Monopole Metrics and Rational Functions

by

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Abstract.

We study the natural L^2-metrics on moduli spaces of monopoles, or alternately of rational maps from $\mathbf{C}P^1$ into flag manifolds. We prove estimates showing that for $SU(2)$ these metrics are asymptotic to flat metrics arising from a "particle" description of these moduli spaces. More precisely, there is a natural separation parameter r for these particles and we prove that the metric is within cr^{-1} of the flat metric, and that the curvature is bounded by cr^{-3}, with c a constant. Also, the spectral curves of the monopoles approach the spectral curves of distinct charge one monopoles exponentially fast. We give some comments on how these estimates should extend to the case $SU(N)$.

We also examine the twistor construction of these metrics for the case of $SU(N)$ monopoles with maximal symmetry breaking. In particular, we calculate the symplectic form associated to this construction.

Résumé.

Nous étudions la metrique L^2 naturelle sur les espaces de modules de monopoles, qui sont aussi des espaces de modules d'applications rationnelles de $\mathbf{C}P^1$ dans des variétés de drapeaux. Nous obtenons des estimés, qui permettent de démontrer que, pour $SU(2)$, ces métriques sont asymptotiques à des métriques plates qui proviennent d'une description de ces espaces de modules en termes de particules. Plus précisément, il existe un paramètre de séparation r pour ces particules, et nous montrons que la métrique est distante de cr^{-1} de la métrique plate, et que la courbure est bornée par cr^{-3}, c une constante. Aussi, les courbes spectrales des monopoles approximent à un taux exponentiel les courbes spectrales des monopoles de charge un. Nous discutons de l'extension de ces estimés au cas $SU(N)$.

Nous considérons aussi le construction twistorielle de ces métriques pour la cas de monopoles $SU(N)$ avec brisure maximale de symétrie. En particulier, nous calculons la forme symplectique associée à cette construction.

Contents

I. Introduction

The concept of a magnetic monopole, as an isolated point-source of magnetic charge was introduced by Dirac [1931]. Such a point-particle is an essential (nonsmoothable) singularity in the magnetic field and as such is physically unacceptable. The basic reason for this is the linearity of Maxwell's equations which follows from their $U(1)$ gauge invariance.

On the other hand the unified theories in which the abelian gauge group $U(1)$ of Maxwell's theory is replaced by non-abelian groups like $SU(2)$ admit solutions to the field equations that look asymptotically like Dirac monopole but are smoothed out and have finite energy. The first such a non-abelian monopole was discovered in 1975 by Bogomolny [1976] and Prassad and Sommerfeld [1975]. It is a *static* monopole in $\mathbf{R}^3$ of charge 1. It is unique up to gauge transformations and translations in $\mathbf{R}^3$. This discovery encouraged the search for multimonopoles, i.e. monopoles of charge greater than 1. Such a multimonopole should be an approximation of a superposition of well separated 1-monopoles. The first proof of their existence was given along these lines by Taubes (Jaffe and Taubes [1980]). The first actual solution was produced by Ward [1981] by twistor methods. The monopole equations, called *Bogomolny* equations, are a dimensional reduction of self-dual Yang-Mills equations on $\mathbf{R}^4$ and therefore the twistor methods of Penrose and Ward can be used. This construction was given a more geometrical form by Hitchin [1982]. He showed in particular that a monopole is determined by an algebraic curve - the *spectral curve* - in TP^1.

In all these approaches checking the non-singularity of a solution was a major difficulty. Nahm [1982] used an infinite-dimensional version of the ADHM construction of instantons to replace the Bogomolny equations by a system of ordinary differential equations for which the non-singularity of the monopole can be seen directly. The full proof of the natural equivalence between the $SU(2)$ monopoles and solutions to Nahm's equations was given by Hitchin [1983].

At the same time there had been an interest in the space of all monopole solutions, modulo gauge equivalence - the *moduli space*. Taubes [1983],[1985] established the existence and smoothness of the moduli space, its dimension and asymptotic properties of the energy functional. The actual structure of the moduli space of $SU(2)$ monopoles was first analysed for charge 2 by Hurtubise [1983] using the spectral curve description. In the case of an arbitrary charge k it was Donaldson [1984] who identified a circle bundle over the moduli space with the space of based rational maps from CP^1

to itself. This isomorphism was given by Donaldson via Nahm's equations and it was Hurtubise [1985] who defined the rational map directly in terms of the monopole. It is worth pointing out that both monopoles and rational maps represent the minima of certain energy functionals which allows a Morse-theoretic approach to their structure (Taubes [1984]).
The monopoles exist also for other classical groups like $SU(N)$ and their construction was first analysed by Murray [1984]. From some of them, that have *maximal symmetry breaking* at infinity, we can expect a particle-like behaviour similar to $SU(2)$ monopoles. For these the equivalence with the solutions to Nahm's equations was established by Hurtubise and Murray [1989] and the isomorphism of the moduli space with the space of based rational maps into flag varieties by Hurtubise [1989].
The interest in moduli spaces of monopoles and in particular in their natural L^2-*metric* was given a major impact by a conjecture concerning the motion of low-energy monopoles made by Manton [1982]. His conjecture, whose proof requires some infinite-dimensional analysis, is that the motion of low-energy monopoles should be given approximately by geodesic motion on the moduli space. Using this assumption, Atiyah and Hitchin [1988] analysed the metric for charge 2 monopoles and found some interesting facts about the scattering of two monopoles.
It is therefore important to know the metric structure for moduli spaces of monopoles. The natural L^2-metric is *hyperkähler* which in particular implies that we can encode the metric data in a holomorphic form, the *twistor space*. Thus to find the hyperkähler metric on a moduli space of monopoles one has only (in principle) to find its corresponding twistor space. This was done for $SU(2)$ monopoles by Atiyah and Hitchin [1988]. It is perhaps worth pointing out that the Donaldson isomorphism gives rise to a hyperkähler metric on the space of rational maps, a fact which is difficult to see directly.

We want, therefore, to study the natural L^2-metric on the moduli spaces of monopoles. This separates into two problems. One is to study the asymptotic behaviour of the metric which, in view of Manton's conjecture, gives an information on the motion of well separated monopoles. In particular we would like to know whether the metric reflects the fact that the multimonopoles are an approximate superposition of charge 1 $SU(2)$ monopoles, i.e. particles. The results of Atiyah and Hitchin [1988] and Connell [1991] show that this is the case for $SU(2)$ monopoles of charge 2 and $SU(3)$ monopoles of charge $(1,1)$ (on the other hand the results of Dancer [1993] seem to indicate this *not* being true for $SU(3)$ monopoles of charge $(2,1)$ with *non-maximal* symmetry breaking). Their results show also that the metric becomes the flat product metric at the rate of the inverse of the separation distance of particles, while the curvature as the cube of it.

We would like to show eventually that this fact is true for $SU(N)$ monopoles (with maximal symmetry breaking) with arbitrary charges. Here, however, we do it for $SU(2)$ monopoles of arbitrary charge. In fact, we show that the Donaldson isomorphism between the moduli space $\mathcal{M}_k$ of monopoles of charge k and the space of based rational maps of degree k on $\mathbf{C}P^1$ is an asymptotic isometry between $\mathcal{M}_k$ and $(\mathbf{C}\times\mathbf{C})^k$ in the natural coordinates provided by this isomorphism.

We are aided in this by the fact that the moduli spaces of $SU(2)$ monopoles and the space of of gauge equivalent solutions to Nahm's equations are isometric (Nakajima [1991]) (the similar fact for $N>2$ is not known). This leads us to study the asymptotic behaviour of solutions to Nahm's equations. We obtain precise estimates, e.g. we show the exponential decay of the solutions. This allows us to show that the metric on the moduli space of monopoles approximates the flat product metric at the rate of the inverse of the separation distance of particles (in the natural coordinates provided by Donaldson's isomorphism), while the curvature as the cube of it.

We also show that the spectral curve of a monopole, which is the "controlling element" in the particle picture becomes the union of spectral curves of 1-monopoles exponentially fast.

We then indicate how our proof of the asymptotic behaviour of the metric on the moduli space of solutions to $SU(2)$ monopoles can be adapted to the $SU(N)$ case. In this case the Nahm's equations are defined on several intervals; our analysis of the metric will carry on each of them - there remains question of matching the solutions on different intervals.

All of this can be seen from the point of view of rational maps. The monopole metrics can be thought of as natural hyperkähler metrics on the spaces of rational maps from $\mathbf{C}P^1$ into flag manifolds. These spaces have many interesting topological properties, e.g. they are stable homotopy equivalent to the spaces of *all* based maps of fixed degree from $\mathbf{C}P^1$ into flag manifolds (Segal [1976]).

The other problem we are dealing with is to give the twistor space description of the moduli spaces of $SU(N)$ monopoles with maximal symmetry breaking. This involves the knowledge of the symplectic form on rational maps given by the Donaldson isomorphism. In the case $N=2$, Atiyah and Hitchin [1988] were able to circumvent it by showing that the metric is irreducible, i.e. the symplectic form must be the desired one. In case of arbitrary N one cannot appeal to a similar fact (there is no unique axisymmetric monopole in every direction). We therefore calculate the symplectic form directly from the Nahm's equations which gives us the full twistor space description of the moduli spaces of $SU(N)$ monopoles with maximal symmetry breaking (modulo a Plancherel type theorem on isometry between the moduli spaces of $SU(N)$ monopoles and Nahm's equations).

1.1 Magnetic monopoles

In this section we recall the basic notions on magnetic monopoles after Atiyah and Hitchin [1988].

The data for an $SU(N)$ magnetic monopole consists of a connection A on the (trivial) principal bundle

$$\begin{array}{c} P \\ \downarrow \\ R^3 \end{array}$$

and a section Φ (*Higgs field*) of the associated adjoint bundle

$$\begin{array}{c} \mathrm{ad}\, P \\ \downarrow\uparrow\ \Phi \\ R^3 \end{array}$$

which minimize the action

$$\int_{R^3} |F_A|^2 + |d_A\,\Phi|^2$$

where F_A is the curvature of A, d_A - the exterior covariant derivative and $|S|^2 = \mathrm{tr}\, S^* S$.

This condition is equivalent to the action being finite and (A, Φ) satisfying the Bogomolny equation

$$*F_A = d_A\,\Phi$$

Finiteness of the action implies in turn that Φ has a limit over S^2_∞ (the sphere of radial directions):

$$\Phi^\infty : S^2 \to su(N)$$

whose image lies in some adjoint orbit of $SU(N)$.

Such an orbit is of the form

$$SU(N)/C(T)$$

where T is a torus in $SU(N)$ and $C(T)$ - its centralizer.
If T is a maximal torus, we say that the monopole has *maximal symmetry breaking at infinity.* We will be considering only monopoles with maximal symmetry breaking.
Since the image of Φ^∞ lies in a single orbit, $-i\Phi^\infty$ has q ($q = \dim T + 1$) distinct *real* eigenvalues

$$\mu_1 < \ldots < \mu_q$$

with some multiplicities. In the case of maximal symmetry breaking $q = N$ and all these multiplicities are 1.
Moreover, the map

$$\Phi^\infty : S^2 \to SU(N)/\mathrm{C(T)}$$

defines an element of the second homotopy group

$$\pi_2\left(SU(N)/C(T)\right)$$

which is isomorphic to Z^{q-1}. Hence we get $q-1$ integers

$$m_1, \ldots, m_{q-1}$$

called *magnetic charges* of the monopole.

We frame the monopoles, i.e. we fix a basis at some point at infinity (say corresponding to the x_3-axis) and consider only the monopoles whose Higgs field is diagonal there with the eigenvalues μ_i ordered as above.
The gauge group acts on the space of all monopoles and the framing condition is preserved by gauge transformations that are 1 at the basepoint.
The *moduli space* is now defined by identifying the gauge equivalent monopoles:

$$\mathcal{M}_{m_1,\ldots,m_{q-1}} = \left\{ \begin{array}{l} \text{framed monopoles} \\ \text{of charge } m_1, \ldots, m_{q-1} \end{array} \right\} \Big/ \left\{ \begin{array}{l} \text{group of based gauge} \\ \text{transformations} \end{array} \right\}$$

In the case of maximal symmetry breaking it is known (Hurtubise [1989]) that $\mathcal{M}_{m_1,\ldots,m_{N-1}}$ is a smooth manifold of dimension $4(m_1 + \ldots + m_N)$.
We can define a natural metric on the moduli space of monopoles.
First of all we have a metric on the space $\mathcal{M}$ of all monopoles: if (A, Φ) is monopole with $A = A_1\,dx_1 + A_2\,dx_2 + A_3\,dx_3$, and (a_1, a_2, a_3, ϕ) is an infinitesimal variation of (A, Φ) ,then we can put

$$\| (a_1, a_2, a_3, \phi) \|^2 = -\int_{R^3} \mathrm{tr}\left(\sum_{i=1}^{3} a_i^2 + \phi^2\right)$$

The gauge group acts on $\mathcal{M}$ as isometries, so we can define the metric on $\mathcal{M}_{m_1,\ldots,m_{N-1}}$ as follows: each element (a,ϕ) of $\mathbf{T}\mathcal{M}_{m_1,\ldots,m_{N-1}}$ has a representative $(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,\tilde{\phi}) \in \mathbf{T}\mathcal{M}$ orthogonal to infinitesimal gauge transformations; we define the norm of (a,ϕ) to be the norm of this representative.
With this choice of metric, $\mathcal{M}_{m_1,\ldots,m_{N-1}}$ becomes a Riemannian manifold. The metric has a very special property: it is *hyperkähler*. This means that it is Kähler with respect to three complex structures I,J,K that define an action of unit imaginary quaternions. It is quite easy to see this action: we can think of (a_1,a_2,a_3,ϕ) as the element $\phi + ia_1 + ja_2 + ka_3$ of a quaternionic vector space.
Consider now $SU(2)$ monopoles of charge 1. Their moduli space has dimension 4 and if we divide out by the circle factor ($U(1)$ action preserves the framing) we get a space diffeomorphic to R^3. We can therefore think of an $SU(2)$ monopole of charge 1 as a point in R^3. Correspondingly we could think of an $SU(N)$ monopole of charge $(m_1,\ldots,m_{N-1})$ as an approximate superposition of $m_1+\ldots+m_{N-1}$ charge 1 $SU(2)$-monopoles, i.e. as a sum of particles, with the approximation getting better if the particles are far apart. Indeed, the results of Taubes[1985] for $N=2$ indicate that there is an asymptotic region of the moduli space in which the energy of a monopole approximates the energy of such superposition.
We proceed to give an equivalent description of the moduli space of the monopoles.

1.2 Nahm's equations

There is a generalized Fourier transform, called *Nahm transform*, obtained by coupling Dirac's equation with a connection. For details we refer to Nahm[1982], Hitchin[1983], Hurtubise and Murray[1989] or Nakajima [1991]. It gives a natural equivalence (in fact a diffeomorphism) between the moduli space $\mathcal{M}_{m_1,\ldots,m_{N-1}}$ and the (moduli) space $\mathcal{N}_{m_1,\ldots,m_{N-1}}$ of gauge equivalent matrix valued functions on $\mathbf{R}$:

$$T_0(t), T_1(t), T_2(t), T_3(t)$$

If the framing for $\mathcal{M}_{m_1,\ldots,m_{N-1}}$ was $\mu_1 < \ldots < \mu_N$, then the matrices T_i have rank m_j on the interval (μ_j,μ_{j+1}), and are 0 outside of (μ_1,μ_N). The matrices $T_i(t)$ are skew-symmetric and they satisfy Nahm's equations:

$$\dot{T}_i + \tfrac{1}{2}[T_0,T_i] + \tfrac{1}{2}\sum_{j,k=1,2,3} \epsilon_{ijk}[T_j,T_k] = 0 \tag{1.1}$$

for $i = 1,2,3$, i.e.

$$\dot{T}_1 + \tfrac{1}{2}[T_0, T_1] + [T_2, T_3] = 0$$

and similar two equations given by a cyclic permutation of indices $1,2,3$.
We can think of $(T_0(t), T_1(t), T_2(t), T_3(t))$ as components of an (anti-self-dual) R^3-invariant connection on R^4. It is then clear that the gauge group of unitary functions $g(t)$ acts as follows:

$$\begin{aligned} &T_0 \mapsto gT_0g^{-1} - \tfrac{1}{2}\dot{g}g^{-1} \\ &T_i \mapsto gT_ig^{-1} \qquad\qquad i = 1,2,3 \end{aligned} \tag{1.2}$$

The final part of Nahm's data are the boundary conditions at points μ_i. In general these can be quite complicated, so we postpone the $SU(N)$-case until the section 3.1 and concentrate for the time being on the case $N = 2$, i.e. $SU(2)$ monopoles.
In this case we have just two eigenvalues μ_1, μ_2 which we can take to be ± 1 and one magnetic charge $m_1 = k$. Therefore T_i-s are now $u(k)$-valued functions on $(-1,1)$. They are analytic in a neighbourhood of $[-1,1]$ with the exception of ± 1 where T_1, T_2, T_3 have simple poles with the residues defining the k-dimensional irreducible representation of $su(2)$. For instance $\operatorname{res} T_1$ is conjugate to

$$\begin{pmatrix} -\frac{k-1}{4} & 0 & \dots & \dots & 0 \\ 0 & -\frac{k-3}{4} & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \frac{k-1}{4} \end{pmatrix} \tag{1.3}$$

The framing condition for monopoles corresponds now to a choice of unit vectors v_-, v_+ at $-1, +1$ the $-\left(\frac{k-1}{4}\right)$-eigenspace of $\operatorname{res}_{\pm 1} T_1$.
The gauge group consists of analytic functions

$$g(t) : [-1,1] \to U(k)$$

which act on T_i-s by (1.2) and on the vectors $v_\pm$ by

$$\begin{aligned} v_- &\to g(-1)v_- \\ v_+ &\to g(1)v_+ \end{aligned}$$

If we now identify the gauge equivalent 6-tuples

$$(T_0, T_1, T_2, T_3, v_-, v_+)$$

we obtain a manifold $\mathcal{N}_k$ diffeomorphic to the moduli space $\mathcal{M}_k$ of charge k $SU(2)$ monopoles.

We would like to define a metric on $\mathcal{N}_k$. The natural candidate is the L^2-metric on $[-1,1]$. First, however, we must fix the residues of T_i-s so that the variations will not have poles. We can use part of gauge freedom to put the residues of T_1, T_2, T_3 into the standard form of the representation of $su(2)$. We can also fix the boundary vectors to be $v_\pm = (1,0,\ldots,0)^T$. We allow then only gauge transformations that are 1 at ± 1. This gives us an alternative description of $\mathcal{N}_k$:

$$\mathcal{N}_k = \left\{ \begin{array}{l} \text{solutions to Nahm's equations} \\ \text{with fixed residues and boundary vectors} \end{array} \right\} \Big/ \left\{ \begin{array}{l} \text{gauge transformations} \\ \text{that are 1 at } \pm 1 \end{array} \right\}$$

Now we can describe the tangent space to $\mathcal{N}_k$ as the space of infinitesimal variations $(t_o, \ldots, t_4)$ (i.e. functions that satisfy linearized Nahm's equations) which are orthogonal to infinitesimal gauge transformations in the L^2-scalar product. This defines a Riemannian metric on $\mathcal{N}_k$:

$$\| (t_o, \ldots, t_4) \|^2 = -\int_{-1}^{1} \operatorname{tr} \sum_{i=0}^{4} t_i^2$$

As for monopoles, it is a hyperkähler metric. In fact, since the Nahm transform is analogous to the Fourier transform it is not surprising that it is an isometry:

Theorem 1.2.1 (Nakajima[1991]) *The Nahm transform*

$$\mathcal{M}_k \longrightarrow \mathcal{N}_k$$

is an isometry.

Maciocia [1992] proved a similar theorem for instantons.

1.3 Complex structure and rational functions

Because of the above theorem of Nakajima, we can study the metric properties of the moduli space of solutions to Nahm's equations instead of the moduli space of magnetic monopoles. We would like to compare the metric on $\mathcal{N}_k$ to the one of a simpler manifold. It is well known (Donaldson [1984]) that $\mathcal{N}_k$ is diffeomorphic to the space of degree k based rational functions from $\mathbf{C}P^1$ into itself. Similar result was shown by Hurtubise [1989] for $SU(N)$ monopoles. Let us indicate how such a diffeomorphism is realised.

We will have, in fact, a 2-dimensional sphere of such natural diffeomorphisms; one for each direction in R^3 (see section 1.4). Choosing such a direction corresponds to choosing an isomorphism (compatible with the usual metrics) $\mathbf{R}^3 = \mathbf{R} \times \mathbf{C}$ which allows to introduce complex coordinates $(t + ip_1, p_2 + ip_3)$ and to write the matrices T_i-s in a similar fashion: if the direction chosen is, say, the positive x_1-axis, we put

$$\alpha \stackrel{\text{def}}{=} \tfrac{1}{2}(T_0 + iT_1)\,,\ \beta \stackrel{\text{def}}{=} \tfrac{1}{2}(T_2 + iT_3) \tag{1.4}$$

Had we chosen the x_2-axis instead, we would have put $\alpha = \frac{1}{2}(T_0+iT_2)\,,\ \beta = \frac{1}{2}(T_3+iT_1)$. Having chosen α and β, we can write Nahm's equations as

$$\frac{d}{dt}\beta = 2[\beta, \alpha] \qquad \text{(complex equation)} \tag{1.5}$$

$$\frac{d}{dt}(\alpha + \alpha^*) + 2[\alpha, \alpha^*] + 2[\beta, \beta^*] = 0 \qquad \text{(real equation)} \tag{1.6}$$

Let us recall from the previous section that we fixed the residues at ± 1. The residues of α, β become

$$\operatorname{res}\alpha = \begin{pmatrix} -\frac{k-1}{4} & 0 & \cdots & \cdots & 0 \\ 0 & -\frac{k-3}{4} & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \frac{k-1}{4} \end{pmatrix}, \quad \operatorname{res}\beta = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \delta_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \delta_{k-1} & 0 \end{pmatrix} \tag{1.7}$$

where $\delta_i = \left(\frac{i(k-i)}{4}\right)^{1/2}$.

We also fixed the boundary vectors to be

$$v_\pm = (1, 0, \ldots, 0)^T \tag{1.8}$$

Following Donaldson [1984] and Hurtubise[1989] we adopt the following definition

Definition 1.3.1 *i) A* real Nahm complex *is a pair* (α, β) *of* $gl(k)$*-valued functions, smooth on* $(-1,1)$*, satisfying there (1.5) and (1.6) and having simple poles at* ± 1 *with residues given by (1.7).*
ii) a Nahm complex is a pair (α,β) *of* $gl(k)$*-valued functions, smooth on* $(-1,1)$*, satisfying there (1.5) and having simple poles at* ± 1 *with residues given by (1.7).*

It is obvious that we have

$$\mathcal{N}_k = \left\{ \text{ real Nahm complexes } \right\} \Big/ \left\{ \begin{array}{l} \text{unitary gauge transformations} \\ \text{that are 1 at } \pm 1 \end{array} \right\}$$

where the action of the unitary gauge group is given by

$$\alpha \mapsto g\alpha g^{-1} - \tfrac{1}{2}\dot{g}g^{-1} \quad , \quad \beta \mapsto g\beta g^{-1} \tag{1.9}$$

On the other hand the complex equation (1.5) is preserved by *complex* gauge transformations, i.e. $gl(k)$-valued functions $g(t)$, even the ones that are singular at ± 1. The essence of Donaldson's proof is the fact that in every complex gauge orbit of a Nahm complex there is a unique orbit of *real* Nahm complexes. To define the map

$$\mathcal{N}_k \longrightarrow \mathrm{Rat}_k\left(\mathbf{C}P^1\right)$$

it is therefore enough to define a complex gauge invariant map from the space of Nahm complexes to the space of rational functions. This is done as follows (we follow here Hurtubise [1989] whose description is equivalent to Donaldson's [1984]):
By a singular gauge transformation we can always transform any Nahm complex into a regular one; then we can solve the equation

$$\dot{g} = 2\alpha g$$

with $g(-1) = 1$ (this is equivalent to making $\alpha = 0$ or, in other words, to choosing a flat connection). The complex equation implies then that β becomes constant. Consider the resulting Nahm complex $(0, \beta, v_-, v_+)$. We can define a covector s by

$$s\beta^i v_- = \delta_{i,k-1} \tag{1.10}$$

Now we define a rational function of degree k on $\mathbf{C}P^1$ by

$$f(z) = s(z-\beta)^{-1}v_+ \tag{1.11}$$

It is clear that $f(\infty) = 0$. As we mentioned before Donaldson showed that $\mathcal{N}_k$ is diffeomorphic to the space $\mathrm{Rat}_k(\mathbf{C}P^1)$ of such based rational functions (in fact Donaldson showed a natural equivalence; the diffeomorphism was proved by Boyer and Mann [1988]).

A function

$$f(z) \in \mathrm{Rat}_k\left(\mathbf{C}P^1\right)$$

can be written as a quotient

$$f(z) = \frac{p(z)}{q(z)}$$

of polynomials p of degree $\leq k-1$ and q of degree k. Moreover p, q cannot have a common factor. Therefore, if β is a pole of f, then $p(\beta) \neq 0$. If the poles of f are distinct, say $\beta_1, \ldots, \beta_k$, then

$$(\beta_1, \ldots, \beta_k, p(\beta_1), \ldots, p(\beta_k)) \tag{1.12}$$

are good local coordinates on $\mathrm{Rat}_k(\mathbf{C}P^1)$ and hence on $\mathcal{N}_k$. Moreover, Atiyah and Hitchin [1988] showed that the symplectic structure on $\mathcal{M}_k$ is, in these coordinates equal to

$$\sum_{i=1}^{k} d\beta_i \wedge \frac{dp(\beta_i)}{p(\beta_i)} \tag{1.13}$$

This suggests that we should be comparing the L^2-metric on $\mathcal{N}_k$ with the flat metric given in coordinates (1.12) by

$$\left(\sum_{i=1}^{k} |d\beta_i|^2 + \left| \frac{dp(\beta_i}{p(\beta_i)} \right|^2 \right)^{1/2} \tag{1.14}$$

This metric has also another natural interpretation. Let a rational function $f(z)$ be defined by (1.11). The poles of f are given by the eigenvalues of β. If the eigenvalues are distinct we can diagonalize β to get, say

$$\mathrm{diag}\,(\beta_1, \ldots, \beta_k)$$

The vector v_- (which is cyclic) can be changed by a constant diagonal gauge transformation to be

$$(1, \ldots, 1)^T \tag{1.15}$$

The covector s, defined by (1.10), is then equal to

$$s = \left(\prod_{j \neq 1} (\beta_1 - \beta_j)^{-1}, \ldots, \prod_{j \neq k} (\beta_k - \beta_j)^{-1} \right)$$

It follows from (1.11) that the vector v_+ must be equal to

$$(p(\beta_1), \ldots, p(\beta_k))^T$$

If we perform the gauge transformation

$$\exp\{-\mathrm{diag}\,(p(\beta_i))(1+t)\}$$

we will get a Nahm complex

$$(\alpha_d, \beta_d, w_\pm) \tag{1.16}$$

where $\alpha_d = \mathrm{diag}\,(\alpha_i)$, $\alpha_i = \frac{1}{2}\ln p(\beta_i)$, $\beta_d = \mathrm{diag}\,(\beta_i)$, $w_\pm = (1, \ldots, 1)^T$.
The space of Nahm complexes of the form (1.16) is acted on only by the permutation matrices and gauge transformations of the form $e^{m\pi i(t+1)}$. As this is a discrete group, $(\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k)$ form local coordinates on the subset of $\mathcal{N}_k$ where the eigenvalues of $\beta = T_2 + iT_3$ are distinct (the complex equation implies that the eigenvalues of β are independent of t). Moreover the flat product metric given by

$$\left(\sum_{i=1}^{k} |d\beta_i|^2 + |d\alpha_i|^2\right)^{1/2} \tag{1.17}$$

is the one given by (1.14).
What is really nice is that a complex of the form (1.16) also satisfies the real Nahm equation (1.6). This will give us a tool when estimating the behaviour of real Nahm complexes.
Let us denote the space of constant diagonal Nahm complexes of the form (1.16) with distinct β_i-s (modulo permutation matrices and gauge transformation $e^{m\pi i(t+1)}$) by $\mathcal{N}_k^d$. It is locally isometric to $(\mathbf{C}\times\mathbf{C})^k$ (it is simply the nonsingular subset of the k-fold symmetric product on $\mathbf{C}\times\mathbf{C}$). We have a map

$$\mathcal{N}_k^d \longrightarrow \mathcal{N}_k$$

which is a local diffeomorphism. We will show that this map is an *asymptotic* isometry where the metric on $\mathcal{N}_k^d$ is flat, given by (1.17) or equivalently (1.14). This corresponds to the particle-like behaviour mentioned at the beginning of this introduction.
It will follow from our results that the metric on $\mathcal{M}_k$ approaches the flat metric as a function of the separation distance between these points, i.e. as the function of $r \stackrel{\text{def}}{=} \min_{i\neq j} |\beta_i - \beta_j| + |\alpha_i - \alpha_j|$. This has been known for charge 2 monopoles (Atiyah and Hitchin [1988]). Their results show also that the rate of the approximation is like the inverse of r, which is therefore the best possible result and which we will match.

1.4 Twistor space construction and the spectral curve

A hyperkähler manifold admits a remarkable way of encoding the metric (local) data in a holomorphic (global) form. This generalizes the Penrose non-linear graviton construction [1976]. It is perhaps worth pointing out that since the holonomy group of a hyperkähler manifold is a subgroup of $Sp(n)$, its Ricci tensor vanishes, so we may think of such a manifold as a (vacuum) solution to Einstein's equations.

We begin by reviewing the results of Hitchin et al. [1987] for an arbitrary hyperkähler manifold M^{4n}. As we already noted in the previous section there is no canonical way of choosing a complex structure on a hyperkähler manifold; there is a 2-sphere of them. The idea of the twistor space is to incorporate all these structures into one complex structure on a larger manifold called the twistor space of M^{4n}.

The twistor space $\mathbf{Z}$ of M^{4n} is defined to be the product manifold

$$\mathbf{Z} = M^{4n} \times S^2$$

equipped with the almost complex structure

$$\mathbf{I} = (aI + bJ + cK, I_o)$$

at the point (m, a, b, c), where I_o is the standard complex structure on S^2. It follows from the Newlander-Nirenberg theorem that this structure is integrable, so $\mathbf{Z}$ is a complex manifold of dimension $2n + 1$. Moreover the projection

$$\mathbf{Z} \xrightarrow{p} \mathbf{C}P^1 \tag{1.18}$$

is holomorphic and each copy $(m, \mathbf{C}P^1)$ of the projective line is a holomorphic section of this projection with normal bundle isomorphic to $\mathbf{C}^{2n} \otimes \mathcal{O}(1)$, where $\mathcal{O}(1)$ is line bundle on $\mathbf{Z}$ pulled back from the unique line bundle of degree 1 on $\mathbf{C}P^1$. With respect to a standard covering of $\mathbf{C}P^1$ by two open affine sets it has transition function ξ^{-1}, where ξ is a complex affine coordinate.

Since the metric is hyperkähler we have three symplectic forms corresponding to the complex structures I, J and K: if $(\ ,\ )$ denotes the scalar product, then $\omega_1(s,t) = (Is,t)$, $\omega_2(s,t) = (Js,t)$, $\omega_3(s,t) = (Ks,t)$. We can construct then a symplectic form on each fibre of (1.18) by setting

$$\omega = (\omega_2 + i\omega_3) - 2i\omega_1\xi - (\omega_2 - i\omega_3)\xi^2 \tag{1.19}$$

Its quadratic dependence on ξ means that globally it is a holomorphic section of $\Lambda^2 T_F^* \otimes \mathcal{O}(2)$, where T_F denotes the tangent bundle along the fibres.

The last piece of information needed to encode the metric is a real structure. The antipodal map on the 2-sphere induces an antiholomorphic involution $\tau : \mathbf{Z} \to \mathbf{Z}$ and the sections of the form (m, S^2) are preserved by this real structure.
It turns out that the above holomorphic data is enough to recover the metric. We have the following

Theorem 1.4.1 (**Hitchin, Karlhede, Lindström, Roček [1987]**) *Let $\mathbf{Z}^{2n+1}$ be a complex manifold such that*
i) $\mathbf{Z}$ is a holomorphic fibre bundle $p : \mathbf{Z} \to \mathbf{C}P^1$ over the projective line;
ii) the bundle admits a family of holomorphic sections each with normal bundle isomorphic to $\mathbf{C}^{2n} \otimes \mathcal{O}(1)$;
iii) there exists a holomorphic section ω of $\Lambda^2 T_F^* \otimes \mathcal{O}(2)$ defining a symplectic structure on each fibre;
iv) $\mathbf{Z}$ has a real structure τ compatible with i), ii), and iii) and inducing the antipodal map on $\mathbf{C}P^1$.
Then the parameter space of real sections is a $4n$-dimensional manifold with a natural hyperkähler metric for which $\mathbf{Z}$ is the twistor space.

□

The twistor space for magnetic monopoles can be described in terms of the space of rational maps. For $SU(2)$ monopoles this was done by Atiyah and Hitchin [1988]. In section 3.1 we will do it for $SU(N)$ monopoles with maximal symmetry breaking.
To give Atiyah and Hitchin's description of the twistor space $\mathbf{Z}_k$ of the moduli space $\mathcal{M}_k$ of charge k $SU(2)$ monopoles, we have to recall the definition of the *spectral curve* of a monopole after Hitchin [1983]. It is a compact algebraic curve Γ in the tangent bundle $\mathbf{T}P^1$ of the projective line which lies in the linear system $|\mathcal{O}(2k)|$. In terms of a monopole it is defined by the scattering of certain differential operator. In terms of Nahm's equations Γ is defined by the equation:

$$\det\left(\eta - 2\beta(t) - (\alpha(t) + \alpha^*(t))\,\zeta + 2\beta^*(t)\zeta^2\right) = 0 \tag{1.20}$$

where ζ is a standard coordinate on $\mathbf{C}P^1$, and $\eta \to \eta(d/d\zeta)$ the associated fiber coordinate in $\mathbf{T}P^1$.
Γ is independent of t, since Nahm's equations are isospectral. An important property of Γ is the fact that the line bundle L^2 over $\mathbf{T}P^1$, defined by the transition function $\exp(2\eta/\zeta)$, is trivial over Γ.

We can recover the monopole from the spectral curve by choosing a special trivialization of L^2 over Γ. We can also recover the solution to Nahm's equations in terms of a flow on the Jacobian over Γ (Krichever method for solving non-linear differential equations). Now the description of the twistor space for $SU(2)$ monopoles can be stated:

Theorem 1.4.2 (Atiyah, Hitchin) *Let $\mathcal{M}_k$ be the moduli space of based $SU(2)$ monopoles of charge k, and $\mathbf{Z}_k$ its twistor space. Then,*

i) $\mathbf{Z}_k$ is obtained by taking two copies of $\mathbf{C}\times Rat_k(\mathbf{C}P^1)$, parametrized by $(\zeta, p(z)/q(z))$ and $\left(\tilde{\zeta}, \tilde{p}(z)/\tilde{q}(z)\right)$, and identifying over $\zeta \neq 0$ by:

$$\tilde{\zeta} = \zeta^{-1} \quad , \quad \tilde{q}\left(\frac{z}{\zeta^2}\right) = \zeta^{-2k} q(z) \quad , \quad \tilde{p}\left(\frac{z}{\zeta^2}\right) = e^{-2z/\zeta} p(z) \ \text{ modulo } q(z)$$

ii) a monopole determines a section of $\mathbf{Z}_k$ by its spectral curve Γ: the equation of the curve gives the denominator of the rational map and a trivialization of L^2 over Γ the numerator,

iii) the holomorphic section ω of $\Lambda^2 T_F^* \otimes \mathcal{O}(2)$ is defined by

$$\omega = \pi \sum d\tilde{\beta}_i \wedge \frac{d\tilde{p}\left(\tilde{\beta}_i\right)}{\tilde{p}\left(\tilde{\beta}_i\right)} = \frac{\pi}{\zeta^2} \sum d\beta_i \wedge \frac{dp(\beta_i)}{p(\beta_i)}$$

where $\tilde{\beta}_i$ (resp. β_i) are the roots of $\tilde{q}$ (resp. q),

iv) the real structure is defined by

$$\tau(\zeta) = -\bar{\zeta}^{-1} \quad , \quad \tau\left(\frac{p(z)}{q(z)}\right) = (-1)^k \bar{\zeta}^{-2k} \frac{\overline{p\left(-\bar{z}\zeta^{-2}\right)}}{\overline{q\left(-\bar{z}\zeta^{-2}\right)}}$$

□

II Asymptotic behaviour of the $SU(2)$ monopole metrics

This entire chapter is devoted to proving that the L^2-metric on the moduli space $\mathcal{N}_k$ of solutions to Nahm's equations is asymptotically the product metric on $(\mathbf{C} \times \mathbf{C})^k$. In section 1.3 we defined the natural coordinates (α_d, β_d) on $\mathcal{N}_k$, provided by the Donaldson isomorphism

$$S^k(\mathbf{C} \times \mathbf{C}) \ni (\alpha_d, \beta_d) \longmapsto (\alpha, \beta) \in \mathcal{N}_k$$

where α_d, β_d are constant diagonal matrices

$$\alpha_d = \operatorname{diag}(\alpha_1, \ldots, \alpha_k) \quad , \quad \beta_d = \operatorname{diag}(\beta_1, \ldots, \beta_k)$$

We will show that, if

$$r \stackrel{\text{def}}{=} \min_{i \neq j} |\beta_i - \beta_j| + |\alpha_i - \alpha_j|$$

then the metric at (α, β) differs less than $\frac{const}{r}$ from the flat product metric at α_d, β_d while the curvature is bounded by $\frac{const}{r^3}$. These results appear in sections 4 and 5 respectively. In section 3 we will get estimates on the behaviour of the spectral curves.

In all proofs the constants are generic, i.e. they may take different values in different formulas.
Let us define symbols which will appear throughout the chapter:

$$R_{ij} = |\beta_i - \beta_j| + |\alpha_i - \alpha_j|$$

for $1 \leq i, j \leq k$;

$$R = \max R_{ij}$$

Notice that

$$r = \min_{i \neq j} R_{ij}$$

Finally:

$$F(\alpha, \beta) = \tfrac{d}{dt}(\alpha + \alpha^*) + 2[\alpha, \alpha^*] + 2[\beta, \beta^*]$$

II.1 An approximate gauge

Consider an element of $S^k(\mathbf{C} \times \mathbf{C})$ with a representative $(\alpha_d, \beta_d) = (\alpha_i, \beta_i)_{i \leq k} \in (\mathbf{C} \times \mathbf{C})^k$. If all β_i-s are different, then a small neighbourhood in $S^k(\mathbf{C} \times \mathbf{C})$ can be identified with a neighbourhood of (α_d, β_d) in $(\mathbf{C} \times \mathbf{C})^k$ or with a neighbourhood of any $(\alpha_{\pi(i)}, \beta_{\pi(i)})_{i \leq k}$ where π is a permutation. We will choose one such permutation, having special properties, and we will be dealing with a neighbourhood of (α_d, β_d) in $(\mathbf{C} \times \mathbf{C})^k$.

Then we will put the right poles in (α_d, β_d) "by hand" (i.e. by an explicitly given gauge transformation $\tilde{g}$) in such a way that the resulting complex $(\tilde{\alpha}, \tilde{\beta})$ almost satisfies the real equation. We also get precise estimates for the entries of $(\tilde{\alpha}, \tilde{\beta})$ which can be differentiated with respect to tangent directions. What we do here therefore is constructing a manifold $\tilde{N}_k$ diffeomorphic to an open neighbourhood U of (α_d, β_d). This manifold with the natural $\mathcal{L}^2$ metric introduced in section 1. is asymptotically (i.e. the bigger r the better the approximation) isometric to $U \subset (\mathcal{C} \times \mathcal{C})^k$. Later on we will be able to show that the diffeomorphism (or rather immersion)

$$\tilde{N}_k \longrightarrow \mathcal{N}_k$$

is also an asymptotic isometry.

We use Taubes' idea of cluster decomposition - we group together (α_i, β_i)-s that are relatively close together. Let us recall the pertinent definitions and facts from [Taubes 1985].

Definition 1.1 *Let* $X = \{x_1, \ldots, x_k\}$ *be a finite subset of a normed space and let* $R = \operatorname{diam} X$. *A* molecule *is a subset* Y *of* X *satisfying the following two conditions:*

i) $\operatorname{diam} Y < R\left(\frac{1}{2}\right)^{k-|Y|+1}$ *where* $|Y|$ *denotes the cardinality of* Y

ii) Y *is maximal among subsets of* X *with property i).*

We have the following fact the proof of which we give for completeness:

Lemma 1.2 (Taubes)

i) *Each* $x \in X$ *is contained in a unique molecule*

ii) *Also,* X *contains at least two molecules*

iii) *If* Y *is a molecule, then* $\operatorname{dist}(Y, X - Y) \geq R\left(\frac{1}{2}\right)^{k-|Y|+1}$

Proof: For an $x \in X$ consider the family $\mathcal{S}$ of subsets of X containing x and satisfying *i)*. As $\{x\} \in \mathcal{S}$, $\mathcal{S}$ is nonempty. Observe that if $Y_1, Y_2 \in \mathcal{S}$ and $Y_1 \neq Y_2$, then $Y_1 \cup Y_2 \in \mathcal{S}$. Indeed, as $x \in Y_1 \cap Y_2$, we have for $z, w \in Y_1 \cup Y_2$:

$$\begin{aligned} |z-w| &\leq |z-x| + |x-w| \leq R\left(\frac{1}{2}\right)^{k-|Y_1|+1} + R\left(\frac{1}{2}\right)^{k-|Y_2|+1} \\ &\leq 2R\left(\frac{1}{2}\right)^{k-\max\left(|Y_1|,|Y_2|\right)+1} \leq R\left(\frac{1}{2}\right)^{k-\max\left(|Y_1|,|Y_2|\right)} \end{aligned}$$

However $|Y_1 \cup Y_2| \geq \max(|Y_1|, |Y_2|) + 1$, since $Y_1 \neq Y_2$.
Hence $Y_1 \cup Y_2 \in \mathcal{S}$ and so there is a unique maximal element of R which must be a molecule.
Since the diameter of a molecule is less than $\frac{1}{2}\operatorname{diam} X$, X must contain at least two molecules.
Finally, if Y is a molecule , $y \in Y$, $x \notin Y$ and $|y-x| < \frac{1}{2}R\left(\frac{1}{2}\right)^{k-|Y|}$, then for any $z \in Y$

$$|z-x| \leq |z-y| + |y-x| < R\left(\frac{1}{2}\right)^{k-|Y|+1} + \frac{1}{2}R\left(\frac{1}{2}\right)^{k-|Y|} \leq R\left(\frac{1}{2}\right)^{k-(|Y|+1)+1}$$

so $Y \cup \{x\}$ satisfies $i)$ which is a contradiction.

□

We will think now of X as the space $\{\operatorname{Re}\alpha_i, \beta_i\}_{i \leq k}$ for a fixed $(\alpha_d, \beta_d) \in (\mathcal{C} \times \mathcal{C})^k$. For notational convenience we will identify X and the set of indices $\{1, \ldots, k\}$. We define inductively a permutation of indices according to the following scheme:
We have two possibilities: either $\operatorname{diam}\{\beta_i\} > \frac{1}{4}\operatorname{diam} X$ or not.
In the first case let us find a molecule Y among $\{\beta_i\}$ and permute the indices so that $Y = 1, \ldots, |Y|$.
In the second case we have $\operatorname{diam}\{\operatorname{Re}\alpha_i\} > \frac{1}{4}\operatorname{diam} X$. There are then $\operatorname{Re}\alpha_{j_1} < \operatorname{Re}\alpha_{j_2}$ such that $|\operatorname{Re}\alpha_{j_1} - \operatorname{Re}\alpha_{j_2}| > \frac{1}{4k}\operatorname{diam} X$ and such that there is no j with $\operatorname{Re}\alpha_j \in (\operatorname{Re}\alpha_{j_1}, \operatorname{Re}\alpha_{j_2})$. Put $Y = \{i;\ \operatorname{Re}\alpha_i \leq \operatorname{Re}\alpha_{j_1}\}$ and again permute the indices so that $Y = 1, \ldots, |Y|$.
Now we repeat the process inside Y and $X - Y$. This way we can show:

Lemma 1.3 *For every (α_d, β_d) there is a permutation after which the set of indices will satisfy the following:*
i) for any $i, j = 1, \ldots, k$, $i < j$, we have either $|\beta_i - \beta_j| \geq cR_{ij}$ or $\operatorname{Re}\alpha_j - \operatorname{Re}\alpha_i > cR_{ij}$,
ii) for any $i, j = 1, \ldots, k$, $i > j$ and any $m \geq i$, $n \leq j$, $R_{m,n} \geq cR_{ij}$,
iii) there is an $l < k$ such that either for all $i > l, j \leq l$, $|\beta_i - \beta_j| \geq cR$ or for all

$i > l,\ j \leq l,$ $\mathrm{Re}\,\alpha_i - \mathrm{Re}\,\alpha_j \geq cR$.
The constant $c > 0$ can be taken to be $\left(\frac{1}{2}\right)^{k+2}$.

□

Let us assume the following definition

Definition 1.4 *A matrix valued function α (resp. β) defined in a neighbourhood of a point $p \in \mathbf{R}$ has $\mathcal{N}_k$-type pole at p if its residue is equal there to the matrix $\mathrm{res}\,\alpha$ (resp. $\mathrm{res}\,\beta$) of (I.1.7).*

Now we will define a gauge transformation which will give correct poles to (α, β) and almost solve the real equation. Let

$$z = 1 \pm t$$

Proposition 1.5 *Let (α_d, β_d) be an element of $(\mathbf{C} \times \mathbf{C})^k$ with β_i-s all different and satisfying assertions i)-iii) of Lemma 1.3. There is a smooth lower-triangular gauge transformation g on $(-1, +1)$ such that the resulting $(\tilde{\alpha}, \tilde{\beta}) := g(\alpha, \beta)$ has $\mathcal{N}_k$-type poles at ± 1 and satisfies for $z \leq 1$:*
i) $|\tilde{\alpha}_{ii}(z) - \alpha_i| \leq \frac{K}{z}$,
ii) $z\,|\tilde{\alpha}_{ij}(z)|,\ z\left|\tilde{\beta}_{ij}(z)\right| \leq Ke^{-sR_{ij}z}$ for some $K, s > 0$ independent of (α_d, β_d),
iii) there is a lower-triangular gauge transformation $\tilde{g}$, smooth on $[-1, 1]$, having real diagonal, with $\tilde{g}(\pm 1) = 1$, such that $(\alpha, \beta) = \tilde{g}(\tilde{\alpha}, \tilde{\beta})$ satisfies the real equation and $\tilde{g}, \tilde{g}^{-1}$ are bounded independently of (α_d, β_d) on $[-1, 1]$.

Observe that ii) implies that for $t \in \left[-1 + \frac{1}{R_{ij}}, 1 - \frac{1}{R_{ij}}\right]$

$$|\tilde{\alpha}_{ij}(t)|\ ,\ \left|\tilde{\beta}_{ij}(t)\right| \leq KR_{ij}e^{-sR_{ij}(1+t)} + KR_{ij}e^{-sR_{ij}(1-t)}$$

Estimates of this type will be crucial in analyzing the asymptotic behaviour of the metric.

We will show in the process of proving the above proposition that the estimates can be differentiated with respect to the tangent directions, namely we will show the following

Proposition 1.6 *The gauge transformation g of Proposition 1.5 can be extended smoothly on a neighbourhood of (α_d, β_d) so that if (a_d, b_d) is an element of norm 1 of the tangent space at (α_d, β_d) then the corresponding*

$$(\tilde{a}, \tilde{b}) = dg\,(a_d, b_d)$$

will satisfy:

i) $|\tilde{a}_{ii}(t) - a_i| \leq Ke^{-sr(1+t)} + Ke^{-sr(1-t)}$

ii) $|\tilde{a}_{ij}(t)| \leq Ke^{-sR_{ij}(1+t)} + Ke^{-sR_{ij}(1-t)}$ *if* $i > j$,

iii) for $1 \geq z = 1 \pm t$, $z\left|\dot{\tilde{a}}_{ij}(z)\right| \leq Ke^{-sR_{ij}z}$,

iv) $\tilde{b}_{ii} = b_i$

v) $\left|\tilde{b}_{ij}(t)\right| \leq Ke^{-sR_{ij}(1+t)} + Ke^{-sR_{ij}(1-t)}$ *if* $i > j$

with constants K, s independent of (α_d, β_d).

Proof of Propositions 1.5 and 1.6: We assume that 1.5 and 1.6 hold for charges less than k (the statements are trivial for $k = 1$). For the purpose of induction we will prove slightly more namely in addition to i)-iii) of Proposition 2.4 $(\tilde{\alpha}, \tilde{\beta})$ will satisfy the following:

iv) for $t \in [-1, -1/2]$,

$$\tilde{\alpha}_{ii}(t) = \alpha_i + \sum_{j \leq k}\left(q_{ij}\frac{r_{ij}e^{-r_{ij}(1+t)}}{1 - e^{-r_{ij}(1+t)}} + p_i(z)\right)$$

with $\sum_{j\leq k} q_{ij} = -(k+1-2i)/2$, $r_{ij} \in \{R_{mn};\, m, n \leq k\}$, *$p_i(z)$ bounded by KR, its derivative bounded by KR^2, $zp_i(z)$ bounded; finally $q_{ij}, p_i(z)$ independent of (α_d, β_d),*

v) for $t \in [1/2, 1]$, $\tilde{\alpha}_{ii}(t) - \alpha_i = -\left(\tilde{\alpha}_{ii}(-t) - \alpha_i\right)$

vi) $|\tilde{\alpha}_{ij}| \leq KR$ *if* $i > j$,

vii) $\left|\dot{\tilde{\alpha}}_{ij}\right| \leq KR^2$ *if* $i > j$,

viii) $\left|\tilde{\beta}_{ij}\right| \leq KR$ *if* $i > j+1$

ix) $\left|\tilde{\beta}_{i+1,i} \mp \frac{\delta_i}{(1\pm t)}\right| \leq KR$

x) the solution $w(t)$ to $\ddot{w} = -\left\|F(\tilde{\alpha}, \tilde{\beta})\right\|$ *with $w(\pm 1) = 0$ is bounded on* $[-1, 1]$.

First of all let us show that x) implies iii). The norm of $\tilde{g}$ can be estimated by the norm of $\tilde{g}^*\tilde{g}$, which, as a hermitian matrix, can be estimated by the maximum of its eigenvalues. It follows from Donaldson's result [1984], Lemma (2.10) that

$$\frac{d^2}{dt^2}\ln\max\{\text{eigenvalues of } \tilde{g}^*\tilde{g}(t)\} \geq -\left\|F(\tilde{\alpha}, \tilde{\beta})(t)\right\|$$

$$-\frac{d^2}{dt^2}\ln \max\{\text{ eigenvalues of } \tilde{g}^{-1}\tilde{g}^{*-1}(t)\} \geq -\left\|F(\tilde{\alpha},\tilde{\beta})(t)\right\|$$

and from the following simple comparison theorem which we will use repeatedly

Lemma 1.7 *Let x, y be two real-valued functions defined on an interval $[a,b]$ such that*

$$x(a) \leq y(a)\,,\quad x(b) \leq y(b)$$

$$\ddot{y} = F(t)y + h(t)$$

$$\ddot{x} \geq F(t)x + h(t)$$

for some functions F, h with F being non-negative. Then, for all $t \in [a,b]$

$$x(t) \leq y(t)$$

Proof: First of all there cannot exist a point at which $x(t) \geq y(t)$ and $\ddot{x}(t) - \ddot{y}(t) < 0$. Therefore the function $x - y$ is convex on any interval on which $x \geq y$, in particular on any *maximal* interval with this property. However at the endpoints of such maximal interval $x - y \leq 0$. Therefore $x - y \leq 0$ everywhere.
□

We are going therefore to prove Proposition 1.5 i),ii), Proposition 1.6 and the inductive assumptions iv)-x).
Let l be the number $< k$ given by Lemma 1.3 iii). Let g' be the gauge transformation of Proposition 1.5 and 1.6 corresponding to $((\alpha_i)_{i\leq l}, (\beta_i)_{i\leq l})$ and g'' similar gauge transformation corresponding to $((\alpha_i)_{i>l}, (\beta_i)_{i>l})$. Let $(\tilde{\alpha}', \tilde{\beta}')$ and $(\tilde{\alpha}'', \tilde{\beta}'')$ be the resulting complexes.
Let us put

$$(\alpha,\beta) := \begin{pmatrix} (\tilde{\alpha}', \tilde{\beta}') & 0 \\ & \\ 0 & (\tilde{\alpha}'', \tilde{\beta}'') \end{pmatrix}$$

Define similarily (a,b) satisfying assertions of Proposition 1.6.
Observe that both (α,β) and (a,b) are lower-triangular and

$$\operatorname{diag}\beta \equiv \beta_d\ ,\quad \operatorname{diag} b \equiv b_d$$

We will consider first the situation near one of the poles, say -1, i.e. we put

$$z = t + 1$$

We will define several gauge transformations, the table of which is provided on page 30.

First of all we create the $\mathcal{N}_k$-type poles in α by the diagonal gauge transformation given by:

$$d(z) = \begin{pmatrix} \left(\frac{R}{1-\exp(-Rz)}\right)^{\frac{l-k}{2}} & 0 \\ & \\ 0 & \left(\frac{R}{1-\exp(-Rz)}\right)^{\frac{l}{2}} \end{pmatrix} \tag{1.1}$$

Observe that this does not change β nor does it change α_{ij} for $i > j$. Call this new complex $(\bar{\alpha}, \bar{\beta})$. Now we create correct poles of correct order in $\bar{\beta}$ by a gauge transformation of the form $1 + n$, with

$$n = \begin{pmatrix} 0 & 0 \\ \bar{n} & 0 \end{pmatrix} \tag{1.2}$$

where $\bar{n}$ is an $(k - l) \times l$ block. Let us show first a simple fact:

Lemma 1.8 *Let α and β be lower-triangular and meromorphic and let α have the N_n-type poles at 0. Assume that for $i - j > 1$ the term $t^{-(i-j)}$ does not occur in the Laurent expansion of β_{ij}. If α,β satisfy the complex equation, then β_{ij} is regular if $i > j + 1$.*

PROOF: From the complex equation we have in this case:

$$\dot{\beta}_{ij}(z) = 2\beta_{ij}(\alpha_{jj} - \alpha_{ii}) + \sum_{s\geq 1} \beta_{i,j+s}\alpha_{j+s,j} - \sum_{s\geq 1} \alpha_{i,j+s}\beta_{j+s,i}$$

Let $p = i - j > 1$ and assume that β_{st} is regular if $1 < s - t < p$. Suppose that $\beta_{ij} = at^{-r}(1 + t \cdot \text{regular function})$. Then the above equation yields $-art^{-r-1} = -apt^{-r-1}$ so we get a contradiction unless $r \leq 0$.

□

Observe also that if n, n' are both of the form (1.2), then $nn' = 0$ and so $(1 + n)^{-1} = 1 - n + n^2 - \ldots = 1 - n$. Therefore under the action of $1 + n$

$$\bar{\alpha} \longmapsto \bar{\alpha} + [n, \bar{\alpha}] - \tfrac{1}{2}\dot{n} \ , \quad \bar{\beta} \longmapsto \bar{\beta} + [n, \bar{\beta}]$$

We choose an n which not only creates poles of correct order in $\bar{\beta}$ but also does not change $\bar{\alpha}$, i.e. n satisfies the equation

$$\dot{n} = 2[n, \bar{\alpha}]$$

The general solution to this equation is given inductively by

$$\left(C_{ij} + \int_0^z \left(2\sum_{s\geq 1} n_{i,j+s}\bar{\alpha}_{j+s,j} - 2\sum_{s\geq 1} \bar{\alpha}_{i,i-s}n_{i-s,j}\right) \left(\sigma(2\bar{\alpha}_{jj} - 2\bar{\alpha}_{ii})\right)^{-1} d\tau\right) \sigma(2\bar{\alpha}_{jj} - 2\bar{\alpha}_{ii})(z) \tag{1.3}$$

where $\sigma(g)$, for $g(z)$ a function having a simple pole with residue p at $z = 0$, is the unique solution of the equation

$$\dot{\sigma} = g \cdot \sigma$$

such that

$$\lim_{z\to 0} z^p \sigma(g)(z) = 1$$

Observe that $\sigma(2\bar{\alpha}_{jj} - 2\bar{\alpha}_{ii})(z)$ has a pole of order $i - j$.
Since we require $[n, \bar{\beta}]_{l+1,l}$ to have a pole of residue δ_l, we must put

$$C_{l+1,l} = \frac{\delta_l}{\beta_l - \beta_{l+1}} \tag{1.4}$$

and the in order to satisfy the condition of Lemma 1.8 we must have

$$C_{ij}(\beta_j - \beta_i) + C_{i,j+1} \operatorname{res} \bar{\beta}_{j+1,j} - \operatorname{res} \bar{\beta}_{i,i-1} C_{i-1,j} = 0 \tag{1.5}$$

Therefore, as $\beta_j \neq \beta_i$ for $j \neq i$, it is possible to find a gauge transformation $1 + n$ with n of the form (1.2) such that $\bar{\alpha}$ does not change and $\bar{\beta}$ becomes $\bar{\beta} + [n, \bar{\beta}]$.

From now on we have to separate the proof in two cases according to the alternative in Lemma 1.3 iii). The reason for this is that if for all $i > l$, $j \leq l$, $\operatorname{Re}\alpha_i - \operatorname{Re}\alpha_j \geq cR$, then n has automatically an exponential decay while in the other case we will have to introduce such a decay artificially. Let us therefore introduce formally two cases:

a) for all $i > l$, $j \leq l$, $\operatorname{Re}\alpha_i - \operatorname{Re}\alpha_j \geq cR$

b) for all $i > l$, $j \leq l$, $|\beta_i - \beta_j| \geq cR$

Continuation of the proof in case a):

We would like to estimate $B = [n, \bar{\beta}]$. Observe that B satisfies

$$\dot{B} = 2[B, \bar{\alpha}]$$

and hence $B_{ij}(z)$ is given by equations (1.3) with $C_{l+1,l} = \delta_l$ and $C_{ij} = 0$ if $i > j+1$ (as B_{ij} is then regular). Hence

$$B_{l+1,l} = \sigma(2\bar{\alpha}_{ll} - 2\bar{\alpha}_{l+1,l+1})$$

so, by integrating the estimate in the inductive assumption iv), and recalling the gauge transformation (1.1),

$$B_{l+1,l} = \delta_l \frac{R}{(1-e^{-Rz})} \left(\frac{R}{1-e^{-Rz}}\right)^{\frac{k}{2}-1} \cdot \prod \left(\frac{1-e^{-r_j z}}{r_j}\right)^{q_j} \cdot e^{2(\alpha_l - \alpha_{l+1})z + p(z)}$$

where $\sum q_j = \frac{k}{2} - 1$, $r_j > 0$, $r \le |r_j| \le R$ and $p(z) = \int_0^z p_l(\tau)$ is a bounded function (as $\tau p_l(\tau)$ is bounded).

We want to estimate the above function. Namely we have:

Lemma 1.9 *Let*

$$f(z) = \left(\frac{R}{1-e^{-Rz}}\right)^q \cdot \prod \left(\frac{1-e^{-r_j z}}{r_j}\right)^{q_j} \cdot e^{-sRz}$$

where $q_j > 0$, $\sum q_j = q$, $r_j > 0$, $|r_j| \le R$, $s > 0$. Then $f, \frac{1}{R}f'$ are bounded with the bounds depending only on q and s.

PROOF: It is enough to consider f with just one j and $q = 1$, i.e.

$$f(z) = \frac{R}{1-e^{-Rz}} \frac{1-e^{-rz}}{r} e^{-sRz}$$

with $R \ge r$, $s > 0$. Let us put $u = Rz$, $\lambda = \frac{r}{R}$ so that

$$f(u) = \frac{1}{\lambda} \frac{1-e^{-\lambda u}}{1-e^{-u}} e^{-su} \qquad , \qquad \lambda \le 1$$

To prove the lemma it is enough to show that $f(u)$, $\frac{d \ln f}{du}$ are bounded independently of λ.

First of all, if $u \le 1$, then there must be a bound independent of λ as the set of u, λ is compact and $f(t)$ is continuous at $\lambda = 0$ or $u = 0$.

If $u \ge 1$, $1 - e^{-u} \ge 1 - e^{-1} \ge \frac{1}{2}$, so $f(u) \le \frac{2}{\lambda}\left(1 - e^{-\lambda u}\right) e^{-su}$. The maximum of the right-hand side occurs at the point u_o where

$$\lambda e^{-\lambda u_o} - \left(1 - e^{-\lambda u_o}\right) s e^{-su_o} = 0$$

Hence $1 - e^{-\lambda u_o} = \frac{\lambda}{c} e^{-\lambda u_o}$, so

$$f(u) \leq \frac{2}{\lambda} \frac{\lambda}{\epsilon} e^{-\lambda u_o} e^{-s u_o} \leq \frac{2}{s}$$

which shows that f is bounded.
Now:

$$(\ln f)'(u) = \frac{\lambda e^{-\lambda u}}{1 - e^{-\lambda u}} - \frac{e^{-u}}{1 - e^{-u}} - s$$

For $u \geq 1$

$$1 - e^{-\lambda u} \geq 1 - e^{-\lambda} \geq \frac{1}{e} \lambda$$

as $\lambda \leq 1$. Hence, for $u \geq 1$, $(\ln f)'(u)$ is bounded.
If $u \leq 1$ then again we have a continuous function defined on the compact set $\lambda \leq 1$, $u \leq 1$.

$\square$

Using this lemma and the fact that $\operatorname{Re} \alpha_{l+1} - \operatorname{Re} \alpha_l \geq cR$ we get (e.g. taking $s = \frac{1}{2} c$) the following estimate

$$B_{l+1,l}(z) = \frac{R e^{-sRz}}{1 - e^{-Rz}} f(z) \tag{1.6}$$

where $f, \frac{1}{R} f'$ are bounded, $f(0) = \delta_l$.
Now we want to estimate

$$B_{ij}(z) = \left(\int_0^z \left(2 \sum_{s \geq 1} B_{i,j+s} \bar{\alpha}_{j+s,j} - 2 \sum_{s \geq 1} \bar{\alpha}_{i,i-s} B_{i-s,j} \right) (\sigma(2\bar{\alpha}_{jj} - 2\bar{\alpha}_{ii}))^{-1} \, d\tau \right) \sigma(2\bar{\alpha}_{jj} - 2\bar{\alpha}_{ii})(z)$$

for $i > j + 1$. First of all we show an estimate similar to (1.6):

$$|B_{ij}(z)| = \frac{R e^{-sRz}}{1 - e^{-Rz}} f_{ij}(z) \tag{1.7}$$

where f_{ij} is bounded.We will show it by induction on $i - j$. We can estimate just one term of the form:

$$\left(\int_0^z B_{i,j+s}(\tau) \bar{\alpha}_{j+s,j}(\tau) (\sigma(2\bar{\alpha}_{jj} - 2\bar{\alpha}_{ii}))^{-1} \, d\tau \right) \sigma(2\bar{\alpha}_{jj} - 2\bar{\alpha}_{ii})(z) \tag{1.8}$$

Observe that $\sigma(2\bar{\alpha}_{jj} - 2\bar{\alpha}_{ii})(z)$ can be written as

$$\frac{R}{(1 - e^{-Rz})} \prod \left(\frac{r_m}{1 - e^{-r_m z}} \right)^{n_m} \left[\left(\frac{R}{1 - e^{-Rz}} \right)^{\frac{k}{2} - 1} \cdot \prod \left(\frac{1 - e^{-r_m z}}{r_m} \right)^{q_m} \cdot e^{p(z)} \right] e^{2(\alpha_j - \alpha_i) z}$$

where $\sum q_m = \frac{k}{2} - 1$, $\sum n_m = i - j - 1$, $n_m \geq 0$, $r_m > 0$, $r \leq r_m \leq R$ and $p(z)$ bounded.

Table of gauge transformations

$$\begin{tikzcd}[row sep=large]
\text{block-diagonal } (\alpha,\beta) \arrow[d, "\substack{\text{diagonal gauge}\\ \text{transformation } \boldsymbol{d}\\ \text{of (1.1)}}"] \arrow[ddd, bend left=60, "\hat{g}"] \\
\substack{(\bar\alpha,\bar\beta) \text{ such that: } \bar\beta=\beta;\\ \text{strictly lower-triangular parts}\\ \text{of } \bar\alpha \text{ and } \alpha \text{ are the same;}\\ \bar\alpha \text{ has } \mathcal{N}_k\text{-type poles}} \arrow[d, "1+n \text{ of (1.2)}"] \\
\substack{\bar\alpha \text{ does not change}\\ \bar\beta \text{ gets poles of correct order}} \arrow[d, "\substack{\text{diagonal}\\ \text{gauge transfor-}\\ \text{mation of (1.11)}}"] \\
(\hat\alpha,\hat\beta) \text{ having } \mathcal{N}_k\text{-type poles}
\end{tikzcd}$$

Call the expression in the square brackets $g_{ij}(z)$. Observe that g_{ij}^{-1} is bounded (as $r_m \leq R$). Hence, using the inductive assumption on $|B_{i,j+s}|$ and the bound $|\bar{\alpha}_{j+s,j}| \leq KR$ we get

$$\left|B_{i,j+s}(\tau)\bar{\alpha}_{j+s,j}(\tau)\left(\sigma(2\bar{\alpha}_{jj}-2\bar{\alpha}_{ii})(\tau)\right)^{-1}\right| \leq KRe^{-sR\tau}\prod\left(\frac{1-e^{-r_m\tau}}{r_m}\right)^{n_m} e^{2\mathrm{Re}(\alpha_i-\alpha_j)\tau}$$

The product $\prod\left(\frac{1-e^{-r_m\tau}}{r_m}\right)^{n_m}$ is an increasing function so it can be majorized by its value at z. Therefore the absolute value of (1.8) can be majorized by

$$\left(\int_0^z KRe^{-sR\tau}e^{2\mathrm{Re}(\alpha_i-\alpha_j)\tau}d\tau\right)\frac{R}{1-e^{-Rz}}g_{ij}(z)e^{2\mathrm{Re}(\alpha_j-\alpha_i)z}$$

which is less than

$$K\left(e^{-sRz}e^{2\mathrm{Re}(\alpha_i-\alpha_j)z}-1\right)\frac{R}{1-e^{-Rz}}g_{ij}(z)e^{2\mathrm{Re}(\alpha_j-\alpha_i)z}$$

which is

$$K\left(1-e^{(2\mathrm{Re}(\alpha_j-\alpha_i)+sR)z}\right)\frac{R}{1-e^{-Rz}}g_{ij}(z)e^{-sRz} \tag{1.9}$$

As Lemma (2.8) guarantees that $g_{ij}(z)e^{-\frac{1}{2}sRz}$ is bounded, taking s small enough proves (1.7). We can say more. We have

$$1-e^{(2\mathrm{Re}(\alpha_j-\alpha_i)+s)z} \leq KRz$$

and

$$\frac{Rz}{1-e^{-Rz}} \leq K+Rz$$

These two facts and (1.9) allow us to improve (1.7) to

$$|B_{ij}| \leq CRe^{-sRz} \tag{1.10}$$

for $i > j+1$.

Observe that the residues of $\tilde{\beta}+B$ are not correct. We can them change by a diagonal gauge transformation of the form

$$e^{\delta e^{-Rz}} \tag{1.11}$$

where δ is a constant diagonal matrix depending only on k and l.

Call the resulting complex $(\hat{\alpha},\hat{\beta})$. It has N_n-type poles and

$$(\hat{\alpha},\hat{\beta}) = (\alpha,\beta)+(\check{\alpha},\check{\beta})$$

where

$$\check{\alpha}_{ii}(z) = q_i \frac{Re^{-Rz}}{1-e^{-Rz}} + p_i Re^{-Rz} \tag{1.12}$$

$$|\check{\alpha}_{ij}(z)| \leq KRe^{-sRz} \quad \text{if } i > j \tag{1.13}$$

$$\left|\dot{\check{\alpha}}_{ij}(z)\right| \leq KR^2 e^{-\epsilon Rz} \quad \text{if } i > j \tag{1.14}$$

$$\check{\beta}_{ii} \equiv 0 \tag{1.15}$$

$$\left|\check{\beta}_{i+1,i}(z) - \tfrac{\delta_i}{z} e^{-sRz}\right| \leq KRe^{-sRz} \tag{1.16}$$

$$\left|\check{\beta}_{i,j}(z)\right| \leq KRe^{-sRz} \quad \text{if } i > j+1 \tag{1.17}$$

where all the constants are independent of (α_d, β_d).
Of all these facts maybe (1.16) in case when $i = l$ requires an explanation. It follows from (1.6), the fact that $|f(z) - 1| \leq KRz$ and the following simple observations:

$$\left|\frac{Rz}{1-e^{-Rz}}\right| \leq K + Rz$$

$$\left|\frac{R}{1-e^{-Rz}} - \frac{1}{z}\right| \leq KR$$

These also allow us to replace (1.12) by

$$\left|\check{\alpha}_{ii}(z) - \frac{q_i}{z} e^{-Rz}\right| \leq KRe^{-Rz} \tag{1.18}$$

$$\left|\dot{\check{\alpha}}_{ii}(z) + \frac{p_i}{z^2} e^{-Rz}\right| \leq \frac{KR}{z} e^{-Rz} + KR^2 e^{-Rz} \tag{1.19}$$

Observe that the inductive assumption and the above estimates imply that $(\hat{\alpha}, \hat{\beta})$ satisfies conditions i),ii) of Proposition 1.5 as well as the conditions iv),vi),vii),viii) and ix) of the inductive assumption. It is not yet our complex $(\tilde{\alpha}, \tilde{\beta})$ as it is defined only on $[-1, -1/2]$.
To estimate $(\hat{a}, \hat{b})$ note that for any i, j

$$dR_{ij} \quad \text{is bounded} \tag{1.20}$$

Now: α_{ij} $i > j$ changed only after the gauge transformation (1.11) which is bounded and its tangent directions derivatives behave like Kze^{-sRz} which together with the inductive assumption vi) gives us 1.6 ii) for $t \in [-1, -1/2]$ and $i \neq j$. Similarily vii) implies 1.6 iii). For 1.6 i) notice that α_{ii} changed by gauge transformations (1.1) and (1.11), differentials of which satisfy 1.6 i) and 1.6 iii) for $i = j$. Finally 2.5 v) will follow from differentiating the formulas (1.3),(1.4),(1.5).

Let $\hat{g}$ denote the gauge transformation that took us from (α,β) to $(\hat{\alpha},\hat{\beta})$. $\hat{g}$ is defined on $[-1,-1/2]$. We want to extend $\hat{g}$ onto $[-1,1]$. Notice that if we put $\hat{g}(t)=\hat{g}(-t)$ fot $t\in[1/2,1]$, then the resulting $\hat{\alpha}$ has $\mathcal{N}_k$-type poles at 1 but the residue of $\hat{\beta}$ is negative of what it should be. We can change this however by a diagonal gauge transformation D which is 1 at -1 and at $+1$ satisfies

$$D_j(+1)=(-1)^{j-1} \tag{1.21}$$

We want to preserve the exponential decay and so we define D_j for even j as

$$D_i(t)=\exp\left\{\frac{1}{2}\pi i(1+t)e^{-R(1-t)}\right\} \tag{1.22}$$

Since $\hat{g}$ and all its derivatives (both in the t-direction as well as in the tangent directions) at $\pm\frac{1}{2}$ are bounded by e^{-sR} we can extend smoothly $\hat{g}$ onto $[-1,1]$ so that the resulting $(\tilde{\alpha},\tilde{\beta})$ satisfies Proposition 1.5 i),ii),iv)-ix) as well as Proposition 1.6. There remains the problem of x).

We want to show that the lower-triangular gauge transformation $\tilde{g}$ that solves the real equation and $\tilde{g}(\pm 1)=1$ has bounded logarithm. Observe that we can first act on $(\tilde{\alpha},\tilde{\beta})$ by the diagonal unitary gauge transformation $u_o=e^{2i\mathrm{Im}\,\alpha_d t}$ which removes the imaginary part of $\operatorname{diag}\tilde{\alpha}$ (but, as it is unitary, it does not change $\left\|F(\tilde{\alpha},\tilde{\beta})\right\|$) and then by $u_o^*\tilde{g}u_o$ which is also 1 at ± 1. Obviously estimates on $\tilde{g}$ and $u_o^*\tilde{g}u_o$ are the same. Therefore in what follows we can assume that $\tilde{\alpha}_{ii}$ is real for all i. We have

$$F(\hat{\alpha},\hat{\beta})=F(\alpha,\beta)+\dot{\check{\alpha}}+\dot{\check{\alpha}}^*+2[\alpha,\check{\alpha}^*]+2[\check{\alpha},\alpha^*]+2[\check{\alpha},\check{\alpha}^*]+2[\beta,\check{\beta}^*]+2[\check{\beta},\beta^*]+2[\check{\beta},\check{\beta}^*]$$

As the poles of $(\tilde{\alpha},\tilde{\beta})$ define a representation of $su(2)$, the second order poles of $F(\tilde{\alpha},\tilde{\beta})$ must vanish. Similarily they vanished in $F(\alpha,\beta)$. Moreover, since we have

$$\left|\frac{1}{z^2}e^{-c_1Rz}-\frac{1}{z^2}e^{-c_2Rz}\right|\le\frac{KR}{z}e^{-\min\{c_1,c_2\}Rz}$$

and because of the inductive estimates iv)-ix) for (α,β) as well as the estimates (1.13)-(1.17),(1.18) and (1.19) for $(\check{\alpha},\check{\beta})$ we have

$$\left\|F(\hat{\alpha},\hat{\beta})(t)-F(\alpha,\beta)(t)\right\|\le\frac{KR}{1+t}e^{-sR(1+t)}+KR^2e^{-sR(1+t)}$$

for $t\in[-1,-1/2]$. Since we extended $(\hat{\alpha},\hat{\beta})$ onto $[1/2,1]$ by symmetry and on $[-1/2,1/2]$ by a gauge transformation which is bounded by e^{-sR} (and so is its derivative), we get

$$\left\|F(\tilde{\alpha},\tilde{\beta})(t)-F(\alpha,\beta)(t)\right\|\le\frac{KR}{(1+t)}e^{-sR(1+t)}+KR^2e^{-sR(1+t)}+\frac{KR}{(1-t)}e^{-sR(1-t)}+KR^2e^{-sR(1-t)}$$

We want to prove x) . Observe that the solution w to $\ddot{w} = -\left\|F(\tilde{\alpha},\tilde{\beta})\right\|$, $w(\pm 1) = 0$ satisfies $|w(z)| \leq |\bar{w}(z)|$ where $\bar{w}$ is the solution to
$\ddot{\bar{w}} = -\|F(\alpha,\beta)\| - \left\|F(\tilde{\alpha},\tilde{\beta}) - F(\alpha,\beta)\right\|$, $\bar{w}(\pm 1) = 0$. $\bar{w}$ can be written as $w_1 + w_2$ where $w_{1,2}(\pm 1) = 0$, $\ddot{w_1} = -\|F(\alpha,\beta)\|$, $\ddot{w_2} = -\left\|F(\tilde{\alpha},\tilde{\beta}) - F(\alpha,\beta)\right\|$. From the inductive assumption w_1 is bounded on $[-1,1]$. On the other hand the above estimate for $\left\|F(\hat{\alpha},\hat{\beta})(t) - F(\alpha,\beta)(t)\right\|$ and Lemma 2.6 give us (writing out the explicit boundary-value solution of $\ddot{w} = -h(t)$) that $|w_2(t)| \leq$

$\frac{1}{2}(1-t)\int_{-1}^{t}(1+\tau)\left(\frac{KR}{(1+\tau)}e^{-sR(1+\tau)} + KR^2e^{-sR(1+\tau)} + \frac{KR}{(1-\tau)}e^{-sR(1-\tau)} + KR^2e^{-sR(1-\tau)}\right)d\tau +$
$\frac{1}{2}(1+t)\int_{t}^{1}(1-\tau)\left(\frac{KR}{(1+\tau)}e^{-sR(1+\tau)} + KR^2e^{-sR(1+\tau)} + \frac{KR}{(1-\tau)}e^{-sR(1-\tau)} + KR^2e^{-sR(1-\tau)}\right)d\tau$

Observe that the integrals

$$\int_{-1}^{t}(1+\tau)\left(\frac{KR}{(1+\tau)}e^{-sR(1+\tau)} + KR^2e^{-sR(1+\tau)}\right)d\tau$$

$$\int_{t}^{1}(1-\tau)\left(\frac{KR}{(1-\tau)}e^{-sR(1-\tau)} + KR^2e^{-sR(1-\tau)}\right)d\tau$$

are bounded on $[-1,1]$. To estimate the remaining part notice that if we write $1+\tau = (1-\tau)+2\tau$ and $1-\tau = (1+\tau)-2\tau$, then we have to estimate

$$(1-t)\int_{-1}^{t}2\tau\left(\frac{KR}{(1-\tau)}e^{-sR(1-\tau)} + KR^2e^{-sR(1-\tau)}\right)d\tau$$

$$(1+t)\int_{t}^{1}-2\tau\left(\frac{KR}{(1+\tau)}e^{-sR(1+\tau)} + KR^2e^{-sR(1+\tau)}\right)d\tau$$

Of course it is enough to estimate one of them. As $1-t \leq 1-\tau$ on $[-1,t]$, it follows that the first expression is $\leq$

$$\int_{-1}t2\tau KRe^{-sR(1-\tau)}\,d\tau + (1-t)\int_{-1}t2\tau KR^2e^{-sR(1-\tau)}\,d\tau$$

The first term is obviously bounded while the second one is equal to

$$(1-t)\frac{2KR}{s}e^{-sR(1-\tau)}\Big|_{-1}^{t}$$

which is also bounded on $[-1,1]$. This shows that $w_2(t)$ is bounded for $t \in [-1,1]$ which proves x).

We still have to consider the boundary vectors. We started with two complexes $(\tilde{\alpha}',\tilde{\beta}')$ and $(\tilde{\alpha}'',\tilde{\beta}'')$. The boundary vector $\tilde{v}'_{\pm} = (1,0,\ldots,0)^T$ at ± 1 for $(\tilde{\alpha}',\tilde{\beta}')$ determines the unique solution (the one with maximal decay rate) to the equation

$$\dot{\tilde{v}}'(z) = -2\tilde{\alpha}'(z)\tilde{v}'(z)$$

such that

$$\lim_{z\to 0} z^{-\frac{l-1}{2}}\tilde{v}'(z) = \tilde{v}'_{\pm}$$

and similarily for $(\tilde{\alpha}'', \tilde{\beta}'')$. We can take linear combination of these two solutions to get a solution $v(z)$ to

$$\dot{v}(z) = -2\alpha(z)v(z) \tag{1.23}$$

We have to show that after acting by $\hat{g}$ (see page 30) the solution $\hat{v}(z)$ satisfies

$$\lim_{z\to 0} z^{-\frac{k-1}{2}}\hat{v}(z) = (1,0,\ldots,0)^T \tag{1.24}$$

In (1.11) we can take the $(1,1)$-entry of δ to be 0. Therefore it is enough to show that we get vector (1.24) after acting by $(1+n)d$ (see page 30). This gauge transformation is determined, up to a constant matrix commuting with α, by the order of poles and the residues it gives to (α, β) and by leaving the strictly lower-triangular part of α unchanged. We express $(1+n)d$ differently, namely as the composition

$$(1+n)d = (1+m)^{-1}(1+M)\ \bar{d}(1+m)$$

and we proceed to explain the meaning of each of them. First of all there is a regular gauge transformation $1+m$ with m strictly lower-triangular such that α becomes diagonal. Moreover $m(0)=0$. Observe that the resulting $v(z)$ satisfies

$$v_i(z) \equiv 0 \quad \text{if} \quad i \neq 1, l+1$$

This follows from the fact that if α is diagonal with N_l-type poles, $v(z)$ satisfies (2.18) and $\lim_{z\to 0} z^{\frac{l-1}{2}-i}v_i(z) = 0$, then $v_i(z) \equiv 0$. Now we act by

$$\bar{d}(z) = \mathrm{diag}(z^{\frac{k-l}{2}},\ldots,z^{\frac{k-l}{2}},z^{-\frac{l}{2}},\ldots,z^{-\frac{l}{2}})$$

giving the $\mathcal{N}_k$-type poles to α. Then we act by the gauge transformation $1+M$, M strictly lower-triangular, that gives the $\mathcal{N}_k$-type poles to β, does not change α (as $\dot{M} = [\alpha, M]$) and keeps $\beta_{ij}(0)$ equal to 0 if $i > j+1$. Observe that $M_{l+1,1}$ has a pole of order l at 0. This means that $M_{l+1,1}v_1(z)$ and $v_{l+1}(z)$ both are of the form $z^{\frac{(k-l)-l-1}{2}} \cdot$ (*analytic function*). Therefore if, before $1+M$ we act by a diagonal gauge transformation C which for $z=0$ has first l entries equal to 1 and next $k-l$ to some appropriate c, then we can make $M_{l+1,1}v_1(z) + cv_{l+1}(z) = z^{\frac{k-2l+1}{2}} \cdot$ (*analytic function*). As α is still diagonal, the previous argument shows that the resulting

$$\bar{v}_i(z) \equiv 0 \quad \text{if} \quad i \neq 1$$

Notice also that the gauge transformation C we defined above commutes with $\bar{d}(1+m)$ and therefore we can replace $(1=n)d$ by $(1+n)dC$. Also observe that C is the same at $+1$ as at -1, so we can act by a *constant* gauge transformation which commutes with α and β, so it will not change them.
Returning to $\bar{v}(z)$ we see that it satisfies (1.24). Therefore $\hat{v}(z) = (1+m)^{-1}(z)\bar{v}(z)$ also satisfies (1.24) (as m is regular). However $(1+m)^{-1}(1+M)\bar{d}(1+m) = (1+n)d$ so we do get the correct boundary vectors.
We have proved Proposition 1.5 and 1.6 in case a).

Continuation of the proof in case b):
To give $(\tilde{\alpha}, \tilde{\beta})$ correct exponential decay we consider the gauge transformation

$$1 + n(z)e^{-R^k z^k}$$

Observe that since $\dot{n} = 2[n, \bar{\alpha}]$, $\bar{\alpha}$ becomes

$$\bar{\alpha} + \tfrac{1}{2}kR^k z^{k-1} e^{-R^k z^k} n(z)$$

while $\bar{\beta}$ becomes

$$\bar{\beta} + [n, \bar{\beta}]e^{-R^k z^k}$$

In order to estimate $\frac{1}{2}kR^k z^{k-1} e^{-R^k z^k} n(z)$ we have to improve the estimates for $n(z)$. We want

$$|n_{ij}(z)| \le \frac{Ke^{CRz}}{(Rz)^{i-j}} \tag{1.25}$$

Observe that since $|\beta_i - \beta_j| \ge cR$ for $i > l$, $j \le l$, (1.4) and (1.5) give us

$$|C_{ij}| \le KR^{-(i-j)}$$

Recall now the expression for $\sigma(2\bar{\alpha}_{jj} - 2\bar{\alpha}_{ii})(z)$ from part a):

$$\frac{R}{(1-e^{-Rz})} \prod \left(\frac{r_s}{1-e^{-r_s z}}\right)^{n_s} \left[\left(\frac{R}{1-e^{-Rz}}\right)^{\frac{k}{2}-1} \cdot \prod \left(\frac{1-e^{-r_s z}}{r_s}\right)^{q_s} \cdot e^{p(z)}\right] e^{2(\alpha_j - \alpha_i)z}$$

where $\sum q_s = \frac{k}{2} - 1$, $\sum n_s = i - j - 1$, $n_s \ge 0$, $r_s > 0$, $r \le |r_s| \le R$ and $p(z)$ bounded. Call the expression in the square brackets $g_{ij}(z)$. As Lemma 1.9 shows $g_{ij}(z)e^{-sRz}$ is bounded. Because of the above estimate on $|C_{ij}|$ as well as the fact that

$$\frac{r_s z}{1-e^{-r_s z}} \le K + Rz\ , \quad Rze^{CRz} \le e^{(C+1)Rz}$$

we get

$$\left|(Rz)^{i-j} C_{ij}\, \sigma(2\bar{\alpha}_{jj} - 2\bar{\alpha}_{ii})(z)\right| \le Ke^{CRz} \tag{1.26}$$

We want similar estimates for the other terms of $n_{ij}(z)$, i.e. for

$$\left(\int_0^z 2n_{i,j+s}\bar{\alpha}_{j+s,j}\left(\sigma(2\bar{\alpha}_{jj}-2\bar{\alpha}_{ii})\right)^{-1}\,d\tau\right)\sigma(2\bar{\alpha}_{jj}-2\bar{\alpha}_{ii})(z)$$

Observe that (1.26) gives us the desired estimate for $n_{k+1,k}$:

$$|n_{k+1,k}| \le \frac{K}{Rz}e^{CRz}$$

Notice that using the Mean Value Theorem we get

$$\int_0^z 2n_{i,j+s}\bar{\alpha}_{j+s,j}\left(\sigma(2\bar{\alpha}_{jj}-2\bar{\alpha}_{ii})\right)^{-1}\,d\tau = 2n_{i,j+s}(\tau)\bar{\alpha}_{j+s,j}(\tau)\left(\sigma(2\bar{\alpha}_{jj}-2\bar{\alpha}_{ii})(\tau)\right)^{-1}\cdot z$$

for some $0<\tau<z$.

Using now the inductive assumption ($|n_{i,j+s}(\tau)| \le \frac{Ke^{CR\tau}}{(R\tau)^{i-j-s}}$), the fact that $|\bar{\alpha}_{j+s,i}| \le KR$, the formula for $\sigma(2\bar{\alpha}_{jj}-2\bar{\alpha}_{ii})(\tau)$ and Lemma 1.8 we get that the norm of

$2n_{i,j+s}(\tau)\bar{\alpha}_{j+s,j}(\tau)\left(\sigma(2\bar{\alpha}_{jj}-2\bar{\alpha}_{ii})(\tau)\right)^{-1}\cdot z$ can be majorized by

$$\frac{Rz}{(R\tau)^p}\frac{1-e^{-R\tau}}{R}\prod\left(\frac{1-e^{-r_s\tau}}{r_s}\right)^{n_s}\cdot Ke^{CR\tau}$$

where $p<i-j$, $\sum n_s = i-j-1$, $n_s \ge 0$, $r_s>0$ and $r \le |r_s| \le R$.

We can rewrite it as

$$Rz\frac{\left(1-e^{-R\tau}\right)^p}{(R\tau)^p}\left[\frac{1}{(1-e^{-R\tau})^p}\frac{1-e^{-R\tau}}{R}\prod\left(\frac{1-e^{-r_s\tau}}{r_s}\right)^{n_s}\right]\cdot Ke^{CR\tau}$$

Now, as $\frac{1-e^{-R\tau}}{R\tau}$ is bounded and both $1-e^{-r_s\tau}$ and $\frac{1-e^{-r_s\tau}}{1-e^{-R\tau}}$ are increasing functions, and the expression in the square brackets is increasing, we get that

$$\left|2n_{i,j+s}(\tau)\bar{\alpha}_{j+s,j}(\tau)\left(\sigma(2\bar{\alpha}_{jj}-2\bar{\alpha}_{ii})(\tau)\right)^{-1}\cdot z\right| \le Rz\frac{1}{R(1-e^{-Rz})^{p-1}}\prod\left(\frac{1-e^{-r_sz}}{r_s}\right)^{n_s}\cdot Ke^{CRz}$$

From this it follows that

$$\left|\left(\int_0^z 2n_{i,j+s}\bar{\alpha}_{j+s,j}\left(\sigma(2\bar{\alpha}_{jj}-2\bar{\alpha}_{ii})\right)^{-1}\,d\tau\right)\sigma(2\bar{\alpha}_{jj}-2\bar{\alpha}_{ii})(z)\right| \le \frac{KRz}{(1-e^{-Rz})^p}e^{CRz}$$

Now we can use the simple facts

$$\frac{1}{1-e^{-Rz}} \le \frac{1}{Rz}e^{CRz}$$

$$1 \le \frac{1}{Rz}e^{CRz}$$

to show finally (1.25).

We also want to estimate $\dot{n}_{ij}$. We know that $\dot{n} = 2[n, \bar{\alpha}]$ and as $|\bar{\alpha}_{ij}| \leq KR$ if $i > j$ and $|\bar{\alpha}_{ii}| \leq \frac{K}{z} e^{CRz} + KR$, it follows that

$$|\dot{n}_{ij}(z)| \leq \frac{KRe^{CRz}}{(Rz)^{i-j+1}} \tag{1.27}$$

Now we would like to estimate $B = [n, \bar{\beta}]$. First of all (1.6) gets replaced by

$$B_{l+1,l} = \frac{R}{1 - e^{-Rz}} f(z) e^{2(\alpha_l - 2\alpha_{l+1})z} \tag{1.28}$$

where $f, \frac{1}{R} f'$ are bounded and $f(0) = \delta_l$.
B_{ij} for $i > j+1$ is given by the formula (1.3) (as $\dot{B} = 2[B, \bar{\alpha}]$) but with $C_{ij} = 0$.
Repeating the arguments for n_{ij} we can show now inductively:

$$|B_{ij}| \leq KRe^{CRz} \tag{1.29}$$

Now if we correct the residues by a diagonal gauge transformation of the form

$$e^{\delta e^{-Rz}}$$

creating thus the complex $(\hat{\alpha}, \hat{\beta})$ which we can write as

$$(\hat{\alpha}, \hat{\beta}) = (\alpha, \beta) + (\check{\alpha}, \check{\beta})$$

then (1.25) and (1.27) together with the fact that $i - j \leq k - 2$ if $i > l, j \leq l$, and together with the following fact

$$e^{c_1 Rz - c_2 R^k z^k} \leq Ke^{-sRz} \tag{1.30}$$

give us:

$$|\check{\alpha}_{ij}(z)| \leq KRe^{-sRz} \quad \text{if } i > j \tag{1.31}$$

$$\left|\dot{\check{\alpha}}_{ij}(z)\right| \leq KR^2 e^{-sRz} \quad \text{if } i > j \tag{1.32}$$

We also have, like in the case a)

$$\left|\check{\alpha}_{ii}(z) - \frac{p_i}{z} e^{-Rz}\right| \leq KRe^{-Rz} \tag{1.33}$$

$$\left|\dot{\check{\alpha}}_{ii}(z) + \frac{p_i}{z^2} e^{-Rz}\right| \leq \frac{KR}{z} e^{-Rz} + KR^2 e^{-Rz} \tag{1.34}$$

(1.29) together with (1.30) lead

$$\left|\check{\beta}_{i,j}(z)\right| \leq KRe^{-sRz} \quad \text{if } i > j+1 \tag{1.35}$$

Finally we also have

$$\check{\beta}_{ii} \equiv 0 \tag{1.36}$$

$$\left|\check{\beta}_{i+1,i}(z) - \tfrac{\delta_i}{z} e^{-Rz}\right| \leq KRe^{-Rz} \tag{1.37}$$

if $i \neq k$.

There remains problem of $\check{\beta}_{l+1,l}(z)$. We know that it is equal to

$$\frac{R}{1-e^{-Rz}} f(z) e^{2(\alpha_l - 2\alpha_{l+1})z} e^{-R^k z^k}$$

We can improve (1.30) to

$$e^{c_1 Rz - c_2 R^k z^k} = e^{2c_1 Rz - c_2 R^k z^k} e^{-c_1 Rz} \equiv g(z) e^{-c_1 Rz}$$

where g, $\frac{1}{R}\dot{g}$ are bounded and $g(0) = 1$. Then in the same way as for (1.16) we get

$$\left|\check{\beta}_{l+1,l}(z) - \tfrac{\delta_l}{z} e^{-sRz}\right| \leq KRe^{-sRz} \tag{1.38}$$

The above estimates for $(\check{\alpha}, \check{\beta})$ are the same as in case a) therefore we can prove Proposition 1.5 i),ii) and iv)-x) in the same way. Differentiating with respect to tangent directions is also done the same way (we use again (1.20) so we can prove Proposition 1.6. The analysis of the boundary vectors is exactly the same which ends the proof of Propositions 1.5 and 1.6.

$\square$

2.2 Exponential decay of the solutions to Nahm's equations

In this section we will analyze Nahm's equations and get estimates on their solutions. This in turn will let us get estimates on the gauge transformation $\tilde{g}$ defined in Proposition 1.5 iii).

We consider $(\alpha,\beta) = \tilde{g}(\tilde{\alpha},\tilde{\beta})$. α and β are lower triangular on $(-1,1)$ and are the result of a smooth (on $(-1,1)$), lower triangular gauge transformation $\tilde{g}g$ (Proposition 1.5) acting on (α_d,β_d). Let us express $\tilde{g}g$ in the form

$$\hat{d}(1+\hat{n})$$

where $\hat{d}$ is diagonal and $\hat{n}$ is strictly lower-triangular. $\hat{d}(1+\hat{n})$ can be also written as

$$u_o^* v_o d(1+n)u_o$$

where u_o, v_o are the diagonal unitary gauge transformations $e^{2i\mathrm{Im}\,\alpha_d}$, $e^{i,\mathrm{Im}\ln\hat{d}}$, $d = v_o^*\hat{d}$ and $n = u_o\hat{n}u_o^*$.

Therefore, instead of α_d, we can start with $u_o(\alpha_d) = \mathrm{Re}\,\alpha_d$ and consider

$$(\alpha,\beta) = d(1+n)(\mathrm{Re}\,\alpha_d,\beta_d) \tag{2.1}$$

(α,β) is lower-triangular, has real diagonal, satisfies the real equation and is complex-gauge equivalent to (α_d,β_d). It differs from (α,β) of Proposition 1.5 iii) by a unitary diagonal gauge transformation $u_o^* v_o$ which does not affect the estimates on the stictly lower-triangular part of (α,β) nor on the real part of the diagonal.

The goal of this section is to prove

Proposition 2.1 *Let $z = 1 \pm t$. There are constants $s > 0, K$, depending only on the charge k, such that for any $i > j$ and $z \leq 1$*

$$z\cdot|\alpha_{ij}(z)|\ ,\ \ z\cdot|\beta_{ij}(z)| \leq Ke^{-sR_{ij}z}$$

Observe that a corollary follows immediately

Corollary 2.2 *There are constants $s > 0, K$, depending only on the charge k, such that for any $i > j$ and $t \in \left[-1+\frac{1}{R_{ij}}, 1-\frac{1}{R_{ij}}\right]$*

$$|\alpha_{ij}(t)|\ ,\ |\beta_{ij}(t)| \leq KR_{ij}e^{-sR_{ij}(1-t)} + KR_{ij}e^{-sR_{ij}(1+t)}$$

We also would like to say something about the diagonal part of α (observe that the diagonal part of β is $\equiv \beta_d$). We have

Proposition 2.3 *Let $z = 1 \pm t$. There is a constant K, depending only on the charge k, such that for $z \le 1$ and $i \le k$*

$$z \cdot |\alpha_{ii}(z) - \operatorname{Re}\alpha_i| \le K$$

We want to remark that Proposition 2.1 is equivalent to the following two facts

Proposition 2.4 *There are constants $s > 0, C, K$, depending only on the charge k, such that for any $i > j$ and $t \in \left[-1 + \frac{C}{R_{ij}}, 1 - \frac{C}{R_{ij}}\right]$*

$$|\alpha_{ij}(t)|\ ,\ |\beta_{ij}(t)| \le K R_{ij} e^{-sR_{ij}(1-t)} + K R_{ij} e^{-sR_{ij}(1+t)}$$

Proposition 2.5 *Let $z = 1 \pm t$. There is a constant K, depending only on the charge k, such that for $z \le 1$ and all $i > j$*

$$z \cdot |\alpha_{ij}(z)|\ ,\ z \cdot |\beta_{ij}(z)| \le K$$

It is clear that Proposition 2.1 implies 2.4 and 2.5. It is also clear that 2.4 implies that the estimates of 2.1 hold for $z \ge \frac{C}{R_{ij}}$. Then however, changing K and s, 2.5 implies they hold for all $0 \le z \le 1$.
Therefore we are going to prove Propositions 2.3, 2.4 and 2.5.
Before starting the proof let us make some remarks. We consider molecules (cf. Def.1.1) in the set

$$X = \{ (\operatorname{Re}\alpha_i, \beta_i)\,;\ i = 1, \ldots, k \}$$

If Y is a subset of X we will identify Y with the set of indices $\{ i\,;\ (\operatorname{Re}\alpha_i, \beta_i) \in Y \}$. We will use the following symbol ($Y \subset X$):

$$I(Y) = X \times X - (Y \times Y \cup Y' \times Y') \tag{2.2}$$

Let Y be a molecule in X. First of all we want to prove Propositions 2.4 and 2.5 for $(i, j) \in I(Y)$. Observe that Lemma 1.2 gives us

$$\text{if } (i,j) \in I(Y), \text{ then } R_{ij} \ge \left(\tfrac{1}{2}\right)^k R \tag{2.3}$$

We have the following lemma that reduces the exponential decay to the $o(1)$ decay:

Lemma 2.6 *There are $\epsilon > 0, s > 0$ such that for every $a < b$ and every molecule Y, if*

$$\|\alpha(t) - \mathrm{Re}\,\alpha_d\| \ , \ \|\beta(t) - \beta_d\| \leq \epsilon R$$

for all $t \in [a, b]$, then

$$|\alpha_{ij}(t)| \ , \ |\beta_{ij}(t)| \leq \epsilon R e^{-sR(t-a)} + \epsilon R e^{-sR(b-t)}$$

for $t \in [a, b]$ and $(i, j) \in I(Y)$.

Note that (2.3) and the fact that $R \geq R_{ij}$ imply that for $(i, j) \in I(Y)$ R-decay is equivalent to R_{ij}-decay.

Proof: In our gauge α is lower triangular with real entries on the diagonal. Therefore the real equation

$$(\alpha \dot{+} \alpha^*) + 2[\alpha, \alpha^*] + 2[\beta, \beta^*] = 0$$

gives us an equation for α. If M is a matrix, put

$$M_\triangle = \text{strictly lower triangular part of } M + \tfrac{1}{2}\text{diagonal of } M$$

The equation for α can be written then as

$$\dot{\alpha} = 2[\alpha^*, \alpha]_\triangle + 2[\beta^*, \beta]_\triangle \tag{2.4}$$

We can differentiate this and the complex equation to get

$$\begin{aligned}
\ddot{\alpha} &= 2\left[(2[\alpha^*, \alpha]_\triangle + 2[\beta^*, \beta]_\triangle)^* , \alpha \right]_\triangle \ + \ 2\left[\alpha^*, 2[\alpha^*, \alpha]_\triangle + 2[\beta^*, \beta]_\triangle \right]_\triangle \\
&\quad +2\left[2[\alpha^*, \beta^*], \beta \right]_\triangle \ + \ 2\left[\beta^*, 2[\beta, \alpha] \right]_\triangle \\
\ddot{\beta} &= 2\left[2[\beta, \alpha], \alpha \right] \ + \ 2\left[\beta, 2[\alpha^*, \alpha]_\triangle + 2[\beta^*, \beta]_\triangle \right]
\end{aligned}$$

The terms on the right are products of three entries of α and β. Let us write down those terms in the equation for $\ddot{\alpha}_{ij}$, $(i, j) \in I(Y)$, in which two of the entries are diagonal. We get none from the first bracket ; the second bracket gives $4\,|\alpha_{ii} - \alpha_{jj}|^2\, \alpha_{ij} \ + \ 4(\alpha_{ii} - \alpha_{jj})(\bar{\beta}_i - \bar{\beta}_j)\beta_{ij}$; there is no such terms in the third bracket and finally the fourth one contributes $4\,|\beta_i - \beta_j|^2\, \alpha_{ij} \ + \ 4(\bar{\beta}_i - \bar{\beta}_j)\beta_{ij}(\alpha_{jj} - \alpha_{ii})$. Therefore the total sum of such terms is

$$4\left(|\alpha_{ii} - \alpha_{jj}|^2 + |\beta_i - \beta_j|^2 \right) \alpha_{ij}$$

Similarily writing out such terms in the equation for $\ddot{\beta}_{ij}$ gives

$$4\left(|\alpha_{ii}-\alpha_{jj}|^2+|\beta_i-\beta_j|^2\right)\beta_{ij} + 4(\beta_i-\beta_j)\alpha_{ij}(\alpha_{jj}-\alpha_{ii}) + 4(\beta_i-\beta_j)(\alpha_{ii}-\alpha_{jj})\alpha_{ij}$$
$$= 4\left(|\alpha_{ii}-\alpha_{jj}|^2+|\beta_i-\beta_j|^2\right)\beta_{ij}$$

All the other terms involve at least two nondiagonal entries. One of them must have coordinates belonging to $I(Y)$ while the other one will have, by assumption, norm $\leq \epsilon R$. Also by assumption $|\alpha_{ii}-\operatorname{Re}\alpha_i| \leq \epsilon R$. We have the exactly symmetric situation for (α^*, β^*). Therefore, if

$$x = (\alpha_{ij}, \beta_{ij}, \bar{\alpha}_{ij}, \bar{\beta}_{ij})_{(i,j)\in I(Y)}$$

then the above equations for $\ddot{\alpha}, \ddot{\beta}, \ddot{\alpha^*}, \ddot{\beta^*}$ give us an equation for x of the form

$$\ddot{x} = Dx + E(t)x \tag{2.5}$$

where D is diagonal and

$$D_{ij,ij} = 4\ |\operatorname{Re}\alpha_i - \operatorname{Re}\alpha_j|^2 + |\beta_i-\beta_j|^2$$

while every entry of E has norm $\leq \epsilon R^2$. This means that, if ϵ is sufficiently small, we get

$$\ddot{\|x\|^2} = 2\operatorname{Re}(\ddot{x},x) + 2(\dot{x},\dot{x}) \geq 2\operatorname{Re}(\ddot{x},x) = 2\operatorname{Re}(\,(D+E)x,x) \geq s^2R^2\,\|x\|^2 \tag{2.6}$$

where we use (2.3) for the last inequality. Now Lemma 2.6 proves the result.

□

Therefore we want to show that $|\beta-\beta_d|, |\alpha - \operatorname{Re}\alpha_d|$ are small for $t \in \left[-1+\frac{C}{R}, 1-\frac{C}{R}\right]$ if C is large. First of all let us prove it for the diagonal part of α.

Lemma 2.7 *For every $\epsilon > 0$ there is a C depending only on ϵ such that for any number M*

$$\|\operatorname{diag}\alpha(t) - \operatorname{Re}\alpha_d\| \leq \epsilon M$$

for every t in $[-1+\frac{C}{M}, 1-\frac{C}{M}]$.

If M is not large enough the interval $[-1+\frac{C}{M}, 1-\frac{C}{M}]$ can be empty.
Proof: We can write, according to Proposition 1.5 iii),

$$\operatorname{diag}\alpha(t) - \operatorname{Re}\alpha_d = \operatorname{Re}\operatorname{diag}\tilde{\alpha}(t) - \operatorname{Re}\alpha_d \;-\; \tfrac{1}{2}\tfrac{d}{dt}\ln\tilde{d}$$

where $\ln \tilde{d}$ is bounded on $[-1,1]$ and independently of (α_d, β_d). It follows from Proposition 1.5 i) that $\operatorname{Re} \operatorname{diag} \tilde{\alpha}(t) - \operatorname{Re} \alpha_d$ satisfies the assertions of our lemma. On the other hand, as $\ln \tilde{d}_k$ is bounded independently of (α_d, β_d), for every $\epsilon > 0$ there is a C independent of (α_d, β_d) such that $|(\frac{d}{dt} \ln \tilde{d}_k)(t)| \leq \epsilon M$ at least once on any interval of length $\frac{C}{M}$.
Hence there are points $t_o \in [-1, -1 + \frac{C}{M}]$, $t_1 \in [1 - \frac{C}{M}, 1]$ such that

$$|(\tfrac{d}{dt} \ln \tilde{d}_k)(t_o)| \leq \epsilon M \quad , \quad |(\tfrac{d}{dt} \ln \tilde{d}_k)(t_1)| \leq \epsilon M$$

Therefore the same conclusion holds for $\alpha_{kk} - \operatorname{Re} \alpha_k$ (i.e. there are points t_o, t_1 at which $\alpha_{kk} - \operatorname{Re} \alpha_k$ satsfies the above inequalities).
Now the diagonal part of the real equation

$$(\alpha \dot{+} \alpha^*) + 2[\alpha, \alpha^*] + 2[\beta, \beta^*] = 0$$

gives, in the case of a lower-triangular (α, β), the following equations

$$-\dot{\alpha_{ii}} = \sum_{i>j} \left(|\alpha_{ij}|^2 + |\beta_{ij}|^2 \right) - \sum_{j>i} \left(|\alpha_{ji}|^2 + |\beta_{ji}|^2 \right) \tag{2.7}$$

This gives, as α and β are lower-triangular,

$$\begin{aligned} -\dot{\alpha_{kk}} &\geq 0 \\ -\dot{\alpha_{kk}} - \dot{\alpha_{k-1,k-1}} &\geq 0 \\ &\vdots \\ -\dot{\alpha_{kk}} - \ldots - \dot{\alpha_{11}} &\geq 0 \end{aligned} \tag{2.8}$$

Therefore we get

$$-\epsilon M \leq -(\alpha_{kk}(t_o) - \operatorname{Re} \alpha_k) \leq -(\alpha_{kk}(t) - \operatorname{Re} \alpha_k) \leq -(\alpha_{kk}(t_1) - \operatorname{Re} \alpha_k) \leq \epsilon M$$

for all $t \in [t_o, t_1]$.
We can repeat now the process with $\ln \tilde{d}_k + \ln \tilde{d}_{k-1}$ (i.e. find two points at which the derivative is $\leq \epsilon M$...) and so on, proving the fact.

□

This lemma implies Proposition 2.3.
Proof of Proposition 2.3: Take $\epsilon = 1$ and the corresponding C given by the above lemma. If $t \in (-1, 0]$, put $z = 1 + t$ and $M = \frac{C}{z}$. It follows that at t, $\|\operatorname{diag} \alpha(t) - \operatorname{Re} \alpha_d\| \leq M$, so $z \cdot \|\operatorname{diag} \alpha(t) - \operatorname{Re} \alpha_d\| \leq C$. The result for $t = 0$ follows from continuity of $z\alpha_(t)$. The arguments for $t \in [0, 1]$ are the same.

□

For nondiagonal entries we have

Lemma 2.8 *For every $\epsilon, \delta > 0$ there is a C depending only on ϵ, δ such that for any number M*

$$\mu\left\{t \in \left[-1+\tfrac{C}{M}, 1-\tfrac{C}{M}\right] \; ; \; \sum_{i>j} |\alpha_{ij}(t)|^2 + |\beta_{ij}(t)|^2 \geq \epsilon M^2\right\} \leq \frac{\delta}{M}$$

$$\int_{-1+\frac{C}{M}}^{1-\frac{C}{M}} \left(\sum_{i>j} |\alpha_{ij}(t)|^2 + |\beta_{ij}(t)|^2\right) dt \leq \delta M$$

where μ denotes the measure of a set.

Proof: Let us take some $\nu > 0$ and a C given by Lemma 2.7 for $\epsilon = \frac{\nu}{k}$. We will show that if ν is small enough, C will satisfy the assertions of this lemma. Let

$$x_i = -(\alpha_{kk} - \operatorname{Re}\alpha_k) - \ldots - (\alpha_{k-i,k-i} - \operatorname{Re}\alpha_{k-i})$$

From (2.8) we know that x_i is an increasing function for $i = 0, \ldots, k-1$. From our choice of C we know that

$$|x_i(t)| \leq \nu M$$

on $[-1+\frac{C}{M}, 1-\frac{C}{M}]$. Therefore if

$$\mu_i = \mu\left\{t \in [-1+\tfrac{C}{M}, 1-\tfrac{C}{M}] \; ; \; \dot{x}_i \geq \sqrt{\nu} M^2\right\}$$

then

$$2\nu M \geq x_i\left(1-\tfrac{C}{M}\right) - x_i\left(1-\tfrac{C}{M}\right) = \int_{-1+\frac{C}{M}}^{1-\frac{C}{M}} \dot{x}_i \geq \sqrt{\nu} M^2 \mu_i$$

Hence

$$\mu_i \leq \frac{2\sqrt{\nu}}{M}$$

However (2.7) gives us

$$\begin{aligned}
\dot{x}_0 &= \sum_{j<k} |\alpha_{kj}|^2 + |\beta_{kj}|^2 \\
\dot{x}_1 &= \sum_{j<k-1} \left(|\alpha_{k-1,j}|^2 + |\beta_{k-1,j}|^2\right) + \sum_{j<k-1} \left(|\alpha_{k,j}|^2 + |\beta_{k,j}|\right) \\
&\vdots
\end{aligned} \tag{2.9}$$

which shows that

$$\mu\left\{t \in [-1+\tfrac{C}{M}, 1-\tfrac{C}{M}] \; ; \; \sum_{i>j} |\alpha_{ij}(t)|^2 + |\beta_{ij}(t)|^2 \geq \sqrt{\nu} M^2\right\} \leq K\frac{\sqrt{\nu}}{M}$$

where K depends only on charge k. This proves that C satisfies the first condition of the lemma if ν is small relative to ϵ, δ.

Now

$$\int_{-1+\frac{C}{M}}^{1-\frac{C}{M}} \dot{x}_i = x_i\left(1-\frac{C}{M}\right) - x_i\left(-1+\frac{C}{M}\right) \leq 2\nu M$$

so (2.9) proves the second condition if ν is small.

□

We will apply the last two lemmas to the equation (2.5). First let us prove a fact in a more general setting:

Lemma 2.9 *Let $y(t)$ be defined on an interval I and satisfies there a differential equation*

$$\dot{y} = L(t)y + G(t)y + h(t)$$

where L, G, h are continuous and for some numbers M, s

$$\|L\| \leq sM \quad , \quad \int_I \|G(t)\|^2 \leq sM \quad , \quad \|h\| \leq sM^2$$

There is a constant K depending only on s, such that if ϵ is such that

$$\mu\{t \in I\,;\, \|y(t)\| \geq \epsilon M\} \leq \frac{\epsilon}{M}$$

then

$$\|y(t)\| \leq K\epsilon M$$

for all $t \in I$.

Proof: Let $t_o \in I$ be such that $\|y(t_0)\| > \epsilon M$. Then it follows from the assumption that there is a point $a \in I$ such that $|t_0 - a| < \frac{\epsilon}{M}$ and $\|y(a)\| \leq \epsilon M$. The basic estimate for linear equations (cf. [Hartman, Lemma IV.4.1]) gives, for any t

$$\|y(t)\| \leq \left\{ \|y(a)\| + \left|\int_a^t \|h(\tau)\|\, d\tau\right| \right\} e^{\left|\int_a^t (\|L(\tau)\| + \|G(\tau)\|)\, d\tau\right|}$$

Since the $\mathcal{L}^1$-norm is bounded by $\mathcal{L}^2$-norm times the square root of the length of the interval, $\int_a^{t_o} \|G(\tau)\|\, d\tau \leq \sqrt{s\epsilon}$. It follows that

$$\|y(t_o)\| \leq (\epsilon + s\epsilon)e^{s\epsilon + \sqrt{s\epsilon}} M$$

□

The next result will, in view of Lemma 2.6, prove Proposition 2.4 for $(i,j) \in I(Y)$.

Lemma 2.10 *For every $\epsilon > 0$ there is a C, depending only on ϵ, such that for all $M \geq R$ and all $t \in \left[-1 + \frac{C}{M}, 1 - \frac{C}{M}\right]$*

$$\|\alpha(t) - \operatorname{Re}\alpha_d\| \ , \ \|\beta(t) - \beta_d\| \leq \epsilon M$$

Proof of Lemma 2.10: Take $\nu < 1$ and let C_1 be given by Lemma 2.3 for $\epsilon = \nu$, C_2 be given by Lemma 2.5 for $\epsilon = \nu$, $\delta = \nu$. Let $C = \max\{C_1, C_2\}$. We will show that if ν is small enough, then C satisfies the assertions of the lemma.
Observe that our choice of ν guarantees that

$$\|\operatorname{diag}\alpha(t) - \operatorname{Re}\alpha_d\| \leq \nu M \qquad \text{for all } t \in \left[-1 + \tfrac{C}{M}, 1 - \tfrac{C}{M}\right] \tag{2.10}$$

$$\mu\left\{ t \in \left[-1 + \tfrac{C}{M}, 1 - \tfrac{C}{M}\right] \ ; \ \sum_{i>j} |\alpha_{ij}(t)|^2 + |\beta_{ij}(t)|^2 \geq \nu M^2 \right\} \leq \nu \tag{2.11}$$

$$\int_{-1+\frac{C}{M}}^{1-\frac{C}{M}} \left(\sum_{i>j} |\alpha_{ij}(t)|^2 + |\beta_{ij}(t)|^2 \right) dt \leq \nu M \tag{2.12}$$

We can write equation (2.4), its conjugate, the complex Nahm's equation and its conjugate as an equation for

$$y = (\alpha_{ij}, \beta_{ij}, \bar{\alpha}_{ij}, \bar{\beta}_{ij})_{i>j}$$

It has the form

$$\dot{y} = L(t)y + G(t)y \tag{2.13}$$

where the entries of L are of the form $\alpha_{pp} - \alpha_{rr}$, $\beta_p - \beta_r$ or their conjugates while the entries of G are of the form α_{pr}, β_{pr} or their conjugates with $p > r$. Hence $\|L\| \leq 2R$. Since $M \geq R$, (2.10)-(2.12) imply that y, L, G satisfy conditions of Lemma 2.9 with $h = 0$ and $\epsilon = P\sqrt{\nu}$ for some P depending only on the charge k. Therefore $\|y\| \leq P\sqrt{\nu}M$ and this together with (2.10) gives

$$\|\alpha(t) - \operatorname{Re}\alpha_d\| \ , \ \|\beta(t) - \beta_d\| \leq P\sqrt{\nu}M$$

$\square$

Remark Proposition 2.5 is now also proved for $(i,j) \in I(Y)$. Indeed, setting $M = \frac{C}{z}$ (the same way as in the proof of Proposition 2.3), we can see that $z \cdot |\alpha_{ij}(z)|$, $z \cdot |\beta_{ij}(z)|$ are bounded for $z \leq \frac{C}{R}$. Then however, by Proposition 2.4, they are bounded for all $z \leq 1$ (as $z\, Re^{-sRz}$ is bounded by $\frac{1}{se}$).

where every entry of h involves as a factor an entry of α or β with coordinates belonging to $I(Y)$. Therefore, arguing as for the function $g(z)$, we can assume that

$$\|h(t)\| \leq \epsilon\, R(Y)^2 \left(e^{-sR(t-a)} + e^{-sR(b-t)}\right)$$

Similarily if an entry of E has a factor with coordinates not in $Y \times Y$, then it must have another factor with coordinates belonging to $I(Y)$. Therefore, again by the above considerations, such an entry of E must satisfy

$$\|E(t)\| \leq \epsilon\, R(Y)^2$$

on $[a, b]$. All other entries of E must satisfy the above inequality because of the assumption.

This time instead of (2.6), we get

$$\tfrac{d}{dt}\left(\|x\|^2\right) = 2\,\mathrm{Re}\,(\ddot{x}, x) + 2(\dot{x}, \dot{x}) \geq 2(\dot{x}, \dot{x}) + s^2 R(Y)^2 \|x\|^2 - 2\,\|h\|\, \|x\| \qquad (2.15)$$

where we used the fact that

$$D_{ij} \geq \left(\tfrac{1}{2}\right)^k R(Y)^2$$

Since, by Cauchy-Schwartz inequality, we have ($'$ denotes $\frac{d}{dt}$)

$$\left(\|x\|^2\right)'' = 2\,\|x\|''\, \|x\| + 2\left(\|x\|'\right)^2 \leq 2\,\|x\|''\, \|x\| + 2\,\|x'\|^2$$

we get from this and (2.15) that

$$\tfrac{d}{dt}\,\|x\| \geq s^2 R(Y)^2\, \|x\| - \|h\|$$

and the result follows from Lemma 1.7 as in Lemma 2.6.

□

Therefore we will get estimates of Propositions 2.4 and 2.5 for $I(Y_1)$ if we can prove Lemma 2.10 with R replaced by $R(Y)$.

Lemma 2.12 *For every $\epsilon > 0$ there is a C such that for all $(i, j) \in Y \times Y$ and all $M \geq R(Y)$, $t \in \left[-1 + \frac{C}{M}, 1 - \frac{C}{M}\right]$*

$$\left\|\left(\alpha(t) - \mathrm{Re}\,\alpha_d\right)_{ij}\right\|, \left\|\left(\beta(t) - \beta_d\right)_{ij}\right\| \leq \epsilon R(Y)$$

Proof: Let us proceed as in the proof of Lemma 2.10. Because of the remark following Lemma 2.10, the result is proven for $M > R$; we can assume that $R(Y) \leq M \leq R$. This time the equation for

$$y = (\alpha_{ij}, \beta_{ij}, \bar{\alpha}_{ij}, \bar{\beta}_{ij})_{(i,j)\in Y\times Y, i\neq j}$$

is of the form

$$\dot{y} = L(t)y + G(t)y + h(t)$$

where the entries of h are given by products of two entries of α, β with coordinates in $I(Y)$. Therefore (cf. the considerations on the function $g(z)$ in the proof of Lemma 2.10) we can assume that $\|h(t)\| \leq \epsilon M^2$ on the interval $\left[-1 + \frac{C}{M}, 1 - \frac{C}{M}\right]$. The result follows, as before, from Lemma 2.9.

□

We have shown the estimates of Propositions 2.4 and 2.5 to hold for $I(Y_1)$. Inductively we can prove the same way that there are constants $C, s > 0$ depending only on the charge k such that for any sequence

$$X \supset Y \supset Y_1 \supset \ldots \supset Y_m \tag{2.16}$$

where Y_{n+1} is a molecule in Y_n the estimates of Propositions 2.4 and 2.5 hold for any $(i, j) \in I(Y_m)$.

However for any $i = 1, \ldots, k$, there is a sequence (2.16) such that $\bigcap Y_n = \{i\}$. Since $I(\{i\}) = \{(i,j)\,;\, j \neq i\}$, this proves Propositions 2.4 and 2.5.

□

As a corollary we get estimates on the gauge transformation $\tilde{g}$ of Proposition 1.5 iii). Let us express $\tilde{g}$ as

$$\tilde{g} = (1 + \tilde{n})\tilde{d}$$

with $\tilde{d}$ real-diagonal and $\tilde{n}$ stristly lower-triangular.

Corollary 2.13 *Let $z = 1 \pm t$. There are constants $s > 0, K$, depending only on the charge k, such that for any $i > j$ and $z \leq 1$*

$$|\tilde{n}_{ij}(z)| \leq Ke^{-sR_{ij}z}$$

$$z\left|\dot{\tilde{n}}_{ij}(z)\right| \leq Ke^{-sR_{ij}z}$$

$$z\left\|\frac{d}{dt}(\ln \tilde{d})\right\| \leq K$$

Proof: The third of these estimates follows immediately from Proposition 1.5 i) and Proposition 2.3 (recall from the beginning of this section how we altered (α, β).
For $\tilde{n}$ let us show first an estimate similar to Proposition 2.4, namely that there is a constant C such that for all $t \in \left[-1 + \frac{C}{R_{ij}}, 1 - \frac{C}{R_{ij}}\right]$

$$|\tilde{n}_{ij}(t)| \leq K e^{-sR_{ij}(1-t)} + K e^{-sR_{ij}(1+t)} \tag{2.17}$$

Observe that as $\ln \tilde{d}$ is bounded, the strictly lower-triangular part of

$$(\hat{\alpha}, \hat{\beta}) = \tilde{d}(\tilde{\alpha}, \tilde{\beta})$$

still satisfies the exponential decay estimates of Proposition 1.5 ii) and therefore it follows from this and Proposition 2.4 that there is a constant C such that for all $t \in \left[-1 + \frac{C}{R_{ij}}, 1 - \frac{C}{R_{ij}}\right]$

$$|\alpha_{ij}(t) - \hat{\alpha}_{ij}(t)| \,,\, \left|\beta_{ij}(t) - \hat{\beta}_{ij}(t)\right| \leq K R_{ij} e^{-sR_{ij}(1-t)} + K R_{ij} e^{-sR_{ij}(1+t)} \tag{2.18}$$

Now recall that (α_d, β_d)were permuted so that Lemma 1.3 is satisfied. If we take now $i = j + 1$, then we have

$$\beta_{i,i-1} - \hat{\beta}_{i,i-1} = \tilde{n}_{i,i-1} \left(\beta_{i-1} - \beta_i\right) \ , \ \alpha_{i,i-1} - \hat{\alpha}_{i,i-1} = \tilde{n}_{i,i-1} \left(\hat{\alpha}_{i-1,i-1} - \hat{\alpha}_{ii}\right) - \tfrac{1}{2}\dot{\tilde{n}}_{i,i-1} \tag{2.19}$$

According to Lemma 2.3 ii) we can either have $|\beta_{i-1} - \beta_i| \geq cR_{i,i-1}$ or $\operatorname{Re}\alpha_{i-1} < \operatorname{Re}\alpha_i$. In the first case the first of the equations (2.19) and the estimate (2.18) imply (2.17). In the second case, as $c \leq \frac{1}{4}$, we must have $\operatorname{Re}\alpha_{i-1} - \operatorname{Re}\alpha_i \leq -cR_{i,i-1}$. As $\hat{\alpha}_{jj} = \alpha_{jj}$, we get from Proposition 2.3 that there is a constant C such that for all $t \in \left[-1 + \frac{C}{R_{i,i-1}}, 1 - \frac{C}{R_{i,i-1}}\right]$

$$\operatorname{Re}\left(\hat{\alpha}_{i-1,i-1} - \hat{\alpha}_{ii}\right) - \left(\operatorname{Re}\alpha_{i-1} - \operatorname{Re}\alpha_i\right) \leq \epsilon R_{i,i-1}$$

with ϵ as small as necessary. Then from the second of equations (2.19), integrating, we get for $t \in \left[-1 + \frac{C}{R_{i,i-1}}, 1 - \frac{C}{R_{i,i-1}}\right]$

$$|\tilde{n}_{i,i-1}(t)| \leq \left(\left|\tilde{n}_{i,i-1}\left(-1 + \frac{C}{R_{i,i-1}}\right)\right| + \int_{-1+\frac{C}{R_{i,i-1}}}^{t} 2\left|\hat{\alpha}_{i,i-1} - \alpha_{i,i-1}\right|\right) e^{-2(c-\epsilon)R_{i,i-1}\left(t+1-\frac{C}{R_{i,i-1}}\right)}$$

Now (2.17) for $i = j + 1$ follows from (2.18) and the fact that $\tilde{n}$ is bounded. As $\tilde{n}$ is bounded on $[-1, 1]$, by changing s and K we can assume that (2.17) holds for $i = j + 1$ on all of $[-1, 1]$. This is equivalent to first of the estimates of our

statement. Now we get from (2.18) and the second of the equations (2.19) the second of the estimates of our corollary for $i = j + 1$.
Now, let $i = j + 2$. We have

$$\beta_{i,i-2} - \hat{\beta}_{i,i-2} = \tilde{n}_{i,i-2}\,(\beta_{i-2} - \beta_i) + (\beta_{i,i-1} - \hat{\beta}_{i,i-1})\tilde{n}_{i-1,i-2}$$

The second term on the right-hand side of this equality is, in view of (2.18) and (2.17) for $i = j + 1$, bounded by

$$K R_{i,i-1} \left(e^{-sR_{i,i-1}(1-t)} + e^{-sR_{i,i-1}(1+t)}\right) \left(e^{-sR_{i-1,i-2}(1-t)} + e^{-sR_{i-1,i-2}(1+t)}\right)$$

This however, because of Lemma 1.3 ii) (and the triangle inequality), is $\leq$

$$K R_{i,i-2} \left(e^{-sR_{i,i-2}(1-t)} + e^{-sR_{i,i-2}(1+t)}\right)$$

so, if $|\beta_{i-2} - \beta_i| \geq cR_{i,i-2}$ we get (2.17) for $i = j + 2$ as before. The case when $\operatorname{Re}\alpha_{i-2} < \operatorname{Re}\alpha_i$ also follows along the same lines. We can see now how the statement follows by induction on $i - j$.

$\square$

2.3 Asymptotic behaviour of the spectral curve

From the estimates of the previous section we immediately get a result on the behaviour of the spectral curve of a monopole. Since this is important, we are going to phrase it separately. Recall from section 1.4 that the spectral curve is defined in terms of Nahm complexes by

$$\det\left(\eta - 2\beta(t) - (\alpha(t) + \alpha^*(t))\,\zeta + 2\beta^*(t)\zeta^2\right) = 0$$

and its definition is independent of t. Therefore, if we take $t = 0$ it follows from Proposition 2.1 and the fact that ζ varies over a compact set that the spectral curve approaches the union of spectral curves of 1-monopoles exponentially fast. This corresponds to the monopole being approximated by a configuration of particles. The precise result is:

Proposition 3.1 *There are constants $K, s > 0$ independent of (α_d, β_d), such that the spectral curve of the monopole corresponding to a Nahm complex (α, β) is within Ke^{-sr} of the union of spectral lines, i.e curves of the form $\eta = a\zeta^2 + b\zeta + c$.*

□

2.4 Asymptotic behaviour of the metric

We would like to estimate how much the metric tensor on $\mathcal{N}_k$ differs from the product metric on $(\mathbf{C} \times \mathbf{C})^k$. Recall from Section 2. that we constructed a manifold $\tilde{N}_k = \tilde{N}_k((\alpha_d, \beta_d))$ diffeomorphic to a neighbourhood of (α_d, β_d) on which the natural metric introduced in Section 1. differs from the product metric by $\frac{const}{r}$. Indeed, from Proposition 1.6 we know that the image $(\tilde{a}, \tilde{b})$ under the tangent diffeomorphism of an element $(a_d, b_d) \in \mathbf{T}_{\alpha_d,\beta_d}(\mathbf{C} \times \mathbf{C})^k$ of norm 1 satisfies pointwise estimates:

$$\|\tilde{a}(t) - a_d\| + \left\|\tilde{b}(t) - b_d\right\| \leq K e^{-sr(1+t)} + K e^{-sr(1-t)} \tag{4.1}$$

for some $K, s > 0$ independent of (α_d, β_d). Integrating this gives, as a_d, b_d are constant and bounded by 1,

$$\left| \int_{-1}^{1} \operatorname{tr}\left(\tilde{a}^*\tilde{a} + \tilde{b}^*\tilde{b}\right) \ - \sum_{i=1}^{k} \left(|a_i|^2 + |b_i|^2\right) \right| \leq \frac{C}{r}$$

for some C independent of (α_d, β_d).
In this section we want to show that the metric on $\mathcal{N}_k$ differs from the one on $\tilde{N}_k$ by $\frac{const}{r}$. Consider the way the tangent space $\mathbf{T}_{(\alpha,\beta)}\mathcal{N}_k$ is obtained. Starting with $(\tilde{\alpha}, \tilde{\beta}) \in \tilde{N}_k$ we have a gauge transformation $\tilde{g}$ such that $\tilde{g}(\tilde{\alpha}, \tilde{\beta}) = (\alpha, \beta)$. We can extend $\tilde{g}$ to a neighbourhood of $(\tilde{\alpha}, \tilde{\beta})$, or as we are interested only in the tangent space at the particular point (α, β), we can extend $\tilde{g}$ on an infinitesimal neighbourhood of $(\tilde{\alpha}, \tilde{\beta})$ and obtain gauge transformations of the form

$$(1 + s\tilde{\rho})\tilde{g} \tag{4.2}$$

where $\tilde{\rho}$ is an infinitesimal gauge transformation and s the infinitesimal tangent directions parameter. By considering the action of (4.2) on $\tilde{\alpha} + s\tilde{a}$, $\tilde{\beta} + s\tilde{b}$ we see that $\tilde{a}$, $\tilde{b}$ change as follows

$$\tilde{a} \longmapsto \tilde{g}\tilde{a}\tilde{g}^{-1} - \tfrac{1}{2}\dot{\tilde{\rho}} - [\alpha, \tilde{\rho}] \quad , \quad \tilde{b} \longmapsto \tilde{g}\tilde{b}\tilde{g}^{-1} - [\beta, \tilde{\rho}] \tag{4.3}$$

$\tilde{\rho}$ must be such that the resulting a, b are in the tangent space to $\mathcal{N}_k$. From the construction of the tangent space (cf. Section 1) we know that (a, b) must be orthogonal to the infinitesimal gauge directions, i.e. if ρ is any infinitesimal gauge transformation with $\rho(\pm 1) = 0$, then

$$\int_{-1}^{1} \operatorname{tr}\left(\left(\tfrac{1}{2}\dot{\rho} + [\alpha, \rho]\right)^* a \ + \ [\beta, \rho]^* b\right) \ = \ 0$$

Integrating by parts gives

$$\int_{-1}^{1} \operatorname{tr}\left(\dot{a} - 2[\alpha^*, a] - 2[\beta^*, b]\right)\rho^* = 0$$

Since this is true for any ρ, we have

$$\dot{a} = 2[\alpha^*, a] + 2[\beta^*, b] \tag{4.4}$$

Observe that if we add this equation to its adjoint, we get the linearization of the real Nahm's equation. (a, b) must also satisfy the linearization of the complex Nahm's equation:

$$\dot{b} = 2[\beta, a] + 2[b, \alpha] \tag{4.5}$$

Finally notice that as all elements of $\tilde{N}_k$ have the correct $\mathcal{N}_k$-type poles and any gauge transformation that preserves them is $= 1$ at ± 1 (up to an irrelevant central matrix) we must have

$$\tilde{\rho}(\pm 1) = 0 \tag{4.6}$$

Equations (4.4),(4.5) and (4.6) determine $\tilde{\rho}$. For notational purposes let us write

$$\hat{a} = \tilde{g}\tilde{a}\tilde{g}^{-1} \quad , \quad \hat{b} = \tilde{g}\tilde{b}\tilde{g}^{-1}$$

our aim is to show that there is a constant C independent of (α_d, β_d) such that

$$\left| \int_{-1}^{1} \operatorname{tr}\left(a^* a + b^* b\right) \; - \; \int_{-1}^{1} \operatorname{tr}\left(\tilde{a}^* \tilde{a} + \tilde{b}^* \tilde{b}\right) \right| \leq \frac{C}{r} \quad .$$

We want estimates on $\tilde{\rho}$. First of all we have

Proposition 4.1 *For* $t \in [-1, 1]$,

$$\|\tilde{\rho}(t)\| \leq \tfrac{K}{r}$$

where K depends only on charge k.

Proof: We want to show

$$\tfrac{d^2}{dt^2} \|\tilde{\rho}\| \geq - \|h(t)\| \tag{4.7}$$

where

$$h(t) = \dot{\hat{a}} - 2[\alpha^*, \hat{a}] - 2[\beta^*, \hat{b}]$$

Let u be a unitary gauge transformation that makes α hermitian. Then a becomes uau^*, $b \mapsto ubu^*$, $\tilde{\rho} \mapsto u\tilde{\rho}u^*$, $h(t) \mapsto uh(t)u^*$. In particular $\|\tilde{\rho}\|$, $\|h(t)\|$ do not change.

Therefore we can assume that $\alpha = \alpha^*$. The equation (4.4) is satisfied and if we plug (4.3) into it, we get

$$\dot{\hat{a}} - \tfrac{1}{2}\ddot{\tilde{\rho}} - [\dot{\alpha}, \tilde{\rho}] - [\alpha, \dot{\tilde{\rho}}] = 2[\alpha^*, \hat{a}] - [\alpha^*, \dot{\tilde{\rho}}] - 2\,[\alpha^*, [\alpha, \tilde{\rho}]\,] + 2[\beta^*, \hat{b}] - 2\,[\beta^*, [\beta, \tilde{\rho}]\,] \quad (4.8)$$

As $\alpha = \alpha^*$ the terms involving $\dot{\tilde{\rho}}$ cancel and $\dot{\alpha} = [\beta^*, \beta]$, so we get

$$\ddot{\tilde{\rho}} = 4\,[\beta^*, [\beta, \tilde{\rho}]\,] + 4\,[\alpha^*, [\alpha, \tilde{\rho}]\,] - 2[\,[\beta^*, \beta], \tilde{\rho}] + h(t) \quad (4.9)$$

Now, if M is a matrix then the adjoint of the linear operator

$$[M,\ \ ]$$

is

$$[M^*,\ \ ]$$

Moreover the commutator of two such operators $[M,\ \ ], [N,\ \ ]$ is

$$[\,[M, N],\ \ ]$$

(Jacobi's identity).
Hence if we put $A = [\alpha,\ \ ]$, $B = [\beta,\ \ ]$ and we split B into the sum of a hermitian H and skew-hermitian S, we can write (4.9) as

$$\ddot{\tilde{\rho}} = (4\,H^*H + 4\,S^*S + 4\,A^*A)\,\tilde{\rho} + h(t)$$

or

$$\ddot{\tilde{\rho}} = D(t)\tilde{\rho} + h(t)$$

where $D \stackrel{\text{def}}{=} 4\,H^*H + 4\,S^*S + 4\,A^*A$ is positive definite. This implies (see (2.6)) that $(\,' = \frac{d}{dt})$:

$$\left(\|\tilde{\rho}\|^2\right)'' \geq 2\,\|\tilde{\rho}'\|^2 - 2\,\|h\|\,\|\tilde{\rho}\|$$

Now (4.7) follows from the fact that

$$\left(\|\tilde{\rho}\|^2\right)'' = 2\,\|\tilde{\rho}\|''\,\|\tilde{\rho}\| + 2\left(\|\tilde{\rho}\|'\right)^2 \leq 2\,\|\tilde{\rho}\|''\,\|\tilde{\rho}\| + 2\,\|\tilde{\rho}'\|^2$$

We get now from Lemma 1.7 and from (4.6) that

$$\|\tilde{\rho}(t)\| \leq \tfrac{1}{2}(1-t)\int_{-1}^{t}(1+\tau)\,\|h(\tau)\|\,d\tau + \tfrac{1}{2}(1+t)\int_{t}^{1}(1-\tau)\,\|h(\tau)\|\,d\tau \quad (4.10)$$

Therefore we need to estimate $\|h(t)\|$. Let us go back to the gauge in which α is lower-triangular. Observe that $\dot{\hat{a}} = \left[\dot{\tilde{g}}\tilde{g}^{-1}, \tilde{g}\hat{a}\tilde{g}^{-1}\right] + \tilde{g}\dot{a}\tilde{g}^{-1}$. The latter term can be estimated from Proposition 1.6 while the first one and all other terms in the formula for $h(t)$ consist only of commutators and so they do not involve products of two diagonal elements. Therefore Propositions 1.6, 2.1 and Corollary 2.13 together with $R_{ij} \geq r$ give us for $z \leq 1$, $z = 1 \pm t$:

$$z \, \|h(z)\| \leq K e^{-srz} \tag{4.11}$$

for some constants K, s depending only on k. From this and 4.10 we get

$$\|\tilde{\rho}(t)\| \leq \int_{-1}^{1} (1+\tau)(1-\tau) \, \|h(\tau)\| \, d\tau \leq \int_{-1}^{0} (1+\tau) 2 \, \|h(\tau)\| \, d\tau + \int_{0}^{1} 2(1-\tau) \, \|h(\tau)\| \, d\tau \leq \frac{K}{r}$$

□

We can improve the estimates as follows:

Proposition 4.2 *There are constants* C, K, *and* s *depending only on charge* k *such that* $i \neq j$ *and* $t \in \left[-1 + \frac{C}{r}, 1 - \frac{C}{r}\right]$

$$\| \tilde{\rho}(t) \| \leq \frac{K}{r} \left(e^{-sR_{ij}(1+t)} + e^{-sR_{ij}(1-t)} \right)$$

Proof: Let us look at the equation (4.8). If we take C very large we can assume (Proposition 2.1) that for $t \in \left[-1 + \frac{C}{r}, 1 - \frac{C}{r}\right]$

$$|(\alpha - \operatorname{Re} \alpha_d)_{ij}(t)| \; , \; |(\beta - \beta_d)_{ij}(t)| \leq r \, \frac{\epsilon r}{R_{ij}} \left(e^{-sR_{ij}(1+t)} + e^{-sR_{ij}(1-t)} \right) \tag{4.12}$$

It will follow then using Proposition 4.1 that the equation (4.8) gives the following equation for

$$x = (\tilde{\rho}_{ij})_{(i,j) \in I(Y)} \tag{4.13}$$

(recall the definition of $I(Y)$ from section 2.2)

$$\ddot{x} = D(t)x + E(t)\dot{x} + G(t)$$

where

$$\operatorname{Re}(Dw, w) \geq s^2 R^2 \|w\|^2 \quad , \quad \| E(t) \| \leq \epsilon R \quad , \quad \|G(t)\| \leq r \left(e^{-sR(1+t)} + e^{-sR(1-t)} \right)$$

Indeed, it follows from the simple observation that in each of the terms on the right-hand side of the equation (4.8) for $(\tilde{\rho}_{ij})_{(i,j) \in I(Y)}$ at least one factor must have coordinates belonging to $I(Y)$ and from (4.12) and Proposition 4.1. Now the result follows for $(i,j) \in I(Y)$ as in Lemma 2.11 from the simple observation that

$$\operatorname{Re}(Dx + E\dot{x}, x) + (\dot{x}, \dot{x}) = \left\|\dot{x} + \tfrac{1}{2}E^* x\right\|^2 + \operatorname{Re}(Dx - \tfrac{1}{4}EE^* x, x)$$

so that

$$\tfrac{d^2}{dt^2}\|x\|^2 \geq 2\operatorname{Re}(Dx - \tfrac{1}{4}EE^* x, x) + (G(t), x)$$

Then we can continue the proof of inductively as for Proposition 2.1.

□

We get the following corollary

Corollary 4.3 *There are constants C, K, and s depending only on charge k such that for $t \in \left[-1 + \frac{C}{r}, 1 - \frac{C}{r}\right]$*

$$\|[\alpha, \tilde{\rho}](t)\|\ ,\ \|[\beta, \tilde{\rho}](t)\| \leq K\left(e^{-sr(1+t)} + e^{-sr(1-t)}\right)$$

Proof: It follows from the last proposition that we can take C large enough so that for $t \in \left[-1 + \frac{C}{r}, 1 - \frac{C}{r}\right]$ we have both (4.12) and

$$|\tilde{\rho}_{ij}(t)| \leq \tfrac{K}{R_{ij}}\left(e^{-sR_{ij}(1+t)} + e^{-sR_{ij}(1-t)}\right) \tag{4.14}$$

Now we can just multiply these estimates noticing that as we are estimating the bracket of two quantities the terms that are products of two diagonal entries cancel.

□

We can also estimate the derivative of $\tilde{\rho}$.

Proposition 4.4 *There are constants C, K independent of (α_d, β_d), such that for all $t \in \left[-1 + \frac{C}{r}, 1 - \frac{C}{r}\right]$*

$$\left\|\dot{\tilde{\rho}}(t)\right\| \leq K$$

Proof: Let us look again at the equation (4.8). Treating $\tilde{\rho}$ as a known function we can write as an equation for $\dot{\tilde{\rho}}$:

$$\ddot{\tilde{\rho}} = [\alpha^* - \alpha\ ,\ \dot{\tilde{\rho}}] + f(t) \tag{4.15}$$

If we take C large enough for (4.12) and (4.14) to hold, we will get the following estimate for $t \in \left[-1+\frac{C}{r}, 1-\frac{C}{r}\right]$

$$\|f(t)\| \leq Kre^{-sr(1+t)} + Kre^{-sr(1-t)} \tag{4.16}$$

From (4.15) we get

$$\tfrac{d}{dt}(\dot{\tilde{\rho}}, \dot{\tilde{\rho}}) = 2\,\mathrm{Re}\,(\ddot{\tilde{\rho}}, \dot{\tilde{\rho}}) = 2\,\mathrm{Re}\,(f(t), \dot{\tilde{\rho}})$$

since $[\alpha^* - \alpha, \ ]$ is skew-hermitian. Hence

$$\left| \tfrac{d}{dt} \left\|\dot{\tilde{\rho}}\right\| \right| \leq 2\,\|f(t)\|$$

This and 4.16 imply that the result will follow if we can show that there is a constant M such that for any (α_d, β_d) and any (a, b) there is a point $t_o \in \left[-1+\frac{C}{r}, 1-\frac{C}{r}\right]$ such that

$$\left\|\dot{\tilde{\rho}}(t_o)\right\| \leq M$$

Let us look again at (4.8). If we change

$$\alpha(t),\, \beta(t) \mapsto \frac{1}{r}\alpha(t/r),\, \frac{1}{r}\alpha(t/r) \quad , \quad \hat{a}(t),\, \hat{b}(t) \mapsto \hat{a}(t/r),\, \hat{a}(t/r)$$

then (4.8) is satisfied on $[-r, r]$ by the function

$$\hat{\rho}(t) = r\tilde{\rho}(t/r)$$

But now our "dilated" α, β are *bounded* uniformly in (α_d, β_d) on $[-r+C, r-C]$ (from (4.12), so is $\hat{\rho}$ (from Proposition 4.1) and so is $h(t)$ (Proposition 1.6). Therefore, for $t \in [-r+C, r-C]$,

$$\left\|\ddot{\hat{\rho}}(t)\right\| \leq K_1 \left\|\dot{\hat{\rho}}(t)\right\| + K_1 \tag{4.17}$$

where K_1 depends only on k. We also know from Proposition 4.1 that

$$\|\hat{\rho}\| \leq K$$

where K depends only on k.

Suppose there is an interval $I \subset [-r+C, r-C]$ of length 1 at every point of which

$$\left\|\dot{\hat{\rho}}(t)\right\| \geq 2K\,K_1$$

Then (4.17) implies that on I

$$\left\|\ddot{\hat{\rho}}(t)\right\| \leq \tfrac{1}{2K} \left\|\dot{\hat{\rho}}(t)\right\|^2 + K_1$$

Then it follows from [Hartman, Remark 1 following Corollary XII 5.1] that there is a constant M depending only on K and K_1 such that *for all* $t \in I$,

$$\left\|\dot{\hat{\rho}}(t)\right\| \leq M$$

Thus we get a point at which

$$\left\|\dot{\hat{\rho}}(t_o)\right\| \leq \max\{M, 2KK_1\}$$

However

$$\dot{\rho}(t) = \dot{\hat{\rho}}(t/r)$$

so the result follows.

□

The last fact together with Corollary 4.3 give us

Corollary 4.5 *There are constants* C, K *independent of* (α_d, β_d), *such that for all* $t \in \left[-1+\frac{C}{r}, 1-\frac{C}{r}\right]$

$$\|a(t)\| \ , \ \|b(t)\| \leq K$$

□

We are finally able to prove the asymptotic behaviour of the metric tensor

Theorem 4.6 *The metric tensor on* $\mathcal{N}_k$ *differs asymptotically from the one on* $\mathcal{S}^k N_1$ *by* $\frac{const}{r}$, *i.e. there is a constant* K *independent of* (α_d, β_d) *such that for all* $(a_d, b_d) \in \mathbf{T}_{(\alpha_d,\beta_d)}\mathcal{S}^k N_1$ *and corresponding* $(a, b) \in \mathbf{T}_{(\alpha,\beta)}\mathcal{N}_k$

$$\left| \frac{1}{2}\int_{-1}^{1} \operatorname{tr}(aa^* + bb^*) \ - \sum_{i=1}^{k}\left(|a_i|^2 + |b_i|^2\right) \right| \ \leq \ \frac{K}{r}\sum_{i=1}^{k}\left(|a_i|^2 + |b_i|^2\right)$$

Proof: Of course we can assume

$$\sum_{i=1}^{k}\left(|a_i|^2 + |b_i|^2\right) = 1$$

Recall that

$$\hat{a} = \tilde{g}\tilde{a}\tilde{g}^{-1} \ , \ \hat{b} = \tilde{g}\tilde{b}\tilde{g}^{-1}$$

and therefore, from (4.3)

$$a = \hat{a} - \tfrac{1}{2}\dot{\tilde{\rho}} - [\alpha, \tilde{\rho}] \quad , \qquad b = \hat{b} - [\beta, \tilde{\rho}] \tag{4.18}$$

The estimates of Proposition 1.6 and Corollary 2.13 show that

$$\| (\hat{a} - a_d)(t) \| + \left\| (\hat{b} - b_d)(t) \right\| \leq K \left(e^{-sr(1+t)} + e^{-sr(1-t)} \right) \tag{4.19}$$

for all $t \in [-1, 1]$ and for some K, s depending only on the charge k.
Now observe that for any t_1, $t_2 \in [-1, 1]$,

$$\begin{aligned} \int_{t_1}^{t_2} \mathrm{tr}\,(\hat{a}\hat{a}^* + \hat{b}\hat{b}^*) \;&=\; \int_{t_1}^{t_2} \mathrm{tr}\,(aa^* + bb^*) \;+\; \tfrac{1}{2}\mathrm{tr}\,(a\tilde{\rho}^* + a^*\tilde{\rho})\Big|_{t_1}^{t_2} \\ &+ \int_{t_1}^{t_2} \mathrm{tr}\,\left(\left(\tfrac{1}{2}\dot{\tilde{\rho}} + [\alpha, \tilde{\rho}]\right) \left(\tfrac{1}{2}\dot{\tilde{\rho}} + [\alpha, \tilde{\rho}]\right)^* + [\beta, \tilde{\rho}]\,[\beta, \tilde{\rho}]^* \right) \end{aligned} \tag{4.20}$$

Indeed it follows by plugging (4.18) into the integral on the left-hand side, integrating by parts the terms $a\dot{\tilde{\rho}}^*$, $a^*\dot{\tilde{\rho}}$, and then grouping together products that involve $\tilde{\rho}$ only once and using (4.4). In particular (4.20) and (4.6) give

$$\int_{-1}^{t} \mathrm{tr}\,(aa^* + bb^*) \leq \int_{-1}^{t} \mathrm{tr}\,(\hat{a}\hat{a}^* + \hat{b}\hat{b}^*) \; - \mathrm{tr}\,\left(a(t)\tilde{\rho}^*(t) + a(t)^*\tilde{\rho}(t)\right) \tag{4.21}$$

$$\int_{t}^{1} \mathrm{tr}\,(aa^* + bb^*) \leq \int_{t}^{1} \mathrm{tr}\,(\hat{a}\hat{a}^* + \hat{b}\hat{b}^*) \; + \mathrm{tr}\,\left(a(t)\tilde{\rho}^*(t) + a(t)^*\tilde{\rho}(t)\right) \tag{4.22}$$

This and (4.19) show that there is a constant K depending only on k such that for any C

$$\int_{-1}^{-1+\frac{C}{r}} \mathrm{tr}\,(aa^* + bb^*) \;+\; \int_{1-\frac{C}{r}}^{1} \mathrm{tr}\,(aa^* + bb^*) \;\leq\; \frac{K}{r} + \frac{2C}{r} \tag{4.23}$$

Since the integral of $\sum \left(|a_i|^2 + |b_i|^2\right)$ over the same set is $\leq \frac{2C}{r}$, we only have to estimate the metric on $\left[-1 + \frac{C}{r}, 1 - \frac{C}{r}\right]$ for a suitable C. Note that if C is large enough for the conclusion of Corollary 4.5 to hold, then (4.20), (4.19) and Proposition 4.1 imply that all we have to estimate is

$$\int_{-1+\frac{C}{r}}^{1-\frac{C}{r}} \mathrm{tr}\,\left(\left(\tfrac{1}{2}\dot{\tilde{\rho}} + [\alpha, \tilde{\rho}]\right) \left(\tfrac{1}{2}\dot{\tilde{\rho}} + [\alpha, \tilde{\rho}]\right)^* + [\beta, \tilde{\rho}]\,[\beta, \tilde{\rho}]^* \right)$$

However, if C is large enough for the conclusion of Corollary 4.3 (as well as Proposition 4.4) to hold, then

$$\left| \int_{-1+\frac{C}{r}}^{1-\frac{C}{r}} \mathrm{tr}\,\left(\left(\tfrac{1}{2}\dot{\tilde{\rho}} + [\alpha, \tilde{\rho}]\right) \left(\tfrac{1}{2}\dot{\tilde{\rho}} + [\alpha, \tilde{\rho}]\right)^* + [\beta, \tilde{\rho}]\,[\beta, \tilde{\rho}]^* - \tfrac{1}{4}\dot{\tilde{\rho}}\dot{\tilde{\rho}}^* \right) \right| \leq \frac{K}{r}$$

for some K depending only on k. Therefore all we have to estimate is

$$\int_{-1+\frac{C}{r}}^{1-\frac{C}{r}} \operatorname{tr} \dot{\tilde{\rho}} \dot{\tilde{\rho}}^*$$

Integrating by parts and using the fact that, due to our choice of C, $\left\| \tilde{\rho}\left(-1+\frac{C}{r}\right) \right\|$, $\left\| \tilde{\rho}\left(1-\frac{C}{r}\right) \right\| \leq \frac{K}{r}$, $\left\| \dot{\tilde{\rho}}\left(-1+\frac{C}{r}\right) \right\|, \left\| \dot{\tilde{\rho}}\left(1-\frac{C}{r}\right) \right\| \leq K$ leaves us with the integral

$$\int_{-1+\frac{C}{r}}^{1-\frac{C}{r}} \operatorname{tr} \ddot{\tilde{\rho}} \tilde{\rho}^*$$

Now recall the equation (4.15) for $\ddot{\tilde{\rho}}$ together with the estimate (4.16). This, Proposition 4.4 and the fact that we can take C large enough for (4.12) to hold, implies that

$$\left\| \ddot{\tilde{\rho}}(t) \right\| \leq Kr \left(e^{-sr(1+t)} + e^{-sr(1-t)} \right)$$

for all $t \in \left[-1+\frac{C}{r}, 1-\frac{C}{r}\right]$ and for some K, s depending only on the charge k. This and Proposition 4.1 imply finally that

$$\left| \int_{-1+\frac{C}{r}}^{1-\frac{C}{r}} \operatorname{tr} \ddot{\tilde{\rho}} \tilde{\rho}^* \right| \leq \frac{K}{r}$$

which proves the result.

□

This is a good place to include pointwise estimates for (a, b) which we will use when estimating the curvature.

Proposition 4.7 *There are constants C, K independent of (α_d, β_d), such that for all $t \in \left[-1+\frac{C}{r}, 1-\frac{C}{r}\right]$ and $i \neq j$*

$$\|a_{ij}(t)\|\ ,\ \|b_{ij}(t)\| \leq Ke^{-sr(1+t)} + Ke^{-sr(1-t)}$$

Proof: Let us put

$$x = (a_{ij}, b_{ij})_{i \neq j}$$

If we differentiate equations (4.4) and (4.5), we get an equation for x of the form

$$\ddot{x} = D(t)x + g(t)$$

If we take now C large enough so that for $t \in \left[-1+\frac{C}{r}, 1-\frac{C}{r}\right]$

$$|(\alpha - \operatorname{Re}\alpha_d)_{ij}(t)| \ , \ |(\beta - \beta_d)_{ij}(t)| \leq \epsilon\, r\left(\tfrac{r}{R_{ij}}\right)^2 \left(e^{-sR_{ij}(1+t)} + e^{-sR_{ij}(1-t)}\right) \tag{4.24}$$

for $i \neq j$ and

$$|(\alpha_{ii}(t) - \operatorname{Re}\alpha_i)| \leq \epsilon\, r \tag{4.25}$$

for $i = 1, \dots, k$, then, using Corollary 4.5, we can assume that for $t \in \left[-1+\frac{C}{r}, 1-\frac{C}{r}\right]$

$$\operatorname{Re}(D(t)x, x) \geq s^2r^2\, \|x\|^2 \quad , \quad \|\, g(t)\,\| \leq Kr^2e^{-sr(1+t)} + Kr^2e^{-sr(1-t)}$$

and the result follows as in Lemma 2.11.

□

2.5 The curvature

In this section we will show that the curvature tensor on $\mathcal{N}_k$ decays asymptotically as $\frac{const}{r^3}$. More precisely we have

Theorem 5.1 *There is a constant K depending only on charge k such that if X_d, Y_d, Z_d, W_d are vector fields of norm 1, defined in a neighbourhood of (α_d, β_d) and X, Y, Z, W are corresponding vector fields on $\mathcal{N}_k$, then*

$$|(R(X,Y)Z\,,\,W)| \leq \frac{K}{r^3}$$

where R denotes the curvature tensor.

We will prove this by comparing the curvature on $\mathcal{N}_k$ with the zero curvature of a certain flat (infinite-dimensional) manifold. Let us recall that the $\mathcal{N}_k$ arises as a quotient manifold of the manifold $\mathcal{N}$ of *all* solutions to Nahm's equations. On the other hand $\mathcal{N}$ is embedded in the flat manifold $\mathcal{A}$ of all paths $\alpha, \beta : [-1, 1] \to gl(k)$ with $\mathcal{N}_k$-type poles at ± 1.
The maps

$$\mathcal{N} \xrightarrow{\pi} \mathcal{N}_k$$

$$\mathcal{N} \overset{i}{\hookrightarrow} \mathcal{A}$$

are a Riemannian submersion and immersion, respectively.

First, let us compare the curvature tensor on $\mathcal{N}_k$ with the one on $\mathcal{N}$. Since π is a Riemannian submersion, the tangent space of $\mathcal{N}$ at (α, β) splits into the horizontal and vertical parts, the horizontal one being the lift of the tangent space of $\mathcal{N}_k$. The proof of the following fact can be found in Klingenberg [1982] (Theorem 1.11.12).

Theorem 5.2 (O'Neill) *If X, Y, Z, W are vector fields on $\mathcal{N}_k$, $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}$ their horizontal lifts to $\mathcal{N}$, $R, \tilde{R}$ the curvature tensors on $\mathcal{N}_k$, $\mathcal{N}$ respectively, then*

$$\begin{aligned}(R(X,Y)Z\,,\,W) \;=\; & \left(R(\tilde{X},\tilde{Y})\tilde{Z}\,,\,\tilde{W}\right) - \tfrac{1}{4}\left([\tilde{X},\tilde{Z}]^v\,,\,[\tilde{Y},\tilde{W}]^v\right) \\ & +\tfrac{1}{4}\left([\tilde{Y},\tilde{Z}]^v\,,\,[\tilde{X},\tilde{W}]^v\right) - \tfrac{1}{2}\left([\tilde{Z},\tilde{W}]^v\,,\,[\tilde{X},\tilde{Y}]^v\right)\end{aligned}$$

where ^{v} denotes the vertical part of a vector field.

□

Therefore we want to show that the norm of the vertical component of the bracket of two horizontal vector fields on $\mathcal{N}$ is $\leq \frac{const}{r^{3/2}}$. More precisely we will show the folloowing

Proposition 5.3 *There is a constant K depending only on charge k such that if X_d, Y_d are vector fields defined in a neighbourhood of (α_d, β_d) with*

$$\| X_d\,(\alpha_d, \beta_d)\,\| = \|\, Y_d\,(\alpha_d, \beta_d)\,\| = 1$$

and X, Y are corresponding vector fields on $\mathcal{N}_k$, then

$$\| \,[X, Y]^v(\alpha, \beta)\,\| \leq \frac{K}{r^{3/2}}$$

Before proving this we will say few things about vector fields on $\mathcal{N}$. Since the value of the vertical component of the bracket of horizontal vector fields at a given point depends only on the values of the vector fields at that point, we can consider infinitesimal version of vector fields.
To have such an infinitesimal version of a vector field X (at (α, β) we must specify the value of X at (α, β), i.e. a tangent vector (a, b), and the differential of X in the tangent directions, i.e. a function

$$\mathbf{T}_{(\alpha,\beta)}\mathcal{N} \ni (u, v) \longmapsto (A, B) \in \mathbf{T}_{(\alpha,\beta,u,v,)}\,(\mathbf{T}\mathcal{N})$$

We can think of X as assigning to a point (s denotes an infinitesimal parameter)

$$\alpha + su\,,\ \beta + sv$$

of an infinitesimal neighbourhood of (α, β) the tangent vector

$$(a + sA, b + sB) \in \mathbf{T}_{(\alpha+su,\beta+sv)}\mathcal{N}$$

The fact that $(a + sA, b + sB)$ is a tangent vector imposes certain conditions. In fact, as we are interested only in horizontal vector fields, we require $(a + sA, b + sB)$ to satisfy equations (4.4) and (4.5) (with α, β replaced by $\alpha + su\,,\ \beta + sv$). This leads to the following equations for $A = A(u, v)$, $B = B(u, v)$:

$$\begin{aligned} \dot{A} &= 2\,[\alpha^*, A] + 2\,[\beta^*, B] + 2\,[u^*, a] + 2\,[v^*, b] \\ \dot{B} &= 2\,[\beta, A] + 2\,[B, \alpha] + 2\,[v, a] + 2\,[b, u] \end{aligned} \tag{5.1}$$

Suppose we have a second horizontal vector field Y given by

$$(\alpha + su, \beta + sv) \longmapsto (a' + sA', b' + sB')$$

Then $X \circ Y$ at (α, β) is given by the differential of X in the direction $Y(\alpha, \beta) = (a', b')$, i.e.

$$X \circ Y\,(\alpha, \beta) = (\,A(a', b'), B(a', b')\,)$$

Therefore we get from (5.1) that the components $A = A(a', b')$, $B = B(a', b')$ of $X \circ Y\,(\alpha, \beta)$ satisfy

$$\begin{aligned} \dot{A} &= 2\,[\alpha^*, A] + 2\,[\beta^*, B] + 2\,[a'^*, a] + 2\,[b'^*, b] \\ \dot{B} &= 2\,[\beta, A] + 2\,[B, \alpha] + 2\,[b', a] + 2\,[b, a'] \end{aligned} \tag{5.2}$$

Hence

$$(P, Q) := [X, Y](\alpha, \beta) = (X \circ Y - Y \circ X)(\alpha, \beta)$$

satisfies the equations

$$\begin{aligned} \dot{P} &= 2\,[\alpha^*, P] + 2\,[\beta^*, Q] + 2\,[a'^*, a] - 2\,[a^*, a'] + 2\,[b'^*, b] - 2\,[b^*, b'] \\ \dot{Q} &= 2\,[\beta, P] + 2\,[Q, \alpha] \end{aligned} \tag{5.3}$$

Note that if we add the first equation to its conjugate, we get that P, Q satisfy the linearized Nahm's equations, i.e. $(P, Q) \in \mathbf{T}_{(\alpha,\beta)}\mathcal{N}$, as it should.
We want to estimate the vertical part of (P, Q). Notice that as $(P, Q)^v$ is a vertical tangent vector on $\mathcal{N}$ it must be of the form

$$(P, Q)^v = \left(\tfrac{1}{2}\dot{\rho} + [\alpha, \rho]\,,\, [\beta, \rho]\right) \tag{5.4}$$

where ρ is an infinitesimal gauge transformation; in particular $\rho(\pm 1) = 0$.
We want to get estimates for ρ similar to the ones of section 2.4.

Lemma 5.4 *There are constants C, K and s depending only on the charge k such that if X, Y correspond to X_d, Y_d - vector fields defined in a neighbourhood of (α_d, β_d) with $\|\, X_d\,(\alpha_d, \beta_d)\,\| = \|\, Y_d\,(\alpha_d, \beta_d)\,\| = 1$, then ρ given by 5.4 satisfies:*

$$\begin{array}{ll} \|\,\rho(t)\,\| \le \frac{K}{r^2} & \textit{for all } t \in [-1, 1] \\ \|\,\rho_{ij}(t)\,\| \le \frac{K}{r^2}\left(e^{-sr(1+t)} + e^{-sr(1-t)}\right) & \textit{for } i \neq j \textit{ and } t \in \left[-1 + \frac{C}{r}, 1 - \frac{C}{r}\right] \\ \|\,[\alpha, \rho](t)\,\|\,,\ \|\,[\beta, \rho](t)\,\| \le \frac{K}{r}\left(e^{-sr(1+t)} + e^{-sr(1-t)}\right) & \textit{for } t \in \left[-1 + \frac{C}{r}, 1 - \frac{C}{r}\right] \\ \|\,\dot{\rho}(t)\,\| \le \frac{K}{r} & \textit{for } t \in \left[-1 + \frac{C}{r}, 1 - \frac{C}{r}\right] \end{array}$$

Proof: First of all, since the horizontal part of $(P,Q) = [X,Y](\alpha,\beta)$ satisfies the homogeneous version of equations (5.3), the vertical part must satisfy the full equations (5.3). If we plug (5.4) in the first of the equations (5.3), we will get a second order equation for ρ which is exactly the same as equation (4.8) except that $h(t)$ is given now by

$$h(t) = 2\,[a'^*, a] - 2\,[a^*, a'] + 2\,[b'^*, b] - 2\,[b^*, b'] \tag{5.5}$$

It follows, the same way as in the proof of Proposition 4.1 that

$$\|\rho(t)\| \le \int_{-1}^{1} (1+\tau)(1-\tau)\,\|h(\tau)\|\,d\tau \quad .$$

Let us estimate this expression. First of all from the Schwartz inequality we have for any C

$$\int_{-1}^{-1+\frac{C}{r}} (1+\tau)(1-\tau)\,\|h(\tau)\|\,d\tau \le \left|\int_{-1}^{-1+\frac{C}{r}} 4(1+\tau)^2\,d\tau\right|^{1/2} \cdot \left|\int_{-1}^{-1+\frac{C}{r}} \|h(\tau)\|^2\,d\tau\right|^{1/2}$$

The $\mathcal{L}^2$-norm of $h(t)$ is majorized by the product of $\mathcal{L}^2$-norms of (a,b) and (a',b'). Since these correspond to elements of $\mathbf{T}_{\alpha_d,\beta_d}(\mathcal{C}\times\mathcal{C})^k$ of norm 1, they satisfy the estimates of section 2.4. It follows from (4.23) that the above expression is $\le \frac{K}{r^2}$ where K depends only on C and k.
Similar result holds for the integral over $\left[1-\frac{C}{r},1\right]$. Therefore to prove the first of our assertions we have to estimate

$$\int_{-1+\frac{C}{r}}^{1-\frac{C}{r}} (1+\tau)(1-\tau)\,\|h(\tau)\|\,d\tau$$

Now notice that as $h(t)$ is defined by brackets, all the terms have as a factor an off-diagonal entry of a, b, a' or b'. Proposition 4.7 and Corollary 4.5) imply then that for $t \in \left[-1+\frac{C}{r}, 1-\frac{C}{r}\right]$

$$\|\, h(t)\,\| \le K\left(e^{-sr(1+t)} + e^{-sr(1-t)}\right) \tag{5.6}$$

it follows now easily that the above integral is bounded by $\frac{K}{r^2}$.
We proved our first statement.
The proofs of the remaining three are completely analogous to the proofs of Proposition 4.2, Corollary 4.3 and Proposition 4.4.

□

Now we can prove Proposition 5.3.

Proof of Proposition 5.3: We have to estimate

$$\left\| \left(\hat{P}, \hat{Q}\right) \right\|$$

where

$$\left(\hat{P}, \hat{Q}\right) \stackrel{\text{def}}{=} \left(\tfrac{1}{2}\dot{\rho} + [\alpha, \rho], [\beta, \rho]\right)$$

We will show that for any $t_1, t_2 \in (-1, 1)$

$$2\int_{t_1}^{t_2} \operatorname{tr}\left(\hat{P}\hat{P}^* + \hat{Q}\hat{Q}^*\right) = \tfrac{1}{2}\operatorname{tr}\left(\hat{P}\tilde{\rho}^* + \hat{P}^*\tilde{\rho}\right)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \operatorname{tr}\left(h(\tau)\rho^* + h(\tau)^*\rho\right) \tag{5.7}$$

where $h(t)$ is given by (5.5). To see this, integrate

$$\operatorname{tr}\left(\left(\hat{P} - \tfrac{1}{2}\dot{\rho} - [\alpha, \rho]\right)\left(\hat{P} - \tfrac{1}{2}\dot{\rho} - [\alpha, \rho]\right)^* + \left(\hat{Q} - [\beta, \rho]\right)\left(\hat{Q} - [\beta, \rho]\right)^*\right)$$

which is obviously $\equiv 0$. On the other hand it can be written as

$$2\operatorname{tr}\left(\hat{P}\hat{P}^* + \hat{Q}\hat{Q}^*\right) - \operatorname{tr}\left(\tfrac{1}{2}\hat{P}\dot{\rho}^* + \hat{P}\,[\alpha, \rho]^* + \hat{Q}\,[\beta, \rho]^*\right) - \operatorname{tr}\left(\tfrac{1}{2}\hat{P}\dot{\rho}^* + \hat{P}\,[\alpha, \rho]^* + \hat{Q}\,[\beta, \rho]^*\right)^*$$

Now integrating by parts $\hat{P}\dot{\rho}^*$ and using the first of equations (5.3) proves (5.7).
Let us integrate now

$$\operatorname{tr}\left(\hat{P}\hat{P}^* + \hat{Q}\hat{Q}^*\right)$$

over $[-1, 1]$. Using (5.7) we can integrate this separately over $\left[-1, -1 + \frac{C}{r}\right]$, $\left[-1 + \frac{C}{r}, 1 - \frac{C}{r}\right]$, $\left[1 - \frac{C}{r}, 1\right]$. Then the estimates of Lemma 5.4, (5.6) and the fact that $\rho(\pm 1) = 0$ show:

$$\int_{-1}^{1} \operatorname{tr}\left(\hat{P}\hat{P}^* + \hat{Q}\hat{Q}^*\right) \leq \frac{K}{r^3}$$

Proposition 5.3 is now proved.

□

We have showed now that the curvature tensor on $\mathcal{N}_k$ differs from the one on N by $\frac{const}{r^3}$. We have to show similar result for the immersion

$$\mathcal{N} \stackrel{i}{\hookrightarrow} \mathcal{A}$$

the curvature of N is related to the curvature of A by the Gauss equation (Kobayashi and Nomizu [1969], Proposition VII.4.1). We recall that for a Riemannian immersion $\mathcal{N} \stackrel{i}{\hookrightarrow} \mathcal{A}$ with connection ∇ on $\mathcal{A}$ the second fundamental form $h(X, Y)$ of two vector fields on $\mathcal{N}$ is the part of $\nabla_X Y$ orthogonal to $\mathbf{T}\mathcal{N}$. The equation of Gauss can be then stated as follows;

Theorem 5.5 (Gauss) *If X, Y, Z, W are vector fields on $\mathcal{N}$, $R, \tilde{R}$ the curvature tensors on $\mathcal{N}$, $\mathcal{A}$ respectively, then*

$$\begin{aligned}(R(X,Y)Z\,,\,W) \;=\; & \left(R(\tilde{X},\tilde{Y})\tilde{Z}\,,\,\tilde{W}\right) - \\ & (\,h(X,Z), h(Y,W)\,) + (\,h(Y,Z), h(X,W)\,)\end{aligned}$$

□

We have to show therefore that the second fundamental form h satisfies

$$\| \, h(X,Y) \, \| \leq \frac{K}{r^{3/2}} \tag{5.8}$$

for some constant K depending only on charge k and for any tangential horizontal vector fields X, Y on N that correspond to vector fields X_d, Y_d defined in a neighbourhood of (α_d, β_d) with $\| X_d\,(\alpha_d, \beta_d) \| = \| Y_d\,(\alpha_d, \beta_d) \| = 1$.
Since A is flat

$$\nabla_X Y = X \circ Y$$

Hence we have to estimate

$$(\, X \circ Y \,)^{\perp}$$

For horizontal vector fields X, Y, $X \circ Y\,(\alpha, \beta) = (A, B)$ satisfies equations (5.2). Let us write these as

$$\dot{A} = 2\,[\alpha^*, A] + 2\,[\beta^*, B] + p(t)$$

$$\dot{B} = 2\,[\beta, A] + 2\,[B, \alpha] + q(t)$$

(A, B) can be written as

$$(A, B) = (A_1, B_1) + (A_2, B_2)$$

in such a way that (A_1, B_1) satisfies:

$$\begin{aligned}\dot{A}_1 &= 2\,[\alpha^*, A_1] + 2\,[\beta^*, B_1] + p(t)\\ \dot{B}_1 &= 2\,[\beta, A_1] + 2\,[B_1, \alpha]\end{aligned} \tag{5.9}$$

while (A_2, B_2) satisfies:

$$\begin{aligned}\dot{A}_2 &= 2\,[\alpha^*, A_2] + 2\,[\beta^*, B_2]\\ \dot{B}_2 &= 2\,[\beta, A_2] + 2\,[B_2, \alpha] + q(t)\end{aligned} \tag{5.10}$$

This means that (A_1, B_1) and $(-B_2^*, A_2^*)$ both satisfy the linearised complex Nahm's equation (4.5) (in the hyperkähler setting this corresponds to writing a vector $v \in \mathbf{T}\mathcal{A}$ as $v_1 + jv_2$ with v_1, v_2 both are in the 0-set of a complex moment map).

However a vector satisfying the linearised complex Nahm's equation can be made to satisfy the equation (4.4) by a *complex* infinitesimal gauge transformation. Indeed such a vector can be thought of as being tangent to the space of Nahm complexes in the sense of Definition 1.3.1. It is the essence of Donaldson's construction that in every complex orbit of a Nahm complex there is (unique) *real* Nahm complex.
Thus we have two complex infinitesimal gauge transformations ρ_1, ρ_2 (with $\rho_1(\pm 1) = \rho_2(\pm 1) = 0$) such that

$$(A_1, B_1) - \left(\tfrac{1}{2}\dot{\rho}_1 + [\alpha, \rho_1],\, [\beta, \rho_1]\right)$$

and

$$(A_2, B_2) - \left(-[\beta, \rho_2]^*,\, \tfrac{1}{2}\dot{\rho}_2^* + [\alpha, \rho_2]^*\right)$$

are horizontal vector fields.
Let us put

$$\left(\hat{A}_1, \hat{B}_1\right) = \left(\tfrac{1}{2}\dot{\rho}_1 + [\alpha, \rho_1],\, [\beta, \rho_1]\right)$$
$$\left(\hat{A}_2, \hat{B}_2\right) = \left(\tfrac{1}{2}\dot{\rho}_2 + [\alpha, \rho_2],\, [\beta, \rho_2]\right)$$

We have

$$\left\|(X \circ Y)^{\perp}\right\| \le \left\| \left(\hat{A}_1, \hat{B}_1\right) \right\| + \left\| \left(\hat{A}_2, \hat{B}_2\right) \right\|$$

Moreover, both $\left(\hat{A}_1, \hat{B}_1\right)$ and $\left(\hat{A}_2, \hat{B}_2\right)$ satisfy equations (5.9) which are essentialy the same as (5.3) with $p(t)$ satisfying the same estimates as $h(t)$ given by (5.5). Therefore the proof of Lemma 5.4 and Proposition 5.3 will carry for $\left(\hat{A}_1, \hat{B}_1\right)$ and $\left(\hat{A}_2, \hat{B}_2\right)$ without any changes. Therefore

$$\left\| (A, B)^{\perp} \right\| \le \| (A_1, B_1) \| + \| (A_2, B_2) \| \le \frac{K}{r^{3/2}}$$

which proves (5.8).
Theorem 5.1 is now proved.

□

III. $SU(N)$ case

3.1 Nahm complexes for $SU(N)$ monopoles with maximal symmetry breaking

The purpose of this section is to give a brief overview of the results of Hurtubise and Murray[1989] and Hurtubise[1989].
Let us consider the moduli space of based $SU(N)$ monopoles with eigenvalues of the Higgs field at infinity being $\mu_1 < \ldots < \mu_N$ and magnetic charges $m_1, \ldots, m_{N-1}$. As we mentioned in section 1.2, such monopoles correspond to solutions T_o, T_1, T_2, T_3 to Nahm's equations on the interval (μ_1, μ_N) with $\operatorname{rank} T_i = m_j$ on the interval (μ_j, μ_{j+1}). We describe now the the boundary behaviour in this case after Hurtubise and Murray [1989].
As in the $SU(2)$case we define α and β by (1.4) so that the Nahm's equations become the "real" equation (1.6) and the complex one (1.5).
The boundary behaviour at μ_j depends on the value of the jump $m_j - m_{j-1}$. We say that μ_j is *superior* (respectively *inferior, neutral*) boundary point of the interval (μ_j, μ_{j+1}), if $m_j > m_{j-1}$ (respectively $<,=$).
We also say that μ_{j+1} is *superior* (respectively *inferior, neutral*) boundary point of the interval (μ_j, μ_{j+1}), if $m_j > m_{j+1}$ (respectively $<,=$).
At a boundary point μ_j, we denote by

- k or k_j the absolute value $|m_j - m_{j-1}|$ of the jump
- $\overline{m}$ or $\overline{m}_j$ the maximum of $\{m_j, m_{j-1}\}$
- $\underline{m}$ or $\underline{m}_j$ the minimum $\{m_j, m_{j-1}\}$
- α_j, β_j the restriction of α, β to $[\mu_j, \mu_{j+1}]$

Remark: The points μ_1, μ_N are superior with $k_1 = m_1$, $k_{N-1} = m_{N-1}$.

We can use a part of gauge freedom to fix the boundary conditions as follows: at a boundary point μ, setting $z = t - \mu$, one has:

- if μ is inferior or neutral, α_j,β_j are analytic at $z = 0$
- if μ is superior, splitting $\mathbf{C}^{m_j}$ as $\mathbf{C}^{\underline{m}} \oplus \mathbf{C}^k$, one has, near $z = 0$:

$$\alpha_j = \begin{pmatrix} Y & z^{(k-1)/2}V \\ z^{(k-1)/2}W & X \end{pmatrix}, \quad \beta_j = \begin{pmatrix} P & z^{(k-1)/2}Q \\ z^{(k-1)/2}R & S \end{pmatrix} \tag{1.1}$$

with: i) Y, V, W and P, Q, R analytic at $z = 0$,

ii) X, S are meromorphic, with simple poles at $z = 0$, and residues (1.7)

Furthermore, α_j, β_j satisfy patching conditions at μ_j:

If μ_j is not neutral, then the limits of α, β from the inferior side are equal to $Y(0), P(0)$ of (1.1)

If μ_j is neutral, one has limits $\alpha_\pm, \beta_\pm$ from both sides of μ_j; there then exist column vectors U, W in $\mathbf{C}^{m_j}$ with

$$\beta_+ - \beta_- = -\tfrac{1}{2} U W^T \tag{1.2}$$

$$\left(\alpha_+ + \alpha_+^*\right) - \left(\alpha_- + \alpha_-^*\right) = \tfrac{1}{2}\left(-U\bar{U}^T + \bar{W}W^T\right) \tag{1.3}$$

The above conditions fully describe Nahm's construction for $SU(N)$ monopoles with maximal symmetry breaking. As in Definition (1.3.1) we define a *real Nahm complex* to be

$$(\,\alpha, \beta, U, W\,) = (\,\alpha, \beta,\ \{U(\mu), W(\mu);\ \mu \text{ a neutral boundary point}\,\}\,)$$

Since we fixed some of the boundary conditions, the space of such Nahm complexes will have to be acted on by a smaller gauge group. Let us define three gauge groups:

- the group $\mathcal{G}$ of complex gauge transformations g which, at a superior boundary point preserve the decomposition $\mathbf{C}^{\overline{m}} = \mathbf{C}^{\underline{m}} \oplus \mathbf{C}^k$ with the upper-diagonal block being equal to the limit of g from the inferior side and the off-diagonal blocks have derivatives $O\left(z^{(k-1)/2}\right)$;
- the subgroup $\mathcal{G}_{\mathbf{C}} \subset \mathcal{G}$ of those that at a superior boundary point have the lower-diagonal block equal to identity;
- $\mathcal{G}_{\mathbf{R}} \subset \mathcal{G}_{\mathbf{C}}$ of unitary gauge transformations.

Note that a $g \in \mathcal{G}$ will act on U, W by

$$U \to gU \ , \quad W \to \left(g^T\right)^{-1} W$$

We put

$$\mathcal{N}_{m_1,\ldots,m_{N-1}} = \left\{ \text{ real Nahm complexes } \right\} \Big/ \mathcal{G}_{\mathbf{R}}$$

We have then (Hurtubise & Murray [1989])

$$\mathcal{M}_{m_1,\ldots,m_{N-1}} \overset{\text{diff}}{\Longleftrightarrow} \mathcal{N}_{m_1,\ldots,m_{N-1}}$$

As for $SU(2)$ monopoles, there is a correspondence between the moduli space of $SU(N)$ monopoles and based rational maps

$$F : \mathbf{C}P^1 \longmapsto SU(N)\big/_T$$

where T is a maximal torus. This was proved by Hurtubise[1989]. From there we also take a description of such rational maps suited to our purposes.
Let E be the trivial rank N bundle over $\mathbf{C}P^1$, with a fixed global basis $\{e_1, \ldots, e_N\}$. One can define a standard flag of subbundles:

$$E_0^+ = \{0\}\,, \quad E_1^+ = (\,e_1\,)\,, \quad E_2^+ = (\,e_1, e_2\,)\,, \quad \ldots\,, E_N^+ = E$$

Let $\bar{E}_i^+$ denote the "anti-standard" flag:

$$\bar{E}_i^+ = (\,e_N, e_{N-1}, \ldots, e_{N-i+1}\,)$$

A based rational map $\mathbf{C}P^1 \to SU(N)/T$ can be thought of as a flag $E_1^- \subset E_2^- \subset \ldots \subset E_{N-1}^- \subset E$ of subbundles of E such that E_i^- coincides with $\bar{E}_i^+$ at ∞, i.e. $F(\infty) = \bar{E}^+$.
If the map F is of degree $m = (m_1, \ldots, m_{N-1})$, one has that E_i^-/E_{i-1}^- is the line bundle $\mathcal{O}(k_{N-i+1})$, where $k_i = (m_i - m_{i-1})$; on the other hand, $E_i^+/E_{i-1}^+ \cong \mathcal{O}$. The sum $E_i^+ + E_{N-i}^-$ is equal to E except over a finite set of points, and so the sheaf

$$Q_i \stackrel{\text{def}}{=} E/(E_i^+ + E_{N-i}^-) \tag{1.4}$$

is supported on a finite set of points. Similarly, the sheaf

$$P_i \stackrel{\text{def}}{=} E/(E_{i-1}^+ + E_{N-i}^+) \tag{1.5}$$

is a line bundle ($\cong \mathcal{O}(m_i)$) away from $\operatorname{supp}(Q_i) \cap \operatorname{supp}(Q_{i-1})$. Futhermore one has exact sequences:

$$0 \to \mathcal{O} \to P_i \xrightarrow{\pi_i} Q_i \to 0 \tag{1.6}$$

$$0 \to \mathcal{O}(k_i) \to P_i \xrightarrow{\rho_i} Q_{i-1} \to 0 \tag{1.7}$$

One has then an exact sequence (Hurtubise [1989], Proposition 3.5):

$$0 \to E \to \begin{array}{c} P_1 \\ \oplus \\ P_2 \\ \oplus \\ \vdots \\ P_{N-1} \\ \oplus \\ P_N \end{array} \longrightarrow \begin{array}{c} Q_1 \\ \oplus \\ Q_2 \\ \vdots \\ \oplus \\ Q_{N-1} \end{array} \to 0 \tag{1.8}$$

where the map between the second and third terms is of the form $(a_1, \dots, a_N) \mapsto (\pi_1(a_1) - \rho_2(a_2), \pi_2(a_2) - \rho_3(a_3), \dots, \pi_{N-1}(a_{N-1}) - \rho_N(a_N))$. This, in fact, is an equivalent way of describing rational maps $\mathbf{C}P^1 \to SU(N)/T$ (Hurtubise [1989], Proposition (3.5)): a based rational map $\mathbf{C}P^1 \to SU(N)/T$ of degree $(m_1, \dots, m_{N-1})$ can be thought of as an equivalence class under automorphisms of pairs (S, e), where S is an exact sequence of $\mathcal{O}$-modules of the form (1.8) with:

- $e = (e_1, \dots, e_N)$ is a basis of E at ∞ with $e_i \in P_i$
- Q_i supported over a finite set of points not including ∞ and $h^0(\mathbf{C}P^1, Q_i) = m_i$
- P_i, Q_i fitting into exact sequences (1.6) and (1.7).

If the rational map is generic, in the sense that every point in $\bigcup_{i=1}^{N-1} \operatorname{supp} Q_i$ has multiplicity 1, then, for each i, we can take a section trivializing P_i away from ∞ and its restriction to Q_i as a basis of $H^0(\mathbf{C}P^1, Q_i)$. In this basis, the map

$$P_i \longrightarrow Q_i$$

is given by

$$(1, \dots, 1)$$

i.e. simple evaluation, while the map

$$P_{i+1} \longrightarrow Q_i$$

is given by

$$\left(p_i^1, \dots, p_i^{m_i}\right)$$

for some $(p_i^1, \dots, p_i^{m_i}) \in \mathbf{C}^*$.

To summarize, we think of a generic rational map as being of the form

$$0 \to E \to \begin{array}{c} P_1 \\ \oplus \\ P_2 \\ \oplus \\ \vdots \\ P_{N-1} \\ \oplus \\ P_N \end{array} \longrightarrow \begin{array}{c} Q_1 \\ \oplus \\ Q_2 \\ \vdots \\ \oplus \\ Q_{N-1} \end{array} \to 0 \ , \quad \text{where} \quad \begin{array}{ccc} P_i & \xrightarrow{1,\dots,1} & Q_i \\ & & \| \\ P_{i+1} & \xrightarrow{p_i^1,\dots,p_i^{m_i}} & Q_i \end{array} \tag{1.9}$$

Finally let us recall from Hurtubise [1989] how do we associate such a rational map to a generic Nahm complex. For any Nahm complex Q_i is supported over the eigenvalues of β_i. We can regularize a Nahm complex at every superior boundary point: if μ is such a point, setting $z = t - \mu$ we act by the gauge transformation

$$\operatorname{diag}\left(1, \ldots, 1, z^{-(k-1)/2}, z^{-(k-3)/2}, \ldots, z^{(k-1)/2}\right) \tag{1.10}$$

Then, for a generic (i.e. with all eigenvalues of multiplicity 1) Nahm complex the above choice of the map $P_i \to Q_i$ corresponds to choosing at each point μ_i a basis in which β_j is diagonal. We have a section of P_i given by the vector

$$(1, \ldots, 1)^T$$

On the other hand to find the map $P_{i+1} \to Q_i$ we have to parallel transport the diagonal basis of β_i from μ_i to μ_{i+1} and evaluate the above section of P_{i+1} in Q_i. This gives us the numbers $p_i^1, \ldots, p_i^{m_i}$.

3.2 The symplectic form and the twistor space

Consider now the moduli space $\mathcal{N}_{m_1,\ldots,m_{N-1}}$ described in the previous section. We can define a natural L^2-metric on this manifold. This time however we have to take into consideration the vectors U, W at neutral boundary points. The metric will have the form

$$\int_{\mu_1}^{\mu_N} \operatorname{tr}(d\alpha^* d\alpha + d\beta^* d\beta) + \sum_{\substack{\text{neutral}\\\text{boundary}\\\text{points}}} \operatorname{tr}\left(dU^* dU + \overline{dW}\, dW^T\right) \tag{2.1}$$

As always with the metric defined on a moduli space, one calculates it on representatives which are orthogonal to infinitesimal gauge transformations. Recall from section 2.4 that in the $SU(2)$ case this meant that an element $(a,b) \in \mathbf{T}_{(\alpha,\beta)}N_k$ had to satisfy the equation (4.4). In the $SU(N)$ case an element of the tangent space will be of the form

$$(a, b, u, w) = (a, b, \{u(\mu), w(\mu);\ \mu \text{ a neutral boundary point}\})$$

In addition to the equation (4.4) the orthogonality condition will also give us now a condition for the jump of a at every neutral point. It is not difficult to see, by a calculation similar to the one in section 2.4, that we get

$$a_+ - a_- = -U^* u + \bar{W} w^T \tag{2.2}$$

Remark: In contrast with the $SU(2)$ casen it is not known whether the manifolds $\mathcal{N}_{m_1,\ldots,m_{N-1}}$ and $\mathcal{M}_{m_1,\ldots,m_{N-1}}$ are isometric.

The above L^2-metric on $\mathcal{N}_{m_1,\ldots,m_{N-1}}$ is hyperkähler. Therefore it can be described by means of the twistor space construction of section 1.4 and that is what we are going to do in this section. The only ingredient that is not immediate is the complex symplectic structure on $\mathcal{N}_{m_1,\ldots,m_{N-1}}$. Since we have three noncommuting complex structures I, J, K on $\mathcal{N}_{m_1,\ldots,m_{N-1}}$, we get three symplectic forms $\omega_1, \omega_2, \omega_3$, where, if $(\ ,\ )$ denotes the scalar product, $\omega_1(s,t) = (Is, t)$ and similarily for the other two ones. Recall from section 1.4 that the complex symplectic form, relevant for the twistor space construction, is

$$\omega = \omega_2 + i\omega_3$$

First of all we should write down the action of I, J, K on an element (a, b, u, w) of the tangent space. We express it as an element of a quaternionic vector space $(a + bj, u + wj)$. This is the same as writing a variation (t_o, t_1, t_2, t_3) of a solution to Nahm's equations as $t_0 + it_1 + jt_2 + kt_3$. We can also write $u = \frac{1}{2}(u_1 + iu_2)$, $w = \frac{1}{2}(w_1 + iw_2)$ with u_1, u_2, w_1, w_2 being purely imaginary. The scalar product given by (2.1) can be written then (up to a constant factor) as

$$\Big((t_i, u_j, w_j), (\hat{t}_i, \hat{u}_j, \hat{w}_j)\Big) = -\int_{\mu_1}^{\mu_N} \left(\operatorname{tr} \sum_{i=0}^{3} t_i \hat{t}_i\right) - \sum_{\mu} \sum_{j=1}^{2} \operatorname{tr} \left(u_j \hat{u}_j^T + w_j \hat{w}_j^T\right) \tag{2.3}$$

where μ varies over neutral boundary points (as in (2.1)).

Now, in general, if $(s_1, s_2, s_3, s_4)^T = s_1 + is_2 + js_3 + ks_4 \in \mathbf{H}^m$, then it is easy to see that j acts by

$$(s_1, s_2, s_3, s_4)^T \longmapsto (-s_3, s_4, s_1, -s_2)^T$$

while k

$$(s_1, s_2, s_3, s_4)^T \longmapsto (-s_4, -s_3, s_2, s_1)^T$$

Hence

$$\begin{aligned} \omega(s, \hat{s}) = (js, \hat{s}) + i(ks, \hat{s}) &= -s_3\hat{s}_1 + s_4\hat{s}_2 + s_1\hat{s}_3 - s_2\hat{s}_4 + i(s_4\hat{s}_1 - s_3\hat{s}_2 + s_2\hat{s}_3 + s_1\hat{s}_4) \\ &= (s_1 + is_2\,,\, \hat{s}_3 + i\hat{s}_4) - (\hat{s}_1 + i\hat{s}_2\,,\, s_3 + is_4) \end{aligned}$$

Therefore, from (2.3)

$$\omega\Big((a, b, u, w), (\hat{a}, \hat{b}, \hat{u}, \hat{w})\Big) = \int_{\mu_1}^{\mu_N} \operatorname{tr}\left(a\hat{b} - \hat{a}b\right) + \operatorname{tr}\left(u\hat{w}^T - w^T\hat{u}\right) \tag{2.4}$$

which can be written as

$$\omega = \int_{\mu_1}^{\mu_N} \operatorname{tr}(d\alpha \wedge d\beta) + \operatorname{tr} dU \wedge dW^T \tag{2.5}$$

We would like to calculate this form in terms of the corresponding rational map. We are going to do it for a generic Nahm complex, i.e. for the open dense set where all eigenvalues of β have multiplicity 1.

Notice that the form is invariant under a global (i.e. depending only on t) conjugation by a complex gauge transformation, i.e. under an element of $\mathcal{G}$.

On the other hand if we act by a gauge transformation changing with (α, β, U, W), the form may change. We are going to choose a sequence of gauge transformation which leave the form (2.5) invariant and which will lead us ultimately to an extremely simple form of (α, β, U, W).

Let us see how does an infinitesimal gauge transformation change the symplectic form. An infinitesimal gauge transformation is of the form

$$1 + s\rho \tag{2.6}$$

where s is an infinitesimal tangent directions parameter. Such a gauge transformation acts on $(a, b, u, w) \in \mathbf{T}_{(\alpha,\beta,U,W)}\mathcal{N}_{m_1,\dots,m_{N-1}}$ by

$$a \longmapsto a - \tfrac{1}{2}\dot{\rho} - [\alpha, \rho] \quad , \quad b \longmapsto b - [\beta, \rho] \quad , \quad u \longmapsto u + \rho U \quad , \quad w \longmapsto w - \rho^T W \tag{2.7}$$

We have

Lemma 2.1 *The symplectic form (2.4) is invariant under gauge transformations which belong to $\mathcal{G}_{\mathbf{C}}$.*

Proof: Since, as we observed, a global (i.e. depending only on t) conjugation does not change the symplectic form, we only have to consider an infinitesimal gauge transformation, i.e. a $(1 + s\rho) \in \mathcal{G}_{\mathbf{C}}$. Such a ρ at a superior boundary point will have its lower-diagonal block equal to 0.
First let us consider how the part of the symplectic form given by the integral changes under any complex infinitesimal gauge transformation, i.e. a $(1 + s\rho) \in \mathcal{G}$. Let

$$\left(\tilde{a}, \tilde{b}, \tilde{u}, \tilde{w}\right) \quad , \quad \left(\tilde{\hat{a}}, \tilde{\hat{b}}, \tilde{\hat{u}}, \tilde{\hat{w}})\right)$$

be the result of the action of $(1 + s\rho)$ on (a, b, u, w), $\left(\hat{a}, \hat{b}, \hat{u}, \hat{w})\right)$ given by (2.7). Let us use the notation

$$\nabla = \tfrac{d}{dt} - 2[\alpha, \ ]$$

We have then

$$\begin{aligned}
&\textstyle\int_{\mu_1}^{\mu_N} \mathrm{tr}\left(\tilde{a}\tilde{\hat{b}} - \tilde{b}\tilde{\hat{a}}\right) - \int_{\mu_1}^{\mu_N} \mathrm{tr}\left(a\hat{b} - \hat{a}b\right) = \\
&\textstyle\int_{\mu_1}^{\mu_N} \mathrm{tr}\left(-\frac{1}{2}\nabla\rho\,\hat{b} - a[\beta, \hat{\rho}] + \frac{1}{2}\nabla\rho\,[\beta, \hat{\rho}]\right) + \int_{\mu_1}^{\mu_N} \mathrm{tr}\left(-\frac{1}{2}\nabla\hat{\rho}\,b - \hat{a}[\beta, \rho] + \nabla\hat{\rho}\,[\beta, \rho]\right) = \\
&\textstyle\int_{\mu_1}^{\mu_N} \mathrm{tr}\left(\frac{1}{2}\rho\,\nabla\hat{b} + \hat{\rho}[\beta, a] - \frac{1}{2}\rho\left([\nabla\beta, \hat{\rho}] + [\beta, \nabla\hat{\rho}]\right)\right) + \\
&\textstyle\int_{\mu_1}^{\mu_N} \mathrm{tr}\left(-\frac{1}{2}\hat{\rho}\,\nabla b - \rho[\beta, \hat{a}] - \frac{1}{2}\nabla\hat{\rho}\,[\beta, \rho]\right) \\
&\textstyle+ \frac{1}{2}\sum_{i=1}^{N} \mathrm{tr}\left(-\rho\hat{b} + \hat{\rho}b + \rho[\beta, \hat{\rho}]\right)\Big|_{\mu_{i_+}}^{\mu_{i_-}}
\end{aligned}$$

which is just

$$\frac{1}{2}\sum_{i=1}^{N} \mathrm{tr}\left(-\rho\hat{b} + \hat{\rho}b + \rho[\beta, \hat{\rho}]\right)\Big|_{\mu_{i_+}}^{\mu_{i_-}}$$

because $\nabla b = 2[\beta, a]$, $\nabla \hat{b} = 2[\beta, \hat{a}]$, $\nabla \beta = 0$.
Therefore

$$\int_{\mu_1}^{\mu_N} \mathrm{tr}\left(\tilde{a}\hat{\tilde{b}} - \tilde{b}\hat{\tilde{a}}\right) = \int_{\mu_1}^{\mu_N} \mathrm{tr}\left(a\hat{b} - \hat{a}b\right) + \tfrac{1}{2}\sum_{i=1}^{N} \mathrm{tr}\left(-\rho\hat{b} + \hat{\rho}b + \rho[\beta, \hat{\rho}]\right)\Big|_{\mu_{i_+}}^{\mu_{i_-}} \tag{2.8}$$

for any $(1 + s\rho) \in \mathcal{G}$.
Notice that if $(1 + s\rho) \in \mathcal{G}_{\mathbf{C}}$, then the expression

$$\left(-\rho\hat{b} + \hat{\rho}b + \rho[\beta, \hat{\rho}]\right)\Big|_{\mu_{i_+}}^{\mu_{i_-}} \tag{2.9}$$

vanishes at any superior (or inferior) boundary point μ_i, as the upper-diagonal block of ρ is continuous and the lower-diagonal one is 0 at μ_i. Therefore the lemma will be proved if we show that at a neutral boundary point μ the expression (2.9) cancels with

$$\mathrm{tr}\left(\tilde{u}\hat{\tilde{w}}^T - \tilde{w}^T\hat{\tilde{u}}\right) - \mathrm{tr}\left(u\hat{w}^T - w^T\hat{u}\right)$$

However, from (2.7), we have that the last expression is equal to

$$\mathrm{tr} - \left(uW^T\hat{\rho} + \rho U\hat{w}^T + W^T\rho\hat{u} - w^T\hat{\rho}U - \rho UW^T\hat{\rho} + W^T\rho\hat{\rho}U\right)$$

Using now (1.2) and its linearization we see that the expression indeed cancels with (2.9).

□

Now we can find a gauge transformation defined on all of $\mathcal{N}_{m_1,\ldots,m_{N-1}}$ which does not change ω and such that the resulting variations b are upper-triangular.
Observe that, that taking any cyclic vector v (this is possible because of the genericity assumption) and setting $g^{-1} = (v, \beta v, \ldots, \beta^{m-1}v)$, where $m = \mathrm{rank}\,\beta$, will give us (cf. Hurtubise[ ], Proposition (1.15)):

$$g\beta g^{-1} = \begin{pmatrix} 0 & & & 0 & p_0 \\ 1 & \ddots & & \vdots & p_1 \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & 0 & p_{m-2} \\ & & & 1 & p_{m-1} \end{pmatrix} \tag{2.10}$$

for some $p_o, \ldots, p_{m-1}$. Such a gauge transformation is not, however, in $\mathcal{G}_{\mathbf{C}}$, so the symplectic form may change. We have to proceed differently.
First we need a technical lemma.

Lemma 2.2 *Let β be of the form (2.10) and suppose an $M \in gl(m)$ satisfies $e_m^T M = 0$, i.e.*

$$M = \begin{pmatrix} M_{11} & \dots & \dots & M_{1m} \\ \vdots & & & \vdots \\ M_{m-1,1} & \dots & \dots & M_{m-1,m} \\ 0 & \dots & \dots & 0 \end{pmatrix}$$

Then:

i) If $[\beta, M]$ is upper-triangular, then M is strictly upper-triangular;

ii) If $[\beta, M]$ is constant, then M is constant;

iii) If $[\beta, M] = 0$, then $M = 0$;

In particular $gl(m)$ decomposes into the commutator of β and the annihilator of e_m^T.

Proof: All three facts follow easily after we write down the commutator for such β, M:

$$[\beta, M] = \begin{pmatrix} 0 & \dots & \dots & 0 \\ M_{11} & \dots & \dots & M_{1m} \\ \vdots & & & \vdots \\ M_{m-1,1} & \dots & \dots & M_{m-1,m} \end{pmatrix} - \begin{pmatrix} M_{12} & \dots & M_{1m} & \sum_i M_{1i} p_i \\ \vdots & & & \vdots \\ M_{m-1,2} & \dots & M_{m-1,m} & \sum_i M_{m-1,i} p_i \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

□

Now we have:

Lemma 2.3 *There is a gauge transformation $g = g(t, \alpha, \beta)$, defined for generic (α, β), which is a composition of a diagonal gauge transformation g_1 which depends only on t and $g_2 \in \mathcal{G}_{\mathbf{C}}$ such that after acting by g the resulting Nahm complex satisfies:*

i) if μ is a superior boundary point, then the limit $\beta(\mu^-)$ of β from the inferior side is of the form (2.10) while the limit from the superior side is

$$\beta(\mu^+) = \left(\begin{array}{cccc|cccc} & \beta(\mu^-) & & & & G & & \\ \hline 0 & \dots & 0 & 1 & * & \dots & \dots & * \\ & & & 0 & 1 & \ddots & & \vdots \\ & 0 & & \vdots & & \ddots & * & \vdots \\ & & & 0 & & 0 & 1 & * \end{array}\right)$$

ii) if μ is a neutral boundary point, then one-sided limits $\beta(\mu_\pm)$ of B are of the form (2.10) and the vector $W = W(\mu)$ is equal to $e_m = (0, \ldots, 0, 1)^T$.

Note that a gauge transformation of the described form will not change the symplectic form and, because of the second assertion in ii), the part $dU \wedge dW^T$ of the symplectic form will be now 0.

Proof: First of all let us define g_1. At a neutral or inferior boundary point μ put $g_1(\mu) = 1$; in a neighbourhood of a superior boundary point μ, setting $z = t - \mu$, put $g_1(z) = \operatorname{diag}\left(1, \ldots, 1, z^{-(k-1)/2}, \ldots, z^{(k-1)/2}\right)$ (see (1.10). Now extend this onto $[\mu_1, \mu_N]$. At a superior boundary point β becomes of the form (cf. Hurtubise[1989], Proposition (1.15)):

$$\beta(\mu^+) = \left(\begin{array}{cccc|cccc} & \beta(\mu^-) & & & & G & & \\ \hline f_1 & \cdots & \cdots & f_m & * & \cdots & \cdots & * \\ & & & 0 & \delta_1 & \ddots & & \vdots \\ & 0 & & \vdots & & \ddots & * & \vdots \\ & & & 0 & 0 & & \delta_{k-1} & * \end{array}\right) \tag{2.11}$$

Obviously we can modify this by a diagonal gauge transformation depending only on t so that the entries below the diagonal in the lower-diagonal block are all 1.

Now we are seeking g_2. Obviously it is enough to define it at boundary points. The definition of $\mathcal{G}_{\mathbf{C}}$ permits arbitrary values at a neutral or inferior boundary point while at a superior boundary point such a gauge transformation is of the form

$$\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \tag{2.12}$$

where h is the limit from the inferior side.

Such a gauge transformation conjugates a matrix of the form

$$\begin{pmatrix} \beta^- & G \\ F & E \end{pmatrix}$$

to

$$\begin{pmatrix} h(\beta^-)h^{-1} & hG \\ Fh^{-1} & E \end{pmatrix}$$

First, by genericity assumption, we can find an h such that $h\beta^- h^{-1}$ becomes of the form (2.10). β^+ is still of the form (2.11). Now we want to find h' so that (2.12) conjugates β^+ to the form described in the statement of our lemma. This equivalent to demanding that h' commutes with β^- and $Fh'^{-1} = (0, \dots, 0, 1)$. The existence of such a h' follows from Lemma 2.2.

There remains problem of a neutral boundary point. We have to know if we can conjugate both β_- and β_+ to the form (2.10) by *the same* matrix.

Let us recall how we find a matrix h that conjugates some β to the form (2.10). We choose any cyclic vector v and put

$$h^{-1} = (v, \beta v, \dots, \beta^{m-1} v)$$

Therefore we can conjugate both β_- and β_+ by the same h providing there is a cyclic (for both β_- and β_+) vector v such that

$$\beta_+^i v = \beta_-^i v \qquad \text{for } i = 1, \dots, m-1$$

Since, by the genericity assumption, β_- and β_+ have disjoint spectra, one of them, say β_-, is invertible. If we diagonalise β_-, we see that W must be cyclic for β_-; otherwise the spectra of β_- and β_+ are not disjoint. Hence the vector

$$v \overset{\text{def}}{=} \beta_-^{-m+1} W$$

is cyclic for β_-. If we write β_- in the basis

$$\left(v, \beta v, \dots, \beta^{m-1} v\right)$$

we see that it is of the form (2.10). However, in this basis

$$W = (0, \dots, 0, 1)$$

so β_+ differs from β_- only in the last column. Hence β_+ is also of the canonical form (2.10).

□

Therefore we are now in a gauge in which

$$\omega = \int_{\mu_1}^{\mu_N} \operatorname{tr} d\alpha \wedge d\beta \tag{2.13}$$

and onesided limits of $d\beta$ at any boundary point are upper-triangular.

Let us calculate this on a particular interval (μ_i, μ_{i+1}). On such an interval we can conjugate β so that it is of the form (2.10). Moreover, since β is of the form described in the last lemma, we can do this by a gauge transformation g that is upper-triangular at boundary points.

Let us take a particular (α_o, β_o) and calculate ω at this point. In an infinitesimal neighbourhood of (α_o, β_o), g can be written as

$$g = (1 + s\rho) g_o$$

where

$$\rho = dg\, g_o^{-1}$$

First of all g_o, as a global conjugation leaves ω invariant, and as it is upper-triangular at the ends, $d\beta$ stays upper-triangular there. On the other hand after acting by $1 + s\rho$:

$$d\beta = \begin{pmatrix} 0 & \begin{matrix} dp_0 \\ \vdots \\ dp_{m-1} \end{matrix} \end{pmatrix}$$

Suppose that we started with $(a, b)\,(\hat{a}, \hat{b})$ in the tangent space at $g_o(\alpha_o, \beta_o)$. We know that $b, \hat{b}$ are upper-triangular at μ_i, μ_{i+1}. If we take the differentials $\rho, \hat{\rho}$ of g in the directions given by $(a, b)\,(\hat{a}, \hat{b})$, we will get

$$b - [\beta, \rho] = \tilde{b}$$

$$\hat{b} - [\beta, \rho] = \hat{\tilde{b}}$$

where $\tilde{b}, \hat{\tilde{b}}$ are the corresponding differentials of β in the form (2.10) and therefore have only last column nonzero.

Since $b, \tilde{b}, \hat{b}, \hat{\tilde{b}}$ are all upper-triangular at μ_i, μ_{i+1}, it follows from Lemma 2.2i) that ρ, $\hat{\rho}$ are strictly upper-triangular at μ_i, μ_{i+1}. However, according to (2.8), we have now

$$\int_{\mu_i}^{\mu_{i+1}} \mathrm{tr}\left(\tilde{a}\hat{\tilde{b}} - \tilde{b}\hat{\tilde{a}}\right) - \int_{\mu_i}^{\mu_{i+1}} \mathrm{tr}\left(a\hat{b} - \hat{a}b\right) = \tfrac{1}{2}\mathrm{tr}\left(-\rho\hat{b} + \hat{\rho}b + \rho[\beta_o, \hat{\rho}]\right)\Big|_{\mu_i}^{\mu_{i+1}} \qquad (2.14)$$

where $\tilde{a} = a - \frac{1}{2}\dot{\rho} - [\alpha_o, \rho]$ and similarily for $\hat{\tilde{a}}$.

We can see that the expression on the right-hand side of the formula is $= 0$, as, at μ_i and μ_{i+1}, ρ, $\hat{\rho}$ are strictly upper-triangular, $b, \hat{b}$ are upper-triangular, so the first two terms vanish and, as β_o is of the form (2.10), the commutator $[\beta_o, \hat{\rho}]$ is upper-triangular, so the third term vanishes as well.

Therefore, on the interval (μ_i, μ_{i+1}), we are now in the gauge, which we will call V-gauge, in which β is of the form (2.10) (in particular constantin t) and the symplectic form is given by

$$\omega = \int_{\mu_i}^{\mu_{i+1}} \operatorname{tr} d\alpha \wedge d\beta \tag{2.15}$$

For a generic α, β we can go to another gauge, which we call D-gauge, in which β is diagonal. The passage from the V-gauge to the D-gauge is given by a gauge transformation that is constant in t. The difference between two integrals is given again by a formula (2.14) and we observe that the expression on the right hand side is the same at both ends, so it vanishes. Therefore the symplectic form in the D-gauge is still given by (2.15). Now, because of the complex equation, α in the D-gauge commutes with B, so it is diagonal. Let the eigenvalues of β on (μ_i, μ_{i+1}) be

$$\beta_i^1, \ldots, \beta_i^{m_i}$$

and let α in this gauge be:

$$\operatorname{diag}\left(\alpha_i^1(t), \ldots, \alpha_i^{m_i}(t)\right)$$

Let us take the gauge transformation p_i that makes $\alpha \equiv 0$ and is 1 at μ_i (i.e. the parallel transport of the diagonal basis from μ_i to μ_{i+1}); setting $z = \frac{t-\mu_i}{\mu_{i+1}-\mu_i}$ we can write this gauge transformation as

$$p_i(z) = \operatorname{diag}\left(p_i^1(z), \ldots, p_i^{m_i}(z)\right) = \operatorname{diag}\left(e^{\int_0^z \alpha_i^1(z)}, \ldots, e^{\int_0^z \alpha_i^{m_i}(z)}\right)$$

It follows that the symplectic form is

$$\omega = \frac{dp_i^1(1)}{p_i^1(1)} \wedge d\beta_i^1 + \ldots + \frac{dp_i^{m_i}(1)}{p_i^{m_i}(1)} \wedge d\beta_i^{m_i}$$

This extends to all of (μ_1, μ_N). Let us identify this expression in terms of the rational map. We have to know what is $(p_i^1(1), \ldots, p_i^{m_i}(1))$. To arrive at this form of ω we did the following: on each interval (μ_i, μ_{i+1}) we made β constant, then we diagonalized it and finally we set α to 0 by a gauge transformation that is 1 at μ_i. Making β constant and diagonalizing it *at* μ_i corresponds, in terms of the description in previous section, to choosing the basis of sections of Q_i which are restrictions of some section trivializing P_i away from ∞. Therefore in this basis the map

$$P_i \longrightarrow Q_i$$

is given by

$$1, \ldots, 1$$

On the other hand to evaluate this section of P_{i+1} in Q_i, we have to parallel transport the basis of Q_i from μ_i to μ_{i+1}. That is what the gauge transformation p_i does. It follows that the map

$$P_{i+1} \longrightarrow Q_i$$

is given by

$$p_i^1(1), \ldots, p_i^{m_i}(1)$$

Therefore, we have

Proposition 2.4 *On the open dense subset of $\mathcal{N}_{m_1,\ldots,m_{N-1}}$ where the eigenvalues of β are distinct, the symplectic form (2.5) is given in terms of the corresponding rational map (1.9) by*

$$\sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{dp_i^j}{p_i^j} \wedge d\beta_i^j$$

$\square$

This result allows us to give the full description of the twistor space for $\mathcal{N}_{m_1,\ldots,m_{N-1}}$. The symplectic form was the only missing ingredient.
Let $\mathbf{T}P^1$ denote the tangent bundle to $\mathbf{C}P^1$ and let ζ be an affine coordinate on $\mathbf{C}P^1$ and η the associated fibre coordinate. Let $\mathcal{O}(k)$ denote the lift to $\mathbf{T}P^1$ of the line bundle $\mathcal{O}(k)$ on $\mathbf{C}P^1$ and let $L^\mu(k)$ be the line bundle over $\mathbf{T}P^1$ with transition function $\exp(\mu\eta/\zeta)\zeta^k$ from $\{\zeta \neq 0\}$ to $\{\zeta \neq \infty\}$. Finally let $\tau : \mathbf{T}P^1 \to \mathbf{T}P^1$ denote the real structure

$$\tau(\eta,\zeta) = \left(-\bar{\eta}/\bar{\zeta}^2, -1/\bar{\zeta}\right) \tag{2.16}$$

The twistor space $\mathbf{Z}$ of the manifold $\mathcal{N}_{m_1,\ldots,m_{N-1}}$ is obtained by taking two copies of based rational maps $\mathbf{C} \times \mathit{Rat}(\mathbf{C}P^1, SU(N)/T)$, parametrized by

$$\left(\zeta,\quad 0 \to E \to \begin{tikzcd}[row sep=small] P_1 \arrow[dr] & \\ \oplus & Q_1 \\ P_2 \arrow[ur] \arrow[dr] & \oplus \\ \oplus & Q_2 \\ \vdots & \vdots \\ P_{N-1} \arrow[dr] & \oplus \\ \oplus & Q_{N-1} \\ P_N \arrow[ur] & \end{tikzcd} \to 0 \right)$$

where P_i, Q_i are sheaves over $\mathbf{C}P^1$ with affine coordinate η, and

$$\left(\tilde{\zeta},\quad 0 \to \tilde{E} \to \begin{array}{c} \tilde{P}_1 \\ \oplus \\ \tilde{P}_2 \\ \oplus \\ \vdots \\ \tilde{P}_{N-1} \\ \oplus \\ \tilde{P}_N \end{array} \begin{array}{c} \searrow \\ \nearrow \\ \searrow \\ \\ \\ \searrow \\ \nearrow \end{array} \begin{array}{c} \tilde{Q}_1 \\ \oplus \\ \tilde{Q}_2 \\ \vdots \\ \oplus \\ \tilde{Q}_{N-1} \end{array} \to 0 \right)$$

with the $\mathbf{C}P^1$ parameter $\tilde{\eta}$. We identify these over $\zeta \neq 0$, $\tilde{\zeta}, \eta, \tilde{\eta} \neq \infty$ by:

$$\tilde{\zeta} = \zeta^{-1} \quad , \quad \tilde{\eta} = \eta/\zeta^2 \quad , \quad \begin{array}{ccc} Q_i\big|_\eta & \xrightarrow{\cdot e^{\mu_i \eta/\zeta} \zeta^{m_i + m_{i-1}}} & \tilde{Q}_i\big|_{\tilde{\eta}} \\ \downarrow & & \downarrow \\ \eta & \longrightarrow & \tilde{\eta} \end{array} \quad , \quad \begin{array}{ccc} P_i\big|_\eta & \xrightarrow{\cdot e^{\mu_i \eta/\zeta} \zeta^{m_i + m_{i-1}}} & \tilde{P}_i\big|_{\tilde{\eta}} \\ \downarrow & & \downarrow \\ \eta & \longrightarrow & \tilde{\eta} \end{array}$$

This also identifies the maps $\pi_i, \tilde{\pi}_i$ of (1.6) and $\rho_i, \tilde{\rho}_i$ of (1.7) by writing down corresponding commuting diagrams.

Next we need a family of sections. According to Hurtubise and Murray [1989] a monopole determines tautologically a bundle E over $\mathbf{T}P^1$ and two flags of subbundles E_i^+, E_i^- over $\mathbf{T}P^1$. Proceeding as in (1.4) and (1.5) we get an exact sequence of sheaves over $\mathbf{T}P^1$ (see Hurtubise and Murray [1989], Proposition 1.12)

$$0 \to E \to \begin{array}{c} P_1 \\ \oplus \\ P_2 \\ \oplus \\ \vdots \\ P_{N-1} \\ \oplus \\ P_N \end{array} \begin{array}{c} \searrow \\ \nearrow \\ \searrow \\ \\ \\ \searrow \\ \nearrow \end{array} \begin{array}{c} Q_1 \\ \oplus \\ Q_2 \\ \vdots \\ \oplus \\ Q_{N-1} \end{array} \to 0$$

with:

- Q_i supported over a compact curve in the linear system $|\mathcal{O}(2m_i)|$
- $\operatorname{Ker}(P_i \to Q_i) = L^{\mu_i}(-k_i)$ while $\operatorname{Ker}(P_i \to Q_{i-1}) = L^{\mu_i}(k_i)$
- the map between the second and third terms is of the form $(a_1, \ldots, a_N) \mapsto (a_1 - a_2, a_2 - a_3, \ldots, a_{N-1} - a_N)$

This is a section of our twistor space. The real structure is the one coming from $\mathbf{T}P^1$ ((2.16)) and it preserves the above sections as Hurtubise and Murray [1989] have shown. The section ω of $\Lambda^2 T_F^* \otimes \mathcal{O}(2)$ is given in each of the trivializations $\zeta, \tilde{\zeta}$ by Proposition 2.4. One can check., using a description of a generic monopole given in Hurtubise and Murray [1989] (eq. (1.14)), that it is indeed a section of $\Lambda^2 T_F^* \otimes \mathcal{O}(2)$.

$\square$

3.3 Asymptotics of the metric in the $SU(N)$ case - an example

This section is an indication of how we plan to extend the results of Chapter II to the case of $SU(N)$ Nahm complexes with maximal symmetry breaking.
In this case we have several intervals and we have to match the solutions to the real equation according to the prescription of section 3.1.
We can do this explicitly; we can construct an approximate gauge in the sense of section II.1 with exponential decay of $(\tilde{\alpha}, \tilde{\beta})$ away from the boundary points. If we could show that Propositions 2.1.5 and 2.4.1 still hold, i.e. $\tilde{g}, \tilde{\rho}$ are bounded (uniformly in (α, β), we could carry out the analysis of sections 2.2, 2.4 and 2.5 on each interval without many changes. Also, it is enough to know that $\tilde{g}$, and $\tilde{\rho}$ are bounded at each boundary point. We are going to show on an example ($SU(3)$ monopoles of charge $(2,1)$) how one could attempt to prove this.

Let us then consider the case of $N=3$, i.e. two intervals, say $(-1,0)$ and $(0,1)$ and two charges $\underline{m} < \overline{m}$. Let us discuss the main problem in this case.
We can construct an approximate gauge in the sense of section II.1 such that $\tilde{\alpha}, \tilde{\beta}$ match at 0 and have the exponential decay on *each interval* $(-1,0)$ and $(0,1)$. However we do not know that the gauge transformation $\tilde{g}$ that solves the real equation (and preserves the matching) is $=1$ at 0. All we know from the results of Hurtubise [1989] is that there is a gauge transformation $\tilde{g} \in \mathcal{G}_{\mathbf{C}}$ which does it. The value of $\tilde{g}^*\tilde{g}$ at 0 is determined by choosing $\tilde{g}(\pm 1) = 1$ but we do not know what it is.
Notice, however, that if $\tilde{\alpha}$ satisfies the matching conditions at 0, then $\tilde{g}$ is C^1 (in the sense that the upper-diagonal block of $\tilde{g}$ is C^1). Indeed, since both $\tilde{\alpha}$ and $\alpha = \tilde{g}\tilde{\alpha}\tilde{g}^{-1} - \frac{1}{2}\dot{\tilde{g}}\tilde{g}^{-1}$ satisfy the matching conditions we get from the equation

$$\dot{\tilde{g}} = 2\left(\tilde{g}\tilde{\alpha} - \alpha\tilde{g}\right)$$

and the fact that $\tilde{g}$ is block-diagonal at 0 that the upper-diagonal block of $\tilde{g}$ is C^1. Therefore we can still apply the Donaldson's convexity argument (see just before Lemma 2.1.7) to the eigenvalues of $\tilde{g}$ (as the lower-diagonal block of $\tilde{g}(0) = 1$). This however is not enough to show that $\tilde{g}(0)$ is bounded independently of (α_d, β_d) since $\left\| F(\tilde{\alpha}, \tilde{\beta}) \right\|$ is decaying exponentially only away from $-1, 0, 1$, not just $-1, 1$. We could, however, try to improve the convexity argument by showing a stronger its version. Let us sketch this in the simplest case of charge $(2,1)$ (higher charges

involve cluster decomposition as for the $SU(2)$ case).
We start with two diagonal complexes, one of charge 2, say

$$(\text{diag}\,\{\alpha_1, \alpha_2\}\,,\ \text{diag}\,\{\beta_1, \beta_2\}$$

another of charge 1, (α_3, β_3).We assume that $\beta_2 \neq \beta_1$. In order not to consider two cases, as in the proof of Proposition 1.5, we will further simplify and consider the region where α_i-s stay bounded while

$$r \stackrel{\text{def}}{=} \min\{\, |\beta_i - \beta_j|\ ;\, i \neq j,\ i, j = 1, 2, 3\,\}$$

tends to ∞. Let us also put

$$R \stackrel{\text{def}}{=} \max\{\, |\beta_i - \beta_j|\ ;\, i \neq j,\ i, j = 1, 2, 3\,\}$$

First we use a permutation:
if $|\beta_3 - \beta_1| > |\beta_3 - \beta_2|$, we permute β_1 and β_2.
Now we would like to match β-s at 0, i.e. we are looking for a 2×2 matrix M such after conjugating the charge 2 complex by M the $(1,1)$-entry becomes β_3. Since we would like eventually to show that the gauge transformation taking us from the diagonal complex to the solution of real equation approaches 1, we choose M to be of the form:

$$\begin{pmatrix} 1 & T \\ S & 1 \end{pmatrix} \tag{3.1}$$

Then we have

$$\beta^0 \stackrel{\text{def}}{=} M \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} M^{-1} = \frac{1}{1 - ST} \begin{pmatrix} \beta_1 - ST\beta_2 & T(\beta_2 - \beta_1) \\ S(\beta_1 - \beta_2) & \beta_2 - ST\beta_1 \end{pmatrix} \tag{3.2}$$

Therefore

$$ST = \frac{\beta_3 - \beta_1}{\beta_3 - \beta_2} \tag{3.3}$$

Because of the above permutation, $|\,ST\,| < 1$. We also have $(1 - ST) = \frac{\beta_2 - \beta_1}{\beta_3 - \beta_2}$, so, writing $\beta_1 - ST\beta_2 = \beta_1 - \beta_2 + (1 - ST)\beta_2$ and similarily for the $(2,2)$-entry, we get:

$$\beta^0 = \begin{pmatrix} \beta_2 + (\beta_2 - \beta_3) & T(\beta_3 - \beta_2) \\ S(\beta_2 - \beta_3) & \beta_1 + (\beta_3 - \beta_2) \end{pmatrix} \quad . \tag{3.4}$$

Now we are looking for a gauge transformation near and on the left of 0 which will be $= M$ at 0 and will give the proper matching conditions to α. We are looking for a gauge transformation of the form (3.1) where S and T are now functions of t. It follows that we must have

$$\dot{T} = (\, ST(\alpha_3 - \alpha_2) + (\alpha_1 - \alpha_3)\,)S^{-1}$$

(we can see how the differences of α_i-s come into play). Therefore $\dot{T}T^{-1}$ is bounded by $(ST)^{-1}$, so by R. We can now multiply the off-diagonal terms of our gauge transformation by $e^{-R^2t^2}$ (same way as in case b) in the proof of Proposition 1.5). It follows that our gauge transformation approaches 1 as e^{sRt} $(t < 0)$ and that the resulting $\tilde{\alpha}, \tilde{\beta}$ satisfy

$$\left\| F(\tilde{\alpha}, \tilde{\beta})(t) \right\| \le R^2 e^{sRt}$$

for $t < 0$. On the other hand near $t = -1$ we can use Proposition 1.5 (or rather its proof) to find a gauge transformation approaching 1 as $e^{sR_{12}(1+t)}$, giving the correct poles and such that

$$\left\| F(\tilde{\alpha}, \tilde{\beta})(t) \right\| \le R_{12}^2 e^{-sR_{12}(1+t)}$$

We can extend these to a gauge transformation on $[-1, 0]$ so that

$$\left\| F(\tilde{\alpha}, \tilde{\beta})(t) \right\| \le R_{12}^2 e^{-sR_{12}(1+t)} + R^2 e^{sRt} \tag{3.5}$$

Now we would like to estimate $\tilde{g}$ that solves the real equation or rather, as in Proposition 1.5, $\ln \max\{\text{eigenvalues of } \tilde{g}^*\tilde{g}(t)\}$ and $\ln \max\{\text{eigenvalues of } \tilde{g}^{-1}\tilde{g}^{*-1}(t)\}$. Let us put

$$h \stackrel{\text{def}}{=} \tilde{g}^*\tilde{g}$$

and let us diagonalize h on $[0, 1]$ (h is diagonal on $[0, 1]$) by a unitary factor, so that

$$h = \operatorname{diag}\,(e^{x_1}, e^{x_2})$$

We can do this if the eigenvalues of h are different and then extend by continuity giving us a weak version of the inequality we are aiming for. A unitary factor acts on $F(\tilde{\alpha}, \tilde{\beta})$ by conjugation, so it does not change its norm. From the real equation we get then

$$\ddot{x}_1 = \left(|\alpha_{12}|^2 + |\beta_{12}|^2\right)\left(e^{x_1 - x_2} - 1\right) - \left(|\alpha_{21}|^2 + |\beta_{21}|^2\right)\left(1 - e^{x_2 - x_1}\right) + F(\tilde{\alpha}, \tilde{\beta})_{11}$$

Since x_2 is 1 at 0, x_2 will be bounded by the arguments of Proposition 1.5. The above equation for x_1 will give us an inequality of the form

$$|\ddot{x}_1| \ge F(t)\,|x_1| - f(t)$$

The problem is that $f(t)$ is decaying exponentially only away from $-1, 0, 1$, and it becomes large ($\approx R^2$) near 0. We are going to argue that $F(t)$ also becomes large near 0, so that a convexity argument of the type given in Lemma 1.7 can be used. Indeed, suppose that $F(t)$ is small (compared to R) for some t near 0. This means that $|\beta_{21}(t)|$ and $|\beta_{12}(t)|$ (in the basis in which $h(t) = \tilde{g}^*\tilde{g}(t)$ is diagonal) are small compared to R. Then, however, $[\beta^*(t), \beta(t)]$ has norm small compared to R^2. As this last expression gets conjugated by a unitary gauge transformation, its norm does not change under such a gauge transformation. It follows that

$$\left\| \left[\beta^{0*}, \beta^0\right] \right\|$$

is small compared to R^2 (as t is close to 0), where β^0 is given by (3.2) or (3.4). However

$$\left\| \left[\beta^{0*}, \beta^0\right] \right\| \geq \left| \left|\beta^0_{21}\right|^2 - \left|\beta^0_{12}\right|^2 \right|$$

From (3.2) we have

$$\left| \left|\beta^0_{21}\right|^2 - \left|\beta^0_{12}\right|^2 \right| \geq \left| |S|^2 - |T|^2 \right| |\beta_2 - \beta_1|^2$$

while from (3.4)

$$\left| \left|\beta^0_{21}\right|^2 - \left|\beta^0_{12}\right|^2 \right| \geq \left| |S|^2 - |T|^2 \right| |\beta_3 - \beta_2|^2$$

Since only the product of S and T is determined, we can choose S, T so that

$$\left| |S|^2 - |T|^2 \right| \geq \tfrac{1}{2}$$

and then we have

$$\left| \left|\beta^0_{21}\right|^2 - \left|\beta^0_{12}\right|^2 \right| \geq \tfrac{1}{2} R^2$$

since $R = |\beta_2 - \beta_1|$ or $R = |\beta_3 - \beta_2|$. This shows that $F(t)$ for t near 0 cannot be small compared to R and so $\tilde{g}$ is bounded. Similar argument will work for $\tilde{\rho}$.

$\square$

It is possible that since we are missing just the boundness of a gauge transformation at one point, a variational argument would be more suitable. The author will not shave until he solves this problem.

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