THE UNIVERSITY OF ALBERTA

PROBLEM OF TIES IN SOME NONPARAMETRIC TESTS

BY

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

IN

MATHEMATICAL STATISTICS

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

FALL, 1974

ABSTRACT

In nonparametric testing based on ranks, the occurence of ties is a fairly common phenomenon. Many methods have been suggested for assigning ranks to tied observations in such problems. In this thesis we review and discuss these methods with regards to their relative merits under different situations. The recent extention of the asymptotic theory of rank statistics from continuous to discontinuous distributions has made it possible to calculate the Asymptotic Relative Efficiency (ARE) of different methods of handling ties. A comparison, based on ARE, among the three main methods of handling ties, namely, the average scores, midranks and randomized ranks methods, has been discussed.

ACKNOWLEDGEMENTS

I wish to thank Professor K.L. Mehra for his guidance and encouragement during the preperation of this thesis. Thanks are also due to Dr. M.S. Rao for some very useful suggestions and discussions.

I would also like to thank The University of Alberta for the financial help and Miss Olwyn Buckland for the care she took in typing this thesis.

Finally, I must thank my wife Kamlesh for all her efforts, patience and encouragement during the course of this study.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Most nonparametric test procedures based on ranks, assume the continuity of underlying distributions. In the case when data consist of independent observations, this assumption makes the (theoretical) considerations of ties unnecessary, since then no tie can occur with positive probability. In practice however, ties do occur even when the underlying distributions may be assumed continuous. This happens due to various reasons, such as rounding off errors, limited refinement of measuring instruments etc.. In the discontinuous case however, ties cannot be ignored even in theoretical considerations. Since the occurence of ties is fairly common in most practical data, it should be of considerable interest to statisticians using nonparametric methods to study the operating characteristics of various methods of dealing with tied observations.

In the present study we discuss this problem when the set of observations are mutually independent and the underlying distributions are either continuous or discontinuous (including discrete). As early as in 1945 this problem was recognized to be of practical importance and methods of treating tied observations in two-sample test were proposed by Wilcoxon (1945). Kendall (1970) and Krushkal and Wallis (1952) have also dealt this problem in their respective tests. Putter (1955) considers the case of purely discrete observations and examines the merits and demerits of *randomized* vs. *non-randomized* methods of treating ties. A study by Pratt (1959) gives intuitive arguments to show that the procedure suggested by Wilcoxon dealing with zeros in Wilcoxon signed rank test has many short comings. Pratt suggested a new method which ranks zero also and then dropping their ranks, whereas zeros are totally ignored in the procedure proposed by Wilcoxon. (See Conover (1973b) for a detailed comparison of these two methods.) After Putter's study many papers dealing with nonparametric tests and their efficiencies in the discrete case have appeared in literature. Chanda (1963) obtains the efficiencies of Wilcoxon two sample test under different discrete distributions. Among others, Bühler (1967), Klotz (1966), Krauth (1971), Taylor (1964) and Verlíčková (1972) should be mentioned. Conover (1973a) discusses the tests of randomness and symmetry under general case (with no continuity assumption).

In Chapter 2 we give an account of different methods of handling ties and demonstrate these by means of an example. Some *major* nonparametric tests in forms appropriate for continuous distributions are also described in this chapter. As pointed out earlier, ties in this situation do not present any problem and may be treated by any of the three methods described in Section 2.2. In Sections 2.3, 2.4 and 2.5, we discuss the common rank tests for randomness, symmetry and independence respectively. Section 2.6 deals with one and two way layouts Analysis of Variance rank tests and also a new class of conditional tests for two way layout Analysis of Variance problem as proposed by Hodges and Lehmann (1962) and discussed at length in Mehra and Sarangi (1967), Mehra (1968) and Sen (1968).

Section 3.1 of Chapter 3 includes a treatment of ties in general for various tests discussed in Chapter 2. The conditional means and variances given a vector of ties, have been given in this section for large sample approximations. In Section 3.2, 3.3 and 3.4, respectively, the Wilcoxon two sample test, the sign test and the Wilcoxon one sample signed rank test are discussed in detail, in case the ties are present in the data. A review of the literature dealing with common one and two sample tests is also given in these sections.

Chapter 4 deals with the asymptotic efficiencies of the tests discussed earlier both with and without the assumption of continuity of the distribution function. In Section 4.1 we introduce the concept of asymptotic efficiency. Section 4.2 gives the asymptotic distributions of the linear rank statistics under different null hypotheses and in different cases arising from different methods of handling ties. In Section 4.3 the asymptotic distribution of linear rank statistics under contiguous alternatives are given. In Section 4.4 expression for asymptotic efficiencies under appropriate conditions are derived. In this section we also show that Putter's (1955) result is a special case of a result due to Conover (1973a). In Section 4.5 we show with the help of two examples that in testing for symmetry using rank tests, the two methods of handling ties at zero, described later, are superior than each other in different situations.

Finally, Chapter 5 gives some general remarks and conclusions which should be of practical value for users of nonparametric methods.

CHAPTER 2

TIES IN CONTINUOUS CASE

In this chapter we introduce some major nonparametric (rank) tests and describe for these tests various methods of handling ties proposed and discussed in the literature.

2.1 NOTATION AND PRELIMINARIES.

Let $x^{(i)}$ denote the ith smallest co-ordinate in n-tuple $x = (x_1, x_2, \ldots, x_n)$ so that $x^{(1)}$ and $x^{(n)}$ denote the minimum and maximum of n-coordinates respectively. If $X = (X_1, X_2, \ldots, X_n)$ the vector of n observations, the statistic $X^{(i)}$ is called the ith order statistic and $X^{(\cdot)} = (X^{(1)}, X^{(2)}, \ldots, X^{(n)})$ is the vector of order statistics.

Now suppose that with probability one no two co-ordinates in X coincide; this is the case, for example, when $X_1, X_2, \ldots, X_n$ are independent observations with common continuous distribution function $F(x) = P\ (X \leq x)$. Then

$$R_i(X) = \# X\text{'s} \leq X_i \qquad (2.1.1)$$

is called rank of X_i. Clearly,

$$X_i = X^{(R_i)} \qquad (2.1.2)$$

The statistic $R = (R_1, R_2, \ldots, R_n)$ denotes the vector of ranks.

The rank tests under consideration can be divided in four major groups.

Group 1. Tests of Randomness

Group 2. Tests of Symmetry

Group 3. Tests of Independence

Group 4. Analysis of Variance Tests.

Let

$$S = \sum_{i=1}^{n} C_i \, a(R_i) \quad , \tag{2.1.3}$$

where $a(\cdot)$ is a function on $\{1,2,\ldots,n\}$ and C_i are the so called regression constants. We shall denote a_i for $a(i)$, $i = 1,2,\ldots,n$; a_i's are called rank scores. S is called linear rank statistic. For different score functions $a(\cdot)$ and appropriate constants C_i , the statistic (2.1.3) covers most of the test statistics under the above four groups.

2.2 RANKING OF TIES.

Although the probability of a tie is zero when observations are independent and have a continuous distribution function, ties do occur in practice as stated earlier. The probability distribution of S , defined by (2.1.3) and the properties of tests based on S , however, remain unchanged when any of the methods described below is used to assign ranks to tied observations. Following are the three most commonly used methods of ranking tied observations.

(a) Randomization method

(b) Averaged score method

(c) Midrank method.

Let

$$(2.2.1) \qquad x^{(1)} = \cdots = x^{(\tau_1)} < x^{(\tau_1+1)} = \cdots$$

$$= x^{(\tau_1+\tau_2)} < \cdots < x^{(\tau_1+\cdots+\tau_{g-1}+1)} = \cdots = x^{(n)} ,$$

where $\tau_1, \tau_2, \ldots, \tau_g$ are the sizes of ties and $\sum_{j=1}^{g} \tau_j = n$.

(a) RANDOMIZATION METHOD.

In the randomized rank procedure, we assign ranks to tied observations on the basis of some random experiment in which each permutation of tied observations has same probability of occurence. This random experiment is introduced only to deal with tied observations and it is in no way related to the basic experiment. The outcome of this experiment on the other hand, does affect the final decision.

(b) AVERAGED SCORE METHOD.

For a given vector of sizes of ties (to be called vector of ties henceforth) $\tau = (\tau_1, \ldots, \tau_g)$ and scores $a_1, a_2, \ldots, a_n$, we shall introduce averaged scores

(2.2.2) $$\bar{a}_i = \bar{a}(i,\tau) = \frac{1}{\tau_k} \sum_{j=\tau_1+\ldots+\tau_{k-1}+1}^{\tau_1+\ldots+\tau_k} a_j \quad ,$$

if $\tau_1 + \ldots + \tau_{k-1} < i \le \tau_1 + \ldots + \tau_k$.

Then (2.1.3) is modified as

(2.2.3) $$\bar{S} = \sum_{i=1}^{n} C_i \, \bar{a}(R_i,\tau) \quad .$$

(c) MIDRANK (AVERAGE RANK) METHOD.

We assign the midrank for all the observations tied. For example, if

$$x^{(\tau_{i-1}+1)} = \ldots = x^{\tau_i} \quad ,$$

we assign rank $\frac{\tau_{i-1}+\tau_i+1}{2}$ to $x^{(\tau_{i-1}+1)},\ldots,x^{(\tau_i)}$ and then also define scores for half integer i .

(2.2.4) $$a(R_i,\tau) = a\left(\frac{\tau_{i-1}+\tau_i+1}{2}\right) \quad ,$$

and modifications similar to (2.2.3) can be incorporated.

Now we give an example to illustrate the above three methods.

EXAMPLE 2.1: Let S be as in (2.1.3) with $C_i = 1$ for $i = 1,2,\ldots,5$,

$$a(R_i) = \psi\left(\frac{R_i}{n+1}\right) \; , \quad n = 10$$

and $X = (X_1,X_2,\ldots,X_{10}) = (15,16,17,18,25,16,17,19,16,20)$, where

$\psi(t)$ denotes the quantile function of the standardized normal distribution function. We have X_2 , X_6 and X_9 tied for ranks 2 , 3 and 4 and X_3 and X_7 for 5 and 6 . In randomization we assign rank 4 to X_2 , 2 to X_6 and 3 to X_5 . Similarly 5 for X_3 and 6 for X_7 . Then we have

$$S_{(ran)} = -.11 ,$$

$$\bar{S} = S_{(ave)} = -.27 , \text{ and}$$

$$S_{(mid)} = -.25 .$$

Though it seems in this case that the above three values of S are quite close but this may not be the case in general. In some cases (see Section 3.1) the midrank and average score procedures are identical.

2.3 TESTS OF RANDOMNESS.

H_o : We shall say random variables $X_1, X_2, \ldots, X_n$ satisfy the hypothesis of randomness H_o , if they are independent and have common distribution function $F(x)$, i.e., if $P = L((X_1, \ldots, X_n))$, $P \in H_o$ iff

$$(2.3.1) \qquad P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n) = F(x_1) \cdot F(x_2) \cdot \ldots \cdot F(x_n) ,$$

$$-\infty < x_1 < x_2 < \ldots < x_n < \infty ,$$

where F is continuous.

$\overline{H}_o$: When F is arbitrary in H_o' , we have $\overline{H}_o$. Clearly $H_o \subset \overline{H}_o$.

Suppose that the observations are recorded under different conditions. We want to know whether the differences in conditions have affected their distributions (X_i's are assumed to be independent). In other words, we want to test

$$(2.3.2) \quad \begin{aligned} &H_o : F_i = F \ , \quad 1 \le i \le n \quad \text{for some } F \\ &\text{vs.} \quad K : F_i \ne F_j \ , \quad \text{for some } i \ne j \ , \quad 1 \le i \ , j \le n \ . \end{aligned}$$

The alternative, say K , may be described as a family of distributions Q for the form (2.3.1) ; $K = \{Q\}$. The following results whose proof is elementary are needed for our later work. Let $\mathcal{R}$ denote the space of all permutations of $1,2,\ldots,n$. If

$$(2.3.3) \qquad \overline{a} = \frac{1}{n}\sum_{i=1}^{n} a_i \ , \quad \overline{C} = \frac{1}{n}\sum_{i=1}^{n} C_i$$

and R is uniformly distributed over $\mathcal{R}$, then

$$(2.3.4) \qquad \begin{aligned} E(S) &= \frac{1}{n}\sum_{i=1}^{n} C_i \sum_{i=1}^{n} a_i \ , \quad \text{and} \\ V(S) &= \frac{1}{n-1}\sum_{i=1}^{n} (C_i-\overline{C})^2 \sum_{i=1}^{n} (a_i-\overline{a})^2 \ , \end{aligned}$$

where S is given by (2.1.3).

Let $X_1, X_2, \ldots, X_{n_1}, X_{n_1+1}, \ldots, X_{n_1+n_2}$ $(n_1+n_2 = n)$ denote two combined samples of sizes n_1 and n_2. Let F_1 and F_2 be the distribution functions of the populations from which the first and the second samples are drawn, respectively. We want to test $F_1 = F_2$. Now suppose F_1 and F_2 differ in location only, i.e., $F_1(x) = F(x-\mu_1)$ and $F_2(x) = F(x-\mu_2)$ holds for some F and some constant μ_1 and μ_2, then we have the two sample location alternative:

$$(2.3.5) \qquad Q(X_1 \le x_1, \ldots, X_n \le x_n) = \prod_{i=1}^{n_1} F(x_i-\mu_1) \prod_{j=1}^{n_2} F(x_{n_1+j}-\mu_2) \quad .$$

If sign of $(\mu_1-\mu_2)$ is known, the alternatives are one sided, otherwise two sided. These alternatives may be summarized as one sided (i) $\mu_1 > \mu_2$, (ii) $\mu_1 < \mu_2$ and two sided $\mu_1 \neq \mu_2$.

Now we discuss some of the important tests. For these tests, tables of exact probability points are available in many books, for example, Hollander and Wolfe (1973), for small sample sizes. In case of samples of larger size the approximations are given. Generally, the test procedure consists of rejecting H_o when S is too large or too small or both depending upon the nature of the alternative.

(a) WILCOXON TWO SAMPLE TEST.

Let $a_i = i$ and $C_i = 1$ for $i \le n_1$ and zero otherwise then

$$(2.3.6) \qquad S = \sum_{i=1}^{n_1} R_i = \text{sum of the ranks for first sample.}$$

By (2.3.4) under H_o we have

$$E(S) = \frac{1}{2} n_1(n+1) \quad , \quad V(S) = \frac{1}{12} n_1 n_2 (n+1) \quad . \tag{2.3.7}$$

We note again that the tied observations may be ranked according to one of the three procedures described in Section 2.2.

(b) MEDIAN TEST.

If

$$a_i = \begin{cases} 0 & \text{if} \quad i \leq \frac{1}{2}(n+1) \\ 1 & \text{if} \quad i > \frac{1}{2}(n+1) \end{cases} \tag{2.3.8}$$

and $C_i = 1$, $1 \leq i \leq n_1$, then

$$S = \sum_{i=1}^{n_1} a(R_i) = \sum_{i=1}^{n_1} u(R_i - \frac{1}{2} n - 1) \quad , \tag{2.3.9}$$

where $u(\cdot)$ is given by

$$u(x) = \begin{cases} 0 & , \quad x < 0 \\ 1 & , \quad x \geq 0 \quad , \end{cases} \tag{2.3.10}$$

is called the median statistic. By (2.3.4) under H_o we have

$$E(S) = \begin{cases} \frac{1}{2} n_1 & \text{if} \quad n \quad \text{even} \\ \frac{1}{2} n_1 \frac{n-1}{n} & \text{if} \quad n \quad \text{odd} \quad , \quad \text{and} \end{cases} \tag{2.3.11a}$$

$$V(S) = \begin{cases} \dfrac{n_1 n_2}{4(n-1)} & \text{if } n \text{ even} \\ \dfrac{n_1 n_2 (n+1)}{4n^2} & \text{if } n \text{ odd} \end{cases} \quad . \tag{2.3.11b}$$

REMARK 2.1: Using a combinatorial argument it can be shown that under H_0 , S follows the hypergeometric distribution (see for example [8]).

(c) THE VAN DER WAERDEN TEST.

In Wilcoxon two-sample test if we set $a_i = \psi(\frac{i}{n+1})$ where $\psi(\cdot)$ is the same as defined in Example 2.1 then we have the Van der Waerden test statistic

$$S = \sum_{i=1}^{n_1} \psi(\frac{R_i}{n+1}) \quad . \tag{2.3.12}$$

By (2.3.4) under H_0 we have

$$E(S) = 0 \quad , \quad V(S) = \frac{n_1 n_2}{n(n-1)} \sum_{i=1}^{n} \psi^2(\frac{i}{n+1}) \quad . \tag{2.3.13}$$

REMARK 2.2: In tests (a), (b) and (c) under certain regularity conditions (see [8]) $S-E(S)/(V(S))^{1/2}$ follows approximately $N(0,1)$ and hence when sample sizes are large we can apply this approximation.

(d) KRUSHKAL-WALLIS k - SAMPLE TEST.

In the setup of (2.3.5) suppose we have k-sample instead of two and want to test the equality of μ_i's , $i = 1,2,\ldots,k$, Krushkal and Wallis (1952) have introduced the following test.

Let us rank the pooled sample and if S_j be the sum of the ranks of jth sample, then the test statistic is defined by

(2.3.14) $$H = \frac{12}{n(n+1)} \sum_{j=1}^{k} \frac{S_j^2}{n_j} - 3(n+1) ,$$

where $n_1 + n_2 + \ldots + n_k = n$. Exact percentage points of H are given in Krushkal and Wallis (1952) for small n_i's . If n_i's are large, H follows Chi-square distribution with (k-1) degrees of freedom (d.f.). The test consists of rejecting the null hypothesis when H is large.

REMARK 2.3. When $k = 2$, the above test reduces to the Wilcoxon two-sample test.

2.4 TESTS OF SYMMETRY.

H_1 : We shall say the random variable $X_1, X_2, \ldots, X_n$ satisfy the hypothesis of symmetry H_1 , if $P \in H_o$ given by (2.3.1), with F satisfying symmetry conditions, i.e.,

(2.4.1a) $$F(x) + F(-x) = 1 \quad \text{for all} \quad x .$$

$\overline{H}_1$: P satisfies all other conditions of H_1 except the continuity of F .

We can confine our attention only to testing symmetry with respect to the origin, as the other cases follow with trivial modifications. Clearly under H_1 if $f(x) = \frac{d\,F(x)}{dx}$ exists at a point x we have

$$f(x) = f(-x) \quad .$$

The alternative against which H_1 is frequently tested is the shift in median from zero to some point Δ. Let the alternative K be a family of distributions Q of the form

$$Q(X \le x_1, \ldots, X_n \le x_n) = \prod_{i=1}^{n} F(x_i - \Delta) \quad . \tag{2.4.1b}$$

If the sign of Δ is known, we have one sided alternatives otherwise two sided. Let

$$R_i^+ = \sum_{j=1}^{n} u(|X_i| - |X_j|)$$

$$= \text{rank of } X_i \text{ when } |X_i|\text{'s are ranked,}$$

where $u(\cdot)$ is defined by (2.3.10). S takes the following form when we use it to test H_1 .

$$S = \sum_{i=1}^{n} u(X_i)\, a(R_i^+) \quad . \tag{2.4.2}$$

Under H_1 , we have

$$E(S) = \frac{1}{2} \sum_{i=1}^{n} a_i \quad \text{and} \quad V(S) = \frac{1}{4} \sum_{i=1}^{n} a_i^2 \quad . \tag{2.4.3}$$

Now we shall describe some of the important special cases of the statistics (2.4.2). Tables for tests (a) and (b) may be found in Hollander and Wolfe [13]. Also, under certain regularity conditions in tests (a) and (b) (see [8]) $\{S-E(S)\}/\{V(S)\}^{1/2}$ follows $N(0,1)$ as $n \to \infty$ and hence when n is large normal tables may be used.

(a) SIGN TEST.

Let $a_i = 1$, $1 \leq i \leq n$. Then (2.4.2) and (2.4.3) give

$$S = \sum_{i=1}^{n} u(x_i) = \# \text{ of positive observations}$$

(2.4.4)

$$E(S) = \frac{n}{2} \quad , \quad V(S) = \frac{n}{4} \qquad (\text{under } H_1) \quad .$$

REMARK 2.4: Note that if there are some zero observations, we may follow one of the two methods outlined in Sections 3.1 and 4.5.

(b) WILCOXON ONE-SAMPLE (SIGNED RANK) TEST.

Set $a_i = i$, $1 \leq i \leq n$ and we have from (2.4.2) and (2.4.3)

$$S = \sum_{i=1}^{n} R_i^+ u(X_i) = \text{sum of ranks of positive observations,}$$

(2.4.5)

$$E(S) = \frac{1}{4} n(n+1) \quad \text{and} \quad V(S) = \frac{1}{24} n(n+1)(2n+1) \quad (\text{under } H_1) \quad .$$

Remark 2.4 is applicable here also.

(c) MEHRA'S TEST FOR PAIRED COMPARISONS (k-sample).

Let us consider a paired comparison experiment involving k-treatments. Suppose that the n_{ij} independent comparisons for the pair (i,j) of treatments $(1 \leq i < j \leq k)$ provide observed comparison differences $Z_{ij\ell}$ $(\ell = 1,2,\ldots,n_{ij})$. Let $G_{ij}(z)$ denote their common distribution function which is assumed to be continuous. The hypothesis of equality among the treatments can be expressed as

H_1' : $G_{ij}(z) + G_{ij}(-z) = 1$ and $G_{ij}(z) = G_{i'j'}(z)$ for any two pairs (i,j) and (i',j') .

The alternative may be 'not H_1''. We rank the absolute values of $n \; (= \sum_{i<j}\sum n_{ij})$ comparison differences $Z_{ij\ell}$ $(1 \leq i < j \leq k$, $\ell = 1,2,\ldots,n_{ij})$ in a pooled sample. In case of ties we may use one of the methods described in Section 2.2. Let $r_{ij\ell}$ be the rank of $|Z_{ij\ell}|$ if $Z_{ij\ell} \geq 0$ and zero otherwise; similarly, let $s_{ij\ell}$ be the rank of $|Z_{ij\ell}|$ if $Z_{ij\ell} < 0$ and zero otherwise. Also, denote $R_n^{(i,j)} = \sum_{\ell=1}^{n_{ij}} r_{ij\ell}$ and $S_n^{(i,j)} = \sum_{\ell=1}^{n_{ij}} s_{ij\ell}$. Under this set up Mehra (1964) suggested the following test statistic.

$$L = 6[(n+1)(2n+1)k]^{-1} \sum_{i=1}^{k} \{\sum_{j\neq i}^{k} V^{(i,j)}/n_{ij}^{1/2}\}^2 , \qquad (2.4.6)$$

where $V_n^{(i,j)} = R_n^{(i,j)} - S_n^{(i,j)}$.

The test consists of rejecting H_1' at level α if $L > \ell_\alpha$, where ℓ_α satisfies $P_{H_1'}[L \geq \ell_\alpha] = \alpha$. No tables for probability distribution of L are available, for small n. But, for large n, L is asymptotically distributed as the Chi-square distribution with $(k-1)$ d.f. under H_1' (see Mehra (1964)).

REMARK 2.5: For $k = 2$, the above test reduces to the Wilcoxon paired comparison test.

2.5 TESTS OF INDEPENDENCE.

H_2 : We shall say that the family $\{(X_1,Y_1),(X_2,Y_2),\ldots,(X_n,Y_n)\}$ satisfies the hypothesis of independence H_2, if all the $2n$ random variables are mutually independent, X_i's having arbitrary continuous distribution function $F(x)$ and Y_i's having an arbitrary continuous distribution function $G(y)$. We may write it as

$$(2.5.1)\quad P(X_1\leq x_1,Y_1\leq y_1,X_2\leq x_2,Y_2\leq y_2,\ldots,X_n\leq x_n,Y_n\leq y_n) = \prod_{i=1}^{n} F(x_i)G(y_i) \quad .$$

$\overline{H}_2$: Continuity assumption regarding $F(x)$ and $G(y)$ in H_2, is dropped. Let us consider alternatives $K = \{Q\}$ against H_2, where Q's are given by

$$(2.5.2)\quad Q(X_1\leq x_1,Y_1\leq y_1,X_2\leq x_2,Y_2\leq y_2,\ldots,X_n\leq x_n,Y_n\leq y_n) = \prod_{i=1}^{n} A(x_i,y_i) \quad ,$$

where $A(x,y)$ is a continuous two-dimensional distribution function. The pairs (X_i,Y_i) are independent under Q but within pairs there is a dependence. Statistics for testing H_2 against alternatives of the form (2.5.2) are given by,

(2.5.3) $$S = \sum_{i=1}^{n} a(R_i)\, b(Q_i) \quad ,$$

where

$$R_i = \sum_{j=1}^{n} u(X_i - X_j) = \text{rank of } X_i \text{ in seperate ranking}$$

and

$$Q_i = \sum_{j=1}^{n} u(Y_i - Y_j) = \text{rank of } Y_i \text{ in seperate ranking;}$$

both scores are non-decreasing, $u(\cdot)$ is given by (2.3.10). Tables for the tests described below are given in [13].

(a) SPEARMAN TEST.

Taking the Wilcox scores $a_i = b_i = i$, we get from (2.5.3)

(2.5.4) $$S = \sum_{i=1}^{n} R_i Q_i \quad ,$$

and it can be easily verified that under H_2

(2.5.5) $$E(S) = \frac{1}{4} n(n+1)^2 \quad \text{and} \quad V(S) = \frac{1}{144} n^2(n+1)^2(n-1) \quad ,$$

and distribution of S is approximately normal for large values of n. If S is linearly transformed so that the minimum and maximum values are -1 and $+1$ respectively, it is called the *Spearman Rank Correlation Coefficient* and is given by

$$(2.5.6)\qquad \rho = \frac{12}{n^3-n}\,[S-E(S)] = \frac{12}{n^3-n}\sum_{i=1}^{n}(R_i - \tfrac{1}{2}n - \tfrac{1}{2})(Q_i - \tfrac{1}{2}n - \tfrac{1}{2}) \quad .$$

The test consists of rejecting H_2 for large $|\rho|$.

REMARK 2.6: If we use median scores as in Section 2.3, the test is called *Quadrant test*.

(b) KENDALL τ-TEST.

Kendall [14] suggests the following statistic for testing H_2 (we use τ^* to distinguish between the vector of ties τ and Kendall τ).

$$(2.5.7)\qquad \tau^* = \frac{1}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\operatorname{sign}(R_i-R_j)\operatorname{sign}(Q_i-Q_j) \quad .$$

The maximum and minimum values of τ^* are +1 and -1 respectively. The test consists of rejecting H_2 when $|\tau^*|$ is large. For large sample approximation, let us note that τ^* is a linear function of

$$(2.5.8)\qquad K = \sum_{i \neq j}\sum u(R_i-R_j)\,u(Q_i-Q_j) \quad ,$$

namely, $n(n-1)(1+\tau^*) = 4K$. Under H_2 we have

$$(2.5.9)\qquad E(K) = \frac{1}{4}n(n-1) \quad , \quad V(K) = \frac{1}{72}n(n-1)(2n+5) \quad ,$$

and the distribution of K is approximately normal for large n (see Kendall [14]).

2.6 ANALYSIS OF VARIANCE TESTS.

We have already described one-way layout Analysis of Variance design (Krushkal-Wallis test), let us consider now two-way layout with m_{ij} independent random observations $X_{ij\ell}$, $\ell = 1,2,\ldots,m_{ij}$ in the (i,j)th cell $j = 1,2,\ldots,k$, $i = 1,2,\ldots,n$. Here k is the number of treatments and n is the number of blocks. Let $X_{ij\ell}$, $1 \leq \ell \leq m_{ij}$, be distributed according to common continuous distribution function

$$F_{ij}(x) = F_j(x+\xi_i) \quad , \tag{2.6.1}$$

where ξ_i may represent the unknown block effects. Then the hypothesis of equality of the treatments effects (the null hypothesis) H_3, can be defined as

$$H_3 : F_1 = F_2 = \ldots = F_k . \tag{2.6.2}$$

$\overline{H}_3$ can be defined as usual dropping the continuity assumption in H_3. In the following, we shall describe two tests: one based on seperate rankings (Friedman test) and the other based on joint ranking of observations after 'alignment'.

REMARK 2.7: The model (2.6.1) is used for aligned rank test only. For Friedman's test we need the following weaker condition

$$F_{ij} = F_i \quad \text{for some } F_i \quad , \quad i = 1,2,\ldots,n \quad .$$

(a) FRIEDMAN'S TEST.

Let $m_{ij} = 1$ for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,k$. Within each block rank the k observations. Let r_{ij} denote the rank X_{ij} in ranking of $X_{i1},\ldots,X_{ik}$. Set $R_j = \sum_{i=1}^{n} r_{ij}$, $R_{\cdot j} = \frac{R_j}{n}$ and $R_{\cdot\cdot} = \frac{k+1}{2}$. Then Friedman's (1937) test statistic is

$$
\begin{aligned}
Q &= \frac{12n}{k(k+1)} \sum_{j=1}^{k} (R_{\cdot j} - R_{\cdot\cdot})^2 \\
&= \left[\frac{12}{nk(k+1)} \sum_{j=1}^{k} R_j^2\right] - 3n(k+1) \quad .
\end{aligned} \tag{2.6.3}
$$

We reject H_3 when Q is large. Tables for exact probability points are available in Hollander and Wolfe (1973). Under H_3 Q has asymptotically, as $n \to \infty$, Chi-square distribution with $(k-1)$ degrees of freedom.

(b) CONDITIONAL ALIGNED RANK TEST.

The approach here is entirely different than that in Friedman's test. We use the rank comparison after the 'alignment' (defined below) as presented in Hodges and Lehmann (1962), Mehra and Sarangi (1967), Mehra (1968) and Sen (1968).

Alignment essentially means removing the block effects ξ_i $(i = 1,2,\ldots,n)$ from observations by subtracting from each observations in a block, say ith, some reasonable function μ of the observations in the block which satisfy the condition

(2.6.4) $$\mu(X_{i_{11}}+a,\ldots,X_{i\ell m_{i_1}}+a,\ldots,X_{ik_1}+a,\ldots,X_{i_k m_{i_k}}+a)$$

$$= \mu(X_{i_{11}},\ldots,X_{i\ell m_{i_1}},\ldots,X_{ik_1},\ldots,X_{ikm_{i_k}}) + a \quad .$$

Let the aligned observations be denoted by $Z_{ij\ell}$, $\ell = 1,2,\ldots,m_{ij}$, $1 \leq j \leq k$, $1 \leq i \leq n$ and $r_{ij\ell}$ be the rank of $Z_{ij\ell}$ in a combined ranking of all the $N = \sum_i N_i = \sum_i \sum_j m_{ij}$ aligned observations. Each conditional situation (given a set of ranks for each block) is referred to as a *configuration*. More precisely, if $r_i^{(1)} < r_i^{(2)} < \ldots < r_i^{(N_i)}$ then the configuration is simply an event $E = (r_1,r_2,\ldots,r_n)$ in our sample sapce. Note that only randomness (given a configuration) that remains is due to independent assignments of ranks to the treatment. Let $a(\cdot)$ be rank scores and

(2.6.5) $$T_{N_j} = \sum_i \sum_\ell a(r_{ij\ell})$$

$$= \text{sum of the rank scores for the } j\text{th treatment.}$$

Also, let $m_{ij} = m_j$, for all i and j (the complete case which also covers the equal observations per cell). Under this scheme N_i's are equal to $N' = \sum_j m_j$ and the proposed test function is (see Mehra [21]):

(2.6.6) $$L_n = [(N'-1)/N' \ (\sum_{i=1}^{n} \alpha_i^2)] \sum_{j=1}^{k} \frac{1}{m_j} \{T_{N_j} - m_j n \, \bar{a}\}^2 \quad ,$$

where $\alpha_i^2 = \sum_j \sum_\ell \{a(r_{ij\ell}) - a(i)\}^2/N_i$ and $\bar{a} = \sum_i a(i)/n$. The test consists of rejecting H_3 for large values of L_n .

In the general case when m_{ij}'s do not satisfy the above condition, whether the design is complete or not, the following statistic is suggested ([21],(2.7)).

$$(2.6.7) \qquad \underset{\sim}{V}' \underset{\sim}{\Delta}^{-1} \underset{\sim}{V} ,$$

where $\underset{\sim}{V} = (V_{N1},\ldots,V_{N,k-1})$ with $V_{N_j} = [T_{N_j} - \tilde{E}(T_{N_j})]$, and $\underset{\sim}{\Delta}$ is the exact covariance matrix of $\underset{\sim}{V}$ (which is required to be non-singular) and $\tilde{E}(T_{N_j})$ and $\tilde{\sigma}_{jj'}$, elements of $\underset{\sim}{\Delta}$, are given by

$$\tilde{E}(T_{N_j}) = \sum_{i=1}^{n} m_{ij}\, a(i)$$

$$\tilde{\sigma}_{jj'} = \sum_{i=1}^{n} (N_i\, \delta_{jj'} - m_{ij})\, m_{ij'} \{\frac{\alpha_i^2}{N_i - 1}\} \quad , \quad \delta_{jj'} = 1$$

or zero according as $j = j'$ or not ($\tilde{E}$ and $\tilde{\sigma}_{jj'}$ represent the conditional expectation and covariance function under H_3 and condition (2.6.4)).

Computational techniques for the evaluation of exact distribution for the statistics of the type L_n are discussed in [12] with Wilcoxon score and $k = 2$. With the help of computers exact distribution tables can be prepared for various values of k , n and m_{ij} . As $n \to \infty$ the conditional statistic L_n , given a configuration, converges in distribution, (under certain regularity conditions, see [21]) to a chi-square variable with $(k-1)$ degrees of freedom.

CHAPTER 3

TIES IN DISCONTINUOUS CASE

In this chapter we discuss the tests described in Chapter 2 without the continuity assumption. Consequently, in this case ties may occur with positive probability. Expressions for conditional (given the vector of ties) expectations and variances are given for large sample approximations. For explicit results on asymptotic distributions, see Section 4.2 and [3].

3.1 TREATMENT OF TIES (GENERAL CASE).

In the following we shall describe the three methods of handling ties introduced in Section 2.2 for the general case.

(a) RANDOMIZATION.

Let $R^* = (R_1^*,\ldots,R_n^*)$ be the vector of ranks after randomization procedure (Section 2.2) applied for tied observations. The statistic $\sum C_i\, a(R_i^*)$ has the same distribution under $\bar{H}_o$ as that of $\sum_{i=1}^{n} C_i\, a(R_i)$ under H_o (Similarly for $\bar{H}_1$, $\bar{H}_2$ and $\bar{H}_3$); and so the same tables may be used. The asymptotic convergence of distributions for particular statistics is the same as described in Chapter 2.

(b) AVERAGED SCORES.

(i) *Tests of Randomness:* Let the scores $\overline{a}_i$ and statistic $\overline{S}$ be the same as in (2.2.2) and (2.2.3).

THEOREM 3.1: *Under* $\overline{H}_o$ *and arbitrary* τ *(the vector of ties) the statistic* $\overline{S}$ *satisfies*

$$E(\overline{S}/\tau) = \frac{1}{n} \sum_{i=1}^{n} C_i \sum_{i=1}^{n} a_i \ ,$$

(3.1.1)

$$V(\overline{S}/\tau) = \frac{1}{n-1} \sum_{i=1}^{n} (C_i-\overline{C})^2 \sum_{i=1}^{n} (\overline{a}_i-\overline{a})^2$$

and

$$\sum_{i=1}^{n} (\overline{a}_i-\overline{a})^2 = \sum_{i=1}^{n} (a_i-\overline{a})^2 - \sum_{i=1}^{n} (\overline{a}_i-a_i)^2 \ .$$

PROOF: It is easy to verify that

$$\overline{a}(R_i^*) = \overline{a}\ (R_i)$$

and hence $\overline{S}$ may be written equivalently as

$$\overline{S} = \sum_{i=1}^{n} C_i\ \overline{a}\ (R_i^*) \ .$$

According to Theorem 29A of Hájek (1969), the vectors R^* and τ are independent and hence

(3.1.2) $$E(\overline{S}/\tau) = E[\sum_{i=1}^{n} C_i\ \overline{a}(R_i^*,\tau)] = \sum_{i=1}^{n} C_i\ E(\overline{a}(R_i^*,\tau)) \ ,$$

where τ is considered fixed.

$$E(\bar{a}(R_i^*,\tau)) = \frac{1}{n}\sum_{i=1}^{n} \bar{a}_i(\tau)$$

$$= \frac{1}{n}\sum a_i = \bar{a} \quad .$$

Therefore by (3.1.2) we have

$$E(\bar{S}/\tau) = \frac{1}{n}\sum_{i=1}^{n} C_i \sum_{i=1}^{n} a_i$$

$$V(\bar{S}/\tau) = V[\sum_{i=1}^{n} C_i \ \bar{a}(R^*,\tau)]$$

$$= \frac{1}{n-1}\sum (C_i-\bar{C})^2 \sum_{i=1}^{n} (\bar{a}_i-\bar{a})^2 \quad ,$$

by Theorem 3B of Hájek (1969). Finally, it suffices for completing the proof to show that $\sum_{i=1}^{n} (\bar{a}_i-a_i)(\bar{a}_i-\bar{a}) = 0$. This follows from the fact that

$$\sum_{i=\tau_1+\ldots+\tau_{k-1}+1}^{\tau_1+\ldots+\tau_k} (\bar{a}_i-a_i)(\bar{a}_i-\bar{a}) = \tau_i(\bar{a}_k-\bar{a}_i)(\bar{a}_i-\bar{a}) = 0 \quad , \quad 1 \leq k \leq g \quad . \quad \square$$

In view of the above theorem, as the distribution of $\bar{S}$ depends upon the vector of ties we need different tables. Hájek (1969) suggests that if ties are few, we can use the same table for $\bar{S}$ as for S noting that the resulting critical levels will be somewhat larger than the exact conditional critical levels.

In case of the two-sample Wilcoxon test, using Theorem 3.1, we have

$$E(\overline{S}/\tau) = E(S)$$

and

$$(3.1.3) \qquad V(\overline{S}/\tau) = \frac{1}{n-1} \sum_{i=1}^{n} (C_i-\overline{C})^2 \sum_{i=1}^{n} (a_i-\overline{a})^2 \quad .$$

But as $a_i = i$

$$\sum_{i=1}^{n} (\overline{a}_i-a_i)^2 = \sum_{k=1}^{g} \Big(\sum_{i=\tau_1+\ldots+\tau_{k-1}+1}^{\tau_1+\ldots+\tau_k} \Big(\frac{\tau_1+\ldots+\tau_{k-1}+1+\tau_1+\ldots+\tau_k}{2} - i\Big)^2\Big)$$

$$= \sum_{k=1}^{g} \Big(\sum_{i=\tau_1+\ldots+\tau_k+1}^{\tau_1+\ldots+\tau_k} \{i^2 - (\tau_1+\ldots+\tau_{k-1} + \frac{\tau_k+1}{2})^2\}$$

$$= \sum_{k=1}^{g} \sum_{i=1}^{\tau_k} \{(x+i)^2 - (x + \frac{\tau_k+1}{2})^2\} \quad ,$$

where $x = \tau_1 + \ldots + \tau_{k-1}$. Thus

$$(3.1.4) \qquad \sum_{i=1}^{n} (\overline{a}_i-a_i)^2 = \sum_{k=1}^{g} \frac{1}{12} (\tau_k-1)\, \tau_k(\tau_k+1) \quad .$$

Therefore, using the last equation of (3.1.1), (3.1.3), (3.1.4) and (2.3.7) we have

$$(3.1.5) \qquad V(S/\tau) = \frac{n_1 n_2}{12} \Big[n+1 - \frac{\sum_{k=1}^{g} \tau_k(\tau_k^2-1)}{n(n-1)}\Big] \quad .$$

Similar expressions can also be obtained in the case of Median and Van der Waerden tests for expected values and conditional variances. For asymptotic distributions of these statistics see Chapter 4.

In k-sample case, we can show as in Wilcoxon test (see (3.1.4)), that the variance is reduced by

$$\sum_{i=1}^{g} \frac{1}{12} \tau_i(\tau_i-1)(\tau_i+1) \quad ,$$

and hence we modify H (2.3.4) accordingly (see [19]):

$$(3.1.6) \quad \overline{H} = 12[n(n+1) - \sum_{i=1}^{g} \frac{\tau_i(\tau_i-1)(\tau_i+1)}{(n-1)}]^{-1} [\sum_{j=1}^{k} \overline{S}_j^2/n_j - n(n+1)^2] \quad .$$

For $\min(n_1,\ldots,n_k) \to \infty$, $\overline{H}$ has Chi-square distribution with (k-1) d.f. . Tables for small n_i's are available in [19].

(ii) *Tests of Symmetry:* In testing $\overline{H}_1$ there are two types of problems; (1) zero observations, and (2) the ties among non-zero absolute values. There are two methods of handling zeros proposed by Wilcoxon (1945) and Pratt (1959). We will compare these two methods with the help of two examples in Section 4.5. Here we use Wilcoxon's method i.e., deleting zero observations altogether.

Let ν be the number of non-zero observations and let us denote them by $\hat{X}_1,\ldots,\hat{X}_\nu$. Let τ be the vector of ties in the sequence $|\hat{X}_1|,\ldots,|\hat{X}_\nu|$. Let

(3.1.7) $$\bar{S} = \sum_{i=1}^{\nu} u(X_i)\, \bar{a}(R_i^+) \quad ,$$

where $R_i^+ = \sum_{j=1}^{\nu} u(|\hat{X}_i|-|\hat{X}_j|)$, $1 \leq i \leq \nu$. It is easy to verify that under $\bar{H}_1$ (see [8] Theorem 30A) we have

$$E(\bar{S}/\nu,\tau) = \frac{1}{2} \sum_{i=1}^{\nu} a_i \quad ,$$

(3.1.8)

$$V(\bar{S}/\nu,\tau) = \frac{1}{4} \sum_{i=1}^{\nu} \bar{a}_i^{\,2} \quad ,$$

and

$$\sum \bar{a}_i^{\,2} = \sum_{i=1}^{\nu} a_i^{\,2} - \sum_{i=1}^{\nu} (\bar{a}_i - a_i)^2 \quad .$$

In the case of sign test, $a_i = 1$ and hence, $\bar{a}_i = 1$ for all τ . And therefore the distribution of $\bar{S}$ is the same as that of S with n replaced by ν . In one sample Wilcoxon test, proceeding the same way as in two sample case, we get

$$E(\bar{S}/\nu,\tau) = \frac{1}{4}\, \nu(\nu+1)$$

(3.1.9)

$$V(\bar{S}/\nu,\tau) = \frac{1}{24} \left[\nu(\nu+1)(2\nu+1) - \frac{1}{2} \sum_{j=1}^{g} \tau_j(\tau_j-1)(\tau_j+1)\right] .$$

For large sample approximations of above tests see Section 4.2.

For Mehra's k-sample test, we do not have the modified form of L , when ties are present. Also no asymptotic result regarding the distribution of L under $\bar{H}_1$ is available.

(iii) *Tests of Independence:* Let τ_x and τ_y be the sizes of ties in $(X_1,\ldots,X_n)$ and $(Y_1,\ldots,Y_n)$ respectively. We define $\bar{a}(i,\tau_x)$ and $\bar{a}(i,\tau_y)$ by (2.2.2), and

$$\bar{S} = \sum_{i=1}^{n} \bar{a}(R_i,\tau_x)\,\bar{a}(Q_i,\tau_y) \quad .$$

We have under $\bar{H}_2$

(3.1.10)
$$E(\bar{S}|\tau_x,\tau_y) = \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2 \quad \text{and}$$
$$V(\bar{S}|\tau_x,\tau_y) = \frac{1}{n}\sum_{i=1}^{n}(\bar{a}(i,\tau_x)-\bar{a})^2 \sum_{i=1}^{n}(\bar{a}(i,\tau_y)-\bar{a})^2 \quad .$$

In Spearman test with $a_i = i$, we have, under $\bar{H}_2$

$$E(\bar{S}) = E(S)$$

and

(3.1.11)
$$V(\bar{S}|\tau_x,\tau_y) = \frac{1}{144(n-1)}\left[n(n+1)(n-1) - \sum_{j=1}^{g^x}\tau_j^x(\tau_j^x+1)(\tau_j^x-1)\right]\left[n(n+1)(n-1) - \sum_{j=1}^{g^y}\tau_j^y(\tau_j^y+1)(\tau_j^y-1)\right] \quad .$$

Under certain regularity conditions (see Theorem 31A of [8]) the distribution of $\bar{S}$ given τ_x and τ_y, is asymptotically normal.

While using the Kendall's test, Kendall (1970) suggests the following argument: If there are τ_1 consecutive ties, all the scores arising from any pair chosen from them are zero. There are $\tau_i(\tau_i-1)$ such pairs and so the sum $n(n-1)$ will be reduced by $\sum_{i=1}^{g^x} \tau_i^x(\tau_i^x-1)$ and $\sum_{i=1}^{g^y} \tau_i^y(\tau_i^y-1)$. Therefore our alternative form of the coefficient τ^* may be written

$$(3.1.12)\quad \overline{\tau}^* = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} \text{sign}(\overline{R}_i-\overline{R}_j)\ \text{sign}(\overline{Q}_i-\overline{Q}_y)}{\sqrt{(n(n-1) - \sum_{i=1}^{g^x} \tau_i^x (\tau_i^x-1)(n(n-1) - \sum_{i=1}^{g^y} \tau_i^y (\tau_i^y-1))}}\ .$$

The expression for conditional variance of K (defined by (2.5.8)) is given in [13] for large sample approximation.

(iv) *Analysis of Variance Tests:* In Friedman test, we rank each block seperately. Let g_i be the number of tied groups in block i and $\tau_{i,j}$ represent the size of jth tied group in block i. The modified Q statistic is (derived similar to $\overline{H}$) given by

$$\overline{Q} = [nk(k+1)-\{1/(k-1) \sum_{i=1}^{n} \{(\sum_{j=1}^{g_i} \tau_{i,j}^3)-k\}]^{-1}\ 12n^2 \sum_{j=1}^{k} (R_{\cdot j}-R_{\cdot\cdot})^2$$

(see [13]). The distribution of $\overline{Q}$ under $\overline{H}_3$ is asymptotically Chi-square with $k-1$ d.f. .

Unfortunately, no similar results are available in the case of aligned rank test.

(c) MIDRANK METHOD.

But for the Van der Waerden and aligned rank tests, all the tests have scores which are equal to either ranks or constant values. Therefore the midrank method (see Section 2.2) of handling ties are the same as average score method.

In the van der Waerden test, we have the test statistic

$$S^* = \sum_{i=1}^{n_1} \psi(\frac{\bar{R}_i}{n+1}) \quad ,$$

where $\bar{R}_i$ represents the midranks. The conditional variance is reduced as in the case of average score and the asymptotic distribution will be given later (see Section 4.2).

In the aligned rank test, we do not have the modified form of the test statistic L_n while using the midrank method as in average score method.

3.2 TIES IN WILCOXON 2-SAMPLE TEST.

In this section we would derive a nonparametric test, similar to Wilcoxon 2-sample test, when the underlying distribution is purely discrete. Let us denote the two independent samples by $X_1,\ldots,X_{n_1}$ and $Y_1,\ldots,Y_{n_2}$ $(n_1+n_2 = n)$ with distribution functions F_1 and F_2 (discontinuous). We want to test $\bar{H}_o : F_1 = F_2 = F$ (say) against location alternatives. It seems reasonable to choose a test based on the following criteria (see [24]):

(i) distribution free under the hypothesis;

(ii) depends on observations only; and

(iii) as close as possible to original Wilcoxon test.

Let us assume that F_1 and F_2 have same discontinuity points and denote them by ξ_k, $k = 1,2,\ldots$. We define

$$p_k = P(X_1=\xi_k) \quad , \quad q_k = P(Y_1=\xi_k) \quad ;$$

$$U_k = \text{\# of } X\text{'s which are equal to } \xi_k \quad ;$$

$$V_k = \text{\# of } Y\text{'s which are equal to } \xi_k \quad ;$$

$$W_k = U_k + V_k \quad ;$$

$$U = (U_1,\ldots,U_k,\ldots) \ , \ V = (V_1,\ldots,V_k,\ldots) \ , \ W = (W_1,\ldots,W_k,\ldots) \quad .$$

The ordered pooled sample is given by the nonzero components of two vectors U and V . Hence, any rank (order) statistic which depends upon the observations only, can be expressed in terms to U and V . According to the criteria (ii), the critical region C can be defined by U and V only. Now we show that W is a sufficient statistic for the vector of parameters.

LEMMA 3.1: $P(u/w)$ *is independent of* p_k's .

PROOF:

$$(3.2.1)\qquad P(u|w) = P(U=u\ ,\ V=w-u\ |\ W=w)$$

$$= \frac{P(U=u)P(V=w-u)}{P(W=w)}\ .$$

Now, $P(U=u) = \sum_{(r_1,\ldots,r_{n_1})} P(U=u \mid \underset{\sim}{X} = (r_1,\ldots,r_{n_1})) \cdot P(r_1,\ldots,r_{n_1})$

where the above sum is over all possible $(r_1,\ldots,r_{n_1})$ in the space $\xi_1,\xi_2,\ldots$.

$$P(r_1,\ldots,r_{n_1}) = p_{r_1}\cdot p_{r_2}\cdot\ldots\cdot p_{r_{n-1}}\ .$$

Also conditional probability in the summation is zero unless $(r_1,r_2,\ldots,r_{n_1})$ has exactly u_1 of ξ_1's , u_2 of ξ_2's etc.. Let $(r_1',r_2',\ldots,r_{n_1}')$ satisfy this. Therefore, we have only $h(1 \leq h \leq n_1)$ u_k's which are nonzero, say $u_{\ell_1},\ldots,u_{\ell_h}$ s.t., $\sum_{i=1}^{h} u_{\ell_i} = n_1$. We therefore have $\frac{n_1!}{u_{\ell_1}!\ldots u_{\ell_h}!}$ non-null events in the space. In this situation, $P(U=u \mid \underset{\sim}{X} = (r_1',\ldots,r_{n_1}')) = 1$.

$$(3.2.2)\qquad \therefore\ P(U=u) = \frac{n_1!}{u_{\ell_1}!u_{\ell_2}!\ldots u_{\ell_h}!}\, p_{\ell_1}^{u_{\ell_1}}\cdot\ldots\cdot p_{\ell_h}^{u_{\ell_h}}\ .$$

Similarly,

$$(3.2.3)\qquad P(V=w-u) = \frac{n_2!\, p_{j_1}^{(w_{j_1}-u_{j_1})}\cdot p_{j_2}^{(w_{j_2}-u_{j_2})}\cdot\ldots\cdot p_{j_g}^{(w_{j_y}-u_{j_g})}}{(w_{j_1}-u_{j_1})!\ \ldots\ (w_{j_g}-u_{j_g})!}$$

where $g \geq h$ and some of the u_{j_i}'s are zero; and

$$(3.2.4) \qquad P(W=w) = \frac{n! \; p_{j_1}^{w_{j_1}} \cdot \ldots \cdot p_{j_g}^{w_{j_g}}}{w_{j_1}! \cdot \ldots \cdot w_{j_g}!} \; .$$

Hence, substituting (3.2.2), (3.2.3) and (3.2.4) in (3.2.1) we have

$$P(u|w) = \frac{n_1!}{u_{\ell_1}! \ldots u_{\ell_h}!} \cdot \frac{n_2!}{(w_{j_1}-u_{j_1})! \ldots (w_{j_g}-u_{j_g})!} \cdot \frac{w_{j_1}! \ldots w_{j_g}!}{n!}$$

$$\cdot \left[\frac{(p_{\ell_1}^{u_{\ell_1}} \ldots p_{\ell_h}^{u_{\ell_h}})(p_{j_1}^{(w_{j_1}-u_{j_1})} \ldots p_{j_g}^{(w_{j_g}-u_{j_g})})}{p_{j_1}^{w_{j_1}} \cdot \ldots \cdot p_{j_g}^{w_{j_g}}} \right] .$$

The quantity in the brackets is 1. Hence the result. □

Let the size of C be α, i.e. $P(C) = \sum_w P(W=w)\; P(C/W=w) = \alpha$

or, $\sum_w \frac{n!}{w_{j_1}! \ldots w_{j_g}!} p_{j_1}^{w_{j_1}} \cdot \ldots \cdot p_{j_g}^{w_{j_g}} \sum_{(u,w-u)\in C} P(u|w) = \alpha$. As $P(C)$ has to be independent of p_k's (requirement (i)), we must have $P(R|W=w) = \alpha$ for every w, which is the usual condition for every distribution free tests. Since for every fixed w we have only finite set of $P(u|w)$, and these sets vary with w, it will in general be impossible to find a region C with exact size α. However, this can be solved by taking some sample points in C not definitely, but with certain given probability.

Now suppose C_o is the rejection region $[S > a]$, of the same size α , given by 'randomized' Wilcoxon test. Then we have $P(C) = P(C_o) = \alpha$ or, $P(C \cap \overline{C}_o) = P(\overline{C} \cap C_o)$, where $\overline{A}$ stands for complement of A . One possible explanation of (iii) above is to choose C such that $P(C \cap \overline{C}_o)$ is minimized. This may be justified as follows: Suppose F is really continuous and ties occur only because of lack of precision of measurements. In this case the randomized test is approximately equal to Wilcoxon test (as randomization procedure is similar to the effect of replacing each discontinuity by an interval of the uniform distribution). It is therefore appropriate to minimize the probability of getting a different result than that of randomizad test ([24]). This probability when the hypothesis is true, is

$$P(C \cap \overline{C}_o) + P(\overline{C} \cap C_o) = 2P(C \cap \overline{C}_o) \quad .$$

The above is achieved, if we minimize

$$P(C \cap [S \leq a] \mid W=w) = \sum_{(u,w-u) \in C} P(u \mid w) \; P(S \leq a \mid U=u, V=w-u)$$

for every w , where S is the same as in (2.2.6); under the condition

$$(3.2.5) \qquad \sum_{(u,w-u)} P(u \mid w) = P(C \mid W=w) = \alpha \quad .$$

As suggested by Putter (1955), we can use the following algorithm. For every w , we order all possible vectors $(u,v) = (u,w-u)$ by the magnitude of $P(S \leq a \mid U=u, V=w-u)$. We take the vector with the smallest probability, then the next smallest etc., until the (conditional) size α as in (3.2.5), is reached. Doing this for all w we get the desired C.

Unfortunately, the above derived test seems to be very difficult to apply and so we modify it without going far from the test, as follows. Instead of rejecting hypothesis when $P(S \leq a | U=u, V=w-u)$ is too small we reject it when $E(S|U=u,V=w-u)$ is too large ([24]). Let

$$S' = E_r(S|U,V) \quad , \tag{3.2.6}$$

where E_r denotes the expectation under randomization. It is easy to see that S' is the same as $\overline{S}$ we had in Section 3.1. Hence under criteria (i), (ii) and (iii) we have derived a test which is the same as the average score test we proposed earlier. However, the cutoff point does depend upon W and the tabulation involved is prohibitive. Klotz (1966) has given an algorithm to calculate the exact distribution given a vector of ties. It is also suggested to use the computer for calculating the significance probabilities. We will compare this test with randomized test in Chapter 4.

3.3 TIES IN SIGN TEST.

Let the number of observations which are positive, negative and zero be n_+, n_- and n_0 respectively. In Section 3.1, we mentioned that we ignore the n_0 zero observations which amounts to omitting ties from the observations. We have

$$\bar{H}_1 : P(X_i > 0) = P(X_i < 0) \quad \text{vs say,}$$

$$K : P(X_i > 0) > P(X_i < 0)$$

and our test procedure is to reject $\bar{H}_1$ whenever n_+ is too large. Let

$$P(X_i > 0|\bar{H}_1) = p_+ \ , \ P(X_i = 0|\bar{H}_1) = p_o \ ;$$

$$P(X_i > 0|K) = q_+ \ , \ P(X_i = 0|K) = q_o \ , \ P(X_i < 0|K) = q_- \ .$$

Let us consider the conditional distribution of n_+ given $n_o = c$. Under $\bar{H}_1$,

(3.3.1) $$P(n_+=x|n_o=c) = P_{\bar{H}_1}(x) = \binom{n-c}{x} (\tfrac{1}{2})^{n-c} \ ;$$

under K

(3.3.2) $$P(n_+=x|n_o=c) = P_K(x) = \binom{n-c}{x} (\frac{q_-}{1-q_o})^{n-c} (\frac{q_+}{q_-})^x \ ,$$

$x = 0,1,\ldots,n-c$. Therefore,

$$\frac{P_K(x)}{P_{\bar{H}_1}(x)} = h(c) \ (\frac{q_+}{q_-})^x \ ,$$

which is strictly increasing function of x. Therefore, by Neyman-Pearson lemma, the unique most powerful (conditional) test is given by

(3.3.3) $$n_+ > k(n_o) \ ,$$

where the cutoff point $k(n_o)$ is, of course, the one corresponding to $B\,(n-c,\frac{1}{2})$.

The test (3.3.3) amounts to "omitting the ties from observations". Let us compare this method with randomization. The n_o zeros are divided into two parts according to the outcome of a random experimen- and suppose n_+^r of them are assigned to the positive part. The random variable $n_+^R = n_+ + n_+^r$ is under $\bar{H}_1$, $B\,(n,\frac{1}{2})$ and we can use the test

(3.3.4) $$n_+^R > k \ ,$$

without any concern about unknown p_o .

THEOREM 3.2: *The non-randomized test* (3.3.3) *is uniformly more powerful conditional (given* n_o=c*) test (against one-sided alternative* K*) than the randomized test* (3.3.4).

PROOF: Let $n_o = c$, and $p(y)$ be the frequency distribution of $B\,(c,\frac{1}{2})$. The joint (conditional) distribution of n_+ and n_+^r is $p_{\bar{H}_1}(x)\cdot p(y)$ under $\bar{H}_1$ and $p_k(x)\cdot p(y)$ under K . The ratio of two is $P_K(x)|P_{\bar{H}_1}(x)$ and hence (3.3.3) is also unique most powerful (conditional) test based on n_+ , n_o and n_+^r . □

In the above, we have omitted the zeros altogether. Recently, Krauth (1973) has proposed a test procedure which does not ignore the zeros.

THEOREM 3.3 (Krauth): *An UMP test for testing* $\bar{H}_1$ *against* K *with a known constant* $p_o = q_o$ *is given by*

$$n_+ + \frac{1}{2} n_o > k_n(p_o) \quad . \tag{3.3.5}$$

PROOF: Let us consider the distribution of

$$n_+ - n_- = 2n_+ + n_o - n = 2(n_+ + \frac{1}{2} n_o) - n \quad ; \tag{3.3.6}$$

under K ,

$$P_K(x) = P(n_+ - n_- = x) = \sum_{n_+ - n_- = x} \frac{n!}{n_+! n_-! n_o!} q_+^{n_+} q_-^{n_-} q_o^{n_o}$$

$$= q_o^n (q_+/q_o)^x \sum_{i=0}^{n} \binom{n}{i} \binom{n-i}{i-x} (q_+ q_- / q_o^2)^{i-x} \quad ,$$

and under $\bar{H}_1$,

$$P_{\bar{H}_1}(x) = P(n_+ - n_- = x) = p_o^n (p_+/p_-)^x \sum_{i=0}^{n} \binom{n}{i} \binom{n-i}{i-x} (p_+ p_- / p_o^2)^{i-x} \quad ,$$

$x = -n, -n+1, \ldots, n$. Therefore,

$$\frac{P_K(x)}{P_{\bar{H}_1}(x)} = \frac{y^x A_n(z,x)}{y_o^x A_n(z_o,x)} \quad , \tag{3.3.7}$$

with

$$A_n(z,x) = \sum_{i=0}^{n} \binom{n}{i} \binom{n-i}{i-x} z^{i-x} \quad , \tag{3.3.8}$$

$$y = q_+|q_o \ , \ y_o = p_+|p_o = 1-p_o|2p_o \ ;$$

$$z = q_+q_-|q_o^2 \ , \ z_o = p_+p_-|p_o^2 = (1-p_o)^2|4p_o^2 \ .$$

If we prove (3.3.7) is strictly increasing function of x (which we do in the following lemma), by Neyman-Pearson lemma the test (3.3.5) is UMP test for known $p_o = q_o$ and hence the result. □

LEMMA 3.2: $P_k(x)|P_{\bar{H}_1}(x)$, (3.3.7), *is strictly increasing function of* x *, for* $x = -n+1,-n,\ldots,n$.

PROOF: As $(\frac{y}{y_o})^x$ is strictly increasing function of x and $z < z_o$, it suffices to prove that

$$A_n(z,x)|A_n(z_o,x) > A_n(z,x-1)|A_n(z_o,x-1) \ , \tag{3.3.8}$$

for $z < z_o$, $x = -n+1,-n,\ldots,n$. Or, equivalently

$$A_n(x,z)|A_n(z,x-1) > A_n(z_o,x)|A_n(z_o,x-1) \tag{3.3.9}$$

for $z < z_o$, $x = -n+1,-n,\ldots,n$. We prove (3.3.9) by showing that the derivative of $H(z,x) = A_n(z,x)|A_n(z,x-1)$ with respect to z is negative for $z > 0$, $x = -n+1,-n,\ldots,n$. For $x \leq 0$ we have

$$A_n(z,x) = \sum_{i=1}^{m} \binom{n}{i}\binom{n-i}{i-x} z^{i-x} \ ,$$

with $m = [(n+x)/2]$. Therefore it is enough to show that

$$\sum_{i=0}^{m} \sum_{j=0}^{m'} (i-j-1)\binom{n}{i}\binom{n-i}{i-x}\binom{n}{j}\binom{n-j}{j-x+1} z^{i+j} < 0$$

with $m' = [(n+x-1)/2]$. Or,

$$\sum_{i=1}^{m} \sum_{j=1}^{m'+1} (i-j)\binom{n}{i}\binom{n-i}{i-x}\binom{n}{j-1}\binom{n-j+1}{j-x} z^{i+j-1} < 0 \quad . \tag{3.3.10}$$

We consider only terms for which $i,j \in \{1,2,\ldots,m\}$, since the terms with $i = 0$ or $j = m+1$, are negative anyhow. For $i = j$, the summation vanishes. For the sum of two terms with $(i_1,j_1) = (s,t)$, $(i_2,j_2) = (t,s)$; $s,t \in \{1,2,\ldots,m\}$, we get

$$-(s-t)\binom{n}{s}\binom{n-s}{s-x}\binom{n}{t}\binom{n-t}{t-x} \times \frac{n+x+1}{(n-2t+x+1)(n-2s+x+1)} \; z^{s+t-1} \quad ,$$

which is negative for all $s,t \in \{1,2,\ldots,m\}$. This completes the proof for $x \le 0$. For $x \ge 0$ we have

$$A_n(z,x) = \sum_{i=x}^{m} \binom{n}{i}\binom{n-i}{i-x} z^{i-x} \quad .$$

We complete the proof proving

$$\sum_{i=1}^{m} \sum_{j=1}^{m'+1} (i-j)\binom{n}{i}\binom{n-i}{i-x}\binom{n}{j-1}\binom{n-j+1}{j-x} z^{i+j-1} < 0 \quad ,$$

the same way as (3.3.11). □

Putter [24] has shown that $T_n = (2n_+ + n_o - n)/(n-n_o)^{1/2}$ is asymptotically $N(0,1)$ as $n \to \infty$. In virtue of Theorem 3.3, we can now state the following result.

THEOREM 3.4. *An asymptotically UMP test for testing* $\bar{H}_1$ *against* K *, under the restriction* $p_o = q_o$ *, is given by*

$$T_n = (2n_+ + n_o - n)/(n - n_o)^{1/2} > k \qquad (3.3.11)$$

where the cutoff point k *corresponds to* N(0,1) *distribution.*

3.4: TREATMENT OF TIES IN WILCOXON 1-SAMPLE (SIGNED RANK) TEST.

In §3.1, we ignored the zeros from the sample and then ranked the rest of the observations as suggested by Wilcoxon (1945). Pratt (1959) has suggested a different procedure in this section we would review these criteria.

The following three requirements have been suggested for a test when these are 0's .

(i) Increasing the observed values shall not make a significantly positive sample insignificant nor an insignificant sample significantly negative.

(ii) Assuming that the distribution of the observations has a center of symmetry μ , those values of μ which are not rejected shall form an interval.

(iii) A sample shall be judged significantly positive if, when the 0's are included in the ranking, the sample is significantly positive whatever signs are attached to the ranks of the 0's ; similarly for significantly negative and not significant.

Pratt (1959) points out that none of the three conditions are satisfied (which are reasonable and are satisfied when there are no zeros) when we use the Wilcoxon's procedure. The two methods of handling zeros have been compared in [4] when the underlying distributions are discontinuous and we would discuss these in Section 4.5.

CHAPTER 4

ASYMPTOTIC RELATIVE EFFICIENCY (ARE)

In this chapter we examine asymptotic efficiencies of the linear rank tests for randomness and symmetry with particular attention paid to the three methods of handling ties, discussed in Section 2.2. This is studied using ARE.

4.1 EFFICIENCY.

Asymptotic power of a test against a given alternative provides a good clue to the large sample operating characteristic of the test. Asymptotic efficiency gives a comparative measure of the asymptotic power of a test relative to a most powerful test or relative to a standard test. In the latter case we call it asymptotic relative efficiency. We consider the asymptotic efficiency as defined in Hájek and Sidák ((1967) p. 267) and asymptotic relative efficiency as in Hodges and Lehmann (1956).

Assume that an asymptotically most powerful test for H_o against q is based on a statistic S_o, where S_o is asymptotically normal $(0,\sigma_o^2)$ under the null hypothesis and asymptotically normal (μ_o,σ_o^2) under the alternative. Further, let us consider another test for H_o against q based on S, which is asymptotically normal $(0,\sigma^2)$ and (μ,σ^2) under H_o and q respectively. Then the asymptotic powers of S_o - test and S - test equal

$$1-\Phi(k_{1-\alpha}-\mu_o\sigma_o^{-1}) \quad \text{and} \quad 1-\phi(k_{1-\alpha}-\mu\sigma^{-1})$$

respectively. The expression

$$(4.1.1) \qquad e = \left(\frac{\mu\sigma_o}{\mu_o\sigma}\right)^2$$

is called asymptotic efficiency of S - test (it is ratio of the two asymptotic powers given above).

Now, let $\beta_n(\theta)$ and $\beta_n^*(\theta)$ denote the power function of two tests, say A and A^* based on same set of n observations, against a family of alternatives labelled by θ and let θ_o be the value of θ specified by the hypothesis. We shall assume that all tests are at the same level of significance α . Let β be a specified power with $\alpha < \beta < 1$. Consider the sequence of alternatives θ_n such that

$$(4.1.2) \qquad \beta_n(\theta_n) \to \beta \quad \text{as} \quad n \to \infty$$

and a sequence $n^* = h(n)$ such that

$$(4.1.3) \qquad \beta^*_{n^*}(\theta_n) \to \beta \quad \text{as} \quad n \to \infty \ .$$

Then if

$$(4.1.4) \qquad e_{A^*,A} = \lim_{n\to\infty} \frac{n}{n^*}$$

exists and is independent of α,β and the particular sequences $\{\theta_n\}$ and $\{h(n)\}$ chosen, $e_{A^*,A}$ is defined to be the asymptotic relative efficiency (ARE) of the test A^* with respect to the test A. Methods of obtaining the limit (4.1.4) in different situations are available in literature (see for example, Hodges and Lehmann (1956)). ARE is useful for problems where optimum tests either do not exist or are not available.

We shall use the form of asymptotic efficiency as described in Hájek and Sidák [9] (pp. 267-70).

4.2 ASYMPTOTIC DISTRIBUTION UNDER NULL HYPOTHESES.

To calculate the asymptotic efficiencies let us first examine the asymptotic distribution of linear rank statistic under $\overline{H}_0$ and $\overline{H}_1$, as discussed in [3].

The following theorems present conditions under which

$$\frac{S - E(S|\tau)}{[V\{S|\tau\}]^{1/2}} \to N(0,1) \quad , \tag{4.2.1}$$

where $E(S|\tau)$ and $V(S|\tau)$ are as in (3.1.1).

Let $\phi(u)$ denote an arbitrary real valued function defined on the interval $0 \le u \le 1$ and

$$0 < \int_0^1 (\phi(u)-\bar{\phi})^2 \, du < \infty \ , \quad \text{where} \quad \bar{\phi} = \int_0^1 \phi(u) \, du \tag{4.2.2}$$

(4.2.3). $$\sum_{i=1}^{n} (C_i-\overline{C})^2 \Big/ \max_{1\le i\le n} (C_i-\overline{C})^2 \to \infty \quad .$$

THEOREM 4.1: *Under* $\overline{H}_o$ *, if conditions* (4.2.2), (4.2.3) *and*

(4.2.4) $$\int_0^1 (a(n\cdot T_n^{-1}(u),\tau) - \phi(u))^2\, du \xrightarrow[P]{} 0$$

hold then (4.2.1) *follows. Here,* $T_n(u) = \frac{1}{n}$ {# of R_i's $\le$ un} *and the inverse is defined by* $f^{-1}(t) = \inf\{x \mid f(x) \ge t\}$ *, for a real valued function* f .

PROOF: Let us consider the random variable $Y_i = F(X_i)$ which under $\overline{H}_o$ are i.i.d. with some cdf $G(u)$. Let $W_1, W_2, \ldots, W_n$ be uniform random variables which are also independent of Y_i . Let $G(\{\cdot\})$ denote the measure induced by $G(u)$ on any set $\{\cdot\}$ of real numbers. Then $G(\{y\}) = P(Y=y)$ at discontinuity points of $G(u)$, and equals zero elsewhere. Now we will prove that the random variables $U_i = Y_i - W_i G(\{Y_i\})$ are mutually independent with uniform distribution on (0,1) . Let

$$a(u) = G(G^{-1}(u)) - G(\{G^{-1}(u)\}) \quad \text{and}$$

(4.2.5)

$$b(u) = G(G^{-1}(u)) \quad .$$

Then

$$P(U_i \le u) = P(Y_i \le b(u), W_i\; G(\{G^{-1}(u)\}) \ge b(u)-u) \quad .$$

If $G(u) = u$ then $b(u) = u$ and $P(U_i \le u) = P(Y_i \le b(u)) = u$. If $G(u) < u$, then $G(u)$ is constant on the interval $[a(u),b(u))$ and $W_i\, G(\{h^{-1}(u)\})$ is uniformly distributed on $(0,b(u)-a(u))$. And we have

$$(4.2.6)\qquad P(U_i \le u) = P(U_i \le a(u)) + P(a(u) < U_i \le u)$$

$$= a(u) + P(Y_i = b(u))\, P(W_i \ge \frac{b(u)-u}{b(u)-a(u)}) = u\ .$$

It is shown in [9], p. 153 that under the assumptions (4.2.2) and (4.2.3) the random variable $T_c | \sigma_c$, where

$$(4.2.7)\qquad T_c = \sum_{i=1}^{n} (C_i - \bar{C})\, \phi(U_i)\ , \qquad \text{and}$$

$$\sigma_c^2 = \sum_{i=1}^{n} (C_i - \bar{C})^2 \int_0^1 (\phi(u) - \bar{\phi})^2\, du$$

has asymptotically standard normal distribution.

It is also shown on p. 160 that $S^{\phi} | \sigma_c$ where

$$(4.2.8)\qquad S^{\phi} = \sum_{i=1}^{n} (C_i - \bar{C})\, a^{\phi}(R_i^*)\ ;\ a^{\phi}(i) = E\{\phi(U_1) | R_1^* = i\}$$

$$R_i^* = \text{rank of } U_i\ ,$$

satisfies

$$(4.2.9)\qquad E\{\frac{(S^{\phi} - T_c)^2}{\sigma_c^2}\} \to 0\ .$$

Consequently, $S^{\phi}|\sigma_c \to N(0,1)$ under (4.2.2) and (4.2.3). By (3.1.1) we have

$$E\{[S-E(S|\tau)-S^{\phi}]^2|\tau\} = E\{[\sum_{i=1}^{n} (C_i-\bar{C})(a(R_i,\tau)-a^{\phi}(R_i^*))]^2|\tau\}$$

(4.2.10) $$\le \frac{1}{n-1} \sum_{i=1}^{n} (C_i-\bar{C})^2 \sum_{j=1}^{n} [a(r_j,\tau)-a^{\phi}(r_j^*)]^2$$

$$= \frac{n}{n-1} \sum_{i=1}^{n} (C_i-\bar{C})^2 \int_0^1 [a(nT_n^{-1}(u),\tau)-a^{\phi}(1+[un])]^2 \, du \; .$$

Now,

(4.2.11) $$E\{\frac{[S-E(S|\tau)-S^{\phi}]^2}{\sigma_c^2}\} \xrightarrow[P]{} 0$$

if the integral in (4.2.10) converge to zero in probability. But the integral in (4.2.10) is less than or equal to

$$2\int_0^1 [a(n\;T_n^{-1}(u),\tau)-\phi(u)]^2 \, du + 2\int_0^1 [a^{\phi}(1+[un])-\phi(u)]^2 \, du \quad .$$

The first integral goes to zero by hypothesis and the second by Theorem b of [9], p. 158. The rest of the proof follows on the same lines as in [9], p. 161. □

When there are tied observations in the data, ranks may be assigned by one of the three methods described in Section 2.1. Let us state the particular forms of Theorem 4.1 in different situations.

Average Score Method: Let $\phi_\alpha(u)$ be $\phi(u)$ averaged over the intervals in which $G(u)$ is constant valued:

$$(4.2.12) \qquad \phi_\alpha(u) = \phi(u) \qquad \text{if} \qquad G(\{G^{-1}(u)\}) = 0$$

$$= \frac{1}{b(u)-a(u)} \int_{a(u)}^{b(u)} \phi(t)\, dt \qquad \text{otherwise}$$

where $a(u)$ and $b(u)$ are the same as in (4.2.5) and are left and right end points of the interval containing u .

COROLLARY 4.1: *Under* $\overline{H}_0$, *if* (4.2.2) *holds, the scores* $a(i)$ *satisfy*

$$(4.2.13) \qquad \int_0^1 (a(1+[un]) - \phi(u))^2\, du \to 0$$

and if $\phi_\alpha(u)$ *is square integrable and non-constant over* (0,1) , *then* (4.2.1) *holds for the average scores defined by* (2.2.2).

PROOF: Proof follows from Theorem 4.1 and the fact that (4.2.13) implies (4.2.4) ([3], p. 1112). □

Midrank Method: Let $\{I_k\}_{k\geq 0}$ denote the countable set of discontinuity intervals $(a(u),b(u)]$, where $a(u)$ and $b(u)$ are defined as in (4.2.5) for each discontinuity point of $G(u)$. Let

$$(4.2.14) \qquad \phi_m = \phi(u) \qquad \text{if } u \text{ is in a continuity interval}$$

$$= \phi(\text{med } I_j) \quad \text{if } u \text{ is in a discontinuity interval}$$

where, med I_j refers to the midpoint of I_j , $(a(u)+b(u))/2$.

COROLLARY 4.2: *Let* $\overline{H}_o$ *be true. If* (4.2.2) *and* (4.2.13) *hold,* $\phi_m(u)$ *is square integrable and non-constant over* (0,1) , $\{\text{med } I_k\}_{k\geq 0}$ *are continuity points of* $\phi(u)$, *and*

$$a\left(\frac{1+[2un]}{2}\right) \to \phi(u) \quad , \text{ for } \quad 0 < u < 1 \tag{4.2.15}$$

then, (4.2.1) *follows for midrank scores* (2.2.4).

PROOF: It suffices to prove that (4.2.3), which takes the form

$$\int_0^1 (a(n\ T_n^{-1}(u);\tau)-\phi_m(u))^2\ du \xrightarrow[P]{} 0 \quad ,$$

holds for the scores defined by (4.2.4). For the outline of the proof of the above, we refer to [3], p. 1113. □

Randomized Ranks:

COROLLARY 4.3: *Under* $\overline{H}_o$ *if* (4.2.2), (4.2.3) *and* (4.2.13) *holds then* (4.2.1) *follows for the scores given by*

$$a(R_i,\tau) = a(R_i^*)$$

where R_i^* *are randomized ranks.*

PROOF: Since $a(n \cdot T_n^{-1}(u),\tau) = a(1+[un])$, (4.2.13) implies (4.2.4) and hence the result. □

Let R_i^+ be as defined in Section 2.4 and

$$T_n^+(u) = \frac{1}{n} \{\#\text{ of } R_i^+\text{'s} \leq un\} \quad .$$

THEOREM 4.2: *Let* $\phi^+(u)$ *be a square integrable (on* $0 \leq u \leq 1$*) function with*

(4.2.16) $$\int_{F^+(0)}^{1} [\phi^+(u)]^2 \, du > 0$$

and let $\overline{H}_1$ *be true. If*

(4.2.17) $$\int_{\tau_o|n}^{1} [a(n\, T_n^{+-1}(u);\tau) - \phi^+(u)]^2 \, du \xrightarrow[P]{} 0$$

holds, then $S|\sigma_n \xrightarrow{L} N(0,1)$ *, where* $\sigma_n^2 = V(S|\tau)$ *and* S *is the same as defined in* (2.4.2).

PROOF: As in Theorem 4.1, let $Y_i^+ = F_i^+(|X_i|)$ where F^+ denotes the cdf of $|X_i|$. Let $U_i^+ = Y_i^+ - W_i\, G(\{Y_i^+\})$, where $G(u)$ is the cdf of Y_i^+ and $W_1,\ldots,W_n$ are iid uniform on $(0,1)$. Let

$$a^+(i) = E[\phi^+(U_1^+) \mid R_1^* = i]$$

and

$$S_\phi = \sum_{i=1}^{n} a^+(R_i^*) \text{ sign } X_i \quad ,$$

where R_i^* is rank of U_i^+ . Then $S_\phi|\sigma \to N(0,1)$ as $n \to \infty$ [see [27], Theorem 2], where

(4.2.18) $$\sigma^2 = n \int_{F^+(0)}^{1} [\phi^+(u)]^2 \, du \quad .$$

Now we show that $E\{\frac{(S-S_\phi)^2}{\sigma^2}\} \xrightarrow[P]{} 0$.

$$E\{(S-S_\phi)^2|\tau\} = \mathrm{Var}(S-S_\phi|\tau)$$

$$= \sum_{r_i^+ > \tau_o} [a(r_i^+,\tau) - a^+(r_i^*)]^2$$

(4.2.19) $$= n \int_{\tau_o|n}^{1} (a(n\, T_n^{+-1}(u),\tau) - a^+(1+[un]))^2 \, du$$

$$\leq 2n \int_{\tau_o|n}^{1} (a(n\, T_n^{+-1}(u),\tau) - \phi^+(u))^2 \, du$$

$$+ 2n \int_{\tau_o|n}^{1} (a^+(1+[un]) - \phi^+(u))^2 \, du \quad .$$

The first integral in (4.2.19) $\xrightarrow{P} 0$ by (4.2.17) and the second converges to zero by Theorem V.1.4 of [9]. Therefore,

$$E\{\frac{(S-S_\phi)^2}{\sigma^2}\} \xrightarrow[P]{} 0 \quad .$$

Proof is completed from the fact that $\sigma_n^2 \to \sigma^2$ (see [9], p. 161). □

As in the case of tests for randomness we can prove results similar to Corollaries 4.1, 4.2, and 4.3 in this case also. Results along these lines for purely discrete distribution functions are given in [27].

REMARK 4.1: No such results for $\overline{H}_2$ and $\overline{H}_3$ are available in literature. In case of k-sample test for randomness we do have similar results (see Conover [3]).

4.3 ASYMPTOTIC DISTRIBUTION UNDER CONTIGUOUS ALTERNATIVES.

Let us first note that the locally most powerful conditional rank test for $\overline{H}_o$ and $\overline{H}_1$ is a linear rank test, under certain regularity conditions [see [3], Theorems 6.1 and 7.1]. Now, we shall discuss the asymptotic distribution of S under contiguous alternatives [in both cases: for testing Randomness and Symmetry].

Let us consider a distribution function $F(x,\theta)$ with parameter θ. Let $f(x,\theta)$ represent the Radon-Nikodym derivative of $F(x,\theta)$ with respect to $F(x,\theta_o)$ and assume this exists. We define the generalized Fisher's information

$$I(F,\theta) = \int_{-\infty}^{\infty} \left[\frac{(\partial/\partial\theta)f(x,\theta)}{f(x,\theta)}\right]^2 dF(x,\theta) \quad . \tag{4.3.1}$$

The distribution function of $Y = F(x;\theta)$ is denoted by $G(u;\theta)$, where $F(x;\theta)$ is the distribution function of X. Let the distribution function of the X's under $\overline{H}_o$ be denoted by $F(x;\theta_o)$ and consider the alternative

$H_{\alpha n}$: $X_1,\ldots,X_n$ are independent and X_i is distributed according to $F(x;\theta_i)$.

The asymptotic distribution of S is found under the conditions

(4.3.2)
$$\max_{1\le i\le n} (\theta_i-\theta_o) \to 0$$

and

(4.3.3)
$$\lim_{n\to\infty} I(F,\theta_0) \sum_{i=1}^{n} (\theta_i-\theta_o)^2 = b^2$$

for $0 < b^2 < \infty$ where I( ) satisfies

$$0 < \lim_{\theta=\theta_o} I(F,\theta) = I(F,\theta_o) < \infty \quad .$$

Let also

(4.3.4)
$$\frac{\partial}{\partial\theta} f(x,\theta)/\theta=\theta_o = \lim_{\theta\to\theta_o} \frac{f(x,\theta)-f(x,\theta_o)}{\theta-\theta_o}$$

exists and

(4.3.5)
$$f(x,\theta_o) = \lim_{\theta\to\theta_o} f(x,\theta)$$

exists almost everywhere with respect to $F(x,\theta_o)$. We shall omit the double subscript implied by conditions (4.3.2) and (4.3.3) in order to take the limit. Let

(4.3.6)
$$L_o = \prod_{i=1}^{n} \frac{f(X_i,\theta_i)}{f(X_i,\theta_o)}$$

be likelihood ratio and consider the statistics

(4.3.7) $$W_o = 2 \sum_{i=1}^{n} \{[\frac{f(X_i,\theta_i)}{f(X_i,\theta_o)}]^{1/2} - 1\}$$

and

(4.3.8) $$T_o = \sum_{i=1}^{n} (\theta_i - \theta_o)\ \phi(Y_i,F,\theta_o) \quad ,$$

where we denote $\phi(u,F,\theta_o)$ for

(4.3.9) $$\phi(u,F,\theta_o) = \frac{(\partial/\partial\theta)\ f(F^{-1}(u;\theta),\theta)\,|\,\theta=\theta_o}{f(F^{-1}(u;\theta_o),\theta_o)} \quad .$$

LEMMA 4.1: *Conditions* (4.3.2) *through* (4.3.5) *imply* $T_o \to N(0,b^2)$ *under* $\overline{H}_o$.

PROOF: Proof is omitted (see [3], Theorem 8.1). □

LEMMA 4.2: *Under conditions of Lemma* 4.1, *we have*

(4.3.10) $$\log L_o - T_o - \frac{1}{2} b^2 \xrightarrow[P]{} 0$$

and

(4.3.11) $$\log L_o \to N(-\frac{1}{2} b^2, b^2)$$

under $\overline{H}_o$.

PROOF: See Conover (1973a), Theorem 8.2. □

Let

$$S' = S - E\{S|\tau\} = \sum_{i=1}^{n} (C_i - \overline{C})\ a(R_i, \tau) \quad . \tag{4.3.12}$$

The limiting distribution of S' is already given in Theorem 4.1.

THEOREM 4.3: *Let* $\phi(u)$ *be a non-constant square integrable function on* $0 \leq u \leq 1$, *and let*

$$\int_0^1 (a(n\ T_n^{-1}(u), \tau) - \phi(u))^2\ du \xrightarrow[P]{} 0 \tag{4.3.13}$$

hold under $\overline{H}_o$. *Then if*

$$\sum_{i=1}^{n} (C_i - \overline{C})^2 / \max_{1 \leq i \leq n} (C_i - \overline{C})^2 \rightarrow \infty \ , \tag{4.3.14}$$

holds, the conditions of Lemma 4.1 imply that S' *is asymptotically* $N(\mu_\theta, \sigma^2)$ *under* H_α , *where*

$$\mu_\theta = \sum_{i=1}^{n} (C_i - \overline{C})(\theta_i - \theta_o) \int_0^1 \phi(u)\ \phi(u, F, \theta_o)\ du \tag{4.3.15}$$

and

$$\sigma^2 = \sum_{i=1}^{n} (C_i - \overline{C})^2 \int_0^1 (\phi(u) - \overline{\phi})^2\ du \quad . \tag{4.3.16}$$

PROOF: We shall outline the proof. From (4.2.9) and (4.2.11) we have S' and T_c asymptotically equivalent under $\overline{H}_o$. This and the first result of Lemma 4.2 imply that the bivariate random variables $(S', \log L_o)$ and $(T_c, T_o - b^2/2)$ converge in probability to the same limit. Under $\overline{H}_o$, by Theorem 4.1 and Lemma 4.1, we have $T_c \to N(0,\sigma^2)$ and $T_o \to N(0,b^2)$. Note also that

$$T_o = \sum_{i=1}^{n} (\theta_i - \theta_o)\, \phi(U_i, F, \theta_o) \quad .$$

The covariance of T_c and T_o is

(4.3.17) $$\operatorname{cov}(T_c, T_o) = \sum_{i=1}^{n} (C_i - \overline{C})(\theta_i - \theta_o) \int_0^1 \phi(u)\, \phi(u, F, \theta_o)\, du \quad .$$

because $E\{T_o\} = 0$. Rest of the proof that (T_c, T_o) is asymptotically bivariate normal is the same as in [9], p. 218. This implies $(S', \log L_o)$ is asymptotically bivariate normal under $\overline{H}_o$ and the parameters satisfy the conditions of LeCam's third lemma, p. 208 of [9] and so S' is asymptotically normal (μ_θ, σ^2). □

Now, we state an analogous result under $\overline{H}_1$, the proof of which is similar to the above theorem. Let

(4.3.18) $$\phi^+(u, F, \theta_o) = \frac{\partial}{\partial\theta} f(F^{-1}(\tfrac{1}{2} + \tfrac{1}{2} u), \theta) \,|\, \theta = \theta_o \quad .$$

Let $F(x,\theta)$ be a symmetric function for $\theta = \theta_o$ (when $\overline{H}_1$ is true) and define likelihood function

(4.3.19) $$L_\Delta = \prod_{i=1}^{n} \frac{f(X_i, \theta_o + \Delta)}{f(X_i, \theta_o)} \quad .$$

Assume

$$\Delta \to 0 \quad , \tag{4.3.20}$$

$$\lim_{n\to\infty} I(F,\theta_o) \cdot n\,\Delta^2 = b^2 \quad \text{for} \quad 0 < b^2 < \infty \tag{4.3.21}$$

and

$$0 < \lim_{\theta\to\theta_o} I(F,\theta) = I(F,\theta_o) < \infty \quad , \tag{4.3.22}$$

where $I(F,\theta)$ is defined by (4.3.1). As T_o in (4.3.8) let

$$T_\Delta = \sum_{i=1}^{n} \Delta\, \phi(F(X_i),F,\theta_o) = \sum_{i=1}^{n} \Delta \frac{\partial}{\partial\theta} f(X_i,\theta)\Big|_{\theta=\theta_o} .$$

THEOREM 4.4. *Let* $F(x,\theta)$ *satisfy* (4.3.4), (4.3.5), (4.3.22) *and*

$$\frac{\partial}{\partial\theta} f(-x,\theta)\Big|_{\theta=\theta_o} = -\frac{\partial}{\partial\theta} f(x,\theta)\Big|_{\theta=\theta_o} .$$

If (4.2.17) *holds under* $\overline{H}_1$ *for some square integrable (on* (0,1)*) function* $\phi^+(u)$ *that satisfies* (4.2.16), *then* (4.3.20) *and* (4.3.21) *imply that the sequence* S *is asymptotically* $N(\mu_\Delta,\sigma^2)$ *under* H_α *, where* σ^2 *is given by* (4.2.18) *and* μ_Δ *by*

$$\mu_\Delta = n\,\Delta \int_{F^+(0)}^{1} \phi^+(u)\, \phi^+(u,F,\theta_o)\, du \tag{4.3.23}$$

and the sequence $S|\sigma_n$ *is asymptotically* $N(\mu_\Delta|\sigma,1)$.

PROOF: The proof is similar to the proof of Theorem 4.3 (see Theorem 9.1 of [3]) and hence we omit it. □

4.4 ASYMPTOTIC EFFICIENCY.

When testing $\overline{H}_o$, if there is convergence

(4.4.1) $$\frac{\sum_{i=1}^{n} (C_i-\overline{C})\ (\theta_i-\theta_o)}{(\sum_{i=1}^{n} (C_i-\overline{C})^2 \sum_{i=1}^{n} (\theta_i-\theta_o)^2)^{1/2}} \to \rho_2 ,$$

then the asymptotic efficiency of the test using S is defined (see [9] p. 268] as

(4.4.2) $$e = \rho_1^2\, \rho_2^2 ,$$

where ρ_1 is given by

(4.4.3) $$\rho_1 = \frac{\int_0^1 \phi(u)\ \phi(u,F,\theta_o)\ du}{(\int_0^1 (\phi(u)-\overline{\phi})^2\ du \int_0^1 \phi^2(u,F,\theta_o)du)^{1/2}} .$$

If we want to compare two tests, for which $\phi(u)$ differs we calculate the asymptotic relative efficiency (ARE). In the usual case C_i's are the same, then the ARE of the test using $\phi_1(u)$, say relative to the test using $\phi_2(u)$ is

(4.4.4) $$\text{ARE}_{\phi_1,\phi_2} = \frac{(\int_0^1 \phi_1(u)\phi(u,F,\theta_o)du)^2 \int_0^1 (\phi_2(u)-\overline{\phi}_2)^2\ du}{(\int_0^1 \phi_2(u)\phi(u,F,\theta_o)du)^2 \int_0^1 (\phi_1(u)-\overline{\phi}_1)^2\ du} .$$

Now let us mention the changed ρ_1's in different methods of handling ties (see [3]). When we use average score method ρ_1 is given by

$$(4.4.5) \qquad \rho_1 = \frac{\int_0^1 \phi_\alpha(u)\phi(u,F,\theta_o)\ du}{\left(\int_0^1 (\phi_\alpha(u)-\bar{\phi})^2 du\ I(F,\theta_o)\right)^{1/2}}$$

where $\phi_\alpha(u)$ is defined by (4.2.12). Using midrank method we have,

$$(4.4.6) \qquad \rho_1 = \frac{\int_0^1 \phi(u)\phi(u,F,\theta_o)du}{\left(\int_0^1 (\phi_m(u)-\bar{\phi}_m)^2 du\ I(F,\theta_o)\right)^{1/2}}$$

where $\phi_m(u)$ is defined by (4.2.14). By using randomized rank, we get

$$(4.4.7) \qquad \rho_1 = \frac{\int_0^1 \phi(u)\ (u,F,\theta_o)du}{\left(\int_0^1 (\phi(u)-\bar{\phi})^2 du\ I(F,\theta_o)\right)^{1/2}} .$$

Let us find out the ARE of an average score test (A) relative to a randomized rank test (R). The numerators of both ρ_1's in (4.4.5) and (4.4.7) are identical, because $\phi(u,F,\theta_o)$ is constant over the same interval in which $\phi(u)$ is averaged to give $\phi_\alpha(u)$. Hence

$$(4.4.8) \qquad ARE_{A,R} = \frac{\int_0^1 (\phi(u)-\bar{\phi})^2\ du}{\int (\phi_\alpha(u)-\bar{\phi})^2\ du} .$$

Note that (4.4.8) is greater than or equal to one with equality only if $\phi(u)$ is constant in the same interval where $G(u)$ is constant. Theorem 6 of Putter (1955) given below as Theorem 4.5, is a special case of (4.4.8).

THEOREM 4.5: *Under the regularity conditions, under which (4.4.8) holds, in the same set up as in Section 3.2, the ARE of randomized test with respect to averaged score (or midrank, as they are the same in Wilcoxon test) is* $1 - \sum_k p_k^3$.

PROOF:

(4.4.9)
$$ARE_{R,A} = \frac{\int_0^1 (\phi_\alpha(u)-\bar{\phi})^2\, du}{\int_0^1 (\phi(u)-\bar{\phi})^2\, du}$$

$\phi(u) = u$, in Wilcoxon test and hence the denominator of (4.4.9) is

(4.4.10)
$$\int_0^1 (u - \tfrac{1}{2})du = \frac{1}{12} .$$

Let $q_i = p_1 + p_2 + \dots + p_i$. The $\phi_\alpha(u)$ is given by (using (4.2.12))

(4.4.11)
$$\phi_\alpha(u) = \frac{1}{p_i}\int_{q_{i-1}}^{q_i} t\, dt \qquad q_{i-1} \le u < q_i$$
$$= \frac{1}{p_i}\,\frac{q_i^2 - q_{i-1}^2}{2} = \frac{q_i + q_{i-1}}{2} = \frac{2q_i - p_i}{2} .$$

Therefore the numberator of (4.4.9)

$$\int_0^1 (\phi_\alpha(u)-\bar{\phi})^2\, du = \sum_i \int_{q_{i-1}}^{q_i} \left(\frac{2q_i - p_i}{2} - \frac{1}{2}\right)^2 du$$
$$= \sum_i \frac{(2q_i - p_i - 1)^2}{4}\; p_i .$$

$$\therefore \quad ARE_{R,A} = 3 \sum_i (2q_i - p_i - 1)^2 \; p_i \; .$$

Now, we will be through if we prove that

$$3 \sum_i (2q_i - p_i - 1)^2 \; p_i = 1 - \sum_i p_i^3 \; , \quad \text{or}$$

(4.4.12)
$$\sum_i p_i^3 = 1 - 3 \sum_i (2q_i - p_i - 1)^2 \; p_i \; .$$

For only one point mass (i.e. $k=1$) (4.4.12) is trivial. Let us assume that it is true for $k = \ell$ (i.e. it holds for all probability distributions with ℓ points having positive probability.

(4.4.13)
$$\therefore \quad \sum_{i=1}^{\ell} p_i^3 = 1 - 3 \sum_{i=1}^{\ell} (2q_i - p_i - 1)^2 \; p_i \; .$$

We want to prove that it is true for any distribution with $(\ell+1)$ points having positive probability

$$\sum_{i=1}^{\ell+1} p_i^3 = q^3 \sum_{i=1}^{\ell} \frac{p_i^3}{q^3} + p_{\ell+1}^3 \quad , \text{ where } \quad q = 1 - p_{\ell+1}$$

$$= q^3 \{1 - 3 \sum_{i=1}^{\ell} (\frac{2q_i}{q} - \frac{p_i}{q} - 1)^2 \frac{p_i}{q}\} + p_{\ell+1}^3$$

using (4.4.13) for probability distribution with $\frac{p_i}{q}$, $i = 1,\ldots,\ell$ as probabilities $(\sum_{i=1}^{\ell} \frac{p_i}{q} = \frac{1-p_{\ell+1}}{q} = 1)$. Therefore,

$$\sum_{i=1}^{\ell+1} p_i^3 = (1-p_{\ell+1})^3 + p_{\ell+1}^3 - 3\sum_{i=1}^{\ell} (2q_i-p_i-q)^2 p_i$$

$$= (1-p_{\ell+1})^3 + p_{\ell+1}^3 - 3\sum_{i=1}^{\ell} (2q_i-p_i-1+p_{\ell+1})^2 p_i$$

$$= (1-p_{\ell+1})^3 + p_{\ell+1}^3 - 3\sum_{i=1}^{\ell} (2q_i-p_i-1)^2 p_i - 3p_{\ell+1}^2(1-p_{\ell+1})$$

$$-6p_{\ell+1}\sum_{i=1}^{\ell} (2q_i-p_i-1)p_i \quad , \text{ or}$$

$$(4.4.14) \quad \sum_{i=1}^{\ell+1} p_i^3 = 1 - 3\sum_{i=1}^{\ell+1} (2q_i-p_i-1)^2\, p_i + 6p_{\ell+1}^3 - 6p_{\ell+1}^2$$

$$- 6p_{\ell+1}\sum_{i=1}^{\ell} (2q_i-p_i-1)\, p_i \quad .$$

It can easily be shown that for every n , with $k = n+1$

$$(4.4.15) \qquad \sum_{i=1}^{n} (2q_i-p_i-1)\, p_i = p_{n+1}^2 - p_{n+1} \quad .$$

Therefore by (4.4.14) and (4.4.15), we have

$$\sum_{i=1}^{\ell+1} p_i^3 = 1 - 3\sum_{i=1}^{\ell+1} (2q_i-p_i-1)^2\, p_i \quad .$$

Hence, by induction hypothesis the result follows. □

In case of tests for symmetry, $\overline{H}_1$, the asymptotic efficiency becomes (Canover (1973a))

$$(4.4.16) \qquad e = \frac{\left[\int_{F^+(0)}^{1} \phi^+(u)\ \phi^+(u,F,\theta_o)du\right]^2}{\int_{F^+(0)}^{1} [\phi^+(u)]^2 du \int_{F^+(0)}^{1} [\phi^+(u,F,\theta_o)]^2 du} .$$

We can discuss as in case of $\overline{H}_o$, the asymptotic relative efficiencies of different methods of handling ties and prove Putter's (1955) Theorem 2 as a special case.

4.5 TWO METHODS OF HANDLING TIES AT ZERO: COMPARISON.

Two methods of handling ties at zero in Wilcoxon signed rank tests has been mentioned in Chapter 3 (Pratt's method and Wilcoxon's method). In this section, we will compare the asymptotic efficiencies of two, as given by (4.4.16) and show that each one performs better in different conditions (see [4]).

Let $X_1,X_2,\ldots,X_n$ be a random sample with discrete distribution function $F(x,\theta)$. Let $p(x,\theta)$ represent the probability function. In order to apply the results of preceding sections, $F(x,\theta)$ should satisfy the following conditions

(i) $p(x,\theta_o) = p(-x,\theta_o)$ for some $\theta = \theta_o$

(ii) $f(x,\theta) = \frac{p(x,\theta)}{p(x,\theta_o)}$ exists almost everywhere with respect to $F(x,\theta_o)$

(iii) $\frac{\partial}{\partial\theta} f(x,\theta)\Big|_{\theta=\theta_o}$ exists almost everywhere with respect to

$F(x,\theta_o)$

(iv) $\lim\limits_{\theta\to 0} f(x,\theta) = 1$ almost everywhere with respect to $F(x,\theta_o)$

(v) $$\lim_{\theta\to\theta_o} \int_{-\infty}^{\infty} \left|\frac{\partial}{\partial\theta} f(x,\theta)\right| dF(x,\theta_o) = \int_{-\infty}^{\infty} \left|\frac{\partial}{\partial\theta} f(x,\theta)\right|_{\theta=\theta_o} \Big| dF(x,\theta_o) < \infty$$

(vi) $$\frac{\partial}{\partial\theta} f(x,\theta)\Big|_{\theta=\theta_o} = -\frac{\partial}{\partial\theta} f(-x,\theta)\Big|_{\theta=\theta_o} .$$

The above conditions do hold for the examples under consideration. Let

(4.5.1) $$T = \sum_{i=1}^{n} a(R_i^+) \operatorname{sign} X_i \Big/ \Big(\sum_{R_i^+>\tau_o} a^2(R_i^+)\Big)^{1/2}$$

where τ_o equals the number of observations which equal zero, $a(R_i^+)$ are scores. If scores satisfy the conditions of Theorem 4.2 then (4.5.1) is asymptotically standard normal (in Conover (1973b), the statistic (2.1) is incorrect and (4.5.1) is corrected form of that). In the following, we will calculate asymptotic efficiency of T from (4.4.16) in different situations.

(a) W : *Wilcoxon's Test with Pratt's method for ties at zero and the Randomized Rank method for other ties:* Here the scores $a(i) = i/n+1$ and converge to $\phi_W^+(u) = u$, $0 < u < 1$. Hence (4.4.16) becomes

(4.5.2) $$e_W = 3\Big[\int_{p_o}^{1} u\, \phi^+(u,F,\theta_o)du\Big]^2 \Big/ (1-p_o^3) \int_{p_o}^{1} [\phi^+(u,F,\theta_o)]^2\, du .$$

(b) $\bar{W}$: *Wilcoxon's test with Pratt's method for ties at zero and averaged (≡ midrank) rank method for other ties:*

Since the midrank method is used, the score does not converge to u because of discontinuities in the distribution function. In this case the scores converge to

$$(4.5.3) \qquad \phi_{\bar{W}}(u) = F[F^{-1}(\tfrac{u+1}{2}, \theta_o);\theta_o] - P[F^{-1}(\tfrac{u+1}{2}, \theta_o),\theta_o]/2 \quad .$$

And (4.5.3) and (4.4.16) gives

$$(4.5.4) \qquad e\bar{w} = \frac{3[\int_{p_o}^{1} u\ p^{+}(u,F,\theta_o)du]^2}{(1-\sum_i p_i^3) \int_{p_o}^{1} [\phi^{+}(u,F,\theta_o)]^2\ du} \quad .$$

(c) W_o : *Wilcoxon's test with zero discarded and randomized rank method for other ties:*

The scores start at $1/(n+1)$ for non-zero observations rather than at about p_o as in the previous case. The scores converge to

$$(4.5.5) \qquad \phi_{W_o}(u) = 0 \qquad 0 < u \le p_o \ ,$$

$$= \frac{u-p_o}{1-po} \qquad p_o < u < 1$$

which gives

$$(4.5.6) \qquad e_{W_o} = \frac{3[\int_{p_o}^{1} u\ \phi^{+}(u,F,\theta_o')du - p_o \int_{p_o}^{1} \phi^{+}(u,F,\theta_o)du]^2}{(1-p_o)^3 \int_{p_o}^{1} [\phi^{+}(u,F,\theta)]^2\ du} \quad .$$

(d) $\overline{W}_o$: *Wilcoxon's Test with zero discarded and midrank method for other ties:*

From (4.5.3) and (4.5.5), we have scores converging to

(4.5.7)
$$\phi_{\overline{W}_o}(u) = 0 \qquad 0 < u \le p_o$$
$$= \frac{\phi_{\overline{W}}(u) - p_o}{1-p_o} \qquad p_o < u \le 1$$

which gives

(4.5.8)
$$e_{\overline{W}_o} = \frac{3\left[\int_{p_o}^{1} u\,\phi^+(u,F,\theta_o)du - p_o\int_{p_o}^{1}\phi^+(u,F,\theta_o)du\right]^2}{\left[(1-p_o)^3 - \sum_{i\neq 0} p_i^3\int_{p_o}^{1}[\phi^+(u,F,\theta_o)^2\,du\right]}$$

Having discussed the asymptotic efficiencies of different methods of handling ties in Wilcoxon's signed rank test, we give two examples; one of which favours Pratt's method and the other discarding zeros.

EXAMPLE 4.1: Let us consider the discrete uniform distribution. Under null hypothesis the probabilities are equal and symmetric about zero, namely

(4.5.9)
$$P(x,0) = 1/(2k+1) \quad \text{for} \quad x = 0,\pm1,\ldots,\pm k \quad ,$$

and zero elsewhere, under the null hypothesis. Let the alternative be

$$P(X=x) = \frac{1+x\theta}{2k+1} \qquad x = 0,\pm1,\ldots,\pm k \quad , \quad 0<|\theta| \le \frac{1}{k}$$
$$= 0 \qquad \text{elsewhere.}$$

Then $\overline{H}_1$ is $\theta \neq \theta_o = 0$. We have,

$$\phi^+(u,F,\theta) = F^{-1}[(u+1)/2] = 0 \quad , \quad 0 < u \leq p_o = \frac{1}{(2k+1)}$$

$$= F^{-1}[\frac{(u+1)}{2}] = i \quad , \quad \frac{(2i-1)}{(2k-1)} < u \leq \frac{(2i+1)}{(2k+1)}$$

$$i = 1,2,\ldots,k \quad .$$

Therefore (2.5.2), (2.5.4), (2.5.6) and (2.5.8) become respectively,

$$e_W = (4k^2+6k+2)/(4k^2+6k+3) \quad ,$$

$$e_{\overline{W}} = 1 \quad ,$$

$$e_{W_o} = (16k^3+8k^2-7k+1)/(16k^3+8k^2) \quad , \text{ and}$$

$$e_{\overline{W}_o} = (16k^3+8k^2-7k+1)/(16k^3+8k^2-4k-2) \quad .$$

For $k = 1$, $\overline{W}$ and $\overline{W}_o$ are equivalent with efficiencies 1. For $k > 1$, we have

$$W_o < W < \overline{W}_o < \overline{W} \quad .$$

Hence Pratt's method seems to be preferred. A table showing different e_W's for different k has been given in [4].

EXAMPLE 4.2: Let

$$P(X=x) = \binom{2k}{k+x} \theta^{k+x} (1-\theta)^{k-x} \quad , \quad x = 0,\pm 1,\ldots,\pm k$$

$$= 0 \quad , \quad \text{elsewhere.}$$

The test of symmetry $\bar{H}_1$ tests $\theta = \frac{1}{2}$.

In this case we have for $0 < u < 1$ and $1 \le i \le k$

$$\phi^+(u,F,\tfrac{1}{2}) = 4F^{-1}\left[\frac{(u+1)}{2}\right] = 0 \quad , \quad 0 < u \le p_o = \binom{2k}{k}\left(\tfrac{1}{2}\right)^{2k}$$

$$= 4i \quad , \quad P_{i-1} < u < P_i$$

where

$$P_o = p_o$$

$$P_i = p_o + 2\sum_{j=1}^{k} \binom{2k}{k+j}\left(\tfrac{1}{2}\right)^{2k} \quad , \quad i \ge 1 \ .$$

Let $I = 2k - 2\sum_{j=0}^{k-1} P_j^2$. Then we have

$$e_W = 3I^2/[8k(1-P_o^3)]$$

$$e_{\bar{W}} = 3I^2/[8k(1 - p_o^3 - \sum_{i\neq o} p_i^3)]$$

$$e_{W_o} = 3(1-4k\ p_o^2)^2/\{8k[(1-p_o)^3 - \sum_{i\neq 0} p_i^3]\} \ .$$

The above formulae do not suggest any obvious ordering but some numerical results for different k (see [4]) yield that $\bar{W}_o$ is preferred.

CHAPTER 5

GENERAL REMARKS

In this chapter we give some general remarks which may be of some use to a practical statistician. The main concern while using a rank test when ties are present is that, the null distribution of statistics depends upon the pattern of ties and is usually difficult to compute. Let us note the following points.

(i) As proved in Section 4.4, the test statistics based on the average score method is more powerful than that based on the randomized rank procedure. But as the tables for each vector of ties are different in the former case, it may not always be practicable to use the average score method. However, we suggest the use of average score method in case of large samples, i.e., whenever a large sample approximation is used and modify the test statistics according to the charge in the variance (for example, see (3.1.6)).

(ii) In case we do not have the modified form of the statistics or limiting distribution of the test statistics, when ties are present is difficult to compute we should note that using the original test statistics and ranking ties by averaged score method, increases the level of significance than the one indicated by tables (Hájek (1969)).

(iii) We also suggest the use of *Computer Tables*. By this we mean, the use of computer programmes, which should be available in readily usable forms, to calculate the probability (given a vector of ties) $P[S > s/\tau]$, where s is the observed value of S and τ is the vector of ties. By this procedure we do not have to print huge amounts of tables which may be used rarely. Klotz (1966) has given an algorithm to compute approximate probabilities in case of Wilcoxon two sample test. The approximation is quite good for large n $(n \geq 5)$ but for $n < 5$ it fails like other approximations.

(iv) In tests of symmetry we have the problem of zero observations apart from the usual ties. Among the two methods Pratt's and Wilcoxon's, as discussed in Section 4.5, it is hard to recommend one over the other. As argued by Pratt, omitting zeros from the observations (Wilcoxon's method) seems to be causing some loss of information. Hence intuitively, we suggest the Pratt's method (ranking zero along with other observations and then dropping them from the rank vector).

(v) If the number of tied observations are very few, it might be much easier to use the randomized rank procedure and hence use the usual tables (see Section 3.1) without losing much power than to use averaged score method or midrank method and so requiring tabulation.

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