$, then we are done, since they induce an independent set. Suppose not, then every five consecutive vertices have total weight more than $1/2$. We define the following halves s_i, for each $i = 1, 2, \ldots, 8$ on the vertices of the graph and prove that there is at least one sparse half among them.

$$s_i(v) = \begin{cases} \omega(v), & \text{if } v = v_{i+1}, v_{i+2}, v_{i+3} \\ \frac{1}{2}\left(\frac{1}{2} - (\omega(v_{i+1}) + \omega(v_{i+2}) + \omega(v_{i+3}))\right), & \text{if } v = v_i \text{ } or \text{ } v = v_{i+4}, \\ 0, & \text{otherwise.} \end{cases}$$

Claim 7.3.1. *For each $i = 1, 2, \ldots, 8$, s_i is a half.*

Proof. It suffices to show that

$$\frac{1}{2}\left(\frac{1}{2} - (\omega(v_{i+1}) + \omega(v_{i+2}) + \omega(v_{i+3}))\right) \leq \omega(v_i),$$

$$\frac{1}{2}\left(\frac{1}{2} - (\omega(v_{i+1}) + \omega(v_{i+2}) + \omega(v_{i+3}))\right) \leq \omega(v_{i+4}).$$

By symmetry it suffices to prove the first inequality. By (7.2) we get that

$$\frac{1}{2}\left(\frac{1}{2} - (\omega(v_{i+1}) + \omega(v_{i+2}) + \omega(v_{i+3}))\right) \leq \frac{1}{2} \cdot \left(\frac{1}{2} - \frac{5}{14}\right) = \frac{1}{14} \leq \omega(v_i).$$

This finishes the proof of the claim. □

We relegate the proof of the following lemma, which ensures that one of the halves s_i is sparse, to the Section 7.7.

Lemma 7.3.2. *Let $1/14 \leq x_i \leq 1$ for $i = 1, 2, \ldots, 8$. If $\sum_{i=1}^{8} x_i = 1$ and $x_i + x_{i+1} + x_{i+2} \geq 5/14$ for every i, then there exists an i such that*

$$\frac{1}{2}\left(\frac{1}{2} - (x_i + x_{i+1} + x_{i+2})\right)(x_i + x_{i+2}) + \frac{1}{4}\left(\frac{1}{2} - (x_i + x_{i+1} + x_{i+2})\right)^2 \leq \frac{1}{50}. \quad (7.3)$$

d=4: Let $F_4 = \{v_1, v_2, \ldots, v_{11}\}$. The minimum degree condition gives us the following inequality

$$\omega(v_i) + \omega(v_{i+1}) + \omega(v_{i+2}) + \omega(v_{i+3}) \geq 5/14, \quad (7.4)$$

for all $i = 1, 2, \ldots, 11$. It follows, as in the case $d = 3$, that $\omega(v_i) \geq 1/14$ for all i. If any of four consecutive vertices have total weight at least $1/2$, then we are done, since they induce an independent set.

For every $i = 1, 2, \ldots, 11$ by (7.4), we have $\omega(v_{i+5})+\omega(v_{i+6})+\omega(v_{i+7})+\omega(v_{i+8}) \geq 5/14$, $\omega(v_{i+9}) \geq 1/14$ and $\omega(v_{i+10}) \geq 1/14$, therefore

$$\omega(v_i) + \omega(v_{i+1}) + \cdots + \omega(v_{i+5}) \leq 1/2.$$

It follows that

$$\omega(v_i) + \omega(v_{i+1}) + \cdots + \omega(v_{i+6}) \geq 1/2,$$

for all $i = 1, 2, \ldots, 11$. This allows us to define halves s_i in the following way:

$$s_i(v) = \begin{cases} \omega(v), & \text{if } v = v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, \\ \frac{1}{2}\left(\frac{1}{2} - (\omega(v_{i+1}) + \omega(v_{i+2}) + \omega(v_{i+3}) + \omega(v_{i+1}))\right), & \text{if } v = v_i \textit{ or } v = v_{i+5}, \\ 0, & \text{otherwise.} \end{cases}$$

As in the previous case, it is easy to verify that each s_i is a half, and the following lemma proved in the Section 7.7 implies that at least one of these halves is sparse.

Lemma 7.3.3. *Suppose given are $x_1, x_2, \ldots, x_{11}$ reals such that $1/14 \leq x_i \leq 1$ for each $i = 1, 2, \ldots, 11$ and $\sum_{i=1}^{11} x_i = 1$. If $x_i + x_{i+1} + x_{i+2} + x_{i+3} > 5/14$ for every i, then there exists an i such that*

$$\frac{1}{2}\left(\frac{1}{2} - (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4})\right)(x_{i+1} + x_{i+4})$$
$$+ \frac{1}{4}\left(\frac{1}{2} - (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4})\right)^2 \leq 1/50.$$

□

Proof of Theorem 7.1.1. Let G be a triangle-free graph on n vertices with minimum degree $\geq 5n/14$. By Lemma 7.2.1 it suffice to prove that the uniformly weighted graph (G, ξ) has a sparse half. By Theorem 7.2.1 and Lemma 7.2.3, the graph G admits a surjective homomorphism φ to F_d for $1 \leq d \leq 4$. Clearly, (F_d, ω_φ) has minimum degree $\geq 5/14$ and thus has a sparse half by Theorem 7.3.1. Theorem 7.1.1 now follows from Lemma 7.2.2. □

7.4. Uniform sparse halves, balanced weights and disturbed graphs

The purpose of this section is to develop a set of tools, which under fairly general circumstances allow us to show that graphs "close" to a fixed graph have sparse halves. A number of technical definitions will be necessary. We start by a variant of the definition of *edit distance* (see e.g. [LS10]).

Definition 7.4.1. *Given a graph G of order n and a graph H, we say that G is ε-close to H, if there exists a blowup B of H on $V(B) = V(G)$ such that*

- $|E(G \triangle B)| \leq \varepsilon n^2$,
- *B is ε-trimmed, that is, if $\{V_1, V_2, \ldots, V_k\}$ is the blowup partition of B, then* $\left||V_i| - \frac{n}{k}\right| \leq \varepsilon n$.

We will also need a stronger notion of distance defined as follows.

Definition 7.4.2. *Given a graph G and $F \subseteq E(G)$, we say that $D \subseteq V(G)$ is an* ε-controlling *set for F, if $|D| \leq \varepsilon |V(G)|$ and every edge in F has at least one end in D. We say that the graph G' is an ε-disturbed graph of G for some $0 < \varepsilon < 1$, if the following conditions hold:*

1. $V(G') = V(G)$,
2. *there exists an ε-controlling set for $E(G') - E(G)$ in G',*
3. $|N_{G'}(v) \setminus N_G(v)| \leq \varepsilon |V(G)|$ *for every vertex $v \in V(G)$.*

Let H be a graph. Let $\mathcal{N}(H)$ be the set of neighborhoods of vertices of H, and let $\mathcal{I}(H)$ be the set of maximum independent sets of H. Let $\mathcal{I}^*(H) = \mathcal{I}(H) - \mathcal{N}(H)$. We construct a graph H^* as follows. Let $V(H^*) = V(H) \cup \mathcal{I}^*(H)$, let H be an induced subgraph of H^*, and let every $I \in \mathcal{I}^*(H)$ be adjacent to every $v \in I$

and no other vertex of H^*. We say that a weighted graph (H^*, ω) is *ε-balanced* if $|\omega(v) - 1/|V(H)|| \leq \varepsilon$ for every $v \in V(H)$ and $\omega(v) \leq \varepsilon$ for every $v \in V(H^*) - V(H)$. We are now ready for our first technical result.

Theorem 7.4.1. *Let H be a maximal triangle-free graph. For every $\varepsilon > 0$ there exists δ such that if G is a triangle-free graph which is δ-close to H then there exists a graph G' and a homomorphism φ from G' to H^* with the following properties*

(i) G is ε-disturbed graph of G',

(ii) (H^*, ω_φ) is ε-balanced,

(iii) φ is a strong homomorphism*, that is $uv \notin E(G')$ implies $\varphi(u)\varphi(v) \notin E(H^*)$ for every pair of vertices $u, v \in V(G')$.*

Proof. We assume that $V(H) = \{1, 2, \ldots, k\}$. We show that $\delta > 0$ satisfies the theorem if $(k+2)\sqrt{\delta} \leq \min(\varepsilon, 1/k)$. Let B be as in Definition 7.4.1 and $\mathcal{V} = (V_1, V_2, \ldots, V_k)$ be the corresponding partition of $V(G)$. Let $n := |V(G)|$, $F := E(G) \triangle E(B)$ and let J be the set of all vertices of F incident to at least $\sqrt{\delta}n$ edges in F'. We have $\delta n^2 \geq |F| \geq \frac{1}{2}\sqrt{\delta}n|J|$. It follows that $|J| \leq 2\sqrt{\delta}n$.

We define a map $\varphi : V(G) \to V(H^*)$, as follows. If $v \in V_i \setminus J$ for some $i \in V(H)$ then $\varphi(v) := i$. Now consider $v \in J$ and let

$$I_0(v) := \{i \in V(G) \mid |N(v) \cap V_i| > \sqrt{\delta}n\}.$$

Then $I_0(v)$ is independent, as otherwise there exist $i, j \in [k]$, such that $ij \in E(H)$, $|N(v) \cap V_i| > \sqrt{\delta}n$ and $|N(v) \cap V_j| > \sqrt{\delta}n$. As G is triangle-free it follows that $|E(B) \triangle E(G)| > \varepsilon n^2$, contradicting the choice of B. Let $I(v)$ be a maximal independent set containing $I_0(v)$, chosen arbitrarily. Let $\varphi(v) = i$, if $I(v) = N_H(i)$ for some $i \in V(H)$, and let $\varphi(v) = I(v)$, otherwise.

Let G' be the graph with $V(G') = V(G)$ and the vertices $uv \in E(G')$ if and only if $\varphi(u)\varphi(v) \in E(H^*)$. Then φ is a strong homomorphism from G' to H^*. For $i \in V(H)$ we have $V_i - J \subseteq \varphi^{-1}(i) \subseteq V_i \cup J$ and, therefore, $||\varphi^{-1}(i)|/n - 1/k| \leq \delta + 2\sqrt{\delta}$. For $I \in V(H^*) \setminus V(H)$ we have $|\varphi^{-1}(I)| \leq |J| \leq 2\sqrt{\delta}n$. Thus (ii) holds, as $\varepsilon \geq \delta + 2\sqrt{\delta}$.

It remains to verify (i). We will show that J is an ε-controlling set for $F' := E(G) \setminus E(G')$ for δ sufficiently small. First, we show that every edge of F' has an end in J. Indeed suppose that $uv \in E(G)$ for some $u \in V_i, v \in V_j$ with $i, j \in V(H)$ not necessarily distinct and $ij \notin E(H)$. Then there exists $h \in V(H)$ adjacent to both i and j, as H is maximally triangle-free. It follows that both v and u have at least $(1/k - \delta - \sqrt{\delta})n$ neighbors in V_h and share a common neighbor if $2(1/k - \delta - \sqrt{\delta}) > 1/k$. Thus our first claim holds, as $\delta + \sqrt{\delta} < 1/k$.

Consider now $v \in J$ and let $N'(v)$ be the set of neighbors of v in $V(G) \setminus V(J)$ joined to v by edges of F'. Then $|N'(v) \cap V_i| \leq \sqrt{\delta}n$ for every $i \in V(H)$ by the choice of $\varphi(v)$. It follows that $|N'(v)| \leq k\sqrt{\delta}n$. Therefore v is incident to at most $(k+2)\sqrt{\delta}n$ edges in F, as $|J| \leq 2\sqrt{\delta}n$. Thus J is an ε-controlling set for F', as $(k+2)\sqrt{\delta}n \leq \varepsilon$. □

We say that H is *entwined* if $\mathcal{I}^*(H)$ is intersecting. Note that the graph F_i is entwined for every i, as $\mathcal{I}^*(H)$ is empty. It is routine to check that the Petersen graph is entwined. We say that the graph G of order n is *c-maximal triangle-free* if it is triangle-free and adding any new edge to G creates at least cn triangles. The following lemma follows immediately from definitions.

Lemma 7.4.3. *Let H be an entwined triangle-free graph, and let $0 < \varepsilon < 1/|V(H)|$. If (H^*, ω) is ε-balanced then it is $(1/|V(H)| - \varepsilon)$-maximal triangle-free.*

Definition 7.4.4. *For a weighted graph (G, ω) we call a distribution $\boldsymbol{s}$ defined on the set of halves of (G, ω) a c-uniform sparse half for some $0 < c \leq 1$, if*

1. *for every edge $e \in E(G)$ $\mathbb{E}[\boldsymbol{s}(e)]) \geq c\,\omega(e)$,*
2. *$\mathbb{E}[\boldsymbol{s}(E(G))] \leq \frac{1}{50}$.*

Whenever we refer to c-uniform sparse halves in unweighted graphs, they are understood as the c-uniform sparse halves in the corresponding uniformly weighted graphs.

Theorem 7.4.2. *Let $0 < c < 1$ be real, let G' be a c-maximal triangle-free graph and let G be a triangle-free $\frac{c^2}{2(1+c)}$-disturbed graph of G'. If G' has a c-uniform sparse half then G has a sparse half.*

Proof. Let $\varepsilon = \frac{c^2}{2(1+c)}$. Let $F = E(G) \setminus E(G')$ and let M be a maximal matching in F. let $|M| = \delta n$ for some $0 \le \delta \le 1/2$. Since G is an ε-disturbed graph of G', it has an ε-controlling set for F. Let D be the minimum one. Let $V(M)$ be the set of ends of the edges in M. By the choice of M every edge of F has at least one end in $V(M)$. It follows that $|D| \le |V(M)| = 2|M|$. By the third condition in the definition of ε-disturbed graph, $|F| \le \varepsilon n|D| \le 2\delta\varepsilon n^2$.

Let $F' := E(G') \setminus E(G)$. For an edge $e \in M$, let $T(e)$ be the set of edges $f \in F'$, such that the only vertex of $V(M)$ that f is incident to is an end of e. Let u, v be the ends of e. By the c-maximality of G', we have $|N_{G'}(u) \cap N_{G'}(v)| \ge cn$ for every pair of vertices $u, v \in V(G')$ non-adjacent in G'. Since $e \in E(G)$ and G is triangle-free, for every vertex $w \in N_{G'}(u) \cap N_{G'}(v)$, either $uw \in F'$ or $vw \in F'$. It follows that $|T_e| \ge |N_{G'}(u) \cap N_{G'}(v)| - |V(M)| \ge (c - 2\delta)n$. Thus $|F'| \ge (c - 2\delta)n \cdot \delta n$.

Let $\mathbf{s}$ be a c-uniform sparse half in the graph G'. Then

$$\begin{aligned}\mathbb{E}\left[\mathbf{s}(E(G)) - \mathbf{s}(E(G'))\right] &= \mathbb{E}\left[\mathbf{s}(F) - \mathbf{s}(F')\right] \\ &= \mathbb{E}\left[\mathbf{s}(F)\right] - \mathbb{E}\left[\mathbf{s}(F')\right],\end{aligned}$$

by linearity of the expectation. We have

$$\mathbb{E}\left[\mathbf{s}(F')\right] = \sum_{e \in F'} \mathbb{E}\left[\mathbf{s}(e)\right] \ge \sum_{e \in F'} c\,\omega(e) = c \sum_{e \in F'} \frac{1}{n^2} \ge \frac{c|F'|}{n^2}.$$

On the other hand,

$$\mathbb{E}\left[\mathbf{s}(F)\right] = \sum_{e \in F} \mathbb{E}\left[\mathbf{s}(e)\right] \le \sum_{e \in F} \omega(e) = \frac{|F|}{n^2}.$$

Finally, note that $\delta \le \varepsilon$, since every edge in M has at least one of its ends in D. Hence,

$$\begin{aligned}\mathbb{E}\left[\mathbf{s}(E(G)) - \mathbf{s}(E(G'))\right] &\le \frac{|F|}{n^2} - \frac{c|F'|}{n^2} \le 2\delta\varepsilon - c(c - 2\delta)\delta \\ &= \delta(2\delta c + 2\varepsilon - c^2) \le 0,\end{aligned}$$

where the last inequality holds, as $2\delta c + 2\varepsilon - c^2 \le 2(1 + c)\varepsilon - c^2 = 0$. Therefore,

$\mathbb{E}[\mathbf{s}(E(G))] \leq \mathbb{E}[\mathbf{s}(E(G'))] \leq \frac{1}{50}$, and the graph G has a sparse half by Lemma 7.2.1.

□

We are now ready to prove the main result of this section.

Theorem 7.4.3. *Let H be an entwined maximal triangle-free graph. Suppose that there exists $\alpha > 0$ such that, if (H^*, ω) is α-balanced, then (H^*, ω) has an α-uniform sparse half. Then there exists $\delta > 0$ such that every triangle-free graph G which is δ-close to H has a sparse half.*

Proof. Let $\varepsilon := \min\left(\frac{1}{3|V(H)|^2}, \frac{\alpha^2}{2(1+\alpha)}\right)$, and let δ be chosen so that there exist φ and G' satisfying the conclusion of Theorem 7.4.1. The weighted graph (H^*, ω_φ) has an α-uniform sparse half, as $\varepsilon \leq \alpha$. Therefore, the graph G' has an α-uniform sparse half. By Lemma 7.4.3, the graph (H^*, ω_φ) is $(1/|V(H)| - \varepsilon)$-triangle-free. Therefore, the graph G' is ε-triangle-free, because φ is a strong homomorphism. Let $c := \min(\alpha, 1/2|V(H)|)$. Then $\varepsilon \leq c^2/(2(1+c))$ and $c \leq 1/|V(H)| - \varepsilon$, by the choice of ε. It follows that the conditions of Theorem 7.4.2 are satisfied for G' and G. Thus G has a sparse half, as desired. □

7.5. Triangle-free graphs with at least $(1/5 - \gamma)n^2$ edges

In order to establish the conjecture for the triangle-free graphs with average degree $\left(\frac{2}{5} - \gamma\right) n$ we separate the cases when the graphs under consideration are close in the sense of Definition 7.4.1 to the blowup of C_5 and when they are not. In the second case, we use the result of Sudakov and Keevash [KS06b] that can be rephrased in the following way.

Theorem 7.5.1. *Let G be a triangle-free graph on n vertices such that one of the following conditions holds*

(a) either $\frac{1}{n}\sum_{v \in V(G)} d^2(v) \geq \left(\frac{2}{5}n\right)^2$ *and* $\Delta(G) < \left(\frac{2}{5} + \frac{1}{135}\right) n$, *or*

(b) $\Delta(G) \geq \left(\frac{2}{5} + \frac{1}{135}\right) n$ *and* $\frac{1}{n}\sum_{v \in V(G)} d(v) \geq \left(\frac{2}{5} - \frac{1}{125}\right) n$.

Then G has a sparse half.

The next theorem represents the main technical step in the proof of Theorem 7.1.2.

Theorem 7.5.2. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds. If G is a triangle-free graph with $|V(G)| = n$ and $|E(G)| \geq (\frac{1}{5} - \delta)n^2$ then either*

(i) G is ε-close to C_5,

(ii) or at least δn vertices of G have degree at least $\left(\frac{2}{5} + \delta\right) n$,

(iii) or there exists an $F \subseteq E(G)$ with $|F| \leq \varepsilon n^2$ such that the graph $G \setminus F$ is bipartite.

The proof of Theorem 7.5.2 uses the following technical lemmas.

Lemma 7.5.1. *For $\delta > 0$ let G be a graph with $|V(G)| = n$ and $|E(G)| \geq (\frac{1}{5} - \delta)n^2$. Then either*

(1) *at least δn vertices have degree at least $(\frac{2}{5} + \delta)n$,*

(2) *or at most $2\sqrt{\delta}n$ vertices have degree at most $(\frac{2}{5} - 2\sqrt{\delta})n$.*

Proof. Suppose that the outcome (1) does not hold. Let sn be the number of vertices that have degree at most $(\frac{2}{5} - 2\sqrt{\delta})n$. Then

$$\begin{aligned} 2\left(\frac{1}{5} - \delta\right) n^2 \leq 2|E(G)| = \sum_{v \in V} d(v) \\ &= \sum_{\substack{v \in V \\ d(v) \leq (\frac{2}{5} - 2\sqrt{\delta})n}} d(v) + \sum_{\substack{v \in V \\ (\frac{2}{5} - 2\sqrt{\delta})n < d(v) < (\frac{2}{5}+\delta)n}} d(v) + \sum_{\substack{v \in V \\ d(v) \geq (\frac{2}{5}+\delta)n}} d(v) \\ &< sn\left(\frac{2}{5} - 2\sqrt{\delta}\right) n + (1 - s)\, n \left(\frac{2}{5} + \delta\right) n + \delta n^2 = \\ &\left(\frac{2}{5} - \sqrt{\delta}s + 2\delta - s\delta\right). \end{aligned}$$

Thus $-2\delta < 2\delta - 2s\sqrt{\delta} - s\delta$, and

$$s < \frac{4\delta}{2\sqrt{\delta} + \delta} < 2\sqrt{\delta},$$

as desired. $\square$

Lemma 7.5.2. *Let H be a graph on n vertices with minimum degree at least $\left(\frac{2}{5}-\delta\right)n$. Let φ be a surjective homomorphism from H to C_5.*

$$\left(\frac{1}{5}-3\delta\right)n \leq |\varphi^{-1}(v)| \leq \left(\frac{1}{5}+2\delta\right)n, \tag{7.5}$$

for every $v \in V(C_5)$.

Proof. Let the vertices of C_5 be labelled $v_1, v_2, \ldots, v_5$ in order and let $V_i := \varphi^{-1}(v_i)$ for $i = 1, 2, \ldots, 5$. Then

$$|V_i| + |V_{i+1}| \geq \left(\frac{2}{5}-\delta\right)n \tag{7.6}$$

$$|V_{i+2}| + |V_{i+3}| \geq \left(\frac{2}{5}-\delta\right)n \tag{7.7}$$

$$|V_{i+4}| + |V_i| \geq \left(\frac{2}{5}-\delta\right)n, \tag{7.8}$$

therefore

$$|V_i| + \left(\frac{2}{5}-\delta\right)n + \left(\frac{2}{5}-\delta\right)n \leq |V_i| + (|V_{i+1}| + |V_{i+2}|) + (|V_{i+3}| + |V_{i+4}|) \leq n,$$

which gives us the desired upper bound.

For the lower bound summing inequalities (7.6)-(7.8) we get

$$n + |V_i| \geq \sum_{j=1}^{5} |V_j| + |V_i| \geq 3\left(\frac{2}{5}-\delta\right)n,$$

which gives $|V_i| \geq \left(\frac{1}{5}-3\delta\right)n$, as desired. □

Proof of Theorem 7.5.2. We show that $\delta := (\varepsilon/40)^2$ satisfies the theorem for $\varepsilon \leq 1$. We apply Lemma 7.5.1. The first outcome of Lemma 7.5.1 corresponds to the outcome (ii) of the theorem. Therefore we assume that the second outcome holds: There exists $|S| \leq \frac{\varepsilon}{20}n$ such that every vertex in $V(G) - S$ has degree at least $\left(\frac{2}{5}-\frac{\varepsilon}{20}\right)n$ in G. It follows that $G' := G \setminus S$ has the minimum degree at least $\left(\frac{2}{5}-\frac{\varepsilon}{10}\right)n$.

Since $\left(\frac{2}{5}-\frac{\varepsilon}{10}\right)n \geq \frac{3}{10}|V(G')|$, the result of Chen,Jin and Koh quoted before Theorem 7.2.1 implies that there exists a homomorphism φ from G' to C_5. Let

$V(C_5) = \{v_1, v_2, \ldots, v_5\}$ and $V_i = \varphi^{-1}(v_i)$ for each $i = 1, 2, \ldots, 5$.

If the homomorphism φ is not surjective then by Lemma 7.2.3 the graph G' is bipartite and therefore the graph $G^* = (V(G), E(G'))$ is also bipartite. We have $|E(G)| - |E(G')| \leq |S|n \leq \varepsilon n^2$. Thus outcome (iii) holds. Hence we can suppose that the homomorphism φ is surjective. Applying Lemma 7.5.2 to the graph G' with $\delta = \frac{\varepsilon}{10}$, we get that

$$\left(\frac{1}{5} - \frac{2\varepsilon}{5}\right) n \leq |V_i| \leq \left(\frac{1}{5} + \frac{\varepsilon}{5}\right) n. \tag{7.9}$$

Let $\mathcal{V} := (V_1 \cup S, V_2, \ldots, V_5)$ be a partition of $V(G)$. From (7.9) we have $\left||V_i| - \frac{n}{5}\right| \leq \varepsilon n$. Let B be as in Definition 7.4.1. Then

$$|E(G \triangle B)| \leq |S|n + (|E(B)| - E(G')) \leq \frac{\varepsilon}{20} n^2 + \frac{n^2}{5} - \frac{1}{2}\left(\frac{2}{5} - \frac{\varepsilon}{10}\right)\left(1 - \frac{\varepsilon}{20}\right) n^2$$
$$\leq \varepsilon n^2.$$

Thus outcome (i) holds. □

If outcome (i) of Theorem 7.5.2 holds our goal is to apply Theorem 7.4.3. To do that we need to ensure that a c-balanced weighting of C_5 has a c-uniform sparse half for some $c > 0$.

Theorem 7.5.3. *A $(1/50)$-balanced weighted graph (C_5, ω) has a $(1/30)$-uniform sparse half.*

Proof. Let $\delta := 1/50$, let $V(C_5) = \{v_1, v_2, \ldots, v_5\}$ and let $E(C_5) = \{v_i v_{i+2}\}_{i=1}^{5}$, as in the proof of Theorem 7.3.1. We define a distribution on the set of halves of the graph C_5. Recall the halves s_i, $1 \leq i \leq 5$ that we have defined earlier in the proof of the Theorem 7.3.1.

Let the probability mass of the distribution $\mathbf{s}$ be $\frac{1}{5}$ on every s_i, $i = 1, 2, \ldots, 5$. We show that $\mathbf{s}$ is a $\frac{1}{30}$-uniform sparse half. Let us begin by showing that $\mathbb{E}[\mathbf{s}(e)]) \geq \frac{1}{30}\,\omega(e)$ for every $e \in E(C_5)$. Let $e = (v_i, v_{i+2})$, then

$$\mathbb{E}[\mathbf{s}(e)] = \frac{1}{5} \cdot \omega(v_i)\left(\frac{1}{2} - (\omega(v_i) + \omega(v_{i+1}))\right).$$

hence

$$\mathbb{E}\left[\mathbf{s}(e)\right] \geq \frac{1}{5}\left(\frac{1}{5}-\delta\right)\left(\frac{1}{2}-2\left(\frac{1}{5}+\delta\right)\right) = \frac{1}{5}\cdot\left(\frac{1}{5}-\delta\right)\left(\frac{1}{10}-2\delta\right).$$

On the other hand,

$$\omega(e) = \omega(v_i)\cdot\omega(v_{i+2}) \leq \left(\frac{1}{5}+\delta\right)^2.$$

Thus it suffices to show that

$$\frac{1}{5}\left(\frac{1}{5}-\delta\right)\left(\frac{1}{10}-2\delta\right) \geq \frac{1}{30}\left(\frac{1}{5}+\delta\right)^2,$$

which can be easily verified. It is shown in the proof of Theorem 7.3.1 that $\mathbb{E}\left[\mathbf{s}(E(G))\right] \leq \frac{1}{50}$. Thus $\mathbf{s}$ is a $\frac{1}{30}$-uniform sparse half of G, as claimed. □

We need a final technical lemma.

Lemma 7.5.3. *For every $\delta > 0$ there exists a $\gamma > 0$ such that if G is a graph on n vertices with at least $\left(\frac{1}{5}-\gamma\right)n^2$ edges and at least δn vertices of degree at least $(\frac{2}{5}+\delta)n$ then*

$$\frac{1}{n}\sum_{v\in V(G)} d^2(v) \geq \left(\frac{2}{5}n\right)^2.$$

Proof. Suppose that G contains αn vertices of degree at most $\frac{2}{5}n$ and βn vertices of degree at least $\left(\frac{2}{5}+\delta\right)n$. We may assume that the average degree of G is less than $\frac{2}{5}n$, as otherwise lemma clearly holds. Thus

$$\frac{2}{5}n > (1-\alpha-\beta)\frac{2}{5}n + \beta\left(\frac{2}{5}+\delta\right)n,$$

hence $\alpha > \frac{5}{2}\beta\delta \geq \frac{5}{2}\delta^2$. We have

$$\begin{aligned} n\sum_{v\in V(G)} d^2(v) - \left(\sum_{v\in V(G)} d(v)\right)^2 &= \frac{1}{2}\sum_{u\neq v}\left(d(u)-d(v)\right)^2 \\ &\geq \alpha\delta\Big(\frac{2}{5}+\delta-\frac{2}{5}\Big)^2 n^4 > \frac{5}{2}\delta^5 n^4. \end{aligned}$$

Hence

$$\frac{1}{n}\sum_{v\in V(G)} d^2(v) \geq \frac{1}{n^2}\left(\sum_{v\in V(G)} d(v)\right)^2 + \frac{5}{2}\delta^5 n^2 \geq 4\left(\frac{1}{5}-\gamma\right)^2 n^2 + \frac{5}{2}\delta^5 n^2$$
$$\geq \frac{4}{25}n^2 + \left(\frac{5}{2}\delta^5 - \frac{8}{5}\gamma\right) n^2 \geq \frac{4}{25}n^2,$$

if we choose $\gamma = \frac{25}{16}\delta^5$. □

Proof of Theorem 7.1.2. By Theorem 7.5.3, $\alpha = 1/50$ satisfies the conditions in the statement of Theorem 7.4.3 for $H := C_5$. Thus by Theorem 7.4.3 there exists $0 < \varepsilon \leq 1/50$ such that every triangle-free graph G that is ε-close to C_5 has a sparse half. Let δ be such that Theorem 7.5.2 holds, and finally let γ be such that Lemma 7.5.3 holds. We show that Theorem 7.1.2 holds for this choice of γ.

We distinguish cases based on the outcome of Theorem 7.5.2 which holds for G.

Case (i): If G is ε-close to C_5 then the theorem holds by the choice of ε.

Case (ii): Now suppose that at least δn vertices of G have degree at least $\left(\frac{2}{5}+\delta\right)n$. If $\Delta(G) \geq \left(\frac{2}{5}+\frac{1}{135}\right)n$ then Theorem 7.5.1 (b) implies that there is a sparse half in G. Therefore we assume that $\Delta(G) < \left(\frac{2}{5}+\frac{1}{135}\right)n$. By Lemma 7.5.3 and the choice of γ we have $\frac{1}{n}\sum_{v\in V(G)} d^2(v) \geq \left(\frac{2}{5}n\right)^2$. Hence Theorem 7.5.1 (a) implies that there is a sparse half in G.

Case (iii): Lastly, suppose there exists an $F \subseteq E(G)$ with $|F| \leq \varepsilon n^2$ such that the graph $G' = (V(G), E(G) \setminus F)$ is bipartite with bipartition (U, V). Then either $|U| \geq \frac{n}{2}$ or $|V| \geq \frac{n}{2}$. Without loss of generality suppose $|U| \geq \frac{n}{2}$. The set U is independent in G', while in G it might not be, but $|E(G[U])| \leq |F| \leq \varepsilon n^2 \leq \frac{n^2}{50}$. Hence U supports a sparse half in graph G. □

7.6. Neighbourhood of the Petersen Graph

The uniform blowup of the Petersen graph, is an extremal example for Conjecture 7.1.1, that is every set of $\lfloor n/2 \rfloor$ vertices spans at least $n^2/50$ edges. Here we show that the Conjecture 7.1.1 holds for any graph that is close to a uniform blowup Petersen graph in the sense of Definition 7.4.1. By Theorem 7.4.3 it is enough to show that sufficiently balanced blowups of P^* have uniform sparse halves.

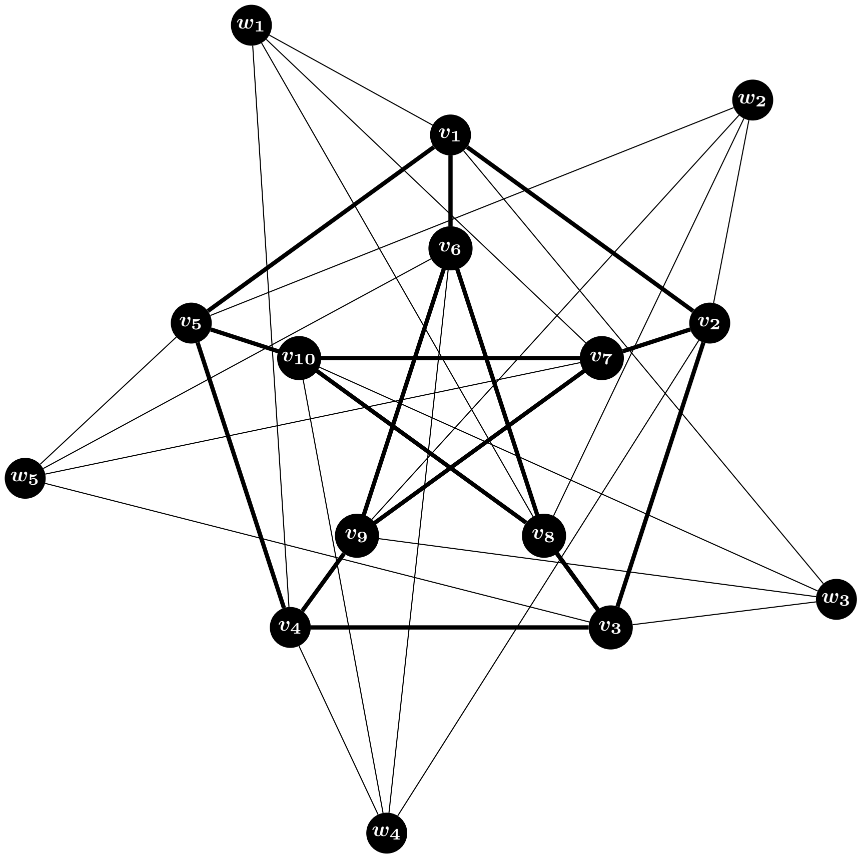

Figure 7.4: The graph P^*.

Lemma 7.6.1. *Let the weighted graph* (P^*, ω) *be* $(1/500)$*-balanced then it has a* $\frac{1}{80}$*-uniform sparse half.*

Proof. For $\delta := 1/500$ and let the vertices of P^* be labeled as on Figure 7.4, where $V := V(P) = \{v_1, v_2, \ldots, v_{10}\}$ and $W := V(P^*) \setminus V(P) = \{w_1, w_2, \ldots, w_5\}$. We define a collection $\{s_{i,j}\}_{i\in[5], j\in[4]}$ of halves of (P^*, ω). Fix $i \in [5]$ and consider the vertex w_i. Let $M_i := V \setminus N(w_i)$ and not that M_i induces a matching of size three. Choose vertices $\{v_{i_1}, v_{i_2}, v_{i_3}\} \subseteq V(M_i)$ such that they are independent and there exists a unique $w_{i_j} \neq w_i$ such that every $v \in V(M_i) \setminus \{v_{i_1}, v_{i_2}, v_{i_3}\}$ is adjacent to w_{i_j}. Note that for every i there exists four such choices of $\{v_{i_1}, v_{i_2}, v_{i_3}\}$, fix one of them and assign

$$s_{i,j}(w_q) = \begin{cases} \omega(w_q), & \text{if } q = i, \\ \frac{1}{4}\omega(w_q), & \text{if } q = i_j, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$s_{i,j}(v_k) = \begin{cases} 0, & \text{if } v_k \notin V(M_i) \\ \omega(v_k), & \text{if } k = j_1, j_2, j_3, \\ \frac{1}{3}\left(\frac{1}{2} - (\omega(v_{i_1}) + \omega(v_{i_2}) + \omega(v_{i_3}) + \omega(w_i) + \frac{1}{4}\omega(w_{i_j}))\right), & \text{otherwise.} \end{cases}$$

It is easy to check that every $s_{i,j}$ is a half. Let $\mathbf{s}$ be a distribution concentrated on $\{s_{i,j}\}_{i\in[5],j\in[4]}$ with each of the halves having the same probability 1/20. We show that $\mathbf{s}$ is a (1/80)-uniform sparse half for ω.

First, we show that $\mathbb{E}[\mathbf{s}(e)]) \geq \frac{1}{80} \cdot \omega(e)$ for every edge $e \in E(P^*)$. It can be routinely checked that for our choice of δ one has

$$\frac{1}{3}\left(\frac{1}{2} - (\omega(v_{i_1}) + \omega(v_{i_2}) + \omega(v_{i_3}) + \omega(w_i) + \frac{1}{4}\omega(w_{i_j}))\right) \geq \frac{1}{3}\omega(v_k), \tag{7.10}$$

for every $v_k \in V(M_i) \setminus \{v_{i_1}, v_{i_2}, v_{i_3}\}$. If both ends v, v' of $e \in E(P^*)$ lie in V then, using (7.10), we have

$$\mathbb{E}[\mathbf{s}(e)] \geq \frac{4}{20} \cdot \frac{1}{3}\omega(v)\omega(v') = \frac{1}{15}\omega(e) \geq \frac{1}{80}\omega(e).$$

If e joins $v \in V$ and $w \in W$ the

$$\mathbb{E}[\mathbf{s}(e)] \geq \frac{3}{20} \cdot \frac{1}{4}\omega(w_i) \cdot \frac{1}{3}\omega(v_j) = \frac{1}{80}\omega(e)$$

It remains to prove that $\mathbb{E}[\mathbf{s}(E(G))] \leq \frac{1}{50}$. Note that,

$$s_{i,j}(E(G)) = \left(\frac{1}{2} - (\omega(v_{i_1}) + \omega(v_{i_2}) + \omega(v_{i_3}) + \omega(w_i) + \frac{1}{4}\omega(w_{i_j}))\right) \times$$
$$\times \left(\frac{1}{4}\omega(w_{i_j}) + \frac{1}{3}\left(\omega(v_{i_1}) + \omega(v_{i_2}) + \omega(v_{i_3})\right)\right).$$

We finish the proof using the following technical lemma, the proof of which is included in the appendix.

Lemma 7.6.2. *Suppose given are $x_1, x_2, \ldots, x_{10}, y_1, y_2, \ldots y_5$ reals and*

$$L(y_i) = \{x_{i+1}, x_{i+2}, x_{i+4}, x_{i+5}, x_{i+8}, x_{i+9}\},$$

for each $1 \leq i \leq 5$ such that $0 \leq x_i \leq 1$, $0 \leq y_j \leq 1$ and $\sum_{i=1}^{10} x_i + \sum_{j=1}^{5} y_j = 1$. If there exists some $0 < \delta \leq \frac{1}{90}$ such that $x_i \geq \frac{1}{10} - \delta$ for each $i = 1, 2, \ldots, 10$ then

$$\sum_{\substack{i \neq j \\ x_{i,j_1}, x_{i,j_2}, x_{i,j_3} \\ \in L(y_i) \cap L(y_j)}} \left(\frac{1}{2} - x_{i,j_1} - x_{i,j_2} - x_{i,j_3} - y_i - \frac{1}{4} y_j\right) \left(\frac{1}{4} y_j \ + \frac{1}{3}\left(x_{i,j_1} + x_{i,j_2} + x_{i,j_3}\right)\right) \leq \frac{2}{5}. \tag{7.11}$$

It is easy to see that $\mathbb{E}\left[\mathbf{s}(\mathbf{E}(\mathbf{G}))\right] \leq \frac{1}{50}$. follows from Lemma 7.6.2 applied with $x_i := \omega(v_i)$ and $y_j := \omega(w_j)$. Thus $\mathbf{s}$ is a $1/80$-uniform sparse half, as claimed. $\square$

As promised, Lemma 7.6.1 implies the main theorem of this section.

Theorem 7.6.1. *There exists $\delta > 0$ such that any triangle-free graph G on n vertices which is δ-close to the Petersen graph P has a sparse half.*

Proof. The theorem follows from Theorem 7.4.3, as the Petersen graph satisfies the rquirements of that theorem with $\alpha = 1/500$ by Lemma 7.6.1. $\square$

7.7. Proofs of Numerical Lemmas

Proof of Lemma 7.3.2

Suppose the Lemma is false. Then for each $i = 1, 2, \ldots, 8$

$$\frac{1}{2}\left(\frac{1}{2} - x_i - x_{i+1} - x_{i+2}\right)(x_i + x_{i+2}) + \frac{1}{4}\left(\frac{1}{2} - x_i - x_{i+1} - x_{i+2}\right)^2 > \frac{1}{50}. \tag{7.12}$$

Summing up these inequalities over all $i = 1, 2, \ldots, 8$ we get that

$$
\begin{aligned}
\frac{8}{50} &< \sum_{i=1}^{8} \frac{1}{2}\left(\frac{1}{2} - x_i - x_{i+1} - x_{i+2}\right)(x_i + x_{i+2}) + \sum_{i=1}^{8} \frac{1}{4}\left(\frac{1}{2} - x_i - x_{i+1} - x_{i+2}\right)^2 \\
&= \frac{1}{2}\sum_{i=1}^{8} x_i - \sum_{i=1}^{8} (x_i + x_{i+1} + x_{i+2})\, x_i + \frac{1}{4}\cdot\frac{1}{4}\cdot 8 - \frac{1}{4}\sum_{i=1}^{8}(x_i + x_{i+1} + x_{i+2}) \\
&\qquad\qquad + \frac{1}{4}\sum_{i=1}^{8}(x_i + x_{i+1} + x_{i+2})^2 \\
&= \frac{1}{2} - \sum_{i=1}^{8}(x_i + x_{i+1} + x_{i+2})x_i + \frac{1}{2} - \frac{3}{4} + \frac{1}{4}\sum_{i=1}^{8}(x_i + x_{i+1} + x_{i+2})^2 \\
&= \frac{1}{4} - \frac{1}{4}\sum_{i=1}^{8} {x_i}^2 - \frac{1}{2}\sum_{i=1}^{8} x_i x_{i+2}.
\end{aligned}
\tag{7.13}
$$

Let us find the maximum value under the conditions of the lemma. To find the maximum value of the expression in (7.13), we need to find the minimum value of

$$S := \frac{1}{4}\sum_{i=1}^{8} {x_i}^2 + \frac{1}{2}\sum_{i=1}^{8} x_i x_{i+2}.$$

Claim 7.7.1. *For every* $1 \le i \le 8$

$$x_{i+1} + x_{i+2} + x_{i+3} < 0.394$$

Proof. By inequality (7.12)

$$\frac{1}{2}\left(\frac{1}{2} - (x_{i+1} + x_{i+2} + x_{i+3})\right)(x_{i+1} + x_{i+3}) + \frac{1}{4}\left(\frac{1}{2} - (x_{i+1} + x_{i+2} + x_{i+3})\right)^2 > \frac{1}{50}.$$

Let $\alpha = x_{i+1} + x_{i+2} + x_{i+3}$. Then $x_{i+1} + x_{i+3} = \alpha - x_{i+2} \le \alpha - \frac{1}{14}$. Therefore

$$\frac{1}{2}\left(\frac{1}{2} - \alpha\right)\left(\alpha - \frac{1}{14}\right) + \frac{1}{4}\left(\frac{1}{2} - \alpha\right)^2 = -\frac{\alpha^2}{4} + \frac{\alpha}{28} + \frac{5}{112} > \frac{1}{50},$$

which reduces to the inequality

$$\frac{\alpha^2}{4} - \frac{\alpha}{28} - \frac{69}{2800} < 0.$$

This quadratic inequality gives us the desired $\alpha < 0.393412$ bound. $\square$

It is easy to check that the following claim is true.

Claim 7.7.2.

$$S \geq \frac{1}{2}\sum_{i=1}^{4} {z_i}^2 + \sum_{i=1}^{4} z_i z_{i+2},$$

where $z_i = (x_i + x_{i+4})/2$ *for all* $1 \leq i \leq 4$.

Claim 7.7.3. *For every* $1 \leq i \leq 4$, $z_i > 0.106$.

Proof. For every $1 \leq i \leq 4$ we have

$$1 = (x_i + x_{i+4}) + (x_{i+1} + x_{i+2} + x_{i+3}) + (x_{i+5} + x_{i+6} + x_{i+7}) \overset{(7.7.1)}{<} 2z_i + 2 \cdot 0.394,$$

therefore $z_i > 0.106$. $\square$

Let $\beta = z_1 + z_3$. Then

$$\begin{aligned} S &\overset{(7.7.2)}{\geq} \frac{1}{2}(z_1 + z_3)^2 + \frac{1}{2}(z_2 + z_4)^2 + z_1 z_3 + z_2 z_4 \\ &= \frac{1}{2}\beta^2 + \frac{1}{2}\left(\frac{1}{2} - \beta\right)^2 + z_1 z_3 + z_2 z_4 \\ &\overset{(7.7.3)}{\geq} \frac{1}{2}\beta^2 + \frac{1}{2}\left(\frac{1}{2} - \beta\right)^2 + 0.106 \cdot (\beta - 0.106) + 0.106\left(\frac{1}{2} - \beta - 0.106\right) \\ &= \beta^2 - \frac{1}{2}\beta + 0.155528 \\ &> 0.093 \end{aligned}$$

The last two inequalities hold because for fixed β the expression $z_1 z_3 + z_2 z_4$ achieves its minimum value when $z_1 = z_2 = 0.106$, $z_3 = \beta - 0.106$ and $z_4 = \frac{1}{2} - \beta - 0.106$. The expression $\beta^2 - \frac{1}{2}\beta + 0.155528$ achieves its minimum for $\beta = \frac{1}{4}$. It follows from the inequality above that

$$\frac{1}{4} - \frac{1}{4}\sum_{i=1}^{8} {x_i}^2 - \frac{1}{2}\sum_{i=1}^{8} x_i x_{i+2} = \frac{1}{4} - S < \frac{8}{50},$$

a contradiction that finishes the proof of the lemma.

Proof of Lemma 7.3.3

Suppose the lemma is false. Then for all $1 \leq i \leq 11$ we have

$$\frac{1}{2}\left(\frac{1}{2} - (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4})\right)(x_{i+1} + x_{i+4})$$
$$+ \frac{1}{4}\left(\frac{1}{2} - (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4})\right)^2 > 1/50.$$

Summing these inequalities for $1 \leq i \leq 11$ we obtain

$$\begin{aligned}
\frac{11}{50} &< \sum_{i=1}^{11} \frac{1}{2}\left(\frac{1}{2} - (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4})\right)(x_{i+1} + x_{i+4}) \\
&+ \sum_{i=1}^{11} \frac{1}{4}\left(\frac{1}{2} - (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4})\right)^2 \\
&= \frac{1}{4}\sum_{i=1}^{11}(x_{i+1} + x_{i+4}) - \sum_{i=1}^{11}(x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4})x_{i+1} \\
&+ \frac{11}{16} - \frac{1}{4}\sum_{i=1}^{11}(x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4}) + \frac{1}{4}\sum_{i=1}^{11}(x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4})^2 \\
&= \frac{3}{16} - \sum_{i=1}^{11}(x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4})x_{i+1} + \frac{1}{4}\sum_{i=1}^{11}(x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4})^2 \\
&= \frac{3}{16} + \frac{1}{2}\sum_{i=1}^{11} x_i(x_{i+1} - x_{i+3}) \\
&\leq \frac{3}{16} + \frac{1}{2}\left(\sum_{i=1}^{11} x_i x_{i+1} - \frac{11}{196}\right).
\end{aligned} \tag{7.14}$$

We claim that

$$f(x_1, \ldots, x_1 1) := \sum_{i=1}^{11} x_i x_{i+1} \leq \frac{77}{784}.$$

Indeed, consider any pair (x_i, x_j), such that $j \neq i \pm 1$, let $\alpha = x_i + x_j$ is fixed. Note that f is linear as a function of (x_i, x_j). Therefore $f(x_i, x_j)$ achieves its maximum value on the region $R := \{0 \leq x_i \leq 1/14 \text{ for } 1 \leq 1 \leq 14, \sum_{i=1}^{1} 1x_i = 1\}$, when $x_i = \frac{1}{14}$ and $x_j = \alpha - \frac{1}{14}$, or when $x_j = \frac{1}{14}$ and $x_i = \alpha - \frac{1}{14}$. Thus f attains its maximum on R when all variables are equal $\frac{1}{14}$ except possibly two of them whose indices are consecutive, without loss of generality, say x_{10} and x_{11}. It is easy to see that the maximum is achieved for $x_{10} = x_{11} = \frac{5}{28}$ and is equal to $\frac{77}{784}$, as claimed.

Thus (7.14) implies

$$\frac{11}{50} < \frac{3}{16} + \frac{1}{2}\left(\sum_{i=1}^{11} x_i x_{i+1} - \frac{11}{196}\right) \leq \frac{3}{16} + \frac{1}{2}\cdot\left(\frac{77}{784} - \frac{11}{196}\right) = \frac{327}{1568} < \frac{11}{50},$$

a contradiction that finishes the proof.

Proof of Lemma 7.6.2

Let $Y := \sum_{i=1}^{5} y_i$. Summing inequalities (7.11) over all i, j such that $i \neq j$ and $x_{i,j_1}, x_{i,j_2}, x_{i,j_3} \in L(y_i) \cap L(y_j)$. We get

$$\begin{aligned}
&\sum_{i,j \neq i}\left(\frac{1}{2} - (x_{i,j_1} + x_{i,j_2} + x_{i,j_3} + y_i + \frac{1}{4}y_j)\right)\left(\frac{1}{4}y_j + \frac{1}{3}\left(x_{i,j_1} + x_{i,j_2} + x_{i,j_3}\right)\right)\\
&= \frac{1}{8}\sum_{i,j \neq i} y_j - \frac{1}{4}\sum_{i,j \neq i} y_j\left(x_{i,j_1} + x_{i,j_2} + x_{i,j_3}\right) + \frac{1}{6}\sum_i (x_{i,j_1} + x_{i,j_2} + x_{i,j_3})\\
&- \frac{1}{3}\sum_{i,j}\left(x_{i,j_1} + x_{i,j_2} + x_{i,j_3}\right)^2 - \frac{1}{3}\sum_{i,j}\left(y_i + \frac{1}{4}y_j\right)\left(x_{i,j_1} + x_{i,j_2} + x_{i,j_3}\right) - \frac{1}{4}\sum_{i,j} y_j\left(y_i + \frac{1}{4}y_j\right)\\
&\leq \frac{1}{2}Y - 3Y\left(\frac{1}{10} - \delta\right) + (1 - Y) - \frac{36(1-Y)^2}{60} - 5Y\left(\frac{1}{10} - \delta\right)\\
&= \frac{2}{5} - \frac{Y}{10} - \frac{3}{5}Y^2 + 8\delta Y\\
&\leq \frac{2}{5},
\end{aligned}$$

since $\delta \leq \frac{1}{90}$.

Chapter 8

Conclusion and Future Research

In this chapter we summarize the work presented in this thesis and shortly discuss some possible further applications of the presented methods.

8.1. Quick Summary

In this thesis we presented a generic method which allows one to obtain the Turán number of a family of hypergraphs or a hypergraph once the corresponding Turán density result is known. This method can be viewed as a generalization of the classical stability method, pionereed by Erdős and Simonovits in 1960's. We do so by utilizing the Lagrangian function of graphs and generalizing the symmetrization procedure, initiated by Zykov and Sidorenko. The method alows us to obtain new Turán numbers of several families which all lie in a general class of graphs, called extensions. We believe the method can be universally applicable to derive new Turán numbers from density results. In the next section we consider one such possible application.

8.2. Future Research

Recall the family Σ_r defined in the Chapter 4; it is the family of all r-graphs with three edges such that two of them share $(r-1)$ vertices and the third edge contains the symmetric difference of the first two. In other words, $\Sigma_r = \mathrm{WExt}(\mathcal{D}_r)$ where $\mathcal{D}_r$ is the r-graph on $[r+1]$ with two edges sharing $(r-1)$ vertices. In [Sid87], Sidorenko considered a generalization of this family. Let $\mathcal{D}_t^{(r)}$ denote the r-graph on $[t+r-1]$ and edges $\{1, \ldots, r-1, i\}$, for every $i \in [t+r-1] \setminus [r-1]$, that is, $\mathcal{D}_t^{(r)}$ is the r-graph with t edges sharing some $(r-1)$-tuple and otherwise being disjoint. Clearly, $\Sigma_r = \mathrm{WExt}(\mathcal{D}_2^{(r)})$. Sidorenko [Sid87] considered the Turán number of the family $\mathrm{WExt}(\mathcal{D}_t^{(r)})$ for general r and t.

To state the main results from [Sid87] let us use the following notation. We denote $\Sigma_t^{(r)} = \mathrm{WExt}(\mathcal{D}_t^{(r)})$ and $\alpha_r(n) = n^{1-r}(n-1)(n-2)\cdot\cdots\cdot(n-r+2)$. It is not hard to see that the function $\alpha_r(n)$ decreases for sufficiently large n. Let N_r be the value of n when the function $\alpha_r(n)$ starts decreasing. The results from [Sid87] on $\mathrm{ex}(n, \Sigma_t^{(r)})$ can be summarized as follows.

Theorem 8.2.1 (Sidorenko, [Sid87]). *For given r, t and n, the largest $\Sigma_r^{(t)}$-free graph on n vertices is $\mathcal{K}_{t+r-2}^{(r)}(n)$, the balanced blowup of $\mathcal{K}_{t+r-2}^{(r)}$, if*

(1) $r \le 3$ and any t, n,

(2) $r = 4$, $t = 2$ and any n,

(3) for any $t \ge N_r$ and any n which is a multiple of $t+r-2$,

(4) for any $t \ge N_r$ and all sufficiently large n.

Note that (1) is equivalent to Mantel's theorem and the result of Bollobás mentioned in Chapter 4, Theorem 4.1.1, taken together. As for (2), we already stated this result of Sidorenko in Chapter 4, Theorem 4.1.3. Now let us discuss (3) and (4). As Sidorenko observed in his paper, the condition of requiring t to be large enough with respect to r for $r \ge 5$ is necessary. Indeed, when $r = 5$, $t = 2$, we get Σ_5 for which the extremal graphs turn out to be the balanced blowups of Steiner system $(11, 5, 4)$, as Frankl and Füredi showed (see Theorem 4.1.6) rather than the balanced blowups of $\mathcal{K}_5^{(5)}$.

Motivated by the family $\Sigma_t^{(r)}$, we suggest to consider the following generalization of the generalized triangle, $\mathcal{T}_t^{(r)} = \mathrm{Ext}(\mathcal{D}_t^{(r)})$. Note that the generalized triangle, $\mathcal{T}_r$ is simply $\mathcal{T}_2^{(r)}$. By Lemma 3.8.1, we know that $\pi(\mathcal{T}_t^{(r)}) = \pi(\Sigma_t^{(r)})$. We suggest the following generalization of Conjecture 4.1.1.

Conjecture 8.2.1. *For any integers $r \geq 2$ and $t \geq 2$, there exists $n_0 := n_0(r)$ such that for all $n \geq n_0$,*

$$\mathrm{ex}(n, \mathcal{T}_t^{(r)}) = \mathrm{ex}(n, \Sigma_t^{(r)}).$$

To support Conjecture 8.2.1, let us show that it is true for the values of parameters that Theorem 8.2.1 covers. Recall that an (m, r, q, λ)-design is an r-graph on m vertices such that every q-tuple is contained in exactly λ edges. It was a long-standing conjecture of Steiner from 1853 that such designs exist for all sufficiently large m if the neccessary divisibility conditions hold (these are $\binom{q-i}{r-i}$ divides $\lambda\binom{m-i}{r-1}$ for every $0 \leq i \leq r-1$). In [Kee14] Keevash proved that Steiner's conjecture is true.

Let $\mathfrak{F}^*$ be the family of r-graphs in $\mathrm{Forb}(\mathcal{D}_t^{(r)})$ that cover pairs. Using our ideas from Chapter 4 we believe that the following result must follow.

Theorem 8.2.2. *For given $m, r \geq 2, t \geq 2$ let D be a $(m, r, r-1, t-1)$-design that is uniquely dense and balanced. If $\mathcal{D}$ is the unique Lagrangian maximizer of $\mathfrak{F}^*$, then* $\mathrm{Forb}(\mathcal{T}_t^{(r)})$ *is $\mathfrak{B}(\mathcal{D})$-stable.*

We shortly discuss here how this result follows along the same lines as the proof of Theorem 4.1.8. Let $\mathfrak{B} = \mathfrak{B}(\mathcal{D})$. Following Theorem 3.8.2, we would like to show that the following two conditions hold.

(C1) $\mathrm{Forb}(\mathcal{T}_t^{(r)})$ is $\mathfrak{B}$-vertex locally stable,

(C2) The family $\mathfrak{F}^*$ is $\mathfrak{B}$-weakly weight-stable.

To prove (C1), one can follow exact same steps as in Section 4.2. It is not hard to see that $\mathcal{T}_t^{(r)}$ is $\mathcal{D}_t^{(r)}$-hom-critical. Hence, the tools developed in Section 3.9 are applicable. Next, we can obtain an embedding lemma that allows to embed every large $\mathcal{T}_t^{(r)}$-free graph into a blowup of $\mathcal{D}$, meaning that if $\mathcal{G}$ is an $\mathcal{T}_t^{(r)}$-free graph with large minimum degree and has a small edit distance to the family $\mathfrak{B}$ then we can find a blowup $\mathcal{B} \in \mathfrak{B}$ on the same vertex set as $\mathcal{G}$ such that every vertex has almost the same neighbourhoods in $\mathcal{G}$ and $\mathcal{B}$. The steps to pursue are along the lines of the proof of Theorem 4.2.3, just a little bit more technical. To finish the proof of (C1), the only next key step is to obtain the analogous statement as in Claim 4.2.8, that is, to show that we can bound the number of bad edges that contain some i-tuple from above by the proportion of the missing edges that contain any of its $(i-1)$-subtuples. This is also true along the same lines as in the proof of Claim 4.2.8, except that it gets more technical in computations.

Now let us discuss the proof of (C2). To this end, we can prove that a result analogous to Theorem 4.3.3 holds. For an integer k, we say that an r-graph is k-*thin* if every $(r-1)$-tuple is contained in at most k edges.

Theorem 8.2.3. *For any fixed integer k, if $\mathfrak{F}^*$ is a k-thin family such that $\lambda(\mathfrak{F}^*) = \lambda(\mathcal{F}^*)$ for some $\mathcal{F}^* \in \mathfrak{F}^*$, then it is $\mathfrak{F}^{**}$-weakly weight-stable, where*

$$\mathfrak{F}^{**} = \{\mathcal{F}^*|_{\mathrm{supp}(\mu)} \mid \mathcal{F}^* \in \mathfrak{F}^*,\ \lambda(\mathcal{F}^*, \mu) = \lambda(\mathfrak{F}^*) \text{ for some } \mu \in \mathcal{M}(\mathcal{F}^*)\}.$$

Given Theorem 8.2.2, we can verify Conjecture 8.2.1 for some cases. Clearly the graph $\mathcal{K}_{t+r-2}^{(r)}$ is $(t+r-2, r, r-1, t-1)$-design and is balanced and uniquely dense. Also note that from the results of [Sid87] it can be implied that the following holds.

Theorem 8.2.4 (Sidorenko, [Sid87])**.** *For positive integers $t, r \geq 2$, $\mathcal{K}_{t+r-2}^{(r)}$ is the unique Lagrangian maximizer of the family $\mathfrak{F}^*$ when $r \leq 4$ or $t \geq N_r$.*

Thus, by Theorem 8.2.2 and Theorem 8.2.4 we can derive the following result which confirms Conjecture 8.2.1 in certain cases.

Theorem 8.2.5. *For integers $r \geq 2, t \geq 2$ such that either $r \leq 4$ or $t \geq N_r$, there exists n_0 such that the largest $\mathcal{T}_t^{(r)}$-free graph on $n \geq n_0$ vertices is the balanced blowup of $\mathcal{K}_{t+r-2}^{(r)}$, that is, $\mathcal{K}_{t+r-2}^{(r)}(n)$.*

Lastly, we would like to discuss the first case that is different from the question of the Turán number of the generalized triangle and also, the extremal graph is not the balanced blowup of $\mathcal{K}^{(r)}_{t+r-2}$. Let $r = 5$, $t = 3$. Let $\mathcal{D}_{11}$ be a design with parameters $(11, 5, 4, 2)$. It is easy to see that

$$\lambda(\mathcal{D}_{11}) = \frac{2\binom{11}{4}}{5} \cdot \frac{1}{11^5} > 6 \cdot \frac{1}{6^5} = \lambda(\mathcal{K}^{(5)}_6).$$

In fact, we suspect that that the balanced blowups of $\mathcal{D}_{11}$ are the extremal graphs for the question of Turán number of $\mathcal{T}^{(5)}_3$ and $\Sigma^{(5)}_3$.

Conjecture 8.2.2. *There exists some n_0 such that for all $n \geq n_0$,*

$$\text{ex}(n, \Sigma^{(5)}_3) = \text{ex}(n, \mathcal{T}^{(5)}_3),$$

and, moreover, for all such n the largest $\Sigma^{(5)}_3$-free and $\mathcal{T}^{(5)}_3$-free r-graphs are the balanced blowups of $\mathcal{D}_{11}$.

To prove the above conjecture, one needs in particular, to show that $\mathcal{D}_{11}$ is the unique Lagrangian maximizer for the corresponding family $\mathfrak{F}^*$. The calculation that shows that the Lagrangian maximizer must have $6 \leq m \leq 11$ vertices, is quite easy. The only m among these for which the divisibility conditions hold with respect to parameters $(m, 5, 4, 2)$ is $m = 6$ and $m = 11$. Further work is required to show that the *partial designs* with these parameters (that is, every 4-tuple is contained in at most two edges) on the vertices $7 \leq m \leq 10$ and which cover pairs have smaller Lagrangian than $\mathcal{D}_{11}$.

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