FACTORIZATIONS OF COVERS

by
Gilles Pesant

School of Computer Science
McGill University, Montréal

December 1989

A THESIS SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

Abstract

A family of sets $\mathcal{S} = (S_1, S_2, \ldots, S_m)$ is a *k-fold cover* for set T if each element of T is contained in at least k members of $\mathcal{S}$. This thesis examines partitions of $\mathcal{S}$ such that each part is a cover for T. It is mainly concerned with extremal problems of the type "find the smallest k such that every k-fold cover can be partitioned into two 1-fold covers".

In general, no such k exists. If elements of T are points in the plane and sets of $\mathcal{S}$ are straight lines then k is at least 4. If T is the interior of a simple polygon and $\mathcal{S}$ a family of convex or star-shaped sets then there does not always exist a "perfect" partition of the cover. This shows that computing a minimum size k-fold cover for a polygon does not reduce to combining minimum size thinner covers.

Résumé

Une famille d'ensembles $\mathcal{S} = (S_1, S_2, \ldots, S_m)$ forme une *k-couverture* d'un ensemble T si chaque élément de T est présent dans au moins k membres de $\mathcal{S}$. Ce mémoire se penche sur les partitions de $\mathcal{S}$ telles que chaque partie forme une couverture de T. Sa préoccupation principale réside en des problèmes du genre "trouvez le plus petit k tel que chaque k-couverture peut être divisée en deux 1-couvertures".

En général, un tel k n'existe pas. Si les éléments de T sont des points dans le plan et les ensembles de $\mathcal{S}$ sont des droites alors k vaut au moins 4. Si T représente l'intérieur d'un polygone simple et $\mathcal{S}$ une famille d'ensembles convexes ou "étoilés" alors il n'existe pas toujours une partition "parfaite" de la couverture. Ceci démontre que trouver une k-couverture minimale pour un polygone n'équivaut pas à combiner des couvertures minimales d'ordre inférieur.

Statement of Originality

All elements of this thesis should be considered original contributions to knowledge, except for some background material and results whose origin is indicated in the text. Furthermore, no assistance outside that acknowledged has been received.

Acknowledgements

The work on this thesis started with chapter 4 while taking an advanced computational geometry course in which we were encouraged to do original research. Our instructor Godfried Toussaint through his good advice, enthusiasm and confidence in our capabilities has stimulated at least three theses among the geometry group.

I thank Tom Shermer for fruitful discussions that oriented my early research and led to some of the basic results. I am also grateful for the stimulating though relaxed atmosphere within the geometry group at McGill, both faculty and students, during my years here — it has been very enjoyable. Part of this enjoyment must be credited to NSERC for its financial support in the last four years.

Finally, my thesis would probably still be on the drawing board without my supervisor David Avis, who taught me that if everyone strived for perfection nothing would ever get written up. The doorway to his office had this magic in that you would come out enlightened where before you had entered so dim.

Contents

Chapter 1

Introduction

Polly had a *diet problem* ([Chv83]). She wanted an economical way of getting the energy, protein and calcium she needed and used linear programming to solve it. She then went shopping and bought foods to meet her requirements for a week. Her shopping list is shown in the table below along with the nutritional contents of each food. (To keep the example simple, a serving either contains the daily requirement or a negligible quantity of each of the three nutrients.) To her great surprise, she is not able to plan seven daily meals. Her problem is known as a *decomposition problem.*

Food	*Nb of servings bought*	*Energy*	*Protein*	*Calcium*
chicken	2	√	√	
whole milk	2	√		√
cherry pie	1	√		
eggs	3		√	√
pork with beans	2	√	√	√

This thesis aims at the study of covering problems, which aside from being an interesting source of mathematical queries ([T77,St78,Pac80,Pac86,Rog64,CS88]) find themselves the abstraction for diverse problems of a more applied nature. They occur in the minimization of switching functions for the logical design of digital machines or in the Dimer problem of crystal physics ([D74]), for example. Computer science also provides a good number of them in the field of robotics or in areas using polygon decomposition such as pattern

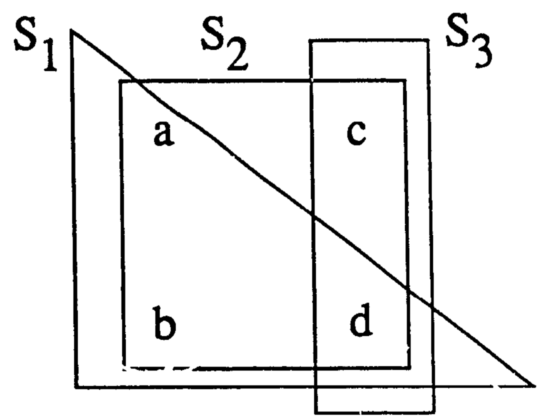

Figure 1: A source of examples for covers.

recognition and computer graphics ([FP75,M72,O82,Pav77,KS85]). The investigation will emphasize certain geometric covering problems which arise in computational geometry, a growing branch of computer science which involves designing algorithms to solve problems of a geometric nature. Covering points with lines in the plane and covering the interior of a simple polygon, known as the "Art Gallery" problem, will be considered.

A family of sets $\mathcal{S} = (S_1, S_2, \ldots, S_m)$ is a *k-fold cover* for set T if each element of T is contained in at least k members of $\mathcal{S}$. The sets $S_1, S_2, \ldots, S_m$ are not necessarily distinct. The size of a k-fold cover is the number of members of $\mathcal{S}$, namely m. A 1-fold cover will often be referred to as a *simple cover*, whereas the term cover used alone should be taken to mean k-fold covers in general.

Consider the set $T = \{a, b, c, d\}$ together with sets $S_1 = \{a, b, d\}$, $S_2 = \{a, b, c, d\}$ and $S_3 = \{c, d\}$ as represented in figure 1. The families (S_1, S_3) and (S_2) form simple covers of size 2 and 1, respectively, for set T. Family (S_1, S_2, S_3) forms a 2-fold cover of size 3 for T; note that it is perfectly valid to say that (S_1, S_2, S_3) forms a simple cover for T, although the strongest statement is usually expected. In the text, it should be assumed that statements made about covers are the strongest possible. A 4-fold cover of size 4 for T can be realized by (S_2, S_2, S_2, S_2).

Various questions can be asked about covers. For instance, quite obviously and as illustrated in some of the previous examples, k-fold covers for a fixed set do not necessarily have the same size. Therefore, finding minimum size covers should deserve some attention. Moreover, applications of covers are often optimization problems looking for such results. That issue will be addressed in chapter 4 when taking up covers of polygons.

Another kind of inquiry involves decompositions of covers. On top of being of high combinatorial interest, as shall be seen from the past and ongoing efforts in this area, it appears in several applied problems which will be discussed throughout. That second type of question will be the main topic of the thesis.

The remainder of this chapter provides the notions and elementary results necessary for the rest. Chapter 2 presents a transformation of the decomposition problem into coloring of hypergraphs, which is used to achieve some of the results. It also presents results on covers in general and gives an account of the related work. The last two chapters contain most of the original material. Chapter 3 looks into decompositions of covers of points by lines. It includes an interesting open problem on the multiplicity needed for a cover to admit a certain decomposition. Chapter 4 investigates prime covers for polygons. The difficulty involved in computing minimum size k-fold covers is illustrated by exhibiting a family of polygons having prime minimum size covers.

What is meant by decomposition will now be made more precise. Let families of sets $\mathcal{S}_1$ and $\mathcal{S}_2$ be respectively k_1- and k_2-fold covers for set T. $\mathcal{S}_1$ is said to be *thinner* than $\mathcal{S}_2$ if k_1 is less than k_2. This definition becomes rather intuitive if a k-fold cover is visualized as k layers on T. Note that this relation between covers only makes sense if the covered set, T, is the same.

Broadly speaking, a decomposition of a cover is simply a partition of the family of sets. The type of decomposition which will be considered here produces thinner covers. This means that instead of any partitioning into components with no connection to covers, the decomposition will in a way preserve the covering property since each part will in turn form a cover.

Let $\{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_r\}$ be a partition of a k-fold cover $\mathcal{S}$ for set T. Define k_i as the largest integer so that $\mathcal{S}_i$ forms a k_i-fold cover for T. If $\mathcal{S}_i$ does not even form a simple cover then k_i is set to zero. Partition $\{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_r\}$ is a *perfect factorization* of $\mathcal{S}$ if $\sum_{i=1}^{r} k_i = k$. This summation cannot possibly be greater than k but may be less than k in which case $\{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_r\}$ will simply be referred to as a factorization. Hence, every partition is considered a factorization of some sort. Finally, $\mathcal{S}$ is said to be a *prime* cover if it does not admit a perfect factorization, aside from the trivial $\{\mathcal{S}\}$.

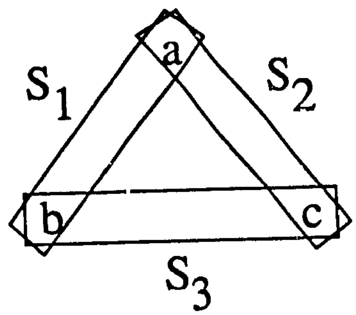

Figure 2: A prime 2-fold cover.

A few examples will help to illustrate these new concepts. Coming back to the sets of figure 1, consider again the 2-fold cover (S_1, S_2, S_3). Partition $\{(S_1, S_3), (S_2)\}$ is a perfect factorization of it since each of the two parts forms a 1-fold cover, whereas $\{(S_1), (S_2, S_3)\}$ is just a factorization because (S_2, S_3) forms a 1-fold cover and (S_1) is not a cover at all. Furthermore, the existence of $\{(S_1, S_3), (S_2)\}$ guarantees that (S_1, S_2, S_3) is not prime. Perfect factorizations do not have to be composed of simple covers: $\{(S_2, S_2, S_2), (S_2)\}$ and $\{(S_2, S_2), (S_2, S_2)\}$ both qualify as perfect factorizations of 4-fold cover (S_2, S_2, S_2, S_2).

Simple covers such as (S_1, S_3) are obviously prime but primality is not restricted to 1-fold covers. Consider sets $S_1 = \{a, b\}$, $S_2 = \{a, c\}$, $S_3 = \{b, c\}$ and $T = \{a, b, c\}$ which are depicted in figure 2. Family (S_1, S_2, S_3) is a prime 2-fold cover for T. A non-trivial factorization of this family will have at most one part containing more than one set but it is easily seen that at least two sets are needed for a cover and so the factorization at best includes a single 1-fold cover.

This may be an indication that for a lot of covers it is impossible to find a perfect factorization. Yet it would be nice to have a way of expressing how abundantly a cover lends itself to decomposition. The existence of a perfect factorization for a cover says that it decomposes efficiently into thinner covers, in the sense that the sum of the parts is as powerful as the whole. If on the contrary only common factorizations are possible, nothing is learned about efficiency because the term factorization is too general: as mentioned previously, any partition meets the requirements.

An evaluation of the efficiency of a factorization based on the number of 1-fold covers, 2-fold covers, 3-fold covers and so forth is both very accurate and very messy, especially when used to compare covers. Imagine deciding between a pair of 9-fold covers, one whose "best" factorization includes a 6-fold cover and a 2-fold cover, the other, a 3-fold cover with two 2-fold covers and a simple cover.

A more reasonable measure might be the number of thinner covers. This is certainly easier to manipulate: a *best factorization* of a k-fold cover will be one with the largest number of thinner covers and the k-fold cover whose best factorization has a larger number will be considered more efficient in its decomposition. Note that under this measure, a k-fold cover with a perfect factorization does not automatically have a more efficient decomposition than a prime k-fold cover.

It may seem a bit arbitrary, even misleading, to ignore the "thickness" of the covers and to consider three 1-fold covers better than two 5-fold covers, but for the type of problem motivating this investigation it is precisely what is desired.

All that is asked of the constituents of a decomposition is that they each form a cover, be it 1-fold or 5-fold, caring only about the transfer of the covering property. For example, in an art gallery you might have video cameras set up so that each painting is monitored by at least four cameras (a 4-fold cover), in case of malfunctions. One day, due to budget cuts, it is decided to reduce the use of these very unreliable cameras by grouping them into shifts required to watch over all the paintings, thus distributing the wear evenly. The problem consists of creating as many shifts (simple covers) as possible to keep the wear to a minimum.

It is now time to formalize things a bit. Define $F(T,\mathcal{S})$ as the largest integer r such that family $\mathcal{S}$ forming a cover for set T can be factorized into r simple covers for T. Tied with this issue are preoccupations of computational efficiency such as designing algorithms to compute $F(T,\mathcal{S})$ and also output the simple covers. A first step in that direction usually involves a study of the extremal properties of the mathematical object to be manipulated, so as to give an idea of the potential difficulties connected with the elaboration of an algorithm or what to expect in terms of worst-case results and time complexity.

The bulk of the thesis will be spent on that first step while looking into the following. Call $f(r)$ the smallest k such that $F(T,\mathcal{S})$ is at least r for every set T and every k-fold cover $\mathcal{S}$ for T. In other words, a $f(r)$-fold cover is guaranteed to be factorizable into r simple covers. Recall the sets associated with figure 2, where family (S_1, S_2, S_3) was a prime 2-fold cover. Since $F(T,(S_1,S_2,S_3)) = 1$ it must be that $f(2)$ exceeds 2.

To conclude this introductory chapter, investigations related to covers but which are not in the main stream of this study will be mentioned. A good number of efforts have been put in along the lines of minimum size covers ([T77]). They deal with a set to be covered which is infinite and so look for minimum density k-fold covers, typically covers of the plane with equal circles. Alternatively, resembling decompositions of k-fold covers, the reconstruction of a smaller $(k-1)$-fold cover with sets obtained through set operations on the original ones leads to a theorem with interesting corollaries by preserving an incidence property ([St78]).

Chapter 2

Covers

2.1 The Hypergraph Approach

This section offers a transformation of covering problems which will greatly facilitate the investigation of $f(\cdot)$. Let T be a finite set $\{t_1, t_2, \ldots, t_n\}$ and $\mathcal{S}$ a finite family $(S_1, S_2, \ldots, S_m)$ of non-empty subsets of T. An instance of a cover can then be viewed as a hypergraph $H = (T, \mathcal{S})$ where T is the set of vertices and $\mathcal{S}$ the family of edges. Now consider its dual hypergraph $H^* = (S, \mathcal{T})$ with $S = \{s_1, s_2, \ldots, s_m\}$ (respectively representing $S_1, S_2, \ldots, S_m$) and $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ (respectively representing $t_1, t_2, \ldots, t_n$) for which we define $T_j = \{s_i : \ t_j \in S_i\}$.

For instance, the dual of the hypergraph represented in figure 1 would have edges $T_a = \{s_1, s_2\}$, $T_b = \{s_1, s_2\}$, $T_c = \{s_2, s_3\}$ and $T_d = \{s_1, s_2, s_3\}$, with each S_i mapped to s_i and each $j \in T$ to T_j.

What does a factorization look like in the dual hypergraph? Consider a factorization of $\mathcal{S}$ into r covers $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_r$. Since each $\mathcal{S}_l$ forms a cover for T, for every $t_j \in T$ there is a $S_{i(j,l)} \in \mathcal{S}_l$ containing it. This means that in the dual, T_j contains $s_{i(j,l)}$ and so has (the dual of) a representative from each $\mathcal{S}_l$. Viewing the factorization as a coloring of the members of $\mathcal{S}$ (and their dual) in r colors, a factorization of $\mathcal{S}$ into r covers, in H, corresponds to a coloring of the elements of S in r colors so that every color is present in each T_j, in H^*. Figure 1 illustrates this parallel on the hypergraph of figure 1 and its dual described earlier.

This duality had already been pointed out in [Pac80] which introduced these decomposition problems. Unfortunately, the condition for a coloring to be valid in the dual problem does not agree with the usual notion of a valid coloring of a hypergraph. A hypergraph is

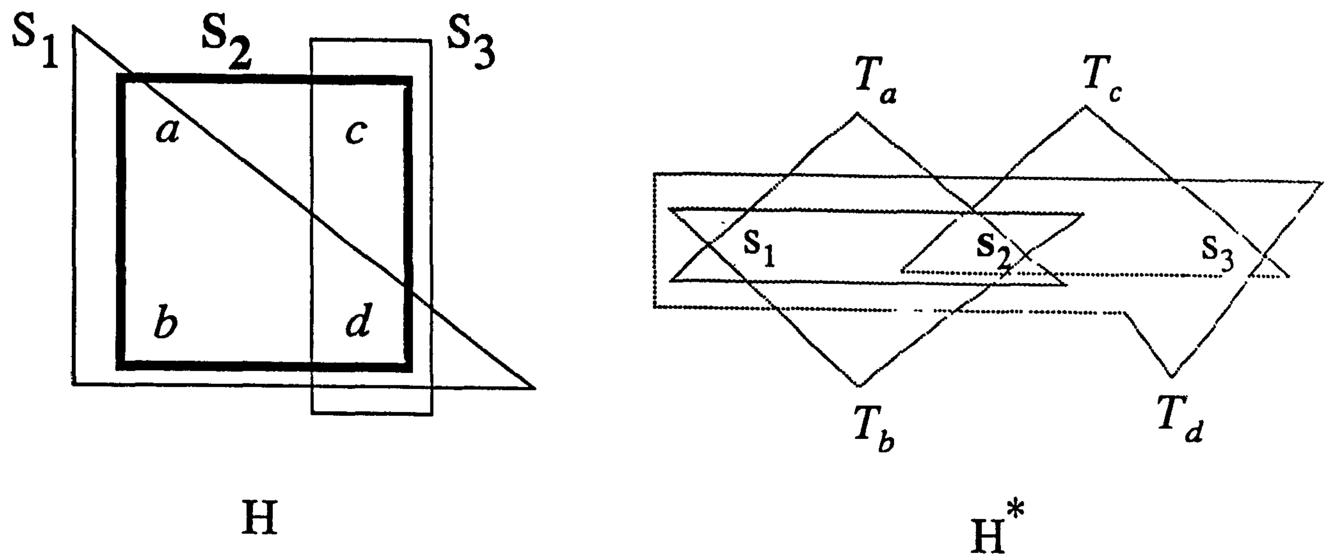

Figure 1: The factorization of 2-fold cover (S_1, S_2, S_3) into **two** simple covers $\{(S_1, S_3), (S_2)\}$ (left) and the dual coloring in **two** colors (right).

r-colorable if we can color its vertices in r colors so that none of its edges is monochromatic. Therefore an edge only needs to include two different colors as opposed to all of them. Note however that for 2-colorability the distinction vanishes. Hence, the following:

Fact 2.1.1 *A cover S for set T is factorizable into two covers iff the corresponding dual hypergraph H^* is 2-colorable.*

What advantage is there in reformulating the problem as a hypergraph coloring and why emphasize the case $r = 2$? Hypergraphs are more familiar than covers, having been studied from various angles ([B73]), and the results that were gathered provide tools to tackle the problem. For instance, extensive work has been done on 2-colorable hypergraphs ([B73,Se74,FL72,FL74]), which can certainly be used for factorizations into two covers since in that case the notion of a valid coloring is the same. Findings will also have a tendency to be more elegant in the dual.

Just as not every cover can be factorized into two covers (fig. 2, chapter 1), not all hypergraphs are 2-colorable. While the question of 2-colorable graphs has long been completely solved by showing them equivalent to graphs without odd-length cycles ([K36]), results on

2-colorable hypergraphs remain partial. They mainly consist of sufficiency conditions. A restriction on H^* can be derived from the nature of covers.

Fact 2.1.2 *Set $\mathcal{S}$ forms a k-fold cover for T iff every edge of H^*, the dual of $H = (T, \mathcal{S})$, has size at least k.*

PROOF It will be directly shown that the two are equivalent, starting in the dual $H^* = (S, \mathcal{T})$.

$$\begin{aligned}
& \forall j \in \{1,2,\ldots,n\} \;\; |T_j| \geq k,\; T_j \in \mathcal{T} \\
\Leftrightarrow\;\; & \forall j \in \{1,2,\ldots,n\} \;\; |\{s_i \in S :\; t_j \in S_i\}| \geq k, \text{by definition of } T_j \\
\Leftrightarrow\;\; & \forall j \in \{1,2,\ldots,n\} \;\; |\{S_i \in \mathcal{S} :\; t_j \in S_i\}| \geq k,\; t_j \in T \\
\Leftrightarrow\;\; & \mathcal{S} \text{ forms a k-fold cover for } T
\end{aligned}$$

□

This means that to look at all k-fold covers, it is equivalent to consider all hypergraphs satisfying the corresponding lower bound on the size of their edges. That will be useful in the upcoming investigation of $f(\cdot)$.

2.2 Unconstrained Covers

First considered are covers in general, i.e. on which no additional constraint is put. What is understood by constraint consists of a restriction on the covering sets. The constraints that will be introduced in the following chapters will have a geometric origin.

As a function of r, $f(\cdot)$ is quite obviously monotone increasing and non-trivial starting at $r = 2$. It therefore makes sense to start with $f(2)$ which will provide a lower bound for the rest. Approaching the problem in the dual, both facts of the preceding section can be combined to express $f(2)$. It was a statement about all k-fold covers and their factorization, which facts 2.1.2 and 2.1.1 respectively dualize. It is now the smallest integer k such that every hypergraph $H = (V, \mathcal{E})$ with $|E| \geq k \;\forall\; E \in \mathcal{E}$, is 2-colorable.

Theorem 2.2.1 $f(r)$, $r > 1$, *does not exist for unconstrained covers.*

PROOF It is sufficient to show that $f(2)$ does not exist, using a construction stated in [Pac80]. Consider hypergraph $(V, \mathcal{E})$ with $V = \{1, 2, \ldots, 2k-1\}$ and $\mathcal{E} = \{E \subseteq V : |E| = k\}$. By definition it obeys the restriction on the size of the edges. Furthermore, regardless of the value of k, it will be shown that no 2-coloring of it exists.

By the Pigeon-Hole principle, at least k elements of V have the same color, say navy blue. By the definition of $\mathcal{E}$, we can find an edge whose elements are all navy blue. Hence it is not 2-colorable and $f(2)$ does not exist. □

Notice that the hypergraph constructed for $k = 2$ in the proof of theorem 2.2.1 yields precisely the 2-fold cover of figure 2 in chapter 1, which was already known not to factorize into two simple covers. Already for $k = 3$ the corresponding cover increases in complexity. Hypergraph $(S, \mathcal{T})$, $S = \{s_1, s_2, s_3, s_4, s_5\}$, $\mathcal{T} = (\; T_1 = \{s_1, s_2, s_3\}, T_2 = \{s_1, s_2, s_4\}$, $T_3 = \{s_1, s_2, s_5\}$, $T_4 = \{s_1, s_3, s_4\}, \ldots, T_{10} = \{s_3, s_4, s_5\})$ dualizes to 3-fold cover $\mathcal{S} = (\{t_1, t_2, t_3, t_4, t_5, t_6\}, \{t_1, t_2, t_3, t_7, t_8, t_9\}, \{t_1, t_4, t_5, t_7, t_8, t_{10}\}, \{t_2, t_4, t_6, t_7, t_9, t_{10}\}, \{t_3, t_5, t_6, t_8, t_9, t_{10}\})$ for $T = \{t_1, t_2, \ldots, t_{10}\}$.

So there is no k large enough for all k-fold covers to be at least factorizable into two simple covers. Nevertheless, a special case of unconstrained covers unveils a more positive outcome, aided by a result on 2-colorability.

Theorem 2.2.2 *If each pair of elements of T must be present in exactly one set of an unconstrained cover $\mathcal{S}$ then* $f(2) = 4$.

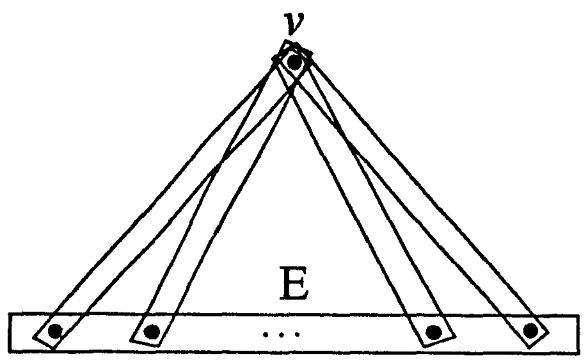

Figure 2: Hypergraphs of the form $(E \cup \{v\},\ \{E\} \cup \{\{v,w\} :\ w \in E\})$, $|E| \geq 2$.

PROOF In the dual hypergraph, the expression of this special case becomes "each pair of edges have exactly one point in common". A result, apparently due to Lovász ([L79][problem 13.35]), describes all such hypergraphs which are not 2-colorable. They consist of hypergraphs of the form $(E \cup \{v\},\ \{E\} \cup \{\{v,w\} :\ w \in E\})$, $|E| \geq 2$ (figure 2) and the Fano plane (figure 3). Since hypergraphs of the first type have edges of size 2 and every edge of the Fano plane has size 3, a lower bound of 4 on the size of the edges will only leave 2-colorable hypergraphs. □

Those hypergraphs of figures 2 and 3 happen to have a isomorphic dual and so represent as well the covers which are not factorizable into two simple covers.

One might find the condition associated with theorem 2.2.2 very restrictive but in fact the class of covers satisfying it seems quite rich. Theorems about the existence of Steiner triple systems and projective planes serve as witnesses to that. These two notions can be unified under balanced block designs. A (m,n,k,r,λ)-*design* consists of a set T of n elements and a family $\mathcal{S}$ of m subsets of T such that:

- each subset consists of exactly r elements.
- each element appears in exactly k subsets.
- each pair of elements appears in exactly λ subsets.

It is easy to see that family $\mathcal{S}$ of a (m,n,k,r,λ)-design forms a k-fold cover for set T.

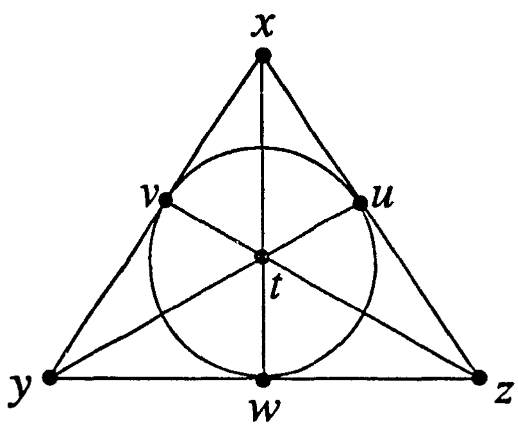

Figure 3: The Fano plane, with edges xuz, xvy, ywz, xtw, ytu, ztv and uvw.

Furthermore if $\lambda = 1$ then it satisfies the condition of the theorem. Balanced block designs do not exist for all combinations of the parameters but there are theorems proving the existence of some of them. Two of these theorems, respectively about Steiner triple systems and projective planes, are now stated in the language of balanced block designs. They can both be found in their original form in [Rob84].

Theorem 2.2.3 *There is a* $(m, n, k, 3, 1)$*-design if and only if* $n = 3$ *or* $n = 6i + 1$ *or* $n = 6i + 3$, $i \geq 1$.

(For Steiner triple systems, $m = \frac{n(n-1)}{6}$ and $k = \frac{n-1}{2}$.)

Theorem 2.2.4 *If* i *is a power of a prime then there is a* $(i^2+i+1, i^2+i+1, i+1, i+1, 1)$*-design.*

Simply judging from these two theorems, theorem 2.2.2 involves an abundance of covers for various values of k.

2.3 Some Known Results on Geometric Covers

The results mentioned in this section appear in [Pac80,Pac86]. There has been some work on constrained covers in which the set T to be covered is a Euclidean space and $\mathcal{S}$ a family of convex sets.

In the one-dimensional case E^1, the Euclidean straight line, is covered by intervals. This constraint leads to $f(r) = r$, i.e. a k-fold cover can always be factorized into k simple covers. It is not hard to see with a construction of the simple covers. Consider sweeping E^1 from left to right with a line L, maintaining k intervals intersecting L and incrementally building the k covers. Initially all the intervals are in $\mathcal{S}$. The k intervals intersecting L are kept in CURRENT and once they are removed from it they are put in OLD. CURRENT is initialized with k intervals whose left endpoint is $-\infty$. Whenever L meets the right endpoint of one of the current intervals (if several of them have the same right endpoint, they are handled one by one), that interval must be "followed" by one covering the immediate right of L. Such an interval must exist in $\mathcal{S}$ or else that part of E^1 will only be $(k-1)$-covered. That new interval cannot already be committed to a simple cover since those either are in CURRENT or have their right endpoint to the left of L and are in OLD. Hence k disjoint covers are formed.

Factorization already becomes much more complicated in two dimensions. A result identical to theorem 2.2.1. namely that $f(r)$ $(r > 1)$ does not exist, is obtained for covers of the plane with convex sets. It essentially uses the same hypergraphs as in the proof of that theorem and chooses a suitable planar embedding. The vertices are put on a circular arc and the edges consist of the convex hull of their elements. Some more sets are added to sufficiently cover the rest of the plane. Figure 4 exhibits a partial embedding of the hypergraph for $k = 3$.

Nevertheless, it has been proved that if the covering sets are translates of a centrally symmetric convex polygonal domain, $f(r)$ does exist. Whether it exists for translates of a circle is still open.

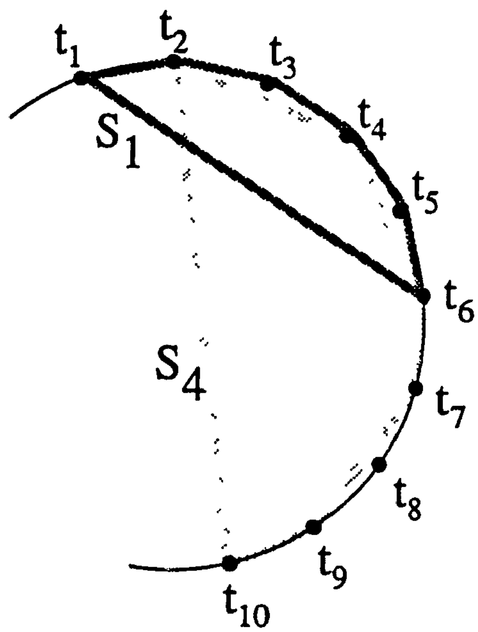

Figure 4: Sets S_1 and S_4, part of a 3-fold cover of the plane by convex sets.

Chapter 3

Covering Points with Lines

3.1 Introduction

In a warehouse, hoops are held in n huge heaps according to hue. A system of m straight rails has been installed on the ceiling, each with a mobile hook to handle the hoops. So that several manipulations could be performed on a heap simultaneously, the system was designed to insure k rails over each hue. Unfortunately, the fact that a hook operator can only do one thing at a time was overlooked. It is decided to distribute the rails among operators in such a way that each may reach all heaps. The foreman would like to know how many workers she will need.

This is another geometrically constrained covering problem, taking place in the plane as in the preceding chapter but where T is now a finite set of points and S a family of straight lines, a special case of convex sets. As a set, a line will contain all the elements of T with which it is incident. A cover obeying that geometric constraint will be called a point-line cover. Figure 1 gives an example of such a cover.

As with unconstrained covers, some admit a perfect factorization while others don't. Partition $\{(L_1, L_2, L_5), (L_3, L_4, L_6, L_7)\}$ represents a perfect factorization for the 3-fold cover of figure 1 while the prime 2-fold cover of figure 2 in chapter 1 can easily be drawn with its covering sets as straight lines.

The constraint might influence results on $f(\cdot)$, though. Since that prime cover can be adapted, $f(2)$ must be greater than 2, but the construction in the proof of theorem 2.2.1 can no longer be generalized to all k. For instance, the 3-fold cover exhibited contains distinct sets $\{t_1, t_2, t_3, t_4, t_5, t_6\}$ and $\{t_1, t_2, t_3, t_7, t_8, t_9\}$ which can only be represented as the same

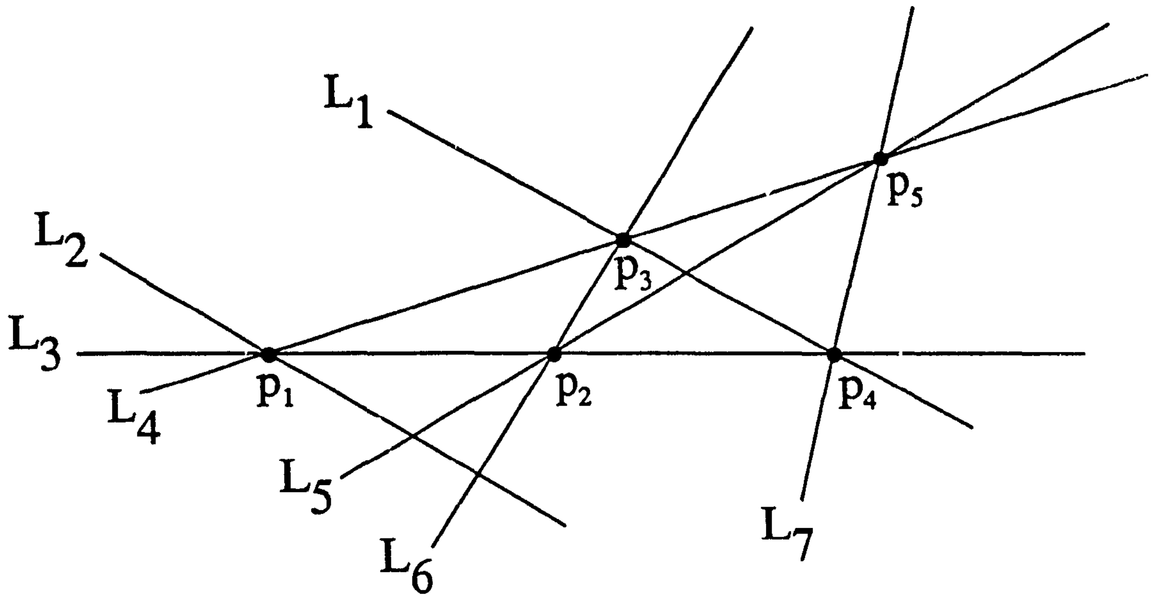

Figure 1: Lines $(L_1, L_2, \ldots, L_7)$ form a 3-fold cover for points $\{p_1, p_2, \ldots, p_5\}$.

line, being both incident with distinct points t_1, t_2 and t_3.

That brings to light a property of the sets of family $\mathcal{S}$ for this constrained covering problem. *Two sets contain either no common element, one common element or all the same elements*, since two points uniquely determine a line. In order to better express the problem in terms of hypergraphs, a lemma about covers in general will be introduced. First, to an instance of a k-fold cover $\mathcal{S}$ for a set T corresponds a *reduced cover* $\mathcal{S}'$ for a set T', obtained through the following procedure:

1. $\mathcal{S}' := \mathcal{S}$; $T' := T$;
2. If a set appears more than once in $\mathcal{S}'$ then remove every occurrence of it and remove from T' all the elements it covered.
3. Update the sets in $\mathcal{S}'$ so that they do not contain elements which have been removed from T'.
4. Repeat from step 2 until $\mathcal{S}'$ remains unchanged.

For example, 2-fold cover $\mathcal{S} = (\{a,b,c\}, \{a,b,d\}, \{c,d\}, \{c,d\}, \{a,e,f\}, \{e,g\}, \{f,g\})$ for set $T = \{a,b,c,d,e,f,g\}$ successively becomes $\mathcal{S}' = (\{a,b\}, \{a,b\}, \{a,e,f\}, \{e,g\}, \{f,g\})$, $T' =$

$\{a,b,e,f,g\}$ and finally $\mathcal{S}' = (\{e,f\},\{e,g\},\{f,g\})$, $T' = \{e,f,g\}$. Note that the reduced cover remains k-fold. (It should be pointed out that the reduced cover may be empty, i.e. $\mathcal{S}' = \emptyset$, $T' = \emptyset$. Nevertheless, this does not pose a problem if it is agreed that such an instance forms a k-fold cover for any k.)

Lemma 3.1.1 *A cover is factorizable into two simple covers iff its corresponding reduced cover is factorizable into two simple covers.*

PROOF As was just introduced, let pairs $\mathcal{S}$,T and $\mathcal{S}'$,T' respectively represent the instance of the cover and of its reduced cover.
$\Rightarrow$: Let $\{\mathcal{S}_1.\mathcal{S}_2\}$ be a factorization of $\mathcal{S}$ into two simple covers for T. Define $\mathcal{S}_1' \stackrel{\text{def}}{=} \{S_i \cap T' : S_i \in \mathcal{S}_1\}$ and $\mathcal{S}_2' \stackrel{\text{def}}{=} \{S_i \cap T' : S_i \in \mathcal{S}_2\}$. Note that $\mathcal{S}' \equiv \{S_i \cap T' : S_i \in \mathcal{S}\}$. It follows from the definitions that $\{\mathcal{S}_1', \mathcal{S}_2'\}$ is a factorization of $\mathcal{S}'$ into two simple covers for T'.
$\Leftarrow$: Let $\{\mathcal{S}_1', \mathcal{S}_2'\}$ be a factorization of $\mathcal{S}'$ into two simple covers for T'. The idea is to complete families $\mathcal{S}_1'$ and $\mathcal{S}_2'$ into covers for T with sets from $\mathcal{S} \setminus \mathcal{S}'$ chosen to cover $T \setminus T'$ since every set in $\mathcal{S} \setminus \mathcal{S}'$ appears at least twice and so identical sets can be distributed to $\mathcal{S}_1'$ and $\mathcal{S}_2'$.

Because the sets are updated while computing the reduced cover, getting the two covers for T actually involves performing the reduced-cover procedure backwards in order to recuperate the original sets from $\mathcal{S}$: sets are gradually reaugmented to their initial form as reintroduced duplicate sets get distributed between the two covers and removed elements reappear. $\square$

According to lemma 3.1.1, if a k-fold cover containing duplicate sets is not factorizable into two simple covers then there exists another k-fold cover (its reduced cover) which cannot be factorized either. It is therefore sufficient when investigating $f(2)$ to consider covers in which the covering sets are distinct. Applying this to the point-line covering problem, two sets now have at most one element in common. The corresponding hypergraph thus becomes easier to formulate: $H = (T, \mathcal{S})$, $|S_i \cap S_j| \leq 1 \quad \forall S_i, S_j \in \mathcal{S}, i \neq j$. This is not a total characterization of the problem though, because some of these hypergraphs do not represent covers of points by straight lines. The geometry inherent to the problem, namely that there exists a placement of the vertices of the hypergraph on the plane in which vertices belonging to the same edge are collinear, does not find a translation in the combinatorial language. A perfect example of this flaw within the hypergraph formulation is the Fano plane already encountered in chapter 2. Any representation of it includes at least one edge which is not straight (see figure 3 in chapter 2) even though no pair of edges have more than one common vertex.

3.2 Bounds on $f(2)$

The hypergraph approach nevertheless embodies necessary conditions for these constrained covers and might thus provide an upper bound for $f(2)$ or at least orient the search for answers. Since the transposition of the problem involves the dual hypergraph, the constraint must be dualized as well. From *two sets have at most one element in common* it becomes *two elements are contained in at most one common set* which really means the same thing. The first version will be used because it is more convenient.

This yields a more restrictive version of what was introduced at the beginning of section 2.2: the smallest integer k such that every hypergraph $H = (V, \mathcal{E})$ with $|E| \geq k \;\; \forall E \in \mathcal{E}$ and $|E_i \cap E_j| \leq 1 \;\; \forall E_i, E_j \in \mathcal{E}, i \neq j$, is 2-colorable.

It has been pointed out before that $f(2) > 2$. The Fano plane would be a good candidate to show $f(2) > 3$ if its dual admitted a straight-line planar embedding but as mentioned earlier it is its own dual. Fortunately, a minor modification of the Fano plane produces a dual which is drawable in straight lines while preserving the relevant features.

Theorem 3.2.1 *$f(2) > 3$ for point-line covers.*

PROOF Consider the hypergraph with edges $(\{x,u,z\}, \{x,v,y\}, \{y,w,x\}, \{x,t,w\}, \{y,t,u\}, \{z,t,v\}, \{u',v,w\}, \{u.v',w\}. \{u,v,w'\}, \{u',v',w'\})$ as represented in figure 2. It can be obtained from the Fano plane by replacing edge $\{u,v,w\}$ by $\{u',v,w\}, \{u,v',w\}, \{u,v,w'\}$ and $\{u',v',w'\}$.

Each of its edges has size 3 and no two share more than one vertex. Furthermore it is not 2-colorable, as will be argued now. Assigning colors to $\{t,u,v,w,x,y,z\}$ first, it is not hard to see that u, v and w must have the same color for the assignment to be valid. But then u', v' and w' must share the other color because of edges $\{u',v,w\}$, $\{u,v',w\}$ and $\{u,v,w'\}$, making $\{u',v',w'\}$ monochromatic and the coloring invalid.

Finally, its dual can be drawn with straight lines in the plane, as shown in figure 3. $\square$

It was suggested before that results were more elegant in the dual and this last situation serves to illustrate it. Obtaining the 3-fold cover directly seems to require a lot more insight.

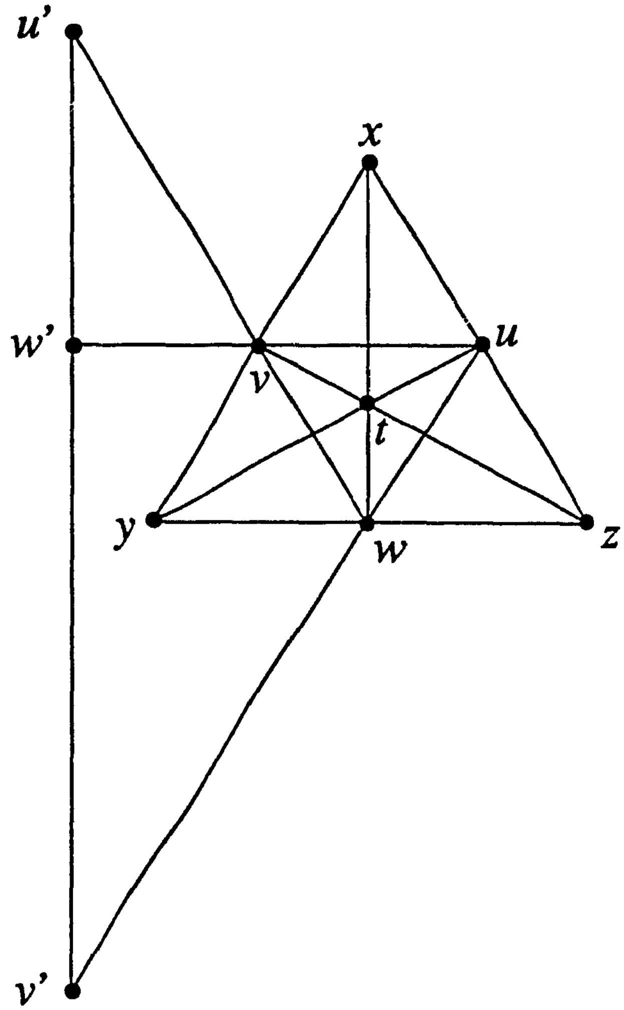

Figure 2: A modification of the Fano plane for theorem 5.2.

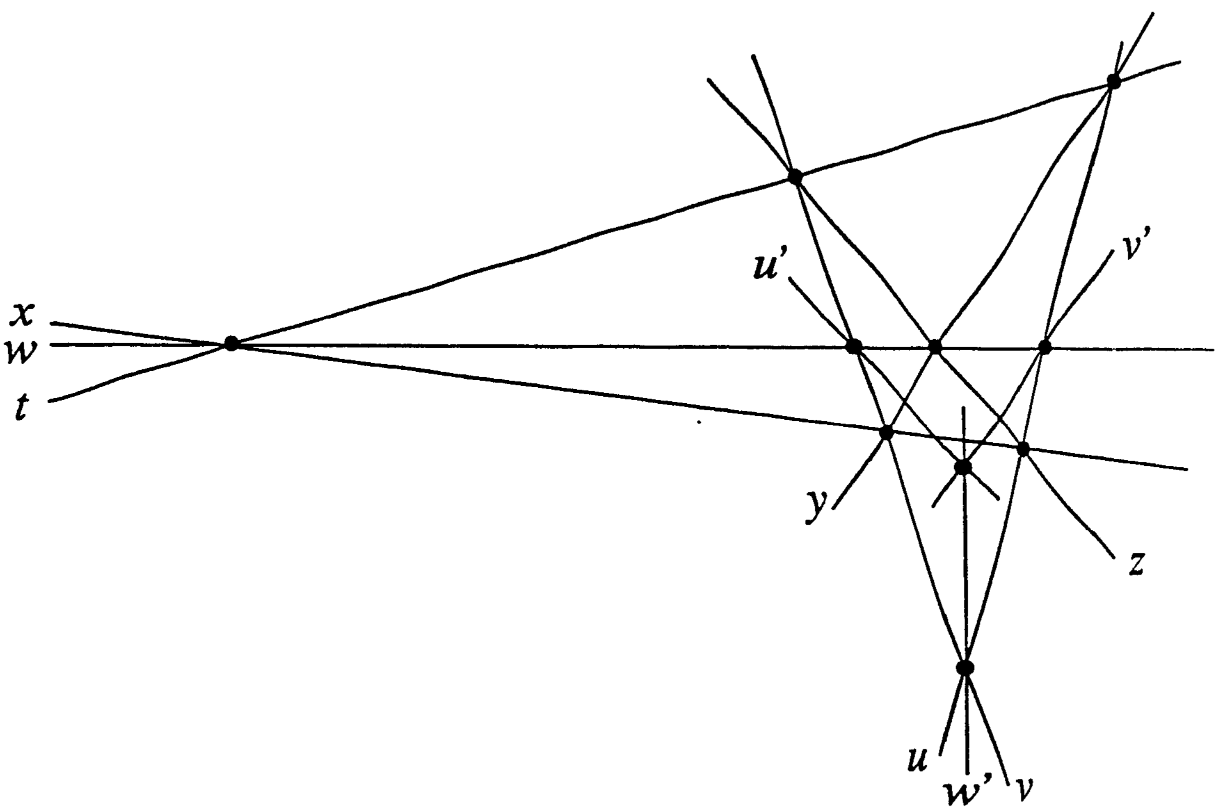

Figure 3: A 3-fold cover of points by lines which is not factorizable into two simple covers.

The next result requires knowledge of Ramsey theory. An *r-subset* is a subset of size r. A K_n^r is a hypergraph on n vertices whose edges are all the r-subsets of its vertex set. A special case of *Ramsey's Theorem* ([G80]) says that given integers n', r, there always exists an integer n large enough that if to each edge of a K_n^r is assigned one of 2 colors, a n'-subset of the vertices whose spanned edges are all of the same color can be found. That integer n is called a *Ramsey number* and denoted $R(n', r)$. An elegant special case of the theorem states that if a graph contains sufficiently many vertices ($\geq R(n', 2)$) then it must contain either a complete set or an independent set of vertices of size n'.

Theorem 3.2.2 *For any integer $k > 1$, there exists a hypergraph $(V, \mathcal{E})$ with $|E| \geq k \ \ \forall E \in \mathcal{E}$ and $|E_i \cap E_j| \leq 1 \ \ \forall E_i, E_j \in \mathcal{E},\ i \neq j$, which is not 2-colorable.*

PROOF This is a direct consequence of a very nice application of Ramsey theory by Lovász [L79, problem 14.24].

Starting from $X = \{1, 2, \ldots, R(k, k-1)\}$, construct $V = \{v \subseteq X : \ |v| = k - 1\}$ and $\mathcal{E} = \{E \subseteq V : |E| = k, |\bigcup_{v \in E} v| = k\}$, the vertices and edges of the hypergraph. In other words, to each $(k-1)$-subset of X corresponds a vertex of V and to each k-subset of X corresponds an edge of E composed of the k vertices ($(k-1)$-subsets) spanned by it.

a) $|E| \geq k \ \ \forall E \in \mathcal{E}$: every edge has size k by definition.

b) $|E_i \cap E_j| \leq 1 \ \forall E_i, E_j \in \mathcal{E},\ i \neq j$: two distinct k-subsets have at most $k-1$ elements in common and so two distinct edges have at most one vertex in common.

c) $(V, \mathcal{E})$ is not 2-colorable, because X was chosen to have size $R(k, k-1)$: according to Ramsey theory, in any 2-coloring of the edges of a $K_{R(k,k-1)}^{k-1}$ (i.e. vertices of V), there exists a k-subset of the vertices of the $K_{R(k,k-1)}^{k-1}$ (i.e. elements of X) all of whose $(k-1)$-subsets (i.e. spanned vertices of V) are of the same color. To this k-subset corresponds an edge of E which is thus monochromatic. □

Theorem 3.2.2 can hopefully be applied to factorizations of covers. The first two of these Ramsey numbers are $R(2,1) = 3$ and $R(3,2) = 6$. Use of the former produces the familiar prime 2-fold cover of figure 2 from chapter 1. The case $k = 3$ will be looked at in more detail.

From $X = \{1, 2, \ldots, 6\}$ are constructed $V = \{\{1,2\}, \{1,3\}, \ldots, \{5,6\}\}$ and $\mathcal{E} = \{ \{\{1,2\}, \{1,3\}, \{2,3\}\}, \{\{1,2\}, \{1,4\}, \{2,4\}\}, \ldots, \{\{4,5\}, \{4,6\}, \{5,6\}\}\}$. In all, $\binom{6}{2} = 15$ vertices and $\binom{6}{3} = 20$ edges. The corresponding 3-fold cover thus involves 20 points and

15 lines, which is considerably larger than the one constructed in theorem 3.2.1. It is a very regular cover in which every point is covered by exactly 3 lines and every line covers exactly 4 points. Nevertheless, to represent a point-line cover an embedding with straight lines must be found.

There are in fact two major problems in using theorem 3.2.2 to improve the lower bound on $f(2)$. First of all, Ramsey numbers $R(k, k-1)$, $k > 3$ are not known. Existence of these numbers would be enough if it weren't for the second problem: showing a suitable embedding. That alone seems to be a considerable task which is only complicated by the lack of information on the size of the cover. The matter thus remains unsettled.

3.3 A Special Case

As in section 2.2, considering a special case of the covering problem proves fruitful.

Theorem 3.3.1 *If each pair of points lies on at least one line then* $f(2) = 3$.

PROOF Using lemma 3.1.1, it is sufficient to consider covers in which each pair of points lies on exactly one line. Then, the situation is identical to that of theorem 2.2.2, except for the added geometric constraint. The result used in the proof of that theorem still applies, giving as before two types of forbidden hypergraphs. Recall they both have a isomorphic dual, so that they can be directly examined for a proper embedding. The Fano plane must be discarded, leaving the hypergraphs of figure 2 which have edges of size 2. Consequently a lower bound of 3 on the size of the edges is sufficient. □

While the exact value of $f(2)$ is known for this special case, the question for point-line covers in general is still open. It has been shown that $f(2)$ must be at least 4 and might not even exist if the covers associated with theorem 3.2.2 can be embedded in the plane with straight lines.

Chapter 4

Covering Simple Polygons

4.1 Introduction

In an art gallery, guards are hired to handle security. They are to be sitting down and watch over everything they see around them. Because in this comfortable position they might fall asleep, guards should be placed so that every part of the art gallery is under surveillance by at least k of them. How many guards are needed?

A few definitions are needed to formalize this minimization problem. The sequence of points $x_1, x_2, \ldots, x_n$ ($n \geq 3$) in E^2 defines a *polygon* $P = [x_1, x_2, \ldots, x_n]$ consisting of its n vertices $x_1, x_2, \ldots, x_n$ and n edges (line segments) $[x_i, x_{i+1}]$, $i = 1, 2, \ldots, n-1$, and $[x_n, x_1]$, i.e. a closed polygonal curve. If no two non-consecutive edges intersect, the polygon is *simple* and has a well-defined interior and exterior. Henceforth, the term "polygon" will designate a simple polygon and its interior.

The covered set T will be an infinite set of points in the plane described by a polygon whose number of vertices n will serve as a measure of the size of T. The constraint on the family $\mathcal{S}$ covering T requires now that it must be composed of subsets of T sharing a specific property. Two such properties will be considered for their relevance to applications and link to other research — they will be described shortly.

The problem consists of finding a minimum size family $\mathcal{S}$ forming a cover for T defined by a polygon — the smallest cover given a polygon, for short. The covering sets will either be convex or star-shaped. Those properties make the problem become non-trivial.

A set of points is *convex* if for every two points x, y in the set, the line segment $[x, y]$ also lies completely in the set. Because convex sets are simple and in a way elegant, they

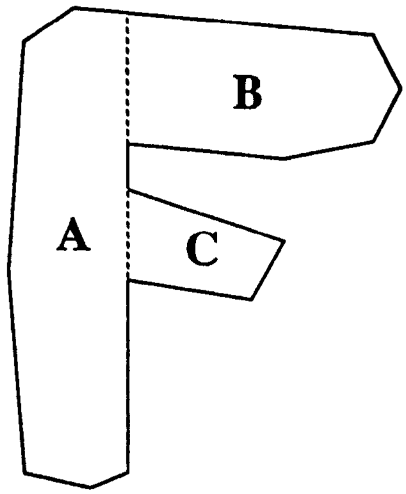

Figure 1: Convex sets A, B and C form a smallest cover for the polygon.

are popular for polygon decomposition. The decomposition of potentially complex polygons into simpler components has been proven very useful in developing efficient algorithms for various problems ([KS85]). As an example, triangulation, the partition of a polygon into triangles, can often be found as a preprocessing step. The smallest convex cover (cover by convex sets) constitutes an ideal decomposition for such things as feature extraction in pattern recognition. Figure 1 shows a smallest convex cover for a rendering of the letter "F".

A set of points is *star-shaped* if there exists a point x such that for every point y in the set, $[x, y]$ lies completely in the set. Star-shaped covers find correspondence in "Art Gallery" problems, which have a somewhat different formulation ([Chv75,F78,O87]). They seek the smallest number of points from which the whole polygon is visible, where two points are visible from each other if the line segment joining them does not intersect the exterior of the polygon. It is said that the polygon is *guarded* by those points. The set of points visible from a point x is star-shaped and said to be *generated* by x. Figure 2 illustrates the equivalence.

As it was hinted in the example with the watchmen, smallest k-fold covers for polygons offer an added security quantified by the choice of k. One way to look at it is that removing any $k-1$ of the covering sets still leaves the polygon fully covered, reminiscent of the motivation behind k-connected graphs ([Har69]).

It will now be seen how factorization comes into play with minimization problems. Computing minimum cardinality convex and star-shaped 1-fold covers for polygons have

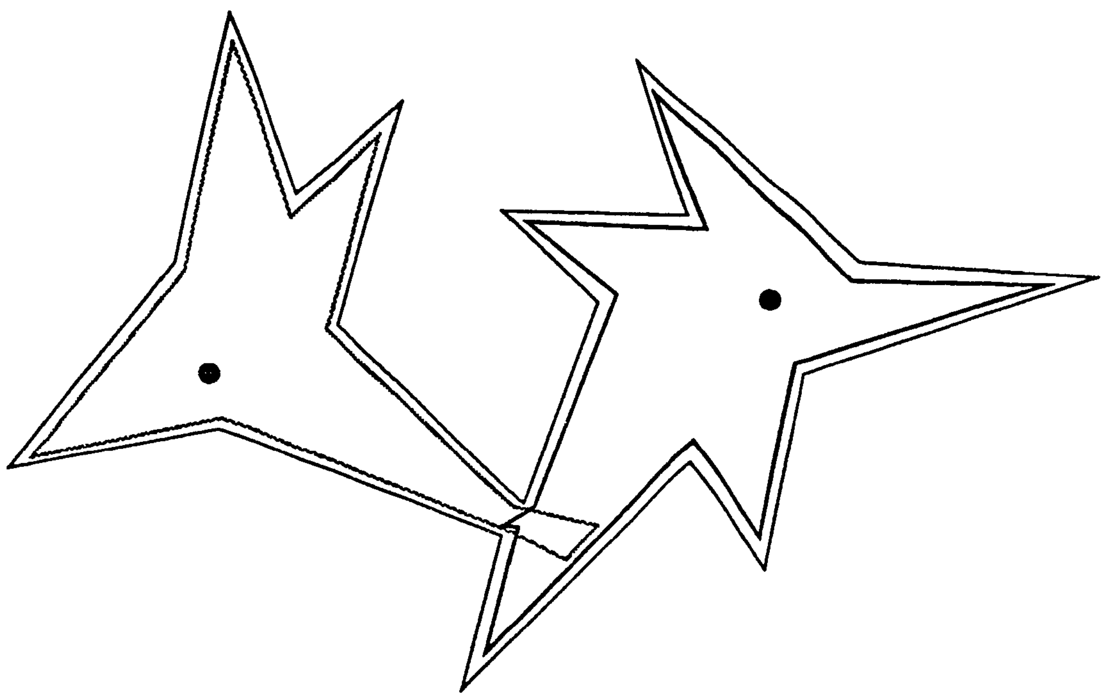

Figure 2: The polygon is guarded by the two points or alternatively covered by the two corresponding star-shaped sets.

both been proven *NP-hard* ([LL86,CR88,Sh89]). Obviously by generalizing to k-fold covers the problems remain at least as hard. Nevertheless if additional information is available, such as thinner covers of minimum size for the same polygon, a solution may be easier to obtain. Can a minimum size k-fold cover for a polygon always be achieved through some combination of the thinner covers? In other words, do all smallest k-fold covers admit a perfect factorization?

4.2 Preliminary Results

Before going on, a few things should be defined. Keeping with the tradition of the "art gallery" approach and the notation appearing in [Pe88], $G_k^c(P)$ and $G_k^s(P)$ will respectively represent the cardinality of the smallest convex and star-shaped k-fold covers for polygon P. *Naive covering*, be it by convex or star-shaped sets, will refer to a covering strategy consisting of building a k-fold cover from a 1-fold cover by making k copies of each set. It is a trivial example of a combination of thinner covers.
Tight upper bounds exist for $G_1^c(\cdot)$ and $G_1^s(\cdot)$ in the literature ([Chv75,F78,Cha80]).

Fact 4.2.1 $n-2$ *convex sets are sometimes necessary and always sufficient to cover a polygon with n vertices.*

Fact 4.2.2 $\lfloor \frac{n}{3} \rfloor$ *star-shaped sets are sometimes necessary and always sufficient to cover a polygon with n vertices.*

The "necessary" part of the statements is easily shown by exhibiting polygons requiring that many sets. The polygon of figure 3 needs a different set for each edge of its concave part. In figure 4, each prong needs its set. It shouldn't be surprising that the bound on convex covers is considerably higher since convex sets are a subclass of star-shaped sets and hence not as powerful.
These bounds can be generalized for $G_k^c(\cdot)$ and $G_k^s(\cdot)$.

Fact 4.2.3 $k(n-2)$ *convex sets are sometimes necessary and always sufficient to form a k-fold cover for a polygon with n vertices.*

Fact 4.2.4 $k \cdot \lfloor \frac{n}{3} \rfloor$ *star-shaped sets are sometimes necessary and always sufficient to form a k-fold cover for a polygon with n vertices.*

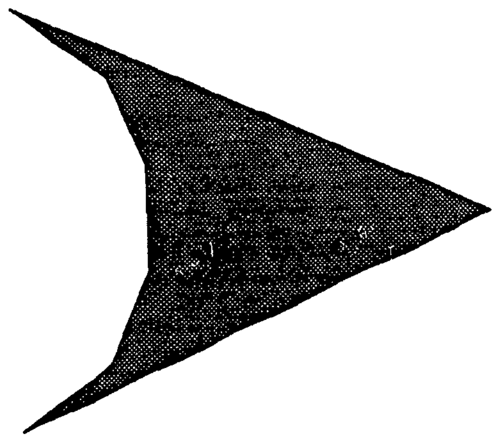

Figure 3: An extremal polygon for convex covers.

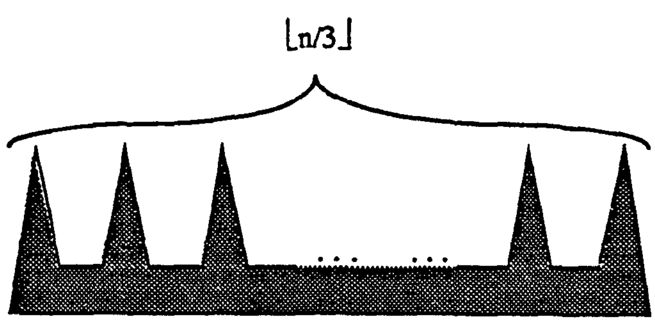

Figure 4: An extremal polygon for star-shaped covers.

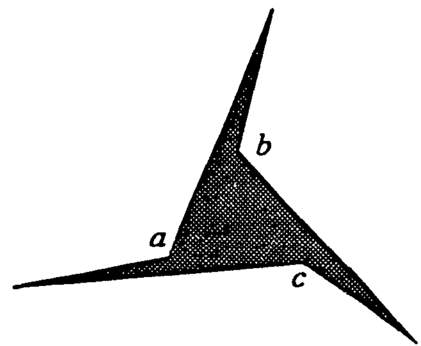

Figure 5: A polygon with a prime star-shaped 2-fold cover.

The same two figures can be used for necessity — sufficiency follows from naive covering using facts 4.2.1 and 4.2.2.

Those extremal cases should not be taken as an indication that finding minimum size k-fold covers reduces to finding minimum size 1-fold covers, though. For example, the polygon of figure 5 is not star-shaped and so requires at least two star-shaped sets to form a cover. Nevertheless, the three star-shaped sets generated by vertices a, b and c form a 2-fold cover for the polygon, a minimum, whereas doubling a 1-fold cover yields a cover of size at least four. This polygon simply embodies the familiar cover of figure 2 from chapter 1.

4.3 $SPUR_{l,m}$ Polygons

Next is described a family of polygons which will be at the source of following results. They resemble the "spur" polygons mentioned in [Sh89] in that they have spikes arranged in a circular fashion, though they carry additional constraints reflected in the width of their spikes.

The polygon $SPUR_{l,m}$ has l spikes and each sequence of m consecutive spikes is visible from some point. The description of $SPUR_{l,m}$ will be much more precise and restrictive than it needs to be — the essential characteristics sought are not exclusive to an exactly determined polygon and many variations will still retain them, but it was chosen to fix a representative which is of convenient manipulation to ease the proofs. A construction is given next, performing correctly for $l \geq m$; $l \geq 3$, $m \geq 2$, the only meaningful range,

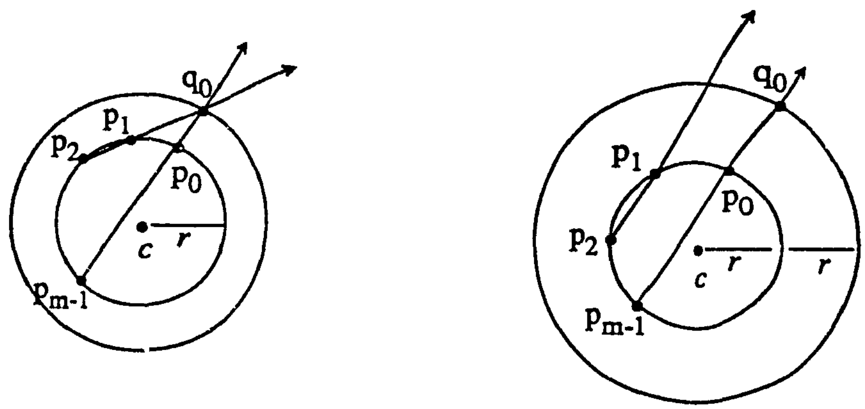

Figure 6: Step 3 of the construction of $SPUR_{l,m}$, $m > 2$.

but proceeding in a slightly different way for $m = 2$. Indices on points and vertices of the construction should be taken *modulo* l. A line going through points p and q will be denoted (p,q) and will have an orientation $\overrightarrow{pq}$. A half-line starting at p and going through q will be denoted $[p,q)$.

Construction:

$m > 2$:

1. Draw a circle of radius r centered at c.

2. Put l points on it, equally spaced. From an arbitrary starting point, label them $p_0, p_1, \ldots, p_{l-1}$, in counterclockwise order.

Because $p_0, p_1, \ldots, p_{l-1}$ are evenly distributed on the circle, $SPUR_{l,m}$ will be symmetrical around c and the construction of each spike will be identical.

3. If half-lines $[p_{m-1}, p_0)$ and $[p_2, p_1)$ intersect, call their point of intersection q_0 and draw the circle through it which is centered at c. Otherwise, draw a circle of radius $2r$ centered at c and call q_0 the intersection of that circle with $[p_{m-1}, p_0)$. (Figure 6 depicts the two alternatives.)

4. For $i = 1, 2, \ldots, l-1$, put point q_i at the intersection of the newly drawn circle and $[p_{i+m-1}, p_i)$.

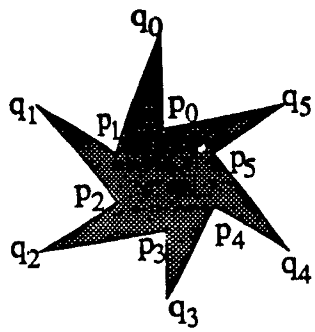

Figure 7: A $SPUR_{6,4}$ with its labelled vertices.

5. $SPUR_{l,m} = [p_0, q_0, p_1, q_1, \ldots, p_{l-1}, q_{l-1}]$. Figure 7 illustrates the case $l = 6$, $m = 4$.

$m = 2$:

1. Draw a circle of radius r centered at c.

2. Put l points on it, equally spaced. From an arbitrary starting point, label them $c_0, c_1, \ldots, c_{l-1}$, in counterclockwise order.

3. Draw a circle of radius $2r$ centered at c.

4. For $i = 0, 1, \ldots, l-1$, put point q_i at the intersection of the newly drawn circle and $[c_{i+1}, c_i)$.

5. For $i = 0, 1, \ldots, l-1$, put point p_i on $[c_{i+1}, c_i)$ just outside the inner circle (see figure 8). Points p_i are introduced to avoid having overlapping edges $[c_i, q_i]$, $[q_i, c_{i+1}]$.

6. $SPUR_{l,2} = [p_0, q_0, p_1, q_1, \ldots, p_{l-1}, q_{l-1}]$. Figure 9 illustrates the case $l = 4$.

For the sake of uniformity, points p_i in the construction for $m > 2$ will also be referred to as c_i. This way, p_i will always refer to some vertex of $SPUR_{l,m}$, and c_i to a point on the circle of radius r, regardless of the value of m.

It is now argued that the polygon resulting from the construction is simple. Define R_i to be the region delimited by $[c_{i+1}, q_{i+1}]$, $[c_i, c_{i+1}]$, $[c_i, q_i]$ and $arc(q_i, q_{i+1})$, but excluding the first segment (see figure 10) — $R_0, R_1, \ldots, R_{l-1}$ are pairwise disjoint.

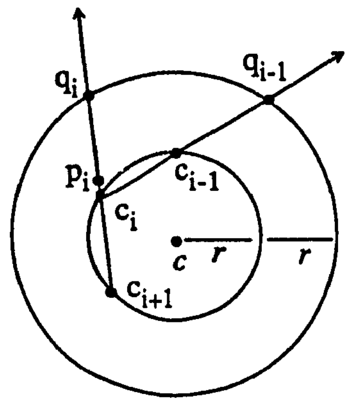

Figure 8: Construction of $SPUR_{l,2}$.

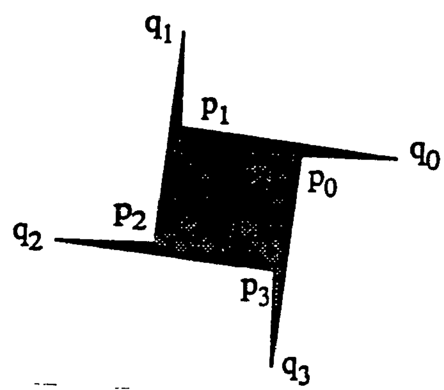

Figure 9: A $SPUR_{4,2}$ with its labelled vertices.

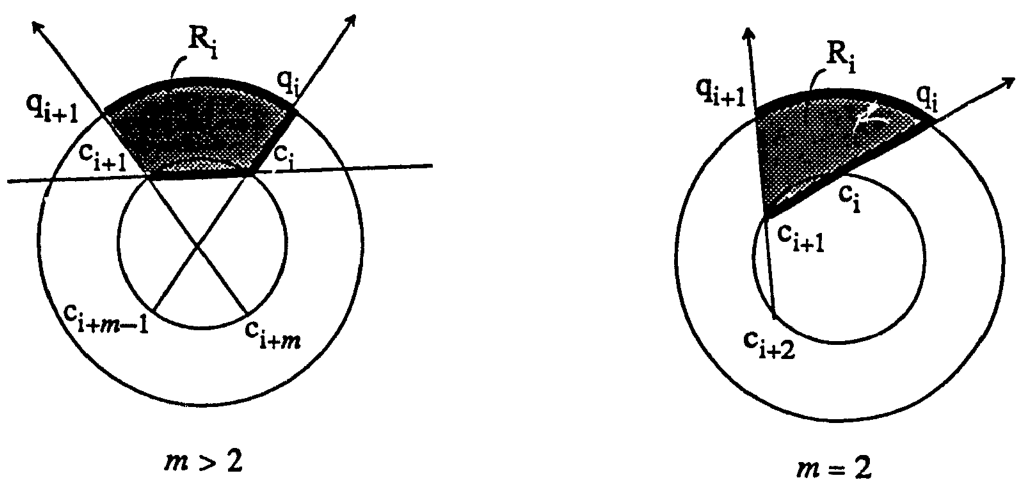

Figure 10: Region R_i for the proof of simplicity. (The light boundary edges are open.)

Edges $[p_i, q_i]$ and $[q_i, p_{i+1}]$ lie in R_i. Hence none of the edges intersect (except at the endpoints for consecutive edges) and $SPUR_{l,m}$ is simple.

4.4 Properties of $SPUR_{l,m}$

The next few results will serve to determine the size of smallest convex and star-shaped covers for $SPUR_{l,m}$. Let W_i be the wedge located in R_i, i.e. the closed region bounded by triangle $[p_i, p_{i+1}, q_i]$. Call O the closed region bounded by polygon $[p_0, p_1, \ldots, p_{l-1}]$. $W_0, W_1, \ldots, W_{l-1}$ and O represent a partition of $SPUR_{l,m}$ (not in a strict sense since they share boundaries, though). A region (i.e. set of points) will be considered visible from a point p if every point of that region is visible from p. $H_L(p,q)$ and $H_R(p,q)$ will respectively represent the left and right half-planes associated with line (p,q).
First, a simple geometric fact is stated without proof.

Fact 4.4.1 *Consider a chord of a circle, splitting it into two arcs. If the arcs have equal length then the center of the circle lies on the chord. Otherwise it lies on the side of the chord corresponding to the longer arc.*

Some properties of the family of polygons are given next.

Lemma 4.4.1 *Let $p \in O$. Wedge W_i is visible from p iff p lies in $H_R(q_i, p_i) \cap H_L(q_i, p_{i+1})$.*

PROOF
$\Rightarrow$: If W_i is visible from p then in particular q_i is. Since $[q_i, p]$ does not intersect the exterior of $SPUR_{l,m}$, it must lie in between edges $[q_i, p_i]$ and $[q_i, p_{i+1}]$. Equivalently p must lie in $H_R(q_i, p_i) \cap H_L(q_i, p_{i+1})$.
$\Leftarrow$: Consider a point $q \in W_i$. Both p and q are contained in $(O \cup W_i) \cap (H_R(q_i, p_i) \cap H_L(q_i, p_{i+1}))$, a convex region included in the polygon and so $[p,q]$ does not intersect the exterior. Hence W_i is visible from p. □

Lemma 4.4.2 *If $l \leq 2m - 2$ then $SPUR_{l,m}$ represents a star-shaped set.*

PROOF Note that since $l \geq 3$, the case $m = 2$ may be discarded. $SPUR_{l,m}$ will be demonstrated star-shaped by proving that it is visible from point c, the center of the construction.

First, convex set O contains c and so is visible from it. Remains to show that W_i is visible from c, $i = 0, 1, \ldots, l-1$. Using lemma 4.4.1, it is equivalent to showing $c \in H_R(q_i, p_i) \cap H_L(q_i, p_{i+1})$. Note that $H_R(q_i, p_i) \cap H_L(q_i, p_{i+1}) \supseteq H_R(q_i, p_i) \cap H_L(q_i, p_{i+1}) \cap O \supseteq H_R(c_i, c_{i+m-1}) \cap H_L(c_{i+1}, c_{i+2}) \cap O$, from the construction. Point c is in O — it is sufficient to show $c \in H_R(c_i, c_{i+m-1}) \cap H_L(c_{i+1}, c_{i+2})$.

Consider directed chord $[c_{i+1}, c_{i+2}]$. Points $c_{i+3}, \ldots, c_{l-1}, c_0, \ldots, c_i$ are all on its left and so, since points are equally spaced, the left arc is longer. Fact 4.4.1 says that $c \in H_L(c_{i+1}, c_{i+2})$.

Consider next directed chord $[c_i, c_{i+m-1}]$. Points $c_{i+1}, \ldots, c_{i+m-2}$ are on its right and $c_{i+m}, \ldots, c_{l-1}, c_0, \ldots, c_{i-1}$ on its left. In all, $m-2$ points on the right arc and $l-(m-2)-2 = l-m \leq m-2$ on the left arc. So either the arcs have equal length and c lies on the chord or the right arc is longer and c lies on that side, still using fact 4.4.1. In either case, $c \in H_R(c_i, c_{i+m-1})$.

Hence $SPUR_{l,m}$ is star-shaped. □

Let $Q = \{q_0, q_1, \ldots, q_{l-1}\}$, the set of vertices of $SPUR_{l,m}$ which are at the apex of spikes.

Lemma 4.4.3 *If $l > 2m-2$ then none of the elements of Q are visible from c.*

PROOF Consider directed chord $[c_i, c_{i+m-1}]$ as in lemma 4.4.2. This time $l - m > m - 2$ and so the left arc is longer. Therefore $c \notin H_R(c_i, c_{i+m-1})$ and $[c, q_i]$ intersects the exterior of the polygon. □

Lemma 4.4.4 *If $l > 2m-2$ then a star-shaped set restricted to $SPUR_{l,m}$ covers at most m elements of Q.*

PROOF By contradiction. Suppose there exists a point x such that the star-shaped set X generated by x covers at least $m+1$ elements from Q.

First will be established the existence, among those elements covered by X, of a *distant pair* $\{q_i, q_j\}$ such that q_j appears between $q_{i+(m-1)}$ and $q_{i-(m-1)}$ in a counterclockwise traversal of the boundary of the circle on which they lie, i.e. $q_j \in \{q_{i+(m-1)}, q_{i+m}, \ldots, q_{i-(m-1)}\}$.

W.l.o.g. say $q_{m-2} \in X$. If one of $\{q_{2m-3}, q_{2m-2}, \ldots, q_{l-1}\}$ is also contained in X then a distant pair is present. Otherwise, X must contain at least m of $Q_1 \cup Q_2$, where $Q_1 = \{q_0, q_1, \ldots, q_{m-3}\}$, $Q_2 = \{q_{m-1}, q_m, \ldots, q_{2m-4}\}$, $|Q_1| = |Q_2| = m-2$. Certainly, $\{q_0, q_{m-1}\}$

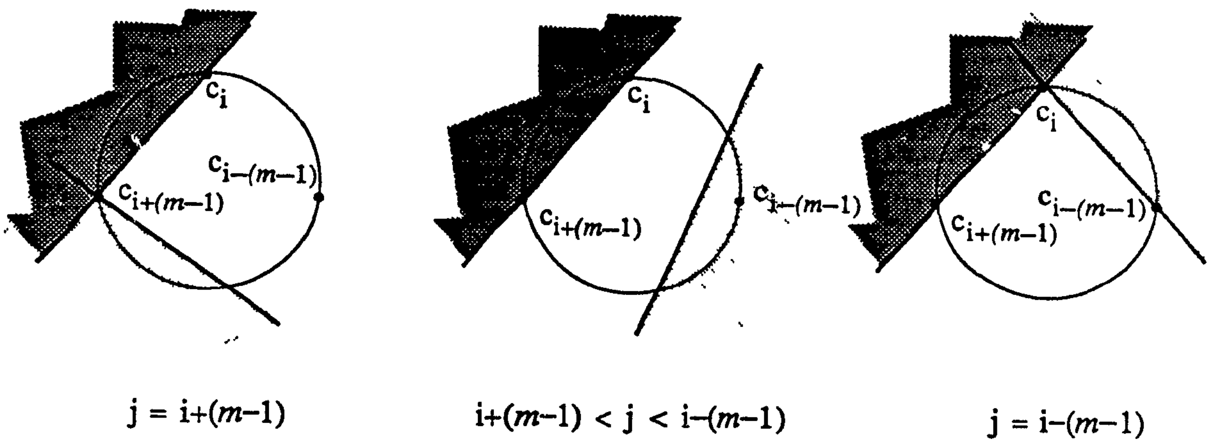

Figure 11: How $H_R(c_i, c_{i+m-1})$ and $H_R(c_j, c_{j+m-1})$ can intersect.

form a distant pair and so do $\{q_1, q_m\}, \ldots, \{q_{m-3}, q_{2m-4}\}$. Now suppose s elements of Q_1 are contained in X — at least $m - s$ of Q_2 must then be part of X but no more than $m-2-s$ can be chosen without introducing a distant pair. Therefore X includes a distant pair — call it $\{q_i, q_j\}$.

Since q_i and q_j are both visible from x, it must lie in $H_R(q_i, p_i) \cap H_R(q_j, p_j) \equiv H_R(c_i, c_{i+m-1}) \cap H_R(c_j, c_{j+m-1})$, in particular. Because of the restrictions on j, those two half-planes do not intersect inside the inner circle of the construction and can only intersect on the circle at c_i or c_j (see figure 11).

Point x cannot lie on c_i or c_j since each of them is included in exactly m of the right half-planes associated with elements of Q, by construction. and so at most m would be visible from it.

For x to lie outside the inner circle, it must be in W_i or W_j (for $m = 2$, a vertex p_i is very close to point c_i) — w.l.o.g. say it is in W_i (see figure 12). But because of edge $[p_i, q_i]$, q_j cannot possibly be visible from x. Hence such an x does not exist. □

Lemma 4.4.5 *A convex set restricted to $SPUR_{l,m}$ covers at most $\lfloor \frac{l}{l-m+2} \rfloor$ elements of Q.*

PROOF It is sufficient to show that $q_{i+1}, q_{i+2}, \ldots, q_{i+l-m+1}$ are not visible from q_i, $i = 0, 1, \ldots, l-1$, for then a convex set at best covers every $(l-m+2)^{th}$ element of Q, i.e. at most $\lfloor \frac{l}{l-m+2} \rfloor$. Let $[p, q)'$ denote the half-line open at p and going away from q, so that

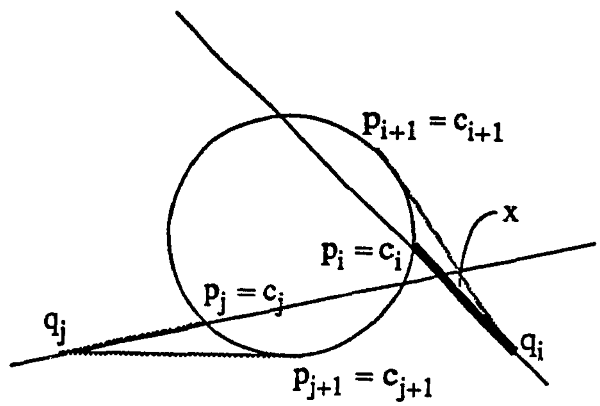

Figure 12: A point x in the intersection of spike W_i and the two half-planes cannot see q_j from behind $[p_i, q_i]$.

$[p,q) \cup [p,q)' = (p,q)$. From the construction, vertex q_j lies on $[c_j, c_{j+m-1})'$ and is only visible from points in $H_R(c_j, c_{j+m-1})$. The strategy will be to argue that

$[c_i, c_{i+m-1})'$ does not lie in $H_R(c_{i+j}, c_{i+j+m-1})$ $(*)$
$\forall j \in \{1, 2, \ldots, l-m+1\}$.

Three useful observations should be made:
(1) $c_{i+m-1} \in H_R(c_{i+j}, c_{i+j+m-1})$, $\forall j \in \{1, 2, \ldots, m-1\}$.
(2) $c_i \notin H_R(c_{i+j}, c_{i+j+m-1})$, $\forall j \in \{1, 2, \ldots, l-m\}$.
(3) $(*)$ does not need to be shown for $j > \lfloor \frac{l}{2} \rfloor$. (Once proven that $q_{i+1}, q_{i+2}, \ldots, q_{i+\lfloor \frac{l}{2} \rfloor}$ are not visible from q_i, it is immediate that $q_{i+\lfloor \frac{l}{2} \rfloor+1}$ isn't either by applying the result with substitution $i \to i + \lfloor \frac{l}{2} \rfloor + 1$, and so forth.)

$m \geq \lceil \frac{l}{2} \rceil + 1$:

Since $l - m + 1 \leq l - (\lceil \frac{l}{2} \rceil + 1) + 1 = \lfloor \frac{l}{2} \rfloor \leq \lceil \frac{l}{2} \rceil \leq m - 1$, (1) gives $c_{i+m-1} \in H_R(c_{i+j}, c_{i+j+m-1})$, $\forall j \in \{1, 2, \ldots, l-m+1\}$. Combination with (2) immediately gives $(*)$ $\forall j \in \{1, 2, \ldots, l-m\}$. For $j = l - m + 1$, the corresponding $H_R(c_{i+l-m+1}, c_i)$ has c_i on its defining line and $(*)$ holds.

$m \leq \lceil \frac{l}{2} \rceil$:

$l - m + 1 \geq l - \lceil \frac{l}{2} \rceil + 1 = \lfloor \frac{l}{2} \rfloor + 1$ so using (3) only $j \in \{1, 2, \ldots, \lfloor \frac{l}{2} \rfloor\}$ will be considered. Since $\lfloor \frac{l}{2} \rfloor \leq l - m$, from (2) $c_i \notin H_R(c_{i+j}, c_{i+j+m-1})$, $\forall j \in \{1, 2, \ldots, \lfloor \frac{l}{2} \rfloor\}$. To obtain $(*)$, it suffices to show that $(c_{i+j}, c_{i+j+m-1})$ does not intersect $[c_i, c_{i+m-1})'$. Because $j \leq \lfloor \frac{l}{2} \rfloor$ and $m \leq \lceil \frac{l}{2} \rceil$, lines $(c_{i+j}, c_{i+j+m-1})$ and (c_i, c_{i+m-1}) are found either parallel or intersecting on $[c_i, c_{i+m-1})$. Hence $(*)$. □

Lemma 4.4.6 *$\forall i = 0, 1, \ldots, l-1$, the star-shaped set generated by p_{i+m-1} covers $W_i, W_{i+1}, \ldots, W_{i+m-1}$ and O.*

PROOF It will equivalently be shown that each of those regions is visible from p_{i+m-1}. Convex set O contains p_{i+m-1} so it is visible from the latter. Observe next that $p_j, p_{j+1}, \ldots, p_{j+m-1}$ lie in $H_R(q_j, p_j) \cap H_L(q_j, p_{j+1})$. It follows that p_{i+m-1} lies in $H_R(q_j, p_j) \cap H_L(q_j, p_{j+1})$, for $j = i, i+1, \ldots, i+m-1$. According to lemma 4.4.1, $W_i, W_{i+1}, \ldots, W_{i+m-1}$ are visible from p_{i+m-1}. □

For convenience, let last $= (\lfloor \frac{l}{l-m+2} \rfloor - 1)(l-m+2)$.
Define C_i as $[p_i, q_i, p_{i+1}, p_{i+l-m+2}, q_{i+l-m+2}, p_{i+l-m+3}, p_{i+2(l-m+2)}, q_{i+2(l-m+2)}, p_{i+2(l-m+2)+1}, \ldots, p_{i+\text{last}}, q_{i+\text{last}}, p_{i+\text{last}+1}] \cup [p_{i+\text{last}+1}, c, p_i]$, the union of two sets described by polygons. If point c lies in the first one then C_i amounts to that set, otherwise it can be described by the first polygon with vertex c added between existing vertices $p_{i+\text{last}+1}$ and p_i. The essential feature of C_i is that it includes every $(l-m+2)^{th}$ wedge of the polygon, starting with W_i.

Lemma 4.4.7 *C_i is a convex set.*

PROOF It will equivalently be shown that for every three consecutive (in counterclockwise order) vertices p, p', p'' of C_i, $p'' \in H_L(p, p')$. Because of the symmetry of the construction, there are only a few cases to consider.
p_j, q_j, p_{j+1}: it follows from $W_j = [p_j, p_{j+1}, q_j]$ being part of $SPUR_{l,m}$.
$q_j, p_{j+1}, p_{j'}$: by construction, $\forall j' \in \{0, 1, \ldots, l-1\}$, $p_{j'} \in H_L(q_j, p_{j+1})$.
$p_{j+1}, p_{j+l-m+2}, q_{j+l-m+2}$: $j+1 \equiv (j+l-m+2)+(m-1)$ so $q_{j+l-m+2}$ lies on $[p_{j+1}, p_{j+l-m+2})$, by construction.
If $c \in$ first set:
$p_{i+\text{last}+1}, p_i, q_i$: by construction, $p_i, p_{i+1}, \ldots, p_{i+m-1} \in H_R(q_i, p_i)$. Since last+1 $= (\lfloor \frac{l}{l-m+2} \rfloor - 1)(l-m+2) + 1 \leq m-1$, $p_{i+\text{last}+1} \in H_R(q_i, p_i)$, i.e. $q_i \in H_L(p_{i+\text{last}+1}, p_i)$.
Otherwise:

$p_{i+\text{last}+1}, c, p_i$: follows from c not being part of the first set.
$q_{i+\text{last}}, p_{i+\text{last}+1}, c$ and c, p_i, q_i: as seen in lemma 4.4.2 $SPUR_{l,m}$ is star-shaped from c, which implies that $c \in H_L(q_{i+\text{last}}, p_{i+\text{last}+1})$ and $q_i \in H_L(c, p_i)$.
Hence C_i is convex. □

With these lemmas in hand, it is possible to characterize smallest covers for $SPUR_{l,m}$.

Theorem 4.4.8 $G_k^s(SPUR_{l,m}) = \begin{cases} k & \textit{if } l \leq 2m-2 \\ \lceil \frac{kl}{m} \rceil & \textit{otherwise} \end{cases}$

PROOF
$l \leq 2m-2$: From lemma 4.4.2, $G_1^s(SPUR_{l,m}) = 1$ so naive covering yields $G_k^s(SPUR_{l,m}) \leq k$, which is obviously tight.
$l > 2m-2$: Recall $Q = \{q_0, q_1, \ldots, q_{l-1}\}$. Each of the l vertices in Q needs to be covered at least k times to achieve a star-shaped k- fold cover of $SPUR_{l,m}$ but from lemma 4.4.4 no more than m of Q can appear in a star-shaped set. Therefore $G_k^s(SPUR_{l,m}) \geq \lceil \frac{kl}{m} \rceil$.

An upper bound of $\lceil \frac{kl}{m} \rceil$ will be obtained for $G_k^s(SPUR_{l,m})$ by exhibiting an algorithm that produces a star-shaped k-fold cover of that size. Because of the previous lower bound, it is in fact a smallest cover.

Algorithm:

- $i := m-1$;
- $S = \emptyset$;
- *for* $j = 1$ *to* $\lceil \frac{kl}{m} \rceil$ *do begin*
  - *add the star-shaped set generated by* p_i *to* S;
  - $i := (i+m)$ *mod* l;
- *end*

Family S obviously has size $\lceil \frac{kl}{m} \rceil$ so it remains to prove it forms a k-fold cover. The result of lemma 4.4.6 will guarantee this.

Region O is covered by every set and so is $\lceil \frac{kl}{m} \rceil$-guarded, which is more than sufficient ($\lceil \frac{kl}{m} \rceil \geq \frac{kl}{m} \geq k$ since $l \geq m$).

The first set put in $\mathcal{S}$ covers $W_0, W_1, \ldots, W_{m-1}$. Each new set added covers the next m wedges in counterclockwise order ("$i := (i+m) \bmod l$;"). Therefore, the number of times each of $W_0, W_1, \ldots, W_{l-1}$ is covered corresponds to the number of cycles around $SPUR_{l,m}$ the algorithm goes through. Each of the $\lceil \frac{kl}{m} \rceil$ sets covers m of a cycle of length l:

$$\begin{aligned} \left\lceil \frac{kl}{m} \right\rceil \cdot \frac{m}{l} &\geq \frac{kl}{m} \cdot \frac{m}{l} \\ &= k \end{aligned}$$

Hence the algorithm achieves a k-fold cover. □

Theorem 4.4.9 $G_k^c(SPUR_{l,m}) = \begin{cases} \left\lceil \frac{kl}{\lfloor \frac{l}{l-m+2} \rfloor} \right\rceil & \text{if } l < 2m-3 \\ kl & \text{if } 2m-3 \leq l \leq 2m-2 \\ k(l+1) & \text{otherwise} \end{cases}$

PROOF Lower bounds are first established. The argument about covering the vertices in Q is the same as in theorem 4.4.8 and this time uses lemma 4.4.5. Hence $G_k^c(SPUR_{l,m}) \geq \left\lceil \frac{kl}{\lfloor \frac{l}{l-m+2} \rfloor} \right\rceil$.

In particular, if $l \geq 2m-3$ then $\frac{l}{l-m+2} < 2$ and so $G_k^c(SPUR_{l,m}) \geq kl$. If in addition $l > 2m-2$ then according to lemma 4.4.3 a convex set covering a vertex in Q cannot cover point c as well. Therefore separate sets are needed for c and $G_k^c(SPUR_{l,m}) \geq k(l+1)$.

Upper bounds matching the lower bounds will again be obtained through covering algorithms. For $l > 2m-2$, naive covering on $\{W_0, W_1, \ldots, W_{l-1}, O\}$ will do nicely. For $2m-3 \leq l \leq 2m-2$, naive covering on $\{[c, p_i, q_i, p_{i+1}] : i \in \{0, 1, \ldots, l-1\}\}$ forms a convex k-fold cover: each $[c, p_i, q_i, p_{i+1}]$ includes W_i and $[c, p_i, p_{i+1}]$ of O and so the family of sets covers the polygon — each set is the union of two triangles sharing edge $[p_i, p_{i+1}]$ and line segment $[c, q_i]$ lies inside the set since for $l \leq 2m-2$ $SPUR_{l,m}$ is star-shaped from c as seen in lemma 4.4.2, so each set is convex.

For $l < 2m-3$, the convex sets C_i of lemma 4.4.7 will be used in an algorithm essentially identical to the one appearing in the proof of theorem 4.4.8.

Algorithm:

- $i := 0;$

- $\mathcal{S} = \emptyset$;
- *for* $j = 1$ *to* $\lceil \frac{kl}{\lfloor \frac{l}{l-m+2} \rfloor} \rceil$ *do begin*
  - $\mathcal{S} := \mathcal{S} \cup \{C_i\}$;
  - $i := (i+1)$ *mod* l;
- *end*

Family $\mathcal{S}$ has size $\lceil \frac{kl}{\lfloor \frac{l}{l-m+2} \rfloor} \rceil$ and is claimed to form a convex k-fold cover for $SPUR_{l,m}$.

Note that for every vertex q_j of C_i, region $[c, p_j, q_j, p_{j+1}]$ is covered by C_i since the four vertices lie in the convex set. Since C_i covers $\lfloor \frac{l}{l-m+2} \rfloor$ such regions, one cycle of l iterations for counter i in the algorithm will construct a family $\mathcal{S}$ covering each region $\lfloor \frac{l}{l-m+2} \rfloor$ times. To form a k-fold cover the algorithm must go through $\lceil \frac{kl}{\lfloor \frac{l}{l-m+2} \rfloor} \rceil$ iterations, as specified. □

4.5 Prime Minimum Size Covers

With these two theorems in hand, it is now possible to determine the cardinality of minimum size convex and star-shaped k-fold covers for $SPUR_{l,m}$ without having to look at its geometry. It should be kept in mind though that there is a geometric construct obeying the formulas. All this will now be used to show that some polygons do not have a smallest k-fold cover which admits a perfect factorization.

Theorem 4.5.1 $\forall k \geq 2$, *there exists a polygon for which no smallest star-shaped k-fold cover admits a perfect factorization.*

PROOF Consider $SPUR_{2k-1,k}$. A smallest star-shaped k-fold cover for it will have size $2k-1$, from theorem 4.4.8:

$$\begin{aligned} G_k^s(SPUR_{2k-1,k}) &= \lceil \frac{k \cdot (2k-1)}{k} \rceil \\ &= 2k-1 \end{aligned}$$

If a perfect factorization of such a cover exists, it can be expressed as a combination of thinner covers satisfying $\sum_{i=1}^{k-1} a_i \cdot i = k$, where a_i stands for the number of times a i-fold

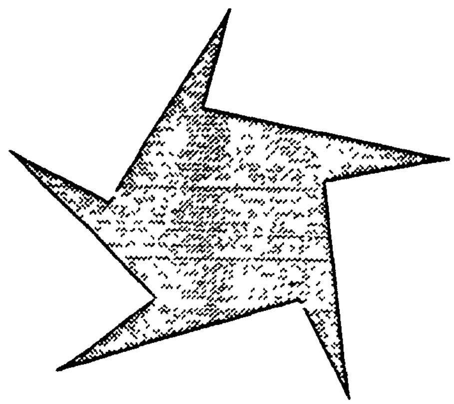

Figure 13: $SPUR_{5,3}$

cover appears in the combination. Still according to theorem 4.4.8, such a combination will add up to a family of size at least $2k$:

$$\begin{aligned}
\sum_{i=1}^{k-1} a_i \cdot G_i^s(SPUR_{2k-1,k}) &= \sum_{i=1}^{k-1} a_i \cdot \left\lceil \frac{i \cdot (2k-1)}{k} \right\rceil \\
&= \sum_{i=1}^{k-1} a_i \cdot \left\lceil 2i - \frac{i}{k} \right\rceil \\
&= \sum_{i=1}^{k-1} a_i \cdot 2i \\
&= 2 \cdot \sum_{i=1}^{k-1} a_i \cdot i \\
&= 2k
\end{aligned}$$

Hence every smallest star-shaped k-fold cover for $SPUR_{2k-1,k}$ is prime. □

For $k = 2$, $SPUR_{3,2}$ is the polygon of figure 5 whose smallest star-shaped 2-fold cover was already known to be prime. Figure 13 shows $SPUR_{5,3}$, for $k = 3$.

Theorem 4.5.2 $\forall k \geq 2$, *there exists a polygon for which no smallest convex k-fold cover admits a perfect factorization.*

PROOF Consider $SPUR_{2k+1,2k+1}$. A smallest convex k-fold cover for it will have size $2k+1$, from theorem 4.4.9:

$$
\begin{aligned}
G_k^c(SPUR_{2k+1,2k+1}) &= \lceil \frac{k\cdot(2k+1)}{k} \rceil \\
&= 2k+1
\end{aligned}
$$

Again, if a perfect factorization of such a cover exists, it can be expressed as a combination of thinner covers satisfying $\sum_{i=1}^{k-1} a_i \cdot i = k$, where a_i stands for the number of times a i-fold cover appears in the combination. Still according to theorem 4.4.9, such a combination will add up to a family of size at least $2k+2$:

$$
\begin{aligned}
\sum_{i=1}^{k-1} a_i \cdot G_i^c(SPUR_{2k+1,2k+1}) &= \sum_{i=1}^{k-1} a_i \cdot \lceil \frac{i\cdot(2k+1)}{k} \rceil \\
&= \sum_{i=1}^{k-1} a_i \cdot \lceil 2i + \frac{i}{k} \rceil \\
&= \sum_{i=1}^{k-1} a_i \cdot (2i+1) \\
&= 2 \cdot \sum_{i=1}^{k-1} a_i \cdot i + \sum_{i=1}^{k-1} a_i \\
&= 2k + \sum_{i=1}^{k-1} a_i \\
&\geq 2k+2
\end{aligned}
$$

($\sum_{i=1}^{k-1} a_i \geq 2$ for otherwise $a_j \leq 1$, $a_i = 0$ $i \neq j$, for some j and so $\sum_{i=1}^{k-1} a_i \cdot i \leq j < k$). Hence every smallest convex k-fold cover for $SPUR_{2k+1,2k+1}$ is prime. $\square$

Figure 14 shows $SPUR_{5,5}$, for $k = 2$.

The last two theorems provide an answer to the question asked at the beginning of the chapter about achieving a minimum size k-fold cover by combining thinner covers. Even if minimum size 1- through $(k-1)$-fold covers are given, for some polygons a minimum size k-fold cover cannot be obtained simply by looking at all combinations of those. Finding a smallest convex or star-shaped k-fold cover for a simple polygon thus is far from reducing to finding smallest thinner covers, no matter how large k is.

Not only is it not always possible to form a smallest k-fold cover by combining thinner ones, but sometimes it produces a lot more sets than needed. In fact for any constant A,

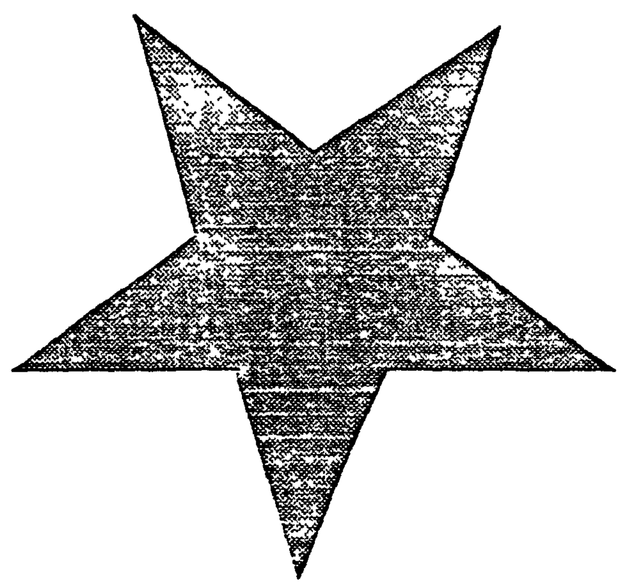

Figure 14: $SPUR_{5,5}$

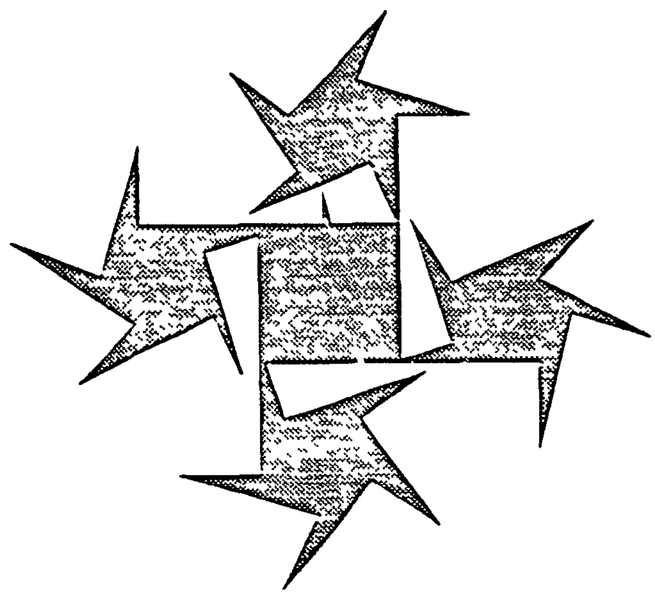

Figure 15: The polygon built from 4 $SPUR_{5,3}$.

a polygon can be constructed for which such a strategy will use at least A more sets than the optimal solution.

The idea behind the construction is to use $SPUR_{2k-1,k}$ and $SPUR_{2k+1,2k+1}$ as building blocks. Knowing that a smallest k-fold cover for these polygons saves at least one set over a combination of thinner ones, they are hooked together in such a manner that the best way to cover the resulting polygon is still to cover each component locally in the best way.

For convex covers the construction is simple. The A $SPUR_{2k+1,2k+1}$ are hooked together in a string, as if they were holding hands. Star-shaped covers require a bit more sophistication. The A $SPUR_{2k-1,k}$ are arranged in a circular fashion around a central hull to which is added a strategically placed spike. Figure 15 gives an example. Because of the very narrow connection between each $SPUR_{2k-1,k}$ and the central hull, no star-shaped set can significantly cover part of more than one $SPUR_{2k-1,k}$. The spike discourages attempts to introduce sets covering the central hull (including the spike) as well as a significant part of some $SPUR_{2k-1,k}$.

Chapter 5

Conclusion

This has been an investigation of the extreme cases of factorizations of covers. Through the duality with hypergraph 2-colorability, it has been possible to determine $f(2)$ — the smallest k such that every instance of a k-fold cover can be factorized into two simple covers — for unconstrained covers and some special cases. For point-line covers, a non-trivial lower bound was obtained but even with a promising result using Ramsey theory the question of determining $f(2)$ exactly remains open. The computation of minimum size convex and star-shaped covers for polygons has been shown not to reduce to combining minimum size thinner covers by introducing the family of polygons $SPUR_{l,m}$ which provides polygons with prime minimum size k-fold covers for every k.

There is still room for a fair amount of work to be done in the area. A few directions for future research will be outlined. The first one is the open problem mentioned above:

Open Problem 5.0.3 *Find the smallest integer k such that every family of straight lines forming a k-fold cover for some set of points in the plane can be factorized into two covers for that set of points.*

The problem can also be generalized to higher dimensions, where points in E^d are covered by hyperplanes. For $d > 2$, the condition $|S_i \cap S_j| \leq 1$ no longer holds for reduced covers though.

Most of the results are on $f(r)$, $r = 2$ — considering $r > 2$ might not be so difficult, particularly for special cases such as the ones that led to theorems 2.2.2 and 3.3.1. The work on r-colorability of hypergraphs can no longer be used, though, since as it has been pointed out before the notion of a valid coloring in the dual problem does not correspond to the usual one for hypergraphs except when $r = 2$.

It is still not clear that knowledge of minimum size thinner covers for a polygon cannot help in devising a polynomial time algorithm computing its minimum size k-fold cover. Blindly combining thinner covers has been shown to get nowhere but the structure of those covers might contain information easing the construction of a k-fold cover.

It would be interesting to know whether restricting the polygon to some subclass or considering covering sets with some property other than convex or star-shaped can lead to minimum size k-fold covers for which there always exists a perfect factorization.

At the end of chapter 4, two subfamilies of $SPUR_{l,m}$ — $SPUR_{2k-1,k}$ and $SPUR_{2k+1,2k+1}$ — were introduced because they had prime minimum size star-shaped and convex k-fold covers, respectively. The first one contains polygons with $4k - 2$ vertices — the other, polygons with $4k + 2$ vertices. They are believed to be the smallest polygons with prime minimum size covers.

Conjecture 5.0.4 $4k - 2$ *vertices are necessary and sufficient to construct a polygon for which every minimum size star-shaped k-fold cover is prime.*

Conjecture 5.0.5 $4k + 2$ *vertices are necessary and sufficient to construct a polygon for which every minimum size convex k-fold cover is prime.*

Finally, the algorithmic side of these problems has been barely touched. It is hoped that this preliminary study of the extremal properties of covers will have shed some light on how to design efficient algorithms to factorize covers, though extremal problems by themselves are fascinating.

Bibliography

[A84] Aggarwal, A., The Art Gallery Theorem: its variations, applications, and algorithmic aspects, Ph.D. thesis, Johns Hopkins Univ., 1984.

[AT81] Avis, D. and Toussaint, G.T., An optimal algorithm for determining the visibility of a polygon from an edge, *IEEE Trans. Comput.* C-30, 1981, pp. 910-914.

[B73] Berge, C., *Graphs and Hypergraphs*, American Elsevier, New York, 1973.

[Cha80] Chazelle, B., Computational geometry and convexity, Ph.D. thesis, Yale Univ., 1980.

[Chv75] Chvátal, V., A combinatorial theorem in plane geometry, *J. Combinatorial Theory, Series B* 18, 1975, pp. 39-41.

[Chv83] Chvátal, V., *Linear Programming*, W. H. Freeman and Co., New York, 1983.

[CR88] Culberson, J.C. and Reckhow, R.A., Covering polygons is hard, *Proc. 29th IEEE Symp. on Fundations of Computer Science*, 1988, pp. 601-611.

[CS88] Conway, J.H. and Sloane, N.J.A. *Sphere Packings, Lattices and Groups*, Springer-Verlag, Berlin, 1988.

[D74] Deo, N., *Graph Theory with Applications to Engineering and Computer Science* , Prentice-Hall, Englewood Cliffs, N.J., 1974.

[DLS88] Dean, J.A., Lingas, A. and Sack, J.-R., Recognizing polygons, or how to spy, *The Visual Computer* 3, 1988, pp. 344-355.

[F78] Fisk, S., A short proof of Chvátal's watchman theorem, *J. Combinatorial Theory, Series B* 24, 1978, p. 374.

[FL72] Fournier, J.C. and Las Vergnas, M., Une classe d'hypergraphes bichromatiques, *Discrete Math.* **2**, 1972, pp. 407-410.

[FL74] Fournier, J.C. and Las Vergnas, M., Une classe d'hypergraphes bichromatiques II, *Discrete Math.* **7**, 1974, pp. 99-106.

[FP75] Feng, H.-Y. F. and Pavlidis, T., Decomposition of polygons into simple components: feature generation for syntactic pattern recognition, *IEEE Trans. Comput.* **C-24**, 1975, pp. 636-650.

[G80] Graham, R.L., Rothschild, B.L. and Spencer, J.H., *Ramsey Theory*, Wiley, New York, 1980.

[Hal70] Halmos, P., How to write mathematics, *L'enseignement Mathématique* **16**, 1970, pp. 123-152.

[Har69] Harary, F., *Graph Theory*, Addison-Westley, Reading, Mass., 1969.

[K36] König, D., *Theorie des endlichen und unendlichen Graphen*, Akademische Verlagsgesellschaft, Leipzig, 1936.

[KS85] Keil, J.M. and Sack, J.R., Minimum decomposition of polygonal objects, *Computational Geometry* edited by G.T. Toussaint, North-Holland, Amsterdam, 1985.

[L79] Lovász, L., *Combinatorial Problems and Exercises*, North-Holland, Amsterdam, 1979.

[LL86] Lee, D.T. and Lin, A.K., Computational complexity of Art Gallery problems, *IEEE Trans. Information Theory* **IT-32**, 1986, pp. 276-282.

[M72] Maruyama, K., A study of visual shape perception, Univ. Illinois tech. report UIUCDCS-R-72-533, 1972.

[M82] Martin, G.E., *Transformation Geometry: An Introduction to Symmetry*, Springer-Verlag, New York, 1982.

[O82] O'Rourke, J., Polygon decomposition and switching function minimization, *Comput. Graphics and Image Processing* **18**, 1982, pp. 382-391.

[O83] O'Rourke, J., Galleries need fewer mobile guards: a variation on Chvátal's theorem, *Geometriae Dedicata* 14, 1983, pp. 273-283.

[O87] O'Rourke, J., *Art Gallery Theorems and Algorithms*, Oxford University Press, Oxford, 1987.

[Pac80] Pach, J., Decomposition of multiple packing and covering, *Diskrete Geometrie 2. Kolloq. Math. Inst. Univ. Salzburg*, 1980, pp. 169-178.

[Pac86] Pach, J., Covering the plane with convex polygons, *Discrete Comput. Geom.* 1, 1986, pp. 73-81.

[Pav77] Pavlidis, T., *Structural Pattern Recognition*. Springer Verlag, Berlin, 1977, pp. 236-241.

[Pe88] Pesant, G., Galleries require more sleepy watchmen: k-guarding simple polygons, Tech. Rep. No. SOCS-88.11, School of Computer Science, McGill University, 1988, pp. 145-165.

[Rob84] Roberts, F.S., *Applied Combinatorics*, Prentice-Hall, Englewood Cliffs, N.J., 1984.

[Rog64] Rogers, C.A., Packing and covering, *Cambridge Tracts in Math. and Math. Physics* 54, Cambridge University Press, Cambridge, 1964.

[Se74] Seymour, P.D., On the two-colouring of hypergraphs, *Quart. J. Math. Oxford (3)* 25, 1974, pp. 303-312.

[Sh89] Shermer, T., Visibility properties of polygons, Ph.D. thesis, McGill Univ., 1989.

[St78] Stahl, S., Reductions of n-fold covers, *Proc. Amer. Math. Soc.* 72, 1978, pp. 422-424.

[ST83a] Szemerédi, E. and Trotter, W.T. Jr., Extremal problems in discrete geometry, *Combinatorica* 3, 1983, pp. 381-392.

[ST83b] Szemerédi, E. and Trotter, W.T. Jr., A combinatorial distinction between the euclidean and projective planes, *Europ. J. Combinatorics* 4, 1983, pp. 385-394.

[T77] Fejes Tóth, G., A problem connected with multiple circle-packings and circle-coverings, *Studia Scientiarum Mathematicarum Hungarica* 12, 1977, pp. 447-456.