Hyperfiniteness of Boundary Actions of Cubulated Hyperbolic Groups

Forte Shinko

Department of Mathematics and Statistics

McGill University, Montreal

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Abstract

A classical result due to Dougherty, Jackson and Kechris states that tail equivalence on Cantor space is a hyperfinite Borel equivalence relation, that is to say that it is the increasing union of finite Borel equivalence relations. This tail equivalence relation is Borel bireducible with the orbit equivalence relation induced by a free group on the boundary of its Cayley graph. We generalize this result to a wider class of hyperbolic groups. Namely, we prove that if a hyperbolic group acts geometrically on a CAT(0) cube complex, then the induced action on the Gromov boundary is hyperfinite.

Abrégé

Un résultat classique dû à Dougherty, Jackson et Kechris affirme que l'équivalence de queue sur l'espace de Cantor est une relation d'équivalence borélienne hyperfinie, c'est-à-dire qu'elle est la réunion croissante de relations d'équivalence boréliennes finies. Cette relation d'équivalence de queue est Borel biréductible avec la relation d'équivalence d'orbite induite par l'action d'un groupe libre sur le bord de son graphe de Cayley. On généralise ce résultat à une classe de groupes hyperboliques plus large. À savoir, on prouve que si un groupe hyperbolique agit géométriquement sur un complexe cubique CAT(0), alors l'action induite sur le bord de Gromov est hyperfinie.

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Preface and contribution of authors

The original material in this document is contained in the final chapter, and is joint work with Jingyin Huang and Marcin Sabok.

Contents

Introduction

Descriptive set theory is classically concerned with studying the structural properties of Borel sets in Polish spaces. More recently, the theory of Borel equivalence relations has emerged as an active area of research, with connections to classification problems in such areas as operator algebras, ergodic theory and topological dynamics.

The notion of Borel reducibility induces a partial order on the class of Borel equivalence relations. The simplest class under this partial order is the class of smooth equivalence relations, which are the Borel equivalence relations which are Borel reducible to the identity relation on the real numbers. A Borel equivalence relation is said to be hyperfinite if it is the increasing union of finite Borel equivalence relations. It was shown by Dougherty, Jackson and Kechris that the class of hyperfinite relations is the minimal class of Borel equivalence relations which properly contains the class of smooth equivalence relations. Thus a hyperfinite relation is relatively tame among all Borel equivalence relations.

In their seminal work on hyperfinite equivalence relations, Dougherty, Jackson and Kechris proved that the tail equivalence relation on 2^ω is hyperfinite. This tail equivalence relation is Borel bireducible with $E_{F_2}^{\partial F_2}$, the orbit equivalence relation induced by the action of the free group on two generators on its Gromov boundary ∂F_2. It is thus natural to wonder if the same hyperfiniteness result can be obtained for other hyperbolic groups. Although this is still an open question for hyperbolic groups in general, we have been able to prove with Jingyin Huang and Marcin Sabok that if the hyperbolic group additionally admits a geometric action on a CAT(0) cube complex, then the induced action on the Gromov boundary is hyperfinite.

In the first chapter, we will give an overview of the classical descriptive set theory, and then present the theory of Borel equivalence relations. In the second chapter, we introduce notions central to metric geometry, including hyperbolic groups and CAT(0) cube complexes. In the final chapter, we present a proof of the main result.

We fix some notation. The natural number n is the Von Neumann ordinal $n = \{0, \ldots, n-1\}$, and the ordinal ω denotes the set of natural numbers. If $f : X \to Y$ is a function, we define $\mathrm{graph}(f) = \{(x, y) \in X \times Y : f(x) = y\}$. The projection $\mathrm{proj}_X : X \times Y \to X$ is defined by $\mathrm{proj}_X(x, y) = x$, and proj_Y is defined analogously. If $P \subset X \times Y$, then for $x \in X$, we define $P_x = \{y \in Y : (x, y) \in P\}$, and P_y is defined analogously. A function $f : X \to Y$ is **countable-to-one** if $f^{-1}(y)$ is countable for every $y \in Y$.

Chapter 1

Descriptive set theory

1.1 Classical descriptive set theory

Descriptive set theory takes place in Polish spaces:

1.1 Definition. A **Polish space** is a second countable completely metrizable space.

1.2 Proposition.

1. *A closed subspace of a Polish space is Polish.*
2. *Let (X_n) be a sequence of Polish spaces. Then $\bigsqcup X_n$ and $\prod X_n$ are Polish spaces.*

Proof. (1) is clear. We prove (2). For each n, fix a complete metric d_n on X_n which is bounded by 1. To get a complete metric on $\bigsqcup X_n$, let $d(x,y) = d_n(x,y)$ if $x, y \in X_n$, and otherwise set $d(x,y) = 2$. To get a complete metric on $\prod X_n$, we can do the following:

$$d((x_n),(y_n)) = \sum_n \frac{d_n(x_n,y_n)}{2^n}$$

□

1.3 Example. Some examples of Polish spaces are as follows:

- $\mathbb{R}$, 2^ω
- Compact metrizable spaces.

- L^p spaces for $1 \leq p < \infty$.

Some non-examples of Polish spaces are as follows:

- Spaces which are not second countable, such as uncountable discrete spaces and L^∞ spaces.
- Spaces which are not completely metrizable, such as $\mathbb{Q}$.

A Polish space cannot be too large:

1.4 Proposition. *Let X be a Polish space. Then $|X| \leq 2^{\aleph_0}$.*

Proof. If $A \subset X$ is a countable dense subset, then the map $X \to \mathbb{R}^A$ defined by $x \mapsto d(x, \cdot)$ is an injection. □

We can actually say more about the possible cardinalities of a Polish space:

1.5 Proposition. *Every uncountable Polish space has cardinality $2^{\aleph_0}$.*

Proof. This follows from the Cantor-Bendixson theorem; see [Kec95, Corollary 6.5]. □

The topology of a Polish space is a bit too rigid for our purposes, so we introduce a more flexible notion.

1.6 Definition.

1. A **σ-algebra** on a set X is a nonempty subset $\mathcal{A} \subset \mathcal{P}(X)$ which is closed under complement and countable union.
2. A **measurable space** is a set equipped with a σ-algebra, whose members are called **measurable sets**.

1.7 Definition. Let X and Y be measurable spaces. A map $f : X \to Y$ is **measurable** if the preimage under f of every measurable set is measurable.

1.8 Definition. Let X be a Polish space. The **Borel σ-algebra** of X is the σ-algebra generated by the open sets in X.

1.9 Definition. A **standard Borel space** is a measurable space which is isomorphic to a Polish space with its Borel σ-algebra. In the context of standard Borel spaces, measurable sets and functions are called **Borel**.

We will usually not need the entire topological structure of a Polish space, and we will most often pass to the underlying standard Borel space.

1.10 Proposition.

1. *If (X_n) is a sequence of standard Borel spaces, then $\bigsqcup X_n$ and $\prod X_n$ are standard Borel spaces.*

2. *If X is a standard Borel space and $B \subset X$, then B is a standard Borel space.*

Proof. (1) follows from Proposition 1.2 (2). (2) follows from Proposition 1.2 (1) since we can assume that B is clopen by [Kec95, Theorem 13.1]. □

1.11 Theorem (Borel isomorphism theorem)**.** *If X and Y are standard Borel spaces with $|X| = |Y|$, then $X \cong Y$.*

Proof. This can be proved by using a measurable version of the Cantor-Schröder-Bernstein theorem; see [Kec95, Theorem 15.6]. □

The following is a typical application of the isomorphism theorem:

1.12 Proposition. *Every standard Borel space admits a Borel linear order.*

Proof. By Theorem 1.11 and Proposition 1.5, every standard Borel space is isomorphic to one of the following:

$$0, 1, 2, \ldots, \mathbb{N}, \mathbb{R}$$

all of which admit a Borel linear order. □

1.13 Definition.

1. A **standard Borel group** is a standard Borel space G with a group operation such that the maps $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are Borel.

2. Given a standard Borel group G, a **Borel G-action** on a standard Borel space X is a group action of G on X such that the map $(g, x) \mapsto gx$ is Borel. We say that X is a **Borel G-space**.

1.14 Proposition. *Let X be a standard Borel space and let $A \subset X$. Then the following are equivalent:*

1. *There is a standard Borel space Y and a Borel set $P \subset X \times Y$ such that $A = \mathrm{proj}_X(P)$.*
2. *There is a standard Borel space Y, a Borel set $B \subset Y$ and a Borel map $f : Y \to X$ such that $A = f(B)$.*
3. *There is a standard Borel space Y and a Borel map $f : Y \to X$ such that $A = f(Y)$.*

Proof. $(1 \implies 2)$ holds since $X \times Y$ is a standard Borel space, and $(2 \implies 3)$ holds since B is a standard Borel space by Proposition 1.10. For $(3 \implies 1)$, we can take $P = \mathrm{graph}(f)$. □

1.15 Definition. Let X be a standard Borel space and let $A \subset X$.

1. A is **analytic** if it satisfies any of the equivalent conditions in Proposition 1.14. $\boldsymbol{\Sigma}^1_1(X) \subset \mathcal{P}(X)$ denotes the collection of analytic sets in X.
2. A is **coanalytic** if its complement is analytic. $\boldsymbol{\Pi}^1_1(X) \subset \mathcal{P}(X)$ denotes the collection of coanalytic sets in X.

Analytic sets can be used to show that maps are Borel:

1.16 Theorem. *Let X and Y be standard Borel spaces and let $f : X \to Y$ be a function. Then the following are equivalent:*

1. *f is Borel.*
2. *$\mathrm{graph}(f)$ is Borel.*
3. *$\mathrm{graph}(f)$ is analytic.*

Proof. This follows from Suslin's theorem; see [Kec95, Theorem 14.12]. □

The following uniformization theorem is of great use when dealing with countable Borel equivalence relations:

1.17 Theorem (Luzin-Novikov uniformization theorem)**.** *Let X and Y be standard Borel spaces and let $P \subset X \times Y$ be Borel such that each P_y is countable. Then $\mathrm{proj}_Y(P)$ is Borel, and there is a Borel map $f : \mathrm{proj}_Y(P) \to X$ such that $f(y) \in P_y$ for all $y \in Y$.*

Proof. See [Kec95, Theorem 18.10]. $\square$

1.18 Corollary. *Let X and Y be standard Borel spaces and let $f : X \to Y$ be a countable-to-one Borel map. Then $f(X)$ is Borel and there is a Borel map $g : f(X) \to X$ with $fg = \mathrm{id}$.*

Proof. Apply Theorem 1.17 to $P = \mathrm{graph}(f)$. $\square$

Finally, we state a reflection theorem, which allows us expand an analytic set to a Borel set sharing the same properties.

1.19 Definition. Let X be a standard Borel space with $\Phi \subset \mathcal{P}(X) \times \mathcal{P}(X)$.

1. Φ is **$\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$** if for any standard Borel space Y and Z and any $A \in \mathbf{\Sigma}^1_1(X \times Y)$ and $B \in \mathbf{\Sigma}^1_1(X \times Z)$, we have $\{(y, z) \in Y \times Z : \Phi(A_y, B_z)\} \in \mathbf{\Pi}^1_1(X)$.

2. Φ is **hereditary** if $\Phi(A, B)$ and $A' \subset A$ and $B' \subset B$ imply $\Phi(A', B')$.

3. Φ is **continuous upward in the second variable** if $\Phi(A, B_n)$ and $B_n \subset B_{n+1}$ imply $\Phi(A, \bigcup_n B_n)$.

1.20 Theorem (Second reflection theorem)**.** *Let X be a standard Borel space with $\Phi \subset \mathcal{P}(X) \times \mathcal{P}(X)$. If Φ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, hereditary and continuous upward in the second variable, then for any $A \in \mathbf{\Sigma}^1_1(X)$ with $\Phi(A, A^c)$, there is a Borel set $B \supset A$ and $\Phi(B, B^c)$.*

Proof. This follows from the existence of coanalytic ranks; see the remark after [Kec95, Theorem 35.16]. $\square$

1.2 Borel equivalence relations

We will always view an equivalence relation E on a set X as a subset $E \subset X^2$. We introduce some important examples:

1.21 Definition. Let X be a set.

1. The **identity relation** on X, denoted $\mathrm{id}(X)$, is the equivalence relation defined as follows:
$$x \,\mathrm{id}(X) y \iff x = y$$

2. The **eventually equality relation** on X^ω, denoted $E_0(X)$, is the equivalence relation defined as follows:
$$xE_0(X)y \iff \exists k \forall n [x_{k+n} = y_{k+n}]$$
We write E_0 and E_1 for $E_0(2)$ for $E_0(2^\omega)$ respectively.

3. The **tail equivalence relation** on X^ω, denoted $E_t(X)$, is the equivalence relation defined as follows:
$$xE_t(X)y \iff \exists k, l \forall n [x_{k+n} = y_{l+n}]$$
We write E_t for $E_t(2)$.

4. Let G be a group acting on X. The **orbit equivalence relation** on X, denoted E_G^X, is the equivalence relation defined as follows:
$$xE_G^X y \iff \exists g \in G [gx = y]$$

We define some basic notions:

1.22 Definition. Let E be an equivalence relation on X and let $B \subset X$.

1. The **restriction** of E to B, denoted $E \restriction B$, is $E \cap B^2$. This is an equivalence relation on B.

2. B is **invariant** if it is a union of equivalence classes.

3. The **saturation** of B, denoted $[B]_E$, is the smallest invariant subset containing B. In particular, we will denote the equivalence class of x by $[x]_E$.

4. A **selector** is a function $s : X \to X$ such that $s(x)Ex$ for every $x \in X$, and $s(x) = s(y)$ whenever xEy.

5. B is a **transversal** if it intersects each equivalence class exactly once.

1.23 Definition. Let $E \subset F$ be equivalence relations on X. We say that E **has finite index** in F if every F-class contains finitely many E-classes.

1.24 Proposition. *Let X be a G-set and let $H \leq G$ be a finite index subgroup. Then E_H^X has finite index in E_G^X.*

Proof. For any $x \in X$, we have

$$Gx = \bigcup_{\text{cosets } Hg} Hgx$$

which is a finite union since $[G : H]$ is finite. $\square$

1.25 Definition. Let X be a standard Borel space. An equivalence relation E on X is **Borel** (resp. **analytic**) if E is Borel (resp. analytic) as a subset of X^2.

1.26 Example.

1. If X is a standard Borel space, then $\mathrm{id}(X)$, $E_0(X)$ and $E_t(X)$ are all Borel equivalence relations.

2. If X is a Borel G-space, then E_G^X is an analytic equivalence relation. Moreover, if G is countable, then E_G^X is Borel.

There is a dichotomy theorem characterizing when an analytic equivalence relation does not have countably many classes:

1.27 Theorem (Silver)**.** *If E is an analytic equivalence relation, then exactly one of the following holds:*

1. *E has countably many classes.*

2. $\mathrm{id}(2^\omega) \leq E$.

Proof. The modern proof is due to Harrington and uses effective descriptive set theory; see [Gao09, Theorem 5.3.5]. □

1.28 Definition. Let E and F be Borel equivalence relations on standard Borel spaces X and Y respectively.

1. A **Borel reduction** from E to F is a Borel map $f : X \to Y$ such that

$$xEx' \iff f(x)Ff(x')$$

2. E is **Borel reducible** to F, denoted $E \leq F$, if there is a Borel reduction from E to F.

3. E and F are **Borel bireducible**, denoted $E \sim F$, if $E \leq F$ and $F \leq E$.

4. E and F are **Borel isomorphic**, denoted $E \cong F$, if there is an isomorphism $f : X \to Y$ which is a reduction witnessing $E \leq F$.

1.29 Example.

1. Every Borel injection $X \hookrightarrow Y$ induces a reduction $X^\omega \hookrightarrow Y^\omega$ witnessing $E_0(X) \leq E_0(Y)$ and $E_t(X) \leq E_t(Y)$.

2. $E_0(2^{<\omega}) \leq E_0$ in the following way: let $\langle \cdot, \cdot \rangle : \omega^2 \to \omega$ be any computable bijection, and define $f : (2^{<\omega})^\omega \to 2^\omega$ by $f(x)_{\langle n,m \rangle} = (x_n)_m$ (where $2^{<\omega}$ is embedded in 2^ω by appending 0^∞). Then f witnesses $E_0(2^{<\omega}) \leq E_0$. Thus we have $E_0(X) \sim E_0(Y)$ for X, Y countable.

3. $E_t(2^{<\omega}) \leq E_t$ in the following way: Define $f : 2^{<\omega} \to 2^{<\omega}$ by $f(s_1 \cdots s_n) = 0s_1 0s_2 \cdots 0s_n 11$. Then the induced map $(2^{<\omega})^\omega \to 2^\omega$ defined by $s_1 s_2 \cdots \mapsto f(s_1)f(s_2)\cdots$ witnesses the reduction $E_t(2^{<\omega}) \leq E_t$. Thus we have $E_t(X) \sim E_0(Y)$ for X, Y countable.

4. The map $2^\omega \to 3^\omega$ defined by $s_1 s_2 \cdots \mapsto s_1 2 s_2 2^2 s_3 2^3 s_4 2^4 \cdots$ is a reduction witnessing $E_0 \leq E_t(3)$. Thus we have $E_0(X) \leq E_t(Y)$ for X, Y countable.

We now introduce the notion of a smooth equivalence relation:

1.30 Theorem. *Let E be a Borel equivalence relation. Then $E \leq \mathrm{id}(2^\omega)$ iff $E \leq \mathrm{id}(X)$ for some X.*

Proof. ($2 \implies 1$) is obvious, and ($1 \implies 2$) holds since every standard Borel space embeds into 2^ω. □

1.31 Definition. E is **smooth** if it satisfies either of the equivalent conditions in Theorem 1.30.

1.32 Proposition. *Let E be a Borel equivalence relation on X.*

1. *E has a Borel selector iff it has a Borel transversal.*
2. *If E has a Borel selector, then E is smooth.*

Proof.

1. If s is a Borel selector, then:
$$\{x \in X : s(x) = x\}$$
is a Borel transversal.

   Conversely, if A is a Borel transversal, then
$$s(x) = y \iff xEy \wedge y \in A$$
defines a selector $s : X \to X$ whose graph is Borel. Then s is Borel by Theorem 1.16.

2. Every Borel selector witnesses a reduction $E \leq \mathrm{id}(X)$.

□

We will show in the next section that E_0 is not smooth. In fact, the following dichotomy theorem shows that E_0 is the only obstruction to smoothness:

1.33 Theorem (Harrington-Kechris-Louveau)**.** *Let E be a Borel equivalence relation. Then exactly one of the following holds:*

1. *E is smooth.*
2. *$E_0 \leq E$*

Proof. The standard proof goes via effective descriptive set theory; see [HKL90, Theorem 1.1]. □

1.3 Countable Borel equivalence relations

1.34 Definition. An equivalence relation E is **finite** (resp. **countable**) if every E-class is finite (resp. countable).

1.35 Example.

1. $\mathrm{id}(X)$ is finite for any X.

2. $E_0(X)$ is countable for any countable X. In particular, E_0 is countable.

3. $E_t(X)$ is countable for any countable X. In particular, E_t is countable.

4. If G is a countable group and X is a Borel G-space, then E_G^X is countable.

There is a lowest countable Borel equivalence relation:

1.36 Proposition. *If E is a countable Borel equivalence relation on an uncountable space X, then $\mathrm{id}(2^\omega) \leq E$.*

Proof. This follows from Theorem 1.27. □

1.37 Proposition. *Let E be a countable Borel equivalence relation. Then the following are equivalent:*

1. *E is smooth.*

2. *E has a Borel selector.*

3. *E has a Borel transversal.*

Proof. By Proposition 1.32, it suffices to show $(1 \implies 2)$.
If $f : X \to 2^\omega$ is a reduction witnessing $E \leq \mathrm{id}(2^\omega)$, then f is countable-to-one so by Corollary 1.18, f has a Borel section $g : f(X) \to X$. Then gf is a selector. □

We can show that E_0 is not smooth.

1.38 Proposition. *E_0 is not smooth.*

Proof. By Proposition 1.37, it suffices to show that E_0 does not have a Borel transversal. Note that $E_0 \cong E_X^G$, where $G = \bigoplus_n \mathbb{Z}/2\mathbb{Z}$ and $X = \prod_n \mathbb{Z}/2\mathbb{Z}$ with the natural G-action. If E_X^G has a Borel transversal $A \subset X$, then since the G-action is free, we can write $X = \bigcup_{g \in G} gA$, so if μ is the Lebesgue measure on X, then $\mu(X) = \sum_{g \in G} \mu(gA) = \sum_{g \in G} \mu(A)$, which is not possible. $\square$

The following is an indispensible tool in the study of countable Borel equivalence relations:

1.39 Theorem (Feldman-Moore)**.** *Let E be a countable Borel equivalence relation on X. Then there is a countable group G and a Borel G-action on X such that $E = E_G^X$.*

Proof. See [FM77, Theorem 1]. $\square$

The following is a typical application:

1.40 Proposition. *Every finite Borel equivalence relation is smooth.*

Proof. By Theorem 1.39, $E = E_G^X$ for some Borel action of a countable group G. Fix a Borel linear order on X by Proposition 1.12. Then

$$\{x \in X : \forall g \in G[x \leq gx]\}$$

is a Borel transversal for E, so the result follows from Proposition 1.32. $\square$

1.41 Proposition. *Let E be a countable Borel equivalence relation on X and let $A \subset X$ be Borel.*

1. *$[A]_E$ is Borel.*

2. *If E is smooth on A, then E is smooth on $[A]_E$.*

Proof.

1. By Theorem 1.39, $E = E_G^X$ for some Borel action of a countable group G. Then

$$[A]_E = \bigcup_{g \in G} gA$$

so $[A]_E$ is Borel.

2. If B is a Borel transversal for $E \restriction A$, then B is a Borel transversal for $E \restriction [A]_E$, so the result follows from Proposition 1.37.

$\square$

1.4 Hyperfinite equivalence relations

1.42 Definition. Let E be a Borel equivalence relation. E is **hyperfinite** if $E = \bigcup_n E_n$ where $(E_n)_n$ is an increasing sequence $(E_n)_n$ of finite Borel equivalence relations.

Note that every hyperfinite relation is countable.

1.43 Example.

1. Every finite relation is hyperfinite.
2. $E_0(X)$ is hyperfinite for any finite X.

1.44 Proposition. *Let E and F be countable Borel equivalence relations on spaces X and Y respectively. Let $A \subset X$ be Borel.*

1. *If $X = Y$, $E \subset F$ and F is hyperfinite, then E is hyperfinite.*
2. *If $X = \bigsqcup_{P \in \mathcal{P}} P$ is a countable partition of X into Borel sets and $E \restriction P$ is hyperfinite for each $P \in \mathcal{P}$, then X is hyperfinite.*
3. *If E is hyperfinite, then $E \restriction A$ is hyperfinite.*
4. *If $E \restriction A$ is hyperfinite and $[A]_E = X$, then E is hyperfinite.*
5. *If $E \leq F$ and F is hyperfinite, then E is hyperfinite.*
6. *If $X = Y$, $E \subset F$, E has finite index in F and E is hyperfinite, then F is hyperfinite.*

Proof.

1. If $F = \bigcup F_n$, then $E = \bigcup (E \cap F_n)$.
2. If $E \restriction P = \bigcup_n E_n^P$, then $E = \bigcup_n \bigcup_P E_n^P$.
3. If $E = \bigcup E_n$ then $E \restriction A = \bigcup (E_n \cap A^2)$.
4. By Theorem 1.39, $E = E_G^X$ for some Borel action of a countable group $G = (g_n)_n$. Let $E \restriction A = \bigcup E_n$ for an increasing union of finite $(E_n)_n$. Define F_n as follows:

$$xF_n y \iff x = y \vee \exists i, j \leq n [g_i x E_n g_j y]$$

   Then $E = \bigcup F_n$.

5. Let f be a reduction witnessing $E \leq F$. By Corollary 1.18, $f(X)$ is Borel and f has a Borel section $g : f(X) \to X$. Now $F \restriction f(X)$ is hyperfinite by (3), but $F \restriction f(X) \cong E \restriction g(f(X))$ via g, so $E \restriction g(f(X))$ is hyperfinite. Thus since $[g(f(X))]_E = X$, we have that E is hyperfinite by (4).

6. By Theorem 1.39, $F = E_G^X$ for some Borel action of a countable group $G = (g_n)_n$. Note that for each $k < \omega$,

$$\{x \in X : [x]_F \text{ contains less than } k \text{ } E\text{-classes}\}$$

is Borel since x is in the above set iff

$$\forall h_1, \ldots, h_k \in G \exists i, j[h_i x E h_j x]$$

Thus by (2), we can assume that every F-class contains exactly k E-classes. We inductively construct Borel maps $f_1, \ldots, f_k : X \to X$ such that $[\{f_1(x), \ldots, f_k(x)\}]_E = [x]_F$. Let $f_i(x) = g_n x$, where n is minimal such that

$$\neg \exists j < i[g_n x E f_j(x)]$$

Now let $E = \bigcup E_n$ where $(E_n)_n$ is an increasing sequence of finite Borel equivalence relations. Define the equivalence relation F_n by:

$$x F_n y \iff \exists \sigma \in \mathrm{Sym}(n)[f_i(x) E_n f_{\sigma(i)}(y)]$$

Then $F = \bigcup F_n$, so F is hyperfinite.

□

1.45 Proposition. *Every smooth equivalence relation is hyperfinite.*

Proof. This follows from Proposition 1.44 (5) since $\mathrm{id}(2^\omega)$ is finite. □

1.46 Proposition. *Let X be a Borel G-space and let $H \leq G$ be a finite index subgroup. If E_H^X is hyperfinite, then E_G^X is hyperfinite.*

Proof. This follows from Proposition 1.24 and Proposition 1.44 (6). □

There are multiple characterizations of hyperfinite equivalence relations:

1.47 Theorem (Dougherty-Jackson-Kechris)**.** *Let E be a countable Borel equivalence relation on a standard Borel space X. Then the following are equivalent:*

1. *E is hyperfinite.*

2. *$E \leq E_0$*

3. *$E = E_{\mathbb{Z}}^X$ for some Borel $\mathbb{Z}$-action on X.*

Proof. See [DJK94, Theorem 5.1]. □

The ideas in the following proof form the basis for the proof of the result presented in the final chapter

1.48 Theorem (Dougherty-Jackson-Kechris)**.** *E_t is hyperfinite.*

Proof. Let $X = 2^\omega$ and let $E = E_t$. For each $x \in X$ and $n < \omega$, let s_n^x be the lexicographically least string of length n which occurs infinitely often in x. Note that we have:

$$s_1^x \prec s_2^x \prec s_3^x \prec \cdots$$

Now let k_n^x be the first index at which s_n^x appears in x. Then since $(s_n^x)_n$ is increasing, we have:

$$k_1^x \leq k_2^x \leq k_3^x \leq \cdots$$

Now let $Z = \{x \in X : k_n^x \not\to \infty\}$. Then the map $Z \to Z$ defined by $x \mapsto \lim_n s_n^x$ is a Borel selector on Z, so E is smooth on $[Z]_E$ by Proposition 1.32 and Proposition 1.41, and thus hyperfinite by Proposition 1.45.

Now let $Y = X \setminus [Z]_E$. Then the map $Y \to (2^{<\omega})^\omega$ defined by $x \mapsto (x \restriction [k_n^x, k_{n+1}^x))_n$ witnesses the reduction $E \restriction Y \leq E_0(2^{<\omega})$. Since $E_0(2^{<\omega})$ is hyperfinite, we get that $E \restriction Y$ is hyperfinite by Proposition 1.44 (5).

Thus by Proposition 1.44 (2), E is hyperfinite. □

1.5 Hypersmooth equivalence relations

We will have to deal with Borel equivalence relations which are not necessarily countable, and for this we need the equivalent notion of hyperfiniteness:

1.49 Definition. Let E be a Borel equivalence relation. E is **hypersmooth** if there is an increasing sequence $\{E_n\}_n$ of smooth Borel equivalence relations on X such that $E = \bigcup_n E_n$.

1.50 Example.

1. Every smooth relation is hypersmooth.

2. By Proposition 1.40, every hyperfinite relation is hypersmooth.

3. $E_0(X)$ is hypersmooth.

There is a dichotomy theorem in this setting as well:

1.51 Proposition. *A Borel equivalence relation E is hypersmooth iff $E \leq E_1$.*

Proof. This is elementary; see [Gao09, Proposition 8.1.4]. □

One can pass from hypersmoothness to hyperfiniteness in the following way:

1.52 Theorem. *If E is a countable hypersmooth equivalence relation, then E is hyperfinite.*

Proof. See [DJK94, Theorem 5.1]. □

Chapter 2

Metric geometry

In this chapter, we develop some fundamental notions in the study of metric nonpositive curvature, in particular the notions of hyperbolic group and CAT(0) cube complexes.

2.1 Coarse geometry

2.1 Definition. Let (X, d) be a metric space and let $\gamma : I \to X$ be an isometric embedding for some space I.

1. If I is an an interval of the form $[0, t]$ in $\mathbb{R}_{\geq 0}$ or $\mathbb{N}$, then γ is a **geodesic from** $\gamma(0)$ **to** $\gamma(t)$.
2. If I is $\mathbb{R}_{\geq 0}$ or $\mathbb{N}$, then γ is a **geodesic ray based at** $\gamma(0)$. In the case that $I = \mathbb{N}$, we may refer to this as a **geodesic sequence**.

2.2 Definition. A metric space X is **geodesic** if for all $x, y \in X$, there is a geodesic from x to y.

2.3 Definition. Let G be a group acting by isometries on a geodesic metric space X.

1. The action is **proper** if for every compact subset $K \subset X$, the set $\{g \in g : K \cap gK \neq \varnothing\}$ is finite.
2. The action is **cocompact** if there is a compact subset $K \subset X$ with $GK = X$.

3. The action is **geometric** if it is proper and cocompact.

2.4 Definition. A metric space is **proper** if every closed ball is compact.

2.2 Hyperbolic spaces and hyperbolic groups

We will use Gromov products to define hyperbolic spaces, but there are many other approaches; see [BH11, Chapter III.H] for alternate characterizations.

2.5 Definition. Let X be a metric space and let $w, x, y \in X$. The **Gromov product** of x and y based at w, denoted $(x \cdot y)_w$, is defined as follows:

$$(x \cdot y)_w = \frac{1}{2}(d(x,z) + d(y,w) - d(x,w))$$

2.6 Definition. Let X be a metric space.

1. X is δ**-hyperbolic** for $\delta \geq 0$ if for all $w, x, y, z \in X$:

$$(x \cdot z)_w \geq \min\{(x \cdot y)_w, (y \cdot z)_w\} - \delta$$

2. X is **hyperbolic** if it is δ-hyperbolic for some $\delta \geq 0$.

2.7 Definition. Let X be a metric space.

1. Given two sequences (x_n) and (y_n) in X and $w \in X$, we define:

$$((x_n) \cdot (y_n))_w = \liminf_{i,j \to \infty} (x_i \cdot y_j)_w$$

2. A sequence (x_n) **converges at infinity** if $((x_n) \cdot (x_n))_w = \infty$ for all $w \in X$.

2.8 Example. Every geodesic (x_n) converges at infinity.

2.9 Proposition. *Let X be a δ-hyperbolic space and let (x_n) and (y_n) be geodesics based at w. Then the following are equivalent:*

1. $\lim x_n = \lim y_n$.

2. $d(x_n, y_n) \leq 4\delta$ *for every* n.

3. *There is some constant D such that for infinitely many n, there exists m with $d(x_n, y_m) \leq D$.*

Proof. For ($1 \implies 2$), let $n < \omega$. Since $((x_n) \cdot (y_n)) \to \infty$, choose $m \geq n$ with $(x_m \cdot y_m) \geq n$. Then:

$$(x_n \cdot y_n) \geq \min\{(x_n \cdot x_m), (x_m \cdot y_m), (y_m \cdot y_n)\} - 2\delta \geq n - 2\delta$$

which is equivalent to $d(x_n, y_n) \leq 4\delta$.
($2 \implies 3$) is obvious.
For ($3 \implies 1$), let $r \geq 0$. Let $n \geq r + D + 2\delta$ such that $d(x_n, y_m) \leq D$ for some m. Note that we have $m \geq n - D$ by the triangle inequality on w, x_n, y_m. Thus for $i \geq n$ and $j \geq m$, we have:

$$(x_i \cdot y_j) \geq \min\{(x_i \cdot x_n), (x_n \cdot y_m), (y_m \cdot y_j)\} - 2\delta \geq \min\{n, n - D, n - D\} - 2\delta = n - D - 2\delta \geq r$$

□

We now introduce the Gromov boundary:

2.10 Proposition. *Let X be a δ-hyperbolic space with fixed $w \in X$. Then for sequences (x_n) and (y_n), define the relation U_r for $r \geq 0$ as follows:*

$$(x_n) U_r (y_n) \iff ((x_n) \cdot (y_n))_w \geq r$$

Then $\{U_r : r \geq 0\}$ is a fundamental system of entourages for the set of sequences converging at infinity, turning it into a uniform space.

Proof. Clearly each U_r is symmetric and contains the diagonal, and $\{U_r\}$ is closed under intersection. Now let $r \geq 0$. Then if $(x_n) U_{r+\delta} (y_n)$ and $(y_n) U_{r+\delta} (z_n)$, then for any i, j, k, we have:

$$(x_i \cdot z_j) \geq \min\{(x_i \cdot y_k), (y_k \cdot z_j)\} - \delta$$

Taking $\liminf_{i,j,k}$, we get:

$$((x_n) \cdot (z_n)) \geq \min\{((x_n) \cdot (y_n)), ((y_n) \cdot (z_n))\} - \delta \geq \min\{r + \delta, r + \delta\} - \delta = r$$

So $(x_n) U_r (z_n)$. □

2.11 Definition. Let X be a hyperbolic space with $w \in X$ fixed. The **Gromov boundary**, denoted ∂X, is the Hausdorff quotient of the uniform space of sequences converging at infinity. We write $\lim x_n$ for the equivalence class of (x_n) in ∂X, and we say that x_n **converges** to $\lim x_n$, denoted $x_n \to \lim x_n$.

2.12 Proposition. *Let X be a proper geodesic δ-hyperbolic space. Then ∂X is compact metrizable. In particular, ∂X is a Polish space.*

Proof. ∂X is metrizable because the uniform structure is separated and countably generated (by $\{U_n : n \in \mathbb{N}\}$). Sequential compactness follows from the Arzelà-Ascoli theorem, and by metrizability, this is equivalent to compactness. $\square$

2.13 Example.

1. If X is a tree whose vertex degrees are at least 3, then ∂X is a Cantor set.

2. $\partial \mathbb{H}^2 \cong S^1$.

2.14 Remark. Every action of a group G by isometries on a hyperbolic space X induces a G-action on ∂X by homeomorphisms.

2.15 Definition. A finitely generated group G is **hyperbolic** if it has a Cayley graph which is hyperbolic.

With regards to Borel complexity, every geometric action of a hyperbolic group on a space induces the same Borel equivalence relation on the boundary:

2.16 Theorem. *Let G be a group acting geometrically on proper geodesic metric spaces X and Y. Then there is a G-equivariant homeomorphism from ∂X to ∂Y. In particular, we have $E_G^{\partial X} \cong E_G^{\partial Y}$.*

Proof. See [Gro87]. $\square$

2.3 Cube complexes

We introduce the notion of a cube complex:

2.17 Definition. A **cube complex** is a polyhedral complex all of whose cells are Euclidean n-cubes. It is naturally endowed with a piecewise Euclidean metric, making it into a geodesic metric space.

To impose a combinatorial nonpositive curvature condition, we introduce the following two notions:

2.18 Definition. A simplicial complex is **flag** if every clique spans a simplex.

2.19 Definition. Let X be a polyhedral complex and let $x \in X^{(0)}$. The **link** of x is the simplicial complex whose vertex set consists of the outgoing edges at x (counted twice if there is a loop), and a collection of vertices spans a simplex iff there is one cell which contains them all.

2.20 Definition. A CAT(0) **cube complex** is a simply-connected cube complex each of whose vertex links is flag.

Note that simply-connectedness is not a large obstacle, as it is usually possible to pass to the universal cover.

2.21 Example. A tree is a CAT(0) cube complex.

2.22 Definition. A group G is **cubulated** if it acts geometrically on a CAT(0) cube complex.

2.23 Example. Some examples of cubulated groups:

- Free groups, since their Cayley graphs are trees.
- More generally, fundamental groups of hyperbolic surfaces and hyperbolic closed 3-manifolds (see [KM12] and [BW12])
- Uniform arithmetic hyperbolic lattices of "simple type" (see [HW12])
- Hyperbolic Coxeter groups (see [NR97] and [CM07])
- $C'(1/6)$ and $C'(1/4)$-$T(4)$ metric small cancellation groups (see [Wis04])
- Certain cubical small cancellation groups (see [Wis17])

- Gromov random groups of density $< \frac{1}{6}$ (see [OW11])
- Hyperbolic free-by-cyclic groups (see [HW16] and [HW15])

2.24 Definition. X is **uniformly locally finite** if the degrees of vertices in X are uniformly bounded.

2.25 Example. Every CAT(0) cube complex which admits a cocompact group action is uniformly locally finite.

Now we introduce the basic notions in the study of CAT(0) cube complexes. For the rest of this section, let X denote a CAT(0) cube complex. First of all, we will describe the notion of a hyperplane.

2.26 Definition. A subset $C \subset X$ is **convex** if every geodesic with endpoints in C is contained in C.

2.27 Definition. A **mid-cube** of $[0,1]^n$ is a subset of form $\pi_i^{-1}(\{\frac{1}{2}\})$, where π_i is a coordinate function.

2.28 Definition. A subset $h \subset X$ is a **hyperplane** if it is connected and for every cell $C \subset X$, $h \cap C$ is either empty or a mid-cube. A hyperplane is **dual** to an edge if they intersect.

2.29 Proposition. *If e is an edge in X, then there is a unique hyperplane h_e dual to e. Moreover, h_e is a convex subset of X and is a* CAT(0) *cube complex with the induced cell structure from X. Finally, $X \setminus h_e$ has exactly two connected components, called **halfspaces**.*

Proof. See [Sag95, Sag12]. □

We can describe geodesics combinatorially by passing to the 1-skeleton:

2.30 Definition. Let $u, v \in X^{(0)}$.

1. The ℓ^1**-distance** between u and v is the length of the shortest path from u to v in the 1-skeleton $X^{(1)}$.

2. A **combinatorial geodesic** between vertices u and v is an edge path in $X^{(1)}$ from u to v which realizes the ℓ^1-distance between them.

2.31 Definition. Two points in X are **separated** by a hyperplane h if they are in different connected components of $X \setminus h$.

2.32 Proposition. *The ℓ^1-distance between two vertices is equal to number of hyperplanes separating them. Thus an edge path γ is a combinatorial geodesic iff $h_{e_1} \neq h_{e_2}$ for every pair of distinct edges e_1, e_2 on γ.*

Proof. See [HW08, Lemma 13.1]. □

The piecewise Euclidean metric and the ℓ^1-metric give the same notion of convexity on X:

2.33 Proposition. *Let $Y \subseteq X$ be a convex subcomplex. Then Y is also convex with respect to the ℓ^1-metric, ie. every combinatorial geodesic with endpoints in Y is contained in Y.*

Proof. See [HW08, Proposition 13.7]. □

2.34 Proposition. *Let $Y \subseteq X$ be a convex subcomplex and let γ be a combinatorial geodesic from a vertex v to a vertex in Y realizing the ℓ^1-distance between v and Y. Then a hyperplane separates v from Y iff it is dual to an edge in γ. Thus $d(v, Y^{(0)})$ is the number of hyperplanes separating v from Y.*

Proof. This is a special case of [HW08, Proposition 13.10]. □

There is a nearest point projection:

2.35 Proposition. *Let $Y \subseteq X$ be a convex subcomplex and let d denote the ℓ^1-metric on $X^{(0)}$. For any vertex $v \in X$, there is a unique vertex $u \in Y$ such that $d(u, v) = d(v, Y^{(0)})$. Thus there is a nearest point projection $\pi_Y : X^{(0)} \to Y^{(0)}$.*

Proof. See [HW08, Lemma 13.8]. □

Chapter 3

Hyperfiniteness of boundary actions

Let $F_2 = \langle a, b \rangle$ be the free group on two generators. We can identify ∂F_2 with the set of infinite reduced words in $\{a, a^{-1}, b, b^{-1}\}$. In this setting, two words are in the same orbit of the F_2-action iff they are tail equivalent. It is not hard to show the following:

3.1 Proposition. *$E_{F_2}^{\partial F_2} \sim E_t$. In particular, $E_{F_2}^{\partial F_2}$ is hyperfinite.*

The following conjecture suggests one direction in which this result could possibly be generalized:

3.2 Conjecture. *If G is a hyperbolic group, then $E_G^{\partial G}$ is hyperfinite.*

In general this question is still open, but we have been able to prove the result for the class of cubulated groups:

3.3 Theorem. *Let G be a hyperbolic group acting geometrically on a $\mathrm{CAT}(0)$ cube complex X. Then the induced action on ∂X is hyperfinite.*

The proof of the theorem uses the idea of Dougherty, Jackson and Kechris for proving hyperfiniteness of tail equivalence. To imitate their proof, we rely on a crucial lemma which is interesting in its own regard:

3.4 Lemma. *Let X be a uniformly locally finite hyperbolic $\mathrm{CAT}(0)$ cube complex. Then for any vertices x and y in X and any $a \in \partial X$, the symmetric difference $\mathrm{Geo}(x, a) \triangle \mathrm{Geo}(y, a)$ is finite, where $\mathrm{Geo}(x, a)$ is defined as follows:*

$$\mathrm{Geo}(x, a) = \{y \in X^{(0)} : y \text{ lies on a combinatorial geodesic from } x \text{ to } a\}$$

All work done in this chapter is joint with Jingyin Huang and Marcin Sabok (see [HSS17]).

3.1 Proof of the main lemma

We start by establishing properties of adjacent vertices in CAT(0) cube complexes.

3.5 Lemma. *Let X be a* CAT(0) *cube complex and let $Y \subseteq X$ be a convex subcomplex. Let u and v be adjacent vertices in X separated by a hyperplane h and let $u' = \pi_Y(u)$ and $v' = \pi_Y(v)$.*

1. *If $h \cap Y = \varnothing$, then $u' = v'$.*
2. *If $h \cap Y \neq \varnothing$, then u' and v' are adjacent vertices in Y separated by h.*

Thus the the nearest point projection $\pi_Y : X^{(0)} \to Y^{(0)}$ extends naturally to $\pi_Y : X^{(1)} \to Y^{(1)}$.

Proof. We can assume $d(v, Y^{(0)}) \leq d(u, Y^{(0)})$. Let γ_v and γ_u be combinatorial geodesics realizing the ℓ^1-distances from v to $Y^{(0)}$ and u to $Y^{(0)}$ respectively.

1. Suppose $h \cap Y = \varnothing$. Then $h \cap \gamma_v = \varnothing$, otherwise we would have $d(v, Y^{(0)}) > d(u, Y^{(0)})$. Thus h separates u from Y. Moreover, each hyperplane dual to an edge in γ_v separates u from Y. By Proposition 2.32, we have $d(v, Y^{(0)}) + 1 \leq d(u, Y^{(0)})$. On the other hand, the concatenation of the edge $\overline{uv}$ with γ_v has length $\leq d(v, Y^{(0)}) + 1$. Thus this concatenation realizes the ℓ^1-distance from u to $Y^{(0)}$. It follows that $u' = v'$.

2. Now suppose $h \cap Y \neq \varnothing$. First, by Proposition 2.34 we get $\gamma_v \cap h = \gamma_u \cap h = \varnothing$ because otherwise h would be dual to some edge in γ_v or γ_u and thus separate u or v from Y and hence be disjoint from Y. Let γ be a geodesic joining v' and u'. Note that γ is contained in Y. The path obtained by concatenating γ_v, γ and γ_u must intersect h because v and u lie on different sides of h. Thus h must intersect γ and thus separate v' and u'. To see that v' and u' are adjacent, it is enough to show that h is the only hyperplane separating u' and v'. Note, however, that if h' is a hyperplane separating u' from v', then h' must intersect the path obtained by contatenating γ_v, the edge from v to u and γ_u. By Proposition 2.34 we get $h' \cap \gamma_v = h' \cap \gamma_u = \varnothing$ as above. Thus, h' intersects the edge from u to v and hence $h' = h$.

□

3.6 Corollary. *Let X be a* CAT(0) *cube complex and let $Y \subseteq X$ be a convex subcomplex. Then the image under π_Y of a combinatorial geodesic γ is the image of a combinatorial geodesic. We will denote this geodesic by $\pi_Y(\gamma)$.*

We will also need the following:

3.7 Lemma. *Let X be a* CAT(0) *cube complex. Let x and y be adjacent vertices in X separated by a hyperplane h and let γ be a combinatorial geodesic ray based at y.*

1. *If γ does not cross h, then $\overline{xy} \cup \gamma$ is a combinatorial geodesic ray, where $\overline{xy}$ is the edge from x to y.*

2. *If γ crosses h and z is a vertex on γ after the crossing, then $\overline{xz} \cup \gamma_z$ is a combinatorial geodesic ray, where $\overline{xz}$ is any combinatorial geodesic from x to z and γ_z is the subgeodesic ray of γ based at z.*

Proof. Throughout this proof, we will be using the characterization of combinatorial geodesics given in Proposition 2.32.

1. Since γ is a geodesic which does not cross h, the hyperplanes dual to distinct edges on $\overline{xy} \cup \gamma$ are distinct, and thus it is a geodesic.

2. Let h' be a hyperplane dual to an edge on γ_z. Now γ is a geodesic, so h' does not cross γ between y and z, and thus y and z are on the same side of h'. Now since γ_z does not cross h, $h \neq h'$ and thus h' does not separate x and y, so x and y are on the same side of h'. Thus x and z are on the same side of h' and thus $\overline{xz}$ does not cross h'. Thus $\overline{xz} \cup \gamma_z$ is a geodesic.

□

We now turn to the proof of Lemma 3.4.

Proof of Lemma 3.4. Suppose that X is δ-hyperbolic. We can assume that x and y are adjacent. Let h be the hyperplane separating x and y. Suppose for a contradiction that

$\mathrm{Geo}(y,a) \triangle \mathrm{Geo}(x,a)$ is infinite. We can assume that $\mathrm{Geo}(y,a) \setminus \mathrm{Geo}(x,a)$ is infinite. Since X is locally finite, there is a sequence $(z_n)_n$ in $\mathrm{Geo}(y,a) \setminus \mathrm{Geo}(x,a)$ with $d(z_n, y) \to \infty$. For each $n < \omega$, let γ_n be a geodesic from y to a containing z_n.

Let h_y be the combinatorial hyperplane containing y. Note that each $z_n \in h_y$ by Lemma 3.7 (2). Now for each n, we have $d(\gamma_0, \gamma_n) \leq 4\delta$ by Proposition 2.9. Thus since $d(y, z_n) \to \infty$, the distance from γ_0 to h_y is bounded by 4δ infinitely often. Thus if $\pi : X^{(1)} \to h_y^{(1)}$ is the nearest point projection, then the distance from γ_0 to $\pi(\gamma_0)$ is bounded by 4δ infinitely often, so by Proposition 2.9, $\pi(\gamma_0)$ converges to a. But $\pi(\gamma_0)$ contains z_0 and it does not cross h_y, so by Lemma 3.7 (1), we have $z_0 \in \mathrm{Geo}(x,a)$, which is a contradiction. □

3.2 Proof of the main theorem

The following lemma allows us to restrict to the case of free actions:

3.8 Lemma. *If a hyperbolic group acts geometrically on a* $\mathrm{CAT}(0)$ *cube complex X, then it has a finite index subgroup acting freely and cocompactly on X.*

Proof. By Agol's theorem [Ago13, Theorem 1.1] (see also [Wis17]) there is a finite index subgroup H acting faithfully and specially on X (see [HW08, Definition 3.4] for the definition of special action). Now H embeds into a right-angled Artin group which is torsion-free, so H is torsion-free. Since every stabilizer is finite by properness of the action, it must be trivial since H is torsion-free, and thus H acts freely on X. □

We will also need the following application of the second reflection theorem:

3.9 Lemma. *Let X be a standard Borel space. Let $A \subset X$ be analytic and let E be an analytic equivalence relation on X such that there is some $n < \omega$ such that every $E \restriction A$-class has size less than n. Then there is a Borel equivalence relation F on X containing $E \restriction A$ such that every F-class has size less than n.*

Proof. We can assume wlog that $A = X$ by replacing E with $E \restriction A \cup \mathrm{id}(X)$. Now consider

$\Phi \subseteq \mathcal{P}(X^2) \times \mathcal{P}(X^2)$ defined as follows:

$$\begin{aligned}(B, C) \in \Phi \iff & \forall x \, \neg x C x \\ & \wedge \forall (x, y) \, \neg x B y \vee \neg y B x \\ & \wedge \forall (x, y, z) \, \neg x B y \vee \neg y B z \vee \neg x C z \\ & \wedge \forall_{i=1}^n x_i \, (\bigvee_{i \neq j} x_i = x_j) \vee (\bigvee_{i \neq j} \neg x_i B x_j)\end{aligned}$$

Note that $\Phi(B, X \setminus B)$ holds iff B is an equivalence relation on X whose classes have size less than n. Now Φ satisfies the conditions of Theorem 1.20 (2), so since E is analytic, there is a Borel equivalence relation $F \supset E$ whose classes have size less than n. □

We now turn to the proof of Theorem 3.3.

Proof of Theorem 3.3. By Proposition 1.46 and Lemma 3.8, we can assume that G acts freely and cocompactly on X. Let $E = E_G^{\partial X}$.

Let $V = X^{(0)}$ be the set of vertices of X. Fix $v_0 \in V$ and fix a total order on V with

$$v \leq w \implies d(v_0, v) \leq d(v_0, w)$$

Fix a transversal $\tilde{V}$ of the action of G on V (note the transversal is finite by cocompactness). For $v \in V$, we denote by $\tilde{v}$ the unique element of $\tilde{V}$ in the orbit of v. By a directed edge of X we mean a pair $(v, v') \in V^2$ such that there is an edge from v to v'.

We colour the directed edges of X as follows: assign a distinct colour to every directed edge (v, v') with $v \in \tilde{V}$, and extend uniquely (by freeness) to a G-invariant colouring on all directed edges.

Let C be the set of colours (which is finite since X is locally finite), and let $c(v, v')$ be the colour of (v, v'). Fix a total order on C. This induces a lexicographical order on $C^{<\omega}$.

For any geodesic $\eta \in V^{<\omega}$ and $m, n < \omega$, define:

$$c(\eta, m, n) = (c(\eta_m, \eta_{m+1}), c(\eta_{m+1}, \eta_{m+2}), \ldots, c(\eta_{m+n-1}, \eta_{m+n})) \in C^{<\omega}$$

For every $a \in \partial X$, define $S^a \subseteq V \times C^{<\omega}$ as follows:

$$\begin{aligned}S^a = \{(\eta_m, c(\eta, m, n)) \in V \times C^{<\omega} : \\ \eta \text{ is a geodesic from } v_0 \text{ to } a \text{ and } m, n < \omega\}\end{aligned}$$

Let $s_n^a \in C^{<\omega}$ be the least string of length n for which there are infinitely many $v \in V$ with $(v, s_n^a) \in S^a$. Note that we have:

$$s_1^a \prec s_2^a \prec s_3^a \prec \cdots$$

Let

$$T_n^a = \{v \in V : (v, s_n^a) \in S^a\}$$

and let $v_n^a = \min T_n^a$. Note that every vertex in T_n^a has an edge coloured by s_1^a leaving it, so every vertex of T_n^a is in the same orbit. Let $k_n^a = d(v_0, v_n^a)$ and note that since (s_n^a) is increasing, we have:

$$k_1^a \leq k_2^a \leq k_3^a \leq \cdots$$

Now let $Z = \{a \in \partial X : k_n^a \not\to \infty\}$. Then for each $a \in Z$, since $k_n^a \not\to \infty$ and V is discrete, there is a finite set containing all v_n^a, so there is some $v \in V$ which is in T_n^a for infinitely many n. Thus the geodesic class determined by the geodesic starting at $\tilde{v}$ (which is determined by k_1^a) and following the colours of $\lim_n s_n^a \in C^\omega$ is a Borel selector on Z. Thus E is smooth on $[Z]_E$ by Proposition 1.32 and Proposition 1.41, and thus hyperfinite by Proposition 1.45.
Now let

$$Y = \partial X \setminus [Z]_E = \{a \in \partial X : \forall b E a \ k_n^b \to \infty\}$$

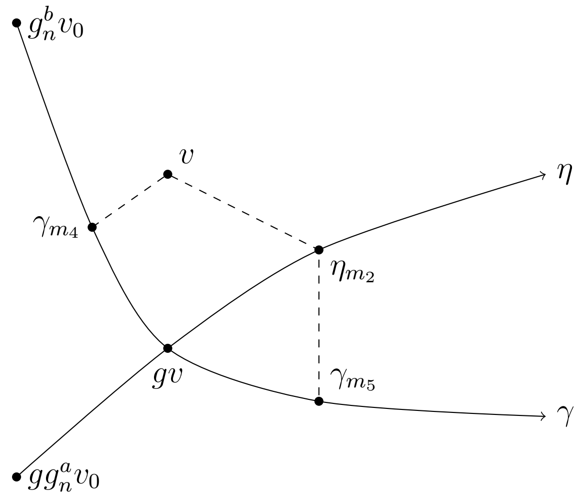
We will show that E is hyperfinite on Y. For each $n < \omega$, define $H_n : Y \to 2^V$ by

$$H_n(a) = g_n^a T_n^a$$

where $g_n^a \in G$ is the unique element with $g_n^a v_n^a \in \tilde{V}$. Let E_n be the equivalence relation on $\operatorname{im} H_n$ which is the restriction of the shift action of G on 2^V. We claim the following:

3.10 Claim. *There exists $K < \omega$ such that the equivalence classes of $E_n \restriction \operatorname{im} H_n$ have size at most K.*

Proof. Let $a, b \in Y$ and suppose $g \in G$ is such that $gH_n(a) = H_n(b)$, i.e. $gg_n^a T_n^a = g_n^b T_n^b$. Since the vertices in both sets are in the same orbit, $g_n^a v_n^a$ and $g_n^b v_n^b$ are elements of $\tilde{V}$ which are in the same orbit, so they are equal, say to some $v \in \tilde{V}$. It suffices to show that $d(v, gv) \leq 12\delta$, since then we can choose any $K < \omega$ larger than $\max_{v \in \tilde{V}} |\{g : d(v, gv) \leq 12\delta\}|$.



Note that since T_n^a and T_n^b are infinite, we have that $gg_n^a a = g_n^b b$, which we will call $c \in \partial X$. Let η be a geodesic from $gg_n^a v_0$ to c with $\eta_{m_1} = gv$. Now $v \in gg_n^a T_n^a$, so there is some m_2 with $d(v, \eta_{m_2}) \leq 4\delta$. Note that by choice of v_n^a, we have $m_2 \geq m_1$. Now let γ be a geodesic from $g_n^b \hat{x}$ to c with $\gamma_{m_3} = gv$. By the choice of v_n^b, there is some $m_4 \leq m_3$ such that $d(v, \gamma_{m_4}) \leq 4\delta$. Also η and γ are 2δ-close after they go through gv, so since $m_2 \geq m_1$, there is some $m_5 \geq m_3$ such that $d(\eta_{m_2}, \gamma_{m_5}) \leq 4\delta$. Thus

$$\begin{aligned}
2d(v, gv) &\leq d(v, \gamma_{m_4}) + d(\gamma_{m_4}, gv) + d(v, \eta_{m_2}) + d(\eta_{m_2}, \gamma_{m_5}) + d(\gamma_{m_5}, gv) \\
&= d(\gamma_{m_4}, \gamma_{m_5}) + d(v, \gamma_{m_4}) + d(v, \eta_{m_2}) + d(\eta_{m_2}, \gamma_{m_5}) \\
&\leq 2(d(v, \gamma_{m_4}) + d(v, \eta_{m_2}) + d(\eta_{m_2}, \gamma_{m_5})) \\
&\leq 2(12\delta),
\end{aligned}$$

where the first equality follows from the fact that γ is a geodesic. $\square$

Now $\operatorname{im} H_n$ is analytic, so by Lemma 3.9 there is a Borel equivalence relation E_n' on 2^V containing E_n whose classes are of size at most K. Let $f_n : 2^V \to 2^\omega$ be a reduction for $E_n' \leq \mathrm{id}_{2^\omega}$, and define $f : Y \to (2^\omega)^\omega$ by $f(a) = (f_n(H_n(a)))_{n<\omega}$. Write E' for the pullback of E_1 via f. Note that since each E_n' is finite, the relation E' is countable. As E' is clearly hypersmooth, we get that E' is hyperfinite by Theorem 1.52. Now, f is a homomorphism from E to E_1. Indeed, if $a, b \in Y$ with aEb, then by Lemma 3.4 there is $N < \omega$ such that

$H_n(a)E_nH_n(b)$ for $n \geq N$, and thus $f(x)E_1f(y)$. Thus $E \subseteq E'$, so since E' is hyperfinite, E is hyperfinite by Proposition 1.44 (1). $\square$

Conclusion

The tail equivalence relation, which was shown to be hyperfinite by Dougherty, Jackson and Kechris, is bireducbile with the action of the free group on the boundary of its Cayley graph. We have been able to successfully extend this hyperfiniteness result to a much wider family of hyperbolic groups, namely the cubulated hyperbolic groups. It is still an open problem if this property holds for every hyperbolic group, and it would be sufficient to prove that a version of Lemma 3.4 holds for Cayley graphs of hyperbolic groups. It seems that the CAT(0) cube complex methods used here cannot be generalized to the setting of Cayley graphs, so proving the result or the main lemma in general will most likely require an entirely different approach.

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