Nonpositive Towers In Compact 3-Manifold Spines

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Contents

Abstract. A 2-complex X has nonpositive towers if every tower map $Y \to X$ from a connected compact 2-complex Y either has $\chi(Y) \leq 0$ or Y contractible. We give a short exposition of the consequences of nonpositive towers and ways to detect the stronger property of nonpositive immersions. We then show that compact contractible 3-manifold spines can fail to have nonpositive immersions but always have nonpositive towers.

Abstract. Un 2-complexe X a la propriété de "nonpositive towers" si chaque "tower map" $Y \to X$ d'un 2-complexe Y qui est connexe et compact satisfait $\chi(Y) \leq 0$ ou a le Y contractile. On donne une courte exposition des conséquences des "nonpositive towers" et des façons de détecter la propriété plus restrictive de "nonpositive immersions". On montre ensuite que les épines connexes, compactes et contractiles de 3-variétés peuvent ne pas avoir de "nonpositive immersions" mais ont toujours des "nonpositive towers".

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1. Introduction

Definition 1.1. A 2-complex X has *nonpositive immersions* if for every combinatorial immersion $Y \to X$ with Y compact and connected, either $\chi(Y) \leq 0$ or Y is contractible.

A 2-complex X has *nonpositive towers* if for every tower map $Y \to X$ with Y compact and connected, either $\chi(Y) \leq 0$ or Y is contractible.

The objective of this paper is to motivate these definitions through their consequences on 2-complexes and their fundamental groups, and show via some recent examples that nonpositive towers do not imply nonpositive immersions.

Tower maps and tower lifts will be defined in Section 2. The former are the main maps of interest in the rest of the paper, and the latter are important in the following section.

In Section 3, we will restate the above definitions and review some of the consequences of nonpositive towers and immersions. In particular, we show that a 2-complex X having nonpositive towers is aspherical with $\pi_1 X$ locally indicable.

While nonpositive immersions are a stronger condition than nonpositive towers, the former are often easier to detect. In Section 4, we give some tests for determining whether a 2-complex has nonpositive immersions, and list some classes of 2-complexes having nonpositive immersions.

In Section 5, we give two sources of counterexamples to the conjecture that contractible 2-complexes have nonpositive immersions. The second of these is the following original theorem [2], which is a special case of Proposition 5.5:

Theorem 1.2. *Every collapsed compact spine of a simply-connected 3-manifold containing a disc has an immersed sphere.*

It was shown in [19] that every aspherical 3-manifold with nonempty boundary has a spine with nonpositive immersions. This utilized that there exists a spine with no near-immersion of a 2-sphere [3]. However, it was shown in [5] that for $n \geq 3$, every PL n-manifold M with $\partial M \neq \emptyset$ has a spine X such that $\partial M \to X$ is an immersion, and moreover, such spines are generic among all spines. So Theorem 1.2 is a variant of the simplest instance of their result.

We show in Section 6 (see Theorem 6.7) that these spines still have nonpositive towers. This is another original result [2]:

Theorem 1.3. *Every compact spine of an aspherical 3-manifold has nonpositive towers.*

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2. Towers

In this section, we define *towers* and *tower lifts*. The latter will be a useful construction in Section 3. We then give a proof by Howie and Wise of the existence of maximal tower lifts.

Definition 2.1. A *combinatorial 2-complex* is a 2-dimensional CW-complex whose 2-cells are attached to the 1-skeleton by a finite concatenation of homeomorphisms onto 1-cells. That is, the attaching maps of each 2-cell is specified up to homotopy by the finite sequence of 1-cells it traverses.

Convention 2.2. All 2-complexes in this paper are assumed to be combinatorial.

Definition 2.3. A *combinatorial map* between 2-complexes is a continuous map that maps open cells homeomorphically onto open cells. A map $Y \to X$ between topological spaces is an *immersion* if every point in Y has an open neighbourhood that is mapped injectively into X. A combinatorial map $Y \to X$ between connected 2-complexes is a *tower map* if it factors as a finite composition of subcomplex inclusions and covering maps. That is,

$$Y \hookrightarrow \widehat{A}_n \twoheadrightarrow A_n \hookrightarrow \widehat{A}_{n-1} \twoheadrightarrow A_{n-1} \hookrightarrow \cdots \hookrightarrow \widehat{A}_1 \twoheadrightarrow A_1 \hookrightarrow X.$$

Since inclusions and covering maps are immersions, so are tower maps. Note also that, if Y is finite, then we can take each A_i to also be finite by restricting our attention to the image of Y at each step.

Tower maps arose in Papakyriakopolous' classical 3-manifold proofs, and also arose naturally in one-relator group theory [10].

Definition 2.4. Let $f : Y \to X$ be a map of connected CW-complexes. Then a map $\tilde{f} : Y \to T$ is a *tower lift* of f if there is a tower map $t : T \to X$ with $t \circ \tilde{f} = f$.

A tower lift $\tilde{f} : Y \to T$ is *maximal* if, for any further tower lift $\tilde{f}' : Y \to T'$ of $\tilde{f}$, the tower map $T' \to T$ is an isomorphism of CW complexes.

Some important properties of maximal tower lifts is that they are surjective and π_1-surjective. Indeed, if a tower map $f : Y \to T$ is not surjective, then it can be factored as $Y \to \mathrm{im}(f) \hookrightarrow T$ where $\mathrm{im}(f) \hookrightarrow T$ is a tower map, and so f is not maximal. Similarly, if $f : Y \to T$ is not

π_1-surjective, then it lifts to a covering space $\widehat{T}$ corresponding to $\mathrm{im}(f_*) \leq \pi_1 T$ with $\widehat{T} \twoheadrightarrow T$ a tower map, and so f is again not maximal.

The surjectivity and π_1-surjectivity of maximal tower lifts will be very convenient for us in the next section. If we want to make the most of them, it would be great to know that such maximal tower lifts always exist. We end this section with the following lemma, proved in the combinatorial case by Howie in [9] and then extended to the case of maps sending open cells to open cells by Wise in [19], which tells us that this is true when the domain is finite:

Lemma 2.5. *Let X and Y be connected CW-complexes with Y finite. Then any map $f : Y \to X$ mapping open cells onto open cells (of possibly lower dimension) has a maximal tower lift.*

Proof. Let $\mathrm{size}(\cdot)$ denote the total number of cells in a finite complex. Let $T_0 = X$, and for each $i \geq 0$ let $Y \to T_{i+1}$ be a surjective tower lift of $Y \to T_i$. Note that $T_i \to X$ is a combinatorial immersion and each open cell in Y maps to a single open cell in X, so each open cell in Y must map to a single open cell in T_i. Since $Y \to T_i$ is surjective, we have $\mathrm{size}(T_i) \leq \mathrm{size}(Y)$ for each T_i. But $\mathrm{size}(T_{i+1}) > \mathrm{size}(T_i)$ whenever $T_{i+1} \to T_i$ fails to be an isomorphism, since $T_{i+1} \to T_i$ is a combinatorial surjection. So $T_{i+1} \to T_i$ can only fail to be an isomorphism at most $\mathrm{size}(Y)$ times, and so a maximal tower lift exists. $\square$

Remark 2.6. For any continuous map $f : Y \to X$, there exists a subdivision of Y and X into simplices so that f is homotopic to a simplicial map by the Simplicial Approximation Theorem. Such a map sends open cells to open cells.

3. Nonpositive Towers and Nonpositive Immersions

In this section, we define *nonpositive towers* and *nonpositive immersions*, which will be the main focus of the rest of this paper. We will then motivate these definitions by looking at some of their consequences.

Definition 3.1. A 2-complex X has *nonpositive immersions* if for every combinatorial immersion $Y \to X$ with Y compact and connected, either $\chi(Y) \leq 0$ or Y is contractible.

Similarly, a 2-complex X has *nonpositive towers* if for every tower map $Y \to X$ with Y compact and connected, either $\chi(Y) \leq 0$ or Y is contractible.

Note that nonpositive immersions imply nonpositive towers. There are several variants of these definitions. For example, one can weaken the requirement on Y to "either $\chi(Y) \leq 0$ or $\pi_1 Y = 1$", or simply "$\chi(Y) \leq 1$". There are also variations requiring $\chi(Y) \leq -c|Y|$ for some "size" $|Y|$ of Y.

These ideas have promise as a contextualizing framework towards Whitehead's asphericity conjecture, as well as towards understanding coherence [19]. The main consequesnces of nonpositive immersions are also enjoyed by complexes with nonpositive towers. The rest of this section will go over some of these consequences.

3.1. **Asphericity.** In this section, we talk about the connection between nonpositive towers and asphericity. We start by showing that any 2-complex having nonpositive towers is aspherical. This was proved in [19] for a more general version of nonpositive towers. The simplified proof below applies only to our stronger definition.

Theorem 3.2. *Let X be a 2-complex with nonpositive towers. Then X is aspherical.*

Proof. Let $f : S^2 \to X$ represent an element of $\pi_2 X$. By Remark 2.6, we can assume that f maps open cells onto open cells after subdividing and homotoping. By Lemma 2.5, f factors as $f : S^2 \xrightarrow{\tilde{f}} T \xrightarrow{t} X$, with $\tilde{f}$ a maximal tower lift and t a tower map. Since $\tilde{f}$ is surjective, T is compact and connected. Since $\tilde{f}$ is π_1-surjective, $\pi_1 T = 1$ and so $\chi(T) = b_0(T) - b_1(T) + b_2(T) = 1 + b_2(T) \geq 1$. So T must be contractible, since t is a tower. So f factors through a contractible space T, and is therefore nullhomotopic. □

Since having nonpositive towers is a property inherited by subcomplexes, they could be a way towards resolving Whitehead's famous asphericity conjecture [16]:

Conjecture 3.3. *Every subcomplex of an aspherical 2-complex is aspherical.*

In hopes of resolving the above, it was conjectured in [19] that all contractible 2-complexes have nonpositive immersions. This will be shown to be false in Section 5, but we are still led to the following:

Conjecture 3.4. *Every contractible 2-complex has nonpositive towers.*

If this were true, then Whitehead's asphericity conjecture would immediately follow. Let X be aspherical, and let $Y \subseteq X$ be a subcomplex. Then there is some covering space $\widehat{Y}$ of Y which embeds in the universal cover $\widetilde{X}$ of X. Since $\widetilde{X}$ is contractible, it would have nonpositive towers. Then $\widehat{Y}$ would have nonpositive towers and so be aspherical, and so Y would also be aspherical.

3.2. **Local Indicability.**

Definition 3.5. A group G is *locally indicable* if every nontrivial finitely generated subgroup $H \leq G$ surjects homomorphically onto $\mathbb{Z}$ (or equivalently, has infinite abelianization $\mathrm{Ab}(H)$).

We end this section with the following algebraic consequence of nonpositive towers [18]:

Theorem 3.6. *Let X be a 2-complex with nonpositive towers. Then $\pi_1 X$ is locally indicable.*

Proof. We wish to show that every nontrivial finitely generated subgroup of $\pi_1 X$ has infinite abelianization. Let $H \leq \pi_1 X$ be finitely generated subgroup with presentation $\langle h_1, \ldots, h_n \mid r_1, r_2, \ldots \rangle$ and having finite abelianization $\mathrm{Ab}(H)$.

We construct a 2-complex Y and a map $f : Y \to X$ as follows: let Y have a single 0-cell mapped by f to some basepoint in X. Let Y have a 1-cell for each h_i, mapped to representative closed paths in X^1. $\mathrm{Ab}(H)$ is determined by a finite subset $\{r_i\}$ of the relators. For each such r_i, attach a corresponding 2-cell to Y. We can extend the map $f|_{\partial r_i}$ to r_i, since the closed path in X corresponding to ∂r_i is nullhomotopic. This gives us a map $f : Y \to X$.

Note that Y is connected and compact, and has $\mathrm{Ab}(\pi_1 Y) = \mathrm{Ab}(H)$. Note also that the induced map $f_* : \pi_1 Y \to \pi_1 X$ surjects onto H.

By Remark 2.6, we can assume that f maps open cells onto open cells after subdividing and homotoping. By Lemma 2.5, f factors as $f : Y \xrightarrow{\tilde{f}} T \xrightarrow{t} X$, with $\tilde{f}$ a maximal tower lift and t a tower map. Since $\tilde{f}$ is surjective, T is compact and connected. Since $\tilde{f}$ is π_1-surjective, $\mathrm{Ab}(\pi_1 T)$ is finite, and so $\chi(T) = 1 - 0 + b_2(T) \geq 1$. So T must be contractible, since X has nonpositive towers and t is a tower. This means that $t_* : \pi_1 T \to \pi_1 X$ is the trivial map. But since f_* surjects onto H, so does t_*. So we have that H is trivial. $\square$

4. Tests for Detecting Nonpositive Immersions

While having nonpositive towers is more general than having nonpositive immersions, the latter property is often more easily detected. In this section, we give a few tests for nonpositive immersions.

4.1. **Sieradski Colouring Test.**

The following is an example of a test that uses angle assignments to detect nonpositive immersions.

Definition 4.1. An *angled 2-complex* X is a 2-complex with an assignment of real-valued angles to all corners of 2-cells.

Definition 4.2. The *link* of a 0-cell v in a 2-complex X, denoted $\mathrm{link}(v)$, is a graph with a vertex for each end of a 1-cell incident with v, and an edge for each corner of a 2-cell incident with v, connecting the vertices corresponding to its two adjacent ends of 1-cells.

Definition 4.3. A 2-complex X satisfies the *colouring test* if it admits an assignment of angles such that

(1) all angles are either 0 or π,

(2) every 2-cell in X has at least two corners with angle 0,

(3) every cycle in the link of each 0-cell in X contains at least two edges with angle π.

This test was used by Sieradski in [15] to detect asphericity, but Wise showed in [17] that it also implies nonpositive immersions. (In fact, it implies "nonpositive generalized sectional curvature", a stronger property which then implies nonpositive immersions.) The proof that we give here will be more streamlined to our purposes, but first, we will need to define a notion of curvature for angled 2-complexes.

Definition 4.4. Let X be an angled 2-complex.

The *curvature* $\kappa(f)$ at a 2-cell f of X is $\kappa(f) = \sum_{c \in f} \measuredangle c - (|\partial f| - 2)\pi$, where $|\partial f|$ denotes the path length of ∂f, and the sum is over the corners of f.

The *curvature* $\kappa(v)$ at a 0-cell v of X is defined as $\kappa(v) = (2 - \chi(\mathrm{link}(v)))\pi - \sum_{c \text{ at } v} \measuredangle c$, where the sum is over all 2-cell corners meeting at v.

The curvature at a 2-cell is just the amount that its angle sum deviates from that of a euclidean polygon. The curvature at a 0-cell can also be thought of as the deviation from the "flat" case. For example, on a surface, the equation reduces to $\kappa(v) = 2\pi - \sum_{c \text{ at } v} \measuredangle c$, corresponding to the deviation from the angle sum around a point in the euclidean plane.

Relating these curvatures to $\chi(X)$, we have the following combinatorial version of the Gauss-Bonnet theorem, which was first proven by Ballmann and Buyalo in [1] and rediscovered by McCammond and Wise in [13].

Theorem 4.5 (Combinatorial Gauss-Bonnet)**.** *Let X be an angled 2-complex. Then*

$$2\pi\chi(X) = \sum_{f \in X} \kappa(f) + \sum_{v \in X} \kappa(v).$$

This is the main tool that we will need to show that the colouring test implies nonpositive immersions. Before the proof, we will need one more property of 2-complexes which will simplify the argument:

Definition 4.6. A 2-complex is *collapsed* if no cell has a free face - i.e. no 0-cell has degree 1, and no 1-cell is incident to a single side of a 2-cell.

If a 2-complex contains a cell with a free face, then it *collapses.* That is, it deformation retracts to a subcomplex missing that open cell and face.

Theorem 4.7. *If a 2-complex X satisfies the colouring test, then X has nonpositive immersions.*

Proof. Let $Y \looparrowright X$ be a combinatorial immersion with Y compact and connected. Since Y collapses to a collapsed subcomplex which is homotopy equivalent to Y, we can assume that Y is collapsed.

Note that the angles in X induce an assignment of angles in Y. If the angles in X satisfy the colouring test, then so do the induced angles in Y. To see this, note that 2-cells of Y are sent to 2-cells in X and so contain at least two corners with angle 0, and cycles in links in Y are sent to cycles in links in X and so contain at least two edges with angle π. We give Y such an assignment of angles.

If Y is a single point, then we are done, so we assume that this is not the case.

For any 0-cell v in Y, we know that $\text{link}(v)$ is nonempty and finite, is not just a single vertex, and has no vertices of degree 1. Indeed, nonempty and finite follow from the fact that Y is connected, compact, and not just a single point. If the link were a single vertex, then v would be a free face of some 1-cell in X. Any vertex of degree 1 in the link would correspond to a free face of a 2-cell in X.

Let v be a 0-cell in Y. Consider the subgraph $\text{link}_0(v) \subseteq \text{link}(v)$ obtained by deleting all edges with angle π. If $\text{link}_0(v)$ has no edges, then it must have at least two connected components. If $\text{link}_0(v)$ contains an edge, then that edge is part of a cycle in $\text{link}(v)$ since there are no vertices with degree 1. This cycle must contain an edge e with angle π. If the endpoints of e lay in the same connected component of $\text{link}_0(v)$, then $\text{link}(v)$ would contain a cycle having only one edge with angle π. So the endpoints of e lie in different components of $\text{link}_0(v)$, and so $\text{link}_0(v)$ again has at least two connected components. Note that each component of $\text{link}_0(v)$ is a tree, and so $\chi(\text{link}_0(v)) \geq 2$.

For any 0-cell v in Y, let n be the number of edges in $\text{link}(v)$ with angle π. Then we have $\kappa(v) = (2 - \chi(\text{link}(v)))\pi - \sum_{c \text{ at } v} \measuredangle c = (2 - \chi(\text{link}_0(v)) + n)\pi - n\pi \leq (2 - 2 + n)\pi - n\pi = 0.$

We also have for any 2-cell f in Y that $\kappa(f) = \sum_{c \in f} \measuredangle c - (|\partial f| - 2)\pi \leq 0$, since f has at most $|\partial f| - 2$ corners with angle π. So by Theorem 4.5, $2\pi\chi(Y) = \sum_{f \in X} \kappa(f) + \sum_{v \in X} \kappa(v) \leq 0.$ $\square$

Example 4.8. Consider the Baumslag-Solitar group $BS(m, n) = \langle a, b \mid ba^m b^{-1} a^{-n} \rangle$ with $m, n \geq 1$. Then its presentation complex has nonpositive immersions. See Figure 1 for an angle assignment satisfying the colouring test.

4.2. **Good Stackings.** The following geometric test for nonpositive immersions was developed by Louder and Wilton in [11]. In contrast with the colouring test, which relied on the combinatorial Gauss-Bonnet theorem, this test computes the Euler characteristic directly.

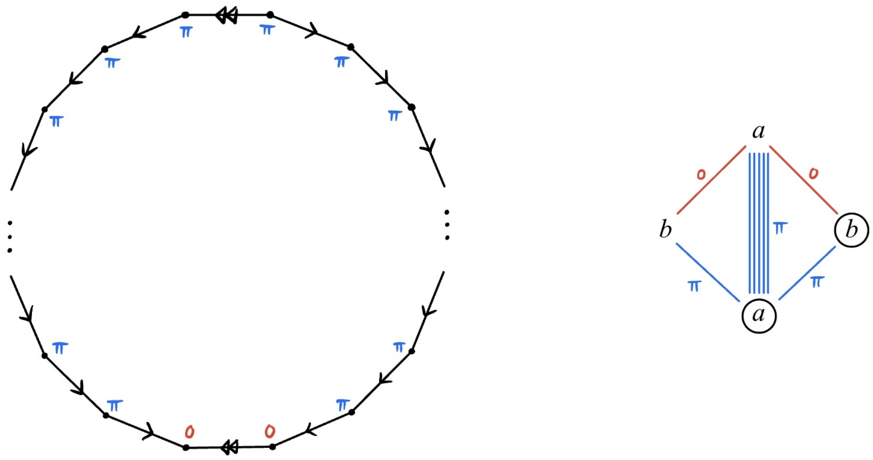

FIGURE 1. An angle assignmtment for the Baumslag-Solitar presentation complex $BS(m,n)$ with $m,n \geq 1$ and the link at its 0-cell.

Definition 4.9. Let X be a 2-complex with 2-cells whose boundary circles $\{w_i\}$ immerse in X^1. Let $\varphi : \bigsqcup_i w_i \to X^1$ be the immersion of these circles, and let p_X and $p_{\mathbb{R}}$ be the projections of $X^1 \times \mathbb{R}$ onto X^1 and $\mathbb{R}$. Then a *stacking* is an embedding $\sigma : \bigsqcup_i w_i \hookrightarrow X^1 \times \mathbb{R}$ such that $p_X \circ \sigma = \varphi$. A stacking is *good* if, for each w_i, there exist points $h_i, \ell_i \in w_i$ such that $p_{\mathbb{R}}(\sigma(h_i))$ is the highest point in $p_{\mathbb{R}}(\sigma(\varphi^{-1}(\varphi(h_i))))$ and $p_{\mathbb{R}}(\sigma(\ell_i))$ is the lowest point in $p_{\mathbb{R}}(\sigma(\varphi^{-1}(\varphi(\ell_i))))$.

A good stacking can be thought of as a stacking where each embedded w_i is "partially visible" to both an observer looking down from $X^1 \times \{+\infty\}$ and one looking up from $X^1 \times \{-\infty\}$, in the sense that their view is not blocked by any other embedded circles.

Continuing with this idea of an observer, we can ask what such an observer would see when, say, looking at the w_i from above. If every edge in X^1 is traversed by some w_i, they would see a decomposition of X^1 into circles and open arcs, with each arc belonging to a single w_i and ending when another arc passes above it. We can use this decomposition to calculate the Euler characteristic.

Theorem 4.10. *Let X be a 2-complex that admits a good stacking. Then X has nonpositive immersions.*

Proof. Let $Y \looparrowright X$ be a combinatorial immersion with Y compact connected. Note that the good stacking on X pulls back to a good stacking on Y. Without loss of generality, we can assume that Y is collapsed. If Y is a graph, we are done, so we also assume that Y contains at least one 2-cell.

Let $\{w_i\}$ be the boundary circles of 2-cells in Y. Consider the subcomplex of $Z \subseteq Y^1$ traversed by the w_i. Then, looking at the w_i in the stacking from above, we get a decomposition of Z into circles and open arcs. This decomposition is actually entirely composed of open arcs. To see this, suppose that this decomposition included a circle. Then there is some w_i that is entirely visible from above. Since it is also partially visible from below, there must be some point in w_i visible from both, and so some point p in Z traversed only once by a w_i. This means that p is contained in a free 1-cell of Y, contradicting the assumption that Y is collapsed.

So Z decomposes into open arcs $\{A_j\}$. Since each w_i is visible, each contributes at least one arc. So $\chi(Y) = \chi(Y^1) + |\{w_i\}| \leq \chi(Z) + |\{w_i\}| = -|\{A_i\}| + |\{w_i\}| \leq 0$. □

Definition 4.11. A group presentation is *staggered* if there is a linear order on the relators and on a subset of the generators such that each relator contains an ordered generator, and for any relators $r_1 < r_2$, $(\min r_1) < (\min r_2)$ and $(\max r_1) < (\max r_2)$, where $(\min r)$ and $(\max r)$ denote the minimal and maximal generators appearing in r.

It can be shown [11] that the standard complexes of 1-relator presentations without torsion admit good stackings, and so have nonpositive immersions. Similarly, staggered presentations yield complexes with good stackings. It was shown using other methods [12] that 1-relator presentation complexes with torsion also have nonpositive immersions, so we get the following:

Theorem 4.12. *If X is a the standard complex of a 1-relator or staggered presentation, then X has nonpositive immersions.*

4.3. **Slimness.** We end this section with one more test, developed by Helfer and Wise in [8]. It uses another method of directly computing the Euler characteristic, based on maximal elements of a particular preorder on 1-cells.

Definition 4.13. A *preorder* on a set S is a reflexive transitive relation on S, denoted by $\preceq$. For a subset $A \subseteq S$, an element $a \in A$ is *strictly maximal* in A if there is no $b \in A - \{a\}$ with $a \preceq b$.

Definition 4.14. Let X be a 2-complex. Then X is *slim* if there is a π_1-invariant preorder on 1-cells of $\widetilde{X}$ such that

(1) each 2-cell r in $\widetilde{X}$ has a unique strictly maximal 1-cell $e_r^+ \in \partial r$, which is traversed exactly once by the boundary path of r,
(2) for distinct 2-cells r_1 and r_2 in $\widetilde{X}$, $e_{r_1}^+ \subseteq \partial r_2 \implies e_{r_1}^+ \prec e_{r_2}^+$.

This can be thought of as follows: the assignment of e_r^+ corresponds to a "direction of flow" out of each 2-cell r of $\widetilde{X}$, and this choice of direction is π_1-invariant. Condition 1 adds the restriction that flow is directed out through a 1-cell of multiplicity one in the attaching map of r, and condition 2 then tells us that this flow from one 2-cell to another cannot double back on itself and cannot form directed cycles.

Definition 4.15. Let X be slim and let $Y \looparrowright X$ be a combinatorial immersion. A *widge* in Y is an open 1-cell that is mapped to some e_r^+ in X. An *isle* in Y is a connected component of Y^1 with the widges removed.

It is shown in [8] that for Y compact connected, if any isle in Y is a tree, then Y collapses to a point. So from that, we get the following:

Theorem 4.16. *Let X be a slim 2-complex. Then X has nonpositive immersions.*

Proof. Let $Y \looparrowright X$ be a combinatorial immersion with Y compact connected. We can assume that Y is collapsed. If any isle in Y is a tree, then Y is a point and we are done. So suppose that this is not the case.

Note that Y has at least one widge for each 2-cell, since each 2-cell is mapped to a 2-cell in X that has an associated e_r^+ and these e_r^+ are distinct by condition 2 of slimness. So $\chi(Y) \leq \chi(Y^1) + |\{widges\}| = \sum_{I \text{ isle}} \chi(I) \leq 0.$ □

While this looks very different from the previous test based on good stackings, it turns out that good stackings imply slimness. In fact, it was proven in [6] that good stackings are equivalent to *bislimness*, a stronger version of slimness also introduced in [8]. So torsion-free 1-relator and staggered presentation complexes are slim. There are however classes of complexes that are slim but do not always admit a good stacking:

Definition 4.17. A group presentation is *reducible* if there is a linear order on the relators and on the generators such that each relator contains an ordered generator, and for any relators $r_1 < r_2$, $(\max r_1) < (\max r_2)$, where $(\max r)$ denotes the maximal generator appearing in r.

Complexes from reducible presentations without torsion do not always admit a good stacking, but it was shown in [8] that such complexes are slim, and so have nonpositive immersions.

5. Contractible Complexes without Nonpositive Immersions

As mentioned in Section 3, there exist contractible CW-complexes failing to have nonpositive immersions. In this section, we examine two such families: certain contractible 3-manifold spines and certain two-relator presentations of the trivial group.

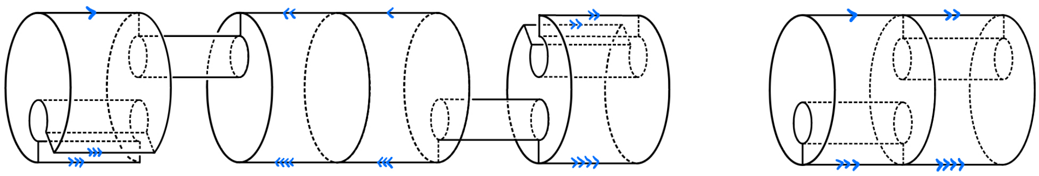

FIGURE 2. An immersed sphere (left) in Bing's house (right).

5.1. **3-Manifold Spines.** In this section, we show that compact collapsed contractible 3-manifold spines containing a 2-cell fail to have nonpositive immersions. This is one of the main results in a paper that I recently coauthored with Wise [2]. Most of this section is taken directly from that paper.

Definition 5.1. A *spine* of a 3-manifold M is an embedded 2-complex $X \subseteq M$ such that M deformation retracts to X.

Let's start with a motivating example: Bing's "house with two rooms" is obtained from a 3-ball by dividing it into two rooms via a pair of collapses, corresponding to entering the left room from the right side of the house and entering the right room from the left side. See Figure 2 for an example of an immersed 2-sphere. The collapses can also be thought of as follows: homeomorphically deform the 3-ball to create the two tunnels and the two rooms. The result is a "thickened" version of the house (still a 3-ball) which then deformation retracts onto it. The immersed 2-sphere corresponds to the boundary 2-sphere of the "thickened" complex. To extend this idea to other 3-manifold spines, we will have to formalize this idea of a thickening.

Construction 5.2. Let X be a compact connected collapsed 2-complex with no isolated vertex or edge that PL-embeds in a 3-manifold M. A *thickening* $T = T(X)$ is given by the following construction, which also gives a cell structure on its boundary ∂T:

∂T^0 For each vertex v in X, consider a regular neighbourhood $N(v) \subseteq M$ of v. Add a 0-cell in each component of $N(v) - X$. The union of these 0-cells is ∂T^0.

∂T^1 For each edge e in X, consider a regular neighbourhood $N(e) \subseteq M$ of e containing the 0-cells in ∂T^0 associated with the endpoints of e. Add a 1-cell in each component of $N(e) - X$ that contains a 0-cell associated to each endpoint of e. That 1-cell joins those 0-cells. If n sides of discs are incident with e in X, then this process yields n 1-cells parallel to e. The union of ∂T^0 with these 1-cells is ∂T^1.

∂T^2 For each disc d of X, consider a regular neighbourhood $N(d) \subseteq M$ of d containing the 1-cells in ∂T^1 associated with the ∂d. Add a 2-cell in each component of $N(d) - X$ containing

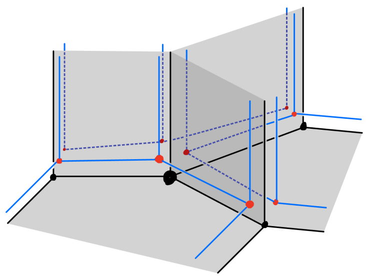

FIGURE 3. Part of $\partial T(X)$ for a complex X. The 0-cells of ∂T are red, and the 1-cells of ∂T are blue. The 2-cells of ∂T run parallel to the grey discs in X, forming a "bubble" hovering around X.

1-cells associated to all edges of ∂d. Attach this 2-cell to those 1-cells according to ∂d. This yields two 2-cells on opposite sides of d in M. The union of ∂T^1 with these 2-cells is ∂T.

The submanifold $T \subseteq M$ is the union of ∂T and the component of $M - \partial T$ containing X.

Lemma 5.3. *$T = T(X)$ deformation retracts to X. The retraction $r : T \to X$ induces an immersion $\partial T \to X$. If X is simply-connected, then ∂T is a union of 2-spheres.*

Proof. Consider the map $\partial T \to X$ sending i-cells in ∂T to their associated i-cells in X. T is homeomorphic to the mapping cylinder of $\partial T \to X$, yielding a deformation retraction.

Let c_1 and c_2 be closed i-cells in ∂T with $c_1 \cap c_2 \neq \varnothing$. Suppose $r(c_1) = r(c_2) = c$. Then $c_1 \cup c_2$ is a connected subset of the neighbourhood $N(c)$ used in the construction of ∂T. At most one i-cell was added for each component of $N(c)$, so $c_1 = c_2$. Thus $r|_{\partial T} : \partial T \to X$ is an immersion.

The map $\partial T \to X$ can also be seen geometrically to be an immersion: in terms of Figure 3, each cycle of 2-cells around a vertex in ∂T is mapped to a cycle of discs in X associated to a corner of $M - X$.

Suppose $\pi_1 X = 1$. Then T is a compact orientable 3-manifold with $\text{rank}(\mathsf{H}_1(T)) = 0$, so $\text{rank}(\mathsf{H}_1(\partial T)) = 0$ by "half lives, half dies" [7, Lem 3.5]. Thus ∂T is a union of 2-spheres. $\square$

Lemma 5.4. *Let X be a compact 2-complex with a collapsed subcomplex Y containing a disc. Then X has a connected collapsed subcomplex Y' with no isolated vertex or edge, whose inclusion map $Y' \hookrightarrow X$ is π_1-injective.*

Proof. If X contains a free i-face, delete that face and its attached $(i+1)$-cell. This deletion does not affect π_1. Repeat this process until the remaining subcomplex is collapsed. Delete all isolated edges leaving a disjoint union X' of collapsed components X'_i without isolated vertices or edges. Each $X_i' \hookrightarrow X$ is π_1-injective. As the disc from Y must be contained in some X_i', we let $Y' = X_i'$. □

Proposition 5.5. *Let X be a compact 2-spine of a simply-connected 3-manifold. Then X has an immersed combinatorial 2-sphere if and only if X has a collapsed subcomplex containing a disc.*

Proof. Suppose $S^2 \looparrowright X$ is a combinatorial immersion. Then $\mathrm{im}(S^2 \looparrowright X)$ is a collapsed subcomplex containing a disc.

Conversely, suppose X has a collapsed subcomplex Y containing a disc. By Lemma 5.4, we can assume Y has no isolated vertex or edge, and $\pi_1 Y \leq \pi_1 X = 1$. Then by Lemma 5.3, $\partial T(Y)$ is a union of 2-spheres that immerses in $Y \subseteq X$. □

In particular, this shows that collapsed compact contractible 3-manifold spines having a 2-cell fail to have nonpositive immersions, since $\chi(S^2) = 2$ and S^2 is not contractible. Section 6 will be dedicated to showing that this family of 2-complexes still has nonpositive towers.

5.2. **Miller-Schupp Presentation Complexes.** The rest of this section will examine another class of contractible 2-complexes failing to have nonpositive immersions. These were found by Fisher in [4] by using *foldings* to generate immersions from compact complexes with high Euler charateristic.

Definition 5.6. Let $f : Y \to X$ be a combinatorial map between finite 2-complexes. Then a *folding* of f is a combinatorial immersion $Y'' \looparrowright X$ obtained as follows [12]:

If any vertex of Y has two incident ends of 1-cells being mapped to the same end of a 1-cell in X, identify those 1-cells in Y. Repeat this until there are no such pairs of 1-cells in Y, and call the resulting complex Y'. We get a new map $Y' \to X$, where the 1-skeleton of Y' immerses in X. If any 2-cells of have the same attaching map in Y' and the same image in X, identify them, and call the resulting complex Y''. The map $Y'' \to X$ is then an immersion.

Note that the map $Y \to Y''$ given by a folding is π_1-surjective. Indeed, every cycle in the 1-skeleton of Y'' comes from a cycle in the 1-skeleton Y, and all 2-cells coning off nullhomotopic cycles in Y are preserved in Y''.

Using this tool, we get the following simple algorithm:

(1) For each 2-cell F in X, take the map from a disc D to $F \subseteq X$ and fold it into an immersion. Call the set of such immersions S_1.

(2) For each immersion $Y \looparrowright X$ in S_n, consider every combinatorial map $Y \vee D \to X$, where $Y \vee D$ is the wedge sum at a 0-cell of Y with a disc D, and fold it into an immersion. Discard any immersions where a pair of 2-cells get identified during the folding process, and call the set of all remaining immersions the *children* of $Y \looparrowright X$. The set of all children of immersions in S_n is then S_{n+1}.

Note that each child $Y' \looparrowright X$ of $Y \looparrowright X$ satisfies $\chi(Y') \geq \chi(Y)$. This happens because all 2-cells of Y survive the folding process, so the second homology cannot decrease, and folding is π_1-surjective, so the rank of the first homology cannot increase. So we get $\chi(Y') = 1 - b_1(Y') + b_2(Y') \geq 1 - b_1(Y) + b_2(Y) = \chi(Y)$. With a bit of luck, the above algorithm will eventually generate an immersion from a complex with Euler characteristic ≥ 2.

Presentations of the form $\langle a, b \mid w,\ ba^n b^{-1} a^{-(n+1)} \rangle$ with w a word with exponent sum 1 in b were shown by Miller and Schupp in [14] to give the trivial group. The exponent sums ensure that the corresponding presentation complexes have trivial second homology, and so the complexes are contractible by the theorems of Hurewicz and Whitehead. The above algorithm was used in [4] to show that the presentation complexes of $\langle a, b \mid w,\ bab^{-1}a^{-2} \rangle$ fail to have nonpositive immersions for $w \in \{ab^2ab^{-1},\ a^{-1}b^2a^{-1}b^{-1},\ \ a^2b^{-1}ab^2,\ ab^{-1}a^{-2}b^2,\ a^2b^2a^{-1}b^{-1}\}$.

Note that, while these examples fail to have nonpositive immersions, they (and any 2-relator simply-connected presentation complex) still have nonpositive towers. Let X be the presentation complex for the trivial presentation $\langle a, b \mid r_1, r_2 \rangle$. Since X has no nontrivial connected covering space, the last step of any tower map $Y \to X$ must be an inclusion. In particular, $Y \to X$ factors as $Y \to Z \hookrightarrow X$, where Z has at most one 2-cell and $Y \to Z$ is a tower map. Since all one-relator groups have nonpositive immersions, either $\chi(Y) \leq 0$ or Y is contractible.

6. Nonpositive Towers in 3-Manifold Spines

The goal of this section is to prove the nonpositive tower property for a compact aspherical 2-complex X embedded in a 3-manifold M. This is the other main result of my paper with Wise [2], and most of this section is taken directly from it.

The idea of the proof is to consider the thickening T of X in M, which deformation retracts to X. The asphericity of X ensures that ∂T has no 2-sphere, which in turn ensures $\chi(X) \leq 0$. Asphericity is preserved by towers, since it is preserved by both covering maps and subcomplexes. The latter is ensured by a simple argument using the Sphere Theorem.

We start by giving another construction of the thickening of X, this time building it up as a handlebody so that we can talk about "thickenings" of cells in X.

Construction 6.1. Let X be a locally finite 2-complex. A *thickening* of X is a 3-manifold $T = T(X)$ with boundary, and a continuous map $\Theta : T \to X$, constructed as follows:

- Let T^0 be a disjoint union of closed 3-balls, one for each vertex in X, and let Θ map each ball to its corresponding vertex.
- For each edge e in X, define $T(e) \cong [0,1] \times D^2$, and identify $\{0\} \times D^2$ and $\{1\} \times D^2$ with discs on the boundary of the components of T^0 corresponding to the endpoints of e. Let Θ map $(0,1) \times D^2$ onto $\mathrm{int}(e)$. The resulting complex is T^1.

  We require that each $T(e)$ embeds and that $T(e_1) \cap T(e_2) = \varnothing$ for $e_1 \neq e_2$.
- For each disc F in X, define $T(F) \cong D^2 \times [0,1]$, and identify the outer cylinder $S^1 \times [0,1]$ with an embedded cylinder on the boundary of T^1 which is mapped by Θ to the attaching loop of F in X. Let Θ map $\mathrm{int}(D^2) \times [0,1]$ onto $\mathrm{int}(F)$. The resulting complex is T.

  We require that each $T(F)$ embeds and that $T(F_1) \cap T(F_2) = \varnothing$ for $F_1 \neq F_2$.

Since $T(X)$ and Θ are defined cell by cell, we use $T(A)$ to denote the thickening $\Theta^{-1}(A) \subseteq T(X)$ for any union of open cells $A \subseteq X$.

Remark 6.2. If a thickening T of X exists, then by construction, T is a 3-manifold with boundary. Furthermore, there is a PL-embedding $X \hookrightarrow \mathrm{int}(T(X))$ such that T deformation retracts to X, with $T(A)$ retracting to A for any subcomplex $A \subseteq X$. By construction, this retraction is homotopic to Θ.

Remark 6.3. If X PL-embeds in a 3-manifold M, then we can take T to be a closed regular neighbourhood of X in M. Then T has a handlebody decomposition whose structure follows that of X. Taking Θ to be a map sending each i-handle to its corresponding i-cell, T and Θ give a thickening of X.

Lemma 6.4. *Let X be a locally finite aspherical connected 2-complex that PL-embeds in a 3-manifold. Then any subcomplex $Y \subseteq X$ is also aspherical.*

Proof. If $T(X)$ is non-orientable, we can consider an orientable double-cover $\widehat{T(X)}$. This induces a double-cover $\widehat{X}$ of X, which is locally finite, aspherical, connected, and PL-embeds in $T(\widehat{X}) = \widehat{T(X)}$. Consider the induced double-cover $\widehat{Y} \subseteq \widehat{X}$ of Y. It suffices to prove that $\widehat{Y}$ is aspherical, since this would imply that Y is aspherical. Since the non-orientable case with X and Y reduces to the orientable case with $\widehat{X}$ and $\widehat{Y}$, we can assume without loss of generality that $T(X)$ is orientable.

Suppose for contradiction that Y is not aspherical. Let $(T(X), \Theta)$ be an orientable thickening of X. By Hurewicz and Whitehead's theorems, $\pi_2(\mathrm{int}(T(Y))) \neq 0$. Since $\mathrm{int}(T(Y))$ is orientable,

$\text{int}(T(Y))$ has an embedded 2-sphere S representing a nontrivial element of $\pi_2(\text{int}(T(Y)))$ by the Sphere Theorem [7, Thm 3.8]. Since X is aspherical, S bounds a contractible submanifold $B \subseteq \text{int}(T(X))$ by [7, Prop 3.10]. Note that $T(X) - S$ has two connected components: $\text{int}(B)$ and the component C containing $\partial T(X)$.

Let $\{U_i\}$ be the set of connected components of $T(X)-T(Y)$. For each U_i, we have $U_i \cap S = \varnothing$, since $S \subseteq T(Y)$. Since U_i is connected, it must lie either entirely in $\text{int}(B)$ or entirely in C. By construction of $T(X)$, U_i is the thickening of some union of open cells of X, so $U_i \cap \partial T(X) \neq \varnothing$. So $U_i \subseteq C$.

Since this is true for all U_i, we have $\text{int}(B) \subseteq T(X) - \bigcup_i U_i = T(Y)$. Since $\text{int}(B)$ is an open submanifold of $T(Y)$, it is contained in $\text{int}(T(Y))$. So S bounds a contractible submanifold B of $\text{int}(T(Y))$, and so is trivial in $\pi_2(\text{int}(T(Y)))$ by [7, Prop 3.10]. This is a contradiction. □

Lemma 6.5. *Let M be a compact orientable 3-manifold with boundary. Then $\chi(M) = \frac{1}{2}\chi(\partial M)$. And if ∂M does not contain a 2-sphere then $\chi(M) \leq 0$.*

Proof. Let $\widetilde{M}$ be the manifold obtained by gluing two copies of M along ∂M. Since $\widetilde{M}$ is a closed orientable 3-manifold, $2\chi(M) - \chi(\partial M) = \chi(\widetilde{M}) = 0$. So $\chi(M) = \frac{1}{2}\chi(\partial M)$. Since M is orientable, each component of ∂M is an orientable surface. Since ∂M contains no 2-sphere, every component of ∂M has nonpositive χ. So $\chi(M) = \frac{1}{2}\chi(\partial M) \leq 0$. □

Lemma 6.6. *Let $X' \to X$ be a finite-sheeted cover of a compact connected 2-complex X. Then X has nonpositive towers if and only if X' has nonpositive towers.*

Proof. Suppose X has nonpositive towers. Let $Y \to X'$ be a tower map. Then $Y \to X' \twoheadrightarrow X$ is a tower map, so either $\chi(Y) \leq 0$ or Y is contractible.

Suppose that X' has nonpositive towers. Let $t : Y \to X$ be a tower map. Let n be the degree of the cover $p : X' \to X$. Then there is an induced tower map $Y' \to X'$, where Y' is an n-sheeted cover of Y, and maps to $p^{-1}(t(Y))$. Either $\chi(Y') \leq 0$ or Y' is contractible.

If $\chi(Y') \leq 0$, then $\chi(Y) = \frac{1}{n}\chi(Y') \leq 0$. If Y' is contractible, then Y' is the universal cover of Y, so Y is a $K(\pi_1(Y), 1)$ complex with $|\pi_1 Y| = n$. A nontrivial finite group does not have a compact $K(\pi, 1)$, so $\pi_1 Y = 1$. Thus $Y = Y'$ is contractible. □

Theorem 6.7. *Let X be an aspherical compact connected 2-complex that PL-embeds in a 3-manifold M. Then X has nonpositive towers.*

Proof. Since X is compact, it PL-embeds in a thickening $T(X)$ in M that is a compact sub-3-manifold with boundary. So without loss of generality, we can assume that $M = T(X)$ is a compact 3-manifold with boundary that deformation retracts to X.

If M is non-orientable, we consider an orientable double cover of M that deformation retracts to a double cover X' of X. By Lemma 6.6, it suffices to show that X' has nonpositive towers. So without loss of generality, we can assume that M is orientable.

X itself is either contractible or has $\chi(X) \leq 0$. Indeed, if ∂M includes a 2-sphere S, then asphericity of M ensures that S bounds a contractible submanifold of M [7, Prop 3.10]. Since M is connected, this submanifold must be M itself, and so M and X are contractible. If ∂M does not include any 2-spheres, then $\chi(X) = \chi(M) \leq 0$ by Lemma 6.5.

Note that any covering map $\widehat{X} \to X$ (with $\widehat{X}$ connected) extends to a covering map $\widehat{M} \to M$, where $\widehat{X}$ PL-embeds in $\operatorname{int}(\widehat{M})$. Then $\widehat{X}$ is locally finite, aspherical, connected, and PL-embeds in a 3-manifold.

Let X' be a compact connected subcomplex of $\widehat{X}$. By Lemma 6.4, X' is aspherical. And X' also PL-embeds in $\operatorname{int}(\widehat{M})$. Therefore, X' satisfies the same hypotheses as X, and is also either contractible or has $\chi(X') \leq 0$.

Any tower map $Y \to X$ is a composition of maps $X' \hookrightarrow \widehat{X} \twoheadrightarrow X$, so we are done. $\square$

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