A LINEARIZATION METHOD
FOR DETERMINING THE EFFECT OF LOADS,
SHUNTS, AND SYSTEM UNCERTAINTIES
ON LINE PROTECTION WITH DISTANCE RELAYS

by

Edward R. Sexton, Eng.
A Linearization Method for Determining the Effect of Loads, Shunts, and System Uncertainties on Line Protection with Distance Relays

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ABSTRACT

Short circuit calculations usually performed for determining the setting of line protection relays neglect factors such as loads and shunts, and assume that all network parameters, such as line impedances, are known with perfect precision, which is not the case. This paper describes a technique for assessing the effect of these factors on the voltages and currents activating the relays. The study is centered on distance relays using the principle of impedance determination.

A computer program was developed using a linearization method which calculates probability densities of the apparent impedance observed by the relays under fault conditions. The results are displayed in the form of equi-probability contours on the R-X diagram of the protected line. A further advantage of this technique is that the sensitivity of the impedance to each of the system parameters is easily obtained. The program runs for a fraction of the cost of a conventional full-scale short circuit program taking loads and shunts into account and could be readily adapted as a tool for assessing ranges of variations of apparent impedances on a systematic basis.

The method is applied to a reduced model of the Hydro-Quebec 735kV grid. Worst case errors are in the order of 25%.

INTRODUCTION

Line protection relays are set according to simplified short circuit calculations with a safety factor added to take uncertainties into account. The goal of this paper was to determine whether these simplifications could result in marked differences between relay settings and actual values seen by the relay. The linearization technique described in this paper was used in developing a computer program which is fast, accurate, and inexpensive. The program operates from certain outputs of conventional short circuit programs (selected entries of the impedance matrix) and data concerning the ranges of variation of the parameters and loading of the protected line or lines. Standard deviations of factors affecting impedances observed by the relays are calculated and then used to determine possible variations for the impedances in the form of probability densities in the complex plane.

The methodology consisted in simplifying the short circuit equations in such a way as to render the analytical determination of probability densities possible for apparent impedances. This approach entailed reducing the network to an equivalent and a linearization of the resulting equations.

The linearization method also permits the assessment of the sensitivity of the impedance perceived by the relay to individual sources of error, which permits the ranking of the sources of error by importance.

The technique was applied to nine lines from the test network. Line-to-line and line-to-ground faults occurring at three different line locations were examined. The following results were obtained at each fault location:

- mean value of apparent impedance
- sensitivity matrix
- equiprobability curves of apparent impedance for 1%, 5%, 90%, 95% probability

DEVELOPMENT OF THE ANALYSIS TECHNIQUE

The development of the linearization technique is described here in brief. A detailed analysis showing the formulation of the equiprobability contours and the computation of the apparent impedance is given in Appendices I and II.

Linearization Method for Computing Sensitivities and Probability Densities of Apparent Impedances

The apparent impedance seen by the distance relay is represented by

\[ Z_{app} = f(x) \]

This impedance is a function of the voltages and currents observed by the relay, which are themselves functions of network variables such as positive and zero sequence impedances of all lines, real and reactive loads, etc... Let the real vector \( x \) represent these network variables and let the vector function \( f \) represent the fundamental relationship between \( Z_{app} \) and \( x \):

\[ Z_{app} = f(x) \]

The objective will be to establish the sensitivities of individual elements of \( Z_{app} \) to individual elements of \( x \), and to obtain a probability density for \( Z_{app} \) from an available probability density for \( x \), characterized by a mean \( \overline{x} \) and a covariance matrix \( \Sigma x \).

The function \( f \) is nonlinear and must be simplified if the above information is to be extracted in a computationally tractable way. This can be achieved by linearizing around a given point \( \overline{x} \). The most logical choice for \( \overline{x} \) is the expectation of \( x \). \( f(\overline{x}) \) can thus be approximated as:

\[ f(x) = f(\overline{x}) + A (x - \overline{x}) \]  \hspace{1cm} (1)

where the sensitivity matrix \( A \) is given by:
The expectation of \( z_{\text{app}} \) can then be computed as:

\[
E(z_{\text{app}}) = E[f(x)]
\]

The covariance matrix of \( z_{\text{app}} \) is given by:

\[
S_x = E[(z_{\text{app}} - \bar{z}_{\text{app}})(z_{\text{app}} - \bar{z}_{\text{app}})^T]
\]

which can be approximated using (1) and (3) to give:

\[
S_x = E[A(x - \bar{x})(x - \bar{x})^T A^T]
\]

With the formulation used herein (see Appendix I), \( x \) is a vector of 39 variables. \( S_x \) is therefore a 39 x 39 matrix and the matrix \( A \) is a 2 x 39. It can be seen that the product in equation (4) reduces to a 2 x 2 matrix for \( S_x \).

The expected value of the apparent impedance seen by the relay can be obtained by:

1) reducing the network to a two-point equivalent as seen from the relay and the fault, for the expected values of the network parameters. This reduction is carried out in Appendix II. This step was necessary to keep the computational requirements of the procedure within reasonable bounds: not using an equivalent would have required the linearization of the inverse of the admittance matrix of the network.

2) computing the apparent impedance from the equivalent network parameters. Formulas for this computation are also derived in Appendix II.

The covariance matrix \( S_x \) for the parameters and initial voltages of the equivalent can be obtained from the ranges of deviations assumed for the parameters of the entire network. Equi-probability contours can then be drawn from the expectation and covariance of the probability density of the apparent impedance as outlined in Appendix I.

### Development of the Computer Program

Using the technique described above and the equations derived in Appendices I and II, a program was written to calculate and plot equi-probability curves for the apparent impedance of the line. The program requires as input, mean values and standard deviations of line parameters, shunts, loads and certain values of the positive sequence and zero sequence bus impedance matrix of the network. The program can be used for either line-to-ground or line-to-line fault conditions. Plots may be obtained for any selected fault location.

### Ranges of Variation of Equivalent Network Parameters

#### Ranges of Variation of System Parameters:

The ranges of variation obtained for the system parameters in [5] used. The only exception was for the case of the impedance matrix entries whose variations had to be determined.

The ranges were given to the program in terms of upper and lower limits of the parameter, expressed as a percentage of the mean value, except for the measurement noises, which were expressed as a percentage of the measured value. The standard deviation was calculated for each parameter from the above ranges of variation. The mean values of the voltages at the extremes of the faulted line, \( V_1 \) and \( V_2 \), were assumed to be 1.0 p.u. in magnitude and ranges of variations of \( \pm 5\% \) were assumed. Variations in \( \phi \), the power angle between the extremes of the line, reflected the effect of load variations, as well as variations in the network matrix entries \( Z_{Is}, Z_{Is'}, Z_{Ps}, Z_{Ps'} \). Variations of the matrix entries are discussed in the following sections. \( \phi \) was assumed to have a mean value corresponding to the loading of the line under average loading conditions, and a percentage variation corresponding to the percentage variation of the total load of the network.

### Calculation of Ranges of Variation of Impedance Matrix Entries:

The matrix entries are very nonlinear functions of line impedances, loads and shunts: they are obtained by inverting the admittance matrix of the network, which is itself a function of the inverses of the individual impedances. A rigorous derivation of the standard deviations of the matrix entries would be to map the probability densities of the line impedances, loads and shunts through the matrix inversion, extract the probability densities of the matrix entries, and compute the standard deviations from these probability densities. Such an approach raised analytical and computational problems, and the following approximation was used instead to obtain an order of magnitude for the variance of the matrix entries.

#### a) Positive Sequence Matrix

Since the effect of the loads and shunts was essentially taken into account by variations of the power angle \( \phi \) between the terminals of the faulted line, and by variations of the shunt elements in the equivalent network, the remaining variations to be taken into account were those attributable to uncertainties on the values of positive sequence line impedance matrix. These uncertainties are very small and variations of positive sequence matrix entries were therefore neglected.

#### b) Zero sequence Matrix Entries

Uncertainties in zero sequence matrix entries due to zero sequence line impedances were taken into account as follows: a diagonal entry of the impedance matrix represents the equivalent impedance of the many parallel paths existing through the network between this point and the ground. An equivalent network between this point and the ground can therefore be said to consist of a number of parallel lines, each one having an impedance equal to the impedance of an average line in the network. If the variance of the impedance of such a line is known, then the variance of the impedance of the parallel equivalent can be computed as follows:

Let \( Z_{eq} \) be the equivalent impedance of \( n \) lines in parallel, and consider only magnitudes:

\[
Z_{eq} (\bar{Z}_1 + \Delta Z_1, \ldots + \bar{Z}_n + \Delta Z_n) = Z_{eq} (\bar{Z}) + \frac{n}{2} \sum_{i=1}^{n} \Delta Z_i
\]

The mean value of the equivalent impedance can be
shown to be equal to:

\[ E(Z_{eq}) = Z_{eq}(\bar{Z}) = Z_{eq} \]

The variance of \( Z_{eq} \) is found by

\[ \sigma^2(Z_{eq}) = E[(Z_{eq} - Z_{eq}(\bar{Z} + \Delta Z))^2] \]

which reduces to:

\[ \sigma^2(Z_{eq}) = \frac{R}{n} \sigma^2(Z) Z_{eq}^4 \]

If we assume all \( Z_i \) and \( \sigma_i \) to be equal, the standard deviation becomes:

\[ \sigma(Z_{eq}) = \frac{1}{\sqrt{n}} \sigma(Z) Z_{eq} \]

(5)

The standard deviation of \( Z_{ii} \) is therefore found from equation (5) by substituting \( Z_{ii} \) for \( Z_{eq} \) and making \( Z_{ii} \) equal to the impedance of an average line in the network.

The variation of the entries of the zero sequence matrix was therefore calculated as follows. The average zero sequence line impedance can be extracted from the network line data. Similarly, the average value of a network driving point impedance can be computed from a printout of the matrix as computed by a conventional short circuit program.

Using these results in equation (5) led to standard deviations of entries of the zero sequence matrix of 3.4%.

Calculation of Covariance Matrix for Equivalent System Parameters.

Network Parameters: The diagonal elements of the covariance matrix are equal to the squares of the standard deviations of the variables. The off-diagonal elements were assumed equal to zero: this assumption corresponds to the case of independent variables. In actuality, the variations may not be completely independent: it is possible that certain factors (i.e., temperature, earth resistances, etc.) could be common to variations in more than one parameter. The associated covariances were neglected in order to simplify the analysis.

Shunt Elements: The bus shunt elements \( Z_{is} \) and \( Z_{ps} \) (see Fig. 11.3) are composed of inductances connected on the line close to the bus and of loads. The load had a variation of \( \pm 33\% \), whereas the inductances were fixed values which could be switched on and off. For simplification we assumed the variation of the inductances to be zero. The ranges of variation of the real parts of the shunt elements \( Z_{is} \) and \( Z_{ps} \) were therefore such that the real parts of the corresponding admittances varied by \( \pm 33\% \).

RESULTS

Test System

The system used was a reduced version of the Hydro-Quebec 735 kV transmission system; a single line diagram of it appears in Fig. 1. Short circuits have been simulated on the lines listed in Table 1. Different probability contours have been obtained for faults at 50%, 80%, (figure traditionally taken as limit of the primary protection zone) and 95% of the line length. A sensitivity matrix was also calculated for each fault location.

Current transformer errors were included for all lines. Voltage transformer errors were included on three lines only.

Mutual coupling between lines was not considered.

Table I: Faulted Lines and Relay Positions

<table>
<thead>
<tr>
<th>Line</th>
<th>Relay Position</th>
<th>Line Length (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>101-103</td>
<td>101</td>
<td>216</td>
</tr>
<tr>
<td>102-117</td>
<td>102</td>
<td>222</td>
</tr>
<tr>
<td>103-105</td>
<td>103</td>
<td>300</td>
</tr>
<tr>
<td>104-106</td>
<td>104</td>
<td>410</td>
</tr>
<tr>
<td>105-109</td>
<td>105</td>
<td>167</td>
</tr>
<tr>
<td>106-109</td>
<td>106</td>
<td>182</td>
</tr>
<tr>
<td>106-117</td>
<td>106</td>
<td>431</td>
</tr>
<tr>
<td>109-110</td>
<td>109</td>
<td>220</td>
</tr>
<tr>
<td>110-111</td>
<td>110</td>
<td>230</td>
</tr>
</tbody>
</table>

Figure 1. Reduced Model of Hydro-Quebec 735 kV System

General Appearance of Equiprobability Contours

Figure 2(a) is a collection of equiprobability contours for line-to-ground faults along line 106-109, for the impedance observed by the relay located at bus 106. The relay was assumed to perform the operation:

\[ \text{Va} = \frac{1}{I_a} \]

Three sets of concentric contours appear, corresponding to faults occurring at 50%, 80% and 95% of the line length. This diagram is a fairly typical case, and the impedance contours of Figure 2(a) are representative of the contours of other lines for the same type of faults. In the diagram, the straight line drawn from the origin represents the line impedance vector. The intersecting arcs show the fault locations on the line. Each arc is the locus of an impedance vector of a magnitude equal to the magnitude of the line impedance from the relay to the fault. For each fault location the contours for 10%, 50% and 95% probability are shown. More precisely, the outermost contour, corresponding to the 95% ellipse, is such that the probability of the apparent impedance falling within its bounds is 95%. In Figure 2(a) potential transformer inaccuracies were assumed to be zero, and current transformer inaccuracies only were considered.
Figure 2. Impedance Contours, Line-to-Ground Faults

Single Line-to-Ground Faults

It can be seen from Figure 2(a) that there are two effects which contribute to the error in impedance measurements. One effect is the offset or bias of the mean value of the impedance seen by the relay from the actual value of line impedance to the fault location. The other effect is the variation around the mean caused by uncertainties in the variables.

Bias of the Mean: For line-to-ground faults, the bias of the mean is due to three factors. The horizontal displacement (error on the real component) is due to the fault resistance. This resistance seems to increase with the distance from the relay to the fault because of the increase in current contribution from the far end of the line. The vertical displacement (error on the reactive component) is caused by the effect of line charging and capacitance. It can be seen in figure 2(a) that the relay is underreaching; the fault appears to be further away along the line than it actually is.

If shunt compensation is neglected the impedance seen by the relay during a fault can be represented by the charging capacitance in parallel with the line reactance as shown in Figure 3.
Figure 3. Equivalent Impedance Seen by the Relay

The equivalent impedance can be reduced to:

\[ Z_{app} = \frac{j \frac{d}{Y_{ls}}}{1 - dX \frac{d}{Y_{ls}}} \]

where:
- \( Z_{app} \): apparent impedance, ohms.
- \( X_{l} \): line series reactance, ohms per km.
- \( Y_{ls} \): line charging susceptance, mhos per km.
- \( d \): distance from relay to fault, km.

The relative error is then:

\[ 1 - \frac{Z_{app}}{j \frac{d}{Y_{ls}}X_{l}} = \frac{-dY_{ls}X_{l}}{1 - \frac{d}{Y_{ls}}X_{l}} \]

This equation points out that an increase in either the charging capacitance \( Y_{ls} \) or the distance from the fault \( d \) will cause a nonlinear increase in the value of the impedance seen by the relay. In other words, the percent bias should be greater for faults occurring closer to the far end of the line. This effect is apparent in Figure 2(a).

From equation (6), it should also be expected that the bias would be more important for longer lines. In that case a fault occurring a given fraction of the line away from the relay will correspond to a larger value of \( d \) in the equation, and therefore to a larger bias. An examination of Figure 2(b) shows this to be the case. Line 104-106 shown in this figure is approximately twice the length of line 106-109 and it can be seen that the bias is correspondingly larger.

The third factor causing the bias of the mean is the effect of single parameter zero sequence compensation setting. It can be seen that the bias affecting the mean value of the impedance is much larger in figure 2(c) than in 2(a). This is because single parameter, or scalar, zero sequence compensation has been assumed in this figure. The effect of the zero sequence compensation setting on the bias can be demonstrated as follows. Theoretically, for a perfect elimination of zero sequence effects, the phase relaying units should perform the operation:

\[ Z_{app} = \frac{V_{a}}{I_{a} + (S-1)I_{o}} \]

where

\[ \frac{Z_{o}/Z_{l}}{S \angle \theta} \]

In practice however the relay setting has no phase component, and the value used is:

\[ \theta = \frac{|Z_{o}/Z_{l}| \angle 0} \]

The absence of the phase \( \theta \) will have an effect on both the phase and magnitude of the apparent impedance. It can be seen that ignoring the phase angle of \( \theta \) displaces the apparent impedance over to the right of the true value. In addition there appears also a difference between the mean measured impedance magnitude and the true magnitude which is especially noticeable in the longer lines. The resultant effect is an increase in the magnitude of the bias.

Variations About the Mean: These variations are due to uncertainties affecting the network parameters, metering errors, and variations in prefault network states.

A quantitative evaluation of the magnitude of the errors can be obtained from Table II, in which a listing of the biases, variations about the mean, and worst case errors in impedance evaluation is given. The values of the worst case errors were computed by measuring the shortest distance between the constant impedance circle corresponding to the fault position and the farthest point on the 95% contour, for the fault occurring at the far end of the line. The errors are shown as a percentage of the actual line impedance between the relay and the fault.

Line-to-Line Faults

The diagram of the impedance observed by unit BC (faulted phases) of the relay protecting line 106-109, end 106, appears in Figure 4. The apparent impedance was assumed given by the operation:

\[ \frac{V_{bc}}{I_{b}} = \frac{V_{bc}}{I_{c}} \]

The errors in impedance measurement made by this unit are summarized in Table III. The biases do not differ significantly from those in the line-to-ground case. The vertical variations about the mean are however noticeably smaller. This is due to the fact that the zero sequence parameters do not come into effect. The sensitivity analysis, described below, shows the zero sequence line impedance to be one of the major sources of uncertainty in the case of line-to-ground faults.

Sensitivity Analysis

Although the apparent impedance might be very sensitive to a certain variable, if the range of variation of that variable is small, the actual effect on the apparent impedance could be negligible. In order to get a more significant picture of the effect of each variable on the apparent impedance, the product of the sensitivity and the standard deviation has been calculated and displayed. By examining this product, the order of importance of each variable can be determined. Figures 5 and 6 show the values of this product for line-to-ground and line-to-line faults. The sensitivities are those of the apparent reactance to each of the variables for a typical line (101-103) for a fault occurring at 80% of the distance from the relay to the end bus. The meaning of the variables is as follows:

ELA, EVA, E10: measurement noises on phase currents, zero phase voltages and zero sequence currents
Im, Re: real and imaginary parts
THP1: prefault voltage angle between line ends
VP, VL: prefault voltage magnitudes at line ends
zf: fault impedance
Zll, Zlo: positive and zero sequence line impedances

Experience indicates that quantities having a sensitivity-variation product of less than 0.5 10E-5 have a negligible effect. Figures 5 and 6 therefore indicate that the main factors contributing to the variations of the apparent impedance around its mean, in order of importance, are:
1) for a line-to-ground fault (also valid for line-line-ground fault):
- uncertainties affecting the zero sequence line impedance values
- instrument transformer errors
- the initial loading of the line (voltage angle difference between end buses)
- uncertainties affecting the magnitude of pre-fault bus voltages

2) for a line-to-line fault (also valid for a three-phase fault):
- instrument transformer errors
- fault resistance

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![Figure 4. Impedance Contours, Line-to-Line Faults](image)

![Figure 5. Sensitivities for L-G Fault](image)

![Figure 6. Sensitivities for L-L Fault](image)

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Table II - Errors in Impedance Measurement for L-G Faults with Single Parameter Zero Sequence Compensation

<table>
<thead>
<tr>
<th>Line #</th>
<th>Range of Uncertainty</th>
<th>Bias</th>
<th>Worst Case Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>101-103</td>
<td>9.7</td>
<td>8.7</td>
<td>17.9</td>
</tr>
<tr>
<td>102-117</td>
<td>8.8</td>
<td>7.3</td>
<td>16.1</td>
</tr>
<tr>
<td>103-105</td>
<td>9.2</td>
<td>17.1</td>
<td>25.9</td>
</tr>
<tr>
<td>104-106</td>
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<td>18.0</td>
<td>26.7</td>
</tr>
<tr>
<td>105-109</td>
<td>9.3</td>
<td>2.8</td>
<td>11.9</td>
</tr>
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<td>106-109</td>
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<td>13.6</td>
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Table III - Errors in Impedance Measurement for L-L Faults

<table>
<thead>
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<th>Line #</th>
<th>Range of Uncertainty</th>
<th>Bias</th>
<th>Worst case Error</th>
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<td>2.0</td>
<td>7.3</td>
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<tr>
<td>110-111</td>
<td>4.8</td>
<td>7.2</td>
<td>0.9</td>
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CONCLUSIONS

The results obtained by the linearization method showed the errors in impedance measurement due to the bias effect are equally as important as those due to variations about the mean. The overall effect of these errors resulted in an underreaching of the relay. The worst case error for the lines examined was approximately 28%, which is higher than had been previously expected in the setting of relays. (Only 15% is normally allowed for). [3].

The results indicate that for faults occurring at the far end of the line, it is impossible to predict with certainty whether the relay will trip or not. This should not be a problem in the primary protection zone of the relay. However, relays are normally assigned a secondary protection zone extending beyond the line length as a backup to other relays. Regions of uncertainty would exist in this zone.

Since the biases and ranges of variation of apparent impedance vary for different relays in the same network, there does not seem to exist easily applied rules of thumb which could be used to adjust relay settings to compensate for these errors. Rather the method of analysis described above should be developed and used to study individual cases. The method should be extended to take into account mutual effects and incorporated into an industrial program available to utilities.

The analysis also shows that the angle of the compensation factor should be taken into account for zero sequence compensation wherever possible.

A reasonable check on the accuracy of this analysis was provided by a comparison of the results with a simulation method [5] which showed both methods to be in reasonably good agreement.

APPENDIX I: DETERMINATION OF EQUIPROBABILITY CONTOURS

Equation of contour

If we assume that the probability density function of \( x \) is Gaussian and that linear equation (1) holds, then the probability density function of Zapp is also Gaussian and it can then be shown that the equiprobability curves are ellipses. It is important to note that even though the assumed probability density functions for the individual variables \( x_i \) are not all Gaussian, the density function of Zapp will, by virtue of the Central Limit Theorem \[\text{CLT}\] tend towards a normal shape since Zapp is a function of a large number of individual variables.

The joint probability density function of a multivariable gaussian is given by [1]

\[
p(x) = \frac{1}{(2\pi)^{K/2} |S_x|^{1/2}} e^{-\frac{1}{2}(z - \bar{z})^T S_x^{-1} (z - \bar{z})}
\]

where \( K \) is the dimension of \( z \), and \( \bar{z} \) the mean value of \( z \).

A constant probability contour is given by the equation:

\[
(z - \bar{z})^T S_x^{-1} (z - \bar{z}) = c^2
\]

which is the equation for an ellipse.

The probability of obtaining \( z \) inside the ellipse is a Chi-squared distribution [1].

\[
p(c) = P(X^2 < c^2)
\]

where

\[
X^2 = (z - \bar{z})^T S_x^{-1} (z - \bar{z})
\]

For \( k = 2 \), the desired probabilities are associated with the following values of \( c \):

<table>
<thead>
<tr>
<th>( p(c) )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.02</td>
</tr>
<tr>
<td>0.5</td>
<td>1.39</td>
</tr>
<tr>
<td>0.9</td>
<td>4.61</td>
</tr>
<tr>
<td>0.95</td>
<td>5.99</td>
</tr>
</tbody>
</table>

Axes of Ellipsoids

The equation of an equiprobability contour can be written as:

\[
(z - \bar{z})^T S_x^{-1} (z - \bar{z}) = 1
\]

where \( S_x^{-1} = c^2 \).

The axes of the ellipsoid are given by the square roots of the eigenvalues of \( S_x \) [1]. The eigenvalues are found by setting the determinant of \( (\lambda I - S_x) \) to:

\[
\begin{vmatrix}
\lambda - \delta_{11} & -\delta_{12} \\
-\delta_{12} & \lambda - \delta_{22}
\end{vmatrix} = 0
\]

Equation (I.2) can be used to find the minor and major axes of the ellipsoid.

Angle of inclination of Ellipse

If \( \lambda \) is an eigenvalue of \( S_x \), there exists a vector \( x \) for which:

\[
\begin{bmatrix}
\delta_{11} - \lambda \\
\delta_{12} \\
\delta_{22} - \lambda
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = 0
\]

By solving this equation we find:

\[
\begin{align*}
x_1 &= \tan \theta = \frac{\lambda - \delta_{11}}{\delta_{12}} \\
x_2 &= \frac{\lambda - \delta_{22}}{\delta_{22} - \lambda}
\end{align*}
\]

where \( \theta \) is the angle associated with a given eigenvalue as shown in Fig. I.1.

![Figure I.1: Equiprobability Contour of Zapp](image)
APPENDIX II: COMPUTATION OF APPARENT IMPEDANCES
IN REDUCED NETWORK.

Methodology

The network can be represented by a two-port equivalent as seen from the terminals of the line on which the fault occurs. The equivalent for the network prior to fault is shown in Fig. II.1. Variables associated with the line (Zl, Zis, Zps, Zls) are known but the variables representing the rest of the network, (Zs, Zim, Zpm) are unknown. However, elements of the bus impedance matrix of the network (Zii, Zip, Zpp) are assumed known since they can be computed by a conventional short circuit program. By forming the bus impedance matrix of the equivalent network, the unknowns can be solved for.

With known values of Zs, Zim, Zpm the equivalent is now formed for the network during a fault. This is done by adding a fictitious bus at the fault location. (see Fig. II.3). The network is again reduced to a two-port equivalent as seen from the relay location and the fault location. The new bus impedance matrix is formed and new entries Zii, Ziq, Zqq are calculated. With these new values, the voltages and currents measured at the relay location can be calculated and we can compute the apparent impedance seen by the relay for any type of fault.

Determination of the System Unknowns

The equivalent network prior to the fault is shown in Fig. II.1. The sources are set to zero and the network is then reduced to that shown in Fig. II.2. The objective is to solve for the unknowns Zs, Zim and Zpm in terms of the other impedances appearing in Fig. II.2 and the known impedance matrix elements of the network. From Fig. II.2 the bus admittance matrix is formed by letting

\[
\begin{align*}
Y_{bus} &= \begin{pmatrix}
Y_a + Y_b & -Y_b \\
-Y_b & Y_b + Y_c \\
\end{pmatrix} \\
&= \begin{pmatrix}
Y_{21} & Y_{22} \\
Y_{23} & Y_{24} \\
\end{pmatrix}
\end{align*}
\]  

(II.1)

Writing Ybus in terms of the known entries of the bus impedance matrix we get:

\[
Y_{bus}^{-1} = Z_{bus} = \begin{pmatrix}
Z_{ii} & Z_{ip} \\
Z_{pi} & Z_{pp} \\
\end{pmatrix}
\]

which allows us to solve for Ya, Yb and Yc. Using the value thus obtained for Ya, it is possible, by considering Figure II.2, to solve for the unknown Zim in terms of known quantities.

\[
Z_{im} = \frac{Z_{ii} Z_{pp} - Z_{ip}^2}{Z_{pp} - Z_{ip} + Z_{is} + Z_{ls} (Z_{ip}^2 - Z_{ii} Z_{pp})} \\
Z_{is} Z_{ls}
\]  

(II.2)

Similarly, using the values for Yb and Yc, Zpm and Zs are solved for as follows:

\[
Z_{pm} = \frac{Z_{ii} Z_{ps} - Z_{ip}^2}{Z_{ps} - Z_{ip} + Z_{is} + Z_{ls} (Z_{ip}^2 - Z_{ii} Z_{ps})} \\
Z_{ps} Z_{ls}
\]  

(II.3)

Determination of New Bus Impedance Matrix During Fault

The equivalent network during the fault is shown in Fig. II.3 with the relay located at bus i and the fault occurring at fictitious bus q.

\[
d = \frac{\text{distance from relay to fault}}{\text{total length of line}}
\]

This network can be reduced to that shown in Fig. II.4.a where:

\[
Z_1 = \frac{1}{Z_{im} + \frac{d}{Z_{is} Z_{ls}}}
\]

\[
Z_2 = d Z_1
\]

\[
Z_3 = \frac{1}{d + \frac{(1-d)}{Z_{ls} Z_{1ls}}}
\]

\[
Z_4 = (1-d) Z_1
\]

\[
Z_5 = \frac{1}{\frac{Z_{pm}}{Z_{ps}} + \frac{1}{\frac{Z_{pp}}{Z_{ps}} + \frac{(1-d)}{Z_{ls}}}}
\]

\[
Z_6 = Z_s
\]

By using a star-delta conversion the network is further reduced to that shown in Fig. II.4.b. A new bus impedance matrix is then calculated as follows:

\[
Z_{bus} = \begin{pmatrix}
Z_{ii} & Z_{iq} \\
Z_{iq} & Z_{qq} \\
\end{pmatrix}
\]

\[
Z_{1l} = \begin{pmatrix}
\frac{Z_{a} Z_{b} + Z_{a} Z_{c}}{Z_{a} Z_{b} + Z_{a} Z_{c}} & \frac{Z_{a} Z_{c}}{Z_{a} Z_{b} + Z_{a} Z_{c}} \\
\frac{Z_{a} Z_{c}}{Z_{a} Z_{b} + Z_{a} Z_{c}} & \frac{Z_{a} Z_{b} + Z_{a} Z_{c}}{Z_{a} Z_{b} + Z_{a} Z_{c}} \\
\end{pmatrix}
\]

(II.5)

Computation of Apparent Impedance Seen by Relay for a Single-Line-to-Ground Fault

For a single-line-to-ground fault on phase a the impedance measured by the distance relay is calculated as:[2]
Zapp = \frac{Va + Ea}{(S - 1)(Ir + Eio)}

(II.6)

where Eva, Eia, Eio are random zero-mean measurement noises, S is a setting on the relay equal to the ratio of the zero to positive sequence line impedances, Va is the voltage on phase a during fault, Ir and Ir(0) are the positive and zero sequence currents through the relay in phase a. Using short circuit equations for calculation of voltages and currents during fault 2, the value Zapp can then be obtained.

![Figure II.1. Network Prior to Fault](image)

![Figure II.2. Reduced Network Prior to Fault](image)

![Figure II.3. Network During Fault](image)

![Figure II.4. Reduced Network During Fault](image)

REFERENCES


4) Mendehall, W.: "Introduction to Probability and Statistics".


A linearization method for determining the effect of loads...
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