THE ELEMENTARY THEORY OF SETS OF POINTS

T H E E L E M E N T A R Y T H E O R Y

of

S E T S O F P O I N T S

With an Introductory Essay on

IRRATIONAL NUMBERS

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Presented in Partial Fulfilment of the Requirements

for the

Degree of Master of Science

of McGill University

by Iveson Miller

Historical Introduction.

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It is quite generally known that arithmetic had its beginning in the counting of groups of things, such as trees and herds of cattle; but it is not nearly so widely understood why the number ten should have been chosen as the basal unit of our initial system of enumeration, the decimal system, rather than any other number. When one considers that counting would naturally take the form of telling off the objects of a group on the fingers of both hands, it is easy to understand how the positive whole numbers were constructed on the decimal system. Clearly the decimal system of notation is not the only one that could be invented. In fact, it is unfortunate that man had not chosen a duodecimal system, that is one based on twelve as unit of notation. For instance, ten is divisible by two and five only, whereas twelve is divisible by two, three, four and six. Again, 2 x 10 is divisible only by 2, 4, 5 and 10, whereas 2 x 12 is divisible by 2, 3, 4, 6, 8, and 12. In ordinary business, where 1/2, 1/3 and 1/4 are used extensively, the advantage of the duo-decimal notation is very clearly put in evidence.

To show that other scales of notation have been invented we need but mention the sexagesimal measure of angles, and of intervals of time, which though constructed many centuries ago remain in common usage at present.

The extension of the number system to include negative numbers, zero, positive and negative fractions, is briefly outlined in the following pages, together with a full discussion of irrational numbers.

The difficulties that presented themselves in a rigorous discussion of the theory of incommensurable quantities were known to the Greeks of the time of Euclid (330-275 B. C.), as is shown by the fifth and tenth books of Euclid's "Elements", which deal with the ratios of magnitudes, the fifth with commensurable, the tenth with incommensurable magnitude (incommensurable magnitudes correspond to irrational numbers), but they were unable to arrive at a satisfactory solution of the difficulties, as were all the mathematicians until within the last century.

The applications of the four fundamental rules--addition, subtraction, multiplication and division--to rational numbers were quite perfectly understood during this long period of time; and as an irrational number taken to any required degree by approximation is a rational number, mathematicians achieved results that were, in the main, correct. Towards the end of the eighteenth century, mathematicians, who were making a careful study of the calculus methods of Newton (1642-1727) and Leitnitz (1646-1716), discovered so many inconsistencies in results and points in the theory that were not rigorously demonstrable by them, that they turned their chief attention to a revision of the fundamentals of the whole mathematical system. These thinkers and logicians were much perturbed that the notion of the approach of a variable to a fixed limiting value and such an obvious thing as the theorem 'that every magnitude which grows continually but not beyond all limits, must approach a limiting value' , could be demonstrated easily by geometrical, but not at all rigorously by arithmetical methods.

The pioneers in this demand for greater accuracy and rigor in arithmetic were Gauss, Lagrange, Cauchy and Abel. The contributions of these man had the effect of clearing up many points that had formerly been obscure, and served as an inspiration to a long list of investigators. This movement towards absolute rigor in the proofs of theorems became in the second half of the nineteenth century very general, with Weierstrass as its greatest exponent. Weierstrass showed that to place mathematical analysis on a satisfactory basis it was necessary to create a theory of irrational numbers, with the same rigor as in the theory of incommensurables contemplated by Euclid, and with as much care as had been bestowed on the system of rational numbers by the early mathematicians.

In 1872 Cantor and Dedekind, accepting the results of Weierstrass' studies in the general theory of functions, constructed two theories that satisfy the rigor demanded by investigators in mathematical analysis, and which although apparently very different can be readily brought into equivalence: Dedekind, 'Essays on Number', Cantor 'Ueber die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen'.

Scope of the Thesis.

This essay deals with the development of the system of rational numbers, not fully as is to be found in Fine's Algebra and Pierpont's 'Theory of Functions of a Real Variable' , but as a foundation upon which to construct Dedekind's theory of the irrational number.

A discussion of limits for sequences of rational numbers follows in preparation for the deduction of Cantor's theory from that of Dedekind.

Cantor's theory is constructed and its equivalence with that of Dedekind demonstrated.

The idea of continuity is developed only in so far as it is necessary to the understanding of the nature and explanation of the arithmetic continuum.

The second part of the thesis deals with sets of points. Many of the newer branches of Pure Mathematics, such as: The Theory of Functions of a Complex Variable, The Calculus of Variations and Differential Equations, make use, either directly or indirectly, of an infinite number of operations. These investigations are extremely complex, and the reasoning is liable to many errors. To one entering upon a careful study of such subjects a knowledge of the fundamental principles of the theory of sets of points is almost indispensable.

Rational Numbers.

As a beginning, I will accept that the human mind holds the conceptions of unity, aggregate, order, and correspondence, as fundamentals.

By unity we mean the consideration of any object, however complex in structure or attributes, as a single thing.

The idea of aggregate is but bearing in mind the two conceptions of unity as a group, and the components of the group as separate unities.

Order is the term by which we designate the conception that enables us to select different objects with respect to some attribute as size or hardness, so that of two objects we can determine which is the larger or the harder. A group of objects of which each two had been examined and the larger or harder one selected would be called an ordered group. Of many ways of arranging the elements of a group in order it will suffice to mention the most usual one, namely, that in which the objects are chosen with respect to time, and are correspondingly labelled 1, 2, 3, 4, -- to any number however great.

The idea of correspondence is that of associating one unity with another unity, where these unities may be aggregates of any degree of complexity. In particular one-to-one correspondence, written (1,1), is used to indicate that we are pairing off individual elements of one group with individual elements of a second group.

To unity as a distinguishing mark we give the symbol 1. To an aggregate (A,B) or (B,A), perceiving that here we have unity together with unity in one whole, we assign an arbitrary symbol 2. Similarly to an aggregate (A, B, C) we assign a new and arbitrary symbol 3; and so on indefinitely. In this process we are performing but the simplest act in passing from an already formed symbol to the consecutive new one to be formed. Thus we create successively the series of integral positive, or natural numbers, which is characterized by the fact that the sequence of numbers has no last number: namely, 1, 2, 3, 4, 5, 6, ---------------- (I).

The Fundamental Rules.

Addition is but the combination into a single act of any arbitrary repetitions of the simplest act mentioned above. For example,

$$\begin{aligned} 7+4 &= 7+(3+1) \\ &= 7+(2+1+1) \\ &= 7+(1+1+1+1) \\ &= (8+1)+1+1 \\ &= (9+1)+1 \\ &= (10+1) \\ &= 11 \end{aligned}$$

Multiplication arises in a similar way from addition, that is, it is but a combination of any arbitrary number of repetitions of the addition process.

Ex. $5 \times 3 = 3+3+3+3+3$, or $= 5+5+5$.

The two processes, addition and multiplication, are always possible in the sense that each application of either process to two numbers of the series results in a number of the series, and obviously the statement is true for any number of applications of these processes.

The process of subtraction when applied to numbers belonging to the series of natural numbers is the reverse of addition.

Ex. $7 - 4 = 3$.

$$\begin{aligned} 7 - 4 &= 7 - (3+1) \\ &= 7 - (2+1+1) \\ &= 7 - (1+1+1+1) \\ &= 6 - (1+1+1) \\ &= 5 - (1+1) \\ &= 4 - 1 = 3 \end{aligned}$$

However, this process would not be applicable in the case where the number to be subtracted is the larger, without an extension of the number system.

Thus 1-1 introduces a new quantity to which the symbol 0, zero, is attached; 0-1 introduces a successive new quantity to which we assign the symbol -1, minus one; and so on indefinitely. The numbers given in order of creation are 0, -1, -2, -3, -4, -----------------

The extended number system is represented by ---------- ----- -3, -2, -1, 0, 1, 2, 3 ----------(LL), the series having no last number on either side.

Division may be considered as the reverse process to multiplication. Thus $3 \times 5 = 15$, and $15 \div 3 = 5$. In general if $\frac{b}{a} = x$, then $b = ax$. Elearly a further extension of the number system is required before division will always be possible among the members of the extended number system(II). To perform this the rule is applied that if $b \div a = x$, then $a \quad x = b$, thus the positive and negative fractions become necessary. From the conception of multiplication as a condensed process of addition it is clear that $a \times (-b) = -ab$, in accordance with which, $(-a) \times (-b)$ is defined to be ab. Hence a negative number divided by a positive number or vice versa results in a negative number, whereas a positive number divided by a positive number or a negative number by a negative number results in a positive number.

The Ordinal Numbers.

The set of symbols, or corpus III, is now complete in the sense that on the application of the processes of addition, subtraction, multiplication and division, every result is itself a member of the sequence, with the exception of division by zero.

Ex. $a + 0 = a$
$a - 0 = a$
$a \times 0 = 0$
$a \div 0$ has not been defined. Let it be x, then $x \times 0 = a$. But any definite number, however great, multiplied by zero, is still zero, therefore $a \div 0$ introduces a new quantity which is called infinity and is assigned the symbol ∞.

We wish now to arrange the corpus III in some definite order, that we may have an ordered set of symbols. The extended number system II is really a set ordered with respect to time, that is, ordered by each successive creative act as indicated in the above reasoning. However, to make the complete reasoning more intelligible a proof will be given based on the fundamental ideas of aggregate and correspondence. By pairing off the elements of two aggregates we can always determine which is the greater unless they should happen to be equal. And if we agree that to indicate that b is greater than a, we use the symbolic notation $b > a$, and write b on the right hand side of a, all the positive integers become ordered when written 1, 2, 3, 4, ---------(A). The negative integers -1, -2, -3, -4, -------------(B) are evidently in an order, since they correspond exactly to the ordered sequence (A). Further, as $-1+1=0$, $-2+1=-1$, and so on, the sequence (B) must be reversed before it will be in accordance with our convention of writing the greater number to the right of every smaller number.

The series ---------- -4, -3, -2, -1, 0, 1, 2, 3, 4, ------------ is now an ordered Corpus II. It should be observed here that every negative number is less than zero, or any positive number whatsoever.

Now, taking the case of any two positive fractions, $\frac{a}{b}$ and $\frac{c}{d}$. These can be written $\frac{ad}{bd}$ and $\frac{bc}{bd}$. Evidently, $\frac{a}{b}$ is greater than, equal to, or less than $\frac{c}{d}$. Written $\frac{a}{b} \gtreqless \frac{c}{d}$, according as $\frac{ad}{bd} \gtreqless \frac{bc}{bd}$, that is, according as $ad \gtreqless bc$. But ad, bc are integral positive numbers and belong to the ordered set (A), hence the greater number may be obtained by inspection. An exactly similar proof will apply to any two negative fractions whatsoever, whereas every negative fraction can be shown to be less than any positive quantity or zero by the same process of making the denominators of the fractions equal.

The complete set of symbols, arranged in ascending order of magnitude, has now become an ordered corpus, with no greatest positive number and no least negative number. This corpus is called the set of rational numbers.

When we have at our disposal the set of natural numbers arranged in order of magnitude, we are in a position to count the elements of any finite group. To determine the ordinal number of the group, or in other words count the group, we impress an order with respect to time, (i.e.) the succession in which they are chosen, on the elements of the group and then place this group in (1,1) correspondence with the rational numbers 1, 2, 3, 4 ------------. Evidently the natural number paired off with the last selected number of the group represents the ordinal number of the group.

Ordinal number is thus regarded as the concept obtained by making abstraction of the nature of the objects, but retaining the order in which they are given in the aggregate.

The System of Rational Numbers.

To the ordered corpus of rational numbers we may apply the following laws:

1. If $a>b$, and $b>c$, then $a>c$.

When the elements of a group b are paired off with the elements of a group a, and when there are elements of a remaining unattached while every element of the group b has been paired off against one of the group a, the group a is said to be greater than the group b, and is expressed symbolically by $a>b$. Similarly, the expression $b>c$ means that when every element of the group c has been placed in (1,1) correspondence with elements of group b , there are elements in group b remaining unattached.

Evidently then if the elements of the group c were placed in (1,1) correspondence with the elements of the group a, there would remain unattached elements in the group a, therefore $a>c$.

By the method employed earlier of reducing two positive fractions to a common denominator and then considering their numerators which are positive integers, the above proof is directly applicable. In the case of negative numbers, one can readily generalize the above statement by considering that the negative numbers are in but the reversed order to that of the positive numbers considered from the number zero.

A more simple proof can be observed by mere inspection, since when we write the ordered set of rational numbers in full every positive and negative rational fraction is included. Hence the expression $a > b$ means that a lies to the right of b, and the expression $b > c$, that b lies to the right of c, that is, $a > c$. Another statement of the above law is that b lies between a and c.

If a and c are two different rational numbers, there are infinitely many different rational numbers lying between a and c.

First proof. Let a and c be two positive integral values, or decimal numbers that have a finite number of decimals only---this excludes such numbers as $.\dot{3}$. Suppose a is given by $a_1 .a_2 a_3 a_4 a_5$ -----a_n and c is given by $c_1 .c_2, c_3, c_4, c_5, c_6$ ----- c_m, and let $a_1 = c_1$, $a_2 = c_2$. -------- $a_n = c_n$. If $m > n$, then $c_m > c_n$, in the sense of appearing in the sequence further to the right.

The number $a_1 . a_2 a_3 a_4 a_5$ ----------a_n --------a_m, a_{m+1} a_{m+2}........a_p

Where $a_{n+1} \leqq c_{n+1}$, $a_{n+2} \leqq c_{n+2}$, ----$a_n < c_n$ is less than c.

Further it is evident that there are infinitely many numbers given by the above sequence, since a_{m+1}, a_{m+2}, a_{m+3} -------- can each be any one of the ten figures. With assigned values of $a_1\ a_2\ a_3\ a_4$ -------- a_m , the number of rational numbers, $a_1\ a_2\ a_3\ a_4$--------a_m----- a_{m+r}, proceeding as far as the term $a_{m+r} = 10^r$; this can be made greater than any assignable number however great by choosing r properly.

To provide for the case of infinite decimals that appear from the division of two integral numbers, and are therefore rational numbers, such as $\frac{1}{3} = .\dot{3}$ and $\frac{1}{6} = .1\dot{6}$, we can reduce the two fractions to a common denominator and then proceed with the numerators as above. Illustration: There are infinitely many rational numbers between $\frac{1}{3}$ and $\frac{1}{6}$; we have but to show by the above proof that there are infinitely many rational numbers between 2 and 1 , then dividing each of these numbers by 6 will give infinitely many rational numbers between $\frac{1}{3}$ and $\frac{1}{6}$.

Second proof. Taking $\frac{a}{b}$ and $\frac{c}{d}$ as two general rational numbers, where a, b, c, d are positive integers, and given that $\frac{a}{b} < \frac{c}{d}$, it is required to prove

$$\frac{a}{b} < \frac{ad\ \ bc}{2bd} < \frac{c}{d}$$

Since $\frac{a}{b} < \frac{c}{d}$, we have $ad < bc$;

$$2ad < bc + ad \quad ;$$

$$2adb < b(bc + ad);$$

$$\frac{a}{b} < \frac{bc + ad}{2bd} \quad .$$

Again , $$ad < bc \quad ;$$
$$ad + bc < abc \quad ;$$
$$d \quad (ad + bc) < 2bd;$$
$$\frac{ad+bc}{2bd} < \frac{c}{d} \; .$$

That is, between any two positive rationals we can insert their arithmetic mean, between this arithmetic mean and each of the initial numbers arithmetic means can again be inserted, and so on indefinitely. If, the given numbers are both negative numbers, the demonstration follows at once from the above proof. In the case where one number is negative and the other positive, by the above proofs we can show that between 0 and either of the numbers there are infinitely many rational numbers.

If <u>a</u> is any definite rational number, the system of rational numbers can be considered as making up two classes A_1 and A_2, which are such that A_1 contains all the rational numbers that are less than <u>a</u>, and A_2 contains all the rational numbers that are greater than <u>a</u>. The number <u>a</u> can be assigned at will to either class, becoming either the first of the class A_2, or the last number of the class A_1. The classes A_1 and A_2 are such that every member of the class A_1 is less than every member of the class A_2 and every member of the class A_2 is greater than every member of the class A_1.

We can thus dintinguish three methods of separating the rational numbers so as to define the rational number <u>a</u>:

(α) A_1 , a , A_2

(β) (A_1 a), A_2

(γ) A_1 , (a A_2)

The Straight Line.

It is required to show that the points of a straight line obey laws that are similar to those that have been applied to the ordered set of rational numbers.

Let L be any straight line and p, q, r any three different points.

The following laws apply:

1. If p lies to the right of q, and q lies to the right of r, than p lies to the right of r; and we say that q lies between the points p and r.

From the diagram this statement is seen to be true.

If p and r are two different points, then there always exist infinitely many points that lie between p and r.

The general conception of continuity is derived from a straight line. We conceive the straight line to be such that between any two points however close there are infinitely many other points.

If <u>p</u> in a definite point in <u>L</u>, then all the points of <u>L</u> fall into two classes P_1 and P_2 , which are such that P_1 includes all the points that lie to the left of <u>p</u>, and P_2 includes all the points that lie to the right of p. The point <u>p</u> can be assigned at will to either class. Moreover every point of the class P_1 lies to the left of every point of the class P_2; and every point of the class P_2 lies to the right of every point of the class P_1.

This analogy between the rational numbers and the points of a straight line becomes a real correspondence if we take some point of L as origin or zero point, and a definite length as unit of the measurement of segments.

The succeeding discussion will establish a (1, 1) correspondence by arithmetical means alone between the rational numbers and definite points of the straight line in such a way that the order is maintained.

One method of establishing this correspondence is by making use of the harmonic ratio of special numbers. If a, b, c, d are any four numbers, they are said to form a harmonic ratio when $\frac{a-b}{b-c} \cdot \frac{c-d}{d-a}$ is equal to -1.

Clearly there are infinitely many sets of four rational numbers which form a harmonic ratio. If a, b, c are given rational numbers, they determine a definite number <u>d</u> according to the equation $d = \frac{ab+bc-2ac}{-a-c+2b}$ which is obtained by simplification of the above harmonic ratio (a, b, c, d). The equation evidently holds for all values of a, b, c except $2b = a+c$. In this case if we take the ratio $(a, \frac{a+c}{2}, c, d)$ where <u>d</u> takes successively increasing values, it is found that

$$\frac{a - \frac{a-c}{2}}{\frac{a+c}{2} - c} \cdot \frac{c-d}{d-a}$$

approaches nearer and nearer to the value of -1 as <u>d</u> is given larger and larger values. That is when <u>d</u> is increased indefinitely, the given cross ratio becomes a harmonic ratio.

The following harmonic ratios will be required in the course of the demonstration. If p is any positive integer, we have

$$(p-1,\quad p\quad ,\quad p+1\quad ,\infty)=-1$$

$$(-p,\quad o\quad ,\quad p\quad ,\infty)=-1$$

$$\left(o,\quad \frac{1}{p+1}\quad ,\quad \frac{1}{p}\quad ,\frac{1}{p-1}\right)=-1$$

Take any straight line LL, unlimited in either direction and let any three points be selected and numbered in order P, 1, Q. We can then establish the (1,1) correspondence as follows:

Let C be any point outside the line LL. Join PC, QC and 1C. In the line PC take any point A. Join QA. We shall call the fourth harmonic point of P with respect to 1 and Q the

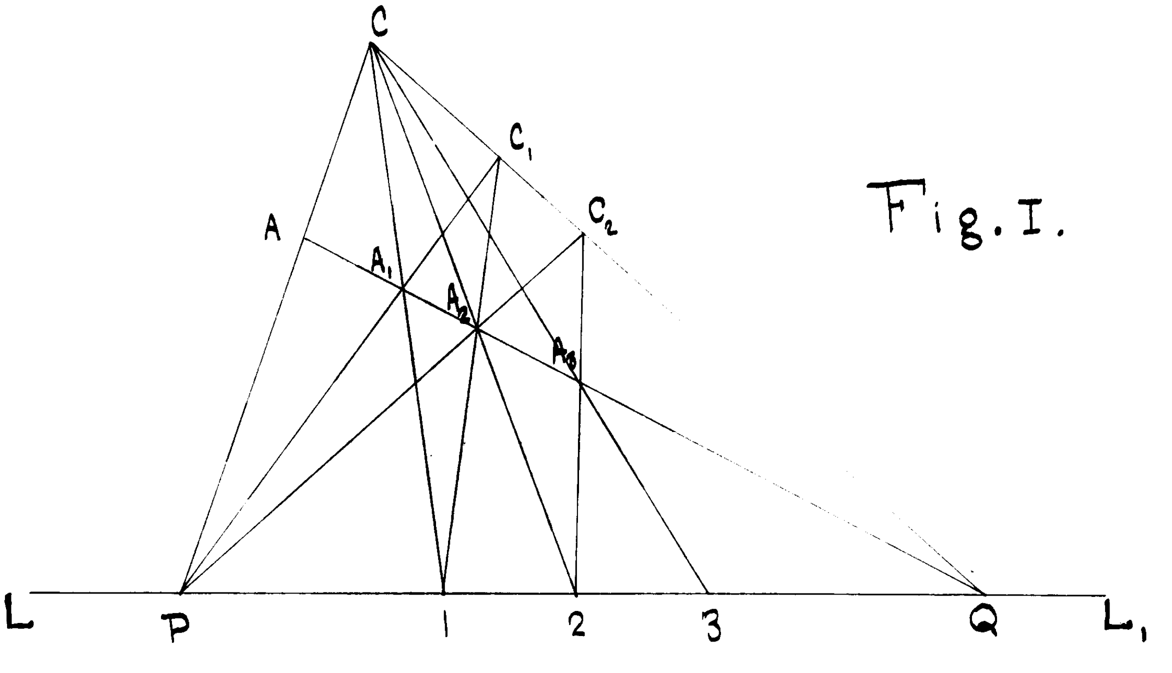


Fig. I.

integral number 2. Similarly we shall call the fourth harmonic of 1 with respect to 2 and Q the integral number 3, and so on indefinitely.

The geometrical construction of these various harmonic points is indicated in Fig. 1. The lines 1C and QA intersect at some point which we will call A_1. Join PA_1 and produce to meet QC in C_1. Join $1C_1$ meeting QA in A_2. Join CA_2 and produce to meet LL_1 in the required harmonic point 2.

Again, join PA_2 and produce to meet QC in C_2. Join $2C_2$ meeting QA in A_3. Join CA_3, meeting LL_1 in the harmonic point 3.

This construction sets up a (1,1) correspondence between all the positive integral numbers and points in the segment PQ of the straight line LL_1.

The above harmonic construction is made more clear by a second figure:

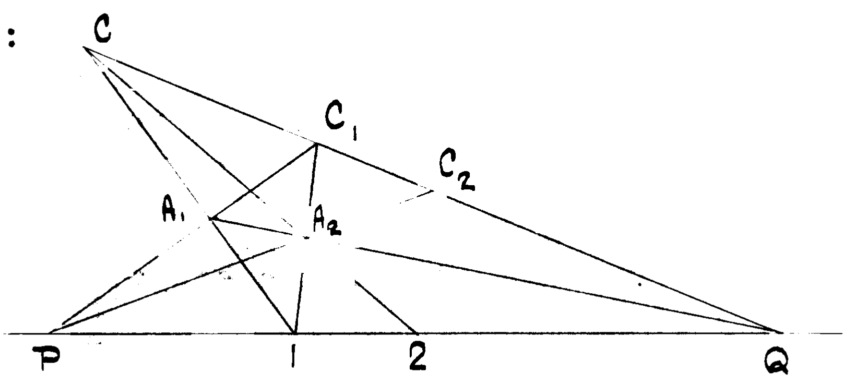


Figure 11.

A_1CQ1A_1 is a quadrilateral. C, P, and A_2 are its diagonal points. The two sets of points (P, 1, Q,2) and (C, C_1, Q, C_2) are in harmonic ratio.

To construct points that will represent the negative integers, we will call the fourth harmonic of the positive integer M with respect to P and Q, the negative integer -M. The geometrical construction represents the point -M as the intersection of AC_m with the straight line LL_1, where C_m is the point in CQ which is used directly in the construction of the positive integer M+1.

For example, the point -1 is represented by the intersection of AC_1 produced with the straight line LL_1; the point -2 is represented by the intersection of AC_2 with the straight line LL_1; and so on indefinitely. That is, all the negative integers are represented on the segment $P\infty Q$ of the straight line LL_1.

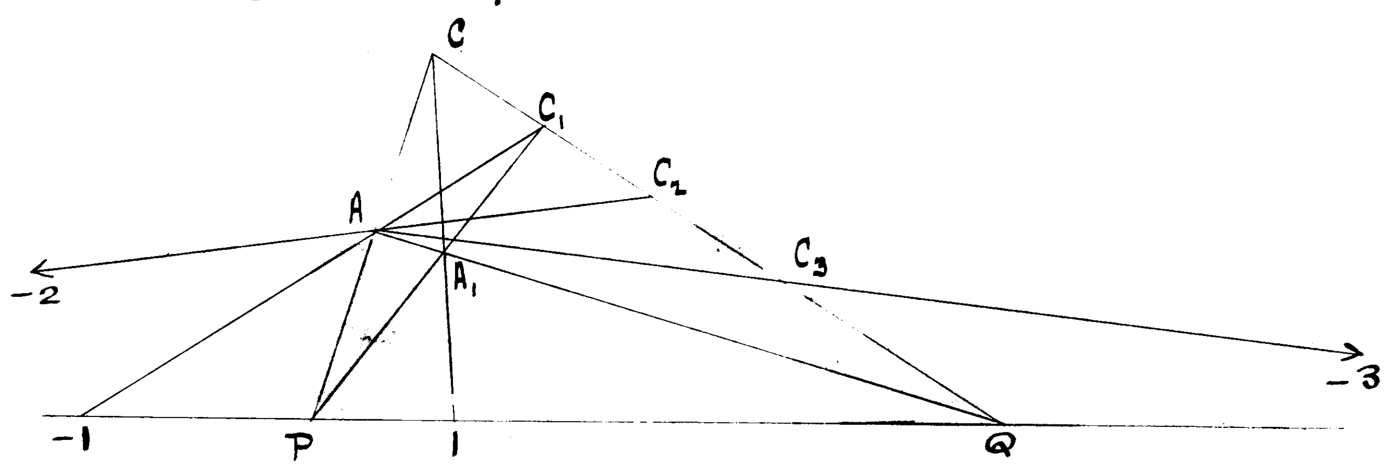


Figure III.

At some period in the construction the line AC_n changes its slope from positive to negative, and all succeeding negative numbers will be represented by points on the straight line LL_1, but to the right of Q. As drawn in Figure III AC_2 produced intercepts the line LL_1 on the left of P, and AC_3 produced intercepts the line LL_1 on the right of Q. The result will be comprehensive if we consider the straight line to be a closed curve. The extended definition of a straight line is that of a great circle on a sphere of infinite radius, which demonstrates that, from our figure, a point at an infinite distance to the left of P will coincide with a point at an infinite distance to the right of Q.

A construction for the inverses of the rational positive numbers is readily obtained by a slight modification of the previous method.

Let us represent the positive number $\frac{1}{2}$ as the point which is the fourth harmonic of Q with respect to P and 1. Join 1A meeting A_1P in $A_{\frac{1}{2}}$. Join $CA_{\frac{1}{2}}$ and produce to meet LL_1 in the point $\frac{1}{2}$. Similarly $\frac{1}{3}$ will be represented as the fourth harmonic of 1, P and $\frac{1}{2}$. Join $\frac{1}{2}$ and A meeting $PA_{\frac{1}{2}}$ in $A_{\frac{1}{3}}$. Join $CA_{\frac{1}{3}}$ to meet the straight line LL_1 in the point $\frac{1}{3}$. Generally, the point $\frac{1}{m-2}$ is the fourth harmonic of $\frac{1}{m}$ with respect to P and $\frac{1}{m-1}$.

The points $1+\frac{1}{r}$, $2+\frac{1}{r}$, $3+\frac{1}{r}$ ----------- can be constructed from P, $\frac{1}{r}$, Q; just as 2, 3, 4, --------- were from P, 1, Q.

Generally, the points $1+\frac{n}{r}$, $2+\frac{n}{r}$, $3+\frac{n}{r}$, -------

can be constructed from P, $\frac{n}{r}$, Q.

The inverse points $\frac{r}{r+1}$, $\frac{r}{2r+1}$, $\frac{r}{3r+1}$, ------- can be constructed from Q, P, $\frac{1}{r}$; just as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, --- were from Q, P. 1.

Generally, the points $\frac{r+n}{r}$, $\frac{2r+n}{r}$, $\frac{3r+n}{r}$, --- can be constructed from Q, P, $\frac{n}{r}$.

When n and r have been given all positive integral values, points to represent all rational positive fractions will have been constructed. By a repetition of the previous construction the negative rational fractions $\frac{-1}{m}$ can be represented by the intersection of the line $C_{\frac{1}{m}}A$ with the straight line LL_1. The point $C_{\frac{1}{m}}$ is the point of intersection of $Pa_{\frac{1}{m}}$ with the line CQ where $A_{\frac{1}{m}}$ is the intersection of $Ca_{\frac{1}{m}}$ with the line AQ. (See Figure IV). In this last paragraph m is used to represent any rational fraction, and $\frac{1}{m}$ the inverse of the general rational fraction.

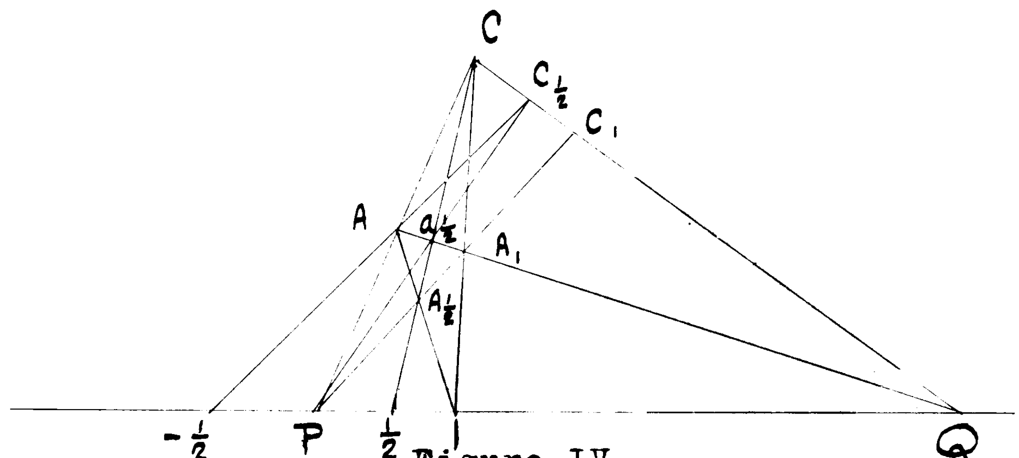


Figure IV.

From the general method of construction it is clear that as we approach P the numbers decrease in absolute value without limit, becoming less than any assignable positive rational quantity, ϵ. Further, the numbers to the left of P are negative, while those to the right of P are positive. That is, P cannot be other than the point 0. Similar reasoning demonstrates that Q is the point ∞.

Supposing that in the infinitely many applications of the above projective scale there is one point which does not represent any rational number, we can plot rational numbers that approach nearer to that point than any assignable magnitude. This will be shown to be the definition of an irrational number. It can be shown that there are infinitely many points of the straight line that do not represent rational numbers. For instance, the diagonal of a square whose sides are positive integers in magnitude is incommensurable with that unit of magnitude.

Proof. If possible let the side be to the diagonal in a commensurable ratio, namely, that of the two integers a and b. Suppose this ratio reduced to its lowest terms so that a and b have no common divisor other than unity, that is, they are prime to each other. Then $b^2 = 2a^2$ (by Euclid I.47); therefore b is an even number; hence, since a is prime to b, a must be an odd number. But since b is an even number, it can be written as 2n; therefore $(2n)^2 = 2a^2$, or $a^2 = 2n^2$; therefore a^2 is an even number; therefore a is an even number. Thus a is both odd and even, which is absurd; therefore the side and diagonal are incommensurable.

Hence, if from the origin O, a length is laid off along the straight line LL, we obtain a point to which no rational number corresponds. It is at once obvious that the number of these incommensurable lengths is infinite; therefore the number of points in a straight line that have no corresponding rational number is infinite. Thus the straight line is richer in points than the system of rational numbers is in symbols.

The Irrational Number.

We wish to follow out arithmetically all phenomena in a straight line, and the domain of rational numbers is evidently insufficient, since we have demonstrated the existence of gaps in the system. Furthermore, we attribute continuity to a straight line.

Before developing the theories of the irrational number it will be helpful to state clearly what is meant by algebraic, rational, and irrational numbers.

The various quantities that will satisfy an algebraic equation of any definite degree, say $a_0x^m + a_1x^{m-1} + a_2x^{m-2}$ -------------- $a_m = 0$, where the a's are any integers, positive or negative, and where the equation is irreducible, are called algebraic numbers.

It can be shown that there are numbers other than algebraic, such as the logarithms of the rational numbers and the Liouville numbers.

Those algebraic numbers which are solutions of an equation of the first degree are called rational numbers, and include the natural numbers, all terminating decimal fractions, as well as some non-terminating decimal fractions that follow a simple recurrence law, as $.\dot{3}$ or $.1\dot{6}$.

Those algebraic numbers that are solutions of equations that are irreducible and of degree higher than the first, are irrational numbers. Furthermore, all logarithmic numbers and the very special Liouville numbers are irrational numbers.

Dedekind asserts that the essence of continuity is embodied in the following principle: "If all the points of a straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two such classes, this severing of the straight line into two portions".

We wish to prove arithmetically that there are numbers other than rational numbers. Let D be a positive rational number but not the square of a rational number. If we place in the class A_2 all these positive numbers whose squares are greater than D, and assign to the class A_1 all the negative numbers and zero, together with all the positive numbers whose squares are less than D. Since there is no rational number whose square is equal to D, we have the domain of rational numbers divided into two classes by a number $\sqrt{D}$ which is not a rational but an irrational number.

It is required to prove that there can exist a positive integer D which is not the square of a rational number. A rational number can always be expressed as a rational fraction of which both the numerator and denominator are integers. Suppose $D = (\frac{u}{t})$, where $\frac{u}{t}$ is a fraction reduced to its lowest terms, that is u is prime to t. We have, then, chosen u as the least positive integer that will satisfy the expression, $t^2 - Du^2 = o$, which is known as Pell's equation.

By Archimedes principle, which I have accepted, no number will be considered in the theory of numbers that is so small that when multiplied by a sufficiently large number it cannot be made to exceed any arbitartily chosen number. That is, infinitesimal numbers are not included in the system of numbers.

Now as u is not an infinitesimal but is rather a positive integer, we know that there exists a positive integer λ, which is such that $\lambda u < t < (\lambda + 1) u$.

Let $u' = t - \lambda u$. Evidently u' is $< u$, and is a positive integer. Let $t' = Du - \lambda t$, t' is likewise a positive integer. We have

$t'^2 - D u'^2 = (\lambda^2 - D)(t^2 - D u^2) = o$, which is contrary to our assumption concerning u. That is, D cannot be expressed as the square of a rational number.

As an example of this type of reasoning, Fine in his Algebra gives a proof that $\sqrt{2}$ is irrational.

There is no integer whose square is 2. Suppose the fraction is such that $(\frac{p}{q})^2 = 2$, where p and q are positive integers. We have $\frac{p^2}{q^2} = \frac{2}{1}$.

That is, q must be 1. and $p^2 = 2$, since p^2 is prime to q^2. But p^2 cannot be equal to 2. That is, $\sqrt{2}$ cannot be expressed as a rational number and is called an irrational number.

Proceeding with the discussion of the division of the system of rational numbers into two classes A_1 and A_2, where the class A_1 includes all the negative numbers, zero, and those positive rational numbers whose squares are less than D, and where the class A_2 includes all those positive numbers whose squares are greater than D, while there is no rational number whose square is equal to D. In this case the class A_1 will have no greatest number and the class A_2 no least number. Proof.

If we put $$y = \frac{x\,(x^2 + 3D)}{3x^2 + D}$$

we have $$y - x = \frac{2x\,(D - x^2)}{3x^2 + D}$$

and $$y^2 - D = \frac{(x^2 - D)^3}{(3x+D)^2}$$

Then, if x is a positive number from the class A_1, we have $x^2 < D$; hence $y > x$ and $y^2 < D$; therefore y likewise belongs to the class A_1. While, if we assume x to be a positive number from the class A_2, we have $x^2 > D$; hence $y < x$, also $y > o$, and $y > D$; therefore y likewise belongs to the class A_2. This cut in the system of rational numbers is produced by no rational number. We are now in a position to define and create the irrational numbers.

Dedekind's theory of the irrational number. When the domain of rational numbers is separated into two classes by a point $\underline{a}$, such that the class A_1 includes all the rational numbers that are less than $\underline{a}$, and the class A_2 includes all the rational numbers that are greater than $\underline{a}$; further, when the class A_1 has no greatest number and the class A_2 has no least number; when every number of the class A_1 is less than every number of the class A_2, and conversely, this division of the rational numbers is said to define the irrational number $\underline{a}$.

Consider now two numbers a, b defined by cuts (A_1, A_2) and (B_1, B_2) in the rational numbers. If every number contained in the class A_1 is likewise contained in the class B_1, and every number contained in the class A_2 is also contained in the class B_2, then the two cuts are identical, which is denoted by $a = b$.

If the cuts are such that there is one number, and only one, in the class A_1 that is not included in the class B_1, denoting this number by a_1', since every other number a_1 of the class A_1 is also contained in the class B_1, we have $a_1 < a_1'$ Thus, a_1' is the greatest number in the class A_1, or in other words, the cut (A_1, A_2) is produced by the rational number $a = a_1'$. By our hypothesis $a_1' = b_2'$, where b_2' is the only number that is contained in the class B_2, which is not con-contained in the class A_2. Since every number b_2 of the class B_2, other than b_2', is also contained in the class A_2; we have $b_2' < b_2$, that is, b_2' is the least number of class B_2, therefore the cut (B_1, B_2) is produced by the rational number $a = b_2'$. Thus the rational number a is defined by two different cuts.

If the cuts (A_1, A_2) and (B_1, B_2) are such that there are two numbers in the class A_1 that are not contained in the class B_1, that is, are contained in the class B_2, we can assert that there are an infinity of numbers in the class A_1 that are contained in the class B_1, because, between any two rational numbers whatsoever there are infinitely many rational numbers. In this case the number a, defined by the cut

(A_1, A_2), is said to be greater than the number b, defined by the cut (B_1, B_2). In symbolic language $a > b$ and $b < a$.

It may advantageously be pointed out that the usual definitions of $>, =, <$ as applied to rational numbers also apply without ambiguity to irrational numbers.

The extended system of numbers consisting of all the rational and irrational numbers is called the domain of real numbers. It forms a well-arranged or ordered domain of one dimension and is subject to the following laws, which will be proved only for irrational numbers, as the rational numbers have been shown to obey these laws.

I. If $\alpha < \beta$, and $\beta < \gamma$, then is $\alpha < \gamma$. β is said to lie between α and γ.

The statement $\alpha < \beta$, asserts that there are infinitely many numbers in the class B_1 that are not included in the class A_1. Similarly the statement $\beta < \gamma$, asserts that there are infinitely many numbers in the class C_1 that are not included in the class B_1. I am considering the irrational numbers α, β, γ to be defined by the cuts (A_1, A_2), (B_1, B_2) and (C_1, C_2) respectively. It is evident that there are infinitely many numbers in the class C_1 that are not included in the class A_1, that is, $\alpha < \gamma$. And β is said to lie between α and γ.

II. If α and γ are two different irrational numbers, there exist infinitely many different numbers β lying between α and γ.

This is incidentally established in the preceding proof.

III. If the entire system of real numbers be separated into two classes R_1, R_2 each containing infinitely many element and such that every number of the class R_1 is less than every number of the class R_2, and conversely, further there is either a last number in the class R_1, or a first number in the class R_2. This separation is said in every case to be produced by the number $\underline{a}$.

In separating the real numbers into two parts R_1, R_2, we likewise separate the rational numbers into two classes A_1, A_2, because we but assign to the class A_1 all the rational numbers that are contained in the class R_1, and assign to the class A_2 all the rational numbers that are contained in the class R_2.

Let $\underline{a}$ be the number defined by the cut(A_1, A_2). If $\underline{a}$ is rational, it must be either the last number in the class A_1, or the first in the class A_2. Also, if $\underline{a}$ is the last number in the class A_1, it must be the last number in the class R_1 as well, because between any two numbers rational or irrational there are infinitely many rational numbers. Hence if $\underline{a}$ is not the last number in the class R_1, between $\underline{a}$ and $\underline{\alpha}$, there would exist infinitely many rational numbers, but all the rational numbers that are less than $\underline{\alpha}$ are contained in the class A_1 and $\underline{a}$ is the last number in the class R_1. Similarly, if $\underline{a}$ is the first number in the class A_2 it can be proved to be the first number in the class R_2.

If a is an irrational number, it must belong either to the class R_1 or to the class R_2; further, a must be the last number in the class R_1 or the first number in the class R_2, because if there was any number after a, there would be rationals between it and a. But all the rationals of the class R_1 are contained in the class A_1; hence there would be rationals of the class A_1 after a, which is impossible. In like manner, it can be proved that if a belongs to the class R_2 it is the first number in that class. Lastly, there cannot be both a last number in the class R_1 and a first in the class R_2, since there would be infinitely many rationals between these two rational or irrational numbers, that is, numbers belonging either to the class A_1 or to the class A_2, which is impossible.

In the previous paragraphs proofs have been given that between two rational numbers, and between two irrational numbers there exist infinitely many rational numbers. To make the statement absolutely general, a proof will be inserted of the case in which one number is rational and the other irrational.

From the definition of an irrational number, it follows that from the lower class infinitely many sequences of numbers can be chosen which will have the irrational number as an upper limit of the sequence, and that from the upper class infinitely many sequences can be chosen which have the irrational number as a lower limit; that is, there are infinitely many sequences that define any irrational number.

Illustration: The number $\sqrt{2}$ is given by the sequences:

$$1, \sqrt{\frac{3}{2}}, \sqrt{\frac{5}{3}}, \sqrt{\frac{7}{4}}, \sqrt{\frac{9}{5}}, -------$$

$$1, \sqrt{\frac{4}{3}}, \sqrt{\frac{3}{2}}, \sqrt{\frac{8}{5}}, \sqrt{\frac{5}{3}}, -------$$

$$\sqrt{3}, \sqrt{\frac{5}{2}}, \sqrt{\frac{7}{3}}, \sqrt{\frac{9}{4}}, \sqrt{\frac{11}{5}}, -------$$

We can choose one special sequence from the lower class that is expressible as a decimal number, for if we write in order, the largest integer whose square is less than 2, the largest number taken to one decimal place whose square is less than 2, the largest number taken to two decimal places whose square is less than 2, and so on, indefinitely, we obtain the sequence:

1 , 1.4 , 1.41 , 1.414 , 1.4142 , 1.41421 , ------

This sequence is unending, that is, the irrational number 2 can be expressed as an unending decimal 1.41421------

Let the irrational number be $a_0 . a_1 a_2 a_3 ---- a_n a_{n+1}$ --------- and the rational number $a_0 . a_1 a_2 a_3 ------ a_{n-1} a_n$.

If we assign to the n+1th decimal place any of the figures $< a_{n+1}$, to the n+2th decimal place any of the ten figures $< a_{n+2}$, and so on to the a_{n+m}th decimal place, where <u>m</u> is indefinitely great, it is clear that every number thus formed is greater than the chosen rational number and less than the chosen irrational number, and that there are infinitely many such numbers.

We attribute the property of continuity to the domainof real numbers, so that to every cut (R_1, R_2) in the domain, such that every element of the class R_1, is less than every element of the class R_2, there exists one and only one number a by which this separation is produced.

Any number rational or irrational which we may denote by a is such that, if any positive number δ, be assigned, it matters not how small, we can always find two rational numbers a_1 in the class A_1 and a_2 in the class A_2, such that $a_1 < a < a_2$, and $a_2 - a_1 < \delta$.

This result follows at once from the statement that between any two numbers whatsoever there are infinitely many rational numbers, and that we can determine any irrational number to a given degree of approximation.

The addition of irrational numbers. Let the numbers α and β be defined respectively by the cuts (A_1, A_2) and (B_1, B_2). Let us arrange in a class C_1 all those rational numbers c for which $a_1 + b_1 \geqq c$, where a_1 is any number contained in the class A_1 and b_1 is any number contained in the class B_1. And place all other rational numbers in the class C_2. We have now separated all the rational numbers into two classes C_1 and C_2, such that every number in the class C_1 is less than every number in the class C_2, and conversely. Therefore the cut (C_1, C_2) determines a number γ.

If both α and β are rational, then every number c_1 contained in C_1 is $\leqq \alpha+\beta$, because $a_1 \leqq \alpha$, $b_1 \leqq \beta$, and therefore $a_1 + b_1 \leqq \alpha+\beta$. Further, every number which is less than $\alpha+\beta$ must lie in C_1, for let c_2 be a number of C_2 and suppose $c_2 < \alpha+\beta$.

We have $c_2 + p = \alpha+\beta$

$$c_2 = (\alpha - \tfrac{1}{2}p) + (\beta - \tfrac{1}{2}p)$$

Where p is a positive quantity

$\alpha - \frac{1}{2}p$ is a number of the class A_1 and

$\beta - \frac{1}{2}p$ is a number of the class B_1, that is, $(\alpha - \frac{1}{2}p) + (\beta - \frac{1}{2}p)$ lies in the class C_1, by our definition.

Thus, every member in the class C_1 is $\leqq \alpha+\beta$. and every number in the class C_2 is $\geqq \alpha+\beta$; therefore the cut (C_1, C_2) defines a number $\gamma = \alpha+\beta$.

If α and β are irrationals they may appear in either of the classes A_1 or A_2, B_1 or B_2, respectively, which makes no difference to the argument, because for any positive number δ, however small, we can choose numbers, a_1, a_2, b_1 and b_2, in the classes A_1, A_2, B_1 and B_2, respectively, such that $a_2 - a_1 < \delta$, and $b_2 - b_1 < \delta$.

In a similar way we can define multiplication, subtraction, division, including both powers and roots.

As an illustration, I will find the product of the two irrational numbers $\sqrt{a} = \sqrt{\alpha}$, $\sqrt{b} = \sqrt{\beta}$. Let $\sqrt{a}$ and $\sqrt{b}$ be defined by the cuts (A_1, A_2) and (B_1, B_2) respectively. We form two classes, C_1 and C_2, such that every member c_1 in the class C_1 is given by $c_1 \leqq a_1 b_1$, where a_1 is any rational number whose square is less than α, and b_1 is any rational number whose square is less than β; and assign to the class C_2 all those numbers c_2 given by $c_2^2 \geqq a_2 b_2$, where a_2 and b_2 are any rational numbers whose squares are respectively greater than α and β. We have then separated the rational numbers into two classes, C_1 and C_2, such that every number in the class C_1 is less than every number in the class C_2; further $c_1^2 \leqq \alpha\beta$ and $c_2^2 \geqq \alpha\beta$; therefore the cut itself (C_1, C_2) defines a number γ, such that $\gamma^2 = \alpha\beta$; therefore $\gamma = \sqrt{\alpha\beta}$.

Dedekind in his theory of the irrational number, represents an irrational number as a separation of all the rational numbers into two classes, such that the inferior class has no greatest number, and the superior class has no least number.

Cantor's theory of the irrational number essentially depends upon the use of convergent simply infinite ascending aggregates, or convergent sequences in which the elements are rational numbers.

In order to connect the theory of Dedekind with that of Cantor, it is necessary to discuss the idea of limits.

We say that a variable magnitude <u>x</u> approaches <u>s</u> as a limiting value when the difference s-x taken in absolute value becomes and remains finally less than any given positive value different from zero.

Theorem I. If a magnitude grows, but not beyond all limits, it approaches a limiting value.

If we denote the variable magnitude by x, by our hypothesis, there is some number a that is greater than x; hence there exist infinitely many numbers that are greater than x. Let all these numbers a_2 that are greater than x make up the class R_2, and put all the other real numbers a_1, in the class R_1. Every number a_1 of the class R_1 is such that as x takes all its allowable values for some of these values $x \geqq a_1$; hence, every number a_1 is less than every number a_2. Consequently, there exists a number α, which is either the greatest in the class R_1, or the least in the class R_2. As x never ceases to grow, by hypothesis, there cannot be a greatest number in the class R_1, so that α must be the least number in the class R_2, That is, whatever value a_1 we choose from the class R_1, we will have finally $a_1 < x < \alpha$, or, in other words, x approaches the limiting value α.

Theorem II. If, as x varies, we can assign a positive quantity δ, such that after some given position x changes by less than δ, then x approaches a limiting value.

Retaining the notation from the proceeding proof, let a be a fixed number the least in the class R_2, and choose in the class R_1 numbers $a_1 < a_2 < a_3$ ---------- all of which are less than a. If x approaches the number a as a limiting value, it is evident that x-a becomes less in absolute value than δ, where δ is any positive quantity different from zero. As δ decreases in absolute value since a is a fixed point the boundaries within which x must lie are given successively by $a_n - a$, $a_{n+1} - a$, --------- that is, the numbers $a_1 < a_2 < a_3$ --------- approach a to within a distance arbitrarily small; therefore a is a limiting value to the variable magnitude x.

Theorem III. If the aggregate (a_1, a_2 ----------) is such that from and after some fixed element, each element is less than the following one, and if all the elements are less than some fixed number N, then the aggregate is a convergent sequence.

For suppose it were not convergent. Then, if δ is any definite positive number such that $|a_{n+1} - a_n|$, $|a_{n+2} - a_{n+1}|$ --------- $\geqq \delta$, we would have $a_{n+r} \geqq a_n + r\delta$ where r can be taken as large as we please. Therefore we can make $a_n + r\delta > N$ which is contrary to the hypothesis; that is, the aggregate cannot be other than convergent.

Theorem IV. If the aggregate (a_1, a_2 ---------) is such that from and after some fixed element, each element is greater than the following one, and if all the elements are greater than some fixed number M, then the aggregate is a comvergent sequence.

For suppose it is not convergent. Then, using the notation from the preceding proof $|a_n - a_{n+1}|$, $|a_{n+1} - a_{n+2}|$ ------------ $\geqq \delta$ and we have $a_{n+r} = a_n - r\delta$. But as $r\delta$ can be made as large as we please by choosing r great enough; therefore $a_n - r\delta$ can be made less than M, which is contrary to the hypothesis; therefore the aggregate must be convergent.

A simply infinite ascending aggregate in which each element is a rational number is said to be convergent, if it is such that corresponding to any fixed arbitarily chosen positive rational number ϵ, as small in the ordinal sense as we please, a number n can be found that $a_n - a_{n+m} < \epsilon$ for $m = 1, 2, 3,$ --------

A sequence of rational numbers is said to form a set when by means of some definite law we can determine whether any given number belongs to the sequence or not. If A is a set of rational numbers, there is a first number a_1, a succeeding number a_2, and in general after a_n follows a certain number a_{n+1}. The set A is then called an infinite sequence and is denoted by $A = a_1, a_2, a_3$----------- or by $A = [a_n]$

Exs. The sequence 1, 2, 3, -------- forms the set $[n]$.

The sequence $1, \frac{1}{2}, \frac{1}{3}$,---------forms the set $\left[\frac{1}{n}\right]$.

The sequence 1, 1, 1,---------forms the set $[1]$.

Let C be any fixed rational number. Then C is said to be the limit of the sequence $A = [a_n]$, when for any positive rational number ϵ, chosen arbitrarily small, there exists an index m, such that $C - a_m < \epsilon$ for every value $n > m$. When C is the limit of a set A, we say that A is a convergent sequence, and that $[a_n]$ converges to C as limiting value.

Infinite sequences can be arranged to represent any number whatsoever. In fact, many sequences can be arranged to represent the same number. A convergent sequence $[a_n]$ of which the elements are rational numbers, is taken to represent a real number, the limit of the convergent sequence.

Illustrative example. To find an infinite sequence that will represent the real number $\sqrt{2}$. That is, required to find an infinite sequence of rational numbers a_1, a_2, a_3--------- such that Lim. $\left[a_n^2\right] = 2$.

Let a_1 be the greatest integer such that $a_1^2 < 2$; then, $a_1 = 1$.

The numbers $a_1 + \frac{1}{10}$, $a_1 + \frac{2}{10}$, ---------- $a_1 + \frac{9}{10}$ must be such that the squares on some two consecutive numbers are respectively less and greater than 2. Let a_2 be the greatest number of this set whose square is less than 2; then, $(a_2)^2 < 2 < (a_2 + \frac{1}{10})^2$.

Suppose $a_2 = a_1 + \frac{x_1}{10}$.

Similarly, the numbers $a_2 + \frac{1}{10^2}$, $a_2 + \frac{2}{10^2}$ -------- $a_2 + \frac{9}{10^2}$ must be such that the squares on some two consecutive numbers is respectively less and greater than 2. Let a_3 be the greatest of these numbers whose square is less than 2; then $(a_3)^2 < 2 < (a_3 + \frac{1}{10^2})^2$.

Suppose $a_3 = a_2 + \frac{x_2}{10^2}$.

Proceeding indefinitely in this way we obtain an infinite sequence of rational numbers, whose squares are always less than 2.

a_1

$a_2 = a_1 + \frac{x_1}{10}$

$a_3 = a_2 + \frac{x_2}{10^2} = a_1 + \frac{x_1}{10} + \frac{x_2}{10^2}$

$a_n = a_1 + \frac{x_1}{10} + \frac{x_2}{10^2} + \frac{x_3}{10^3} + \text{---------} + \frac{x_{n-1}}{10^{n-1}}$

Where x_1, x_2, x_3, --------- are respectively some one of the nine figures, 1, 2, 3, ------------9, we have

$$(a_n)^2 < 2 < \left\{a_n + \frac{1}{10^{n-1}}\right\}^2$$

$$\therefore\ 0 < 2 - (a_n)^2 < \left\{a_n + \frac{1}{10^{n-1}}\right\}^2 - (a_n)^2$$

$$\therefore\ 2 - (a_n)^2 < \frac{4}{10^{n-1}} + \frac{1}{10^{2(n-1)}},$$

since $a_n < 2$. Obviously we can choose an index m, that will make $\frac{4}{10^{n-1}} + \frac{1}{10^{2(n-1)}} < \epsilon$, where ϵ is an arbitrarily small positive quantity.

$\therefore\ 2 - (a_n)^2 < \epsilon$, for $n > m$.

$\therefore\ \text{Lim.} \left[a_n^2\right] = 2.$

When the numerical work has been accomplished, we obtain the infinite sequence 1, 1.4, 1.41, 1.414, 1.4142 ------ as representing the irrational number $\sqrt{2}$.

As an illustration of the way in which a number may be represented by different infinite sequences, consider the sets

(1) 2, 2, 2, 2, -----------

(2) $1\frac{1}{2}$, $1\frac{2}{3}$, $1\frac{3}{4}$, $1\frac{4}{5}$, --------

(3) $2\frac{1}{2}$, $2\frac{1}{3}$, $2\frac{1}{4}$, $2\frac{1}{5}$ --------

(4) 1.9, 1.99, 1.999, ------

It is evident that each of these sets has the number 2 as limiting value. Therefore each sequence is considered as a representation of the number 2.

Cantor's Theory of the Irrational Number.

Let $[a_n]$ and $[b_n]$ be two convergent sequences of rational numbers, that define the real numbers a and b, respectively. $[a_n]$ and $[b_n]$ are said to define the same number, represented by $a = b$, provided they satisfy the condition that for any arbitrarily chosen positive quantity ϵ, however small, an index n can be found, such that,

$|a_{n+m} - b_{n+m}| < \epsilon$, where m can have any of the values 0. 1, 2, 3, ----------

Again, if after a definite number of elements n , the sequences $[a_n]$, and $[b_n]$, are such that $|a_{n+m} - b_{n+m}| \geqq \delta$ where δ is an arbitrarily small chosen positive number, for all values of m, then $[a_n]$ is said to be greater than $[b_n]$ which is denoted by $a > b$, or $b < a$.

If on the other hand δ be any minus quantity whatsoever $[b_n]$ is said to be greater than $[a_n]$, denoted by $b > a$, or $a < b$.

We have thus defined the application of the ideas of equality, greater than and less than, to infinite sequences.

An aggregate (x, x, x ------) or $[x]$, since it is a convergent sequence, defines the number x .

The sum of a and b, represented by the infinite sequences $[a_n]$ and $[b_n]$, is defined to be the number represented by the sequence $[a_n + b_n]$. It is therefore required to prove $[a_n + b_n]$ convergent.

Since $[a_n]$ is convergent, we can choose an index m such that $a_m - a_{m+p} < \frac{\epsilon}{2}$. Also since $[b_n]$ is convergent, we can choose an m_1, such that $b_{m_1} - b_{m_1+p} < \frac{\epsilon}{2}$. Substituting the larger value of m and m_1 in both equations, we obtain $(a_m + b_m) - (a_{m+p} + b_{m+p}) < \epsilon$. That is, $[a_n + b_n]$ is convergent.

The difference of any two numbers a and b, denoted by $a-b$, is defined to be the number represented by the sequence $[a_n - b_n]$, which can be shown to be convergent, by a proof similar to the preceding one.

The product of any two numbers a and b, written ab, is defined to be the number represented by the sequence $[a_n b_n]$. It is required to prove this convergent. Since both $[a_n]$ and $[b_n]$ are convergent, a number m can be found such that, $b_m - b_{m+p} < \delta$, and $a_m - a_{m+p} < \delta$. Now, $\left| a_m b_m - a_{m+p} b_{m+p} \right|$

$$= a_m (b_m - b_{m+p}) + b_m (a_m - a_{m+p})$$
$$< a(b_m - b_{m+p}) + b(a_m - a_{m+p})$$
$$\therefore a_m b_m - a_{m+p} b_{m+p} < (a+b)\,\delta$$

Writing $\delta = \frac{\epsilon}{a+b}$

we have,

$a_m b_m - a_{m+p} b_{m+p} < \epsilon$; therefore the sequence $[a_n b_n]$ is convergent.

The quotient of a and b, written $\frac{a}{b}$, is defined to be the number represented by $\left[\frac{a}{b}\right]$. The sequence $\left[\frac{a_n}{b_n}\right]$ can be shown to be convergent by a method very similar to the above, except in the case where $\left[b_n\right]$ is zero.

If $\left[b_n\right]$ is zero, let $\frac{a}{b}=x$, $\therefore$ $a = bx$. This equation has no solution unless $a=o$, because when $b=o, bx=o$, for all finite values of x. If $a=o$, the equation $a=bx$ is satisfied for all finite values in the real domain. So we can state that when the divisor is zero, division is either impossible or entirely indeterminate.

Throughout this discussion we have considered both the real and rational systems of numbers to be Archimedean. That is, there is no positive number a in the system so small but that some multiple of a , say na is greater than any prescribed number b in the system. Secondly, however large the number c may be, there is a positive integer m such that $\frac{c}{m} < d$, where d is arbitrarily small.

In an absolutely rigorous proof of the laws of addition, subtraction, multiplication and divison, it would be necessary to prove that these different methods obey the associative, distributive and commutative laws.

Assuming that these laws have been established for rational numbers, I will prove the associative law for sequences.

It is required to prove that $\alpha\cdot\beta\gamma = \alpha\beta\cdot\gamma$. We have

$\beta\gamma = \left[b_n\ c_n\right] = (b_1c_1, b_2c_2, b_3c_3 \text{------------})$, and

$\alpha\cdot\beta\gamma = (a_1, a_2, a_3, \text{----------})(b_1c_1, b_2c_2, \text{------------})$

$= a_1 . b_1c_1, a_2 . b_2c_2, a_3 . b_3c_3, \text{---------}$

Similarly, we have

$\alpha\beta\cdot\gamma = a_1 b_1 . c_1, a_2b_2 . c_2, a_3b_3 . c_3, \text{-------}$

But the associative law holds for rational numbers, and

$(a_1, a_2, a_3$---------$)$, $(b_1, b_2, b_3$--------$)$, $(c_1, c_2, c_3$------$)$ are all rational numbers.∴ The associative law is established for numbers represented by infinite sequences.

That the communitative law holds in the case of infinite sequence is very evident because our proofs are based on discussions in which we use only rational numbers, for which the law holds.

For instance, $a+b$ is given by $[a_n+b_n]$ or $(a_1+b_1, a_2+b_2, a_3+b_3$-------------$)$.

Since the commutative law holds for rational numbers, each term of this sequence can be reversed, and written

$(b_1+a_1, b_2+a_2, b_3+a_3,$--------------$)$ or

$[b_n+a_n]$, which is convergent, as can be shown by a proof identical with that given for $[a_n+b_n]$.

Between any two real numbers a and b defined by $[a_n]$ and $[b_n]$, respectively, there lie infinitely many rational numbers.

Suppose $a > b$; then we know that $|a_{n+m} - b_{n+m}| > \delta$ for $n > \mu$. Also since $[a_n]$ and $[b_n]$ are convergent, we have $|a_n - a_{n+m}| < \epsilon$, and $|b_n - b_{n+m}| < \epsilon$, for $n > p$, where ϵ is a rational number, which for simplicity we will choose to be $\frac{1}{2}\delta$.

If we choose x to be such that $\epsilon < x < \delta$, we have a number definied by $[a_n - x]$, which can be proved to lie between a and b.

Since $a_n - a_{n+m} < \epsilon$, for $n > \mu$, we have

$a_{n+m} - (a_n - x) > x - \epsilon$, ∴ $a > a_n - x$.

Also $(a_n - x) - b_{n+m} = (a_n - b_n) + (b_n - b_{n+m}) - x$
$$> \delta + \epsilon - x$$

Therefore, provided x is $< \delta + \epsilon$, the real number which corresponds to $a_n - x$ is greater than b, and thus lies between a and b.

But $\delta + \epsilon$ and ϵ are both rational numbers, hence there are infinitely many rational numbers lying between $\delta + \epsilon$ and ϵ Whenever x assumes any of these infinitely many values between $\delta + \epsilon$ and ϵ, the number $a_n - x$ lies between a and b. Therefore there are infinitely many rational numbers lying between the real numbers represented by the sequences $[a_n]$ and $[b_n]$.

We wish now to apply the idea of bounded aggregates to the determination of rational and irrational numbers.

Since a (1,1) correspondence has been established between the real numbers and the points of a straight line, we can use the terms "aggregate of numbers" and "aggregate of points" indiscriminately.

An aggregate is said to be bounded on the right (or left) when there is no point of the aggregate to the right (or left) of some fixed point.

When an aggregate (E) is bounded on the right there exists a number M such that there is no number of (E) greater than M, and if any number M' whatsoever be chosen smaller than M, then there exists one number at least of (E) which is greater than M!

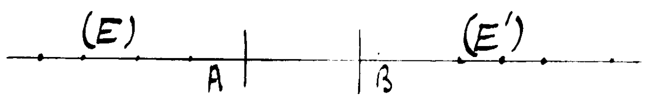
If an aggregate has a greatest number this is called the maximum of the aggreagte and enjoys the properties enumerated above.

When an aggregate has no greatest number, separate the rational numbers into two classes R_1 and R_2, placing in the class R_1 every rational number that belongs to the set, as well as all the rational numbers that are less than or equal to numbers of the set, and place all the other rational numbers in the class R_2. The cut (R_1, R_2) determines a number M which may be rational or irrational. Now, the aggregate (E) has no greatest term and as all the numbers in the class R_1 are either equal to or less than the elements of (E), evidently the class R_1 cannot have a greatest term. If there is a lowest number M in the class R_2, it is the rational number defined by the cut (R_1, R_2). If there is no lowest number in the class R_2, the cut (R_1, R_2) must define an irrational number M. It is evident from the definition of M, that there is no number of (E) greater than M. Let M' be any number less than M. We have proved that between any two real numbers there lie infinitely many rational numbers, which in this case, being all less than M, belong to the lower class R_1.

Similarly, if (E) is bounded on the left, there exists a number M such that there is no number of (E) less than M, and if M' be any number whatsoever greater than M, there exists one number at least of (E) that is smaller than M', that is, lies to the left of M'.

If (E) and (E') be two aggregates such that every number of (E) is less than every number of (E'); (E) having no maximum, and (E') no minimum; and for any chosen positive quantity ϵ, however small, there exists a number a_1 in (E) and a number a_2 in (E') such that $a_2 - a_1 < \epsilon$, then these two aggregates define a definite number, rational or irrational.

Since (E) has no maximum, let A be the upper boundary of the aggregate, and since (E') has no minimum, let B be the lower boundary of the aggregate, as in the accompanying figure.



To insure that (E) and (E') define the same number or the same point, we must have A and B coincident points. That A and B are coincident points is required by the latter condition of the theorem, since we can choose ϵ arbitrarily small and therefore less than A-B in absolute value, however close these two points may be. That is, A and B must exactly coincide.

Illustrative Example.

Let (E) 1, 1.4, 1.41, 1.414, 1.4142, -----and let (E') 2, 1.5, 1.42, 1.415, 1.4143,--------. These two aggregates fulfill the conditions imposed by the theorem; every number of (E) is less than every number of (E'); (E) has no maximum, as we have chosen successive approximations of the irrational number $\sqrt{2}$, and (E') likewise has no minimum; further we can find two numbers a_1 in (E) and a_2 in (E) which differ by less than ϵ, however small ϵ may be in absolute value. Therefore, the aggregate (E) approximates to the value of $\sqrt{2}$ by defect, the error being in every case less than $\frac{1}{10^p}$ where there are p+1 terms considered in the aggregate; and the aggregate (E') approximates to the value of $\sqrt{2}$ by excess, the error being in every case less than $\frac{1}{10^p}$ where there are p+1 terms considered in the aggregate.

Example: $\frac{1}{3}$ is represented by the two aggregates, (E) .3,.33, .333, .3333---------and (E') 4, .34, .334, .3334 ----------

The examples just given are illustrations of infinite sequences in decimal form of rational or irrational numbers. That an irrational number can always be represented by an infinite series is clear from the theory of irrationals as developed by Dedekind. Further that the idea can be extended to rational numbers is shown by:

$$3 = (3, 3, 3, 3 \text{----------})$$
$$3 = (2\tfrac{1}{2}, 2\tfrac{2}{3}, 2\tfrac{3}{4}, 2\tfrac{4}{5}, \text{------})$$
$$3 = (3\tfrac{1}{2}, 3\tfrac{1}{3}, 3\tfrac{1}{4}, 3\tfrac{1}{5}, 3\tfrac{1}{6} \text{-------})$$
$$3 = (2.9, 2.99, 2.999, \text{--------})$$

and
$$0 = (1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4} \text{------})$$
$$= (1, -\tfrac{1}{2}, -\tfrac{1}{3}, -\tfrac{1}{4}, \text{----})$$
$$= (-1, -\tfrac{1}{2}, -\tfrac{1}{3}, -\tfrac{1}{4}, \text{----})$$
$$= (0, 0, 0, \text{-------})$$

Hobson in his 'Theory of Functions of a Real Variable', shows that every real number can be represented by means of a non-terminating series of radix-fractions, of which r, the radix, is any integer $\geqq 2$.

Let N be any real number. The series o, r, 2r, 3r,------- ultimately becomes $> N$.

Let $rN > C_0 r$ and $rN < (C_0 + 1)r$

$\therefore N = C_0 + \frac{N_1}{r}$, where $N_1 < r$;

Similarly $N_1 = C_1 + \frac{N_2}{r}$, where $N_2 < r$; and $N_2 = C_2 + \frac{N_3}{r}$, where $N_3 < r$;

$$\therefore N = C_0 + \frac{C_1}{r} + \frac{C_2}{r^2} + \frac{C_3}{r^3} + \text{-------} \frac{C_n}{r^n} + \frac{N_{n+1}}{r^{n+1}}$$

and since $N_{n+1} < r$, we have

$$N - \left(C_0 + \frac{C_1}{r} + \frac{C_2}{r^2} + \text{------} \frac{C_n}{r^n}\right) < \frac{1}{r^n}$$

and as $\frac{1}{r^n}$ tends to the limit zero when n is indefinitely increased, it is evident that the sequence whose nth term is $C_0 + \frac{C_1}{r} + \text{-----} \frac{C_n}{r^n}$ is convergent and represents the real number N. Therefore writing $N = C_0 + \frac{C_1}{r} + \frac{C_2}{r^2} + \text{-----}$ we have the real number N represented by a non-terminating radix-fraction.

The case in which N is a ratinnal number $\frac{a}{b}$ in its lowest terms, requires special treatment.

$a = C_0 b + B_0$, where $B_0 < b$

$rB_0 = C_1 b + B_1$, where $B_1 < b$

$rB_1 = C_2 b + B_2$, where $B_2 < b$

$rB_{n-1} = C_n b + B_n$

Case 1. If $B_n = 0$

We have , $N = \frac{a}{b} = C_0 + \frac{c_1}{r} + \frac{c_2}{r^2} + \text{--------} \frac{c_n}{r^n}$

We wish to form a non-ending radix-fraction; therefore writing $C_n - 1$ for C_n, we have $rB_{n-1} = (C_n - 1)b + b$

$\therefore B_n = b$.

$r \; b = (r-1) \; b + b \qquad \therefore \qquad B_{n+1} = b$.

Similarly, for all the succeeding numbers B_{n+2}, B_{n+3}, --------- we can substitute the value b. That is, C_{n+1}, C_{n+2}, C_{n+3}, --------are each equal to r-1.

Writing $N = C_0 + \frac{C_1}{r} + \frac{C_2}{r^2} + \text{---} + \frac{C_{n-1}}{r^n} + \frac{r-1}{r^{n+1}} + \frac{r-1}{r^{n+2}} + \text{---}$

we have represented the rational number N by a non-ending radix-fraction.

Case 2. If none of the numbers B_1, B_2, -------B_n, B_{n+1}, --------- = 0, since they are all integers and less than b, they cannot be all different. Suppose $B_n = B_{n+m}$; then we have,

$rB_n = C_{n+1} \; b + B_{n+1}$

and $rB_{n+m} = C_{n+1} \; b + B_{n+m+1}$

$\therefore B_{n+1} = B_{n+m+1}$

That is, the sequence of radix-fractions becomes recurring.

Among the many ways of representing numbers, only one more will be treated here----Cantor's Sequence of Products. Cantor has shown that any number $N > 1$ can be uniquely represented in the form,

$(1+\frac{1}{a})\ (1+\frac{1}{b})\ (1+\frac{1}{c})\ (1+\frac{1}{d})$ --------
where a, b, c, d-----are integers such that $b \leqq a^2$, $c \leqq b^2$, $d \leqq c^2$--------

The number a is determined as the integral part of $\frac{N}{N-1}$;

writing $\frac{Na}{a+1}$ as B, b is determined as the integral part of $\frac{B}{B-1}$; writing $\frac{Bb}{b+1}$ as C, c is the integral part of $\frac{C}{c-1}$; and so on indefinitely.

Examples:

$\sqrt{2}$ is uniquely represented by the indefinite product.

$(1+\frac{1}{3})\ (1+\frac{1}{17})\ (1+\frac{1}{577})\ (1+\frac{1}{665857})$ --------

$\sqrt{3}$ is represented by the infinite product $(1+\frac{1}{2})(1+\frac{1}{7})(1+\frac{1}{106})$---.

Cantor was able to prove that when all the numbers a, b, c, d,-------which are positive integers, are such that after some fixed number m all the numbers m+1, m+2, -------were each the square of the preceding number of the sequence, then the sequence represents a rational number, otherwise the number represented is an irrational number.

It is easy to construct sequences of never-ending decimals. For instance, let each successive element of the sequence be constructed by adding to the immediately preceding element the next prime in order from those already requisitioned. .1,.12, .123, .1235, 12357, .1235711, .123571113,---

Or again, let an infinite fractional sequence be constructed, where the numerator of any term becomes the denominator of the next, while the sum of the denominators of any two consecutive terms forms the numerator of the last term used in the computation,

$$\frac{1}{2}, \frac{3}{1}, \frac{4}{3}, \frac{7}{4}, \frac{11}{7}, \text{-----} \quad .$$

Sequences of this type may be rational or irrational, and very often it is very difficult to determine from a given sequence whether the number represented is rational or irrational. Other sequences as the two numerical ones given above obviously determine irrational numbers.

Liouville's theorem. If $\frac{p_1}{q_1}, \frac{p_2}{q_2}, ---$ is a sequence of rational fractions in their lowest terms, defining an algebraic number b of rank m, then for every element $\frac{p}{q}$ from and after an assignable stage, we have

$$\left|\frac{p}{q} - b\right| > \frac{1}{q^{m+1}}.$$

This theorem, the proof of which is given by Young, enables one to determine whether any given series can represent an algebraic or a rational number, but does not furnish a means of determining the number actually represented by the sequence.

A distinction between rational and irrational numbers is readily drawn in the theory of irrational numbers as developed by Dedekind, but when considered from an algebraic point of view these numbers become much more similar.

All equations containing a variable with rational integral coefficients require but the system of rational numbers for their solution. While all the general equations of higher orders introduce irrational numbers, and many special equations of these high orders involve the use of rational numbers as well.

There is another type of irrational number that must be considered, namely, that based on the exponential number ϵ.

The exponential number ϵ is determined from a discussion of the expression $(1+\frac{1}{x})^x$. It has been shown that two rational numbers A and B can be found, differing from one another by as small a quantity as we please, such that $(1+\frac{1}{x})^x$ lies between them. Moreover ϵ, which is defined as the limit of $(1+\frac{1}{x})^x$, where $x \longrightarrow \infty$, has been shown to be irrational. ϵ can also be shown to be non-algebraic. To distinguish ϵ and π from algebraic irrationals, they are called transcendental irrationals. The Naperian logarithms are based on ϵ as logarithmic base; they are therefore likewise transcendental, and are evidently infinitely many in number.

Another group of transcendental irrational numbers, infinitely many in number, is represented by the Liouville numbers, $\frac{x_1}{10}+\frac{x_2}{10^{1\cdot 2}}+\frac{x_3}{10^{1\cdot 2\cdot 3}}+$ ------- ,where x_1, x_2, x_3, ---- --------may be respectively any of the ten figures.

Example:

$\frac{1}{10}+\frac{2}{10^{1\cdot 2}}+\frac{3}{10^{1\cdot 2\cdot 3}}+\frac{4}{10^{1\cdot 2\cdot 3\cdot 4}}+$ -------- is given by the the decimal,

.120003000000000000000004---------

THE THEORY OF SETS OF POINTS

---ooOoo---

If in any aggregate there is a point such that in the neighborhood of this point there are infinitely many points of the aggregate, then that point is called a limiting point.

Ex. In the aggregate , $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$,----- ---, there are infinitely many elements of this aggregate in any interval however small extending to the right of the origin. Therefore o, is a limiting point, although it does not belong to the sequence.

The domain of rational numbers does not possess the property of containing all the limiting points of particular aggregates chosen from among the rational numbers, for we have seen that the irrational number is defined by a sequence of rational numbers.

The domain of real numbers, however, does possess the property that all limiting points of particular aggregates of real numbers are themselves members of the real domain. This idea is expressed by saying that the domain of real numbers is perfect.

The domain of rational numbers and the domain of real numbers both possess the property that between any two numbers whatsoever of the domain, there lie infinitely many rational numbers; this is expressed by saying that these aggregates are dense.

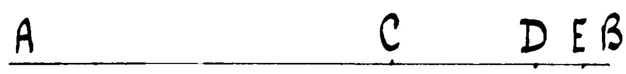
All the real numbers x, such that $a \leqq x \leqq b$, in the ordinary sense of the symbols $<, =, >$, are said to form an *interval* (a,b). Such an interval is frequently described as a *closed interval*, whereas all the numbers x, given by $a < x < b$, are said to form an *open interval*.

The Dedekind-Cantor axiom states that to every number, rational or irrational, there corresponds a definite point of the straight line; that is, we but assign to a straight line the same degree of continuity that we conceive to belong to the domain of real numbers.

If one set of numbers is contained in a second set, the first is called a *component* of the second set; and if the latter contains points that do not belong to the former set, then the first is called a *proper component* of the second set.

We have established a (1,1) correspondence between all the rational numbers and the points of a straight line, and by the Dedekind-Cantor axiom we attribute to the straight line a continuity comparable with that of the real numbers; hence, we can use the terms sequence of numbers and aggregate of points indiscriminately.

In order to construct a convergent infinite sequence of number or set of points we have but to select any segment of a straight line, trisect, let us say, this segment AB at C, then trisect the segment CB at D, again trisect the segment DB at E, and so on to an infinite number of divisional points, as indicated in the accompanying figure:



These points are infinitely many in number and as they continually approach the fixed point B, we can say, with reference to the theorems already proved, that B is the limit of the set (A, C, D, E, F, ---------).

Theorem 1. If we take any series of closed segments, each lying entirely within the preceding; and if the length of the segments decrease without limit, then the end points of these segments form an infinite sequence, and the segments determine one and only one point L, internal to all the segments.

It is evident from the accompanying figures that the end-points of an infinite series of segments, each lying within the preceding, form sequences which have boundaries.

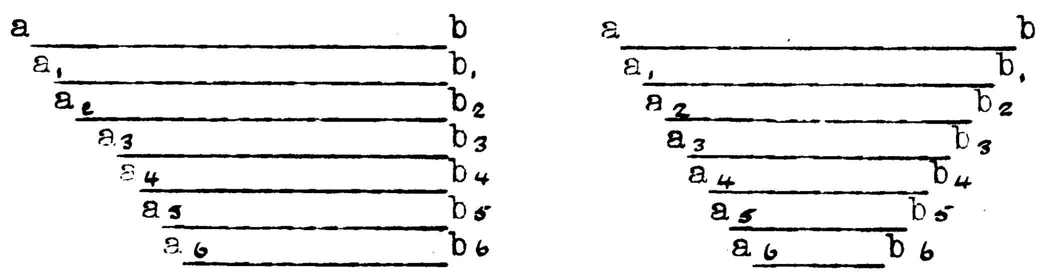


Fig. 2.

The end-points that are denoted by a's form a series of infinitely many elements, such that $a_1 \leqq a_2 \leqq a_3 \leqq a_4$ -------------- and further, every a is less than every b; that is, the series of a's has an upper boundary A . Similarly the end-points that are denoted by b's form a series of infinitely many elements, such that, $b_1 \geqq b_2 \geqq b_3$ ------- and every b is greater than every a; therefore the series by a previous theorem must have a lower boundary B. Since there are infinitely many intervals, and as our hypothesis rules out such a case as $a_n = a_{n+1}$, and $b_n = b_{n+1}$ at the same time, we know that the interval $b_n - a_n$ can be made smaller than any assignable positive magnitude ϵ; therefore A and B, (the upper boundary of the b-series respectively), must coincide exactly. That is, the given series of segments represents one definite number.

Any point P, of a set which is not a limiting point of the set is called an isolated point of that set. The set is such that we can find a neighborhood of P, that is a small interval having P as an interval point, which does not contain any point of the set other than P; therefore P is not a limiting point of the set.

An example will suffice to illustrate this definition. Consider the sequence; $1, \frac{1}{3}, \frac{1}{9}. \frac{1}{27}, \frac{1}{81},$ ---------------- every point of the set is an isolated point, and 0 is obviously a limiting point of the set; for choosing the point $\frac{1}{27}$, the point of the sequence which is nearest to the chosen point is $\frac{1}{81}$, and within the interval $\frac{1}{27} - \frac{1}{81}$ there are no points of the set; therefore the point $\frac{1}{27}$ cannot be a limiting point and must be a isolated point, because if ϵ were chosen $< \frac{1}{27} - \frac{1}{81}$, the condition for a limiting point fails.

A set of points is said to be an isolated set when no point of the set is a limiting point. For instance, omitting the point zero from the previous example we obtain an isolated set, or more obviously, breaking off the sequence after a finite number of terms, since for one limiting point infinitely many terms are required.

A set, of whose limiting points all belong to the set itself, is said to be closed.

Ex.1. Any finite set, because such a set has no limiting point or points.

Ex. 2. $1\frac{1}{2}$, $1\frac{2}{3}$, $1\frac{3}{4}$, ---------2.

A set which is such that every point of the set is a limiting point is said to be dense-in-itself.

A set which is both closed and dense-in-itself is said to be perfect.

Ex. All the numbers in any segment including both end-points, which we have defined as a closed interval and denoted by (a,b), or (2,3).

A set of points (E), contained in an interval (a,b), is said to be everywhere -dense, if in every sub-interval (a , b) however small there are points of the set (E).

The set of points (E) is said to be nowhere-dense, if in every subinterval (a , b) a second subinterval can be found which contains no points of the set (E).

It is evident from the definition of a limiting point, that no finite set of points can contain or have a limiting point. On the other hand every infinite set of points has a limiting point which may or may not belong to the set.

The interval (a,b) contains infinitely many points; therefore, bisecting the interval at the point c, we obtain two intervals (a,c) and (c,b) one at least of which must contain infinitely many points, and both intervals may have infinitely many points; again bisecting the two intervals at d and ϵ, we obtain four intervals (a,d), (d,c)(c,ϵ)

and (ϵ, b), one of which must and as many as the four intervals may contain infinitely many points. For simplicity, suppose that the right hand intervals (a,b), (c, b). (e, b) ---- contain infinitely many points, since by this construction we obtain a series of intervals that conforms to the requirements of Theorem 1, the intervals determine a point that is the limiting point of either sequence of end-points. Now returning to the case where the interval (a,c) contained infinitely many points, it likewise has at least one limiting point; and the reasoning applies generally, however numerous may be the segments containing an infinity of points.

If (G) is a set of points that has but one limiting point, we can construct a sequence of points of the set which have the same limiting point. Let A be the limiting point of the set (G), and suppose it to be an upper boundary. Let a_1 be any point of the set chosen to the left of the point A. That some such restriction of the position of the point a_1 is shown by such a set as

$$1\tfrac{1}{2},\ 1\tfrac{2}{3},\ 1\tfrac{3}{4},\ 1\tfrac{4}{5}\ \text{--------}\ 2,\ 3,\ 4,\ 5.$$

The interval (a .b) contains infinitely many points of the set from our definition of a limiting point. In (a_1, A) choose any point a_2 belonging to the set (G) ; clearly the interval $(a_2 A)$ contains infinitely many points of (G). Proceeding in this way we construct a sequence of points a_1, a_2, a_3 ------ such that $a_1 < a_2 < a_3$, --------- and all $<$ A. That is, the interval $A - a_n$ can be made smaller than any assignable positive however small; therefore A is the limiting point of the sequence.

Before discussing further the properties of a closed finite interval, it is instructive to notice that by a suitable transformation, it is possible to set up a (1,1) correspondence between the points of two different intervals.

For instance, the transformation $x' = \frac{x}{\sqrt{x^2+1}}$, where the positive value of the radical is always taken, converts all the points of the unlimited interval $(-\infty, +\infty)$ into points of the limited interval (-1, 1). Further, it is obvious that the order of the points is conserved during the transformation since $x' \gtreqless x''$ according as $x_1 \gtreqless x_2$.

In order to transform the finite interval (a, b) into the finite interval (0,1) so that any arbitrary point c in the interval (a, b) will coincide with a chosen point in the interval (0,1) say $\frac{1}{10}$, we can use the transformation equation:

$$\frac{9x'}{x'-1} = -\frac{x-a}{x-b} \cdot \frac{c-b}{c-a}$$

Closed and Perfect Sets.

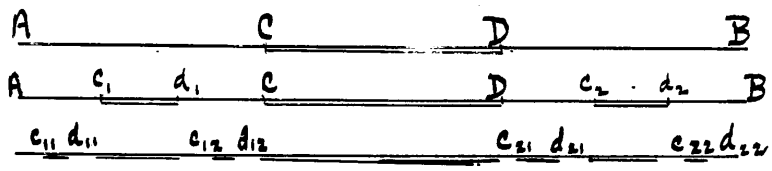
In considering the interval (a, b), since all the real numbers are included we have an interval which is dense, dense-in-itself, and also closed; therefore any closed segment of a line, representing a closed interval of real numbers is a perfect set. The interval (a,b) is such that no sub-interval (a', b') can be chosen in (a, b), however small, that does not contain real numbers. This is expressed by saying that the real numbers in any segment of the real domain is <u>dense</u>. But if we consider only the rational numbers of the interval (a,b) they will be closed---all the rational numbers of the closed interval (1,2) for instance, form a closed set; and since between any two rational numbers however near together, there lie infinitely many rational numbers, the rational numbers in any closed interval form a dense set; but the set is not dense-in-itself, because the irrational numbers are limiting points of rational numbers.

Non-dense perfect sets.

It is possible to construct a set which is perfect and yet nowhere dense. The example to be given in detail is Cantor's tertiary set of points.

Divide any straight line AB into three equal parts and darken the middle part. This will be considered as void of points of our set. Divide each of the two remaining segments into three equal parts and darken the middle part in each case. Continue this divisional process indefinitely and in every case consider the dark interval to be void of point of the set (E). This process creates an infinite set of <u>non-overlapping</u> intervals, because each two consecutive intervals belonging to the set are separated by a black interval.

Two intervals are said to <u>overlap</u> when they have one internal point at least in common. If two intervals have the same common end-point only, they are said to <u>abut</u>. Hence two intervals are said to be <u>non-overlapping</u> when they have no points whatever in common.



It is evident from the diagram that the segments between any two black intervals are all equal at any stage of the subdivision, and by successive division these segments can be made less than any assignable magnitude. That is, every point that is external to the black intervals after indefinite subdivision is a limiting point of the end-points of these black intervals on both sides, since between any two points, however close, there are infinitely many points and therefore infinitely many intervals constructed according to the above law. Thus we can choose a set of intervals, such as AC , d_1C , d_1C_{12} ------ that conforms to the conditions of Theorem I, and therefore defines a definite limiting point α , which must be internal to all of these infinitely many intervals, that is, must be an external point of the black intervals. Further, by Theorem I, the point defined by a series of intervals each lying within the immediately preceding one, is a limiting point of the end-points of the intervals on both sides.

The end-points of all these intervals A, C, c_1, d_1, c_2, c_{11} , d_{11} ---------are limiting points of end points on one side only. For consider the sequence of intervals (A,C) , (A,C_1) , (A,C_{11}) . (A,C_{111}) ---------, since in the neighborhood of A and on the right a member of this sequence can be found which is less than any assignable quantity, however small, therefore the given sequence defines the point A.

Consider the set made up of the end-points of these intervals and the external points of the black intervals, every point of the set is a limiting point, therefore the set is dense-in-itself. We require to prove it a closed set. None of the black intervals include points that belong to the set (E) that we are discussing, and if P is any point internal to one of these black intervals we can construct an interval with P as centre so small as to include none of the points of the set. Perhaps a clearer idea will be arrived at if we mention that the points A and B the upper and lower boundary of the segment may be considered to be rational points, then all the other divisional points will likewise be rational points, and clearly between any rational point and a rational or irrational point there is an interval of definite magnitude. Thus the point P cannot be a limiting point of the set (E); therefore the set (E) is closed; therefore (E) is a perfect set.

The set (E) however is not dense. In fact, since by indefinite subdivision the black intervals can be made to approach closer to each other than any assignable magnitude, the set of black intervals satisfies the condition for being everywhere-dense; that is, the set (E) is nowhere-dense.

Thus we may have a perfect set that is everywhere-dense or one that is nowhere-dense, and the most general set will contain a combination of these two groups of points. For instance, consider the set of points in the segment (0.2), where we include all the points of the interval (o,1), and then add the points of the interval (1,2), after it has been sub-divided after the manner of Cantor's typical tertiary set just discussed.

If we take the set (E) and omit all the end-points of intervals, we obtain a set dense-in-itself and closed, therefore perfect. Moreover each point of the set is a limiting point on both sides. If we include the end-points that are limiting points on the right (or left) only, we obtain a set that is perfect and dense-in-itself such that every point is a limiting point on the right (or left) side.

Example of a non-dense perfect set.

Let x be a number given by

$$x = \frac{C_1}{3} + \frac{C_2}{3^2} + \frac{C_3}{3^3} + \text{--------} + \frac{C_n}{3^n}$$

, where the numbers C_1, C_2, C_3 -------C_n are either 0 or 2, and where n has every integral value any may also be indefinitely great. Clearly no number lies between

$$\frac{C_1}{3} + \frac{C_2}{3^2} + \text{-----} + \frac{C_{n-1}}{3^{n-1}} + \frac{0}{3^n} + \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} + \text{-------}$$

the largest number we can form having 0 in the nth place, and

$$\frac{C_1}{3} + \frac{C_2}{3^2} + \text{-------------} + \frac{C_{n-1}}{3^{n-1}} + \frac{2}{3^n},$$

the smallest number we can form with $C_n = 2$. But the geometrical series, $\frac{2}{3^{n+1}}$ $\frac{2}{3^{n+2}} + \frac{2}{3^{n+3}} +$ ------- is equivalent to $\frac{1}{3^n}$; therefore, the interval between $\frac{C_1}{3} + \frac{C_2}{3^2} + \frac{C_3}{3^3} + \text{-----} + \frac{1}{3^n}$, and $\frac{C_1}{3} + \frac{C_2}{3^2} + \frac{C_3}{3^3} + \text{---------------} + \frac{2}{3}$. that is of magnitude $\frac{1}{3^n}$ contains no points of the given set.

This complementary interval of magnitude $\frac{1}{3^n}$ can be made as small as we please since <u>n</u> can be indefinitely great.

The numbers C_1, C_2, --------C_n can be either o or 2, by hypothesis, and none other, so that the number of complementary intervals of length $\frac{1}{3^n}$ is 2^{n-1}. The sum of all the complementary intervals is given by $\sum_{n=1}^{n} \frac{2^{n-1}}{3^n}$, which in the limit where n becomes indefinitely great, is unity. That is, the set of complementary intervals is everywhere-dense, therefore the set of points is nowhere dense.

Derived Sets.

It is clear that not all sets of points are dense-in-themselves, for infinitely manysets of rational numbers have irrational numbers as their limiting poihts, and the irrational number cannot belong to the set of rational numbers.

Ex. the sets of rational numbers,

1, 1.4, 1.41, 1.414, 1.4142,-------- and

2, 1.5, 1.42, 1.415, 1.4143, ------- both

define the irrational number $\sqrt{2}$.

Other sets may contain several or infinitely many limiting points.

Ex. The points of the interval (o,1) contain infinitely many limiting points.

It is convenient to separate the limiting points of a set (E) into a class by themselves. This new class is called the first derived set of (E) and is denoted by (E_1). In case (E_1) has limiting points, these are pbaced in a class (E_2), which is called the second derived set of (E). Proceeding thus we consstruct the third, fourth,---------- derived sets of (E), denoted by (E_3), (E_4),----------(E_n). Any point of (E_n) which does not appear in (E_{n+1}) is called a limiting point of (E) of the nth order.

Ex. Let (E) $= \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$, --------- . In this case (E_1) contains the single point 1, and 1 is called a limiting point of (E) of the first order.

Ex. Let the points of (E) be given by $\frac{1}{3^{x_1}}+\frac{1}{5^{x_2}}+\frac{1}{7^{x_3}}+\frac{1}{11^{x_4}}+\frac{1}{13^{x_5}}$. Let us first consider x_1 as a variable and x_2, x_3, x_4, x_5 as constants; by giving successively greater values to x_1 it is evident that $\frac{1}{3^{x_1}}$ becomes 0 in the limit $x \longrightarrow \infty$; therefore the points given by (1) $\frac{1}{5^{x_2}}+\frac{1}{7^{x_3}}+\frac{1}{11^{x_4}}+\frac{1}{13^{x_5}}$ where x_2, x_3, x_4, and x_5 are free to assume any definite rational value, are limiting points of the set (E). Likewise by considering x_2, x_3, x_4, and x_5, to be the variable successively, and (x_1, x_3, x_4, x_5), (x_1, x_2, x_4, x_5) and (x_1, x_2, x_3, x_4) the constants respectively, we obtain five groups of numbers of the type (1).

Next we may consider two numbers x_1 and x_2, say, to be variable and to assume any value whatsoever approaching the limiting value ∞, while the remaining numbers x_3, x_4, x_5, remain constants, we therefore obtain 10 groups of the type $\frac{1}{7^{x_3}}+\frac{1}{11^{x_4}}+\frac{1}{13^{x_5}}$ (2).

On considering three numbers, say x_1, x_2, x_3, as variables, we obtain 10 groups of the type $\frac{1}{11^{x_4}}+\frac{1}{13^{x_5}}$ (3).

On considering four numbers, say x_1, x_2, x_3, x_4, as variables, we obtain 5 groups of numbers, namely:

$\frac{1}{3^{x_1}}, \frac{1}{5^{x_2}}, \frac{1}{7^{x_3}}, \frac{1}{11^{x_4}}, \frac{1}{13^{x_5}}$. (4).

Lastly considering the five numbers, x_1, x_2, x_3, x_4, x_5, as variables, and taking the limit where each becomes indefinitely great, we obtain a group containing the single element 0.

Then collecting all the numbersof these several groups into one set we form the first derived set of (E), which we will call (E_1). The second derived set of (E), (E_2) consists of the 10 groups of type (2), the 10 groups of type (3), the 5 groups of type (4), and the single point zero. All those numbers that appear in (E_1),but not in (E_2) are called limiting points of (E) of the 1st order. (E_3), the third derived set of (E), consists of the last 15 groups of the above classification, together with the point 0. (E_4)contains the last 5 groups of the classification, together with 0, and(E_5)contains the single point 0. Thus the set (E)has a limiting point of the fifth order, the single point zero.

It is obvious from the above reasoning that a set (E) of which the component points are given by $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+$ ------ $+\frac{1}{a_n}$, where a_1, a_2, a_3, ----------a_n are free to take all integral values, is a set of order <u>n</u>.

Again, a set (E) which contains all the points of the interval (0.1), since every point is a limiting point of the set, and since the set is closed, therefore the first derived set (E_1) is identical with the original set (E), as are the derived sets of higher orders. That is, the set (E) is of infinite order.

An example of derived sets of any order. The zeros of the function $\sin\frac{1}{x}$ form a set simply infinite in number. In this case (E) is given by $\frac{1}{x} = \pm\pi, \pm 2\pi, \pm 3\pi, ----$ (E_1) consists of the single point 0.

The zeros of the function $\sin\left(\frac{1}{\sin\frac{1}{x}}\right)$, form a set of the second order. The numbers of the set (E) are given by the values of x, where $\sin\frac{1}{x} = \frac{1}{\pm\pi}, \frac{1}{\pm 2\pi}, \frac{1}{\pm 3\pi}$,----------, that is, by $\frac{1}{x} = \sin^{-1}\frac{1}{\pm\pi}, \sin^{-1}\frac{1}{\pm 2\pi}, \sin^{-1}\frac{1}{\pm 3\pi}$,---------.

(E_1) consists of the zeros of the function $\sin\frac{1}{x}$, and (E_2) consists of the single point 0.

The zeros of the function $\sin\left[\frac{1}{\sin\frac{1}{\sin\frac{1}{x}}}\right]$ form a set of the third order. Those of the function $\sin\left[\frac{1}{\sin\frac{1}{\sin\frac{1}{\sin\frac{1}{x}}}}\right]$ a set of the fourth order and so on indefinitely.

If we choose as initial set (E) any closed interval (a,b), that is, the set of points is given by $a \leqq x \leqq b$, or any open interval (a,b), that is the set of points is given by $a < x < b$, in the case of the closed interval obviously every limiting point of any set of points chosen from the interval is a point of the interval, and in the case of the open interval, the points a and b are limiting points of sets chosen in the interval; therefore, in both cases we have (E_1), the first derived set, containing all the points of the closed interval (a,b), as does every succeeding derived set.

Theorem: Every derived set is a closed set; the first derived set may introduce new members not contained in the original set, but no further numbers can be introduced by other derivations; each derived set is contained in all the preceding ones.

It is required to prove that a limiting point of limiting points is itself a limiting point of the initial set. Let L_1, L_2, L_3 ------------be limiting points of (E) and let **P** be a limiting point of the series L_1, L_2, L_3 -------. From our definition of a limiting point there must exist a point of (E) between L_1 and L_2. Call this point P_1, This applies both to a set with isolated limiting points, and to the case of an interval that is dense-in-itself, for in the former case the infinite series L_1, L_2, L_3,--------would of necessity have a limiting point, and in the latter case, it being a dense interval, there are infinitely many points between any two points. Therefore between the pairs of limiting points L_1 and L_2, L_2 and L_3,------ we can choose points P_1, P_2, -------which being infinite in number have some limiting point P. That is, P is a limiting point of (E).

This conclusion that a limiting point of limiting points is a limiting point of the initial set, leads us to construct a new series of numbers, the transfinite.

Transfinite Numbers.

Consider the series $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, $1\frac{15}{16}$,-------. This is convergent and its limiting point, 2, is not an element of the series. If we represent the points by P_1, P_2 ,--------it is clear that the limiting point can be represented by P_ω, where ω is the first ordinal number that comes after the series 1, 2, 3,-----------. The number ω is called the first transfinite ordinal number. In the (1,1) correspondence that we have established between the sequence P_1, P_2, P_3, --------P_ω, and the series $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, --------------2, P_ω corresponds to the limiting value 2. It is evident that the system of transfinites can be extended, thus we create the transfinite numbers of the first order, P_ω, $P_{\omega+1}$, $P_{\omega+2}$, --------- which may be finite or infinite in number.

If finite the points are fully represented by

P_1, P_2, P_3 ---------- P_ω, $P_{\omega+1}$, $P_{\omega+2}$, --------$P_{\omega+m}$.

If the number of points is infinite, assuming that the points lie in a finite interval, for we have given a simple transformation by which the interval $(-\infty, +\infty)$ can be changed into any required finite interval, in this case, the interval (2,4), there will necessarily be a limiting point to the transfinite ordinal numbers of the first order, which is called the first transfinite number of the second order and is denoted by $P_{\omega+\omega}$ or $P_{\omega.2}$. Again we recognize the possibility of the existence of points represented by $P_{\omega.2+1}$, $P_{\omega.2+2}$, $P_{\omega.2+3}$ ---------. If the number of these transfinite ordinal numbers of the second order, is finite, our extended number system is represented by the indices 1, 2, 3, ---------- ω, $\omega+1$, $\omega+2$, -------

----- $\omega.2$, $\omega.2+1$, $\omega.2+2$,----$\omega.2+m$.

If infinite, there must be a limiting points which is called the first transfinite number of the third order, and is denoted by $P_{\omega+\omega+\omega}$, or $P_{\omega.3}$.

We can conceive of these transfinite numbers appearing in all finite orders, hence the series P_{ω}, $P_{\omega.2}$, $P_{\omega.3}$ ------- being infinite in number, it must have a limiting point, denoted by $\omega.\omega$, or ω^2.

The system, then, lends itself to extension as follows:

$$\omega^2+1,\ \omega^2+2,\ \omega^2+3, \text{-----------}\ \omega^2+\omega.$$
$$\omega^2+\omega+1,\ \omega^2+\omega+2, \text{-----------}\ \omega^2+\omega.2$$
$$\omega^2+\omega.2+1,\ \omega^2+\omega.2+2 \text{-----------}\ \omega^2+\omega.3$$
$$\omega^2+\omega.3+1, \text{---}$$

The general type of ordinal numbers is indicated by

$$\omega^n.p_n + \omega^{n-1}.p_{n-1} + \text{-------} + \omega^1.p_1 + p_0$$

If the set of points ω^1, ω^2, ω^3, ------- is infinite, we denote the limiting point of this set by ω^{ω}. With this new number we can build up a greatly extended number system as denoted by the notation:

$$\omega^{\omega^n.p_n + \omega^{n-1}.p_{n-1} + \text{--------} + \omega^1.p_1 + p_0}$$

Further, if the series ω^{ω}, ω^{ω^2}, ω^{ω^3}, ------- is infinite, we denote the limiting point of this set by $\omega^{\omega^{\omega}}$. The system can evidently be very greatly extended. Cantor conceives a new number Ω which is the limit of all these transfinite numbers, and from this a vastly greater extension can be imagined.

The formation of the transfinite numbers is accomplished by an application of Cantor's two principles of generation.

(1) After any number the immediately succeeding number is formed by the addition of unity; as

$1, 2, 3, \text{---------}$

$\omega+1, \omega+2, \omega+3, \text{--------}$

$\omega\cdot n+1, \omega\cdot n+2, \text{--------}$

$\omega^{\omega^{n}}+1, \omega^{\omega^{n}}+2, \text{-------}$

(2) After any endless sequence of numbers, a new number is formed which succeeds all the numbers of the sequence, and is distinguished by the fact that it has no number immediately preceding it.

Exs. $\omega, \omega.2, \omega^2, \omega^\omega, \omega^{\omega^n}, \omega^{\omega^\omega}$.

To indicate more clearly the principle that is involved, let us divide up any finite interval by successively bisecting each right hand interval. By this method we obtain a set of points to which we can assign numerical values if any definite interval is chosen initially. For instance, the interval (0.1) will upon indefinite bisection produce the series $0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}$, ----------1. Let us place this series in (1,1) correspondence with the sequence P_1, P_2, P_3, --------, which must have a limiting point P that corresponds to 1.

Now if we repeat a similar subdivision with every interval of the above sequence, for instance the interval $(\frac{7}{8}, \frac{15}{16})$ or (P_3, P_4), we obtain an infinite sequence of divisional points: $\frac{7}{8}, \frac{29}{32}, \frac{59}{64}$, ---------which has the limiting point $\frac{15}{16}$. Therefore we can represent the points of successive bisection of the infinite series of intervals $(P_1, P_2)(P_2, P_3)$---- as follows:

The points of (P_0, P_1) by the indices 1, 2, 3, -------- ω,

(P_1, P_2) $\omega+1, \omega+2$, ------ $\omega.2$,

(P_2, P_3) $\omega.2+1, \omega.2+2$-----$\omega.3$

and so on for the infinite number of intervals. By the notation a (1,1) correspondence is set up such that the points $0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}$, --------- are paired off with the transfinite numbers $0, \omega, \omega.2, \omega.3, \omega.4, \omega.5$------ respectively. Hence the point 1 can be represented as the limiting point of the series $o, \omega, \omega.2, \omega.3$, ------- that is, by the transfinite number ω^2.

By a similar subdivision of each of the infinitely many intervals of each of the intervals $(P_0,P_1), (P_1, P_2)$, --------, we obtain a (1,1) correspondence in which the points $0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}$ --------- are paired off respectively with the sequence $o, \omega^2, \omega^2.2, \omega^2.3, \omega^2.4, \omega^2.5$, --------. Hence 1 is in (1,1) correspondence with the transfinite ordinal number ω^3.

By repeated applications of this method of subdivision, the number 1 may be represented by a transfinite of any finite order, say ω^n. If we imagine the subdivision to be performed infinitely many times, then the number 1 becomes the transfinite number ω^ω.

A second example on the representation of transfinite numbers by rational numbers is the following:

A set of numbers that can be placed in (1,1) correspondance with the natural numbers is said to be countably infinite: It will be shown in the course of a few pages that the prime numbers are countably infinite in number, assuming this, it is clear that the series of prime numbers, 1, 2, 3, 5, 7, 11-------, can be placed in (1,1) correspondence with the natural numbers 1, 2, 3, -----------. Then taking the squares of all the prime numbers, omitting 1, no number is repeated and these can be placed in (1,1) correspondence with the transfinite numbers of the first order $\omega, \omega+1, \omega+2, \omega+3$, ---------. We may then set up a (1,1) correspondence between the cubes of the prime numbers, $2^3, 3^3, 5^3, 7^3, 11^3$ --------and the transfinites of the second order, $\omega.2, \omega.2+1, \omega.2+2$, ---------, and in general the numbers $2^{h+1}, 3^{h+1}, 5^{h+1}$,------corresponding to the transfinites of the rth order, $\omega.r, \omega.r+1, \omega.r+2$, ------ We may then take the numbers (a.b) which consist of the product of two prime factors; these arranged in ascending order of magnitude may be placed in (1,1) correspondence with the transfinites $\omega^2, \omega^2+1, \omega^2+2, \omega^2+3$, ------.

Next, taking the numbers $(a^2 b^2)$, which are the squares of the numbers in the set last considered, these we may place in (1,1) correspondence with the numbers,

$\omega^2+\omega, \omega^2+\omega+1, \omega^2+\omega+2$, -------

and by taking the successive sets of numbers $(a^3 b^3)$, $(a^4 b^4)$, ---------, we obtain groups of numbers that may be taken to correspond with the following transfinite sets respectively.

(1) $\omega^2+\omega.2, \quad \omega^2+\omega.2+1, \quad \omega^2+\omega.2+2$, ---

(2) $\omega^2+\omega.3, \quad \omega^2+\omega.3+1, \quad \omega^2+\omega.3+2$, ---

----

(p-1) $\omega^2+\omega p, \quad \omega^2+\omega.p+1, \quad \omega^2+\omega.p+2$, ---

All of these numbers are ordinally less than the transfinite $\omega^2 .2$.

Proceeding, we may next place in (1,1) correspondence the sets of numbers

$(a^2 b)$, $(a^2 b)^2$, $(a^2 b)^3$, -----------

$(a^3 b)$, $(a^3 b)^2$, $(a^3 b)^3$, ------------

$(a^4 b)$, $(a^4 b)^2$, $(a^4 b)^3$,------------

and so on indefinitely.

Afterwards we may make use of produce of 3, 4, 5, ------- prime factors.

It is thus proved that the system of transfinites may be carried to a degree of magnitude, that seems unlimited.

An Application of the Conception of Transfinite Numbers to Derived Sets.

If (E) is any set of points, we require to show that the derived sets (E_1) , (E_2), (E_3),------are closed sets, and that each set after (E_1) consists only of points belonging to the preceding sets.

Proof. Suppose P is a point in (E_n) that is not contained in (E_1), that is the point P is not a limiting point of the set (E), or in other words there is a neighborhood of P which contains at most but a finite number of points of the set (E). Hence within this neighborhood there are no points of (E_1), nor of (E_2), since (E_2) contains only the limiting points of (E_1). Likewise this neighborhood cannot contain any point of the sets (E_3), (E_4),--------, which is contrary to our hypothesis. Therefore every set (E_{n+1}) must contain only points of the set (E_n), where $n \geqq 1$.

By the definition of a derived set we can form the derived set (E_n) of any set (E), where $\underline{n}$ is any finite number. But if the number of derived sets is infinite we define that set which contains those points common to all the sets (E_1), (E_2), (E_3), (E_4), ------------(E_n), where $\underline{n}$ can be indefinitely great, as the set (E_ω).

It is required to show that (E_ω) is a set of at least one point and is closed. Let p_1 be any point in (E_1); p_2 any point in (E_2) and so on. The points p_1, p_2, p_3, -----form a set $[p_n]$ of infinitely many elements and therefore must have a limiting point which we will call p. Now, p belongs to the set (E_n) whatever valye $\underline{n}$ may have, because all but a finite number of the points of $[p_n]$ belong to (E_n); therefore p is a point of (E_ω).

Choose any sequence of points in (E_n), let the sequence be a_1, a_2, -----having the limiting point $\underline{a}$, since (E_n) is a closed set, $\underline{a}$ is a point of (E_n) whatever value $\underline{n}$ may have, therefore $\underline{a}$ is a point of (E_ω), that is (E_ω) is a closed set.

Then, proceeding in the regular way, we can form the successive derived sets, $(E_{\omega+1})$, $(E_{\omega+2})$, $(E_{\omega+3})$-------; these may be finite or infinite in number. If infinite in number, a similar process of reasoning will prove that $(E_{\omega \cdot 2})$ is a set of at least one point and is closed.

It has been remarked earlier that a set whose points are given by $\frac{1}{a_1}+\frac{1}{a_2}+ \text{----} + \frac{1}{a_n}$ where the a's are free to take every integral value, has $\underline{n}$ derived sets. The following is an example of a set which has infinitely many derivatives, yet (E_ω) does not exist. Suppose we have a set (E) such that (E_n) consists of the points $\frac{1}{n}$, $\frac{1}{n\ 1}$, $\frac{1}{n\ 2}$, ---------, when n becomes indefinitely great, (E_n) tends towards 0. Hence (E_ω) does not exist since ω is the next original number following the series 1, 2, 3, 4, 5, -----------.

POTENCY.

A very helpful conception in the study of sets of points is that of potency, with applications to the general idea of cardinal number.

Definition. Any two sets that can be brought into (1,1) correspondence are said to be equivalent in that they have the same potency.

If we take a single point as unit, we can construct the natural numbers as far as any given number however great, and by means of these count any finite set of points. Thus every finite set is countable and corresponds in potency with that set of the natural numbers used in the process of counting.

Theorem 1. A sequence or simply infinite group of numbers can be brought into (1,1) correspondence with the domain of natural numbers.

Let L be the only limiting point of a sequence that extends over the whole straight line or over any part; choose any point M and take any interval (A,B) including M, this interval contains but a finite number of points n which we may may pair off with the integers 1, 2, 3, --------n.

In the figure, the line represents one unlimited in length, therefore closed.

Bisect the segments (B,L) and (A,L not including the point M), at B' and A'. Then the segments BB' and AA' contain each but a finite number of points.

Suppose the segment BB' contains n_1 points, these can be made to correspond with the integers, $n+1$, $n+2$, -------- $n+n_1$; and if the segment AA' contains n_2 points, these can be paired off with the integers, $n+n_1+1$, $n+n_1+2$, $n+n_1+3$, --------$n+n_1+n_2$.

Continuing this subdivision and always choosing those segments that have not L as an end-point, we can place every point of the simply infinite sequence in (1,1) correspondence with one of the natural numbers. If L happens to be a point of the sequence we begin by pairing off the point L with the number 1, and then proceed as before.

Therefore since any simply infinite sequence can be brought into (1,1) correspondence with the system of natural numbers, obviously any two simply infinite sequences can be brought in (1,1) correspondence with one another.

Thus the potency of the system of natural numbers, is likewise the potency of all simply infinite sequences and it is convenient to give to it a symbol. The potency of the natural numbers will be denoted by <u>a</u>.

Examples:

(1) The even numbers have the potency a. Proof, they can be paired off in the following manner to any indefinite number, that is a complete (1,1) correspondence ie set up.

1, 2, 3, 4, 5, ---------

2, 4, 6, 8,10, ---------

Similarly it can be shown that

(2) The odd numbers have the potency a.

(3) The prime numbers have the potency a.

(4) The numbers that are the squares of the natural numbers have the potency a.

A set which can thusly be brought into (1,1) correspondence with the natural numbers is said to be countably infinite.

Theorem 2. If we have a finite number of countably infinite sets, their sum is likewise a countably infinite set. Let the sets be:

a_1, a_2, a_3, -------

b_1, b_2, b_3,--------

c_1, c_2, c_3,--------

We can rearrange the sets in the following way:

a_1, b_1, c_1, d_1, -----------a_2, b_2, c_2, d_2, -----------

a_3, b_3, c_3, d_3, -----------, and the composite set thus arranged can be placed in (1,1) correspondence with the natural numbers. Conversely, if the composite set consisting of all the points of any finite number of sets is countable, then each of the compound sets is countable.

As every limiting points of a set is determined by a simply infinite sequence, therefore a set which has but a finite number of limited points is countable.

Theorem 3. Any set which can be divided into a countable number of countable sets is itself countable.

Let the sets be

(1,1)(1,2)(1,3)(1,4)(1,5)(1,6)(1,7) ----
(2,1)(2,2)(2,3)(2,4)(2,5)(2,6)(2,7) ----
(3,1)(3,2)(3,3)(3,4)(3,5)(3,6)---- Fig.
(4,1)(4,2)(4,3)(4,4)(4,5)(4,6) ----
(5,1)(5,2)(5,3)(5,4)----

And rearranging

[(1,1)],[(1,2), (2,1)] , [(1,3),(2,2),(3,1)] ,-------

where the sum of the indices of each element of any bracket [ ] are equal. This rearrangement of the composite sets can be placed in (1,1) correspondence with the natural numbers since the elements of each group are finite in number.

From the theorems that we have proved, assuming that the associative and commutative laws hold as for ordinary multiplication and addition:

$$\underline{a} + n = n + \underline{a} = \underline{a}$$

$$n.\underline{a} = \underline{a}.n = \underline{a}$$

$$\underline{a}^n = \underline{a}$$

Theorem 4. Every set of intervals on a straight line is countable provided no two overlap.

We can assume the intervals to be contained in a finite segment since we can readily transform as infinite segment into points contained in a given segment.

Choose e_1, e_2, e_3, -------- a series that has zero as its limiting point, then arrange in one group all the intervals that are $\geqq$ the interval e_1 e_2, . Call the number of these intervals a_1 which must be finite since we are dealing with Archimedian quantities. Similarly , those intervals that are $\geqq$ e_2 e_3 and $<$ e_1 e_2 , in number a_2, must be finite, and so on indefinitely. This arrangement includes all the intervals and according to Theorem 3, the intervals are countable.

As a corollary to the preceding theorem we can state that every set of isolated points is countable. Proof : we can describe intervals each containing but one isolated point and such that the intervals do not overlap, though they may abut in special cases, hence any set of isolated points is countable.

Theorem 5. If (E) is a set of points, those points of (E) that are not included in the first derived set (E_1) are countable.

These points cannot form an interval that is dense, for if so, the interval would necessarily contain limiting points of (E); therefore the points of (E) other than those included in (E_1) from an isolated set, that is, they are countable. Further, if (E_1) is a countable set, then (E) is likewise a countable set.

The reasoning is genera]. Those points of (E_1) that are not countained in (E_2) from an isolated and thereforea countable set, and so on.

Thus, if (E_n) is a countable set we can state , at once, that (E_{n-1})------(E_1) and (E) are likewise countable sets.

The rational numbers are countable, since every rational can be expressed in the form (a,b), where a and b are integral numbers, the numerator and denominator of a proper fraction respectively. We can evidently arrange the rational numbers in a sequence of the form:

$$[(1,1)], [(1,2),(2,1)], [(1,3),(2,2),(3,1)], \text{--------}$$

which is countable by Theorem 3.

Any set of intervals whose end-points are rational numbers, where the raional numbers can be chosen in any order whatsoever, may be overlapping in a very complex manner, yet they are countable as is shown by the proof in the preceding paragraph.

Theorem 6. Given a set of intervals overlapping in any way, we can determine a countable set of intervals from among them such that every point internal to any interval of the given set is also interval to an interval of the countable set.

Let the overlapping intervals be contained in the finite segment (A,B), this does not impair the generality of the proof. In (A,B) take any point P corresponding to a rational number; choose rational numbers Q and R in (A, P) and (P,B); choose rational numbers a_1, a_2, a_3, a_4 in the segments

(A,Q), (Q,P),(P,R) and (RB) respectively, and so on indefinitely.

If A and B are rational points, we obtain a set of overlapping intervals whose end-points are rational numbers, which we have proved to be countable, and the segment (AB) is such that every point is internal to some interval of the countable set of overlapping intervals. If A and B are irrational points, and $A > B$, we have but to choose $A_1 < A$ and $B_1 > B$, where A_1 and B_1 are rational points in order to obtain a set of overlapping intervals with rational end-points that contain every point of the segment (A B), and that is countable.

The Heine-Borel Theorem.

Given any closed sets of points on a straight line and a set of intervals so that each point of the closed set of points is an internal point of at least one of the intervals, there exists a finite number of intervals which have the property of including as an internal point every point of the closed set of points.

In Theorem 6 it was proved that any set of overlapping intervals could be replaced by a countable set which likewise included every point as an internal point.

Let us call these intervals d_1, d_2, d_3 ,------.
Three cases may arise:

(1) d_2 may not overlap d_1; if so, call the two intervals δ_2 and δ_1, respectively.

(2) d_2 may overlap d_1 on one side only; in this case, denote the interval d_1 by δ_1, and by δ_2, the non-overlapping part of d_2.

(3) d_2 may overlap d_1 on both sides; if so, denote the interval d_1 by δ_1, and by δ_2 and δ_3 the non-overlapping parts of d_2.

Proceeding with the interval d_3, denote its whole, part, or parts by δ_4, δ_4 or (δ_4, δ_5) respectively; and so on indefinitely.

We get a set of non-overlapping intervals. It is required to prove them finite in number. There cannot be any points of the closed set of points that are external to the set of δ--intervals, because the intervals abut, nor can there be any semi-external points.

A semi-external point is one which is a limiting point of a sequence of intervals on one side and an end point of one or more intervals on the other.

Ex. The sequence of intervals $(0,\frac{1}{2})$ $(\frac{1}{2},\frac{3}{4})$ $(\frac{3}{4},\frac{7}{8})$ ----- together with the interval (1,2) determine the semi-external point 1.

If any end-point were a semi-external point, necessarily it would be a limiting point of points of the closed set and therefore a point of the set. But by our definition, the limiting point of a set of intervals is a point such that any interval whatsoever having this point as an internal point contains intervals of the set, Therefore limiting points and semi-external points cannot exist in this set of non-overlapping intervals. Further, it has been shown that a set of infinitely many points must have at least one limiting point. Hence considering the right-hand end-points as forming a set, the two sets obviously determine a limiting point or a limiting interval. But a set of non-overlapping intervals cannot have a limiting interval. And as we have shown that in this case the δ-intervals cannot have a limiting point, therefore the δ-intervals are finite in number. Likewise the d-intervals are finite in number since they can be chosen so as to be less than the δ-intervals.

It is required to prove that the system of algebraic numbers is countable. Algebraic numbers are those that arise in the complete solutions of algebraic equation of any finite degree, whereas rational numbers arise from the solution of equations of the first degree.

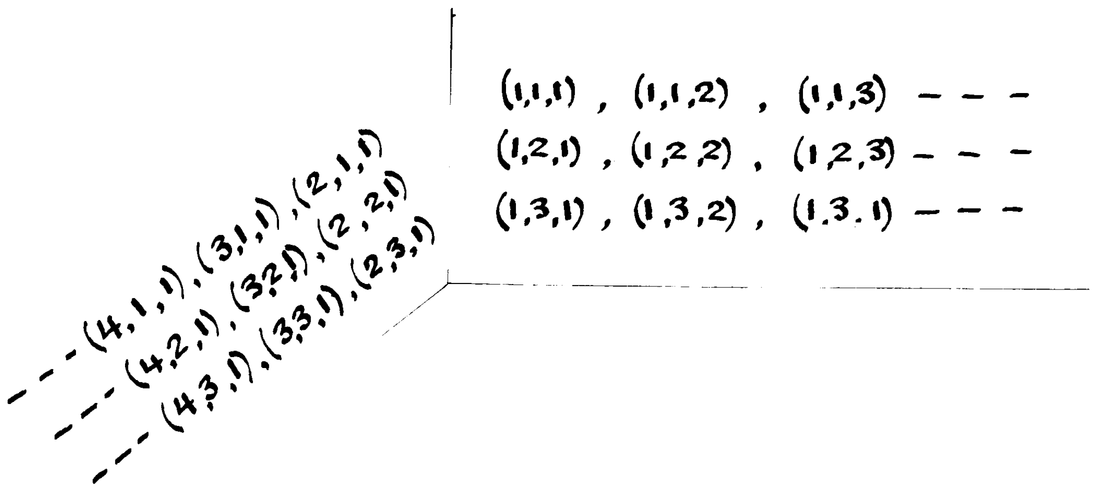
Let the general equation be written as

$$a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \text{---------------} + a_m = 0 \quad (1).$$

Since $a_1, a_2, \text{-------} a_m$ are necessarily rational integral numbers, therefore

$$|a_0| + |a_1| + \text{-------} + |a_m| = n \qquad (2)$$

The number of solutions of equation (2) is finite, because the symbols a_0, a_1, ----a_m are free to take only integral values, both positive and negative, and when we add their absolute values, evidently there are but a finite number of permutations that satisfy equation (2) for any given value of n. Further, an equation of degree m has but m roots, that is, the number of roots is always finite. If we now arrange the different solutions or values for a given m and n in some definite order, and denote by p the position in this ordered sequence of any number, we can fully represent every algebraic number by a triply infinite system of indices, as indicated in the diagram :

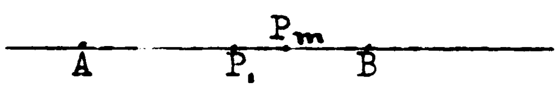


We thus obtain a solid formation of elements, which we may separate off into sections whose aggregate of indices are the same for any section and change by unity from any section to the succeeding one. Arranging the above sequence of numbers in this manner [(1,1,1)], [(1,1,2),(1,2,1),(2,1,1)] ,------ we can place the sequence in (1,1) correspondence with the natural numbers, therefore the system of algebraic numbers is countable.

The algebraic numbers comprise both rational and irrational numbers. We have shown that the whole system of algebraic numbers is countable, and likewise the rational numbers are countably infinite, therefore the irrational algebraic numbers are countable.

It has been proved that a set of isolated points it a closed set and is countable infinite. We can prove that a set of points that is dense-in-itself and nowhere-dense is countable. For consider the right-hand end-points of the black intervals of Cantor's tertiary set, which is such that any given interval is divided into three equal parts, and the middle one is darkened, then the remaining two parts are similarly divided and their middle parts blackened, the process being carried on indefinitely;----it has been shown that the end-points of all of these infinitely many black intervals are limiting points of end-points, but on one side only. Therefore the set of end-points is dense-in-itself; further between any two end points there lies either a black interval or an interval containing external points of black intervals, so that the set is nowhere-dense. Again it has been shown that any set of non-overlapping intervals is countable, the black intervals form a non-overlapping set, therefore the set comprising their end-points is countable.

A countable set is never perfect. Let the countable set be P_1, P_2, P_3,------arranged in countable order, and we will suppose it to be a perfect set.



With P_1 as centre point take an interval δ_1 and let P_m be the first point in the countable order following P_1, therefore $m > 1$. Then, with P_m as centre point take an interval δ_2 not containing P_1 and lying wholly within δ_1, and in δ_2 let P_n be the first point in the countable order following P_m. Continuing this process indefinitely we obtain a set of intervals δ_1, δ_2, ------ each contained within all the preceding ones, and the limit of the set of intervals $\delta_1, \delta_2, \delta_3$ ------is zero. Hence the set of intervals determine a limiting point P_x. We have $1 < m < n$ ------$< x$, and since we have chosen a countable set and but a finite interval of that set, therefore x is finite, that is, the number of intervals is finite. But by hypothesis P_1 is a limiting point of the set, therefore within the successive intervals $(P_1 B), (P_1 P_m)$ $(P_1 P_n)$-------which we have denoted by the series $\delta_1, \delta_2, \delta_3,$ ------- there must be infinitely many points of the set. This is a direct contradiction of the result of the preceding reasoning, hence we may state that no countable set is perfect.

It follows from the last result that a closed countable set of points cannot contain any part that is dense-in-itself, and as every segment of the domain of real number, as the interval (0,1) say, is both closed and dense-in-itself, we may conclude that the aggregate of real points is not countable.

The Linear Continuum.

The linear continuum is not countable. We assign to it a newpotency c.

Direct proof: All the rational numbers in the interval (0,1) can be expressed as decimals of the type:

$0.\ a_{11}\ a_{12}\ a_{13}\ a_{14}$ -------

$0.\ a_{21}\ a_{22}\ a_{23}\ a_{24}$-------

$0.\ a_{31}\ a_{32}\ a_{33}\ a_{34}$-------

----

and we have proved that they are countable in number. If we can define a number lying in the segment (0,1) different from every member of the above group, since every real number that is conceivable belongs to the linear continuum, we will have proved that the linear continuum is not countable.

Let the number b be defined by

o, b_1 b_2 b_3 b_4--------, where b_n is never the same as a_{nn}, which we can assure by choosing $b = a + 1$ if $a < 9$, and $b = o$ if $a = 9$. The number b evidently lies in the segment (0,1) and can be seen to be different from every rational number by at least one place in the decimal. Therefore we have proved that the linear continuum is not countable.

It is required to show that any closed or open segment can be brought into (1,1) correspondence with the whole straight line, or with any segment of that line.

Consider the open interval (1,2) and the same interval closed; choose out two sets, $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{4}{5}$---------- having 2 as its limit, and $1\frac{1}{4}$, $1\frac{1}{8}$, $1\frac{1}{16}$, ----------having 1 as its limit; and let every number not included in these sets be placed in correspondence with itself. Then if we pair off consecutively the numbers 1 and $1\frac{1}{4}$, $1\frac{1}{4}$ and $1\frac{1}{8}$, $1\frac{1}{8}$ and $1\frac{1}{16}$,------ and also the numbers 2 and $1\frac{1}{2}$, $1\frac{1}{2}$ and $1\frac{3}{4}$, -------- evidently every point of the closed segment is placed in (1,1) correspondence with a point of the open segment.

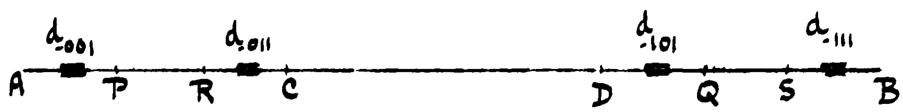
Let us place the segment (-5,5) in (1,1) correspondence with the whole straight line, making use of the equation $xx' = 4$, every number > 1 and < -1 is placed in correspondence with the points of the segment $(-4,4)$. The points of the closed interval $(0,1)$ are placed in correspondence with the segment $(4,5)$ by means of the equation

$x' - 4 = 1 - x$; and the points lying between -1 and 0, -1 included only is placed in correspondence with the interval $(-4,-5)$ by means of the transformation.

$x' + 4 = -1 - x$, which completes the correspondence. Therefore every segment of the linear continuum has the potency c.

Every set of non-overlapping intervals is countable in number, and the points of each interval whether open or closed has the potency c. Therefore the points of any set of non-overlapping intervals can be brought into (a,1) correspondence with the whole straight line, that is, the set of points has the potency c. In order to prove generally that every perfect set has the potency c, we must discuss the case of a perfect set dense-nowhere.

Suppose the perfect set to be inclosed in a segment (A,B) which is the smallest possible, then A and B must be points of the set and hence limiting points of points of the set. In Cantor's tertiary set, perfect and dense-nowhere, the black intervals are dense-everywhere, but as A and B are limiting points they are not internal points of black intervals, but must be external points. Returning to Cantor's typical perfect set dense-nowhere, divide the segment into two equal parts at C and D, and denote the black interval CD by $d_{.1}$ where .1 is the binary number corresponding to its middle point. Proceed in the same manner with the segments AC and DB, numbering their middle black intervals $d_{.01}$ and $d_{.11}$ respectively. We have now four intervals AP, RC, DQ and SB, whose middle blackened intervals would be numbered $d_{.001}$, $d_{.011}$, $d_{.101}$, and $d_{.111}$, where the suffixes are the binary numbers corresponding to the middle points of each black interval.



This method of subdivision can be carried out indefinitely. We have proved that the resulting set of black intervals are countable infinite, and that the end-points and external points of black intervals form a perfect set dense-nowhere.

Every number of the continuum (0,1) is expressible in the dyad scale by means of a sequence .p , .pq , .pqr, -------
where p, q, r, -----are either 0 or 1. And we have proved that it is possible to place any segment (a,b) in (1,1) correspondence with any other segment, say (0, 1). Further, every number determined by the above sequence of numbers is unique, except those in which after some certain decimal digit all the digits are 1, for those can be expressed by a second sequence, in which after some certain decimal digit all the digits are zero.

Now, let us put the white intervals into a notation similar to that of the black intervals, for instance, let the interval (AB) be denoted by $1_{.1}$ the two intervals formed by the first subdivision by $1_{.01}$ and $1_{.11}$, then the four intervals arising from the next subdivision by $1_{.001}$, $1_{.011}$ $1_{.101}$ and $1_{.111}$, the eight intervals from the next subdivision by the indices: .0001, .0011, .0101, .0111, .1001, .1011, .1101, .1111, and so on indefinitely. If we denote the perfect set of points by (E) evidently we can represent every point of (E) by a dyad sequence. For let a be any point of the set (E), a must lie in the interval; then in either of the intervals $1_{.01}$, or $1_{.11}$; then in one of the four $1_{.001}$, $1_{.011}$, $1_{.101}$ or $1_{.111}$; then in one of the eight intervals, and so on; that is, we obtain a dyad sequence, .p, .pq, .pqr, ------- which represents the point a; as the intervals from an infinite set each enclosed within the preceding, therefore they determine a definite limiting point.

In the case where after some fixed number, all the decimals of the suffix are either all 0 or all 1, the point represented by the sequence is a common end-point of all the intervals after a certain fixed one. Therefore the sequence $1_{.p}$, $1_{.pq}$, $1_{.pqr}$, ------determines in every case a point of the set (E) and determines each point uniquely except in the case of end-points of black intervals which are countably infinite in number. We have thus shown that the points of a perfect set can be placed in (1,1) correspondence with the dyad scale which represents all the numbers of the continuum (0,1), because if from any set of potency c a countable infinite set be removed the remaining points of the set will still have the potency c.

That this proof will apply to the general perfect set dense-nowhere, is seen from the following consideration. Divide the segment (AB) into three equal parts by the points C and D. If D is enclosed in a black interval call it d_1 as in our previous notation. If CD is not enclosed in a black interval then it must either from a part of a black interval or it must include black intervals; if the former is the case, denote by d_1 the interval of which CD forms a part, and if the latter is the case, choose out the largest black interval enclosed and denote it by d_1. In every case we have two white intervals terminated by A and B respectively and to these we apply a similar subdivison, where A and B being points of the set cannot be other than external points of the black intervals that are dense-everywhere. Clearly the notation can be made to conform exactly to that of the regular case. The statement about infinitely many sets of intervals each enclosed within the preceding, and therefore determining a definite limiting point, follows directly as does the remainder of the proof.

We have just shown that every perfect set has the potency c. Now, by an example we will illustrate that a countably infinite set can be taken away and the remaining points will still have the potency c.

Let us take the set of points $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \text{--------}$ together with the points of the interval (0,-2). The whole set has obviously the potency c, while the set $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, ----- has the potency a, and the set remaining after subtracting the set of potency a is the set (0,-2) which has the potency c.

It has been proved that the linear continuum has the potency c, and that the rational numbers of the continuum has the potency a. Therefore the irrational numbers in the whole continuum or in any segment has the potency c. It has been shown that the set of algebraic irrationals has the potency c.

It is required to show that the Liouville numbers in the continuum has the potency c. We may place the Liouville numbers $\frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^{1\cdot 2\cdot 3}} +$ ------ in (1,1) correspondence with the numbers $\frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} +$ ------ where a_1, a_2, a_3, -------- may have for values any of the ten figures. But the set of numbers given by $\frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} +$ ----- makes up the rational and irrational numbers of the real continuum, that is has the potency c.

The reader is referred to 'Sets of Points' by Young for a discussion on the content and measure of a set of points, also for the extension of these elementary ideas to include the properties of plane sets of points, points in three dimensions and generally points in n dimensions.

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