Iterated Medial Triangle Subdivision in Surfaces of Constant Curvature

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Abstract

Consider a geodesic triangle on a surface of constant curvature and subdivide it recursively into 4 triangles by joining the midpoints of its edges. We show the existence of a uniform $\delta > 0$ such that, at any step of the subdivision, all the triangle angles lie in the interval $(\delta, \pi - \delta)$. Additionally, we exhibit stabilising behaviours for both angles and lengths as this subdivision progresses.

Abrégé

Considérons un triangle géodésique sur une surface de courbure constante et subdivisons-le de manière récursive en 4 triangles en joignant les milieux de ses arêtes. On montre alors l'existence d'un $\delta > 0$ uniforme tel que, à n'importe quelle étape de la subdivision, tous les angles des triangles de la subdivision appartiennent à l'interval $(\delta, \pi - \delta)$. De plus, on étudie le comportement et la stabilisation des angles ainsi que des longueurs quand la subdivision progresse.

Acknowledgements

My sincere gratitude goes to both of my supervisors, Professor Piotr Przytycki and Christophe Hohlweg, for their continued support, encouragement and direction throughout my masters.

Contribution of Authors

Each chapter presented in this thesis is the result of my own original work.

Table of Contents

Chapter 1

Introduction

In this section, we motivate the study of this subdivision in a non-Euclidean setting, introduce some notation, and state our main results.

We first define what we call the *iterated medial triangle subdivision* (see Figure 1.1). In our setting, all geodesics will be taken to be minimal. A geodesic triangle T in the surface M_κ^2 of constant curvature κ is defined as a triple of points of M_κ^2, together with a choice of three geodesic segments joining each pair of points. If $\kappa \leqslant 0$, M_κ^2 is uniquely geodesic and any triple of points defines a unique geodesic triangle. If $\kappa > 0$ however, there exists a unique geodesic between two points if and only if the distance between them is strictly less than $\frac{\pi}{\sqrt{\kappa}}$ ([BH99]). For our subdivision to be well-defined in the positive curvature case, we then require the three vertices of our triangle to lie in the same open hemisphere (the largest uniquely geodesic convex set in M_κ^2, see [BH99]). Equivalently, we could require the perimeter of our triangles to be strictly less than 2π. In the positive curvature setting, we shall then understand the meaning of "geodesic triangle" to include these restrictions on the possible triples of points. For all κ, we define the iterated medial triangle subdivision of a geodesic triangle $T \subset M_\kappa^2$ inductively, as the following sequence $T_0, T_1, T_2, \ldots$ of refining triangulations:

- $T_0 = T$

- T_{n+1} is obtained from T_n by adding the midpoints of the edges of T_n and, within each triangle of T_n, pairwise connecting its 3 midpoints by geodesic segments (this creates 4 new sub-triangles for each triangle of T_n).
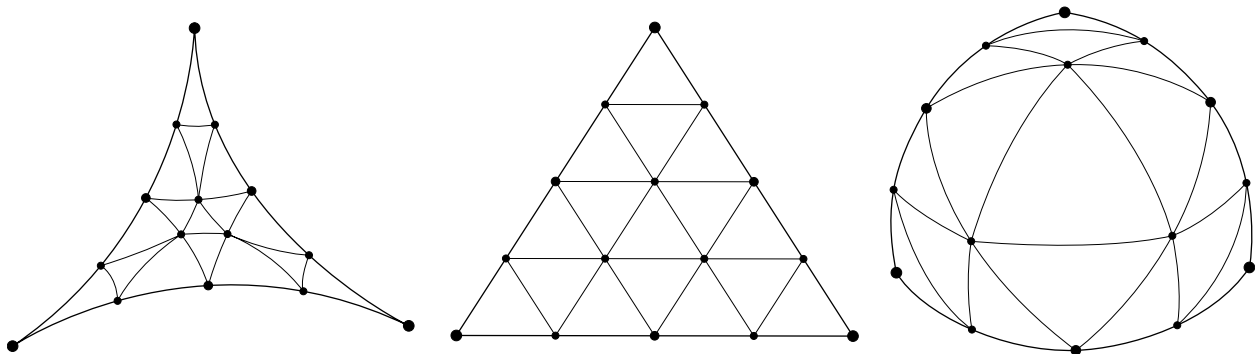

FIGURE 1.1: The first two medial triangle subdivisions of a triangle in $\mathbb{H}^2$, $\mathbb{E}^2$ and $\mathbb{S}^2$.

Our main work is to show that this elementary subdivision behaves "nicely" with respect to both lengths and angles, in a sense made precise by our three main theorems, A, B and C. Before stating our theorems, we point out that it is enough to prove each theorem in the specific case where $\kappa = \pm 1$. The general case can be reduced to the previous one by rescaling the spherical/hyperbolic metric by a constant $|\kappa|^{-\frac{1}{2}}$. Indeed, doing so the curvature becomes κ, but angles are not affected.

Theorem A. *For any geodesic triangle T in M_κ^2, there exists $\delta > 0$ such that, for all n, all the angles of T_n lie in the interval $(\delta, \pi - \delta)$.*

This theorem will be derived as an immediate consequence of our Theorem B. However, we point out to the reader that Theorem A does not require the full strength of Theorem B and can be obtained through a faster route (see Remark 5.0.2). Before stating our remaining theorems, we first need to introduce some notation to make clear the statements of Theorems B and C.

To fix our notation, we will consider a sequence of *nested* triangles $t_0, t_1, \ldots$ with $t_i \in T_i, i \in \mathbb{N}$, such that t_{n+1} is one of the 4 triangles of the medial triangle subdivision

of t_n. We name the sides of t_0 as a_0, b_0, and c_0 and use the following notation scheme, by analogy with the Euclidean case (c.f. Figure 1.2):

(a) If t_{n+1} is obtained as the innermost triangle in the medial decomposition of t_n (as seen on the left diagram), then we name each of its edges according to the only edge of t_n it does not intersect, e.g. a_{n+1} denotes the side of t_{n+1} not intersecting a_n. In this case, we call a_{n+1} the *parallel side* of a_n in t_{n+1}.

(b) If t_{n+1} is obtained from t_n as one of the three outer triangles of the medial subdivision (as seen on the right diagram), then two of its sides are contained in t_n. Those sides inherit their letter from the associated side in t_n, e.g. the side of t_{n+1} contained in the side labelled a_n of t_n is named a_{n+1}. The remaining side of t_n is to be named according to the convention of case (a).

As for angles, α_n (resp. β_n, γ_n) will denote the angle opposite a_n (resp. b_n, c_n).
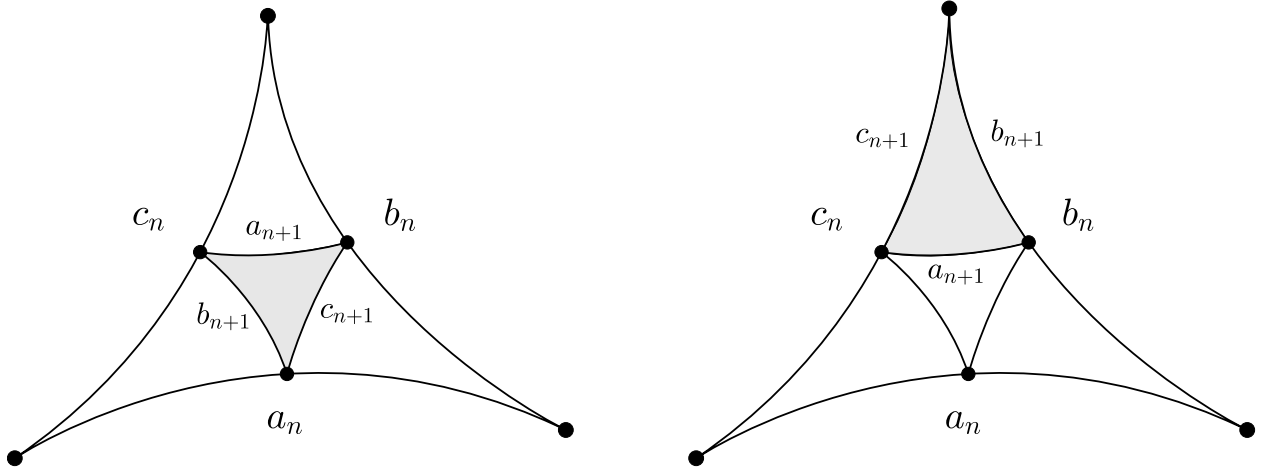

FIGURE 1.2: Case (a) on the left and (b) on the right. In both cases t_n is the outer triangle and the nested triangle t_{n+1} is highlighted in grey.

Theorem B. *For any sequence of nested triangles $t_0, t_1, \ldots$ and for all $n \in \mathbb{N}$, there exists $l_\alpha, L_\alpha > 0$ such that:*

$$\alpha_0 \cdot l_\alpha < \alpha_n < \alpha_0 \cdot L_\alpha$$

In addition, l_α (resp. L_α) approaches 1 from below (resp. above) as all the side lengths of t_0 become smaller.

This theorem itself will be obtained as a consequence of Theorem C and a similar statement regarding the heights of triangles in our triangulations (Proposition 4.0.1).

Theorem C. *For any sequence of nested triangles $t_0, t_1, \ldots$ and for all $n \in \mathbb{N}$, there exists $l_a, L_a > 0$ such that:*

$$a_0 \cdot l_a \leqslant 2^n \cdot a_n \leqslant a_0 \qquad\qquad (Hyperbolic\ geometry)$$

$$a_0 \leqslant 2^n \cdot a_n \leqslant a_0 \cdot L_a \qquad\qquad (Spherical\ geometry)$$

In addition, in the non-trivial cases where there exists at least some integer $n \in \mathbb{N}$ such that a_{n+1} is obtained as the parallel side of a_n in t_{n+1}, the inequalities are strict and l_a (resp. L_a) approaches 1 from below (resp. above) in the hyperbolic (resp. spherical) setting as all the side lengths of t_0 become smaller.

In addition to bringing light on a very natural and elementary object, the study of the medial triangle subdivision in the non-Euclidean setting is motivated by our ongoing work on *acute triangulations* of *Riemannian triangle complexes*. A Riemannian triangle complex is a 2-dimensional simplicial complex in which each simplex is given its own individual Riemannian metric. While acute triangulations have been extensively studied in the Euclidean planar setting, the only existing result in the Riemannian setting is a highly non-constructive existence result for 2-dimensional Riemannian manifolds ([CdVM90]). Even in the constant curvature setting, it is currently unknown whether complexes of spherical or hyperbolic polygons can be acutely triangulated or not. In a follow-up article, we will lay out a constructive existence result for the class of spherical and hyperbolic triangle complexes with finite isometry types. Our methods will exploit Theorem A to explicitly transport a new particular Euclidean acute triangulation scheme due to Chris Bishop ([Bis21]) onto a fine enough medial triangle subdivision of the triangle complex to control the angle distortion occurring during the transport (Fig. 1.3 illustrates this heuristic for a hyperbolic triangle complex).
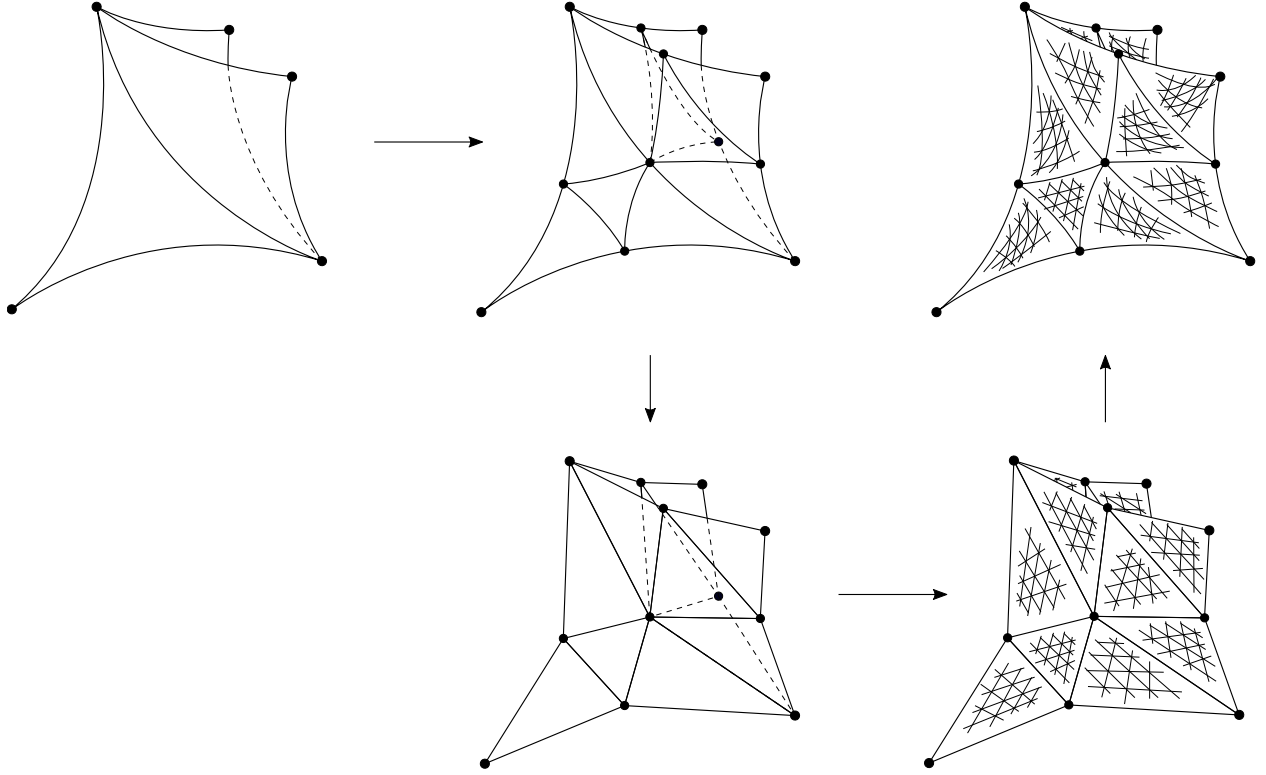
FIGURE 1.3

Before we proceed with the proof, we provide examples to convince the reader of the unusual behaviours exhibited by the medial triangle subdivision in the non-Euclidean cases. Indeed, in the Euclidean case, this subdivision yields 4 congruent triangles that are obtained from the original one by a similarity with scale factor one half. Therefore each iteration preserves the angles and halves the lengths, making our results trivial remarks. However, in the presence of non-zero curvature, the situation is not so straightforward. In fact, as we set off to showcase, a surprising fact about this subdivision in curved spaces is that a single step of the subdivision can be "arbitrarily degenerate". This should make our result perhaps slightly more surprising to the reader.

Chapter 2

Examples

Example 1. We first show how to construct a family of triangles for which the ratio α_1/α_0 is unbounded (for one of the 4 possible choices of α_1). More strikingly, α_0 can be taken arbitrarily small and α_1 arbitrarily close to π. Consider the following example of an isosceles triangle t_0 in the hyperbolic plane (seen in the Poincaré disk model on Figure 2.1). We label its vertices A, B, C and its midpoints D, E, F, as indicated on the figure. In the Euclidean case, the angle α_0 and α_1 would be equal. Here however, fixing A, we can extend the geodesic segment BC to a line and have the points B and C move further apart from each other at equal speed on this line. Doing so will drag the midpoints F and E arbitrarily close to B and C (in the Euclidean metric), making the angle α_1 arbitrarily close to π. This construction is valid for any choice of A, and in particular we can choose A to be arbitrarily close to the boundary, making the angle α_0 arbitrarily close to 0.
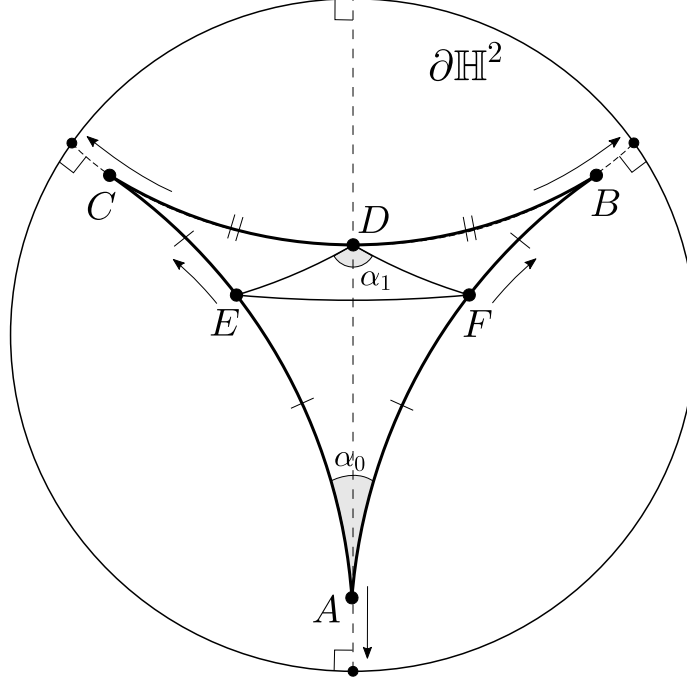
FIGURE 2.1

Example 2. The second surprising phenomenon about the medial triangle subdivision in non-Euclidean geometries is that its behaviour with respect to the angles depends on the initial triangle, namely: in certain triangles it will increase the corresponding angles while it will decrease it in others. To showcase this behaviour, we will provide a quantitative geometric criterion for isosceles triangles to exhibit either one behaviour or the other. Consider an isosceles triangle with vertices A, B, C and sides of length $2a$ and $2b$ (see Figure 2.2). Let D, E, F be the midpoints of the sides BC, CA and AB. Let Q be the point of intersection of the geodesic segments FE and AD; let $\alpha, \beta, \alpha', \beta'$ be the angles described on Figure 2.2.

By symmetry, the triangle DCA has a right angle at D. For the same reason, DEQ has a right angle at Q. Let u denote the length of the geodesic segment DE and h that of the segment DA. Hyperbolic trigonometry identities ([Thu97], p81) in the right-angled triangle DEQ give us:

$$\cot \beta' \, \cot \alpha' = \cosh u \tag{1}$$

Likewise, the hyperbolic sine rule in the triangles DCE and DEA give us the two identities:

$$\frac{\sin\beta}{\sinh u} = \frac{\sin(\frac{\pi}{2} - \alpha')}{\sinh a} = \frac{\cos\alpha'}{\sinh a}$$

$$\frac{\sin\alpha}{\sinh u} = \frac{\sin\alpha'}{\sinh a}$$

which, combined, give us:
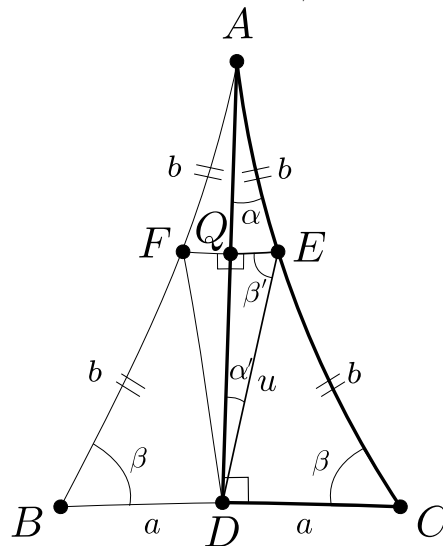
$$\tan\alpha' = \frac{\sin\alpha}{\sin\beta} \tag{2}$$

FIGURE 2.2

Finally, the dual hyperbolic cosine rule ([Thu97]) applied to the triangle DBC informs us that:

$$\cos\beta = \sin\alpha \, \sin\frac{\pi}{2} \, \cosh h - \cos\frac{\pi}{2} \, \cos\alpha = \sin\alpha \, \cosh h \tag{3}$$

We are looking for a condition on the isosceles triangle ABC to ensure that either $\beta' > \beta$ or its converse is true, or equivalently that $\tan\beta' > \tan\beta$ or otherwise. Combining equations (1), (2) and (3), we obtain:

$$\tan \beta' > \tan \beta \iff \left(\frac{\sin \alpha}{\sin \beta} \cosh u \right)^{-1} > \frac{\sin \beta}{\cos \beta} \iff \cosh h > \cosh u$$

There are thus two distinct and opposite scenarios possible, in which either:

- $u > h$, in which case the medial subdivision produces a smaller angle, i.e. $\beta' < \beta$ (left diagram of Figure 2.3 shows an example in the Poincaré disk model).

- $u < h$, in which case the medial subdivision produces a larger angle, i.e. $\beta' > \beta$ (right diagram of Figure 2.3).
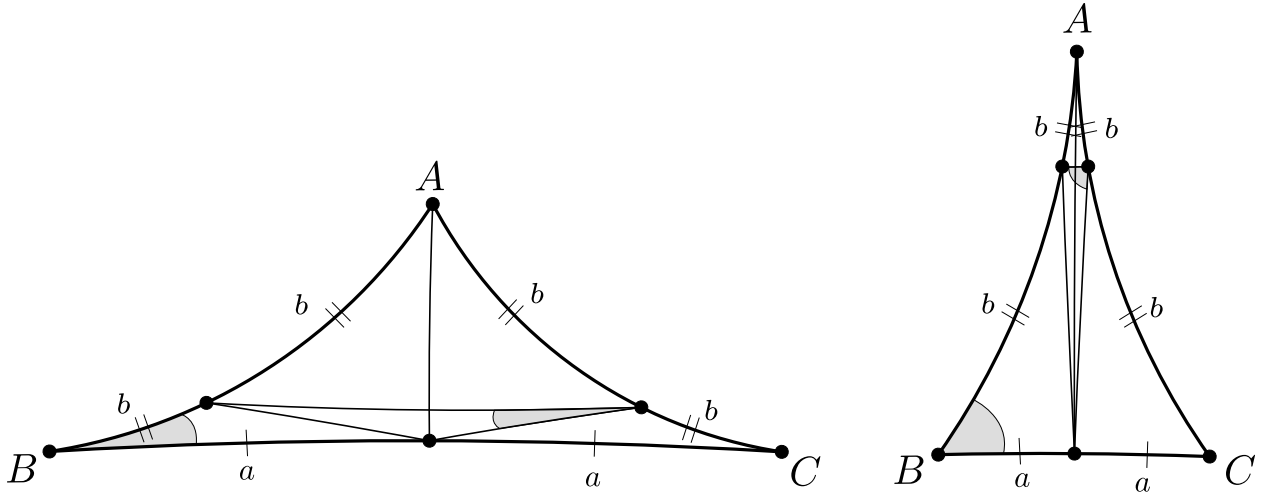

FIGURE 2.3

Remark 2.0.1. The exact same reasoning works transposed in the spherical setting but yields the opposite inequalities and behaviours, namely the two cases on Fig. 2.3 are reversed. This is because equalities (2),(3) and (1) all remain the same with regular sines and cosines instead of their hyperbolic counterpart. The two scenarios previously highlighted are however to be swapped, since the cosine function is decreasing on the interval $[0, \pi]$, while the hyperbolic cosine function is increasing on that same interval.

To briefly address how the subdivision behaves with respect to lengths, we provide two particularly striking examples in the spherical case.

Example 3. We first show that, for certain spherical triangles, lengths can be arbitrarily distorted by the medial subdivision. Consider a spherical isosceles triangle ABC, with right angle at A and equal sides $|AB| = |AC|$. As B and C approach the antipodal point of A, $|BC|$ becomes arbitrarily small, while $|EF|$ will approach a quarter of the equatorial circle between those two poles. In that sense, the spherical upper bound of Theorem B should perhaps appear less natural, as we can create triangles in which one of the sides will have its corresponding side in the next step of the medial triangle subdivision arbitrarily larger.
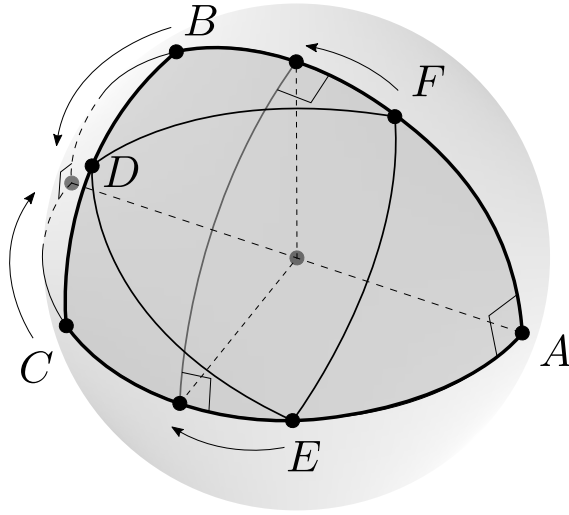

FIGURE 2.4

Example 4. Lastly, we provide an example to show that, for any $\epsilon > 0$ and for any given $N \in \mathbb{N}$, we can find a triangle T_0^ϵ for which every edge of T_N^ϵ lies in the ϵ-neighbourhood of the union of the sides of T_0^ϵ. Moreover, $\mathrm{diam}(T_N^\epsilon) > \frac{\pi}{2}$. We first remark, that, while our subdivision is not defined for a triangle on the equator circle, one can nevertheless imagine what the subdivision would resemble in the case where the three vertices A, B and C are equidistributed on the equator, as there is still a unique geodesic between all midpoints in this case. It is easy to see that taking the midpoints of this triangle gives another triple of points of the equator which are also equidistributed. By continuity, if we consider a triangle T_0^ϵ whose vertices are all equidistributed on a latitude circle close to the equator and let its vertices approach A, B and C respectively, we see that its midpoints will also stay close to

10

the midpoints of ABC. By induction, for any finite number N of subdivisions and any $\epsilon > 0$, choosing T_0^ϵ to lie on a latitude circle sufficiently close to the equator then guarantees that all edges of T_N^ϵ remain within the ϵ-neighbourhood of the union of the sides of T_0^ϵ. Because of this, it is clear that for ϵ sufficiently small, $\mathrm{diam}(T_N^\epsilon) > \frac{\pi}{2}$.
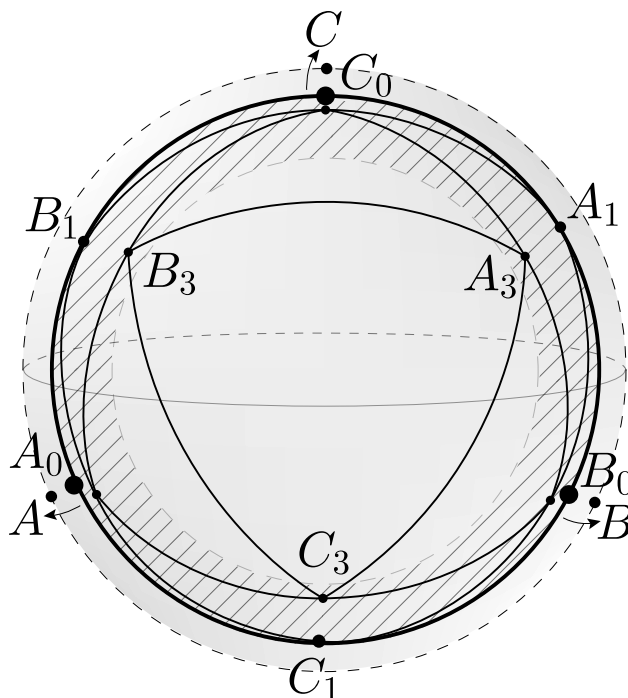

FIGURE 2.5: This figure illustrates Example 4, with the hatched annulus corresponding to the ϵ-neighbourhood of the union of the sides of T_0 and $N = 3$. Smaller values of ϵ and larger values of N are achieved by choosing the vertices of T_0 closer to those of ABC.

Chapter 3

Stabilisation of Lengths

Despite the various unusual behaviours showcased by our previous examples, we claim that as n grows, the refining triangulations eventually "stabilise" to the limiting Euclidean case, in the sense of Theorem B and Theorem C. In this section, we establish our notation and focus on the behaviour of the edge-lengths of the subdivision. The core of this section is our proof of Theorem C, which gives a precise sense to the "stabilisation of lengths" observed in the medial triangle subdivision. Theorem C will also play a crucial part in our proofs of Proposition 4.0.1 and Theorems A and B.

While the study of the behaviour of the heights in the subdivision is delayed to Section 4, the constructions used towards the proof of Theorem C rely heavily on taking orthogonal projections and measuring heights. We thus begin by making clear the meaning of "height" in the positive curvature setting. Indeed, while there is a unique orthogonal projection from any point to any line in the non-positive curvature setting, the situation is slightly more subtle in the spherical case. In the spherical setting, if we fix a point p and a great circle $\mathcal{C}$, there are two possible cases, depending on whether p is a pole of the sphere for $\mathcal{C}$ considered to be the equator circle. If it is not (left diagram of Fig. 3.1), there is a unique geodesic arc which realises the distance of p to $\mathcal{C}$. This arc is what we refer to as the *altitude* drawn from p onto $\mathcal{C}$ and we call its length the *height* of p to $\mathcal{C}$. The point of intersection between this

arc and $\mathcal{C}$ is referred to as the *orthogonal projection* of p onto $\mathcal{C}$. Note that, in this case, the height is always strictly less than $\frac{\pi}{2}$. If p is a pole of the sphere for $\mathcal{C}$ considered to be the equator circle (right diagram of Fig. 3.1), the height of p to $\mathcal{C}$ is defined to be equal to $\frac{\pi}{2}$, while both the altitude from p to $\mathcal{C}$ and its orthogonal projection on $\mathcal{C}$ are undefined. Notice that in both cases, the height is the minimal distance from p to any point on the line $\mathcal{C}$. Fortunately, all our proofs relying on altitudes and orthogonal projections will take place in a setting where the distance between any two points is strictly less than $\frac{\pi}{2}$, therefore ensuring the second case where p is a pole with respect to the equator circle $\mathcal{C}$ cannot happen.
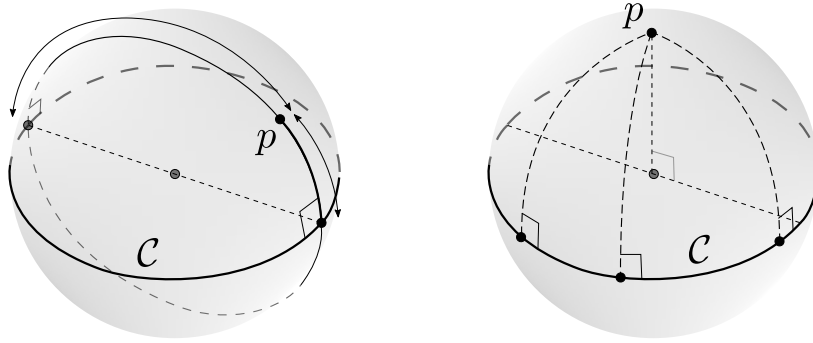

FIGURE 3.1: A point on the sphere has either two (left diagram) or uncountably many (right diagram) orthogonal projections to a line.

We begin by stating the following lemma:

Lemma 3.0.1. *For any sequence of nested triangles $t_0, t_1, \ldots$, and for all $n \in \mathbb{N}$, the following inequalities hold:*

$$a_{n+1} \leqslant \frac{a_n}{2} \qquad\qquad (Hyperbolic\ geometry)$$

$$a_{n+1} \geqslant \frac{a_n}{2} \qquad\qquad (Spherical\ geometry)$$

(and similarly for b_{n+1} and c_{n+1}). The inequalities are strict when a_{n+1} is obtained as the parallel side of a_n in t_{n+1}.

Proof. The two cases where a_{n+1} is contained in a_n are trivial. The only case of interest is thus when a_{n+1} is obtained as the parallel side of a_n in t_{n+1}. In that case, it is a consequence

of the observation that the hyperbolic plane (resp. the Euclidean plane) is a CAT(-1) space (resp. a CAT(0) space), and thus also CAT(0) space (resp. a CAT(1) space) ([BH99, II.1.13]). The lemma follows directly from the CAT(κ) inequality. $\qquad\square$

While this observation is a well known fact, a later construction of ours will provide an elegant alternative proof of Lemma 3.0.1 later on in this section, see Remark 3.0.8 in our proof of Theorem C.

Lemma 3.0.2. *In the spherical case, there exists an integer N and a positive constant $C < 2$ depending only on t_0, such that, for all $n > N$, we have $a_{n+1} \leqslant C \frac{a_n}{2}$.*

We now note that Lemma 3.0.2 clearly holds in the hyperbolic case because of Lemma 3.0.1, choosing $C = 1$ and $N = 0$ in the statement. But we also wish to establish an upper bound in the spherical case. For that purpose, we will think of a geodesic triangle T on the sphere as a Jordan curve and define its *interior* as the connected component of $\mathbb{S}^2 - T$ that is contained in the open hemisphere containing T.

Lemma 3.0.3. *Given a geodesic $c : [0, l] \to \mathbb{S}^2$ joining two points $u = c(0)$ and $v = c(l)$ on a geodesic triangle T, the restriction of c to the open interval $(0, l)$ lies in the interior of T.*

Proof. To fix the notation, PQR will denote a geodesic triangle and X, Y, Z the midpoints of its three edges QR, RP and PQ. Note that we have that $d_{\mathbb{S}^2(X,Y)} < \pi$ and likewise for the other two pairs. There is then a unique minimal arc joining joining X and Y and this arc is the intersection of $\mathbb{S}^2$ with the positive cone in $\mathbb{E}^3$ spanned by X and Y, seen as unit vectors in $\mathbb{E}^3$. Thus all the points of this geodesic arc are of the form $xX + yY$, with $x, y \geqslant 0$ and $x + y > 1$. Since X is the midpoint of QR, it can be expressed as $\lambda(Q + R)$, with $\lambda > \frac{1}{2}$. Likewise Y can be expressed as $\mu(P + R)$, $\mu > \frac{1}{2}$. This shows that each point on the geodesic segment joining X and Y can be written as a sum $x\lambda(P + Q) + y\mu(Q + R) = \alpha P + \beta Q + \gamma R$ with $\alpha + \beta + \gamma > 1$ and $\alpha, \beta, \gamma \geqslant 0$. The entire geodesic segment thus lies in the intersection between the positive cone in $\mathbb{E}^3$ spanned by P, Q and R and $\mathbb{S}^2$ and is thus contained within

14

the triangle PQR. For any points of the geodesic distinct from X and Y, we have $\alpha, \beta, \gamma > 0$, which shows that these points lie in the interior of the triangle PQR. □

We shall now need the following important lemma to prove Lemma 3.0.2:

Lemma 3.0.4. *For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that all the edge-lengths of t_n, for $n > N$, are smaller than ϵ.*

We point out here that this is still a weaker statement than that of Lemma 3.0.2, which tells us that any sequence of edge lengths $(a_n)_{n\in\mathbb{N}}$ not only converges to 0 but is also bounded above by a geometric sequence.

Proof. Given a sequence of nested triangles $t_0, t_1, \ldots$, we define the sequences of points $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$ consisting, for each n, of the vertices of t_n incident to the angles α_n, β_n and γ_n respectively. In our setting, t_0 is a closed compact subset of the open hemisphere, thus each of the three sequences (A_n), (B_n) and (C_n) have subsequences (A_{n_k}), (B_{n_k}) and (C_{n_k}) converging to A, B and C respectively, with all three points lying in t_0 (by Lemma 3.0.3).
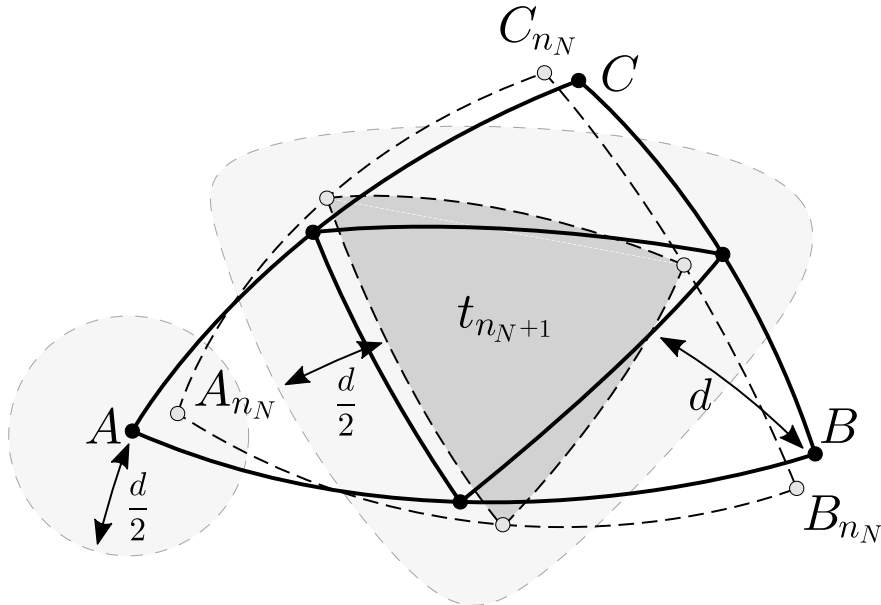

FIGURE 3.2

15

Suppose now by contradiction that the sequences (A_{n_k}), (B_{n_k}) and (C_{n_k}) do not converge to the same point. There are two possible cases.

Case 1: $A \neq B \neq C$. Lemma 3.0.3 guarantees that the vertices of $T = ABC$ and the innermost triangle T' of T in the medial triangle subdivision form two disjoint closed sets. Denote then by d the minimum of the three distances between T' and each of the points A, B and C. Then the $d/2$-neighbourhoods of A and T' are disjoint (likewise for B and C). Since geodesics between any two points of the open hemisphere are unique and continuous with respect to their endpoints, we have that there exists N such that the $d/2$-neighbourhood of each edge in the medial triangle subdivision of T contains the corresponding edge of $A_{n_N} B_{n_N} C_{n_N}$ (see Fig. 3.2). Assuming that $t_{n_{N+1}}$ is the innermost triangle of t_{n_N}, we can then guarantee that A and $t_{n_{(N+1)}}$ are disjoint. But this is impossible as A lies in $t_{n_{(N+1)}}$. If instead t_{n_N+1} were not the innermost triangle but (for example) the triangle $B_{n_N+1} A_{n_N} C_{n_N+1}$ (the bottom-left triangle on Fig. 10), then we could now guarantee that B (or C) and $t_{n_{(N+1)}}$ are disjoint. This is again impossible as B (and C) lies in $t_{n_{(N+1)}}$. The other two cases are dealt with in the exact same fashion.
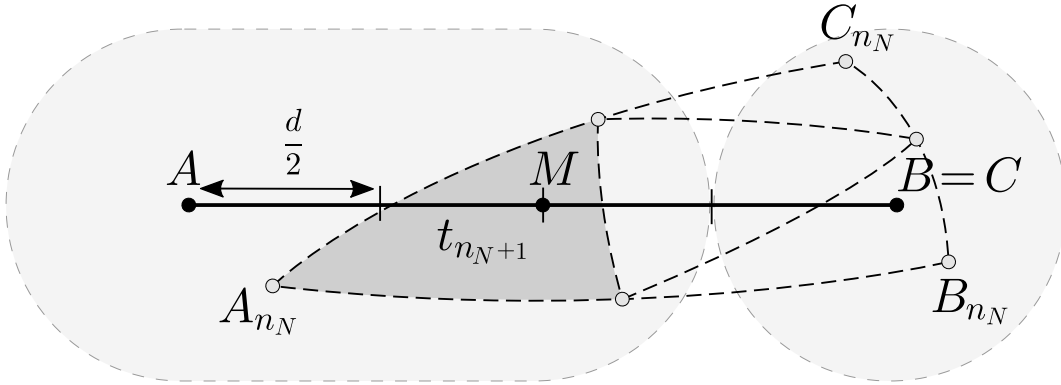

FIGURE 3.3

Case 2: $A \neq B$ and $B = C$ (the other two possible cases are symmetric up to a relabelling of the vertices). We proceed with a similar argument as in the first case, replacing T by the geodesic segment joining A and B, T' by the geodesic segment joining the midpoint M of AB to $B = C$ and letting $d = d_{\mathbb{S}^2}(A, B)/2$. In this degenerate case, we consider the

"medial triangle subdivision" of T where the midpoints of AB and AC coincide with M, the midpoint of BC coincides with B, and the tree edges joining the three midpoints of BC, CA and AB are the point M and the edge MB (counted twice). We then similarly obtain an integer N such that the $d/2$-neighbourhood of each edge in the "medial triangle subdivision" of T contains the corresponding edge of $A_{n_N}B_{n_N}C_{n_N}$. For the three possible choices of t_{n_N+1} that do not contain A_{n_N}, we can consider the $d/2$-neighbourhoods of T' and A to guarantee that A and $t_{n_{(N+1)}}$ are disjoint. But this is impossible since A lies in $t_{n_{(N+1)}}$. If instead t_{n_N+1} was the triangle containing A_{n_N} (see Fig. 3.3), then we could now consider the $d/2$-neighbourhoods of AM and $B = C$ to guarantee that $B = C$ and $t_{n_{(N+1)}}$ are disjoint. This is again impossible since $B = C$ lie in $t_{n_{(N+1)}}$. This concludes the proof of Case 2. $\square$

Remark 3.0.5. Owing to Lemmas 3.0.3 and 3.0.4, we now know that, in the spherical setting, we can assume our triangles to be small enough to lie inside an open ball of radius $\frac{\pi}{4}$, so that the distance between any two points is strictly less than $\frac{\pi}{2}$. This guarantees that the height from any point lying on such a triangle to any line (great circle) intersecting the triangle must be strictly less than $\frac{\pi}{2}$. Indeed, as we noted before, the height from a point to a line is the minimal distance from this point to any point on the line. The second case of Figure 3.1 will thus be safely averted from there on.

We now introduce some notation. For convenience, we write a_n (resp. b_n, c_n) as BC (resp. CA, AB) and the midpoints of BC, CA and AB by D, E and F (see Fig. 3.4). In the following proofs, we let A', B', C', D' be the orthogonal projections of A, B, C, D on the geodesic line (i.e. the great circle in the spherical setting) FE. We also consider the orthogonal projections F' and E' of F and E on the line BC. Note that these are all well-defined in light of Remark 3.0.5.
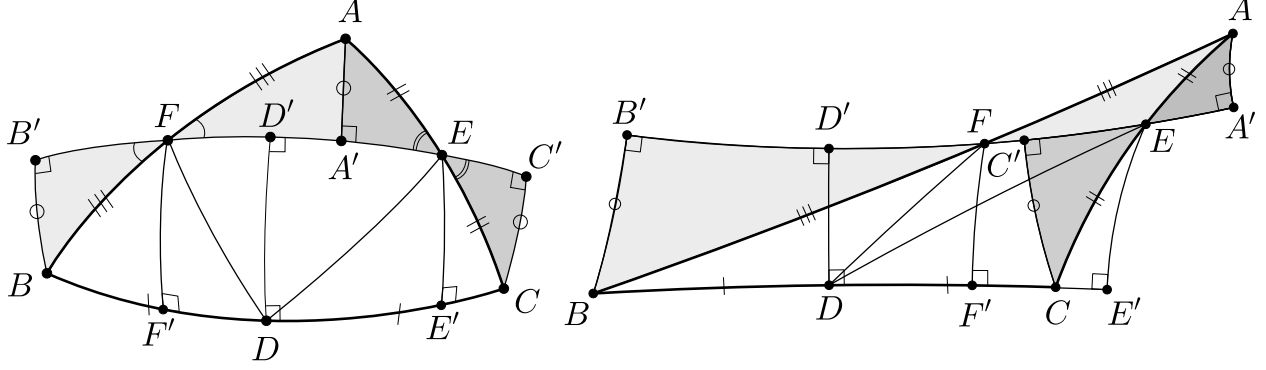
FIGURE 3.4: The construction for an acute spherical triangle (left) and an obtuse hyperbolic triangle (right).

In the following discussion, we define a *quadrilateral $XYY'X'$* as the union of the geodesic segments XY, YY', $Y'X'$ and $X'X$, provided that every pairwise intersection of the four interiors of these segments is empty. We remind the reader that a quadrilateral $XYY'X'$ for which the oriented angles $\angle Y'YX$ and $\angle YXX'$ are right angles (for the two possible orientations of the hyperbolic plane or the sphere) and the lengths of its sides XX' and YY' are equal is called a *Saccheri quadrilateral* with *base XY* and *summit $X'Y'$* (sometimes called a Saccheri isosceles birectangle). Saccheri quadrilaterals have a unique line of symmetry cutting both their base and summit sides perpendicularly through their midpoints (see for example [Mar96, §21]). Likewise, a quadrilateral $XYY'X'$ in which the angles at X, Y and X' are right is called a *Lambert* quadrilateral (sometimes called Lambert trirectangle) with *apex* at Y'.

We first give a short proof of a trigonometric identity in Lambert quadrilaterals which will prove to be very useful in our proof of the main theorem and which we shall use throughout this article.

Lemma 3.0.6. *In a hyperbolic Lambert quadrilateral $XYY'X'$ with apex Y', the following identity holds:*

$$\sinh |X'Y'| = \sinh |XY| \cdot \cosh |YY'|$$

18

The identity in spherical geometry is obtained by replacing hyperbolic trigonometric functions by spherical trigonometric functions:
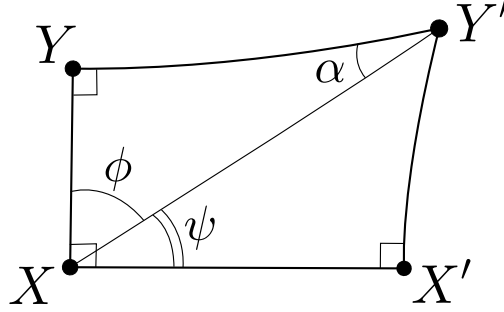
$$\sin|X'Y'| = \sin|XY| \cdot \cos|YY'|$$

FIGURE 3.5

Proof. From the dual hyperbolic law of cosines in the right triangle $XY'Y$ ([Thu97, 2.4.9]), we obtain $\cosh|YY'| = \frac{\cos\phi}{\sin\alpha}$. Morevoer, from the hyperbolic law of sines we obtain that $\frac{\sin\alpha}{\sinh|XY|} = \frac{1}{\sinh|XY'|}$. Combining the two and using the fact that $\cos\phi = \sin\psi$, we get: $\sinh|XY|\cosh|YY'| = \sin\psi\sinh|XY'|$. Using the hyperbolic law of sines a second time in the right triangle $XX'Y'$, we obtain $\frac{\sin\psi}{\sinh|X'Y'|} = \frac{1}{\sinh|XY'|}$. Substituting for $\sin\psi$ using this identity, we reach the desired equality. $\qquad\square$

We now get back to our proof of Lemma 3.0.2 and start by establishing the following fact:

Lemma 3.0.7. *There exists $N \in \mathbb{N}$ such that, for all $n > N$, the quadrilateral $C'B'BC$ is a Saccheri quadrilateral with base $B'C'$ of length $2|FE|$, symmetry line DD' and base and summit midpoints D' and D.*

Proof. In the spherical setting, we refer to Remark 3.0.5 to select $N \in \mathbb{N}$ such that all the edge-lengths of t_n and all heights are strictly less than $\frac{\pi}{2}$. In the hyperbolic setting, we select $N = 0$.

19

By construction, the triangles $FB'B$ and $FA'A$ share an angle and two edge-lengths, and are therefore congruent. Likewise for the triangles EAA' and ECC'. This shows that $|AA'| = |CC'| = |BB'|$. Since B' and C' are the orthogonal projections of B and C on the line FE, there only remains to show that $C'B'BC$ is indeed a quadrilateral. This is clear in the hyperbolic case, but requires more care in the spherical case.

In the spherical setting, we first observe that $B'C'$ and BC cannot intersect. Indeed, $|B'C'| \leqslant \pi$, $|B'B|, |C'C| < \frac{\pi}{2}$ and $\angle C'B'B = \angle B'C'C = \frac{\pi}{2}$. We cannot have $B' = B$ or $C = C'$ as it would imply that all points in fact lie on a single great circle. Therefore, it must be that B and C are two distinct interior points of the same right-angled spherical lune with one of its half great circles passing through B' and C' (see Fig. 3.6). However, the geodesic joining any two interior points of a right-angled spherical lune does not cross either boundary edge (a right-angled spherical lune can be completed to a hemisphere sharing either half great circle of the lune as its boundary great circle, and open hemispheres are convex). The only possibility is then that BB' and CC' intersect. But since $\angle C'B'B = \angle B'C'C = \frac{\pi}{2}$, this would imply that both $|BB'|$ and $|CC'|$ are greater than $\frac{\pi}{2}$, which contradicts our definition of orthogonal projections. This argument proves that $C'B'BC$ is a Saccheri quadrilateral.
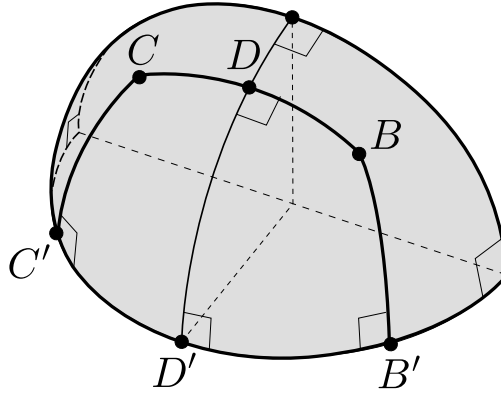

FIGURE 3.6

We now prove that D' is the midpoint of $B'C'$ and not its antipodal point. Denote by D'' the midpoint of $B'C'$. In the previous paragraph, we have shown all four points B, C, B' and C' to all lie in the same right-angled spherical lune. Since $C'B'BC$ is Saccheri, we

now also know that the line through the midpoints of its base and summit is perpendicular to both base and summit (and is its only line of symmetry). Because of this, we can choose the right-angled lune containing all four points to have D'' as the midpoint of one of its two boundary half great circles. The edge DD'' can then be seen as a strict sub arc of the lune's equatorial arc (see Fig 3.6), which proves that $|DD''| < \frac{\pi}{2}$ and confirms that $D'' = D'$.

Lastly, we justify why the base $B'C'$ has length $2|FE|$. By construction, we have that $|B'C'| = |B'A'| + |A'C'|$ (resp. $|B'C'| = -|B'A'| + |A'C'|$, $|B'C'| = |B'A'| - |A'C'|$) when β_n and γ_n are acute (resp. when β_n is obtuse, γ_n is obtuse). Note that the case where both β_n and γ_n are obtuse is impossible since a spherical triangle with all its edge-lengths smaller than $\frac{\pi}{2}$ has at most one obtuse angle (this is a direct consequence of the spherical law of cosines). Observe then that $|FE| = |FA'| + |A'E|$ (resp. $|FE| = -|FA'| + |A'E|$, $|FE| = |FA'| - |A'E|$) and we have $|B'A'| = 2|FA'|$ and $|A'C'| = 2|A'E|$ in all cases, since $2|FA'|$, $2|A'E| < \frac{\pi}{2}$. Thus, $|B'C'| = 2|FA'| + 2|A'E| = 2|FE|$ (resp. $|B'C'| = -2|FA'| + 2|A'E| = 2|FE|$, $|B'C'| = 2|FA'| - 2|A'E| = 2|FE|$). $\qquad\square$

Proof of Lemma 3.0.2. Using Lemma 3.0.4, we first show that the statement of Lemma 3.0.2 is true for the sines of the edge lengths and the added constraint that $C \geqslant 1$, namely:

Claim. *There exists an integer N and a positive constant $1 \leqslant C < 2$ depending only on t_0 such that, for all $n > N$, $\sin a_{n+1} \leqslant C \sin \frac{a_n}{2}$.*

Proof of Claim. The only cases of interest are the two non-trivial cases where a_{n+1} is obtained as the parallel side of a_n in t_{n+1}. In both of these cases, we can appeal to Remark 3.0.5 and use Lemma 3.0.7 and the formulae of Lemma 3.0.6 to obtain:

$$\sin \frac{a_n}{2} = \sin |DC| = \sin |D'C'| \cdot \cos |CC'| = \sin a_{n+1} \cdot \cos |CC'| \qquad (\star)$$

If we now suppose by contradiction that our claim is false, then for all $N \in \mathbb{N}$ and for all $1 < C < 2$, there exists $n > N$ such that $\sin a_{n+1} > C \sin \frac{a_n}{2}$. Using $(\star)$, this implies that:

$$\cos |CC'| = \frac{\sin \frac{a_n}{2}}{\sin a_{n+1}} < C^{-1}$$

which in turn implies that:

$$|CC'| > \arccos C^{-1} > 0$$

However, in positive curvature, the sides of a Lambert quadrilateral incident to the apex are strictly smaller than than their opposite side in the quadrilateral. Therefore, $|CC'| < |DD'|$ (using the Lambert quadrilateral $C'D'DC$). Applying the spherical version of Pythagoras' theorem in the right angled triangle $DD'F$, we see that $|DD'| \leqslant |FD|$, since $|DD'|, |DF| \leqslant \frac{\pi}{2}$. As Lemma 3.0.4 guarantees that $|FD| \to 0$ when $n \to \infty$, we have shown that $|CC'| \to 0$ when $n \to \infty$. This contradicts $|CC'| > \arccos C^{-1}$ and proves the claim.

To get back to the proof of the lemma, we first note that, for all $x > 0$, we have $\sin x < x$. On the other hand, for any $\epsilon > 0$, we have $(1 - \epsilon)x < \sin x$, for x small enough. Using the particular value of C and N given by the previous claim, we can choose an $\epsilon > 0$ small enough to guarantee that $C/(1 - \epsilon) < 2$. There is then an integer $N' > N$ large enough to guarantee that, for all $n > N'$:

$$(1 - \epsilon)a_{n+1} < \sin a_{n+1} < C \sin \left(\frac{a_n}{2} \right) < C \frac{a_n}{2}$$

which gives the following desired inequality:

$$a_{n+1} < \frac{C}{1 - \epsilon} \frac{a_n}{2}$$

and finishes the lemma, as $C/(1 - \epsilon) < 2$.

□

We now get back to the proof of Theorem C, which we state again below.

Theorem C. *For any sequence of nested triangles $t_0, t_1, \ldots$ and for all $n \in \mathbb{N}$, there exists $l_a, L_a > 0$ such that:*

$$a_0 \cdot l_a \leqslant 2^n \cdot a_n \leqslant a_0 \qquad\qquad (\textit{Hyperbolic geometry})$$

$$a_0 \leqslant 2^n \cdot a_n \leqslant a_0 \cdot L_a \qquad\qquad (\textit{Spherical geometry})$$

and similarly for b_n, c_n. In addition, in the non-trivial cases where there exists at least some integer $n \in \mathbb{N}$ such that a_{n+1} is obtained as the parallel side of a_n in t_{n+1}, the inequalities are strict and l_a (resp. L_a) approaches 1 from below (resp. above) in the hyperbolic (resp. spherical) case as all the side lengths of t_0 become smaller.

Proof of Theorem C (Hyperbolic Setting). **Hyperbolic Upper Bound.** Note first that Lemma 3.0.1 tells us that the sequence $(2^n a_n)_{n \in \mathbb{N}}$ is decreasing as $\frac{2^{n+1} a_{n+1}}{2^n a_n} = \frac{2 a_{n+1}}{a_n} \leqslant 1$. Since the sequence is bounded below by 0, this guarantees its convergence to a non-negative limit. Applying Lemma 3.0.1 n times also directly gives us the a_0 upper bound: $2^n a_n = 2^{n-1}(2 a_n) \leqslant 2^{n-1} a_{n-1} \leqslant \ldots \leqslant 2 a_1 \leqslant a_0$.

Hyperbolic Lower Bound. Our task is to show that the sequence $(2^n a_n)_{n \in \mathbb{N}}$ converges to a strictly positive limit. Let us start by rewriting the previous limit as the following infinite product:

$$\lim_{n \to \infty} 2^n \cdot a_n = \lim_{n \to \infty} \frac{2 a_n}{a_{n-1}} \cdot \frac{2 a_{n-1}}{a_{n-2}} \cdot \ldots \cdot \frac{2 a_1}{a_0} \cdot a_0$$

The convergence to a strictly positive value of this infinite product is equivalent to the finiteness of the infinite sum of the logarithm of its factors. Namely:

$$a_0 \prod_{n=1}^{\infty} \frac{2 a_n}{a_{n-1}} > 0 \iff |\ln(a_0)| + \sum_{n=1}^{\infty} \left| \ln\left(\frac{2 a_n}{a_{n-1}} \right) \right| = |\ln(a_0)| + \sum_{n=1}^{\infty} \ln\left(\frac{a_{n-1}}{2 a_n} \right) < \infty$$

Note that Lemma 3.0.1 tells us the sign of the ratio inside the absolute value. In order to

prove that this sum is indeed finite, we will show that the following inequality holds:

$$\ln\left(\frac{a_{n-1}}{2a_n}\right) < \frac{b_n}{2} \qquad (\star)$$

The convergence of the geometric series $(b_0 \, 2^{-(n+1)})_{n\in\mathbb{N}}$ then ensures the convergence of the logarithm sum, as $\frac{b_n}{2} \leqslant b_0 \, 2^{-(n+1)}$. It then also provides us with a uniform bound:

$$\sum_{n=1}^{\infty} \ln\left(\frac{2a_n}{a_{n-1}}\right) = -\sum_{n=1}^{\infty} \ln\left(\frac{a_{n-1}}{2a_n}\right) > -\sum_{n=1}^{\infty} \frac{b_n}{2} > -\frac{1}{2}\sum_{n=1}^{\infty} b_0 \cdot 2^{-n} = -\frac{b_0}{2}$$

This gives us that $l_a = e^{-\frac{b_0}{2}}$ is a valid choice, since after taking the exponential on both sides in the previous inequality, we obtain:

$$2^n \cdot a_n > \lim_{n\to\infty} 2^n \cdot a_n = a_0 \prod_{n=1}^{\infty} \frac{2a_n}{a_{n-1}} > a_0 \, e^{-\frac{b}{2}}$$

Note that l_a indeed approaches 1 from below when all side lengths of t_0 become smaller (thus approaching the Euclidean case for small triangles). We now proceed to prove inequality $(\star)$. Note that this inequality is trivially satisfied for the terms of the sequences where a_{n+1} is contained in a_n, since the logarithms of their corresponding ratio are then each equal to zero and do not contribute to the sum. It is thus enough to prove it in the case where a_{n+1} is obtained as the parallel side of a_n in t_{n+1}.

We shall use the same notation as on Figure 3.4, with the added simplification that we will write a in place of a_n and a' in place of a_{n+1}, and similarly for the other sides. We begin by extending the geodesic segment joining E and F on either side of the triangle and introduce points G and H such that $|GE| = |EF| = |FH| = a'$. The resulting figure, resembling a "jester hat", can be seen on Figure 3.7. The geodesic triangle GEC and FEA share two equal sides and an angle and are therefore congruent. For the same reason, the triangle FHB is also congruent to FEA and therefore also to GEC.
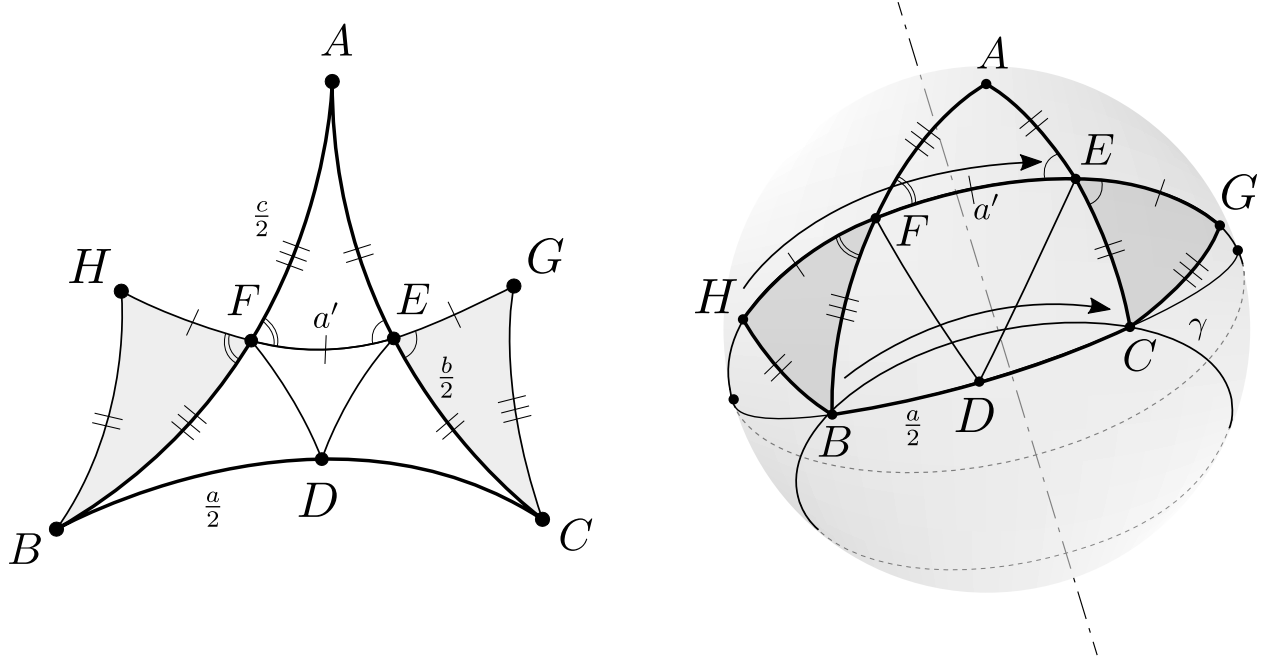
FIGURE 3.7: The hyperbolic and spherical "jester hats".

Remark 3.0.8. We take this opportunity to give a short elegant proof of Lemma 3.0.1, stating that $2a' < a$ in hyperbolic triangles and $2a' > a$ in spherical triangles. Indeed, with this construction we know that there is a hyperbolic translation, with axis the geodesic line going through E and F and translation distance $2a'$, taking the triangle GEC to the triangle FHB. Since the minimum translation distance of a hyperbolic translation is realised for points on its axis, we can conclude that $a > 2a'$. We can of course replicate the exact same construction in the spherical case, where the hyperbolic translation along the line EF now becomes the rotation of the sphere whose axis has the great circle passing through EF as its equator circle. Since the maximum translation distance of a rotation of the sphere is realised along its associated equator circle (see Fig. 3.7), this concludes the proof of Lemma 3.0.1.

We now turn our attention to the geodesic quadrilateral $EHBC$. There is a unique equidistant curve γ staying within a fixed distance h from the geodesic line EF and passing through B and C (Figure 3.8). Since $h < \frac{b}{2}$ and the length $L_\gamma(B,C)$ of γ between B and C is more than that of the geodesic segment joining B and C, it is enough to show that
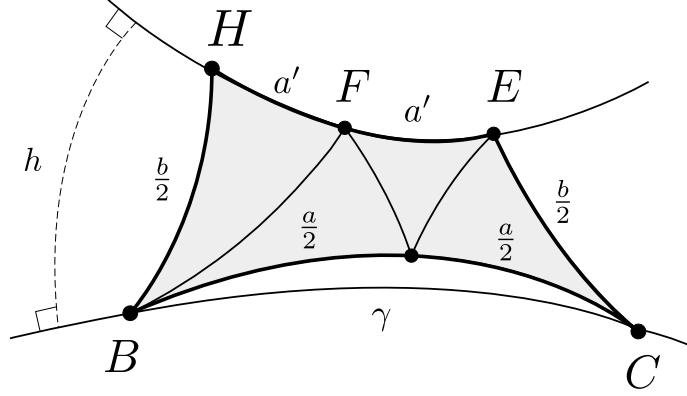
$$\ln\left(\frac{L_\gamma(B,C)}{2a'}\right) < h$$

FIGURE 3.8

in order to show (⋆). The key to the demonstration is now to notice that the figure in invariant under translation along (EH), so that we may look at the ratio of infinitesimal displacements along (EH) and γ instead of that between $L_\gamma(B,C)$ and $2a'$. If we consider the quadrilateral $EHBC$ in the upper half-plane model (Figure 3.9), the equidistant curve γ is now a line meeting the geodesic line (EH) on the boundary. Denote by θ the angle between (EH) and γ. Since the metric in the upper half-plane is scaled by the inverse of the y-coordinate, the ratio between infinitesimal displacements along (EH) and γ is simply given by $\cos^{-1}\theta$.
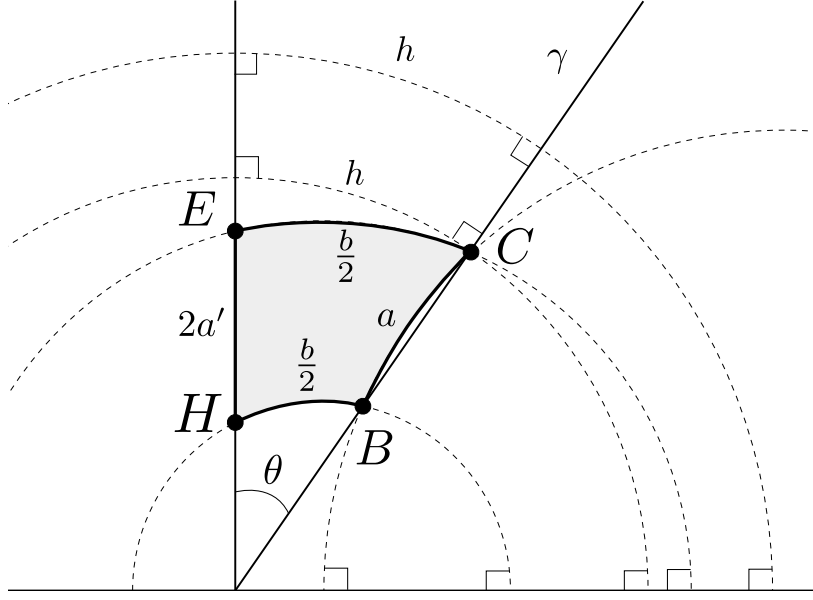
FIGURE 3.9

On the other hand, we can compute the distance h between the geodesic line (EH) and the equidistant curve γ using the metric of the upper half-plane model:

$$h = \int_0^\theta \frac{1}{\cos t} dt$$

Putting both together, it then remains to prove the following inequality:

$$\ln\left(\frac{L_\gamma(B,C)}{2a'}\right) = \ln\left(\cos^{-1}\theta\right) < \int_0^\theta \frac{1}{\cos t} dt = h$$

To obtain that inequality, it is enough to differentiate both sides with respect to θ and show that the derivatives verify the inequality. After differentiating and simplifying, we reach the following equivalent inequality:

$$\sin\theta < 1$$

Note that the previous inequality need in fact only be strict at one point to guarantee the

27

strict inequality after integration (the point $\theta = 0$, for example, is then enough). Alternatively, we can also observe that the only possibility for equality is when $\theta = \frac{\pi}{2}$. However, the line EH and γ are at finite distance h from each other, which implies precisely that $\theta < \frac{\pi}{2}$.

$\square$

Proof of Theorem C (Spherical Setting). **Spherical Lower Bound.** We start with the opposite observation than in the hyperbolic case, namely that Lemma 3.0.1 here gives us the lower bound instead of the upper bound: $2^n a_n \geqslant a_0$. Likewise, it also informs us that the sequence $(2^n a_n)_{n\in\mathbb{N}}$ is increasing as $\frac{2^{n+1} a_{n+1}}{2^n a_n} = \frac{2a_{n+1}}{a_n} \geqslant 1$. Our task is thus to show that the sequence $(2^n a_n)_{n\in\mathbb{N}}$ is bounded above.

Spherical Upper Bound. In the spherical case, we give a shorter, purely trigonometric proof (note that this proof is also available in the hyperbolic case). It will prove easier to exhibit an upper bound for the sequence $(2^n \sin a_n)_{n\in\mathbb{N}}$ instead, which will transfer to our original sequence via a small correction term. As in the hyperbolic setting, we start by rewriting the n-th term of the sequence as the following product:

$$2^n \cdot \sin a_n = \frac{2\sin a_n}{\sin a_{n-1}} \cdot \frac{2\sin a_{n-1}}{\sin a_{n-2}} \cdot \ldots \cdot \frac{2\sin a_1}{\sin a_0} \cdot \sin a_0$$

Finding an upper bound for this product of n terms is again equivalent to finding an upper bound to the sum of the logarithm of its factors. Namely:

$$a_0 \prod_{i=1}^{n} \frac{2\sin a_i}{\sin a_{i-1}} > \delta > 0 \iff |\ln(a_0)| + \sum_{i=1}^{n} \left| \ln\left(\frac{2\sin a_i}{\sin a_{i-1}}\right) \right| < \Delta < \infty$$

Where $e^\Delta = \delta$. There again, Lemma 3.0.1 tells us the sign of the ratio inside the absolute value. To prove an upper bound on this sum of logarithms, we take up again the notation of Figure 3.4 and focus on the non-trivial case where a_{n+1} is obtained as the parallel side of a_n in t_{n+1}. Lemma 3.0.7 guarantees that for a large enough n, the quadrilateral $C'B'BC$ is Saccheri and the quadrilateral $C'D'DC$ is Lambert. For such an n, we can now use the spherical Pythagoras theorem in the triangle $D'CC'$ and Lemma 3.0.6 in $C'D'DC$ to obtain

the two following identities:

$$\cos |D'C| = \cos |CC'| \cdot \cos |D'C'| \tag{3.1}$$

$$\sin |DC| = \sin |D'C'| \cdot \cos |CC'| \tag{3.2}$$

Using equation (3.2) together with the double angle formula for the sine, we reach the following inequality:

$$\frac{2 \sin \frac{a_{n+1}}{2}}{\sin \frac{a_n}{2}} = \left(\cos \left(\frac{a_{n+1}}{2} \right) \cos |CC'| \right)^{-1} > 1 \tag{3.3}$$

Combining equation (3.3) with equation (3.1), we obtain, for t_n small enough:

$$\frac{2 \sin \frac{a_{n+1}}{2}}{\sin \frac{a_n}{2}} = \frac{\cos a_{n+1}}{\cos |D'C| \cdot \cos \frac{a_{n+1}}{2}} < \left(\cos \frac{a_{n+1}}{2} \right)^{-1} \tag{3.4}$$

The last inequality is obtained by noticing that, for small enough spherical triangles, the length of the hypotenuse $|D'C|$ in the right-angled spherical triangle $D'CC'$ is larger than that of the leg $|D'C'| = a_{n+1}$. This can be shown to be true of any spherical triangle contained in a spherical octant and thus, in particular, for any triangle with side lengths smaller than $\frac{\pi}{3}$. By Lemmas 3.0.4 and 3.0.2, we can pick $N \in \mathbb{N}$ such that for all $n > N$, all three sides of t_n are smaller than $\frac{\pi}{3}$ and the inequality $a_{n+1} \leqslant Ca_n/2$ holds. Since all side-lengths are strictly less than $\frac{\pi}{2}$, our choice of N was large enough to guarantee that the quadrilateral $C'B'BC$ is Saccheri, allowing us to make use of the previous derivations. Writing $C_N = \sum_{i=1}^{N} \left(\frac{2 \sin a_{i+1}}{2} / \sin \frac{a_i}{2} \right)$, we can now write the following inequality:

$$\sum_{i=1}^{n} \ln \left(\frac{2 \sin \frac{a_{i+1}}{2}}{\sin \frac{a_i}{2}} \right) < - \sum_{i=N}^{n} \ln \cos \frac{a_{i+1}}{2} + C_N < \sum_{i=N}^{n} \frac{a_{i+1}}{2} + C_N \tag{3.5}$$

where the last inequality stems from the observation that $- \ln \cos(x) < x$ for $x \in (0, \frac{\pi}{3})$.

29

Indeed, the first function is strictly convex and has vanishing derivative at 0. It is then enough to check that the inequality is true in $\frac{\pi}{3}$: a quick computation gives $\cos\frac{\pi}{3} = \frac{1}{2} > \frac{1}{e} > \frac{1}{e^{\frac{\pi}{3}}}$, which is the desired inequality after taking logarithms and changing signs. Using Lemma 3.0.2 and after multiplying inequality (3.5) by $\ln\sin\frac{a_0}{2}$ and taking the exponential, we obtain the following chain of inequalities:

$$2^n \sin\frac{a_n}{2} < \sin\frac{a_0}{2}\exp\left(\frac{a_0}{2}\sum_{i=N}^{n}\left(\frac{C}{2}\right)^i + C_N\right) < \sin\frac{a_0}{2}\exp\left(a_0\frac{(C/2)^N}{2-C} + C_N\right)$$

For all $n \in \mathbb{N}$, let us consider the quantity $\epsilon_n = a_n/\sin a_n - 1$. it is easy to see that this quantity is always strictly positive, goes to 0 as $n \to \infty$ and $(\epsilon_n)_{n\in\mathbb{N}}$ is a monotone decreasing sequence. Noting that $\sin a_n < 2\sin\frac{a_n}{2}$ and $\sin\frac{a_0}{2} < \frac{a_0}{2}$, we obtain:

$$2^n a_n = 2^n(1 + \epsilon_n)\sin a_n < (1 + \epsilon_0)\, a_0\, \exp\left(a_0\frac{(C/2)^N}{2-C} + C_N\right)$$

From this, we conclude that $l_a = (1+\epsilon_0)\exp\left(a_0\,(C/2)^N/(2-C) + C_N\right)$ is a valid choice. Indeed, by construction, both N and C_N become zero for a_0 small enough. Since C is fixed, we also have that both $(1 + \epsilon_0)$ and the exponential term approach 1 from above when a_0 becomes small. $\square$

Chapter 4

Stabilisation of Heights

In this section, we investigate the behaviour of heights in the subdivision and show that they also "stabilise" to the Euclidean case as our triangulations refine. The exact meaning of this expression is made precise by the statement of the following proposition:

Proposition 4.0.1. *For any sequence of nested triangles $t_0, t_1, \ldots$, and for all $n \in \mathbb{N}$, there exists $l_h, L_h > 0$ such that:*

$$h_0 \cdot l_h \leqslant 2^n \cdot h_n \leqslant h_0 \cdot L_h$$

where h_n denotes the height from the vertex incident to α_n onto the line prolonging the side of length a_n in t_n. In addition, in the non-trivial cases where there exists at least some integer $n \in \mathbb{N}$ such that a_{n+1} is obtained as the parallel side of a_n in t_{n+1}, the inequalities are strict and l_h (resp. L_h) approaches 1 from below (resp. above) as all the side lengths of t_0 become smaller.

The following lemma will prove to be useful to simplify our proof of Proposition 4.0.1:

Lemma 4.0.2. *In the hyperbolic setting (resp. in the spherical setting, for n large enough), the height of the vertex incident to α_{n+1} to the line prolonging the side of length a_{n+1} in t_{n+1} is minimal among all 4 choices of t_{n+1} (resp. maximal) when the innermost triangle of t_n is selected.*

Proof. We stated the theorem and give the proof only for the height associated to α_{n+1} in the hyperbolic setting, the spherical proof is obtained by simply reversing each conclusion/inequality and the other cases are derived in the same fashion. Recall that $|A'A| = |C'C|$ and consider the Lambert quadrilateral $D'C'CD$. In the hyperbolic case, we have that $|C'C| > |DD'|$, as the sides incident to the apex are larger than their opposite sides in hyperbolic Lambert quadrilaterals (this opposite conclusion is true for spherical Lambert quadrilaterals). The same inequality can be derived for each of the other two heights by using the Lambert quadrilaterals $FDD'F$ and $DE'ED'$. $\square$

Proof of Proposition 4.0.1 (Hyperbolic Setting). There again, we shall only give the proof regarding the heights associated to α_n, the proofs of the other cases are derived in the same fashion. We once again take up our notation for Lemma 3.0.2. In addition, let A'' be the orthogonal projection of A onto the line BC and denote by E'' the midpoint of $A''C$ (see Fig. 4.1). For the sake of brevity, we shall disregard the similar but easier case where $A'' = C$.
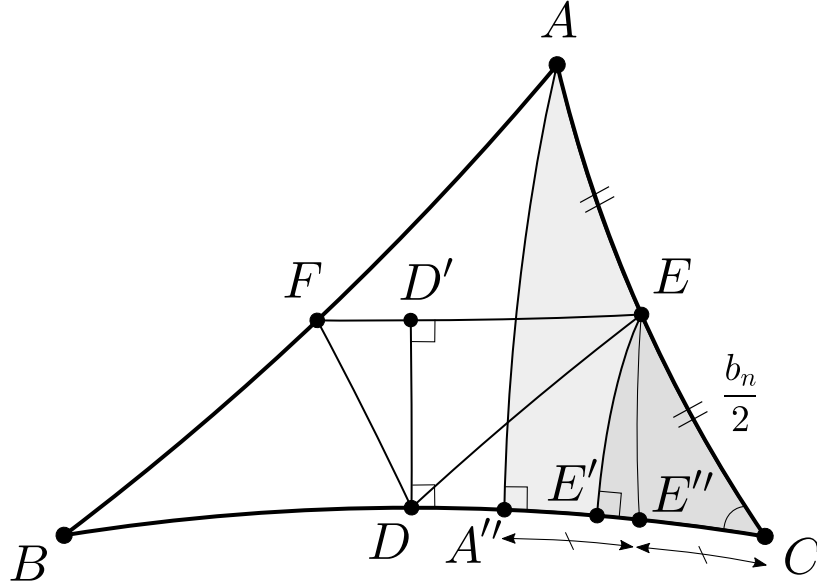

FIGURE 4.1

Once again, we begin by writing the quantity $2^n h_n$ as the following product:

$$2^n \cdot h_n = \frac{2h_n}{h_{n-1}} \cdot \frac{2h_{n-1}}{h_{n-2}} \cdot \ldots \cdot \frac{2h_1}{h_0} \cdot h_0 = h_0 \prod_{n=1}^{n} \frac{2h_i}{h_{i-1}}$$

Hyperbolic Upper Bound. Note first that, $\frac{1}{2}|A''A| > |EE''|$ by Lemma 3.0.1. Using the hyperbolic Pythagoras theorem we then have that $|EE''| > |EE'|$. Lemma 4.0.2 then guarantees that $|EE'| > |DD'|$. Putting everything together, we obtain that $\frac{1}{2}h_n = \frac{1}{2}|A''A| > |EE''| > |EE'| > |DD'|$, which shows that each factor (other than h_0) in the previous product is strictly less than 1. This shows the upper bound in the two cases where $t_{n+1} = CED$ (where $h_{n+1} = |EE'|$) and $t_{n+1} = FDE$ (where $h_{n+1} = |DD'|$). The case where $t_{n+1} = DFB$ is symmetrical to the case $t_{n+1} = CED$, but a slightly more circumvoluted argument is required to derive the upper bound when $t_{n+1} = EAF$. Let us denote by γ the angle $\angle E'CA$ and by γ' the angle $\angle A'EA = \angle C'EC$ (see Fig. 4.2). Note that $\gamma = \gamma_n$ and $\gamma' = \angle FEA$ if $\gamma_n \leqslant \frac{\pi}{2}$, but otherwise we instead have $\gamma = \pi - \gamma_n$ and $\gamma' = \pi - \angle FEA$. Applying hyperbolic trigonometric identities in the triangles $EE'C$ and ECC', we obtain:

$$\sinh |EE'| = \sin \gamma \cdot \sinh |EC| \tag{4.1}$$

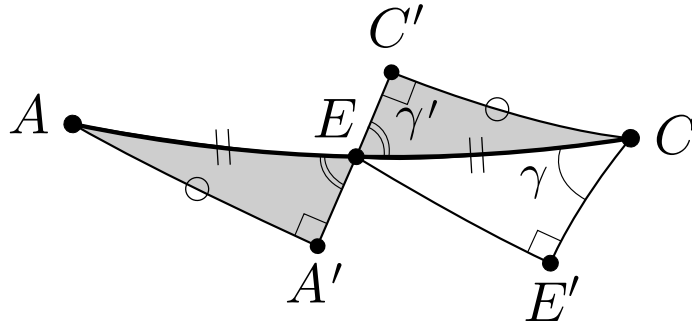$$\sinh |CC'| = \sin \gamma' \cdot \sinh |EC| \tag{4.2}$$

FIGURE 4.2

Using the hyperbolic sine law in the triangles ABC and AFE and making use on their shared angle α_n, we get:

$$\sin \gamma' = \frac{\sinh \frac{c_n}{2}}{\sinh c_n} \frac{\sinh a_n}{\sinh a_{n+1}} \sin \gamma \tag{4.3}$$

Let us denote by ρ_{n+1} the factor preceding $\sin \gamma$ in the previous equation, i.e. such that we have $\sin \gamma' = \rho_{n+1} \sin \gamma$. Combining equations (4.1), (4.2) and (4.3) now gives us:

$$\sinh |CC'| = \rho_{n+1} \sinh |EE'| \tag{4.4}$$

Using the observation that for all $x > 0$, we have $\sinh \frac{x}{2} < \frac{1}{2} \sinh x$, we reach:

$$\rho_{n+1} < \frac{\sinh \frac{a_n}{2}}{2 \sinh \frac{a_{n+1}}{2}} \cosh \frac{a_n}{2} \tag{4.5}$$

Using inequality $(\star)$ in the proof of Theorem C and noting that, for all $x > 0$, we have $\sinh x > x$ and $\sinh x / x < \exp x$, we obtain that:

$$\frac{\sinh \frac{a_n}{2}}{2 \sinh \frac{a_{n+1}}{2}} < \frac{\sinh \frac{a_n}{2}}{a_{n+1}} = \frac{a_n}{2\, a_{n+1}} \frac{\sinh \frac{a_n}{2}}{\frac{a_n}{2}} < \exp\left(\frac{b_{n+1}}{2}\right) \frac{\sinh \frac{a_n}{2}}{\frac{a_n}{2}} < \exp\left(\frac{b_{n+1}}{2} + \frac{a_n}{2}\right)$$

which leads us to the following upper bound on ρ_n (noting that for all $x > 0$, we have $\cosh x < \exp x$):

$$\rho_{n+1} < \exp\left(\frac{b_{n+1}}{2} + \frac{a_n}{2}\right) \cosh \frac{a_n}{2} < \exp\left(\frac{b_{n+1}}{2} + a_n\right)$$

Observing that, for all $n > 0$, we have $\exp\left(\frac{b_{n+1}}{2} + a_n\right) > 1$, we obtain the following inequality (for any choice of t_{n+1}):

$$2 \sinh h_{n+1} < 2 \max\{\rho_{n+1}, 1\} \sinh |EE'| < \exp\left(\frac{b_{n+1}}{2} + a_n\right) \sinh h_n$$

From this we conclude that:

$$2^n h_n < 2^n \sinh h_n = \sinh h_0 \prod_{i=0}^{n} \frac{2 \sinh h_{i+1}}{\sinh h_i} < \sinh h_0 \prod_{i=0}^{n} \exp\left(\frac{b_{i+1}}{2} + a_i\right)$$

Making use of Lemma 3.0.1, we obtain:

$$2^n h_n < \sinh h_0 \cdot \exp\left(\sum_{i=0}^{n} \left(\frac{b_{i+1}}{2} + a_i\right)\right) < \frac{\sinh h_0}{h_0} e^{\frac{b_0}{2} + 2a_0} h_0$$

We have thus shown that $L_h = \frac{\sinh h_0}{h_0} e^{\frac{b_0}{2} + 2a_0} > 1$ is a valid choice as L_h indeed approaches 1 from above as the edge lengths of t_0 become smaller.

Hyperbolic Lower Bound. We now want a lower bound for the product $\prod_{n=1}^{n} \frac{2h_i}{h_{i-1}}$. Because of Lemma 4.0.2, we know that it is enough to derive the lower bound in the case where t_{n+1} is obtained as the innermost triangle. Once again, it is equivalent to derive an upper bound for the sum of the absolute values of the logarithm of its factors, namely:

$$2^n \cdot h_n = h_0 \prod_{i=1}^{n} \frac{2h_i}{h_{i-1}} > \delta > 0 \iff |\ln(h_0)| + \sum_{i=1}^{n} \left|\ln\left(\frac{2h_i}{h_{i-1}}\right)\right| < \Delta < \infty$$

Where $e^\Delta = \delta$. However, it will prove easier to bound the ratio $\sinh h_n / \sinh h_{n-1}$ instead. We start by applying Lemma 3.0.6 in the Lambert quadrilateral $DE'ED'$:

$$\sinh h_{n+1} = \sinh |DD'| = \frac{\sinh |E'E|}{\cosh |D'E|} \tag{4.6}$$

We can also establish the following identity through the hyperbolic sine rule applied to the triangles CAA'' and CEE':

$$\frac{\sinh h_n}{\sinh |EE'|} = \frac{\sinh |A''A|}{\sinh |EE'|} = \frac{\sinh |CA|}{\sinh |CE|} = \frac{\sinh 2b_n}{\sinh b_n} \tag{4.7}$$

Combining both identities (4.6) and (4.7), we obtain:

$$\frac{2 \sinh h_{n+1}}{\sinh h_n} = \frac{2 \sinh b_n \cdot \sinh |EE'|}{\sinh 2b_n \cdot \sinh |EE'| \cdot \cosh |ED'|} = \frac{1}{\cosh b_n \cosh |ED'|} \tag{4.8}$$

This shows that $2 \sinh h_{n+1} / \sinh h_n < 1$ and tells us that we need a lower bound on the ratios $2 \sinh h_i / \sinh h_{i-1}$ instead of an upper bound as $|\ln(2 \sinh h_i / \sinh h_{i-1})| = \ln(\sinh h_{i-1}/2 \sinh h_i)$. Since $|ED'| \leqslant b_{n+1}$, as the leg is always less than the hypotenuse in hyperbolic geometry, we obtain:

$$\frac{2 \sinh h_{n+1}}{\sinh h_n} \geqslant \frac{1}{\cosh b_n \cdot \cosh b_{n+1}} \geqslant (\cosh b_n)^{-2} \tag{4.9}$$

As $\ln \cosh b_n < b_n$ and the sequence $(b_n)_{n \in \mathbb{N}}$ is bounded above by the geometric series $(b_0 \cdot 2^{-n})_{n \in \mathbb{N}}$, inequality (4.9) gives us the desired logarithm convergence criterion for $\sinh h_n$:

$$\sum_{i=0}^{n} \left| \ln \left(\frac{2 \sinh h_{i+1}}{\sinh h_i} \right) \right| < 2 \sum_{i=0}^{n} \ln(\cosh b_i) < 2 \sum_{i=0}^{n} b_i$$

Taking the exponential of both sides and multiplying by $\sinh h_0$, we now get back to the original product:

$$2^n \sinh h_n = \sinh h_0 \prod_{i=0}^{n} \frac{2 \sinh h_{i+1}}{\sinh h_i} > \sinh h_0 \, \exp \left(-2b_0 \sum_{i=1}^{n} 2^{-i} \right) > \sinh h_0 \, e^{-2b_0}$$

For any $n \in \mathbb{N}$, we can choose $\epsilon_n = (h_n - \sinh h_n)/\sinh h_n$ so that $h_n > (1 - \epsilon_n) \sinh h_n$. It is easy to see that, for all $n \in \mathbb{N}$, $\epsilon_n > 0$ and $\lim_{n \to \infty} \epsilon_n \to 0$. Moreover, one can show the sequence $(\epsilon_n)_{n \in \mathbb{N}}$ to be monotone decreasing. This allows us to write, for all $n \in \mathbb{N}$:

$$2^n h_n > 2^n \sinh h_n (1 - \epsilon_n) > \lim_{n \to \infty} 2^n \sinh h_n (1 - \epsilon_0) > \sinh h_0 \, e^{-2b_0}(1 - \epsilon_0)$$

As t_0 becomes smaller, both $\exp(-2b_0)$ and $(1 - \epsilon_0)$ approach 1 from below. Taking $l_h = \exp(-2b_0)(1 - \epsilon_0)$ and noticing that $\sinh h_0 > h_0$ thus finishes the proof. $\qquad \square$

Proof of Proposition 4.0.1 (Spherical Setting). Some of the arguments used to derive the hyperbolic upper bound unfortunately do not translate to the spherical lower bound, which is why we take a slightly different approach here.

Spherical Lower Bound. In the proof of the hyperbolic lower bound, we showed equality (4.8) using only hyperbolic trigonometry, this result is thus also valid in the spherical case (switching hyperbolic functions for the spherical ones) as long as Lemma 3.0.7 holds. Selecting the integer N given by Lemma 3.0.7, we have that, for all $n > N$ for which t_{n+1} is obtained as the innermost triangle of t_n:

$$\frac{2\sin h_{n+1}}{\sin h_n} = \frac{1}{\cos b_n \cos |ED'|} \tag{4.10}$$

For such values of n, we thus have $2\sin h_{n+1}/\sin h_n > 1$.

In the case where $t_{n+1} = CED$, we make use of the spherical equivalent of identity (4.6) and the spherical Pythagoras theorem in triangle DED' to obtain, for all $n > N$:

$$\sin h_{n+1} = \sin |DD'| \cos |D'E| > \frac{\sin h_n}{2} \frac{\cos |DE|}{\cos |DD'|} > \frac{\sin h_n}{2} \cos c_{n+1} \tag{4.11}$$

which shows that for such values of n, we have $2\sin h_{n+1}/\sin h_n > \cos c_{n+1}$. The case where $t_{n+1} = DFB$ is symmetrical. The case where $t_{n+1} = EAF$ is dealt with in the exact same way as the hyperbolic upper bound case, substituting spherical functions for the hyperbolic ones, reversing all the inequalities used and making use of (3.4) instead of ($\star$). In this fashion, we reach the inequality:

$$\sin h_{n+1} > \sin |EE'| \cos \frac{a_n}{2} \cos \frac{a_{n+1}}{2} > \frac{\sin h_n}{2} \cos c_{n+1} \left(\cos \frac{a_n}{2}\right)^2$$

Putting all the cases together and observing that, for all $n > N$, we have $\cos c_{n+1} \left(\cos \frac{a_n}{2}\right)^2 < \cos c_{n+1} < 1$, we obtain the following inequality (for all choices of t_{n+1}):

$$2\sinh h_{n+1} > \min\left\{\cos c_{n+1} \left(\cos \frac{a_n}{2}\right)^2, \cos c_{n+1}, 1\right\} \sin h_n = \cos c_{n+1} \left(\cos \frac{a_n}{2}\right)^2 \sin h_n$$

Noting that for all $x > 0$, we have $x > \sin x$, we obtain:

$$2^n h_n > 2^n \sin h_n = \sin h_0 \prod_{i=0}^{N-1} \frac{2 \sin h_{i+1}}{\sin h_i} \prod_{i=N}^{n} \frac{2 \sin h_{i+1}}{\sin h_i} > C_N \sin h_0 \prod_{i=N}^{n} \frac{2 \sin h_{i+1}}{\sin h_i}$$

where $C_N = \prod_{i=0}^{N-1} \frac{2 \sin h_{i+1}}{\sin h_i}$. Combining this last inequality with our lower bound for $2 \sin h_{i+1} / \sin h_i$, we obtain:

$$2^n h_n > C_N \sin h_0 \prod_{i=N}^{n} \cos c_{i+1} \left(\cos \frac{a_i}{2} \right)^2$$

Adjusting our choice of N to ensure that, for all $n > N$, we have $\cos c_{n+1}$, $\cos \frac{a_n}{2} < \frac{\pi}{3}$, we can make use of the inequality $\cos x > e^{-x}$, for all $x \in (0, \frac{\pi}{3})$. Using Theorem C, we obtain:

$$2^n h_n > C_N \sin h_0 \exp \left(-\sum_{i=0}^{n} (c_{i+1} + a_i) \right) > \frac{\sin h_0}{h_0} C_N e^{-(c_0 L_c + 2 a_0 L_a)} h_0$$

We have thus shown that $l_h = \frac{\sin h_0}{h_0} C_N e^{-(c_0 L_c + 2 a_0 L_a)} < 1$ is a valid choice as l_h indeed approaches 1 from below as the edge lengths of t_0 become smaller.

Spherical Upper Bound. Just like for the lower bound in the hyperbolic setting, Lemma 4.0.2 allows us to deal only with the case where t_{n+1} is the innermost triangle of t_n. Since we showed that for all $n > N$, we have $2 \sin h_{n+1} / \sin h_n > 1$, we now need an upper bound as $|\ln (2 \sin h_{i+1} / \sin h_i)| = \ln (2 \sin h_i / \sin h_{i-1})$. But since the cosine function is decreasing on $[0, \pi]$, we still desire an upper bound on $|ED'|$ in (4.10). Our previous remark that $|ED'| \leqslant b_{n+1}$, for all $n > N$, is again enough to yield the desired upper bound:

$$\frac{2 \sin h_{n+1}}{\sin h_n} \leqslant \frac{1}{\cos b_n \cdot \cos b_{n+1}} \leqslant (\cos b_{n+1})^{-2} \tag{4.12}$$

Lemma 3.0.2 allows us to adjust our choice of N such that the series $(b_n)_{n>N}$ is bounded above by the geometric series $(b_0 \cdot (\frac{C}{2})^n)_{n>N}$. Adjusting one last time our choice of N to ensure that all edge lengths are smaller than $\pi/3$, we can once again guarantee that $-\ln \cos b_n < b_n$, for all $n > N$. Writing $C'_N = \sum_{n=0}^{N} \ln (2 \sin h_{n+1} / \sin h_n)$, we can write:

$$\sum_{i=0}^{n} \left| \ln \left(\frac{2 \sin h_{i+1}}{\sin h_i} \right) \right| < -2 \sum_{i=N}^{n} \ln(\cos b_{i+1}) + C'_N < 2 \sum_{i=N}^{n} b_{i+1} + C'_N$$

which gives us:

$$|\ln(h_0)| + \sum_{i=0}^{n} \left| \ln \left(\frac{2 \sin h_{i+1}}{\sin h_i} \right) \right| < |\ln(h_0)| + 2 b_0 \sum_{i=N}^{n} \left(\frac{C}{2} \right)^i + C'_N$$

After taking the exponential of both sides, we retrieve:

$$2^n \sin h_n = h_0 \prod_{i=0}^{n} \frac{2 \sin h_{i+1}}{\sin h_i} < h_0 \exp \left(4 b_0 \frac{(C/2)^N}{2 - C} + C'_N \right)$$

Let then $\epsilon_n = h_n / \sin h_n - 1$. It is straightforward to check that for all n, we have $\epsilon_n > 0$ and $\lim_{n \to \infty} \epsilon_n \to 0$. It is also easy to establish that $(\epsilon_n)_{n \in \mathbb{N}}$ is a monotone and decreasing sequence. Because of these observations, we can write:

$$2^n h_n < (1 + \epsilon_n) \, 2^n \sin h_n < h_0 \, (1 + \epsilon_0) \exp \left(4 b_0 \frac{(C/2)^N}{2 - C} + C'_N \right)$$

Choosing $l_h = (1 + \epsilon_0) \exp \left(4 b_0 \frac{(C/2)^N}{2-C} + C'_N \right)$ finishes the proof. Indeed, both N and C'_N converges to zero for t_0 small enough and the exponential goes to 1 as b_0 grows smaller and C is fixed. Both $(1 + \epsilon_0)$ and the exponential approach 1 (the exponent is always strictly positive). $\qquad \square$

Chapter 5

Stabilisation of Angles

In this section, we quickly derive the proof of our main result as a corollary of Proposition 4.0.1 and Theorem C and show that the angles in the subdivision behave "nicely" in the limit, in a sense made precise by the following proposition:

Theorem B. *For any sequence of nested triangles $t_0, t_1, \ldots$ and for all $n \in \mathbb{N}$, there exists $l_\alpha, L_\alpha > 0$ such that:*

$$\alpha_0 \cdot l_\alpha < \alpha_n < \alpha_0 \cdot L_\alpha$$

In addition, l_α (resp. L_α) approaches 1 from below (resp. above) as all the side lengths of t_0 become smaller.

Proof. We derive the proof in the hyperbolic case, the spherical case is obtained in the exact same fashion by swapping hyperbolic trigonometric functions for spherical ones and reversing the appropriate inequalities.

Let h_n denote the height of the altitude drawn from the vertex incident to α_n onto the line prolonging the side of t_n of length c_n. Trigonometric identities ([Thu97], p.81) give us that $\sin \alpha_n$ is simply the ratio of $\sinh h_n$ by $\sinh b_n$. Let $\epsilon_n = 1 - \sinh(h_0 \, l_h \, 2^{-n})/(l_h \, 2^{-n} \sinh h_0)$. It is easy to check that $\epsilon_n > 0$ for all $n \in \mathbb{N}$, and $\lim_{n \to \infty} \epsilon_n = 0$. Similarly, we define ϵ'_n as the quantity $1 - \sinh(b_0 \, l_b \, 2^{-n})/(l_b \, 2^{-n} \sinh b_0)$ and make the same observations. We remark as

well, that, for all $x > 0$ and all $0 < \epsilon < 1$, we have $\sinh(\epsilon x) < \epsilon \sinh x$. Putting the previous observations together with Theorem C and Proposition 4.0.1 gives us the two following two chains of inequalities:

$$\sin \alpha_n = \frac{\sinh h_n}{\sinh b_n} < \frac{\sinh(h_0 \, 2^{-n})}{\sinh(b_0 \, l_b \, 2^{-n})} < \frac{(\sinh h_0) \, 2^{-n}}{(1 - \epsilon'_n) \, l_b \, 2^{-n} \sinh b_0} = \sin \alpha_0 (1 - \epsilon'_n)^{-1} l_b^{-1}$$

$$\sin \alpha_n = \frac{\sinh h_n}{\sinh b_n} > \frac{\sinh(h_0 \, l_h \, 2^{-n})}{\sinh(b_0 \, 2^{-n})} > \frac{(1 - \epsilon_n) \, l_h \, 2^{-n} \sinh h_0}{(\sinh b_0) \, 2^{-n}} = \sin \alpha_0 (1 - \epsilon_n) l_h$$

It is easy to show that both $(\epsilon_n)_{n \in \mathbb{N}}$ and $(\epsilon'_n)_{n \in \mathbb{N}}$ are monotone sequences decreasing to 0. Putting together both previous inequalities, we obtain:

$$\sin \alpha_0 (1 - \epsilon_0) l_h < \sin \alpha_n < \sin \alpha_0 \left((1 - \epsilon'_0) l_b\right)^{-1} \tag{5.1}$$

Choosing $l'_\alpha = (1 - \epsilon_0) l_h$ and $L'_\alpha = \left((1 - \epsilon'_0) l_b\right)^{-1}$ then yields the desired inequality on the sines of the angles. Indeed, both $1 - \epsilon_0$ and l_h approach 1 from below as t_0 becomes smaller. Likewise, both $(1 - \epsilon'_0)$ and l_b approach 1 from above as t_0 becomes smaller.

While the sine function is monotonous and continuous, each element of the open interval $(0, 1)$ has two pre-images under it, one in the interval $(0, \frac{\pi}{2})$ and the other in the interval $(\frac{\pi}{2}, \pi)$. The following claim will allow us to differentiate between these two pre-images and establish the equivalent of inequality (5.1) for the angles:

Claim. *In the hyperbolic setting, for any sequence of nested triangles $t_0, t_1, \ldots$ such that $\alpha_0 > \frac{\pi}{2}$, and for all $n \in \mathbb{N}$, we have $\alpha_n > \frac{\pi}{2}$. In the spherical setting, for any sequence of nested triangles $t_0, t_1, \ldots$, there exists $N \in \mathbb{N}$ such that if $\alpha_N < \frac{\pi}{2}$, for all $n > N$, we have $\alpha_n < \frac{\pi}{2}$.*
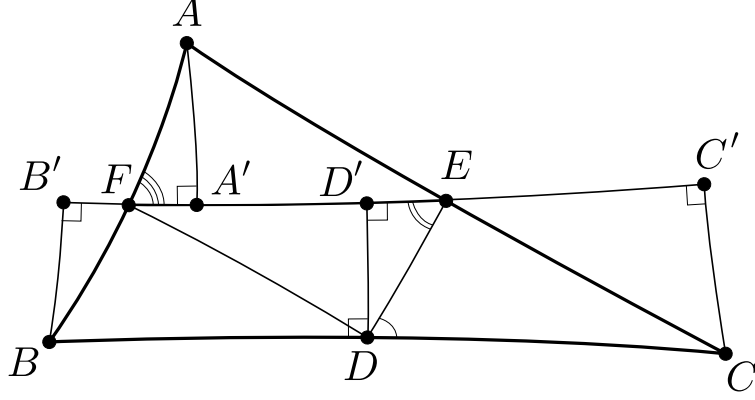
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FIGURE 5.1

Proof of Claim. In order to allow us to reuse some of our previous constructions and notations, we prove our claim for β_n instead of α_n. It is enough to show that, for all $n \in \mathbb{N}$ and any choice of t_{n+1}, we have that $\beta_n > \frac{\pi}{2}$ implies $\beta_{n+1} > \frac{\pi}{2}$. Suppose not, and there exists an integer n and a choice of t_{n+1} such that $\beta_n > \frac{\pi}{2}$ but $\beta_{n+1} \leqslant \frac{\pi}{2}$. There are three possible cases. Let us first investigate the case where t_{n+1} was chosen as the innermost triangle of t_n. Following the notation of Figure 3.4, where $t_{n+1} = DEF$, we see that if $\beta_{n+1} = \angle FED \leqslant \frac{\pi}{2}$, D' must be positioned on the closed half-line starting at E and containing the edge FE (see Fig. 5.1). Since $|B'D'| = |FE|$, this entails that B' lies on the closed half-line starting at F and not containing the open edge FE. But since β_n is obtuse and the angle at the apex of a hyperbolic Saccheri quadrilateral is acute, B' must in fact lie on the open half-line starting at F and containing the open edge FE. A similar contradiction is derived for the other two cases: if $t_{n+1} = EDC$, D' must also be positioned on the half-line starting at E and containing the edge FE and the same contradiction is derived; the case where $t_{n+1} = AFE$ is symmetric with the previous case, exchanging A and C.

The exact same argument solves the spherical case, selecting $N \in \mathbb{N}$ according to Lemma 3.0.7 and reversing all the inequalities. To derive the contradiction, this time, we appeal to the fact that β_n is acute and that the angle at the apex of a spherical Saccheri quadrilateral is obtuse.
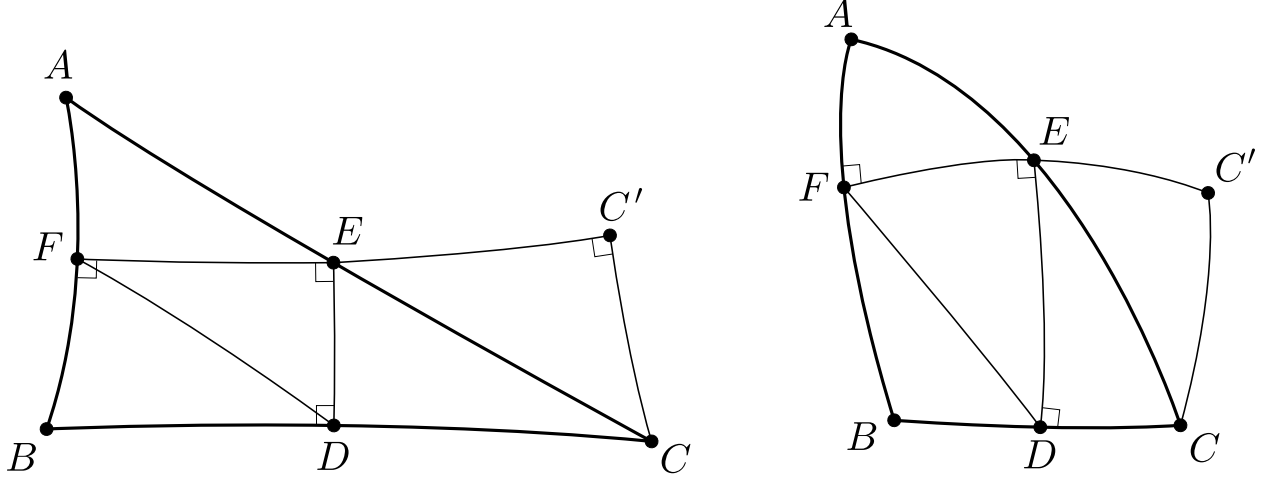
FIGURE 5.2

Remark 5.0.1. Interestingly, in both the hyperbolic case and the spherical case (as long as $|DD''| \neq \frac{\pi}{2}$), whenever one of the angles $\angle FED$, $\angle EFA$ or $\angle CDE$ is right, they must in fact all be simultaneously right (see Fig. 5.2). Indeed, if $\angle EFA$ is right, then $B' = F$ and thus $D' = E$ which forces both other angles to be right. Likewise, if $\angle CDE$ (resp. $\angle FED$) is a right angle, we have $D' = E$ which means that $\angle FED$ (resp. $\angle EFA$) is a right angle and also forces $B' = F$, which implies $\angle EFA$ is also right.

$\square$

As an immediate consequence of Theorem B, we deduce our main theorem:

Theorem A. *For any geodesic triangle T in M_κ^2, there exists $\delta > 0$ such that, for all $n \in \mathbb{N}$, all the angles of T_n lie in the interval $(\delta, \pi - \delta)$.*

Proof. Let $t_0 = T_0$ and $\delta_\alpha = \min\{\alpha_0 l_\alpha, (\pi - \alpha_0 L_\alpha)\}$, $\delta_\beta = \min\{\beta_0 l_\beta, (\pi - \beta_0 L_\beta)\}$ and $\delta_\gamma = \min\{\gamma_0 l_\gamma, (\pi - \gamma_0 L_\gamma)\}$. By construction, $\delta := \min\{\delta_\alpha, \delta_\beta, \delta_\gamma\} > 0$ and every angle of T_n lies in the interval $(\delta, \pi - \delta)$. $\square$

Remark 5.0.2. Both in the hyperbolic and spherical case, there is an easier route to proving Theorem A without the full strength of Theorem B. We briefly explain here why, in both cases, a lower bound on the angles in fact also gives an upper bound. Indeed, in the hyperbolic

case, to establish the upper bound in Theorem A we need only remember that the sum of the angles of a hyperbolic triangle is always *less* than π. Therefore if we assume by contradiction that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\alpha_N > \pi - \epsilon$, since for all $n \in \mathbb{N}$, we have $\beta_n, \gamma_n > \delta$, we are lead to a contradiction for $\epsilon < 2\delta$ as we would then obtain that $\alpha_n + \beta_n + \gamma_n > 2\delta + \pi - 2\delta > \pi$ for some choice of n. In the spherical case, the sum of the angles of a triangle is allowed to exceed π, but the amount by which it does is exactly the area of the triangle. To derive a contradiction we thus need to consider a second parameter to utilise the key property that our triangulations are getting arbitrarily small. Let A_n denote the area of the triangle t_n. Suppose then by contradiction that, for all $\epsilon, \epsilon' > 0$, there exists $N \in \mathbb{N}$ such that $\alpha_N > \pi - \epsilon$ and $A_n < \epsilon'$. Let us pick $\epsilon' < 2\delta$ and $\epsilon < 2\delta - \epsilon'$. Since for all $n \in \mathbb{N}$, we have $\beta_n, \gamma_n > \delta$, this leads us to the following contradiction:

$$\alpha_n + \beta_n + \gamma_n > 2\delta + \pi - \epsilon > \pi + \epsilon'$$

Because of this observation, and using our approach to proving Theorem B, readers interested solely in proving Theorem A need only refer to the proofs of the lower bounds in the proof of Proposition 4.1 and the proofs of the upper bounds in the proof of Theorem C.

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