ABBREVIATED TITLE

AN ABERRATION IN A MICROWAVE OPTICAL SYSTEM

AN ABERRATION DUE TO THE LIMITED APERTURE OF A MICROWAVE OPTICAL SYSTEM

by

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TABLE OF CONTENTS

TABLE OF CONTENTS (continued)

ABSTRACT

The limitation in the diameter of the exit pupil of a microwave optical system produces an aberration in the field of the system which is hereafter called mutilation. The present work extends that initiated by Woonton to include two dimensional sources and circular mutilating apertures. Observations were made of the distant field patterns of a series of horns and paraboloids when mutilated by apertures ranging in diameter from 15 wave-lengths to 40 wave-lengths. A theory has been developed to include radiators of dimensions greater than 5 wave-lengths having a planar surface or aperture distribution. Calculations were made for a number of patterns and agreement with observation obtained. Spherical aberrations, coma, and other optical aberrations are discussed. Methods of approximate calculation, and conclusions relevant to the design of microwave systems are presented.

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GLOSSARY OF SYMBOLS

a	(1) radius of circular source aperture
	(2) side of square source aperture
a_{ij}	coefficient in expansion of paraboloid function in Chapter IV
A	constant in derivation of integrals, Chapter I and Appendix II
A_E, A_H	$1/A = 1/q + 1/s$ in Chapter III
b	distance from source to mutilating aperture
b_i	coefficients in expansion of distant paraboloid field in Chapter IV
B	$(x_1'^2 + y_1'^2 \cos^2\alpha)^{1/2}$
c	radius of mutilating aperture
C	(1) $ik/2\pi R_2$
	(2) part of Fresnel integral in $(C + iS)$
D	quantity defined by Eq. (A3.2)
D_{ij}	integrals in Appendix II and Chapter V
$e(0)$	mutilation
$\vec{E}$	electric field
f	function: $f(\ )$
f_{2n}	Fourier component of unmutilated Fresnel field
f_{2mn}	Fourier component of mutilated Fresenl field
f_{en}	Fourier component of the emergent field
F	function: $F(\ )$
$F(0)$	unmutilated field at $z = s + b$
$F_a(0)$	the field including optical aberrations
F_{mn}	Fourier component of mutilated field at $z = s + b$

g	azimuthal component of Poynting vector divided by z-component of Poynting vector
G_{mn}	Fourier component specifying the distant field
h	r-component of Poynting vector divided by z-component of Poynting vector
h_i	coefficients in paraboloid analysis in Chapter IV
$\vec{H}$	mangetic field
$\vec{H}^i$	incident magnetic field
H	quantity defined in Chapter V
i	$\sqrt{-1}$ operator
i,j,k	unit vectors in X, Y, Z directions
i',j',k'	unit vectors in X', Y', Z' directions
$I_1, I_2, ..$	integrals in Appendic III
J_n	Bessel function
$\mathcal{J}_n$	$(1/2\pi)\exp(-iMZ^2/2) \oint \cos n\tau \ \exp(iQ) \, d\tau$
k	propogation constant
K	$(\sec\alpha/(s+b))\exp\left[2\pi i(s+b) + \pi i c^2(1/s + 1/b)\right]$
l	direction cosine
l'	direction cosine in inclined system
L	quantity defined by Eq. (A3.3)
m	direction cosine
m'	direction cosine in inclined system
M	quantity defined by Eq. (A3.3)
$P(0)$	power
P_i	summation term defined in Chapter IV
q	(1) length of horn
	(2) phase shift constant in Chapter V

Q	exponential coefficient in Appendix III
r	radial co-ordinate in a cylindrical system
r_1	radial co-ordinate in plane $z = 0$
r_1'	radial co-ordinate in plane $z' = 0$
r_2	radial co-ordinate in plane $z = b$
r_3	radial co-ordinate in plane $z = b + s$
R	(1) spherical co-ordinate
	(2) quantity defined in Appendix III
R_2	the distance $(0,0,b)$ to (x,y,z)
s	(1) distance from mutilating screen to plane of observation
	(2) aberration coefficient
S	part of Fresnel integral $(C + iS)$
S_i	aperture function series in Chapter IV
$\vec{S}$	Poynting vector
t	(1) $2\pi c^2 \left[1/s + \dfrac{1}{b + y_1' \sin \alpha} \right]$
	(2) aberration constant in Chapter V
t_o	$2\pi c^2 \left[1/s + 1/b \right]$
T	$(L^2 + N^2)^{1/2}$
u	(1) the scalar field
	(2) $2\pi a \sin \alpha$
	(3) aberration constant in Chapter V
u_e	the emergent field in Chapter V
u_m	the mutilated field
u_2^i	the incident field in the plane $z = b$
u_{2m}	the mutilated Fresnel field in the plane $z = b$
u_o	aberration constant

$U_o, U_1, ..$	Lommel functions
v	aberration constant in Chapter V
V_o, V_1 ..	Lommel functions
W	the aberration function Chapter V
W_n	the Lommel function $V_n + iV_{n-1}$
w	(1) $\dfrac{2\pi c\sqrt{x_1'^2 + y_1'^2\cos^2\alpha}}{b + y_1'\sin\alpha}$
	(2) aberration constant in Chapter V
x,y,z	co-ordinates of point in X,Y,Z system
x',y',z'	co-ordinates of point in X',Y',Z' system
x_1	co-ordinates of point in X-direction measured in aperture plane of source
x_1'	co-ordinates of point in X'-direction measured in aperture plane of source
x_2,y_2	co-ordinates of point in plane $z = b$
X	(1) co-ordinate system designation
	(2) $\pi\left[1/s + \dfrac{1}{b + y_1'\sin\alpha}\right]$
	(3) $Z\cos\tau$
X'	co-ordinate system designation
y_1	co-ordinates of point in Y-direction measured in aperture plane of source
y_1'	co-ordinates of point in Y'-direction measured in aperture plane of source
Y	co-ordinate system designation
	(2) $Z\sin\tau$
	(3) phase shift function in Chapter V
Y'	co-ordinate system designation
z	$2\pi r_1 \sin\alpha$ in Chapter IV

Z	$\dfrac{b\sqrt{x_1'^2 + y_1'^2 \cos^2 \alpha}}{b + y_1' \sin \alpha}$	Appendix III
α	angle of rotation	
β	dummy variable for Υ	
Υ	$\sin \Theta$	
δ	(1) increment operator	
	(2) constants in horn theory	
	(3) displacement of paraboloid feed in Appendix IV	
Δ	(1) increment operator	
	(2) displacement from focus in Appendix IV	
ε	(1) $\tan \varepsilon = x_1'/y_1' \sin \alpha$	
	(2) $\tan \varepsilon = L/N$	
Θ	spherical co-ordinate	
ϕ	(1) parameter in incomplete Bessel function $\int_0^{\phi} \cos \sin (TX)\, dx$	
	(2) angle in paraboloid analysis in Appendix IV	
ϕ_o	$(1/\rho)\exp(ik\rho)$	
	(1) distance from point in plane to field point	
	(2) radial distance in horn function	
σ	aberration constant in Chapter V	
τ	variable introduced in Appendix III	
ψ	spherical co-ordinate	
ψ_1	azimuthal co-ordinate at source	
ψ_2	azimuthal co-ordinate in plane $z = b$	
λ	(1) wave-length	
	(2) focal length of paraboloid in Appendix IV	
ξ	(1) co-ordinate in paraboloid analysis Appendix IV	
	(2) azimuthal angle in Chapter V	

ξ'	azimuthal angle in Chapter V
μ	aberration constant in Chapter V

INTRODUCTION

An ideal optical system is one in which a point, line, or plane has as its image a point, line, or plane. Any departure from this ideal is called an aberration. All practical systems have aberrations to some extent; indeed, it can be shown generally that this must be so for a finite object. The classification and minimization of these defects is now a long established art.

The above remarks are entirely within the scope of geometrical optics where light is treated in terms of rays, the ray being a minimal path between two points. But, physically, light is a wave, and, therefore, all optical relations must ultimately be consistent with the wave equation. As a consequence it can be shown that the ray concept is valid only in the limit of extremely high frequencies. It is the extension beyond the ray concept which introduces all diffraction phenomena.

The diffraction theory of aberrations, initiated by Hamilton, has received increased attention in recent years. Of particular note is the reclassification by Nijboer[1] and the many calculations based thereon, in the case where the aberrations are only a fraction of a wave-length. Not only are the standard type of aberration, coma, astigmatism, etc., modified by diffraction, but a new distortion of the field is introduced by the periphery into what would be, geometrically, a perfect system. The investigation of this effect goes back to the historical papers of Airy and Lommel and includes such recent work as that of Zernike and Nijboer[2] on the intensity near the focus of a lens.

The peripheral aberration just introduced is of some importance in optics, in regard to resolving power, for instance. In microwave optics it must be considered of major importance on account of the much greater wave-length involved. Work on this effect was initiated by Woonton[3] and his collaborators[4] who called it "mutilation". They investigated the one dimensional case, applicable to slits and horns, both experimentally and theoretically.

The present work extends that of Woonton to include two dimensional sources when the mutilating aperture is circular. Experiments on the distant field of horns and paraboloids were conducted with a view to isolating the effect here considered. A resumé of the experimental arrangement is given in the second last paragraph of this introduction. A theoretical treatment of the problem from the physical optics point of view is given in Chapter I, and, within this limitation, it is sufficiently general to include many optical systems having a circular aperture. Calculations based on theory were found to be in agreement with experiment. The same type of mutilation calculation was used by Hogg[5] in his investigation of back scattering by discs.

Some discussion is presented in Chapter V regarding the place of ordinary optical aberrations in microwave systems. Their experimental investigation must await the development of microwave lenses of high quality, and even then their isolation may be a matter of great difficulty.

A typical microwave system is illustrated schematically in Fig. 1. The dimensions of the source, A, may vary from a fraction of a wave-length to possibly 100 wave-lengths, and the source will invariably be coherent. B is a lens, although it may be generalized to include a number of reflecting and refracting elements. The relatively large conducting screen, B', prevents stray effects from reaching the receiver C. The receiver will generally be polarized to one plane and the present work is limited to this case.

If the distant field of the source is under investigation the receiver should be placed in the focal surface of the lens since the focal point is optically conjugate to infinity. A great contraction in the physical layout of apparatus is thereby effected. Pattern or scattering measurements, normally requiring a siting range of 100 yards can be made in the laboratory[5].

The arrangement just described attempts to simulate the distant field of the source, while it does, in fact, place the receiver in the near field of the optical system. The complexity of the near field of an open aperture was discovered by Andrews[6] and his work was confirmed and extended by Woonton and Bekefi[7]. Hogg[5] has studied the field of a lens placed in a circular aperture, and his findings indicate that the lens produces a further increase in the field perturbation. These phenomena can be ascribed, in part, to the fact that the relation between E and H is complicated in the near field but not in the distant field, and, in part, to the boundary conditions which form the basis of a rigorous electromagnetic theory. The latter aspect is not included in the present treatment but will be briefly considered below.

Microwave systems, of which an idealized example is represented in Fig. 1, have a great variety of forms. The particular arrangement used in this work is illustrated in three photographs, Figs. 4 to 6, where the disposition of transmitter, aperture, screen, and receiver is shown. The mutilation effect was isolated so far as possible by removing the lens, thereby reducing optical system simply to an aperture in a large diffraction screen. It was then necessary to place the receiver in the distant field, and this had the advatange of eliminating the Andrews' ripples. The arrangement was further simplified by placing the receiver in a fixed position on the axis of the system as shown in Fig. 6, and schematically in Fig. 2. The pattern of the source was obtained by rotating the source. The alternate arrangement of fixed source and movable receiver is more difficult theoretically and harder to realize experimentally. However one calculation for this case is presented.

The results and conclusions of this work are discussed later. They are difficult to summarize on account of the great variety of results and the large number of parameters involved. A direct comparison of mutilated and unmutilated patterns is shown in Figs. 10 to 31 where the nature of the effect may be observed. Small radiators, or radiators of moderate size containing a point source, including paraboloids, require special consideration. Mutilation depends on the non-uniformity of phase across the transmitter aperture. It varies greatly in detail as the diameter of the mutilating aperture is increased. The mutilation also increases with the separation of transmitter and mutilating aperture.

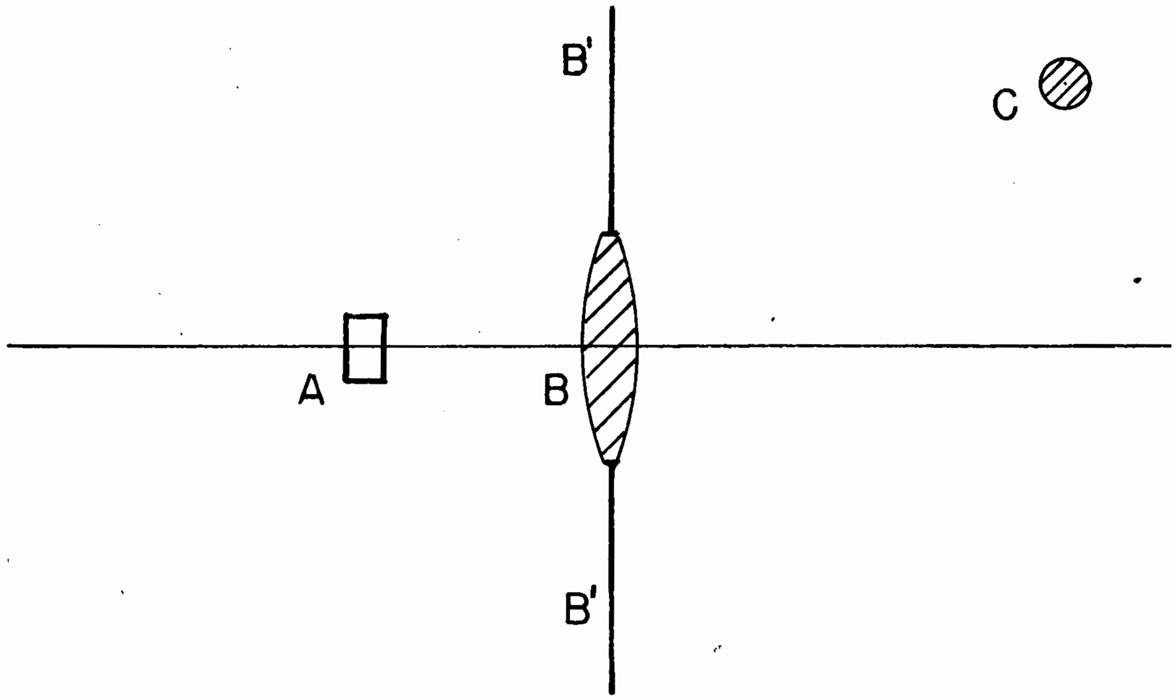


Fig. 1 Schematic of Microwave Optical System

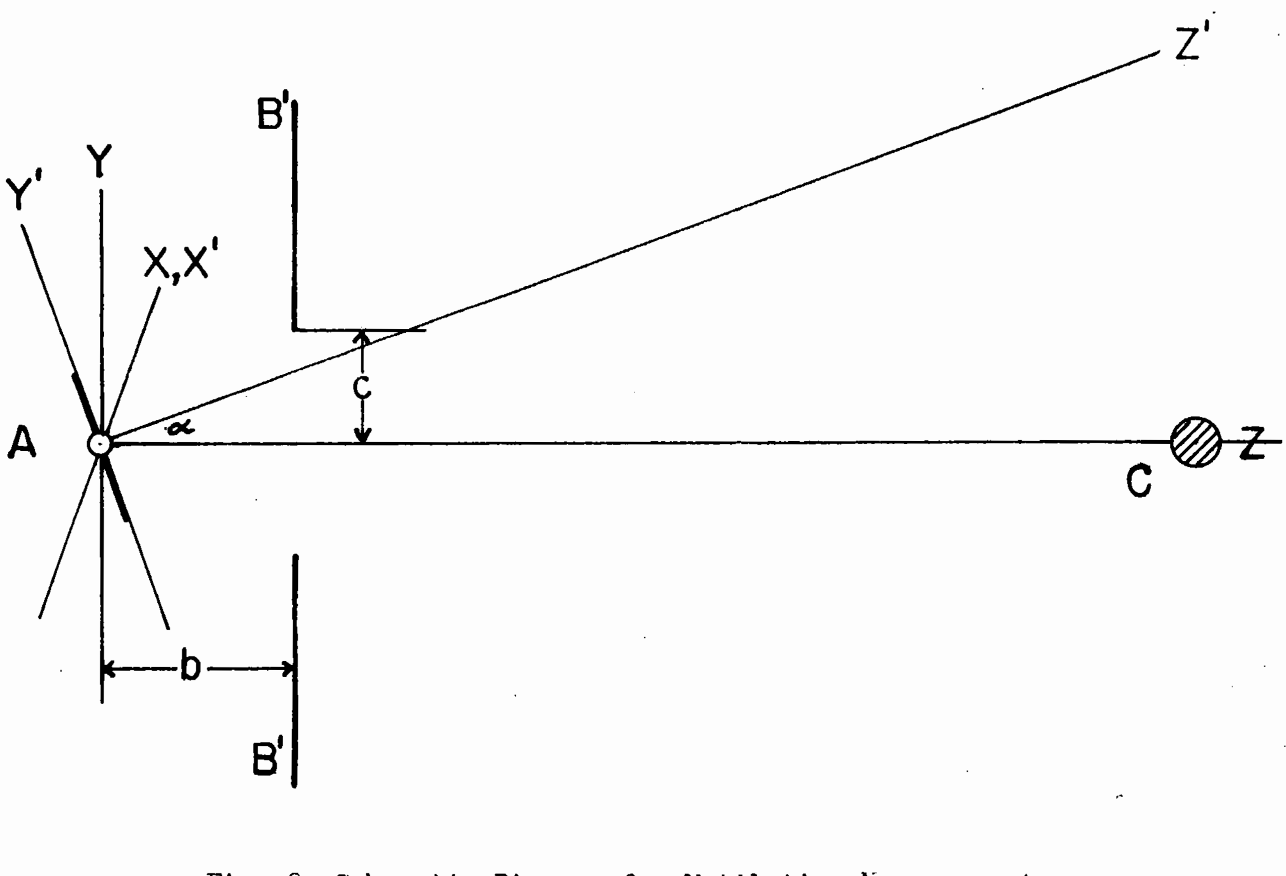


Fig. 2 Schematic Diagram for Mutilation Measurement

I. THEORY OF THE MUTILATED FIELD

The physical lay-out of apparatus for mutilations measurements has been briefly indicated in the Introduction. As the transmitter is rotated the main lobe of its radiation pattern is directed at an angle with the axis which becomes increasingly larger until eventually the main portion of the radiation is directed toward the metallic screen. This will indicate the complex nature of the field which is incident in the mutilating aperture. In the analysis which follows attention will be directed first toward the field which exists in the plane of the screen. This field will be considered for both cases, namely, for the screen absent, and for the screen physically present. Afterward the field will be considered as a function of the distribution over the transmitter aperture. And, finally, formulas will be obtained for the mutilation expressed in terms of an integral over the transmitter aperture.

1. The mutilating aperture is one of moderate size in a large conducting screen. From the point of view of physical optics based on the Kirchhoff approximation[8], a part of the wave front has been cut off, and that remaining is equal to that which would exist at the opening if the screen were entirely removed. The application of Huygens' Principle to the aperture results in the Kirchhoff field. But Babinet's Principle leads to a consideration of the complementary problem in which a disc replaces the aperture; then Huygens' Principle results in the Kirchhoff mutilation: that is, the quantity which must be subtracted from the original field to give the Kirchhoff field. These assumptions lead to results which are substantially in accord with experiment when the exit pupil has a diameter of thousands of wave-lengths, but they must be re-examined when the diameter is reduced to the order of 30 wave-lengths.

Much work has been done in recent years on the rigorous theory of diffraction, both scalar and electromagnetic. The mutilating aperture poses a diffraction problem of such generality that it is beyond the scope of any existing theory. Nevertheless, the current formulations can exhibit the relation between Kirchhoff theory and rigorous theory, thereby giving physical significance to the quantities involved in the former, and an explicit physical representation of what is neglected. The content of such theories pertinent to the present work is as follows.

Electromagnetic radiation is incident on a plane perfectly conducting thin screen which is of infinite extent except for one or more apertures. Then the tangential components of the magnetic field, H_x and H_y, and the normal component of the electric field, E_z, are equal, respectively, to H^i_x, H^i_y, and E^i_z where the latter are the components of the incident field, i.e., the field which would exist in the plane of the screen if the screen were removed[9].

Copson's formulation[10] is based on the Rayleigh formula,

$$u(x,y,z) = (1/2\pi) \int u(x_2,y_2,b) \; \frac{\partial \phi_o}{\partial z} \; dx_2dy_2 \tag{1.1}$$

where (x,y,z) is a field point, and x_2, y_2 specify position over the infinite plane $z = b$, and $\phi_o = (1/\rho)\exp(ik\rho)$ where ρ is the distance from an element in the $z = b$ plane to (x,y,z). Copson shows that this formula may be applied to the field components in the aperture problem. Thus,

$$\vec{H}(x,y,z) = (1/2\pi) \int_{S_1+S_2} \vec{H}(x_2,y_2,b) \; \frac{\partial \phi_o}{\partial z} \; dx_2dy_2 \tag{1.2}$$

where S_1 is the aperture area and S_2 the metal in the screen. But it is also true that,

$$\vec{H^i}(x,y,z) = (1/2\pi) \int_{S_1+S_2} \vec{H^i}(x_2,y_2,b) \; \frac{\partial \phi_o}{\partial z} \; dx_2dy_2$$

Therefore, since $H_{tang} = H^i_{tang}$ in S_1,

$$H_{tang}(x,y,z) = H^i_{tang}(x,y,z) - (1/2\pi)\int_{S_2} H^i_{tang}(x_2,y_2,b)\,\frac{\partial\phi_o}{\partial z}\,dx_2dy_2$$
$$+(1/2\pi)\int_{S_2} H_{tang}(x_2,y_2,b)\,\frac{\partial\phi_o}{\partial z}\,dx_2dy_2 \qquad (1.3)$$

Eq. (1.1) may also be written,

$$H_{tang}(x,y,z) = (1/2\pi)\int_{S_1} H^i_{tang}(x_2,y_2,b)\,\frac{\partial\phi_o}{\partial z}\,dx_2dy_2 + (1/2\pi)$$
$$\int_{S_2} H_{tang}(x_2,y_2,b)\,\frac{\partial\phi_o}{\partial z}\,dx_2dy_2 \qquad (1.4)$$

In (1.3) the first term is the unmutilated field, the second is the Kirchhoff mutilation, and the third is an electromagnetic perturbation due to currents generated on the shadow side of the screen. The first term of (1.4), which combines the first two of (1.3) is a Kirchhoff-like field. It is identical with the scalar Kirchhoff formulation provided the components of $\vec{H}$ are identified with the scalar field. Therefore it can be stated that, except for the effect of the last terms of (1.3) and (1.4), Kirchhoff theory may be used to derive the magnetic field at all points in space.

The electric field can be derived from the magnetic field by use of Maxwell's equations. The result near the aperture is totally different from what would be obtained from a Kirchhoff formulation based on the electric field. Eq. (1.4), neglecting the last term, is equivalent to theories advanced by Silver[11], Neugebauer[12], and Bekefi[13] to explain the large fluctuations, discovered by Andrews, in the electric field near the aperture. The success of these approximate theories is, to some extent, a justification for neglecting the last term in the present work. However, it cannot be definitely concluded that the last term of (1.4) is insignificant because the aperture fields here considered are of much greater complexity than those treated by Bekefi. An application of the Sommerfeld solution should indicate the order of magnitude of the quantities involved.

On this basis the magnetic field is about 25% of the incident field at a distance one wave-length inside the shadow, and 5% of the incident field at 25 wave-lengths inside the shadow.

For distant axial points the magnetic field is directly proportional to the electric field, and the latter is the quantity to which the receiver responds. Hence, in the following section, a field u will be introduced which can be identified with the magnetic field, or, at a great distance, with either the tangential magnetic field or the tangential electric field. Thus the finally derived field will be such that its magnitude squared will be proportional to the receiver response when the receiver is in the distant field.

2. A general expression for the Kirchhoff field will be obtained in this section. A further development toward forms suitable for calculation will be given in the following section, and in Appendix III, and in Chapters III and IV.

Two sets of co-ordinate systems will be used throughout the work. Their common origin is at O as shown in Fig. 2, and the X', Y', Z' system is fixed in the antenna and is capable of rotation about the X,X' axis. The X,Y,Z system is so oriented that the diffraction screen and mutilating aperture are in the plane z=b. Points in the source region near O will be designated by the subscript "1". Thus points in the antenna aperture will be represented (x_1', y_1'). Transformation to spherical co-ordinates (R,θ,ψ) and (R,θ',ψ') will be made where necessary in either system. A radial direction will be given direction cosines l, m, cos θ or l', m', cos θ'. R_2 will be the distance between (0,0,b) and (x,y,z).

If (x,y,z) is in the distant field (1.1) reduces to,

$$u(x,y,z) = u(R,\theta,\psi) = \frac{ik \cos \theta}{2\pi R_2} \exp(ikR_2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_2,y_2,b)\exp\left[-ik(lx_2+my_2)\right] dx_2 dy_2 \tag{1.5}$$

provided $u(x_2,y_2,b)$ is confined to a finite region. Great distance implies that the l, m here used do not differ significantly from those defined above. The field at z=b is not confined to a finite region either with the screen present or with it absent. However, it is approximately so limited provided the radiator is reasonably large in comparison with the wave-length. In view of these considerations the inclination factor, $\cos \theta$, cannot be regarded as significant, and it will be dropped.

Therefore the distant unmutilated field can be written,

$$u(R,\theta,\psi) = C\exp(ikR_2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_2^i(x_2,y_2,b)\exp\left[-ik(lx_2+my_2)\right] dx_2 dy_2 \ldots (1.6)$$

where $u_2^i(x_2,y_2,b)$ is the incident field in the plane z=b and $C = \frac{ik}{2\pi R_2}$.

With the screen present the mutilated field will be,

$$u_m(R,\theta,\psi) = C\exp(ikR_2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_{2m}(x_2,y_2,b)\exp\left[-ik(lx_2+my_2)\right] dx_2 dy_2 \tag{1.7}$$

$$\left.\begin{aligned} \text{where } u_{2m} &= u_2^i \text{ in the aperture} \\ &= 0 \text{ outside the aperture} \end{aligned}\right\} \text{---------------------} (1.8)$$

The Fresnel fields of the antenna may be expanded in a Fourier series since they are periodic in ψ_2. Thus,

$$u_2^i = \sum_{n=-\infty}^{\infty} f_{2n}(r_2)\exp(in\psi_2) \tag{1.9}$$

$$u_{2m} = \sum_{n=-\infty}^{\infty} f_{2mn}(r_2)\exp(in\psi_2) \tag{1.10}$$

$$\left.\begin{aligned} \text{where } f_{2mn} &= f_{2n} \text{ for } r_2 < c \\ &= 0 \quad \text{for } r_2 > c \end{aligned}\right\} \text{---------------------} (1.11)$$

c being the radius of the mutilating aperture, and (r_2,ψ_2) polar co-ordinates in the plane z=b.

$$x_2 = r_2 \cos \Psi_2 \qquad\qquad y_2 = r_2 \sin \Psi_2$$

$$l = \sin \Theta \cos \Psi \qquad\qquad m = \sin \Theta \sin \Psi$$

$$dx_2 dy_2 = r_2 dr_2 d\Psi_2$$

Substituting in (1.6),

$$u(R,\Theta,\Psi) = C \exp(ikR_2)$$
$$\int_0^\infty \int_0^{2\pi} \sum_{-\infty}^{\infty} f_{2n}(r_2) \exp\left[in\Psi_2 - ikr_2 \sin \Theta \cos (\Psi - \Psi_2)\right] r_2 d\Psi_2 dr_2$$

But $\exp\left[-ikr_2 \sin \Theta \cos (\Psi-\Psi_2)\right] = \exp\left[ikr_2 \sin \Theta \sin (\Psi - \Psi_2 - \pi/2)\right]$

$$= \sum_{p=-\infty}^{\infty} J_p(kr_2 \sin \Theta) \exp\left[ip(\Psi - \Psi_2 - \pi/2)\right]$$

$$\therefore\ u(R,\Theta,\Psi) = C \exp(ikR_2) \int_0^\infty \int_0^{2\pi} \sum_{n=-\infty}^{\infty} f_{2n}(r_2) \exp(in\Psi_2)$$
$$\sum_{p=-\infty}^{\infty} J_p(kr_2 \sin \Theta) \exp\left[ip(\Psi-\Psi_2-\pi/2)\right] r_2 d\Psi_2 dr_2$$

Integration in Ψ_2 eliminates all terms except those for which $p = n$. It is always justifiable to integrate the Fourier series by term. Moreover, since the distant field is continuous, the series itself must be uniformly convergent[14]. Therefore integration and summation may be commuted giving,

$$u(R,\Theta,\Psi) = 2\pi\ C \exp(ikR_2) \sum_{-\infty}^{\infty} \exp\left[in(\Psi-\pi/2)\right] \int_0^\infty f_{2n}(r_2) J_n(kr_2 \sin \Theta) r_2 dr_2 \quad ..(1.12)$$

Similarly,

$$u_m(R,\Theta,\Psi) = 2\pi C \exp(ikR_2) \sum_{-\infty}^{\infty} \exp\left[in(\Psi-\pi/2)\right]$$
$$\int_0^\infty f_{2mn}(r_2) J_n(kr_2 \sin \Theta) r_2 dr_2 \qquad (1.13)$$

Equations (1.12) and (1.13) are seen to be Fourier expansions of the distant field. Such expansions can be obtained directly in terms of the radiation source. For it has been shown[15] generally that the distant field of a finite source distribution can be written,

$$u(R,\Theta,\Psi) = \frac{ik}{2\pi R} \exp(ikR) \quad F(\Theta,\Psi).$$

which, on expansion, becomes,

$$u(R,\theta\psi) = \frac{ik}{2\pi R} \exp(ikR)\sum_{-\infty}^{\infty} G_n(\theta) \exp(in\psi)$$

$$= C \exp(ikR)\sum_{-\infty}^{\infty} G_n(\theta) \exp(in\psi) \tag{1.14}$$

since $\frac{R}{R_2} \rightarrow 1$ in the distant field.

Similarly,

$$u_m(R,\theta,\psi) = C \exp(ikR)\sum_{-\infty}^{\infty} G_{mn}(\theta) \exp(in\psi) \tag{1.15}$$

Distances will now be expressed in wave-length units, and the substitution $\sin\theta = \gamma$ will be made. Then, equating coefficients between (1.14) and (1.12) and between (1.15) and (1.13),

$$G_n(\gamma) = \exp\left[2\pi i(R_2-R) - \frac{in\pi}{2}\right]\int_0^{\infty} f_{2n}(r_2)\, J_n(2\pi r_2\gamma)\, r_2 dr_2 \tag{1.16}$$

$$G_{mn}(\gamma) = \exp\left[2\pi i(R_2-R) - \frac{in\pi}{2}\right]\int_0^{\infty} f_{2mn}(r_2)\, J_n(2\pi r_2\gamma)\, r_2 dr_2 \tag{1.17}$$

But $R - R_2 = b\cos\theta = b\sqrt{1-\gamma^2}$

$\therefore$ from (1.16),

$$\int_0^{\infty} f_{2n}(r_2)J_n(2\pi r_2\gamma)\, r_2 dr_2 = G_n(\gamma)\exp\left[2\pi i b\sqrt{1-\gamma^2} + \frac{in\pi}{2}\right] \tag{1.18}$$

This equation permits a component of the Fresnel field to be expressed in terms of a corresponding component of the distant field. This inversion is effected by applying a Hankel transform to (1.18),

$$f_{2n}(r_2) = 4\pi^2 \exp\left(\frac{in\pi}{2}\right)\int_0^{\infty} G_n(\gamma)\exp\left[2\pi i b\sqrt{1-\gamma^2}\right] J_n(2\pi r_2\gamma)\gamma d\gamma \tag{1.19}$$

Taking account of (1.11), (1.19) may be substituted into (1.17) giving,

$$G_{mn}(\gamma) = 4\pi^2\exp\left[-2\pi i b\sqrt{1-\gamma^2}\right]\int_0^{c}\int_0^{\infty} G_n(\beta)\exp\left[2\pi i b\sqrt{1-\beta^2}\right]$$
$$J_n(2\pi r_2\beta)\, J_n(2\pi r_2\gamma)\, \beta r_2 d\beta dr_2 \tag{1.20}$$

where β is a dummy variable.

(1.20) expresses a component of the mutilated distant field in terms of a component of the unmutilated distant field. But the receiver in a microwave system may be at a moderate rather than a great distance. Therefore it is useful to obtain the Fourier components, $F_{mn}(r_3)$, of the field in the plane $z = b+s$.

This quantity is obtained at once from (1.20) by expressing it in a form parallel to (1.19),

$$F_{mn}(r_3) = 4\pi^2 \exp\left(\frac{in\pi}{2}\right) \int_0^\infty G_{mn}(\gamma) \exp(2\pi is \sqrt{1-\gamma^2})J_n(2\pi r_3\gamma)\gamma d\gamma$$

$$= 16\pi^4 \exp\left(\frac{in\pi}{2}\right) \int_0^c \int_0^\infty \int_0^\infty G_{n(\beta)} \exp\left[2\pi is\sqrt{1-\gamma^2} + 2\pi ib\sqrt{1-\beta^2}\right]$$

$$J_n(2\pi r_2\beta)\, J_n(2\pi r_2\gamma)\, J_n(2\pi r_3\gamma)\; \beta\gamma r_2 d\beta d\gamma dr_2 \qquad (1.21)$$

The coefficients $G_n(\beta)$ are derived in Appendix I in terms of an integral over the aperture,

$$G_n(\beta) = \frac{1}{2\pi} \exp(-in\pi) \int_{(1)} f(x_1', y_1')\, J_n(2\pi B\beta)$$

$$\exp\left[in\varepsilon + 2\pi y_1' \sqrt{1-\beta^2} \sin\alpha\right] dx_1' dy_1' \qquad (1.22)$$

where $f(x_1', y_1')$ is the aperture function for an antenna or a current density function in the case of a scatterer. And $B^2 = x_1'^2 + y_1'^2 \cos^2\alpha$, $\sin\varepsilon = x_1'/B$.

Therefore,

$$G_{mn}(\gamma) = 2\pi(-1)^n \exp\left[-2\pi ib\sqrt{1-\gamma^2}\right] \int_{(1)} \int_0^c \int_0^\infty f(x_1', y_1')$$

$$\exp\left[in\varepsilon + 2\pi i\sqrt{1-\beta^2}\,(b+y_1' \sin\alpha)\right] J_n(2\pi B\beta)J_n(2\pi r_2\beta)J_n(2\pi r_2\gamma)\beta r_2 d\beta dr_2 dx_1' dy_1'. \,(1.23)$$

$$F_{mn}(r_3) = 8\pi^3 \exp\left(-\frac{in\pi}{2}\right) \int_{(1)} \int_0^c \int_0^\infty \int_0^\infty f(x_1', y_1') \exp\left[in\varepsilon + 2\pi i(b+y_1' \sin\alpha)\sqrt{1-\beta^2}\right]$$

$$J_n(2\pi r_2\beta)J_n(2\pi B\beta) \exp\left[2\pi is\sqrt{1-\gamma^2}\right] J_n(2\pi r_2\gamma)\; J_n(2\pi r_3\gamma)\beta\gamma r_2 d\beta d\gamma dr_2 dx_1' dy_1' \ldots\ldots (1.24)$$

3. If observations are made on the axis of the mutilating aperture $r_3=0$. Then $F_{mn} = 0$ for $n \neq 0$, and,

$$F_{mo}(0) = 8\pi^3 \int_{(1)} \int_0^c \int_0^\infty \int_0^\infty f(x_1', y_1') \exp\left[2\pi i(b+y_1' \sin\alpha)\sqrt{1-\beta^2}\right] J_o(2\pi B\beta)$$

$$J_o(2\pi r_2\beta) \exp(2\pi is\sqrt{1-\gamma^2})\; J_o(2\pi r_2\gamma)\; \beta\gamma r_2 d\beta d\gamma dr_2 dx_1' dy_1' \qquad (1.25$$

With radiation which is fairly directive the integrals will be negligible for $\beta > \beta_o$, $\beta_o \ll 1$ and $\gamma > \gamma_o$, $\gamma_o \ll 1$. Then by binomial expansion,

$$\sqrt{1-\beta^2} \doteq 1-\beta^2/2 \quad \text{and} \quad \sqrt{1-\gamma^2} \doteq 1-\frac{\gamma^2}{2} \qquad (1.26)$$

Substitution in (1.25) gives,

$$F_{mo}(0) = -\frac{2\pi}{s}\exp\left[2\pi i(s+b)\right]\int_{(1)}\int_0^c\int_0^\infty\int_0^\infty f(x_1',y_1')\exp(2\pi i y_1' \sin\alpha)$$
$$\exp\left[-\pi i\beta^2(b+y_1' \sin\alpha)\right] J_o(2\pi r_2\beta)J_o(2\pi B\beta)\exp(-\pi i s\gamma^2)J_o(2\pi r_2\gamma)\beta\gamma r_2 d\beta d\gamma dr_2 dx_1' dy_1'$$

The integrals with respect to β and γ are now of a standard type[16], and the expression reduces to,

$$F_{mo}(0) = -\frac{2\pi}{s}\exp\left(2\pi i(s+b)\right)\int_{(1)}\int_0^c f(x_1'y_1')\exp\left[2\pi i y_1' \sin\alpha + \frac{\pi i B^2}{b+y_1' \sin\alpha}\right]$$
$$\exp\left[\pi i r_2^{\,2}\left(\frac{1}{s}+\frac{1}{b+y_1' \sin\alpha}\right)\right] J_o\left(\frac{2\pi r_2 B}{b+y_1' \sin\alpha}\right) r_2 dr_2 dx_1' dy_1' \qquad (1.27)$$

The integral with respect to r_2 is of a kind which leads to a Lommel function[17]. The reduction of several integrals of this type is given in Appendix II. Accordingly,

$$F_{mo}(0) = -\frac{2\pi}{s}\exp\left[2\pi i(s+b)\right]\int_{(1)} f(x_1',y_1')\exp\left[2\pi i y_1' \sin\alpha + \frac{\pi i B^2}{b+y_1' \sin\alpha}\right] \qquad (1.28)$$
$$\left[\frac{i}{2X}\exp(-iA^2/4X) - (W_1(t,w)/2X)\exp(iXc^2)\right] dx_1' dy_1'.$$

where $t = 2Xc^2$, $w = Ac$. $A = \dfrac{2\pi B}{b+y_1' \sin\alpha}$, $X = \pi\left[1/s + \dfrac{1}{b+y_1' \sin\alpha}\right]$

$$W_1(t,w) = V_1(t,w) + i\,V_o(t,w)$$

and $V_o(t,w) = J_o(w) - (\frac{w}{t})^2 J_2(w) + (\frac{w}{t})^4 J_4(w)\ldots\ldots.$

$V_1(t,w) = (\frac{w}{t}) J_1(w) - (\frac{w}{t})^3 J_3(w) + (\frac{w}{t})^5 J_5(w)\ldots\ldots.$

The first term of (1.28) is independent of c and represents the unmutilated field. The exponent $A^2/4X$ may be expanded and for $s \gg b$ it gives,

$$F(o) = -i\exp\left[2\pi i(s+b)\right]\int_{(1)}\left[f(x_1',y_1')/(s+b+y_1' \sin\alpha)\right]\exp\left[2\pi i y_1' \sin\alpha + XiB^2/s\right] dx_1' dy_1' \qquad (1.29)$$

The reversal of signs from the previous form (1.5) can be attributed to the fact that α has been chosen so that the positive Y' direction turns away from the axis of the aperture.

The mutilation ,e(o), may be written

$$e(0) = F_{mo}(0) - F(0)$$
$$= \exp\left[2\pi i(s+b) + \pi i c^2/s\right] \int_{(1)} f(x_1', y_1')\, W_1(t,w)/(s+b+y_1' \sin\alpha)$$
$$\exp\left[\pi i \left[2y_1' \sin\alpha + (B^2+c^2)/(b+y_1' \sin\alpha)\right]\right] dx_1' dy_1' \qquad (1.30)$$

Equation (1.30) is the starting point for practically all of the computations which have been made. It expresses the mutilation in terms of a single integral over the aperture or surface of the radiator with no restriction as to the shape of this surface provided it is planar.

The related problem of fixed source and variable field point is of some interest. It has not been possible to obtain a general formula corresponding to (1.30). If, however, the aperture function is specialized to $f(x_1', y_1') = 1$, a solution is possible. Then with $\alpha = 0$, $G_{mn}(\gamma) = 0$ for $n = o$, and,

$$G_{mo}(\gamma) = 4\pi^2 \exp\left[-2\pi i b\sqrt{1-\gamma^2}\right] \int_o^a\int_o^c\int_o^\infty \exp\left[2\pi i b\sqrt{1-\beta^2}\right]$$
$$J_o(2\pi r_1\beta) J_o(2\pi r_2\beta) J_o(2\pi r_2\gamma)\ \beta r_1 r_2\, d\beta\, dr_2\, dr_1$$
$$= -(2\pi i/b)\exp(\pi i b\gamma^2) \int_o^c\int_o^a \exp\left[(\pi i/b)(r_1^2 + r_2^2)\right] J_o(2\pi r_1 r_2/b) J_o(2\pi r_2\gamma)$$
$$dr_1 dr_2$$

Using the results of Appendix II the integration in r_1 may be performed,

$$G_{mo}(\gamma) = -i \exp(\pi i a^2/b + \pi i b\gamma^2) \int_o^c \exp(\pi i r_2^2/b)\left[U_1(t,w) - iU_2(t,w)\right]$$
$$J_o(2\pi r_2\gamma)\ r_2 dr_2$$

where $U_1(t,w) = (\frac{t}{w})J_1(w) - (\frac{t}{w})^3 J_3(w) + (\frac{t}{w})^5 J_5(w) \ldots\ldots\ldots\ldots$

$$U_2(t,w) = (\tfrac{t}{w})^2 J_2(w) - (\tfrac{t}{w})^4 J_4(w) + (\tfrac{t}{w})^6 J_6(w) \ldots\ldots\ldots$$
$$t = 2\pi a^2/b \quad ; \qquad w = 2\pi a r_2/b$$

Therefore,

$$u_m(\gamma) = 2\pi/R \exp\left[2\pi i R + \pi i a^2/b + \pi i b\gamma^2\right]$$
$$\int_o^c \exp(\pi i r_2^2/b)\left[U_1(t,w) - iU_2(t,w)\right] J_o(2\pi r_2\gamma)\ r_2 dr_2 \qquad (1.31)$$

One computation based on (1.31) is presented in Appendix III.

4. A detailed treatment of Eq. (1.30) to reduce it to a form suitable for computation is given in Appendix III. The present section is a survey which is intended to be a preface to Chapters III and IV. It is there applied to horns and paraboloids and comparison is made with experiment.

The integrand of (1.30) contains the Lommel function $W_1(w,t)$. The argument, w, is large so that the function oscillates rapidly, and it depends in a complicated way on x_1' and y_1'. It is apparent that the major difficulty centers around this function. A transformation to new variables (Z,τ) is introduced where $Z = bw/2\pi c$. (Z has no connection with the previously defined co-ordinate system.) This transformation introduces new terms and complicates existing ones. But the Lommel function can be integrated and the whole reduced to a manageable summation.

For $\alpha = 0$, (Z,τ) is identical with the polar co-ordinates (r_1', ψ_1'). With increase in α the two sets of variables become increasingly different. Nevertheless, it is a useful concept to regard (Z,τ) as radial and polar co-ordinates over a slightly distorted aperture. (1.30) is shown to reduce to a series of integrals of which the dominant one is,

$$I_1 = K \int_{(1)} f(Z,\tau)\ \exp(iD)\ W_1(t_o,w)\ ZdZd\tau \ \ldots\ldots\ldots\ldots \qquad (1.32)$$

It is possible to put $f(Z,\tau)\exp(iD)$ equal to one or more terms of the form $f_1(Z)\exp(iQ)\cos n\tau$, where,

$$Q = MZ^2/2 + LZ\sin\tau + NZ\cos\tau - RZ^2\cos 2\tau \qquad (1.33)$$

$\oint_n(Z)$ is defined by the equation,

$$\oint \exp(iQ)\ \cos n\tau\ d\tau = 2\pi \oint_n(Z)\exp(iMZ^2/2) \qquad (1.34)$$

This gives,

$$I_1^{(n)} = 2\pi K \int_{(1)} f_1^{(n)}(Z)\exp(iMZ^2/2)\oint_n(Z)\; W_1(t_o,w)\; ZdZ$$
$$= \frac{b(s+b)}{s}\; K \int_{(1)} f_1^{(n)}(Z)\exp(iMZ^2/2)\oint_n(Z)\,\frac{dW_2}{dw}\; dw$$

where $I_1 = I_1' + I_1'' + \ldots\ldots$

This may be replaced by the equivalent summation,

$$I_1^{(n)} = \frac{b \sec \alpha}{s} \exp\left[\pi i\left[2(s+b) + c^2(1/s + 1/b)\right]\right]$$
$$\sum_i f_1^{(n)}(Z_i)\oint_n(Z_i)\exp(iMZ^2/2)\;\delta W_2 \tag{1.35}$$

where the value of the constant K has been inserted.

The summation is then over a number of zones, that is, regions of approximately constant Z. The τ integration within a zone is given in (1.34). $\oint$ is equal to $\int_0^{2\pi}$ for inner zones but splits up into a number of regions of integration for outer zones which cut the Z, τ perimeter. It will be noted that the transformation deforms the physical perimeter of the radiator. The zones and the deformation of perimeter are shown in Fig. 7.

The τ integration yields $\oint_n(Z)$. δW_2 is readily obtained by series summation. The final operation is to add up all of the individual complex terms graphically on the drawing board. Such summations are shown in Fig. 8 (starting at O and following the arrows) for three values of α. The quantities which depend on the individual aperture function are the M, L, N, R (Eq. 1.33). Examples are given in Appendix III and Chapter III.

II. EXPERIMENTAL APPARATUS

The experimental part of the present work consisted essentially in recording the power patterns of a number of microwave antennas. Such patterns can be taken in the following way. The antenna is mounted on a turntable and suitably connected to the source of power and the necessary auxiliary apparatus. The receiver is sufficiently remote to ensure that the distance from any point in the transmitting aperture to any point in the receiving aperture does not vary any more than a small fraction of a wave-length. The site and elevation of the apparatus must be such that the received power is not modified by reflection from obstacles or the ground. The pattern is a graph of the received power as a function of the angle through which the transmitter is rotated.

Mutilated patterns were obtained in this investigation simply by interposing a specially constructed diffraction screen between transmitter and receiver. The screen containing the mutilating aperture was rigidly installed on the roof of the Eaton Electronics Research Laboratory, and the transmitting equipment was set up before it in the desired position. The receiver was located on the roof of the McGill University Biology Building at a distance of 250 feet. Photographs of the apparatus are shown in Figs. 3 to 6.

The section which follows contains a detailed account of the site, the screen, and other apparatus. Consideration is given, in the next section, to the transmitting antennas, and to the experimental procedure used in their measurement. Finally, there is some discussion of the patterns themselves with particular reference to the conditions of measurement and the precision attained.

1. The Site and Apparatus: The transmitter-receiver site was chosen so that the main lobe of the receiver pattern was unobstructed by roof obstacles, trees, etc. Tests were made in the vicinity of both transmitter and receiver by placing scatterers or absorbers near points which could be suspected of reflecting secondary radiation into the receiving paraboloid. These were all negative. When a horn under test was turned over a very slight change in the asymmetry of the pattern was observed; however, further investigation in this regard was judged to be unnecessary. The siting distance of 250 feet corresponded to a phase shift of 30° across the largest aperture. Formulas were obtained in such a form that corrections for finite distance could be made and this was done in some cases. Fig. 6 is a photograph showing transmitter, receiver, and screen.

The function of the diffraction screen was to provide a conducting surface of considerable extent, and to support the apertures. It approached the ideal of an infinite screen: the power was not measurable with the mutilating aperture closed. The screen was designed to have a minimum of framework in the vicinity of the aperture consistent with mechanical strength. It extended a minimum of 7 feet in all directions from the centre of the aperture, except downward, where a copper parapet provided adequate screening. The screen itself consisted of two layers of fine copper mesh.

The screen was placed parallel to the side of the building and positioned ($\pm$ 6 in) so that the receiver was on the perpendicular line through the centre of the aperture. The centres of receiver, mutilating aperture and transmitter were at the same horizontal level ($\pm$ 1 in). A transit was levelled on the turntable and the turntable axis adjusted to the vertical so that the transit did not change on rotation. A carpenter's level was used for fixing the diffraction screen to a vertical position, and also for adjusting the face of the radiators to vertical. The centre of the face of the radiator was brought to a position on the axis of rotation of the turntable to within 1/8 inch.

The apertures were made from #12 Gauge B. & S. (0.080") sheet aluminum. It was necessary to cut the 40 wave-length and 35 wave-length apertures each in two pieces and fasten them together with screws. Their outside dimension fitted the 5 foot square wooden framework. The other apertures were made in one piece. Those of 25 wave-lengths and 30 wave-lengths, with outside dimensions 4 foot square, were screwed to the 35 wave-length aperture. The smallest apertures of 15 and 20 wave-lengths were screwed to the 25 wave-length aperture. The circular holes were sawn and finished to within 1/2 mm (0.016 wave-lengths). Screen, aperture, and transmitting assembly are shown in Fig. 4.

Power for transmission was generated by a Varian X13 klystron which was square wave modulated at approximately 1000 cycles per second. Accessory tuner, flap attenuators, and wave meter are sho n in Fig. 3. The transmitting antenna was connected to this assembly through a turntable capable of 360° rotation. Standard X-band guide was used except in the turntable where circular guide is required for mechanical reasons and the usual couplings made to the rectangular guide.

The microwave signal was received by a paraboloid of 30 inches diameter placed on the axis of the aperture. It was then carried, via a double dipole feed of standard design[18], to a Sperry Type 821 bolometer for detection. The bolometer has less sensitivity than a crystal detector, but is accurately square law over a large range of incident power and, therefore, does not require calibration. The audio signal was fed to a battery operated amplifier especially designed for pattern measurements[19].

This pre-amplifier which employs a tuned inverse feed-back circuit, was linear over a large range of signal strengths. It had a noise input of 2×10^{-7} volts, and its voltage gain was several thousand when its output was connected to the 300 feet of cable returning the signal to the Eaton Laboratory for recording. The effect of noise induced in the cable was eliminated by use of a balanced line and by the pre-amplification. The amplifier heaters and bolometer battery were turned on by a relay whose switch was located in the penthouse of the Eaton Laboratory along with the recorder, klystron power supply, and other equipment for the control of the apparatus.

The recorder used was an Airborne Instruments Rectangular Recorder, Type 373-4. It included within its assembly a high gain selective amplifier having a band width of 40 cps and a rated noise input of 3×10^{-8} volts. A variable GR audio frequency attenuator preceded the input to this amplifier to reduce the signal to an appropriate level. The recorder proper gave a direct indication of power with decibels plotted on a linear scale. The chart advance was coupled to the rotation of the turntable by means of a dual Selsyn system. This permitted the choice of three chart advance speeds. However only one, 20 inches for 60° of rotation, was used. The turntable was powered by a D.C. motor which was amplidyne controlled, and, therefore, capable of close adjustment.

<u>2. The Transmitting Antennas and Experimental Procedure:</u> Complete sets of patterns (except for c = 20) were taken for three horns whose dimensions are given in Chapter III. All horns were constructed from sheet copper.

Complete measurements were obtained for a series of paraboloids all having focal length equal to one-quarter the diameter. The feed was of the same double dipole design as was used in the receiver. The 14 inch paraboloid was made from spun copper, and the others, of diameters 12 inches, 18 inches, 24 inches, and 30 inches, from spun aluminum.

The horns and 14 inch parabaloid were mounted on a wooden support and could be adjusted horizontally and vertically. Impregnated wooden frames were attached to the other paraboloids and these could be bolted to a universal mount which contained the feed. The mount permitted rotation and longitudinal displacement of the feed, but was closely machined from heavy stock to prevent lateral motion.

Patterns of wave guide mouth and of back radiation from the feed were obtained. For these measurements, in which radiation proceeded backward as well as forward, the mount was covered by pads of an absorbing material.

The klystron was tuned for maximum signal. The receiving paraboloid was oriented and its feed adjusted for maximum signal. Transmitting paraboloids were also focussed for maximum signal. The transmission wavelength was 3.20 $\pm$ 0.00 cm, based on a PRD standard wave meter. In all measurements of mutilated patterns 5 to 10 db of microwave attenuation was inserted to prevent coupling between klystron and the diffraction screen. Otherwise the screen could conceivably change the output impedance of the Klystron by an amount sufficient to affect its power output.

The linearity of the pre-amplifier was tested by inserting 15 db of microwave attenuation and raising the signal to the recorder to the original level by means of the a.f. attenuator. The pattern remained unchanged. The linearity of the recorder was similarly checked by leaving the a.f. attenuation unchanged.

The monitoring of the apparatus was not thought to be feasible. Not only was some of the apparatus remote, but, to be fully useful, a monitor would have to provide a stable reference over a period of several weeks accurate to 0.1 db. Such a monitor would be extremely helpful in the interpretation of mutilated patterns.

3. Patterns, Unmutilated and Mutilated: The recording of patterns on a decibel scale tends to emphasize the details of their side structure. Reference to Figs. 10 to 31 will illustrate this variety of detail for both horn and paraboloid patterns; and in Chapter III it will be related to a variation of amplitude and phase in the antenna aperture.

The field of all antennas is affected by at least some asymmetry, this being a property of the radiator construction. For horns this is normally small and the averaged pattern can be related to a symmetric aperture function. Paraboloids are sensitive to the positioning of the feed, and especially to its lateral displacement which introduces a marked asymmetry. In the case of mutilated patterns proper centering and alignment are important and, if not maintained, will lead to further asymmetry.

Noise in the amplifiers, etc limits the observable power range. This range varied from over 40 db for the larger paraboloids to about 20 db for wave guide mouth, and was correspondingly less when transmission power was reduced by microwave attenuation. However, no radiator for which calculations were made had an effective range of less than 35 db even when so attenuated.

An estimate of the error of measurement depends on the type of pattern under review, and the basis of comparison. It is believed that power levels are accurate to within 0.2 db on the upper portion of the main lobe and to within 0.5 db elsewhere, except where the antenna was altered by dismantling and reassembling. Two patterns taken in a set, within an hour of each other, would generally differ by less than half the error specified above. Errors which resulted from dismantling and reassembling were more serious. These were sometimes observable even with careful alignment and mounts of rigid construction. A considerable number of patterns was taken through the 5 foot square hole provided by the framework of the screen. From observation of a series of these it was concluded that they differ from the unmutilated patterns by a negligible amount even at large angles. Therefore, some paraboloid patterns were compared with those taken through the 5 foot opening. The latter could be taken immediately, while the free space patterns were necessarily delayed.

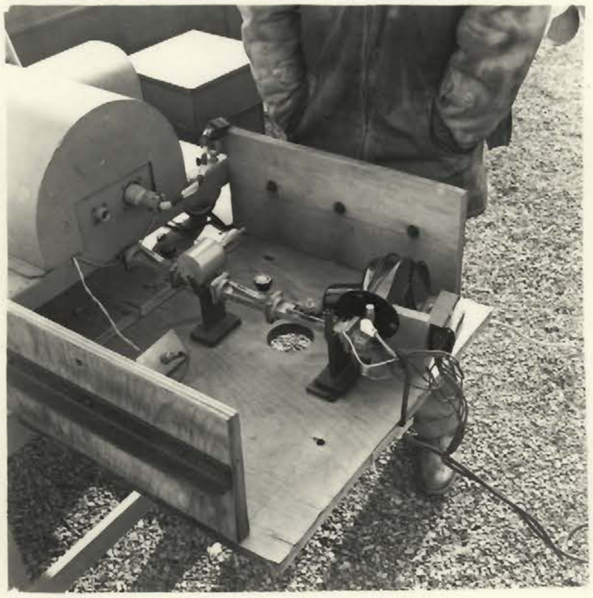


Fig. 3 Microwave Assembly

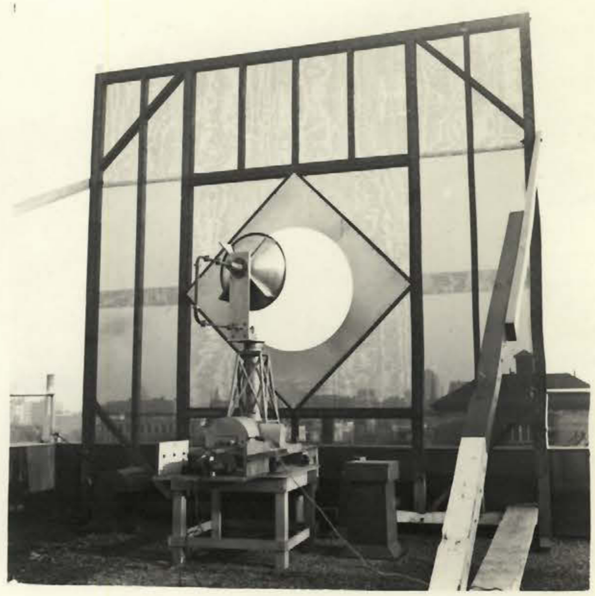


Fig. 4 Transmitter and Screen

Fig. 5 Receiver

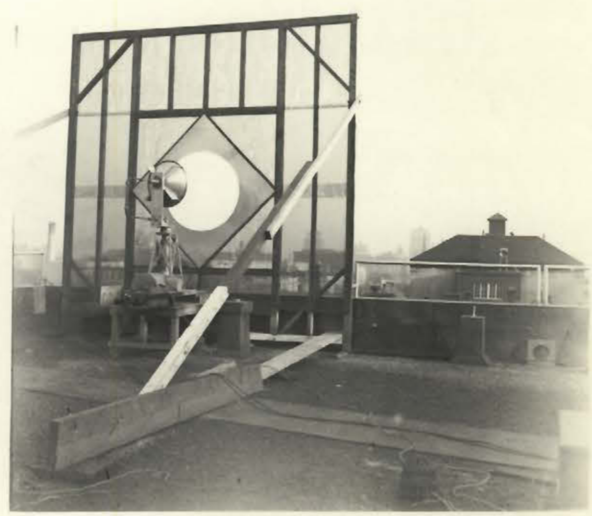


Fig. 6 Transmitter, Screen, and Receiver

III. HORNS AND WAVE GUIDE MOUTH.

The present chapter is concerned largely with the details of horn calculations, first of the unmutilated pattern, and, finally, for the mutilation. Representative horn patterns are shown in Figs. 10 to 19. The concluding section gives a descriptive account of horn patterns and their mutilation.

1. The Unmutilated Pattern: It is necessary to know both the magnitude and phase of the unmutilated field in order to apply the methods of Appendix III. The magnitude is readily obtained from the measured pattern, but the phase can be determined only by a complete calculation.

A horn is a flared wave guide which has been terminated in free space after a certain distance. The radial function for such a system is a Hankel function, the argument of which is large for a horn of considerable length. Consequently, there will be an $e^{ik\rho}$ radial phase term, and one may treat the radiation at the mouth of the horn as though it originated at a point in the throat. The lateral variation of field will depend on the horn modes. It will be assumed that all modes other than TE_{01} have been completely attenuated. Therefore the field function at the mouth of a square horn of side a is,

$$\exp\left[2\pi i(\rho-q)\right] \cos\ (\pi y_1'/a) \qquad \text{for H-plane}$$

and $$\exp\left[2\pi i(\rho-q)\right] \cos\ (\pi x_1'/a) \qquad \text{for E-plane}$$

where q is the length of the horn along its axis. Since the flare angle is assumed small one may put $\rho - q = \dfrac{x_1'^2 + y_1'^2}{2q}$. However, q differs slightly in E and H plane, so the final form of the aperture function will be written,

$$\left.\begin{array}{ll}\exp\left[\pi i({x'_1}^2/q_E + {y'_1}^2/q_H)\right] \cos\ (\pi y'_1/a) & \text{for H-Plane} \\ \exp\left[\pi i({x'_1}^2/q_H + {y'_1}^2/q_E)\right] \cos\ (\pi x'_1/a) & \text{for E-Plane}\end{array}\right\} \qquad (3.1)$$

The field at a distant point is obtained by substituting in (1.29) with approximations appropriate to $s \gg y'_1$,

$$F_H(0) = -(i/(s+b))\exp\left[2\pi i(s+b)\right]\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\exp\left[\pi i({x'_1}^2/A_E + {y'_1}^2/A_H) + 2\pi i y'_1 \sin\alpha\right] \cos\ (\pi y'_1/a)\ dx'_1 dy'_1$$

in H-plane, and

$$F_E(0) = -(i/s+b)\exp\ \left[2\pi i(s+b)\right]\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\exp\left[\pi i({x'_1}^2/A_H + {y'_1}^2/A_E) + 2\pi i y'_1 \sin\alpha\right] \cos\ (\pi x'_1/a)\ dx'_1 dy'_1$$

in E-plane, where $1/A = 1/s + 1/q$.

Integration leads to,

$$F_H(0) = \frac{-i\sqrt{A_E A_H}}{2(s+b)} \exp\left[2\pi i(s+b)\right] (C+iS)_{\delta_E}\Bigg[\exp\left[(-\pi i/2)(\delta_1+\delta_2)^2\right]\left[(C+iS)_{\delta+\delta_1+\delta_2} + (C+iS)_{\delta-\delta_1-\delta_2}\right] + \exp\left[(-\pi i/2)(\delta_1-\delta_2)^2\right]\left[(C+iS)_{\delta+\delta_1-\delta_2} + (C+iS)_{\delta-\delta_1+\delta_2}\right]\Bigg]_H \qquad \ldots\ldots.(3.2)$$

$$F_E(0) = \frac{-i\sqrt{A_E A_H}}{2(s+b)} \exp\left[2\pi i(s+b)\right]\ \exp\left[(-\pi i/2)(\delta^2_{1_H} + \delta^2_{2_E})\right] \left[(C+iS)_{\delta+\delta_1} + (C+iS)_{\delta-\delta_1}\right]_H \left[(C+iS)_{\delta+\delta_2} + (C+iS)_{\delta-\delta_2}\right]_E \qquad \ldots\ldots\ldots.(3.3)$$

where $\delta = \dfrac{a}{\sqrt{2A}}$; $\quad \delta_1 = \dfrac{1}{a}\sqrt{\dfrac{A}{2}}$; $\quad \delta_2 = \sqrt{2A}\ \sin\alpha$

and $C + iS$ is the Fresnel integral with argument equal to the indicated subscript.

It has been noted above that (1.29) contains no inclination factor. There would be some justification for the $(1 + \cos\alpha)/2$ factor in the case of the horn provided the termination at the mouth did not affect the fields. Such an assumption seems unwarrented; and it would appear that wide angle fields cannot be obtained in any simple way. The 2.5

wave-length horn mentioned below had a considerable amplitude at 90° in E-plane, in agreement with Stratton-Chu theory. But the Stratton-Chu correction proved to be contrary to observation for the 5 wave-length horn at intermediate angles. Since the inclination factor affects the intensity and not the phase it is of no importance in the present work. Mutilations were referred to the observed intensity where it differed significantly from the calculated value.

2. <u>The Horn Mutilation</u>: The aperture function (3.1) must be written in exponential form. Then,

$$f(Z,\tau)\exp(iD) = 1/2(\exp(i\psi') + \exp(i\psi''))$$

Applying the transformation (A3.1) leads to the following values of the quantities in (A3.3).

<u>E-Plane</u>

$$\frac{M'}{2} = \frac{M''}{2} = \pi\left[\frac{1}{b} + \frac{1}{q} + \left[\frac{2}{b} + \frac{1}{2q} - \frac{c^2}{2b^3}\right]\tan^2\alpha\right]$$

$$L' = 2\pi\tan\alpha\left[1 - \frac{c^2}{2b^2} + \frac{Z^2}{2b^2} + \frac{Z^2}{bq}\sec^2\alpha + \frac{Z}{2ba}\right]$$

$$L'' = 2\pi\tan\alpha\left[1 - \frac{c^2}{2b^2} + \frac{Z^2}{2b^2} + \frac{Z^2}{bq}\sec^2\alpha - \frac{Z}{2ba}\right]$$

$$N' = \frac{\pi}{a} - \frac{2\pi Z\tan^2\alpha}{b}\left(1 - \frac{c^2}{2b^2}\right)$$

$$N'' = -\frac{\pi}{a} - \frac{2\pi Z\tan^2\alpha}{b}\left(1 - \frac{c^2}{2b^2}\right)$$

$$R = \frac{\pi\tan^2\alpha}{2}\left(\frac{c^2}{b^3} + \frac{1}{q}\right)$$

<u>H-Plane</u>

$$\frac{M'}{2} = \pi\left[\frac{1}{b} + \frac{1}{q} + \left(\frac{2}{b} + \frac{1}{2q} - \frac{c^2}{2b^3}\right)\tan^2\alpha + \frac{\tan\alpha\sec\alpha}{ab}\right]$$

$$\frac{M''}{2} = \pi\left[\frac{1}{b} + \frac{1}{q} + \left(\frac{2}{b} + \frac{1}{2q} - \frac{c^2}{2b^3}\right)\tan^2\alpha - \frac{\tan\alpha\sec\alpha}{ab}\right]$$

$$L' = 2\pi \tan\alpha \left[1 - \frac{c^2}{2b^2} + \frac{Z^2}{2b^2} + \frac{Z^2 \sec^2\alpha}{bq} \right] + \frac{\pi \sec\alpha}{a}$$

$$L'' = 2\pi \tan\alpha \left[1 - \frac{c^2}{2b^2} + \frac{Z^2}{2b^2} + \frac{Z^2 \sec^2\alpha}{bq} \right] - \frac{\pi \sec\alpha}{a}$$

$$N' = -\frac{2\pi Z}{b} \left[\tan^2\alpha \left(1 - \frac{c^2}{2b^2}\right) + \frac{\tan\alpha \sec\alpha}{2a} \right]$$

$$N'' = -\frac{2\pi Z}{b} \left[\tan^2\alpha \left(1 - \frac{c^2}{2b^2}\right) - \frac{\tan\alpha \sec\alpha}{2a} \right]$$

$$R = \frac{\pi \tan^2\alpha}{2} \left(\frac{c^2}{b^3} + \frac{1}{q}\right)$$

It is permissible to put $R = 0$ in all cases. It will be seen that usually $N' \doteq N'' \doteq 0$ and $M' \doteq M''$ in H-plane. Data sheets can be set up in a straightforward manner for each value of α.

The major difficulty arises in dealing with the partial zones. It is necessary to obtain $\oint_0(Z) = 1/2\pi \oint e^{iTZ\sin(\tau+\varepsilon)}\, d\tau$ where $T^2 = L^2 + N^2$ and $\tan\varepsilon = N/L$. It will be seen from Fig. 7 that the integral is in four parts and that the range of integration is determined from the intersections of the zones with the Z, τ perimeter, taking account of the angular displacement ε. Evaluation can be performed by planimeter using curves of $\cos(TZ \sin x)$ and $\sin(TZ \sin x)$, and can include the four parts in one operation. Curves were prepared for integral values of TZ ranging from 0 to 21, and interpolation made as required for the particular value of TZ.

3. Discussion of the Calculation: The dimensions of the three horns used in this work are presented in Table I. The 10λ and 5λ horns were the same as previously used by Woonton for mutilation by a rectangular aperture and designated[4] respectively AN18WON and AN4WON.

TABLE I Horn Dimensions.

	a	q_E	q_H
Horn #1 (AN18WON)	10	30.68	32.00
Horn #2 (AN4WON)	5	31.08	33.92
Horn #3	2-1/2	18.8	21.6

$s = 2380$

Patterns were taken for each horn in E-plane and H-plane with a range of mutilating aperture varying from $c = 7\text{-}1/2$ to $c = 17\text{-}1/2$. Comparison of some of these patterns with the unmutilated pattern is exhibited in Figs. 10 to 19. For those patterns where computations have been made, the computed value was assumed to be correct at $\alpha = 0$, and the patterns displaced accordingly. Otherwise the patterns have been made to coincide at $\alpha = 0$.

The unmutilated amplitude and phase have been calculated from formulas (3.2) and (3.3) and are presented in Tables II and III.

TABLE II Horn #1 (AN18WON)

α (Deg)	E-plane Db down calc.	E-plane Db down Obs.	E-plane Phase (Deg)	H-plane Db down calc.	H-plane Db down Obs.	H-plane Phase (Deg)
0	0	0	-19	0	0	-19
2.5	1.9	1.8	-33	1.2	1.3	-25
5	4.6	4.4	-82	4.3	4.6	-44
7.5	5.1	5.0	-126	8.7	8.9	-82
10	10.0	9.6	-171	12.0	12.3	-124
12.5	12.1	12.1	-255	17.1	17.4	-172
15	13.1	12.9	-298	21.3	21.6	-236
17.5	19.9	19.2	-384	23.9	23.9	-295
20	16.5	16.6	-457	29.0	29.4	-358
22.5	21.8	21.8	-498	30.2	30.5	-438
25	21.2	22.0	-619	31.7	33.6	-486

TABLE III Horn #2 (AN4WON)

α (Deg)	E-plane Db down calc	E-plane Db down Obs.	E-plane Phase (Deg)	H-plane Db down calc.	H-plane Db down Obs.	H-plane Phase (Deg)
0	0	0	-71	0	0	-71
5	2.8	2.8	-75	1.5	1.5	-72
10	14.4	14.2	-109	6.4	6.4	-86
15	12.7	13.0	-219	17.4	16.7	-107
20	16.3	17.9	-240	22.0	22.4	-207
25	23.5	20.9	-383	24.3	24.5	-237
30	17.4	17.6	-409	40.1	37.7	-325
35	28.2	28.8	-429	30.1	32.3	-404

Calculations were made as follows:

Horn #1:

	c	Range of Computation	ΔZ
E-plane	10	$\alpha = 0$ to $\alpha = 25^{\circ}$	1
"	12.5	$\alpha = 0$ to $\alpha = 10^{\circ}$	1
"	15	$\alpha = 0$ to $\alpha = 10^{\circ}$	1
"	17.5	$\alpha = 0$ to $\alpha = 25^{\circ}$	1
H-plane	10	$\alpha = 0$ to $\alpha = 25^{\circ}$	1
"	17.5	$\alpha = 0$ to $\alpha = 25^{\circ}$	1

Horn #2:

E-plane	10	$\alpha = 0$ to $\alpha = 30^{\circ}$	1/2
	10	$\alpha = 0$ to $\alpha = 30^{\circ}$	1/2

The computations at c = 17.5 used a number of short-cuts. The square periphery of the horn was replaced by an equivalent circle of radius 5.7. Also some of the terms in the expressions for L, M, and N were lumped and given approximate values.

A general feature of the mutilated patterns is that they oscillate about the unmutilated. The number of oscillations is greater in E-plane than in H-plane, and the number also increases with c. This behaviour can be explained in semi-physical terms. Reference to Tables I and II shows that the horn phases rotate in a clockwise direction with increasing α. On the other hand, reference to Fig. 8 will show that the mutilation phases rotate in a counter-clockwise direction, a trend which is found to be general. But in each case the rate of rotation is less in H-plane than in E-plane.

The calculated points are seen to be in fair agreement with observation in most cases. The factors which can conceivably cause disagreement may be listed as follows:

(a) Experimental error as discussed in Chapter II.

(b) Neglect of integrals I_2, I_3, etc.

(c) The substitution of summation for integral in Eq. (A3.5)

(d) Error in the calculated value of phase.

(e) Electromagnetic perturbation as discussed in Chapter I.

(f) The binomial approximation leading to Eq. (1.27).

All factors tend to become increasingly important with increasing α. In addition (f) is known to be insignificant only when the field is negligible outside a relatively small angle about the axis of the mutilating aperture. Nothing of a quantitative nature can be said regarding (e). Referring to (1.20) the mutilation can be written,

$$e(0) = \text{const.} \int_c^\infty \int_0^\infty G_0(\beta) \exp\left[2\pi i b \sqrt{1-\beta^2}\right] J_0(2\pi r_2 \beta)\, \beta r_2 \, d\beta \, dr_2$$

If the theory of stationary phase can be applied to this integral, one is led to the conclusion that the binomial approximation begins to break down for values of c/b greater than about 1/3. However this theory for two variables, as given by Van Kampen[20], appears to be intuitive rather than rigorous, and its validity when applied outside its physical context is not obvious.

Errors have been indicated in Figs. 16 and 17, based on an error of $\pm$ 10% in the magnitude of the mutilation and an error of $\pm$ 10° in the phase relation. The 5λ horn, to which reference is here made, has a pattern sufficiently broad to cast doubt on the validity of the calculation. It will be noted that agreement for E-plane is about as good as for any pattern, but that agreement in H-plane is poor.

4. Wave Guide Mouth: Unmutilated wave guide mouth patterns are shown in Figs. 20 and 21. The mutilated patterns in the same figures are represented with levels at $\alpha = 0$ as recorded. It is evident that this is an extreme case of mutilation. For illumination of the mutilating aperture by an isotropic point source on the axis, the Kirchhoff formula for distant axial field points can be integrated exactly, giving,

$$u = u_o \exp(2\pi ib)\left[1 - 1/2\left(1 + \frac{b}{\sqrt{b^2+c^2}}\right) \exp\left[2\pi i\left(\sqrt{b^2+c^2} - b\right)\right]\right]$$

As c is increased great fluctuations in amplitude result. Even as $c \longrightarrow \infty$ the power should still fluctuate over a range of 9.5 decibels. Observed power was in agreement with that predicted by Eq. (3.4), except for c = 20, where the observed value was 4 db less than computed.

No explanation has been offered for the ripples at the sides of the patterns.

5. Horn Patterns and Their Mutilation: The infinitely long horn has the characteristic slit patterns with minima of zero intensity. Finite length, which introduces a phase term as discussed in Section 1, tends to fill in the minima. This process is readily observable in E-plane. In H-plane patterns the amplitude distribution combined with the phase term tends to eliminate the side lobes entirely. H-plane patterns are broader than those in E-plane near the axis, but the H-plane power becomes negligible at a much smaller angle than in E-plane.

The characteristic oscillation of mutilated pattern over unmutilated will be observed in the figures shown, particularly in E-plane. The mutilations were not negligible with the largest aperture used, viz., 35 wave-lengths diameter. In E-plane they were 0.5 db, 2.4 db, and 7 db, for the 10λ, 5λ, and 2.5λ horns, respectively, at the beginning of the first side lobe, and correspondingly larger at greater angles. In H-plane the qualitative features of the pattern seemed frequently to be more radically altered than in E-plane. No attempt was made to calculate mutilations for the 2.5 wave-length horn. Its patterns all had a common feature: the main lobe was broadened. The aperture of 15 wave-lengths diameter produced large mutilations. For instance, the gain of the 10 wavelength horn was increased 60% by interposing this aperture.

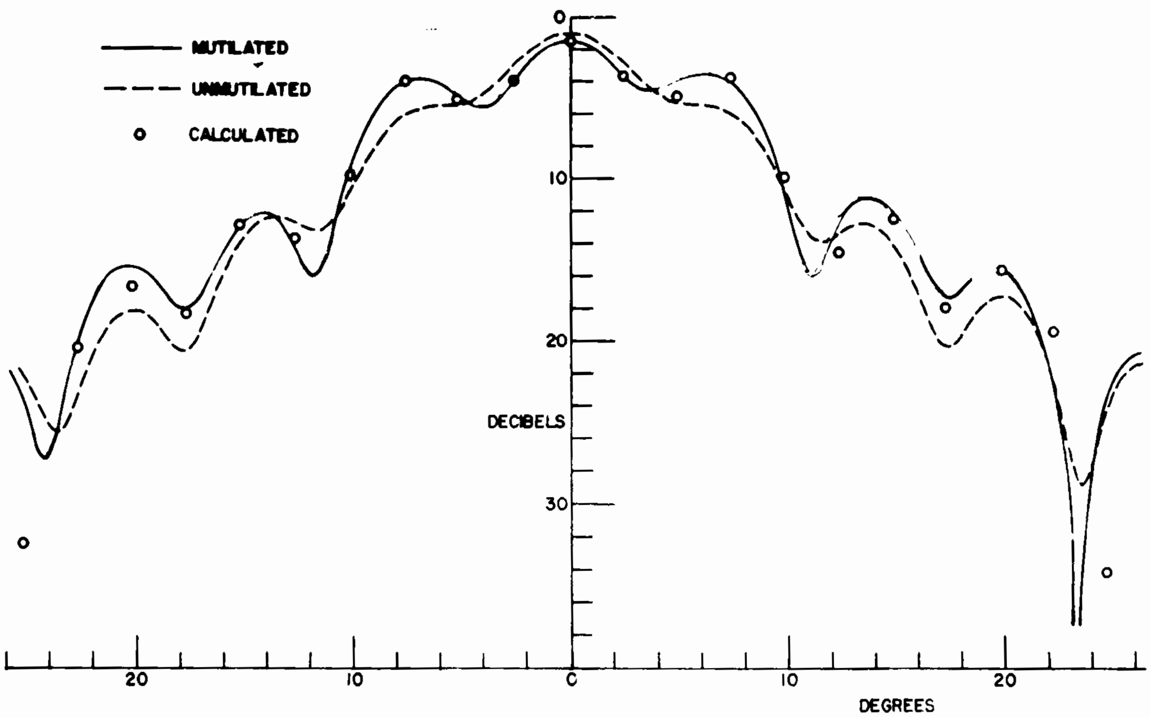


Fig. 10 Horn No. 1 c = 10 E-plane

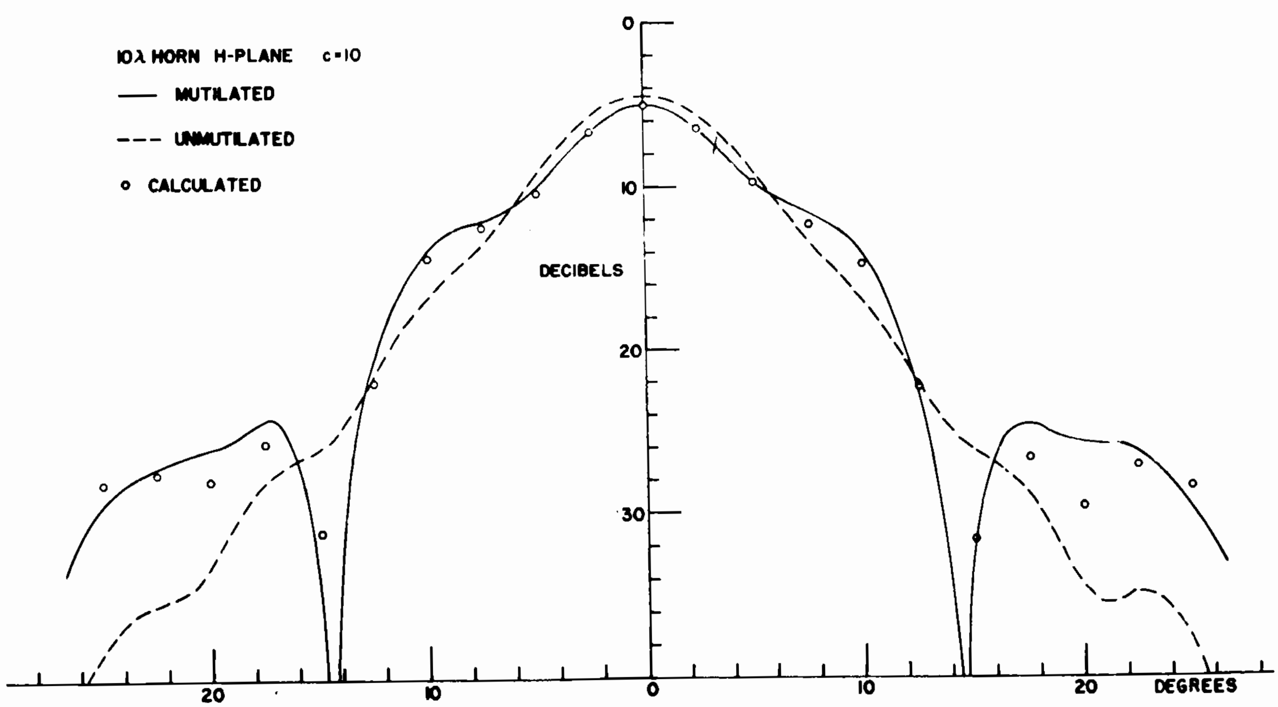


Fig. 11 Horn No. 1 c = 10 H-plane

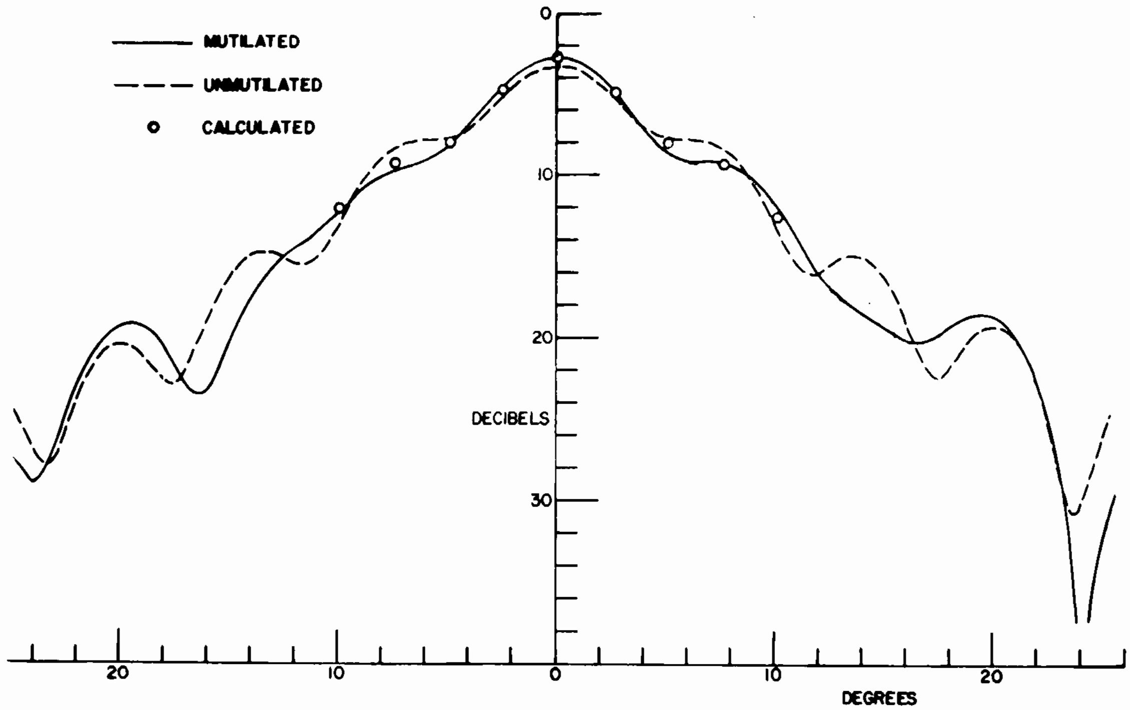


Fig. 12 Horn No. 1 c = 12.5 E-plane

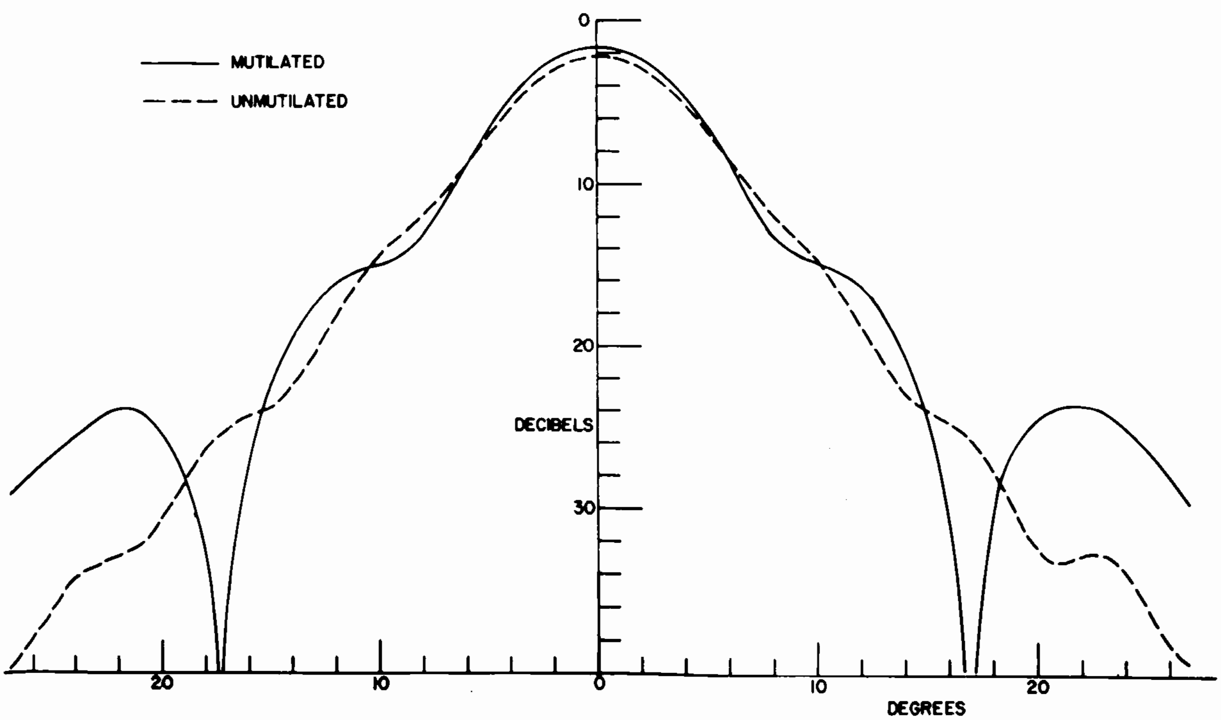


Fig. 13 Horn No. 1 c = 12.5 H-plane

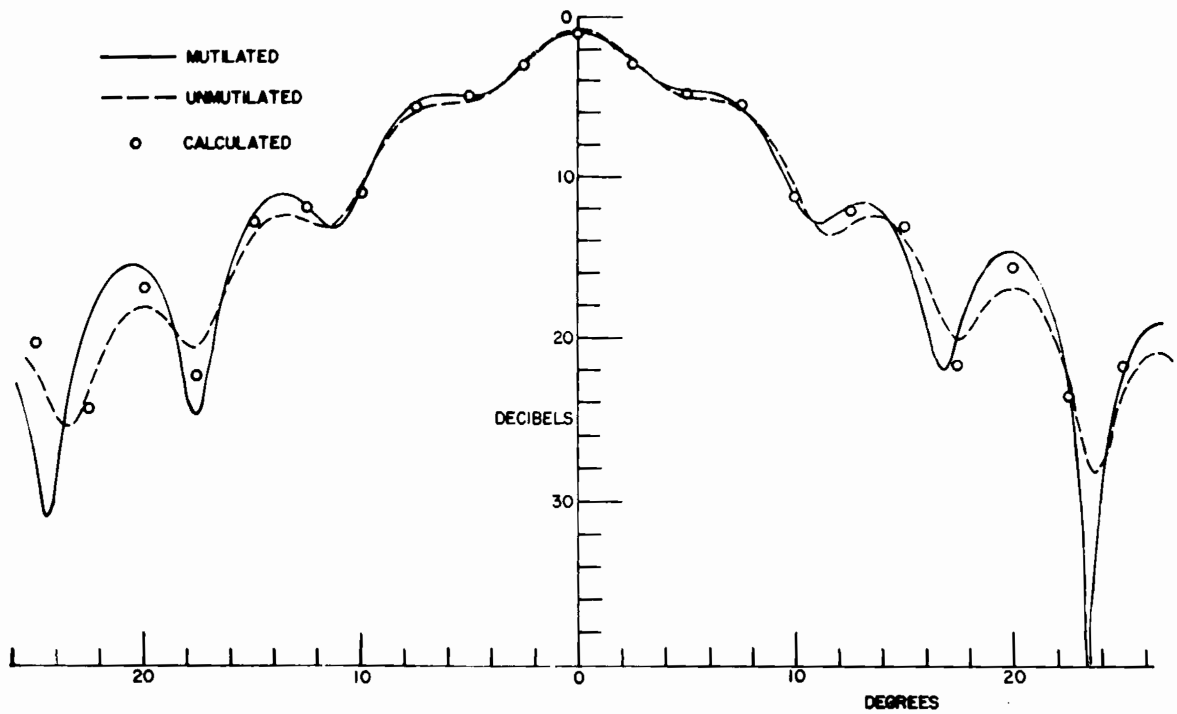


Fig. 14 Horn No. 1 c = 17.5 E-plane

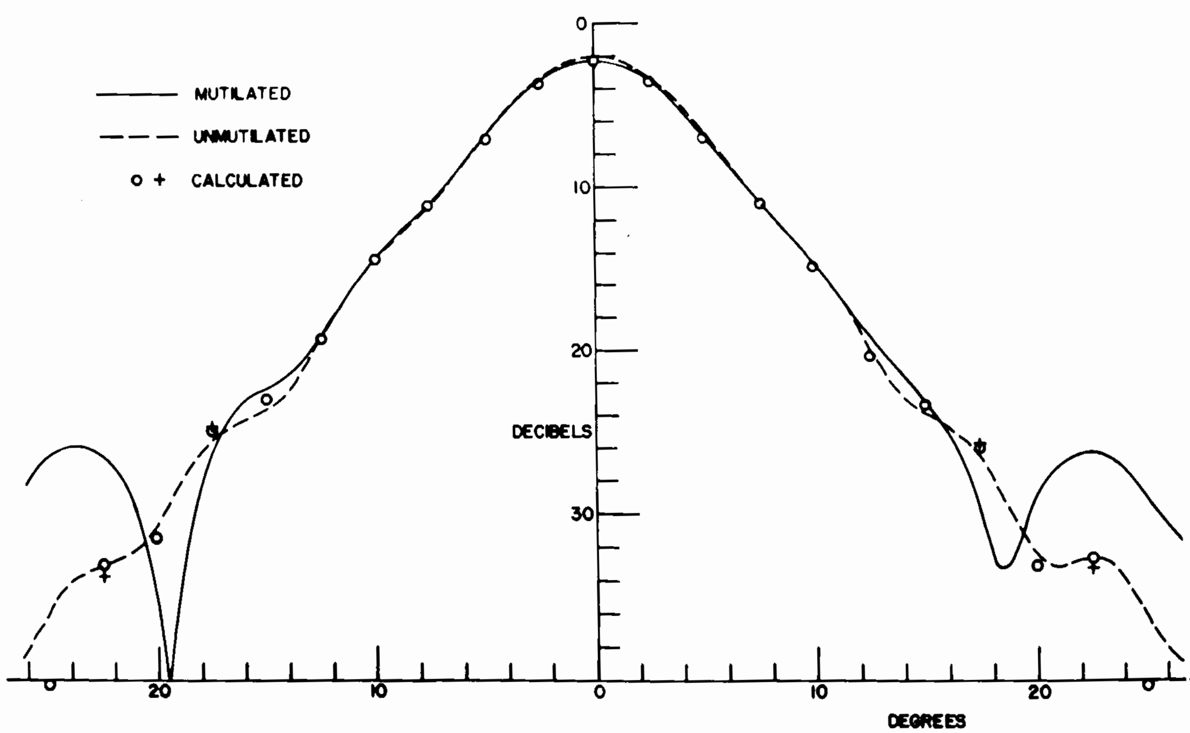


Fig. 15 Horn No. 1 c = 17.5 H-plane

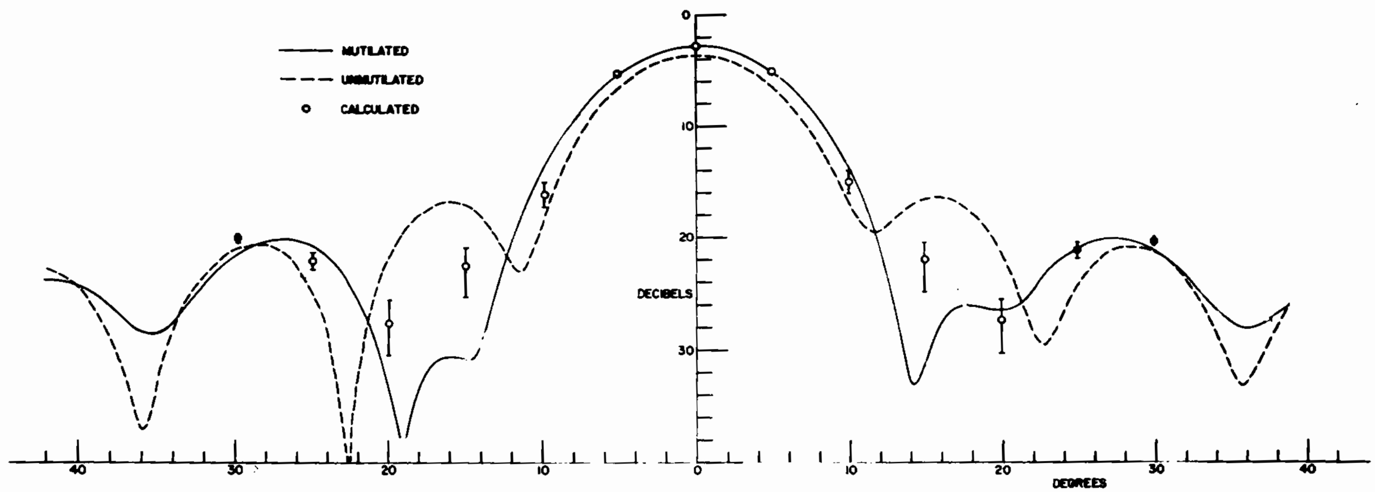


Fig. 16 Horn No. 2 c = 10 E-plane

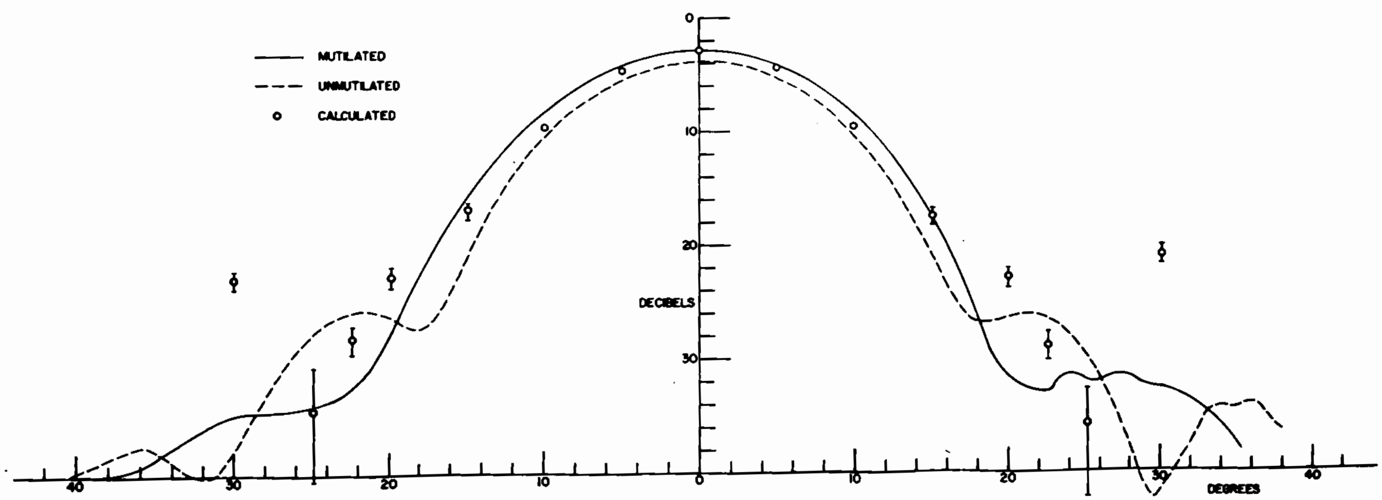


Fig. 17 Horn No. 2 c = 10 H-plane

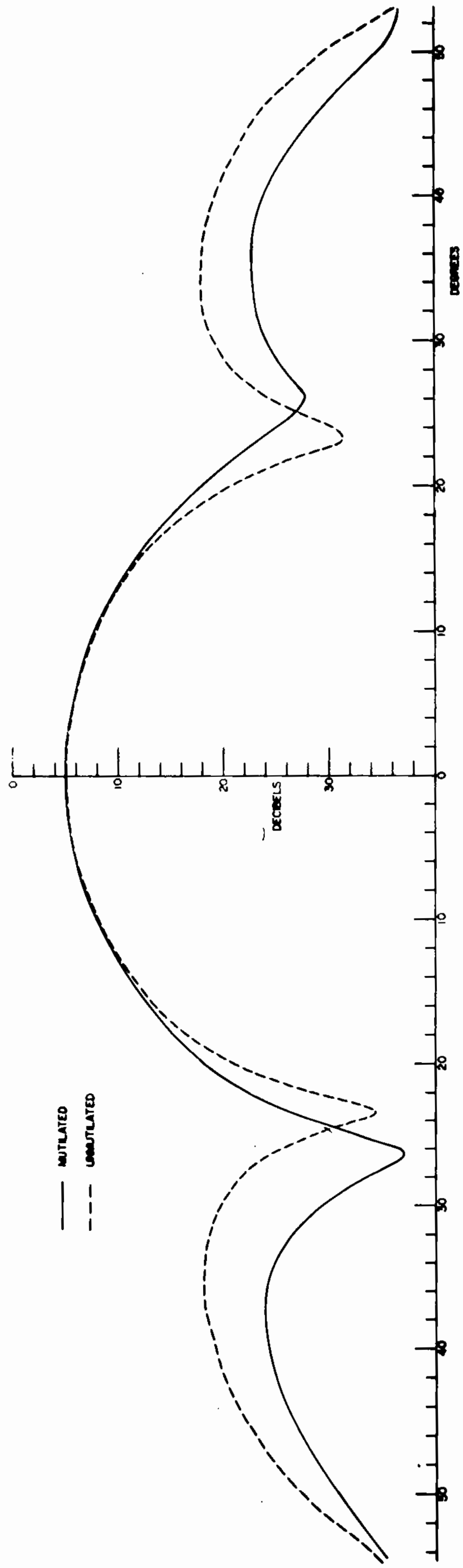


Fig. 18 Horn No. 3 c = 17.5 E-plane

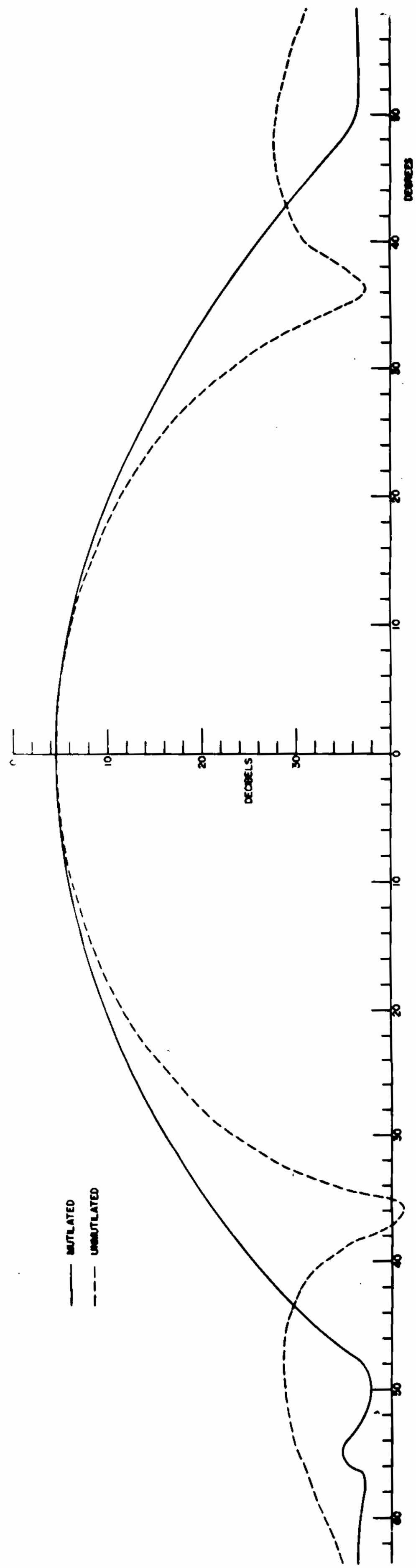


Fig. 19 Horn No. 3 c = 17.5 H-plane

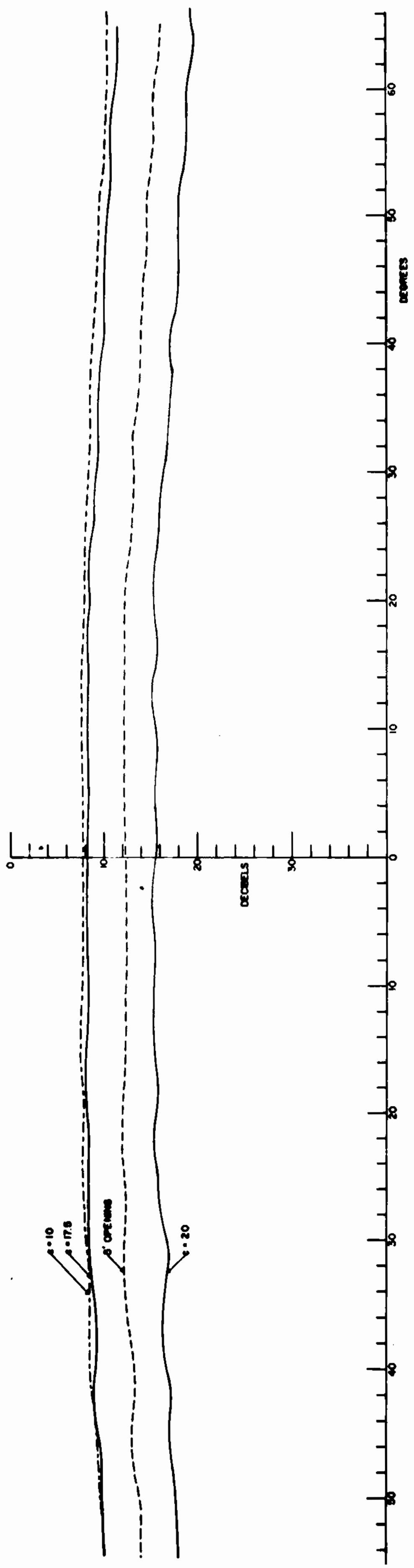


Fig. 20 Wave Guide Mouth E-plane

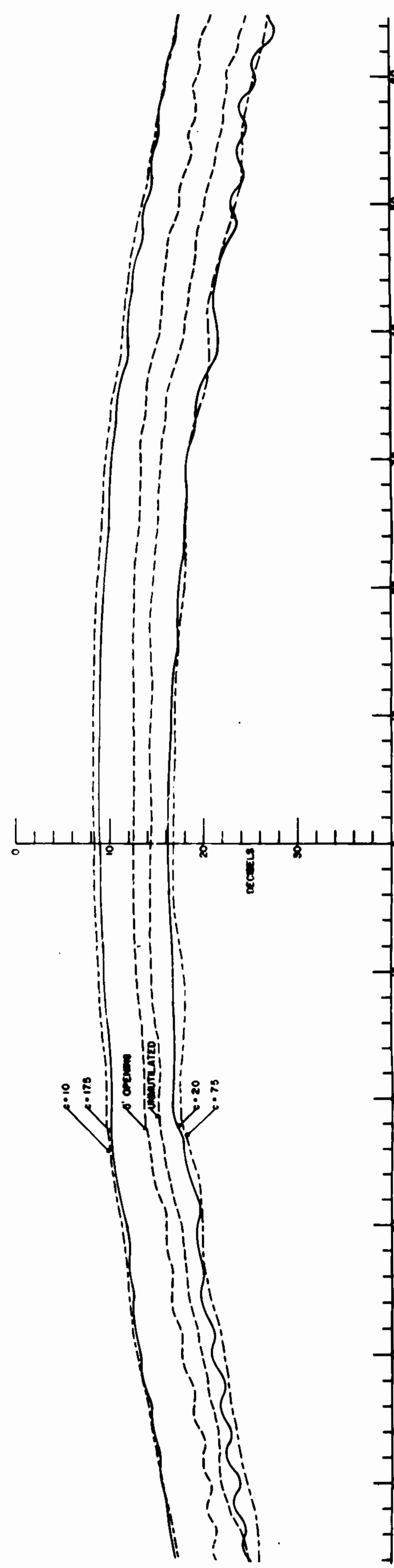


Fig. 21 Wave Guide Mouth H-plane

IV. PARABOLOIDS

The present chapter is concerned largely with the details of paraboloid calculation. First, a distant field function is derived. Then consideration is given to the calculation of the aperture function and of the mutilation. Representative paraboloid patterns are shown in Figs. 22 to 31. The concluding section gives a descriptive account of paraboloid patterns and their mutilation.

1. The Unmutilated Pattern: The principal difficulty in the analysis of paraboloid patterns lies in the fact that their aperture function is unknown. Practical feeds do not conform to the ideal of a point source: hence an unpredictable phase error is introduced. The geometry of the reflector, as discussed in Appendix IV, suggests that the function will have the form,

$$f(r_1,\ \Psi_1) = \sum_{m=0}^{\infty} A_m(\Psi_1)\ (4\lambda^2 + r^2)^{-m}\ P_m(r_1^2)$$

where λ is the focal length and P_m is a polynomial in r_1^2. The individual terms, however, cannot be integrated.

A paraboloid field should be symmetrical about two planes at right angles. Hence,

$$f(r_1,\Psi_1) = f(r_1,-\Psi_1) = f(r_1,\pi+\Psi_1) = f(r_1,\pi-\Psi_1)$$

It follows that the aperture function can be expressed in the Fourier series,

$$f(r_1,\Psi_1) = S_o(r_1) + S_1(r_1)\cos 2\Psi_1 + S_2(r_1)\cos 4\Psi_1 \ \ldots\ldots \qquad (4.1)$$

There appeared to be sufficient reason for using the simplified expansion,

$$S_m = \sum_{p=m}^{\infty} a_{mp}(r_1/a)^{2p} \qquad (4.2)$$

This form may be justified on physical grounds because the feed pattern has nearly circular symmetry near the axis, but depends increasingly on Ψ_1 with displacement from the axis. Then, from (1.29)

$$F(0) = -i/(s+b)\ \exp\left[2\pi i(s+b)\right] \int_{(1)} f(r_1, \Psi_1) \exp(2\pi i y_1' \sin\alpha)\ r_1 dr_1 d\Psi_1$$

The relation between rectangular and polar co-ordinates in the aperture depends on the plane of observation and will be given the following values:-

$$y_1' = r_1 \sin \Psi_1 \qquad \text{in H-plane}$$

$$y_1' = r_1 \sin(\Psi_1 + 45^\circ) \qquad \text{in } 45^\circ\text{-plane}$$

$$y_1' = r_1 \sin(\Psi_1 + 90\ ) \qquad \text{in E-plane}$$

Integration with respect to Ψ_1 leads to,

$$F_H(0) = -\ 2\pi i/(s+b)\ \exp\left[2\pi i(s+b)\right] \int_0^a \Big[S_o J_o(z) + S_1 J_1(z) + S_2 J_2(z) + S_3 J_3(z) + \ldots\ldots\Big] r_1 dr_1$$

in H-plane

$$F_{45}(0) = -\ 2\pi i/(s+b)\ \exp\left[2\pi i(s+b)\right] \int_0^a \Big[S_o J_o(z) - S_2 J_2(z) + S_4 J_4(z) \ldots\ldots\Big] r_1 dr_1$$

in 45°-plane, and

$$F_E(0) = -\ 2\pi i/(s+b)\ \exp\left[2\pi i(s+b)\right] \int_0^a \Big[S_o J_o(z) - S_1 J_1(z) + S_2 J_2(z) - S_3 J_3(z) \ldots\ldots\Big] r_1 dr_1$$

in E-plane, where $z = 2\pi r_1 \sin\alpha$

Substitution of (4.2) into the above equations gives an elementary integral for each term. Carrying out the integrations and using the recurrence relations where necessary, the following expressions for the distant field are obtained.

$$F_H(0) = \frac{-2\pi i a^2 \exp[2\pi i(s+b)]}{u(s+b)} \Bigg[J_1(u)\Big[a_{oo} + \frac{1}{2}a_{o1} + \frac{1}{3}a_{o2} + \frac{1}{4}a_{o4} + \frac{1}{5}a_{o5} + \cdots\cdots\Big]$$
$$+ J_3\Big[-\frac{1}{2}a_{o1} - \frac{1}{2}a_{o2} - \frac{9}{20}a_{o3} + a_{11} + \frac{3}{4}a_{12} + \frac{3}{5}a_{13} - \frac{4}{5}a_{o4} + \frac{1}{2}a_{14} + \cdots\Big]$$
$$+ J_5\Big[\frac{1}{6}a_{o2} - \frac{2}{3}a_{o4} - \frac{1}{4}a_{12} - \frac{1}{3}a_{13} + \frac{5}{21}a_{14} + a_{22} + \frac{5}{6}a_{23} + \frac{5}{7}a_{24} \quad \cdots\Big]$$
$$+ J_7\Big[-\frac{1}{20}a_{o3} - \frac{1}{15}a_{o4} + \frac{1}{15}a_{13} + \frac{1}{8}a_{14} - \frac{1}{6}a_{23} - \frac{1}{4}a_{24} + \frac{7}{8}a_{34} + a_{33}\Big]$$
$$+ J_9\Big[\frac{2}{105}a_{o4} + \frac{13}{42}a_{14} + \frac{1}{28}a_{24} - \frac{1}{7}a_{34} + a_{44} + \cdot\cdot\Big]\cdots\Bigg] \qquad (4.3)$$

$$F_E(0) = \frac{-2\pi i a^2 \exp[2\pi i(s+b)]}{(s+b)} \cdot \frac{1}{u}$$
$$\Bigg[J_1\Big[a_{oo} + \frac{1}{2}a_{o1} + \frac{1}{3}a_{o2} + \frac{1}{4}a_{o3} + \frac{1}{5}a_{o4} \cdots\Big]$$
$$+ J_3\Big[-\frac{1}{2}a_{o1} - \frac{1}{2}a_{o2} - \frac{9}{20}a_{o3} - \frac{4}{5}a_{o4} - a_{11} - \frac{3}{4}a_{12} - \frac{3}{5}a_{13} - \frac{1}{2}a_{14} \cdots\cdots\Big]$$
$$+ J_5\Big[\frac{1}{6}a_{o2} - \frac{2}{3}a_{o4} + \frac{1}{4}a_{12} + \frac{1}{3}a_{13} - \frac{5}{21}a_{14} + a_{22} + \frac{5}{5}a_{23} + \frac{5}{7}a_{24} \cdots\Big]$$
$$+ J_9\Big[\frac{2}{105}a_{o4} - \frac{13}{42}a_{14} + \frac{1}{28}a_{24} + \frac{1}{7}a_{34} - a_{14} \cdots\cdots\cdots\Big] + \cdots\Bigg] \qquad (4.4)$$

$$F_{45}(0) = \frac{-2\pi i a^2 \exp[2\pi i(s+b)]}{(s+b)} \cdot \frac{1}{u}$$
$$\Bigg[J_1(u)\Big[a_{oo} + \frac{1}{2}a_{o1} + \frac{1}{3}a_{o2} + \frac{1}{4}a_{o3} + \frac{1}{5}a_{o4} + \cdots\cdots\cdots\Big]$$
$$+ J_3(u)\Big[-\frac{1}{2}a_{o1} - \frac{1}{2}a_{o2} - \frac{9}{20}a_{o3} - \frac{4}{5}a_{o4} \quad \cdots\cdots\cdots\Big]$$
$$+ J_5(u)\Big[\frac{1}{6}a_{o2} - \frac{2}{3}a_{o4} - a_{22} - \frac{5}{6}a_{23} - \frac{5}{7}a_{24} \cdots\cdots\cdots\Big]$$
$$+ J_7(u)\Big[-\frac{1}{20}a_{o3} - \frac{1}{15}a_{o4} + \frac{1}{6}a_{23} + \frac{1}{4}a_{24} \quad \cdots\cdots\cdots\Big]$$
$$+ J_9(u)\Big[\frac{2}{105}a_{o4} - \frac{1}{28}a_{24} + a_{44} \cdots\cdots\cdots\Big] + \cdots\cdots\Bigg] \qquad (4.5)$$

where $u = 2\pi a \sin\alpha$.

It is seen that each amplitude function has the form,

$$F(0) = \frac{\text{Const.}}{u}\Big[J_1(u) + b_3J_3(u) + b_5J_5(u) + b_7J_7(u) + b_9J_9(u) + \cdots\Big] \qquad (4.6)$$

Eq. (4.6) suggests the function which will give the best agreement with the measured power pattern provided the constants can be determined. Any power pattern will be fitted by the actual amplitude function and by its complex conjugate. The extent to which the amplitude is indeterminate on the basis of the power pattern alone is a question of some doubt. If in fitting the pattern over a limited region it is allowed to take an arbitrary number of terms of a series with no regard to their convergence, it may be shown that the constants can be chosen with a large degree of arbitrariness. On the other hand, if the amplitude function is known to have the form (4.6) with coefficients b_m forming a convergent sequence, and if b_3 is known, then an attempt to fit the succeeding coefficients with values other than the correct ones will result in a strong divergence.

In fitting Eq. (4.6) to the 14" paraboloid patterns all of the planes measured, viz., H-plane, E-plane, and 45^o-plane, were taken into account. It happened that the E-plane and 45^o-plane patterns were almost identical over the main lobe. Although the exactness of agreement in this case must be fortutitous, there was a marked tendency toward identity in the E and 45^o patterns of all paraboloids for which measurements were taken. Reference to Eqs. (4.3), (4.4) and (4.5) shows that,

$b_3^{45} = 1/2(b_3^H + b_3^E)$. If $(b_3^{45})^2 = (b_3^E)^2$, and $b_3^E \neq b_3^H$, then

$$b_3^{45} = b_3^{E*}$$

$$\text{and,}\quad (b_3^H)/2 = b_3^{E*} - (b_3^E)/2$$

$$= (b_3^{E*})/2 - \mathrm{Im}(b_3^E)$$

$$\text{Therefore,}\quad \mathrm{Re}(b_3^H) = \mathrm{Re}(b_3^E) \tag{4.7}$$

$$\text{and}\quad \mathrm{Im}(b_3^H) = -3\mathrm{Im}(b_3^E)$$

The measured pattern and (4.7) make it possible to determine b_3.

An important feature of the expansion (4.6) is now apparent: if the coefficients are of reasonable magnitude the terms become effective <u>successively</u>, and it is possible to fit the coefficients in order, one at a time. It is tedious, but quite possible, to determine the constants up to and including b_9, and the process could presumably be continued.

The distant field formulas so obtained were as follows:

<u>E-plane</u>:

$$\frac{1}{u}\Big[J_1(u) + (0.81 + 0.25i)\,J_3(u) + (0.026 - 0.097i)\,J_5(u) + (0.26 + 1.48i)\,J_7(u) - (0.4 + 0.8i)\,J_9(u) \quad \ldots\ldots\Big] \tag{4.8}$$

<u>H-plane</u>:

$$\frac{1}{u}\Big[J_1(u) + (0.81 - 0.75i)\,J_3(u) + (0.10 + 0.39i)\,J_5(u) + (0.86 - 1.23i)\,J_7(u) + (1.06 + 1.06i)\,J_9(u) + \ldots\ldots\Big] \tag{4.9}$$

Power and Phase based on these formulas are given below in Table IV with observed power also indicated for comparison. The foregoing analysis is not capable of distinguishing (4.8) and (4.9) from their complex conjugates. The mutilations, as discussed below, require the forms given

TABLE IV 14" PARABOLOID

α (Deg)	E-plane Db Down Calc.	E-plane Db Down Obs.	E-plane Phase (Deg)	H-plane Db Down Calc.	H-plane Db Down Obs.	H-plane Phase (Deg)
0	0	0	0	0	0	0
1	0.3	0.3	0	0.3	0.3	-1
2	1.3	1.3	1	1.3	1.2	-3
3	2.9	2.8	2	2.8	2.8	-7
4	5.3	5.4	5	5.2	5.0	-13
5	8.8	8.7	8	8.3	8.3	-22
6	13.6	13.6	16	12.1	12.3	-37
7	20.3	20.4	37	16.4	16.7	-60
8	25.0	24.3	94	20.5	20.5	-94
9	23.1	22.8	128	23.5	22.8	-128
10	22.4	22.6	129	27.1	26.4	-158
11	23.1	23.1	118	33.8	33.8	-199
12	23.3	23.1	98	35.7	36.6	-306
13	23.0	22.9	81	29.9	30.0	-348
14	23.1	23.3	73	28.1	28.6	-369
15	24.9	25.0	70	29.7	30.2	-390
16	29.9	28.5	75	34.0	34.8	-435

2. The Paraboloid Mutilation: The mutilation is given by

$$e(0) = bK\sum{}'\sum_i f(Z_i)\oint(Z_i)\exp(iMZ_i^2/2)\ \delta W_2$$

where $\sum{}'$ indicates the summation of terms $I_1' + I_1'' + I_1''' \ldots\ldots$ as given in Appendix III. Deformation of periphery will not be taken into account. Then for angles not too great,

$$\begin{aligned}\sum{}' f(Z)\oint(Z) = &\left[a_{oo} + a_{o1}(Z/a)^2 + a_{o2}(Z/a)^4 + \ldots\ldots\right]J_o(LZ)\\ &+\left[a_{11}(Z/a)^2 + a_{12}(Z/a)^4 + a_{13}(Z/a)^6 + \ldots\ldots\right]J_2(LZ)\\ &+\left[a_{22}(Z/a)^4 + a_{23}(Z/a)^6 + a_{24}(Z/a)^8 + \ldots\ldots\right]J_4(LZ)\\ &+\ \ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

in H-plane, with corresponding expressions in E-plane. Therefore,

$$\begin{aligned}e_H(0) = bK\Big[&P_{oo}\beta_{oo}a_{oo} + P_o\left[\beta_{o1}a_{o1} + \beta_{o2}a_{o2} + \beta_{o3}a_{o3}\ldots\ldots\right]\\ &+ P_1\left[\beta_{11}a_{11} + \beta_{12}a_{12} + \beta_{13}a_{13} + \ldots\ldots\right]\\ &+ P_2\left[\beta_{22}a_{22} + \beta_{23}a_{23} + \beta_{24}a_{24} + \ldots\ldots\right]\\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\Big]\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}P_{oo}\beta_{oo} &= \sum J_o(LZ)\exp(iMZ^2/2)\ \delta W_2\\ P_o\beta_{o1} &= \sum J_o(LZ)(Z/a)^2\exp(iMZ^2/2)\ \delta W_2\\ P_o\beta_{o2} &= \sum J_o(LZ)(Z/a)^4\exp(iMZ^2/2)\ \delta W_2\\ P_1\beta_{11} &= \sum J_2(LZ)(Z/a)^2\exp(iMZ^2/2)\ \delta W_2\\ &\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned} \tag{4.11}$$

It will be assumed that the β's are independent of α. It is found that, excluding β_{oo}, this is a fairly good approximation. Of the coefficients, $\beta_{m,m}$, $\beta_{m,m+1}$, $\beta_{m,m+2}$, etc., only their ratio is significant. Hence it will be convenient to take,

$$\beta_{oo} = \beta_{o1} = \beta_{12} = \beta_{23} = \beta_{33} = \beta_{44} = 1$$

and

$$\begin{aligned}
&\beta_{oo}a_{oo} = h_{oo} \\
&\beta_{o1}a_{o1} + \beta_{o2}a_{o2} + \beta_{o3}a_{o3} + \cdots\cdots = h_o \\
&\beta_{11}a_{11} + \beta_{12}a_{12} + \beta_{13}a_{13} + \cdots\cdots = h_1 \\
&\beta_{22}a_{22} + \beta_{23}a_{23} + \beta_{24}a_{24} + \cdots\cdots = h_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{4.12}$$

then

$$e_H(0) = bK\left[P_{oo}h_{oo} + P_oh_o + P_1h_1 + P_2h_2 + \cdot\cdot\right]\ldots\ldots \tag{4.13}$$

and

$$e_E(0) = bK\left[P_{oo}h_{oo} + P_oh_o - P_1h_1 + P_2h_2 - \ldots\right]\ldots\ldots \tag{4.14}$$

Of the quantities appearing in (4.13) and (4.14), b and K are known. The P's are computed by graphical integration as discussed in Appendix III, and the h's are constants related to the aperture function which must be chosen to fit the mutilated patterns. The possibility of such a choice depends on the validity of the mutilation theory, and more particularly, on the validity of the paraboloid analysis.

An estimate of the β's is readily obtainable. Therefore, equations may be set up for the a's, the constants giving the aperture function, provided they converge properly. These equations, in terms of experimentally determined quantities, are:

$$a_{oo} + \frac{1}{2}a_{o1} + \frac{1}{3}a_{o2} + \frac{1}{4}a_{o3} + \frac{1}{5}a_{o4} + \cdots\cdots\cdots = 1$$

$$-\frac{1}{2}a_{o1} - \frac{1}{2}a_{o2} - \frac{9}{20}a_{o3} - \frac{4}{5}a_{o4} \cdots\cdots\cdots = \frac{b_3^E + b_3^H}{2}$$

$$\frac{1}{6}a_{o2} - \frac{2}{3}a_{o4} + a_{22} + \frac{5}{6}a_{23} + \frac{5}{7}a_{24} \cdots\cdots\cdots = \frac{b_5^E + b_5^H}{2}$$

$$-\frac{1}{20}a_{o3} - \frac{1}{15}a_{o4} - \frac{1}{6}a_{23} - \frac{1}{4}a_{24} \cdots\cdots\cdots = \frac{b_7^E + b_7^H}{2}$$

$$\frac{2}{105}a_{o4} + \frac{1}{28}a_{24} - a_{44} \cdots\cdots\cdots = \frac{b_9^E + b_9^H}{2} \qquad (4.15)$$

$$a_{oo} = h_{oo}$$

$$\beta_{o1}a_{o1} + \beta_{o2}a_{o2} + \beta_{o3}a_{o3} + \beta_{o4}a_{o4} + \cdots\cdots\cdots = h_o$$

$$\beta_{22}a_{22} + \beta_{23}a_{23} + \beta_{24}a_{24} + \cdots\cdots\cdots = h_2$$

$$a_{44} + \ldots = h_4$$

and,

$$a_{11} + \frac{3}{4}a_{12} + \frac{3}{5}a_{13} + \frac{1}{2}a_{14} + \cdots\cdots\cdots = \frac{b_3^H - b_3^E}{2}$$

$$-\frac{1}{4}a_{12} - \frac{1}{3}a_{13} + \frac{5}{21}a_{14} \cdots\cdots\cdots = \frac{b_5^H - b_5^E}{2}$$

$$\frac{1}{15}a_{13} + \frac{1}{8}a_{14} + \frac{7}{8}a_{34} + a_{33} \cdots\cdots\cdots = \frac{b_7^H - b_7^E}{2}$$

$$\frac{13}{42}a_{14} - \frac{1}{7}a_{34} + \ldots .n. \cdots\cdots\cdots = \frac{b_9^H - b_9^E}{2}$$

$$\beta_{11}a_{11} + \beta_{12}a_{12} + \beta_{13}a_{13} + \beta_{14}a_{14} \cdots\cdots\cdots = h_1$$

$$\beta_{33}a_{33} + \beta_{34}a_{34} + \cdots\cdots\cdots = h_3 \qquad (4.16)$$

A solution of these equations makes it possible to determine the constants for the 45^o-plane pattern, thereby providing a check with observation.

Another condition is imposed on the aperture function by taking observations with the paraboloid focussed for maximum signal strength.

This condition is expressed in the following equation, when the focal length equals a/2,

$$\text{Im}\left[(1 - \ln 2)\, a_{oo} + (-1/2 + \ln 2)\, a_{o1} + (5/6 - \ln 2) a_{o2} + (-7/12 + \ln 2) a_{o3} + (47/60 - \ln 2)\, a_{o4} + \cdots\cdots\right] = 0 \qquad (4.17)$$

Eq. (4.17) is derived in appendix IV.

3. <u>Discussion of the Calculation</u>: Detailed calculations were made for only one paraboloid, that of diameter 14 in, for which the results of the unmutilated calculation were given in Section 1.

The double dipole feed used in all paraboloid observations had a significant back radiation. It was roughly a constant over the range of observation, and had a power level of the order 35 db below that of the peak of the paraboloid. Its source can be located approximately, but not exactly at the dipole. The back radiation was considerably influenced by the adjustment of the mount and by the placing of absorbing materials.

This radiation does not affect the form of Eq. (4.6), but will modify the constants. The change in the constants can be calculated from the Neumann expansion,

$$1 = 2/u\left[J_1(u) + 3J_3(u) + 5J_5(u) + 7J_7(u) + \cdots\cdots\cdots\right]$$

The effect of the back radiation on the mutilated pattern was simplified for the measurements here considered due to the fact that the double dipole was approximately on the axis of rotation. The feed radiation could, therefore, be treated on the basis of Chapter III, Section 4, and Eq. (3.4) applied. Considering the fact that the source was not exactly localized, it was not considered necessary to adhere strictly to Eq. (3.4). u_o was given the value 0.0050 based on normalizing the maximum amplitude to

0.5000, thereby placing the power at 40 db down, and the phase was retarded by 55°. With these ad hoc modifications Eq. (3.4) was applied exactly to all apertures in both planes.

The determination of all of the h's according to the analysis of Section 2 has not been carried out. A serious attempt was made to do so on the basis of H-plane patterns with $c = 7.5$, $c = 10$, $c = 12.5$, and $c = 17.5$. This failure can be ascribed, first, to the many uncertain factors involved. And, secondly, to the fact that the higher constants, h_1, h_2, etc., have a rather small effect within the angular range under consideration. It is believed that it may be possible to determine these constants by working with both E-plane and H-plane patterns. The determination should be easier when applied to paraboloids having a greater focal length. For, then, the series should converge more rapidly.

Calculations were based on h_{oo} and h_o alone, the other constants being taken equal to zero. They were chosen to fit in H-plane for $c = 7.5$, $c = 10$, and $c = 12.5$, and to be consistent with Eqs. (4.15) and (4.17). These values were,

$$h_{oo} = 2.0/0^{\circ}$$
$$h_o = 2.05/172^{\circ}$$

They were used in the computations for all other patterns, and such agreement with experiment as was obtained is a measure of the validity of the entire work. It is this calculation for the mutilated patterns which indicates the direction of phase rotation, a property of the unmutilated field which cannot be determined from the power pattern alone. The agreement of the calculated points with the observed patterns appears, further, to confirm the correctness of the procedure leading to the formula for the unmutilated patterns, viz., Eq. (4.6). The comparison of mutilated with unmutilated patterns is exhibited in Figs. 22 to 31 with calculated points also indicated.

4. Paraboloid Patterns and Their Mutilation: The greater part of the energy in a paraboloid field is concentrated in the main lobe which corresponds to the geometrical beam. If the aperture were uniformly illuminated there would be a series of well defined side lobes. The taper which is normally present tends to reduce this side structure. It is further modified by the phase and amplitude characteristics of the feed in use, and by the focussing. The paraboloids used in this work had an E-plane side lobe at about 22 db down, which was sometimes broad. In H-plane the side structure was about 28 db down. Some paraboloids had a wide angle background with some structural features but no clear-cut lobes. Occasionally this background was more than 38 db down and was not recorded.

The mutilation of the main lobe was found generally to be relatively small for 30λ, 35λ, and 40λ apertures. This observation was not invariably true, as can be seen in Fig. 27. The mutilation in this case was mainly due to back radiation from the feed. As with horns, mutilation was greater for smaller radiators. All apertures produced considerable mutilation of the side structure except when the larger paraboloids were used with the 40λ aperture. It was noted that wide angle characteristics were fairly well preserved in patterns mutilated by the larger apertures. The average wide angle background was increased by the mutilation except for the 15λ aperture which sometimes decreased it. A consistent feature of the 12 in and 14 in paraboloids, but not of the others, was a prominent side lobe which was generated by the mutilation.

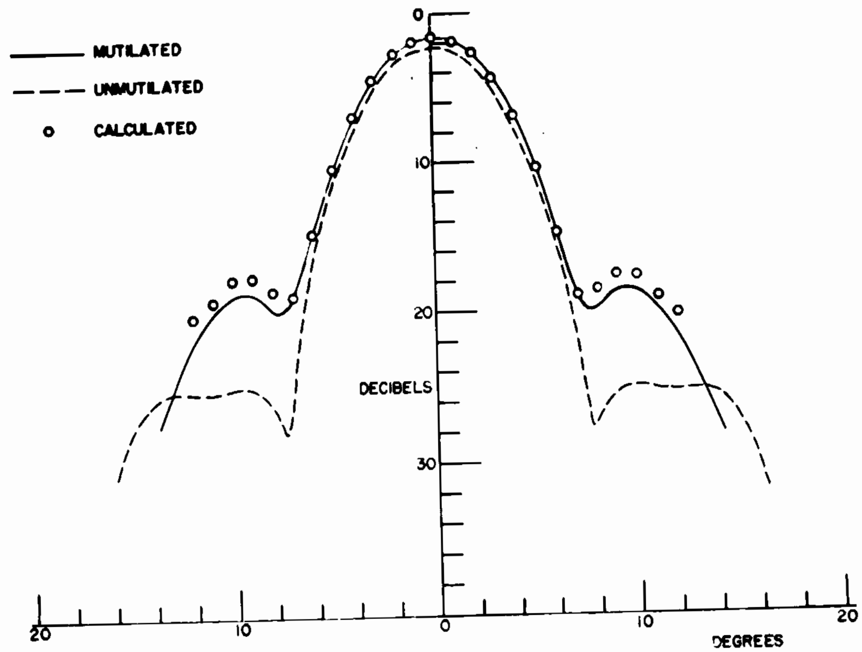


Fig. 22 14" Paraboloid c = 7.5 E-plane

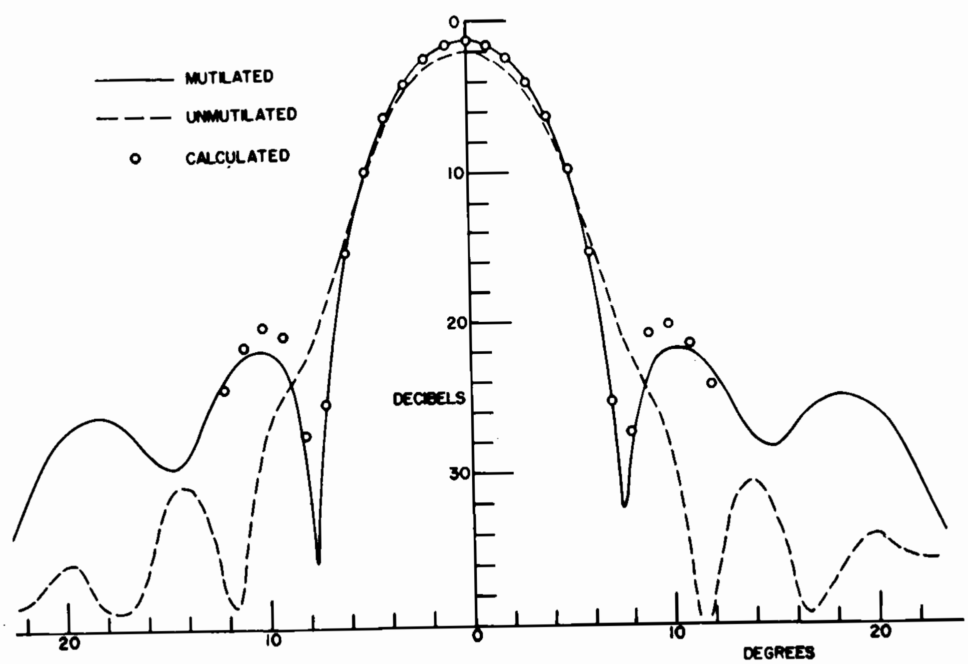


Fig. 23 14" Paraboloid c = 7.5 H-plane

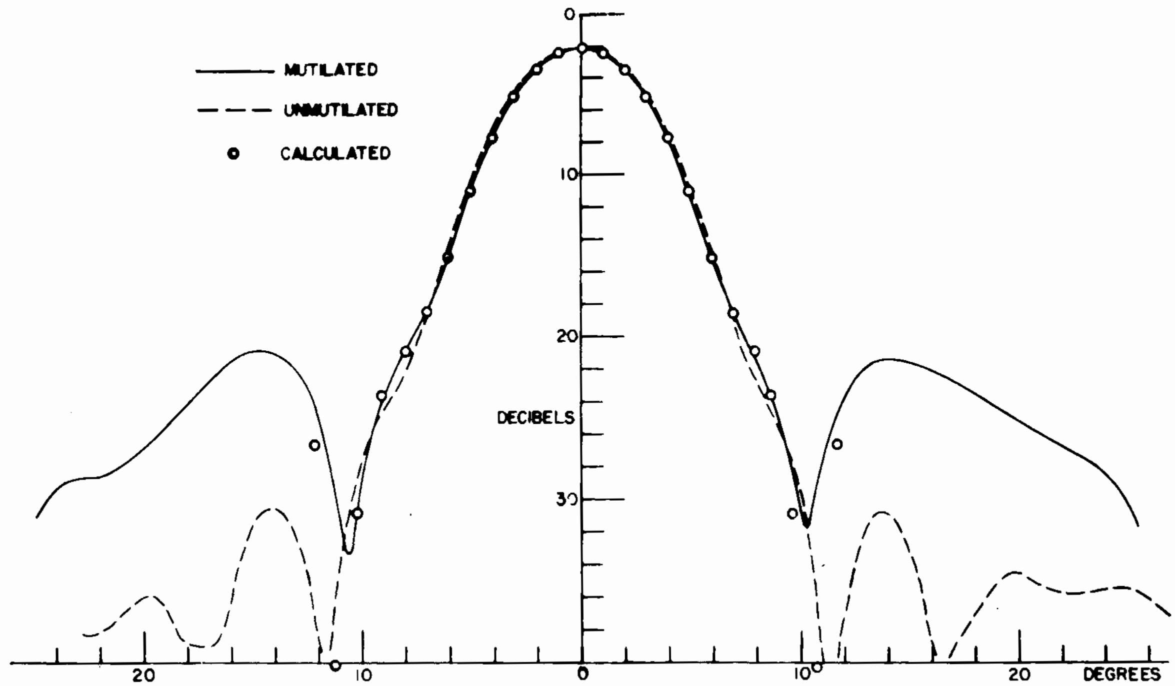


Fig. 24 14" Paraboloid c = 10 H-plane

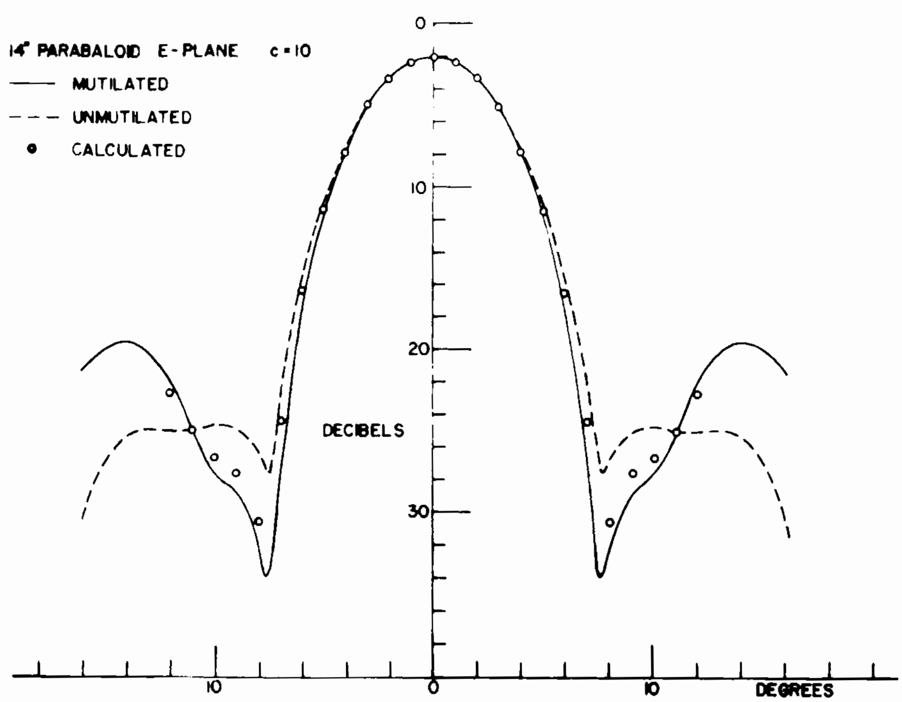
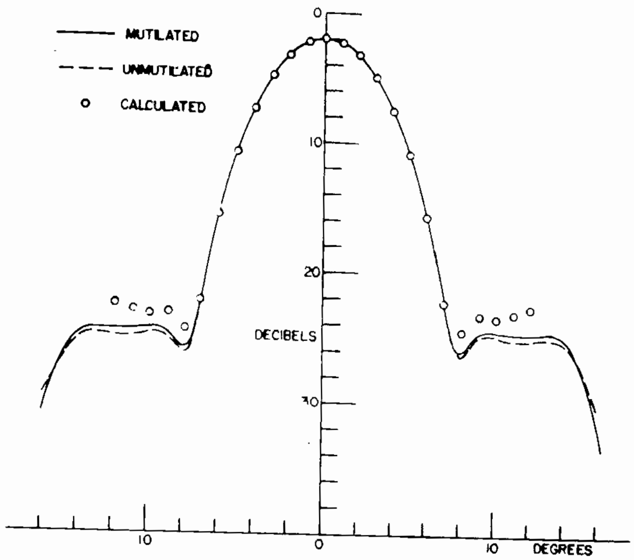


Fig. 25 14" Paraboloid c = 10 E-plane



F'g. 2ᐧ 1⁄." Paraboloid c = 17.5 ᛬-plane

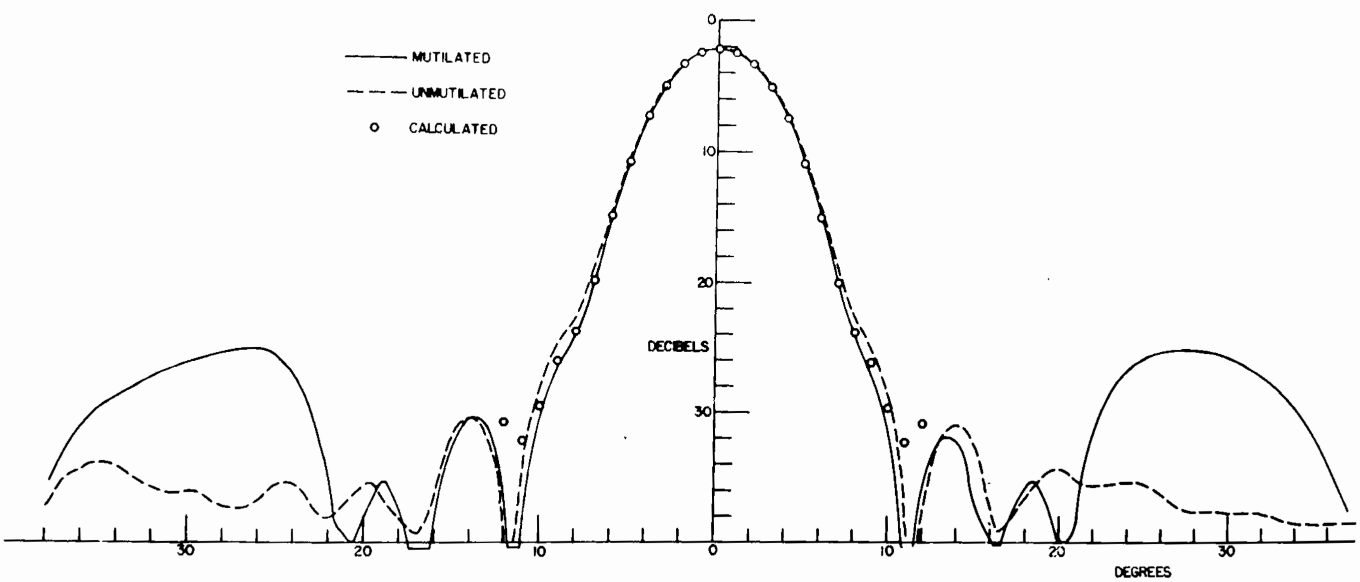


Fig. ?7 14" Paraboloid ᛬ = 17.5 H-᛬ane

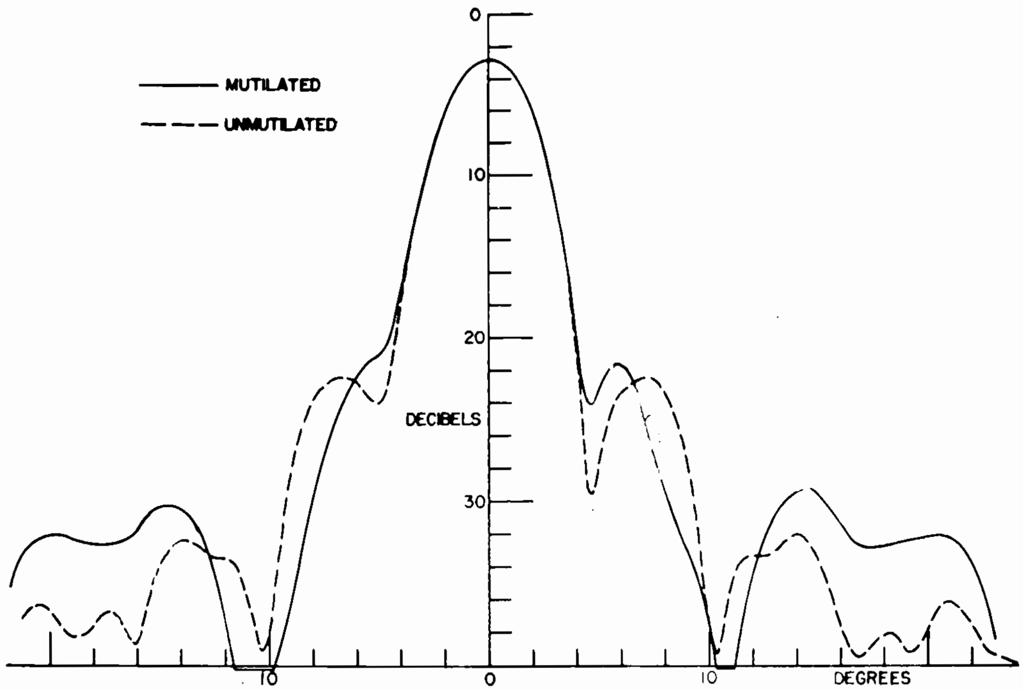


Fig. 28 24" Paraboloid c = 10 E-plane

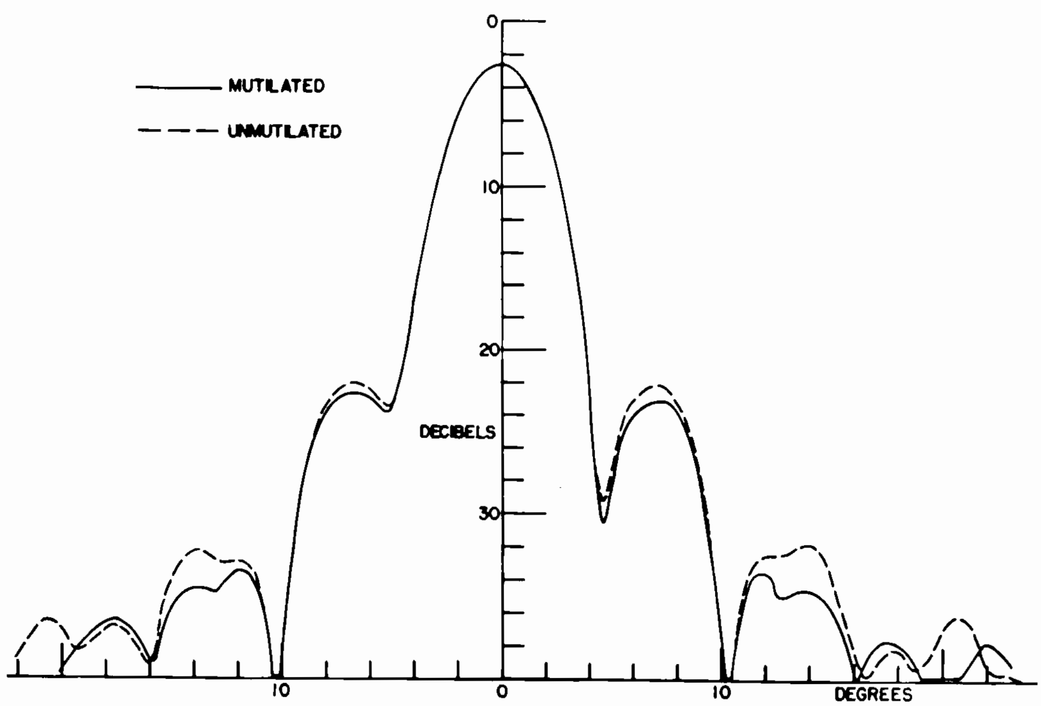


Fig. 29 24" Paraboloid c = 20 E-plane

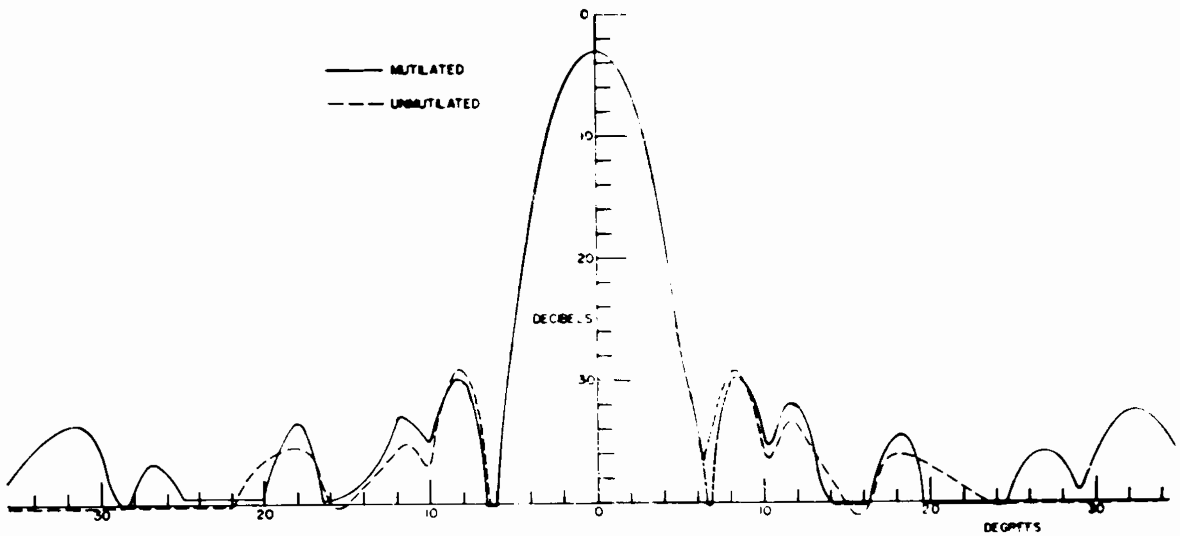


Fig. 30 24" Paraboloid c = 20 H-plane

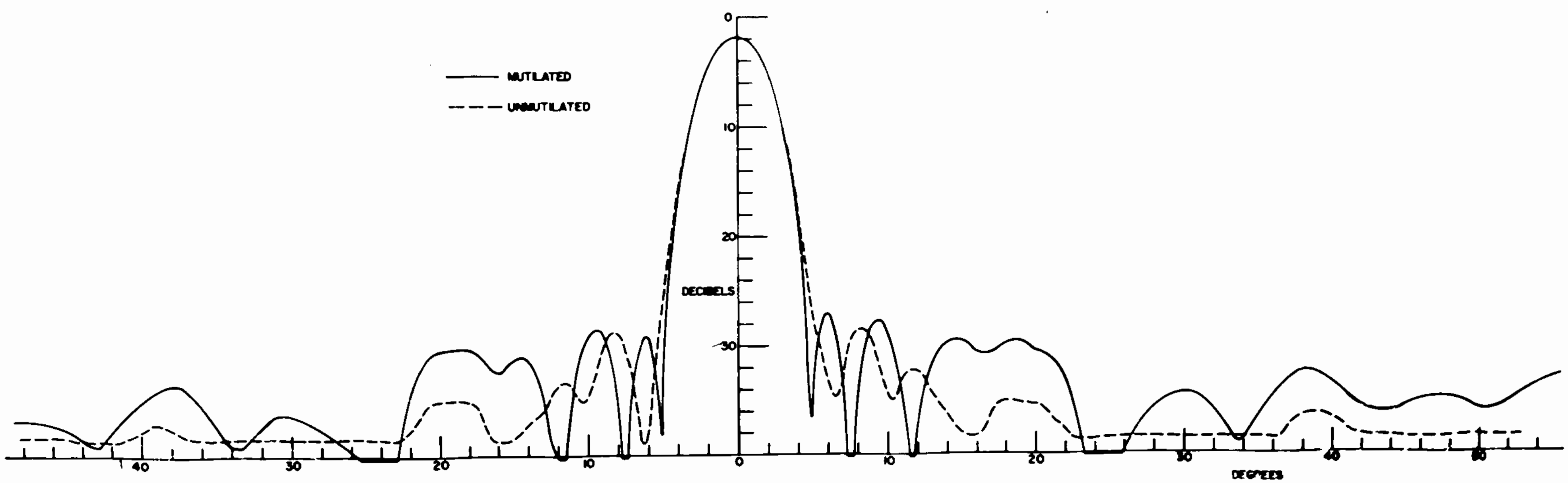


Fig. 31 24" Paraboloid c = 10 H-plane

V. OPTICAL ABERRATIONS

From the standpoint of electromagnetism the lens presents a boundary value problem, but one of such complexity that the possibility of a solution is remote. From the point of view of geometrical optics a lens is a phase shifting device. Ideally, a phase factor $\exp(i\pi r^2/q)$ is introduced where r is the distance from the axis. This concept of phase shift can be carried over into microwave optics as a limiting case subject to the additional effect of electromagnetic perturbation. The aspect of optical theory which is peculiar to microwaves is the directivity of sources due to their coherence. This, together with a somewhat different physical set-up, raises some doubt regarding the usual concepts of optical aberrations. This question will be examined in Section.2. The formal development based on a modified aberration function is presented in Section 1.

1. The aberration function, in the Nijboer classification[1], may be written,

$$W = sr^4 + u_o r^3 \sigma \cos \xi + vr^2\sigma^2 - tr^2\sigma^2 \cos 2\xi$$

where the coefficients, s, u_o, v, -t, refer respectively to spherical aberration, coma, distortion, and astigmatism. As indicated in Section 2, another term $wr^3\sigma \sin\xi$, is added, and the angle ξ is related to the angle Ψ, already used, by $\Psi = \pi/2 - \xi$. The phase shift function can now be written,

$$\begin{aligned} Y &= W + \pi r^2/q \\ &= \pi r^2/q + sr^4 + v\sigma^2 r^2 + u\sigma r^3 \sin(\Psi + \mu) + t\sigma^2 r^2 \cos 2\Psi \ldots \end{aligned} \qquad (5.1)$$

where $u^2 = w^2 + u_o^2$ and $\tan \mu = w/u_o$.

The field incident on the lens is given by (1.9), (1.19) and (1.22). The emergent field u_e will be modified by the phase shift* so that $u_e = u^i \exp(iY)$. (The subscript "2" is omitted). Then,

$$u_e = \exp\left[i\pi r^2/q + isr^4 + iv\sigma^2 r^2\right] \sum_{-\infty}^{\infty} \exp(im(\Psi + \mu))\, J_m\,(ur^3\sigma)$$

$$\sum_{-\infty}^{\infty} \exp\left[ip(2\Psi + \pi/2)\right] J_p(tr^2\sigma^2) \sum_{-\infty}^{\infty} f_n(r)\,\exp(in\Psi) \qquad (5.2)$$

The emergent field may also be written,

$$u_e = \sum_{j=-\infty}^{\infty} f_{ej}(r)\exp(ij\Psi)$$

The axial field, to which this treatment is limited, will be determined by f_{eo} alone. Therefore we choose only those terms of the above expansion for which $m + n + 2p = 0$, giving

$$f_{eo} = \exp\left[i\pi r^2/q + isr^4 + iv\sigma^2 r^2\right] \Big[f_o(r) + J_1(u\sigma r^3)\exp(i\mu)\, f_{-1}$$
$$+ J_{-1}(u\sigma r^3)\exp(-i\mu)\, f_{+1} + J_2(u\sigma r^3)\exp(2i\mu)\, f_{-2} + J_{-2}(u\sigma r^3)\exp(-2i\mu)$$
$$f_{+2} + J_1(t\sigma^2 r^2)\exp(i\pi/2)\, f_{-2} + J_{-1}(t\sigma^2 r^2)\exp(-i\pi/2) f_2 + \ldots\ldots \Big] \qquad (5.3)$$

The higher omitted terms all contain powers of σ greater than 2. These will be neglected since σ is equal to $\sin\alpha$. The bracketed term now reduces to

$$f_o(r) + J_1(u\sigma r^3)\left[\exp(i\mu)\, f_{-1} - \exp(-i\mu) f_{+1}\right] + J_2(u\sigma r^3)$$
$$\left[\exp(2i\mu)\, f_{-2} + \exp(-2i\mu)\, f_{+2}\right] + i\, J_1(t\sigma^2 r^2)\left[f_{-2} + f_{+2}\right]$$

It is readily shown from Chapter I that,

$$f_n = (-1)^{n+1} \int_{(1)} H(x_1', y_1', \alpha)\, \exp\left[in\varepsilon + (\pi i r^2)/(b + y_1' \sin\alpha)\right]$$
$$J_n\left(\frac{2\pi r B}{B + y_1' \sin\alpha}\right)\, dx_1' dy_1' \qquad (5.4)$$

---

* The inclusion of optical aberrations within the framework of mutilation theory was first suggested by Dr. D.C. Hogg.

where

$$H(x_1',y_1',\alpha) = \frac{f_1(x_1',y_1')}{b+y_1' \sin\alpha} \exp\left[\pi i\left[2b + y_1' \sin\alpha + \frac{B^2}{b+y_1' \sin\alpha}\right]\right]$$

Therefore

$$\exp(i\mu)\, f_{-1} - \exp(-i\mu)\, f_{+1} = -2i\int_{(1)} H \sin(\varepsilon-\mu) \exp(iXr^2)\, J_1(Ar)\, dx_1'dy_1'$$

$$\exp(2i\mu)\, f_{-2} + \exp(-2i\mu)\, f_{+2} = 2i\int_{(1)} H \cos 2(\varepsilon-\mu) \exp(iXr^2)\, J_2(Ar)\, dx_1'dy_1'$$

$$f_2 + f_{-2} = 2i\int_{(1)} H \cos 2\varepsilon \exp(iXr^2)\, J_2(Ar)\, dx_1'dy_1'$$

where

$$X = \frac{\pi}{b+y_1' \sin\alpha} \qquad\qquad A = \frac{2\pi B}{b+y_1' \sin\alpha}$$

For small aberrations (5.3) may be written in the approximate form,

$$\begin{aligned} f_{eo}(r) &= \exp(i\pi r^2/q)\, f_o(r) + isr^4 \exp(i\pi r^2/q)\, f_o(r) + iv\sigma^2 r^2 \exp(i\pi r^2/q)\, f_o(r) \\ &+ \frac{u\sigma r^3}{2} \exp(i\pi r^2/q)(-2i)\int_{(1)} H \sin(\varepsilon-\mu) \exp(iXr^2)\, J_1(Ar)\, dx_1'dy_1' \\ &+ \frac{(u\sigma r^3)^2}{8} \exp(i\pi r^2/q)\; 2i\int_{(1)} H \cos 2(\varepsilon-\mu) \exp(iXr^2)\, J_2(Ar)\, dx_1'dy_1' \\ &+ \frac{it\sigma^2 r^2}{2}\; 2i\int_{(1)} H \cos 2\varepsilon \exp(iXr^2)\, J_2(Ar)\, dx_1'dy_1' \end{aligned} \tag{5.5}$$

The field at a distance s will be,

$$\begin{aligned} F_a(\theta) &= 4\pi^2\int_0^c\int_0^\infty f_{eo}(r) \exp\left[2\pi is\sqrt{1-\gamma^2}\right] J_o(2\pi r\gamma)\, r\gamma d\gamma dr \\ &= -\frac{2\pi i}{s} \exp(2\pi is)\int_0^c f_{eo}(r) \exp(\pi ir^2/s)\, rdr \end{aligned} \tag{5.6}$$

At the focus $\pi/q + \pi/s = 0$, and, when the first term of (5.5) is substituted in (5.6) there results the mutilated field previously investigated. The remaining terms lead to the following aberration formulas,

$$\frac{2\pi s}{s} \exp(2\pi is)\int_{(1)} H\, D_{o5}(c,A,X)\, dx_1'dy_1' \tag{5.7}$$

$$\frac{2\pi v\sigma^2}{s} \exp(2\pi is)\int_{(1)} H\, D_{o3}(c,A,X)\, dx_1'dy_1' \tag{5.8}$$

$$- \frac{2\pi u \sigma}{s} \exp(2\pi i s) \int_{(1)} H \sin(\varepsilon - \mu) D_{14}(c,A,X)\, dx_1' dy_1' \tag{5.9}$$

$$\frac{\pi (u\sigma)^2}{2s} \exp(2\pi i s) \int_{(1)} H \cos 2(\varepsilon - \mu) D_{27}(c,A,X)\, dx_1' dy_1' \tag{5.10}$$

$$\frac{2\pi i t \sigma^2}{s} \exp(2\pi i s) \int_{(1)} H \cos 2\varepsilon\, D_{23}(c,A,X)\, dx_1' dy_1' \tag{5.11}$$

where the integrals $D_{ab}(c,A,X) = \int_0^c \exp(iXr^2)\, J_a(Ar)\, r^b dr$

are given in terms of Lommel functions in Appendix II. They have been treated differently by Nijboer (loc. cit.)

2. The microwave system under consideration will be that which has occupied the most of this thesis, namely, a circular system with both transmitter and receiver on the axis, and the transmitter capable of rotation. If the optical system were in the distant field of the transmitter, energy would be propogated radially and the only optical aberration would be spherical aberration. The system is, in practise, in the near field, and such a simplification is not valid.

Referring to Fig. 32, X,Y,Z is a co-ordinate system with OZ along the axis of the optical system, and X', Y', Z' a system attached to the transmitter which, as previously, will be supposed to rotate about OX, OX'. The optical system will be assumed to have circular symmetry. Therefore, radiation passing through at P will, from the optical point of view, suffer a phase shift which depends on r and the direction of the Poynting vector at P. If the Poynting vector is given in cylindrical co-ordinates by,

$$\vec{S} = \vec{r}_1 S_r + \vec{\xi}_1 S_\xi + k S_z$$

Then $Y = Y(r,h,g)$ where $h = S_r/S_z$ and $g = S_\xi/S_z$.

(h has no relation to the coefficients of the previous chapter.)

It is evident from symmetry that Y is even in g. Hence,

$$Y = Y(r,h,g^2)$$

Now,

$$x' = x \qquad r = \sqrt{x^2 + y^2}$$

$$y' = y \cos\alpha - z \sin\alpha \qquad \xi = \tan^{-1} x/y$$

$$z' = z \cos\alpha + y \sin\alpha$$

$$x = r \sin\xi \qquad r' = \sqrt{x'^2 + y'^2}$$

$$y = r \cos\xi \qquad \xi' = \tan^{-1} \frac{x'}{y'}$$

$$\sin\xi' = \frac{r \sin\xi}{r'}$$

$$\cos\xi' = \frac{r \cos\xi \cos\alpha}{r'} - \frac{z \sin\alpha}{r'}$$

$$\vec{r}_1' = \nabla'(r') = i' \sin\xi' + j' \cos\xi'$$

$$\vec{\xi}_1 = r'\nabla'(\xi') = i' \cos\xi' - j' \sin\xi'$$

$$i' = i = \vec{r}_1 \sin\xi + \vec{\xi}_1 \cos\xi$$

$$j' = j \cos\alpha - k \sin\alpha = \vec{r}_1 \cos\xi \cos\alpha - \vec{\xi}_1 \sin\xi \cos\alpha - k \sin\alpha$$

$$k' = k \cos\alpha + j \sin\alpha = \vec{r}_1 \cos\xi \sin\alpha - \vec{\xi}_1 \sin\xi \sin\alpha + k \cos\alpha$$

$$\vec{S} = \vec{r}_1 S_r + \vec{\xi}_1 S_\xi + k S_z$$

$$= \vec{r}_1' S_r' + \vec{\xi}_1' S' + k S_z'$$

$$= i'\left[S_r' \sin\xi' + S_\xi' \cos\xi'\right] + j'\left[S_r' \cos\xi' - S_\xi' \sin\xi'\right] + k' S_z'$$

Thus, finally, $\vec{S} = \vec{r}_1 \Big[S_r' (\sin\xi \sin\xi' + \cos\xi \cos\xi' \cos\alpha)$

$\quad + S_\xi'(\sin\xi \cos\xi' - \cos\xi \sin\xi' \cos\alpha) + S_z' \cos\xi \sin\alpha\Big]$

$+ \vec{\xi}_1 \Big[S_r'(\sin\xi' \cos\xi - \cos\xi' \sin\xi \cos\alpha) + S_\xi'(\cos\xi' \cos\xi$

$\quad + \sin\xi' \sin\xi \cos\alpha) - \sin\xi \sin\alpha\, S_z'\Big]$

$+ k\left[S_r' \cos\xi \sin\alpha + S_\xi' \sin\alpha \sin\xi' + S_z' \cos\alpha\right]$

Making the substitutions, and neglecting terms $O(\sin^3\alpha)$

$$S_r = \frac{S_r'}{r'}\left[r(1 - \cos^2\xi \sin^2\alpha) - z \sin\alpha \cos\xi\right]$$

$$- \frac{S_\xi'}{r'} z \sin\xi \sin\alpha + S_z' \sin\alpha \cos\xi$$

$$S_\xi = \frac{S_r'}{r'} \sin\alpha \sin\xi (r \sin\alpha \cos\xi + z)$$

$$+ \frac{S_\xi'}{r'} (r \cos\alpha - z \sin\alpha \cos\xi) - S_z' \sin\alpha \sin\xi$$

$$S_z = \frac{S_r'}{r'} (r \sin\alpha \cos\xi - z \sin^2\alpha) + \frac{S_\xi'}{r'} \sin\alpha \sin\xi$$

$$+ S_z'(1-\tfrac{1}{2} \sin^2\alpha)$$

and $r'^2 = z^2 \sin^2\alpha + r^2 - r^2 \sin^2\alpha \cos^2\xi - 2zr \sin\alpha \cos\xi$

The discussion is now limited to radiators having symmetry about X'OX' and Y'OY'. Such radiators must produce a symmetrical field. Hence $\frac{S_r'}{S_z'}$ and $\frac{S_\xi'}{S_z'}$ are odd in x' and y'. This implies that $\frac{S_r'}{r'S_z'}$ and $\frac{S_\xi'}{r'S_z'}$, involve only even powers of r'. Examination of the above expressions now reveals that h and g^2 have only terms in, r, $\sin\alpha \cos\xi$, $\sin^2\alpha$, $\sin^2\alpha \cos^2\xi$, and $\sin\alpha \sin\xi$. And, furthermore, the term in $\sin\alpha \sin\xi$ will disappear if $S_\xi' = 0$.

These terms lead, respectively, to the coefficients of spherical aberration, coma, distortion, and astigmatism, with $\sin\alpha$ being identified with σ. If S_ξ' is present an additional coma effect is introduced. The latter may be appropriately called "azimuthal coma". If this aberration is absent $\mu = 0$ in (5.9) and (5.10). Then since $\sin\varepsilon = x_1'/B$ is odd in x_1' and the rest of the integrand is even in x_1' it follows that (5.9) vanishes if $\mu = 0$. Therefore, azimuthal coma introduces a first degree coma term, whereas its absence reduces coma to a second degree effect.

Since the physical significance of formulas (5.7) to (5.11) has now been established, it may be stated that they lead to definite modifications of the observed pattern which can, in principle, be calculated. The magnitude of the coefficients affects the magnitude but not the nature of the pattern modification. Detailed computation of these formulas would be laborious, although an estimate of their order of magnitude would not be difficult. In addition to these formulas there are others representing spherical aberration, coma, etc., of higher orders, and it is probable that these would be significant.

Of the coefficients themselves the one in spherical aberration is likely to be close to the optical constant since it is largely determined by the radial component. This aberration can, therefore, be minimized by the usual optical techniques. All of the others arise in a way which is peculiar to microwaves and their magnitude is dependent on the radiator pattern. There is even less possibility than in optics of computing the constants on basis of physical parameters.

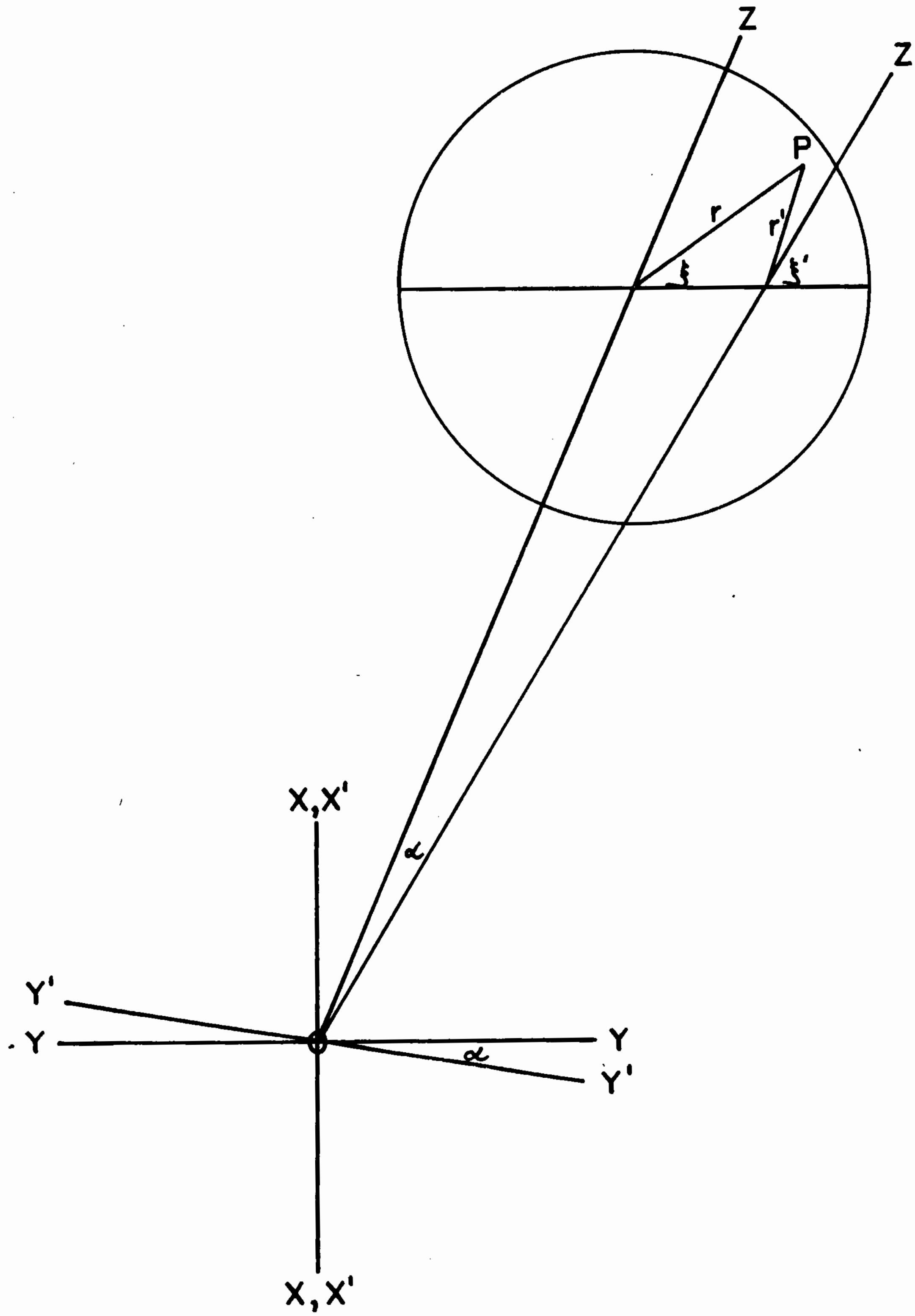


Fig. 32 Optical Aberrations. The relation among the co-ordinate systems used in Chapter V.

VI. A REVIEW

1. Computation Procedure: The complexity of antenna patterns has already been indicated. Aside from a linear element a horn is the simplest antenna. Yet variation of horn length and plane of observation admits of a great variety of patterns. It is customary to specify paraboloids in terms of taper, but this concept does not do justice to the great difference in side structure and phase which can occur in the various planes, and also the not-insignificant effects of back radiation from the feed.

Mutilations also depend greatly on the aperture function and the plane of observation, but generally to a less extent than the unmutilated fields. For instance, the same mutilations were used in the calculations for E-plane and H-plane paraboloid patterns. However, when the mutilation is combined with the unmutilated field there is seldom any obvious relation between the patterns in various planes. Thus it can be concluded that a prediction of the mutilation effect in detail is possible only through a complete calculation taking account of the aperture function and the plane of observation.

If the aperture function is known and if the radiator periphery is circular a detailed calculation by the method of Appendix III is comparatively easy. If the periphery is not circular an exact calculation requires the tedious computation of incomplete Bessel functions. If the aperture function is unknown no detailed calculation is possible. It is of interest to consider the usefulness of abbreviated calculations: first, whether a non-circular periphery can be replaced by an equivalent circular one; and secondly, whether a calculation based on an elementary aperture function rather than the actual one will give predictions which are useful from the point of view of design even though incorrect in detail.

In carrying out the computations it was found that the integration over the partial zones did not affect the mutilations greatly, although there were exceptional points where they yielded relatively large contributions. It was found that the replacing of a square periphery by an equivalent circle gave results which were correct, not only in order of magnitude, but also approximately in detail. Figs. 14 and 15 show points based on such a calculation for the 10 wave-length horn, c = 17.5 where the square perimeter was replaced by an equivalent circle of radius 5.7. (Neither this approximation nor an accurate calculation is in good agreement with observation for H-plane at c = 17.5.) Precision of computation is not required in calculating mutilations. Therefore it is concluded that when the greater part of the area of a radiator can be enclosed by complete zones, it is justifiable and useful to treat the entire radiator as though it contained only an equivalent number of complete zones.

Attempts were made in the use of an elementary aperture function to determine if results having some correspondence with observation could be obtained. A uniform aperture distribution was assumed for the 14 in, 18 in, and 24 in paraboloids. The calculated mutilation was then considered in two parts: first, that due to the back radiation from the feed which, for these paraboloids, is constant with angle; and, secondly, the mutilation proper as calculated according to Appendix III. The magnitude of these two quantities is shown separately, one above the other, in Figs. 34, 35 and 36 for c = 17.5. Points shown are observed differences in $(\text{power})^{1/2}$ between mutilated and unmutilated patterns. If the calculation is meaningful the points should all be below the upper curve, and the curve should indicate the order of magnitude of the points. On this basis the prediction is a conservative estimate of actual observation and would appear to have some value. The same procedure was applied to the horn calculation at $c = 17.5$ using the horn functions. A definite correlation between curve and points will be observed in Fig. 33. This type of calculation is easy to carry out provided an estimate of the effect of point sources can be made. It will be noted that, in the paraboloid examples, the point source contributed more to the mutilation than all of the remainder of the antenna.

2. Results and Conclusions: A direct comparison of mutilated and unmutilated patterns is shown in Figs. 10 to 31. Chapters III and IV close with a brief discussion of experimental results. The following indented statements are intended to summarize the entire work.

(1) The mutilation increases with decreasing size of the radiator, becoming extreme for axial point sources.

(2) Patterns having a pronounced main lobe are subject to relatively small mutilation on this lobe. An exception is horn H-plane patterns where several lobes are merged into one due to phase error.

(3) Mutilation persists in the side lobe region to large diameters of the mutilating aperture. A quantitative estimate requires consideration of wave-length, size of radiator, and separation of radiator from screen.

(4) Patterns with characteristic features extending to

low power levels tend to have these features radically altered. Pronounced side lobes may also be generated in such patterns.

(5) The main features of patterns at wide angles are retained in the mutilated patterns. The average background is increased.

Fig. 8 illustrates the reason for Conclusion 4. The mutilation is a complex term to be added to the unmutilated amplitude. The latter is a decreasing oscillating term while the former is, on the whole, non-decreasing. Theoretical considerations do not extend to the angles contemplated in Conclusion 5.

In addition to the above remarks, which are based on the experimental and descriptive aspects of the problem, some insight can be gained from a consideration of the formula used in computation, and from the computation procedure itself.

The effect of phase distribution over the antenna aperture was illustrated by carrying through a horn computation using a negative horn length such that $1/b + 1/q = 0$. All quantities in (A3.3) were substantially unaffected except M. Thus in the final graphical integration the terms had the same magnitude as for a normal horn calculation but different phases. It was found that the mutilations nearly vanished at all angles. This example corresponds physically to radiation converging toward the center of the mutilating aperture, and the result is plausible. On the other hand, a phase non-uniformity corresponding to a diverging wave front leads to an "opening out" of the graphical computation and hence to greater mutilation. Therefore it can be stated,

(6) Phase non-uniformity in the aperture function corresponding to a diverging wave-front leads to increased mutilation; phase non-uniformity corresponding to a converging wave-front results in decreased mutilation.

The effect of amplitude variation of the aperture function is complex, and no conclusion can be given in regard to it.

The way in which the mutilation depends on the parameters b and c (respectively, distance from antenna to diffraction screen, and radius of mutilating aperture) is revealed by examining the computation formula,

$$I_1 = (b/s) \sec \alpha \exp\left[2\pi i(s+b) + \pi i c^2(1/s + 1/b)\right]$$
$$\sum_i f_1(Z_i) \; \mathcal{J}(Z_i) \exp(iMZ_i^2/2) \, \delta W_2 \qquad \text{(A3.5)}$$

In W_2, $t_o \propto c^2/b$, $w \propto cZ/b$, and $w/t_o = Z/c$. Of the terms within the summation only δW_2 and $\exp(iMZ_i^2/2)$ are significantly affected by an increase in b, the latter tending to decrease I_1 and the former term, which dominates, tending to increase it. Hence it may be stated that,

(7) Mutilation increases with the separation of transmitter and mutilating aperture according to a power of the separation which is greater than unity.

Nothing within the summation tends to nullify the strong effect of $\exp(\pi i c^2/b)$ in (A3.5). Hence it may be stated that,

(8) The detail of the mutilation is affected to a marked degree by change in the radius of the mutilating aperture, and to a lesser extent by a change in the separation of transmitter and mutilating aperture.

The dependence of the magnitude of the mutilation on c in (A3.5) is limited to δW_2, all other parts of the formula being independent of

c. The leading term of δW_2 is $(iZ/c)J_1(2\pi cZ/b)$ which is asymptotically proportional to $c^{-3/2}$. But the phase part of δW_2 tends to wind the mutilation terms into a spiral having total angle approximately proportional to the argument of the Lommel function. This introduces a further c^{-1} dependence. It cannot be assumed that (A3.5) is valid for large values of c; nevertheless, it is reasonable to state tentatively that,

> (9) The magnitude of the mutilation decreases with the increase of the mutilating aperture according to a power of the radius which is less than -3/2 and probably in the neighbourhood of -5/2.

3. Comments on Precision Measurement: In the experimental part of this work the physical parameters were so chosen that the mutilations were considerable. On the other hand it is of interest to extend the investigation to include every small mutilation. For then definite criteria could be established regarding precision measurements at microwave frequencies. Experimentally, such an extension in X-band involves impractically large apertures. Theoretically there are two obstacles. First the validity of the derived formula is questionable for large apertures. Secondly, the difference between the Kirchhoff solution and a rigorous solution is not known to be negligible.

The conclusions of the preceding section can be used to predict the aperture size necessary for a certain degree of precision, although such prediction is subject to the qualifications already mentioned. If, for example, a mutilation 1/10 that observed for the 10λ horn at

35λ aperture can be tolerated, the required aperture diameter is $35 \times 10^{1/2.5} = 88$ wave-lengths.

This thesis has been exclusively concerned with systems having a well defined circular aperture. But it is possible that, for a given area and general shape, the circular aperture may produce the maximum mutilation. This conclusion is true for axial point sources. Further evidence in the same direction is suggested by the fact that there was no measurable mutilation through the 5 foot square framework except, possibly, for the wave guide mouth. All mutilations whether Kirchhoff or rigorous can be ascribed to the aperture edge. It is conceivable that they would be reduced significantly by staggering the edge, thereby eliminating the systematic phase relation from the edge contributions.

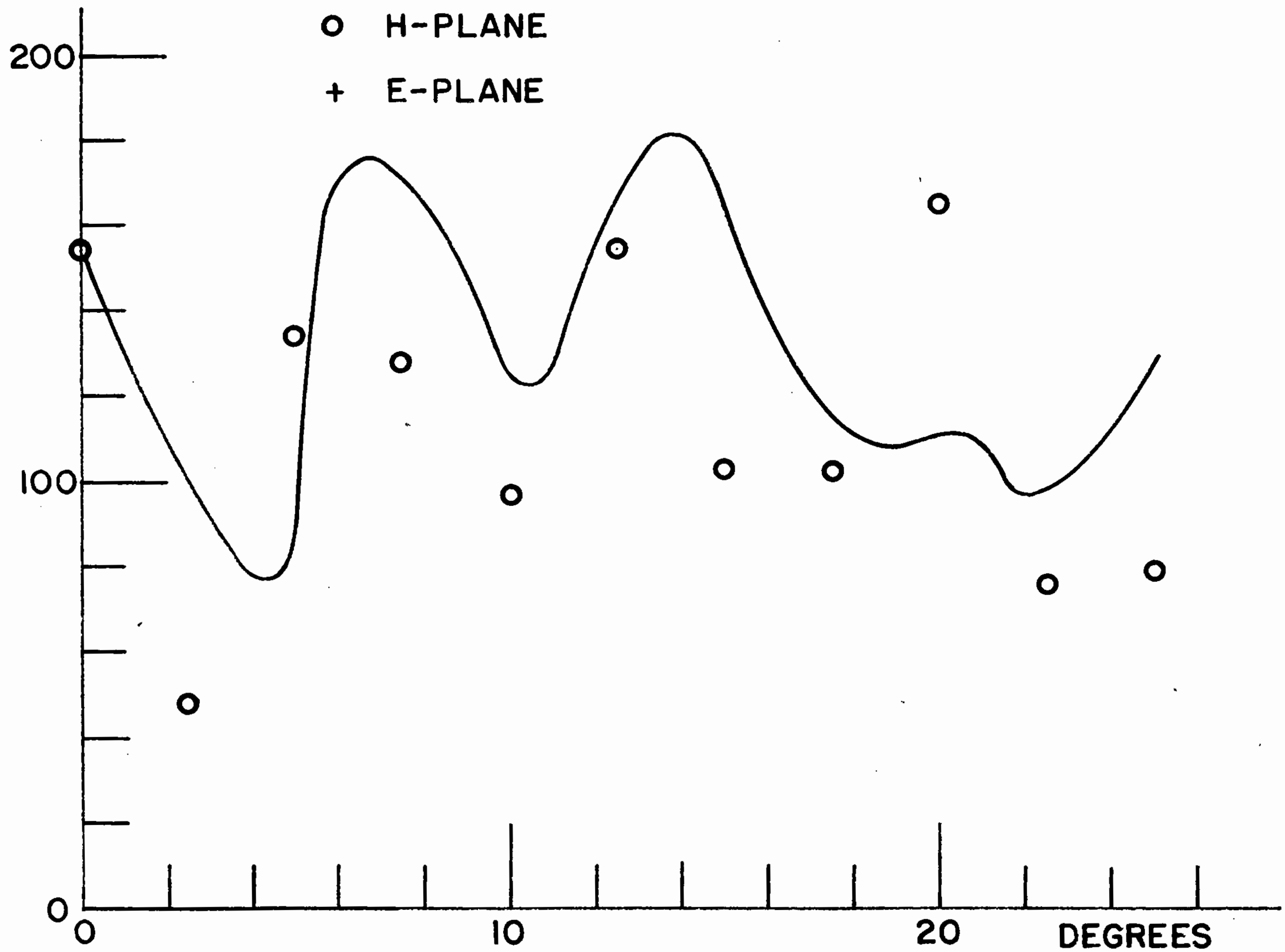


Fig. 33 Approximate Calculation for Horn. No. 1. c = 17.5 Amplitude of unmutilated signal equals 5000 at $\alpha = 0^{\circ}$.

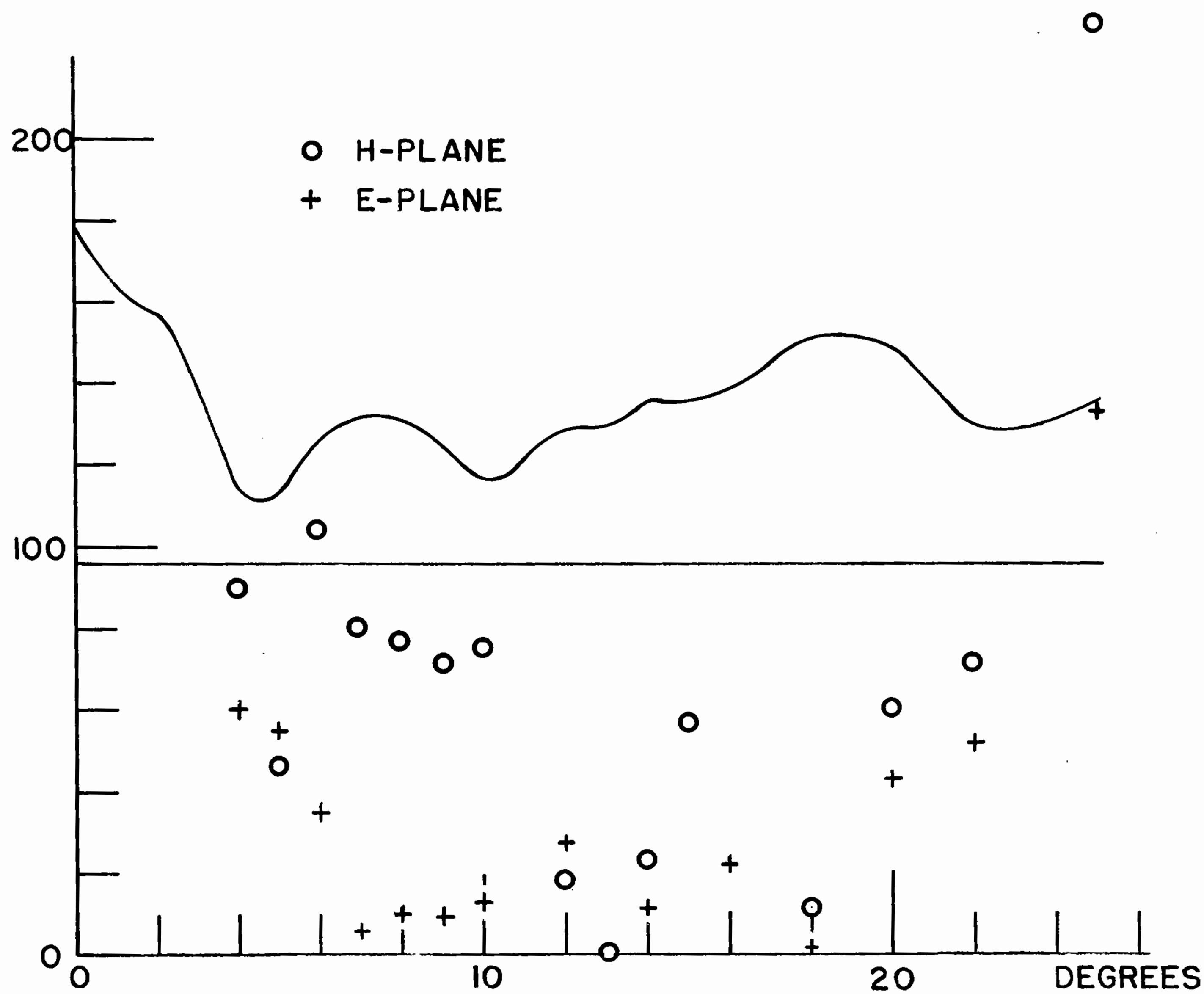


Fig. 34 Approximate Calculation for 14" Paraboloid. $c = 17.5$ Amplitude of unmutilated signal equals 5000 at $\alpha = 0^{\circ}$.

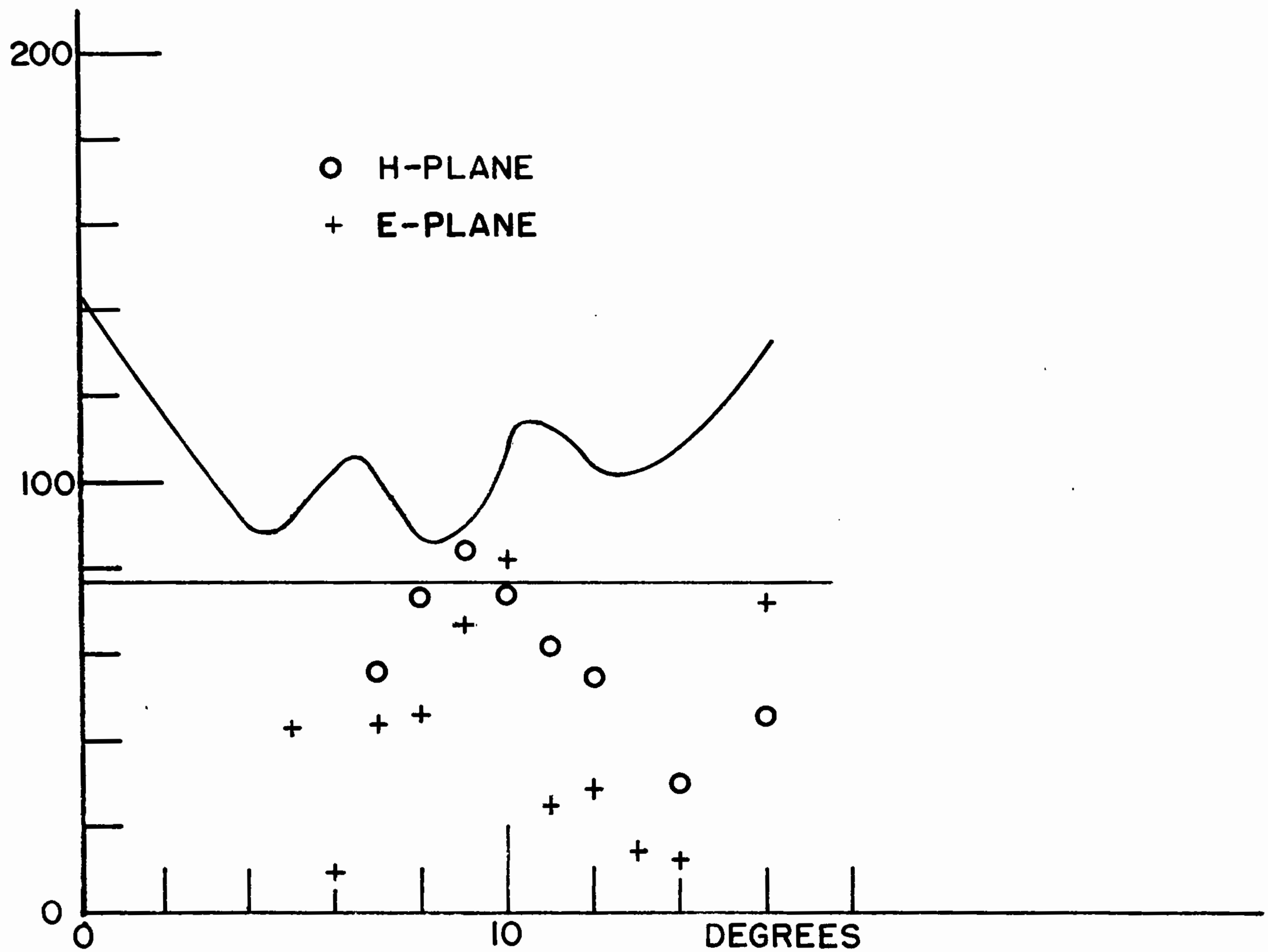


Fig. 35 Approximate Calculation for 18" Paraboloid. c = 17.5 Amplitude of unmutilated signal equals 5000 at $\alpha = 0°$.

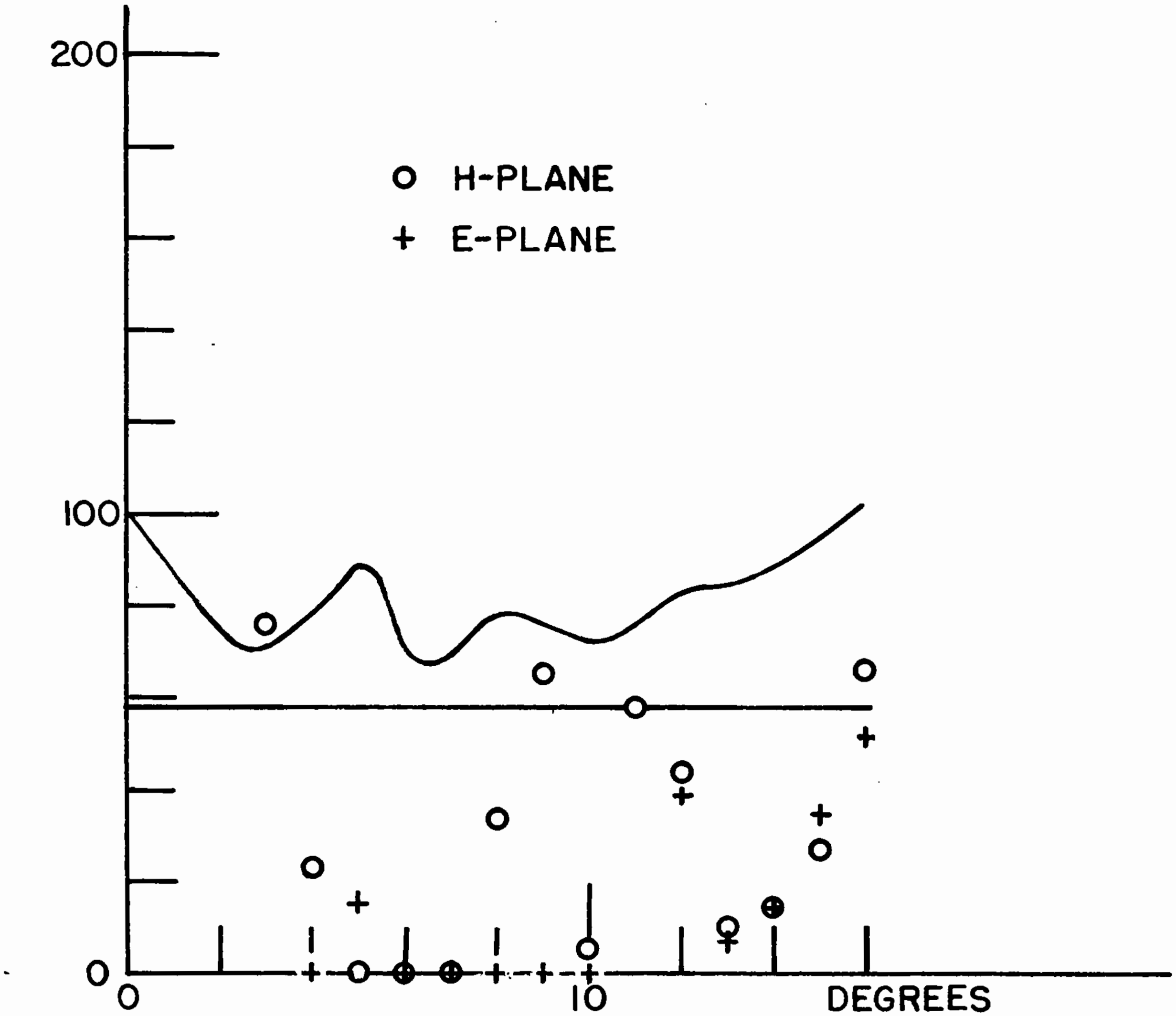


Fig. 36 Approximate Calculation for 24" Paraboloid. c = 17.5 Amplitude of unmutilated signal equals 5000 at $\alpha = 0^{o}$.

APPENDIX I. DISTANT FIELD OF INCLINED SOURCE.

The x', y', z' co-ordinates are fixed in the radiator. The radiation will be supposed to originate in a plane surface or aperture area which lies in the X',Y' plane. Following Eq. (1.6) the distant field may be written,

$$u(x',y',z') = C \exp(ikR) \int_{(1)} f(x_1',y_1') \exp\left[-ik(l'x_1' + m'y_1')\right] dx_1' dy_1'$$

where $f(x_1',y_1')$ is an aperture function or current density function.

The inclined co-ordinate system is related to the non-inclined system as follows, (Fig. 2)

$$x' = x \qquad x = r\cos\psi$$

$$y' = y\cos\alpha - z\sin\alpha \qquad y = r\sin\psi$$

$$z' = z\cos\alpha + y\sin\alpha \qquad z = \sqrt{R^2 - r^2}$$

$$l' = x'/R \qquad r/R = \gamma$$

$$m' = y'/R = 1/R(y\cos\alpha - z\sin\alpha)$$

Then,

$$l'x_1' + m'y_1' = 1/R\left[x_1' r\cos\psi + y_1'(r\cos\alpha\sin\psi - \sqrt{R^2-r^2}\sin\alpha)\right]$$

$$= \gamma B\sin(\psi+\varepsilon) - y_1'\sqrt{1-\gamma^2}\sin\alpha$$

where $B^2 = x_1'^2 + y_1'^2\cos^2\alpha$

$\sin\varepsilon = x_1'/B$

Hence,

$$u(R,\gamma,\psi) = C\exp(ikR)\int_{(1)} f(x_1',y_1')\exp\left[2\pi i y_1'\sqrt{1-\gamma^2}\sin\alpha - 2\pi i\gamma B\sin(\psi+\varepsilon)\right] dx_1' dy_1'$$

$$= C\exp(ikR)\sum_{-\infty}^{\infty}\exp\left[in(\psi-\pi)\right]\int_{(1)} f(x_1',y_1')\exp\left[in\varepsilon + 2\pi i y_1'\sqrt{1-\gamma^2}\sin\alpha\right] J_n(2\pi\gamma B)\, dx_1' dy_1'$$

Equating coefficients with (1.14),

$$G_n(\gamma) = (1/2\pi)\exp(-in\pi) \int_{(1)} f(x_1',y_1')\, J_n(2\pi B\gamma) \exp\left[in\varepsilon + 2\pi i y_1' \sqrt{1-\gamma^2}\ \sin\alpha\right] dx_1' dy_1' \tag{A1.1}$$

APPENDIX II. INTEGRALS EVALUATED IN TERMS OF LOMMEL FUNCTIONS.

The integral

$$\int_0^c \exp(iXu^2)\, u^n \; J_{n-1}(Au)du$$

will be evaluated. The change of variable, $u = v/A$ is made, from which the integral becomes

$$\frac{1}{A^{n+1}} \int_0^{cA} \exp\,(iXv^2/A^2)\; v^n J_{n-1}(v)\; dv$$

$$= \frac{1}{A^{n+1}} \int_0^{cA} \exp(iXv^2/A^2)\; \frac{d}{dv}\left[v^n J_n(v)\right]$$

Integrating by parts and continuing the process one obtains,

$$\exp(iXc^2)/(A^{n+1}) \left[(Ac)^n\, J_n(cA) - \frac{2iX}{A^2}\,(Ac)^{n+1}\, J_{n+1}(cA) + \left(\frac{2iX}{A^2}\right)^2 \;(Ac)^{n+2}\; J_{n+2}(cA) + \dots\dots \right]$$

$$= \frac{A^{n-1}}{(2X)^n} \exp(iXc^2)\; \left[U_n(t,w) - i\; U_{n+1}(t,w)\right]$$

where $t = 2Xc^2$, $w = Ac$, and

$$U_n(t,w) = (t/w)^n J_n(w) - (t/w)^{n+2} J_{n+2}(w) + (t/w)^{n+4} J_{n+4}(w)$$

The function

$$V_n(t,w) = (w/t)^n J_n(w) - (w/t)^{n+2} J_{n+2}(w) + (w/t)^{n+4} J_{n+4}(w)$$

is related to U_n by the equation[17],

$$U_n - i\; U_{n+1} = (-1)^n\left[V_{-n+2} + i\; V_{-n+1}\right] + i^n \exp\left[-i(t/2 + w^2/2t)\right]$$

$$= (-1)^n W_{-n+2} + i^n \exp\left[-i(t/2 + w^2/2t)\right]$$

where $W_n = V_n + iV_{n-1}$

Therefore, $\int_0^c \exp(iXu^2)\; u^n\; J_{n-1}(Au)\; du$

$$= \frac{A^{n-1}}{(2X)^n}\;\; \exp(iXc^2)\left[(-1)^n\; W_{-n+2} + i^n \exp\left[-i(t/2 + w^2/2t)\right]\right]$$

Integrals of the type,

$$D_{ab}(c,A,X) = \int_0^c \exp(iXc^2)\ u^b J_a(Au)\ du$$

may be evaluated by use of the recurrence Formula,

$$J_{n+1}(z) = -J_{n-1}(z) + (2n/z)J_n(z)$$

where b - a is odd.

One obtains in this way,

$$D_{o1}(c,A,X) = \int_0^c \exp(iXu^2)\ J_o(Au)udu = \frac{1}{2X}\exp(iXc^2)\left[-W_1+i\ \exp\left[-i(t/2+w^2/2t)\right]\right]$$

$$D_{o3}(c,A,X) = \int_0^c \exp(iXu^2)J_o(Au)u^3du = \exp\left[-i(t/2+w^2/2t)\right]\left[i\frac{A^2}{(2X)^3} - \frac{2}{(2X)^2}\right]\exp(iXc^2)$$

$$+ \exp(iXc^2)\left[\frac{A^2}{(2X)^3}W_{-1} + \frac{2}{(2X)^2}W_o\right]$$

$$D_{o5}(c,A,X) = \int_0^c \exp(iXu^2)J_o(Au)\ u^5du$$

$$= \exp\left[-i(t/2+w^2/2t)\right]\left[i\frac{A^4}{(2X)^5} - \frac{8A^2}{(2X)^4} - i\frac{8}{(2X)^3}\right]\exp(iXc^2)$$

$$-\left[\frac{A^4}{(2X)^5}W_{-3} + \frac{8A^2}{(2X)^4}W_{-2} + \frac{8}{(2X)^3}W_{-1}\right]\exp(iXc^2)$$

$$D_{14}(c,A,X) = -\exp\left[-i(t/2+w^2/2t)\right]\left[\frac{A^3}{(2X)^4} + \frac{4A}{(2X)^3}\right]\exp(iXc^2)$$

$$-\left[\frac{A^3}{(2X)^4}W_{-2} + \frac{4A}{(2X)^3}W_{-1}\right]\exp(iXc^2)$$

$$D_{23}(c,A,X) = \int_0^c \exp(iXu^2)\ J_2(Au)u^3du = -\frac{A^2}{(2X)^3}\exp(iXc^2)\left[W_{-1}+i\ \exp\left[-i(t/2+w^2/2t)\right]\right]$$

$$D_{27}(c,A,X) = \int_0^c \exp(iXu^2)\ J_2(Au)u^7du$$

$$= \exp\left[-i(t/2+w^2/2t)\right]\left[-i\frac{A^6}{(2X)^7} + \frac{16A^4}{(2X)^6} + i\ \frac{48A^2}{(2X)^5}\right]\exp(iXc^2)$$

$$-\left[\frac{A^6}{(2X)^7}W_{-5} + \frac{16A^4}{(2X)^6}W_{-4} + \frac{48A^2}{(2X)^5}W_{-3}\right]\exp(iXc^2)$$

APPENDIX III. THE CALCULATION OF THE MUTILATION.

1. Derivation of the Basic Formula: An expression for the mutilation has been derived in Chapter I,

$$e(0) = \exp\left[2\pi i(s+b) + \pi i c^2/s\right] \int_{(1)} f(x_1',y_1') W_1(t,w)/(s+b+y_1' \sin\alpha)$$
$$\exp\left[\pi i\left(2y_1' \sin\alpha + (B^2+c^2)/(b+y_1' \sin\alpha)\right)\right] dx_1' dy_1' \qquad (1.30)$$

The following transformation is introduced,

$$x_1' = Z\cos\tau + (Z^2/b)\tan\alpha \sin\tau$$
$$y_1' = Z + (Z^3/2b^2)\tan^2\alpha \ \sec\alpha \sin\tau - (Z^2/b)\tan\alpha \sec\alpha \cos\tau \qquad (A3.1)$$
$$+ (Z^2/b)\tan\alpha \sec\alpha$$

This transformation has been so chosen that

$$w = \frac{2\pi c\sqrt{x_1'^2 + y_1'^2 \cos^2\alpha}}{b+y_1' \sin\alpha} = (2\pi c/b)Z + \text{high order terms.}$$

The t which appears in the Lommel functions depends slightly on y_1'. The Lommel functions will, therefore, be expanded giving,

$$V_0(t,w) = V_0(t_0,w) + \Delta V_0$$
$$V_1(t,w) = V_1(t_0,w) + \Delta V_1$$

where $t_0 = 2\pi c^2(1/s + 1/b)$ is now a constant. It may be shown that

$$\Delta V_0 = -(2Z^3/bc^2)\tan\alpha \sin\tau J_2(w)$$
$$\Delta V_1 = -(Z^2/bc)\tan\alpha \sin\tau \ J_1(w)$$

Also,

$$\frac{\partial(x_1',y_1')}{\partial(Z,\tau)} = Z\sec\alpha + (2Z^2/b)\tan\alpha \sec\alpha \sin\tau$$
$$+ (Z^3/b^2)\sec\alpha \tan^2\alpha \ (5/2 + \sin^2\tau - 2\cos\tau) + \ldots\ldots$$

Substitution in (1.30) results in a series of integrals of which the first three are:

$$I_1 = K \int_{(1)} f(Z,\tau)\, e^{iD}\, W_1(t_o,w)\, ZdZd\tau \quad \text{..........} \qquad (A3.2)$$

$$I_2 = \frac{2s-b}{b(s+b)}\, K \tan\alpha \int_{(1)} f(Z,\tau)\, e^{iD} W_1(t_o,w)\, Z^2 \sin\tau\, dZd\tau.$$

$$I_3 = -(K\tan\alpha/bc)\int_{(1)} f(Z,\tau)\, e^{iD} Z^3 \left[J_1(w) + i\frac{2\pi Z}{c} J_2(w)\right] \sin\tau\, dZd\tau$$

where $K = \left[\sec\alpha/(s+b)\right] \exp\left[2\pi i(s+b) + \pi i c^2(1/s+1/b)\right]$

and $D = (\pi Z^2/b)(1 + (2-c^2/2b^2)\tan^2\alpha)$

$$+ 2\pi Z \tan\alpha\,(1-c^2/2b^2 + Z^2/2b^2)\sin\tau$$

$$- (2\pi Z^2/b)(1-c^2/2b^2)\tan^2\alpha\cos\tau$$

$$- (\pi c^2 Z^2/2b^3)\tan^2\alpha\cos 2\tau$$

It is seen that I_1 is the dominant term. The others will be significant only for large angles if at all. One computation of I_2 was carried out and it proved to be small. All other computations have been confined to I_1, and no further consideration will be given to the other terms.

The aperture functions of Chapters III and IV are of such a type that $f(Z,\tau)\exp(iD)$ is equal to one or more terms of the form $f_1(Z)\cos n\tau$ $\exp(iQ)$ where

$$Q = MZ^2/2 + LZ\sin\tau + NZ\cos\tau - RZ^2\cos 2\tau \qquad (A3.3)$$

The function $\mathcal{J}_n(Z)$ is defined by,

$$\oint \cos n\tau\, \exp(iQ)\, d\tau = 2\pi\, \mathcal{J}_n(Z)\exp(iMZ^2/2) \qquad (A3.4)$$

Then $I_1^{(m)} = 2\pi K \int f_1^{(m)}(Z)\, \exp(iMZ^2/2)\, \mathcal{J}_n(Z) W_1(t_o,w)\, ZdZ$

$$= (Kb^2/2\pi c^2)\int f_1^{(m)}(Z)\exp(iMZ^2/2)\,\mathcal{J}_n(Z) W_1(t_o,w)\, w\, dw$$

where $I_1 = I_1' + I_1'' + \ldots\ldots.$

But $\quad W_1(t_o,w)\, w\, dw = t_o W_2(t_o,w)$

That is, $w\, W_1(t_o,w) = t_o \frac{d}{dw}\left[W_2(t_o,w)\right]$

$$\therefore \quad I_1^{(m)} = (bK/s)(s+b)\int f_1^{(m)}(Z)\exp(iMZ^2/2)\,\mathcal{J}_n(Z)\,\frac{dW_2}{dw}\, dw$$

This may be approximated by a summation,

$$I_1^{(m)} \doteq (bK/s)(s+b) \sum_i f_1^{(m)}(Z_i)\exp(iMZ_i^2/2) \oint_n(Z_i)\ \delta W_2$$

where $\delta W_2 = W_2(Z_i + \Delta Z/2) - W_2(Z_i - \Delta Z/2)$

and the region of integration has been divided into intervals of width ΔZ. This leads finally to,

$$I_1^{(m)} = (b/s)\sec\alpha \exp\left[2\pi i(s+b) + \pi i c^2(1/s+1/b)\right] \sum_i f_1^{(m)}(Z_i) \oint_n(Z_i)\exp(iMZ_i^2/2)\delta W_2 \tag{A3.5}$$

The result is a summation over zones (Fig. 7). The integral in τ over a particular zone reduces to a Bessel function for the inner zones which do not cut the Z, τ perimeter, provided R is negligibly small. That is,

$$\oint_n(Z_i) = J_n(TZ_i) \text{ for inner zones}$$

where $T^2 = L^2 + N^2$

For outer zones, which cut the perimeter , the integration with respect to τ results in "incomplete Bessel functions". Those of order zero may be expressed in terms of,

$$\int_0^{\phi} \cos\ (\sin Tx)dx \quad \text{and} \quad \int_0^{\phi} \sin\ (\sin Tx)dx$$

Since those functions have been tabulated for only a small range of the parameters T and ϕ, their values were computed, where necessary, by graphical integration using curves of cos (sin Tx) and sin (sin Tx). Higher order functions occur in paraboloid analysis but the angle α is sufficiently small that deformation of periphery is not taken into account.

2. <u>Deformation of the Area of Integration</u>: The co-ordinates (Z,τ) are identical with (r_1', ψ_1') for $\alpha = 0$, but they differ increasingly as α increases. Accordingly, the perimeter of the antenna, and hence the area of integration, becomes deformed. Consideration will be given first to a

rectangular radiator specified by the lines,

$$x_1' = \pm a_1/2 \text{ and } y_1' = \pm a_2/2$$

X and Y are introduced as follows:

$$X = Z \cos \tau \qquad \text{and } Y = Z \sin \tau .$$

These result in ,

$$\left.\begin{aligned} Y &= \frac{\pm (a_2/2) \cos \alpha - \frac{Z \tan \alpha}{b} (Z - X)}{1 + \frac{Z^2 \tan^2 \alpha}{2b^2}} \\ \text{and,} \quad X &= \pm (a_1/2) - \frac{YZ \tan \alpha}{b} \end{aligned}\right\} \qquad (A3.6)$$

The deformation depends on the angle α but is independent of the diameter of the mutilating aperture. An example is given in Fig. 7 for $\alpha = 20^\circ$ and a = 10. Summation zones with $\Delta Z = 1$ are also shown.

If the perimeter of the antenna is circular it may be written,

$$x_1'^2 + y_1'^2 = a^2$$

Substitution leads approximately to the ellipse,

$$(X^2/a^2) + (Y + (a^2/2b) \sin 2\alpha)^2/a^2 \cos^2 \alpha = 1$$

3. Computation of Lommel Functions: Equation (A3.5) involves the Lommel function $W_2(t_o,w) = V_2(t_o,w) + iV_1(t_o,w)$, where

$$V_1(t_o,w) = (w/t_o)J_1(w) - (w/t_o)^3J_3(w) + (w/t_o)^5J_5(w) - \ldots..$$

$$V_2(t_o,w) = (w/t_o)^2J_2(w) - (w/t_o)^4J_4(w) + (w/t_o)^6J_6(w) - \ldots..$$

$$w = (2\pi c/b)Z; \qquad t_o = 2\pi c^2(1/s+1/b)$$

V_1 and V_2 are readily evaluated from the series for all parameters normally encountered. The series converges more slowly as the size of the antenna approaches that of the mutilating aperture, particularly if both are large. Tables need not be computed beyond four decimal places. Those used in this work are reproduced in Appendix V. They contain also δV_1,

δV_2, and $\delta V_2 + i\delta V_1$ in terms of magnitude and angle.

4. Validity of the Summation: It is necessary to examine the justification for replacing the integral by a summation in Section 1, and to form some basis for estimating the required smallness of subdivisions. The error, on the basis of complete zones has been derived, and is found to be,

$$\frac{\pi^2 c}{6b^2} K(\Delta Z)^3 \sum_i f_1(Z_i) \exp(iMZ_i{}^2/2) W_1(t_o,w)$$
$$\left[-M^2 Z_i J_o(TZ_i) + 2iMTZ_i J_1(TZ_i) + \frac{T^2}{2}(J_o(TZ_i) - J_2(TZ_i))\right]$$

A conservative estimate of the order of magnitude of each term is, then, $(\pi^2 c)6b^2)$ K $(\Delta Z)^3$. For the values used this has the order of 1/50 $K(\Delta Z)^3$, whereas the terms of I_1 vary from about 0.1K to 2K.

It was concluded that $\Delta Z = 1$ is a reasonable compromise between accuracy and labor of computation. Most of the computations were carried out at $\Delta Z = 1$. However several were done at $\Delta Z = 0.5$, and comparison with those at $\Delta Z = 1$ showed that their difference could be expected to be less than 10%.

5. The Calculation for a Uniformly Illuminated Source: As an example of the method outlined in this chapter, the mutilation will be calculated for a circular source uniformly illuminated. The deformation of perimeter will be neglected. The parameters will be chosen as follows:

$$a = 10 \qquad s \rightarrow \infty$$
$$b = 10\pi \qquad \Delta Z = 1$$
$$c = 10 \qquad f(x_1', y_1') = 1$$

Then,

$$M/2 = (1/10)\left[1 + (2 - 1/2\pi^2)\tan^2\alpha\right]$$
$$L = 2\pi\left[1 - 1/2\pi^2 + Z^2/200\pi^2\right]\tan\alpha$$
$$N \doteq 0 \qquad\qquad R \doteq 0$$
$$I_1 = 10\pi K \sum \exp(iMZ_i^{\,2}/2)\, J_o(LZ_i)\, \delta W_2(20,\ 2Z)$$

There are only 5 zones and only 5 terms in the summation. The procedure is to combine all of the phase terms and all of the magnitude terms to give a phase and magnitude for each zone. Then these complex quantities are summed graphically on the drawing board. Fig. 8 shows the details of such a summation for $\alpha = 0^\circ$, $\alpha = 6^\circ$, and $\alpha = 12^\circ$. A continuous plot of the magnitude and phase of the mutilation as a function of angle is also given in this figure.

The mutilations are relative to a maximum unmutilated amplitude of 25π. The manner in which these mutilations would affect a power pattern is shown in Fig. 9.

6. <u>An Off-Axis Calculation:</u> It is of interest to calculate the mutilated field for the same source as a function of Θ with α fixed at 0°.

$$\begin{aligned} u_m(\Upsilon) &= (2\pi/R)\exp\left[2\pi iR + \pi ia^2/b + \pi ib\Upsilon^2\right] \\ &\quad \int_0^c \exp(\pi ir_2^2/b)\left[U_1(t,w) - iU_2(t,w)\right] J_o(2\pi r_2\Upsilon)\, r_2 dr_2 \\ &\doteq -(b/R)\exp\left[2\pi iR + \pi ia^2/b + \pi ib\Upsilon^2\right] \\ &\quad \sum \exp(\pi ir_2^{\,2}/b)\, J_o(2\pi r_2\Upsilon)\, \delta(U_o - i\, U_1) \end{aligned} \tag{1.31}$$

where the integral has been replaced by the equivalent sum as in Section 1. The Lommel U-functions are readily computed, but the graphical integrations over the mutilating aperture are more laborious than the previous ones over the source. Moreover, they have to be more accurate since it is the field itself and not the mutilation which is being computed. The result is plotted in Fig. 9 for comparison with the mutilations which result from a variation in α.

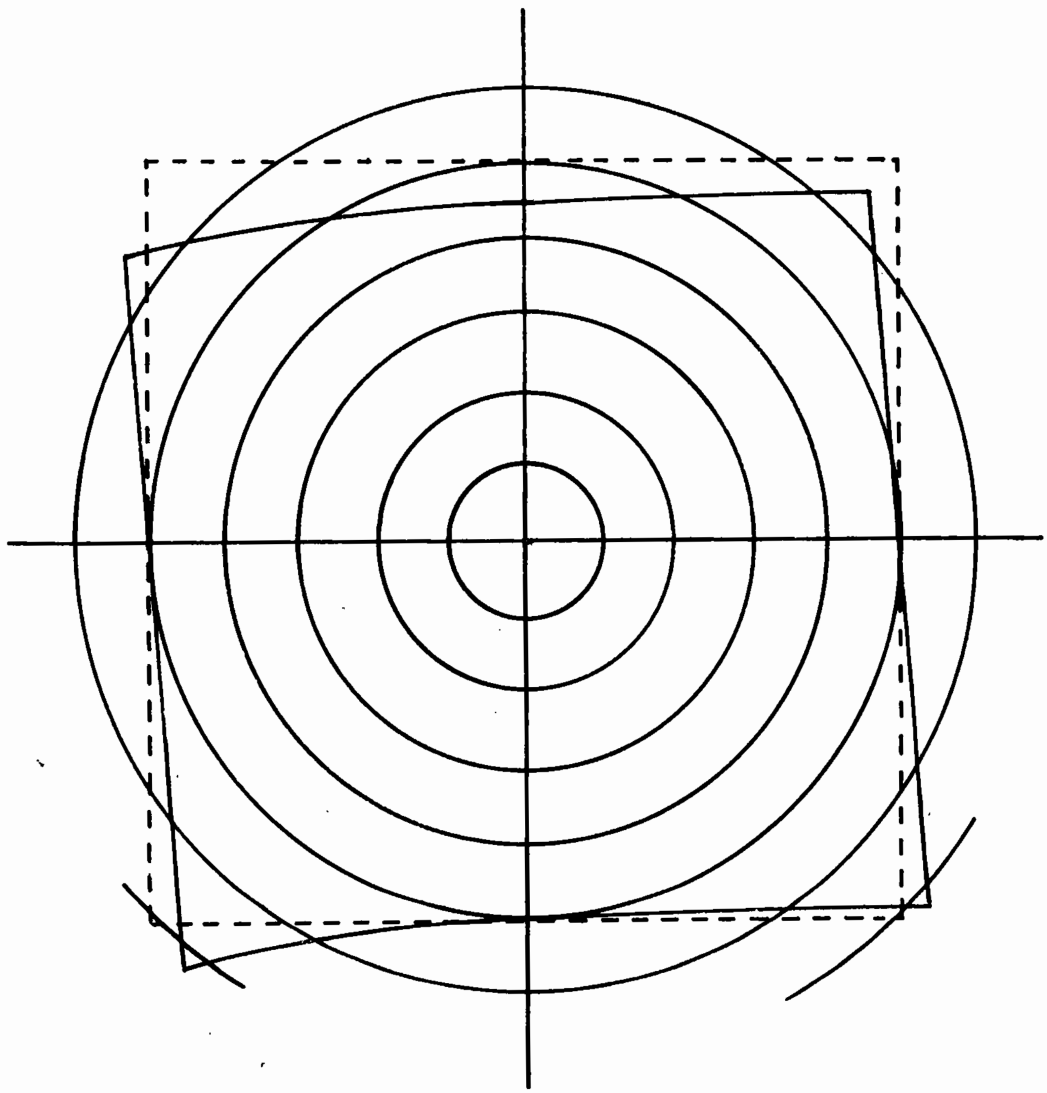


Fig. 7 Zones of Integration. Also shown is the deformation of a square periphery of side 10. $a = 10$ $b = 10\pi$ $\alpha = 20^{\circ}$ $\Delta Z = 1$

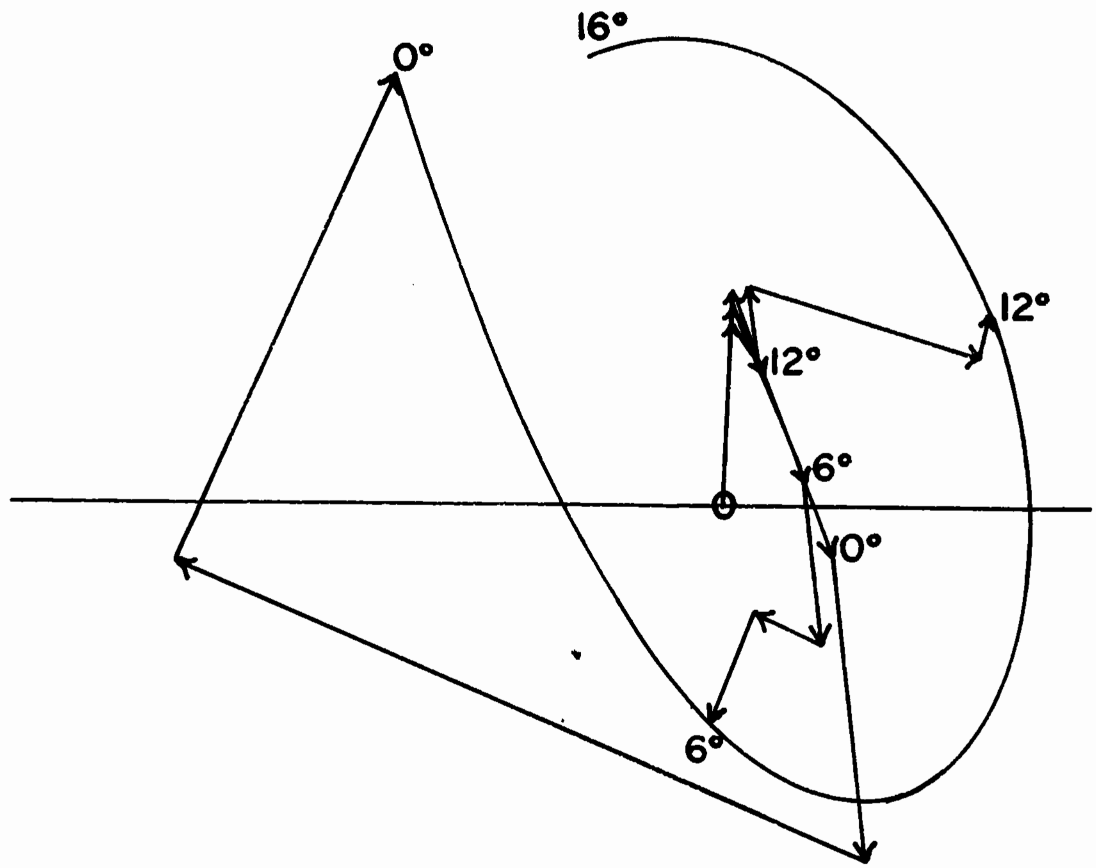


Fig. 8 Magnitude and Phase of Summation. Uniformly illuminated circular aperture of radius 5. $b = 10\pi$ $c = 10$ $f(x_1', y_1') = 1$ Scale: 1 cm = 0.02

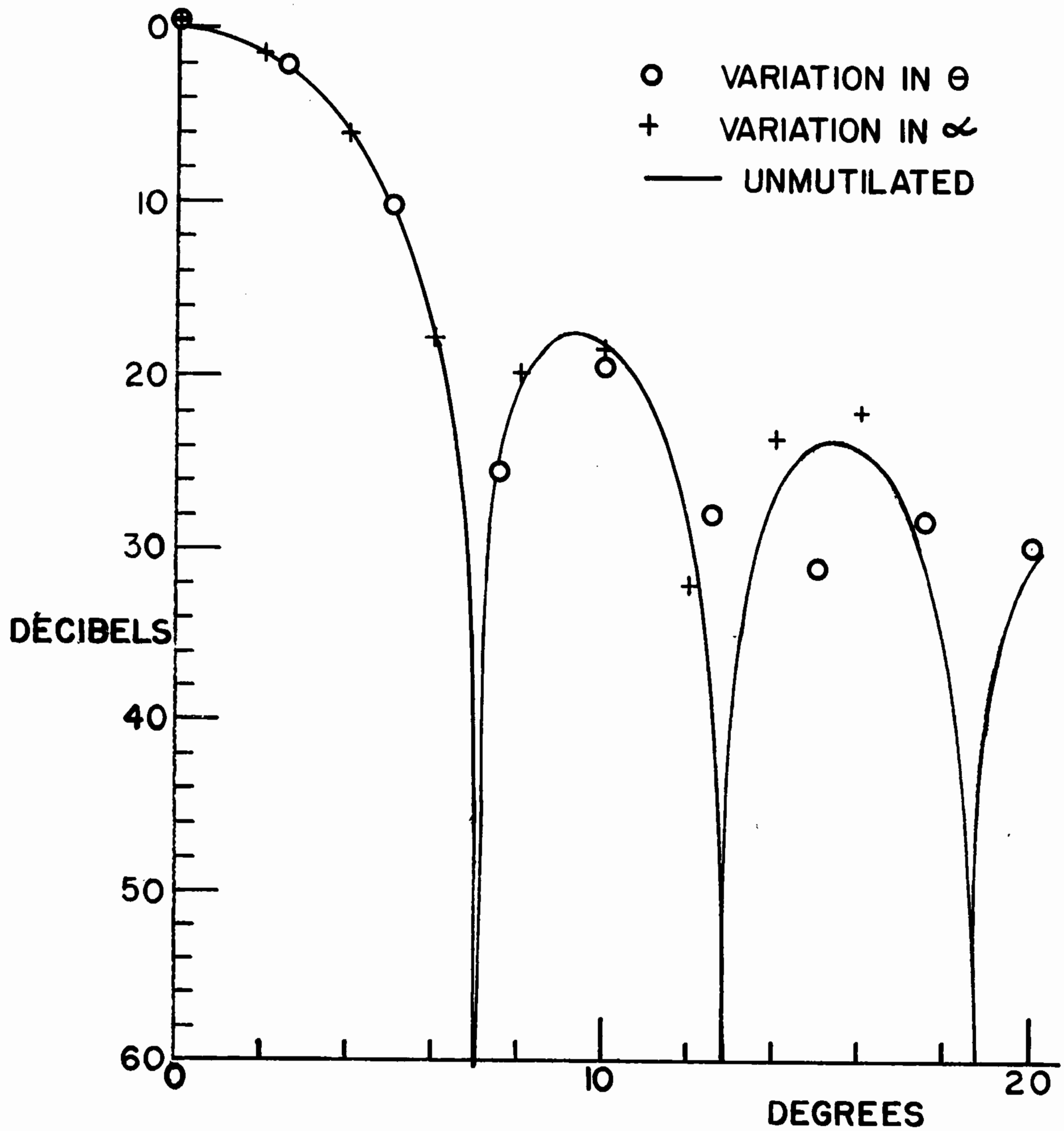


Fig. 9 Pattern for Uniformly Illuminated Circular Aperture.

$a = 5 \quad b = 10\pi \quad c = 10$

APPENDIX IV. CONDITION OF MAXIMUM SIGNAL.

Experimentally, it is convenient to adjust the feed of a paraboloid for maximum signal strength. As the feed is displaced from this position along the axis, the change of pattern is, at first, slight, then marked. It is desired to find the condition imposed on the aperture function by this adjustment to maximum signal.

A paraboloid of radius a and focal length λ will be considered. Let ξ be the distance from the plane through the vertex perpendicular to the axis of the paraboloid and r the radial distance from the axis. Let a line be drawn through a point on the axis near the focus cutting the parabaloid at (ξ, r) and making an angle ϕ with the axis. Then the paraboloid equation is,

$$r^2 = 4\lambda\xi$$

and

$$\cos\phi = \frac{\lambda - \xi}{\left[(\lambda - \xi)^2 + r^2\right]^{1/2}} = \frac{\lambda - \xi}{\lambda + \xi}$$

and $1 - \cos\phi = \dfrac{2\xi}{\lambda + \xi} = \dfrac{2r^2}{4\lambda^2 + r^2}$

Radiation which leaves the focus travels a distance $\lambda + \frac{a^2}{4\lambda}$ in reaching the aperture. If it originates at a point P displaced a distance Δ along the axis away from the focus, it travels a distance which is closely $\lambda + \frac{a^2}{4\lambda} + \Delta\cos\phi$. If the feed is displaced a distance δ from P, the aperture phase will, therefore, change by $2\pi\delta\cos\phi$.

It is assumed, as in Chapter V, that an aperture function can be written,

$$S_o(r) + \cos 2\Psi\, S_1(r) + \cos 4\Psi\, S_2(r) + \ldots\ldots\ldots\ldots$$

It is clear that $S(r)$ should contain terms in $(4\lambda^2 + r^2)^{-n}$, but these can be exp$_a$nded if λ is sufficiently great.

Then,

$$F(0) = -\frac{2\pi i \exp[2\pi i(b+s)]}{s+b} \int_0^a \left[S_o J_o(z) + S_1 J_2(z) + \ldots\ldots\ldots \right] rdr$$

where $z = 2\pi r \sin \alpha$. At $\alpha = 0$,

$$F(0) = -\frac{2\pi i \exp[2\pi i(s+b)]}{s+b} \int_0^a S_o rdr$$

With defocussing the expression will become,

$$F(0) = -\frac{2\pi i \exp(2\pi i(s+b))}{s+b} \int_0^a S_o \exp(2\pi i\delta \cos \emptyset)\, rdr$$

and the power will be given by

$$P(0) = \frac{4\pi^2}{(s+b)^2} \int_0^a \int_0^a S_o(r)\, S_o^*(r')\exp(2\pi i\delta(\cos \emptyset - \cos \emptyset'))rr'drdr'$$

The condition of maximum signal requires that $\frac{dP}{d\delta} = 0$

Differentiating under the integral sign one obtains, since S_o is the function at $\delta = 0$,

$$\int_0^a \int_0^a (\cos \emptyset - \cos \emptyset')\, S_o(r)\, S_o^*(r')\, rr'drdr' = 0,$$

$$\text{i.e.,} \quad \int_0^a S_o^*(r')r'dr' \int_0^a \cos \emptyset\, S_o(r)\, rdr - \int_0^a S_o(r)\, rdr \int_0^a \cos \emptyset\, S_o(r')r'dr' = 0$$

As in Chapter IV,

$$S_o(r) = a_{oo} + a_{o1}(r/a)^2 + a_{o2}(r/a)^4 + \ldots\ldots.$$

Then $\int_0^a S_o(r)\, rdr = a^2 \left[a_{oo} + 1/2a_{o1} + 1/3a_{o2} + \ldots\ldots \right]$

This was put equal to a^2 in Chapter IV, which is permissible, since the phase is arbitrary.

Substitution above gives,

$$\int_0^a \cos \emptyset\, (S_o - S_o^*)\, rdr = 0$$

that is,

$$\mathrm{Im} \int_0^a \cos \emptyset\, S_o(r)\, rdr = 0$$

which is the same as,

$$\mathrm{Im} \int_0^a (1-\cos\phi)\, S_o(r) r dr = 0$$

Therefore, substituting,

$$\mathrm{Im} \sum_{m=0}^{\infty} a_{om} \int_0^a \frac{r^2}{4\lambda^2+r^2} (r/a)^{2m}\, r dr = 0.$$

Integration of the individual terms gives,

$$\int_0^a \frac{r^2}{4\lambda^2+r^2} (r/a)^{2m}\, r dr$$

$$= \frac{1}{2a^{2m}} \left[\frac{y^{m+1}}{m+1} - \frac{m+1}{m} 4\lambda^2 y^m + \frac{(m+1)m}{2\,(m-1)} (4\lambda^2)^2 y^{m-1} - \cdots \right.$$
$$\left. -(-1)^m (4\lambda^2)^{m+1} \ln y \right]_{y=a^2}^{y=a^2+4\lambda^2}$$

Putting $\lambda = a/2$ leads to

$$\mathrm{Im} \Big[(1 - \ln 2)a_{oo} + (-1/2 + \ln 2)\, a_{o1} + (5/6 - \ln 2)\, a_{o2}$$
$$+ (-7/12 + \ln 2)\, a_{o3} + (47/60 - \ln 2) a_{o4} + \cdots\cdots \Big] = 0$$

APPENDIX V. LOMMEL FUNCTIONS.

$c = 7.5 \quad b = 10\pi \quad t = 11.25 \quad w = 1.5Z$

Z	V_2	V_1	δV_2	δV_1	$\lvert\delta W_2\rvert$	Angle (Degrees)
0	.0000	.0000				
1	.0041	.0743	.0041	.0743	.0743	87
2	.0339	.0846	.0298	.0103	.0315	19
3	.0262	-.1177	-.0077	-.2023	.2023	-92
4	-.0927	-.1509	-.1189	-.0332	.1234	-164
5	-.0820	+.1894	+.0107	+.3403	.3403	88
5.70	+.1215	.2974	.2035	.1080	.2303	28

$c = 10 \quad b = 10\pi \quad t = 20 \quad w = 2Z$

Z	V_2	V_1	δV_2	δV_1	$\lvert\delta W_2\rvert$	Angle (Degrees)
0	.0000	.0000				
1	.0035	.0575	.0035	.0575	.0575	87
2	.0141	-.0166	.0106	-.0741	.0749	-82
3	-.0246	-.0853	-.0387	-.0687	.0792	-119
4	-.0142	+.1139	+.0104	+.1992	.1992	87
5	+.0761	.0060	.0903	-.1079	.1407	-50
5.70	-.0005	-.1798	-.0766	-.1858	.2010	-112
6	-.0651	-.1755	-.1412	-.1815	.2300	-128
7	-.0703	+.1962	-.0052	+.3717	.3717	91

$C = 10 \quad b = 10\pi \quad t = 20 \quad w = 2Z$

Z	V_2	V_1	δV_2	δV_1	$\lvert\delta W_2\rvert$	Angle (Degrees)
0	.0000	.0000				
0.5	.0003	.0220	.0003	.0220	.0220	89
1.0	.0035	.0575	.0032	.0355	.0355	85
1.5	.0109	.0499	.0074	-.0073	.0104	-45
2.0	.0141	-.0166	.0032	-.0668	.0668	-87
2.5	.0014	-.0873	-.0127	-.0707	.0718	-100
3.0	-.0246	-.0853	-.0232	+.0020	.0232	175
3.5	-.0387	+.0073	-.0140	.0930	.0940	99
4.0	-.0142	.1139	+.0244	.1062	.1089	77
4.5	+.0414	.1248	.0556	.0109	.0566	11
5.0	.0761	.0060	.0347	-.1188	.1238	-74
5.5	.0366	-.1462	-.0395	-.1522	.1572	-105
6.0	-.0651	-.1755	-.1017	-.0293	.1057	-164
6.5	-.1336	-.0201	-.0685	+.1554	.1698	114
7.0	-.0703	+.1962	+.0633	.2163	.2256	74
7.5	+.1060	.2459	.1763	.0497	.1830	16

c = 12.5 b = 10π t = 31.25 w = 2.5Z

Z	V_2	V_1	δV_2	δV_1	$\lvert\delta W_2\rvert$	Angle (Degrees)
0	.0000	.0000				
1	.0029	.0397	.0029	.0397	.0397	86
2	.0009	-.0539	-.0020	-.0936	.0936	-91
3	-.0133	.0362	-.0142	.0901	.0912	99
4	+.0284	.0139	.0417	-.0223	.0472	-28
5	-.0343	-.0725	-.0627	-.0864	.0937	-126
6	+.0188	+.1227	+.0531	+.1952	.2022	75
7	+.0237	-.1358	.0049	-.2585	.2585	-89

c = 15 b = 10π t = 45 w = 3Z

Z	V_2	V_1	δV_2	δV_1	$\lvert\delta W_2\rvert$	Angle (Degrees)
0	.0000	.0000				
1	.0022	.0225	.0022	.0225	.0225	84
2	-.0044	-.0371	-.0066	-.0596	.0599	-96
3	+.0062	+.0505	+.0106	+.0876	.0883	82
4	-.0070	-.0634	-.0132	-.1139	.1148	-97
5	+.0064	+.0760	+.0134	+.1394	.1394	85
6	-.0039	-.0888	-.0103	-.1648	.1648	-94
7	-.0011	+.1018	+.0028	+.1906	.1906	89

			$c = 17.5$	$b = 10\pi$	$t = 61.25$	$w = 3.5Z$
Z	V_2	V_1	δV_2	δV_1	$\lvert\delta W_2\rvert$	Angle (Degrees)
0	.0000	.0000				
1	.0019	.0078	.0019	.0078	.0080	76
2	-.0040	-.0003	-.0059	-.0081	.0100	-126
3	+.0066	-.0144	+.0106	-.0143	.0178	-54
4	-.0081	+.0327	-.0147	+.0471	.0493	107
5	+.0070	-.0571	+.0151	-.0898	.0912	-80
5.70	-.0190	+.0229	-.0260	+.0800	.0841	98
6	-.0018	.0494	-.0088	.1065	.1065	95
7	-.0084	-.0748	-.0066	-.1242	.1242	-93
7.10	-.0194	-.0690	-.0110	+.0058	.0124	152
8	+.0236	+.0726	+.0320	+.1474	.1507	78
9	-.0425	-.0580	-.0661	-.1306	.1464	-117
9.5	-.0208	+.0341	+.0217	+.0921	.0947	77

t = 5

w	U_1	U_o	δU_1	δU_o	$\|\delta U_1 + i\delta U_o\|$	Angle(Degrees)
0	.5985	-.8011				
0.5	.5541	-.8156	-.0444	-.0145	.0467	198
1	.4277	-.8523	-.1264	-.0367	.1316	196
1.5	.2388	-.8937	-.1889	-.0414	.1934	192
2.0	.0166	-.9155	-.2222	-.0218	.2233	185
2.5	-.2055	-.8929	-.2221	+.0226	.2233	175
3	-.3954	-.8084	-.1899	.0845	.2079	156
3.5	-.5273	-.6561	-.1319	.1523	.2015	131
4	-.5854	-.4447	-.0581	.2114	.2193	105
4.5	-.5665	-.1974	+.0189	.2473	.2480	86
5	-.4794	+.0531	.0871	.2505	.2652	71
5.5	-.3435	.2701	.1359	.2170	.2561	58
6	-.1839	.4215	.1596	.1514	.2200	44
6.5	-.0270	.4860	.1569	.0645	.1697	22
7	+.1029	.4580	.1299	-.0280	.1329	-12
7.5	.1895	.3484	.0866	-.1096	.1397	-52
8	.2252	.1843	.0357	-.1641	.1679	-78
8.5	.2118	.0009	-.0134	-.1834	.1839	-94
9	.1601	-.1639	-.0517	-.1648	.1727	-107
9.5	.0854	-.2782	-.0747	-.1143	.1365	-123
10	.0059	-.3220	-.0795	-.0438	.0908	-151

REFERENCES

1. Nijboer, B.R.A., Physica, 10, 679(1943); Physica, 13, 605 (1947).

2. Zernike, F., and Nijboer, B.R.A., Contribution in La Théorie des Images Optique p.227 (1949)

3. Woonton, G.A., J.Appl.Phys. 21, 577 (1950)

4. Woonton, G.A., Carruthers, J.A., Elliott, H.A., and Rigby, E.C., J.Appl.Phys. 22, 390 (1951)

5. Hogg, D.C., McGill University Ph.D. Thesis (1953)

6. Andrews, C.L., Phys. Rev. 71, 777 (1947)

7. Bekefi, G. and Woonton, G.A., Eaton Electronics Research Laboratory Report No. b7 (1952)

8. Baker, B.B., and Copson, E.T., The Mathematical Theory of Huygens' Principle 2nd Ed. Oxford University Press Chap. II, (1950)

9. Silver, S., University of California, Antenna Laboratory, Report No. 163 (1949)

10. Copson, E.T., Proc. Roy. Soc. (A), 186, 100 (1946)

11. Silver, S., and Ehrlich, M.J., Antenna Laboratory Report No. 181, Dept. of Engineering, Univ. of California (1951)

12. Neugebauer, H.E.J., Eaton Electronics Research Laboratory Report No. b6 (1952)

13. Bekefi, G., Eaton Electronics Research Laboratory Report No. b5, (1952)

14. Whittaker, E.T., and Watson, G.N., Modern Analysis 3rd Ed., Cambridge University Press, p. 179

15. Silver, S., Microwave Antenna Theory and Design, McGraw-Hill (1949) Section 3.11

16. Watson, G.N., Theory of Bessel Functions, 2nd Ed. Macmillan, (1945)

17. Lommel, E., Bayerisch. Akad. d. Wiss., 15, 233 (1884); English translation by G. Bekefi and G.A. Woonton, Eaton Electronics Research Laboratory Report No. 4

18. Silver, S., Microwave Antenna Theory and Design, McGraw-Hill (1949) p.255

19. Tyson, O.A., RL-MIT Report No. 601-3 (1944)

20. vanKampen, N.G., Physica, 14, 575 (1949)