ESTIMATION FOR HOMOGENEOUS POISSON PROCESSES

by

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ABSTRACT

This thesis gives an overview of the methods of estimation available for homogeneous Poisson processes. The main objective is to compare the different estimators in terms of bias and mean-squared error. Several numerical results are given for the problem of estimating the parameter of the process and its reliability function.

RÉSUMÉ

Dans cette thèse on présente une revue détaillée des méthodes d'estimation relatives aux processus homogènes de Poisson. L'objectif premier est de comparer le biais et l'erreur quadratique moyenne des divers estimateurs proposés. Plusieurs résultats numériques sont présentés, concernant l'estimation du paramètre du processus ainsi que de sa fonction de fiabilité.

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INTRODUCTION

Very often, one has to deal with phenomena in which events of some type occur randomly in time. The Poisson process is the formal model of such phenomena. If the rate at which the events occur does not change with time, the process is said to be homogeneous.

The purpose of this thesis is twofold. First, a number of statistical methods are described for the problem of estimating the parameter λ of the homogeneous Poisson process. Second, various estimators for the reliability function of the process are described and their bias and mean-squared error are compared numerically.

In the reliability context, the events are often called "failures", and the reliability function of the process represents the probability that the process will continue without failure throughout a period of duration x, say. Interest in this probability is aroused by its frequent invocation to describe the "reliability" of a piece of equipment, or of a system.

Some of the methods studied in this thesis were developed recently, while others are well known but are described here more briefly for comparison and completeness.

For the problem of estimating the reliability function of the process, the data on which the estimation is based may be available in two forms. In the first form, only the number of events in given time periods is observed, while in the second, the respective positions of the events in the time periods are also recorded. The method of estimation depends on the form in which the data are given. The analysis using "counts" will be discussed in the sections entitled "Poisson analysis" , while for "intervals" , the sections will be denoted by "exponential analysis". The thesis concludes with some numerical results for both types of analyses.

It seems to the author that many distributional results (so far not available) for the statistics discussed hereafter may be obtained by means of the bootstrap method. However, this issue is not addressed here.

Discussion is restricted to homogeneous Poisson processes for which the literature is quite extensive. A further useful contribution to the field would be the unification and assessment of the work on non-homogeneous Poisson processes.

1. Preliminaries

Consider a stochastic process $(\Omega, \mathcal{B}, \mathbb{P})$ defined on a sample space Ω and let $\omega \in \Omega$ be a realization of the process. Define $N_t(\omega)$ as the number of events in the time interval $(0,t]$ for the realization ω.

Definition 1 : The counting process $\{N_t \ ; \ t \geq 0\}$ is called a homogeneous Poisson process if the following conditions are satisfied:

(a) for almost all ω, with respect to the probability measure $\mathbb{P}$, each increment of $t \to N_t(\omega)$ is of unit magnitude,

(b) for any $s,t \geq 0$, the random variable $(N_{t+s} - N_t)$ is independent of $\{N_u \ ; \ u \leq t\}$,

(c) for any $s,t \geq 0$, the distribution of $(N_{t+s} - N_t)$ is independent of t.

The random variable $N_t(\omega)$ will be denoted hereafter by N_t, assuming a fixed realization ω of the process. The following results can be derived from Definition (1); the proofs will be omitted but can be found in Çinlar [1975, p.74].

Proposition 1 : Let $\{N_t \ ; \ t \geq 0\}$ be a homogeneous Poisson process. Then, for all $t \geq 0$,

$$\Pr\left\{N_t = n\right\} = \frac{(\lambda t)^n \, e^{-\lambda t}}{n!} , \qquad n = 0,1,\ldots,$$

for some constant $\lambda \geq 0$, called the mean rate of occurrence.

<u>Proposition 2</u> : Let $T_1, T_2, \ldots$ be the times of occurrence of successive events in a homogeneous Poisson process. Then, for any $k \geq 0$,

$$\Pr\left\{T_{k+1} - T_k \leq t \mid T_0, T_1, \ldots, T_k\right\} = 1 - e^{-\lambda t} \ , \quad t \geq 0 \ ;$$

in other words, the process has independent and exponentially distributed increments.

Some of the methods of estimation discussed here are based on the Poisson distribution of the number of events in a given interval whereas others make use of the fact that the intervals between successive events are exponentially distributed.

2. Estimation of the parameter λ

Let $\{N_t \; ; \; t\geq 0\}$ be a homogeneous Poisson process with unknown parameter $\lambda\geq 0$. In this chapter, various methods for estimating λ will be discussed. The first section assumes that the data are available in the form of counts whereas the second explores the analysis based on intervals between events.

2.1 Poisson analysis.

Suppose that the process has been observed over the interval $(0,t_o]$ and let n_o denote the number of events observed in this interval. In the context of point estimation, three estimators will now be studied: the maximum likelihood estimator $(\hat{\lambda}_{ML})$, the minimum variance unbiased estimator $(\hat{\lambda}_{MVU})$, and a minimax estimator $(\hat{\lambda}_{MINMAX})$. It turns out that these three estimators are the same.

2.1.1 The maximum likelihood and minimum variance unbiased estimators of λ.

The maximum likelihood estimator of λ is well known to be

$$\hat{\lambda}_{ML} = \frac{n_o}{t_o} \; .$$

This estimator is unbiased and moreover, since it is based on the complete sufficient statistic n_o , it is also the minimum variance unbiased estimator of λ, by the Lehmann-Scheffé Theorem [Lehmann, 1983, p.80]. Thus,

$$\hat{\lambda}_{MVU} = \frac{n_o}{t_o} \; .$$

As an aside it is worth noting that an unbiased estimator for λ can also be found by seeking a function $f(n_o)$ that satisfies the identity

$$\sum_{n_o=0}^{\infty} f(n_o) \cdot \frac{(\lambda t_o)^{n_o} e^{-\lambda t_o}}{n_o!} = \lambda .$$

But this implies

$$\sum_{n_o=0}^{\infty} f(n_o) \cdot \frac{(\lambda t_o)^{n_o}}{n_o!} = \lambda e^{\lambda t_o}$$

and

$$f(n_o) = \frac{n_o}{t_o} ,$$

since

$$\sum_{n_o=0}^{\infty} \frac{n_o}{t_o} \cdot \frac{(\lambda t_o)^{n_o}}{n_o!} = \sum_{n_o=0}^{\infty} \lambda \cdot \frac{(\lambda t_o)^{n_o-1}}{(n_o-1)!} = \lambda e^{\lambda t_o} .$$

2.1.2 The minimax estimator of λ.

The derivation of the minimax estimator, which is perhaps not as well known as the derivation of the two previous estimators, is presented here for the sake of completeness. This discussion can be found in the paper written by Dvoretzky, Kiefer and Wolfowitz [1953]. The authors consider two different risk functions: the first one incorporates the cost function and the second ignores it.

Let $R^*_\lambda(\hat{\lambda}, t_o)$ denote the risk function associated with the estimator $\hat{\lambda}$ when the process is observed over the interval $(0,t_o]$.

If $c(t_\circ)$ represents the cost of observing the process over $(0,t_\circ]$ and $L(\lambda,\hat{\lambda})$ is the loss function

$$L(\lambda,\hat{\lambda}) = \frac{1}{\lambda}(\hat{\lambda} - \lambda)^2 \quad , \tag{1}$$

then $R^*_\lambda(\hat{\lambda},t_\circ)$ is defined as follows:

$$R^*_\lambda(\hat{\lambda},t_\circ) = E\left\{c(t_\circ) + L(\lambda,\hat{\lambda})\right\} \quad , \tag{2}$$

where $E\{\cdot\}$ denotes the expectation with respect to the distribution with parameter λ. The loss function in (1) was proposed by Hodges and Lehmann [1951], and by Girshick and Savage [1951]. The main reasons for using (1) instead of the classical loss function $(\hat{\lambda} - \lambda)^2$ are the following:

(a) the loss function in (1) measures the seriousness of errors in terms of the difficulty of estimation, expressed by the variance, λ,

(b) the classical loss function $(\hat{\lambda} - \lambda)^2$ gives infinite minimax risk and when this happens, every estimator is minimax.

The minimax estimator is derived from the Bayes estimator, $\hat{\lambda}_B$, which minimizes the average risk

$$R^*_F(\hat{\lambda},t_\circ) = \int_\varphi R^*_\lambda(\hat{\lambda},t_\circ)\, dF(\lambda) \quad , \tag{3}$$

where F is the prior distribution of λ and φ is the parameter space. The minimax estimator should minimize the maximum risk, not just the average risk.

However, if for every $t_o \geq 0$ there exists a distribution F of λ such that

$$\int_{\varphi} R_\lambda^*(\hat{\lambda}_B, t_o)\ dF(\lambda) = \sup_{\lambda} R_\lambda^*(\hat{\lambda}_B, t_o) \quad , \qquad (4)$$

then $\hat{\lambda}_B$ is also a minimax estimator of λ [Lehmann, 1983, p.249]. The result in (4) corresponds to a situation where the average risk of the Bayes estimator, $\hat{\lambda}_B$, is equal to its maximum risk. This happens, for instance, when the risk is constant for all values of λ in the parameter space φ.

The relationship between minimax estimators and Bayes estimators is now obvious. If the maximum risk of an arbitrary estimator $\hat{\lambda}$ is less than or equal to the risk of a Bayes estimator satisfying (4), then $\hat{\lambda}$ is also minimax. The same result holds for sequences of Bayes estimators.

The foregoing is used to construct a minimax estimator of λ based on a fixed observation period $(0,t_o]$. The value of t that minimizes the cost function, namely t_o, will be determined later on. For the moment, the cost function is left out of the derivation and the risk function considered becomes

$$R_\lambda(\hat{\lambda}) = E_\lambda \left\{ L(\lambda,\hat{\lambda}) \right\} \quad ; \qquad (5)$$

the average risk with respect to the distribution F is then

$$R_F(\hat{\lambda}) = \int_{\varphi} R_\lambda(\hat{\lambda})\ dF(\lambda) \quad . \qquad (6)$$

Observe that the only difference between the risk functions defined in (2) and (5) is the presence (or absence) of the cost function.

A Bayes estimator can be obtained by minimizing the risk defined in (6), but a more convenient way of deriving it consists in minimizing the posterior risk, defined as follows:

$$R_F(\hat{\lambda} \mid n) = E_\lambda \left\{ L(\lambda, \hat{\lambda}(n)) \mid N = n \right\}$$

$$= \int_\varphi R_\lambda(\hat{\lambda}) \, dF(\lambda \mid n) \quad ,$$

where $E_\lambda \{ \bullet \}$ denotes the expectation with respect to $F(\lambda \mid n)$ and where $F(\lambda \mid n)$ is the posterior distribution of λ given $N = n$. If $R_F(\hat{\lambda} \mid n)$ is independent of n, the posterior risk is said to be independent of the sample. Working with the posterior risk instead of the risk itself simplifies the derivation of the Bayes estimator in many situations. The fact that the estimator thus obtained is Bayes follows easily from the next theorem [Lehmann, 1983, p.239]:

Theorem 1 : Let λ have distribution F and let N have distribution P_λ . Suppose, in addition, that the following assumptions hold for the problem of estimating λ with nonnegative loss function $L(\lambda, \hat{\lambda})$:

(a) there exists an estimator $\hat{\lambda}$ with finite risk,

(b) for almost all n, there exists a value $\hat{\lambda}_F(n)$ minimizing

$$E_\lambda \left\{ L(\lambda, \hat{\lambda}(n)) \mid N = n \right\} \quad .$$

Then $\hat{\lambda}_F(n)$ is a Bayes estimator.

The first condition in Theorem (1) follows easily from the convexity of the loss function. Since

$$\left(\frac{b - \hat{\lambda}}{b - a}\right) + \left(\frac{\hat{\lambda} - a}{b - a}\right) = 1 \quad ,$$

the convexity of $L(\lambda,\hat{\lambda})$ implies that for all $\hat{\lambda}$ in a given interval (a,b),

$$L(\lambda,\hat{\lambda}) = \frac{1}{\lambda}(\hat{\lambda}-\lambda)^2 \leq \left(\frac{b-\hat{\lambda}}{b-a}\right)\frac{1}{\lambda}(a-\lambda)^2 + \left(\frac{\hat{\lambda}-a}{b-a}\right)\frac{1}{\lambda}(b-\lambda)^2$$

and hence the risk function $R_\lambda(\hat{\lambda}) = E_\lambda\{L(\lambda,\hat{\lambda})\}$ satisfies

$$R_\lambda(\hat{\lambda}) \leq \left(\frac{b-\hat{\lambda}}{b-a}\right)R_\lambda(a) + \left(\frac{\hat{\lambda}-a}{b-a}\right)R_\lambda(b) .$$

The risk function is therefore convex and thus continuous on (a,b). Consequently, $R_\lambda(\hat{\lambda})$ is finite for all $\hat{\lambda}$ in (a,b) and condition (a) is satisfied.

Dvoretzky, Kiefer and Wolfowitz [1953] make use of the following theorem from decision theory to show that the estimator $\frac{n_o}{t_o}$ is not only the maximum likelihood and minimum variance unbiased estimator of λ, but also its minimax estimator.

__Theorem 2__ : Suppose that for every $t \geq 0$, there exists a sequence of distributions $F_k (k = 1,2,...)$ for which there are corresponding Bayes solutions $\hat{\lambda}_k^t$ with the property that the posterior risk associated with F_k and $\hat{\lambda}_k^t$ is independent of the sample $n(t)$, and suppose that there exists a $\hat{\lambda}^t$ for which

$$R(t) = \sup_\lambda R_\lambda(\hat{\lambda}^t) = \lim_{k\to\infty} R_{F_k}(\hat{\lambda}_k^t) . \tag{7}$$

If there exists a t_o $(0 \leq t_o \leq \infty)$ for which

$$c(t_0) + R(t_0) = \min_{t\geq 0} [c(t) + R(t)]$$

holds, then the fixed-time estimator $\hat{\lambda}^{t_0}$ is minimax.

<u>Proof of Theorem 2</u> : To prove this theorem, consider any other estimator, $\hat{\lambda}^*$ say, with associated time of observation t^*. Then,

$$\begin{aligned} \sup_{\lambda} R_\lambda(\hat{\lambda}^*) &\geq \int_{\varphi} R_\lambda(\hat{\lambda}^*)\, dF_k(\lambda) \\ &\geq \int_{\varphi} R_\lambda(\hat{\lambda}_k^t)\, dF_k(\lambda) \end{aligned} \tag{8}$$

for all k, since $\hat{\lambda}_k^t$ is the Bayes solution, that is, the estimator which minimizes the right-hand side of (8). Hence,

$$\begin{aligned} \sup_{\lambda} R_\lambda(\hat{\lambda}^*) &\geq \lim_{k\to\infty} \int_{\varphi} R_\lambda(\hat{\lambda}_k^t)\, dF_k(\lambda) \\ &= \lim_{k\to\infty} R_{F_k}(\hat{\lambda}_k^t) \\ &= \sup_{\lambda} R_\lambda(\hat{\lambda}^t) \qquad , \text{ from (7).} \end{aligned}$$

Thus,

$$\sup_{\lambda} R_\lambda(\hat{\lambda}^*) \geq \sup_{\lambda} R_\lambda(\hat{\lambda}^{t_0}) \quad ,$$

since t_0 minimizes $\sup_{\lambda} R_\lambda(\hat{\lambda}^t)$ over all values of t. Therefore, $\hat{\lambda}^{t_0}$ is a minimax estimator for λ.

■

Referring now to Theorem (2), Dvoretzky, Kiefer and Wolfowitz [1953] propose the sequence of distribution functions $F_k(\lambda)$, $k = 1,2,\ldots$ on the half-line $\lambda > 0$ with density:

$$f_k(\lambda) = \frac{1}{k} e^{-\lambda/k} \quad , \quad (0 < \lambda < \infty) \quad . \tag{9}$$

This sequence of priors seems to be a convenient choice as it produces a sequence of Bayes estimators with constant risk. However, the authors give no other justification to their choice of prior distribution. In principle, a prior distribution should be selected by combining experience (knowledge about the parameter) and convenience, but the selection in this case seems to be based on mathematical convenience only.

Corresponding to the sequence of priors given in (9), is a sequence of posterior density functions which follows easily using Bayes Theorem:

$$\begin{aligned}
f_k(\lambda \mid n) &= \frac{f_k(\lambda) \cdot f_k(n \mid \lambda)}{\int_{\varphi} f_k(\lambda) \cdot f_k(n \mid \lambda)\, d\lambda} \\
&= \frac{\left\{\frac{1}{k} e^{-\lambda/k}\right\} \cdot \left\{\frac{(\lambda t)^n}{n!} e^{-\lambda t}\right\}}{\int_0^\infty \left\{\frac{1}{k} e^{-\lambda/k}\right\} \cdot \left\{\frac{(\lambda t)^n}{n!} e^{-\lambda t}\right\} d\lambda} \\
&= \frac{\lambda^n e^{-\lambda(t+1/k)}}{\int_0^\infty \lambda^n e^{-\lambda(t+1/k)}\, d\lambda} \\
&= \frac{\lambda^n e^{-\lambda(t+1/k)}}{\left\{\frac{n!}{(t+1/k)^{n+1}}\right\}} \\
&= \frac{\lambda^n}{n!} (t+1/k)^{n+1} e^{-\lambda(t+1/k)} \quad , \quad 0 < \lambda < \infty ,
\end{aligned}$$

which is a Gamma density with parameters $\alpha = n+1$ and $\beta = (t+1/k)^{-1}$. The posterior risk then becomes

$$R_k(\hat{\lambda}^t \mid n) = \int_{\varphi} E_\lambda \left\{ \frac{1}{\lambda} (\hat{\lambda}^t(n) - \lambda)^2 \mid n \right\} dF_k(\lambda \mid n) ,$$

but since the sample value n is now fixed, the integrand becomes the loss function itself, so that

$$R_k(\hat{\lambda}^t \mid n) = \int_0^\infty \frac{1}{\lambda} (\hat{\lambda}^t - \lambda)^2 f_k(\lambda \mid n)\, d\lambda .$$

Lehmann [1983, p.239] shows that if the two assumptions of Theorem (1) hold and if the loss function is equal to

$$L(\lambda,\hat{\lambda}) = \omega(\lambda)\, [\hat{\lambda} - g(\lambda)]^2 ,$$

then the Bayes estimator is given by

$$\hat{\lambda}_B = \frac{E\left\{\omega(\lambda) \cdot g(\lambda) \mid n\right\}}{E\left\{\omega(\lambda) \mid n\right\}} . \tag{10}$$

Here, $\omega(\lambda) = \frac{1}{\lambda}$ and $g(\lambda) = \lambda$ so that (10) becomes

$$\hat{\lambda}_k^t = \frac{E\left\{1 \mid n\right\}}{E\left\{1/\lambda \mid n\right\}} = \frac{1}{\int_0^\infty \frac{1}{\lambda} f_k(\lambda \mid n)\, d\lambda}$$

$$= \frac{1}{\int_0^\infty \frac{1}{\lambda} \cdot \frac{\lambda^n}{n!} \cdot (t+1/k)^{n+1} \; e^{-\lambda(t+1/k)} \; d\lambda}$$

$$= \frac{n!}{(t+1/k)^{n+1} \int_0^\infty \lambda^{n-1} \; e^{-\lambda(t+1/k)} \; d\lambda}$$

$$= \frac{n!}{(t+1/k)^{n+1} \cdot \frac{(n-1)!}{(t+1/k)^n}}$$

$$= \frac{n}{t + 1/k}.$$

The corresponding posterior risk is next shown to be equal to

$$R_k(\lambda_k^t \mid n) = \frac{1}{t + 1/k} \quad .$$

Since

$$R_k(\lambda_k^t \mid n) = \int_0^\infty \frac{1}{\lambda} \left\{\frac{n}{t+1/k} - \lambda\right\}^2 f_k(\lambda \mid n) \; d\lambda$$

$$= \int_0^\infty \frac{1}{\lambda} \left\{\frac{n}{t+1/k} - \lambda\right\}^2 \cdot \frac{\lambda^n}{n!} \; (t+1/k)^{n+1} \; e^{-\lambda(t+1/k)} \; d\lambda$$

$$= \int_0^\infty \frac{\lambda^{n-1}}{n!} \; (t+1/k)^{n-1} \; e^{-\lambda(t+1/k)} \; n^2 \; d\lambda$$

$$- 2 \int_0^\infty \frac{\lambda^n}{n!} \; (t+1/k)^n \; e^{-\lambda(t+1/k)} \; n \; d\lambda$$

$$+ \int_0^\infty \frac{\lambda^{n+1}}{n!} \; (t+1/k)^{n+1} \; e^{-\lambda(t+1/k)} \; d\lambda$$

$$= \frac{n^2}{n!} (t+1/k)^{n-1} \int_0^\infty \lambda^{n-1} e^{-\lambda(t+1/k)} d\lambda$$

$$- \frac{2n}{(t+1/k)} \int_0^\infty \frac{\lambda^n}{n!} (t+1/k)^{n+1} e^{-\lambda(t+1/k)} d\lambda \qquad (11)$$

$$+ \frac{(n+1)}{(t+1/k)} \int_0^\infty \frac{\lambda^{n+1}}{(n+1)!} (t+1/k)^{n+2} e^{-\lambda(t+1/k)} d\lambda ,$$

but since the last two integrands in (11) constitute the posterior density, the corresponding integrals are equal to one. Thus,

$$R_k(\hat{\lambda}_k^t \mid n) = \frac{n^2}{n!} (t+1/k)^{n-1} \frac{(n-1)!}{(t+1/k)^n} - \frac{2n}{(t+1/k)} + \frac{(n+1)}{(t+1/k)}$$

$$= \frac{n}{t+1/k} - \frac{2n}{t+1/k} + \frac{n+1}{t+1/k}$$

$$= \frac{1}{t + 1/k} .$$

The posterior risk is independent of n and hence of the sample since n is a sufficient statistic. The posterior risk being constant over all values of λ, the Bayes estimator $\hat{\lambda}_k^t$ is also minimax for all $k \geq 1$, by the result described earlier in (4).

Consider now the usual estimator

$$\hat{\lambda}^t = \frac{n}{t} .$$

Referring to Theorem (2), $\hat{\lambda}^t$ is minimax if, first of all,

$$R(t) = \sup_\lambda R_\lambda(\hat{\lambda}^t) = \lim_{k \to \infty} R_k(\hat{\lambda}_k^t) . \qquad (12)$$

The left-hand side of (12) is given by

$$\begin{aligned} R(t) &= \sup_{\lambda} E_{\lambda}\left\{L(\lambda,\hat{\lambda}^{t})\right\} \\ &= \sup_{\lambda} E_{\lambda}\left\{\frac{1}{\lambda}\left(\frac{n}{t}-\lambda\right)^{2}\right\} \\ &= \sup_{\lambda} \frac{1}{\lambda}\,\mathrm{Var}\left\{\frac{n}{t}\right\} \\ &= \sup_{\lambda} \frac{1}{\lambda t^{2}}\,\mathrm{Var}\left\{n\right\} \\ &= \sup_{\lambda}\left[\frac{\lambda t}{\lambda t^{2}}\right] \\ &= \frac{1}{t} \quad . \end{aligned}$$

The risk function of the sequence of Bayes estimators was shown earlier to be

$$R_k(\hat{\lambda}_k^{t}) = \frac{1}{t+1/k} \quad ,$$

so that the right-hand side of (12) becomes

$$\begin{aligned} \lim_{k\to\infty} R_k(\hat{\lambda}_k^{t}) &= \lim_{k\to\infty} \frac{1}{t+1/k} \\ &= \frac{1}{t} \quad , \end{aligned}$$

and the first condition of Theorem (2) is satisfied. The next step consists in choosing a t_* $(0\le t_*<\infty)$ for which

$$c(t_*) + R(t_*) = \min_{t\ge 0}\left\{c(t) + R(t)\right\}$$

holds, for the cost function $c(t)$ of interest. The following theorem can now be stated:

Theorem 3 : For the Poisson process with $0 < \lambda < \infty$ and loss function (1), a minimax estimation procedure is to take a single observation n_o at time $t = t_o$ for which $c(t) + 1/t$ becomes a minimum, and to estimate λ by

$$\hat{\lambda}_{MINMAX} = \frac{n_o}{t_o} \quad ,$$

where n_o is the number of events in $(0,t_o]$.

2.1.3 Estimators of bounded relative error.

The point estimator just described, $\frac{n_o}{t_o}$, has many desirable properties. Its main advantages are that

(a) it maximizes the likelihood of the observed sample,

(b) it is unbiased for λ,

(c) it is minimax,

(d) it is convenient to use since only the number of events in $(0,t_o]$ is needed, not their respective positions in the interval.

However, the main disadvantage of this estimator is that it is not possible to make a confidence statement about its reliability. Indeed, it will be shown shortly that the distribution of the ratio $\frac{\hat{\lambda}}{\lambda}$ depends on λ. This ratio is a reasonable indicator of the reliability of an estimator $\hat{\lambda}$. Therefore let

$$\alpha = Pr\left\{(1 - \gamma) \le \frac{\hat{\lambda}}{\lambda} \le (1 + \gamma)\right\} . \qquad (13)$$

The present section is concerned with estimators of "bounded relative error", using a sampling scheme where the number of events, n_0 , to be observed is fixed in advance. An estimator $\hat{\lambda}$ of λ is said to be of "bounded relative error" if it is possible to say with confidence coefficient α that $\hat{\lambda}$ does not differ from λ by more than 100γ percent of λ, where neither α nor γ depend on the true value of λ. In other words, the probability in (13) can be evaluated and does not depend on λ.

For the estimator $\hat{\lambda} = \frac{n_0}{t_0}$, it is easy to show that this probability depends on λ. Indeed,

$$\Pr\left\{(1-\gamma) \leq \frac{\frac{n_0}{t_0}}{\lambda} \leq (1+\gamma)\right\} = \Pr\left\{\lambda t_0(1-\gamma) \leq n_0 \leq \lambda t_0(1+\gamma)\right\}$$

$$= \sum_{j=a}^{b} \frac{(\lambda t_0)^j e^{-\lambda t_0}}{j!} ,$$

where $a = \lambda t_0(1-\gamma)$ and $b = \lambda t_0(1+\gamma)$, but this is a function of λ.

Girshick, Rubin and Sitgreaves [1952] propose some alternative estimation procedures which yield estimators of bounded relative error. The procedures are discussed in terms of a problem of particle counting, in which a set of inert particles is randomly distributed over a microscope slide of area A . It is assumed that the probability of n particles falling in a subset of area a is

$$\Pr\left\{N_a = n\right\} = \frac{(\lambda a)^n e^{-\lambda a}}{n!} , \quad 0 < \lambda < \infty .$$

The following discussion is a restriction of their results to the case where the continuous variable is time instead of area. The results can also be extended to problems in higher dimensional Euclidean spaces.

Consider the procedure in which the Poisson process is observed until a fixed number of events, n , has occured. The time required to observe n events becomes a random variable and its distribution depends only on λ, which appears as a scale parameter.

Theorem 4 : If a Poisson process is observed until a specified number n of events is counted, and if T_n is the time required to observe n events, then the random variable $2\lambda T_n$ has a chi-squared distribution with $2n$ degrees of freedom.

Proof of Theorem 4 : To prove this result, notice that the event $T_n \geq t$ is equivalent to the event "the number of events observed in $(0,t]$ is less than n". Thus,

$$\begin{aligned} \Pr\left\{T_n \leq t\right\} &= \Pr\left\{N > n\right\} \\ &= 1 - \sum_{j=0}^{n-1} \frac{(\lambda t)^j e^{-\lambda t}}{j!} \\ &= F_\lambda(t) \ . \end{aligned}$$

Differentiating $F_\lambda(t)$ with respect to t gives

$$\begin{aligned} f_\lambda(t) &= \lambda e^{-\lambda t} + \sum_{j=1}^{n-1} \frac{(\lambda t)^j \lambda e^{-\lambda t} - (\lambda t)^{j-1} j\lambda e^{-\lambda t}}{j!} \\ &= \frac{(\lambda t)^{n-1} \lambda e^{-\lambda t}}{(n-1)!} \ , \quad t \geq 0 \ , \quad n > 0 \ . \end{aligned}$$

It now follows easily that the moment generating function of $2\lambda T_n$ is $(1-2x)^{-n}$ which is the moment generating function of a chi-squared random variable with $2n$ degrees of freedom. ∎

The following theorem ensures the existence of an estimator of bounded relative error, with the additional property that it is minimax:

<u>Theorem 5</u> : If λ is a scale parameter and is the only unknown parameter in a distribution, then there always exists a minimax estimator of it which is of bounded relative error, with a confidence coefficient α which is independent of λ.

The proof of this theorem can be found in Blackwell and Girshick [1954, p.318].

Suppose now that an observation T_n is made, where T_n is, as before, the time required to observe n events. Let

$$\hat{\lambda}_n^{(1)} = \frac{b}{T_n}$$

be an estimator for λ, where b is a given positive number. If γ is the desired bound on the relative error, then

$$\Pr\left\{\lambda(1-\gamma) \le \frac{b}{T_n} \le \lambda(1+\gamma)\right\} = \Pr\left\{\frac{b}{1+\gamma} \le \lambda T_n \le \frac{b}{1-\gamma}\right\}$$

$$= \int_c^d \frac{x^{n-1}\, e^{-x}}{(n-1)!}\, dx$$

$$= G_\gamma(b,n), \text{ say, independent of } \lambda,$$

where $c = \frac{b}{1+\gamma}$ and $d = \frac{b}{1-\gamma}$.

Clearly one would like to find that value of b, say b^*, for which $G_\gamma(b,n)$ is a maximum. Because of the shape of the Gamma distribution, as b varies from zero to infinity, the value of $G_\gamma(b,n)$ increases continuously from zero to a maximum value $G_\gamma(b^*,n)$, and then decreases again to zero. To determine the maximizing value b^*, consider

$$\frac{d}{db}\left[G_\gamma(b,n)\right] = \frac{d}{db}\left[\int_c^d \frac{x^{n-1}e^{-x}}{(n-1)!}\,dx\right]$$

$$= \frac{1}{(n-1)!}\,\frac{d}{db}\left[\int_0^{\frac{b}{1-\gamma}} x^{n-1}e^{-x}\,dx - \int_0^{\frac{b}{1+\gamma}} x^{n-1}e^{-x}\,dx\right].$$

Using the Fundamental Theorem of Calculus,

$$\frac{d}{db}\left[G_\gamma(b,n)\right] = \frac{b^{n-1}}{(n-1)!}\left[\frac{e^{-\frac{b}{1-\gamma}}}{(1-\gamma)^n} - \frac{e^{-\frac{b}{1+\gamma}}}{(1+\gamma)^n}\right].$$

The maximizing value b^* is the single finite positive value of b for which

$$\frac{d}{db}\left[G_\gamma(b,n)\right] = 0 ,$$

that is for which

$$\frac{b^{n-1}}{(n-1)!}\left[\frac{e^{-b/1-\gamma}}{(1-\gamma)^n} - \frac{e^{-b/1+\gamma}}{(1+\gamma)^n}\right] = 0 . \qquad (14)$$

The nonzero solution to (14) is given by

$$b^* = \frac{n(1-\gamma^2)}{2\gamma} \ln\left\{\frac{1+\gamma}{1-\gamma}\right\}$$

and the corresponding value of $G_\gamma(b^*,n)$ becomes

$$G_\gamma(b^*,n) = \int_{\mathcal{A}} \frac{x^{n-1}\, e^{-x}}{(n-1)!}\, dx$$

$$= G_\gamma^*(n) \quad , \text{ say },$$

where $\mathcal{A} = \left\{\frac{n(1-\gamma)}{2\gamma} \ln\left[\frac{1+\gamma}{1-\gamma}\right], \frac{n(1+\gamma)}{2\gamma} \ln\left[\frac{1+\gamma}{1-\gamma}\right]\right\}$.

For a fixed value of γ, the function $G_\gamma^*(n)$ is a single-valued monotone increasing function of n. The monotonicity of $G_\gamma^*(n)$ implies that, defining n_o to be the least integer such that $G_\gamma^*(n) \geq \alpha$, the function $n_o = H_\gamma(\alpha)$ is a single-valued monotone increasing function of α. In other words, the number of events that need to be observed increases with the confidence level required for estimating λ.

Choosing n_o to be the least integer such that

$$G_\gamma^*(n) = \int_{\mathcal{A}} \frac{x^{n-1}\, e^{-x}}{(n-1)!}\, dx \geq \alpha \quad , \tag{15}$$

an exact sequential procedure can be applied by taking observations in sequence, and by computing after each observation ($n=1,2,\ldots$) the value of $G_\gamma^*(n)$. The procedure terminates when the inequality in (15) is satisfied, and the resulting value of n is called n_o.

The estimator then becomes

$$\hat{\lambda}^{(1)}_{n_o} = \frac{b^*}{T_{n_o}} .$$

To measure the efficiency of this procedure, and to avoid the repeated evaluation of the integral in (15), one may approximate the number n_o of observations required to achieve the desired degree of accuracy, for different values of γ and α.

For $n > 40$, the distribution of $\left\{\sqrt{4\lambda T_n} - \sqrt{4n-1}\right\}$ is approximately Normal with zero mean and unit variance. To prove this result, consider the random variable $2\lambda T_n$ which has a chi-squared distribution with $2n$ degrees of freedom. Then, by the Central Limit Theorem, the asymptotic distribution of the random variable

$$\frac{2\lambda T_n - 2n}{\sqrt{4n}}$$

is the Normal distribution with zero mean and unit variance. Now, since the range of $2\lambda T_n$ is the set of all positive real numbers, then

<u>Theorem 6</u> : As $n \to \infty$,

$$\Pr\left\{\sqrt{4\lambda T_n} - \sqrt{4n-1} \le k\right\} \simeq \Pr\left\{\frac{2\lambda T_n - 2n}{\sqrt{4n}} \le k\right\} .$$

The proof of Theorem (6) is given here for the sake of completeness.

<u>Proof of Theorem 6</u> :

$$\lim_{n\to\infty} \Pr\left\{\sqrt{4\lambda T_n} - \sqrt{4n-1} \le k\right\} = \lim_{n\to\infty} \Pr\left\{\sqrt{4\lambda T_n} - \sqrt{4n} \le k\right\}$$

$$= \lim_{n\to\infty} \Pr\left\{\sqrt{4\lambda T_n} \le k + \sqrt{4n}\right\}$$

$$= \lim_{n\to\infty} \Pr\left\{4\lambda T_n \le k^2 + 2k\sqrt{4n} + 4n\right\}$$

$$= \lim_{n\to\infty} \Pr\left\{2\lambda T_n \le \frac{k^2}{2} + k\sqrt{4n} + 2n\right\}$$

$$= \lim_{n\to\infty} \Pr\left\{\frac{2\lambda T_n - 2n}{\sqrt{4n}} \le \frac{k^2}{2\sqrt{4n}} + k\right\}$$

$$= \lim_{n\to\infty} \Pr\left\{\frac{2\lambda T_n - 2n}{\sqrt{4n}} \le k\right\} .$$

■

Thus,

$$G^*_\gamma(n) = \Pr\left\{\frac{n(1-\gamma)}{2\gamma} \ln\left\{\frac{1+\gamma}{1-\gamma}\right\} \le \lambda T_n \le \frac{n(1+\gamma)}{2\gamma} \ln\left\{\frac{1+\gamma}{1-\gamma}\right\}\right\}$$

$$= \Pr\left\{\sqrt{\frac{4n(1-\gamma)}{2\gamma} \ln\left\{\frac{1+\gamma}{1-\gamma}\right\}} - \sqrt{4n-1} \le X \le \sqrt{\frac{4n(1+\gamma)}{2\gamma} \ln\left\{\frac{1+\gamma}{1-\gamma}\right\}} - \sqrt{4n-1}\right\} ,$$

$$\simeq \Phi\left\{\sqrt{\frac{4n(1+\gamma)}{2\gamma} \ln\left\{\frac{1+\gamma}{1-\gamma}\right\}} - \sqrt{4n-1}\right\} - \Phi\left\{\sqrt{\frac{4n(1-\gamma)}{2\gamma} \ln\left\{\frac{1+\gamma}{1-\gamma}\right\}} - \sqrt{4n-1}\right\} ,$$

where $X = \sqrt{4\lambda T_n} - \sqrt{4n-1}$ and

$$\Phi\{u\} = \int_{-\infty}^{u} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt .$$

Since $0 < \gamma < \frac{1}{2}$, a further approximation can be obtained by replacing $\frac{1}{2\gamma} \ln\left\{\frac{1+\gamma}{1-\gamma}\right\}$ by its Taylor series expansion, to get

$$\frac{1}{2\gamma} \ln\left\{\frac{1+\gamma}{1-\gamma}\right\} \simeq \frac{1}{2\gamma}\left\{2\left[\gamma + \frac{\gamma^3}{3} + \frac{\gamma^5}{5} + \frac{\gamma^7}{7}\right]\right\}$$

$$= 1 + \frac{\gamma^2}{3} + \frac{\gamma^4}{5} + \frac{\gamma^6}{7} \quad .$$

Now, since n is large, $\sqrt{4n-1}$ can be replaced by $\sqrt{4n}$ to get

$$\sqrt{\frac{4n(1+\gamma)}{2\gamma} \ln\left\{\frac{1+\gamma}{1-\gamma}\right\}} - \sqrt{4n-1} \simeq \sqrt{4n(1+\gamma)\left\{1 + \frac{\gamma^2}{3} + \frac{\gamma^4}{5} + \frac{\gamma^6}{7}\right\}} - \sqrt{4n}$$

$$= \sqrt{4n}\left\{\left[(1+\gamma)\left(1 + \frac{\gamma^2}{3} + \frac{\gamma^4}{5} + \frac{\gamma^6}{7}\right)\right]^{1/2} - 1\right\}$$

$$\simeq \sqrt{4n}\left\{\left[1 + \frac{\gamma}{2} + \frac{\gamma^2}{24} + \frac{7}{48}\gamma^3\right] - 1\right\}$$

$$= \sqrt{4n}\left\{\frac{\gamma}{2} + \frac{\gamma^2}{24} + \frac{7}{48}\gamma^3\right\} \quad .$$

and equivalently,

$$\sqrt{\frac{4n(1-\gamma)}{2\gamma} \ln\left\{\frac{1+\gamma}{1-\gamma}\right\}} - \sqrt{4n-1} \simeq \sqrt{4n}\left\{\frac{-\gamma}{2} + \frac{\gamma^2}{24} - \frac{7}{48}\gamma^3\right\} \quad .$$

Therefore, $G_\gamma^*(n)$ can be approximated by

$$G_\gamma^*(n) \simeq \Phi\left\{\sqrt{4n}\left[\frac{\gamma}{2} + \frac{\gamma^2}{24} + \frac{7}{48}\gamma^3\right]\right\} - \Phi\left\{\sqrt{4n}\left[\frac{-\gamma}{2} + \frac{\gamma^2}{24} - \frac{7}{48}\gamma^3\right]\right\} .$$

For values of γ and α which are generally of practical interest, a good approximation for the required value of n is given by the relation

$$\sqrt{4n}\left\{\frac{\gamma}{2} + \frac{7}{48}\gamma^3\right\} = Z_\alpha , \tag{16}$$

where Z_α is the value for which

$$\Phi\left\{Z_\alpha\right\} - \Phi\left\{-Z_\alpha\right\} = \alpha .$$

Table [1] gives the value of $n_\circ$ for $\gamma = 0.01,(0.01),0.10$ and $\alpha = 0.90,0.95,0.99$, using the approximation in (16).

If the loss function is taken to be

$$L(\lambda,\hat{\lambda}_{n_\circ}^{(1)}) = \begin{cases} 0 & \text{if } \left|\dfrac{\hat{\lambda}_{n_\circ}^{(1)}}{\lambda} - 1\right| \leq \gamma , \\ 1 & \text{otherwise,} \end{cases}$$

then the estimator $\hat{\lambda}_{n_\circ}^{(1)}$ is the best invariant. Since the loss function is bounded, it is also minimax [Girshick and Savage, 1951]. The risk, defined as

$$R_\lambda(\hat{\lambda}_{n_\circ}^{(1)}) = E\left\{L(\lambda,\hat{\lambda}_{n_\circ}^{(1)})\right\}$$

is then equal to

$$R_\lambda(\hat{\lambda}_{n_\circ}^{(1)}) = \Pr\left\{\left|\frac{\hat{\lambda}_{n_\circ}^{(1)}}{\lambda} - 1\right| > \gamma\right\}$$

$$= 1 - G_\gamma^*(n_\circ) .$$

Girshick, Rubin and Sitgreaves [1952] present a table of values of $G^*_\gamma(n_0)$ for four values of γ (0.01, 0.05, 0.10 and 0.20) and for $n_0 \leq 40$. For values of n_0 larger than 40, one can use the Normal approximation described earlier.

Interval sampling

The authors also propose another procedure of bounded relative error. Instead of observing the process continuously until n events have occurred, it may be more convenient to adopt a sampling procedure consisting of observing counts in k subintervals. In the j-th subinterval, the process is observed until a fixed number n_j of events is counted, with $\sum_{j=1}^{k} n_j = n$, and with the provision that the subintervals are nonoverlapping.

This procedure will be useful in some situations where the process cannot be observed continuously for long periods of time. The information collected in different non-adjacent time intervals will be sufficient to estimate λ. Another situation where this procedure may be more convenient is when many realizations of short duration are available from the Poisson process, instead of one realization over a long time interval.

If T_j is the time required in the j-th subinterval to observe n_j events, then λ is estimated by

$$\hat{\lambda}_k^{(2)} = \frac{b^*}{\sum_{j=1}^{k} T_j} ,$$

where b^* is determined as before.

Since each of the λT_j is independently distributed in a Gamma distribution with parameters $\alpha = n_j$ and $\beta = 1$, their sum has a Gamma distribution with parameters $\alpha = n$ and $\beta = 1$, because of the additive property of the Gamma distribution. The theory then goes through as before.

The two main classes of estimators proposed in the previous sections, namely the fixed-time and fixed-count estimators, are somewhat difficult to compare. In the fixed-time procedure, the number of events, n_o, is a random variable, whereas in the fixed-count procedure, n_o is a constant whose value depends on the desired degree of accuracy.

In the first procedure, the time of observation, t_o, is chosen to minimize the cost of observing the process. Even though the second procedure produces an estimator of bounded relative error, an obvious criticism of this technique is that it does not take into account the cost function. Once the confidence coefficient α is selected, n_o is chosen to maximize the probability that the estimator lies within the predetermined bounds on the relative error.

A variation of the bounded relative error criterion just discussed is given by Birnbaum's [1954] suggestions of two estimators which, instead of minimizing the relative error, minimize the absolute error.

2.1.4 <u>Estimators of bounded absolute error.</u>

The absolute error of an estimator $\hat{\lambda}$ of λ is defined as

$$\alpha = \Pr\left\{\lambda - \gamma \le \hat{\lambda} \le \lambda + \gamma\right\} .$$

The first method proposed by Birnbaum [1954] consists in observing the process until a specified number, n_o , of events have been observed. If T_{n_o} denotes the time required to observe n_o events, it is possible to find values of L and U such that

$$\Pr\left\{L \le 2\lambda T_{n_o} \le U\right\} = \Pr\left\{\frac{L}{2T_{n_o}} \le \lambda \le \frac{U}{2T_{n_o}}\right\} = \alpha \quad . \tag{17}$$

The values of L and U can be found in a table of the chi-squared distribution with $2n_o$ degrees of freedom. According to (17), α is the probability that the true value of λ satisfies the inequality

$$\frac{L}{2T_{n_o}} \le \lambda \le \frac{U}{2T_{n_o}} \quad . \tag{18}$$

Given this inequality, the maximum percentage deviation of λ from an estimator $\hat{\lambda}$ is minimized by taking

$$\hat{\lambda} = \frac{L + U}{4T_{n_o}} \quad ,$$

and this maximum percentage deviation is equal to

$$\max_{\lambda} \left|\frac{\hat{\lambda} - \lambda}{\hat{\lambda}}\right| = \max_{\lambda} \left|\frac{\frac{L + U}{4T_{n_o}} - \lambda}{\frac{L + U}{4T_{n_o}}}\right| \cdot 100 \ \%$$

$$= \left|\frac{\frac{L + U}{4T_{n_o}} - \frac{L}{2T_{n_o}}}{\frac{L + U}{4T_{n_o}}}\right| \cdot 100 \ \%$$

$$= \left\{ \frac{U - L}{U + L} \right\} \cdot 100\%$$

$$= \varepsilon \quad , \text{say} .$$

The absolute error is therefore equal to

$$\hat{\lambda} \left\{ \frac{U - L}{U + L} \right\} .$$

However, the magnitude of this bound cannot be used to determine the value of n_o necessary to satisfy it. The reason for this is that the bound itself depends on the value of $\hat{\lambda}$.

The second method has the advantage of producing an estimator with a prescribed absolute error, that is an estimator $\hat{\lambda}$ such that

$$\Pr \left\{ \lambda - \gamma \leq \hat{\lambda} \leq \lambda + \gamma \right\} \geq \alpha \quad , \tag{19}$$

where α and γ are fixed positive constants.

Let n be a positive integer. Observe T_n , the time required for the occurrence of n events. Let

$$c = \frac{(1 - \alpha)\,\gamma^2}{2n} .$$

Perform additional observation of the process for $\left\{ \frac{1}{2cT_n} \right\}$ units of time, and let N' be the number of events observed in this period. Consider the estimator

$$\hat{\lambda}' = 2cT_n N' .$$

The fact that $\hat{\lambda}'$ satisfies the condition in (19) is demonstrated here. Notice that

$$
\begin{aligned}
E\left\{\hat{\lambda}'\right\} &= E\left\{2cT_n N'\right\} \\
&= 2c\, E\left\{E\left\{T_n N' \mid T_n\right\}\right\} \\
&= 2c\, E\left\{T_n\, E\left\{N' \mid T_n\right\}\right\} \\
&= 2c\, E\left\{T_n\left(\frac{\lambda}{2cT_n}\right)\right\} \\
&= 2c\, E\left\{\frac{\lambda}{2c}\right\} \\
&= \lambda
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Var}\left\{\hat{\lambda}'\right\} &= 4c^2\left\{E\left\{E\left\{(T_n N')^2 \mid T_n\right\}\right\} - E^2\left\{E\left\{T_n N' \mid T_n\right\}\right\}\right\} \\
&= 4c^2\left\{E\left\{T_n^2\, E\left\{(N')^2 \mid T_n\right\}\right\} - E^2\left\{T_n\, E\left\{N' \mid T_n\right\}\right\}\right\} \\
&= 4c^2\left\{E\left\{T_n^2\left[\frac{\lambda}{2cT_n} + \frac{\lambda^2}{4c^2T_n^2}\right]\right\} - E^2\left\{T_n\left[\frac{\lambda}{2cT_n}\right]\right\}\right\} \\
&= 4c^2\left\{E\left\{\frac{\lambda T_n}{2c}\right\} + E\left\{\frac{\lambda^2}{4c^2}\right\} - E^2\left\{\frac{\lambda}{2c}\right\}\right\} \\
&= 4c^2\left\{\frac{1}{4c}\, E\left\{2\lambda T_n\right\} + \frac{\lambda^2}{4c^2} - \frac{\lambda^2}{4c^2}\right\}
\end{aligned}
$$

but since $2\lambda T_n$ has a chi-squared distribution with $2n$ degrees of freedom,

$$
\begin{aligned}
\mathrm{Var}\left\{\hat{\lambda}'\right\} &= 4c^2\left\{\frac{1}{4c}(2n)\right\} \\
&= 2cn\ .
\end{aligned}
$$

Then Tchebycheff's inequality gives

$$\Pr\left\{\left|\hat{\lambda}' - \lambda\right| \leq \gamma\right\} \geq 1 - \frac{2cn}{\gamma^2} = \alpha .$$

No further investigation has been made so far to determine the optimal choice of n, that is, the value that will minimize the additional amount of observation $\frac{1}{2cT_n}$ without destroying the desired properties of the resulting estimator.

All the methods described so far were based on the analysis of the number of events, n_o, in fixed or random time intervals. None of them made use of the particular instants of time at which the events occurred. The fact that n_o is sufficient for λ explains the frequent use of the Poisson analysis, and the following considerations justify it as well.

2.2 Exponential analysis.

If $T_1, T_2, \ldots, T_n$ represent the times of occurrence of n successive events and if X_i is defined as

$$X_i = T_i - T_{i-1} , \quad i = 2,\ldots,n ,$$

with $X_1 = T_1$, it is well known that the X_i's are independent identically distributed random variables having an exponential distribution with parameter $\theta = \frac{1}{\lambda}$. Then the usual estimator is

$$\hat{\lambda}^E = \frac{n}{\sum_{i=1}^{n} X_i} .$$

This estimator will have the same value as the estimator $\hat{\lambda} = \frac{n_o}{t_o}$, apart from a marginal difference due to the fact that the time stretch considered may not be precisely the same in the two cases.

The analysis based on intervals between events is likely in many cases to be more expensive than counting events falling in assigned intervals, whether it is done by continuous observation with a stop-watch or by automatic recording. Since both estimators are essentially the same, the analysis based on counts is usually preferred because of its practical convenience and its lower cost.

The following sections are specially concerned with estimation of the reliability function associated with a homogeneous Poisson process.

3. Estimation of the reliability function

If $(0,\tau]$ denotes a fixed time interval, then the reliability function is defined as

$$r_\tau = \Pr\left\{N_\tau = 0\right\} = e^{-\lambda\tau}$$
$$= \Pr\left\{X_1 > \tau\right\} ,$$

where X_1 denotes the time to the first event. In the reliability context, the parameter λ represents the constant failure rate and the "events" are called "failures".

It is desired to estimate r_τ (denoted hereafter by r) for a specified value of τ , based on a record of past events over a fixed time t_o. The intervals $(0,\tau]$ and $(0,t_o]$ will be called hereafter "mission period" and "observation period" , respectively. (It is also possible to estimate r based on $\bar{X}_k$, which denotes the mean of the time intervals between the first k failures, or to combine information on counts from k intervals (non-overlapping) of length $(0,t_o]$. This particular case will be the subject of a later section. For the moment, it is assumed that only one interval is available for observation.)

3.1 Estimation based on one interval.

In the absence of any prior information about the unknown failure rate λ , the maximum likelihood estimator $\hat{r}_{ML}$ or the minimum variance unbiased estimator $\hat{r}_{MVU}$ may be used.

3.1.1 <u>The maximum likelihood and minimum variance unbiased estimators of r.</u>

The maximum likelihood estimator of λ was shown earlier to be

$$\hat{\lambda}_{ML} = \frac{n_o}{t_o},$$

so that by the invariance property, the maximum likelihood estimator of r is given by

$$\hat{r}_{ML} = e^{-\hat{\lambda}_{ML}\tau}$$

$$= e^{-\frac{n_o}{t_o}\tau} .$$

The minimum variance unbiased estimator is equal to

$$\hat{r}_{MVU} = \left\{1 - \frac{\tau}{t_o}\right\}^{n_o} , \quad \tau < t_o .$$

To prove the unbiasedness we have

$$E\left\{\hat{r}_{MVU}\right\} = E\left\{\left\{1 - \frac{\tau}{t_o}\right\}^{n_o}\right\}$$

$$= \sum_{j=0}^{\infty}\left\{1 - \frac{\tau}{t_o}\right\}^{j} \frac{(\lambda t_o)^j e^{-\lambda t_o}}{j!}$$

$$= e^{-\lambda t_o} \sum_{j=0}^{\infty} \frac{\{(\lambda t_o)\ (1 - \tau/t_o)\}^j}{j!}$$

$$= e^{-\lambda t_o}\, e^{\lambda t_o(1 - \tau/t_o)} , \quad \text{provided } \tau/t_o < 1 ,$$

$$= e^{-\lambda\tau} .$$

The fact that this estimator is also the minimum variance unbiased estimator for r follows from the Lehmann–Scheffé Theorem [Lehmann, 1983, p.80].

If the assumption of a prior density $g(\lambda)$ is justified, the Bayes estimator $\hat{r}_B$, by definition, has the smallest mean risk and should therefore be used.

3.1.2 The Bayes estimator of r.

Throughout this discussion, the loss function is assumed to be

$$L(r,\hat{r}) = (r - \hat{r})^2 \qquad (20)$$

and the prior density of the random variable Λ is taken to be

$$g(\lambda) = \frac{1}{a} e^{-\lambda/a} \quad , \quad 0 < \lambda < \infty \ , \qquad (21)$$

where $a > 0$ is known. This prior was proposed by Beg and Alam [1977] without any justification, other than mathematical convenience.

The Bayes estimator corresponding to the loss function in (20) is, according to Lehmann [1983, p.240],

$$\hat{r}_B = E\left\{ r \middle| N_o = n_o \right\} \ , \qquad (22)$$

where n_o is, as before, the observed number of failures in the observation period $(0,t_o]$. Hence,

$$\begin{aligned} \hat{r}_B &= \int_0^\infty r \ f(r|n_o) \ dr \\ &= \int_0^\infty e^{-\lambda\tau} \ f(\lambda|n_o) \ d\lambda \ , \end{aligned}$$

where λ represents the value of the random variable Λ for this particular realization of the process. The posterior density $f(\lambda|n_\circ)$ was shown in the previous chapter to be

$$f(\lambda|n_\circ) = \frac{\lambda^{n_\circ}}{n_\circ!}\left[t_\circ + \frac{1}{a}\right]^{n_\circ+1} e^{-\lambda(t_\circ + 1/a)} \quad ,$$

so that the Bayes estimator becomes

$$\hat{r}_B = \int_0^\infty e^{-\lambda\tau}\,\frac{\lambda^{n_\circ}}{n_\circ!}\left[t_\circ + \frac{1}{a}\right]^{n_\circ+1} e^{-\lambda(t_\circ + 1/a)}\,d\lambda$$

$$= \frac{(t_\circ + 1/a)^{n_\circ+1}}{n_\circ!}\int_0^\infty \lambda^{n_\circ}\,e^{-\lambda(t_\circ + 1/a + \tau)}\,d\lambda$$

$$= \frac{(t_\circ + 1/a)^{n_\circ+1}}{n_\circ!}\quad\frac{n_\circ!}{(t_\circ + 1/a + \tau)^{n_\circ+1}}$$

$$= \left[\frac{(t_\circ + 1/a)}{(t_\circ + 1/a) + \tau}\right]^{n_\circ+1} \quad , \quad \tau > 0 \quad .$$

3.1.3 <u>Comparison of $\hat{r}_{ML}$, $\hat{r}_{MVU}$ and $\hat{r}_B$ in terms of mean risk.</u>

The risk function associated with the maximum likelihood estimator $\hat{r}_{ML}$ is by definition

$$R_r(\hat{r}_{ML}) = E\left\{\hat{r}_{ML} - r\right\}^2$$

$$= E\left\{e^{-\frac{n_\circ}{t_\circ}\tau} - e^{-\lambda\tau}\right\}^2 \quad .$$

Let $\delta = \frac{\tau}{t_\circ}$ so that the risk function becomes

$$R_r(\hat{r}_{ML}) = E\left\{e^{-2\delta}\right\}^{n_\circ} - 2e^{-\lambda\tau}\,E\left\{e^{-\delta}\right\}^{n_\circ} + e^{-2\lambda\tau} \quad ,$$

$$= \sum_{j=0}^{\infty} e^{-2\delta j} \frac{(\lambda t_0)^j \, e^{-\lambda t_0}}{j!} - 2e^{-\lambda\tau} \sum_{j=0}^{\infty} e^{-\delta j} \frac{(\lambda t_0)^j \, e^{-\lambda t_0}}{j!} + e^{-2\lambda\tau}$$

$$= e^{-\lambda t_0[1 - e^{-2\delta}]} - 2e^{-\lambda[\tau + t_0(1 - e^{-\delta})]} + e^{-2\lambda\tau} .$$

The mean risk with respect to the prior density (21) is equal to

$$R_g(\hat{r}_{ML}) = E_g \left\{ e^{-\lambda t_0[1 - e^{-2\delta}]} - 2e^{-\lambda[\tau + t_0(1 - e^{-\delta})]} + e^{-2\lambda\tau} \right\}$$

$$= \int_0^{\infty} e^{-\lambda t_0[1-e^{-2\delta}]} \frac{1}{a} e^{-\lambda/a} \, d\lambda - 2\int_0^{\infty} e^{-\lambda[\tau + t_0(1-e^{-\delta})]} \frac{1}{a} e^{-\lambda/a} \, d\lambda + \int_0^{\infty} e^{-2\lambda\tau} \frac{1}{a} e^{-\lambda/a} \, d\lambda$$

$$= \frac{1}{a} \left\{ \int_0^{\infty} e^{-\lambda[1/a + t_0(1-e^{-2\delta})]} d\lambda - 2\int_0^{\infty} e^{-\lambda[1/a + \tau + t_0(1-e^{-\delta})]} \, d\lambda + \int_0^{\infty} e^{-\lambda[1/a + 2\tau]} \, d\lambda \right\}$$

$$= \frac{1}{a} \left\{ \frac{1}{1/a + t_0(1 - e^{-2\delta})} - \frac{2}{1/a + \tau + t_0(1 - e^{-\delta})} + \frac{1}{1/a + 2\tau} \right\}$$

$$= \frac{1}{1 + \Psi_0(1 - e^{-2\delta})} - \frac{2}{1 + \Psi + \Psi_0(1 - e^{-\delta})} + \frac{1}{1 + 2\Psi} , \quad (23)$$

where $\Psi = a\tau$ and $\Psi_0 = at_0$.

The risk function for the minimum variance unbiased estimator is equal to

$$R_r(\hat{r}_{MVU}) = E\left\{ \hat{r}_{MVU} - r \right\}^2$$

$$= E\left\{\left[1 - \frac{\tau}{t_\circ}\right]^{n_\circ} - e^{-\lambda\tau}\right\}^2$$

$$= E\left\{(1 - \vartheta)^2\right\}^{n_\circ} - 2e^{-\lambda\tau}E\left\{(1 - \vartheta)\right\}^{n_\circ} + e^{-2\lambda\tau},$$

where $\vartheta = \frac{\tau}{t_\circ}$ as before;

$$R_r(\hat{r}_{MVU}) = \sum_{j=0}^{\infty}(1-\vartheta)^{2j}\frac{(\lambda t_\circ)^j e^{-\lambda t_\circ}}{j!} - 2e^{-\lambda\tau}\sum_{j=0}^{\infty}(1-\vartheta)^j\frac{(\lambda t_\circ)^j e^{-\lambda t_\circ}}{j!} + e^{-2\lambda\tau}$$

$$= e^{-\lambda t_\circ} e^{(1-\vartheta)^2(\lambda t_\circ)} - 2e^{-\lambda(\tau+t_\circ)} e^{(1-\vartheta)(\lambda t_\circ)} + e^{-2\lambda\tau}$$

$$= e^{-\lambda\tau(2-\vartheta)} - e^{-2\lambda\tau}$$

$$= e^{-2\lambda\tau}\left\{e^{\lambda\vartheta\tau} - 1\right\}.$$

The mean risk of the minimum variance unbiased estimator becomes

$$R_g(\hat{r}_{MVU}) = E_g\left\{e^{-\lambda\tau(2-\vartheta)} - e^{-2\lambda\tau}\right\}$$

$$= \int_0^\infty e^{-\lambda\tau(2-\vartheta)}\frac{1}{a}e^{-\lambda/a}\,d\lambda - \int_0^\infty e^{-2\lambda\tau}\frac{1}{a}e^{-\lambda/a}\,d\lambda$$

$$= \frac{1}{a}\left[\int_0^\infty e^{-\lambda[1/a + \tau(2-\vartheta)]}\,d\lambda - \int_0^\infty e^{-\lambda(1/a + 2\tau]}\,d\lambda\right]$$

$$= \frac{1}{a}\left\{\frac{1}{1/a + \tau(2-\vartheta)} - \frac{1}{1/a + 2\tau}\right\}$$

$$= \frac{1}{1 + \Psi(2-\vartheta)} - \frac{1}{1 + 2\Psi}$$

$$= \frac{\vartheta\Psi}{(1 + 2\Psi)\,[1 + \Psi(2-\vartheta)]}. \qquad (24)$$

The risk function associated with the Bayes estimator is given by

$$
\begin{aligned}
R_r(\hat{r}_B) &= E\left\{\hat{r}_B - r\right\}^2 \\
&= E\left\{\left[\frac{(t_\circ + 1/a)}{(t_\circ + 1/a) + \tau}\right]^{n_\circ+1} - e^{-\lambda\tau}\right\}^2 \\
&= E\left\{\left(\frac{(t_\circ + 1/a)}{(t_\circ + 1/a) + \tau}\right)^{2(n_\circ+1)} - 2\left(\frac{(t_\circ + 1/a)}{(t_\circ + 1/a) + \tau}\right)^{n_\circ+1} \cdot e^{-\lambda\tau} + e^{-2\lambda\tau}\right\} \\
&= \mu^2 E\left\{\mu\right\}^{2n_\circ} - 2\mu e^{-\lambda\tau} E\left\{\mu\right\}^{n_\circ} + e^{-2\lambda\tau} ,
\end{aligned}
$$

$$
\text{where} \quad \mu = \left(\frac{(t_\circ + 1/a)}{(t_\circ + 1/a) + \tau}\right) ,
$$

$$
\begin{aligned}
&= \mu^2 \sum_{j=0}^{\infty} \mu^{2j} \frac{(\lambda t_\circ)^j e^{-\lambda t_\circ}}{j!} - 2\mu e^{-\lambda\tau} \sum_{j=0}^{\infty} \mu^j \frac{(\lambda t_\circ)^j e^{-\lambda t_\circ}}{j!} + e^{-2\lambda\tau} \\
&= \mu^2 e^{-\lambda t_\circ(1 - \mu^2)} - 2\mu e^{-\lambda\tau - \lambda t_\circ(1 - \mu)} + e^{-2\lambda\tau} .
\end{aligned}
$$

The mean risk with respect to the prior density (21) is then equal to

$$
\begin{aligned}
R_g(\hat{r}_B) &= E_g\left\{\mu^2 e^{-\lambda t_\circ(1 - \mu^2)} - 2\mu e^{-\lambda\tau - \lambda t_\circ(1 - \mu)} + e^{-2\lambda\tau}\right\} \\
&= \mu^2 \int_0^{\infty} e^{-\lambda t_\circ(1-\mu^2)} \frac{1}{a} e^{-\lambda/a} d\lambda - 2\mu \int_0^{\infty} e^{-\lambda\tau-\lambda t_\circ(1-\mu)} \frac{1}{a} e^{-\lambda/a} d\lambda \\
&\quad + \int_0^{\infty} e^{-2\lambda\tau} \frac{1}{a} e^{-\lambda/a} d\lambda .
\end{aligned}
$$

Omitting the lengthy details involved in the simplification, the mean risk becomes

$$R_g(\hat{r}_B) = \frac{(t_0 + 1/a)^2}{(t_0 + 1/a + \tau)^2\left\{1 + at_0\left\{1 - \frac{(t_0 + 1/a)^2}{(t_0 + 1/a + \tau)^2}\right\}\right\}}$$

$$- \frac{2(t_0 + 1/a)}{(t_0 + 1/a + \tau)\left\{1 + a\tau + at_0\left\{1 - \frac{(t_0 + 1/a)}{(t_0 + 1/a + \tau)}\right\}\right\}} + \frac{1}{1 + 2a\tau}$$

$$= \frac{\Psi^2}{(1 + 2\Psi)\ [(1 + \Psi)^2 + \Psi_0(1 + 2\Psi)]} \quad . \tag{25}$$

Of the three estimators considered above, the Bayes estimator, by definition, gives the smallest mean risk, provided the chosen prior density does correctly represent the true density of λ . To compare the two classical estimators $\hat{r}_{ML}$ and $\hat{r}_{MVU}$, consider the situation where the mission time , τ , is much smaller than the observation time t_0 . Under these circumstances, $e^{-\delta}$ can be approximated by

$$e^{-\delta} \simeq \left\{1 - \delta + \frac{\delta^2}{2}\right\} \quad . \tag{26}$$

In practice, one should always be wary of estimating r from an observation period t_0 less than mission period τ so that this assumption agrees with common sense.

Using the approximation in (26), the mean risk for the maximum likelihood estimator satisfies

$$R_g(\hat{r}_{ML}) \simeq \frac{1}{1 + \Psi_0[1-(1-2\delta+2\delta^2)]} - \frac{2}{1 + \Psi + \Psi_0\left\{1-(1-\delta+\frac{\delta^2}{2})\right\}} + \frac{1}{1 + 2\Psi}$$

$$= \frac{1}{1 + 2\Psi(1 - \vartheta)} - \frac{4}{2 + \Psi(4 - \vartheta)} + \frac{1}{1 + 2\Psi}$$

$$= \frac{2\vartheta\Psi\ (1 + 2\Psi + \vartheta\Psi)}{(1 + 2\Psi)\ (2 + 4\Psi - \vartheta\Psi)\ (1 + 2\Psi - 2\vartheta\Psi)}$$

$$= \frac{2\varepsilon\ (2\varepsilon + 1)}{(1 + 2\Psi)\ (1 - \varepsilon)\ (1 - 4\varepsilon)} ,$$

where $\varepsilon = \dfrac{\vartheta\Psi}{2(1 + 2\Psi)}$.

The mean risk for the minimum variance unbiased estimator can also be written in terms of Ψ and ε as follows. According to (24),

$$R_g(\hat{r}_{MVU}) = \frac{\vartheta\Psi}{(1 + 2\Psi)\ [1 + \Psi(2 - \vartheta)]}$$

$$= \frac{2\varepsilon}{[1 + \Psi(2 - \vartheta)]}$$

$$= \frac{2\varepsilon}{(1 + 2\Psi)\left\{1 - \dfrac{\vartheta\Psi}{(1 + 2\Psi)}\right\}}$$

$$= \frac{2\varepsilon}{(1 + 2\Psi)\ (1 - 2\varepsilon)} .$$

To compare $\hat{r}_{ML}$ and $\hat{r}_{MVU}$ for the situation considered, let

$$Q = \frac{R_g(\hat{r}_{MVU})}{\tilde{R}_g(\hat{r}_{ML})} ,$$

where $\tilde{R}_g$ denotes the approximate value of R_g.

Then,

$$Q = \left\{ \frac{\dfrac{2\varepsilon}{(1 + 2\Psi)\ (1 - 2\varepsilon)}}{\dfrac{2\varepsilon\ (2\varepsilon + 1)}{(1 + 2\Psi)\ (1 - \varepsilon)\ (1 - 4\varepsilon)}} \right\}$$

$$= \frac{(1 - \varepsilon)\ (1 - 4\varepsilon)}{(1 - 2\varepsilon)\ (1 + 2\varepsilon)}$$

$$= \frac{1 - 5\varepsilon + 4\varepsilon^2}{(1 - 4\varepsilon^2)}$$

$$= 1 - \frac{\varepsilon(5 - 8\varepsilon)}{(1 - 4\varepsilon^2)} ,$$

but

$$0 < \varepsilon \leq \frac{\vartheta}{4}$$

since

$$\varepsilon = \frac{\vartheta\Psi}{2(1 + 2\Psi)} \leq \frac{\vartheta}{4} .$$

This shows that for the situation considered (τ/t_o small), the maximum likelihood estimator is almost as good as the minimum variance unbiased estimator, when the basis for comparison is taken to be the mean risk.

3.1.4 Comparison of $\hat{r}_{ML}$ and $\hat{r}_{MVU}$ in terms of bias and mean-squared error.

In the absence of any prior information about λ, it may be more appropriate to compare the maximum likelihood and minimum variance unbiased estimators from another point of view.

The numerical results given in the paper by Gaver and Hoel [1970], and expanded here to a higher degree of accuracy, may perhaps furnish a guide to choice of an estimator under given circumstances. Comparison is based on bias and mean-squared error and is carried out numerically for different values of the ratio mission time : observation time (τ/t_o).

It is assumed again that only one interval $(0,t_o]$ is available for observation and that the estimation is based on the number , n_o , of failures in this interval. The case where more than one interval can be observed will be discussed in a later section.

Consider the maximum likelihood estimator

$$\hat{r}_{ML} = e^{-\frac{n_o}{t_o}\tau} ,$$

whose m-th moment is given by

$$\begin{aligned} E\left\{\hat{r}_{ML}\right\}^m &= E\left\{e^{-\frac{n_o}{t_o}\tau}\right\}^m \\ &= \sum_{j=0}^{\infty} e^{-\frac{j\tau m}{t_o}} \frac{(\lambda t_o)^j e^{-\lambda t_o}}{j!} \\ &= e^{-\lambda t_o} \sum_{j=0}^{\infty} \frac{\left\{\lambda t_o e^{-\frac{\tau m}{t_o}}\right\}^j}{j!} \\ &= e^{-\lambda t_o} e^{\lambda t_o e^{-\frac{\tau m}{t_o}}} \\ &= \exp\left\{-\lambda t_o\left\{1 - e^{-\frac{\tau m}{t_o}}\right\}\right\} . \end{aligned} \tag{27}$$

Putting $m = 1$ in (27), it is possible to infer from

$$1 - e^{-\frac{\tau}{t_o}} \simeq 1 - \left\{ 1 - \frac{\tau}{t_o} + \frac{\tau^2}{2t_o^2} \right\}$$

$$= \frac{\tau}{t_o} - \frac{\tau^2}{2t_o^2}$$

$$< \frac{\tau}{t_o} ,$$

that

$$E\left\{\hat{r}_{ML}\right\} = \exp\left\{-\lambda t_o(1 - e^{-\tau/t_o})\right\} > \exp\left\{-\lambda t_o(\tau/t_o)\right\} = r .$$

Hence the maximum likelihood estimator has a positive bias. However, as the observation time, t_o, becomes large, the bias approaches zero. Similarly, the variance of $\hat{r}_{ML}$ is derived from (27) :

$$\mathrm{Var}\left\{\hat{r}_{ML}\right\} = E\left\{\hat{r}_{ML}\right\}^2 - E^2\left\{\hat{r}_{ML}\right\}$$

$$= \exp\left\{-\lambda t_o(1 - e^{-2\tau/t_o})\right\} - \exp\left\{-2\lambda t_o(1 - e^{-\tau/t_o})\right\} .$$

Observe that the variance also tends to zero as t_o becomes large, which together with the asymptotic unbiasedness, implies the consistency of the maximum likelihood estimator. It should be noticed that the consistency also follows, under certain regularity conditions, from the fact that the estimator is based on independent and identically distributed random variables.

The mean-squared error of the maximum likelihood estimator was determined earlier in chapter (3), p. 38 :

$$MSE(\hat{r}_{ML}) = E\left\{\hat{r}_{ML} - e^{-\lambda\tau}\right\}^2$$

$$= \exp\left\{-\lambda t_o(1-e^{-2\tau/t_o})\right\} - 2\exp\left\{-\lambda[\tau+t_o(1-e^{-\tau/t_o})]\right\} + \exp\left\{-2\lambda\tau\right\} .$$

The minimum variance unbiased estimator was shown earlier to be

$$\hat{r}_{MVU} = \left\{1 - \frac{\tau}{t_o}\right\}^{n_o} , \tau \leq t_o .$$

It is perhaps appropriate at this point to notice that when $\tau = t_o$, that is when mission time equals observation time, the estimator $\hat{r}_{MVU}$ reduces to

$$\hat{r}_{MVU} = \begin{cases} 1 & \text{if } n_o = 0 \\ 0 & \text{if } n_o \geq 1 \end{cases}$$

The latter has the advantage of being distribution-free, that is it does not depend on the assumption that the Poisson model is true. However, it tends to be inefficient if the Poisson assumption is justified. It should also be noticed that if $\tau > t_o$, that is if mission time is larger than observation time, the estimator $\hat{r}_{MVU}$ becomes negative for odd values of n_o. In this case, one must define

$$\hat{r}^*_{MVU} = \max(0 , \hat{r}_{MVU}) .$$

However, this situation is not likely to be encountered for reasons mentioned earlier.

The m-th moment of $\hat{r}_{MVU}$ can be found directly :

$$
\begin{aligned}
E\left\{\hat{r}_{MVU}\right\}^m &= E\left\{(1-\tau/t_o)^{n_o}\right\}^m \\
&= \sum_{j=0}^{\infty}(1-\tau/t_o)^{mj}\,\frac{(\lambda t_o)^j\,e^{-\lambda t_o}}{j!} \\
&= e^{-\lambda t_o}\sum_{j=0}^{\infty}\frac{[(\lambda t_o)\,(1-\tau/t_o)^m]^j}{j!} \\
&= e^{-\lambda t_o}\,e^{\lambda t_o(1-\tau/t_o)^m} \\
&= \exp\left\{-\lambda t_o[1-(1-\tau/t_o)^m]\right\} \quad . \qquad (28)
\end{aligned}
$$

The mean-squared error (or variance) of $\hat{r}_{MVU}$ is then equal to

$$
\begin{aligned}
MSE(\hat{r}_{MVU}) &= E\left\{\hat{r}_{MVU} - e^{-\lambda\tau}\right\}^2 \\
&= \exp\left\{-2\lambda\tau(e^{\lambda\tau^2/t_o} - 1)\right\}
\end{aligned}
$$

which approaches zero as the observation time t_o becomes large.

The values in Tables (2), (3), (4), and (5) illustrate the behavior of the maximum likelihood and minimum variance unbiased estimators. The ratio mission time : observation time was chosen to be 0.1 but the results are also given for τ/t_o equal to 1.0 . It can be seen that in order to decide which estimator should be used, one must have an idea of the true value of the reliability function r .

If the ratio τ/t_o is equal to 1.0, which represents a rather undesirable situation, the maximum likelihood estimator $\hat{r}_{ML}$ should definitely be used, unless r is expected to be very small $(r \leq 0.1)$. Although it exhibits a larger bias than $\hat{r}_{MVU}$ (which by definition has no bias at all), it compensates by having a substantially smaller mean-squared error.

However, if τ/t_o is taken to be 0.1, the maximum likelihood estimator should only be used if high reliability values are expected. Otherwise, the minimum variance unbiased estimator should be adopted as it exhibits a slightly smaller mean-squared error.

3.1.5 The jack-knifed estimator of r.

In addition to the two classical estimators described in the previous section, Gaver and Hoel [1970] also propose a new estimator which is a modified (jack-knifed) version of the maximum likelihood estimator.

The jack-knifed procedure proposed by the authors consists in dividing the interval $(0,t_o]$ into c equal, non-overlapping intervals of duration $t_i = \frac{t_o}{c}$, $i = 1,2,\ldots,c$. Let n_i be the observed number of failures in the i-th interval. The random variable N_i has a Poisson distribution with parameter $\frac{\lambda t_o}{c}$, and N_i is independent of N_j for $i \neq j$.

Let $\Gamma^{(i)}$ denote the maximum likelihood estimator of r obtained by deleting the count for the i-th interval; for instance,

$$\Gamma^{(2)} = e^{-\left\{\frac{n_1 + n_3 + \ldots + n_c}{t_o}\right\}\left\{\frac{c}{c-1}\right\}\tau} .$$

Next, form the "pseudovalues"

$$\hat{r}_{ML}^{(i)} = c\hat{r}_{ML} - (c-1)\Gamma^{(i)} \quad .$$

Under some regularity conditions given in Brillinger [1964], the pseudovalues will reduce the bias of the regular maximum likelihood estimator. Finally, average the pseudovalues to obtain the jack-knifed estimator

$$\hat{r}_{JK}(c) = \frac{1}{c}\sum_{i=1}^{c}\hat{r}_{ML}^{(i)}$$

$$= c\hat{r}_{ML} - \frac{(c-1)}{c}\sum_{i=1}^{c}\Gamma^{(i)} \quad .$$

Defining

$$\hat{\lambda}_{ML}^{(i)} = \frac{c}{(c-1)t_o}\sum_{\substack{j=1 \\ j\neq i}}^{c} n_j \quad ,$$

the jack-knifed estimator can now be written as

$$\hat{r}_{JK}(c) = ce^{-\hat{\lambda}_{ML}\tau} - \frac{(c-1)}{c}\sum_{i=1}^{c} e^{-\hat{\lambda}_{ML}^{(i)}\tau} \quad .$$

Let $d = \frac{c-1}{c}$ and $\delta = \frac{\tau}{t_o}$. Then the expected value of $\hat{r}_{JK}(c)$ is given by

$$E\left\{\hat{r}_{JK}(c)\right\} = E\left\{ce^{-\hat{\lambda}_{ML}t_o\delta} - d\sum_{i=1}^{c} e^{-\hat{\lambda}_{ML}^{(i)}t_o\delta}\right\}$$

$$= c\exp\left\{-\lambda t_\circ(1 - e^{-\delta})\right\} - dcE\left\{e^{-\hat{\lambda}_{ML}^{(i)}\tau}\right\} .$$

Looking at the expectation of $\Gamma^{(1)}$ for example,

$$E\left\{e^{-(n_2+n_3+\ldots+n_c)\delta/d}\right\} = E\left\{e^{-n_2\delta/d}\, e^{-n_3\delta/d} \cdots e^{-n_c\delta/d}\right\}$$

$$= \left\{\sum_{j=0}^{\infty} e^{-j\delta/d}\ \frac{(\lambda t_\circ/c)^j\, e^{-\lambda t_\circ/c}}{j!}\right\}^{c-1}$$

$$= \exp\left\{-d\lambda t_\circ(1 - e^{-\delta/d})\right\} ,$$

so that the expectation of the jack-knifed estimator becomes

$$E\left\{\hat{r}_{JK}(c)\right\} = c\exp\left\{-\lambda t_\circ(1 - e^{-\delta})\right\} - (c - 1)\exp\left\{-d\lambda t_\circ(1 - e^{-\delta/d})\right\} .$$

The mean-squared error of $\hat{r}_{JK}(c)$ is then equal to

$$\mathrm{MSE}(\hat{r}_{JK}(c)) = E\left\{\hat{r}_{JK}(c) - r\right\}^2$$

$$= E\left\{\hat{r}_{JK}(c)\right\}^2 - 2e^{-\lambda\tau}\, E\left\{\hat{r}_{JK}(c)\right\} + e^{-2\lambda\tau} .$$

Now,

$$E\left\{\hat{r}_{JK}(c)\right\}^2 = E\left\{c^2e^{-2\hat{\lambda}_{ML}\tau} - 2(c-1)e^{-\hat{\lambda}_{ML}\tau}\sum_{i=1}^{c} e^{-\hat{\lambda}_{ML}^{(i)}\tau} + d^2\left\{\sum_{i=1}^{c} e^{-\hat{\lambda}_{ML}^{(i)}\tau}\right\}^2\right\} .$$

The expression on the right-hand side will now be divided into three parts and each term will be dealt with separately as follows:

$$E\left\{\hat{r}_{JK}(c)\right\}^2 = c^2\exp\left\{-\lambda t_o(1 - e^{-2\delta})\right\}$$

$$- 2(c-1)\, E\left\{ e^{-\hat{\lambda}_{ML}\tau}\sum_{i=1}^{c} e^{-\hat{\lambda}_{ML}^{(i)}\tau}\right\} \tag{29}$$

$$+ d^2\, E\left\{\sum_{i=1}^{c} e^{-2\hat{\lambda}_{ML}^{(i)}\tau} + \sum_{\substack{i=1\\ i\neq j}}^{c}\sum_{j=1}^{c} e^{-\hat{\lambda}_{ML}^{(i)}\tau}\, e^{-\hat{\lambda}_{ML}^{(j)}\tau}\right\} . \tag{30}$$

Looking at (29),

$$E\left\{ e^{-\hat{\lambda}_{ML}\tau}\sum_{i=1}^{c} e^{-\hat{\lambda}_{ML}^{(i)}\tau}\right\} = E\left\{ e^{-\hat{\lambda}_{ML}\tau}\, e^{-\hat{\lambda}_{ML}^{(1)}\tau} + \ldots + e^{-\hat{\lambda}_{ML}\tau}\, e^{-\hat{\lambda}_{ML}^{(c)}\tau}\right\}$$

$$= c\, E\left\{ e^{-\delta[n_1 + (1+\frac{c}{c-1})n_2 + \ldots + (1+\frac{c}{c-1})n_c]}\right\} ,$$

and using the fact that

$$1 + \frac{c}{c-1} = 1 + \frac{1}{d} = \frac{d+1}{d} ,$$

the expectation in (29) becomes

$$c\, E\left\{e^{-\delta n_1}\right\}\left(E\left\{e^{-\delta\left[\frac{d+1}{d}\right]n_2}\right\}\right)^{c-1} = c\exp\left\{-\frac{\lambda t_o}{c}(1 - e^{-\delta})\right\}$$

$$\cdot \exp\left\{-d\lambda t_o(1 - e^{-\delta(d+1)/d})\right\}$$

$$= c\exp\left\{-\lambda t_o\left\{\frac{1}{c}(1-e^{-\delta}) + d(1-e^{-\delta(d+1)/d})\right\}\right\} .$$

Looking now at the expectation in (30) , notice that it can also be written as follows :

$$E\left\{\sum_{i=1}^{c} e^{-2\hat{\lambda}_{ML}^{(i)}\tau} + \sum_{\substack{i=1\\ i\neq j}}^{c}\sum_{j=1}^{c} e^{-\hat{\lambda}_{ML}^{(i)}\tau}\, e^{-\hat{\lambda}_{ML}^{(j)}\tau}\right\}$$

$$= cE\left\{e^{-2\hat{\lambda}_{ML}^{(i)}\tau}\right\} + c(c-1)E\left\{e^{-\hat{\lambda}_{ML}^{(1)}\tau}\, e^{-\hat{\lambda}_{ML}^{(2)}\tau}\right\}$$

$$= c\exp\left\{-d\lambda t_{\circ}(1 - e^{-2\delta/d})\right\}$$

$$+ \ c(c-1)\ E\left\{e^{-(n_2+n_3+\ldots+n_c)\delta/d}\ e^{-(n_1+n_3+\ldots+n_c)\delta/d}\right\}$$

$$= c\exp\left\{-d\lambda t_{\circ}(1 - e^{-2\delta/d})\right\} + c(c-1)\ E\left\{e^{-\delta/d(n_1+n_2+2n_3+\ldots+2n_c)}\right\}$$

$$= c\exp\left\{-d\lambda t_{\circ}(1 - e^{-2\delta/d})\right\} + c(c-1)\left\{E\left\{e^{-2\delta n_3/d}\right\}\right\}^{c-2} \cdot \left\{E\left\{e^{-\delta n_1/d}\right\}\right\}^{2}$$

$$= c\exp\left\{-d\lambda t_{\circ}(1 - e^{-2\delta/d})\right\}$$

$$+ c(c-1)\exp\left\{-\frac{(c-2)}{c}\lambda t_{\circ}(1 - e^{-2\delta/d})\right\}\exp\left\{-\frac{2}{c}\lambda t_{\circ}(1 - e^{-\delta/d})\right\}$$

$$= c\exp\left\{-d\lambda t_{\circ}(1 - e^{-2\delta/d})\right\}$$

$$+ c(c-1)\exp\left\{-\frac{\lambda t_{\circ}}{c}\left[(c-2)\ (1 - e^{-2\delta/d}) + 2(1 - e^{-\delta/d})\right]\right\} .$$

Hence,

$$\begin{aligned} E\left\{\hat{r}_{JK}(c)\right\}^2 = \ & c^2\exp\left\{-\lambda t_{\circ}(1 - e^{-2\delta})\right\} \\ & - 2c(c-1)\exp\left\{-\lambda t_{\circ}\left[\frac{1}{c}(1 - e^{-\delta}) + d(1 - e^{-\delta(d+1)/d})\right]\right\} \\ & + d(c-1)\exp\left\{-d\lambda t_{\circ}(1 - e^{-2\delta/d})\right\} \\ & + d^2(c-1)\exp\left\{-\frac{\lambda t_{\circ}}{c}\left[(c-2)\ (1 - e^{-2\delta/d}) + 2(1 - e^{-\delta/d})\right]\right\}. \end{aligned} \tag{31}$$

Correcting the minor errors in the result given by Gaver and Hoel [1970], it should be noticed that the last line of (31) appeared with the coefficient $(c-1)$ instead of $d^2(c-1)$ and with a minus sign instead of a plus sign.

The mean-squared error of $\hat{r}_{JK}(c)$ is obtained by adding to (31) the expression

$$e^{-2\lambda\tau} - 2e^{-\lambda\tau}\left[c\exp\left\{-\lambda t_{\circ}(1 - e^{-\delta})\right\} - (c-1)\exp\left\{-d\lambda t_{\circ}(1 - e^{-\delta/d})\right\}\right] .$$

The authors performed several tabulations for reasonable values of the parameter and they observed that the bias of the jack-knifed estimator decreased when the number of intervals c increased. These results led them to define a new estimator obtained by letting c tend to infinity.

In practice, it is unreasonable to think of an infinitely jack-knifed estimator, but the next best solution is to look for an estimator whose expectation is equal to the limiting expectation of the jack-knifed estimator.

Writing the expectation of $\hat{r}_{JK}(c)$ as follows :

$$E\left\{\hat{r}_{JK}(c)\right\} = c\left[\exp\left\{-\lambda t_{\circ}(1-e^{-\tau/t_{\circ}})\right\} - \exp\left\{-\lambda t_{\circ}\cdot\frac{(c-1)}{c}\,(1 - e^{-c\tau/(c-1)t_{\circ}})\right\}\right] + \exp\left\{-\lambda t_{\circ}\,\frac{(c-1)}{c}\,(1 - e^{-c\tau/(c-1)t_{\circ}})\right\}$$

and letting $c \to \infty$, or $\varepsilon = \frac{t_{\circ}}{c} \to 0$, the limiting expectation becomes

$$\lim_{\varepsilon\to 0}\ \frac{t_{\circ}}{\varepsilon}\left[\exp\left\{-\lambda t_{\circ}(1-e^{-\tau/t_{\circ}})\right\} - \exp\left\{-\lambda t_{\circ}(1-\varepsilon/t_{\circ})\ (1-e^{-\tau/t_{\circ}(1-\varepsilon/t_{\circ})})\right\}\right] + \lim_{\varepsilon\to 0}\ \exp\left\{-\lambda t_{\circ}(1-\varepsilon/t_{\circ})\ (1-e^{-\tau/t_{\circ}(1-\varepsilon/t_{\circ})})\right\} .$$

Applying L'Hôpital's rule, this expression becomes

$$\lim_{\varepsilon\to 0}\ -t_{\circ}\left[\lambda(1 - e^{-\tau/t_{\circ}(1-\varepsilon/t_{\circ})}) - \lambda t_{\circ}(1 - \varepsilon/t_{\circ})\ e^{-\tau/t_{\circ}(1-\varepsilon/t_{\circ})}\ \frac{\tau}{t_{\circ}^{2}(1-\varepsilon/t_{\circ})^{2}}\right] \cdot \left[\exp\left\{-\lambda t_{\circ}(1-\varepsilon/t_{\circ})\ (1-e^{-\tau/t_{\circ}(1-\varepsilon/t_{\circ})})\right\}\right] + \exp\left\{-\lambda t_{\circ}(1 - e^{-\tau/t_{\circ}})\right\}$$

and simplifies to

$$\lim_{c\to\infty} E\left\{\hat{r}_{JK}(c)\right\} = -t_\circ \exp\left\{-\lambda t_\circ(1 - e^{-\tau/t_\circ})\right\}\left[\lambda(1 - e^{-\tau/t_\circ}) - \frac{\lambda\tau}{t_\circ} e^{-\tau/t_\circ}\right] + \exp\left\{-\lambda t_\circ(1 - e^{-\tau/t_\circ})\right\}$$

$$= \exp\left\{-\lambda t_\circ(1-e^{-\tau/t_\circ})\right\}\left[1 - \lambda t_\circ(1-e^{-\tau/t_\circ}) + \lambda\tau e^{-\tau/t_\circ}\right] . \quad (32)$$

It is now desired to find an estimator whose expectation is identical to the limiting expectation given in (32). The following definition of the infinitely jack-knifed estimator achieves that purpose and was proposed by the authors without any justification:

$$\hat{r}_{JK}(\infty) = \exp\left\{-\frac{n_\circ\tau}{t_\circ}\right\}\left[1 - n_\circ(e^{\tau/t_\circ} - 1 - \tau/t_\circ)\right] . \quad (33)$$

Indeed,

$$E\left\{\hat{r}_{JK}(\infty)\right\} = E\left\{e^{-n_\circ\tau/t_\circ} - n_\circ e^{\tau/t_\circ} e^{-n_\circ\tau/t_\circ} + n_\circ e^{-n_\circ\tau/t_\circ} + \frac{n_\circ\tau}{t_\circ} e^{-n_\circ\tau/t_\circ}\right\}$$

$$= \sum_{j=0}^{\infty} e^{-j\tau/t_\circ} \frac{(\lambda t_\circ)^j e^{-\lambda t_\circ}}{j!} + (1-e^{\tau/t_\circ}+\tau/t_\circ)\sum_{j=0}^{\infty} j e^{-j\tau/t_\circ}\frac{(\lambda t_\circ)^j e^{-\lambda t_\circ}}{j!}$$

$$= e^{-\lambda t_\circ} e^{\lambda t_\circ e^{-\tau/t_\circ}} + (1 - e^{\tau/t_\circ} + \tau/t_\circ)\, e^{-\lambda t_\circ}\, \lambda t_\circ e^{-\tau/t_\circ}\, e^{\lambda t_\circ e^{-\tau/t_\circ}}$$

$$= \exp\left\{-\lambda t_\circ(1 - e^{-\tau/t_\circ})\right\}\left[1 - \lambda t_\circ + (1 + \tau/t_\circ)\,\lambda t_\circ e^{-\tau/t_\circ}\right]$$

$$= \exp\left\{-\lambda t_\circ(1 - e^{-\tau/t_\circ})\right\}\left[1 - \lambda t_\circ(1 - e^{-\tau/t_\circ}) + \lambda\tau e^{-\tau/t_\circ}\right]$$

$$= \lim_{c\to\infty} E\left\{\hat{r}_{JK}(c)\right\} .$$

It can be shown that the infinitely jack-knifed estimator reduces the bias. However, for some specific values of the parameter, the mean-squared error increases with c, so that $\hat{r}_{JK}(\infty)$ becomes inefficient.

3.2 Estimation based on the interval $(0,kt_0]$.

In this section, the effect of increasing the period of observation from $(0,t_0]$ to $(0,kt_0]$ will be examined. The question addressed in the sequel is to what extent the maximum likelihood and minimum variance unbiased estimators will exhibit a substantial reduction in their mean-squared errors.

The first part is based on observation of the number of failures in k non-overlapping intervals of duration t_0 (Poisson analysis), while the second uses the time intervals between k successive failures (exponential analysis). We will distinguish the estimators obtained in the exponential analysis by the superscript E.

3.2.1 Poisson analysis.

Let N_i, $i = 1, \ldots, k$, denote the number of failures observed in the i-th interval of duration t_0. It is well known that the random variable

$$S = \sum_{i=1}^{k} N_i$$

is a sufficient statistic for λ (and hence for $r = e^{-\lambda\tau}$) , and has a Poisson distribution with parameter $k\lambda$, denoted by $Po(k\lambda)$.

Based on these records of past events over the observation periods

$$(0,t_0] \; , \; (t_0,2t_0] \; , \; \ldots \; , \; ((k-1)t_0,kt_0] \; ,$$

an estimator of r is desired for the mission period $(0,\tau]$.

Zacks and Even [1966] compare the relative efficiency of the maximum likelihood estimator $\hat{r}_{ML}$ and minimum variance unbiased estimator $\hat{r}_{MVU}$. The relative efficiency considered by the authors is the ratio of the Cramer-Rao lower bound of the variance of unbiased estimators, to the mean-squared error of the considered estimator. The comparison will be made here in terms of bias and mean-squared error.

In order to simplify the notation, let

$$\vartheta = \frac{\tau}{kt_o} \quad \text{and} \quad \mu = k\lambda t_o \quad .$$

An unbiased estimator for $r = e^{-\vartheta\mu}$ is given by

$$\hat{r} = (1 - \vartheta)^S$$

since

$$\begin{aligned} E\left\{\hat{r}\right\} &= E\left\{(1 - \vartheta)^S\right\} \\ &= \sum_{j=0}^{\infty}(1 - \vartheta)^j \frac{\mu^j e^{-\mu}}{j!} \\ &= e^{-\mu}\sum_{j=0}^{\infty}\frac{\left\{\mu(1 - \vartheta)\right\}^j}{j!} \\ &= e^{-\mu} e^{\mu(1-\vartheta)} \\ &= e^{-\vartheta\mu} \quad . \end{aligned}$$

The fact that $\hat{r}$ is also the unique minimum variance unbiased estimator again follows from the Lehmann-Scheffé Theorem [Lehmann, 1983, p.80].

Hence,

$$\hat{r}_{MVU} = (1 - \vartheta)^S$$

The m-th moment of $\hat{r}_{MVU}$ $(m = 1,2,\ldots)$ is the value of the probability generating function of $Po(\mu)$ at $(1 - \vartheta)^m$ since

$$E\left\{\hat{r}_{MVU}\right\}^m = E\left\{(1 - \vartheta)^S\right\}^m$$

$$= E\left\{(1 - \vartheta)^m\right\}^S$$

$$= \sum_{j=0}^{\infty}(1 - \vartheta)^{mj} \frac{\mu^j e^{-\mu}}{j!}$$

$$= e^{-\mu} e^{\mu(1 - \vartheta)^m}$$

$$= \exp\left\{-\mu[1 - (1 - \vartheta)^m]\right\} .$$

Accordingly, the mean-squared error (variance) of $\hat{r}_{MVU}$ is given by

$$MSE(\hat{r}_{MVU}) = \exp\left\{-\mu[1 - (1 - \vartheta)^2]\right\} - \exp\left\{-2\vartheta\mu\right\}$$

$$= \exp\left\{-2\vartheta\mu\right\} \cdot \left[\exp\left\{\vartheta^2\mu\right\} - 1\right] . \tag{34}$$

Since the reliability function $r = e^{-\lambda\tau}$ is a one-to-one function of λ its maximum likelihood estimator is simply

$$\hat{r}_{ML} = e^{-\hat{\lambda}_{ML}\tau} = e^{-\frac{S}{kt_0}\tau} = e^{-\vartheta S} .$$

The m-th moment of $\hat{r}_{ML}$ $(m = 1,2,\ldots)$ is equal to

$$E\left\{\hat{r}_{ML}\right\}^m = E\left\{e^{-\hat{\delta}S}\right\}^m$$

$$= \sum_{j=0}^{\infty} e^{-\hat{\delta}jm} \frac{\mu^j e^{-\mu}}{j!}$$

$$= \exp\left\{-\mu(1 - e^{-\hat{\delta}m})\right\} . \qquad (35)$$

The mean-squared error of $\hat{r}_{ML}$ is then given by

$$MSE(\hat{r}_{ML}) = E\left\{\hat{r}_{ML} - r\right\}^2$$

$$= \exp\left\{-\mu(1 - e^{-2\hat{\delta}})\right\} - 2\exp\left\{-\mu(1 + \hat{\delta} - e^{-\hat{\delta}})\right\} + \exp\left\{-2\hat{\delta}\mu\right\} .$$

3.2.2 Exponential analysis.

Now let $X_1, X_2, \ldots, X_k$ represent the time intervals between k successive failures. Let θ denote the mean time between failures (MTBF) of the process. It was mentioned in section (2.2) that the X_i's are independent identically distributed random variables having an exponential distribution with parameter θ, where

$$\theta = \frac{1}{\lambda} . \qquad (36)$$

The reliability function can now be expressed as

$$r = e^{-\tau/\theta} , \quad 0 \leq \tau < \infty .$$

Let

$$T = \sum_{i=1}^{k} X_i \quad .$$

Since the random variable T has a Gamma distribution with parameters k and θ, the minimum variance unbiased estimator of r is given by the unique solution to

$$\int_0^\infty \hat{r}(x) \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-x/\theta} dx = e^{-\tau/\theta} \quad . \tag{37}$$

The equation in (37) simplifies to

$$\frac{1}{\theta^k (k-1)!} \int_0^\infty \hat{r}(x)\, x^{k-1} e^{-x/\theta} dx = e^{-\tau/\theta}$$

$$\longrightarrow \quad \frac{e^{-\tau/\theta}}{\theta^k (k-1)!} \int_0^\infty \hat{r}(x)\, x^{k-1} e^{-(x-\tau)/\theta} dx = e^{-\tau/\theta}$$

$$\longrightarrow \quad \int_0^\infty \hat{r}(x)\, x^{k-1} e^{-(x-\tau)/\theta} dx = \theta^k (k-1)!$$

$$\longrightarrow \quad \hat{r}(T)\, T^{k-1} = (T - \tau)^{k-1} \quad , \quad \text{(cf. pp. 12 and 14)}$$

which leads to the unique solution

$$\hat{r}_{MVU}^{E} = \left[1 - \frac{\tau}{T}\right]^{k-1} \quad .$$

The mean-squared error (variance) of $\hat{r}^E_{MVU}$ is given by the expression

$$MSE(\hat{r}^E_{MVU}) = E\left\{\hat{r}^E_{MVU}\right\}^2 - e^{-2\tau/\theta} \quad , \tag{38}$$

where the expected value on the right-hand side of (38) is equal to

$$E\left\{\hat{r}^E_{MVU}\right\}^2 = \int_0^\infty \left[1 - \frac{\tau}{x}\right]^{2k-2} \frac{x^{k-1}}{\theta^k \Gamma(k)} e^{-x/\theta}\, dx \quad . \tag{39}$$

Letting $u = \frac{x}{\theta}$, the integral in (39) becomes

$$E\left\{\hat{r}^E_{MVU}\right\}^2 = \int_{\tau/\theta}^\infty \left[1 - \frac{\tau}{\theta u}\right]^{2k-2} \frac{(\theta u)^{k-1}}{\theta^k \Gamma(k)} e^{-u}\, \theta du$$

$$= \frac{1}{(k-1)!} \int_{\tau/\theta}^\infty \left[1 - \frac{\tau}{\theta u}\right]^{2k-2} u^{k-1} e^{-u}\, du$$

$$= \frac{1}{(k-1)!} \int_{\tau/\theta}^\infty \sum_{i=0}^{2k-2} \binom{2k-2}{i} (-\tau/\theta u)^i u^{k-1} e^{-u}\, du$$

$$= \frac{1}{(k-1)!} \sum_{i=0}^{2k-2} \binom{2k-2}{i} (-\tau/\theta u)^i \int_{\tau/\theta}^\infty u^{k-i-1} e^{-u}\, du$$

$$= \frac{1}{(k-1)!} \sum_{i=0}^{k-1} \binom{2k-2}{i} (-\tau/\theta u)^i \int_{\tau/\theta}^\infty u^{k-i-1} e^{-u}\, du \quad ,$$

$$+ \frac{1}{(k-1)!} \sum_{i=k}^{2k-2} \binom{2k-2}{i} (-\tau/\theta u)^i \int_{\tau/\theta}^\infty \frac{e^{-u}}{u^{i-k+1}}\, du \quad . \tag{40}$$

The k integrals in the first sum of (40) are related to the Poisson distribution according to the relationship

$$\int_{\tau/\theta}^{\infty} u^{k-i-1} e^{-u} du = (k-i-1)! \; Po(k-i-1;\tau/\theta) \quad , \quad i = 0,\ldots,k-1 \quad ,$$

where $$Po(x;\lambda) = \sum_{j=0}^{[x]} \frac{\lambda^j e^{-\lambda}}{j!} \quad ,$$

$[x]$ being the integer part of x .

The $(k-1)$ integrals in the second sum of (40) are related to the exponential integral as follows :

$$E_i(-\tau/\theta) = -\int_{\tau/\theta}^{\infty} \frac{e^{-u}}{u} du \quad . \tag{41}$$

The values of $E_i(-\tau/\theta)$ can be found in Jahnke and Emde [1945]. Looking only at

$$\sum_{i=k}^{2k-2} \int_{\tau/\theta}^{\infty} \frac{e^{-u}}{u^{i-k+1}} du \quad , \tag{42}$$

and expanding the terms of this summation gives

$$\sum_{i=k}^{2k-2} \int_{\tau/\theta}^{\infty} \frac{e^{-u}}{u^{i-k+1}} du = \int_{\tau/\theta}^{\infty} \frac{e^{-u}}{u} du + \int_{\tau/\theta}^{\infty} \frac{e^{-u}}{u^2} du + \ldots + \int_{\tau/\theta}^{\infty} \frac{e^{-u}}{u^{k-2}} du + \int_{\tau/\theta}^{\infty} \frac{e^{-u}}{u^{k-1}} du \quad .$$

The value of the first integral on the right-hand side can be found using (41) and the second integral can be written in terms of the first one as follows :

$$- E_i(-\tau/\theta) = \int_{\tau/\theta}^{\infty} \frac{e^{-u}}{u}\, du = \frac{e^{-\tau/\theta}}{(\tau/\theta)} - \int_{\tau/\theta}^{\infty} \frac{e^{-u}}{u^2}\, du$$

so that

$$\int_{\tau/\theta}^{\infty} \frac{e^{-u}}{u^2}\, du = \frac{e^{-\tau/\theta}}{(\tau/\theta)} + E_i(-\tau/\theta) \quad .$$

Similarly, the third integral of the summation is equal to

$$\int_{\tau/\theta}^{\infty} \frac{e^{-u}}{u^3}\, du = \frac{1}{2}\left[\frac{e^{-\tau/\theta}}{(\tau/\theta)^2} - \int_{\tau/\theta}^{\infty} \frac{e^{-u}}{u^2}\, du\right]$$

$$= \frac{e^{-\tau/\theta}}{2(\tau/\theta)^2} - \frac{e^{-\tau/\theta}}{2(\tau/\theta)} - \frac{1}{2} E_i(-\tau/\theta) \quad .$$

Proceeding in a similar way for remaining (k-4) integrals, the value of (42) becomes

$$\sum_{i=k}^{2k-2} \int_{\tau/\theta}^{\infty} \frac{e^{-u}}{u^{i-k+1}}\, du = e^{-\frac{\tau}{\theta}} \sum_{i=k}^{2k-2} \frac{1}{(i-k)!}\left[\sum_{j=1}^{i-k} (-1)^{j+1} \frac{(i-k-j)!}{\left(\frac{\tau}{\theta}\right)^{i-k-j+1}} + (-1)^{i-k+1} E_i(-\tau/\theta)\right]$$

so that , finally, the expectation given in (40) is equivalent to the following expression :

$$E\left\{\hat{r}^{E}_{MVU}\right\}^2 = \sum_{i=0}^{k-1}(-1)^i \frac{\binom{2k-2}{i}}{\binom{k-1}{i}} \frac{(\tau/\theta)^i}{i!} Po(k-i-1;\tau/\theta)$$

$$+ \frac{e^{-\tau/\theta}}{(k-1)!} \sum_{i=k}^{2k-2} (-1)^i \frac{\binom{2k-2}{i}}{(i-k)!} (\tau/\theta)^i$$

$$\cdot \left[\sum_{j=1}^{i-k}(-1)^{j+1} \frac{(i-k-j)!}{(\tau/\theta)^{i-k-j+1}} + (-1)^{i-k+1} E_i(-\tau/\theta)\right] . \quad (43)$$

The mean-squared error of $\hat{r}^{E}_{MVU}$ is obtained by subtracting $e^{-2\tau/\theta}$ from (43).

Evaluation of the mean-squared error using (43) only requires the use of tables. However, the integrals in (40) may be evaluated numerically with little cost on the computer. In the appendix, where some numerical results are given, the Gauss-Laguerre quadrature method was used successfuly as a method of numerical integration.

The maximum likelihood estimator of r is given by

$$\hat{r}^{E}_{ML} = e^{-\tau/\overline{X}_k} ,$$

where

$$\overline{X}_k = \frac{T}{k} = \sum_{i=1}^{k} \frac{X_i}{k} .$$

Since $\overline{X}_k$ follows a Gamma distribution with parameters k and θ/k ,

$$E\left\{\hat{r}_{ML}^{E}\right\} = \int_0^\infty e^{-\tau/x} \frac{x^{k-1} e^{-kx/\theta}}{(\theta/k)^k \Gamma(k)} dx$$

$$= \left(\frac{k}{\theta}\right)^k \frac{1}{(k-1)!} \int_0^\infty \exp\left\{-\frac{\tau}{x} - \frac{kx}{\theta}\right\} x^{k-1}\, dx \quad . \tag{44}$$

With $t = \frac{kx}{\theta}$, the expectation in (44) becomes

$$E\left\{\hat{r}_{ML}^{E}\right\} = \left(\frac{k}{\theta}\right)^k \frac{1}{(k-1)!} \int_0^\infty \exp\left\{-\frac{\tau k}{\theta t} - t\right\} t^{k-1} \left(\frac{\theta}{k}\right)^{k-1} \frac{\theta}{k}\, dt$$

$$= \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp\left\{-\left[\frac{\tau k}{\theta t} + t\right]\right\} dt \quad .$$

Similarly,

$$E\left\{\hat{r}_{ML}^{E}\right\}^2 = \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp\left\{-\left[\frac{2\tau k}{\theta t} + t\right]\right\} dt \quad .$$

Since

$$\int_0^\infty x^{-n} \exp\left\{-\left[ax + \frac{b}{x}\right]\right\} dx = 2\left[\frac{a}{b}\right]^{\frac{1}{2}(n-1)} K_{n-1}(2\sqrt{ab}) \quad ,$$

where $K_n(y)$ is the modified Bessel function of the second kind of order n at the point y, it follows that

$$\int_0^\infty t^{k-1} \exp\left\{-\left[\frac{\tau k}{\theta t} + t\right]\right\} dt = 2\left[\frac{\tau k}{\theta}\right]^{\frac{1}{2}(k)} K_{-k}(2\sqrt{\tau k/\theta}) \quad .$$

Now, using the identity

$$K_{-n}(y) = K_n(y) \quad \text{for all } n = 0,1,2,\ldots \quad ,$$

the expectation in (44) can be written as

$$E\left\{\hat{r}_{ML}^{E}\right\} = \frac{2}{(k-1)!}\left[\frac{\tau k}{\theta}\right]^{k/2} K_k(2\sqrt{\tau k/\theta}) \quad . \tag{45}$$

The bias of the maximum likelihood estimator is obtained by subtracting $e^{-\tau/\theta}$ from (45).

Let $M_k(\tau/\theta)$ denote the expectation given in (45). The mean-squared error of the maximum likelihood estimator can be expressed as

$$\begin{aligned} MSE(\hat{r}_{ML}^{E}) &= E\left\{\hat{r}_{ML}^{E} - r\right\}^2 \\ &= E\left\{\hat{r}_{ML}^{E}\right\}^2 - 2e^{-\tau/\theta} E\left\{\hat{r}_{ML}^{E}\right\} + e^{-2\tau/\theta} \\ &= M_k(2\tau/\theta) - 2e^{-\tau/\theta} M_k(\tau/\theta) + e^{-2\tau/\theta} \quad . \end{aligned} \tag{46}$$

3.2.3 Comparison of $\hat{r}_{ML}$, $\hat{r}_{ML}^{E}$, $\hat{r}_{MVU}$ and $\hat{r}_{MVU}^{E}$ in terms of bias and mean-squared error.

The bias and mean-squared error of the maximum likelihood and minimum variance unbiased estimators are given in Tables (6) to (9), and (10) to (13), respectively, for the Poisson analysis. The equivalent results for the exponential analysis can be found in Tables (14) to (16). Notice that the results for the minimum variance unbiased estimator are only given for $k = 4$, since the value $k = 8$ caused overflow, and brought no further insight to the comparison.

The computations were carried out on the McGill computer, using the Gauss-Laguerre quadrature method to solve the integrals in (40). The results are given for k equal to 4 and 8 , where k is the number of intervals of observation, and for values of r between 0.01 and 0.99 .

The numerical results given in Tables (6),(8),(10),(12) and (14) to (16) can be found in the paper by Zacks and Even [1966], where they are presented in the form of graphs. The authors only considered the case where the ratio mission time : observation time (τ/t_0) is equal to 1.0 . Since this represents a very risky situation, the ratio τ/t_0 equal to 0.1 is studied here and the results are given in Tables (7),(9),(11), and (13) for the Poisson analysis.

Even though the choice of estimator is not influenced by this change of ratio τ/t_0 , the reduction in bias and mean-squared error is substantial for both estimators. For instance, the mean-squared error of the minimum variance unbiased estimator for $k = 4$ and $r = 0.1$ goes from 0.008 to 0.0006 when the ratio τ/t_0 is taken to be 0.1 instead of 1.0 .
These results reinforce the fact that the accuracy of estimation is highly dependent on the length of the observation period, compared to that of the mission period.

Comparison in terms of mean-squared error can be made from two different points of view :

(1) comparing the maximum likelihood and minimum variance unbiased estimators for a given type of analysis (Poisson or exponential),

(2) comparing the two types of analyses (Poisson and exponential) for a given estimator (MLE or MVUE).

Zacks and Even [1966] only considered the first type of comparison and their results are summarized in Table (17), for the ratio τ/t_o equal to 1.0 . Observe that, despite the inverse relationship between the parameters λ and θ ($\lambda = \frac{1}{\theta}$) , there is no such correspondence between intervals of r over which the minimum variance unbiased estimator is superior to the maximum likelihood estimator. In the Poisson analysis, the maximum likelihood estimator is superior when the expected number of failures during the mission period , $\lambda\tau$, is smaller than one ($r \gtrsim \sim 0.4$) . Notice that in situations where high reliability values are expected, the value of $\lambda\tau$ is usually smaller than one.

If the analysis is based on times between failures (exponential analysis) and high reliability values are expected, the choice of estimator requires a precise knowledge of the expected value of r .

For values of r between 0.6 and 1.0 , the minimum variance unbiased estimator performs better in terms of mean-squared error. The maximum likelihood estimator should be used if r is expected to lie between 0.03 and 0.6 . The superiority of the minimum variance unbiased estimator reappears for very small values of the reliability function ($r < \sim 0.03$) .

Most of the time it is not possible to choose between the Poisson and exponential analyses. The type of analysis will depend on the data available, in the form of actual counts of the number of failures or in the form of time intervals between failures. In that case one may use Table (17) to decide which type of estimator is appropriate.

If the analyst is consulted before data collection and is able to predict the approximate range of values for the reliability function, then the following considerations should be kept in mind to determine which sampling scheme is to be adopted.

The data shown in Tables (6) to (16) indicate that the exponential analysis, based on the observation of times between failures, should be performed if high reliability values are expected $(r > 0.4)$, whether one uses the maximum likelihood estimator or the minimum variance unbiased estimator. The bias of the maximum likelihood estimator, in the exponential analysis, becomes negative for high reliability values, and is slightly larger (in absolute value) than in the Poisson analysis. However, a negative bias is more appropriate than a positive bias when high reliability values are expected, as it provides a conservative estimate in the reliability sense.

These results agree with common sense in the following way : the Poisson analysis is based on counts only, from k intervals, and does not take into account the respective positions of the failures in the time intervals considered. The exponential analysis described here is based on observation of time intervals between k successive failures. When the reliability function is large, the number of failures in any given interval is small and may even be zero if the interval is too small. In that case, the Poisson analysis must be performed on the basis of a very large observation period to avoid underestimation of the reliability function. On the other hand, in the exponential analysis, the period of observation is not determined in advance; the observation process continues until the first k failures have occurred, and then stops.

Looking at the maximum likelihood estimator more closely, it can be observed that for values of r smaller than 0.08 and for $k = 4$, the Poisson analysis yields the best results in terms of mean-squared error, the difference in bias being negligible. If $k = 8$ intervals are considered for estimation, the Poisson analysis can be used for values of r up to 0.2. As the reliability function increases, the difference in mean-squared error becomes highly significant and should convince one that the Poisson analysis becomes inadequate in those circumstances. For instance, with $k = 4$ and $r = 0.99$, the mean-squared error in the Poisson analysis is approximately equal to 0.0019, while in the exponential analysis, it is only 0.000097.

If the minimum variance unbiased estimator is being used, the Poisson analysis can be performed for values of r up to 0.3, and as the reliability function increases, the exponential analysis becomes more appropriate. The reduction in mean-squared error becomes very significant for values of r larger than 0.9. For instance, when $k = 4$ and $r = 0.99$, the mean-squared error goes from 0.0025 to 0.000047 when the Poisson analysis is replaced by the analysis based on intervals between failures.

The above discussion and a comparison of Tables (14) and (15) indicate that if a high reliability value is suspected ($r > 0.6$), the best design is the exponential analysis and the best strategy is to use the minimum variance unbiased estimator.

APPENDIX

Table 1 : Values of n_0 for estimators of bounded relative error, using the approximation in (16) of chapter (2).

γ \ α	0.90	0.95	0.99
0.01	27059	38414	66302
0.02	6763	9602	16573
0.03	3005	4266	7363
0.04	1690	2399	4140
0.05	1081	1534	2648
0.06	750	1065	1838
0.07	551	782	1349
0.08	421	598	1032
0.09	333	472	815
0.10	269	382	659

Table 2 : Bias and mean-squared error of the maximum likelihood estimator in the Poisson analysis based on $k = 1$ interval with ratio $\tau/t_o = 1.0$.

R	BIAS	MEAN-SQUARED ERROR
0.01	0.4442007E-01	0.1766126E-01
0.02	0.6434238E-01	0.3098576E-01
0.03	0.7898253E-01	0.4258034E-01
0.04	0.9071743E-01	0.5297997E-01
0.05	0.1005193E 00	0.6244533E-01
0.06	0.1089058E 00	0.7113463E-01
0.07	0.1161929E 00	0.7915509E-01
0.08	0.1225914E 00	0.8658594E-01
0.09	0.1282505E 00	0.9348720E-01
0.10	0.1332811E 00	0.9990746E-01
0.20	0.1615486E 00	0.1440517E 00
0.30	0.1671728E 00	0.1627858E 00
0.40	0.1603429E 00	0.1645345E 00
0.50	0.1452273E 00	0.1539465E 00
0.60	0.1240438E 00	0.1340945E 00
0.70	0.9814769E-01	0.1072116E 00
0.80	0.6844270E-01	0.7501996E-01
0.90	0.3556983E-01	0.3890157E-01
0.91	0.3212643E-01	0.3511959E-01
0.92	0.2865767E-01	0.3131086E-01
0.93	0.2516282E-01	0.2747655E-01
0.94	0.2164227E-01	0.2361774E-01
0.95	0.1809639E-01	0.1973480E-01
0.96	0.1452559E-01	0.1582956E-01
0.97	0.1093024E-01	0.1190233E-01
0.98	0.7310569E-02	0.7956074E-02
0.99	0.3667057E-02	0.3986955E-02

Table 3 : **Bias and mean-squared error of the maximum likelihood estimator in the Poisson analysis based on $k = 1$ interval with ratio $\tau/t_0 = 0.1$.**

p	BIAS	MEAN-SQUARED ERROR
0.01	0.2495345E-02	0.3701376E-04
0.02	0.4166655E-02	0.2656316E-03
0.03	0.5545925E-02	0.5029617E-03
0.04	0.6739508E-02	0.7846861E-03
0.05	0.7797103E-02	0.1101787E-02
0.06	0.8747585E-02	0.1447832E-02
0.07	0.9609580E-02	0.1817979E-02
0.08	0.1039572E-01	0.2207961E-02
0.09	0.1111853E-01	0.2614796E-02
0.10	0.1179265E-01	0.3035624E-02
0.20	0.1619357E-01	0.7595107E-02
0.30	0.1799142E-01	0.1197098E-01
0.40	0.1812863E-01	0.1545316E-01
0.50	0.1704937E-01	0.1760769E-01
0.60	0.1501113E-01	0.1813167E-01
0.70	0.1218241E-01	0.1679665E-01
0.80	0.8682370E-02	0.1342338E-01
0.90	0.4598796E-02	0.7865965E-02
0.91	0.4161119E-02	0.7184625E-02
0.92	0.3718317E-02	0.6480336E-02
0.93	0.3270566E-02	0.5752802E-02
0.94	0.2817750E-02	0.5002558E-02
0.95	0.2360106E-02	0.4228592E-02
0.96	0.1897573E-02	0.3430426E-02
0.97	0.1430273E-02	0.2608776E-02
0.98	0.9582043E-03	0.1763403E-02
0.99	0.4814267E-03	0.8939505E-03

Table 4 : Mean-squared error of the minimum variance unbiased estimator in the Poisson analysis based on $k = 1$ interval with ratio $\tau/t_0 = 1.0$.

R	MEAN-SQUARED ERROR
0.01	0.9900004E-02
0.02	0.1960000E-01
0.03	0.2910000E-01
0.04	0.3840001E-01
0.05	0.4750000E-01
0.06	0.5640001E-01
0.07	0.6509995E-01
0.08	0.7359999E-01
0.09	0.8189994E-01
0.10	0.8999997E-01
0.20	0.1600000E 00
0.30	0.2100000E 00
0.40	0.2399999E 00
0.50	0.2499999E 00
0.60	0.2399999E 00
0.70	0.2099997E 00
0.80	0.1599994E 00
0.90	0.8999968E-01
0.91	0.8189976E-01
0.92	0.7359958E-01
0.93	0.6509912E-01
0.94	0.5639969E-01
0.95	0.4749983E-01
0.96	0.3839942E-01
0.97	0.2909984E-01
0.98	0.1959953E-01
0.99	0.9899367E-02

Table 5 : Mean-squared error of the minimum variance unbiased estimator in the Poisson analysis based on k = 1 interval with ratio $\tau/t_0 = 0.1$.

R	MEAN-SQUARED ERROR
0.01	0.5848931E-04
0.02	0.1915029E-03
0.03	0.3779985E-03
0.04	0.6075676E-03
0.05	0.8732071E-03
0.05	0.1169659E-02
0.07	0.1492726E-02
0.08	0.1838929E-02
0.09	0.2205296E-02
0.10	0.2589246E-02
0.20	0.6984748E-02
0.30	0.1151499E-01
0.40	0.1535325E-01
0.50	0.1794316E-01
0.60	0.1886732E-01
0.70	0.1779247E-01
0.80	0.1444153E-01
0.90	0.8579124E-02
0.91	0.7846046E-02
0.92	0.7086322E-02
0.93	0.6299250E-02
0.94	0.5484074E-02
0.95	0.4640844E-02
0.96	0.3769629E-02
0.97	0.2869605E-02
0.98	0.1941727E-02
0.99	0.9851698E-03

Table 6 : Bias and mean-squared error of the maximum likelihood estimator in the Poisson analysis based on $k = 4$ intervals with ratio $\tau/t_o = 1.0$.

P	BIAS	MEAN-SQUARED ERROR
0.01	0.6998353E-02	0.4716497E-03
0.02	0.1138749E-01	0.1263005E-02
0.03	0.1493260E-01	0.2214333E-02
0.04	0.1795713E-01	0.3270304E-02
0.05	0.2060780E-01	0.4399840E-02
0.06	0.2296830E-01	0.5582586E-02
0.07	0.2509248E-01	0.6803948E-02
0.08	0.2701819E-01	0.8052889E-02
0.09	0.2877283E-01	0.9320769E-02
0.10	0.3037757E-01	0.1060060E-01
0.20	0.4074293E-01	0.2311859E-01
0.30	0.4463434E-01	0.3355193E-01
0.40	0.4453284E-01	0.4079765E-01
0.50	0.4156375E-01	0.4434091E-01
0.60	0.3636885E-01	0.4390311E-01
0.70	0.2936220E-01	0.3932488E-01
0.80	0.2083206E-01	0.3051138E-01
0.90	0.1099062E-01	0.1741248E-01
0.91	0.9940922E-02	0.1586545E-01
0.92	0.8879960E-02	0.1427561E-01
0.93	0.7807791E-02	0.1264179E-01
0.94	0.6724417E-02	0.1096529E-01
0.95	0.5630255E-02	0.9246051E-02
0.96	0.4525304E-02	0.7483304E-02
0.97	0.3409684E-02	0.5677760E-02
0.98	0.2283514E-02	0.3827989E-02
0.99	0.1146853E-02	0.1935601E-02

Table 7 : **Bias and mean-squared error of the maximum likelihood estimator in the Poisson analysis based on $k = 4$ intervals with ratio $\tau/t_o = 0.1$.**

R	BIAS	MEAN-SQUARED ERROR
0.01	0.5873814E-03	0.1365323E-04
0.02	0.9936281E-03	0.4505552E-04
0.03	0.1332603E-02	0.9932245E-04
0.04	0.1628105E-02	0.1440016E-03
0.05	0.1891416E-02	0.2074253E-03
0.06	0.2129115E-02	0.2783514E-03
0.07	0.2345622E-02	0.3557589E-03
0.08	0.2544045E-02	0.4388355E-03
0.09	0.2725555E-02	0.5268641E-03
0.10	0.2894998E-02	0.6191805E-03
0.20	0.4029810E-02	0.1680583E-02
0.30	0.4510522E-02	0.2779901E-02
0.40	0.4568577E-02	0.3715098E-02
0.50	0.4314184E-02	0.4349053E-02
0.60	0.3811061E-02	0.4578650E-02
0.70	0.3101349E-02	0.4323065E-02
0.80	0.2215624E-02	0.3512800E-02
0.90	0.1176059E-02	0.2088368E-02
0.91	0.1064420E-02	0.1910031E-02
0.92	0.9512901E-03	0.1725912E-02
0.93	0.8369088E-03	0.1534283E-02
0.94	0.7211566E-03	0.1335621E-02
0.95	0.6041527E-03	0.1130283E-02
0.96	0.4858375E-03	0.9179711E-03
0.97	0.3662705E-03	0.6995797E-03
0.98	0.2453923E-03	0.4730225E-03
0.99	0.1233220E-03	0.2402663E-03

Table 8 : Mean-squared error of the minimum variance unbiased estimator in the Poisson analysis based on k = 4 intervals with ratio τ/t_0 = 1.0 .

P	MEAN-SQUARED ERROR
0.01	0.2162279E-03
0.02	0.6636593E-03
0.03	0.1262531E-02
0.04	0.1977710E-02
0.05	0.2786858E-02
0.06	0.3673859E-02
0.07	0.4626237E-02
0.08	0.5633924E-02
0.09	0.6688505E-02
0.10	0.7782787E-02
0.20	0.1981391E-01
0.30	0.3160797E-01
0.40	0.4118929E-01
0.50	0.4730156E-01
0.60	0.4903889E-01
0.70	0.4569966E-01
0.80	0.3671754E-01
0.90	0.2161853E-01
0.91	0.1975608E-01
0.92	0.1782841E-01
0.93	0.1583432E-01
0.94	0.1377423E-01
0.95	0.1164687E-01
0.96	0.9453516E-02
0.97	0.7191956E-02
0.98	0.4862558E-02
0.99	0.2464794E-02

Table 9 : Mean-squared error of the minimum variance unbiased estimator in the Poisson analysis based on k = 4 intervals with ratio τ/t_o = 0.1 .

P	MEAN-SQUARED ERROR
0.01	0.1220181E-04
0.02	0.4109689E-04
0.03	0.9245944E-04
0.04	0.1340761E-03
0.05	0.1944211E-03
0.06	0.2623231E-03
0.07	0.3368300E-03
0.08	0.4171450E-03
0.09	0.5025805E-03
0.10	0.5925372E-03
0.20	0.1542227E-02
0.30	0.2750106E-02
0.40	0.3707430E-02
0.50	0.4369736E-02
0.60	0.4626621E-02
0.70	0.4388418E-02
0.80	0.3579713E-02
0.90	0.2135897E-02
0.91	0.1954601E-02
0.92	0.1766132E-02
0.93	0.1570482E-02
0.94	0.1367648E-02
0.95	0.1157630E-02
0.96	0.9404297E-03
0.97	0.7160550E-03
0.98	0.4845157E-03
0.99	0.2458249E-03

Table 10 : **Bias and mean-squared error of the maximum likelihood estimator in the Poisson analysis based on $k = 8$ intervals with ratio $\tau/t_o = 1.0$.**

P	BIAS	MEAN-SQUARED ERROR
0.01	0.3181074E-02	0.1253214E-03
0.02	0.5288713E-02	0.3736261E-03
0.03	0.7021710E-02	0.6976853E-03
0.04	0.8517951E-02	0.1077596E-02
0.05	0.9841144E-02	0.1501343E-02
0.06	0.1102842E-01	0.1960327E-02
0.07	0.1210386E-01	0.2448041E-02
0.08	0.1308459E-01	0.2959345E-02
0.09	0.1398301E-01	0.3490027E-02
0.10	0.1480901E-01	0.4036523E-02
0.20	0.2026784E-01	0.9849995E-02
0.30	0.2246380E-01	0.1529455E-01
0.40	0.2259707E-01	0.1953179E-01
0.50	0.2122396E-01	0.2206731E-01
0.60	0.1866663E-01	0.2256554E-01
0.70	0.1513541E-01	0.2078009E-01
0.80	0.1077843E-01	0.1651973E-01
0.90	0.5705118E-02	0.9635448E-02
0.91	0.5161822E-02	0.8796930E-02
0.92	0.4612327E-02	0.7932186E-02
0.93	0.4056633E-02	0.7039236E-02
0.94	0.3494799E-02	0.6117344E-02
0.95	0.2926946E-02	0.5168438E-02
0.96	0.2353251E-02	0.4191220E-02
0.97	0.1773596E-02	0.3186166E-02
0.98	0.1188099E-02	0.2152026E-02
0.99	0.5968809E-03	0.1090467E-02

Table 11 : **Bias and mean-squared error of the maximum likelihood estimator in the Poisson analysis based on k = 8 intervals with ratio τ/t_0 = 0.1 .**

R	BIAS	MEAN-SQUARED ERROR
0.01	0.2907962E-03	0.6262213E-05
0.02	0.4929602E-03	0.2096361E-04
0.03	0.6619990E-03	0.4191203E-04
0.04	0.8095540E-03	0.6797141E-04
0.05	0.9410083E-03	0.9838003E-04
0.06	0.1059934E-02	0.1325209E-03
0.07	0.1168251E-02	0.1699254E-03
0.08	0.1267612E-02	0.2101921E-03
0.09	0.1359046E-02	0.2529621E-03
0.10	0.1443505E-02	0.2979636E-03
0.20	0.2013505E-02	0.8206442E-03
0.30	0.2256632E-02	0.1369119E-02
0.40	0.2287745E-02	0.1840949E-02
0.50	0.2161801E-02	0.2165496E-02
0.60	0.1910746E-02	0.2289176E-02
0.70	0.1555741E-02	0.2167940E-02
0.80	0.1111984E-02	0.1766384E-02
0.90	0.5903840E-03	0.1053751E-02
0.91	0.5343556E-03	0.9638071E-03
0.92	0.4776120E-03	0.8711219E-03
0.93	0.4201531E-03	0.7746816E-03
0.94	0.3620386E-03	0.6743670E-03
0.95	0.3033280E-03	0.5704165E-03
0.96	0.2439022E-03	0.4634857E-03
0.97	0.1838803E-03	0.3536344E-03
0.98	0.1232028E-03	0.2388954E-03
0.99	0.6192923E-04	0.1220107E-03

Table 12 : **Mean-squared error of the minimum variance unbiased estimator in the Poisson analysis based on $k = 8$ intervals with ratio $\tau/t_0 = 1.0$.**

R	MEAN-SQUARED ERROR
0.01	0.7782795E-04
0.02	0.2522755E-03
0.03	0.4950904E-03
0.04	0.7925578E-03
0.05	0.1135539E-02
0.06	0.1517215E-02
0.07	0.1932171E-02
0.08	0.2375941E-02
0.09	0.2844720E-02
0.10	0.3335210E-02
0.20	0.8913763E-02
0.30	0.1461699E-01
0.40	0.1941651E-01
0.50	0.2262689E-01
0.60	0.2373667E-01
0.70	0.2234023E-01
0.80	0.1810242E-01
0.90	0.1073819E-01
0.91	0.9819601E-02
0.92	0.8867789E-02
0.93	0.7881280E-02
0.94	0.6860148E-02
0.95	0.5804501E-02
0.96	0.4714452E-02
0.97	0.3588352E-02
0.98	0.2428074E-02
0.99	0.1231930E-02

Table 13 : Mean-squared error of the minimum variance unbiased estimator in the Poisson analysis based on $k = 8$ intervals with ratio $\tau/t_0 = 0.1$.

R	MEAN-SQUARED ERROR
0.01	0.5925379E-05
0.02	0.2004624E-04
0.03	0.4032585E-04
0.04	0.6568919E-04
0.05	0.9539143E-04
0.06	0.1288561E-03
0.07	0.1656160E-03
0.08	0.2052797E-03
0.09	0.2475090E-03
0.10	0.2919959E-03
0.20	0.8128355E-03
0.30	0.1364710E-02
0.40	0.1843110E-02
0.50	0.2175333E-02
0.60	0.2305757E-02
0.70	0.2189303E-02
0.80	0.1787110E-02
0.90	0.1066790E-02
0.91	0.9761162E-03
0.92	0.8822589E-03
0.93	0.7844162E-03
0.94	0.6834026E-03
0.95	0.5783844E-03
0.96	0.4702148E-03
0.97	0.3580274E-03
0.98	0.2417999E-03
0.99	0.1224452E-03

Table 14 : Bias and mean-squared error of the maximum likelihood estimator in the exponential analysis based on k = 4 intervals with ratio τ/t_o = 1.0 .

R	BIAS	MEAN-SQUARED ERROR
0.01	0.1134650E-01	0.1289121E-02
0.02	0.1361173E-01	0.2359509E-02
0.03	0.1435579E-01	0.3367020E-02
0.04	0.1432871E-01	0.4337154E-02
0.05	0.1382509E-01	0.5281456E-02
0.06	0.1299686E-01	0.6206479E-02
0.07	0.1193476E-01	0.7116229E-02
0.08	0.1069808E-01	0.9013312E-02
0.09	0.9328246E-02	0.8899316E-02
0.10	0.7854998E-02	0.9775508E-02
0.20	-0.9231448E-02	0.1802141E-01
0.30	-0.2610177E-01	0.2498925E-01
0.40	-0.4007441E-01	0.2989972E-01
0.50	-0.4994671E-01	0.3192948E-01
0.60	-0.5445093E-01	0.3041327E-01
0.70	-0.5296350E-01	0.2506214E-01
0.80	-0.4434830E-01	0.1631802E-01
0.90	-0.2730685E-01	0.6105304E-02
0.91	-0.2507871E-01	0.5167365E-02
0.92	-0.2274346E-01	0.4264414E-02
0.93	-0.2030009E-01	0.3412127E-02
0.94	-0.1775408E-01	0.2631068E-02
0.95	-0.1509130E-01	0.1913190E-02
0.96	-0.1231581E-01	0.1286209E-02
0.97	-0.9424508E-02	0.7662177E-03
0.98	-0.6409287E-02	0.3606081E-03
0.99	-0.3269315E-02	0.9673834E-04

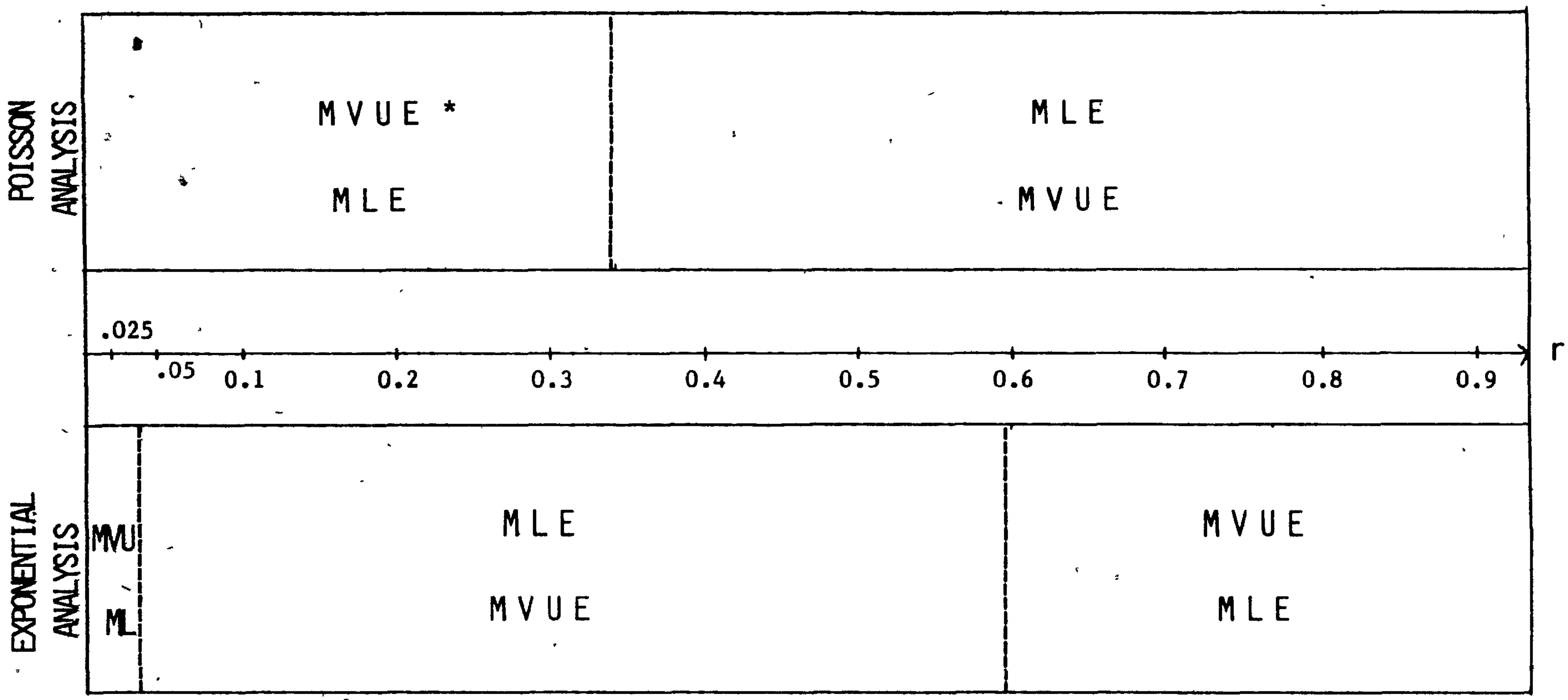
Table 15 : Mean-squared error of the minimum variance unbiased estimator in the exponential analysis based on $k = 4$ intervals with ratio $\tau/t_o = 1.0$.

R	MEAN-SQUARED ERROR
0.01	0.9261533E-03
0.02	0.2180642E-02
0.03	0.3559381E-02
0.04	0.5018495E-02
0.05	0.6511264E-02
0.06	0.8028142E-02
0.07	0.9545304E-02
0.08	0.1106273E-01
0.09	0.1256422E-01
0.10	0.1404212E-01
0.20	0.2691957E-01
0.30	0.3507417E-01
0.40	0.3807992E-01
0.50	0.3628272E-01
0.60	0.3049231E-01
0.70	0.2192903E-01
0.80	0.1228160E-01
0.90	0.3861904E-02
0.91	0.3202677E-02
0.92	0.2591729E-02
0.93	0.2032340E-02
0.94	0.1529932E-02
0.95	0.1089096E-02
0.96	0.7146597E-03
0.97	0.4121661E-03
0.98	0.1872778E-03
0.99	0.4738569E-04

Table 16 : Bias and mean-squared error of the maximum likelihood estimator in the exponential analysis based on $k = 8$ intervals with ratio $\tau/t_o = 1.0$.

P	BIAS	MEAN-SQUARED ERROR
0.01	0.6394781E-02	0.5054469E-03
0.02	0.7817909E-02	0.1046725E-02
0.03	0.8290052E-02	0.1601417E-02
0.04	0.8277133E-02	0.2163048E-02
0.05	0.7965513E-02	0.2728085E-02
0.06	0.7452555E-02	0.3294115E-02
0.07	0.6796658E-02	0.3859267E-02
0.08	0.6036043E-02	0.4421961E-02
0.09	0.5197227E-02	0.4980918E-02
0.10	0.4299819E-02	0.5534820E-02
0.20	-0.5797148E-02	0.1059583E-01
0.30	-0.1517642E-01	0.1430315E-01
0.40	-0.2235931E-01	0.1619959E-01
0.50	-0.2679676E-01	0.1609081E-01
0.60	-0.2819526E-01	0.1404107E-01
0.70	-0.2636594E-01	0.1042789E-01
0.80	-0.2114707E-01	0.5975783E-02
0.90	-0.1240617E-01	0.1899719E-02
0.91	-0.1133317E-01	0.1572967E-02
0.92	-0.1022375E-01	0.1274289E-02
0.93	-0.9075105E-02	0.9992123E-03
0.94	-0.7891715E-02	0.7487535E-03
0.95	-0.6672204E-02	0.5375147E-03
0.96	-0.5412579E-02	0.3496408E-03
0.97	-0.4117429E-02	0.2037883E-03
0.98	-0.2784014E-02	0.9888411E-04
0.99	-0.1414001E-02	0.3111362E-04

Table 17 : Comparison of mean-squared error for the maximum likelihood estimator (MLE) and minimum variance unbiased estimator (MVUE), in the Poisson and exponential analyses.



* The estimator on top represents the one with the smallest mean-squared error (most efficient).

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