Hamiltonian Structures for Evolution Equations Describing Pseudo-Spherical Surfaces

Amanda Maria Petcu

Department of Mathematics and Statistics
McGill University, Montreal
October 2021

A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of a Master of Science in Mathematics.

Acknowledgements

I would like to express my deepest gratitude to my supervisor Niky Kamran for the many hours he spent working with me to facilitate my understanding of the subject at hand, and for believing in me even when I didn't. I would also like to thank the professors I have had during my master's degree for a learning experience that I will never forget. Lastly, I would like to thank my friends, my family and my partner Jacob for supporting me every step of the way and for happily listening to me go on and on about my thesis and related work without having any clue what it all means.

Abstract

The notion of an evolution equation which admits a multi-Hamiltonian structure is introduced. We review the algorithm given in Magri's theorem to compute an infinite hierarchy of conservation laws for the evolution equation. Then the notion of an evolution equation that describes pseudo-spherical surfaces is introduced and we use an algorithm given in [2] to compute an infinite sequence of conservation laws for the equation. Since these two classes of evolution equations share this property, the question of whether or not there exist evolution equations that describe pseudo-spherical surfaces and also admit a multi-Hamiltonian structure is explored in the case of the KdV equation and another quintic evolution equation.

Résumé

La notion d'équation d'évolution qui admet une structure multi-hamiltonienne est introduite. Nous utilisons l'algorithme donné dans le théorème de Magri pour calculer une hiérarchie infinie de lois de conservation pour une telle équation d'évolution. Ensuite, la notion d'équation d'évolution décrivant une surface pseudo-sphérique est introduite et nous utilisons un algorithme donné dans [2] pour calculer une suite infinie de lois de conservation pour l'équation. Puisque ces deux classes d'équations d'évolution partagent cette propriété la question de savoir s'il existe ou non des équations d'évolution qui décrivent des surfaces pseudo-sphériques et admettent également une structure multi-hamiltonienne est explorée pour l'exemple de l'équation KdV et d'une autre équation d'évolution quintique.

Contents

1 Introduction

The calculation of conservation laws for a differential equation has been a problem of interest for many researchers. The conservation laws that arise naturally from physics such as conservation of mass and momentum are but a drop in a bucket. This is why we are very interested in algorithms that could provide an infinite sequence of conservation laws for certain classes of evolution equations. This thesis explores two classes of evolution equations for which there exist algorithms that create an infinite hierarchy of conservation laws for the equation. The first class, explained in section 3 are evolution equations that describe pseudo-spherical surfaces. The second class, explained in section 2 are evolution equations which admit a multi-Hamiltonian structure. The fact that these two classes of evolution equations both admit an infinite sequence of conservation laws begs the question of whether or not there exists a large overlap between these classes. One such example is the KdV equation

$$u_t = u_{xxx} + uu_x$$

which describes a pseudo-spherical surface and also admits a multi-Hamiltonian structure. This will be used as the running example in this thesis. Following Chern and Tenenblat in [1] an evolution equation for $u(x,t)$ describes pseudo-spherical surfaces if there exist smooth real valued functions $f_{ab},\ 1 \leq a \leq 3,\ 1 \leq b \leq 2$ depending on u and finitely many derivatives, such that the 1-forms

$$\omega_a = f_{a1}dx + f_{a2}dt \qquad a = 1, 2, 3$$

satisfy the relations:

$$d\omega_1 = \omega_3 \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_3, \quad d\omega_3 = \omega_1 \wedge \omega_2$$

where ω_1, ω_2 define the metric $g = (\omega_1)^2 + (\omega_2)^2$ and ω_3 is the connection form that determines the Gaussian curvature to be $K = -1$ making these the structure equations for a pseudo-spherical surface. In this thesis we require that $f_{21} = \eta$ be a real parameter. By allowing all the other differential functions to be analytic in η we use the algorithm given in [2] to create an infinite sequence of conservation laws for the equation. This algorithm, explained in section 3 works by creating a single conservation law that is analytic in η and then expanding that law in a power series thus giving us an infinite sequence. In the paper [1] by Chern and Tenenblat a few theorems that help in creating evolution equations that describe pseudo-spherical surfaces are provided. These theorems made it possible to describe large classes of p.s.s. equations leading to a very large list of evolution equations for which we can compute an infinite sequence of conservation laws. The same cannot be said about the second class of evolution equations, those which admit a multi-Hamiltonian structure. These equations are much harder to find, this in part due to the many requirements for the existance of a Hamiltonian structure. A Hamiltonian structure depends upon a differential operator $\mathscr{D}$ that we use to define a Poisson bracket on the space of functionals $\mathscr{F}$ as

$$\{\mathscr{P}, \mathscr{Q}\} = \int \delta\mathscr{P} \cdot \mathscr{D}\delta\mathscr{Q}\ dx$$

where $\mathscr{P} = \int P\ dx$, $\mathscr{Q} = \int Q\ dx$ are functionals in $\mathscr{F}$ and δ is the variational derivative. The Poisson bracket for an operator $\mathscr{D}$ has requirements that are further explained in section 2. We call such an operator $\mathscr{D}$ a Hamiltonian operator. Then we can define an evolution equation $u(x,t) = K[u]$ to be Hamiltonian if there exists a Hamiltonian operator $\mathscr{D}$ and a functional $\mathscr{H} = \int H\ dx$ such that

$$u_t = K[u] = \mathscr{D}\delta\mathscr{H}.$$

Remarkably an evolution equation can admit more than one structure, such an evolution equation is said to be multi-Hamiltonian. For example the KdV equation that is further explored in section 2 admits a bi-Hamiltonian structure, thus it can be written in the form above using two distinct Hamiltonian operators and functionals. An algorithm for the computation of an infinite hierarchy of conservation laws for an evolution equation admitting a multi-Hamiltonian structure relies on a ladder-like application between the Hamiltonian symmetries of the evolution equation and the Poisson bracket of each operator. This ladder-like application rests heavily upon a Noether relation between the Hamiltonian symmetries and the conservation laws of the evolution equation that is further explored in section 2.

These two methods of computing an infinite sequence of conservation laws are quite different and share no obvious overlap. However the fact that we are even able to find an infinite number of conservation laws for both these classes of evolution equations is a very unique property. It is thus an interesting problem to see if these two classes overlap. Therefore in addition to the KdV equation, in section 4 of this thesis we explore an example of a quintic evolution equation that is p.s.s and also bi-Hamiltonian. This thesis focuses on just these two examples and this is due in part to the difficulty of finding not only one but at least two Hamiltonian structures for an evolution equation. However, the hypothesis that there exists a large subset of evolution equations which are both p.s.s and multi-Hamiltonian remains.

This thesis is organized as follows: in section 2 we review the background material on Hamiltonian structures for evolution equations. We introduce Poisson brackets, Hamiltonian operators, bi-Hamiltonian systems and Magri's fundamental theorem on the generation of an infinite hierarchy of conservation laws through the above ladder-like relation. In section 3 we introduce the class of evolution equations describing pseudo-spherical surfaces following the foundational work of Chern and Tenenblat in [1]. We review the algorithm due to Cavalcante and Tenenblat in [2] for generating infinite sequences of conservation laws for p.s.s. equations. Section 4 contains the original contribution of this thesis which explores a quintic p.s.s. equation that is also bi-Hamiltonian. The conservation laws for this equation are computed by both the above methods. The thesis concludes in section 5 with some perspectives for further research.

2 Hamiltonian Structures for Evolution Equations

2.1 Background Material

In this section we will give some background information for the set up of this thesis, specifically we will provide key definitions and theorems that will help us in constructing Hamiltonian structures for evolution equations. The information in this section is paraphrased or directly quoted from [4].

Suppose we have a system of partial differential equations $\mathscr{S}$ that involves p independent variables $\mathbf{x} = (x^1, ..., x^p)$ and q dependent variables $\mathbf{u} = (u^1, ..., u^q)$ where $u^a = f^a(x^1, ..., x^p)$ for $a = 1, ...q$. Then we can let $X \simeq \mathbb{R}^p$ with coordinates $\mathbf{x}$ be the space representing the space of independent variables and let $U \simeq \mathbb{R}^q$ with coordinates $\mathbf{u}$ be the space representing the space of dependent variables. Furthermore, for each u^a there are $p_k = \binom{p+k-1}{k}$ different k-th order partial derivatives, we employ the multi-index notation

$$u^a_J = \partial_J f^a = \frac{\partial^k f^a}{\partial x^{j_1}, ..., \partial x^{j_k}}$$

to represent these derivatives. In this notation $J = (j_1, .., j_k)$ is an unordered k-tuple of integers with $1 \leq j_k \leq p$, where the order of the multi-index indicates how many derivatives are being taken. With this notation, we let $U_k \simeq \mathbb{R}^{q \cdot p_k}$ with coordinates u^a_J corresponding to $a = 1, .., q$ and all multi-indices J of order k be the space that represents all the different k-th order derivatives of the components of $\mathbf{u}$ at a point $\mathbf{x}$. For example in the case where $p = 2, q = 1$, we have $u = f(x, y)$, then $U_1 \simeq \mathbb{R}^2$ with coordinates (u_x, u_y) and $U_2 \simeq \mathbb{R}^3$ with coordinates (u_{xx}, u_{xy}, u_{yy}).

Given a smooth function $u = f(x)$, $f : X \to U$, the n-th prolongation of f denoted $u^{(n)} = pr^{(n)} f(x)$, can be thought of as lifting the function $f : X \to U$ to a function from X to $U^{(n)}$ where

$$U^{(n)} = U \times U_1 \times ... \times U_n$$

is the space whose coordinates represent all the derivatives of the function $u = f(x)$ from 0 to n. For example in the case where $p = 2$ and $q = 1$, $u^{(2)} = pr^{(2)} f$ is given by

$$(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$$

evaluated at (x, y). The space $X \times U^{(n)}$ whose coordinates represent the independent variables, the dependent variables and the derivatives of the dependent variables up to order n is called the n-th order jet space of the underlying space $X \times U$. If $M \subset X \times U$ is an open subset then we define

$$M^{(n)} = M \times U_1 \times ... \times U_n$$

to be n-jet space of M.

Let $(\mathbf{x}, \mathbf{u}^{(n)}) \in M^{(n)}$, $L(\mathbf{x}, \mathbf{u}^{(n)}) : M^{(n)} \to \mathbb{R}$ is a smooth function of $\mathbf{x}, \mathbf{u}$ and the derivatives of $\mathbf{u}$ up to order n, $L(\mathbf{x}, \mathbf{u}^{(n)})$ is called a *differential function*. We use the notation $L[u]$ if we do not care as to precisely how many derivatives of u the function L depends on.

Given a differential function $L(x, u^{(n)})$ and an open, connected subset $\Omega \subset X$ with smooth boundary $\partial\Omega$, a variational problem consist of finding the extrema of a functional

$$\mathscr{L}[u] = \int_\Omega L(x, u^{(n)})\, dx$$

in some class of functions u over Ω. In this notation $dx = dx^1 \cdot ... \cdot dx^p$ where p is the dimension for X. The integrand $L(x, u^{(n)})$ is called the *Lagrangian* of $\mathscr{L}[u]$. In finite dimensions, the extrema of a smooth real valued function $f(x)$ are determined by looking at the points where the gradient vanishes. The gradient itself is found by seeing how the function f changes under variations in x. For functionals $\mathscr{L}[u]$ the role of the gradient is played by the *variational derivative* and is constructed by looking at how $\mathscr{L}[u]$ changes under small variations in u. To define the variational derivative we must first define the total derivative and the Euler operator.

Definition 2.1.1. The *total derivative* D_iL of $L[u]$ is defined by

$$D_iL = \frac{dL}{dx^i} + \sum_{a=1}^{q}\sum_{J} u^a_{J,i}\frac{dL}{du^a_J}$$

$$u^a_{J,i} = \frac{du^a_J}{dx^i}$$

where J is the multi-index for the different order derivatives with respect to the independent variables. We sum over all J's from order 0 to the highest order derivative appearing in L.

Definition 2.1.2. For $1 \leq a \leq q$, the a-th Euler Operator is given by

$$E_a = \sum_J (-D)_J \frac{\partial}{\partial u^a_J}$$

where J sums over all multi-indices.

Let $u = f(x)$ be a smooth function defined over Ω and $n(x) = (n^1(x), ..., n^q(x))$ a smooth function with compact support in Ω. We observe how $\mathscr{L}[u]$ changes under u by the following computation

$$\begin{aligned}\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathscr{L}[f+\epsilon n] &= \int_\Omega \Big\{ \frac{d}{d\epsilon}\Big|_{\epsilon=0} L(x, \mathrm{pr}^{(\mathrm{n})}(f+\epsilon n)(x))\Big\}dx \\ &= \int_\Omega \Big\{ \sum_{a,J}\frac{\partial L}{\partial u^a_J}(x, \mathrm{pr}^{(\mathrm{n})}f(x)) \cdot \partial_J n^a(x)\Big\}dx.\end{aligned}$$

Since n has compact support we can use the divergence theorem and integrate by parts so that the terms evaluated on $\partial\Omega$ vanish. The partial derivative $\frac{\partial}{\partial x_i}$ when applied to the derivatives $\frac{\partial L}{\partial u^a_J}$ becomes the total derivative D_i. Therefore we obtain

$$\begin{aligned}\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathscr{L}[f+\epsilon n] &= \int_\Omega \Big\{ \sum_{a=1}^{q}\Big[\sum_J (-D)_J \frac{\partial L}{\partial u^a_J}(x, \mathrm{pr}^{(\mathrm{n})}f(x))\Big] n^a(x)\Big\}dx \\ &= \int_\Omega \Big\{ \sum_{a=1}^{q} E_a(L)\cdot n^a(x)\Big\}dx \\ &= \int_\Omega \Big\{ E(L)\cdot n(x)\Big\}dx.\end{aligned}$$

Using this and the Euler Operator we are now ready to define the variational derivative of $\mathscr{L}[u]$.

Definition 2.1.3. Let $\mathscr{L}[u]$ be a variational problem. The variational derivative of $\mathscr{L}[u]$ is the unique q-tuple

$$\delta\mathscr{L}[u] = (E_1(L), ..., E_q(L))$$

with the property that

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathscr{L}[f+\epsilon n] = \int_\Omega \Big\{ \sum_{a=1}^{q} E_a(L)\cdot n^a(x)\Big\}\, dx$$

whenever $u = f(x)$ is a smooth function defined over Ω, and $n(x) = (n^1(x), ..., n^q(x))$ is a smooth function with compact support in Ω. The component $E_a(L)$ is the a-th Euler operator applied to the Lagrangian L.

The Euler operator applied to the Lagrangian $L(x, u^{(n)})$ is denoted $E(L) = (E_1(L), ..., E_2(L))$ and by our definition above is equal to the variational derivative $\delta\mathscr{L}[u]$. Therefore we will interchangeably use the notation $\delta\mathscr{L}[u]$ and $E(L)$ to mean the same thing.

Finding the possible extrema of a variational problem boils down to solving the Euler-Lagrange equations

$$\delta \mathscr{L}[u] = E(L) = 0$$

for $u = f(x)$. Of course, not every solution to the Euler-Lagrange equations is an extremal, the other solutions will correspond to other types of critical points for the functional. However functionals $L[u]$ whose Euler-Lagrange equations vanish identically are of interest.

Definition 2.1.4. A function $L(x, u^{(n)})$ defined everywhere on $X \times U^{(n)}$ is called a *null Lagrangian* if the Euler-Lagrange equations $E(L) = 0$ are satisfied identically for all x, u.

Example. Let $\mathscr{L}[u] = \int_a^b uu_x \, dx$ $a, b \in \mathbb{R}$ be a variational problem, then $L = uu_x$ is a null Lagrangian since

$$E(L) = u_x - D_x(u) = 0$$

for all x, u.

Definition 2.1.5. Let $P(x, u^{(n)}) = (P_1(x, u^{(n)}), ..., P_p(x, u^{(n)}))$ be a p-tuple of smooth functions of x, u and the derivatives of u. We define the total divergence of P to be

$$\mathrm{Div} P = D_1 P_1 + D_2 P_2 + ... + D_p P_p$$

where D_i is the total derivative with respect to x^i.

The total divergence and the Euler-Lagrange equations are very closely related. We won't use this idea very often in practice, however it will be used in many of the proofs we give in this section. It is for that reason that we quote the following theorem from [4].

Theorem 2.1.1. A function $L(x, u^{(n)})$ defined on $X \times U^{(n)}$ is a null Lagrangian if and only if it is a total divergence: $L = \mathrm{Div} P$ for some p-tuple of functions $P = (P_1, ..., P_2)$ of x, u and the derivatives of u.

Now we have all the tools to introduce the spaces we will primarily be working with in this paper. We define $\mathscr{A}$ to be the space of differential functions, we note that $\mathscr{A}$ is an algebra and we can add and multiply differential functions together. We extend this to $\mathscr{A}^q$, the space of q-tuples of differential functions. Each differential function $L \in \mathscr{A}$, determines a functional $\mathscr{L}[u] = \int_\Omega L[u] \, dx$ defined over any region $\Omega \subset X$ in it's domain. In this thesis we will ignore the boundary conditions and only consider functions $u = f(x)$ that vanish sufficiently rapidly near the boundary. This, in combination with the theorem above allows us to say that two differential functions $L, L' \in \mathscr{A}$ determine the same functional i.e. $\int_\Omega L[u] \, dx = \int_\Omega L'[u] \, dx$ if and only if they differ by a total divergence:

$$L' = L + \mathrm{Div}\ P, \quad \text{for some} \quad P \in \mathscr{A}^p.$$

We can now define an equivalence relation on $\mathscr{A}$ by the above condition. It is then reasonable to define $\mathscr{F}$ as the set of equivalence classes of $\mathscr{A}$ under the image of total divergence, thus $\mathscr{F} = \mathscr{A}/\mathrm{Div}(\mathscr{A}^p)$. The natural projection from $\mathscr{A}$ to $\mathscr{F}$ associates to each differential function L an equivalence class of functionals which will be denoted by $\int L \, dx \in \mathscr{F}$. Now that we have an understanding of our underlying spaces we are ready for more definitions.

Definition 2.1.6. A differential operator $\mathscr{D} : \mathscr{A} \to \mathscr{A}$ is a finite sum

$$\mathscr{D} = \sum_{i=0}^{n} P_i[u] D_J^i$$

where the coefficients $P_i[u] \in \mathscr{A}$. The set of differential operators is a ring under composition [4]. A $q \times q$ matrix differential operator $\mathscr{D} : \mathscr{A}^q \to \mathscr{A}^q$ is a matrix whose entries are differential operators.

Definition 2.1.7. Let $\mathscr{D} = \sum_J P_J[u]D_J,\ P_J \in \mathscr{A}$ be a differential operator, its *adjoint* is the unique differential operator $\mathscr{D}^*$ which satisfies

$$\int_\Omega P \cdot \mathscr{D}Q \, dx = \int_\Omega Q\mathscr{D}^* P \, dx \tag{2.1.1}$$

for every pair of differential functions $P, Q \in \mathscr{A}$ which vanish when $u \equiv 0$, every domain $\Omega \subset \mathbb{R}^p$ and every function $u = f(x)$ of compact support in Ω. Using integration by parts, the formula for the adjoint becomes

$$\mathscr{D}^* = \sum_J (-D)_J \cdot P_J.$$

If $\mathscr{D} : \mathscr{A}^q \to \mathscr{A}^q$ is a matrix differential operator with entries $\mathscr{D}_{ij}$, the adjoint $\mathscr{D}^* : \mathscr{A}^q \to \mathscr{A}^q$ has entries $\mathscr{D}^*_{ij} = (\mathscr{D}_{ji})^*$, the adjoint of the transposed entries of $\mathscr{D}^*$.

Definition 2.1.8. A differential operator $\mathscr{D}$ is *skew-adjoint* if $\mathscr{D}^* = -\mathscr{D}$ and *self-adjoint* if $\mathscr{D}^* = \mathscr{D}$.

Example. Let $\mathscr{E} = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x$, then the adjoint of $\mathscr{E}$ is

$$\begin{aligned} \mathscr{E}^* &= -D_x^3 - \frac{2}{3}uD_x - \frac{2}{3}u_x + \frac{1}{3}u_x \\ &= -D_x^3 - \frac{2}{3}uD_x - \frac{1}{3}u_x. \end{aligned}$$

Furthermore, $\mathscr{E}$ is skew-adjoint since $\mathscr{E}^* = -\mathscr{E}$.

From here it is important to introduce certain vector fields and the differential forms of interest on the space $M^{(n)}$. Much like the spaces $\mathscr{A}$ and $\mathscr{F}$, we will introduce two kinds of differential forms, the later being the quotient of the first under the image of total divergence.

Definition 2.1.9. *A generalized vector field* is an expression of the form

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i[u] \frac{\partial}{\partial x^i} + \sum_{a=1}^{q} \phi^a[u] \frac{\partial}{\partial u^a} \tag{2.1.2}$$

where ξ^i and ϕ^a are smooth differential functions that depend on x, u and the derivatives of u.

Given a vector field $\mathbf{v}$ on $X \times U$, the n-th prolongation of $\mathbf{v}$ can be thought of as lifting the vector field $\mathbf{v}$ to a vector field on $X \times U^{(n)}$. Suppose $\mathbf{v}$ is of the form (2.1.2) where the coefficients ξ^i and ϕ^a depend only on x and u, we define the n-th prolongation of v to be the vector field

$$\mathrm{pr}^{(\mathrm{n})}\, \mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{a=1}^{q} \sum_J \phi_J^a \frac{\partial}{\partial u_J^a} \tag{2.1.3}$$

whose coefficients are determined by the formula

$$\phi_J^a = D_J \Big(\phi_a - \sum_{i=1}^{p} \xi^i u_i^a \Big) + \sum_{i=1}^{p} \xi u_{J,i}^a \quad \text{where} \quad u_i^a = \frac{\partial u^a}{\partial x^i}, \qquad u_{J,i}^a = \frac{\partial u_J^a}{\partial x^i} \tag{2.1.4}$$

and J sums over all multi-indices up to order n and ϕ_J^a are functions of x, u and derivatives of u up to order n.

In analogy with the finite dimensional case, we can also prolong a generalized vector field. Given that it is infinite-dimensional it makes sense for the prolongation to be an infinite sum. Suppose $\mathbf{v}$ is a generalized vector field as in (2.1.2) the infinite prolongation (prolongation for short) of $\mathbf{v}$ is the infinite vector field

$$\mathrm{pr}\, \mathbf{v} = \sum_{i=1}^{p} \xi^i[u] \frac{\partial}{\partial x^i} + \sum_{a=1}^{q} \sum_J \phi_J^a \frac{\partial}{\partial u_J^a} \tag{2.1.5}$$

whose coefficients are determined by the same formula (2.1.4) where now J sums over all multi-indices.

Definition 2.1.10. Let $Q[u] = (Q_1[u], ..., Q_q[u]) \in \mathscr{A}^q$ be a q-tuple of differential functions. An *evolutionary vector field* is a generalized vector field of the form

$$\mathbf{v}_Q = \sum_{a=1}^{q} Q_a[u] \frac{\partial}{\partial u^a.} \tag{2.1.6}$$

The differential function Q is called its *characteristic.*

Note that the prolongation of an evolutionary vector field takes the simple form

$$\text{pr } \mathbf{v}_Q = \sum_{a,J} D_J(Q_a) \frac{\partial}{\partial u^a_J}. \tag{2.1.7}$$

Any generalized vector field as in (2.1.2) has an associated *evolutionary representative* $\mathbf{v}_Q$ in which the characteristic Q has entries

$$Q_a = \phi^a - \sum_{i=1}^{p} \xi^i u^a_i \quad a = 1, ..., q$$

where $u^a_i = \frac{\partial u^a}{\partial x^i}$.

If $\mathbf{v}$ is a generalized vector field it is important to address the issue of exponentiating $\mathbf{v}$, which arises due to the fact that the exponential is not defined in general when working in infinite dimensions. The easiest way to resolve this is to define the action of the group $\exp(\epsilon \mathbf{v})$ on the space of smooth functions as follows: we first replace $\mathbf{v}$ by its evolutionary representative $\mathbf{v}_Q$ and consider the system of equations

$$\frac{\partial u}{\partial \epsilon} = Q(x, u^{(n)}),$$

the solution (provided it exists) to the Cauchy problem $u(x, 0) = f(x)$ determines the group action:

$$[\exp(\epsilon \mathbf{v}) f](x) \equiv u(x, \epsilon).$$

We assume that the solution to this Cauchy problem is uniquely determined by choosing $f(x)$ in an appropriate space of functions for at least ϵ sufficiently small. Then the resulting flow $\exp(\epsilon \mathbf{v})$ will determine a local one-parameter group of transformations on the given function space.

Definition 2.1.11. A *vertical k-form* is a finite sum

$$\hat{\omega} = \sum P^a_J[u] du^{a_1}_{J_1} \wedge .... \wedge du^{a_k}_{J_k}, \tag{2.1.8}$$

in which the coefficients $P^a_J \in \mathscr{A}$. We use $\hat{\bigwedge}^k$ to denote the space of vertical k-forms.

The analogue of the differential of the de Rham complex on $\hat{\bigwedge}^k$ is the *vertical differential* defined as

$$\hat{d}\hat{\omega} = \sum \frac{\partial P^a_J[u]}{\partial u^b_K} du^b_K \wedge du^{a_1}_{J_1} \wedge .... \wedge du^{a_k}_{J_k}. \tag{2.1.9}$$

Example. $\hat{\omega} = x u_{xx} du \wedge du_x$ is a vertical 2-form and its vertical differential is $\hat{d}\hat{w} = x du_{xx} \wedge du \wedge du_x$.

The vertical differential holds much of the same properties as the de Rham differential, $\hat{\omega}$ lives on the finite jet space $M^{(n)}$ and the vertical differential can be thought of as the de Rham differential on these variables. Then, a vertical k-form $\hat{\omega}$ determines an alternating k-linear map from the space T_0 of evolutionary vector fields to the space $\mathscr{A}$ of differential functions. When $\hat{\omega}$ is of the form (2.1.8), this map is written explicitly as

$$< \hat{\omega}; \text{pr v}_{Q^1}, ..., \text{pr v}_{Q^k} >= \sum_{a,J} P^a_J \det(< du^{a_i}_{J_i}, \text{pr v}_{Q^j} >) \tag{2.1.10}$$

where the determinant is taken over the $k \times k$ matrix with (i,j) entry

$$< du^{a_i}_{J_i}, \text{pr } \mathrm{v}_{Q^j} >= du^{a_i}_{J_i}(\text{pr } \mathrm{v}_{Q^j}) = D_{J_i}(Q^j_{a_i}).$$

Furthermore, we can let the total derivative D_J act on vertical k-forms. For each $i = 1, ..., p$ D_i acts on the vertical one-forms du^a_J by

$$D_i(du^a_J) = d(D_i u^a_J) = du^a_{J,i}. \tag{2.1.11}$$

We allow it to act as a kind of Lie-derivative on general vertical k-forms, determined by the following rules:

1. $D_i(c\hat{\omega} + c'\hat{\omega'}) = cD_i(\hat{\omega}) + c'D_i(\hat{\omega'})$
2. $D_i(\hat{\omega} \wedge \hat{\eta}) = (D_i\hat{\omega}) \wedge \hat{\eta} + \hat{\omega} \wedge (D_i\hat{\eta})$
3. $D_i(\hat{d}\hat{\omega}) = \hat{d}(D_i\hat{\omega})$

for all $c, c' \in \mathbb{R}$ and $\hat{\omega}$, $\hat{\eta} \in \hat{\bigwedge}^k$.

Example. Let $\hat{\omega} = xu_{xx}du \wedge du_x$, then

$$\begin{aligned} D_x(\hat{\omega}) &= D_x(xu_{xx})du \wedge du_x + xu_{xx}D_x(du) \wedge du_x + xu_{xx}du \wedge D_x(du_x) \\ &= (u_{xx} + xu_{xxx})du \wedge du_x + xu_{xx}du_x \wedge du_x + xu_{xx}du \wedge du_{xx} \\ &= (u_{xx} + xu_{xxx})du \wedge du_x + xu_{xx}du \wedge du_{xx}. \end{aligned}$$

The middle term vanishes for the reason that $du^{a_1}_{J_1} \wedge du^{a_1}_{J_2} = 0$ if $J_1 = J_2$, same as in the de Rham complex.

Defining total derivatives over vertical forms allows us to extend the equivalence relation we defined on the space of differential functions to vertical k-forms. We define an equivalence relation on the space $\hat{\bigwedge}^k$ by saying $\hat{\omega} \sim \hat{\omega'}$ if they differ by a total divergence i.e.

$$\hat{\omega} = \hat{\omega'} + \text{Div } \hat{\eta} = \hat{\omega'} + \sum_{i=1}^{p} D_i\hat{\eta}_i \tag{2.1.12}$$

where $\hat{\eta}_i \in \hat{\bigwedge}^k$.

Definition 2.1.12. Let $\hat{\omega}, \hat{\omega'} \in \hat{\bigwedge}^k$ and let $\hat{\omega} \sim \hat{\omega'}$ if they satisfy (2.1.12). The space of equivalence classes denoted

$$\bigwedge^k_* = \hat{\bigwedge}^k / \text{Div}\Big(\hat{\bigwedge}^k\Big)^p \tag{2.1.13}$$

is the space of *functional k-forms.* The natural projection is denoted by an integral sign. So $\omega = \int \hat{\omega}\ dx$ stands for the equivalence class containing $\hat{\omega} \in \hat{\bigwedge}^k$. This notation is reasonable since the divergence theorem tells us that $\int \text{Div}(\hat{\eta}) = 0$ and a functional k-form has coefficients in $\mathscr{F}$.

This definition, along with the second rule of the total derivative above, allows us to state following formula for integration by parts

$$\int \hat{\omega} \wedge D_i(\hat{\eta})\ dx = -\int (D_i\hat{\omega}) \wedge \hat{\eta}\ dx \tag{2.1.14}$$

for $\hat{\omega} \in \hat{\bigwedge}^k$ and $\hat{\eta} \in \hat{\bigwedge}^l$.

Similar to vertical k-forms, functional k-forms determine an alternating k-linear map from the space T_0 of evolutionary vector fields to the space of functionals $\mathscr{F}$, defined so that

$$< \omega; \mathrm{v}_1, ..., \mathrm{v}_k >= \int < \hat{\omega}; \text{pr } \mathrm{v}_1, ..., \text{pr } \mathrm{v}_k >\ dx \quad \mathrm{v}_i \in T_0 \tag{2.1.15}$$

whenever $\omega = \int \hat{\omega}\ dx$, $\hat{\omega} \in \hat{\bigwedge}^k$. In fact, functional k-forms are uniquely determined by their action on T_0, this is stated precisely in the following lemma quoted from [4].

Lemma 2.1.2. If ω and ω' are functional k-forms, then $\omega = \omega'$ if and only if

$$< \omega, \mathrm{v}_1, ..., \mathrm{v}_k > = < \omega', \mathrm{v}_1, ..., \mathrm{v}_k >$$

for all evolutionary vector fields $\mathrm{v}_1, ...\mathrm{v}_k \in T_0$.

Now, we finish this background section by focusing on functional one-forms and two-forms. This is because they have a nice canonical form. Any functional one-form

$$\omega = \int \Big\{ \sum_{a,J} P_a^J[u] du_J^a \Big\} \, dx$$

can be written in multiple ways, this is due to the fact that we can write $du_J^a = D_J(du^a)$ and integrate by parts. We want to find a way to write the functional one-forms in a unique way and to do this to ω we integrate each summand by parts which leads to the expression

$$\omega = \int \Big\{ \sum_{a=1}^{q} P_a[u] du^a \Big\} \, dx = \int \Big\{ P \cdot u \Big\} \, dx \tag{2.1.16}$$

where $P_a = \sum_J (-D)_J P_a^J$. The expression (2.1.16) is called the *canonical form* of ω. Each functional one-form has a uniquely determined canonical form, this is stated in the following proposition we quote from [4].

Proposition 2.1.1. Let $\omega = \int \{P \cdot du\} \, dx$ and $\omega' = \int \{P' \cdot du\} \, dx$ be functional one-forms in canonical form with $P, P' \in \mathscr{A}^q$. Then $\omega = \omega'$ if and only if $P = P'$.

Then in a similar fashion we take a functional two-form and apply integration by parts on each summand of our two-form to be able to write it in a canonical form. The details of this can be found in section 5.4 of [4]. We omit these details at the moment since we will give an example in the next section and things should be clear then. For now, we write that if ω is a functional two-form, then ω has *canonical form*

$$\omega = \frac{1}{2} \int \{du \wedge \mathscr{D} du\} \, dx \tag{2.1.17}$$

where $\mathscr{D}$ is a skew-adjoint $q \times q$ matrix differential operator that depends on the coefficients in ω and its derivatives. Lastly, the canonical form above is unique to ω and we quote the following proposition from [4].

Proposition 2.1.2. Let $\omega = \frac{1}{2} \int \{du \wedge \mathscr{D} du\} \, dx$ and $\omega' = \frac{1}{2} \int \{du \wedge \mathscr{D}' du\} \, dx$ be functional two forms in canonical form, so $\mathscr{D}$ and $\mathscr{D}'$ are skew-adjoint $q \times q$ matrix differential operators. Then $\omega = \omega'$ if and only if $\mathscr{D} = \mathscr{D}'$.

2.2 Poisson Brackets and Hamiltonian Operators

In this section we will start by defining Poisson brackets on the space of functionals $\mathscr{F}$. A Poisson bracket of this kind is defined using a differential operator $\mathscr{D}$ satisfying certain properties. We will then define Hamiltonian operators as differential operators which define a true Poisson bracket by our definition. We will use this to explain what it means for an evolution equation to be Hamiltonian. The information in this section is again paraphrased or directly quoted from [4].

A Poisson bracket on $\mathscr{F}$ is infinite dimensional, so we shall begin by recalling Poisson brackets in finite dimensions and what it means for a system of ordinary differential equations to be Hamiltonian.

Definition 2.2.1. A Poisson Bracket on a smooth manifold M is an operation that assigns a smooth real-valued function $\{F, H\}$ on M to each pair of smooth, real-valued functions F, H, with the basic properties:

1. Bilinearity: $\{cF + P, H + c'T\} = c\{F, H\} + \{P, H\} + cc'\{F, T\} + c'\{P, T\}$ for $c, c' \in \mathbb{R}$.
2. Skew Symmetry: $\{F, H\} = -\{H, F\}$
3. Jacobi Identity: $\{\{F, H\}, P\} + \{\{P, F\}, H\} + \{\{H, P\}, F\} = 0$
4. Leibniz' Rule: $\{F, H \cdot P\} = \{F, H\} \cdot P + H \cdot \{F, P\}$

where F, H, P, and T are arbitrary smooth real valued functions on M. A manifold equipped with a Poisson bracket is called a *Poisson manifold*.

Definition 2.2.2. Let M be a Poisson manifold, let $x = (x^1, ..., x^m)$ be local coordinates on M. The basic brackets

$$J^{ij} = \{x^i, x^j\}, \quad i, j = 1, ..., m$$

are called the structure functions of M relative to the given local coordinates. We assemble the structure functions into a skew-symmetric $m \times m$ matrix $J(x)$ called the *structure matrix* for M.

The local coordinate form for the Poisson bracket is

$$\{F, H\} = \sum_{i=1}^{m} \sum_{j=1}^{m} \{x^i, x^j\} \frac{\partial F}{\partial x^i} \frac{\partial H}{\partial x^j} \tag{2.2.1}$$

thus by using ∇H to denote the column gradient vector for H, (2.2.1) can be written as

$$\{F, H\} = \nabla F \cdot J \nabla H. \tag{2.2.2}$$

Definition 2.2.3. Let M be a Poisson manifold and $H : M \to \mathbb{R}$ a smooth function. The *Hamiltonian vector field* associated with H is the unique smooth vector field $\hat{\mathrm{v}}_H$ on M satisfying

$$\hat{\mathrm{v}}_H(F) = \{F, H\}$$

for every smooth function $F : M \to \mathbb{R}$.

The *Hamiltonian flow* corresponding to H is obtained by exponentiating the vector field $\hat{\mathrm{v}}_H(F)$ and thus leads to a natural definition of a Hamiltonian system of ordinary differential equations.

Definition 2.2.4. A system of first order ordinary differential equations is said to be a *Hamiltonian system* if there exists a smooth function $H(x)$ and a skew-symmetric matrix of functions $J(x)$ that determine a Poisson bracket (2.2.2) whereby the system takes the form

$$\frac{dx}{dt} = J(x) \nabla H(x).$$

We wish to build on this idea and determine what it means for a system of partial differential equations to be Hamiltonian, more specifically what it means for a system of evolution equations

$$u_t = K[u] \quad K \in \mathscr{A}^q$$

to be Hamiltonian. Here K depends only on the spatial variables x and the spatial derivatives of u. Let $M \subset X \times U$ be an open subset of the space of independent and dependent variables $x = (x^1, ..., x^p)$ and $u = (u^1, ..., u^q)$. Recall $\mathscr{A}$ is the algebra of differential functions $P(x, u^{(n)}) = P[u]$ over M and $\mathscr{F}$ is the quotient space of $\mathscr{A}$ under the image of the total divergence. In analogy with the finite dimensional case, we define the Poisson bracket on the space of functionals to be to be a skew symmetric, bi-linear map from $\mathscr{F} \times \mathscr{F} \to \mathscr{F}$ that satisfies the Jacobi identity. We replace the gradient operation ∇H by the variational derivative $\delta\mathscr{H}$, for $\mathscr{H} \in \mathscr{F}$ and the skew-symmetric matrix $J(x)$ by a skew-adjoint linear differential operator $\mathscr{D} : \mathscr{A}^q \to \mathscr{A}^q$, leading to the expression for a Poisson bracket between $\mathscr{P} = \int P\, dx$ and $\mathscr{Q} = \int Q\, dx \in \mathscr{F}$ to be

$$\{\mathscr{P}, \mathscr{Q}\} = \int \delta\mathscr{P} \cdot \mathscr{D}\delta\mathscr{Q}\, dx. \tag{2.2.3}$$

In order for this to be a true Poisson bracket, meaning that its definition coincides with our definition in finite dimensions, we must put some restrictions on the operator $\mathscr{D}$. Specifically, we require that $\mathscr{D}$ be a Hamiltonian operator according to the following definition.

Definition 2.2.5. A linear operator $\mathscr{D} : \mathscr{A}^q \to \mathscr{A}^q$ is called *Hamiltonian* if its Poisson bracket (2.2.3) satisfies the following conditions

1. Skew-Symmetry:
$$\{\mathscr{P}, \mathscr{Q}\} = -\{\mathscr{Q}, \mathscr{P}\} \tag{2.2.4}$$
2. Jacobi Identity:
$$\{\{\mathscr{P}, \mathscr{Q}\}, \mathscr{R}\} + \{\{\mathscr{R}, \mathscr{P}\}, \mathscr{Q}\} + \{\{\mathscr{Q}, \mathscr{R}\}, \mathscr{P}\} = 0 \tag{2.2.5}$$

for all $\mathscr{P}, \mathscr{Q}, \mathscr{R} \in \mathscr{F}$.

The differences between the definition above and the finite-dimensional definition 2.2.1 are the lack of bilinearity and Leibniz' rule. We omit bilinearity since it is obvious from the form (2.2.3) as $\mathscr{D}$ and the variational derivative are linear. There is no Leibniz rule because there is no well defined multiplication rule between functionals.

Before we define what it means for an evolution equation to be Hamiltonian, we state a few propositions and definitions that we will need.

Proposition 2.2.1. Let $\mathbf{v}_Q$ be an evolutionary vector field, $Q \in \mathscr{A}^q$ and let $\mathscr{L}[u] = \int L(x, u^{(n)})\, dx \in \mathscr{F}$. Then,

$$\mathrm{pr}\, \mathbf{v}_Q(L) = Q \cdot E(L) + \mathrm{Div}\, A \tag{2.2.6}$$

where $A = (A_1, ..., A_p)$ is some p-tuple depending on Q, L and their derivatives.

Proof. Since v_Q is evolutionary, its prolongation is of the form:

$$\mathrm{pr}\, \mathbf{v}_Q(L) = \sum_{a,J} D_J(Q_a) \frac{\partial L}{\partial u^a_J}.$$

Using integration by parts, we get the following

$$\int \left\{ D_J(Q_a) \frac{\partial L}{\partial u^a_J} \right\} dx = Q_a \frac{\partial L}{\partial u^a_J} - \int \left\{ Q_a D_J(\frac{\partial L}{\partial u^a_J}) \right\} dx$$
$$\implies D_J(Q_a) \frac{\partial L}{\partial u^a_J} = -Q_a D_J(\frac{\partial L}{\partial u^a_J}) + D_J(Q_a \frac{\partial L}{\partial u^a_J}),$$

and thus we have

$$\begin{aligned}\text{pr }\mathbf{v}_Q(L) = \sum_{a,J} D_J(Q_a)\frac{\partial L}{\partial u^a_J} &= \sum_{a=1}^{q} Q_a \sum_J (-D)_J(\frac{\partial L}{\partial u^a_J}) + \text{Div}A \\ &= \sum_{a=1}^{q} Q_a E_a(L) + \text{Div}A \\ &= Q \cdot E(L) + \text{Div}A\end{aligned}$$

where A is a p-tuple of differential functions that depends on Q, L and their derivatives. The precise form of A is not required in this thesis so we omit it, however it is provided in section 5.4 of [4]. □

Proposition 2.2.2. Let $\mathscr{D}$ be a Hamiltonian Operator with Poisson bracket (2.2.3). For each functional $\mathscr{H} = \int H\ dx \in \mathscr{F}$, there is an evolutionary vector field $\hat{\text{v}}_{\mathscr{H}}$ called the *Hamiltonian vector field associated with* $\mathscr{H}$, which satisfies

$$\text{pr }\hat{\text{v}}_{\mathscr{H}}(\mathscr{P}) = \{\mathscr{P}, \mathscr{H}\}$$

for all functionals $\mathscr{P} \in \mathscr{F}$. The characteristic of $\hat{\text{v}}_{\mathscr{H}}$ is $\mathscr{D}\delta\mathscr{H} = \mathscr{D}E(H)$.

Proof.
Let $\mathscr{P} = \int P\ dx$, $P \in \mathscr{A}$ and let $\mathscr{H} = \int H \in \mathscr{F}$. The Poisson bracket between these two functionals is given by

$$\{\mathscr{P}, \mathscr{H}\} = \int \delta\mathscr{P} \cdot \mathscr{D}\delta\mathscr{H}\ dx = \int E(P)\mathscr{D}E(H)\ dx.$$

Using proposition 2.2.1. we can write the last integral in the following way

$$\int E(P)\mathscr{D}E(H)\ dx = \int \Big\{\text{pr }\mathbf{v}_{\mathscr{D}E(H)}(P) - \text{Div }A\ \Big\}dx.$$

Since we are working in $\mathscr{F}$ and two functional are the same if they differ by a total divergence, we obtain

$$\int \Big\{\text{pr }\mathbf{v}_{\mathscr{D}E(H)}(P) - \text{Div }A\Big\}\ dx = \int \text{pr }\mathbf{v}_{\mathscr{D}E(H)}(P)\ dx = \text{pr }\mathbf{v}_{\mathscr{D}E(H)}\Big(\int P\ dx\Big).$$

Therefore

$$\{\mathscr{P}, \mathscr{H}\} = \text{pr }\mathbf{v}_{\mathscr{D}E(H)}(\mathscr{P}) = \text{pr }\hat{\text{v}}_{\mathscr{H}}(\mathscr{P}).$$

□

In analogy with the finite dimensional case, we wish to define the Hamiltonian flow corresponding to $\mathscr{H}[u]$ as the flow that is obtained by exponentiating the corresponding Hamiltonian vector field $\hat{\text{v}}_{\mathscr{H}}$. This leads us to a natural definition of a Hamiltonian system of evolution equations.

Definition 2.2.6. A system of evolution equations

$$\frac{\partial u}{\partial t} = K[u] \quad K \in \mathscr{A}^q$$

is *Hamiltonian* if there exists a functional $\mathscr{H}[u] = \int H\ dx$, $H \in \mathscr{A}$ and a Hamiltonian operator $\mathscr{D}$ such that the equation can be written in the form

$$\frac{\partial u}{\partial t} = \mathscr{D} \cdot \delta\mathscr{H}$$

where δ is the variational derivative. Here $\mathscr{H}$ is called a *Hamiltonian functional.*

Out of the large class of differential operators, we would like to know which ones satisfy our definition of Hamiltonian. Thus we will discuss some properties that will help us in discerning if a differential operator is Hamiltonian.

Proposition 2.2.3. Let $\mathscr{D}$ be a $q \times q$ matrix operator with bracket (2.2.3) on $\mathscr{F}$. Then the bracket is skew-symmetric if and only if $\mathscr{D}$ is skew-adjoint.

Proof.

The bracket being skew-symmetric implies

$$\{\mathscr{P}, \mathscr{Q}\} = -\{\mathscr{Q}, \mathscr{P}\} \quad \forall \mathscr{Q}, \mathscr{P} \in \mathscr{F}$$
$$\implies \int E(P)\mathscr{D}E(Q)\, dx = -\int E(Q)\mathscr{D}E(P)\, dx$$

We rewrite the right hand side of the equation using the definition 2.1.7 of the adjoint $\mathscr{D}^*$.

$$\implies \int E(P)\mathscr{D}E(Q)\, dx = -\int E(P)\mathscr{D}^*E(Q)\, dx$$
$$\implies \int E(P)(\mathscr{D} + \mathscr{D}^*)E(Q)\, dx = 0$$

The integral above is 0 for all $Q, P \in \mathscr{A}$ if and only if $\mathscr{D}^* + \mathscr{D} = 0$ which implies $\mathscr{D}^* = -\mathscr{D}$ and therefore $\mathscr{D}$ is skew-adjoint. □

Example 2.2.1. The KdV equation

$$u_t = u_{xxx} + uu_x$$

is Hamiltonian and can be written in two distinct Hamiltonian forms. The first form is

$$u_t = \mathscr{D}\delta\mathscr{H}_1$$

where $\mathscr{D} = D_x$ and

$$\mathscr{H}_1 = \int \Big(-\frac{1}{2}u_x^2 + \frac{1}{6}u^3\Big)\, dx$$

with associated Poisson bracket

$$\{\mathscr{P}, \mathscr{Q}\} = \int \delta\mathscr{P} D_x(\delta\mathscr{Q})\, dx.$$

The second form is

$$u_t = \mathscr{E}\delta\mathscr{H}_0$$

where $\mathscr{E} = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x$ and

$$\mathscr{H}_0[u] = \int \frac{1}{2}u^2\, dx$$

with associated Poisson bracket

$$\{\mathscr{P}, \mathscr{Q}\} = \int \delta\mathscr{P}(D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x)\delta\mathscr{Q}\, dx.$$

The bracket is skew symmetric, since obviously $\mathscr{D} = D_x$ is skew-adjoint and $\mathscr{E} = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x$ was the differential operator from our example in the previous section where we showed the skew-symmetry of $\mathscr{E}$. To truly claim that these are Hamiltonian operators we must show the Jacobi identity (2.2.5) is satisfied. However, this computation is quite tedious and thus we will put this on hold until we can state an easier version of the Jacobi identity.

Before we dive into a few different versions of the Jacobi identity, we must introduce a few definitions and theorems to use in our proofs later on.

Definition 2.2.7. Let $P[u] \in \mathscr{A}^r$, the *Fréchet derivative* of P is the differential operator $\mathrm{D}_P : \mathscr{A}^q \to \mathscr{A}^r$ defined as

$$\mathrm{D}_P(Q) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} P[u + \epsilon Q] \tag{2.2.7}$$

for $\mathscr{Q} \in \mathscr{A}^q$.

Example. For $P = u_x u_{xx}$, $\mathrm{D}_P = u_{xx}D_x + u_x D_x^2$.

Let $P[u] \in \mathscr{A}^r$. If we evaluate D_P at $Q = (Q_1, ..., Q_q)$ a q-tuple of differential functions, we get the r-tuple with entries

$$(\mathrm{D}_P(Q))_i = \sum_{a=1}^{q} \sum_J \frac{\partial P_i}{\partial u_J^a} D_J(Q_a) \quad \text{for} \quad i = 1, ..., r \tag{2.2.8}$$

If $Q[u] = (Q_1, ..., Q_q)$ is the characteristic of an evolutionary vector field, then $\mathrm{pr}\,\mathbf{v}_Q = \sum_{a,J} D_J(Q_a)\frac{\partial}{\partial u_J^a}$. If we apply the prolongation to the r-tuple of differential functions $P = (P_1, ..., P_r)$ we obtain the r-tuple with entries

$$(\mathrm{pr}\,\mathbf{v}_Q(P))_i = \sum_{a=1}^{q} \sum_J D_J(Q_a)\frac{\partial P_i}{\partial u_J^a}. \quad \text{for} \;\; i = 1, ..., r. \tag{2.2.9}$$

Comparing (2.2.8) and (2.2.9), we conclude that

$$\mathrm{D}_P(Q) = \mathrm{pr}\,\mathbf{v}_Q(P) \tag{2.2.10}$$

for $P[u] \in \mathscr{A}^r$ and $Q[u] \in \mathscr{A}^q$. We also remark that we can find the adjoint of the Fréchet derivative as well. If $P[u] \in \mathscr{A}^r$, the adjoint $\mathrm{D}_P^* : \mathscr{A}^r \to \mathscr{A}^q$ is a $r \times q$ matrix differential operator with entries

$$(\mathrm{D}_P^*)_{ji} = \sum_J (-D)_J \cdot \frac{\partial P_j}{\partial u_J^j} \quad j = 1, ..., q \;\text{ and }\; i = 1, ..., r.$$

Theorem 2.2.1. Let $P[u] \in \mathscr{A}^p$ be defined over a vertically starshaped region $M \subset X \times U$. Then $P = E(L)$ for some variational problem $\mathscr{L}[u] = \int L\, dx$ if and only if the Fréchet derivative D_P is self adjoint. i.e. $\mathrm{D}_P = \mathrm{D}_P^*$.

We omit the proof of this theorem, it can be found in section 5.4 of [4].

In finite dimensions, we can evaluate the Lie derivative of certain geometric objects, such as functions and differential forms with respect to a vector field $\mathbf{v}$. This represents the infinitesimal change in the object under the flow $\exp(\epsilon\mathbf{v})$ induced by $\mathbf{v}$. An analogous concept exists for generalized vector fields, we are particularly interested in the Lie derivative of a differential operator $\mathscr{D}$ with respect to an evolutionary vector field.

Definition 2.2.8. Let $\mathbf{v}_Q$ be an evolutionary vector field and $\mathscr{D} = \sum P_K[u]D_K$ a differential operator. The *Lie derivative* of $\mathscr{D}$ is a differential operator computed by evaluating the time derivative of $\mathscr{D}$

$$\mathscr{D}_t = \sum_K D_t(P_K)D_K$$

on solutions $u_t = Q$, which leads to the formula

$$\mathrm{pr}\,\mathbf{v}_Q(\mathscr{D}) = \sum_K \mathrm{pr}\,\mathbf{v}_Q(P_K)D_K. \tag{2.2.11}$$

The Lie derivative extends to matrix differential operators by having $\mathrm{pr}\,\mathbf{v}_Q$ act on the individual entries of the matrix. Furthermore, the Lie derivative satisfies the following Leibniz rule

$$\mathrm{pr}\,\mathbf{v}_Q(\mathscr{D}P) = \mathrm{pr}\,\mathbf{v}_Q(\mathscr{D})P + \mathscr{D}(\mathrm{pr}\,\mathbf{v}_Q(P)). \tag{2.2.12}$$

We will use the definition of the Fréchet derivative and the remarks made above in the proof of the following theorem. This theorem provides an easier formula for satisfying the Jacobi identity.

Theorem 2.2.2. Let $\mathscr{D}$ be a skew-adjoint $q \times q$ matrix differential operator. The bracket

$$\{\mathscr{P}, \mathscr{Q}\} = \int \delta\mathscr{P} \cdot \mathscr{D}\delta\mathscr{Q}\, dx.$$

satisfies the Jacobi identity if and only if

$$\int \Big\{P \cdot \mathrm{pr}\,\mathbf{v}_{\mathscr{D}R}(\mathscr{D})Q + R \cdot \mathrm{pr}\,\mathbf{v}_{\mathscr{D}Q}(\mathscr{D})P + Q \cdot \mathrm{pr}\,\mathbf{v}_{\mathscr{D}P}(\mathscr{D})R\Big\}\, dx = 0 \tag{2.2.13}$$

for all q-tuples $P, Q, R \in \mathscr{A}^q$.

Proof.
Let $\mathscr{P}, \mathscr{Q}, \mathscr{R}$ be functionals, with variational derivatives $\delta\mathscr{P} = P$, $\delta\mathscr{Q} = Q$ and $\delta\mathscr{R} = R$. The first term in the Jacobi identity (2.2.5) is

$$\begin{aligned}
\{\{\mathscr{P}, \mathscr{Q}\}, \mathscr{R}\} &= \text{pr } \hat{\text{v}}_{\mathscr{R}}(\{\mathscr{P}, \mathscr{Q}\}) \\
&= \text{pr } \hat{\text{v}}_{\mathscr{R}}\Big(\int P \cdot \mathscr{D}(Q)\, dx \Big) \\
&= \int \text{pr } \text{v}_{\mathscr{D}(R)}(P \cdot \mathscr{D}Q)\, dx \\
&= \int \Big\{ \text{pr } \text{v}_{\mathscr{D}(R)}(P) \cdot \mathscr{D}Q + P \cdot \text{pr } \text{v}_{\mathscr{D}(R)}(\mathscr{D}Q) \Big\}\, dx \\
&= \int \Big\{ \text{pr } \text{v}_{\mathscr{D}(R)}(P) \cdot \mathscr{D}Q + P \cdot \text{pr } \text{v}_{\mathscr{D}(R)}(\mathscr{D})Q + P \cdot \mathscr{D}(\text{pr } \text{v}_{\mathscr{D}(R)}(Q)) \Big\}\, dx \quad \text{by (2.2.12)} \\
&= \int \Big\{ \text{D}_P(\mathscr{D}R) \cdot \mathscr{D}Q + P \cdot \text{pr } \text{v}_{\mathscr{D}(R)}(\mathscr{D})Q + P \cdot \mathscr{D}(\text{pr } \text{v}_{\mathscr{D}(R)}(Q)) \Big\}\, dx \quad \text{by (2.2.10)} \\
&= \int \Big\{ \text{D}_P(\mathscr{D}R) \cdot \mathscr{D}Q + P \cdot \text{pr } \text{v}_{\mathscr{D}(R)}(\mathscr{D})Q + \text{pr } \text{v}_{\mathscr{D}(R)}(Q)) \cdot \mathscr{D}^* P \Big\}\, dx \quad \text{by (2.1.1)}
\end{aligned}$$

By our assumption that $\mathscr{D}$ is skew-adjoint, we can rewrite last term in the integral and obtain:

$$\begin{aligned}
\{\{\mathscr{P}, \mathscr{Q}\}, \mathscr{R}\} &= \int \Big\{ \text{D}_P(\mathscr{D}R) \cdot \mathscr{D}Q + P \cdot \text{pr } \text{v}_{\mathscr{D}(R)}(\mathscr{D})Q - \text{pr } \text{v}_{\mathscr{D}(R)}(Q)) \cdot \mathscr{D}P \Big\}\, dx \\
&= \int \Big\{ \text{D}_P(\mathscr{D}R) \cdot \mathscr{D}Q + P \cdot \text{pr } \text{v}_{\mathscr{D}(R)}(\mathscr{D})Q - \text{D}_Q(\mathscr{D}R) \cdot \mathscr{D}P \Big\}\, dx \quad \text{by (2.2.10).}
\end{aligned}$$

We can do this for the other two brackets as well.

$$\begin{aligned}
\{\{\mathscr{R}, \mathscr{P}\}, \mathscr{Q}\} &= \int \Big\{ \text{D}_R(\mathscr{D}Q) \cdot \mathscr{D}P + R \cdot \text{pr } \text{v}_{\mathscr{D}(Q)}(\mathscr{D})P - \text{D}_P(\mathscr{D}Q) \cdot \mathscr{D}R \Big\}\, dx \\
\{\{\mathscr{Q}, \mathscr{R}\}, \mathscr{P}\} &= \int \Big\{ \text{D}_Q(\mathscr{D}P) \cdot \mathscr{D}R + Q \cdot \text{pr } \text{v}_{\mathscr{D}(P)}(\mathscr{D})R - \text{D}_R(\mathscr{D}P) \cdot \mathscr{D}Q \Big\}\, dx
\end{aligned}$$

Since $\delta\mathscr{P} = P$, $\delta\mathscr{Q} = Q$ and $\delta\mathscr{R} = R$, theorem 2.2.1 tells us that D_P, D_Q and D_R are self-adjoint. This allows us to write the brackets in the following way:

$$\begin{aligned}
\{\{\mathscr{P}, \mathscr{Q}\}, \mathscr{R}\} &= \int \Big\{ \text{D}_P(\mathscr{D}R) \cdot \mathscr{D}Q + P \cdot \text{pr } \text{v}_{\mathscr{D}(R)}(\mathscr{D})Q - \mathscr{D}R \cdot \text{D}_Q(\mathscr{D}P) \Big\}\, dx, \\
\{\{\mathscr{R}, \mathscr{P}\}, \mathscr{Q}\} &= \int \Big\{ \text{D}_R(\mathscr{D}Q) \cdot \mathscr{D}P + R \cdot \text{pr } \text{v}_{\mathscr{D}(Q)}(\mathscr{D})P - \mathscr{D}Q \cdot \text{D}_P(\mathscr{D}R) \Big\} dx, \\
\{\{\mathscr{Q}, \mathscr{R}\}, \mathscr{P}\} &= \int \Big\{ \text{D}_Q(\mathscr{D}P) \cdot \mathscr{D}R + Q \cdot \text{pr } \text{v}_{\mathscr{D}(QP}(\mathscr{D})R - \mathscr{D}P \cdot \text{D}_R(\mathscr{D}Q) \Big\}\, dx.
\end{aligned}$$

Adding all the integrals above together to form the left hand side of the Jacobi identity we obtain some cancellations in the first and third terms of the integrals above. The last term in the first integral cancels with the first term of the last integral, the last term in the second integral cancels with the first term in the first integral and the last term in the last integral cancels with the first term of the second integral. This gives us the following

$$\begin{aligned}
\{\{\mathscr{P}, \mathscr{Q}\}, \mathscr{R}\} + \{\{\mathscr{R}, \mathscr{P}\}, \mathscr{Q}\} + \{\{\mathscr{Q}, \mathscr{R}\}, \mathscr{P}\} = \int \Big\{ & P \cdot \text{pr } \mathbf{v}_{\mathscr{D}R}(\mathscr{D})Q + R \cdot \text{pr } \mathbf{v}_{\mathscr{D}Q}(\mathscr{D})P \\
& + Q \cdot \text{pr } \mathbf{v}_{\mathscr{D}P}(\mathscr{D})R \Big\}\, dx.
\end{aligned}$$

If the equality above is equal to 0 on all variational derivatives, it must be equal to 0 on all differential functions in $\mathscr{A}^q$. Since this integral depends only on P, Q, R and their total derivatives, we obtain that this vanishes on all variational derivatives if and only if it vanishes on all q-tuples of differential functions $P, Q, R \in \mathscr{A}^q$. Therefore the Jacobi identity is satisfied if and only if

$$\{\{\mathscr{P},\mathscr{Q}\},\mathscr{R}\}+\{\{\mathscr{R},\mathscr{P}\},\mathscr{Q}\}+\{\{\mathscr{Q},\mathscr{R}\},\mathscr{P}\}=0 \quad \text{for all functionals } \mathscr{P},\mathscr{Q},\mathscr{R}\in\mathscr{F}$$

$$\iff \int \Big\{P\cdot \mathrm{pr}\,\mathbf{v}_{\mathscr{D}R}(\mathscr{D})Q + R\cdot \mathrm{pr}\,\mathbf{v}_{\mathscr{D}Q}(\mathscr{D})P + Q\cdot \mathrm{pr}\,\mathbf{v}_{\mathscr{D}P}(\mathscr{D})R\Big\}\,dx = 0 \quad \forall\, \delta\mathscr{P}=P,\ \delta\mathscr{Q}=Q,\ \delta\mathscr{R}=R$$

$$\iff \int \Big\{P\cdot \mathrm{pr}\,\mathbf{v}_{\mathscr{D}R}(\mathscr{D})Q + R\cdot \mathrm{pr}\,\mathbf{v}_{\mathscr{D}Q}(\mathscr{D})P + Q\cdot \mathrm{pr}\,\mathbf{v}_{\mathscr{D}P}(\mathscr{D})R\Big\}\,dx = 0 \quad \forall P,Q,R\in\mathscr{A}^q.$$

□

The theorem above provides a nice formula for the computation of the Jacobi identity, however the right hand side of equation (2.2.13) still involves a very tedious computation.

Recall the vertical and functional k-forms we introduced in the previous subsection. A vertical k-form determines an alternating k-linear map from the space T_0 of evolutionary vector fields to the space $\mathscr{A}$ of differential functions. Similarly a functional k-form determines an alternating k-linear map from T_0 to the space $\mathscr{F}$ of functionals. Now we will introduce the idea of a vertical and functional multi-vector. A functional k-vector is defined like the dual object of a functional k-form, however it is important to note that they are not naturally dual vector spaces for $k > 1$. Functional k-forms and k-vectors have coefficients in $\mathscr{F}$ but $\mathscr{F}$ is not a ring, let alone a field due to our inability to define multiplication of functionals. Therefore, the space of functional k-forms and k-vectors cannot be dual vector spaces since they are not vector spaces to begin with.

A *vertical k-vector* is an alternating k-linear map from the space $\bigwedge_*^1$ of functional 1-forms to the space $\mathscr{A}$ of differential functions. As mentioned in the previous section, each functional one form can be uniquely determined by its canonical form, therefore we can identify $\bigwedge_*^1$ with $\mathscr{A}^q$. We also introduce a new notation and let θ_J^a denote the vertical 1-vector corresponding to the 1-form du_J^a, it defines the linear map

$$<\theta_J^a; P>= D_J(P_a) \quad \text{whenever} \quad P=(P_1,....,P_q)\in\mathscr{A}^q. \tag{2.2.14}$$

Similar to vertical k-forms the total derivatives act as Lie derivatives on the space of k-vectors with

$$D_i(\theta_J^a)=\theta_{J,i}^a.$$

This allows us to write certain vertical k-vectors as images of total divergence. Since each functional k-form arises from a vertical k-form, we would like that each functional k-vector arises from a vertical k-vector. Therefore, we define the space $\bigwedge_k^*$ of *functional k-vectors* to be the quotient space of vertical k-vectors under the image of total divergence. A general functional k-vector is thus a finite sum

$$\Theta=\int\Big\{\sum_{a,J} R_J^a[u]\ \theta_{J_1}^{a_1}\wedge ...\wedge \theta_{J_k}^{a_k}\Big\}\,dx$$

with $R_J^a\in\mathscr{A}$ and defines a k-linear map on the space of functional 1-forms $\bigwedge_*^1$

$$<\Theta;P^1,...,P^k>=\int<\sum_{a,J}R_J^a[u]\ \theta_{J_1}^{a_1}\wedge...\wedge\theta_{J_k}^{a_k},P^1,...,P^k>\ dx=\int\Big[\sum_{a,J}R_J^a\det(D_{J_i}P_{a_i}^j)\Big]\,dx \tag{2.2.15}$$

where $P^j\in\mathscr{A}^q$ (here we have replaced $\bigwedge_*^1$ by $\mathscr{A}^q$).

Example. For $\Theta=\int\{\theta\wedge\theta_{xxx}+u\theta\wedge\theta_x\}\,dx$

$$\begin{aligned}<\Theta;P,Q> &= \int\Big\{\begin{vmatrix}P & Q\\ D_x^3(P) & D_x^3(Q)\end{vmatrix}+u\cdot\begin{vmatrix}P & Q\\ D_x(P) & D_x(Q)\end{vmatrix}\Big\}\,dx\\ &=\int\Big\{PD_x^3(Q)-QD_x^3(P)+uPD_x(Q)-uQD_x(P)\Big\}\,dx.\end{aligned}$$

At this point, we must note the uncanny resemblance between functional k-vectors and functional k-forms. A functional k-form is defined on the space T_0 of evolutionary vector fields. Since each evolutionary vector field is uniquely determined by its characteristic [4], we can identity T_0 with $\mathscr{A}^q$ the space of q-tuples of differential functions on M. In [4] Olver mentions that all the theorems that hold for functional k-forms also hold for functional k-vectors, once we replace du^a_J by θ^a_J. In particular this allows us to give functional uni-vectors and bi-vectors a canonical form. Before we continue we must mention that in this thesis, the later sections will only depend on evolution equations with a single spatial variable x and a single dependent variable $u(x,t)$. Therefore, at this point we restrict to this situation.

In the situation where we have a single spatial variable x and a single dependent variable $u(x,t)$, a functional uni-vector will be of the form

$$\gamma = \int \Big\{ \sum_J R_J \theta_J \Big\}\, dx = \int \Big\{ \sum_J R_J D_J(\theta) \Big\}\, dx. \tag{2.2.16}$$

Then each term in the summand can be integrated by parts and put into *canonical form*

$$\gamma = \int \Big\{ R \cdot \theta \Big\}\, dx \quad \text{where} \quad R = \sum_J (-D)_J(R_J). \tag{2.2.17}$$

Similarly, any functional bi-vector has the canonical form

$$\Theta = \frac{1}{2} \int \Big\{ \theta \wedge \mathscr{D}\theta \Big\}\, dx \tag{2.2.18}$$

where $\mathscr{D}$ is a skew-adjoint differential operator.

Example. Let $\omega = \int \Big\{ \frac{1}{2}\theta \wedge \theta_{xxx} + \frac{1}{3} u\theta \wedge \theta_x \Big\}\, dx$, we can write ω in two ways

$$\omega = \int \Big\{ \theta \wedge (\frac{1}{2} D_x^3 + \frac{1}{3} u D_x)(\theta) \Big\}\, dx$$

$$\begin{aligned} \omega &= \int \Big\{ -\frac{1}{2} D_x^3(\theta) \wedge \theta - (\frac{1}{3} u_x \theta + \frac{1}{3} u \theta_x) \wedge \theta \Big\}\, dx \\ &= \int \Big\{ -(\frac{1}{2} D_x^3 + \frac{1}{3} u D_x + \frac{1}{3} u_x)(\theta) \wedge \theta \Big\}\, dx \end{aligned}$$

$$\begin{aligned} \implies \omega &= \frac{1}{2} \int \Big\{ \theta \wedge (\frac{1}{2} D_x^3 + \frac{1}{3} u D_x)(\theta) \Big\}\, dx + \frac{1}{2} \int \Big\{ -(\frac{1}{2} D_x^3 + \frac{1}{3} u D_x + \frac{1}{3} u_x)(\theta) \wedge \theta \Big\}\, dx \\ \omega &= \frac{1}{2} \int \Big\{ \theta \wedge (D_x^3 + \frac{2}{3} u D_x + \frac{1}{3} u_x)(\theta) \Big\}\, dx \\ \omega &= \frac{1}{2} \int \Big\{ \theta \wedge \mathscr{E}(\theta) \Big\}\, dx \end{aligned}$$

where $\mathscr{E} = D_x^3 + \frac{2}{3} u D_x + \frac{1}{3} u_x$ is the skew-adjoint operator for the KdV equation we mentioned earlier. It is important to notice that the operator $\mathscr{E} = D_x^3 + \frac{2}{3} u D_x + \frac{1}{3} u_x$, was created by taking an operator $\mathscr{D} = \frac{1}{2} D_x^3 + \frac{1}{2} u D_x$ and subtracting its adjoint $\mathscr{D}^* = -\frac{1}{2} D_x^3 - \frac{1}{2} u D_x - \frac{1}{2} u_x$ thus giving us a skew-adjoint differential operator.

A functional bi-vector in canonical form (2.2.18), defines a bilinear map for $P, Q \in \mathscr{A}$

$$< \Theta; P, Q > = \frac{1}{2} \int (P \cdot \mathscr{D} Q - Q \mathscr{D} P)\, dx = \frac{1}{2} \int (P \cdot \mathscr{D} Q - P \mathscr{D}^* Q)\, dx = \int P \cdot \mathscr{D} Q\, dx \tag{2.2.19}$$

where we used the skew-adjoint property of $\mathscr{D}$ to get the last equality. Notice that if we let $P = \delta \mathscr{P}$, $Q = \delta \mathscr{Q}$, for $\mathscr{P}, \mathscr{Q} \in \mathscr{F}$, then

$$< \Theta; \delta\mathscr{P}, \delta\mathscr{Q} > = \int (\delta\mathscr{P} \cdot \mathscr{D} \delta\mathscr{Q})\, dx \tag{2.2.20}$$

reproduces the Poisson bracket $\{\mathscr{P}, \mathscr{Q}\}$ for the skew-adjoint operator $\mathscr{D}$.

The reader might start to notice at this point that the left hand side of the Jacobi identity in the form

$$\int \Big\{ P \cdot \text{pr } \mathbf{v}_{\mathscr{D}R}(\mathscr{D})Q + R \cdot \text{pr } \mathbf{v}_{\mathscr{D}Q}(\mathscr{D})P + Q \cdot \text{pr } \mathbf{v}_{\mathscr{D}P}(\mathscr{D})R \Big\}\, dx = 0$$

is an alternating, tri-linear function of $P, Q, R \in \mathscr{A}$. This determines a functional tri-vector, which we will denote by

$$\Psi = \frac{1}{2} \int \{\theta \wedge \text{pr } \mathrm{v}_{\mathscr{D}\theta}(\mathscr{D}) \wedge \theta\}\, dx$$

where $< \Psi; P, Q, R >$ is the left hand side of the Jacobi identity above. Here $\mathrm{v}_{\mathscr{D}\theta}$ is the evolutionary vector field with $\mathscr{D}\theta$ as it's characteristic. If $\mathscr{D} = \sum_K P_K[u] D_K$ is a skew-adjoint differential operator then

$$\text{pr } \mathrm{v}_{\mathscr{D}\theta}(\mathscr{D}) = \sum_K \text{pr } \mathrm{v}_{\mathscr{D}\theta}(P_k) D_K \tag{2.2.21}$$

is a skew-adjoint differential operator [4]. The coefficients here are functional uni-vectors in that they involve the θ_J's. The notation for Ψ is deceiving, so we will run through an example to better understand the integral Ψ.

Example 2.2.2. Let $\mathscr{E} = D_x^3 + \frac{2}{3} u D_x + \frac{1}{3} u_x$ be the skew-adjoint operator from the KdV example.

$$\begin{aligned}
\text{pr } \mathrm{v}_{\mathscr{E}\theta}(\mathscr{E}) &= \text{pr } \mathrm{v}_{\mathscr{E}\theta}(1) D_x^3 + \frac{2}{3} \text{pr } \mathrm{v}_{\mathscr{E}\theta}(u) D_x + \frac{1}{3} \text{pr } \mathrm{v}_{\mathscr{E}\theta}(u_x) \\
&= \frac{2}{3} \mathscr{E}\theta \cdot D_x + \frac{1}{3} D_x(\mathscr{E}\theta) \\
&= \frac{2}{3}(\theta_{xxx} + \frac{2}{3} u\theta_x + \frac{1}{3} u_x \theta) \cdot D_x + \frac{1}{3}(\theta_{xxxx} + \frac{2}{3} u\theta_{xx} + u_x\theta_x + \frac{1}{3} u_{xx}\theta)
\end{aligned}$$

Now the second wedge in Ψ, is telling us to apply $\text{pr } \mathrm{v}_{\mathscr{E}\theta}(\mathscr{E})$ to θ by evaluating the differential operator $\text{pr } \mathrm{v}_{\mathscr{E}\theta}(\mathscr{E})$ at θ and then wedging the coefficients of the operator since they are functional uni-vectors. Thus the tri-vector Ψ for $\mathscr{E}$ is

$$\begin{aligned}
\Psi &= \frac{1}{2} \int \{\theta \wedge \text{pr } \mathrm{v}_{\mathscr{D}\theta}(\mathscr{D}) \wedge \theta\}\, dx \\
&= \frac{1}{2} \int \Big\{ \theta \wedge \Big(\frac{2}{3}(\theta_{xxx} + \frac{2}{3} u\theta_x + \frac{1}{3} u_x\theta) \wedge D_x(\theta) + \frac{1}{3}(\theta_{xxxx} + \frac{2}{3} u\theta_{xx} + u_x\theta_x + \frac{1}{3} u_{xx}\theta) \wedge \theta \Big) \Big\}\, dx \\
&= \int \Big\{ \frac{1}{3}\theta \wedge (\theta_{xxx} + \frac{2}{3} u\theta_x + \frac{1}{3} u_x\theta) \wedge \theta_x + \frac{1}{6}\theta \wedge (\theta_{xxxx} + \frac{2}{3} u\theta_{xx} + u_x\theta_x + \frac{1}{3} u_{xx}\theta) \wedge \theta \Big\}\, dx \\
&= \int -\frac{1}{3}\theta \wedge \theta_x \wedge \theta_{xxx}\, dx
\end{aligned}$$

where we used the skew-symmetry of the wedge product and the fact that $\theta_{J_1} \wedge \theta_{J_2} = 0$ if $J_1 = J_2$. We can further simplify the integral, by integrating by parts

$$\begin{aligned}
\int -\frac{1}{3}\theta \wedge \theta_x \wedge \theta_{xxx}\, dx &= \frac{1}{3} \int D_x(\theta \wedge \theta_x) \wedge \theta_{xx}\, dx \\
&= \frac{1}{3} \int \Big\{ \theta_x \wedge \theta_x \wedge \theta_{xx} + \theta \wedge \theta_{xx} \wedge \theta_{xx} \Big\}\, dx \\
&= 0.
\end{aligned}$$

Therefore the tri-vector Ψ is trivial for $\mathscr{E}$, this is a unique property of $\mathscr{E}$ being Hamiltonian, not all differential operators will give a trivial tri-vector Ψ.

Using the example above as a guide for how to compute the tri-vector Ψ, we state a proposition that gives us a new way of computing the Jacobi identity.

Proposition 2.2.4. Let $\mathscr{D}$ be a skew-adjoint differential operator. Then $\mathscr{D}$ is Hamiltonian if and only if the functional tri-vector

$$\Psi = \frac{1}{2}\int \{\theta \wedge \text{pr v}_{\mathscr{D}\theta}(\mathscr{D}) \wedge \theta\}\, dx \tag{2.2.22}$$

is trivial i.e. $\Psi = 0$.

Proof.
Recall that all the theorems we mentioned earlier about functional k-forms carry over to functional k-vectors. Therefore we can say that the functional tri-vector Ψ is uniquely determined by its action on differential functions P, Q, R. Thus $\Psi = 0$ if and only if $< \Psi; P, R, Q >= 0$ for all $P, R, Q \in \mathscr{A}$. Following the same notation for Ψ that we used in the example above, if we evaluate Ψ on $P, Q, R \in \mathscr{A}$ we obtain

$$\begin{aligned} < \Psi; P, Q, R >= \frac{1}{2}\int \Big\{ & P \cdot \text{pr }\mathbf{v}_{\mathscr{D}R}(\mathscr{D})Q - Q \cdot \text{pr }\mathbf{v}_{\mathscr{D}R}(\mathscr{D})P + R \cdot \text{pr }\mathbf{v}_{\mathscr{D}Q}(\mathscr{D})P - P \cdot \text{pr }\mathbf{v}_{\mathscr{D}Q}(\mathscr{D})R \\ & + Q \cdot \text{pr }\mathbf{v}_{\mathscr{D}P}(\mathscr{D})R - R \cdot \text{pr }\mathbf{v}_{\mathscr{D}P}(\mathscr{D})Q \Big\}\, dx. \end{aligned}$$

Since $\mathscr{D}$ is skew-adjoint so is pr $\mathrm{v}_Q(\mathscr{D})$ for any evolutionary vector field v_Q. Thus we replace the prolonged operator above by its adjoint to obtain

$$\begin{aligned} < \Psi; P, Q, R > &= \frac{1}{2}\int \Big\{ P \cdot \text{pr }\mathbf{v}_{\mathscr{D}R}(\mathscr{D})Q - P \cdot [\text{pr }\mathbf{v}_{\mathscr{D}R}(\mathscr{D})]^* Q + R \cdot \text{pr }\mathbf{v}_{\mathscr{D}Q}(\mathscr{D})P - R \cdot [\text{pr }\mathbf{v}_{\mathscr{D}Q}(\mathscr{D})]^* P \\ &\qquad + Q \cdot \text{pr }\mathbf{v}_{\mathscr{D}P}(\mathscr{D})R - Q \cdot [\text{pr }\mathbf{v}_{\mathscr{D}P}(\mathscr{D})]^* R \Big\}\, dx \\ &= \frac{1}{2}\int \Big\{ P \cdot \text{pr }\mathbf{v}_{\mathscr{D}R}(\mathscr{D})Q + P \cdot \text{pr }\mathbf{v}_{\mathscr{D}R}(\mathscr{D})Q + R \cdot \text{pr }\mathbf{v}_{\mathscr{D}Q}(\mathscr{D})P + R \cdot \text{pr }\mathbf{v}_{\mathscr{D}Q}(\mathscr{D})P \\ &\qquad + Q \cdot \text{pr }\mathbf{v}_{\mathscr{D}P}(\mathscr{D})R + Q \cdot \text{pr }\mathbf{v}_{\mathscr{D}P}(\mathscr{D})R \Big\}\, dx \\ &= \int \Big\{ P \cdot \text{pr }\mathbf{v}_{\mathscr{D}R}(\mathscr{D})Q + R \cdot \text{pr }\mathbf{v}_{\mathscr{D}Q}(\mathscr{D})P + Q \cdot \text{pr }\mathbf{v}_{\mathscr{D}P}(\mathscr{D})R \Big\}\, dx \end{aligned}$$

the left hand side of (2.2.13), the second version of our Jacobi identity. Therefore given a skew-adjoint operator $\mathscr{D}$, $\mathscr{D}$ is Hamiltonian if and only if $< \Psi; P, Q, R >= 0$ for all $P, Q, R \in \mathscr{A}$, however this is true if and only if $\Psi = 0$. Therefore the triviality of Ψ is equivalent to satisfying the Jacobi identity. $\square$

This proposition leaves us with an easier way to compute the Jacobi identity. However, it might be easier to use the functional bi-vector (2.2.18) which determines the Poisson bracket (2.2.20) in our computation of the Jacobi identity. In order to do this we extend the definition of a prolonged vector field pr $\mathrm{v}_{\mathscr{D}\theta}$ to the space of vertical uni-vectors by setting

$$\text{pr v}_{\mathscr{D}\theta}(\theta_J) = 0$$

for all multi-indices J and we require it to act on vertical multi-vectors as a derivation. Therefore if $\Phi = \int \tilde{\Phi}\, dx$ is any functional k-vector , then

$$\text{pr v}_{\mathscr{D}\theta}(\Phi) = \int \text{pr v}_{\mathscr{D}\theta}(\tilde{\Phi})\, dx \tag{2.2.23}$$

is a functional $(k+1)$-vector. Now we are ready to introduce the last version of the Jacobi identity. This theorem will be the one used the most in this thesis to show the Jacobi identity is satisfied.

Theorem 2.2.3. Let $\mathscr{D}$ be a skew-adjoint differential operator, and let $\Theta = \frac{1}{2}\int \{\theta \wedge \mathscr{D}\theta\}\, dx$ be the corresponding functional bi-vector. Then $\mathscr{D}$ is Hamiltonian if and only if

$$\text{pr v}_{\mathscr{D}\theta}(\Theta) = 0 \tag{2.2.24}$$

Proof.
Let $\mathscr{D}$ be a skew-adjoint operator, then $\mathscr{D} = \sum_K P_K[u]D_K$. Evaluating the left hand side of (2.2.24) gives

$$\begin{aligned}
\text{pr v}_{\mathscr{D}\theta}(\Theta) = \text{pr v}_{\mathscr{D}\theta}\Big(\frac{1}{2}\int \theta \wedge \mathscr{D}\theta \; dx\Big) &= \frac{1}{2}\int \text{pr v}_{\mathscr{D}\theta}(\theta \wedge \mathscr{D}\theta) \; dx \\
&= \frac{1}{2}\int \Big\{\text{pr v}_{\mathscr{D}\theta}\Big(\theta \wedge (\sum_K P_K\theta_K)\Big)\Big\} \; dx \\
&= \frac{1}{2}\int \Big\{\text{pr v}_{\mathscr{D}\theta}\Big(\sum_K P_K\theta \wedge \theta_K\Big)\Big\} \; dx \\
&= \frac{1}{2}\int \Big\{\sum_K \text{pr v}_{\mathscr{D}\theta}(P_K) \wedge \theta \wedge \theta_K\Big\} \; dx \\
&= \frac{1}{2}\int \Big\{-\theta \wedge \Big(\sum_K \text{pr v}_{\mathscr{D}\theta}(P_k) \wedge \theta_K\Big)\Big\} \; dx \\
&= \frac{1}{2}\int \Big\{-\theta \wedge \text{pr v}_{\mathscr{D}\theta}(\mathscr{D}) \wedge \theta\Big\} \; dx \\
&= -\Psi
\end{aligned}$$

where Ψ is the same as in proposition 2.2.4. Therefore by proposition 2.2.4, we conclude that $\mathscr{D}$ is Hamiltonian if and only if $\text{pr v}_{\mathscr{D}\theta}(\Theta) = 0$. □

Example 2.2.3. Now using the theorem above, we can finally return to our example of the KdV equation. It remains to prove that the operators $\mathscr{D} = D_x$ and $\mathscr{E} = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x$ satisfy the Jacobi identity. We will show that (2.2.24) is satisfied for both operators. Starting with $\mathscr{D} = D_x$,

$$\text{pr v}_{\mathscr{D}\theta}(\Theta) = \int \text{pr v}_{\theta_x}(\theta \wedge \theta_x) \; dx = 0.$$

Let us do the same for $\mathscr{E} = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x$. Recall from our earlier example,

$$\Theta = \frac{1}{2}\int \theta \wedge \mathscr{E}(\theta) \; dx = \int \Big\{\frac{1}{2}\theta \wedge \theta_{xxx} + \frac{1}{3}u\theta \wedge \theta_x\Big\} \; dx.$$

Then verifying (2.2.24) we get

$$\begin{aligned}
\text{pr v}_{\mathscr{E}\theta}\Big(\int \frac{1}{2}\theta \wedge \theta_{xxx} + \frac{1}{3}u\theta \wedge \theta_x \; dx\Big) &= \int \Big\{\text{pr v}_{\mathscr{E}\theta}(\frac{1}{2}\theta \wedge \theta_{xxx} + \frac{1}{3}u\theta \wedge \theta_x)\Big\} \; dx \\
&= \int \frac{1}{3}\text{pr v}_{\mathscr{E}\theta}(u) \wedge \theta \wedge \theta_x \; dx \\
&= \frac{1}{3}\int \mathscr{E}(\theta) \wedge \theta \wedge \theta_x \; dx \\
&= \frac{1}{3}\int \Big\{\theta_{xxx} \wedge \theta \wedge \theta_x + \frac{2}{3}u\theta_x \wedge \theta \wedge \theta_x + \frac{1}{3}u_x\theta \wedge \theta \wedge \theta_x\Big\} \; dx \\
&= \frac{1}{3}\int \theta \wedge \theta_x \wedge \theta_{xxx} \\
&= 0
\end{aligned}$$

by our calculation in example 2.2.2. Since $\mathscr{E}, \mathscr{D}$ are skew-symmetric and (2.2.24) holds, we can conclude that $\mathscr{E}, \mathscr{D}$ are Hamiltonian. Therefore, we can finally truly say that the KdV equation $u_t = u_{xxx} + uu_x$ is Hamiltonian with two distinct Hamiltonian operators.

This concludes our section on Poisson Brackets and Hamiltonian structures. We have multiple theorems which help us in discerning if our differential operator is Hamiltonian. We will use these theorems in the later sections when we need to show that a particular pde admits a Hamiltonian Structure.

2.3 A Noether Theorem

Calculating the symmetries and conservation laws for a partial differential equation has been a problem of interest for many researchers. A theorem that helps researchers in computing conservation laws is Noether's theorem. Noether's theorem creates a relation between the conservation laws and symmetries of a partial differential equation. In this section we will introduce another Noether theorem that relates the Poisson bracket, the generalized symmetries and conservation laws of a Hamiltonian evolution equation. Furthermore, as we mentioned earlier we will assume that we are working with a single spatial variable and a single dependent variable $u(x,t)$. We begin by recalling some important definitions.

Definition 2.3.1. Consider a system of differential equations $\Delta = 0$. A *conservation law* is an identity of the form

$$\text{Div } P = 0 \tag{2.3.1}$$

which holds for all solutions $u = f(x)$ of the system. Where $P = (P_1, ..., P_p)$ is a p-tuple of differential functions.

A conservation law can trivially hold for a differential equation in two different ways. A trivial conservation law of the first kind is an expression like (2.3.1) that vanishes for all solutions $u = f(x)$ of the system. In this case, we use the differential equation $\Delta = 0$ and make substitutions for different partial derivatives. For example in a system $u_t = K[u]$, we substitute all the partial derivatives of u with respect t with $K[u]$ and it's various derivatives.

Example. The system $u_t = v_x, \;\; v_t = u_x$ has trivial conservation law

$$D_t(\frac{1}{2}u_t^2 - \frac{1}{2}v_x^2) + D_x(v_x u_x - u_t u_x) = D_t(\frac{1}{2}v_x^2 - \frac{1}{2}v_x^2) + D_x(v_x u_x - v_x u_x) = 0$$

which was obtained by substituting the time derivatives with the equations of our system.

A trivial conservation law of second kind occurs when (2.3.1) holds for all functions $u = f(x)$ and does not depend on the structure of the given differential equation.

Example. In the case where we have two independent variables (x, y) and one dependent variable u,

$$D_x(u_y) - D_y(u_x) = 0$$

is trivial conservation law as it holds true for all smooth functions $u = f(x, y)$. We did not even need to define a system to obtain this conservation law.

In the case of an evolution equation, we can separate the time and spatial variables and give an alternate form for the conservation law.

Definition 2.3.2. In an evolution equation, where one of the independent variables is distinguished as time t and the remaining variable x is a spatial variable, the conservation law takes the form

$$D_t(T) + \text{Div}(X) = 0. \tag{2.3.2}$$

Here Div is the spatial divergence of X with respect to the variable x. T is called the *conserved density* and X is called the *conserved flux*, they are both functions of x, t, u and the derivatives of u with respect to both t and x.

Trivial conservation laws of the first and second kind are still conservation laws of the system of evolution equations, even though the form is written differently. In general, we can say that two conservation laws are equivalent is they differ by a trivial conservation law. Therefore we are only interested in finding conservation laws up to an equivalence class. Furthermore, if an evolution equation has a Hamiltonian form, then there is a way to obtain conservation laws for the system just by observing its Poisson bracket.

Definition 2.3.3. Let $\mathscr{D}$ be a Hamiltonian differential operator. A distinguished functional for $\mathscr{D}$ is a functional $\mathscr{C} \in \mathscr{F}$ satisfying

$$\mathscr{D}\delta\mathscr{C} = 0 \tag{2.3.3}$$

for all x, u.

The Hamiltonian operator corresponding to a distinguished functional is trivial i.e. $u_t = 0$. If we combine (2.3.3) with the definition of a Poisson bracket, we obtain that a functional $\mathscr{C}$ is distinguished if and only if

$$\{\mathscr{C}, \mathscr{H}\} = 0 \tag{2.3.4}$$

for all functionals $\mathscr{H} \in \mathscr{F}$. Observing this we can see the conserved nature of the distinguished functionals, leading us to quote the following proposition from [4].

Proposition 2.3.1. Let $\mathscr{D}$ be a Hamiltonian operator. If $\mathscr{C}$ is a distinguished functional for $\mathscr{D}$, then $\mathscr{C}$ determines a conservation law for every Hamiltonian system $u_t = \mathscr{D}\delta\mathscr{H}$ relative to $\mathscr{D}$.

Now we can introduce a new criteria for a functional to give rise to a conservation law.

Proposition 2.3.2. Let $u_t = P[u]$ be an evolution equation, and let $\mathscr{T}[t; u] = \int_\Omega T(x, t, u^{(n)})\ dx$ be a functional for $\Omega \subset X$. Then T is the density for a conservation law of our equation if and only if it's associated functional $\mathscr{T}$ satisfies

$$\frac{\partial \mathscr{T}}{\partial t} + \text{pr v}_P(\mathscr{T}) = 0 \tag{2.3.5}$$

where $\text{pr v}_P(\mathscr{T}) := \int_\Omega \text{pr v}_P(T)\ dx$.

Proof.
Let us start with the forward implication, assume that T is a conserved density for a conservation law and let $\mathscr{T}[t; u] = \int_\Omega T(x, t, u^{(n)})\ dx$ be its associated functional. T can be assumed to depend only on the x-derivatives of u since we can substitute the time derivatives by our equation. If u is a solution to the evolution equation $u_t = P[u]$ then

$$\begin{aligned} D_t(T) &= \frac{\partial T}{\partial t} + u_t \frac{\partial T}{\partial u} + u_{tt} \frac{\partial T}{\partial u_t} + u_{tx} \frac{\partial T}{\partial u_x} + ... \\ &= \frac{\partial T}{\partial t} + P \frac{\partial T}{\partial u} + (P)_t \frac{\partial T}{\partial u_t} + (P_x) \frac{\partial T}{\partial u_x} + ... \\ &= \frac{\partial T}{\partial t} + \text{pr v}_P(T) \end{aligned}$$

here $\frac{\partial T}{\partial t}$ denotes the partial t-derivative. Since T is a conserved density we must have that

$$D_t(T) + \text{Div} A = 0$$

for some $A \in \mathscr{A}$, where Div is the spatial divergence. Integrating both sides over Ω we obtain using the divergence theorem

$$\begin{aligned} \int_\Omega \Big\{ D_t(T) + \text{Div } A \Big\}\ dx = 0 \\ \implies \int_\Omega D_t(T)\ dx = 0. \end{aligned}$$

If we expand the $D_t(T)$ under the integral using our computation we obtain (2.3.5) as follows

$$\begin{aligned} \int_\Omega D_t(T)\ dx = 0 \\ \implies \int_\Omega \Big\{ \frac{\partial T}{\partial t} + \text{pr v}_P(T) \Big\}\ dx = 0 \\ \implies \frac{\partial}{\partial t}\Big(\int_\Omega T\ dx \Big) + \text{pr v}_P\Big(\int_\Omega T\ dx \Big) = 0 \\ \implies \frac{\partial \mathscr{T}}{\partial t} + \text{pr v}_P(\mathscr{T}) = 0. \end{aligned}$$

Now we prove the reverse direction, assume that (2.3.5) holds, if u is a solution to the evolution equation $u_t = P[u]$, we can expand the left hand side of (2.3.5) and obtain the following

$$\begin{aligned} &\frac{\partial \mathscr{T}}{\partial t} + \text{pr v}_P(\mathscr{T}) = 0 \\ \implies &\frac{\partial}{\partial t}\Big(\int_\Omega T\, dx\Big) + \text{pr v}_P\Big(\int_\Omega T\, dx\Big) = 0 \\ \implies &\int_\Omega D_t(T)\, dx = 0. \end{aligned}$$

Since we are working over the space of functionals $\mathscr{F}$ and the integral above is equal to 0, we can say that the integrand is equivalent to a total divergence Div A. Therefore

$$D_t(T) = \text{Div } A \implies D_t(T) + \text{Div }(-A) = 0$$

on solutions u of of $u_t = P[u]$ and thus T is a conserved density. ☐

Now that we have established a criteria for conservation laws, we wish to determine a relation between conservation laws and symmetries of the evolution equation. Before we introduce this Noether theorem we will prove a proposition that relates the Lie bracket of two Hamiltonian vector fields and their corresponding Poisson bracket. This will be used later in the proof of our Noether theorem.

Definition 2.3.4. Let v and w be generalized vector fields. Then their Lie bracket $[\text{v}, \text{w}]$ is the unique generalized vector field satisfying

$$\text{pr }[\text{v}, \text{w}]\ (P) = \text{pr v}[\text{pr w}(P)] - \text{pr w}[\text{pr w}(P)] \tag{2.3.6}$$

for all $P \in \mathscr{A}$.

Proposition 2.3.3. Let $\{\cdot, \cdot\}$ be a Poisson bracket determined by a Hamiltonian operator $\mathscr{D}$. Let $\mathscr{P}, \mathscr{Q} \ \in \mathscr{F}$ be functionals with corresponding Hamiltonian vector fields $\hat{\text{v}}_{\mathscr{P}}$ and $\hat{\text{v}}_{\mathscr{P}}$. Then the Hamiltonian vector field corresponding to the Poisson bracket $\{\mathscr{P}, \mathscr{Q}\}$ is the Lie bracket of the two vector fields;

$$\hat{\text{v}}_{\{\mathscr{P},\mathscr{Q}\}} = [\hat{\text{v}}_{\mathscr{Q}}, \hat{\text{v}}_{\mathscr{P}}]. \tag{2.3.7}$$

Proof.
Let $\mathscr{R}$ be an arbitrary functional, we begin by applying the prolongation of $\hat{\text{v}}_{\{\mathscr{P},\mathscr{Q}\}}$ to $\mathscr{R}$ and using proposition 2.2.2 to obtain the following:

$$\begin{aligned} \text{pr } \hat{\text{v}}_{\{\mathscr{P},\mathscr{Q}\}}(\mathscr{R}) &= \{\mathscr{R}, \{\mathscr{P}, \mathscr{Q}\}\} \\ &= \{\{\mathscr{R}, \mathscr{P}\}, \mathscr{Q}\} + \{\{\mathscr{Q}, \mathscr{R}\}, \mathscr{P}\} \quad \text{By the Jacobi identity and skew-symmetry of the bracket.} \\ &= \{\{\mathscr{R}, \mathscr{P}\}, \mathscr{Q}\} - \{\{\mathscr{R}, \mathscr{Q}\}, \mathscr{P}\} \\ &= \text{pr } \hat{\text{v}}_{\mathscr{Q}}(\{\mathscr{R}, \mathscr{P}\}) - \text{pr } \hat{\text{v}}_{\mathscr{P}}(\{\mathscr{R}, \mathscr{Q}\}) \\ &= \text{pr } \hat{\text{v}}_{\mathscr{Q}} \circ \text{pr } \hat{\text{v}}_{\mathscr{P}}(\mathscr{R}) - \text{pr } \hat{\text{v}}_{\mathscr{P}} \circ \text{pr } \hat{\text{v}}_{\mathscr{Q}}(\mathscr{R}) \\ &= (\text{pr } \hat{\text{v}}_{\mathscr{Q}} \circ \text{pr } \hat{\text{v}}_{\mathscr{P}} - \text{pr } \hat{\text{v}}_{\mathscr{P}} \circ \text{pr } \hat{\text{v}}_{\mathscr{Q}})(\mathscr{R}) \\ &= \text{pr } [\hat{\text{v}}_{\mathscr{Q}}, \hat{\text{v}}_{\mathscr{P}}](\mathscr{R}). \qquad \text{By definition 2.3.4.} \end{aligned}$$

The above implies that

$$\begin{aligned} &\text{pr } \hat{\text{v}}_{\{\mathscr{P},\mathscr{Q}\}}(\mathscr{R}) - \text{pr } [\hat{\text{v}}_{\mathscr{Q}}, \hat{\text{v}}_{\mathscr{P}}](\mathscr{R}) = 0 \\ \implies &\text{pr } (\hat{\text{v}}_{\{\mathscr{P},\mathscr{Q}\}} - [\hat{\text{v}}_{\mathscr{Q}}, \hat{\text{v}}_{\mathscr{P}}])(\mathscr{R}) = 0 \end{aligned}$$

for all functionals $\mathscr{R} \in \mathscr{F}$. This can only happen if the two vector fields are equal, therefore we can conclude that $\hat{\text{v}}_{\{\mathscr{P},\mathscr{Q}\}} = [\hat{\text{v}}_{\mathscr{Q}}, \hat{\text{v}}_{\mathscr{P}}]$. ☐

The reader is probably familiar with symmetries of differential equations. We now introduce the idea of generalized symmetries. Generalized symmetries are very much like classical symmetries only that the infinitesimal generator of the symmetry is a generalized vector field. Thus its coefficients can depend on the various derivatives of u, up to any order. The definition of a generalized symmetry is a direct analogue of the definition of a point symmetry (classical symmetry).

Definition 2.3.5. A generalized vector field v is a *generalized infinitesimal symmetry* of the differential equation $\Delta[u] = 0$ if and only if

$$\mathrm{pr}\ \mathrm{v}(\Delta) = 0 \tag{2.3.8}$$

for every smooth solution u of $\Delta = 0$.

It is important to note that any classical symmetry of a differential equation is also a generalized symmetry. By classical symmetry we mean that the symmetry group has infinitesimal generator with coefficients depending only x, t, and u. Now let us quote another proposition from [4] that gives a new criteria for checking that an evolutionary vector field is a symmetry of an evolution equation.

Proposition 2.3.4. An evolutionary vector field v_Q is a symmetry of the evolution equation $u_t = P[u]$ if and only if

$$\frac{\partial\ \mathrm{v}_Q}{\partial t} + [\mathrm{v_P}, \mathrm{v_Q}] = 0 \tag{2.3.9}$$

holds identically in $(x, t, u^{(n)})$. Here $\frac{\partial\ \mathrm{v}_Q}{\partial t}$ is the evolutionary vector field with characteristic $\frac{\partial Q}{\partial t}$.

Remark. By symmetry, we mean generalized symmetry, however since the vector field is evolutionary we could also call it an evolutionary symmetry. These are interchangeable since evolutionary vector fields are also generalized vector fields.

Finally, we quote one last proposition from [4].

Proposition 2.3.5. A generalized vector field $\mathbf{v}$ is a symmetry of a differential equation if and only if its evolutionary representative $\mathbf{v}_Q$ is.

Now we have all the tools to state and prove the main theorem in this section. It is a Noether theorem in the sense that it creates a relation between the generalized symmetries of an evolution equation and its conservation laws. If a Hamiltonian vector field $\hat{\mathrm{v}}_{\mathscr{H}}$ is a symmetry of an evolution equation, we call it a Hamiltonian symmetry. The theorem we are about to state uses the Hamiltonian nature of the evolution equation to create a relation between Hamiltonian symmetries and conservation laws.

Theorem 2.3.1. Let $u_t = \mathscr{D}\delta\mathscr{H}$ be a Hamiltonian evolution equation. A Hamiltonian vector field $\hat{\mathrm{v}}_{\mathscr{P}}$ with characteristic $\mathscr{D}\delta\mathscr{P}$, $\mathscr{P} \in \mathscr{F}$, determines a generalized symmetry of the equation if and only if there is an equivalent functional $\mathscr{P}' = \mathscr{P} - \mathscr{C}$, differing only from $\mathscr{P}$ by a time-dependent distinguished functional $\mathscr{C}[t; u]$, such that $\mathscr{P}'$ determines a conservation law.

Proof.
A time dependent distinguished functional means a functional

$$\mathscr{C}[t; u] = \int C(t, x, u^{(n)})\ dx$$

where C depends on t, x, u and the x-derivatives of u such that for each fixed t_0, $\mathscr{C}[t_0; u]$ is a distinguished functional i.e. $\mathscr{D}\delta\mathscr{C} = 0$.

Assume that the Hamiltonian vector field $\hat{\mathrm{v}}_{\mathscr{P}}$ determines a symmetry of $u_t = \mathscr{D}\delta\mathscr{H}$. By proposition 2.3.4 $\hat{\mathrm{v}}_{\mathscr{P}}$ is a symmetry of $u_t = \mathscr{D}\delta\mathscr{H}$ if and only if

$$\frac{\partial \hat{\mathrm{v}}_{\mathscr{P}}}{\partial t} + [\hat{\mathrm{v}}_{\mathscr{H}}, \hat{\mathrm{v}}_{\mathscr{P}}] = 0 \tag{2.3.10}$$

where $\hat{\mathrm{v}}_{\mathscr{H}}$ is the Hamiltonian vector field associated with $\mathscr{H}$ and $\frac{\partial \hat{\mathrm{v}}_{\mathscr{P}}}{\partial t}$ is the Hamiltonian vector field associated to the functional $\frac{\partial \mathscr{P}}{\partial t}$. According to proposition 2.3.3., $[\hat{\mathrm{v}}_{\mathscr{H}}, \hat{\mathrm{v}}_{\mathscr{P}}]$ is the Hamiltonian vector field for the Poisson bracket $\{\mathscr{P}, \mathscr{H}\}$. Since the vector field in (2.3.10) is equal to 0 its characteristic is equal to 0, namely

$$\mathscr{D}\delta\Big(\frac{\partial \mathscr{P}}{\partial t} + \{\mathscr{P}, \mathscr{H}\}\Big) = 0.$$

However, this is precisely the definition of a distinguished functional for $\mathscr{D}$. Hence we can say

$$\frac{\partial \mathscr{P}}{\partial t} + \{\mathscr{P}, \mathscr{H}\} = \mathscr{C}'$$

where $\mathscr{C}'$ is a time distinguished functional

$$\mathscr{C}'[t;u] = \int C'(t, x, u^{(n)})\, dx.$$

Now we set

$$\mathscr{C}[t;u] = \int_{t_0}^{t} \mathscr{C}[s;u]\, ds = \int \Big(\int_{t_0}^{t} C'(s, x, u^{(n)})\, ds\Big)\, dx.$$

Let $\mathscr{P}' = \mathscr{P} - \mathscr{C}$, this differs from $\mathscr{P}$ by a time dependent distinguished functional. We are left to prove that $\mathscr{P}'$ determines a conservation law. Using Proposition 2.3.2 all we need to show is

$$\frac{\partial \mathscr{P}'}{\partial t} + \mathrm{pr}\, \hat{\mathrm{v}}_{\mathscr{H}}(\mathscr{P}') = 0 \tag{2.3.11}$$

in order to claim $\mathscr{P}'$ determines a conservation law. Let us do the computations.

$$\frac{\partial \mathscr{P}'}{\partial t} = \frac{\partial \mathscr{P}}{\partial t} - \frac{\partial \mathscr{C}}{\partial t} = \frac{\partial \mathscr{P}}{\partial t} - \mathscr{C}'$$

$$\{\mathscr{P}', \mathscr{H}\} = \{\mathscr{P} - \mathscr{C}, \mathscr{H}\} = \{\mathscr{P}, \mathscr{H}\} - \{\mathscr{C}, \mathscr{H}\} = \{\mathscr{P}, \mathscr{H}\}$$

Therefore

$$\begin{aligned}\frac{\partial \mathscr{P}'}{\partial t} + \mathrm{pr}\, \hat{\mathrm{v}}_{\mathscr{H}}(\mathscr{P}') &= \frac{\partial \mathscr{P}}{\partial t} - \mathscr{C}' + \{\mathscr{P}', \mathscr{H}\} \\ &= \frac{\partial \mathscr{P}}{\partial t} - \mathscr{C}' + \{\mathscr{P}, \mathscr{H}\} \\ &= \mathscr{C}' - \mathscr{C}' \\ &= 0\end{aligned}$$

proving that $\mathscr{P}'$ determines a conservation law.

Now we prove the reverse implication, assume that $\mathscr{P}' = \mathscr{P} - \mathscr{C}$ determines a conservation law, where $\mathscr{C}$ is a distinguished functional. Then by proposition 2.3.2

$$\frac{\partial \mathscr{P}'}{\partial t} + \mathrm{pr}\, \hat{\mathrm{v}}_{\mathscr{H}}(\mathscr{P}') = \frac{\partial \mathscr{P}'}{\partial t} + \{\mathscr{P}', \mathscr{H}\} = 0. \tag{2.3.12}$$

We need to prove that $\hat{\mathrm{v}}_{\mathscr{P}}$ is a generalized symmetry of $u_t = \mathscr{D}\delta\mathscr{H}$, thus we need to show

$$\frac{\partial \hat{\mathrm{v}}_{\mathscr{P}}}{\partial t} + [\hat{\mathrm{v}}_{\mathscr{H}}, \hat{\mathrm{v}}_{\mathscr{P}}] = 0.$$

The characteristic of the vector field above is

$$\mathscr{D}\delta\Big(\frac{\partial \mathscr{P}}{\partial t} + \{\mathscr{P}, \mathscr{H}\}\Big).$$

Since $\mathscr{P} = \mathscr{P}' + \mathscr{C}$, if we explicitly calculate the characteristic above we obtain

$$\frac{\partial \mathscr{P}}{\partial t} = \frac{\partial \mathscr{P}'}{\partial t} + \frac{\partial \mathscr{C}}{\partial t}$$

$$\{\mathscr{P}, \mathscr{H}\} = \{\mathscr{P}' + \mathscr{C}, \mathscr{H}\} = \{\mathscr{P}', \mathscr{H}\} + \{\mathscr{C}, \mathscr{H}\} = \{\mathscr{P}', \mathscr{H}\}$$

$$\begin{aligned}
\implies \mathscr{D}\delta\Big(\frac{\partial \mathscr{P}}{\partial t} + \{\mathscr{P}, \mathscr{H}\}\Big) &= \mathscr{D}\delta\Big(\frac{\partial \mathscr{P}'}{\partial t} + \frac{\partial \mathscr{C}}{\partial t} + \{\mathscr{P}', \mathscr{H}\}\Big) \\
&= \mathscr{D}\delta\Big(\frac{\partial \mathscr{C}}{\partial t}\Big) && \text{by 2.3.12} \\
&= \frac{\partial}{\partial t}\Big(\mathscr{D}\delta\mathscr{C}\Big) \\
&= 0
\end{aligned}$$

where the second last equality is obtained because $\mathscr{D}$ does not explicitly depend on t and the last equality is true because $\mathscr{C}$ is a distinguished functional. Therefore the characteristic of the vector field on the left hand side of equation (2.3.10) is equal to 0 and thus the vector field itself is equal to 0. Therefore we obtain the equality in (2.3.10), which implies that $\hat{\mathrm{v}}_{\mathscr{P}}$ is a generalized symmetry of $u_t = \mathscr{D}\delta\mathscr{H}$. □

Example 2.3.1. Let us return to our running KdV example and see how this Noether theorem is applied. Recall the KdV equation

$$u_t = u_{xxx} + uu_x$$

that had two Hamiltonian structures corresponding to differential operators

$$\mathscr{D} = D_x \quad \text{and} \quad \mathscr{E} = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x.$$

In section 2.4 of [4], Olver computes the algebra of classical symmetries of the KdV equation, it is spanned by

$$\mathrm{v}_1 = \partial_x, \qquad \mathrm{v}_2 = \partial_t, \qquad \mathrm{v}_3 = t\partial_x - \partial_u, \qquad \mathrm{v}_4 = x\partial_x + 3t\partial_t - 2u\partial_u.$$

with replacing x by $-x$. We compute their evolutionary representatives to be

$$\mathrm{v}_{Q_1} = -u_x\partial_u, \quad \mathrm{v}_{Q_2} = (-u_{xxx} - uu_x)\partial_u, \quad \mathrm{v}_{Q_3} = (-1 - tu_x)\partial_u, \quad \mathrm{v}_{Q_4} = (-2u - xu_x - 3tu_{xxx} - 3tuu_x)\partial_u.$$

Now let us determine which of these symmetries are Hamiltonian and thus lead to a conserved density. These symmetries are Hamiltonian if their characteristic Q_i can be written in a Hamiltonian structure with respect to either operator $\mathscr{D}$ or $\mathscr{E}$. Let us start with the first operator $\mathscr{D} = D_x$, out of the four vector fields above, the first three have characteristics which are Hamiltonian

$$Q_i = D_x\delta\mathscr{P}_i, \quad i = 1, 2, 3$$

with conserved functionals

$$\mathscr{P}_1 = \int -\frac{1}{2}u^2 \, dx, \quad \mathscr{P}_2 = \int \Big\{\frac{1}{2}u_x^2 - \frac{1}{6}u^3\Big\} \, dx, \quad \mathscr{P}_3 = \int \Big\{ - xu - \frac{1}{2}tu^2\Big\} \, dx.$$

For the second operator $\mathscr{E} = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x$, Q_1, Q_2 and Q_4 are Hamiltonian,

$$Q_i = \mathscr{E}\delta\mathscr{N}_i \quad i = 1, 2, 4$$

with conserved functionals

$$\mathscr{N}_1 = \int -3u \, dx, \quad \mathscr{N}_2 = \int -\frac{1}{2}u^2 \, dx, \quad \mathscr{N}_4 = \int \Big\{ - 3xu - \frac{3}{2}tu^2\Big\} \, dx.$$

The functional $\mathscr{N}_1$ is a distinguished functional for $\mathscr{D}$ since $\mathscr{D}\delta\mathscr{N}_1 = 0$ and thus cannot give us a new symmetry for the system. The other conserved functionals for $\mathscr{E}$ are either the same or a multiple of the conserved functionals for $\mathscr{D}$. However, the functional $\mathscr{P}_2$ did not arise from one of the classical symmetries in the case of $\mathscr{E}$ but it still determines a conservation law for the KdV equation. Therefore by our Noether theorem 2.3.1, this should give rise to a Hamiltonian symmetry $\hat{\mathrm{v}}_{\mathscr{P}_2}$ that has characteristic

$$\begin{aligned}\mathscr{E}\delta\mathscr{P}_2 &= (D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x)(-u_{xx} - \frac{1}{2}u^2) \\ &= -u_{xxxxx} - \frac{5}{3}uu_{xxx} - \frac{10}{3}u_xu_{xx} - \frac{5}{6}u^2u_x.\end{aligned}$$

We start to notice a pattern at this point, if the characteristic above could be written in Hamiltonian form with respect to the operator $\mathscr{D}$ then this would provide another conserved functional for the KdV equation. Then we could apply $\mathscr{E}$ to this new conserved functional and so on. This phenomenon is not necessarily true for all evolution equations that admit a Hamiltonian structure. For the KdV equation it is due in part to the fact that we have two distinct Hamiltonian structures. In the next section we will discuss this phenomenon and explain why the operators $\mathscr{D}$ and $\mathscr{E}$ for the KdV equation are special enough to do this.

2.4 Bi-Hamiltonian Systems and Magri's Theorem

In the previous section we saw what it means for an evolution equation to be Hamiltonian. We examined the example of the KdV equation, which had two distinct Hamiltonian structures. This leads to the natural question, does an evolution equation that has multiple Hamiltonian structures have more interesting properties? The answer to this question is yes. In this section, we will explore what happens when an evolution equation can be written in two distinct Hamiltonian forms, such an evolution equation is called bi-Hamiltonian. The consequence of this gives rise to the main theorem in this section, Magri's theorem. This theorem allows us to create an infinite hierarchy of conservation laws by applying our Noether theorem from the previous section to the different symmetries of both Hamiltonian structures. Again, the information in this section is paraphrased or directly quoted from [4].

Definition 2.4.1. Let $\mathscr{D}$ and $\mathscr{E}$ be a pair of skew-adjoint $q \times q$ matrix differential operators. They are said to form a *Hamiltonian pair* if every linear combination $a\mathscr{D} + b\mathscr{E}$, for $a, b \in \mathbb{R}$ is a Hamiltonian operator.

Definition 2.4.2. A system of evolution equations $u_t = K[u]$, $K[u] \in \mathscr{A}^q$ is *bi-Hamiltonian* if it can be written in the form

$$u_t = K[u] = \mathscr{D}\delta\mathscr{H}_0 = \mathscr{E}\delta\mathscr{H}_1 \tag{2.4.1}$$

where $\mathscr{H}_0, \mathscr{H}_1$ are Hamiltonian functionals and $\mathscr{D}$ and $\mathscr{E}$ form a Hamiltonian pair.

Lemma 2.4.1. Let $\mathscr{D}$ and $\mathscr{E}$, be skew adjoint differential operators, they form a Hamiltonian pair if and only if $\mathscr{D}, \mathscr{E}$ and $\mathscr{D} + \mathscr{E}$ are all Hamiltonian operators.

Proof.
The forward implication is trivial and comes from the definition of a Hamiltonian pair. So we assume that $\mathscr{D}, \mathscr{E}$ and $\mathscr{D} + \mathscr{E}$ are all Hamiltonian operators and we need to prove every linear combination of $\mathscr{D}, \mathscr{E}$ also is a Hamiltonian operator. We know that a linear combination of skew-adjoint operators is skew-adjoint, so what is left to prove is that the Jacobi identity is satisfied. As we have shown earlier there are multiple ways of stating the Jacobi identity. Let $P, Q, R \in \mathscr{A}^q$ and let $J(\mathscr{D}, \mathscr{D}; P, Q, R)$ be the left hand side of the equation (2.2.13). Recall we can write this as a functional tri-vector evaluated at P, Q, R, this allows us to write

$$\begin{aligned} J(\mathscr{D}, \mathscr{D}; P, Q, R) = \frac{1}{2} \int \Big\{ & P \cdot \mathrm{pr}\, \mathbf{v}_{\mathscr{D}R}(\mathscr{D})Q + R \cdot \mathrm{pr}\, \mathbf{v}_{\mathscr{D}Q}(\mathscr{D})P + Q \cdot \mathrm{pr}\, \mathbf{v}_{\mathscr{D}P}(\mathscr{D})R \\ & + P \cdot \mathrm{pr}\, \mathbf{v}_{\mathscr{D}R}(\mathscr{D})Q + R \cdot \mathrm{pr}\, \mathbf{v}_{\mathscr{D}Q}(\mathscr{D})P + Q \cdot \mathrm{pr}\, \mathbf{v}_{\mathscr{D}P}(\mathscr{D})R \Big\}\, dx. \end{aligned}$$

Thus we can define a symmetric bilinear form $J(\mathscr{D}, \mathscr{E}; P, Q, R)$ by

$$\begin{aligned} J(\mathscr{D}, \mathscr{E}; P, Q, R) = \frac{1}{2} \int \Big\{ & P \cdot \mathrm{pr}\, \mathbf{v}_{\mathscr{D}R}(\mathscr{E})Q + R \cdot \mathrm{pr}\, \mathbf{v}_{\mathscr{D}Q}(\mathscr{E})P + Q \cdot \mathrm{pr}\, \mathbf{v}_{\mathscr{D}P}(\mathscr{E})R \\ & + P \cdot \mathrm{pr}\, \mathbf{v}_{\mathscr{E}R}(\mathscr{D})Q + R \cdot \mathrm{pr}\, \mathbf{v}_{\mathscr{E}Q}(\mathscr{D})P + Q \cdot \mathrm{pr}\, \mathbf{v}_{\mathscr{E}P}(\mathscr{D})R \Big\}\, dx \end{aligned}$$

this is obviously symmetric and bilinear, where bilinearity comes from the prolongation formula (2.2.11). This allows us to say that for any $a, b \in \mathbb{R}$

$$\begin{aligned} J(a\mathscr{D} + b\mathscr{E}, a\mathscr{D} + b\mathscr{E}; P, Q, R) &= J(a\mathscr{D} + b\mathscr{E}, a\mathscr{D}; P, Q, R) + J(a\mathscr{D} + b\mathscr{E}, b\mathscr{E}; P, Q, R) \\ &= a \cdot J(a\mathscr{D} + b\mathscr{E}, \mathscr{D}; P, Q, R) + b \cdot J(a\mathscr{D} + b\mathscr{E}, \mathscr{E}; P, Q, R) \\ &= a^2 \cdot J(\mathscr{D}, \mathscr{D}; P, Q, R) + ab \cdot J(\mathscr{E}, \mathscr{D}; P, Q, R) \\ &\quad + ba \cdot J(\mathscr{D}, \mathscr{E}; P, Q, R) + b^2 \cdot J(\mathscr{E}, \mathscr{E}; P, Q, R) \end{aligned}$$

$$\implies J(a\mathscr{D} + b\mathscr{E}, a\mathscr{D} + b\mathscr{E}; P, Q, R) = a^2 \cdot J(\mathscr{D}, \mathscr{D}; P, Q, R) + 2ab \cdot J(\mathscr{E}, \mathscr{D}; P, Q, R) + b^2 \cdot J(\mathscr{E}, \mathscr{E}; P, Q, R).$$

Since $J(\mathscr{D}, \mathscr{D}; P, Q, R)$ is the left hand side of the Jacobi identity, it is equal to 0 when $\mathscr{D}$ is a Hamiltonian operator. Since $\mathscr{D}, \mathscr{E}$ and $\mathscr{D} + \mathscr{E}$ are all Hamiltonian operators we can say

$$J(\mathscr{D}, \mathscr{D}; P, Q, R) = J(\mathscr{E}, \mathscr{E}; P, Q, R) = J(\mathscr{D} + \mathscr{E}, \mathscr{D} + \mathscr{E}; P, Q, R) = 0.$$

Expanding the last equality using the bilinearity property of $J(\mathscr{D},\mathscr{E};P,Q,R)$ we obtain the following result

$$\begin{aligned} J(\mathscr{D}+\mathscr{E},\mathscr{D}+\mathscr{E};P,Q,R)&=0\\ \implies J(\mathscr{D},\mathscr{D};P,Q,R)+2\cdot J(\mathscr{E},\mathscr{D};P,Q,R)+J(\mathscr{E},\mathscr{E};P,Q,R)&=0\\ \implies J(\mathscr{E},\mathscr{D};P,Q,R)&=0. \end{aligned}$$

Using this result, the fact that $\mathscr{D}$ and $\mathscr{E}$ are Hamiltonian and the bilinearity of J, allows us to say for any $a,b\in\mathbb{R}$

$$J(a\mathscr{D}+b\mathscr{E},a\mathscr{D}+b\mathscr{E};P,Q,R)=a^2\cdot J(\mathscr{D},\mathscr{D};P,Q,R)+2ab\cdot J(\mathscr{E},\mathscr{D};P,Q,R)+b^2\cdot J(\mathscr{E},\mathscr{E};P,Q,R)=0.$$

Since $J(a\mathscr{D}+b\mathscr{E},a\mathscr{D}+b\mathscr{E};P,Q,R)$ is the left hand side of the Jacobi identity with respect to the operator $a\mathscr{D}+b\mathscr{E}$, we conclude that the Jacobi identity is satisfied for all linear combinations $a\mathscr{D}+b\mathscr{E},\ a,b\in\mathbb{R}$. Therefore $\mathscr{D}$ and $\mathscr{E}$ form a Hamiltonian pair. □

Corollary 2.4.1. Let $\mathscr{D}$ and $\mathscr{E}$ be Hamiltonian operators. Then $\mathscr{D},\mathscr{E}$ form a Hamiltonian pair if and only if

$$\mathrm{pr}\ \mathrm{v}_{\mathscr{D}\theta}(\Theta_{\mathscr{E}})+\mathrm{pr}\ \mathrm{v}_{\mathscr{E}\theta}(\Theta_{\mathscr{D}})=0 \tag{2.4.2}$$

where

$$\Theta_{\mathscr{D}}=\frac{1}{2}\int\{\theta\wedge\mathscr{D}\theta\}\,dx\quad\text{and}\quad\Theta_{\mathscr{E}}=\frac{1}{2}\int\{\theta\wedge\mathscr{E}\theta\}\,dx$$

are the functional bi-vectors that represent the Poisson bracket of each operator.

Proof.
Let $P,Q,R\in\mathscr{A}^q$, recall from section 2.2 that the functional tri-vector evaluated at P,Q,R

$$\Big\langle\mathrm{pr}\ \mathrm{v}_{\mathscr{D}\theta}(\Theta_{\mathscr{D}});P,Q,R\Big\rangle$$

is the negative of the left hand side of the Jacobi identity. Therefore we can equate the above with the form we used in the lemma above to say

$$\begin{aligned} -J(\mathscr{D},\mathscr{D};P,Q,R)&=\Big\langle\mathrm{pr}\ \mathrm{v}_{\mathscr{D}\theta}(\Theta_{\mathscr{D}});P,Q,R\Big\rangle\\ -J(\mathscr{D},\mathscr{E};P,Q,R)&=\Big\langle\mathrm{pr}\ \mathrm{v}_{\mathscr{D}\theta}(\Theta_{\mathscr{E}});P,Q,R\Big\rangle\\ \implies -2\cdot J(\mathscr{D},\mathscr{E};P,Q,R)&=\Big\langle\mathrm{pr}\ \mathrm{v}_{\mathscr{D}\theta}(\Theta_{\mathscr{E}})+\mathrm{pr}\ \mathrm{v}_{\mathscr{E}\theta}(\Theta_{\mathscr{D}});P,Q,R\Big\rangle. \end{aligned}$$

By the proof of the lemma above, we know that if $\mathscr{D},\mathscr{E}$ form a Hamiltonian pair then

$$J(\mathscr{D},\mathscr{E};P,Q,R)=0$$

$$\implies\Big\langle\mathrm{pr}\ \mathrm{v}_{\mathscr{D}\theta}(\Theta_{\mathscr{E}})+\mathrm{pr}\ \mathrm{v}_{\mathscr{E}\theta}(\Theta_{\mathscr{D}});P,Q,R\Big\rangle=0$$

since a functional k-vector is uniquely determined by its action on elements of $\mathscr{A}^q$, we can conclude that $\mathrm{pr}\ \mathrm{v}_{\mathscr{D}\theta}(\Theta_{\mathscr{E}})+\mathrm{pr}\ \mathrm{v}_{\mathscr{E}\theta}(\Theta_{\mathscr{D}})=0$.

Similarly, if (2.4.2) holds and $\mathscr{D},\mathscr{E}$ are Hamiltonian operators then for all $P,Q,R\in\mathscr{A}^q$

$$\Big\langle\mathrm{pr}\ \mathrm{v}_{\mathscr{D}\theta}(\Theta_{\mathscr{E}})+\mathrm{pr}\ \mathrm{v}_{\mathscr{E}\theta}(\Theta_{\mathscr{D}});P,Q,R\Big\rangle=0$$

$$\implies J(\mathscr{D},\mathscr{E};P,Q,R)=0$$

$$\implies J(\mathscr{D}+\mathscr{E},\mathscr{D}+\mathscr{E};P,Q,R)=J(\mathscr{D},\mathscr{D};P,Q,R)+J(\mathscr{E},\mathscr{D};P,Q,R)+J(\mathscr{E},\mathscr{E};P,Q,R)=0.$$

Therefore the Jacobi identity is satisfied for $\mathscr{D}+\mathscr{E}$, and skew-symmetry is obvious thus $\mathscr{D},\mathscr{E}$ and $\mathscr{D}+\mathscr{E}$ are Hamiltonian operators. By the lemma above $\mathscr{D}$ and $\mathscr{E}$ form a Hamiltonian pair. □

Example 2.4.1. Let us return to our earlier example of the KdV equation, we recall that it had two distinct Hamiltonian forms

$$u_t = u_{xxx} + uu_x = \mathscr{D}\delta\mathscr{H}_1 = \mathscr{E}\delta\mathscr{H}_0$$

where

$$\mathscr{D} = D_x \qquad \mathscr{H}_1 = \int \Big\{ -\frac{1}{2}u_x^2 + \frac{1}{6}u^3 \Big\}\, dx$$

and

$$\mathscr{E} = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x \qquad \mathscr{H}_0[u] = \int \frac{1}{2}u^2 \, dx.$$

This leads to the natural question, does this gives us a bi-Hamiltonian system? To answer this question we must determine if these operators form a Hamiltonian pair. We can do this by checking that the operators satisfy the previous corollary. Recall the functions $\Theta_{\mathscr{E}}$ and $\Theta_{\mathscr{D}}$ and the prolongations from our examples in the earlier sections and compute

$$\begin{aligned} \text{pr } \mathrm{v}_{\mathscr{D}\theta}(\Theta_{\mathscr{E}}) + \text{pr } \mathrm{v}_{\mathscr{E}\theta}(\Theta_{\mathscr{D}}) &= \int \Big\{ \text{pr } \mathrm{v}_{\theta_x}(\frac{1}{2}\theta \wedge \theta_{xxx} + \frac{1}{3}u\theta \wedge \theta_x) + \text{pr } \mathrm{v}_{\mathscr{E}\theta}(\theta \wedge \theta_x) \Big\}\, dx \\ &= \int \frac{1}{3}\theta_x \wedge \theta \wedge \theta_x \, dx \\ &= 0 \end{aligned} \tag{2.4.3}$$

which gives us that $\mathscr{E}$ and $\mathscr{D}$ form a Hamiltonian pair and thus the KdV equation is bi-Hamiltonian.

Definition 2.4.3. Let $\mathscr{D}$ be a $q \times q$ matrix differential operator. The $(1,1)$- Lie -derivative of $\mathscr{D}$ with respect to a generalized vector field v_Q is the differential operator

$$\mathbf{v}_Q[[\mathscr{D}]] = \mathscr{D}_t + [\mathscr{D}, \mathrm{D}_Q] = \text{pr } \mathrm{v}_Q(\mathscr{D}) + [\mathscr{D}, \mathrm{D}_Q] \tag{2.4.4}$$

where $[\mathscr{D}, \mathrm{D}_Q]$ is the Lie bracket and $\mathscr{D}_t$ is being evaluated on solutions of $u_t = Q[u]$, as in definition 2.2.8.

Definition 2.4.4. Let $\Delta[u] = 0$ be a system of evolution equations. A *recursion operator* for $\Delta = 0$ is a linear operator $\mathscr{R} : \mathscr{A}^q \to \mathscr{A}^q$ such that whenever $\mathbf{v}_Q$ is a generalized symmetry for $\Delta = 0$, so is $\mathrm{v}_{Q'}$, where $Q' = \mathscr{R}(Q)$.

This begs the question of which differential operators are recursion operators, for that we quote the following criteria for recursion operators, this is theorem 5.29 in [4].

Theorem 2.4.2. Suppose $\Delta[u] = 0$ is a differential equation. If $\mathscr{R} : \mathscr{A} \to \mathscr{A}$ is a linear operator such that

$$\mathrm{D}_\Delta \cdot \mathscr{R} = \tilde{\mathscr{R}} \cdot \mathrm{D}_\Delta \tag{2.4.5}$$

for all solutions u of $\Delta = 0$, where $\tilde{\mathscr{R}} : \mathscr{A} \to \mathscr{A}$ is a linear differential differential operator, then $\mathscr{R}$ is a recursion operator for the system.

In the case where $\Delta = u_t - K[u]$, corresponding to an evolution equation we have $\mathrm{D}_\Delta = D_t + \mathrm{D}_K$. In this case if $\mathscr{R}$ is a recursion operator, we see that the operator in (2.4.5) must be the same as $\mathscr{R}$. Thus the condition (2.4.5) reduces to

$$\mathrm{D}_\Delta \cdot \mathscr{R} = \mathscr{R} \cdot \mathrm{D}_\Delta \iff (D_t + \mathrm{D}_K)(\mathscr{R}) - \mathscr{R}(D_t + \mathrm{D}_K) \iff \mathscr{R}_t - [\mathrm{D}_K, \mathscr{R}] = 0 \iff \mathscr{R}_t = [\mathrm{D}_K, \mathscr{R}] \tag{2.4.6}$$

where $\mathscr{R}_t$ is as in definition 2.2.8, and $[\cdot, \cdot]$ is the regular Lie bracket.

Before we state the main theorem of this section, Magri's theorem, we must prove a few more propositions. The last section introduced a Noether theorem that created a relation between the Hamiltonian symmetries of our differential equation and its conservation laws. In particular, if we have a bi-Hamiltonian system

$$u_t = K_1[u] = \mathscr{D}\delta\mathscr{H}_1 = \mathscr{E}\delta\mathscr{H}_0$$

then we can apply our Noether theorem (theorem 2.3.1) to both the differential operators. This creates a ladder-like algorithm that bounces between the symmetries and conservation laws with respect to both operators, meaning that if $\mathscr{P}$ is any conserved functional for our equation above then both Hamiltonian vector fields $\mathrm{v}_{\mathscr{D}\delta\mathscr{P}}$ and $\mathrm{v}_{\mathscr{E}\delta\mathscr{P}}$ are symmetries of our equation. Since $\mathscr{H}_1$ and $\mathscr{H}_0$ are conserved functionals for the equation above, the vector field $\mathrm{v}_{K_1} = \mathrm{v}_{\mathscr{D}\delta\mathscr{H}_1} = \mathrm{v}_{\mathscr{E}\delta\mathscr{H}_0}$ as well as the vector fields $\mathrm{v}_{\mathscr{D}\delta\mathscr{H}_0}$, $\mathrm{v}_{\mathscr{E}\delta\mathscr{H}_1}$ are generalized symmetries. Now if say one of these new vector fields is a Hamiltonian vector field for the other Hamiltonian structure i.e.

$$\mathscr{E}\delta\mathscr{H}_1 = \mathscr{D}\delta\mathscr{H}_2$$

for some functional $\mathscr{H}_2$. Then $\mathscr{H}_2$ is a conserved functional and thus again by our Noether theorem we obtain another symmetry $\mathrm{v}_{\mathscr{E}\delta\mathscr{H}_2}$. We should start to notice the recursive pattern here. If we repeat this algorithm at the n-th stage, we determine a new functional $\mathscr{H}_n$ where

$$K_n = \mathscr{D}\delta\mathscr{H}_n = \mathscr{E}\delta\mathscr{H}_{n-1} \tag{2.4.7}$$

thus giving us a new conservation law for the original system, and another symmetry with characteristic $K_{n+1} = \mathscr{E}\delta\mathscr{H}_n$. So we can imagine that if we define an operator $\mathscr{R} = \mathscr{E}\cdot\mathscr{D}^{-1}$, then we write

$$K_{n+1} = \mathscr{R}K_n$$

and here since K_n and K_{n+1} are symmetries, we suspect that $\mathscr{R}$ is probably a recursion operator. However it is important to note that the algorithm above relies on the fact that at each step we can find a functional $\mathscr{H}_n$ such that (2.4.7) holds. This is essentially the content of Magri's theorem, however before we explicitly prove and state this, we must first prove that $\mathscr{R}$ is truly a recursion operator. Furthermore, we must introduce the requirement of non-degeneracy for at least one of the operators, and we know the operators are invertible since they form a ring.

Definition 2.4.5. A differential operator $\mathscr{D}$ is degenerate if there is a non-zero differential operator $\mathscr{D}'$ such that $\mathscr{D}'\cdot\mathscr{D} \equiv 0$.

Lemma 2.4.3. Let $u_t = K = \mathscr{D}\delta\mathscr{H}$ be a Hamiltonian evolution equation with corresponding vector field $\mathrm{v}_K = \hat{\mathrm{v}}_{\mathscr{H}}$. Then

$$\mathrm{pr}\ \hat{\mathrm{v}}_{\mathscr{H}}(\mathscr{D}) = \mathrm{D}_K\cdot\mathscr{D} + \mathscr{D}\cdot\mathrm{D}_K^*. \tag{2.4.8}$$

Proof.
Following the notation in [4], we let $L = \delta\mathscr{H}$, so $K = \mathscr{D}L$. Let $P = \delta\mathscr{P}$, $Q = \delta\mathscr{Q}$ be arbitrary variational derivatives. Then using the Jacobi identity in the form (2.2.13) and recalling the fact that the Hamiltonian vector field $\hat{\mathrm{v}}_{\mathscr{H}}$ has characteristic $\mathscr{D}\delta\mathscr{H}$ we obtain

$$\begin{aligned}
\int P\cdot\mathrm{pr}\ \hat{\mathrm{v}}_{\mathscr{H}}(\mathscr{D})\cdot Q\,dx &= \int\Big\{-L\cdot\mathrm{pr}\ \hat{\mathrm{v}}_{\mathscr{Q}}(\mathscr{D})L - Q\cdot\mathrm{pr}\ \hat{\mathrm{v}}_{\mathscr{P}}(\mathscr{D})\cdot L\Big\}\,dx \\
&= \int\Big\{P\cdot\mathrm{pr}\ \hat{\mathrm{v}}_{\mathscr{Q}}(\mathscr{D})\cdot L - Q\cdot\mathrm{pr}\ \hat{\mathrm{v}}_{\mathscr{P}}(\mathscr{D})\cdot L\Big\}\,dx \quad \text{Since } \mathscr{D} \text{ is skew-adjoint, so is } \mathrm{pr}\ \hat{\mathrm{v}}_{\mathscr{Q}}(\mathscr{D}). \\
&= \int\Big\{P\cdot[\mathrm{pr}\ \hat{\mathrm{v}}_{\mathscr{Q}}(\mathscr{D}L) - \mathscr{D}(\mathrm{pr}\ \hat{\mathrm{v}}_{\mathscr{Q}}(L)] \qquad \text{By (2.2.12).} \\
&\qquad - Q\cdot[\mathrm{pr}\ \hat{\mathrm{v}}_{\mathscr{P}}(\mathscr{D}L) - \mathscr{D}(\mathrm{pr}\ \hat{\mathrm{v}}_{\mathscr{P}}(L))]\Big\}\,dx \\
&= \int\Big\{P\cdot[\mathrm{D}_K(\mathscr{D}Q) - \mathscr{D}(\mathrm{D}_L(\mathscr{D}Q))] \qquad \text{By (2.2.10).} \\
&\qquad - Q\cdot[\mathrm{D}_K(\mathscr{D}P) - \mathscr{D}(\mathrm{D}_L(\mathscr{D}P))]\Big\}\,dx \\
&= \int\Big\{P\cdot\mathrm{D}_K(\mathscr{D}Q) - P\cdot\mathscr{D}\mathrm{D}_L(\mathscr{D}Q) - Q\cdot\mathrm{D}_K(\mathscr{D}P) + Q\cdot\mathscr{D}\mathrm{D}_L(\mathscr{D}P)\Big\}\,dx \\
&= \int\Big\{P\cdot\mathrm{D}_K(\mathscr{D}Q) - Q\mathrm{D}_K(\mathscr{D}P)\Big\}\,dx
\end{aligned}$$

the last equality coming from the fact that $\mathscr{D}$ is skew adjoint and since L is a variational derivative D_L is self adjoint by theorem 2.2.1. Thus

$$\int Q\mathscr{D}\mathrm{D}_L(\mathscr{D}P)\,dx = \int -\mathscr{D}(Q)\cdot \mathrm{D}_L(\mathscr{D}P)\,dx = \int -\mathrm{D}_L(\mathscr{D}Q)\mathscr{D}(P)\,dx = \int \mathscr{D}\mathrm{D}_L(\mathscr{D}Q)\cdot P\,dx.$$

Returning back to the integral, we obtain

$$\begin{aligned}
\int P\cdot \mathrm{pr}\,\hat{\mathrm{v}}_{\mathscr{H}}(\mathscr{D})\cdot Q\,dx &= \int \Big\{P\cdot \mathrm{D}_K(\mathscr{D}Q) - Q\mathrm{D}_{\mathrm{K}}(\mathscr{D}P)\Big\}\,dx \\
&= \int \Big\{P\cdot \mathrm{D}_K(\mathscr{D}Q) - \mathscr{D}P\cdot \mathrm{D}_K^*(Q)\Big\}\,dx \\
&= \int \Big\{P\cdot \mathrm{D}_K(\mathscr{D}Q) + P\cdot \mathscr{D}\mathrm{D}_K^*(Q)\Big\}\,dx \\
&= \int \Big\{P\cdot (\mathrm{D}_K\mathscr{D} + \mathscr{D}\mathrm{D}_K^*)\cdot Q\Big\}\,dx \\
\implies \int P\cdot(\mathrm{pr}\,\hat{\mathrm{v}}_{\mathscr{H}}(\mathscr{D}) - \mathrm{D}_K\mathscr{D} - \mathscr{D}\mathrm{D}_K^*)\cdot Q\,dx = 0 &
\end{aligned}$$

and since P, Q were arbitrary variational derivatives, the above holds if and only if

$$\mathrm{pr}\,\hat{\mathrm{v}}_{\mathscr{H}}(\mathscr{D}) = \mathrm{D}_K\cdot\mathscr{D} + \mathscr{D}\cdot\mathrm{D}_K^*.$$

□

Theorem 2.4.4. Let $u_t = K = \mathscr{D}\delta\mathscr{H}_1 = \mathscr{E}\delta\mathscr{H}_0$ be a bi-Hamiltonian evolution equation. Then the operator $\mathscr{R} = \mathscr{E}\mathscr{D}^{-1}$ is a recursion operator for the equation.

Proof.
In order to show $\mathscr{R}$ is a recursion operator for u_t, we must show it satisfies (2.4.6). On solutions u of our evolution equation we have the following

$$\begin{aligned}
\mathscr{R}_t &= \mathrm{pr}\,\mathrm{v}_K(\mathscr{R}) \qquad \text{By definition 2.2.8} \\
&= \mathrm{pr}\,\mathrm{v}_K(\mathscr{E})\cdot\mathscr{D}^{-1} - \mathscr{E}\cdot\mathscr{D}^{-1}\cdot\mathrm{pr}\,\mathrm{v}_K(\mathscr{D})\cdot\mathscr{D}^{-1} \\
&= (\mathrm{D}_K\cdot\mathscr{E} + \mathscr{E}\cdot\mathrm{D}_K^*)\cdot\mathscr{D}^{-1} - \mathscr{E}\cdot\mathscr{D}^{-1}(\mathrm{D}_K\cdot\mathscr{D} + \mathscr{D}\cdot\mathrm{D}_K^*)\mathscr{D}^{-1} \qquad \text{By the last lemma.} \\
&= \mathrm{D}_K\cdot\mathscr{E}\cdot\mathscr{D}^{-1} - \mathscr{E}\cdot\mathscr{D}^{-1}\mathrm{D}_K \\
&= \mathrm{D}_K\mathscr{R} - \mathscr{R}\mathrm{D}_K \\
&= [\mathrm{D}_K, \mathscr{R}]
\end{aligned}$$

which satisfies (2.4.6). Therefore $\mathscr{R} = \mathscr{E}\cdot\mathscr{D}^{-1}$ is a recursion operator for our evolution equation. □

Now that we can actually prove that $\mathscr{R}$ is a recursion operator, we quote one more technical lemma from [4] that we use in the proof of Magri's theorem. This is lemma 7.25 from [4].

Lemma 2.4.5. Suppose $\mathscr{D}, \mathscr{E}$ form a Hamiltonian pair, with $\mathscr{D}$ non-degenerate. Let $P, Q, R \in \mathscr{A}$ satisfy

$$\mathscr{E}P = \mathscr{D}Q, \qquad \mathscr{E}Q = \mathscr{D}R. \tag{2.4.9}$$

If $P = \delta\mathscr{P}$, $Q = \delta\mathscr{Q}$ are variational derivatives of functionals $\mathscr{P}, \mathscr{Q} \in \mathscr{F}$, then so is $R = \delta\mathscr{R}$ for some $\mathscr{R} \in \mathscr{F}$.

Finally, we have all the tools to state and prove Magri's Theorem, which is theorem 7.24 in [4].

Theorem 2.4.6 (Magri's Theorem)**.** Let

$$u_t = K_1[u] = \mathscr{D}\delta\mathscr{H}_1 = \mathscr{E}\delta\mathscr{H}_0 \tag{2.4.10}$$

be a bi-Hamiltonian system of evolution equations. Assume that the operator $\mathscr{D}$ of the Hamiltonian pair is non-degenerate. Let $\mathscr{R} = \mathscr{E} \cdot \mathscr{D}^{-1}$ be the corresponding recursion operator, and let $K_0 = \mathscr{D}\delta\mathscr{H}_0$. Assume that for each $n = 1, 2, ...$ we can recursively define

$$K_n = \mathscr{R}K_{n-1} \quad n \geq 1 \tag{2.4.11}$$

meaning that for each n, K_{n-1} lies in the image of $\mathscr{D}$. Then there exists a sequence of functionals $\mathscr{H}_0, \mathscr{H}_1\mathscr{H}_2, ...$ such that

1. for each $n \geq 1$, the evolution equation

$$u_t = K_n[u] = \mathscr{D}\delta\mathscr{H}_n = \mathscr{D}\delta\mathscr{H}_{n-1} \tag{2.4.12}$$

   is a bi-Hamiltonian system;

2. the corresponding evolutionary vector fields $\mathrm{v}_n = \mathrm{v}_{K_n}$ all mutually commute;

$$[\mathrm{v}_n, \mathrm{v}_m] = 0 \quad n, m \geq 0 \tag{2.4.13}$$

3. the Hamiltonian functionals $\mathscr{H}_n$ are all in involution with respect to either Poisson bracket:

$$\{\mathscr{H}_n, \mathscr{H}_n\}_{\mathscr{D}} = 0 = \{\mathscr{H}_n, \mathscr{H}_m\}_{\mathscr{E}}, \quad n, m \geq 0 \tag{2.4.14}$$

   and hence provide and infinite collection of conservation laws for each of the bi-Hamiltonian systems (2.4.12).

Proof.
(1): Since by our assumption $\mathscr{D}, \mathscr{E}$ form a Hamiltonian pair, the only thing we need to prove is the equality (2.4.12). Let us prove this by induction on n. First we let $K_n = \mathscr{D}Q_n$ where $Q_n \in \mathscr{A}$. Let us start with our base case of $n = 1$. For $n = 1$, let $Q_0 = \delta\mathscr{H}_0$ and $Q_1 = \delta\mathscr{H}_1$. Then by our assumptions

$$u_t = K_2[u] = \mathscr{R}K_1 = \mathscr{E} \cdot \mathscr{D}^{-1}\mathscr{D}\delta\mathscr{H}_1 = \mathscr{E}\delta\mathscr{H}_1 = \mathscr{D}Q_2$$

Now since

$$\mathscr{E}\delta\mathscr{H}_0 = \mathscr{D}\delta\mathscr{H}_1 \quad \text{and} \quad \mathscr{E}\delta\mathscr{H}_1 = \mathscr{D}Q_2$$

lemma 2.4.5 tells us that Q_1 is a variational derivative of some functional. Therefore $Q_2 = \delta\mathscr{H}_2$ for $\mathscr{H}_2 \in \mathscr{F}$ and

$$u_t = K_2[u] = \mathscr{D}\delta\mathscr{H}_2 = \mathscr{E}\delta\mathscr{H}_1.$$

Now, let us assume that

$$u_t = K_n[u] = \mathscr{D}\delta\mathscr{H}_n = \mathscr{E}\delta\mathscr{H}_{n-1}$$

is a bi-Hamiltonian system, we must show the same is true for K_{n+1}. By our assumption $K_{n+1} = \mathscr{D}Q_{n+1}$ and

$$u_t = K_{n+1} = \mathscr{R}K_n = \mathscr{E}\cdot\mathscr{D}^{-1} \cdot \mathscr{D}\delta\mathscr{H}_n = \mathscr{E}\delta\mathscr{H}_n = \mathscr{D}Q_{n+1}$$

then since

$$\mathscr{E}\delta\mathscr{H}_{n-1} = \mathscr{D}\delta\mathscr{H}_n \quad \text{and} \quad \mathscr{E}\delta\mathscr{H}_n = \mathscr{D}Q_{n+1}$$

lemma 2.4.5 implies that $Q_{n+1} = \mathscr{D}\delta\mathscr{H}_{n+1}$. Therefore

$$u_t = K_{n+1}[u] = \mathscr{D}\delta\mathscr{H}_{n+1} = \mathscr{E}\delta\mathscr{H}_n$$

is a bi-Hamiltonian system. We can conclude that (1) is true for all $n = 1, 2, ...$

(3): The proof for the second point following from part (3), therefore we will prove (3) first. Recall the notation from the theorem $\mathrm{v}_m = \mathrm{v}_{K_m} = \mathrm{v}_{\mathscr{D}\delta\mathscr{H}_m} = \mathrm{v}_{\mathscr{E}\delta\mathscr{H}_{m-1}}$, then according to proposition 2.2.2

$$\{\mathscr{H}_n, \mathscr{H}_m\}_{\mathscr{D}} = \mathrm{pr}\ \mathrm{v}_{\mathscr{D}\delta\mathscr{H}_m}(\mathscr{H}_n) = \mathrm{pr}\ \mathrm{v}_m(\mathscr{H}_n)$$
$$\{\mathscr{H}_n, \mathscr{H}_m\}_{\mathscr{E}} = \mathrm{pr}\ \mathrm{v}_{\mathscr{E}\delta\mathscr{H}_m}(\mathscr{H}_n) = \mathrm{pr}\ \mathrm{v}_{m+1}(\mathscr{H}_n)$$

therefore,

$$\{\mathscr{H}_n, \mathscr{H}_m\}_{\mathscr{D}} = \mathrm{pr}\ \mathrm{v}_m(\mathscr{H}_n) = \mathrm{pr}\ \mathrm{v}_{\mathscr{D}\delta\mathscr{H}_m}(\mathscr{H}_n) = \mathrm{pr}\ \mathrm{v}_{\mathscr{E}\delta\mathscr{H}_{m-1}}(\mathscr{H}_n) = \{\mathscr{H}_n, \mathscr{H}_{m-1}\}_{\mathscr{E}}.$$

Without loss of generality assume that $n < m$. We will use the equality above we created and the skew symmetry of the Poisson bracket to obtain the sequence of equalities

$$\begin{aligned}\{\mathscr{H}_n, \mathscr{H}_m\}_{\mathscr{D}} = \{\mathscr{H}_n, \mathscr{H}_{m-1}\}_{\mathscr{E}} &= -\{\mathscr{H}_{m-1}, \mathscr{H}_n\}_{\mathscr{E}} \\ &= -\{\mathscr{H}_{m-1}, \mathscr{H}_{n+1}\}_{\mathscr{D}} = \{\mathscr{H}_{n+1}, \mathscr{H}_{m-1}\}_{\mathscr{D}} = \ldots\ldots = \{\mathscr{H}_k, \mathscr{H}_k\} = 0.\end{aligned}$$

By subtracting 1 from m and adding 1 to n every time, we will eventually reach a middle point k where the integers are equal. The bracket we get at the end depends on the integers m and n, but we will either have the $\mathscr{E}$ or $\mathscr{D}$ Poisson bracket evaluated at the same functional $\mathscr{H}_k$ and thus the Poisson brackets are all equal to 0. Therefore the third part of the proof is proved.

(2): Now we finally conclude with the proof of the second part. Let $m, n \geq 0$ be arbitrary then proposition 2.3.3 tell us

$$[\mathrm{v}_n, \mathrm{v}_m] = [\hat{\mathrm{v}}_{\mathscr{H}_n}, \hat{\mathrm{v}}_{\mathscr{H}_m}] = \hat{\mathrm{v}}_{\{\mathscr{H}_n, \mathscr{H}_m\}_{\mathscr{D}}} = 0$$

and

$$[\mathrm{v}_n, \mathrm{v}_m] = [\hat{\mathrm{v}}_{\mathscr{H}_n}, \hat{\mathrm{v}}_{\mathscr{H}_m}] = \hat{\mathrm{v}}_{\{\mathscr{H}_n, \mathscr{H}_m\}_{\mathscr{E}}} = 0$$

where the equality to 0 is thanks to the third part of the theorem. Therefore we have proved all three parts of Magri's theorem.

□

We should mention a technical point at this stage. In Magri's theorem we required that K_{n-1} lies in the image of $\mathscr{D}$, this is quite a powerful assumption. As Olver mentioned in [4], in most of the examples to date this seems to be true, however we don't quite have a theorem to state the effect that we can always find a differential function in the image of $\mathscr{D}$ that gives us the equality to K_n.

Example 2.4.2. Let us return to our example of the KdV equation, earlier in this section we showed that the KdV equation is a bi-Hamiltonian system. Now we wish to apply Magri's theorem to the KdV equation. We start by recalling the bi-Hamiltonian form

$$u_t = u_{xxx} + uu_x = \mathscr{D}\delta\mathscr{H}_1 = \mathscr{E}\delta\mathscr{H}_0$$

where

$$\mathscr{D} = D_x \qquad \mathscr{H}_1 = \int \Big\{ -\frac{1}{2}u_x^2 + \frac{1}{6}u^3 \Big\}\, dx$$

and

$$\mathscr{E} = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x \qquad \mathscr{H}_0[u] = \int \frac{1}{2}u^2\ dx.$$

The recursion operator $\mathscr{R} = \mathscr{E} \cdot \mathscr{D}^{-1} = D_x^2 + \frac{1}{3}u_x D_x^{-1} + \frac{2}{3}u$, then we apply $\mathscr{R}$ recursively to the KdV equation to obtain new symmetries. The first stage in this recursion is

$$\begin{aligned} u_t = \mathscr{R}(u_{xxx} + uu_x) = \mathscr{E} \cdot \mathscr{D}^{-1}\mathscr{D}\delta\mathscr{H}_1 = \mathscr{E}\delta\mathscr{H}_1 &= (D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x)\delta(-\frac{1}{2}u_x^2 + \frac{1}{6}u^3) \\ &= (D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x)(u_{xx} + \frac{1}{2}u^2) \\ &= u_{xxxxx} + \frac{10}{3}u_x u_{xx} + \frac{5}{3}uu_{xxx} + \frac{5}{6}u^2 u_x \\ &= D_x(u_{xxxx} + \frac{5}{3}uu_{xx} + \frac{5}{6}u_x^2 + \frac{5}{18}u^3) \\ &= D_x(\delta\mathscr{H}_2) \\ &= \mathscr{D}\delta\mathscr{H}_2 \end{aligned}$$

where $\mathscr{H}_2[u] = \int \left\{ \frac{1}{2}u_{xx}^2 - \frac{5}{6}uu_x^2 + \frac{5}{72}u^4 \right\} dx$, thus by Magri's theorem this is a conservation law.

3 Pseudo-Spherical Surfaces

In the previous section, given a multi-Hamiltonian evolution equation we used an algorithm to compute an infinite hierarchy of conservation laws for the system. However finding evolution equations that admit a multi-Hamiltonian structure is not easy. This is why when Cavalcante and Tenenblat in [2] provided an algorithm for finding a sequence of conservation laws for an evolution equation describing pseudo-spherical surfaces the question of whether or not such equations are also multi-Hamiltonian arose naturally. In this chapter we will focus on the algorithm provided in [1] and [2] that creates an infinite sequence of conservation laws for evolution equations describing pseudo-spherical surfaces.

3.1 What is a Pseudo-Spherical Surface?

In this section we will introduce the notion of a pseudo-spherical surface. The material in this section will be paraphrased or directly quoted from [1], [2], [3], [7] and [8].

Let (M, g) be a Riemannian surface, the structure equations for M are

$$\begin{aligned} d\omega_1 &= \omega_3 \wedge \omega_2 \\ d\omega_2 &= \omega_1 \wedge \omega_3 \\ d\omega_3 &= K\omega_2 \wedge \omega_1 \end{aligned} \tag{3.1.1}$$

where ω_1 and ω_2 are the one-forms that determine the metric $g = (\omega_1)^2 + (\omega_2)^2$. Furthermore, ω_3 is the connection form determined by the Levi-Civita connection and K is the Gaussian curvature. Thanks to Gauss's Theorema Egregium, which says that the Gaussian curvature K depends only on g and its derivatives, we know that Gaussian curvature is an isometry invariant. Now we are ready to define what we mean by a pseudospherical surface.

Definition 3.1.1. Let M be a two dimensional C^∞ manifold. M is a pseudo-spherical surface (p.s.s.) if M has constant negative Gaussian curvature $K = -1$.

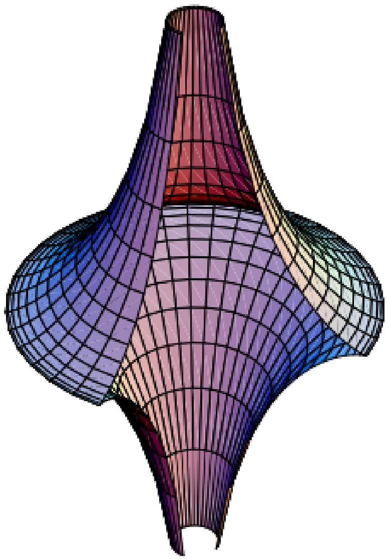
In this case the structure equations (3.1.1) become

$$\begin{aligned} d\omega_1 &= \omega_3 \wedge \omega_2 \\ d\omega_2 &= \omega_1 \wedge \omega_3 \\ d\omega_3 &= -\omega_2 \wedge \omega_1 = \omega_1 \wedge \omega_2. \end{aligned} \tag{3.1.2}$$

Furthermore, Minding's theorem [7] says that any two surfaces with the same constant Gaussian curvature are locally isometric, therefore all p.s.s are locally isometric. It is important to note, when we locally embed these surfaces isometrically into $\mathbb{E}^3$ that even though all pseudo-spherical surfaces are isometric, they are not necessarily congruent under the rigid motions of $\mathbb{E}^2$.

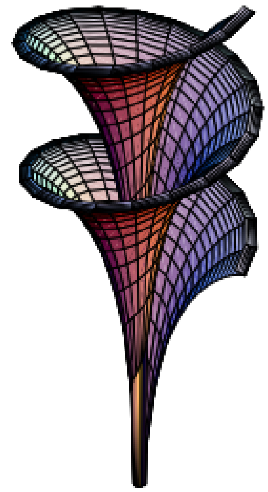
For example two of the most common pseudo-spherical surfaces are the pseudo-sphere and Dini's surface, shown in Figure 1 and 2 below. Dini's surface and the pseudo-sphere of radius 1 both have Gaussian Curvature $K = -1$. Dini's surface is obtained by twisting the pseudo-sphere [3], obviously it cannot be transformed by using any of the rigid motions of $\mathbb{E}^2$ therefore the surfaces are not congruent.

Figure 1: **The Pseudo-Sphere**



From "Modern Differential Geometry of Curves and Surfaces with Mathematica" (pg. 480) by Gray, A., Abbena, E., and Salamon, S.Chapman and Hall CRC. Copyright 2006 by Taylor and Francis Group.

Figure 2: **Dini's Surface**



From "Modern Differential Geometry of Curves and Surfaces with Mathematica" (pg. 482) by Gray, A., Abbena, E., and Salamon, S.Chapman and Hall CRC. Copyright 2006 by Taylor and Francis Group.

3.2 Evolution Equations Describing Pseudo-Spherical Surfaces

In the previous section we defined a pseudo-spherical surface as having constant Gaussian curvature $K = -1$. In this section, we will introduce the notion of a differential equation that describes a pseudo-spherical surface. In the paper [1] by Chern and Tenenblat, there exists a large classification of differential equations which are p.s.s., we will examine one such example, the KdV equation. The information and examples in this section will be paraphrased or directly quoted from [1] and [2]. Let us first begin with some definitions.

Definition 3.2.1. Let M be a two-dimensional differential (C^∞) manifold, with coordinates (x,t). We say that a differential equation for a real valued function $u(x,t)$ describes a pseudo-spherical surface if there exists differential functions $f_{ab},\ 1 \leq a \leq 3,\ 1 \leq b \leq 2$ depending on u and finitely many derivatives, such that the 1-forms

$$\omega_a = f_{a1}dx + f_{a2}dt \qquad a = 1,2,3 \tag{3.2.1}$$

satisfy the structure equations of a p.s.s.

This definition is motivated in part by the inverse scattering problem introduced by Ablowitz, Kaup, Newell and Segur in paper [12] where they considered the integrability of the system

$$dv = \Omega v \tag{3.2.2}$$

where v is a vector valued function and Ω is a traceless matrix of one-forms given by

$$\Omega = \begin{bmatrix} -i\xi dx + Adt & qdx + Bdt \\ rdx + Cdt & i\xi dx - Adt \end{bmatrix}$$

where q, r, A, B, C are functions of (x,t) and ξ is a spectral parameter which is central to the solution of evolution equations by method of inverse scattering. The integrability condition for the system is then $d\Omega - \Omega \wedge \Omega = 0$. Now suppose we had the 1-forms

$$\begin{aligned} \omega_1 &= (r+q)dx + (C+B)dt \\ \omega_2 &= \eta dx + 3Adt \\ \omega_3 &= (r-q)dx + (C-B)dt \end{aligned}$$

where $\eta = i\xi$, then the integrability condition of (3.2.2) is equivalent to saying that these one-forms satisfy the equations (3.1.2). If we suppose that M is a two dimensional C^∞ manifold with coordinates (x,t) and require that $\omega_1 \wedge \omega_2 \neq 0$ we can define a metric on M as $g = (\omega_1)^2 + (\omega_2)^2$. Then the first two equations in (3.1.2) define the connection form ω_3 and the last equation determines the Gaussian curvature of M to be $K = -1$, so M is a p.s.s.. This justifies our definition 3.2.1.

In [1], Chern and Tenenblat restrict to the case where $f_{21} = \eta$ where η is a real parameter (which can be interpreted as a spectral parameter in the method of inverse scattering). We will do the same thing in this thesis, thus giving us the following definition.

Definition 3.2.2. A differential equation $u(x,t)$ *describes a pseudo-spherical surface* if there exists differential functions $f_{ab},\ 1 \leq a \leq 3,\ 1 \leq b \leq 2,\ f_{21} = \eta$, depending on u and its derivatives such that the 1-forms

$$\begin{aligned} \omega_1 &= f_{11}dx + f_{12}dt \\ \omega_2 &= \eta dx + f_{22}dt \\ \omega_3 &= f_{31}dx + f_{31}dt \end{aligned} \tag{3.2.3}$$

satisfy the structure equations (3.1.2) with the further imposition that $\omega_1 \wedge \omega_2 \neq 0$. We call such a differential equation a p.s.s. equation.

Remark. In the definition above, we restrict to the solutions $u(x,t)$ of the differential equation for which $\omega_1 \wedge \omega_2 \neq 0$ in order for the metric $g = (\omega_1)^2 + (\omega_2)^2$ to be positive definite.

The other differential functions f_{ab} can also depend on η. In fact if the functions f_{ab} are analytic in η using the algorithm in [2] we can compute a infinite sequence of conservation laws for the differential equation. This will be done in the next section but at this moment we regard η as a constant and that is all. Let us work through an example from [1] so that we can understand how differential equations that describe psuedo-spherical surfaces arise.

Example 3.2.1. Let M be a Riemmanian surface parametrized by coordinates x, t. Consider the 1-forms given by

$$\begin{aligned} \omega_1 &= (1-u)dx + (-u_{xx} + \eta u_x - \eta^2 u - 2u^2 + \eta^2 + 2u)dt \\ \omega_2 &= \eta dx + (\eta^3 + 2\eta u - 2u_x)dt \\ \omega_3 &= -(1+u)dx + (-u_{xx} + \eta u_x - \eta^2 u - 2u^2 - \eta^2 - 2u)dt. \end{aligned} \tag{3.2.4}$$

We plug these into the structure equations (3.1.2), since we want these equations to determine some pseudo-spherical surface. We start with the last structure equation.

$$\begin{aligned} d\omega_3 &= -u_t dt \wedge dx + (-u_{xxx} + \eta u_{xx} - \eta^2 u_x - 4uu_x - 2u_x)dx \wedge dt \\ &= (u_t - u_{xxx} +_{xx} - \eta^2 u_x - 4uu_x - 2u_x)\ dx \wedge dt \end{aligned}$$

$$\begin{aligned} \omega_1 \wedge \omega_2 &= (1-u)(\eta^3 + 2\eta u - 2u_x)dx \wedge dt + \eta(-u_{xx} + \eta u_x - \eta^2 u - 2u^2 + \eta^2 + 2u)dt \wedge dx \\ &= (-2u_x + 2uu_x + \eta u_x x - \eta^2 u_x)dx \wedge dt \end{aligned}$$

The structure equations will be satisfied if and only if

$$\begin{aligned} d\omega_3 &= \omega_1 \wedge \omega_2 \\ (u_t - u_{xxx} +_{xx} - \eta^2 u_x - 4uu_x - 2u_x)\ dx \wedge dt &= (-2u_x + 2uu_x + \eta u_x x - \eta^2 u_x)dx \wedge dt \\ \iff u_t - u_{xxx} +_{xx} - \eta^2 u_x - 4uu_x - 2u_x &= -2u_x + 2uu_x + \eta u_x x - \eta^2 u_x \\ \iff u_t &= u_{xxx} + 6uu_x \end{aligned}$$

therefore, the last structure equation is satisfied if and only if u satisfies the KdV equation. We also check the first two structure equations.

$$d\omega_1 = (u_t - u_{xxx} + \eta u_x - \eta^2 u_x - 4uu_x + 2u_x)dx \wedge dt$$

$$\omega_3 \wedge \omega_2 = (2u_x + 2uu_x + \eta u_{xx} - \eta^2 u_x)dx \wedge dt$$

$d\omega_1 = \omega_3 \wedge \omega_2$ if and only if

$$u_t - u_{xxx} + \eta u_x - \eta^2 u_x - 4uu_x + 2u_x = 2u_x + 2uu_x + \eta u_{xx} - \eta^2 u_x$$

$$\iff u_t = u_{xxx} + 6uu_x$$

the first structure equation is satisfied if and only if u is a solution of the KdV equation. Computing the pieces of the second structure equation

$$d\omega_2 = (2\eta u_x - 2u_{xx})dx \wedge dt$$

$$\omega_1 \wedge \omega_3 = (-2u_{xx} + 2\eta u_x)dx \wedge dt$$

we see that the second structure equation is trivially satisfied. Therefore since there exists 1-forms that satisfy the structure equations (3.1.2) on solutions u of the KdV equation, we satisfy definition 3.2.2 and thus the KdV equation is a p.s.s. equation.

There are many more examples in [1] of differential equations which describe p.s.s., many of these equations are well known, such as the Sine Gordon equation and the MKdV equation. Furthermore in [1], Chern and Tenenblat introduce a variety of theorems that can help describe classes of evolution equations which describe pseudo-spherical surfaces. Many papers since then use these theorems to write explicit formulas for evolution equations which describe pseudo-spherical surfaces. In the later sections we will examine an example from [5], this paper uses the theorems in [1] to classify some fifth order evolution equations which describe a p.s.s.

3.3 P.S.S. Equations and Conservation Laws

In [1], Chern and Tenenblat explain that a differential equation which describes a p.s.s. can give rise to conservation laws. They use an algorithm that is further explained by Cavalcante and Tenenblat in [2] to find such conservation laws. If we restrict to the case where the functions f_{ab} are analytic in η where η is a parameter we can create an infinite sequence of conservation laws. In this section we will go through the algorithm explained in [2] which gives rise to an infinite hierarchy of conservation laws.

Let M be a two dimensional Riemannian manifold with coordinates (x, t) and Gaussian curvature $K = -1$. Let e_1, e_2 and v_1, v_2 be two orthonormal frame fields. Then

$$\begin{aligned} e_1 &= \cos\phi v_1 + \sin\phi v_2 \\ e_2 &= -\sin\phi v_1 + \cos\phi v_2 \end{aligned} \tag{3.3.1}$$

where ϕ is the rotation angle of the frames. Suppose a differential equation for $u(x, t)$ describes a p.s.s then there exists differentiable functions $f_{ij},\ 1 \le i \le 3,\ 1 \le j$ such that the one forms

$$\omega_i = f_{i1}dx + f_{i2}dt$$

satisfy the structure equations for a p.s.s.. These structure equations are satisfied if and only if

$$\begin{aligned} d(f_{11}dx + f_{12}dt) &= (f_{31}dx + f_{32}dt) \wedge (f_{21}dx + f_{22}dt) \\ (f_{12,x} - f_{11,t})\ dx \wedge dt &= (f_{31} \cdot f_{22} - f_{21} \cdot f_{32})\ dx \wedge dt \\ \implies\ f_{12,x} - f_{11,t} &= f_{31} \cdot f_{22} - f_{21} \cdot f_{32} \\ \\ d(f_{21}dx + f_{22}dt) &= (f_{11}dx + f_{12}dt) \wedge (f_{31}dx + f_{32}dt) \\ \implies\ f_{22,x} - f_{21,t} &= f_{11} \cdot f_{32} - f_{31} \cdot f_{12} \\ \\ d(f_{31}dx + f_{32}dt) &= (f_{11}dx + f_{12}dt) \wedge (f_{21}dx + f_{22}dt) \\ \implies\ f_{32,x} - f_{31,t} &= f_{11} \cdot f_{22} - f_{21} \cdot f_{12}. \end{aligned} \tag{3.3.2}$$

We now have the proper set up we need to quote the main theorem from [2], that serves as the building blocks to the algorithm which allows us to compute an infinite hierarchy of conservation laws. In particular if we allow the differential functions f_{ab} to be analytic in a parameter η, a single conservation law is analytic in η and thus every coefficient in the infinite series gives us a new conservation law.

Theorem 3.3.1. Let $\Delta[u] = 0$ be a differential equation that describes a pseudo-spherical surface with associated one forms

$$\begin{aligned} \omega_1 &= f_{11}dx + f_{12}dt \\ \omega_2 &= f_{21}dx + f_{22}dt \\ \omega_3 &= f_{31}dx + f_{32}dt \end{aligned} \tag{3.3.3}$$

then, the following statements are true.

1. For every solution $u(x, t)$ of the differential equation, the system of equations for $\phi(x, t)$

$$\begin{aligned} \phi_x &= f_{31} + f_{11}\sin\phi + f_{21}\cos\phi \\ \phi_t &= f_{32} + f_{12}\sin\phi + f_{22}\cos\phi \end{aligned} \tag{3.3.4}$$

   is completely integrable.

2. For any solution ϕ of (3.3.4),

$$\omega = (f_{11}\cos\phi - f_{21}\sin\phi)dx + (f_{12}\cos\phi - f_{22}\sin\phi)dt \tag{3.3.5}$$

   is a closed one-form.

3. If the f_{ij}'s are analytic functions of a parameter η at zero, then the solutions $\phi(x,t,\eta)$ of (3.3.4) and the one form ω are also analytic in η at zero.

Proof.
(1): To prove the system (3.3.4) is integrable, we need only show $\phi_{t,x} = \phi_{x,t}$ in view of the Frobenius theorem. Let us verify this.

$$\begin{aligned}
\phi_{x,t} &= f_{31,t} + f_{11,t}\sin\phi + f_{11}\cos\phi\cdot\phi_t + f_{21,t}\cos\phi - f_{21}\sin\phi\cdot\phi_t \\
&= f_{31,t} + f_{11,t}\sin\phi + f_{21,t}\cos\phi + (f_{11}\cos\phi - f_{21}\sin\phi)(f_{32} + f_{12}\sin\phi + f_{22}\cos\phi) \\
&= f_{31,t} - f_{13}f_{21} + \sin\phi(f_{11,t} - f_{21}f_{32}) + \cos\phi(f_{11}f_{32} + f_{21,t}) + \cos^2\phi(f_{11}f_{22} + f_{21}f_{12}) \\
&\quad + \cos\phi\sin\phi(f_{11}f_{12} - f_{21}f_{22})
\end{aligned}$$

$$\begin{aligned}
\phi_{t,x} &= f_{32,x} + f_{12,x}\sin\phi + f_{12}\cos\phi\cdot\phi_x + f_{22,x}\cos\phi - f_{22}\sin\phi\cdot\phi_x \\
&= f_{32,x} + f_{12,x}\sin\phi + f_{22,x}\cos\phi + (f_{12}\cos\phi - f_{22}\sin\phi)(f_{31} + f_{11}\sin\phi + f_{21}\cos\phi) \\
&= f_{32,x} - f_{11}f_{22} + \sin\phi(f_{12,x} - f_{22}f_{31}) + \cos\phi(f_{22,x} + f_{12}f_{31}) + \cos^2\phi(f_{12}f_{21} + f_{22}f_{11}) \\
&\quad + \sin\phi\cos\phi(f_{12}f_{11} - f_{22}f_{21})
\end{aligned}$$

Here we make a substitution with some of the equations in (3.3.2) and we obtain

$$\begin{aligned}
\phi_{t,x} &= f_{31,t} - f_{12}f_{21} + \sin\phi(f_{11,t} - f_{21}f_{32}) + \cos\phi(f_{11}f_{32} + f_{21,t}) + + \cos^2\phi(f_{12}f_{21} + f_{22}f_{11}) \\
&\quad + \sin\phi\cos\phi(f_{12}f_{11} - f_{22}f_{21}) \\
&= \phi_{x,t}
\end{aligned}$$

which tells us the system (3.3.4) is integrable.

(2): To show that ω is a closed one-form, we need to show $d(\omega) = 0$.

$$\begin{aligned}
d(\omega) &= (f_{11,t}\cos\phi - f_{11}\sin\phi\cdot\phi_t - f_{21,t}\phi - f_{21}\cos\phi\cdot\phi_t)\, dt\wedge dx \\
&\quad + (f_{12,x}\cos\phi - f_{12}\sin\phi\cdot\phi_x - f_{22,x}\sin\phi - f_{22}\cos\phi\cdot\phi_x)\, dx\wedge dt \\
&= (f_{12,x}\cos\phi - f_{12}f_{31}\sin\phi - f_{12}f_{11}\sin^2\phi - f_{12}f_{21}\sin\phi\cos\phi - f_{22,x}\sin\phi - f_{22}f_{31}\cos\phi \\
&\quad - f_{22}f_{11}\cos\phi\sin\phi - f_{22}f_{21}\cos^2\phi - f_{11,t}\cos\phi + f_{11}f_{32}\sin\phi + f_{11}f_{12}\sin^2\phi + f_{11}f_{22}\cos\phi\sin\phi \\
&\quad + f_{21,t}\sin\phi + f_{21}f_{32}\cos\phi + f_{21}f_{12}\cos\phi\sin\phi + f_{21}f_{22}\cos^2\phi)\, dx\wedge dt \\
&= \Big(f_{11}f_{12} - f_{12}f_{11} + \cos\phi(f_{12,x} - f_{22}f_{31} - f_{11,t} + f_{21}f_{32}) + \sin\phi(f_{21,t} + f_{11}f_{32} - f_{22,x} - f_{12}f_{31}) \\
&\quad \cos^2\phi(f_{12}f_{11} - f_{22}f_{21} - f_{11}f_{12} + f_{21}f_{22}) + \sin\phi\cos\phi(f_{11}f_{22} + f_{21}f_{12} - f_{12}f_{21} - f_{22}f_{11})\Big)\, dx\wedge dt \\
&= 0
\end{aligned}$$

by substituting the equations in (3.3.2) when necessary. Thus ω is a closed one-form.

(3): Assume that the f_{ij}'s are analytic functions of the parameter η at zero. Then, we can view the equations in (3.3.4) as two ordinary differential equations whose right hand side is analytic in (ϕ,η). The first part of the theorem says the solution $\phi(x,t,\eta)$ exists, therefore by theorem 8.4 in [6], which relates the solutions of an ode to its parameters, we obtain that $\phi(x,t,\eta)$ is an analytic function in η for an appropriate neighbourhood of zero. Since both ϕ and all the $f'_{ij}s$ are analytic in η, it follows that ω is also analytic in η at zero just by substituting the $f'_{ij}s$ and ϕ with their respective power series. □

Here is where idea of conservation laws come into play. Note that in the theorem above, the second statement about ω being a closed one-form is equivalent to the statement

$$D_t(f_{11}\cos\phi - f_{21}\sin\phi) + D_x(f_{12}\cos\phi - f_{22}\sin\phi) = 0$$

on all solutions ϕ of (3.3.4) and all solutions $u(x,t)$ of our differential equation. Therefore this is a conservation law for our differential equation $\Delta[u] = 0$ that describes a p.s.s. We will assume that the functions f_{ij} are analytic in η, then we can write out (3.3.4) and ω as a power series of η. This allows us to describe the solutions ϕ as a power series of η and to define ω as a power series of η for which the coefficients are conservation laws of our p.s.s. differential equation $\Delta = 0$. We will summarize this all in a corollary, but first we introduce the notation used in [2] and do a few calculations to write ϕ_x, ϕ_t and ω as a power series of η.

Since f_{ij} and ϕ are analytic in η we can write them both as

$$f_{ij}(x,t,\eta) = \sum_{k=0}^{\infty} f_{ij}^k(x,t)\eta^k \qquad \phi(x,t,\eta) = \sum_{j=0}^{\infty} \phi_j(x,t)\eta^j. \tag{3.3.6}$$

For fixed x, t, we consider the following functions of η.

$$\begin{aligned} C(\eta) &= \cos(\phi) = \cos\Big(\sum_{j=0}^{\infty} \phi_j \eta^j\Big) \\ S(\eta) &= \sin(\phi) = \sin\Big(\sum_{j=0}^{\infty} \phi_j \eta^j\Big) \end{aligned} \tag{3.3.7}$$

We evaluate these functions and their derivatives at $\eta = 0$.

$$\begin{aligned} C(0) &= \cos(\phi_0) \\ S(0) &= \sin(\phi_0) \\ \frac{d^kC}{d\eta^k}(0) &= -(k-1)!\sum_{i=0}^{k-1} \frac{k-i}{i!}\frac{d^iS}{d\eta^i}(0)\phi_{k-i} \\ \frac{d^kS}{d\eta^k}(0) &= (k-1)!\sum_{i=0}^{k-1} \frac{k-i}{i!}\frac{d^iC}{d\eta^i}(0)\phi_{k-i} \end{aligned} \tag{3.3.8}$$

We write $C(\eta)$ and $S(\eta)$ as a Taylor series centered at $\eta = 0$ and then substitute this and the power series of the f_{ij}'s into the equations for ϕ_x and ϕ_t to write them as analytic functions at $\eta = 0$.

$$\begin{aligned} \phi_x &= \sum_{k=0}^{\infty} f_{31}^k \eta^k + \Big(\sum_{k=0}^{\infty} f_{11}^k \eta^k\Big) \cdot \Big(\sum_{n=0}^{\infty} \frac{S^{(n)}(0)}{n!}\eta^n\Big) + \Big(\sum_{k=0}^{\infty} f_{21}^k \eta^k\Big) \cdot \Big(\sum_{n=0}^{\infty} \frac{C^{(n)}(0)}{n!}\eta^n\Big) \\ &= \sum_{j=0}^{\infty} \Big(f_{31}^j + \sum_{a=0}^{j} f_{11}^a \frac{S^{(j-a)}}{(j-a)!}(0) + \sum_{a=0}^{j} f_{21}^a \frac{C^{(j-a)}}{(j-a)!}(0)\Big) \cdot \eta^j \end{aligned}$$

$$\implies \phi_{0,x} = f_{31}^0 + f_{11}^0 S(0) + f_{21}^0 C(0)$$

and

$$\begin{aligned} \phi_{j,x} &= f_{31}^j + \sum_{a=1}^{j} \frac{1}{(j-a)!}[f_{11}^a S^{(j-a)}(0) + f_{21}^a C^{(j-a)}(0)] + \frac{1}{j!}\cdot(j-1)!\sum_{i=0}^{j-1} \frac{(j-i)}{i!}[f_{11}^0 \cdot C^{(i)}(0) - f_{21}^0 S^{(i)}(0)]\phi_{j-i} \\ &= f_{31}^j + \sum_{a=1}^{j} \frac{1}{(j-a)!}[f_{11}^a S^{(j-a)}(0) + f_{21}^a C^{(j-a)}(0)] + [f_{11}C(0) - f_{21}S(0)] \cdot \phi_j \\ &\quad + \frac{1}{j}\sum_{i=i}^{j-1} \frac{(j-i)}{i!}[f_{11}^0 \cdot C^{(i)}(0) - f_{21}^0 S^{(i)}(0)]\phi_{j-i} \end{aligned}$$

for $j \geq 1$. If we repeat this process for ϕ_t we obtain

$$\begin{aligned}\phi_t &= \sum_{k=0}^{\infty} f_{32}^k \eta^k + \Big(\sum_{k=0}^{\infty} f_{12}^k \eta^k\Big) \cdot \Big(\sum_{n=0}^{\infty} \frac{S^{(n)}(0)}{n!} \eta^n\Big) + \Big(\sum_{k=0}^{\infty} f_{22}^k \eta^k\Big) \cdot \Big(\sum_{n=0}^{\infty} \frac{C^{(n)}(0)}{n!} \eta^n\Big) \\ &= \sum_{j=0}^{\infty} \Big(f_{32}^j + \sum_{a=0}^{j} f_{12}^a \frac{S^{(j-a)}}{(j-a)!}(0) + \sum_{a=0}^{j} f_{22}^a \frac{C^{(j-a)}}{(j-a)!}(0)\Big) \cdot \eta^j\end{aligned}$$

$$\implies \phi_{0,t} = f_{32}^0 + f_{12}^0 S(0) + f_{22}^0 C(0)$$

and

$$\begin{aligned}\phi_{j,t} = f_{32}^j &+ \sum_{a=1}^{j} \frac{1}{(j-a)!}[f_{12}^a S^{(j-a)}(0) + f_{22}^a C^{(j-a)}(0)] + [f_{12} C(0) - f_{22} S(0)] \cdot \phi_j \\ &+ \frac{1}{j} \sum_{i=i}^{j-1} \frac{(j-i)}{i!}[f_{12}^0 \cdot C^{(i)}(0) - f_{22}^0 S^{(i)}(0)]\phi_{j-i}\end{aligned}$$

for $j \geq 1$. If we do the same thing to the one-form ω we obtain

$$\begin{aligned}\omega &= \Big[\Big(\sum_{k=0}^{\infty} f_{11}^k \eta^k\Big) \cdot \Big(\sum_{n=0}^{\infty} \frac{C^{(n)}(0)}{n!} \eta^n\Big) - \Big(\sum_{k=0}^{\infty} f_{21}^k \eta^k\Big) \cdot \Big(\sum_{n=0}^{\infty} \frac{S^{(n)}(0)}{n!} \eta^n\Big)\Big] dx \\ &\quad + \Big[\Big(\sum_{k=0}^{\infty} f_{12}^k \eta^k\Big) \cdot \Big(\sum_{n=0}^{\infty} \frac{C^{(n)}(0)}{n!} \eta^n\Big) - \Big(\sum_{k=0}^{\infty} f_{22}^k \eta^k\Big) \cdot \Big(\sum_{n=0}^{\infty} \frac{S^{(n)}(0)}{n!} \eta^n\Big)\Big] dt \\ &= \Big[\sum_{j=0}^{\infty} \Big(\sum_{i=0}^{j} \frac{1}{(j-i)!} \Big(f_{11}^i C^{(j-i)}(0) - f_{21}^i S^{(j-i)}(0)\Big)\Big) \cdot \eta^j\Big] dx \\ &\quad + \Big[\sum_{j=0}^{\infty} \Big(\sum_{i=0}^{j} \frac{1}{(j-i)!} \Big(f_{12}^i C^{(j-i)}(0) - f_{22}^i S^{(j-i)}(0)\Big)\Big) \cdot \eta^j\Big] dt \\ &= \sum_{j=0}^{\infty} \Big(\sum_{i=0}^{j} \frac{1}{(j-i)!} \Big(\Big[f_{11}^i C^{(j-i)}(0) - f_{21}^i S^{(j-i)}(0)\Big] dx + \Big[f_{12}^i C^{(j-i)}(0) - f_{22}^i S^{(j-i)}(0)\Big] dt\Big)\Big) \cdot \eta^j\end{aligned}$$

$$\implies \omega^j = \sum_{i=0}^{j} \frac{1}{(j-i)!} \Big(\Big[f_{11}^i C^{(j-i)}(0) - f_{21}^i S^{(j-i)}(0)\Big] dx + \Big[f_{12}^i C^{(j-i)}(0) - f_{22}^i S^{(j-i)}(0)\Big] dt\Big)$$

and since ω is a closed one form, each of the coefficients ω^j in the power series is also a closed one-form. Before we summarize this all into a corollary, for the sake of efficiency we will define some functions to represent pieces of the equations we computed above. In order to be cohesive, we use the same notation as in [2] for this and we define

$$\begin{aligned}H_k^{ij} &= f_{1k}^i \frac{d^{j-i}C}{d\eta^{j-i}}(0) - f_{2k}^i \frac{d^{j-i}S}{d\eta^{j-i}}(0) \\ L_k^{ij} &= f_{1k}^i \frac{d^{j-i}S}{d\eta^{j-i}}(0) + f_{2k}^i \frac{d^{j-i}C}{d\eta^{j-i}}(0) \\ F_{1k} &= f_{3k}^1 + L_k^{11} \\ F_{lk} &= f_{3k}^l + \frac{1}{l} \cdot \sum_{r=1}^{l-1} \frac{l-r}{r!} H_k^{0r} \phi_{l-r} + \sum_{r=1}^{l} \frac{1}{(l-r)!} L_k^{rl}\end{aligned}$$

where i, j, l are non-negative integers such that $j \geq i$, $l \geq 2$ and $k = 1, 2$. Then

$$\phi_{0,x} = f_{31}^0 + L_1^{00} \quad \text{and} \quad \phi_{0,t} = f_{32}^0 + L_2^{00}$$

$$\phi_{j,x} = H_1^{00}\phi_j + F_{j1} \quad \text{and} \quad \phi_{j,t} = H_2^{00}\phi_j + F_{j2}$$

and

$$\omega^j = \sum_{i=0}^{j} \frac{1}{(j-i)!}(H_i^{ij}\ dx + H_2^{ij}\ dt).$$

We summarize this all in the following corollary which is quoted from [2].

Corollary 3.3.1. Let $f_{ij}(x, t, \eta)$, $1 \leq i \leq 3$, $1 \leq j \leq 2$ be differentiable functions of x, t that are analytic at $\eta = 0$ and satisfy the equations (3.3.2), then using the notation above, the following two statements are true.

1. The solutions ϕ of (3.3.4) are analytic at $\eta = 0$ and ϕ_0 is determined by

$$\phi_{0,x} = f_{31}^0 + L_1^{00} \quad \text{and} \quad \phi_{0,t} = f_{32}^0 + L_2^{00} \tag{3.3.9}$$

   and, for $j \geq 1$, ϕ_j is recursively determined by

$$\phi_{j,x} = H_1^{00}\phi_j + F_{j1} \quad \text{and} \quad \phi_{j,t} = H_2^{00}\phi_j + F_{j2}. \tag{3.3.10}$$

2. For any solution ϕ and any integer $j \geq 0$

$$\omega^j = \sum_{i=0}^{j} \frac{1}{(j-i)!}(H_i^{ij}\ dx + H_2^{ij}\ dt) \tag{3.3.11}$$

   is a closed one form.

If we have an evolution equation $\Delta[u] = 0$ for which its solution $u(x, t)$ is a p.s.s, and we let the f_{ij}'s be analytic functions of η, then the previous theorem and corollary say that ω^j in (3.3.11) are closed one forms. Therefore for each j, we obtain a conservation law for our evolution equation, where the conserved densities and fluxes are given by

$$\mathbb{D}_j = \sum_{i=0}^{j} \frac{1}{(j-i)!} H_1^{ij} \quad \text{and} \quad \mathbb{F}_j = -\sum_{i=0}^{j} \frac{1}{(j-i)!} H_2^{ij}$$

for $j \geq 0$, thus giving us a infinite sequence of conservation laws.

4 Classes of Pseudo-Spherical Surface Equations that are Multi-Hamiltonian

4.1 A Quintic P.S.S. Equation that is Bi-Hamiltonian

In the previous sections we explained how using the algorithm in [2] allows us to create an infinite hierarchy of conservation laws for a differential equation describing a p.s.s.. We also showed how differential equations that have a bi-Hamiltonian structure can admit an infinite hierarchy of conservation laws. This begs the question of whether there is a large class of p.s.s. equations that are bi-Hamiltonian. Throughout this thesis we have followed the running example of the KdV equation. We saw that it is bi-Hamiltonian, and it is also a p.s.s. equation. In this section we examine an example of a quintic p.s.s. equation from [5] that is bi-Hamiltonian. We will apply the algorithm from [2] and the algorithm provided by Magri's theorem to determine an infinite hierarchy of conservation laws for the equation.

In this chapter we will interchangeably use two different kinds of notation for the partial derivatives of u.

$$z_0 = u, \qquad z_{0,t} = u_t, \qquad z_1 = \frac{\partial u}{\partial x}, ..., \qquad z_k = \frac{\partial^k u}{\partial x^k}$$

The evolution equation

$$u_t = u_{xxxxx} - 5uu_{xxx} - 10u_x u_{xx} + \frac{15}{2}u^2 u_x \tag{4.1.1}$$

obtained from [5] describes a pseudo-spherical surface. The associated 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$ for $1 \leq i \leq 3$, are given by

$$\begin{aligned}
f_{11} &= \frac{z_0}{2} + 1 \qquad f_{21} = \eta \neq 0 \qquad f_{31} = \frac{z_0}{2} - 1 \\
f_{12} &= \frac{1}{2}\Big[z_4 - \eta z_3 + z_2(-4z_0 + \eta^2 - 2) - 3z_1^2 - \eta z_1(-3z_0 + \eta^2) - 2(\frac{z_0}{2} + 1)(-\frac{3}{2}z_0^2 + \eta^2 z_0 - \eta^4)\Big] \\
f_{22} &= z_3 - 3z_0 z_1 + \eta^2 z_1 - \eta(z_2 - \frac{3}{2}z_0^2 + \eta^2 z_0 - \eta^4) \\
f_{32} &= f_{12} + 2z_2 - 3z_0^2 + 2\eta^2 z_0 - 2\eta^4.
\end{aligned}$$

where η is a parameter. This equation is example 2.8 from [5], which was obtained by using theorem 2.3 from [5]. This theorem proves that the equation (4.1.1) is a p.s.s equation. We will check that the one-forms above satisfy the structure equations of a p.s.s.

$$\begin{aligned}
d\omega_1 &= \Big[\frac{d}{dt}(f_{11}) - \frac{d}{dx}(f_{12})\Big]dt \wedge dx \\
&= \Big[-\frac{1}{2}z_0 z_3 + \frac{3}{2}z_0^2 z_1 + z_3 - 3z_0 z_1 + \eta(\frac{1}{2}z_4 - \frac{3}{2}z_1^2 - \frac{3}{2}z_0 z_2) + \eta^2(-\frac{1}{2}z_3 + z_0 z_1 + z_1) \\
&\quad + \frac{1}{2}\eta^3 z_2 - \frac{1}{2}\eta^4 z_1\Big]dt \wedge dx
\end{aligned}$$

$$\begin{aligned}
\omega_3 \wedge \omega_2 &= \Big[f_{31} \cdot f_{21} - f_{31} \cdot f_{22}\Big]dt \wedge dx \\
&= \Big[-\frac{1}{2}z_0 z_3 + \frac{3}{2}z_0^2 z_1 + z_3 - 3z_0 z_1 + \eta(\frac{1}{2}z_4 - \frac{3}{2}z_1^2 - \frac{3}{2}z_0 z_2) + \eta^2(-\frac{1}{2}z_3 + z_0 z_1 + z_1) \\
&\quad + \frac{1}{2}\eta^3 z_2 - \frac{1}{2}\eta^4 z_1\Big]dt \wedge dx
\end{aligned}$$

Therefore the first structure equation $d\omega_1 = \omega_3 \wedge \omega_2$ is satisfied.

$$
\begin{aligned}
d\omega_2 &= \Big[\frac{d}{dt}(f_{21}) - \frac{d}{dx}(f_{22})\Big] dt\wedge dx \\
&= \Big[z_4 - 3z_1^2 - 3z_0z_2 + \eta(3z_0z_1 - z_3) + \eta^2 z_2 - \eta^3 z_1\Big] dx\wedge dt
\end{aligned}
$$

$$
\begin{aligned}
\omega_1\wedge\omega_3 &= \Big[f_{11}\cdot f_{31} - f_{31}\cdot f_{12}\Big] dx\wedge dt \\
&= \Big[z_4 - 3z_1^2 - 3z_0z_2 + \eta(3z_0z_1 - z_3) + \eta^2 z_2 - \eta^3 z_1\Big] dx\wedge dt
\end{aligned}
$$

Therefore the second structure equation $d\omega_2 = \omega_1\wedge\omega_3$ is satisfied.

$$
\begin{aligned}
d\omega_3 &= \Big[\frac{d}{dt}(f_{31}) - \frac{d}{dx}(f_{32})\Big] dt\wedge dx \\
&= \Big[-\frac{1}{2}z_0z_3 + \frac{3}{2}z_0^2z_1 + 3z_0z_1 - z_3 + \eta(\frac{1}{2}z_4 - \frac{3}{2}z_1^2 - \frac{3}{2}z_0z_2) + \eta^2(-\frac{1}{2}z_3 + z_0z_1 - z_1) \\
&\quad + \frac{1}{2}\eta^3 z_2 - \frac{1}{2}\eta^4 z_1\Big] dt\wedge dx
\end{aligned}
$$

$$
\begin{aligned}
\omega_1\wedge\omega_2 &= \Big[f_{12}f_{21} - f_{11}f_{22}\Big] dt\wedge dx \\
&= \Big[-\frac{1}{2}z_0z_3 + \frac{3}{2}z_0^2z_1 + 3z_0z_1 - z_3 + \eta(\frac{1}{2}z_4 - \frac{3}{2}z_1^2 - \frac{3}{2}z_0z_2) + \eta^2(-\frac{1}{2}z_3 + z_0z_1 - z_1) \\
&\quad + \frac{1}{2}\eta^3 z_2 - \frac{1}{2}\eta^4 z_1\Big] dt\wedge dx
\end{aligned}
$$

Finally the last structure equation $d\omega_3 = \omega_1\wedge\omega_2 \neq 0$ is satisfied.

The equation (4.1.1) has bi-Hamiltonian form

$$u_t = \mathscr{D}\delta\mathscr{H}_0 = \mathscr{E}\delta\mathscr{H}_1$$

where

$$\mathscr{D} = D_x \quad , \quad \mathscr{H}_0 = \int \Big\{\frac{5}{8}u^4 + \frac{1}{2}u_{xx}^2 + \frac{5}{2}uu_x^2\Big\}\, dx$$

and,

$$\mathscr{E} = D_x^3 - 2uD_x - u_x \quad , \quad \mathscr{H}_1 = \int \Big\{-\frac{1}{2}u_x^2 - \frac{1}{2}u^3\Big\}\, dx$$

with associated Poisson brackets

$$
\begin{aligned}
\{\mathscr{P},\mathscr{Q}\}_{\mathscr{D}} &= \int \delta\mathscr{P}\cdot D_x\delta\mathscr{Q}\, dx \\
\{\mathscr{P},\mathscr{Q}\}_{\mathscr{E}} &= \int \delta\mathscr{P}\cdot (D_x^3 - 2uD_x - u_x)\delta\mathscr{Q}\, dx
\end{aligned}
\tag{4.1.2}
$$

Proof.

$$
\begin{aligned}
\delta\mathscr{H}_0 &= \frac{5}{2}u^3 + \frac{5}{2}u_x^2 - D_x(5uu_x) + D_x^2(u_{xx}) \\
&= \frac{5}{2}u^3 - \frac{5}{2}u_x^2 - 5uu_{xx} + u_{xxxx}
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{D}\delta\mathscr{H}_0 = D_x(\delta\mathscr{H}_0) &= D_x(\frac{5}{2}u^3 - \frac{5}{2}u_x^2 - 5uu_{xx} + u_{xxxx}) \\
&= \frac{15}{2}u^2u_x - 10u_xu_{xx} - 5uu_{xxx} + u_{xxxxx}
\end{aligned}
$$

This is the left hand side of equation (4.1.1), thus $u_t = \mathscr{D}\delta\mathscr{H}_0$.

$$\begin{aligned}\delta\mathscr{H}_1 &= -\frac{3}{2}u^2 - D_x(-u_x)\\ &= u_{xx} - \frac{3}{2}u^2\end{aligned}$$

$$\begin{aligned}\mathscr{E}\delta\mathscr{H}_1 &= (D_x^3 - 2uD_x - u_x)(u_{xx} - \frac{3}{2}u^2)\\ &= u_{xxxxx} - 2uu_{xxx} - u_xu_{xx} - D_x^3(\frac{3}{2}u^2) + 2uD_x(\frac{3}{2}u^2) + \frac{3}{2}u^2u_x\\ &= u_{xxxxx} - 5uu_{xxx} - 10u_xu_{xx} + \frac{15}{2}u^2u_x\end{aligned}$$

This is the left hand side of equation (4.1.1), thus $u_t = \mathscr{E}\delta\mathscr{H}_1$. In order to claim that this truly is a bi-Hamiltonian system, we must show that the operators $\mathscr{D}$ and $\mathscr{E}$ are a Hamiltonian pair. We start by showing that they are both Hamiltonian operators. As we mentioned in an earlier section this amounts to showing that the operators are skew-adjoint and that the Jacobi identity is satisfied. We begin by showing they are skew adjoint.

$$\mathscr{D} = D_x \quad \text{then,} \quad \mathscr{D}^* = -D_x. \implies \mathscr{D}^* = -\mathscr{D}$$

$$\begin{aligned}\mathscr{E} = D_x^3 - 2uD_x - u_x \quad \text{then,} \quad \mathscr{E}^*(Q) &= -D_x^3(Q) - D_x(-2uQ) - u_xQ\\ &= -D_x^3(Q) + 2uD_x(Q) + 2u_xQ - u_xQ\\ &= -D_x^3(Q) + 2uD_x(Q) + u_xQ\end{aligned}$$

Therefore $\mathscr{E}^* = -D_x^3 + 2uD_x + u_x \implies \mathscr{E}^* = -\mathscr{E}$.

Now we must show that the Jacobi identity is satisfied for both operators. As we mentioned earlier this amounts to showing

$$\text{pr v}_{\mathscr{D}\theta}(\Theta_{\mathscr{D}}) = 0 \quad \text{and} \quad \text{pr v}_{\mathscr{E}\theta}(\Theta_{\mathscr{E}}) = 0$$

where

$$\Theta_{\mathscr{D}} = \frac{1}{2}\int\{\theta \wedge \mathscr{D}\theta\}\,dx \quad \text{and} \quad \Theta_{\mathscr{E}} = \frac{1}{2}\int\{\theta \wedge \mathscr{E}\theta\}\,dx.$$

Let us start with $\mathscr{D} = D_x$.

$$\mathscr{D}(\theta) = D_x(\theta) = \theta_x \implies \Theta_{\mathscr{D}} = \frac{1}{2}\int\{\theta \wedge \theta_x\}\,dx$$

$$\begin{aligned}\text{pr v}_{\mathscr{D}\theta}(\Theta_{\mathscr{D}}) &= \text{pr v}_{\theta_x}\Big(\frac{1}{2}\int\{\theta \wedge \theta_x\}\,dx\Big)\\ &= \frac{1}{2}\int \text{pr v}_{\theta_x}(\theta \wedge \theta_x)\,dx\\ &= 0\end{aligned}$$

We show the same is true for $\mathscr{E} = D_x^3 - 2uD_x - u_x$.

$$\mathscr{E}(\theta) = \theta_{xxx} - 2u\theta_x - u_x\theta$$

$$\begin{aligned}\Theta_{\mathscr{E}} &= \frac{1}{2}\int \{\theta \wedge \mathscr{E}\theta\}\, dx. \\ &= \frac{1}{2}\int \Big\{\theta \wedge (\theta_{xxx} - 2u\theta_x - u_x\theta)\Big\}\, dx \\ &= \frac{1}{2}\int \Big\{\theta \wedge \theta_{xxx} - 2u\theta \wedge \theta_x\Big\}\, dx\end{aligned}$$

$$\begin{aligned}\text{pr v}_{\mathscr{E}\theta}(\Theta_{\mathscr{E}}) &= \text{pr v}_{\mathscr{E}\theta}\Big(\frac{1}{2}\int \theta \wedge \theta_{xxx} - 2u\theta \wedge \theta_x\, dx\Big) \\ &= \frac{1}{2}\int -2(\mathscr{E}(\theta)) \wedge \theta \wedge \theta_x\, dx \\ &= -\int \theta_{xxx} \wedge \theta \wedge \theta_x\, dx \\ &= -\int \theta \wedge \theta_x \wedge D_x(\theta_{xx})\, dx \\ &= \int D_x(\theta \wedge \theta_x) \wedge \theta_{xx}\, dx \qquad \text{By integrating by parts.} \\ &= \int \Big\{(\theta_x \wedge \theta_x + \theta \wedge \theta_{xx}) \wedge \theta_{xx}\Big\}\, dx \\ &= 0\end{aligned}$$

We can conclude that both $\mathscr{D}$ and $\mathscr{E}$ are skew-adjoint and their Poisson brackets satisfy the Jacobi identity, therefore they are both Hamiltonian operators. The last thing we must show is that $\mathscr{D}$ and $\mathscr{E}$ form a Hamiltonian pair. To do this we need to show that

$$\text{pr v}_{\mathscr{D}\theta}(\Theta_{\mathscr{E}}) + \text{pr v}_{\mathscr{E}\theta}(\Theta_{\mathscr{D}}) = 0.$$

$$\begin{aligned}\text{pr v}_{\mathscr{D}\theta}(\Theta_{\mathscr{E}}) + \text{pr v}_{\mathscr{E}\theta}(\Theta_{\mathscr{D}}) &= \text{pr v}_{\mathscr{D}\theta}\Big(\frac{1}{2}\int \Big\{\theta \wedge \theta_{xxx} - 2u\theta \wedge \theta_x\Big\}\, dx\Big) + \text{pr v}_{\mathscr{E}\theta}\Big(\frac{1}{2}\int \{\theta \wedge \theta_x\}\, dx\Big) \\ &= -\int \mathscr{D}(\theta) \wedge \theta \wedge \theta_x\, dx \\ &= -\int \theta_x \wedge \theta \wedge \theta_x\, dx \\ &= 0\end{aligned}$$

Therefore $\mathscr{D}$ and $\mathscr{E}$ form a Hamiltonian pair. We can now safely say that equation (4.1.1) is bi-Hamiltonian and can be written in the form below.

$$u_t = \mathscr{D}\delta\mathscr{H}_0 = \mathscr{E}\delta\mathscr{H}_1$$

□

Since $\mathscr{D}$ and $\mathscr{E}$ form a Hamiltonian pair, we can now create the recursion operator $\mathscr{R} = \mathscr{E}\mathscr{D}^{-1}$ and use Magri's theorem to compute an infinite hierarchy of conservation laws for (4.1.1). We will compute a few in the next section.

4.2 Calculation of Conservation Laws by the Method of Magri's Theorem

In the previous section we saw that the equation

$$u_t = u_{xxxxx} - 5uu_{xxx} - 10u_x u_{xx} + \frac{15}{2}u^2 u_x$$

has a bi-Hamiltonian form

$$u_t = \mathscr{D}\delta\mathscr{H}_0 = \mathscr{E}\delta\mathscr{H}_1$$

where

$$\mathscr{D} = D_x \quad \text{and} \quad \mathscr{H}_0 = \int \Big\{\frac{5}{8}u^4 + \frac{1}{2}u_{xx}^2 + \frac{5}{2}uu_x^2\Big\}\, dx$$

and

$$\mathscr{E} = D_x^3 - 2uD_x - u_x \quad \text{and} \quad \mathscr{H}_1 = \int \Big\{-\frac{1}{2}u_x^2 - \frac{1}{2}u^3\Big\}\, dx.$$

We will now compute the recursion operator $\mathscr{R} = \mathscr{E}\mathscr{D}^{-1}$ and a few conservation laws for the system. To begin the first conserved densities for the system are namely

$$\mathscr{H}_0 = \int \Big\{\frac{5}{8}u^4 + \frac{1}{2}u_{xx}^2 + \frac{5}{2}uu_x^2\Big\}\, dx \quad \text{and} \quad \mathscr{H}_1 = \int \Big\{-\frac{1}{2}u_x^2 - \frac{1}{2}u^3\Big\}\, dx.$$

The recursion operator $\mathscr{R} = \mathscr{E}\mathscr{D}^{-1}$ is

$$\mathscr{R} = \mathscr{E}\mathscr{D}^{-1} = (D_x^3 - 2uD_x - u_x)D_x^{-1} = D_x^2 - u_x D_x^{-1} - 2u.$$

Now we can apply $\mathscr{R}$ successively to the left hand side of our equation to obtain new conservation laws. The first step in this recursion is

$$u_t = \mathscr{R}(u_{xxxxx} - 5uu_{xxx} - 10u_x u_{xx} + \frac{15}{2}u^2 u_x) = \mathscr{R}(\mathscr{D}\delta\mathscr{H}_0) = \mathscr{E}\mathscr{D}^{-1}\mathscr{D}\delta\mathscr{H}_0 = \mathscr{E}\delta\mathscr{H}_0$$

$$\begin{aligned}
\mathscr{E}\delta\mathscr{H}_0 &= (D_x^3 - 2uD_x - u_x)(\frac{5}{2}u^3 - \frac{5}{2}u_x^2 - 5uu_{xx} + u_{xxxx}) \\
&= D_x^3(\frac{5}{2}u^3 - \frac{5}{2}u_x^2 - 5uu_{xx} + u_{xxxx}) - 2uD_x(\frac{5}{2}u^3 - \frac{5}{2}u_x^2 - 5uu_{xx} + u_{xxxx}) \\
&\quad - \frac{5}{2}u^3 u_x + \frac{5}{2}u_x^3 + 5uu_x u_{xx} - u_x u_{xxxx} \\
&= u_{xxxxxxx} + \frac{35}{2}u_x^3 + 70uu_x u_{xx} + \frac{35}{2}u^2 u_{xxx} - 35u_{xx}u_{xxx} - 21u_x u_{xxxx} - 7uu_{xxxxx} - \frac{35}{2}u^3 u_x
\end{aligned}$$

According to Magri's theorem we should be able to write $\mathscr{E}\delta\mathscr{H}_0$ as $\mathscr{D}\delta\mathscr{H}_2$ for some functional $\mathscr{H}_2 \in \mathscr{F}$. We claim that

$$u_t = \mathscr{E}\delta\mathscr{H}_0 = \mathscr{D}\delta\mathscr{H}_2$$

where

$$\mathscr{H}_2 = \int \Big\{-\frac{1}{2}u_{xxx}^2 - \frac{35}{4}u^2 u_x^2 + \frac{7}{2}uu_x u_{xxx} - \frac{7}{8}u^5\Big\}\, dx.$$

$$\begin{aligned}
\delta\mathscr{H}_2 &= \frac{7}{2}u_x u_{xxx} - \frac{35}{2}uu_x^2 - \frac{35}{8}u^4 - D_x(\frac{7}{2}uu_{xxx} - \frac{35}{2}u^2 u_x) - D_x^3(\frac{7}{2}uu_x - u_{xxx}) \\
&= u_{xxxxxx} + \frac{35}{2}uu_x^2 - 14u_x u_{xxx} - 7uu_{xxxx} + \frac{35}{2}u^2 u_{xx} - \frac{21}{2}u_{xx}^2 - \frac{35}{8}u^4
\end{aligned}$$

$$
\begin{aligned}
\mathscr{D}\delta\mathscr{H}_2 &= D_x(u_{xxxxxx} + \frac{35}{2}uu_x^2 - 14u_xu_{xxx} - 7uu_{xxxx} + \frac{35}{2}u^2u_{xx} - \frac{21}{2}u_{xx}^2 - \frac{35}{8}u^4) \\
&= u_{xxxxxxx} + \frac{35}{2}u_x^3 + 35uu_xu_{xx} - 14u_{xx}u_{xxx} - 14u_xu_{xxxx} - 7u_xu_{xxxx} - 7uu_{xxxxx} \\
&\quad + 35uu_xu_{xx} + \frac{35}{2}u^2u_{xxx} - 21u_{xx}u_{xxx} - \frac{35}{2}u^3u_x \\
&= u_{xxxxxxx} + \frac{35}{2}u_x^3 + 70uu_xu_{xx} + \frac{35}{2}u^2u_{xxx} - 35u_{xx}u_{xxx} - 21u_xu_{xxxx} - 7uu_{xxxxx} - \frac{35}{2}u^3u_x \\
&= \mathscr{E}\delta\mathscr{H}_0
\end{aligned}
$$

Thus the first step of the recursion gives us

$$
\begin{aligned}
u_t &= \mathscr{E}\delta\mathscr{H}_0 = \mathscr{D}\delta\mathscr{H}_2 \\
&= u_{xxxxxxx} + \frac{35}{2}u_x^3 + 70uu_xu_{xx} + \frac{35}{2}u^2u_{xxx} - 35u_{xx}u_{xxx} - 21u_xu_{xxxx} - 7uu_{xxxxx} - \frac{35}{2}u^3u_x
\end{aligned}
$$

with consequent conserved density

$$
\mathscr{H}_2 = \int \Big\{ -\frac{1}{2}u_{xxx}^2 - \frac{35}{4}u^2u_x^2 + \frac{7}{2}uu_xu_{xxx} - \frac{7}{8}u^5 \Big\}\, dx.
$$

The next step would be to apply $\mathscr{R}$ to $\mathscr{D}\delta\mathscr{H}_2$ and follow in the same fashion as we have above to obtain another conserved density for the system. By Magri's theorem, we know that we can recursively do this and thus we obtain an infinite hierarchy of conservation laws for our equation.

4.3 Calculation of Conservation Laws Using P.S.S. algorithm

Given a p.s.s. equation, we can use the algorithm in [1] and [2] to compute an infinite hierarchy of conservation laws. The equation we gave in the previous two subsections is a p.s.s. equation, thus we will use this algorithm to compute a few of the conserved densities. Recall our equation

$$u_t = u_{xxxxx} - 5uu_{xxx} - 10u_x u_{xx} + \frac{15}{2}u^2 u_x$$

we rewrite it as

$$z_{0,t} = z_5 - 5z_0 z_3 - 10z_1 z_2 + \frac{15}{2}z_0^2 z_1$$

and its associated 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$ for $1 \leq i \leq 3$.

$$\begin{aligned}
f_{11} &= \frac{z_0}{2} + 1 \quad\quad f_{21} = \eta \neq 0 \quad\quad f_{31} = \frac{z_0}{2} - 1\\
f_{12} &= \frac{1}{2}\Big[z_4 - \eta z_3 + z_2(-4z_0 + \eta^2 - 2) - 3z_1^2 - \eta z_1(-3z_0 + \eta^2) - 2(\frac{z_0}{2} + 1)(-\frac{3}{2}z_0^2 + \eta^2 z_0 - \eta^4)\Big]\\
&= \frac{1}{2}z_4 - 2z_0 z_2 - z_2 - \frac{3}{2}z_1^2 + \frac{3}{4}z_0^3 + \frac{3}{2}z_0^2 + \eta(-\frac{1}{2}z_3 + \frac{3}{2}z_0 z_1) + \eta^2(\frac{1}{2}z_2 - \frac{1}{2}z_0^2 - z_0) + \eta^3(-\frac{1}{2}z_1)\\
&\quad + \eta^4(\frac{1}{2}z_0 + 1)\\
f_{22} &= z_3 - 3z_0 z_1 + \eta^2 z_1 - \eta(z_2 - \frac{3}{2}z_0^2 + \eta^2 z_0 - \eta^4)\\
&= z_3 - 3z_0 z_1 + \eta(\frac{3}{2}z_0^2 - z_2) + \eta^2 z_1 + \eta^3(-z_0) + \eta^5\\
f_{32} &= f_{12} + 2z_2 - 3z_0^2 + 2\eta^2 z_0 - 2\eta^4\\
&= \frac{1}{2}z_4 - 2z_0 z_2 + z_2 - \frac{3}{2}z_1^2 + \frac{3}{4}z_0^3 - \frac{3}{2}z_0^2 + \eta(-\frac{1}{2}z_3 + \frac{3}{2}z_0 z_1) + \eta^2(\frac{1}{2}z_2 - \frac{1}{2}z_0^2 + z_0) + \eta^3(-\frac{1}{2}z_1)\\
&\quad + \eta^4(\frac{1}{2}z_0 - 1)
\end{aligned}$$

Since these are all analytic in η we can apply the algorithm in [2] to find the conserved densities. Recall the algorithm from the earlier section. Suppose we have two orthonormal frames for our surface, then recall ϕ is the rotation angle of the frames. Recall ϕ and the f_{ij}'s are analytic in η so

$$f_{ij}(x,t,\eta) = \sum_{k=0}^{\infty} f_{ij}^k(x,t)\eta^k \quad \text{and} \quad \phi(x,t,\eta) = \sum_{j=0}^{\infty} \phi_j(x,t)\eta^j.$$

Now we recall all the pieces of the algorithm. For fixed x, t

$$C(\eta) = \cos(\phi) = \cos\Big(\sum_{j=0}^{\infty}\phi_j(x,t)\eta^j\Big) \quad\quad \text{and} \quad\quad S(\eta) = \sin(\phi) = \sin\Big(\sum_{j=0}^{\infty}\phi_j(x,t)\eta^j\Big)$$

$$C(0) = \cos(\phi_0) \quad \text{and} \quad S(0) = \sin(\phi_0)$$

$$\begin{aligned}
\frac{d^k C}{d\eta^k}(0) &= -(k-1)!\sum_{i=0}^{k-1}\frac{k-i}{i!}\frac{d^i S}{d\eta^i}(0)\phi_{k-i}\\
\frac{d^k S}{d\eta^k}(0) &= (k-1)!\sum_{i=0}^{k-1}\frac{k-i}{i!}\frac{d^i C}{d\eta^i}(0)\phi_{k-i}
\end{aligned}$$

for $k \geq 1$, and,

$$
\begin{aligned}
H_k^{ij} &= f_{1k}^i \frac{d^{j-i}C}{d\eta^{j-i}}(0) - f_{2k}^i \frac{d^{j-i}S}{d\eta^{j-i}}(0) \\
L_k^{ij} &= f_{1k}^i \frac{d^{j-i}S}{d\eta^{j-i}}(0) + f_{2k}^i \frac{d^{j-i}C}{d\eta^{j-i}}(0) \\
F_{1k} &= f_{3k}^1 + L_k^{11} \\
F_{lk} &= f_{3k}^l + \frac{1}{l}\sum_{r=1}^{l-1} \frac{l-r}{r!} H_k^{0r}\phi_{l-r} + \sum_{r=1}^{l} \frac{1}{(l-r)!} L_k^{rl}
\end{aligned}
$$

where i, j, l are non-negative integers such that $j \geq i$, $l \geq 2$ and $k = 1, 2$. ϕ_0 is determined by

$$\phi_{0,x} = f_{31}^0 + L_1^{00} \quad \text{and} \quad \phi_{0,t} = f_{32}^0 + L_2^{00}$$

and, for $j \geq 1$, ϕ_j is recursively defined by

$$\phi_{j,x} = H_1^{00}\phi_j + F_{j1} \quad \text{and} \quad \phi_{j,t} = H_2^{00}\phi_j + F_{j2}.$$

Lastly the conserved densities and fluxes are given by

$$\mathbb{D}_j = \sum_{i=0}^{j} \frac{1}{(j-i)!} H_1^{ij} \quad \text{and} \quad \mathbb{F}_j = -\sum_{i=0}^{j} \frac{1}{(j-i)!} H_2^{ij}$$

for $j \geq 0$.

We will use the above algorithm to compute a few of the conserved densities for our equation (4.1.1).

$$
\begin{aligned}
\phi_{0,x} &= f_{31}^0 + L_1^{00} \\
&= f_{31}^0 + f_{11}^0 S(0) + f_{21}^0 C(0) \\
&= \frac{z_0}{2} - 1 + (\frac{z_0}{2} + 1)\sin(\phi_0)
\end{aligned}
$$

$$
\begin{aligned}
\phi_{0,t} &= f_{32}^0 + L_2^{00} \\
&= f_{32}^0 + f_{12}^0 S(0) + f_{22}^0 C(0) \\
&= \frac{1}{2}z_4 - 2z_0z_2 + z_2 - \frac{3}{2}z_1^2 + \frac{3}{4}z_0^3 - \frac{3}{2}z_0^2 + \sin(\phi_0)(\frac{1}{2}z_4 - 2z_0z_2 - z_2 - \frac{3}{2}z_1^2 + \frac{3}{4}z_0^3 + \frac{3}{2}z_0^2) \\
&\quad + \cos(\phi_0)(z_3 - 3z_0z_1)
\end{aligned}
$$

For $j \geq 1$,

$$
\begin{aligned}
\phi_{j,x} &= H_1^{00}\phi_j + F_{j1} \\
&= \Big(\frac{z_0}{2} + 1\Big)\cos(\phi_0)\cdot\phi_j + f_{31}^j + \frac{1}{j!}\sum_{r=1}^{j-1}\Big[\Big(\frac{z_0}{2}+1\Big)\cdot\frac{j-r}{r!}\cdot\frac{d^rC}{d\eta^r}(0)\cdot\phi_{j-r}\Big] + \frac{1}{(j-1)!}\cdot\frac{d^{j-1}C}{d\eta^{j-1}}(0)
\end{aligned}
$$

$$
\begin{aligned}
\phi_{j,t} &= H_2^{00}\phi_j + F_{j2} \\
&= \Big[\Big(\frac{1}{2}z_4 - 2z_0z_2 - z_2 - \frac{3}{2}z_1^2 + \frac{3}{4}z_0^3 + \frac{3}{2}z_0^2\Big)\cos(\phi_0) - \Big(z_3 - 3z_0z_1\Big)\sin(\phi_0)\Big]\phi_j + f_{31}^j \\
&\quad + \frac{1}{j}\sum_{r=1}^{j-1}\frac{j-r}{r!}\Big[\Big(\frac{1}{2}z_4 - 2z_0z_2 - z_2 - \frac{3}{2}z_1^2 + \frac{3}{4}z_0^3 + \frac{3}{2}z_0^2\Big)\frac{d^rC}{d\eta^r}(0) - \Big(z_3 - 3z_0z_1\Big)\frac{d^rS}{d\eta^r}(0)\Big]\phi_{j-r} \\
&\quad + \sum_{r=1}^{j}\Big[f_{12}^r\frac{d^{j-r}S}{d\eta^{j-r}}(0) + f_{22}^r\frac{d^{j-r}C}{d\eta^{j-r}}(0)\Big]
\end{aligned}
$$

These equations make it quite difficult to solve for ϕ_0 or ϕ_j for any $j \geq 1$. However we know a solution exists by the Frobenius theorem, this was proved in section 3.3. The first few terms in the sequence of conserved densities for (4.1.1) using the algorithm are given by

$$\begin{aligned}
\mathbb{D}_0 &= H_1^{00} \\
&= f_{11}^0 C(0) - f_{21}^0 S(0) \\
&= (\frac{z_0}{2} + 1)\cos(\phi_0)
\end{aligned}$$

$$\begin{aligned}
\mathbb{D}_1 &= H_1^{01} + H_1^{11} \\
&= f_{11}^0 \frac{d^1 C}{d\eta^1}(0) - f_{21}^0 \frac{d^1 S}{d\eta^1}(0) + f_{11}^1 C(0) - f_{21}^1 S(0) \\
&= (\frac{z_0}{2} + 1)(-S(0)\phi_1) - \sin(\phi_0) \\
&= -\sin(\phi_0)(1 + \frac{1}{2} z_0 \phi_1 + \phi_1)
\end{aligned}$$

$$\begin{aligned}
\mathbb{D}_2 &= \frac{1}{2} H_1^{02} + H_1^{12} + H_1^{22} \\
&= \frac{1}{2} f_{11}^0 \frac{d^2 C}{d\eta^2}(0) - \frac{1}{2} f_{21}^0 \frac{d^2 S}{d\eta^2}(0) + f_{11}^1 \frac{d^1 C}{d\eta^1}(0) - f_{21}^1 \frac{d^1 S}{d\eta^1}(0) + f_{11}^2 C(0) - f_{21}^2 S(0) \\
&= \frac{1}{2}(\frac{z_0}{2} + 1)\frac{d^2 C}{d\eta^2}(0) - \frac{d^1 S}{d\eta^1}(0) \\
&= \frac{1}{2}(\frac{z_0}{2} + 1)(-2\sin(\phi_0)\phi_2 + \phi_1^2 \cos(\phi_0)) - \cos(\phi_0)\phi_1.
\end{aligned}$$

To check that these are indeed conserved densities for our equation (4.1.1), we need to check that they satisfy

$$D_t(\mathbb{D}_j) + D_x(\mathbb{F}_j) = 0 \qquad j = 0, 1, 2...$$

for all solutions u where $\mathbb{F}_j$ is the respective conserved flux from the algorithm in [2] shown above. We omit this since we proved this condition is true in general in the previous chapter. What is more interesting is whether or not the conservation laws computed using this algorithm from [2] are the same as the conservation laws we computed in the previous subsection using Magri's theorem. If we could have solved the earlier equations for ϕ_0 or ϕ_1 in closed form, then we might have been able to determine if this is true. However the Frobenius Theorem that we applied to the system of first-order pdes governing ϕ_0 and ϕ_1 is an existence theorem rather than a method for obtaining closed form solutions. Such solutions are generally very rare and difficult to obtain. Furthermore, since there is potential for an enormous amount of conservation laws for some evolution equations the chance that these conservation laws are the same is probably quite low.

5 Perspectives

Up until this point, we have only considered evolution equations whose solutions are smooth real valued functions. If we loosen our restrictions and consider evolution equations whose solutions are smooth complex valued functions we can actually apply the algorithm from [2] to evolution equations describing spherical-surfaces, that is surfaces of Gaussian curvature $K = -1$. This is exactly what the authors in [9] did and thus we quote their definition below.

Definition 5.0.1. A complex evolution equation for $q(x,t)$, or equivalently a system of evolution equations for real valued functions $u(x,t)$ and $v(x,t)$ describe pseudo-spherical surfaces (respectively spherical-surfaces) if and only if there exists smooth real valued functions f_{ab}, $1 \leq a \leq 3$, $1 \leq b \leq 2$ depending only on u, v and their derivatives, such that the one-forms $\omega_a = f_{a1}dx + f_{a2}dt$, $a = 1,2,3$ satisfy the relations

$$d\omega_1 = \omega_3 \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_3, \quad d\omega_3 = c\omega_1 \wedge \omega_2 \neq 0 \tag{5.0.1}$$

where $c = 1$ (respectively $c = -1$).

For example in [9] the authors consider the non-linear Schrödinger equation

$$iq_t + q_{xx} + 2k|q|^2 q = 0$$

where k is a real constant. This equation can be written in real form using two evolution equations

$$u_t + v_{xx} + 2k(u^2+v^2)v = 0$$
$$-v_t + u_{xx} + 2k(u^2+v^2)u = 0$$

when $q = u + iv$. For $k = 1$ this system describes spherical-surfaces with associated one-forms:

$$\omega_1 = (2v)dx + (-4\eta v + 2u_x)dt,$$
$$\omega_2 = (2\eta)dx + (-4\eta^2 + 2u^2 + 2v^2)dt,$$
$$\omega_3 = (-2u)dx + (4\eta u + 2v_x)dt.$$

where η is a real parameter (which can be interpreted as a spectral parameter in the method of inverse scattering).

The class of evolution equations that describe a s.s. arise by generalizing the following set of integrability conditions satisfied by the forms $\omega_1, \omega_2, \omega_3$ that satisfy the structure equations of a p.s.s.. Indeed the structure equations (3.1.2) are equivalent to the integrability of the linear system

$$da = \left(\Omega_{(c=1)}\right)a \tag{5.0.2}$$

where a is a vector valued function and $\Omega_{(c=1)}$ is the real valued traceless matrix of one-forms

$$\Omega_{(c=1)} = \frac{1}{2}\begin{bmatrix} \omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2 \end{bmatrix}.$$

If we allow for complex evolution equations, then the linear system (5.0.2) can alternatively depend on the complex valued traceless matrix of one-forms

$$\Omega_{(c=-1)} = \frac{1}{2}\begin{bmatrix} i\omega_2 & \omega_1 + i\omega_3 \\ -\omega_1 + i\omega_3 & -i\omega_2 \end{bmatrix}$$

and the linear system (5.0.2) extends to either of the linear systems

$$da = \left(\Omega_{(c=\pm 1)}\right)a$$

whose integrability conditions

$$d\Omega_{(c=\pm 1)} - \Omega_{(c=\pm 1)} \wedge \Omega_{(c=\pm 1)} = 0$$

determine a pseudo-spherical surface if the relations (5.0.1) are satisfied for $c = 1$ and a spherical-surface if they are satisfied for $c = -1$. This reflects in part the work of Ablowitz, Kaup, Newell and Segur in [12].

In [9] the authors build upon the work of Chern and Tenenblat in [1] and provide theorems for classifying systems of evolution equations of two real functions

$$z_{0,t} = F(z_0, ..., z_k, y_0, ..., y_r)$$
$$y_{0,t} = G(z_0, ..., z_k, y_0, ..., y_r)$$

that describe pseudo-spherical and spherical-surfaces. This categorizes systems of two evolution equations describing surfaces with constant non-zero Gaussian curvature. In [10] the authors further expand the work done in [9] by looking at evolution equations that describe 3-dimensional hyperbolic space (see [10] for the definitions of these classes of equations). They examine the non-linear Schrödinger equations in (2+1)-dimensions

$$iq_t + q_{xy} \pm 2q\partial_x^{-1}\partial_y|q|^2 = 0$$

and use a geometric approach in a similar manner as Cavalcante and Tenenblat did in [2] to give rise to an infinite sequence of conservation laws. This naturally begs the question of the overlap between these new classes of differential equations admitting new conservation laws and the existance of a multi-Hamiltonian structure.

In this thesis we only focused on a single evolution equation, but recall our definitions for Hamiltonian structures included systems of evolution equations. Thus it would be very interesting to explore these papers and ideas more to see if these classes of systems of evolution equations admit some Hamiltonian structure.

Finally in [11] Krichever and Phong provide a construction for the symplectic form which arises in the solutions of both the $N = 2$ supersymmetric Yang-Mills theories and soliton equations. They show that the reductions of $N = 2$ supersymmetric gauge theory provide the Poisson brackets for a set of partial differential equations, many of which describe pseudo-spherical surfaces. An interesting avenue to explore would be to see if there is a more precise link between the class of evolution equations describing pseudo-spherical surfaces and the geometric framework of Krichever and Phong.

References

[1] Chern, S. S., Tenenblat, K. (1986). Pseudospherical Surfaces and Evolution Equations. Studies in Applied Mathematics, 74(1), 151-179. doi:10.1002/sapm198674155

[2] Cavalcante, J. A., Tenenblat, K. (1988). Conservation laws for nonlinear evolution equations. Journal of Mathematical Physics, 29(4), 1044-1049. doi:10.1063/1.528020

[3] Gray, A., Abbena, E., Salamon, S. (2006). Modern differential geometry of curves and surfaces with mathematica (3rd ed. /, Ser. Studies in advanced mathematics). Chapman Hall CRC.

[4] Olver, P. J. (1993). Applications of lie groups to differential equations (Second, Ser. Graduate texts in mathematics, 107). Springer-Verlag.

[5] Neto, V. G. (2010). Fifth-order evolution equations describing pseudospherical surfaces. Journal of Differential Equations, 249(11), 2822-2865. doi:10.1016/j.jde.2010.05.016

[6] Coddington, E. A., Levinson, N. (1955). Theory of ordinary differential equations (Ser. International series in pure and applied mathematics). McGraw-Hill Book Company.

[7] Carmo Manfredo Perdigao do. (1976). Differential geometry of curves and surfaces. Prentice-Hall.

[8] Lee, J. M. (2018). Introduction to riemannian manifolds (Second, Ser. Graduate texts in mathematics, 176). Springer. https://doi.org/10.1007/978-3-319-91755-9

[9] Ding, Q., Tenenblat, K. (2002). On Differential Systems Describing Surfaces of Constant Curvature. Journal of Differential Equations, 184(1), 185-214. doi:10.1006/jdeq.2001.4141

[10] Jun-Yi, W., Qing, D., Tenenblat, K. (2006). On Differential Equations Describing 3-Dimensional Hyperbolic Spaces. Communications in Theoretical Physics, 45(1), 135-142. doi:10.1088/0253-6102/45/1/026

[11] Krichever, I. M., Phong, D. H. (1997). On the integrable geometry of soliton equations and $N = 2$ supersymmetric gauge theories. Journal of Differential Geometry, 45(2), 349-389. doi:10.4310/jdg/1214459802

[12] Ablowitz, M. J., Kaup, D. J., Newell, A. C., Segur, H. (1974). The Inverse Scattering Transform-Fourier Analysis for Nonlinear Problems. Studies in Applied Mathematics, 53(4), 249-315. doi:10.1002/sapm1974534249