Equivalence of 2-D Multitopic Category and Ana-bicategory

by
Parama Jyothi Reddappagari

Department of Mathematics and Statistics
McGill University, Montreal

June 2003

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for MS degree.

Acknowledgements

I am grateful to my supervisor Prof. M. Makkai who taught me Category Theory and Logic.

I thank the department of mathematics and statistics of McGill university for their collaboration and for their financial support.

A special thanks goes to Prof. V. S. Borkar who changed my perception about Mathematics during my study at Indian Institute of Science, Bangalore. I would like to thank Prof. M. A. L. Thathachar, Prof. P. S. Sastry who gave me encouragement during those days.

Abstract-English

In this thesis equivalence of the concepts of ana-bicategory and 2D-multitopic category is proved. The equivalence is FOLDS equivalence of the FOLDS-Specifications of the two concepts. Two constructions for transforming one category to another are given and it is shown that we get a structure equivalent to the original one when we compose two constructions.

Abstract-French

Dans cette thèse, l'équivalence de la catégorie de l'ana-bicategory et de la catégorie 2D-multitopic est prouvée. L'équivalence est une équivalence FOLDS. Deux constructions pour transformer une catégorie en l'autre est donnée. Il est montré qu'on obtient une structure équivalente à l'original loisqu'on compose les deux constructions.

Contents

Chapter I
Introduction and Preliminaries

I.1 Introduction

In category theory there is an emergence of higher dimensional categories. There are two distinct flavors of higher dimentional categories:

Pure algebraic: In these the composition of cells is defined by a composition function. Then there are huge coherence conditions about the way composition works. Examples include bicategory, tricategory, 2-category etc.

Virtual: In these composition of cells is defined by universal property of certain special cells called "universals". Examples are multitopic category, opetopic category etc.

Even for case $n = 3$ the pure algebraic version becomes intractable with lots of isomorphisms and coherence diagrams. Virtual version does not have the same problem and in certain sense it is "scalable". Another point to be remembered is that virtual version defines the composition "up to isomorphism" in true categorical spirit. In view of these advantages its tempting to propose virtual definitions of categories. To take the step of proposing virtual definitions of categories, we first need to show that for the case of $n = 1, 2$, virtual definition reduces to the ordinary definitions of category and bicategory respectively. In this thesis we consider multitopic category as the basic virtual definition for higher dimensional categories. We call multitopic category for case $n = 2$, which is the case being dealt here, as 2D-multitopic category.

In parallel there is another point to be made about category theory. Though it is emphasized in category theory that concepts should be defined "up to isomorphism", this does not go beyond structures internal to the category like limits, colimits etc. For example, in defining functor we do not say that it takes certain value up to isomorphism. In [4], Makkai has proposed a version of category theory in which external concepts like functors and natural transformations are also defined up to isomorphism. The functor there called anafunctor is defined up to isomorphism. This was extended to define bicategory as ana-bicategory. Another concept introduced in the same paper was saturation, which more or less means that functor can take any of the isomorphic copy of object as its value.

In light of what has been said, I feel that correct concept of bicategory is ana-bicateogry, and it is actually found to be the case that the 2D-multitopic category is equivalent to an ana-bicategory with saturation. This should not be surprising because the horizontal composition internal to 2D-multitopic category is defined by universal property hence up to isomorphism.

In next section we give the formal definitions that will be used. In chapter 2 we show how to get ana-bicategory from 2D-multitopic category. In chapter 3 the way to get 2D-multitopic category from ana-bicateogry is given. In chapter 4 the equivalence of these two definitions is shown.

I.2 Preliminaries

In this section we give the mathematical definitions that are required for subsequent chapters.

I.2.1 Ana-bicategory:

The concept of ana versions of categorical definitions was introduced in [4].

First the concept of AnaFunctor and Natural AnaTransformation are to be given. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.

AnaFunctor: An AnaFunctor F between categories $\mathcal{C}$ and $\mathcal{D}$ is given by following data (1,2) and conditions (3,4,5):

1) A class $|F|$, with two maps $\sigma : |F| \longrightarrow \mathcal{O}(\mathcal{C})$ (source) and $\tau : |F| \longrightarrow \mathcal{O}(\mathcal{D})$ (target). We use the following notation, for $X \in \mathcal{O}(\mathcal{C})$ we denote $|F|(X) = \{s \in |F| : \sigma(s) = X\}$, and for $s \in |F|(X)$, we denote $\tau(s)$ by $F_s(X)$. $|F|$ is called class of specifications.

2) For each $X, Y \in \mathcal{O}(\mathcal{C})$, $s \in |F|(X)$, $t \in |F|(Y)$ and $f : X \longrightarrow Y$, an arrow $F_{s,t}(f) : F_s(X) \longrightarrow F_t(Y)$ in $\mathcal{D}$.

3) For every $X \in \mathcal{O}(\mathcal{C})$, $|F|(X)$ is non-empty.

4) For all $X \in \mathcal{O}(\mathcal{C})$ and $s \in |F|(X)$, $F_{s,s}(Id_X) = Id_{F_s(X)}$.

5) For all $X, Y, Z \in \mathcal{O}(\mathcal{C})$, $s \in |F|(X)$, $t \in |F|(Y)$, $u \in |F|(Z)$, $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, we have $F_{s,u}(f \cdot g) = F_{s,t}(f) \cdot F_{t,u}(g)$.

Saturated Anafunctor: Given an anafunctor F, F is said to be saturated, if $F_s(X) = A$ and $i : A \cong B$ in $\mathcal{D}$ then there is unique $t \in |F|(X)$ such that $F_t(X) = B$ and $F_{s,t}(Id_X) = i$.

Saturation is something external on anafunctor but usually anafunctors that arise naturally have this saturation property, for example product AnaFunctor.

Natural AnaTransformation: A Natural AnaTransformation ϕ between functors F and G is given by following data (1) and condition (2):

1) A family $< \phi_{X,s,t} : F_s(X) \longrightarrow G_t(X) >_{X \in \mathcal{O}(\mathcal{C}), s \in |F|(X), t \in |G|(X)}$.

2) For every $f : X \longrightarrow Y$ in $\mathcal{C}(X, Y)$, and for every $s \in |F|(X)$, $t \in |G|(X)$, $u \in |F|(Y)$, $v \in |G|(Y)$, then following diagram commutes

$$\begin{array}{ccc} F_s(X) & \xrightarrow{F_{s,u}(f)} & F_u(Y) \\ \downarrow{\scriptstyle \phi_{X,s,t}} & & \downarrow{\scriptstyle \phi_{Y,u,v}} \\ G_t(X) & \xrightarrow[G_{t,v}(f)]{} & G_v(Y) \end{array}$$

Natural Anaisomorphism: Natural Anaisomorphism is a Natural AnaTransformation is which the family $< \phi_{X,s,t} : F_s(X) \longrightarrow G_t(X) >_{X \in \mathcal{O}(\mathcal{C}), s \in |F|(X), t \in |G|(X)}$ consists of isomorphisms.

Ana-bicategory: An ana-bicategory $\mathcal{A}$ consists of following data (1,2,3,4,5,6,7) and conditions (8,9):

1) Collection $\mathcal{O}(\mathcal{A})$ of objects (0-cells).

2) For any pair of objects $A, B \in \mathcal{O}(\mathcal{A})$, a category $\mathcal{A}(A, B)$ (1-cells as its objects and 2-cells as its arrows).

3) For any object $A \in \mathcal{O}(\mathcal{A})$, an identity anaobject in $\mathcal{A}(A, A)$, determined by anafunctor

$$1_A : \mathbf{1} \longrightarrow \mathcal{A}(A, A)$$

4) For any three objects $A, B, C \in \mathcal{O}(\mathcal{A})$, composition anafunctor

$$\circ_{A,B,C} : \mathcal{A}(A, B) \times \mathcal{A}(B, C) \longrightarrow \mathcal{A}(A, C)$$

5) Associativity natural anaisomorphism

$$\alpha_{A,B,C,D} : ((-) \circ (-)) \circ (-) \overset{\cong}{\Longrightarrow} (-) \circ ((-) \circ (-))$$

where $((-) \circ (-)) \circ (-) = (\circ_{A,B,C}, Id_{\mathcal{A}(C,D)}) \cdot \circ_{A,C,D}$ and $(-) \circ ((-) \circ (-)) = (Id_{\mathcal{A}(A,B)}, \circ_{B,C,D}) \cdot \circ_{A,B,D}$.

6) Left identity natural anaisomorphism

$$\lambda_{A,B} : (-) \circ 1_B \overset{\cong}{\Longrightarrow} Id_{\mathcal{A}(A,B)}$$

where $Id_{\mathcal{A}(A,B)}$ is Identity Functor and $(-) \circ 1_B = (Id_{\mathcal{A}(A,B)}, ! \cdot 1_B) \cdot \circ_{A,B,B} : \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, B)$.

7) Right identity natural anaisomorphism

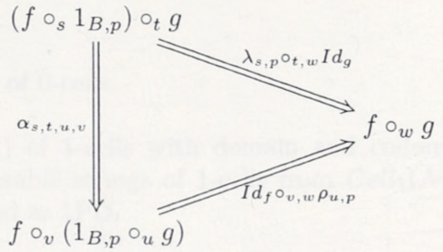
$$\rho_{A,B} : 1_A \circ (-) \overset{\cong}{\Longrightarrow} Id_{\mathcal{A}(A,B)}$$

where $Id_{\mathcal{A}(A,B)}$ is Identity Functor and $1_A \circ (-) = (! \cdot 1_A, Id_{\mathcal{A}(A,B)}) \cdot \circ_{A,A,B} : \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, B)$.

8) For any five objects $A, B, C, D, E \in \mathcal{O}(\mathcal{A})$, and four 1-cells $f \in \mathcal{O}(\mathcal{A}(A, B))$, $g \in \mathcal{O}(\mathcal{A}(B, C))$, $h \in \mathcal{O}(\mathcal{A}(C, D))$, $i \in \mathcal{O}(\mathcal{A}(D, E))$, the coherence pentagon:

$$\begin{array}{ccccc}
((f \circ_1 g) \circ_2 h) \circ_3 i & \xrightarrow{\alpha_{1,2,4,5} \circ_{3,6} Id_i} & (f \circ_5 (g \circ_4 h)) \circ_6 i & \xrightarrow{\alpha_{5,6,7,8}} & f \circ_8 ((g \circ_4 h) \circ_7 i) \\
\Big\Downarrow \alpha_{2,3,10,12} & & & & \Big\Downarrow Id_f \circ_{8,9} \alpha_{4,7,10,11} \\
(f \circ_1 g) \circ_{12} (h \circ_{10} i) & & \xrightarrow[\alpha_{1,12,11,9}]{} & & f \circ_9 (g \circ_{11} (h \circ_{10} i))
\end{array}$$

9) For any two objects $A, B \in \mathcal{O}(\mathcal{A})$, and two 1-cells $f \in \mathcal{O}(\mathcal{A}(A,B))$, $g \in \mathcal{O}(\mathcal{A}(B,C))$, the coherence triangle:



Remark Saturation:. Ana-bicategory is said to be saturated if anafunctors 1_A and $\circ_{A,B,C}$ are.

I.2.2 2D-Multitopic category:

The concept of multitopic category was introduced in **[1,2,3]**. Since here we are concerned with only 2 dimensional case we simplify the definition by removing all the amalgamation mechanism that was built into its definition. First we define what a multicategory is. Then we go to definition of 2D-multitopic category.

Multicategory: Multicategory $\mathcal{M}$ consists of following data (1,2,3) and conditions (4,5,6):

1) A collection $\mathcal{O}(\mathcal{M})$ of objects. We denote by $\mathcal{O}(\mathcal{M})^*$ to be collection of all tuples (strings) of objects.

2) A collection $\mathcal{A}(\mathcal{M})$ of arrows with domain in $\mathcal{O}(\mathcal{M})^*$ and codomain in $\mathcal{O}(\mathcal{M})$.

3) An natural number indexed partially defined composition $\cdot_{\_}$ of arrows in $\mathcal{A}(\mathcal{M})$. Composition of $\alpha, \beta \in \mathcal{A}(\mathcal{M})$ is defined if and only if the codomain of α fits into domain of β. Formally, if the domain and codomain of α are $\bar{f}_\alpha$ and g_α and the domain and codomain of β are $\bar{f}_\beta$ and g_β such that $\bar{f}_\beta(i) = g_\alpha$ then composite $\alpha \cdot_i \beta$ is defined; it is an arrow in $\mathcal{A}(\mathcal{M})$ with domain $\bar{f}_\beta[\bar{f}_\alpha/(i, i+1)]$ and codomain g_β.

Remark 3. In here and subsequent places $s[t/(i,j)]$ is string formed by replacing i^{th} to $j-1^{th}$ substring of s by t.

Remark 4. From here on for convenience the subscript for composition is removed. But it should be kept in mind that composition is placed.

4) Composition is associative i.e. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

5) Composition is commutative i.e. $\alpha \cdot (\beta \cdot \gamma) = \beta \cdot (\alpha \cdot \gamma)$

6) For any $f \in \mathcal{O}(\mathcal{M})$, there is an identity in $\mathcal{A}(\mathcal{M})$ with domain the string $< f >$ and codomain f denoted as Id_f such that, for any appropriate α, $Id_f \cdot \alpha = \alpha = \alpha \cdot Id_f$.

2D-multitopic category: A 2D-Multitopic category consists of following data (1,2,3,4) and condition (5):

1) A collection $Cell_0(\mathcal{M})$ of 0-cells.

2) A collection $Cell_1(\mathcal{M})$ of 1-cells with domain and codomain in $Cell_0(\mathcal{M})$. We denote by $Cell_1(\mathcal{M})^*$ collection of all composable strings of 1-cells from $Cell_1(\mathcal{M})$. $Cell_1(\mathcal{M})$ is referred to as (1-) pasting diagrams, also abbreviated as 1PD.

3) A collection $Cell_2(\mathcal{M})$ of 2-cells with domain in $Cell_1(\mathcal{M})^*$ and codomain in $Cell_1(\mathcal{M})$, such that their initial and terminal 0-cells match.

4) The collections $Cell_1(\mathcal{M})$ and $Cell_2(\mathcal{M})$ form a multicategory with composition $\cdot$.

5) For every $\bar{f} \in Cell_1(\mathcal{M})^*$, there exists a 2-cell say $\alpha \in Cell_2(\mathcal{M})$, with domain $\bar{f}$, such that for every $\beta \in Cell_2(\mathcal{M})$ with domain containing the string $\bar{f}$, there is a unique $\gamma \in Cell_2(\mathcal{M})$, for which $\alpha \cdot \gamma = \beta$. Such an α is called universal arrow.

I.2.3 FOLDS Equivalence:

FOLDS stands for First Order Logic with Dependent Sorts. Here just a short overview will be given. Details are in [5], [7].

A FOLDS theory (L, Σ) consists of a signature L and set of axioms Σ. The FOLDS signature L is a one way category. The objects in this category are the sorts. Each sort is dependent on all the sorts that are below it (an arrow to it). For an object A in L, let $L \downarrow A$ be set of all arrows in L with domain A. The axioms in Σ are first order sentences with restriction. The restriction is that equality is disallowed and all the statements are about existence of certain elements in sorts. For example instead of saying $g \circ f = h$, we would say $\exists \tau \in T(X, Y, Z; f, g, h).T$. The advantage is that all axioms turn out to be asserting existence of certain element that represents the truth of axiom.

Now FOLDS structure S is a functor from L to any category, that satisfies the axioms in Σ. Given two L structures S, T, a homomorphism p is a natural transformation from S to T.

Two FOLDS structures S, T with same signature are said to be equivalent if there is a span

$$S \xleftarrow{p} Q \xrightarrow{q} T$$

where p, q are natural transformations and are fiberwise surjective. This is denoted as $S \simeq_L T$.

$p : S \longrightarrow T$ is fiberwise surjective if the following diagram is a weak pullback, for all objects K in L.

$$\begin{array}{ccc} S(K) & \xrightarrow{p_K} & T(K) \\ {\scriptstyle \pi_{K,S}}\downarrow & & \downarrow{\scriptstyle \pi_{K,T}} \\ S(\dot{K}) & \xrightarrow[p_{\dot{K}}]{} & T(\dot{K}) \end{array}$$

$\dot{K}$ is the context of sort K. Intuitively context is the sorts on which the sort K depends. $\pi_{K,T}$ is projection of context values from the sort K.

Up to this point we have seen the equivalence of two structures with same signatures. To compare two structures with different signatures we need something more **[6]**.

Suppose we have two theories $T_1 = (L_1, \Sigma_1)$. To say that T_1 and T_2 are equivalent we need two constructions, one taking any T_1-model S_1 to a T_2-model S_1^* and another taking any T_2-model S_2 to T_1-model $S_2^{\#}$. Now we say that T_1 and T_2 are equivalent whenever $S_1 \simeq_{L_1} S_1^{*\#}$ and $S_2 \simeq_{L_2} S_2^{\#*}$ for all S_1 and S_2.

The constructions $(-)^*$ and $(-)^{\#}$ are canonical; in particular, they do not use the axiom of choice. More over, the data for the equivalences $S_1 \simeq_{L_1} S_1^{*\#}$ and $S_2 \simeq_{L_2} S_2^{\#*}$ are also canonically constructed from S_1 respectively S_2. In fact, the combined constructions add up to an equivalence of the two concepts: ana-bicategory and 2D-multitopic category, in the sense of **[6], Section 6**.

In Chapter II, the construction of an ana-bicategory from a 2D-multitopic category is given. In Chapter III, the construction of a 2D-multitopic category from the ana-bicategory is given. In Chapter IV, FOLDS Equivalence of ana-bicategory and 2D-multitopic category is proved, using the constructions in Chapter II and Chapter III.

Chapter II
2D-Multitopic Category to Ana-BiCategory

In this chapter, the construction of an ana-bicategory from a 2D-multitopic category is given. This construction will be dentoted as $\mathcal{M} \overset{(-)^*}{\longmapsto} \mathcal{M}^*$, where $\mathcal{M}$ is the given 2D-multitopic category. For simplicity, in this chapter $\mathcal{M}^*$ will be denoted by $\mathcal{A}$. The construction first involves extraction of ana-bicategory data from a 2D-multitopic category and then proving the axioms of ana-bicategory.

II.1 Data Definitions

Given a multitopic category $\mathcal{M}$, data of the associated ana-bicategory $\mathcal{A}$ is defined as follows:

Objects: $\mathcal{O}(\mathcal{A}) = Cell_0(\mathcal{M})$.

Category $\mathcal{A}(A, B)$**:** For $A, B \in \mathcal{O}(\mathcal{A})$, category $\mathcal{A}(A, B)$

Objects: $\mathcal{O}(\mathcal{A}(A, B)) = \{f : f \text{ is 1 cell of the form } A \xrightarrow{f} B \in Cell_1(\mathcal{M})\}$.

Arrows: $\mathcal{A}(A, B)[f, g] = \{\beta : \beta \text{ is 2 cell of the form } A \underset{g}{\overset{f}{\Downarrow \beta}} B \in Cell_2(\mathcal{M})\}$.

Identity: $Id_f = A \underset{f}{\overset{f}{\Downarrow Id_f}} B \in Cell_2(\mathcal{M})$

Composition: Composition of arrows is defined as composition of 2-cells in $\mathcal{M}$ restricted to $\mathcal{A}(A, B)$.

Identity: Identity AnaObject for A in $\mathcal{A}(A, A)$, AnaFunctor

$$1_A : \mathbf{1} \longrightarrow \mathcal{A}(A, A)$$

0-Specifications: $|1_A|(1) = \{p : p \in Cell_2(\mathcal{M}) \text{ is universal from empty pd } A\}$. These are of the form shown in **Figure 1**.

Remark 1. In all the figures here on the universal cells will be denoted by $\odot$ in the center.

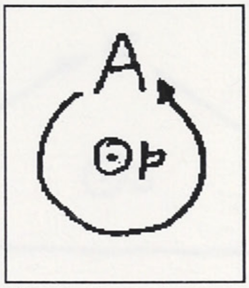


Figure 1.

AnaFunctor on Objects: $1_{A,p}(1) = codom(p)$ where $p \in |1_A|(1)$. Since 1 is the only object in **1**, we denote $1_{A,p}(1)$ by $1_{A,p}$.

AnaFunctor on Arrows: $1_{A,p,q}(Id_1) = \delta$, where $p, q \in |1_A|(1)$, and $\delta \in \mathcal{A}(A,A)[1_{A,p}, 1_{A,q}]$ such that $p \cdot \delta = q$. Since Id_1 is only arrow in **1**, we denote $1_{A,p,q}(Id_1)$ by $1_{A,p,q}$. See **Figure 2**.

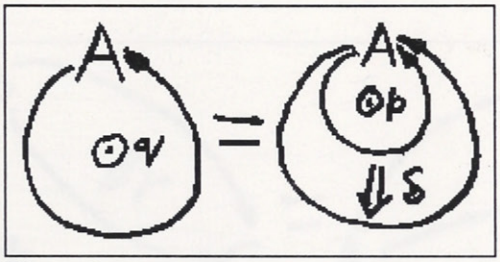


Figure 2.

Horizontal Composition: Composition AnaFunctor,

$$\circ_{A,B,C} : \mathcal{A}(A,B) \times \mathcal{A}(B,C) \longrightarrow \mathcal{A}(A,C)$$

Since A, B, C will be clear from the context, $\circ_{A,B,C,-}$ and $\circ_{A,B,C,-,-}$ will be refered to as $\circ_-$ and $\circ_{-,-}$ respectively. Furthermore, $\circ_-(f,g)$ and $\circ_{-,-}(\beta,\gamma)$ will be denoted in the *infix* form as $f \circ_- g$ and $\beta \circ_{-,-} \gamma$ respectively.

2-Specifications: $|\circ_{A,B,C}|(f,g) = \{s : s \in Cell_2(\mathcal{M})$ is universal from $A \xrightarrow{f} B \xrightarrow{g} C\}$. These are of the form shown in **Figure 3**.

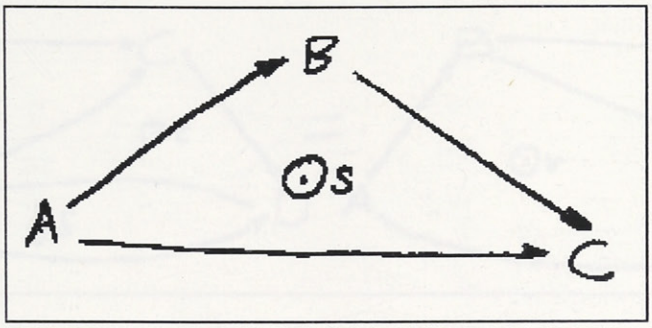


Figure 3.

AnaFunctor on Objects: $f \circ_s g = codom(s)$ where $s \in |\circ_{A,B,C}|(f,g)$.

AnaFunctor on Arrows: $\beta \circ_{s,t} \gamma = \delta$, where $s \in |\circ_{A,B,C}|(f_1,g_1)$, $t \in |\circ_{A,B,C}|(f_2,g_2)$, $(f_1,g_1),(f_2,g_2) \in \mathcal{A}(A,B) \times \mathcal{A}(B,C)$, $(\beta,\gamma) : (f_1,g_1) \Longrightarrow (f_2,g_2)$, and $\delta \in \mathcal{A}(A,C)[f_1 \circ_s g_1, f_2 \circ_t g_2]$ such that $\beta \cdot (\gamma \cdot t) = s \cdot \delta$. See **Figure 4**.

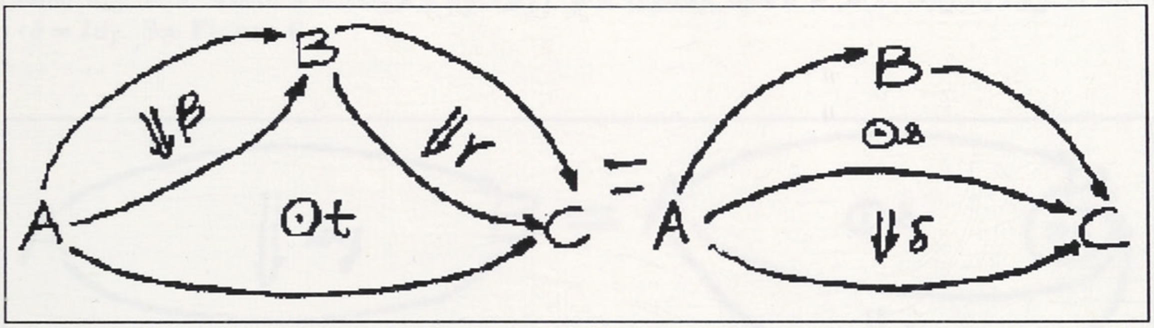


Figure 4.

Associativity Isomorphisms: Natural AnaIsomorphism

$$\alpha_{A,B,C,D} : ((-) \circ (-)) \circ (-) \xrightarrow{\cong} (-) \circ ((-) \circ (-))$$

where $((-) \circ (-)) \circ (-) = (\circ_{A,B,C}, Id_{\mathcal{A}(C,D)}) \cdot \circ_{A,C,D}$ and $(-) \circ ((-) \circ (-)) = (Id_{\mathcal{A}(A,B)}, \circ_{B,C,D}) \cdot \circ_{A,B,D}$. Since A,B,C,D will be clear from the context $\alpha_{A,B,C,D}$ will be denoted by α.

Define $\alpha_{s,t,u,v} = \delta$, where $s \in |\circ_{A,B,C}|(f,g)$, $t \in |\circ_{A,C,D}|(f \circ_s g, h)$, $u \in |\circ_{B,C,D}|(g,h)$, $v \in |\circ_{A,B,D}|(f, g \circ_u h)$ and $\delta \in \mathcal{A}(A,D)[(f \circ_s g) \circ_t h, f \circ_v (g \circ_u h)]$ such that $(s \cdot t) \cdot \delta = u \cdot v$. See **Figure 5**.

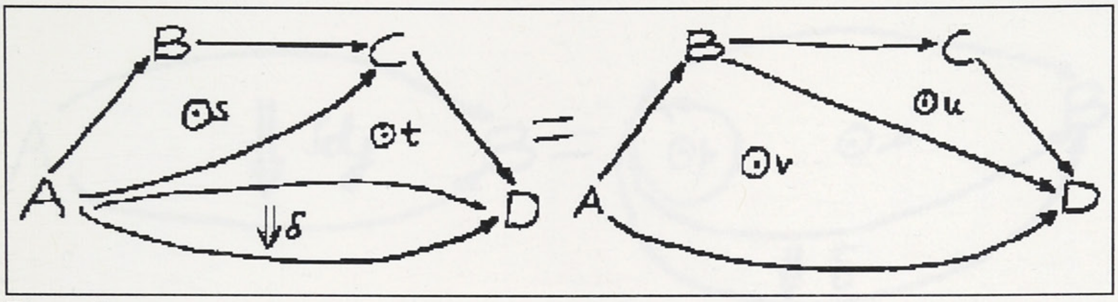


Figure 5.

Left Identity Isomorphisms: Natural AnaIsomorphism

$$\lambda_{A,B} : (-) \circ 1_B \overset{\cong}{\Longrightarrow} Id_{\mathcal{A}(A,B)}$$

where $Id_{\mathcal{A}(A,B)}$ is Identity Functor and $(-) \circ 1_B = (Id_{\mathcal{A}(A,B)}, ! \cdot 1_B) \cdot \circ_{A,B,B} : \mathcal{A}(A,B) \longrightarrow \mathcal{A}(A,B)$. Since A, B will be clear from the context $\lambda_{A,B}$ will be denoted by λ.

Define $\lambda_{s,p} = \delta$, where $s \in |\circ_{A,B,B}|(f, 1_{B,p})$, $p \in |1_B|(1)$, and $\delta \in \mathcal{A}(A,B)[f \circ_s 1_{B,p}, f]$ such that $(p \cdot s) \cdot \delta = Id_f$. See **Figure 6**.

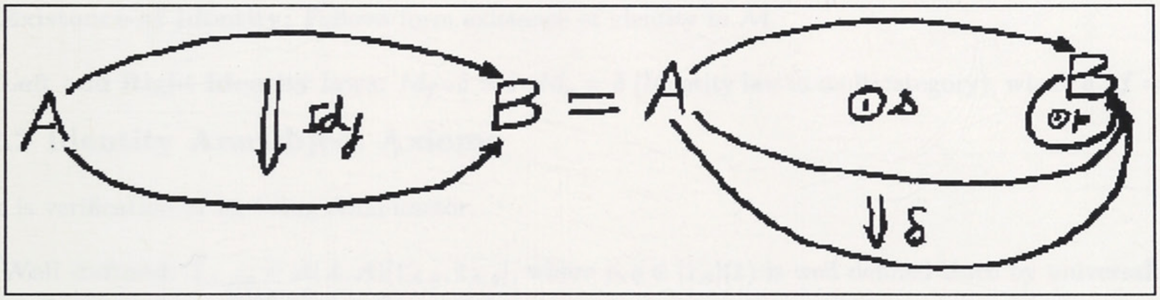


Figure 6.

Right Identity Isomorphisms: Natural AnaIsomorphism

$$\rho_{A,B} : 1_A \circ (-) \overset{\cong}{\Longrightarrow} Id_{\mathcal{A}(A,B)}$$

where $Id_{\mathcal{A}(A,B)}$ is Identity Functor and $1_A \circ (-) = (! \cdot 1_A, Id_{\mathcal{A}(A,B)}) \cdot \circ_{A,A,B} : \mathcal{A}(A,B) \longrightarrow \mathcal{A}(A,B)$. Since A, B will be clear from the context $\rho_{A,B}$ will be denoted by ρ.

Define $\rho_{s,p} = \delta$, where $s \in |\circ_{A,A,B}|(1_{A,p}, f)$, $p \in |1_A|(1)$, and $\delta \in \mathcal{A}(A,B)[1_{A,p} \circ_s f, f]$ such that $(p \cdot s) \cdot \delta = Id_f$. See **Figure 7**.

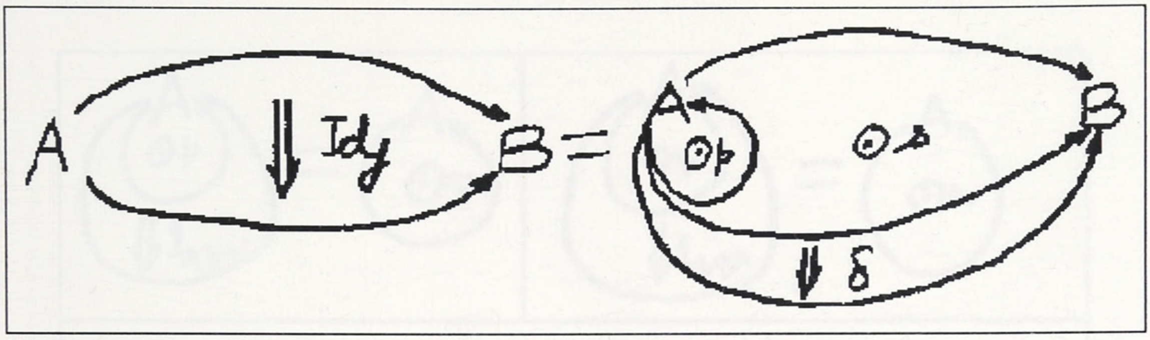


Figure 7.

II.2 Ana-BiCategory Axioms

In this section the axioms of Ana-BiCategory are verified for the data defined the the previous section.

II.2.1 Category Axioms

This is verification of $\mathcal{A}(A, B)$ being a Category.

Associativity: Follows from associativity in multicategory $\mathcal{M}$.

Existence of Identity: Follows form existence of identity in $\mathcal{M}$.

Left and Right Identity laws: $Id_f \circ \delta = \delta \circ Id_g = \delta$ (Identity law in multicategory), where $\delta : f \Longrightarrow g$.

II.2.2 Identity AnaObject Axioms

This is verification of 1_A being AnaFunctor.

Well defined: $1_{A,p,q} \in \mathcal{A}(A, A)[1_{A,p}, 1_{A,q}]$, where $p, q \in |1_A|(1)$ is well defined since by universality of p there is unique $1_{A,p,q}$ such that $p \cdot 1_{A,p,q} = q$.

Inhabitedness: $|1_A|(1)$ is nonempty from existence of universal from every PD, in particular from PD A.

Identity: $1_{A,p,p}(Id_1) = Id_{1_{A,p}}$, since $p \cdot Id_{1_{A,p}} = p$, where $p \in |1_A|(1)$.

Composition: Need to show $1_{A,p,q} \cdot 1_{A,q,r} = 1_{A,p,r}$, where $p, q, r \in |1_A|(1)$. We have $p \cdot 1_{A,p,q} = q$ and $q \cdot 1_{A,q,r} = r$, $p \cdot 1_{A,p,r} = r$ from definition. See **Figure 8**.

$$\begin{aligned} p \cdot (1_{A,p,q} \cdot 1_{A,q,r}) &= (p \cdot 1_{A,p,q}) \cdot 1_{A,q,r} \\ &= q \cdot 1_{A,q,r} \\ &= r \\ &= p \cdot 1_{A,p,r} \end{aligned}$$

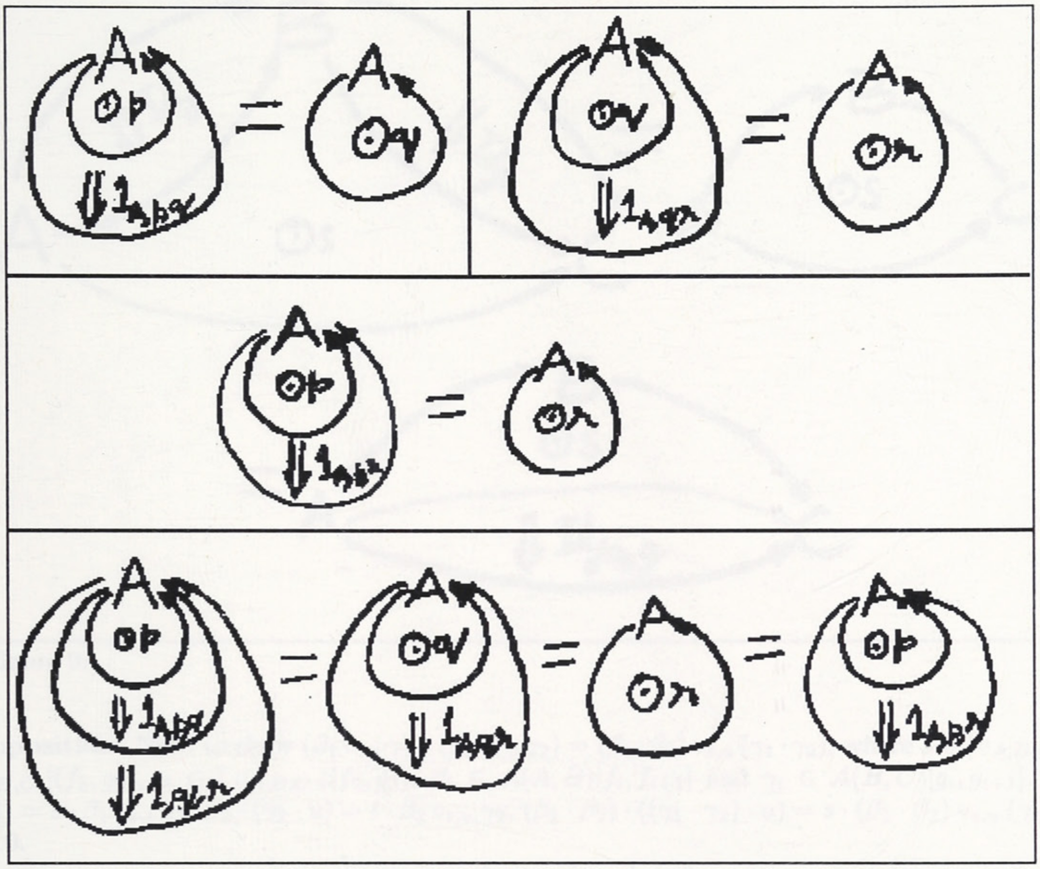


Figure 8.

Since universals are left cancellable, we have $1_{A,p,q} \cdot 1_{A,q,r} = 1_{A,p,r}$.

II.2.3 Composition AnaFunctor:

This is verification of $\circ$ being an AnaFunctor.

Well defined: $\beta \circ_{s,t} \gamma \in \mathcal{A}(A,C)[f_1 \circ_s g_1, f_2 \circ_t g_2]$, where $s \in |\circ_{A,B,C}|(f_1, g_1)$, $t \in |\circ_{A,B,C}|(f_2, g_2)$, $\beta \in \mathcal{A}(A,B)[f_1, f_2]$ and $\gamma \in \mathcal{A}(B,C)[g_1, g_2]$ is well defined since by universality of s, there is unique $\beta \circ_{s,t} \gamma$ such that $\beta \cdot (\gamma \cdot t) = s \cdot \beta \circ_{s,t} \gamma$.

Inhabitedness: $|\circ_{A,B,C}|(f,g)$ is non empty from existence of universal from every PD, in particular from PD $A \xrightarrow{f} B \xrightarrow{g} C$

Identity: $Id_f \circ_{s,s} Id_g = Id_{f \circ_s g}$, since $Id_f \cdot (Id_g \cdot s) = Id_f \cdot s = s = s \cdot Id_{f \circ_s g}$. See **Figure 9**.

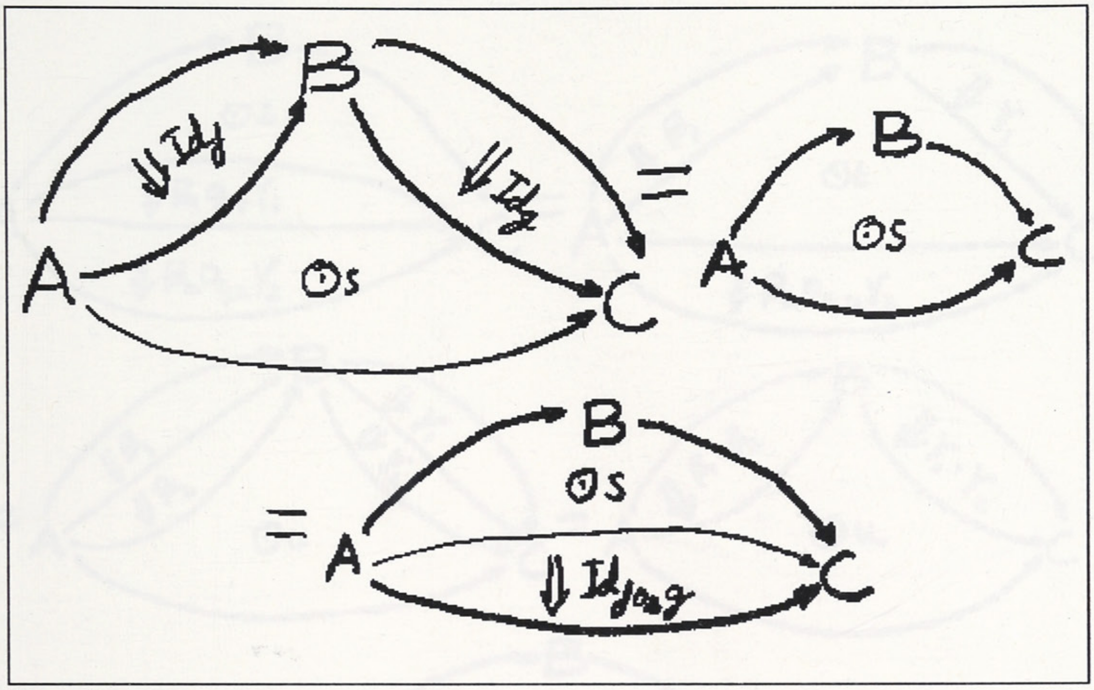


Figure 9.

Composition: Need to show $(\beta_1 \circ_{s,t} \gamma_1) \cdot (\beta_2 \circ_{t,u} \gamma_2) = (\beta_1 \cdot \beta_2) \circ_{s,u} (\gamma_1 \cdot \gamma_2)$, where $s \in |\circ_{A,B,C}|(f_1, g_1)$, $t \in |\circ_{A,B,C}|(f_2, g_2)$, $u \in |\circ_{A,B,C}|(f_3, g_3)$, $\beta_i \in \mathcal{A}(A,B)[f_i, f_{i+1}]$ and $\gamma_i \in \mathcal{A}(B,C)[g_i, g_{i+1}]$. We have $\beta_1 \cdot (\gamma_1 \cdot t) = s \cdot \beta_1 \circ_{s,t} \gamma_1$, $\beta_2 \cdot (\gamma_2 \cdot u) = t \cdot \beta_2 \circ_{t,u} \gamma_2$, $(\beta_1 \cdot \beta_2) \cdot ((\gamma_1 \cdot \gamma_2) \cdot u) = s \cdot (\beta_1 \cdot \beta_2) \circ_{s,u} (\gamma_1 \cdot \gamma_2)$. See **Figure 10**.

$$\begin{aligned}
s \cdot ((\beta_1 \circ_{s,t} \gamma_1) \cdot (\beta_2 \circ_{t,u} \gamma_2)) &= (s \cdot (\beta_1 \circ_{s,t} \gamma_1)) \cdot (\beta_2 \circ_{t,u} \gamma_2) \\
&= (\beta_1 \cdot (\gamma_1 \cdot t)) \cdot (\beta_2 \circ_{t,u} \gamma_2) \\
&= \beta_1 \cdot ((\gamma_1 \cdot t) \cdot (\beta_2 \circ_{t,u} \gamma_2)) \\
&= \beta_1 \cdot (\gamma_1 \cdot (t \cdot (\beta_2 \circ_{t,u} \gamma_2))) \\
&= \beta_1 \cdot (\gamma_1 \cdot (\beta_2 \cdot (\gamma_2 \cdot u))) \\
&= \beta_1 \cdot (\beta_2 \cdot (\gamma_1 \cdot (\gamma_2 \cdot u))) \\
&= \beta_1 \cdot (\beta_2 \cdot ((\gamma_1 \cdot \gamma_2) \cdot u)) \\
&= (\beta_1 \cdot \beta_2) \cdot ((\gamma_1 \cdot \gamma_2) \cdot u) \\
&= s \cdot (\beta_1 \cdot \beta_2) \circ_{s,u} (\gamma_1 \cdot \gamma_2)
\end{aligned}$$

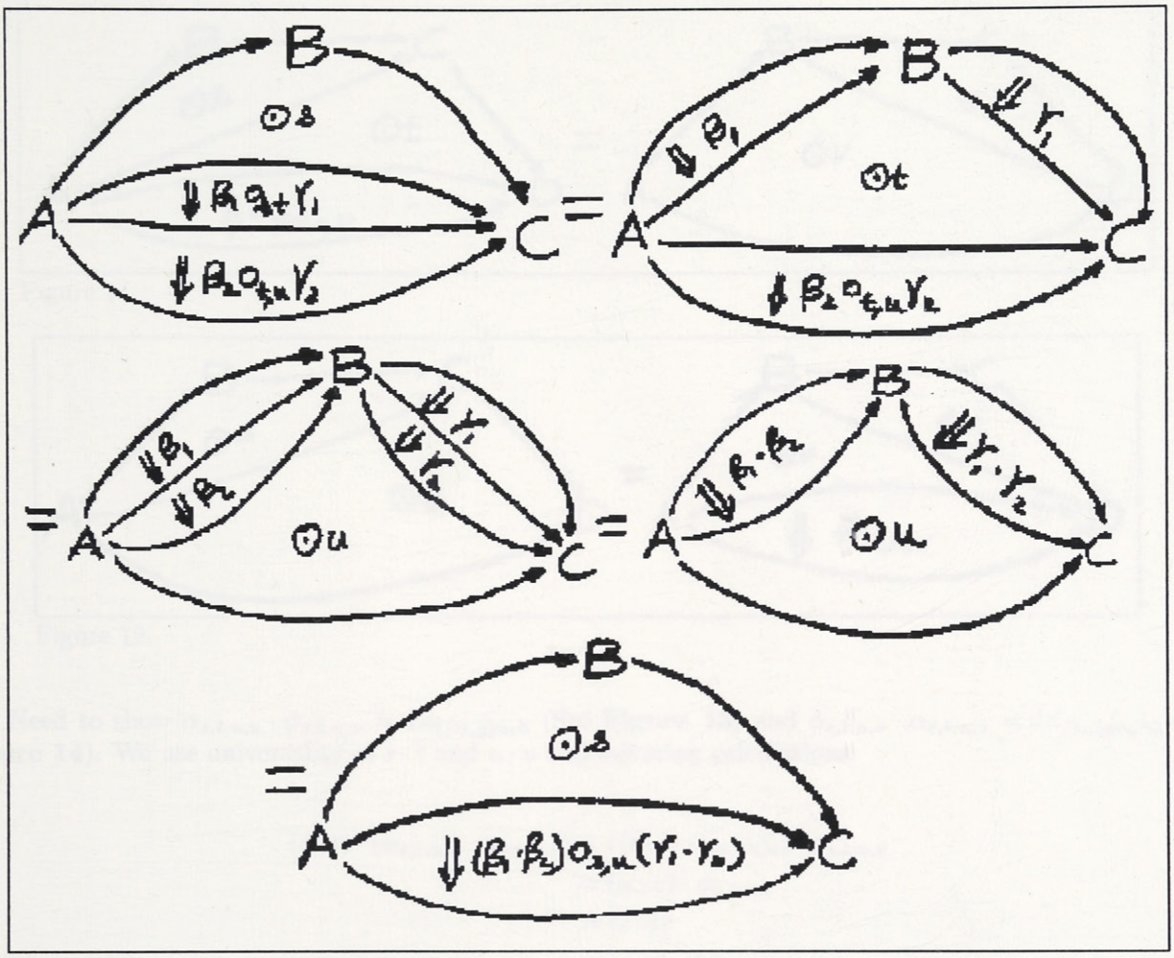


Figure 10.

Since universals are left cancellable, we have $(\beta_1 \circ_{s,t} \gamma_1) \cdot (\beta_2 \circ_{t,u} \gamma_2) = (\beta_1 \cdot \beta_2) \circ_{s,u} (\gamma_1 \cdot \gamma_2)$.

II.2.4 Associativity Isomorphims

Well defined: $\alpha_{s,t,u,v} \in \mathcal{A}(A, D)[(f \circ_s g) \circ_t h, f \circ_v (g \circ_u h)]$, where $s \in |\circ_{A,B,C}|(f, g)$, $t \in |\circ_{A,C,D}|(f \circ_s g, h)$, $u \in |\circ_{B,C,D}|(g, h)$, $v \in |\circ_{A,B,D}|(f, g \circ_u h)$ is well defined, since $s \cdot t$ is composite of universals hence a universal. So, there is unique $\alpha_{s,t,u,v}$ such that $(s \cdot t) \cdot \alpha_{s,t,u,v} = u \cdot v$.

Isomorphism: $\alpha_{s,t,u,v}$ is invertible. Its inverse $\phi_{s,t,u,v}$ is such that $s \cdot t = (u \cdot v) \cdot \phi_{s,t,v,u})$. $\phi_{s,t,u,v}$ is well defined, since $u \cdot v$ is composite of universals, so there is unique $\phi_{s,t,u,v}$ such that $(u \cdot v) \cdot \phi_{s,t,u,v} = s \cdot t$. See **Figure 11, 12**.

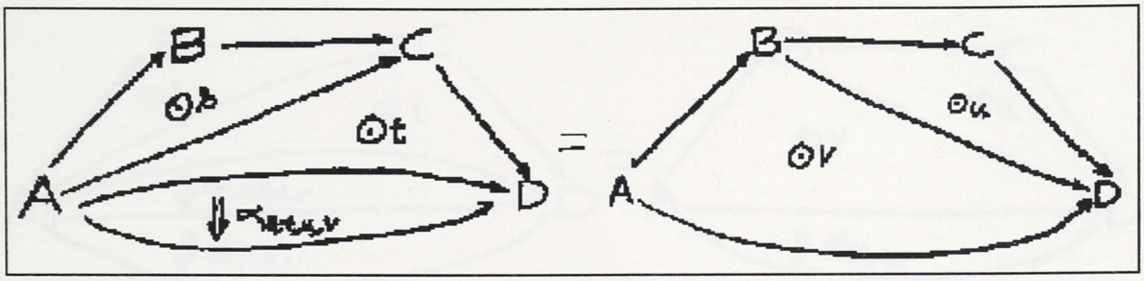


Figure 11.

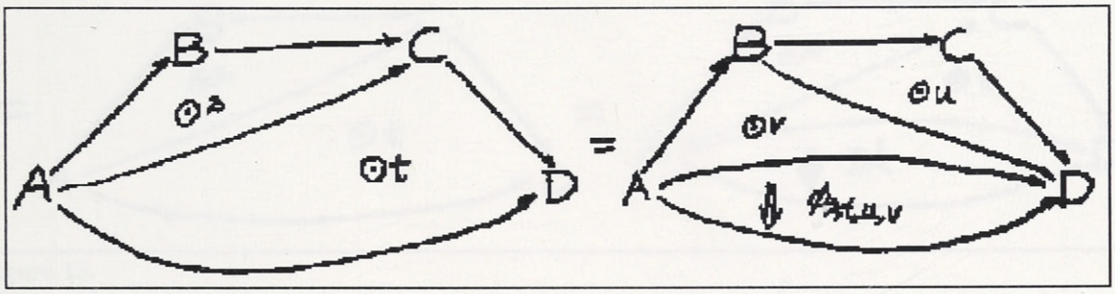


Figure 12.

Need to show $\alpha_{s,t,u,v} \cdot \phi_{s,t,u,v} = Id_{(f \circ_s g) \circ_t h}$ (See **Figure 13**) and $\phi_{s,t,u,v} \cdot \alpha_{s,t,u,v} = Id_{f \circ_v (g \circ_u h)}$ (See **Figure 14**). We use universality of $s \cdot t$ and $u \cdot v$ and following calculations.

$$\begin{aligned}(s \cdot t) \cdot (\alpha_{s,t,u,v} \cdot \phi_{s,t,u,v}) &= ((s \cdot t) \cdot \alpha_{s,t,u,v}) \cdot \phi_{s,t,u,v} \\ &= (u \cdot v) \cdot \phi_{s,t,u,v} \\ &= (s \cdot t) \\ &= (s \cdot t) \cdot Id_{(f \circ_s g) \circ_t h}\end{aligned}$$

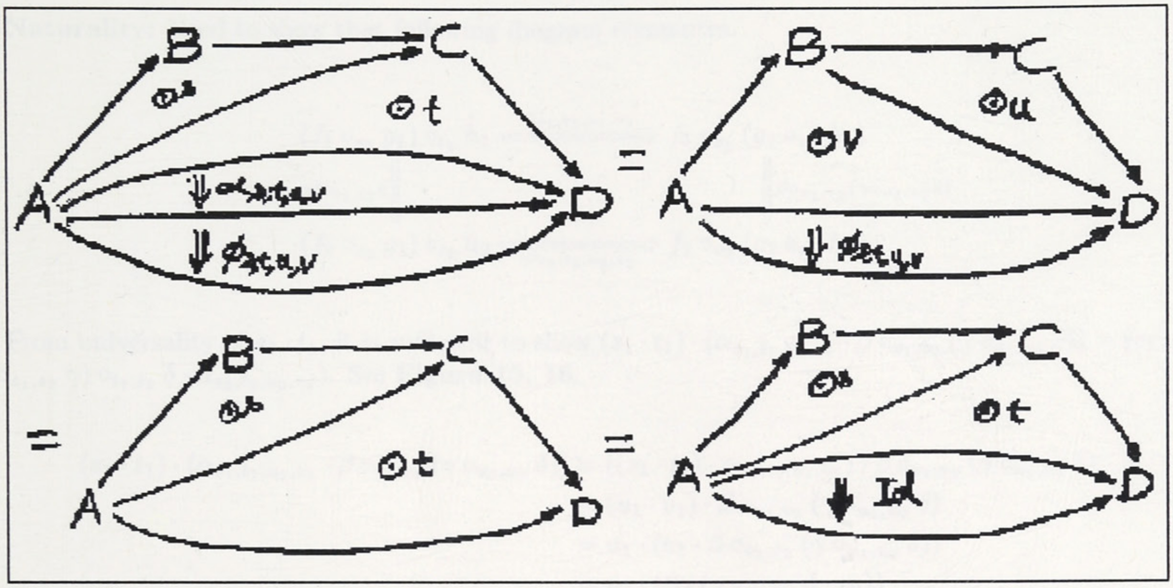


Figure 13.

$$
\begin{aligned}
(u \cdot v) \cdot (\phi_{s,t,u,v} \cdot \alpha_{s,t,u,v}) &= ((u \cdot v) \cdot \phi_{s,t,u,v}) \cdot \alpha_{s,t,u,v} \\
&= (s \cdot t) \cdot \alpha_{s,t,u,v} \\
&= (u \cdot v) \\
&= (u \cdot v) \cdot Id_{f \circ_v (g \circ_u h)}
\end{aligned}
$$

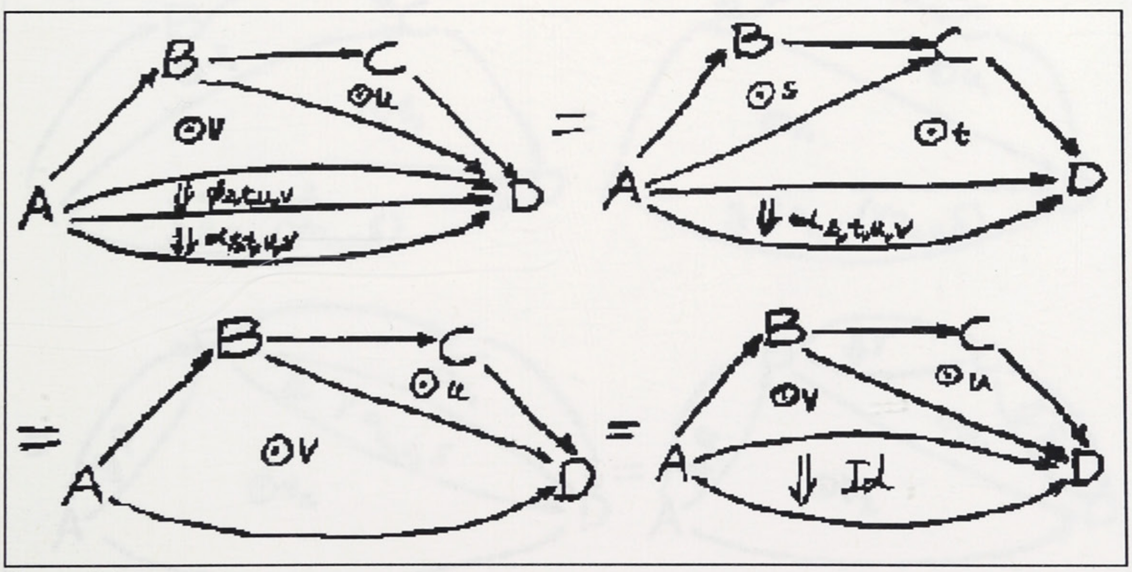


Figure 14.

Naturality: Need to show that following diagram commutes.

$$\begin{array}{ccc}
(f_1 \circ_{s_1} g_1) \circ_{t_1} h_1 & \xRightarrow{\alpha_{s_1,t_1,u_1,v_1}} & f_1 \circ_{v_1} (g_1 \circ_{u_1} h_1) \\
{\scriptstyle (\beta \circ_{s_1,s_2} \gamma) \circ_{t_1,t_2} \delta} \Big\Downarrow & & \Big\Downarrow {\scriptstyle \beta \circ_{v_1,v_2} (\gamma \circ_{u_1,u_2} \delta)} \\
(f_2 \circ_{s_2} g_2) \circ_{t_2} h_2 & \xRightarrow[\alpha_{s_2,t_2,u_2,v_2}]{} & f_2 \circ_{v_2} (g_2 \circ_{u_2} h_2)
\end{array}$$

From universality of $s_1 \cdot t_1$, it is sufficient to show $(s_1 \cdot t_1) \cdot (\alpha_{s_1,t_1,u_1,v_1} \cdot \beta \circ_{v_1,v_2} (\gamma \circ_{u_1,u_2} \delta)) = (s_1 \cdot t_1) \cdot ((\beta \circ_{s_1,s_2} \gamma) \circ_{t_1,t_2} \delta \cdot \alpha_{s_2,t_2,u_2,v_2})$. See **Figure 15, 16**.

$$\begin{aligned}
(s_1 \cdot t_1) \cdot (\alpha_{s_1,t_1,u_1,v_1} \cdot \beta \circ_{v_1,v_2} (\gamma \circ_{u_1,u_2} \delta)) &= ((s_1 \cdot t_1) \cdot \alpha_{s_1,t_1,u_1,v_1}) \cdot \beta \circ_{v_1,v_2} (\gamma \circ_{u_1,u_2} \delta) \\
&= (u_1 \cdot v_1) \cdot \beta \circ_{v_1,v_2} (\gamma \circ_{u_1,u_2} \delta) \\
&= u_1 \cdot (v_1 \cdot \beta \circ_{v_1,v_2} (\gamma \circ_{u_1,u_2} \delta)) \\
&= u_1 \cdot (\beta \cdot (\gamma \circ_{u_1,u_2} \delta \cdot v_2)) \\
&= \beta \cdot (u_1 \cdot (\gamma \circ_{u_1,u_2} \delta \cdot v_2)) \\
&= \beta \cdot ((u_1 \cdot \gamma \circ_{u_1,u_2} \delta) \cdot v_2) \\
&= \beta \cdot ((\gamma \cdot (\delta \cdot u_2)) \cdot v_2) \\
&= \beta \cdot (\gamma \cdot ((\delta \cdot u_2) \cdot v_2)) \\
&= \beta \cdot (\gamma \cdot (\delta \cdot (u_2 \cdot v_2)))
\end{aligned}$$

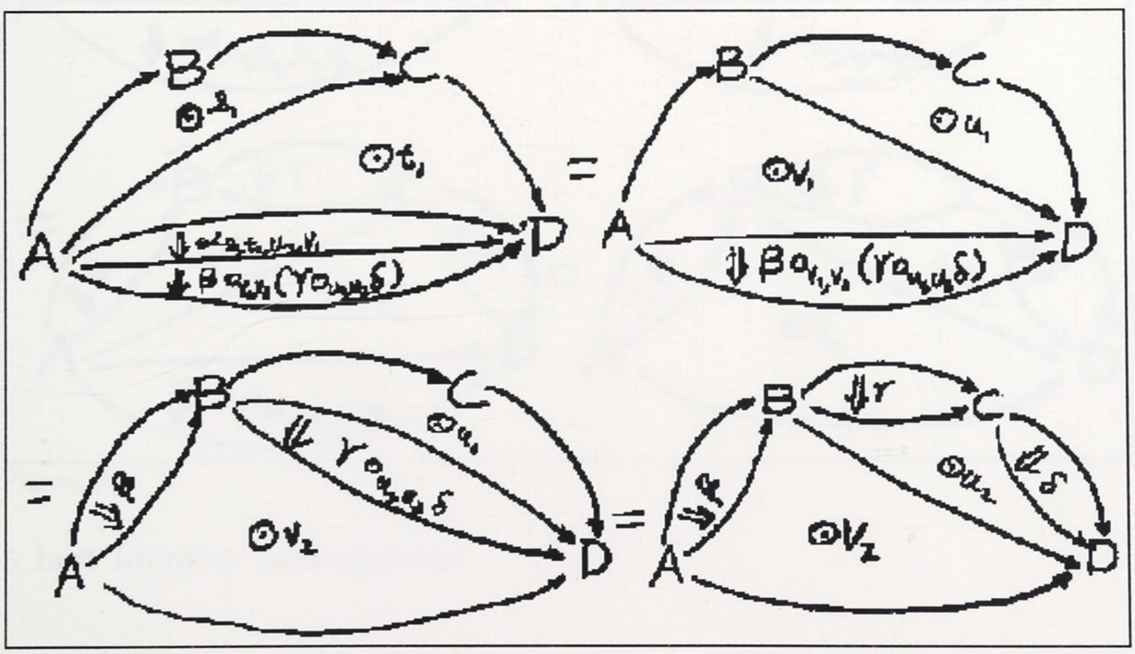


Figure 15.

$$
\begin{aligned}
(s_1 \cdot t_1) \cdot ((\beta \circ_{s_1,s_2} \gamma) \circ_{t_1,t_2} \delta \cdot \alpha_{s_2,t_2,u_2,v_2}) &= ((s_1 \cdot t_1) \cdot (\beta \circ_{s_1,s_2} \gamma) \circ_{t_1,t_2} \delta) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= (s_1 \cdot (t_1 \cdot (\beta \circ_{s_1,s_2} \gamma) \circ_{t_1,t_2} \delta)) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= (s_1 \cdot (\beta \circ_{s_1,s_2} \gamma \cdot (\delta \cdot t_2))) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= ((s_1 \cdot \beta \circ_{s_1,s_2} \gamma) \cdot (\delta \cdot t_2)) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= ((\beta \cdot (\gamma \cdot s_2)) \cdot (\delta \cdot t_2)) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= (\beta \cdot ((\gamma \cdot s_2) \cdot (\delta \cdot t_2))) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= (\beta \cdot (\delta \cdot ((\gamma \cdot s_2) \cdot t_2))) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= (\beta \cdot (\delta \cdot (\gamma \cdot (s_2 \cdot t_2)))) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= (\beta \cdot (\gamma \cdot (\delta \cdot (s_2 \cdot t_2)))) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= \beta \cdot ((\gamma \cdot (\delta \cdot (s_2 \cdot t_2))) \cdot \alpha_{s_2,t_2,u_2,v_2}) \\
&= \beta \cdot (\gamma \cdot ((\delta \cdot (s_2 \cdot t_2)) \cdot \alpha_{s_2,t_2,u_2,v_2})) \\
&= \beta \cdot (\gamma \cdot (\delta \cdot ((s_2 \cdot t_2) \cdot \alpha_{s_2,t_2,u_2,v_2}))) \\
&= \beta \cdot (\gamma \cdot (\delta \cdot (u_2 \cdot v_2)))
\end{aligned}
$$

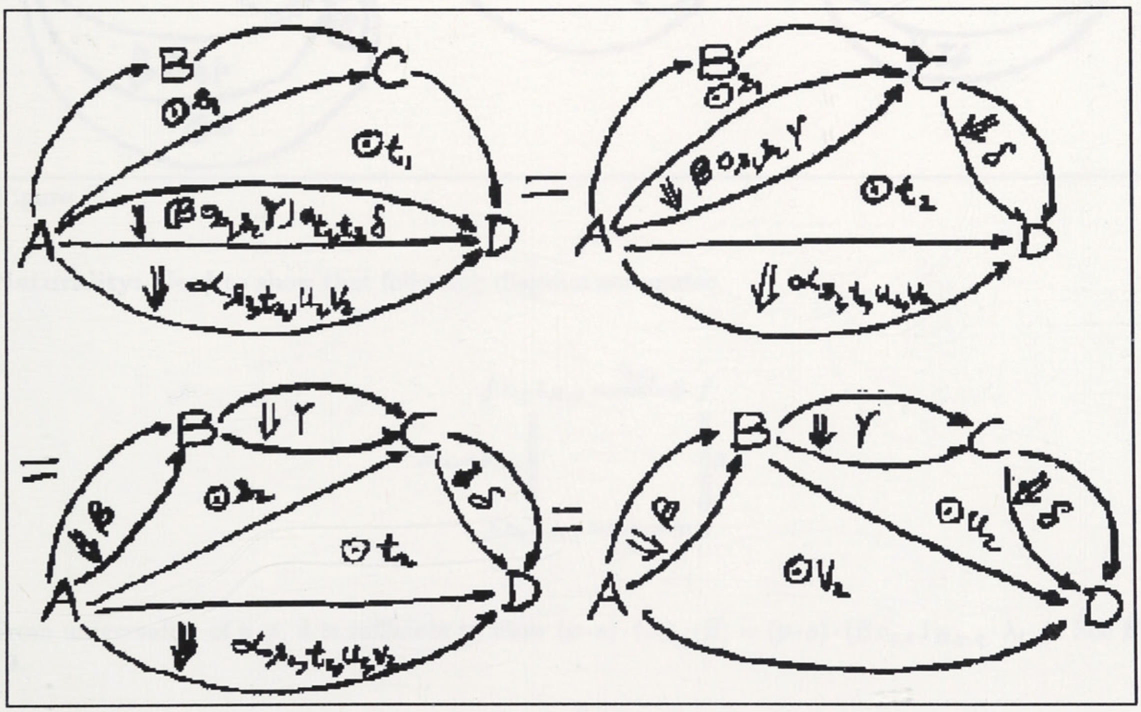


Figure 16.

II.2.5 Left Identity Isomorphims

Well defined: $\lambda_{s,p} \in \mathcal{A}(A,B)[f \circ_s 1_{B,p}, f]$ where $s \in \mid \circ_{A,B,B} \mid (f, 1_{B,p})$, $p \in |1_B|(1)$ is well defined, since $s \cdot p$ is composite of universals hence a universal. So, there is unique $\lambda_{s,p}$ such that $(p \cdot s) \cdot \lambda_{s,p} = Id_f$.

Isomorphism: $\lambda_{s,p}$ is invertible. Its inverse is $p \cdot s$. One side $(p \cdot s) \cdot \lambda_{s,p} = Id_f$ was verified above. Now (See **Figure 17**),

$$
\begin{aligned}
&(p \cdot s) \cdot \lambda_{s,p} = Id_f \\
\Longrightarrow &((p \cdot s) \cdot \lambda_{s,p}) \cdot (p \cdot s) = Id_f \cdot (p \cdot s) \\
\Longrightarrow &(p \cdot s) \cdot (\lambda_{s,p} \cdot (p \cdot s)) = (p \cdot s) \\
\Longrightarrow &(p \cdot s) \cdot (\lambda_{s,p} \cdot (p \cdot s)) = (p \cdot s) \cdot Id_f \\
\Longrightarrow &\lambda_{s,p} \cdot (p \cdot s) = Id_f
\end{aligned}
$$

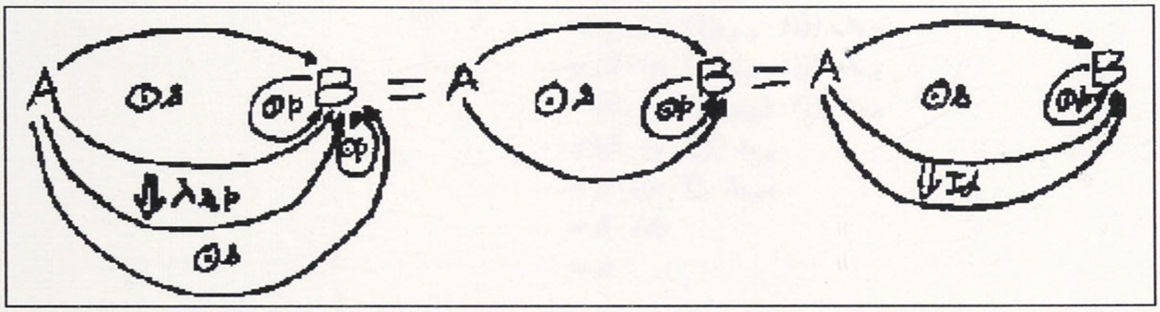


Figure 17.

Naturality: Need to show that following diagram commutes.

$$
\begin{array}{ccc}
f \circ_s 1_{B,p} & \xRightarrow{\lambda_{s,p}} & f \\
{\scriptstyle \beta \circ_{s,t} 1_{B,p,q}} \Downarrow & & \Downarrow {\scriptstyle \beta} \\
f \circ_t 1_{B,q} & \xRightarrow[\lambda_{t,q}]{} & g
\end{array}
$$

From universality of $p \cdot s$, it is sufficient to show $(p \cdot s) \cdot (\lambda_{s,p} \cdot \beta) = (p \cdot s) \cdot (\beta \circ_{s,t} 1_{B,p,q} \cdot \lambda_{t,q})$. See **Figure 18, 19**.

$$
\begin{aligned}
(p \cdot s) \cdot (\lambda_{s,p} \cdot \beta) &= ((p \cdot s) \cdot \lambda_{s,p}) \cdot \beta \\
&= Id_f \cdot \beta \\
&= \beta
\end{aligned}
$$

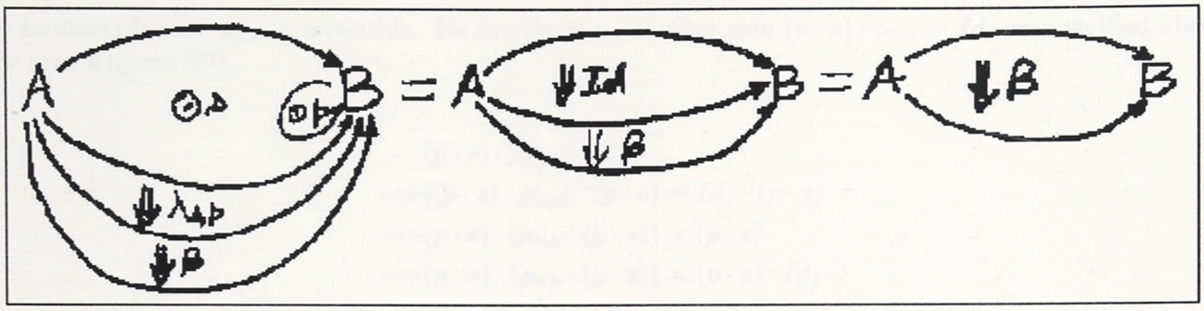


Figure 18.

$$
\begin{aligned}
(p\cdot s)\cdot(\beta \circ_{s,t} 1_{B,p,q}\cdot \lambda_{t,q}) &= ((p\cdot s)\cdot \beta \circ_{s,t} 1_{B,p,q})\cdot \lambda_{t,q}\\
&= (p\cdot(s\cdot \beta \circ_{s,t} 1_{B,p,q}))\cdot \lambda_{t,q}\\
&= (p\cdot(\beta\cdot(1_{B,p,q}\cdot t)))\cdot \lambda_{t,q}\\
&= (\beta\cdot(p\cdot(1_{B,p,q}\cdot t)))\cdot \lambda_{t,q}\\
&= (\beta\cdot((p\cdot 1_{B,p,q})\cdot t))\cdot \lambda_{t,q}\\
&= (\beta\cdot(q\cdot t))\cdot \lambda_{t,q}\\
&= \beta\cdot((q\cdot t)\cdot \lambda_{t,q})\\
&= \beta\cdot Id_f\\
&= \beta
\end{aligned}
$$

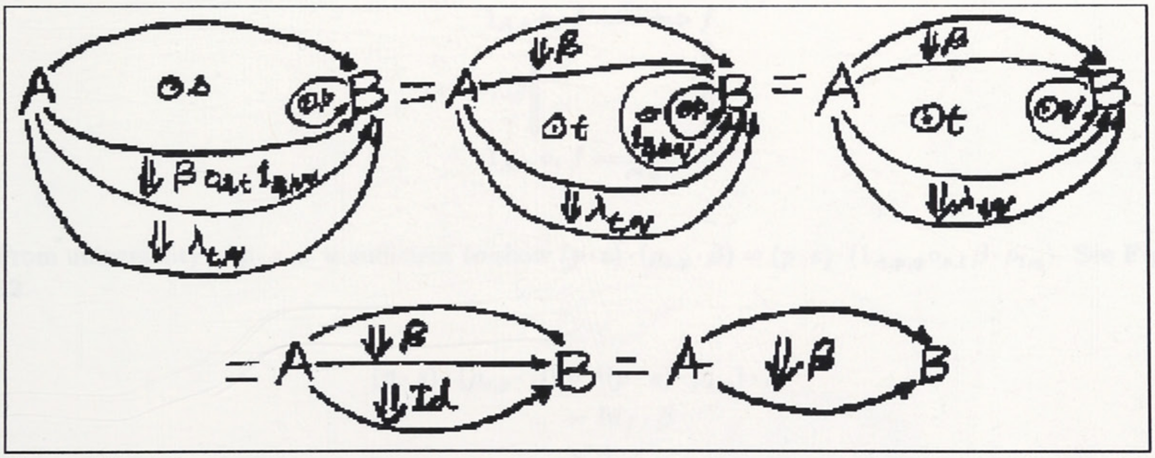


Figure 19.

II.2.6 Right Identity Isomorphims

Well defined: $\rho_{s,p} \in \mathcal{A}(A,B)[1_{A,p} \circ_s f, f]$ where $s \in |\circ_{A,A,B}|(1_{A,p}, f)$, $p \in |1_A|(1)$ is well defined, since $s\cdot p$ is composite of universals hence a universal. So, there is unique $\rho_{s,p}$ such that $(p\cdot s)\cdot \rho_{s,p} = Id_f$.

Isomorphism: $\rho_{s,p}$ is invertible. Its inverse is $p \cdot s$. One side $(p \cdot s) \cdot \rho_{s,p} = Id_f$ was verified above. Now (See **Figure 20**),

$$\begin{aligned}
&(p \cdot s) \cdot \rho_{s,p} = Id_f \\
\Longrightarrow&((p \cdot s) \cdot \rho_{s,p}) \cdot (p \cdot s) = Id_f \cdot (p \cdot s) \\
\Longrightarrow&(p \cdot s) \cdot (\rho_{s,p} \cdot (p \cdot s)) = (p \cdot s) \\
\Longrightarrow&(p \cdot s) \cdot (\rho_{s,p} \cdot (p \cdot s)) = (p \cdot s) \cdot Id_f \\
\Longrightarrow&\rho_{s,p} \cdot (p \cdot s) = Id_f
\end{aligned}$$

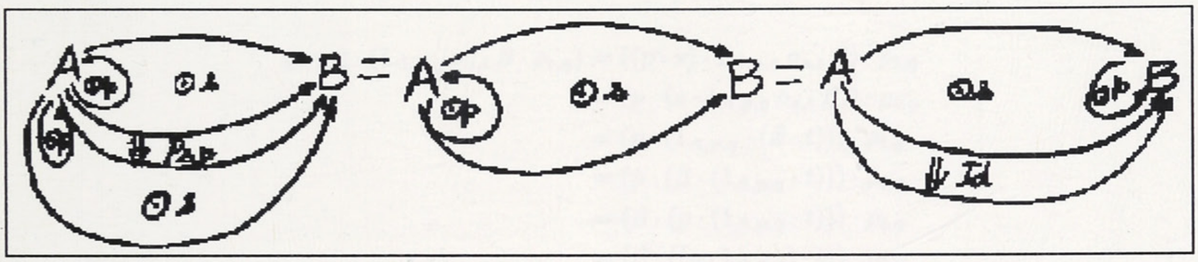


Figure 20.

Naturality: Need to show that following diagram commutes.

$$\begin{array}{ccc}
1_{A,p} \circ_s f & \stackrel{\rho_{s,p}}{\Longrightarrow} & f \\
{\scriptstyle 1_{A,p,q} \circ_{s,t} \beta} \Downarrow & & \Downarrow {\scriptstyle \beta} \\
1_{A,q} \circ_t f & \underset{\rho_{t,q}}{\Longrightarrow} & g
\end{array}$$

From universality of $p \cdot s$, it is sufficient to show $(p \cdot s) \cdot (\rho_{s,p} \cdot \beta) = (p \cdot s) \cdot (1_{A,p,q} \circ_{s,t} \beta \cdot \rho_{t,q})$. See **Figure 21, 22**.

$$\begin{aligned}
(p \cdot s) \cdot (\rho_{s,p} \cdot \beta) &= ((p \cdot s) \cdot \rho_{s,p}) \cdot \beta \\
&= Id_f \cdot \beta \\
&= \beta
\end{aligned}$$

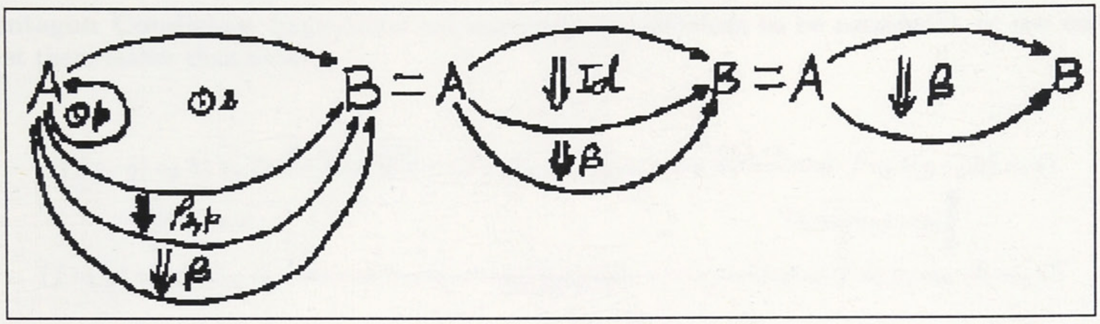


Figure 21.

$$
\begin{aligned}
(p \cdot s) \cdot (1_{A,p,q} \circ_{s,t} \beta \cdot \rho_{t,q}) &= ((p \cdot s) \cdot 1_{A,p,q} \circ_{s,t} \beta) \cdot \rho_{t,q} \\
&= (p \cdot (s \cdot 1_{A,p,q} \circ_{s,t} \beta)) \cdot \rho_{t,q} \\
&= (p \cdot (1_{A,p,q} \cdot (\beta \cdot t))) \cdot \rho_{t,q} \\
&= (p \cdot (\beta \cdot (1_{A,p,q} \cdot t))) \cdot \rho_{t,q} \\
&= (\beta \cdot (p \cdot (1_{A,p,q} \cdot t))) \cdot \rho_{t,q} \\
&= (\beta \cdot ((p \cdot 1_{A,p,q}) \cdot t)) \cdot \rho_{t,q} \\
&= (\beta \cdot (q \cdot t)) \cdot \rho_{t,q} \\
&= \beta \cdot ((q \cdot t) \cdot \rho_{t,q}) \\
&= \beta \cdot Id_f \\
&= \beta
\end{aligned}
$$

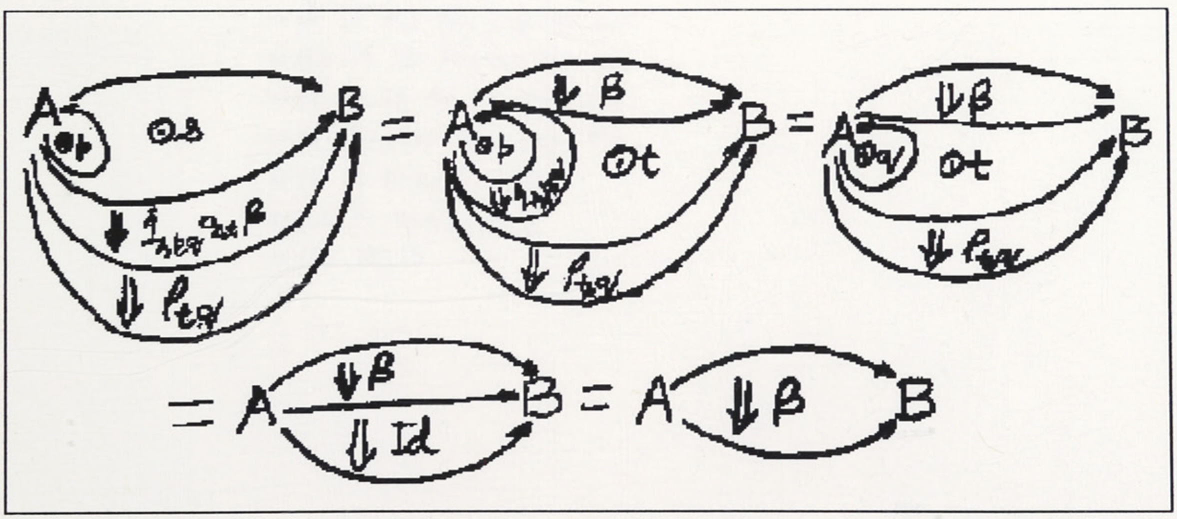


Figure 22.

II.2.7 Coherence

Pentagon Condition: Since there are too many specifications to be considered we use number to represent them rather than letters.

$$\begin{array}{ccccc}
((f \circ_1 g) \circ_2 h) \circ_3 i & \xRightarrow{\alpha_{1,2,4,5} \circ_{3,6} Id_i} & (f \circ_5 (g \circ_4 h)) \circ_6 i & \xRightarrow{\alpha_{5,6,7,8}} & f \circ_8 ((g \circ_4 h) \circ_7 i) \\
\Downarrow \alpha_{2,3,10,12} & & & & Id_f \circ_{8,9} \alpha_{4,7,10,11} \Downarrow \\
(f \circ_1 g) \circ_{12} (h \circ_{10} i) & & \xRightarrow[\alpha_{1,12,11,9}]{} & & f \circ_9 (g \circ_{11} (h \circ_{10} i))
\end{array}$$

Sufficient to show $((1 \cdot 2) \cdot 3) \cdot (\alpha_{1,2,4,5} \circ_{3,6} Id_i \cdot (\alpha_{5,6,7,8} \cdot Id_f \circ_{8,9} \alpha_{4,7,10,11})) = ((1 \cdot 2) \cdot 3)) \cdot (\alpha_{2,3,10,12} \cdot \alpha_{1,12,11,9})$. See **Figure 23, 24**.

$$\begin{aligned}
&((1 \cdot 2) \cdot 3) \cdot (\alpha_{1,2,4,5} \circ_{3,6} Id_i \cdot (\alpha_{5,6,7,8} \cdot Id_f \circ_{8,9} \alpha_{4,7,10,11})) \\
&= (((1 \cdot 2) \cdot 3) \cdot \alpha_{1,2,4,5} \circ_{3,6} Id_i) \cdot (\alpha_{5,6,7,8} \cdot Id_f \circ_{8,9} \alpha_{4,7,10,11}) \\
&= ((1 \cdot 2) \cdot (3 \cdot \alpha_{1,2,4,5} \circ_{3,6} Id_i)) \cdot (\alpha_{5,6,7,8} \cdot Id_f \circ_{8,9} \alpha_{4,7,10,11}) \\
&= ((1 \cdot 2) \cdot (\alpha_{1,2,4,5} \cdot (Id_i \cdot 6))) \cdot (\alpha_{5,6,7,8} \cdot Id_f \circ_{8,9} \alpha_{4,7,10,11}) \\
&= ((1 \cdot 2) \cdot (\alpha_{1,2,4,5} \cdot 6)) \cdot (\alpha_{5,6,7,8} \cdot Id_f \circ_{8,9} \alpha_{4,7,10,11}) \\
&= (((1 \cdot 2) \cdot \alpha_{1,2,4,5}) \cdot 6) \cdot (\alpha_{5,6,7,8} \cdot Id_f \circ_{8,9} \alpha_{4,7,10,11}) \\
&= ((4 \cdot 5) \cdot 6) \cdot (\alpha_{5,6,7,8} \cdot Id_f \circ_{8,9} \alpha_{4,7,10,11}) \\
&= (4 \cdot (5 \cdot 6)) \cdot (\alpha_{5,6,7,8} \cdot Id_f \circ_{8,9} \alpha_{4,7,10,11}) \\
&= ((4 \cdot (5 \cdot 6)) \cdot \alpha_{5,6,7,8}) \cdot Id_f \circ_{8,9} \alpha_{4,7,10,11} \\
&= (4 \cdot ((5 \cdot 6) \cdot \alpha_{5,6,7,8})) \cdot Id_f \circ_{8,9} \alpha_{4,7,10,11} \\
&= (4 \cdot (7 \cdot 8)) \cdot Id_f \circ_{8,9} \alpha_{4,7,10,11} \\
&= ((4 \cdot 7) \cdot 8) \cdot Id_f \circ_{8,9} \alpha_{4,7,10,11} \\
&= (4 \cdot 7) \cdot (8 \cdot Id_f \circ_{8,9} \alpha_{4,7,10,11}) \\
&= (4 \cdot 7) \cdot (Id_f \cdot (\alpha_{4,7,10,11} \cdot 9)) \\
&= (4 \cdot 7) \cdot (\alpha_{4,7,10,11} \cdot 9) \\
&= ((4 \cdot 7) \cdot \alpha_{4,7,10,11}) \cdot 9 \\
&= (10 \cdot 11) \cdot 9
\end{aligned}$$

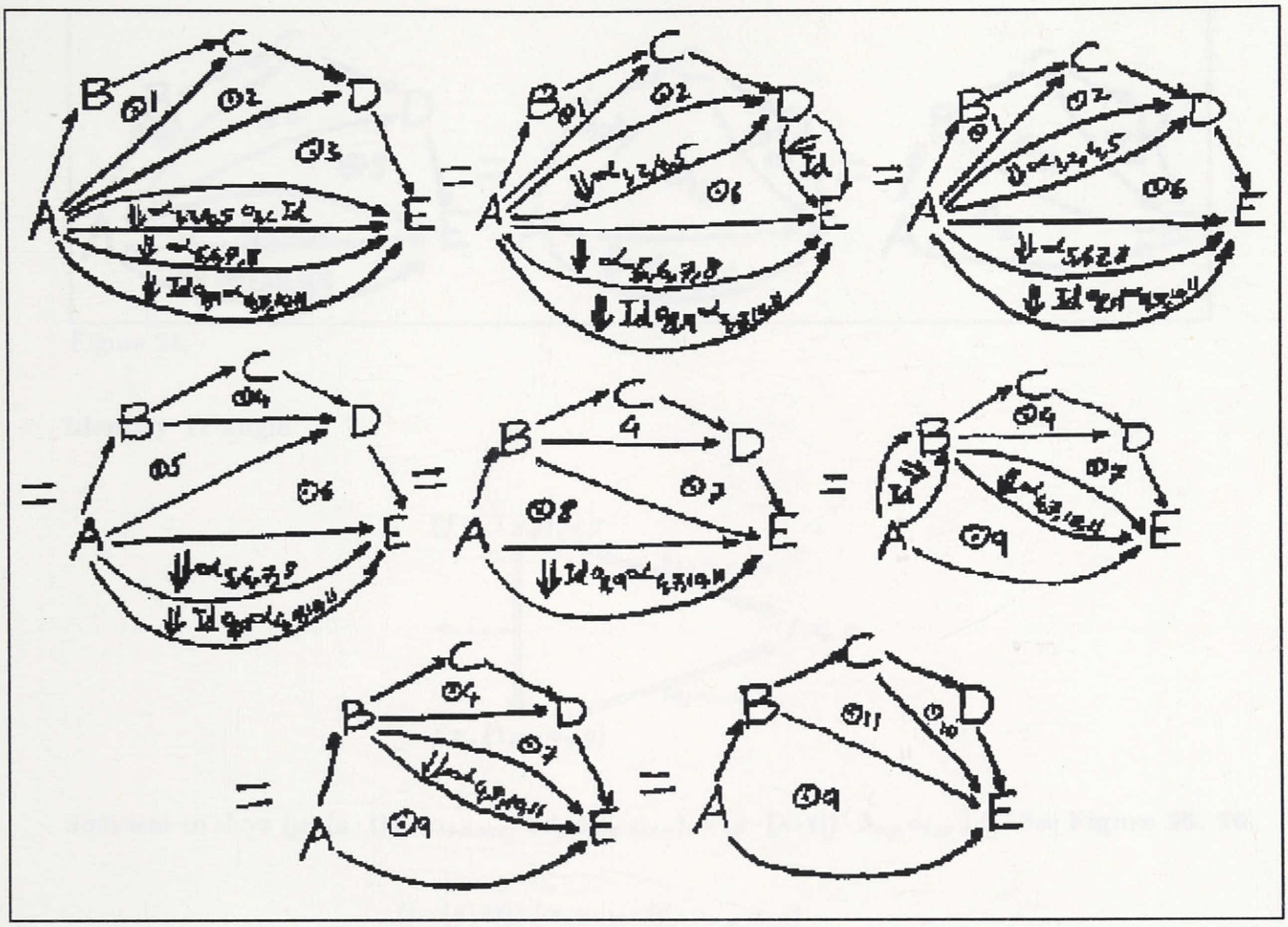


Figure 23.

$$\begin{aligned}
&((1\cdot 2)\cdot 3)\cdot(\alpha_{2,3,10,12}\cdot\alpha_{1,12,11,9})\\
&\qquad=(1\cdot(2\cdot 3))\cdot(\alpha_{2,3,10,12}\cdot\alpha_{1,12,11,9})\\
&\qquad=((1\cdot(2\cdot 3))\cdot\alpha_{2,3,10,12})\cdot\alpha_{1,12,11,9}\\
&\qquad=(1\cdot((2\cdot 3)\cdot\alpha_{2,3,10,12}))\cdot\alpha_{1,12,11,9}\\
&\qquad=(1\cdot(10\cdot 12))\cdot\alpha_{1,12,11,9}\\
&\qquad=(10\cdot(1\cdot 12))\cdot\alpha_{1,12,11,9}\\
&\qquad=10\cdot((1\cdot 12)\cdot\alpha_{1,12,11,9})\\
&\qquad=10\cdot(11\cdot 9)\\
&\qquad=(10\cdot 11)\cdot 9
\end{aligned}$$

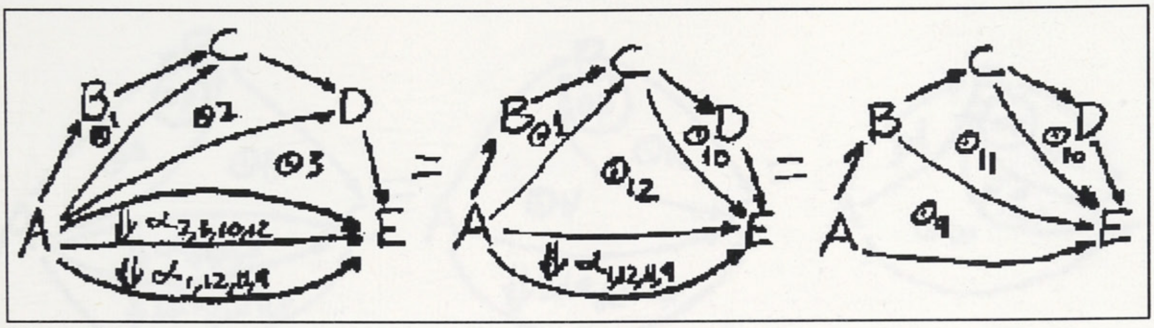
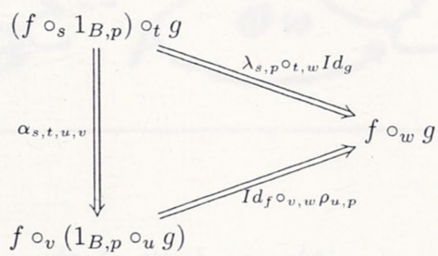


Figure 24.

Identity Triangle:



Sufficient to show $(p \cdot (s \cdot t)) \cdot (\alpha_{s,t,u,v} \cdot Id_f \circ_{v,w} \rho_{u,p}) = (p \cdot (s \cdot t)) \cdot \lambda_{s,p} \circ_{t,w} Id_g$. See **Figure 25, 26**.

$$
\begin{aligned}
&(p \cdot (s \cdot t)) \cdot (\alpha_{s,t,u,v} \cdot Id_f \circ_{v,w} \rho_{u,p}) \\
&\quad = ((p \cdot (s \cdot t)) \cdot \alpha_{s,t,u,v}) \cdot Id_f \circ_{v,w} \rho_{u,p} \\
&\quad = (p \cdot ((s \cdot t) \cdot \alpha_{s,t,u,v})) \cdot Id_f \circ_{v,w} \rho_{u,p} \\
&\quad = (p \cdot (u \cdot v)) \cdot Id_f \circ_{v,w} \rho_{u,p} \\
&\quad = ((p \cdot u) \cdot v) \cdot Id_f \circ_{v,w} \rho_{u,p} \\
&\quad = (p \cdot u) \cdot (v \cdot Id_f \circ_{v,w} \rho_{u,p}) \\
&\quad = (p \cdot u) \cdot (Id_f \cdot (\rho_{u,p} \cdot w)) \\
&\quad = (p \cdot u) \cdot (\rho_{u,p} \cdot w) \\
&\quad = ((p \cdot u) \cdot \rho_{u,p}) \cdot w \\
&\quad = Id_g \cdot w \\
&\quad = w
\end{aligned}
$$

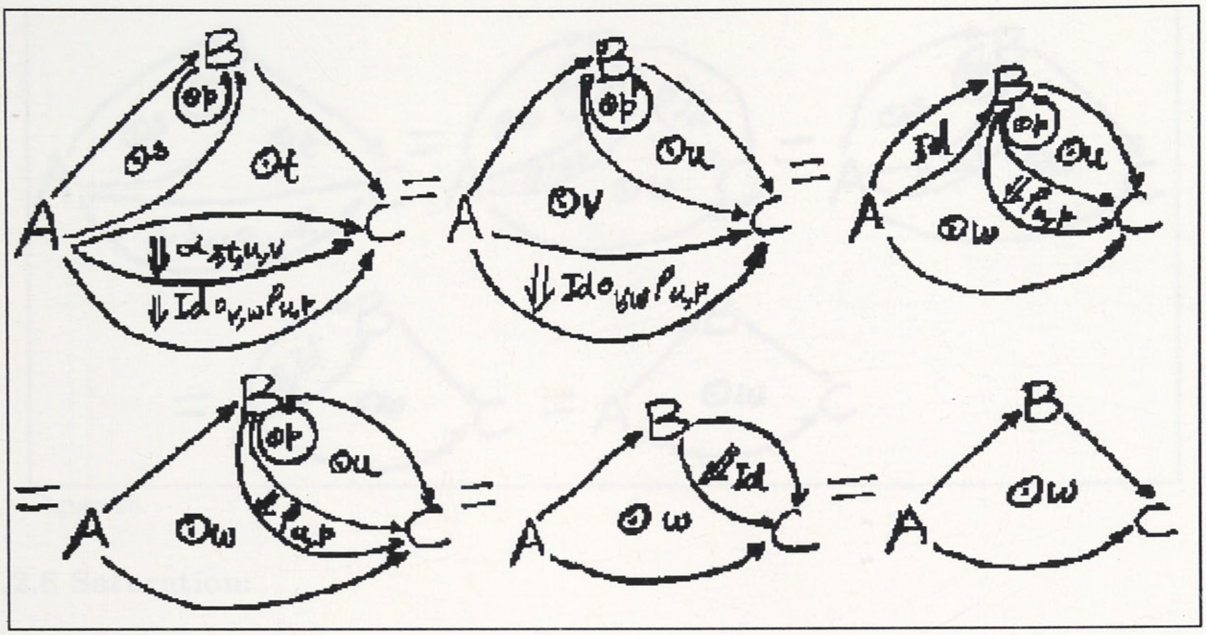


Figure 25.

$$
\begin{aligned}
&(p \cdot (s \cdot t)) \cdot \lambda_{s,p} \circ_{t,w} Id_g \\
&= ((p \cdot s) \cdot t) \cdot \lambda_{s,p} \circ_{t,w} Id_g \\
&= (p \cdot s) \cdot (t \cdot \lambda_{s,p} \circ_{t,w} Id_g) \\
&= (p \cdot s) \cdot (\lambda_{s,p} \cdot (Id_g \cdot w)) \\
&= (p \cdot s) \cdot (\lambda_{s,p} \cdot w) \\
&= ((p \cdot s) \cdot \lambda_{s,p}) \cdot w \\
&= Id_f \cdot w \\
&= w
\end{aligned}
$$

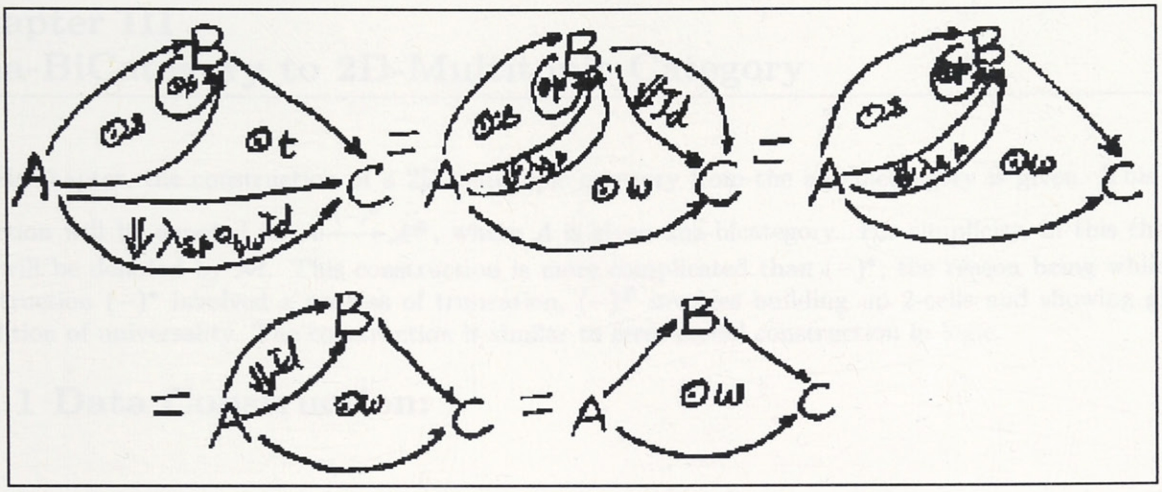


Figure 26.

II.2.8 Saturation:

The ana-bicategory $\mathcal{A}$ constructed is saturated.

Consider $p \in |1_A|(1)$, $1_{A,p} = f$, and $\phi : f \cong g$, then $p \cdot \phi$ is an universal. Hence, there is a (*unique*) $q \in |1_A|(1)$, such that $q = p \cdot \phi$.

Similarly, consider $s \in |\circ_{A,B,C}|(f,g)$, $f \circ_s g = h$, and $\phi : h \cong i$, then $s \cdot \phi$ is an universal. Hence, there is a (*unique*) $t \in |\circ_{A,B,C}|(f,g)$, such that $t = s \cdot \phi$.

II.3 Conclusion:

Theorem 1. *Construction $(-)^*$ transforms 2D-multitopic category to ana-bicategory.*

Chapter III
Ana-BiCategory to 2D-Multitopic Category

In this chapter, the construction of a 2D-multitopic category from the ana-bicategory is given. This construction will be denoted as $\mathcal{A} \overset{(-)^{\#}}{\longmapsto} \mathcal{A}^{\#}$, where $\mathcal{A}$ is given ana-bicategory. For simplicity, in this chapter $\mathcal{A}^{\#}$ will be denoted by $\mathcal{M}$. This construction is more complicated than $(-)^{*}$, the reason being while the construction $(-)^{*}$ involved a process of truncation, $(-)^{\#}$ involves building up 2-cells and showing global condition of universality. The construction is similar to term-model construction in logic.

III.1 Data Construction:

III.1.1 0 and 1 Cells:

0 Cells:

$$Cell_0(\mathcal{M}) = \mathcal{O}(\mathcal{A})$$

1 Cells:

$$Cell_1(\mathcal{M}) = \bigcup_{A,B \in \mathcal{O}(\mathcal{A})} \mathcal{O}(\mathcal{A}(A,B))$$

III.1.2 2 Cells:

2 cells are defined as equivalence classes of ordered labelled typed trees. The set of such trees is denoted as Υ and the equivalence relation as $\simeq \subset \Upsilon \times \Upsilon$.

The ordering means that children of any node has left to right ordering that can not be permuted. The labels on the nodes are either 2-cells, 0-Specifications or 2-Specifications. Labels on the edges are the 1-Cells or 0-Cells (if node below is 0-Specification). The labels on edges will not be shown except for the outermost ones since the can be recovered from the node labels. The degree of each node is at most 2. We will think of degree of the node with 0-Specification to be 0 even though it has an edge coming in.

Each tree has a type. The set of types is $\Theta \subset Cell_1(\mathcal{M})^{*} \times Cell_1(\mathcal{M})$, where $Cell_1(\mathcal{M})^{*}$ is the set of composable strings of $Cell_1(\mathcal{M})$ (*pasting diagrams*). If τ is the type of the tree T, then, define $dom(T) = \pi_1(\tau)$ and $codom(T) = \pi_2(\tau)$. $\cdot$ is the concatenation operation on the strings in $Cell_1(\mathcal{M})^{*}$.

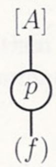
Υ and the type of trees in Υ are recursively defined as follows:

(1) If $f \in \mathcal{A}(A, B)$, then

$$\begin{array}{c}(f)\\ |\\ (f)\end{array}$$

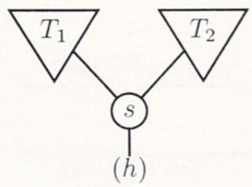
is a tree of type $(A \xrightarrow{f} B, A \xrightarrow{f} B)$.

(2) If $A \in \mathcal{O}(\mathcal{A})$, $p \in |1_A|(1)$ and $f = 1_{A,p}$, then



is a tree of type $(A \overset{\epsilon}{- -} A, A \xrightarrow{f} A)$.

(3) If T_1 is a tree of type $(A \xrightarrow{l_1} B, A \xrightarrow{f} B)$, T_2 a tree of type $(B \xrightarrow{l_2} C, B \xrightarrow{g} C)$ and $s \in |\circ_{A,B,C}|(f, g)$, where $f \in \mathcal{A}(A, B)$, $g \in \mathcal{A}(B, C)$, $A, B, C \in \mathcal{O}(\mathcal{A})$ and $h = f \circ_s h$, then



is a tree of type $(A \xrightarrow{l_1 \cdot l_2} C, A \xrightarrow{h} C)$.

(4) If T is a tree of type $(A \xrightarrow{l} B, A \xrightarrow{f} B)$, and $\beta \in \mathcal{A}(A, B)[f, g]$, where $f, g \in \mathcal{A}(A, B)$, $A, B \in \mathcal{O}(\mathcal{A})$, then

is a tree of type $(A \xrightarrow{l} B, A \xrightarrow{g} B)$.

Remark 1. These trees can be thought of 2 dimensional pasting diagrams in multitopic category in which 0-cells and 1-cells are as in $\mathcal{A}$ and 2-cells are 0-Specifications, 2-Specifications and 2-cells.

Now the equivalence relation $\simeq \subset \Upsilon \times \Upsilon$ is defined. This is done in terms of elementary tree transformations $T_1 \longrightarrow T_2$. We define $\simeq$ to be transitive closure of $\longrightarrow$, hence $\simeq = \longrightarrow^*$. Each elementary tree transformation step is invertible. Each step is labelled as $\overline{XX}$ and its inverse as $\underline{XX}$. The elementary steps are classified according to their origin in ana-bicategory.

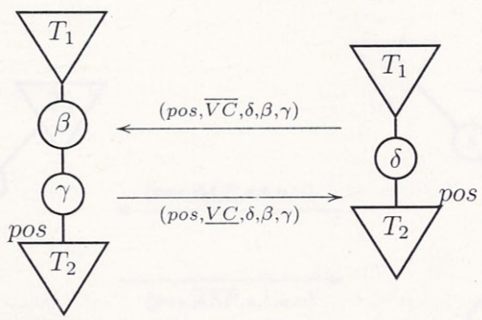
In these elementary steps the position ($pos \in \{u, l, r\}^*$) indicates where the transformation is applied. So *pos* is string from alphabet $\{u, l, r\}$, which gives the position relative to the root node. The logic is simple.

Start from the root node of the tree. Read *pos* from left to right and on seeing u move up, on l move left and on r move right. If such a move is not possible for any part of the string *pos* then position is invalid. If *pos* is a string and pos' is a prefix for it, then $pos - pos'$ denotes string such that $pos' \cdot (pos - pos') = pos$, where $\cdot$ is string concatenation operation.

If T is a tree and *pos* a valid position in T, then $T[pos]$ is subtree of T at position *pos*. Denote by $\wp(T)$ set of all valid positions in T. The notation remind us the fact that *pos* is index for the trees, just like numbers are index for the sequences.

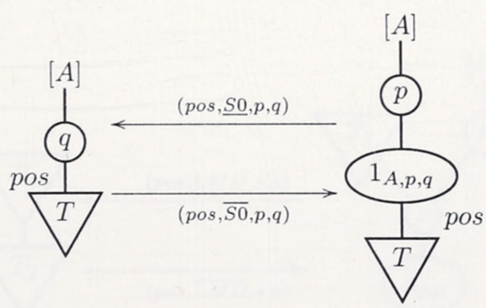
Composition Law: This comes from composition in Category $\mathcal{A}(-,-)$.

(VC) This transformation is replacement of two nodes representing arrows in Category $\mathcal{A}(-,-)$, with their composite. Let $\beta \cdot \gamma = \delta$ in $\mathcal{A}(-,-)$.

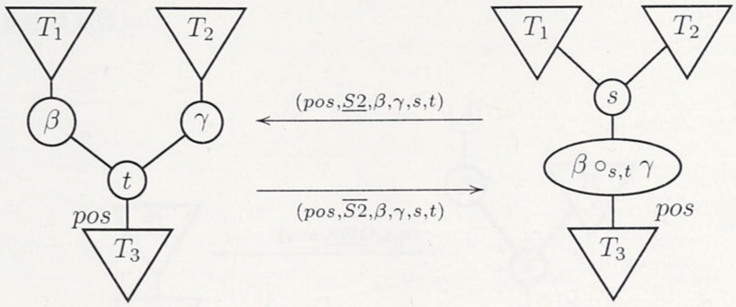


Structural Laws: These are the laws that change the specifications used and the skeleton of the tree. There are two laws for the 0-Specifications and 2-Specifications, and three laws for the three Natural AnaIsoMorphisms α, λ and ρ.

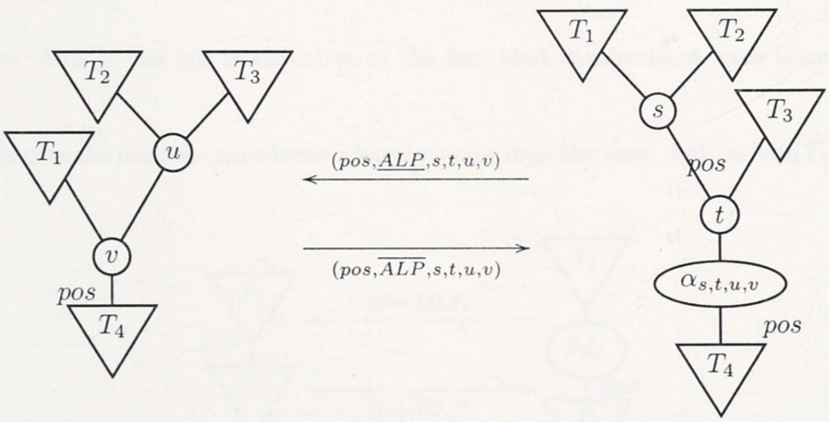
(S0) This transformation is for changing the 0-Specifications. Let $p, q \in |1_A|(1)$.



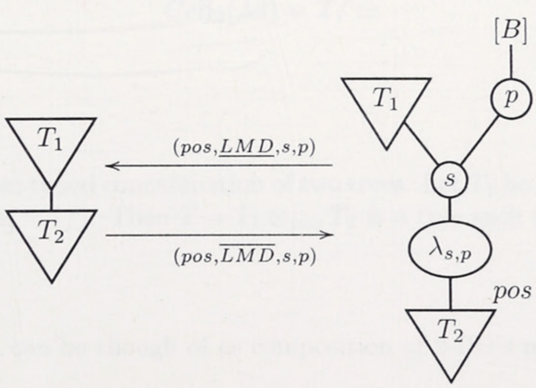
(S2) This transformation is for changing the 2-Specifications. Let $s \in |\circ_{A,B,C}|(f_1, g_1)$, $t \in |\circ_{A,B,C}|(f_2, g_2)$, $\beta \in \mathcal{A}(A,B)[f_1, f_2]$ and $\gamma \in \mathcal{A}(B,C)[g_1, g_2]$.



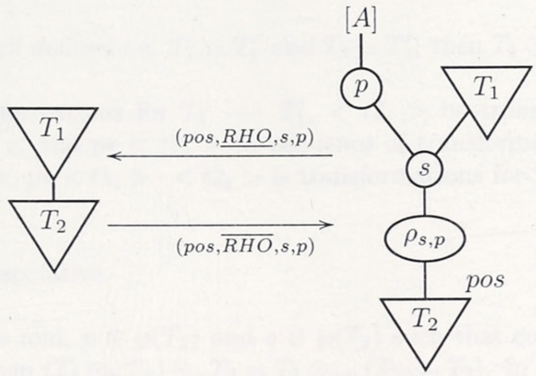
(ALP) This transformation changes the shape of tree. It changes it from right oriented one to left oriented. Let $s \in |\circ_{A,B,C}|(f,g)$, $t \in |\circ_{A,C,D}|(f \circ_s g, h)$, $u \in |\circ_{B,C,D}|(g,h)$, $v \in |\circ_{A,B,D}|(f, g \circ_u h)$.



(LMD) This transformation eliminates the 0-Specification on the right of a 2-Specification. Let $s \in |\circ_{A,B,B}|(f, 1_{B,p})$, $p \in |1_B|(1)$.

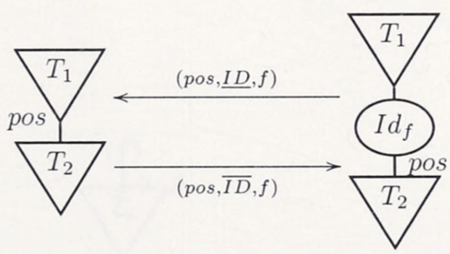


(RHO) This transformation eliminates the 0-Specification on the left of a 2-Specification. Let $s \in |\circ_{A,A,B}|(1_{A,p}, f)$, $p \in |1_A|(1)$.



Identity Law: This is the law is reflective of the fact that composition with identity does not affect the two cell.

(ID) This transformation introduces Identity node into the tree. Let $codom(T_1) = f$, then



Now we can define set of two cells to be

$$Cell_2(\mathcal{M}) = \Upsilon / \simeq$$

III.2 Composition:

Composition of trees is defined as typed concatenation of two trees. Let T_1 be a tree such that $codom(T_1) = f$, T_2 and pos be such that $T_2[pos] = (f)$. Then $T = T_1 \odot_{pos} T_2$ is a tree such that at position pos in T_2, T_1 is attached.

Remark 2. The composition can be though of as composition of 2-PD's mentioned in **Remark 1**.

Formally,

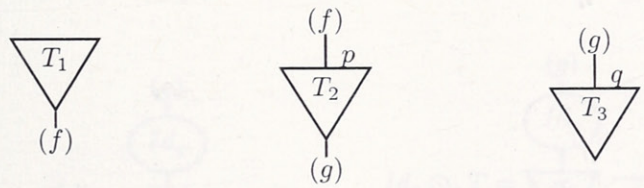
$$(\forall pos' \in \wp(T_2)) \qquad T[pos'] = \begin{cases} T_2[pos'] & \text{if } pos' \text{ is not a prefix of } pos \\ T_1 \odot_{pos''} T_2[pos'] & \text{otherwise, where } pos'' = pos - pos' \end{cases}$$

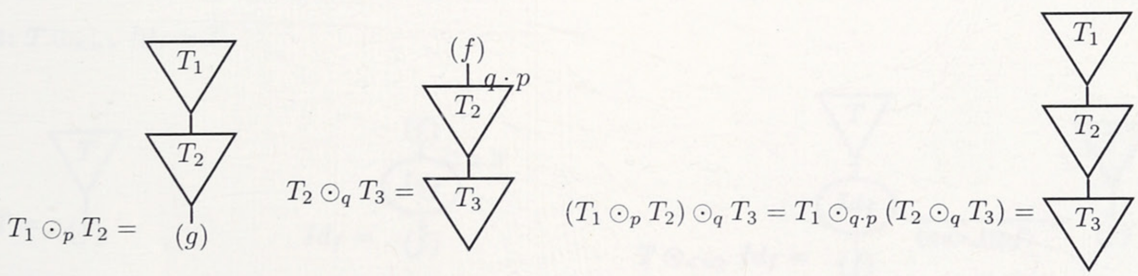
Lemma 1. *Composition is well defined i.e.* $T_1 \simeq T_1'$ *and* $T_2 \simeq T_2'$*, then* $T_1 \odot_p T_2 \simeq T_1' \odot_q T_2'$.

Proof. Let $< t1_i >$ be transformations for $T_1 \longrightarrow T_1'$, $< t2_i >$ be transformations for $T_2 \longrightarrow T_2'$. $< t2_i >$ transforms position p to q, and $p* < t1_i >$ be sequence of transformations with p prefixed to every transformation in $< t1_i >$. Now, $p* < t1_i > \cdot < t2_i >$ is transformations for $T_1 \odot_p T_2 \longrightarrow T_1' \odot_q T_2'$. ∎

Lemma 2. *Composition is associative.*

Proof. Let T_1, T_2, T_3 be trees and, $p \in \wp(T_2)$ and $q \in \wp(T_3)$ such that $codom(T_1) = f$, $codom(T_2) = g$, $T_2[p] = (f)$ and $T_3[q] = (g)$. Then $(T_1 \odot_p T_2) \odot_q T_3 = T_1 \odot_{q \cdot p} (T_2 \odot_q T_3)$. In pictures,

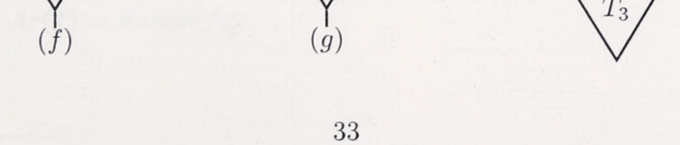


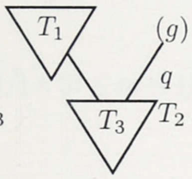


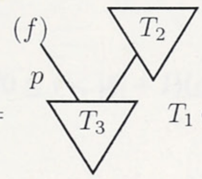
∎

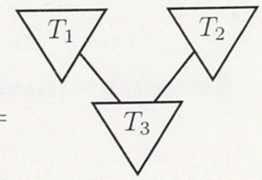
Lemma 3. *Composition is commutative.*

Proof. Let T_1, T_2, T_3 be trees and, $p, q \in \wp(T_3)$ such that $codom(T_1) = f$, $codom(T_2) = g$, $T_3[p] = (f)$ and $T_3[q] = (g)$. Then $T_1 \odot_p (T_2 \odot_q T_3) = T_2 \odot_q (T_1 \odot_p T_3)$. In pictures,



$T_1 \odot_p T_3$ 

$T_2 \odot_q T_3 =$ 

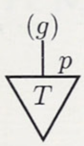
$T_1 \odot_p (T_2 \odot_q T_3) = T_2 \odot_q (T_1 \odot_p T_3) =$ 

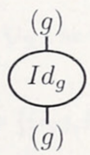
∎

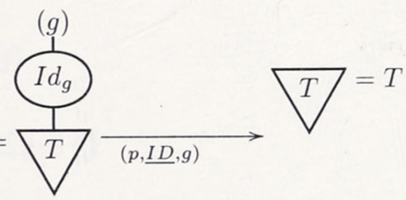
Lemma 4. *Composition respects identity laws.*

Proof. Let T be a tree and $p \in \wp(T)$ such that $codom(T) = f$ and $T[p] = (g)$. Also Id_f and Id_g be identity trees for f and g respectively.

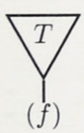
1: $Id_g \odot_p T \simeq T$.

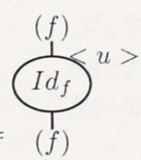
$T =$ 

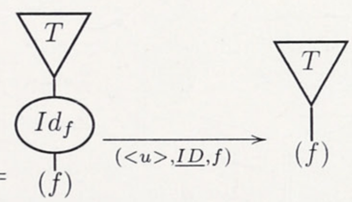
$Id_g =$ 

$Id_g \odot_p T =$ 

2: $T \odot_{<u>} Id_f \simeq T$.

$T =$ 

$Id_f =$ 

$T \odot_{<u>} Id_f =$ 

∎

III.3 Complete set of specifications:

Definition Pasting diagrams:. Pasting diagram (PD) is a triple $\bar{f} = (n, f_O, f_A)$, where $f_O : n+1 \longrightarrow Cell_0(\mathcal{M})$ and $f_A : n \longrightarrow Cell_1(\mathcal{M})$ such that $(\forall 0 \leq i < n)(f_A(i) : f_O(i) \longrightarrow f_O(i+1))$. Define $|\bar{f}| = n$, $Ob(\bar{f}) = Range(f_O)$ and $Ar(\bar{f}) = Range(f_A)$.

Also, we say $\bar{g} \leq_m \bar{f}$, for $\bar{g}$ is substring of $\bar{f}$ at position m, i.e.

$$|\bar{g}| \leq |\bar{f}| \wedge (0 \leq m \leq |\bar{f}|) \wedge ((\forall 0 \leq i < |\bar{g}|+1)(g_O(i) = f_O(i+m)) \wedge (\forall 0 \leq i < |\bar{g}|)(g_A(i) = f_A(i+m))$$

and

$$\bar{g} \leq \bar{f} \longleftrightarrow (\exists 0 \leq m < |\bar{f}|+1)(\bar{g} \leq_m \bar{f})$$

This defines a partial order on PDs. Define $\bar{g} = \bar{f} \upharpoonright_{(m,m')} = (m'-m, g_O, g_A)$, as $g_O(i) = f_O(m+i), 0 \leq i \leq m'-m$ and $g_A(i) = f_A(m+i), 0 \leq i < m'-m$.

We now define complete set of specifications forgiven PD. Intuitively this is a collection of 0-Specifications and 2-Specifications that fit together to define the composition of 1-cells such that between any pair of 0-cells there is a unique 1-cell. So the basic idea in this definition is that all α, ρ, λ's associated with these specifications are identities.

For $I \subset \mathbf{N}$, define

$$\nabla_{2,I} = \{(i,j) | (i,j) \in I \times I \wedge i \leq j\}$$

and

$$\nabla_{3,I} = \{(i,j,k) | (i,j,k) \in I \times I \times I \wedge i \leq j \leq k\}$$

Definition I Indexed set of specifications:. Given a PD $\bar{f}$, let $N = |\bar{f}|$, $I = \{0, \ldots, N\}$. Define I indexed set of specifications $S = (\Theta, \mathcal{F}, S0, S2)$ for $\bar{f}$ as,

$$\begin{aligned} \Theta &: I \longrightarrow Ob(\bar{f}) \\ \mathcal{F} &: \nabla_{2,I} \longrightarrow Ar(\bar{f}) \\ S0 &: I \longrightarrow |1_{-}| \\ S2 &: \nabla_{3,I} \longrightarrow |\circ_{-,-,-}| \end{aligned}$$

such that

1) $\Theta = f_O$,

2) $\mathcal{F}(i,j) : \Theta(i) \longrightarrow \Theta(j)$,

3) $\mathcal{F}(i,i+1) = f_A(i)$.

4) $S0(i) \in |1_{\Theta(i)}|(1)$, $\quad \mathcal{F}(i,i) = 1_{\Theta(i),S0(i)}$,

5) $S2(i,j,k) \in |\circ_{\Theta(i),\Theta(j),\Theta(k)}|(\mathcal{F}(i,j), \mathcal{F}(j,k))$, $\quad \mathcal{F}(i,k) = \mathcal{F}(i,j) \circ_{S2(i,j,k)} \mathcal{F}(j,k)$, Given I indexed set of specifications S, for any non empty $J \subseteq I$, the subsystem $S|_J = (\Theta', \mathcal{F}', S0', S2')$ is

$$\begin{aligned} \Theta' &: J \longrightarrow Ob(\bar{f}) \\ \mathcal{F}' &: \nabla_{2,J} \longrightarrow Ar(\bar{f}) \\ S0' &: J \longrightarrow |1_{-}| \\ S2' &: \nabla_{3,J} \longrightarrow |\circ_{-,-,-}| \end{aligned}$$

such that

1) $\Theta'(i) = \Theta(i)$, for all $i \in J$,

2) $\mathcal{F}'(i,j) = \mathcal{F}(i,j)$, for all $i,j \in J$,

4) $S0'(i) = S0(i)$, for all $i \in J$,

5) $S2'(i,j,k) = S2(i,j,k)$, for all $i,j,k \in J$,

Definition α-Coherent Systems:. I indexed set of specifications S is said to be α-Coherent system if for all non-empty $J \subseteq I$ such that $0 < |J| \leq 4$, the subsystem $S|_J = (\Theta', \mathcal{F}', S0', S2')$ satisfies

$$\alpha_{S2'(i,j,k),S2'(i,k,l),S2'(j,k,l),S2'(i,j,l)} = Id_{\mathcal{F}'(i,l)}$$

for $i,j,k,l \in J$ such that $i \leq j \leq k \leq l$.

Definition ρ-Coherent Systems:. I indexed set of specifications S is said to be ρ-Coherent system if for all non-empty $J \subseteq I$ such that $0 < |J| \leq 2$, the subsystem $S|_J = (\Theta', \mathcal{F}', S0', S2')$ satisfies

$$\rho_{S2'(i,i,j),S0'(i)} = Id_{\mathcal{F}'(i,j)}$$

for $i,j \in J$ such that $i \leq j$.

Definition λ-Coherent Systems:. I indexed set of specifications S is said to be λ-Coherent system if for all non-empty $J \subseteq I$ such that $0 < |J| \leq 2$, the subsystem $S|_J = (\Theta', \mathcal{F}', S0', S2')$ satisfies

$$\lambda_{S2'(i,j,j),S0'(j)} = Id_{\mathcal{F}'(i,j)}$$

for $i,j \in J$ such that $i \leq j$.

Definition Complete set of Specification (CSS) for PD $\bar{f}$:. Given a PD $\bar{f}$, let $N = |\bar{f}|$, $I = \{0, \ldots, N\}$. Then we say I indexed set of specifications $S = (\Theta, \mathcal{F}, S0, S2)$ is complete set of specification for PD $\bar{f}$ whenever its α, ρ, λ-Coherent System.

Examples of complete specifications

1) For $n = 0$,

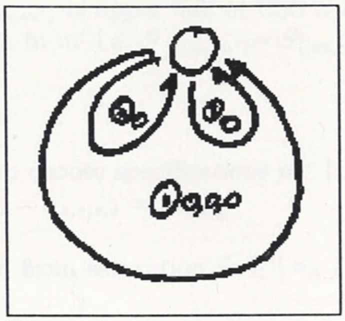


Figure 1.

2) For $n = 1$,

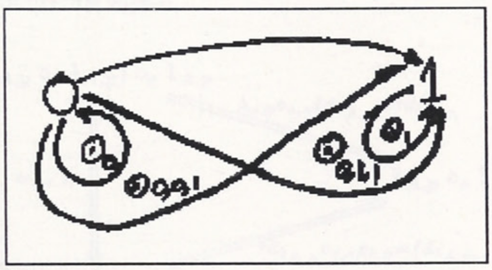


Figure 2.

3) For $n = 2$,

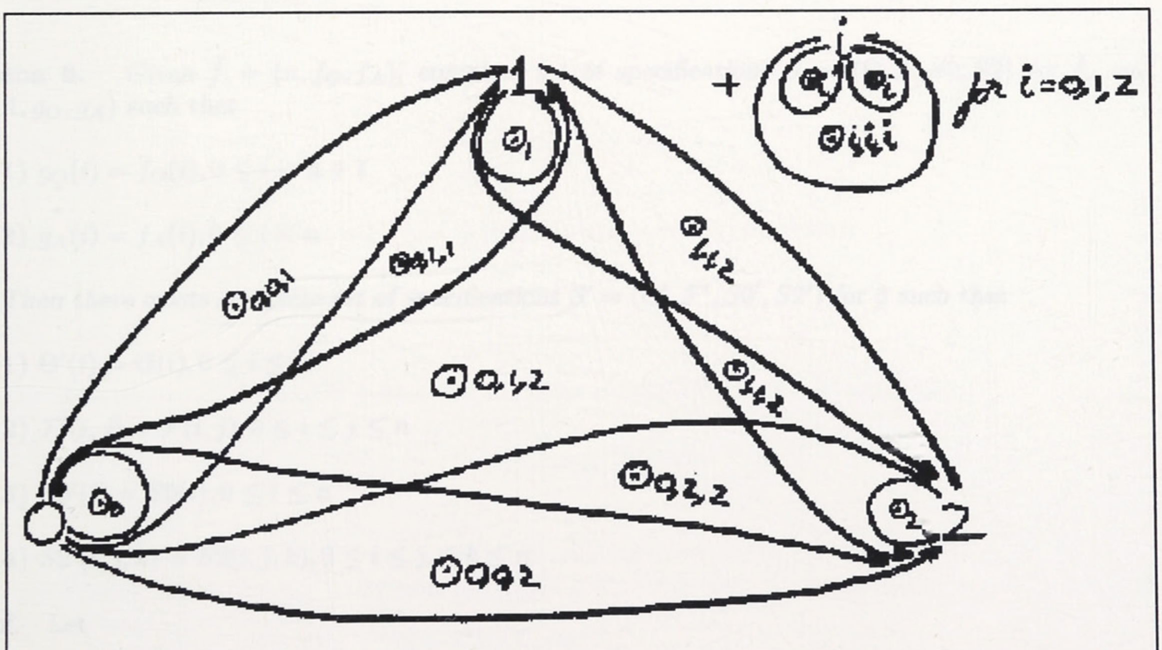


Figure 3.

Given $I \subset \mathbf{N}$, $m, m' \in I$ such that $m < m'$, then define $[m, m'] = \{i \in I | m \leq i \wedge i \leq m'\}$ and $]m, m'[= \{i \in I | i \leq m \vee m' \leq i\}$. $S \uparrow_{(m,m')}$ is upper half of CSS S from m to m', i.e. $S \uparrow_{(m,m')} = S|_{[m,m']}$. $S \downarrow_{(m,m')}$ is lower half of CSS S from m to m' i.e. $S \downarrow_{(m,m')} = S|_{]m,m'[}$. Note we might readjust the indexes whenever necessary.

Lemma 5. *Given an object A, we can choose specifications $p \in |1_A|(1)$ and $s \in | \circ_{A,A,A} |(1_{A,p}, 1_{A,p})$ such that $1_{A,p} \circ_s 1_{A,p} = 1_{A,p}$ and $\rho_{s,p} = \lambda_{s,p} = \alpha_{s,s,s,s} = Id_{1_{A,p}}$*

Proof. Choose any $p \in |1_A|(1)$. Then from saturation $\exists s \in | \circ_{A,A,A} |(1_{A,p}, 1_{A,p})$ such that $\rho_{s,p} = Id_{1_{A,p}}$. Using coherence we have

$$1_{A,p} \circ_s 1_{A,p} \underset{\lambda_{s,p}}{\overset{\rho_{s,p}}{\Longrightarrow}} 1_{A,p}$$

So, $\lambda_{s,p} = \rho_{s,p} = Id_{1_{A,p}}$. Using coherence again,

$$(1_{A,p} \circ_s 1_{A,p}) \circ_s 1_{A,p} \xrightarrow{\lambda_{s,p} \circ_{s,s} Id_{1_{A,p}} = Id_{1_{A,p}}} 1_{A,p} \circ_s 1_{A,p}$$

$$(1_{A,p} \circ_s 1_{A,p}) \circ_s 1_{A,p} \xrightarrow{\alpha_{s,s,s,s}} 1_{A,p} \circ_s (1_{A,p} \circ_s 1_{A,p})$$

$$1_{A,p} \circ_s (1_{A,p} \circ_s 1_{A,p}) \xrightarrow{Id_{1_{A,p}} \circ_{s,s} \rho_{s,p} = Id_{1_{A,p}}} 1_{A,p} \circ_s 1_{A,p}$$

So, $\alpha_{s,s,s,s} = Id_{1_{A,p}}$. ■

Lemma 6. *Given $\bar{f} = (n, f_O, f_A)$, complete set of specifications $S = (\Theta, \mathcal{F}, S0, S2)$ for $\bar{f}$, and $\bar{g} = (n+1, g_O, g_A)$ such that*

1) $g_O(i) = f_O(i), 0 \leq i < n+1$

2) $g_A(i) = f_A(i), 0 \leq i < n$

Then there exists complete set of specifications $S' = (\Theta', \mathcal{F}', S0', S2')$ for $\bar{g}$ such that

1) $\Theta'(i) = \Theta(i), 0 \leq i \leq n$

2) $\mathcal{F}'(i,j) = \mathcal{F}(i,j), 0 \leq i \leq j \leq n$

3) $S0'(i) = S0(i), 0 \leq i \leq n$

4) $S2'(i,j,k) = S2(i,j,k), 0 \leq i \leq j \leq k \leq n$

Proof. Let

1) $\Theta'(i) = \Theta(i), 0 \leq i \leq n$, and $\Theta'(n+1) = g_O(n+1)$

2) $\mathcal{F}'(i,j) = \mathcal{F}(i,j), 0 \leq i \leq j \leq n$, and $\mathcal{F}(n, n+1) = g_A(n)$

3) $S0'(i) = S0(i), 0 \leq i \leq n$

4) $S2'(i,j,k) = S2(i,j,k), 0 \leq i \leq j \leq k \leq n$

In this proof locally we use following abbreviations:

$$\begin{aligned}
\mathcal{F}'_{i,j} &:= \mathcal{F}'(i,j) \\
\circ_{i,j,k} &:= \circ_{S2'(i,j,k)} \\
\circ_{i,j,k} &:= \circ_{S2'(i,j,k),S2'(i,j,k)} \\
\rho_{i,j} &:= \rho_{S2'(i,i,j),S0'(i)} \\
\lambda_{i,j} &:= \lambda_{S2'(i,j,j),S0'(j)} \\
\alpha_{i,j,k,l} &:= \alpha_{S2'(i,j,k),S2'(i,k,l),S2'(j,k,l),S2'(i,j,l)} \\
1_{\Theta'(i)} &:= 1_{\Theta'(i),S0'(i)} \\
Id_{i,j} &:= Id_{\mathcal{F}'(i,j)}
\end{aligned}$$

Base Step: Using **Lemma 5**, we get $S0'(n+1)$ and $S2'(n+1, n+1, n+1)$. From saturation we have unique $S2'(n,n,n+1)$ such that $\rho_{n,n+1} = Id_{n,n+1}$ and $S2'(n, n+1, n+1)$ such that $\lambda_{n,n+1} = Id_{n,n+1}$.

Induction Step: For any $0 \leq i < n$, suppose we completed the following process for all $i < j \leq n+1$, we choose $S2'(i, n, n+1) \in |\circ_{\Theta'(i),\Theta'(n),\Theta'(n+1)}|(\mathcal{F}'(i,n), \mathcal{F}'(n, n+1))$, and let $\mathcal{F}'(i, n+1) = \mathcal{F}'(i,n) \circ_{i,n,n+1} \mathcal{F}'(n, n+1)$.

Now we need to define $S2'(i, j, n+1)$ for $i \leq j \leq n+1 \wedge j \neq n$.

(I1) For $j = n+1$, from saturation we have unique $S2'(i, n+1, n+1)$ such that

$$\lambda_{i,n+1} = Id_{i,n+1}$$

(I2) For $j = i$, from saturation we have unique $S2'(i,i,n+1)$ such that

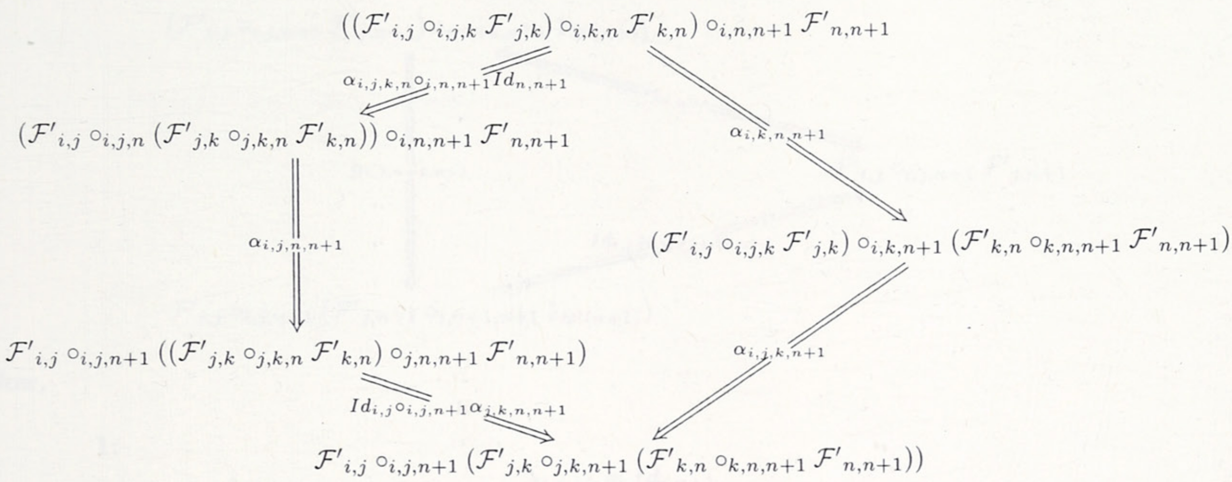
$$\rho_{i,n+1} = Id_{i,n+1}$$

(I3) For any $i < j < n$, from saturation we have unique $S2'(i,j,n+1)$ such that

$$\alpha_{i,j,n,n+1} = Id_{i,n+1}$$

The selection satisfies ρ and λ coherence. We need to show the α coherence for $i \leq j \leq k \leq l = n+1$. This has 8 cases depending on where equalities and inequalities lay. All of the following diagrams commute due to coherence.

Case 1: $i < j < k < l = n+1,\ k \neq n$



Now,

1:

$$\alpha_{i,j,k,n} \circ_{i,n,n+1} Id_{n,n+1} = Id_{i,n+1}$$

because $\alpha_{i,j,k,n} = Id_{i,n}$ from specifications S.

2:

$$\alpha_{i,j,n,n+1} = Id_{i,n+1}$$

from choice in item **(I3)** above.

3:

$$Id_{i,j} \circ_{i,j,n+1} \alpha_{j,k,n,n+1} = Id_{i,n+1}$$

because $\alpha_{j,k,n,n+1} = Id_{j,n+1}$, by induction as $j > i$.

4:

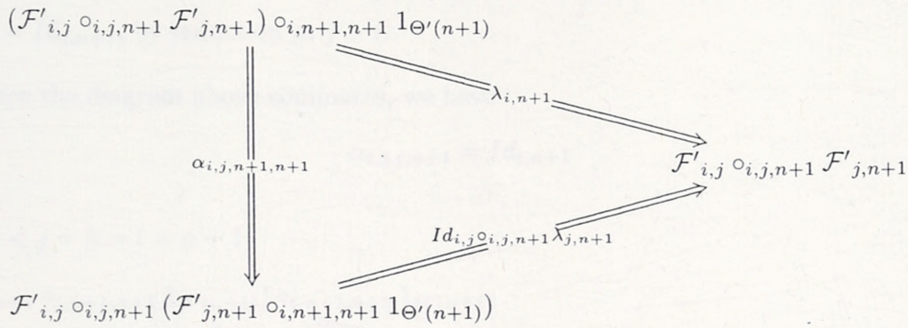
$$\alpha_{i,k,n,n+1} = Id_{i,n+1}$$

from choice in item **(I3)** above.

5: Since the diagram above commutes, we have

$$\alpha_{i,j,k,n+1} = Id_{i,n+1}$$

Case 2: $i < j < k = l = n+1$



Now,

1:

$$\lambda_{i,n+1} = Id_{i,n+1}$$

from choice in item **(I1)** above.

2:

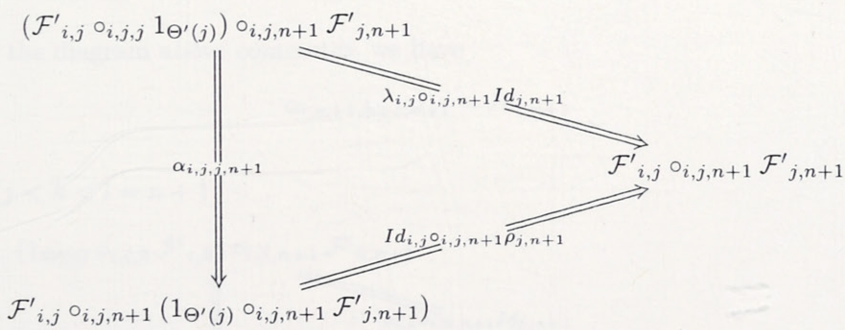
$$Id_{i,j} \circ_{i,j,n+1} \lambda_{j,n+1} = Id_{i,n+1}$$

because $\lambda_{j,n+1} = Id_{j,n+1}$, by induction as $j > i$.

3: Since the diagram above commutes, we have

$$\alpha_{i,j,n+1,n+1} = Id_{i,n+1}$$

Case 3: $i < j = k < l = n+1$



Now,

1:

$$\lambda_{i,j} \circ_{i,j,n+1} Id_{j,n+1} = Id_{i,n+1}$$

because $\lambda_{i,j} = Id_{i,j}$ from choice in item **(I1)** above.

2:

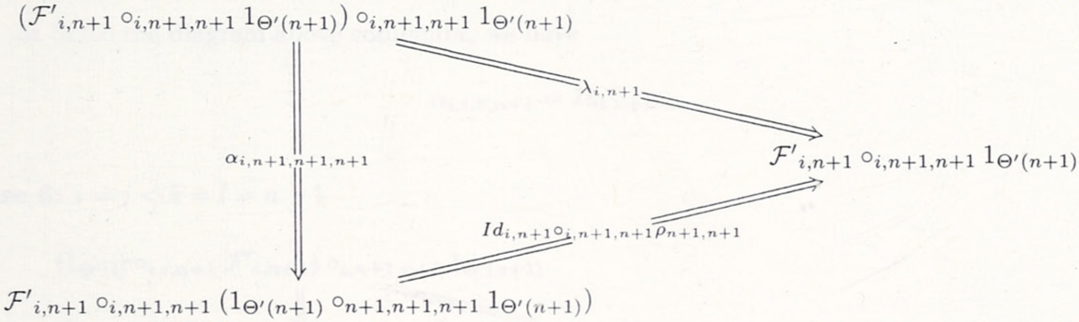
$$Id_{i,j} \circ_{i,j,n+1} \rho_{j,n+1} = Id_{i,n+1}$$

because $\rho_{j,n+1} = Id_{j,n+1}$, by induction as $j > i$.

3: Since the diagram above commutes, we have

$$\alpha_{i,j,j,n+1} = Id_{i,n+1}$$

Case 4: $i < j = k = l = n+1$



Now,

1:

$$\lambda_{i,n+1} = Id_{i,n+1}$$

from choice in item **(I1)** above.

2:

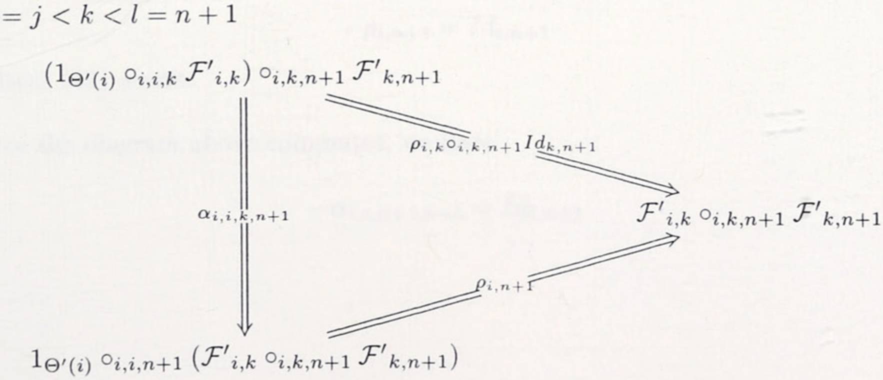
$$Id_{i,j} \circ_{i,n+1,n+1} \rho_{n+1,n+1} = Id_{i,n+1}$$

because $\rho_{n+1,n+1} = Id_{j,n+1}$ from **Lemma 5**.

3: Since the diagram above commutes, we have

$$\alpha_{i,n+1,n+1,n+1} = Id_{i,n+1}$$

Case 5: $i = j < k < l = n+1$



Now,

1:

$$\rho_{i,k} \circ_{i,k,n+1} Id_{k,n+1} = Id_{i,n+1}$$

because $\rho_{i,k} = Id_{i,k}$ from specifications S.

2:

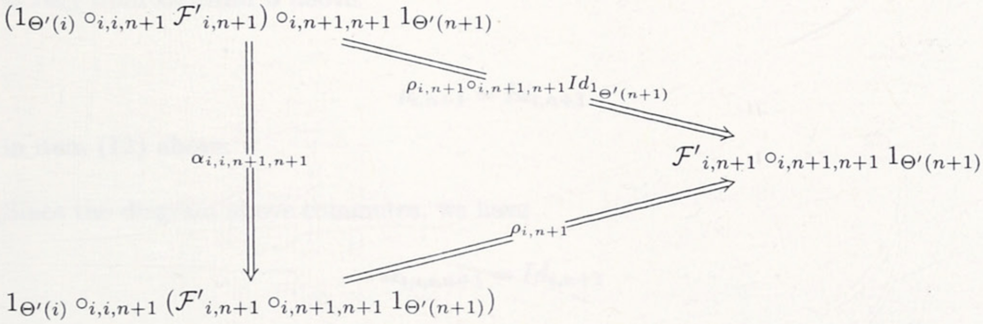
$$\rho_{i,n+1} = Id_{i,n+1}$$

from choice in item **(I2)** above.

3: Since the diagram above commutes, we have

$$\alpha_{i,i,k,n+1} = Id_{i,n+1}$$

Case 6: $i = j < k = l = n+1$



Now,

1:

$$\rho_{i,n+1} \circ_{i,n+1,n+1} Id_{1_{\Theta'(n+1)}} = Id_{i,n+1}$$

because $\rho_{i,n+1} = Id_{i,n+1}$ from choice in item **(I2)** above.

2:

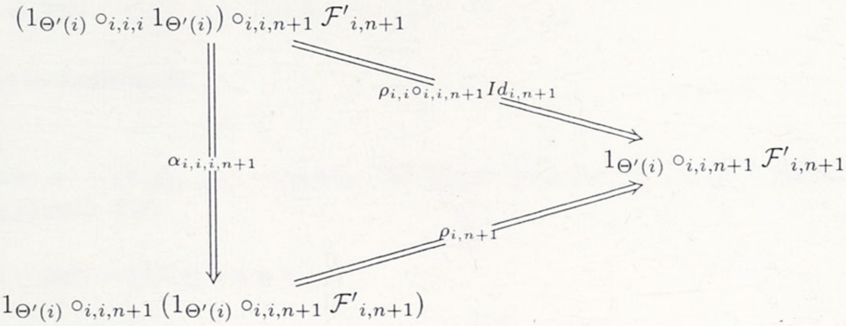
$$\rho_{i,n+1} = Id_{i,n+1}$$

from choice in item **(I2)** above.

3: Since the diagram above commutes, we have

$$\alpha_{i,i,n+1,n+1} = Id_{i,n+1}$$

Case 7: $i = j = k < l = n + 1$



Now,

1:

$$\rho_{i,i} \circ_{i,i,n+1} Id_{i,n+1} = Id_{i,n+1}$$

because $\rho_{i,i} = Id_{i,i}$ from **Lemma 5** above.

2:

$$\rho_{i,n+1} = Id_{i,n+1}$$

from choice in item **(I2)** above.

3: Since the diagram above commutes, we have

$$\alpha_{i,i,i,n+1} = Id_{i,n+1}$$

Case 8: $i = j = k = l = n + 1$

$$\alpha_{n+1,n+1,n=1,n+1} = Id_{n+1,n+1}$$

from **Lemma 5**. ∎

Lemma 7. *Given $\bar{f} = (n, f_O, f_A)$, complete set of specifications $S = (\Theta, \mathcal{F}, S0, S2)$ for $\bar{f}$, and $\bar{g} = (n+1, g_O, g_A)$ such that*

1) $g_O(i) = f_O(i-1), 0 < i \leq n+1$

2) $g_A(i) = f_A(i-1), 0 < i \leq n$

Then there exists complete set of specifications $S' = (\Theta', \mathcal{F}', S0', S2')$ for $\bar{g}$ such that

1) $\Theta'(i) = \Theta(i-1), 0 < i \leq n+1$

2) $\mathcal{F}'(i,j) = \mathcal{F}(i-1, j-1), 0 < i \leq j \leq n+1$

3) $S0'(i) = S0(i-1), 0 < i \leq n+1$

4) $S2'(i,j,k) = S2(i-1, j-1, k-1), 0 < i \leq j \leq k \leq n+1$

Proof. Symmetric to **Lemma 6**. ■

Lemma 8. *Given $\bar{f} = (n, f_O, f_A)$, complete set of specifications $S = (\Theta, \mathcal{F}, S0, S2)$ for $\bar{f}$, and $\bar{g} = (n_0 + n + n_1, g_O, g_A)$ such that*

1) $g_O(n_0 + i) = f_O(i-1), 0 \leq i < n+1$

2) $g_A(n_0 + i) = f_A(i-1), 0 \leq i < n$

Then there exists complete set of specifications $S' = (\Theta', \mathcal{F}', S0', S2')$ for $\bar{g}$ such that

1) $\Theta'(n_0 + i) = \Theta(i), 0 \leq i < n+1$

2) $\mathcal{F}'(n_0 + i, n_0 + j) = \mathcal{F}(i,j), 0 \leq i \leq j \leq n$

3) $S0'(n_0 + i) = S0(i), 0 \leq i \leq n$

4) $S2'(n_0 + i, n_0 + j, n_0 + k) = S2(i,j,k), 0 \leq i \leq j \leq k \leq n$

Proof. Use **Lemma 6** to extend to the right and **Lemma 7** to extend to the left. ■

Lemma 9. *Given $\bar{f} = (n, f_O, f_A)$, there is complete set of specifications $S = (\Theta, \mathcal{F}, S0, S2)$ for $\bar{f}$.*

Proof. We use induction on length $|\bar{f}|$.

1) If $|\bar{f}| = 0$, use **Lemma 5** to get CSS for $\bar{f}$.

2) For $|\bar{f}| = n+1$, where $n \geq 0$, suppose we have CSS S for $\bar{f} \upharpoonright_{(0,n)}$, we use **Lemma 6** to extend it to CSS for $\bar{f}$. ■

III.4 Universal Arrows:

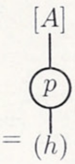
III.4.1 Normal form for complete specification trees:

Given a PD $\bar{f}$, and a CSS S for $\bar{f}$, we have trees with only specification nodes that are constructed from S; they use only specification appearing in S, placed in the same way as in S and contains no 2-Cell nodes. Call such a tree a *Specification Tree*, and denote set of such trees as Υ_S ($dom(T) = \bar{f}$ whenever $T \in \Upsilon_S$).

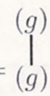
All such $T \in \Upsilon_S$ are equivalent to a unique tree in normal form in Υ_S as shown below. Denote normal form tree for S by $\coprod_S$.

Definition Normal form:. A tree $T \in \Upsilon_S$ is said to be in normal form if and only if .

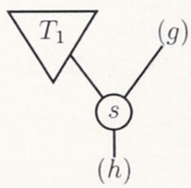
(1) $|\bar{f}| = 0$, and

$T =$ 

(2) or, $|\bar{f}| = 1$, and

$T =$ 

(3) or, $|\bar{f}| > 1$, and

$T =$ 

and T_1 is in normal form.

Lemma 10. *Given complete specification set S and $T \in \Upsilon_S$, then $T \simeq \coprod_S$, i.e. $T \longrightarrow^* \coprod_S$.*

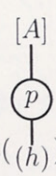
Proof. An algorithm for converting a specification tree into its normal form is given. It actually produces a witness for converting T into its normal form. This is called $Norm$ and has type

$$Norm : \Upsilon_S \longrightarrow (\rightarrow^*)$$

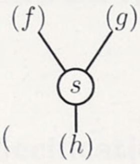
where $(\rightarrow^*)$ is set of sequences of elementary transformations.

$Norm$ is defined recursively as:

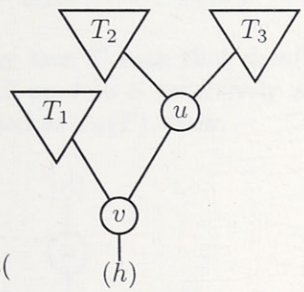
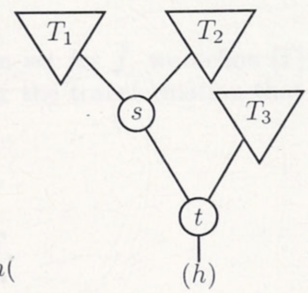
1:

$Norm($  $) = \epsilon$

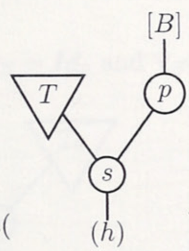
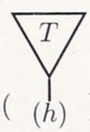
2:

$Norm($  $) = \epsilon$

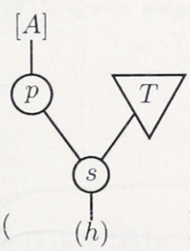
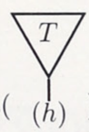
3:

$Norm($  $) = (\epsilon, \overline{ID}, h) \cdot (\epsilon, \overline{ALP}, s, t, u, v) \cdot Norm($  $)$

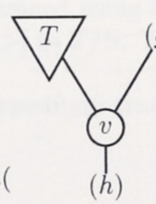
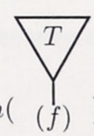
4:

$Norm($  $) = (\epsilon, \overline{ID}, h) \cdot (\epsilon, \underline{LMD}, s, p) \cdot Norm($  $)$

5:

$Norm($  $) = (\epsilon, \overline{ID}, h) \cdot (\epsilon, \underline{RHO}, s, p) \cdot Norm($  $)$

6:

$Norm($  $) =< l > *Norm($  $)$

Let $< t_i >= Norm(T)$. Then $< t_i >$ is transformation for $T \longrightarrow \coprod_S$. Hence, $T \simeq \coprod_S$. ■

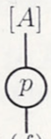
Corollary 10.1. *Given complete specification set S and $T_1, T_2 \in \Upsilon_S$ then $T_1 \simeq T_2$.*

Proof. $T_1 \simeq \coprod_S \simeq T_2$. ■

III.4.2 Residue modulo complete specifications of a tree:

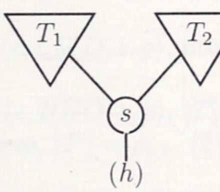
Given a tree T, let number of 0-specifications in T be $\#_0(T)$ and number of 2-specifications in T be $\#_2(T)$. Let $\#(T) = \#_2(T) + 1 - \#_0(T)$.

For any tree T such that $dom(T) = \bar{f}$ and S is complete specification set for $\bar{f}$, we define $[T]_S$ the residue of T modulo S recursively as below. We also produce a witness for the transformation that finds $[T]_S$, denoted as $\vee_S(T)$. Now,



1) If $T =$, then $[T]_S = 1_{A,S0(0),p}$ and $\vee_S(T') = (\epsilon, \overline{S0}, S0(0), p)$.

2) If $T =$ then $[T]_S = Id_g$ and $\vee_S(T) = (\epsilon, \overline{ID}, f)$.



3) If $T =$, then $[T]_S = [T_1]_{S\uparrow dom(T_1)} \circ_{S2(0,\#(T_1),\#(T_1)+\#(T_2)),s} [T_2]_{S\uparrow dom(T_2)}$ and $\vee_S(T) =< l > * \vee_{S\uparrow dom(T_1)}(T_1) \cdot < r > * \vee_{S\uparrow dom(T_2)}(T_2) \cdot (\epsilon, \overline{S2}, [T_1]_{S\uparrow dom(T_1)}, [T_2]_{S\uparrow dom(T_2)}, S2(0, \#(T_1), \#(T_1) + \#(T_2)), s)$.

4) If $T =$ then $[T]_S = [T_1]_S \cdot \beta$ and $\vee_S(T) =< u > * \vee_S (T_1) \cdot (\epsilon, \overline{VC}, [T_1]_S \cdot \beta, [T_1]_S, \beta)$.

Denote by $< T >_S$, the tree obtained using transformations $\vee_S(T)$ on T. Then note that $< T >_S [< u >] \in \Upsilon_S$. We denote $< T >_S [< u >]$ as T/S. Thus we have an equation $T \simeq < T >_S = T/S \odot_{<u>} [T]_S$.

Let $\longrightarrow_r$ be restriction of $\longrightarrow$ (elementary transformations) to case where $pos = \epsilon$ i.e. the transformation is applied at the root only.

Lemma 11. *$T \longrightarrow_r T'$ and S is complete set of specifications for $dom(T)$, then $[T]_S = [T']_S$.*

Proof. For simplicity in this proof subscript S is removed from $[T]_S$.

$\overline{\textbf{VC}}$ For $(\epsilon, \overline{VC}, \delta, \beta, \gamma)$, $[T] = [T_1] \cdot \beta \cdot \gamma$ and $[T'] = [T_1] \cdot \delta = [T_1] \cdot \beta \cdot \gamma$. Hence, $[T] = [T']$.

$\underline{\textbf{VC}}$ For $(\epsilon, \underline{VC}, \delta, \beta, \gamma)$, proof is same as above.

$\overline{\textbf{S0}}$ For $(\epsilon, \overline{S0}, p, q)$, $[T] = 1_{\Theta(i),S0(i),q}$ and $[T'] = 1_{\Theta(i),S0(i),p} \cdot 1_{\Theta(i),p,q} = 1_{\Theta(i),S0(i),q}$. Hence, $[T] = [T']$.

$\underline{\textbf{S0}}$ For $(\epsilon, \underline{S0}, , p, q)$, proof is same as above.

$\overline{\textbf{S2}}$ For $(\epsilon, \overline{S2}, \beta, \gamma, s, t)$, $[T] = ([T_1] \cdot \beta) \circ_{S2(i,j,k),t} ([T_2] \cdot \gamma)$ and $[T'] = ([T_1] \circ_{S2(i,j,k),s} [T_2]) \cdot (\beta \circ_{s,t} \gamma)$. Hence, $[T] = [T']$.

$\underline{\textbf{S2}}$ For $(\epsilon, \underline{S2}, \beta, \gamma, s, t)$, proof is same as above.

$\overline{\textbf{ALP}}$ For $(\epsilon, \overline{ALP}, s, t, u, v)$, $[T] = [T_1] \circ_{S2(i,j,l),v} (\gamma \circ_{S2(j,k,l),u} \delta)$ and $[T'] = ([T_1] \circ_{S2_{i,j,k},s} [T_2]) \circ_{S2(i,k,l),t} [T_3]$. From naturality we have, $[T'] = \alpha_{i,j,k,l} \cdot [T]$ and from α coherence, we have $\alpha_{i,j,k,l} = Id_{i,l}$. Hence, $[T] = [T']$.

$\underline{\textbf{ALP}}$ For $(\epsilon, \underline{ALP}, s, t, u, v)$, proof is same as above.

$\overline{\textbf{LMD}}$ For $(\epsilon, \overline{LMD}, s, p)$, $[T] = [T_1] \cdot Id_f = [T_1]$ and $[T'] = ([T_1] \circ_{S2(i,j,j),s} 1_{\Theta(j),S0(j),p}) \cdot \lambda_{s,p}$. From naturality we have, $[T'] = \lambda_{i,j} \cdot [T]$ and λ coherence we have $\lambda_{i,j} = Id_{i,j}$. Hence, $[T] = [T']$.

$\underline{\textbf{LMD}}$ For $(\epsilon, \underline{LMD}, s, p)$, proof is same as above.

$\overline{\textbf{RHO}}$ For $(\epsilon, \overline{RHO}, s, p)$, $[T] = [T_1] \cdot Id_f = [T_1]$ and $[T'] = (1_{\Theta(j),S0(j),p} \circ_{S2(i,i,j),s} [T_1]) \cdot \rho_{s,p}$. From naturality we have, $[T'] = \rho_{i,j} \cdot [T]$ and ρ coherence we have $\rho_{i,j} = Id_{i,j}$. Hence, $[T] = [T']$.

$\underline{\textbf{RHO}}$ For $(\epsilon, \underline{RHO}, s, p)$, proof is same as above.

$\overline{\textbf{ID}}$ For $(\epsilon, \overline{ID}, f)$, $[T] = [T_1]$, $[T'] = [T_1] \cdot Id_f = [T_1]$.

$\underline{\textbf{ID}}$ For $(\epsilon, \underline{ID}, f)$, proof is same as above. ∎

Lemma 12. *If $[T_1]_S = [T_1']_S$, then $[T_1 \odot_p T_2]_S = [T_1' \odot_p T_2]_S$*

Proof. Since T_1 is subtree of $T_1 \odot_p T_2$ and T_1' is a subtree of $T_1' \odot_p T_2$, while evaluating $[T_1 \odot_p T_2]_S$ and $[T_1' \odot_p T_2]_S$, at certain point we need to evaluate $[T_1]_S$ and $[T_1']_S$. But then $[T_1]_S = [T_1']_S$, and rest of the evaluation is same for $[T_1 \odot_p T_2]_S$ and $[T_1' \odot_p T_2]_S$. Hence, $[T_1 \odot_p T_2]_S = [T_1' \odot_p T_2]_S$ ∎

Corollary 12.1. *$T \simeq T'$ and S is complete set of specifications for $dom(T)$, then $[T]_S = [T']_S$.*

Proof. Using **Lemma 11** and **Lemma 12** we have $T \longrightarrow T' \implies [T]_S = [T']_S$. Using induction on number of steps in $\simeq = \longrightarrow^*$, we get the required result. ∎

Lemma 13. *Suppose $T_1, T_2 \in \Upsilon$ such that $\tau(T_1) = \tau(T_2)$, and S is complete set of specifications for $dom(T_1) = dom(T_2)$, then*

$$T_1 \simeq T_2 \Longleftrightarrow [T_1]_S = [T_2]_S$$

Proof. $(\Longleftarrow)$

$$T_1 \xrightarrow{\vee_S(T_1)} \; < T_1 >_S = T_1/S \odot_{<u>} [T_1]_S \xrightarrow{(\epsilon, \overline{ID}, codom(T_1))} (T_1/S \odot_{<u>} [T_1]_S) \odot_{<u>} Id_{codom(T_1)}$$

$$\big\downarrow \; Associativity$$

$$T_1/S \odot_{<uu>} ([T_1]_S \odot_{<u>} Id_{codom(T_1)})$$

$$\big\downarrow \; (\epsilon, \overline{VC}, [T_2]_S, [T_1]_S, ID_{codom(T_1)})$$

$$T_2 \xrightarrow[\vee_S(T_2)]{} \; < T_2 >_S = T_2/S \odot_{<u>} [T_2]_S \xrightarrow[\textbf{Corollary10.1}]{\simeq} T_1/S \odot_{<u>} [T_2]_S$$

$(\Longrightarrow)$ **Corollary 12.1** ∎

III.4.3 Universal arrows:

Definition Universal Arrow:. A tree T is defined to be universal if and only if $[T]_S$ is isomorphism, where S is CSS for $dom(T)$.

Existence of universal arrow follows from **Lemma 9**. We denote such tree by U. Now we prove the universal property of such an arrow.

Lemma 14. *Given any 2-Cell T such that $dom(T) = \bar{f}$ and a universal arrow U such that $dom(U) = \bar{g} = \bar{f} \uparrow_{(m,m')}$, then there is a 2-Cell T', such that $U \odot_{pos'} T' \simeq T$.*

Proof. Let S be set of CSS for $\bar{g} = dom(U)$. Since, $\bar{g} = \bar{f} \uparrow_{(m,m')}$, we extend S to S' such that S' is CSS for $\bar{f}$. Now, let $S'' = S' \downarrow_{(m,m')}$, $\bar{h} = \bar{f}[codom(U)/(m, m')]$, $pos = < l^{|\bar{h}|-m-1} \cdot r^{if(m=0)(0)else(1)} >$, $pos' = u \cdot pos$

and, $pos'' = u \cdot pos'$. Then $U/S \odot_{pos} \coprod_{S''} \in \Upsilon_{S'}$ with $dom(U \odot_{pos} \coprod_{S''}) = \bar{f}$. Now we have,

$$\begin{aligned}
T &\simeq T/S' \odot_{<u>} [T]_{S'} \\
&\simeq (U/S \odot_{pos} \coprod_{S''}) \odot_{<u>} [T]_{S'} \\
&\simeq (((U/S \odot_{<u>} [U]_S) \odot_{<u>} [U]_S^{-1}) \odot_{pos} \coprod_{S''}) \odot_{<u>} [T]_{S'} \\
&\simeq ((U \odot_{<u>} [U]_S^{-1}) \odot_{pos} \coprod_{S''}) \odot_{<u>} [T]_{S'} \\
&\simeq ((U \odot_{pos} \coprod_{S''}) \odot_{<u>} \delta) \odot_{<u>} [T]_{S'} \\
&= (U \odot_{pos'} (\coprod_{S''} \odot_{<u>} \delta)) \odot_{<u>} [T]_{S'} \\
&= U \odot_{pos''} ((\coprod_{S''} \odot_{<u>} \delta) \odot_{<u>} [T]_{S'})
\end{aligned}$$

where $\delta = (Id_{\mathcal{F}''(0,m)} \circ_{S2''(0,m,m+1),S2''(0,m,m+1)} [U]_S^{-1}) \circ_{S2''(0,m+1,|\bar{h}|),S2''(0,m+1,|\bar{h}|)} Id_{\mathcal{F}''(m+1,|\bar{h}|)}$. Thus $T' = (\coprod_{S''} \odot_{<u>} \delta) \odot_{<u>} [T]_{S'}$ satisfies the lemma. ∎

Lemma 15. *Given a universal arrow U, and two 2-cells T_1 and T_2 such that, $dom(U) = \bar{g}$, $dom(T_1) = dom(T_2) = \bar{f}$, $codom(T_1) = codom(T_2)$ and $codom(U) = f_A(m)$. Then $U \odot_{p1} T_1 \simeq U \odot_{p2} T_2$, implies $T_1 \simeq T_2$.*

Proof. Let S be CSS for $\bar{g}$. Let $\bar{h} = \bar{f}[\bar{g}/(m, m+1)]$. Let S' be extension of S to $\bar{h}$ and $S'' = S' \downarrow_{m,m+|\bar{g}|}$. Let $p1' = \vee_{S''}(T_1)(p1)$, $p2' = \vee_{S''}(T_2)(p2)$, and $p =< l^{|\bar{f}|-k-1} \cdot r^{if(k=0)(0)else(1)} >$. Then

$$\begin{aligned}
& U \odot_{p1} T_1 \simeq U \odot_{p2} T_2 \\
&\Rightarrow U \odot_{p1'} (T_1/S'' \odot_{<u>} [T_1]_{S''}) \simeq U \odot_{p2'} (T_2/S'' \odot_{<u>} [T_2]_{S''}) \\
&\Rightarrow U \odot_{p'} (\coprod_{S''} \odot_{<u>} [T_1]_{S''}) \simeq U \odot_{p'} (\coprod_{S''} \odot_{<u>} [T_2]_{S''}) \\
&\Rightarrow (U \odot_p \coprod_{S''}) \odot_{<u>} [T_1]_{S''} \simeq (U \odot_p \coprod_{S''}) \odot_{<u>} [T_2]_{S''} \\
&\Rightarrow ((U/S \odot_{<u>} [U]_S) \odot_p \coprod_{S''}) \odot_{<u>} [T_1]_{S''} \simeq ((U/S \odot_{<u>} [U]_S) \odot_p \coprod_{S''}) \odot_{<u>} [T_2]_{S''} \\
&\Rightarrow ((U/S \odot_p \coprod_{S''}) \odot_{<u>} \delta) \odot_{<u>} [T_1]_{S''} \simeq ((U/S \odot_p \coprod_{S''}) \odot_{<u>} \delta) \odot_{<u>} [T_2]_{S''} \\
&\Rightarrow \delta \cdot [T_1]_{S''} = \delta \cdot [T_2]_{S''} \\
&\Rightarrow [T_1]_{S''} = [T_2]_{S''} \\
&\Rightarrow T_1 \simeq T_2
\end{aligned}$$

where $\delta = (Id_{\mathcal{F}''(0,m)} \circ_{S2''(0,m,m+1),S2''(0,m,m+1)} [U]_S^{-1}) \circ_{S2''(0,m+1,|\bar{h}|),S2''(0,m+1,|\bar{h}|)} Id_{\mathcal{F}''(m+1,|\bar{h}|)}$. Since $[U]_S$ is isomorphism, so is δ. ∎

III.5 Conclusion:

Theorem 1. *Construction $(-)^{\#}$ transforms ana-bicategory to 2D-multitopic category.*

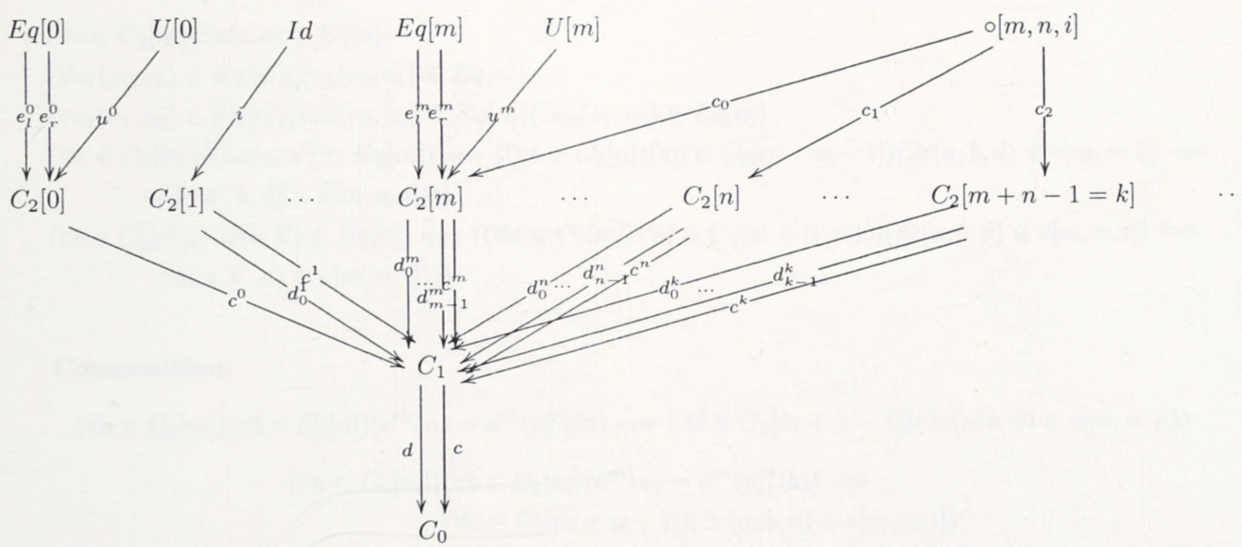
Chapter IV
Equivalence of 2D-Multitopic Category and Ana-BiCategory

In chapter 2 and chapter 3 two constructions $(-)^*$ and $(-)^\#$ were described. In this chapter we show that these constructions form adjoint pairs in the sense of FOLDS. First we take two composites $\mathcal{M} \xmapsto{(-)^*} \mathcal{M}^* \xmapsto{(-)^\#} \mathcal{M}^{*\#}$ and $\mathcal{A} \xmapsto{(-)^\#} \mathcal{A}^\# \xmapsto{(-)^*} \mathcal{A}^{\#*}$. $\mathcal{A} \simeq \mathcal{A}^{\#*}$ is obvious because all the data is preserved. In fact this is equality.

Non obvious equivalence is that of $\mathcal{M} \simeq \mathcal{M}^{*\#}$, on which we start to work now.

IV.1 FOLDS signature

The FOLDS signature for 2-D multitopic category (L_{2-DMlt}) is



The following equations hold for the arrows in the above one way category.

$$(\forall n \in \{1,2,\ldots\})(\forall 1 \leq i < n)(d_i^n \cdot d = d_{i-1}^n \cdot c)$$
$$c^0 \cdot d = c^0 \cdot c$$
$$(\forall n \in \{1,2,\ldots\})(d_0^n \cdot d = c^n \cdot d)$$
$$(\forall n \in \{1,2,\ldots\})(d_{n-1}^n \cdot c = c^n \cdot c)$$
$$i \cdot d_0^1 = i \cdot c^1$$
$$(\forall n \in \{0,1,\ldots\})(\forall p \in C_2[n] \downarrow L_{2-DMlt})(eq_l \cdot p = eq_r \cdot p)$$
$$(\forall m \in \{0,1,\ldots\})(\forall n \in \{0,1,\ldots\})(\forall 0 \leq i < n)(c_1 \cdot d_i^n = c_0 \cdot c^m)$$
$$(\forall m \in \{0,1,\ldots\})(\forall n \in \{0,1,\ldots\})(\forall 0 \leq i < n)(\forall 0 \leq j < i)(c_1 \cdot d_j^n = c_2 \cdot d_j^{m+n-1})$$
$$(\forall m \in \{0,1,\ldots\})(\forall n \in \{0,1,\ldots\})(\forall 0 \leq i < n)(\forall i < j < n)(c_1 \cdot d_j^n = c_2 \cdot d_{m+j-1}^{m+n-1})$$
$$(\forall m \in \{0,1,\ldots\})(\forall n \in \{0,1,\ldots\})(\forall 0 \leq i < n)(\forall 0 \leq j < m)(c_0 \cdot d_j^m = c_2 \cdot d_{i+j}^{m+n-1})$$
$$(\forall m \in \{0,1,\ldots\})(\forall n \in \{0,1,\ldots\})(\forall 0 \leq i < n)(c_1 \cdot c^m = c_2 \cdot c^{m+n-1})$$

The 2-D Multitopic Category is L_{2-DMlt} structure that satisfies the following Axioms (Σ_{2-DMlt}).

Equality:

$(\forall c \in C_2[n])(\exists e(c,c) \in Eq[n])$

$(\forall e_1(c_1,c_2) \in Eq[n])(\exists e_2(c_2,c_1) \in Eq[n])$

$(\forall e_1(c_1,c_2) \in Eq[n])(\forall e2(c_2,c_3) \in Eq[n])(\exists e_3(c_1,c_3) \in Eq[n])$

$(\forall a \in C_2[m])((\exists e(a,a') \in Eq[m]) \Longrightarrow ((\forall b \in C_2[n])(\forall d \in C_2[m+n-1])(\exists c(a,b,d) \in \circ[m,n,i]) \Longrightarrow$
$(\exists c(a',b,d) \in \circ[m,n,i])))$

$(\forall b \in C_2[n])((\exists e(b,b') \in Eq[n]) \Longrightarrow ((\forall a \in C_2[m])(\forall d \in C_2[m+n-1])(\exists c(a,b,d) \in \circ[m,n,i]) \Longrightarrow$
$(\exists c(a,b',d) \in \circ[m,n,i])))$

Composition:

$$(\forall a \in C_2[m])(\forall b \in C_2[n])(c^m(a) = d^m(p_i^m(b)) \Longrightarrow (\exists d \in C_2[m+n-1])(\exists c(a,b,d) \in \circ[m,n,i]))$$

$$\begin{aligned}&(\forall a \in C_2[m])(\forall b \in C_2[n])(c^m(a) = d^m(p_i^m(b)) \Longrightarrow\\&\quad(\forall d \in C_2[m+n-1])(\exists c(a,b,d) \in \circ[m,n,i])\\&\quad(\forall d' \in C_2[m+n-1])(\exists c'(a,b,d') \in \circ[m,n,i])\\&\quad(\exists e(d,d') \in Eq[m+n-1]))\end{aligned}$$

Commutativity:

$$\begin{aligned}&(\forall a \in C_2[m])(\forall a' \in C_2[m'])(\forall b \in C_2[n])(c^m(a) = d^m(p_i^m(b)) \wedge c^m(a') = d^m(p_j^m(b)) \wedge i < j \Longrightarrow\\&\quad(\forall ab \in C_2[m+n-1])(\forall a'b \in C_2[m'+n-1])(\forall a'ab \in C_2[m'+m+n-2])\\&\quad(\forall aa'b \in C_2[m+m'+n-2])(\exists c(a,b,ab) \in \circ[m,n,i])(\exists c'(a',b,a'b) \in \circ[m',n,i])\\&\quad(\exists c''(a',ab,a'ab) \in \circ[m',m+n-1,j+m-1])(\exists c'''(a,a'b,aa'b) \in \circ[m,m'+n-1,i])\\&\quad(\exists e(a'ab,aa'b) \in Eq[m+m'+n-2]))\end{aligned}$$

Associativity:

$$(\forall a \in C_2[l])(\forall b \in C_2[m])(\forall d \in C_2[n])(c^l(a) = d^m(p_j^m(b)) \wedge c^m(b) = d^n(p_i^n(d)) \Longrightarrow$$
$$(\forall ab \in C_2[l+m-1])(\forall bd \in C_2[m+n-1])(\forall (ab)d \in C_2[l+m+n-2])$$
$$(\forall a(bd) \in C_2[l+m+n-1])(\exists c(a,b,ab) \in \circ[l,m,j])(\exists c'(b,d,bd) \in \circ[m,n,i])$$
$$(\exists c''(ab,d,(ab)d) \in \circ[l+m-1,n,l+m+n-2])(\exists c'''(a,bd,a(bd)) \in \circ[l,m+n-1,i+j])$$
$$(\exists e((ab)d,a(bd)) \in Eq[l+m+n-2]))$$

Identity:

$$(\forall f \in C_1)(\exists Id_f \in I)($$
$$(\forall a \in C_2[m])(\forall 0 \le i < n)(d^n(p_i^n(a)) = f \Longrightarrow (\exists c(i(Id_f),a,a) \in \circ[1,n,i]))$$
$$\wedge (\forall a \in C_2[m])(c^n(a) = f \Longrightarrow (\exists c(a,i(Id_f),a) \in \circ[n,1,0])))$$

Universality:

$$Univ(u \in U[n])$$
$$\textbf{iff}$$
$$(\forall m \ge n)(\forall a \in C_2[m])(\forall 0 \le i < m)(\forall 0 \le j < n)((d_j^n(u^n(u)) = d_{i+j}^m(a)) \Longrightarrow$$
$$(\exists b \in C_2[m-n+1])((\exists c(u^n(u),b,a) \in \circ[n,m-n+1,i])$$
$$\wedge (\forall b' \in C_2[m-n+1])(\exists c'(u^n(u),b',a) \in \circ[n,m-n+1,i]) \Longrightarrow$$
$$(\exists e(b,b') \in Eq[m-n+1])))$$

$$(\forall A \in C_0)(\exists u \in U[0])(c(d^0(u^0(u))) = A \wedge Univ(u))$$

$$(\forall n \in \{1,2,\ldots\})(\forall f_0 \in C_1)\prod_{i=1}^{i<n}((\forall f_i \in C_1)(d(f_i) = c(f_{i-1})))$$
$$(\exists u \in U[n])((\forall 0 \le i < n)(d^n(u^n(u)) = f_i) \wedge Univ(u))$$

IV.2 Structure

We define $\mathcal{M}$ and $\mathcal{M}^{*\#}$ as two L_{2-DMlt} structures. The meaning of arrows will be common for both and will be described after filling in the object descriptions.

Definition $\mathcal{M}$:. C_0 and C_1 are $Cell_0(\mathcal{M})$ and $Cell_1(\mathcal{M})$. $C_2[i]$ is 2-cells with length of domain i. $U[i]$ are universals of domain length i. I is identity 2-cells. $Eq[i] = \{(c,c) | c \in C_2[i]\}$. $\circ[m,n,i] = \{(\alpha,\beta,\gamma) \in C_2[m] \times C_2[n] \times C_2[m+n-1] | \alpha \cdot \beta = \gamma\}$.

Definition $\mathcal{M}^{*\#}$**:.** C_0 and C_1 are same as above. $C_2[i] = \{T \in \Upsilon || dom(T)| = i\}$. $Eq[i] = \{(T_1, T_2)|T_1, T_2 \in C_2[i] \wedge T_1 \simeq T_2\}$. $I = \{T|T \in C_2[1] \wedge T \simeq Id_f$ for some $f \in C_1\}$. $U[i] \subset C_2[i]$ are the universal arrows as defined in last chapter. $\circ[m,n,i] = \{(T_1, T_2, T_3) \in C_2[m] \times C_2[n] \times C_2[m+n-1]|T_1 \odot T_2 \simeq T_3\}$.

c and d map 1-cells to their domain and codomain 0-cells. d_i^m and c^m maps 2-cell to its i^{th} place in domain and to its codomain 1-cell. e_l^m and e_r^m are left and right sides of equality on 2-cells. u^m is injection of universals into 2-cells and i is injection of identities into $C_2[1]$.

All the axioms in Σ_{2-DMlt} are true for the structure $\mathcal{M}^{*\#}$ as has been verified in previous chapter. For $\mathcal{M}$ they are automatic from the axioms of 2D-Multitopic Category.

IV.3 Equivalence:

IV.3.1 Evaluation:

0 and 1 cells of these two structures coincide as was give by the constructions in the previous chapters. For 2-cells, we define a map from $\mathcal{M}^{*\#}$ to $\mathcal{M}$ called ev, an abbreviation for evaluation, remembering the fact that trees in Υ come from 2 cells in $\mathcal{M}$ which has composition defined in it.

$$ev : \Upsilon \longrightarrow Cell_2(\mathcal{M})$$

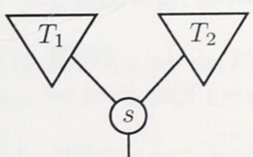
This is defined inductively on structure of trees (in Υ) and we show it is invariant under the equivalence relation $\simeq$ defined in previous chapter.

1) If $T =$ 

, then $ev(T) = Id_f$.

2) If $T =$ 

, then $ev(T) = p$.

3) If $T =$ 

, $ev(T_1) = \alpha$ and $ev(T_2) = \beta$, then $ev(T) = \alpha \cdot \beta \cdot s$

4) If $T =$  and $ev(T_1) = \alpha$, then $ev(T) = \alpha \cdot \beta$.

To show that ev is invariant under $\simeq$, we show it is invariant under each elementary step.

Lemma 1.

$$T \longrightarrow_r T' \Longrightarrow ev(T_1) = ev(T_2)$$

Proof. To show that ev is invariant under $\simeq$, we show it is invariant under each elementary step. Let the resulting tree after elementary transformation of T be T'.

$\overline{\mathbf{VC}}$ For $(\epsilon, \overline{VC}, \delta, \beta, \gamma)$, $ev(T) = ev(T_1)\cdot\beta\cdot\gamma$ and $ev(T') = ev(T_1)\cdot\delta = ev(T_1)\cdot\beta\cdot\gamma$. Hence, $ev(T) = ev(T')$.

$\underline{\mathbf{VC}}$ For $(\epsilon, \underline{VC}, \delta, \beta, \gamma)$, proof is same as above.

$\overline{\mathbf{S0}}$ For $(\epsilon, \overline{S0}, p, q)$, $ev(T) = q$ and $ev(T') = p \cdot 1_{A,p,q} = q$. Hence, $ev(T) = ev(T')$.

$\underline{\mathbf{S0}}$ For $(\epsilon, \underline{S0}, p, q)$, proof is same as above.

$\overline{\mathbf{S2}}$ For $(\epsilon, \overline{S2}, \beta, \gamma, s, t)$, $ev(T) = (ev(T_1) \cdot \beta) \cdot (ev(T_2) \cdot \gamma) \cdot t$ and $ev(T') = ev(T_1) \cdot ev(T_2) \cdot s \cdot (\beta \circ_{s,t} \gamma) = ev(T_1) \cdot ev(T_2) \cdot \beta \cdot \gamma \cdot t$. Hence, $ev(T) = ev(T')$.

$\underline{\mathbf{S2}}$ For $(\epsilon, \underline{S2}, \beta, \gamma, s, t)$, proof is same as above.

$\overline{\mathbf{ALP}}$ For $(\epsilon, \overline{ALP}, s, t, u, v)$, $ev(T) = ev(T_1) \cdot (ev(T_2) \cdot ev(T_3) \cdot u) \cdot v$, and $ev(T') = (ev(T_1) \cdot ev(T_2) \cdot s) \cdot ev(T_3) \cdot t \cdot \alpha_{s,t,u,v}$. By using definition of $\alpha_{s,t,u,v}$, we have $ev(T) = ev(T')$.

$\underline{\mathbf{ALP}}$ For $(\epsilon, \underline{ALP}, s, t, u, v)$, proof is same as above.

$\overline{\mathbf{LMD}}$ For $(\epsilon, \overline{LMD}, s, p)$, $ev(T) = ev(T_1)$ and $ev(T') = (ev(T_1) \cdot p \cdot s \cdot \lambda_{s,p}$. By using definition of $\lambda_{s,p}$, we have $ev(T) = ev(T')$.

$\underline{\mathbf{LMD}}$ For $(\epsilon, \underline{LMD}, s, p)$, proof is same as above.

$\overline{\mathbf{RHO}}$ For $(\epsilon, \overline{RHO}, s, p)$, $ev(T) = ev(T_1)$ and $ev(T') = p \cdot ev(T_1) \cdot s \cdot \rho_{s,p} = ev(T_1) \cdot (p \cdot s) \cdot \rho_{s,p}$. By using definition of $\rho_{s,p}$, we have $ev(T) = ev(T')$.

$\underline{\mathbf{RHO}}$ For $(\epsilon, \underline{RHO}, s, p)$, proof is same as above.

$\overline{\mathbf{ID}}$ For $(\epsilon, \overline{ID}, f)$, $ev(T) = ev(T_1)$, $ev(T') = ev(T_1) \cdot Id_f$. Hence, $ev(T) = ev(T')$.

$\underline{\mathbf{ID}}$ For $(\epsilon, \underline{ID}, f)$, proof is same as above. ∎

Lemma 2. *If $ev(T_1) = ev(T_1')$, then $ev(T_1 \odot_p T_2) = ev(T_1' \odot_p T_2)$*

Proof. Since T_1 is subtree of $T_1 \odot_p T_2$ and T_1' is a subtree of $T_1' \odot_p T_2$, while evaluating $ev(T_1 \odot_p T_2)$ and $ev(T_1' \odot_p T_2)$, at certain point we need to evaluate $ev(T_1)$ and $ev(T_1')$. But then $ev(T_1) = ev(T_1')$, and rest of the evaluation is same for $ev(T_1 \odot_p T_2)$ and $ev(T_1' \odot_p T_2)$. Hence, $ev(T_1 \odot_p T_2) = ev(T_1' \odot_p T_2)$ ∎

Corollary 2.1.

$$T \simeq T' \Longrightarrow ev(T) = ev(T')$$

Proof. Using **Lemma 1** and **Lemma 2** we have $T \longrightarrow T' \Longrightarrow ev(T) = ev(T')$. Using induction on number of steps in $\simeq = \longrightarrow^*$ we get the required result. ∎

Lemma 3.

$$T_1 \simeq T_2 \Longleftrightarrow ev(T_1) = ev(T_2)$$

Proof. ($\Longrightarrow$) **Corollary 2.1**

($\Longleftarrow$) Let S be complete set of specifications for $dom(T_1)$. Then we have $T_1 \simeq T_1/S \odot [T_1]_S$ and $T_2 \simeq T_2/S \odot [T_2]_S$. Since $T_1/S \simeq T_2/S$, we have $ev(T_1/S) = ev(T_2/S)$. Also since T_1/S is composed of only specifications(universals), $ev(T_1/S)$ is universal, hence left cancellable. Thus,

$$\begin{aligned}
&eq(T_1) = ev(T_2)\\
\Rightarrow &ev(T_1/S) \cdot [T_1]_S = ev(T_2/S) \cdot [T_2]_S\\
\Rightarrow &[T_1]_S = [T_2]_S\\
\Rightarrow &T_1 \simeq T_2
\end{aligned}$$

∎

IV.3.2 The Span:

To show FOLDS equivalence for $\mathcal{M}$ and $\mathcal{M}^{*\#}$ we need to find tuple (S, p, q) as $\mathcal{M} \xleftarrow{p} S \xrightarrow{q} \mathcal{M}^{*\#}$ such that p, q are fiberwise surjective. We will show that actually $S = \mathcal{M}^{*\#}$, $q = Id$ and p is constructed using ev for 2-Cells. Since p, q are natural transformations, we use p_{C_0} etc to denote its components.

Surjectivity of q is immediate. Now we list the components of p.

$$\begin{aligned}
p_{C_0} &= Id_{C_0}\\
p_{C_1} &= Id_{C_1}\\
p_{C_2[m]} &= ev|_{C_2[m]}\\
p_{Id} &= ev|_{Id}\\
p_{U[m]} &= ev|_{U[m]}\\
p_{Eq[m]} &= (ev|_{C_2[m]} \circ \pi_1, ev|_{C_2[m]} \circ \pi_2)\\
p_{\circ[m,n,i]} &= (ev|_{C_2[m]} \circ \pi_1, ev|_{C_2[m]} \circ \pi_2, ev|_{C_2[m]} \circ \pi_3)
\end{aligned}$$

p being natural transformation is obvious. p_{C_0} and p_{C_1} are obviously surjective as they are identities.

Lemma 4. *$p_{C_2[m]} = ev|_{C_2[m]}$ is fiberwise surjective on $\mathcal{M}(C_2[m])$.*

Proof. Since p is identity on $C_2[m]$ and $p_{C_2[m]}$ preserves the frame for $C_2[m]$, surjectivity will imply fiberwise surjectivity.

$m = 0$: For any $\beta \in \mathcal{M}(C_2[0])$, let $p \in \mathcal{M}(U[0])$ such that $dom(p) = dom(\beta)$, then there is unique $\gamma \in \mathcal{M}(C_2[1])$ such that $\beta = p \cdot \gamma$. Now, consider $T = p \odot_{<u>} p \in \mathcal{M}^{*\#}(C_2[0])$, then $ev(T) = p \cdot \gamma = \beta$.

$m = 1$: For any $\beta \in \mathcal{M}(C_2[1])$, consider $T = \beta \in \mathcal{M}^{*\#}(C_2[1])$, then $ev(T) = \beta$.

$m \geq 2$: We use induction.

Base case $n = 2$: For any $\gamma \in \mathcal{M}(C_2[2])$, let $s \in \mathcal{M}(U[2])$ such that $dom(s) = dom(\gamma)$. Then there is $\beta \in \mathcal{M}(C_2[1])$ such that $\gamma = s \cdot \beta$. Now, tree $T = s \odot_{<u>} \beta \in \mathcal{M}^{*\#}(C_2[2])$, is such that $ev(T) = s \cdot \beta = \gamma$.

Induction Step: Suppose for all $\alpha \in \mathcal{M}(C_2[n])$, there is T_α such that $ev(T_\alpha) = \alpha$. Now consider $\gamma \in \mathcal{M}(C_2[n+1])$, and $s \in \mathcal{M}(U[2])$ such that $dom(s) \leq_0 dom(\gamma)$. Then there is $\beta \in \mathcal{M}(C_2[n])$ such that $\gamma = s \cdot \beta$. By induction hypothesis, there is a tree T_β such that $ev(T_\beta) = \beta$. Let pos be such that $T_\beta[pos] = codom(s)$. Then tree $T = s \odot_{pos} T_\beta \in \mathcal{M}^{*\#}(C_2[n+1])$ is such that $ev(T) = s \cdot \beta = \gamma$. ∎

Lemma 5. *$p_{Id} = ev|_{Id}$ is fiberwise surjective.*

Proof. Let $Id_f \in \mathcal{M}(Id)$ and $T \in \mathcal{M}^{*\#}(C_2[1])$ such that $p_{C_2[1]}(T) = Id_f$. Since, $\mathcal{M}(i)(Id_f) = Id_f = p_{C_2[1]}(T)$, we need to show that $T \in \mathcal{M}^{*\#}(Id)$, $\mathcal{M}^{*\#}(i)(T) = T$, and $p_{Id}(T) = Id_f$.

Since $p_{C_2[1]}(T) = Id_f$, we have $T \simeq Id_f$, hence $T \in \mathcal{M}^{*\#}(Id)$. Since $\mathcal{M}^{*\#}(i)$ is injection, we have $\mathcal{M}^{*\#}(i)(T) = T$. Now, $p_{Id}(T) = ev|_{Id}(T) = Id_f$. ∎

Lemma 6. *A 2-cell $\alpha : f \Longrightarrow g$ is universal in $\mathcal{M}$ if and only if α is isomorphism in $\mathcal{M}^*$.*

Proof. Isomorphisms are universals is obvious (for any β, consider $\alpha^{-1} \cdot \beta$).

Suppose α is universal in $\mathcal{M}$. Then let β be such that $\alpha \cdot \beta = Id_f$. Now,

$$\begin{aligned}\alpha \cdot (\beta \cdot \alpha) &= (\alpha \cdot \beta) \cdot \alpha \\ &= Id_f \cdot \alpha \\ &= \alpha \\ &= \alpha \cdot Id_g\end{aligned}$$

Since, universals are left-cancellable, $\beta \cdot \alpha = Id_g$. ∎

Lemma 7. *If s, u are two universals in a multitopic category $\mathcal{M}$ such that $dom(s) \leq dom(u)$, then there is a universal t such that $u = s \cdot t$.*

Proof. Existence of t satisfying $u = s \cdot t$ follows since s is universal. Let α, be such that $dom(t) \leq dom(\alpha)$. we need to show existence and uniqueness of β such that $\alpha = t \cdot \beta$.

Since $dom(u) = dom(s \cdot t) \leq dom(s \cdot \alpha)$, there exists unique β such that $s \cdot \alpha = u \cdot \beta$. Hence,

$$\begin{aligned} &s \cdot \alpha = u \cdot \beta \\ \Longrightarrow\ &s \cdot \alpha = s \cdot t \cdot \beta \\ \Longrightarrow\ &\alpha = t \cdot \beta \end{aligned}$$

■

Lemma 8. *Given a $T \in \mathcal{M}^{*\#}(C_2[m])$,*

$$T \in \mathcal{M}^{*\#}(U[m]) \Longleftrightarrow ev(T) \in \mathcal{M}(U[m])$$

Proof. Let S be CSS for $dom(T)$. Now $T \simeq T/S \odot_{<u>} [T]_S$, hence $ev(T) = ev(T/S) \cdot [T]_S$. As T/S is tree made of only specifications (universals), $ev(T/S)$ is universal.

($\Longrightarrow$) Since $T \in \mathcal{M}^{*\#}(U[m])$, $[T]_S$ is an isomorphism. So, from **Lemma 6**, $ev([T]_S)$ is universal. Hence the composite $ev(T/S) \cdot ev([T]_S) = ev(T)$ is universal.

($\Longleftarrow$) Now since $ev(T)$ and $ev(T/S)$ are universals, from **Lemma 7**, $ev([T]_S)$ is universal. Now from **Lemma 6**, $[T]_S$ is isomorphism. Hence, $T \in \mathcal{M}^{*\#}(U[m])$. ■

Lemma 9. *$p_{U[m]} = ev|_{U[m]}$ is fiberwise surjective.*

Proof. Let $u \in \mathcal{M}(U[m])$ and $T \in \mathcal{M}^{*\#}(C_2[m])$ such that $p_{C_2[m]}(T) = u$. Since, $\mathcal{M}(u^m)(u) = u = p_{C_2[m]}(T)$, we need to show that $T \in \mathcal{M}^{*\#}(U[m])$, $\mathcal{M}^{*\#}(u^m)(T) = T$, and $p_{U[m]}(T) = u$.

Since u is universal and $ev(T) = u$, $T \in \mathcal{M}^{*\#}(U[m])$ from **Lemma 8**. Since $\mathcal{M}^{*\#}(u^m)$ is injection, we have $\mathcal{M}^{*\#}(u^m)(T) = T$. Now, $p_{U[m]}(T) = ev|_{U[m]}(T) = u$. ■

Lemma 10. *$p_{Eq[m]} = (ev|_{C_2[m]} \circ \pi_1, ev|_{C_2[m]} \circ \pi_2)$ is fiberwise surjective.*

Proof. Let $(\alpha, \alpha) \in \mathcal{M}(Eq[m])$ and $T, T' \in \mathcal{M}^{*\#}(C_2[m])$ such that $p_{C_2[m]}(T) = p_{C_2[m]}(T') = \alpha$. Since, $\mathcal{M}(e_l^m)((\alpha,\alpha)) = \alpha = p_{C_2[m]}(T)$, and $\mathcal{M}(e_r^m)((\alpha,\alpha)) = \alpha = p_{C_2[m]}(T')$, we need to show that $(T, T') \in \mathcal{M}^{*\#}(Eq[m])$, $\mathcal{M}^{*\#}(e_l^m)((T,T')) = T$, $\mathcal{M}^{*\#}(e_r^m)((T,T')) = T'$, and $p_{Eq[m]}((T,T')) = (\alpha, \alpha)$.

Since $ev(T) = ev(T')$, $T \simeq T'$ (**Lemma 3**), hence $(T, T') \in \mathcal{M}^{*\#}(Eq[m])$. Since $\mathcal{M}^{*\#}(e_l^m)$ and $\mathcal{M}^{*\#}(e_r^m)$ are projections, we have $\mathcal{M}^{*\#}(e_l^m)((T,T')) = T$, $\mathcal{M}^{*\#}(e_r^m)((T,T')) = T'$. Now, $p_{Eq[m]}((T,T')) = (ev|_{C_2[m]} \circ \pi_1, ev|_{C_2[m]} \circ \pi_2)(T,T') = (ev|_{C_2[m]}(T), ev|_{C_2[m]}(T')) = (\alpha, \alpha)$. ∎

Lemma 11. *If T_1 and T_2 are two composable trees at position pos, then $ev(T_1 \odot_{pos} T_2) = ev(T_1) \cdot ev(T_2)$.*

Proof. We use induction on structure of T_2.

1) T_2 is empty tree. Then, $ev(T_1 \odot_{pos} T_2) = ev(T_1) = ev(T_1) \cdot Id_{codom(T_1)} = ev(T_1) \cdot ev(T_2)$.

2) $T_2 = T' \odot_{<l>} (T'' \odot_{<r>} s)$. Here we have two cases.

a) *pos* begins with l. Then,

$$\begin{aligned} ev(T_1 \odot_{pos} T_2) &= ev(T_1 \odot_{pos\ <l>} T') \cdot ev(T'') \cdot s \\ &= ev(T_1) \cdot ev(T') \cdot ev(T'') \cdot s \\ &= ev(T_1) \cdot ev(T_2) \end{aligned}$$

In here, $ev(T_1 \odot_{pos\ <l>} T') = ev(T_1) \cdot ev(T')$ as tree T' is less complex than T_2.

b) *pos* begins with r. Then,

$$\begin{aligned} ev(T_1 \odot_{pos} T_2) &= ev(T') \cdot ev(T_1 \odot_{pos\ <r>} T'') \cdot s \\ &= ev(T') \cdot ev(T_1) \cdot ev(T'') \cdot s \\ &= ev(T_1) \cdot ev(T') \cdot ev(T'') \cdot s \\ &= ev(T_1) \cdot ev(T_2) \end{aligned}$$

In here, $ev(T_1 \odot_{pos\ <r>} T'') = ev(T_1) \cdot ev(T'')$ as tree T'' is less complex than T_2.

3) $T_2 = T' \odot_{<u>} \beta$. Then,

$$\begin{aligned} ev(T_1 \odot_{pos} T_2) &= ev(T_1 \odot_{pos\ <u>} T') \cdot \beta \\ &= ev(T_1) \cdot ev(T') \cdot \beta \\ &= ev(T_1) \cdot ev(T_2) \end{aligned}$$

∎

Lemma 12. *$p_{\circ[m,n,i]} = (ev|_{C_2[m]} \circ \pi_1, ev|_{C_2[m]} \circ \pi_2, ev|_{C_2[m]} \circ \pi_3)$ is fiberwise surjective.*

Proof. Let $(\alpha, \beta, \gamma) \in \mathcal{M}(\circ[m,n,i])$, $T_1 \in \mathcal{M}^{*\#}(C_2[m])$, $T_2 \in \mathcal{M}^{*\#}(C_2[n])$, and $T_3 \in \mathcal{M}^{*\#}(C_2[m+n-1])$ such that $p_{C_2[m]}(T_1) = \alpha$, $p_{C_2[n]}(T_2) = \beta$, and $p_{C_2[m+n-1]}(T_3) = \gamma$. Since, $\mathcal{M}(c_0)((\alpha,\beta,\gamma)) = \alpha = p_{C_2[m]}(T_1)$, $\mathcal{M}(c_1)((\alpha,\beta,\gamma)) = \beta = p_{C_2[n]}(T_2)$, and $\mathcal{M}(c_0)((\alpha,\beta,\gamma)) = \gamma = p_{C_2[m+n-1]}(T_3)$, we need to show that $(T_1, T_2, T_3) \in \mathcal{M}^{*\#}(\circ[m,n,i])$, $\mathcal{M}^{*\#}(c_0)((T_1,T_2,T_3)) = T_1$, $\mathcal{M}^{*\#}(c_1)((T_1,T_2,T_3)) = T_2$, $\mathcal{M}^{*\#}(c_2)((T_1,T_2,T_3)) = T_3$, and $p_{\circ[m,n,i]}((T_1,T_2,T_3)) = (\alpha,\beta,\gamma)$.

Since $ev(T_3) = \gamma = \alpha \cdot \beta = ev(T_1) \cdot ev(T_2) = ev(T_1 \odot T_2)$, we have $T_1 \odot T_2 \simeq T_3$, hence $(T_1, T_2, T_3) \in \mathcal{M}^{*\#}(\circ[m,n,i])$. Since $\mathcal{M}^{*\#}(c_0)$, $\mathcal{M}^{*\#}(c_1)$, and $\mathcal{M}^{*\#}(c_2)$ are projections, we have $\mathcal{M}^{*\#}(c_0)((T_1, T_2, T_3)) = T_1$, $\mathcal{M}^{*\#}(c_1)((T_1, T_2, T_3)) = T_2$ and $\mathcal{M}^{*\#}(c_2)((T_1, T_2, T_3)) = T_3$. Now, $p_{\circ[m,n,i]}((T_1, T_2, T_3)) = (ev|_{C_2[m]} \circ \pi_1, ev|_{C_2[m]} \circ \pi_2, ev|_{C_2[m]} \circ \pi_3)((T_1, T_2, T_3)) = (ev|_{C_2[m]}(T_1), ev|_{C_2[m]}(T_2), ev|_{C_2[m]}(T_3)) = (\alpha, \beta, \gamma)$. ■

IV.4 Conclusion:

Theorem 1. *2D-multitopic category and ana-bicategory are equivalent.*

Bibliography

[1] C.Hermida, M.Makkai, A.J.Power, On weak higher dimensional categories I, Part 1. *Journal of pure and applied algebra* 153 (2000), 221-246.

[2] C.Hermida, M.Makkai, A.J.Power, On weak higher dimensional categories I, Part 2. *Journal of pure and applied algebra* 157 (2001), 247-277.

[3] C.Hermida, M.Makkai, A.J.Power, On weak higher dimensional categories I, Part 3. *Journal of pure and applied algebra* 166 (2002), 83-104.

[4] M.Makkai, Avoiding the axiom of choice in general category theory. *Journal of pure and applied algebra* 108 (1996), 109-173.

[5] M.Makkai, First order logic with dependent sorts, with applications to category theory. Preprint: http://www.math.mcgill.ca/makkai (1995).

[6] M.Makkai, On comparing definitions of weak n-category. Preprint: http://www.math.mcgill.ca/makkai (2001).