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DEPARTMENT OF CIVIL ENGINEERING AND APPLIED MECHANICS

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INTRODUCTION

In problems where specification of material properties under conditions of dynamic loading is necessary, the question of material sensitivity to stress and strain paths becomes a most important consideration. In a soil material where immediate and subsequent performance characteristics are intimately related to past histories of stress and strain, an assessment of the pertinent material properties becomes difficult unless a proper accounting of previous histories is made. The problem is all the more complicated if dynamic and transient loadings are introduced as external conditions, since time response characteristics of soils now demand that the inelastic behaviour of the material be studied in addition to the well established hysteretic performance characteristics.

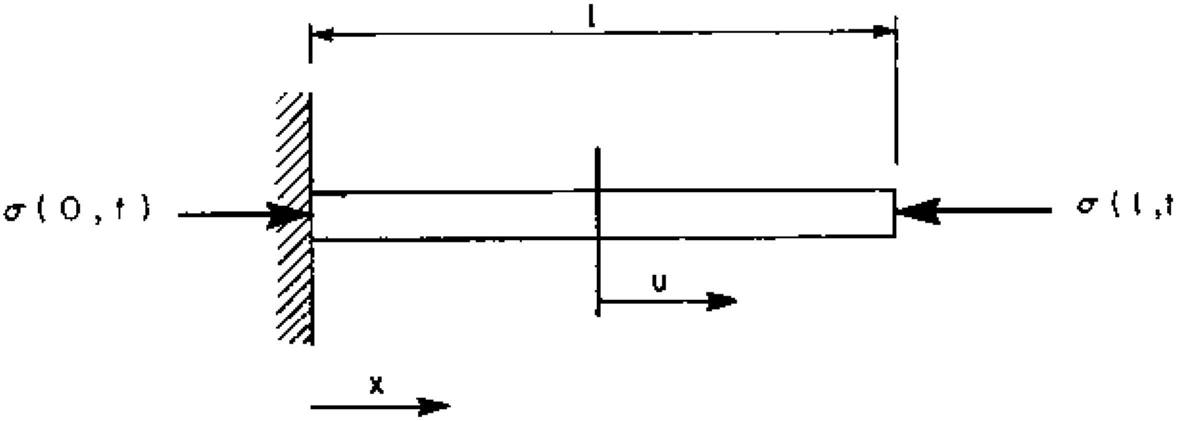
The problem under investigation relates directly to the measurement and evaluation of dynamic properties of clays under conditions of loading such as those provided by moving surface loads [wheels, vehicles, tracks, etc.] or through penetrating devices into the soil [wedge and cone indentation into soil]. The behaviour of the soil directly below the surface in the case of surficial loadings, or directly adjacent to the penetrating wedge or cone, can be described in terms of either transient or dynamic load-response characteristics. As an immediate requirement, various material parameters and moduli are needed for appropriate analyses to be made.

This report presents a numerical method which allows one to compute the transfer function of a two dimensional system composed of a frequency dependent material - soil. This arises from initial considerations of the analytical study which examines a finite length of the specimen resting on a fixed base - see Fig.1 in Section A. A simple model is assumed based upon available experimental experience, and the fact that a pressure discontinuity cannot be properly transmitted through a clay material [Yong, Krizek and Dutertre (1973)]. The transfer function which relates the pressure at one end of the specimen due to an input pressure at the free top surface allows one to compute the response of the specimen subjected to an impulse, a stepload or any arbitrary transient load function. The speed of propagation of an impulse is known to be closely related to the speed of propagation of the maximum disturbance in materials possessing little internal damping. However in a dispersive material, the properties measured in classical transient wave tests are found to be related to the lower resonant frequency of the specimen tested. Thus it becomes important to identify the proper characteristics of material performance, and especially the actual speed of propagation in the dispersive material. The measurement of time of arrival of a pulse and the associated calculations for the evaluation of the various moduli which constitute the experimental and analytical phase of this kind of study have been reported previously [Yong et al. (1973)]. In this report, we present the numerical method which allows for the examination of the transfer function in the solution of the problem of wave propagation in a visco-elastic material which is assumed to model the performance characteristics of clay.

SECTION A

ANALYTICAL SOLUTION OF A FINITE LENGTH ROD RESTING ON A FIXED BASE

A.1 Problem definition



Equation of state for the visco-elastic material is given as

$$\sigma = E\,\epsilon + \eta\,\overset{o}{\epsilon} \qquad \text{A-1}$$

where $$\epsilon = \frac{\partial u}{\partial x} \qquad \text{A-2}$$

and $$\overset{o}{\epsilon} = \frac{\partial \epsilon}{\partial t} = \frac{\partial^2 u}{\partial t \partial x} \qquad \text{A-3}$$

The input $\sigma(\ell,t)$ is known and $\sigma(0,t)$ must be computed.

A.2 Transfer function

The transfer function relating $\sigma(0,t)$ to $\sigma(\ell,t)$ can first be obtained :

The equation of motion is

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \qquad \text{A-4}$$

Using the equation of state A-1 and differentiating with respect to x yields

$$\frac{\partial^2 u}{\partial x^2} + \frac{\eta}{E} \frac{\partial^3 u}{\partial x \partial t^2} - \frac{\rho}{E} \frac{\partial^2 u}{\partial t^2} = 0 \qquad \text{A-5}$$

let $\quad \alpha = \frac{\eta}{E}$

and $\quad c^2 = \frac{E}{\rho}$

Then A-5 becomes

$$\frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial^3 u}{\partial x \partial t^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0 \qquad \text{A-6}$$

The Laplace transform of equation A-6 is

$$\frac{\partial^2 \bar{u}}{\partial x^2} + \alpha p \frac{\partial^2 \bar{u}}{\partial x^2} - \frac{p^2}{c^2} \bar{u} = 0 \qquad \text{A-7}$$

where $\bar{u}$ is the transform of u and p is the Laplace variable. Rearranging A-7 yields

$$\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{p^2}{c^2(1 + \alpha p)} \bar{u} = 0 \qquad \text{A-8}$$

Using x as a new variable the Laplace transform can again be applied to equation A-8.

$$\left(S^2 - \frac{p^2}{c^2(1 + \alpha p)}\right) \bar{\bar{u}} = \frac{\partial \bar{u}(0,p)}{\partial x} + \cancelto{0}{S\,\bar{u}(0,p)} \qquad \text{A-9}$$

where the fixed end boundary condition has been introduced

$$\bar{\bar{u}} = \frac{\partial \bar{u}(0,p)}{\partial x} \; \frac{1}{S^2 + \dfrac{p^2}{c^2(1 + \alpha p)}} \qquad \text{A-10}$$

Equation A-10 can be inverted yielding

$$\bar{u}(x,p) = \frac{\partial \bar{u}(0,p)}{\partial x} \frac{c\sqrt{1+\alpha p}}{p} \sinh \frac{p}{c\sqrt{1+\alpha p}} x \qquad \text{A-11}$$

Equation A-11 is the equation of motion in the Laplace domain. It is now possible to introduce the boundary condition at the free end where the stress is $\sigma(\ell,t)$. Using equation A-1 yields :

$$\sigma(\ell,t) = E \frac{\partial u(\ell,t)}{\partial x} + \eta \frac{\partial^2 u(\ell,t)}{\partial x \partial t} \qquad \text{A-12}$$

In the Laplace domain A-12 becomes

$$\bar{\sigma}(\ell,p) = E \frac{\partial \bar{u}(\ell,p)}{\partial x} + \eta p \frac{\partial \bar{u}(\ell,p)}{\partial x} \qquad \text{A-13}$$

which reduces to

$$\frac{\partial \bar{u}(\ell,p)}{\partial x} = \frac{\bar{\sigma}(\ell,p)}{E+\eta p} \qquad \text{A-14}$$

$\frac{\partial \bar{u}(\ell,p)}{\partial x}$ can be computed from equation A-11

$$\frac{\partial \bar{u}(x,p)}{\partial x} = \frac{\partial \bar{u}(0,p)}{\partial x} \cosh \frac{p}{c\sqrt{1+\alpha p}} x \qquad \text{A-15}$$

and

$$\frac{\partial \bar{u}(\ell,p)}{\partial x} = \frac{\partial \bar{u}(0,p)}{\partial x} \cosh \frac{p\ell}{c\sqrt{1+\alpha p}} \qquad \text{A-16}$$

Equation A-16 can be used in conjunction with the boundary condition A-14

$$\frac{\bar{\sigma}(\ell,p)}{E+\eta p} = \frac{\partial \bar{u}(0,p)}{\partial x} \cosh \frac{p\ell}{c\sqrt{1+\alpha p}} \qquad \text{A-17}$$

or

$$\frac{\partial \bar{u}(0,p)}{\partial x} = \frac{\bar{\sigma}(\ell,p)}{E+\eta p} \frac{1}{\cosh \frac{p\ell}{c\sqrt{1+\alpha p}}} \qquad \text{A-18}$$

which can be used in equation A-11 to obtain the transformed displacement as a function of the transformed input

$$\bar{u}(x,p) = \frac{\bar{\sigma}(\ell,p)}{E + \eta p} \; \frac{c\sqrt{1+\alpha p}}{p} \; \frac{1}{\cosh \frac{p\ell}{c\sqrt{1+\alpha p}}} \sinh \frac{p}{c\sqrt{1+\alpha p}} \; x$$

A-19

$\bar{u}(x,p)$ being known it is possible to compute $\bar{\sigma}(0,p)$

$$\sigma = E\,\frac{\partial u}{\partial x} + \eta\,\frac{\partial^2 u}{\partial x \partial t} \qquad \text{A-1}$$

Taking the Laplace transform of equation A-1 and replacing $\bar{u}(x,p)$ by its value given by equation A-19

$$\bar{\sigma}(x,p) = \frac{\bar{\sigma}(\ell,p)}{\cosh \frac{p\ell}{c\sqrt{1+\alpha p}}} \; \cosh \frac{px}{c\sqrt{1+\alpha p}} \qquad \text{A-20}$$

Letting x = 0 yields the desired stress at the bottom

$$\boxed{\bar{\sigma}(0,p) = \frac{1}{\cosh \frac{p\ell}{c\sqrt{1+\alpha p}}}\,\bar{\sigma}(\ell,p)} \qquad \text{A-21}$$

Equation A-21 gives the transfer function $\bar{H}(p)$ in the Laplace domain.

$$\bar{H}(p) = \frac{1}{\cosh \frac{p\ell}{c\sqrt{1+\alpha p}}} \qquad \text{A-22}$$

Letting $p = i\omega$ yields the transfer function in the frequency domain :

$$\boxed{\bar{H}(\omega) = \frac{1}{\cos \frac{\omega\ell}{c\sqrt{1+i\omega\alpha}}}} \qquad \text{A-23}$$

A.3 Response to an impulse

If $\sigma(\ell,t) = \delta(t)$ where $\delta(t)$ is Dirac's delta function, then

$$\bar{\sigma}(0,t) = \frac{1}{\cosh \frac{p\ell}{c\sqrt{1+\alpha p}}} = \bar{h}(p) \qquad \text{A-24}$$

and the inverse of the transfer function gives the response of the system.

$$h(t) = \sigma(0,t) = \mathcal{L}^{-1}\left(\frac{1}{\cosh \frac{p\ell}{c\sqrt{1+\alpha p}}}\right) \qquad \text{A-25}$$

h(t) can be obtained using

$$h(t) = \Sigma \text{ Residues of } \{f(p)\} \qquad \text{A-26}$$

where
$$f(p) = \frac{e^{pt}}{\cosh \frac{p\ell}{c\sqrt{1+\alpha p}}} = \frac{P(p)}{Q(p)} \qquad \text{A-27}$$

and the residues of f(p) can be calculated from $R = \frac{P(p)}{Q'(p)}$ for each pole of f(p).

$$R = \frac{e^{pt}}{\frac{\delta}{\delta p}\left(\frac{p\ell}{c\sqrt{1+\alpha p}}\right) \sinh \frac{p\ell}{c\sqrt{1+\alpha p}}} \qquad \text{A-28}$$

$$R = \frac{c}{\ell}\ \frac{2(1+\alpha p)^{3/2}}{2+\alpha p}\ \frac{e^{pt}}{\sinh \frac{p\ell}{c\sqrt{1+\alpha p}}} \qquad \text{A-29}$$

The poles of $f(p)$ are obtained for

$$\cosh \frac{p\ell}{c\sqrt{1+\alpha p}} = 0 \qquad \text{A-30}$$

i.e.

$$\frac{p\ell}{c\sqrt{1+\alpha p}} = \frac{2n-1}{2}\pi i \qquad \text{A-31}$$

which allows one to simplify equation A-29, giving

$$\boxed{R = -(-1)^{n-1}\frac{4}{\pi}\frac{1}{2n-1}\frac{p(1+\alpha p)}{2+\alpha p}e^{pt}} \qquad \text{A-32}$$

Solution of equation A-31 yields the poles :

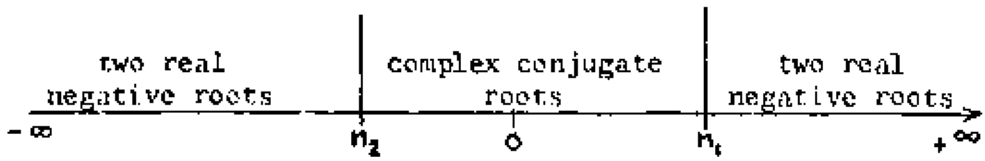
$$p^2 + p\alpha\left(\frac{c}{\ell}\frac{2n-1}{2}\pi\right)^2 + \left(\frac{c}{\ell}\frac{2n-1}{2}\pi\right)^2 = 0 \qquad \text{A-33}$$

$$\boxed{p = \alpha\frac{\gamma^2}{2}\left(-1 \pm \sqrt{1-\frac{4}{\gamma^2\alpha^2}}\right)} \qquad \text{A-34}$$

where $\gamma = \frac{c}{\ell}\frac{2n-1}{2}\pi$

Not all roots of equation A-33 are valid because of the square root in equation A-31. Solutions given by equation A-34 must be checked against equation A-31 thus eliminating unwanted roots.

As n varies from $-\infty$ to $+\infty$, roots of equation A-34 can be classified as follow:



constants n_1 and n_2 can be evaluated from :

$$1 - \frac{4}{\gamma^2 \alpha^2} \cong 0 \qquad \text{A-35}$$

i.e.

$$\gamma_1 \cong + \frac{2}{\alpha} \qquad \text{A-36}$$

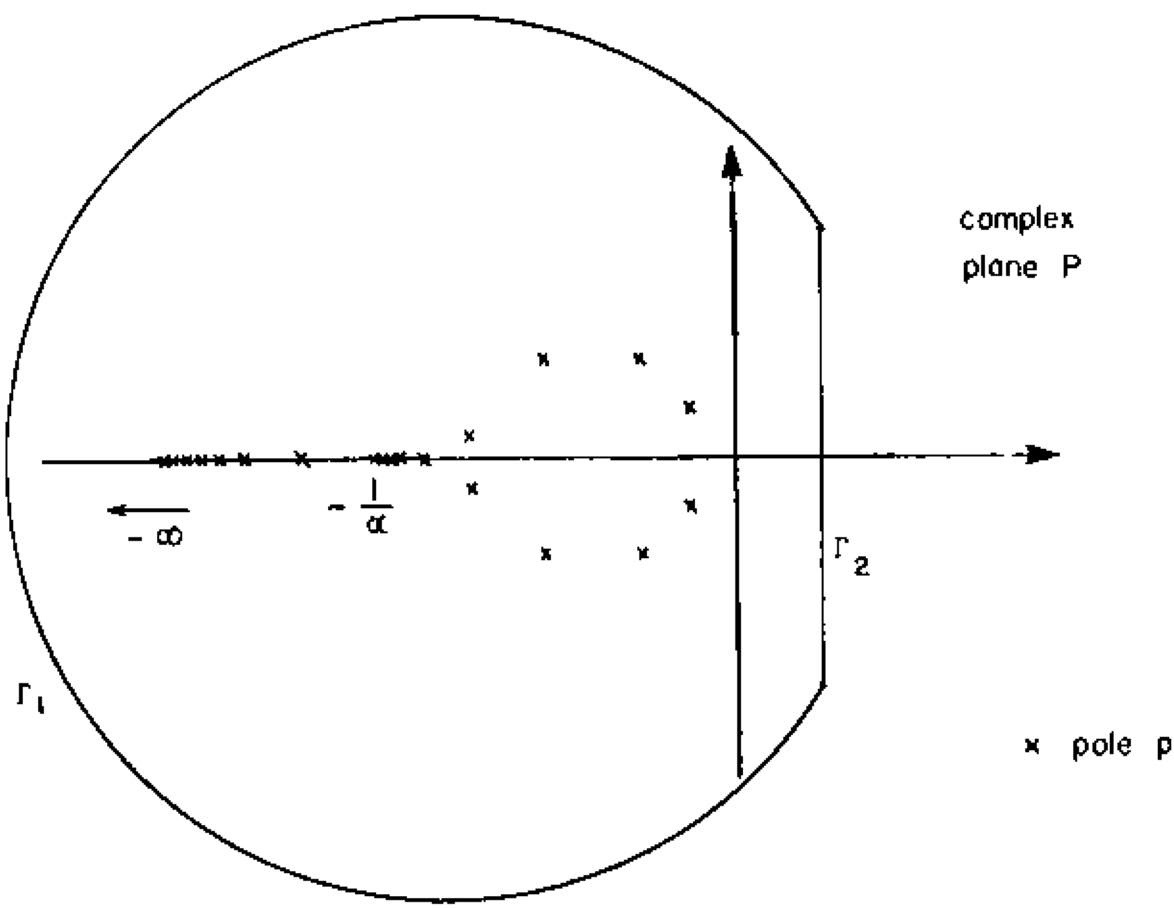
$$\gamma_2 \cong - \frac{2}{\alpha}$$

or

$$n_1 \cong + \frac{2}{\alpha} \; \frac{1}{\pi} \; \frac{\ell}{c} + \frac{1}{2} \qquad \text{A-37}$$

$$n_2 \cong - \frac{2}{\alpha} \; \frac{1}{\pi} \; \frac{\ell}{c} + \frac{1}{2} \qquad \text{A-38}$$

After elimination of the unwanted roots, the poles p in the complex plane, look as follows:



For $n \to \infty$ all poles accumulate on the real axis, one set at a point $-\frac{1}{\alpha}$ and the other set tending to $-\infty$. Because the poles are negative, equation A-32 shows that the residues will converge to zero with increasing $|n|$.

This does not prove, though, that the series of the sum of the residues converges. For the latter to be so, it should be shown that (Doetsch 1961)

$$\int_{\Gamma_1} e^{pt}\, \bar{h}(p)\, dp \to 0 \qquad \text{A-39}$$

for $\Gamma_1 \to \infty$.

Because numerical results actually converge very fast with a small number of poles it was not found necessary to prove equation A-39.

Numerical computations to solve equation A-25 are carried out in a FORTRAN subroutine REPIMP the text of which follows on the next page.

A.4 Response to an arbitrary input function

Knowing the response to an impulse the response to any forcing function can be obtained

$$\bar{\sigma}(0,p) = \bar{h}(p)\ \bar{\sigma}(\ell,p) \qquad \text{A-30}$$

whose inverse is readily obtained using Duhamel's integral

$$\sigma(o,t) = \int_0^t h(t-\tau)\ \sigma(\ell,\tau)\ d\tau \qquad \text{A-31}$$

Integral A-31 is evaluated in the FORTRAN subroutine BOREL.

SECTION B

TWO DIMENSIONAL FINITE ELEMENT PROGRAM

B.1 Introduction

The dynamic analysis (modal analysis and integration) is done by the program EL-2D. This program was developed both with static and dynamic capabilities. It is a fairly general two dimensional finite-element program which can be used either for production or teaching. Some of its characteristics are :

- input characteristics : input has been simplified to a maximum. Complete free format is used, text can figure on the cards (comments or title or option choices). This eases the user's work. Options are selected at the user's discretion and default options are always provided. All options, conventions, data-deck structure, error messages... are defined in the user's manual which follows as paragraph B-3 (special dynamic options for this thesis are presented after the manual).
- computational characteristics : The program was developped on a computer with only approximately 12K words of 24 bits available. Great care had to be taken to save memory space. An overlay structure had to be used and is shown on figure B-1. The assembled stiffness matrix is kept in memory as a whole allowing an iterative method to be employed for the eigenvalue determination (see section B-2). The stiffness matrix is banded diagonally and is first decomposed using a Cholewsky

ELEMENTS FINIS 2D

STRUCTURE

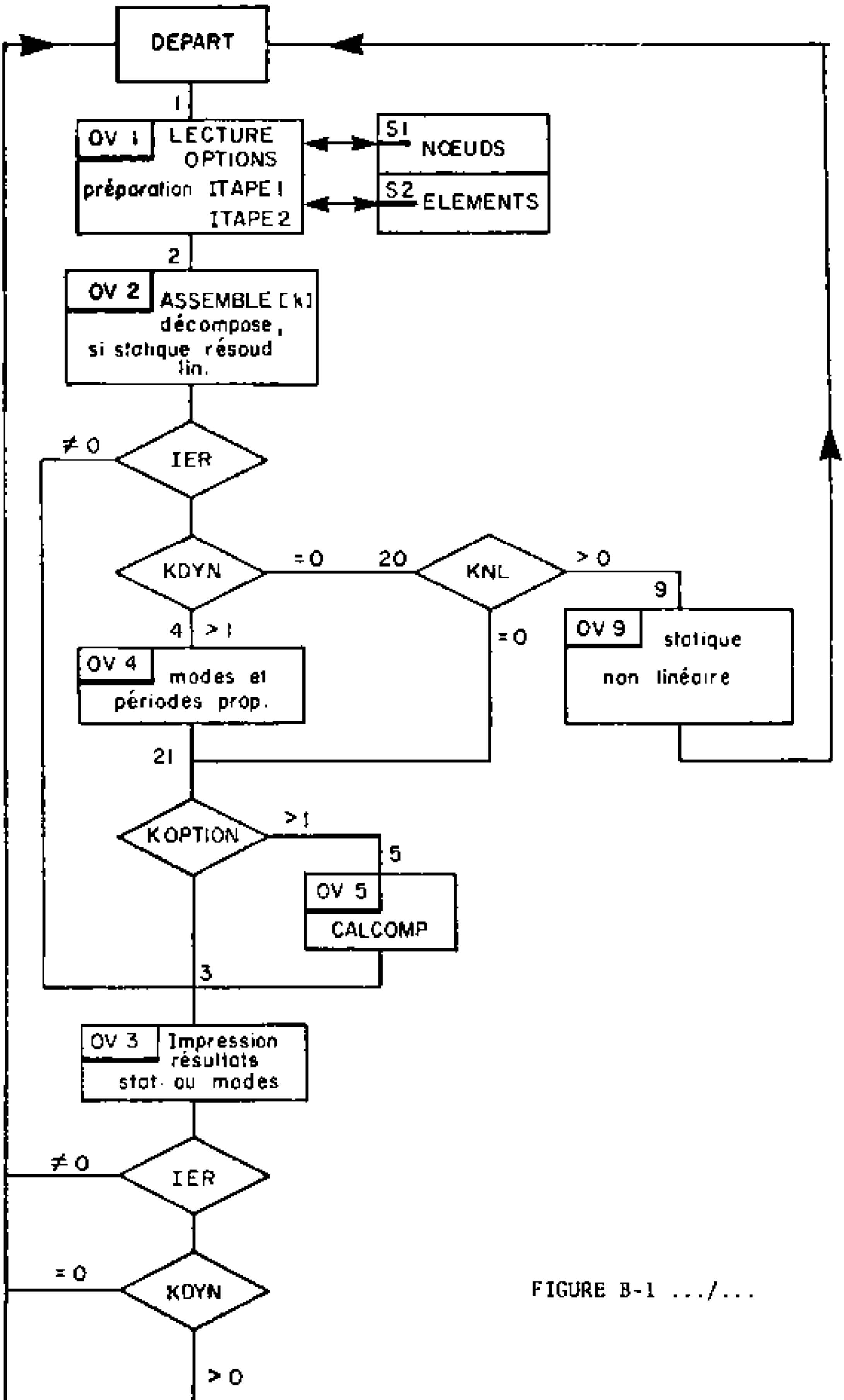
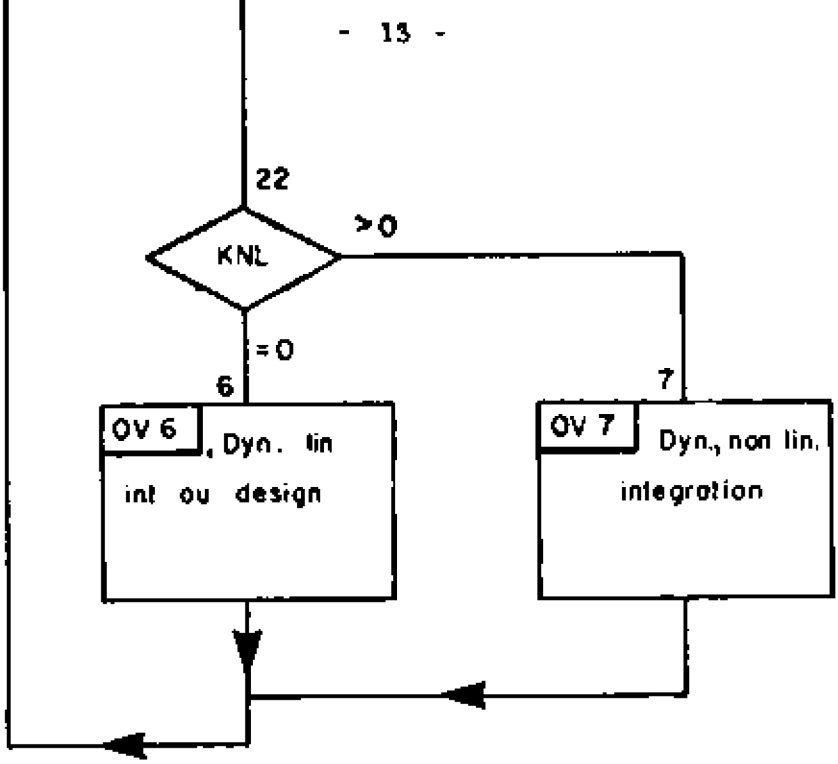


FIGURE B-1 .../...



FLOW CHART FOR EL-2D PROGRAM

FIGURE B-1

algorithm. The complete program text is not given, only the eigen-value and eigen-vector determination overlay is presented.

B.2 Eigen-value and eigen-vector determination

B.2.1 Introduction

The generalized characteristic-value problem of the form

$$\underline{Mx} = \lambda \, \underline{Kx} \qquad \text{B-1}$$

is encountered in dynamic analysis of structures where $\underline{M}$ and $\underline{K}$ are respectively the mass and stiffness matrices. Many publications present the problem and its solution. However only a few take advantage of the particular forms of matrices $\underline{M}$ and $\underline{K}$. [Bronlund 1969; Gupta 1970; Peters and Wilkinson 1969].

$\underline{K}$ is a symmetric positive definite banded matrix and $\underline{M}$ is a symmetric, often non-definite matrix. It is banded like $\underline{K}$ in the case of a continuum and diagonal in many engineering applications. A procedure which preserves the band form of matrices $\underline{M}$ and $\underline{K}$ is based on the iterative method as described by Frazer [1938] and Hurty [1964].

This procedure enables one to obtain the p + 1st eigenvector and eigenvalue providing the first p ones are available. Theoretically, all eigenvectors can be obtained, yet, because of numerical computations, the processus becomes unstable unless great care is taken to avoid numerical errors propagating.

The purpose of this section is to present an algorithm which minimizes numerical errors. After presenting the technique, a hand example will be carried out. Numerical results will be presented and compared to that of earlier workers.

B.2.2 Numerical technique

B.2.2.1 First mode

Equation B-1 can be used as such and the r + 1 iteration step is :

$$\underline{K} (\lambda \underline{x}^{(r+1)}) = \underline{M} \underline{x}^{(r)} \qquad \text{B-2}$$

which involves

- premultiplication of guess vector $\underline{x}^{(r)}$ by $\underline{M}$
- solution of the system of linear equations leading to $(\lambda \underline{x}^{(r+1)})$
- extraction of the value λ from $(\lambda \underline{x}^{(r+1)})$ vector by subjecting it to the same normalizing process as guess vector $\underline{x}^{(r)}$ was subjected to. This is usually done by arbitrarily setting equal to one the largest component of $\underline{x}$.

This sequence of operation enables one to keep $\underline{M}$ and $\underline{K}$ in their original forms. The time required for one iteration cycle is greatly reduced if a decomposition technique, like that of Cholewsky (Weaver 1965), is used to solve the system of linear equations.

Even when applied to large systems, this iteration converges very fast on the first (largest) eigenvalue and eigenvector.

B.2.2.2 Following modes

Equation B-2 must be modified to oblige the process to converge on higher modes. The iterative process becomes :

$$\underline{K} (\lambda \underline{x}^{(r+1)}) = \underline{M} \underline{S} \underline{x}^{(r)} \qquad \text{B-3}$$

where sweeping matrix $\underline{S}$ contains the orthogonality equations existing between the p + 1st vector desired and the p ones already obtained.

The iterative process as described by equation B-3 converges providing matrix $\underline{S}$ is accurate enough.

B.2.2.3 Construction of sweeping matrix (standard form)

Let us suppose that p eigenvalues λ_i and eigenvectors are known (i = 1, 2 ...p) and that we are looking for the p + 1st one. We can write p orthogonality equations of the form :

$$\underline{q}_i^T \, \underline{M} \, \underline{q}_{p+1} = 0 \qquad (i = 1, \ldots p) \qquad \text{B-4}$$

let
$$\underline{Q}_i^T = \underline{q}_i^T \, \underline{M} \qquad \text{B-5}$$

with
$$\underline{Q}_i^T = [Q_{i1}, Q_{i2}, Q_{i3} \ldots, Q_{in}]$$

where n is the number of unknowns in the linear system.

A matrix $\underline{W}_p$ can be formed

$$\underline{W}_p = \left[\begin{array}{ccc|ccc} Q_{11} & \cdots & Q_{1p} & Q_{1,p+1} & \cdots & Q_{1n} \\ Q_{21} & \cdots & Q_{2p} & Q_{2,p+1} & \ldots & Q_{2n} \\ Q_{p} & \cdots & Q_{pp} & Q_{p,p+1} & \cdots & Q_{pn} \end{array}\right] \qquad \text{B-6}$$

or simply

$$\underline{W}_p = \left[\begin{array}{c|c} \underline{Q}_p^1 & \underline{Q}_p^2 \end{array}\right] \qquad \text{B-7}$$

Matrix $\underline{S}$ can then be constructed

$$\underline{S}_p = \left[\begin{array}{c|c} \dfrac{\underline{0}}{\underline{0}} & \dfrac{\underline{Q}_p^{1^{-1}} \, \underline{Q}_p^2}{\underline{I}} \end{array}\right] \qquad \text{B-8}$$

B.2.2.4 Construction of sweeping matrix (modified form)

Equation B-8 shows the mathematical form of the process. If some of the components of matrix $\underline{Q}_p^1$ are small with respect to those of $\underline{Q}_p^2$, matrix $\underline{S}_p$ is bound to contain large numerical errors. In order to prevent the latter, operation in equation B-8 can be replaced by a two phase manipulation process enabling one:

- to select the largest pivotal element, associated with each vector $\underline{Q}_i$,
- to replace the matrix product

$$\underline{x}_c = \underline{S}_p \, \underline{x} \qquad \text{B-9}$$

where $\underline{x}$ is the guess vector and $\underline{X}_c$ the "constrained" guess vector, by a backward elimination type solution of the form

$$\underline{S}_p^* \, \underline{x}_c = \underline{x}^* \qquad \text{B-10}$$

Where $\underline{x}^*$ is a modified guess vector :

$$\underline{x}^* = [0 \,\ldots\ldots 0, \; x_{p+1} \ldots . x_n]^T \qquad \text{B-11}$$

and $\underline{S}_p^*$ is the modified sweeping matrix.

Matrix $\underline{S}_p^*$ is directly obtained from $\underline{W}_p$ matrix of equations B-6 and B-7: Every time a new eigenvector is obtained, one row is added to matrix $\underline{W}_p$. Submatrix $\underline{Q}_p^1$ is kept upper triangular using a downward elimination process applied to a whole row. The p row is then normalized so that a value equal to unity appears on the diagonal of submatrix $\underline{Q}_p^1$. After these transformations, $\underline{W}_p$ matrix is called $\underline{W}_p^*$.

$$\underline{W}_p^* = \left[\begin{array}{ccccc|ccc} 1 & Q_{12}^* & Q_{13}^* & \cdots & Q_{1p}^* & Q_{1p+1}^* & \cdots & Q_{1n}^* \\ 0 & 1 & Q_{23}^* & & Q_{2p}^* & Q_{2p+1} & & Q_{2n}^* \\ 0 & 0 & 1 & \cdots & & & \cdots & \\ - & - & - & \cdots & & & \cdots & \\ 0 & - & - & \cdots & 1 & Q_{p\ p+1}^* & \cdots & Q_{pn}^* \end{array}\right] \qquad \text{B-12}$$

or in simpler form :

$$\underline{W}_p^* = \left[\underline{Q}_p^{1*} \,\middle|\, \underline{Q}_p^{2*} \right] \qquad \text{B-13}$$

where $\underline{Q}_p^{1*}$ is upper triangular.

Matrix $\underline{S}_p^*$ is then obtained :

$$\underline{S}_p^* = \left[\begin{array}{c|c} \underline{Q}_p^{1*} & \underline{Q}_p^{2*} \\ \hline \underline{0} & \underline{I} \end{array}\right] \qquad \text{B-14}$$

During the downward elimination, the largest pivotal element of a row can be used, thus minimizing numerical errors. An index vector is constructed to memorize pivotal positions selected allowing to access them through indirect addressing. It is important to note that very few manipulations are necessary to obtain $\underline{S}_p^*$ matrix.

It is of course unnecessary to construct matrix $\underline{S}_p^*$ when $\underline{W}_p^*$ matrix is available. Equation B-10 can be solved with an algorithm built directly on $\underline{W}_p^*$ matrix.

B.2.2.5 Iteration control

Convergence must be checked on eigenvectors as they converge slower than their associated eigenvalues. Very precise tolerances must be given on the first vectors to prevent the system from becoming unstable on higher modes.

If matrix $\underline{M}$ is m times singular, (i.e. contains m zeros on the diagonal) iteration control need not be done on the m corresponding components of the eigenvectors. m rows can therefore be eliminated from matrix $\underline{W}_p^e$ thus saving a little more computer space.

B.2.3 Numerical examples

B.2.3.1 Hand calculations

A generalized characteristic-value problem of the form

$$\underline{K}^{-1}\,\underline{M}\,\underline{q} = \lambda\,\underline{q} \qquad \text{B-15}$$

is solved by Hurty (1964) using a standard iteration technique. The same problem will be dealtwith, keeping equation B-15 in its initial form.

$$\underline{K}\,\lambda\,\underline{q} = \underline{M}\,\underline{q} \qquad \text{B-16}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \lambda \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \qquad \text{B-17}$$

The first eigenvalue and eigenvector do not present any problem.

$$\lambda_1 = 5.0489 \qquad \underline{q}^{(1)} = \begin{Bmatrix} 1 \\ .8019 \\ .445 \end{Bmatrix}$$

Matrix $\underline{W}^*$ can be constructed :

$$W^* = \begin{bmatrix} 1. & .8019 & .445 \end{bmatrix}$$

and iteration on the second mode can proceed :

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \lambda \quad \underline{q}^{r+1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{q}_c^r \qquad \lambda \qquad 5.043917$$

with

$$\begin{bmatrix} 1 & .801938 & .445042 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{q}_c^r = \begin{Bmatrix} 0 \\ q_2^r \\ q_3^r \end{Bmatrix}$$

A few steps will be illustrated :

$\underline{q}^r$	$\underline{q}_c^r$	$\lambda\ \underline{q}^{r+1}$	$\underline{q}^{r+1}$	λ
1	.445	.335	1	
0	0	-.110	-.3284	.335
-1	-1	-.555	-1.6567	
1	1	.6865	1	
-.3284	-.3284	-.3135	-.4566	.6865
-1.6567	-1.6567	-.9851	-1.435	
1	1.005	.6668	1	
-.4566	-.4566	-.3382	-.5072	.6668
-1.435	-1.435	-.8866	-1.3296	
1	1	.643184	1	
-.554958	-.554958	-.356816	-.554765	.643184
-1.246980	-1.24690	-.801858	-1.246701	

Matrix $\underline{S}_2^*$ can be constructed :

$$\underline{S}_2^* = \begin{bmatrix} 1 & .801938 & .445042 \\ 0 & .801956 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

where pivotal element number 3 has been selected for the second eigenvector. For a 3 x 3 only one step of iteration is necessary leading to :

$\underline{q}^r$	$\underline{q}_c^r$		$\lambda\, \underline{q}^{r+1}$	$\underline{q}^{r+1}$	λ
0	-.445	c	-.136956	-.444599	
1	1		.308044	1	.308044
0	-.801956		.24656	-.801691	

B.2.3.2 Electronic computations

The multiframe building described by Hurby and Rubinstein [1964] and used by Gupta [1970] was chosen to test the algorithm.

It is a 19 storeys 4 bay frame which presents the interesting particularity of having been computed with different assumptions and the results published.

Even though symmetrical, it was entered as a whole to work on a bigger numerical system. Three coordinates were used for each joint allowing vertical accelerations and axial deformations, horizontal beams included, to be taken care of.

Table B-1 shows values of the frequency obtained for the first seven modes. A relative precision of 10^{-5} was imposed on each eigenvector. Results agree well, slightly lower frequencies must be attributed to the effect of axial deformation of the horizontal beams.

MODE	f_{Hz}	Gupta (1970) f_{Hz}
I	.293	.295
2	.809	.816
3	1.401	1.413
4	1.989	2.002
5	2.613	2.627
6	3.325	3.338
7	4.070	4.082

LOWER FREQUENCIES OBTAINED BY EL-2D PROGRAM

COMPARED TO THAT OF GUPTA (1970)

TABLE B-1

The complete text of the algorithm presented can be found in Dutertre (1972). When this program is called COMMON, the following variables are contained:

N = Number of unknowns (or coordinates)

KDYN = Number of modes requested

D(300) = Mass matrix in kip (force)

S(1) = First address of decomposed stiffness matrix (diagonally banded).

Each new eigenvector is output on disk and the corresponding eigenvalue is placed in VP(I). In order to save memory space the stiffness matrix, the eigenvectors used during one iteration step and the sweeping matrix are stored one after the other in the same vector with no gap in between. The same name is therefore used for these variables in the main program. Indirect addressing is obtained when using a CALL to a subroutine.

PART II - SECTION C

FROM NORMAL MODES TO TRANSFER FUNCTION

C.1 Introduction

The transfer function of a system can be computed if the normal modes are known using the equations

$$\zeta_r = \zeta_{r_{st}} \frac{1}{|Z_r|} e^{i(\omega t - \theta_r)} \qquad C.1$$

where $$|Z_r| = \sqrt{(1 - \Omega_r^2)^2 + 4\beta_r^2 \Omega_r^2} \qquad C.2$$

with $$\Omega_r = \frac{\omega}{\omega_r} \qquad C.3$$

and $$\underline{x} = \underline{\phi}\, \underline{\zeta} \qquad C.4$$

$\zeta_{r_{st}}$ can be computed from

$$\zeta_{r_{st}} = \underline{\phi}_r^T \underline{p} \frac{d}{K_r} \qquad C.5$$

C.2 Problem definition and solution

Consider a test sample with physical properties $E = 6000$ lbs/in^2, $\rho = 1.6\ 10^{-4}$, $L = 3$ in, a section of unity and a viscosity $\eta = 0.04$. For computation purposes the sample is divided into 20 discrete elements as indicated on figure C.1. Modes are numbered from 1 to 21. Node number 1 is fixed and all the others are free to move horizontally. The modulus is assumed to be independent of frequency and $\sigma(o,t)$, pressure at the bottom, is proportional to the displacement of node 2.

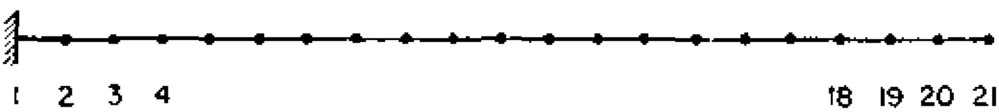
$$\sigma(o,\ t) = E\ u_2 \qquad \text{C.6}$$

if x_2 is the complex representation of u_2, according to equation I.2.60

$$x_2 = e^{i\omega t} \sum_{r=1}^{n} \phi_{ir}\ \zeta_{r_{st}}\ \frac{1}{|Z_r|}\ e^{-i\theta_r} \qquad \text{C.7}$$

Computations are limited to the five first modes. The 21 x 21 mass matrix is diagonal

$$\underline{m} = .15\ 1.6\ 10^{-4} \begin{bmatrix} \frac{1}{2} & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & 1 & \\ & & & & & & & \frac{1}{2} \end{bmatrix}$$



TABULATION DES NOEUDS

NOEUD	X	Y	RX	RY	RZ
1	0	0	-1	-1	-2
2	.20	0	0	-1	-2
3	.40	0	0	-1	-2
4	.60	0	0	-1	-2
5	.80	0	0	-1	-2
6	1.00	0	0	-1	-2
7	1.20	0	0	-1	-2
8	1.40	0	0	-1	-2
9	1.60	0	0	-1	-2
10	1.80	0	0	-1	-2
11	2.00	0	0	-1	-2
12	2.20	0	0	-1	-2
13	2.40	0	0	-1	-2
14	2.60	0	0	-1	-2
15	2.80	0	0	-1	-2
16	3.00	0	0	-1	-2
17	3.20	0	0	-1	-2
18	3.40	0	0	-1	-2
19	3.60	0	0	-1	-2
20	3.80	0	0	-1	-2
21	4.00	0	0	-1	-2

MODEL DISCRETIZATION

FIGURE C-1

Normal modes and periods as computed by EL-2D program are presented in table C.1.

Damping on each mode is computed from equation I.2.41. Table C.2 presents values of the complex impedance of each mode for different values of frequency ω.

Load vector $\underline{p}$ depends on the assumed loading. To obtain the transfer function of the bottom pressure for a top input pressure than $\underline{p} = \underline{0}$ but for coordinate number 21. Computations implied by equations C.5 and C.7 are presented on table C.3. Last column of the table is $|x_2|/|x_{st}|$ the modulus of the actual transfer function where

$$|x_{st}| = \frac{\sigma}{E} \times .15 = 2.5\ 10^{-5}\ \sigma$$

Figure C.2 shows the amplitude of the transfer function versus frequency.

Coord. Numb.	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
1	0.	0.	0.	0.	0.
2	0.785	0.233	0.383	0.522	-0.649
3	0.156	0.454	0.707	0.891	-0.988
4	0.233	0.649	0.924	0.997	-0.853
5	0.309	0.809	1.	0.809	-0.309
6	0.383	0.924	0.924	0.383	0.353
7	0.454	0.988	0.707	-0.156	0.891
8	0.522	0.997	0.383	-0.649	0.972
9	0.588	0.951	0.	-0.951	0.588
10	0.649	0.853	-0.383	-0.972	-0.078
11	0.707	0.707	-0.707	-0.707	-0.707
12	0.760	0.522	-0.924	-0.233	-0.997
13	0.809	0.309	-1.	0.309	-0.809
14	0.853	0.075	-0.924	0.760	-0.233
15	0.891	-0.156	-0.707	0.988	0.454
16	0.924	-0.383	-0.383	0.924	0.924
17	0.951	-0.588	0	0.588	0.951
18	0.972	-0.760	0.383	0.078	0.522
19	0.988	-0.891	0.707	-0.454	-0.156
20	0.997	-0.972	0.924	-0.853	-0.76
21	1.	-1.	1.	-1.	-1.
Period	$1.9601\ 10^{-3}$	$6.547\ 10^{-4}$	$3.944\ 10^{-4}$	$2.835\ 10^{-4}$	$2.223\ 10^{-4}$
$\omega_{r/s}$	3205.5	9596.7	15929.	22162.	28260.
$(\phi^T M \phi)_r$	$2.34\ 10^{-4}$	$2.34\ 10^{-4}$	$2.34\ 10^{-4}$	$2.34\ 10^{-4}$	$2.34\ 10^{-4}$

NORMAL MODES AND CORRESPONDING FREQUENCIES

TABLE C-1

		Mode 1		Mode 2		Mode 3		Mode 4		Mode 5	
	ω_r	3205.5		9596.7		15929.		22163.		26260.	
	M_r	$2.34\ 10^{-4}$		$2.34\ 10^{-4}$		$2.34\ 10^{-4}$		$2.34\ 10^{-4}$		$2.34\ 10^{-4}$	
	ζ_r	0.0107		0.0321		0.0535		0.0749		0.0963	
ω	$1/\lvert Z\rvert$ $\quad \alpha^{\circ}$										
1000		1.105	.424°	1.011	.397°	1.004	.386°	1.002	.39°	1.001	.39°
2000		1.637	1.252°	1.045	.501°	1.016	.78°	1.008	.78°	1.005	.79°
3205.5		46.73	90.00°	1.125	1.383°	1.042	1.29°	1.021	1.27°	1.013	1.27°
5000		.698	178.67°	1.371	2.63°	1.109	2.13°	1.053	2.04°	1.032	2.02°
7000		.265	179.29°	2.126	5.72°	1.237	3.34°	1.109	3.01°	1.064	2.91°
9596.7		.126	179.54°	15.576	90.°	1.562	5.78°	1.227	4.56°	1.127	4.23°
12000		.077	179.65°	1.757	171.9°	2.273	10.56°	1.406	6.55°	1.214	5.7°
14000		.055	179.7°	.883	175.3°	4.062	22.46°	1.644	6.95°	1.315	7.21°
15929		.042	179.74°	.569	176.5°	9.346	90.00°	2.019	12.56°	1.447	9.04°

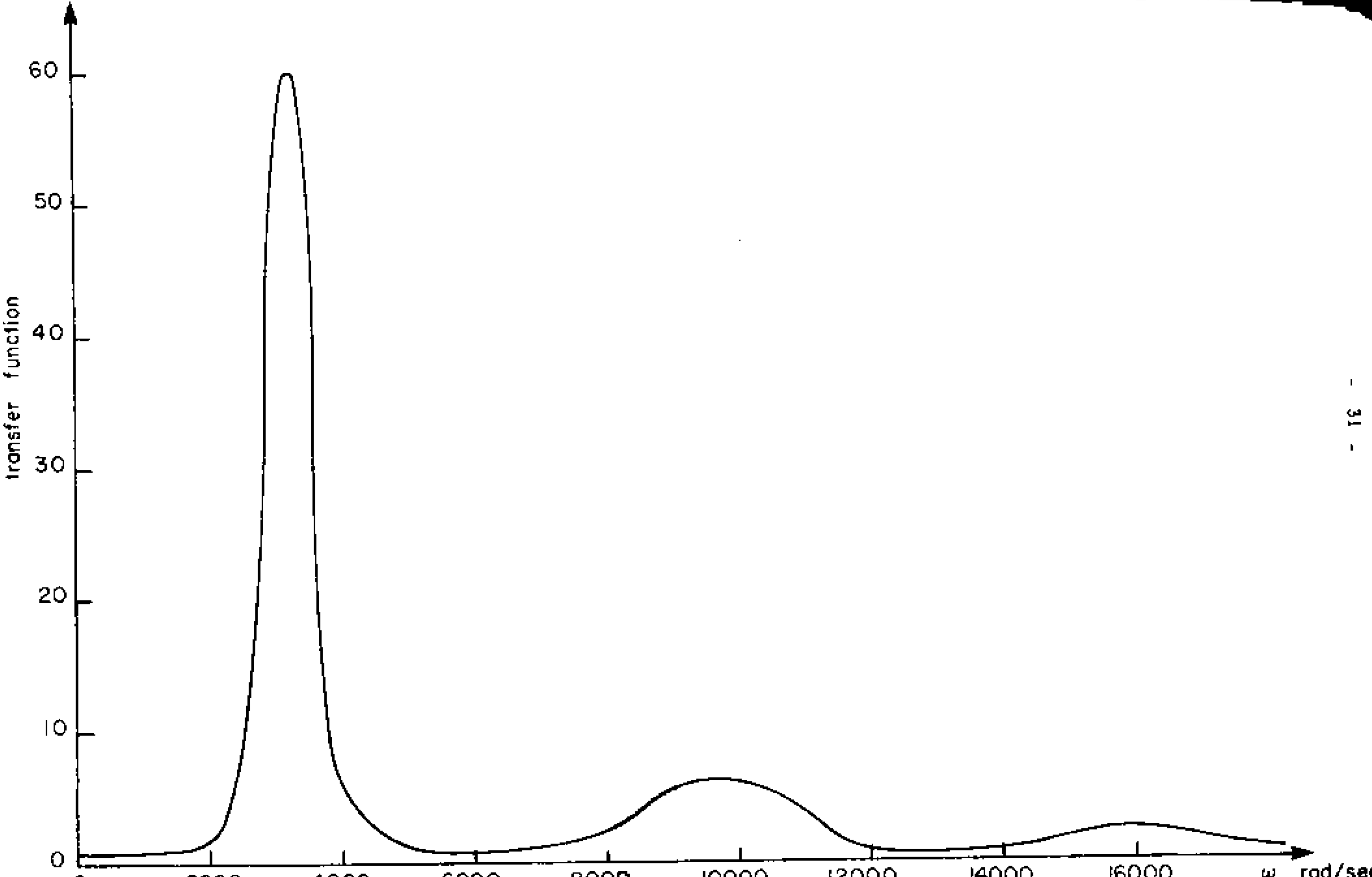
COMPLEX IMPEDANCE AS A FUNCTION OF FREQUENCY

TABLE C-2

		Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	x_2	$\lvert x_2\rvert/\lvert x_1\rvert$
	$\phi_r^T\ P$	1	-1	1	-1	-1		
	k_r	$2.403\ 10^{3}$	$2.154\ 10^{4}$	$5.935\ 10^{4}$	$1.149\ 10^{5}$	$1.866\ 10^{5}$		
ω								
1000	$\phi_{1r}\xi_{r_{st}}\frac{1}{\lvert Z_r\rvert}$	$3.62\ 10^{-5}$	$-1.09\ 10^{-5}$	$6.48\ 10^{-6}$	$-4.55\ 10^{-6}$	$-3.46\ 10^{-6}$	$2.37\ 10^{-5}$ $.44^{o}$	$.95 \simeq 1$
2000		$5.35\ 10^{-5}$	$-1.13\ 10^{-5}$	$6.56\ 10^{-6}$	$-4.58\ 10^{-6}$	$-3.49\ 10^{-6}$	$4.06\ 10^{-5}$ 1.39^{o}	1.62
3205		$1.53\ 10^{-3}$	$-1.22\ 10^{-5}$	$6.73\ 10^{-6}$	$-4.64\ 10^{-6}$	$-3.52\ 10^{-6}$	$1.53\ 10^{-3}$ 90.5^{o}	61.2
5000		$2.28\ 10^{-5}$	$-1.48\ 10^{-5}$	$7.16\ 10^{-6}$	$-4.78\ 10^{-6}$	$-3.58\ 10^{-6}$	$3.88\ 10^{-5}$ 179.7^{o}	1.55
7000		$8.66\ 10^{-6}$	$-2.3\ 10^{-5}$	$7.93\ 10^{-6}$	$-5.04\ 10^{-6}$	$-3.7\ 10^{-6}$	$3.24\ 10^{-5}$ 176.1^{o}	1.3
9596		$4.12\ 10^{-6}$	$-1.68\ 10^{-4}$	$1.00\ 10^{-5}$	$-5.57\ 10^{-6}$	$-3.92\ 10^{-6}$	$1.68\ 10^{-4}$ 91.2^{o}	6.7
12000		$2.52\ 10^{-6}$	$-1.9\ 10^{-5}$	$1.47\ 10^{-5}$	$-6.39\ 10^{-6}$	$-4.22\ 10^{-6}$	$2.02\ 10^{-5}$ 3.186^{o}	.81
14000		$1.80\ 10^{-6}$	$-9.55\ 10^{-6}$	$2.62\ 10^{-5}$	$-7.47\ 10^{-6}$	$-4.57\ 10^{-6}$	$2.14\ 10^{-5}$ 20.51	.86
15929		$1.37\ 10^{-6}$	$-6.15\ 10^{-6}$	$6.03\ 10^{-5}$	$-9.17\ 10^{-6}$	$-5.03\ 10^{-6}$	$5.79\ 10^{-5}$ 99.09	2.32

COMPUTATION OF THE TRANSFER FUNCTION

TABLE C-3



MODULUS OF THE COMPUTED TRANSFER FUNCTION

FIGURE C-2

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