Spectral Asymptotics of Heisenberg Manifolds

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To my parents with love and gratitude

for all their support and inspiration.

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ABSTRACT

Let $R(t)$ be the error term in Weyl's law for $(2n+1)$-dimensional Heisenberg manifolds. We prove that in the 'rational' case, $R(\lambda)$ is of order $O(t^{n-7/41})$. In the 'irrational' case, for generic $(2n+1)$-dimensional Heisenberg manifolds with $n > 1$, we prove that the error term is of the order $O_\delta(t^{n-1/4+\delta})$, for every positive δ. The polynomial growth is optimal. We also prove that for arithmetic Heisenberg metrics, $\int_1^T |R(t)|^2 dt = cT^{2n+\frac{1}{2}} + O_\delta(T^{2n+\frac{1}{4}+\delta})$, where c is a specific nonzero constant and δ is an arbitrary small positive number. In the three dimensional case, this is consistent with the conjecture of Petridis and Toth [PT] stating that $R(t) = O_\delta(t^{\frac{3}{4}+\delta})$.

ABRÉGÉ

Résumé. Soit $R(t)$ le terme d'erreur de la loi de Weyl pour la variété riemannienne d'Heisenberg à $(2n+1)$-dimensions . Nous prouvons que dans le cas 'rationel', $R(\lambda)$ est d'ordre $O(t^{n-7/41})$. Dans le cas 'irrationel', pour des variétés riemanniennes d'Heisenberg génériques à $(2n+1)$-dimensions avec $n > 1$, nous prouvons que le terme d'erreur est d'ordre $O_\delta(t^{n-1/4+\delta})$, pour tout δ positif . La croissance polynômiale est optimale. Nous prouvons aussi que $\int_1^T |R(t)|^2 dt = cT^{2n+\frac{1}{2}} + O_\delta(T^{2n+\frac{1}{4}+\delta})$, où c est une constante spécifique non nulle et δ est un nombre positif arbitrairement petit. Ce résultat est avancée vers la conjecture de Petridis et Toth [PT] qui énonce que pour $n = 1$, nous avons $R(t) = O_\delta(t^{\frac{3}{4}+\delta})$.

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CHAPTER 1
Introduction

1.1 A short history of Weyl's law

Let (M, g) be a closed n-dimensional Riemannian manifold with metric g and Laplace-Beltrami operator Δ. Let its eigenvalues be $0 = \lambda_0 < \lambda_1 \leq \cdots$. For the spectral counting function $N(t) = \#\{j, \lambda_j \leq t\}$ we have Hörmander's theorem

$$N(t) = \frac{vol(B_n)vol(M)}{(2\pi)^n} t^{n/2} + O(t^{(n-1)/2}),$$

where $vol(B_n)$ is the volume of the n-dimensional unit ball and by $O(t^{(n-1)/2})$ we mean a term which grows no faster than $t^{(n-1)/2}$ as t tends to infinity.

The estimate of the error term in the Hörmander's theorem, defined by

$$R(t) = N(t) - \frac{vol(B_n)vol(M)}{(2\pi)^n} t^{n/2},$$

is in general sharp, as the well-known example of the sphere S^n with its canonical metric shows [Hö]. However, the question of determining the optimal bound for this error term in any given example is a difficult problem which depends on the properties of the associated geodesic flow. In many cases, this is an open problem. Nevertheless, for certain types of manifolds some improvements have been obtained and in a few cases the conjectured optimal bound has been attained.

The results obtained in this direction can be classified in three categories: (i) The first type of results deal with the upper bound for the rate of the growth of the error term(i.e. the O-results). (ii) The second type deal with finding a lower bound for this growth(i.e. the Ω-results). (iii) Finally, the third type are results about the averages and the moments of the error term.

One of the first results on pointwise estimates is due to Duistermaat and Guillemin [DG] which asserts that in the case where the geodesic flow is clean and the set of unit-speed geodesics in S^*M has null Liouville measure, then one can improve the Hörmander bound to

$$R(t) = o(t^{(n-1)/2}).$$

Subsequently, Ivrii [Iv] gave a different proof of this result and extended it to manifolds with boundary. There are some additional improvements in $R(t)$ that are known in some specific examples. For instance, in the case of hyperbolic manifolds, a result of Bérard [Bé] gives:

$$R(t) = O(t^{(n-1)/2}/\log t).$$

This result is likely far from optimal. Indeed, it has been conjectured that, at least in the non-arithmetic case $R(t) = O(t^{\delta})$ for all $\delta > 0$. In fact, even for noncompact arithmetic surfaces with cusps, such as $H/SL_2(Z)$, one has

$$R(t) = c\, t^{1/2} \log t + O(t^{1/2}),$$

see [Hej2]. For compact arithmetic surfaces arising from quaternion algebras Selberg proved that $R(t) = \Omega(t^{1/4}/\log t)$; see [Hej1].

In the opposite case of completely integrable geodesic flow there are several cases where improved error terms are known: For generic convex surfaces of revolution, Colin de Verdière [Co] showed that

$$R(t) = O(t^{1/3}),$$

which agrees with a result of Van der Corput and Sierpinski [Si] for the classical circle problem in 2 dimensions, i.e. for the torus R^2/Z^2.

There are also additional more general results of Volovoy [Vo] under long-time recurrence estimates for the geodesic flow, but they are difficult to quantify. The geometrically simplest example of an integrable geodesic flow on a surface is 2-dimensional flat torus. In this case, Hardy's conjecture [Ha2] states that it is not likely that

$$R(t) = O_\delta(t^{\frac{1}{4}+\delta})$$

for all $\delta > 0$. Hardy [Ha3] also proved that for T^2 this is the best possible upper bound, i.e. $R(t) = \Omega(t^{1/4}(\log t)^{1/4})$. See [Hf] for the best Ω-result. Sarnak [Sa] has generalized this upper bound and gave a geometric interpretation for it using the trace formula. He showed that if the geodesic flow on a two dimensional manifold has the property that, for some fixed T, the fixed point set of the flow for time T is two dimensional in the three dimensional unit cotangent sphere, then $R(t) = \Omega(t^{1/4})$.

There is much evidence, both numerical and otherwise to suggest that Hardy's bound is optimal. For instance, a classical result of Cramér [Cr] states that for T^2:

$$\lim_{T\to\infty} \frac{1}{T^{\frac{3}{2}}} \int_1^T |R(t)|^2 dt = C,$$

where $C = \frac{1}{6\pi^3} \sum_1^\infty \frac{r(n)^2}{n^{3/2}}$ with $r(n) = \#\{(a,b) \in Z^2; n = a^2 + b^2\}$.

As the first natural non-commutative generalization of T^2 consider H_1, the 3-dimensional Heisenberg manifold, which has completely integrable geodesic flow [Bu]. Petridis and Toth [PT] have proved that, for certain 'arithmetic' Heisenberg metrics on H_1, $R(t) = O(t^{5/6+\delta})$. Later in [CPT] the exponent was improved and the result was extended to all left-invariant Heisenberg metrics. It was conjectured in [PT] that for H_1

$$R(t) = O_\delta(t^{3/4+\delta}). \tag{1.1}$$

Moreover, Petridis and Toth [PT] have also proved the following L^2 result for H_1, by averaging over perturbations of the metric g and defining $M(u) = (H_1/\Gamma, g(u))$,

$$\int_{I^3} |R(t,u)|^2 du \le Ct^{3/2+\delta},$$

where $I = [1-\epsilon, 1+\epsilon]$.

The conjecture (1.1) follows from the standard conjectures on the growth of exponential sums, see [CPT]. The exponential sums that show up have convex phase and, consequently, van der Corput's method and the method of exponent pairs can be applied. In the case of $2n+1$-dimensional Heisenberg manifolds with $n > 1$, we face multiple sums with linear dependence on $n-1$ variables.

1.2 The statement of the results on Heisenberg manifolds

As we have mentioned in section 1.1, according to a conjecture by Petridis and Toth [PT], for three dimensional Heisenberg manifolds:

$$R(t) = O_\delta(t^{\frac{3}{4}+\delta}), \tag{1.2}$$

for every positive δ. Moreover, this conjecture would follow from the standard conjectures on the growth of exponential sums.

In higher dimensions, i.e. $(H_n/\Gamma, g)$ where $n > 1$, in joint work with Petridis [KP] we have proved the following pointwise estimates:

Theorem 1.2.1 *Let $(H_n/\Gamma, g)$ be the $(2n+1)$-dimensional Heisenberg manifold where $n > 1$ and the metric g is in the orthogonal form*

$$g = \begin{pmatrix} h & 0 \\ 0 & g_{2n+1} \end{pmatrix}.$$

Let J_n be the standard symplectic matrix

$$J_n = \begin{pmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{pmatrix}$$

Denote the eigenvalues of $h^{-1}J_n$ by $\pm\sqrt{-1}d_j^2, 1 \leq j \leq n$. If the ratios ${d_j}^2/{d_i}^2$ are rational, then

$$R(t) = O(t^{n-7/41}).$$

Remark 1 *Conjecturally, in the 'rational' case the best estimate, following from (3.17), is*

$$R(t) = O_\delta(t^{n-1/4+\delta}). \tag{1.3}$$

Theorem 1.2.2 *Let $(H_n/\Gamma, g)$ and $\{\pm\sqrt{-1}d_j^2; 1 \leq j \leq n\}$ be as defined in Theorem 1.2.1. If there exists at least one irrational coefficient ${d_j}^2/{d_n}^2$, then for almost all* *metrics g, which are the ones where this irrational coefficient θ satisfies the Diophantine condition $\|j\theta\| \gg 1/(j \log^2 j)$, we have*

$$R(t) = O_\delta(t^{n-\frac{1}{4}+\delta}),$$

for every $\delta > 0$.

Our second result concerns the average behavior of the error term which has the same structure as Cramér's theorem in the case of two-dimensional tori. This L^2-result can be considered as evidence for the pointwise conjecture (1.1).

Theorem 1.2.3 *Let $M = (H_1/\Gamma, g)$ be the 3-dimensional Heisenberg manifold where the metric g is in the form $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2\pi \end{pmatrix}$. Then, there exists a positive constant c such that*

$$\int_1^T |R(t)|^2 dt = cT^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}), \tag{1.4}$$

for every $\delta > 0$.

Remark 1 *Theorem 1.2.3 holds for all left-invariant Riemannian metrics on H_1/Γ. The proof is similar.*

For Heisenberg manifolds of dimension $n > 3$ we prove:

Theorem 1.2.4 *For the $(2n+1)$-dimensional Heisenberg manifold with metric $g = \begin{pmatrix} I_{2n\times 2n} & 0 \\ 0 & 2\pi \end{pmatrix}$, where $I_{2n\times 2n}$ is the identity matrix, one can similarly prove that there* *exists a positive constant c such that*

$$\int_1^T |R(t)|^2 dt = cT^{2n+\frac{1}{2}} + O_\delta(T^{2n+\frac{1}{4}+\delta}), \tag{1.5}$$

for every $\delta > 0$.

Remark 2 *The proof of Theorem 1.2.4 is very similar to the case* $n = 3$ *and we include it in section 4.3. We are currently unable to extend Theorem 1.2.4 to all left-invariant Riemannian metrics on* H_n/Γ*, but we hope to return to this question elsewhere.*

CHAPTER 2
Background on Heisenberg manifolds

2.1 Heisenberg canonical commutation relations

As the structure in Heisenberg algebras originates naturally from the canonical commutation relations in Hamiltonian and quantum mechanics, we devote this section to review these relations.

Hamiltonian mechanics: In classical or equivalently Hamiltonian mechanics, one is interested in investigating the motion of a particle moving in R^n. According to Newton's principle of determinacy, all motions of the system are uniquely determined by their initial positions and momentums(which is mass$\times$velocity). Therefore, by using the notation $p(t)$ and $q(t)$ in order for the momentum and position of the particle at time t, Newton's equation turns out as a system of first-order differential equations, more precisely a Hamiltonian equation on the phase space $\{(p,q); p,q \in R^n\}$. The physical observables are real-valued smooth functions on the phase space.

Concerning the Hamiltonian system correspondent to Newton's equation, the time evolutions of the particle are the symplectomorphisms or the canonical transformations which are the diffeomorphisms of R^{2n} preserving the canonical 2-form $\omega = dp \wedge dq$.

Shortly, in Hamiltonian mechanics we are working with smooth observables on a symplectic space (R^{2n}, ω). As a result of the non-degeneracy of ω, there is a one to one correspondence between 1-forms and vector fields and therefore to every observable

f we can correspond a Hamiltonian vector field X_f where $\omega(Y, X_f) = df(Y)$ for every vector field Y. In the other words, $X_f = \sum(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i})$.

On the space of smooth observables, we can define a Lie algebra structure obtained from the following Poisson bracket operation:

$$\{f, g\} = \omega(X_f, X_g) = \sum_i (\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}).$$

Specifically, for the coordinate functions from the basis $\{p_i, q_i; i = 1, 2, ..., n\}$ we have the following identities:

$$\{p_i, p_j\} = 0, \ \{q_i, q_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij}, \tag{2.1}$$

which are called Heisenberg canonical commutation relations in Hamiltonian mechanics.

Quantum mechanics: As we mentioned before, in classical mechanics, the future behavior of the particle depends only on its initial position and momentum. However, the physical experiments in most of the quantum systems show that this is no longer true. In quantum mechanics which has been shaped during the last century one works with the systems where instead of finding the precise position of the particle, one can only predict the probability of having the particle in a certain region of the state space. In other words, the state space of the system is no longer R^n but the Borel subsets of R^n and the observables are the expectations of seeing the particle in a specific Borel set, which gives us the probability distributions. In a precise mathematical formulation, one thinks of the observables as the self-adjoint operators from $L^2(R^n)$ to $L^2(R^n)$. In this setting, the coordinate functions q_j and p_j correspond to the operators:

$$X_j f(x) = Q_j f(x) = x_j f(x) \text{ and } \hbar D_j f(x) = P_j f(x) = \frac{\hbar}{2\pi i}\frac{\partial f}{\partial x_j}(x).$$

By formally defining the commutator of every two self-adjoint operators A and B as $[A, B] = AB - BA$, we obtain a Lie algebra structure on the set of the quantum observables. For operators P_j and Q_j we have:

$$[P_j, P_k] = 0,\ [Q_j, Q_k] = 0,\quad [P_j, Q_k] = \frac{\hbar\delta_{jk}}{2\pi i}I, \tag{2.2}$$

which are called the canonical quantum commutation relations.

2.2 Heisenberg group and algebra

As we saw in the previous section, the Heisenberg canonical commutation relations given by (2.1) and (2.2) are the same Lie algebra structures and introducing a third 1-variable coordinate t, we can rewrite them in one equation:

$$[(p, q, t), (p', q', t')] = (0, 0, p.q' - q.p') \tag{2.3}$$

where $p, q \in R^n$ and $t \in R$.

Therefore, taking R^{2n+1} with the natural vector-space structure and the Lie bracket given by (2.3), defines a Lie algebra called the Heisenberg algebra denoted by h_n. Now, letting $P_1, \ldots, P_n, Q_1, \ldots, Q_n, T$ to be the standard basis for R^{2n+1}, the Lie algebra structure of h_n is given by:

$$[P_j, P_k] = 0,\ [Q_j, Q_k] = 0,\ [P_j, T] = 0,\ [Q_j, T] = 0,\qquad [P_j, Q_k] = \delta_{jk}T. \tag{2.4}$$

Thus, according to (2.1) and (2.2), classical and quantum mechanics have Lie algebra structures isomorphic to Heisenberg algebra h_n. An equivalent way of defining h_n is through the matrix representations:

$$h_n = \{X(x, y, t) : x, y \in R^n, t \in R\},$$

where $X(x, y, t)$ is defined in (2.5). This gives a Lie subalgebra of $gl_{n+2}(R)$.

Notation: For a row vector x and a column vector y in R^n, let $X(x, y, t)$ and $\gamma(x, y, t)$ be the $(n+2) \times (n+2)$ matrices

$$X(x,y,t) = \begin{pmatrix} 0 & x & t \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, \quad \gamma(x,y,t) = \begin{pmatrix} 1 & x & t \\ 0 & I_n & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.5}$$

Let $z_n = \{X(0, 0, t), t \in R\}$, then z_n is both the center and the derived subalgebra of h_n. If we also identify the subspace $\{X(x, y, 0), x, y \in R^n\}$ of h_n with R^{2n}, then h_n is the direct sum of these subspaces: $h_n = R^{2n} \oplus z_n$.

Define $Z = X(0, 0, 1)$, then the standard basis of h_n is given by $\delta = \{X_1, X_2, ..., Y_1, ..., Y_n, Z\}$, where the first $2n$ elements are the standard basis of R^{2n}. The nonzero brackets among the elements of δ are thus given by $[X_i, Y_i] = Z$ for $1 \leq i \leq n$, which as we mentioned before are just the standard commutation relations in Hamiltonian and quantum mechanics.

As the matrices $X(x, y, t)$ are nilpotent of degree 3, we get:

$$\exp(X(x,y,t)) = I + X(x,y,t) + X^2(x,y,t) = \gamma(x,y,t+\frac{1}{2}x \cdot y) =: \gamma^*(x,y,t). \tag{2.6}$$

Taking the group law as:

$$\gamma^*(x,y,t)\cdot\gamma^*(x',y',t')=\gamma^*\left(X(x,y,t)+X(x',y',t')+\frac{1}{2}[X(x,y,t),X(x',y',t')]\right), \tag{2.7}$$

we get a Lie group H_n^*:

$$H_n^*=\{\gamma^*(x,y,t):x,y\in R^n,t\in R\},$$

which is called the $(2n+1)$-dimensional unpolarized Heisenberg group. One may instead of taking (2.7) as the group law, think of the natural matrix multiplication between matrices of the form $\gamma(x,y,t)$. This defines the Heisenberg group $H_n = \{\gamma(x,y,t);x,y\in R^n,t\in R\}$ as the Lie subgroup of $\mathrm{Gl}_{n+2}(R)$. The group law is given by:

$$\gamma(x,y,t)\cdot\gamma(x',y',t')=\gamma(x+x',y+y',t+t'+x\cdot y').$$

Now, the matrix exponential maps h_n diffeomorphically onto H_n and satisfies

$$\begin{cases}\exp: h_n\longmapsto H_n,\\ X(x,y,t)\longmapsto\gamma(x,y,t+(1/2)x\cdot y).\end{cases}$$

Automorphisms of the Heisenberg group: Denote by $Aut(H_n^*)$, $Aut(H_n)$ and $Aut(h_n)$, in order the automorphism groups of the Heisenberg group as a Lie group and the Heisenberg algebra as a Lie algebra. Since H_n^* and H_n are isomorphic Lie groups, their automorphism groups are trivially equal. Also since H_n is a simply connected Lie group, it has the same automorphisms as its Lie algebra. Therefore, $Aut(H_n)=Aut(h_n)$.

To identify $Aut(H_n^*)$, we define the principal automorphisms as in the following four types:

1. Symplectic maps: These are the automorphisms which act on the symplectic subspace R^{2n} of h_n:

$$Sp_{2n}(R) = \{S \in Gl_{2n}(R); SJS^t = J\}$$

for $J = \begin{pmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{pmatrix}$. For every $S \in Sp_{2n}(R)$, we define an automorphism of H_n^* by:

$$S(\gamma(x, y, t)) = \gamma(S(x, y), t).$$

2. Dilations: For every $r > 0$, define δ_r by:

$$\delta_r(\gamma(x, y, t)) = \gamma(rx, ry, r^2 t).$$

3. Inner automorphisms: These are the inner automorphisms of the group H_n^*, i.e. for every $\gamma(a, b, c)$ we get an automorphism by:

$$\gamma_{a,b,c}\gamma(x, y, t) = \gamma(a, b, c)\gamma(x, y, t)\gamma^{-1}(a, b, c) = \gamma(x, y, t + a \cdot y - b \cdot x).$$

4. Inversion and identity: They take every group element to its inverse or every element to itself.

Theorem 2.2.1 *Every automorphism of H_n^* has a unique expression as $\alpha_1\alpha_2\alpha_3\alpha_4$, where α_i belongs to the i-th type of automorphisms introduced above.*

2.3 Heisenberg manifolds

Definition 2.3.1 *A Riemannian Heisenberg manifold is a pair* $(H_n/\Gamma, g)$ *where* Γ *is a uniform discrete subgroup of* H_n*, i.e. the quotient* H_n/Γ *is compact, and* g *is a Riemannian metric on* H_n/Γ *whose lift to* H_n *is left* H_n*-invariant.*

Heisenberg manifolds are circle bundles over tori.

2.3.1 Classification of the uniform discrete subgroups of H_n

For every n-tuple $r = (r_1, r_2, ..., r_n) \in Z_+^n$ such that $r_i | r_{i+1}$ for every i, let rZ^n denote the n-tuples $x = (x_1, x_2, ..., x_n)$ where $x_i \in r_i Z$. Define

$$\Gamma_r = \{\gamma(x, y, t) : x \in rZ^n, y \in Z^n, t \in Z\}.$$

It is clear that Γ_r is a uniform discrete subgroup of H_n.

Theorem 2.3.2 *[GW] The subgroups* Γ_r *classify the uniform discrete subgroups of* H_n *up to automorphism. In other words for every uniform discrete subgroups of* H_n *there exists a unique n-tuple* r *and an automorphism of* H_n *which maps* Γ *to* Γ_r*. Also if two subgroups* Γ_r *and* Γ_s *are isomorphic then* r *and* s *are equal.*

Corollary 2.3.3 *[GW] Given any Riemannian Heisenberg manifold* $M = (H_n/\Gamma, g)$ *there exists a unique n-tuple* r *as before and a left-invariant metric* $\tilde{g}$ *on* H_n *such that* M *is isometric to* $(H_n/\Gamma_r, \tilde{g})$.

Since every left-invariant metric g on H_n is uniquely determined by an inner product on h_n, we can identify the left-invariant metrics with their matrices relative to the standard basis δ of h_n.

For any g we can choose an inner automorphism φ of H_n such that R^{2n} is orthogonal to z_n with respect to φ^*g. Therefore $(H_n/\Gamma, g)$ will be isometric to $(H_n/\Gamma, \varphi^*g)$

and we can replace every left-invariant metric g by $\varphi^* g$ and always assume that the metric g is in the following form

$$g = \begin{pmatrix} h & 0 \\ 0 & g_{2n+1} \end{pmatrix},$$

where h is a positive-definite $2n \times 2n$ matrix and g_{2n+1} is a positive real number.

The volume of the Heisenberg manifold is given by $\mathrm{vol}(H_n/\Gamma_r, g) = |\Gamma_r|\sqrt{\det(g)}$, where $|\Gamma_r| = r_1 \cdot r_2 \cdots r_n$ for $r = (r_1, r_2, ..., r_n)$.

2.3.2 The spectrum of Heisenberg manifolds.

Assume that $M = (H_n/\Gamma, g)$ is a Heisenberg manifold and $C^\infty(M)$ is the set of the smooth functions on M. We can view the functions on M as the left Γ-invariant functions on H_n. So, the Laplace-Beltrami operator on $C^\infty(M)$ is given by

$$\Delta f = -\sum_{i=1}^{2n+1} U_i^2 f,$$

where $U_1, U_2, ..., U_{2n+1}$ is any g-orthonormal basis of h_n.

The action of U_i is defined by

$$U_i f(\gamma) = \frac{d}{dt}_{|t=0} f(\gamma \exp(tU_i)) = (R_* U_i) f(\gamma),$$

where R is the quasi-regular representation of H_n on $L^2(H_n/\Gamma)$, that is $R(\gamma')f(\gamma) = f(\gamma\gamma')$. Thus the extension of Δ to an unbounded operator on $L^2(H_n/\Gamma)$ is defined as

$$\Delta f = -\sum_{i=1}^{2n+1} (R_* U_i)^2 f.$$

Notation: Let $r = (r_1, r_2, \ldots, r_n)$ such that for every i, $r_i | r_{i+1}$, then by δ_r we mean a matrix in the following form

$$\delta_r = \begin{pmatrix} r_{n\times n} & 0 \\ 0 & I \end{pmatrix},$$

where $r_{n\times n}$ is the diagonal matrix with the vector $r = (r_1, r_2, ..., r_n)$ as the main diagonal and I is the $n \times n$ identity matrix. Also as before without loss of generality we can assume that the metric g is of the form

$$g = \begin{pmatrix} h_{2n\times 2n} & 0 \\ 0 & g_{2n+1} \end{pmatrix}.$$

Define J_n to be the $2n \times 2n$ matrix

$$J_n = \begin{pmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{pmatrix}.$$

The matrix $h^{-1}J_n$ is similar to the skew-symmetric matrix $h^{-1/2}J_n h^{-1/2}$, so it has pure imaginary eigenvalues which we denote by $\pm\sqrt{-1}d_j^2; 1 \le j \le n$.

Denote the spectrum of $M = (H_n/\Gamma_r, g)$ by $\Sigma(r, g)$, that is the collection of all eigenvalues of the Laplacian, counting the multiplicities. Then $\Sigma(r, g) = \Sigma_1 \cup \Sigma_2$, where Σ_1 contains the eigenvalues of the first type corresponding to the $2n$-dimensional tori as a submanifold of M, and Σ_2 is the second part resulting from the non-commutative structure of the Heisenberg manifold.

More precisely, let $L_r = \{X(x, y, z), x_i \in r_i Z, y \in Z^n, z \in Z\}$ be a lattice in the Lie algebra h. Then $\Sigma_1(r, h)$ is the spectrum of the Laplace operator on the flat torus

$(R^{2n}/L_r, h)$, see [GW, p. 259]. To be more precise, for every two n-tuples $a, b \in Z^n$, define

$$\lambda(a,b) = 4\pi^2[a,b](\delta_r h \delta_r)^{-1}[a,b]^t,$$

where by $[a,b]$ we mean the vector concatenating a and b, and $[a,b]^t$ stands for its transpose. Then, we have

$$\Sigma_1(r,h) = \{\lambda(a,b) : (a,b) \in Z^{2n}\},$$

where $\lambda(a,b)$ is counted once for each pair $(a,b) \in Z^{2n}$ such that $\lambda = \lambda(a,b)$.

The second part of the spectrum, Σ_2 contains the eigenvalues of the form:

$$\mu(c,k) = 4\pi^2 c^2/g_{2n+1} + \sum_{i=1}^{n} 2\pi c d_i^2(2k_i + 1)$$

and

$$\Sigma_2(r,g) = \{\mu(c,k) : c \in Z_+, k \in (Z_+ \cup \{0\})^n\},$$

where every $\mu(c,k)$ is counted with the multiplicity $2c^n|\Gamma_r|$.

CHAPTER 3
Pointwise Estimates Of The Spectral Counting Function

3.1 Computation of the error term.

The spectral counting function corresponding to type II eigenvalues is defined by

$$N_{II}(t) = \#\{\mu(c,k);\ \mu(c,k) \leq t\}, \tag{3.1}$$

where every $\mu(c,k)$ on the right-hand side of (3.1) is counted $2c^n|\Gamma_r|$ times, for each pair (c,k) such that $\mu = \mu(c,k)$.

In the calculations for $N_{II}(t)$, without loss of generality, we assume that $r = (1,1,,...,1)$. In the general case, the only change is a coefficient $|\Gamma_r|$ in $2c^n|\Gamma_r|$, for the multiplicity of each $\mu(c,k)$, which also appears in the coefficients of vol(M) and vol$(R^{2n}/L_r, h) = |\Gamma_r|det(h)$. Therefore, we continue with the computation of $N_{II}(t)$ only for $r = (1,1,,...,1)$ and we count every $\mu(c,k)$ with multiplicity $2c^n$. We compute asymptotics with 2 terms in the expansions, since we need to see that the second term of order t^n cancels the contribution of the main term of type I (torus) eigenvalues. The calculations with 2 terms require the Euler summation formula [GK], which we quote in its only use in this paper:

$$\sum_{n\leq u} n^a = \frac{u^{a+1}}{a+1} - \psi(u)u^a + O(u^{a-1}). \tag{3.2}$$

Here $\psi(u)$ is the first Bernoulli function (row of teeth function) defined by $\psi(u) = u - [u] - 1/2$. Now $\mu(c,k) \leq t$ if and only if

$$c\left(c + \sum d_i^2 g_{2n+1} k_i/\pi + \sum d_i^2 g_{2n+1}/(2\pi)\right) \leq t g_{2n+1}/4\pi^2.$$

Let $b_i = d_i^2 g_{2n+1}/(2\pi)$. Then $\mu(c,k) \leq 4\pi^2 t/g_{2n+1}$ if and only if $c(c + \sum 2b_i k_i + \sum b_i) \leq t$. So

$$N_{II}(4\pi^2 t/g_{2n+1}) = \sum_{c(c+2\sum b_i k_i + \sum b_i) \leq t} 2c^n = 2\sum_{c \leq \sqrt{t}} c^n \sum_{\sum b_i k_i \leq \frac{t}{2c} - \frac{c}{2} - \frac{\sum b_i}{2}} 1.$$

Define

$$\alpha = \frac{t}{2c} - \frac{c}{2} - \frac{1}{2}\sum_{i=1}^{n} b_i, \quad \text{and} \quad s_i = \sum_{j=1}^{i} b_j k_j. \tag{3.3}$$

We adopt the following notation. When a sum is indexed by the variable k_i, this means that the range of k_i is $0 \leq k_i \leq (\alpha - s_{i-1})/b_i$. We have

$$N_2(t) = \frac{1}{2} N_{II}(4\pi^2 t/g_{2n+1}) = \sum_{0<c\leq\sqrt{t}} c^n \sum_{k_1} \sum_{k_2} \cdots \sum_{k_n} 1. \tag{3.4}$$

Evaluating the last sum on the right-hand side of (3.4), we get

$$\sum_{k_n} 1 = \frac{(\alpha - s_{n-2})}{b_n} - \frac{b_{n-1}k_{n-1}}{b_n} - \psi\left(\frac{\alpha - s_{n-1}}{b_n}\right) + \frac{1}{2}. \tag{3.5}$$

Continuing with the next summation in (3.4), we get:

$$\sum_{k_{n-1}} \sum_{k_n} 1 = \left(\frac{\alpha - s_{n-2}}{b_n} + \frac{1}{2}\right) \sum_{k_{n-1}} 1 - \sum_{k_{n-1}} \frac{b_{n-1}k_{n-1}}{b_n} - \sum_{k_{n-1}} \psi\left(\frac{\alpha - s_{n-1}}{b_n}\right).$$

Evaluating $\sum_{k_{n-1}} 1$ and $\sum_{k_{n-1}} k_{n-1}$, using Euler summation (3.2), we obtain:

$$\sum_{k_{n-1}} \sum_{k_n} 1 = \frac{(\alpha - s_{n-2})^2}{2b_n b_{n-1}} + \frac{1}{2}(\alpha - s_{n-2})\frac{(b_n + b_{n-1})}{b_n b_{n-1}} - \sum_{k_{n-1}} \psi\left(\frac{\alpha - s_{n-1}}{b_n}\right) + O(1).$$

By induction we get

$$\sum_{k_1,\dots,k_n} 1 = \frac{\alpha^n}{n! b_1 b_2 \dots b_n} + \frac{(b_1 + \dots + b_n)\alpha^{n-1}}{2(n-1)! b_1 b_2 \dots b_n} - \sum_{k_1,\dots,k_{n-1}} \psi\left(\frac{\alpha - s_{n-1}}{b_n}\right) + O(\alpha^{n-2}).$$

We set $\beta = \sum_i^n b_i$. Hence,

$$\begin{aligned} \sum_{\substack{0<c\leq\sqrt{t} \\ k_1,\dots,k_n}} c^n &= \sum_{0<c\leq\sqrt{t}} \frac{c^n\alpha^n}{n! b_1 b_2 \dots b_n} + \sum_{0<c\leq\sqrt{t}} \frac{\beta c^n \alpha^{n-1}}{2(n-1)! b_1 b_2 \dots b_n} \\ &\quad - \sum_{\substack{0<c\leq\sqrt{t} \\ k_1,\dots,k_n}} c^n \psi\left(\frac{\alpha - s_{n-1}}{b_n}\right) + \sum_{0<c\leq\sqrt{t}} c^n \cdot O(\alpha^{n-2}). \end{aligned} \tag{3.6}$$

For the first sum on the right-hand side of (4.39) we substitute $\alpha = t/(2c) - c/2 - \beta/2$, use the binomial theorem and (3.2), and obtain

$$\sum_{0<c\leq\sqrt{t}} c^n \alpha^n = \frac{t^{n+1/2}}{2^n}$$

Here we notice that the sums involving the binomial coefficients can actually be calculated explicitly. By plugging $x = 1$ into the expansion of $(1-x)^n$ we get that $1 - \binom{n}{1} + \binom{n}{2} - \dots = 0$, which shows that the term with the row-tooth function disappears. By integration over $[0,1]$ the expansion of $(1-x^2)^n$ we get

$$1 - \binom{n}{1}/3 + \binom{n}{2}/5 - \dots = \int_0^1 (1-x^2)^n \, dx = \int_0^{\pi/2} \sin^{2n+1} u \, du = \frac{(2n)!!}{(2n+1)!!},$$

see [GR, 3.621.4, p. 412]. This gives

$$\sum_{0<c\le\sqrt{t}} c^n\alpha^n = t^{n+\frac{1}{2}}\frac{2^n n! n!}{(2n+1)!} + \frac{t^n}{2^{n+1}}(-1-\beta) + O(t^{n-\frac{1}{2}}). \tag{3.7}$$

In the second summation in (4.39) we use $\frac{1}{n} = \int_0^1 (1-x)^{n-1}\,dx = \sum_{i=0}^{n-1} \frac{(-1)^i}{i+1}\binom{n-1}{i}$ to get

$$\sum_{0<c\le\sqrt{t}} c^n\alpha^{n-1} = \sum_{0<c\le\sqrt{t}} c^n\left(\left(\frac{t}{2c}\right)^{n-1} - \binom{n-1}{1}\left(\frac{t}{2c}\right)^{n-2}\left(\frac{c}{2}+\frac{\beta}{2}\right)+\cdots\right) = \frac{t^n}{n2^n}+O(t^{n-\frac{1}{2}}). \tag{3.8}$$

For the fourth summation in (4.39), using $\alpha \le t/(2c)$, we have

$$\sum_{0<c\le\sqrt{t}} c^n\alpha^{n-2} = O\left(\sum_{0<c\le\sqrt{t}} t^{n-2}c^2\right) = O(t^{n-\frac{1}{2}}). \tag{3.9}$$

Substituting the results from (3.7), (3.8) and (3.9), back into (4.39), we have,

$$N_2(t) = \frac{2^n n!}{(2n+1)!b_1b_2...b_n}t^{n+\frac{1}{2}} - \frac{t^n}{2^{n+1}n!b_1b_2...b_n} - \sum_{\substack{0<c\le\sqrt{t}\\ k_1,\ldots,k_{n-1}}} c^n\psi\left(\frac{\alpha - s_{n-1}}{b_n}\right) + O(t^{n-\frac{1}{2}}).$$

Since $N_{II}(t) = 2N_2(g_{2n+1}t/(4\pi^2))$ and $b_j = d_j^2 g_{2n+1}/(2\pi)$, we have proved that

$$N_{II}(t) = t^{n+1/2}\frac{\sqrt{g_{2n+1}}2^{n+1}n!}{(2\pi)^{n+1}(2n+1)!d_1^2d_2^2...d_n^2} - t^n\frac{1}{(2\pi)^n2^n n! d_1^2d_2^2...d_n^2} - R(t) + O(t^{n-\frac{1}{2}}),$$

where

$$R(t) = \sum_{0<c\le\sqrt{t}} c^n \sum_{k_1}\sum_{k_2}\cdots\sum_{k_{n-1}}\psi\left(\frac{\alpha - s_{n-1}}{b_n}\right). \tag{3.10}$$

On the other hand, we denote the spectral counting function, corresponding to type I eigenvalues, by $N_I(t)$. Since $N_I(t)$ represents the spectral counting function of the

$2n$-dimensional torus T equipped with the metric h, we have

$$N_I(t) = \frac{\pi^n}{n!}\sqrt{\det(h)}\frac{t^n}{(2\pi)^{2n}} + O(t^{n-\frac{1}{2}}) = \frac{1}{2^{2n}\pi^n n! d_1^2 d_2^2 \ldots d_n^2} t^n + O(t^{n-\frac{1}{2}}). \quad (3.11)$$

Therefore, if $N(t)$ denotes the spectral counting function of the Heisenberg manifold (M, g), then $N(t) = N_I(t) + N_{II}(t)$. From (3.10) and (3.11), we have

$$N(t) = t^{n+1/2}\frac{\sqrt{g_{2n+1}}2^{n+1}n!}{(2\pi)^{n+1}(2n+1)!d_1^2 d_2^2 \ldots d_n^2} - R(t) + O(t^{n-\frac{1}{2}}),$$

where $R(t)$ is defined by (3.10). Since $\mathrm{vol}(H_n/\Gamma) = \sqrt{\det(h)\cdot g_{2n+1}}$, we get the correct constant in the main term in Weyl's law for a $(2n+1)$-dimensional manifold.

3.2 Proof of theorem 1.2.1.

Suppose that b_{n-1}/b_n is a rational number, i.e. $b_{n-1}/b_n = p_{n-1}/q_{n-1}$ where p_{n-1} and q_{n-1} are two positive integers such that $(p_{n-1}, q_{n-1}) = 1$. Then, using the fact that $\psi(u)$ has period 1, we get

$$\begin{aligned} \sum_{k_{n-1}} \psi((\alpha - s_{n-1})/b_n) &= \sum_{k_{n-1}} \psi\left(\frac{\alpha - s_{n-2} - b_{n-1}k_{n-1}}{b_n}\right) \\ &= \sum_{j=0}^{q_{n-1}-1} \psi\left(\frac{\alpha - s_{n-2} - jb_{n-1}}{b_n}\right) \times \left(\left[\frac{\alpha - s_{n-2}}{q_{n-1}b_{n-1}}\right] + O(1)\right). \end{aligned}$$

We substitute back into (3.10). The $O(1)$ term contributes $O(t^{n-3/2})$ as it gives the sum in (4.39) with 2 variables less. We get

$$R(t) = \sum_{\substack{0<c\leq\sqrt{t} \\ k_1,\ldots,k_{n-2}}} \sum_{j=0}^{q_{n-1}-1} c^n \left(\frac{\alpha - s_{n-2}}{b_n}\right) \psi\left(\frac{\alpha - s_{n-2} - jb_{n-1}}{q_{n-1}b_{n-1}}\right) + O(t^{n-\frac{3}{2}}). \quad (3.12)$$

Without loss of generality, we continue with estimating the first summation on the right-hand side of (3.12) with $j = 0$:

$$\sum_{k_{n-2}} c^n (\alpha - s_{n-2})\, \psi(\frac{\alpha - s_{n-2}}{b_n}) = \sum_{k_{n-2}} c^n (\alpha - s_{n-3}) \psi(\frac{\alpha - s_{n-2}}{b_n}) - \sum_{k_{n-2}} c^n b_{n-2} k_{n-2} \psi(\frac{\alpha - s_{n-2}}{b_n}). \tag{3.13}$$

To evaluate the first term on the right-hand side of (3.13), we proceed as in (3.12). That is, since b_{n-2}/b_n is a rational number, we can write it as $b_{n-2}/b_n = p_{n-2}/q_{n-2}$, where p_{n-2} and q_{n-2} are two relatively prime, positive integers. So

$$\sum_{k_{n-2}} \psi\left(\frac{\alpha - s_{n-2}}{b_n}\right) = \sum_{j=0}^{q_{n-2}-1} \psi\left(\frac{\alpha - s_{n-3} - jb_{n-2}}{b_n}\right) \times \left(\left[\frac{\alpha - s_{n-3}}{q_{n-2} b_{n-2}}\right] + O(1)\right). \tag{3.14}$$

For the second term in (3.13) summation by parts gives

$$\begin{aligned} \sum_{k_{n-2}} k_{n-2} \psi\left(\frac{\alpha - s_{n-2}}{b_n}\right) &= \frac{\alpha - s_{n-3}}{b_{n-2}} \sum_{k_{n-2}} \psi\left(\frac{\alpha - s_{n-2}}{b_n}\right) \\ &\quad - \int_1^{\frac{\alpha - s_{n-3}}{b_{n-2}}} \Big(\sum_{0 \le k_{n-2} \le x} \psi(\frac{\alpha - s_{n-2}}{b_n}) \Big) dx. \end{aligned} \tag{3.15}$$

The first sum on the right-hand side of (3.15) has been evaluated in (3.14). The second term is equal to

$$\begin{aligned} &\int_1^{\frac{\alpha - s_{n-3}}{b_{n-2}}} \Big(\sum_{0 \le k_{n-2} \le x} \psi(\frac{\alpha - s_{n-3} - b_{n-2} k_{n-2}}{b_n}) \Big) dx \\ &= \int_1^{\frac{\alpha - s_{n-3}}{b_{n-2}}} \Big(\sum_{j=0}^{q_{n-2}-1} \psi(\frac{\alpha - s_{n-3} - jb_{n-2}}{b_n}) \Big) \times \Big([\frac{x}{q_{n-2}}] + O(1) \Big) dx \\ &= \Big(\sum_{j=1}^{q_{n-2}-1} \psi(\frac{\alpha - s_{n-3} - jb_{n-2}}{b_n}) \Big) \times \Big(\frac{1}{2q_{n-2}} \left(\frac{\alpha - s_{n-3}}{b_{n-2}}\right)^2 + O(\alpha) \Big). \end{aligned}$$

Taking the results from the last equation and (3.14), (3.15) back into (3.13), we have proved that

$$\sum_{\substack{0<c\leq\sqrt{t}\\k_1,\ldots,k_{n-2}}} c^n\left(\alpha-s_{n-2}\right)\psi\left(\frac{\alpha-s_{n-2}}{b_n}\right)=\sum_{\substack{0<c\leq\sqrt{t}\\k_1,\ldots,k_{n-3}}}\sum_{j=0}^{q_{n-2}-1} c^n\psi\left(\frac{\alpha-s_{n-3}-jb_{n-2}}{b_n}\right)O((\alpha-s_{n-3})^2).$$

We use the last result in (3.12) to get

$$R(t) = O\left(\sum_{0<c\leq\sqrt{t}} c^n\sum_{k_1}\sum_{k_2}\cdots\sum_{k_{n-3}}(\alpha-s_{n-3})^2\left(\psi(\frac{\alpha-s_{n-3}}{b_n})\right)\right).$$

Finally, by induction, after $n-1$ steps, and given $\alpha = t/(2c)-c/2-\beta/2$ we get:

$$R(t)=O\left(\sum_{0<c\leq\sqrt{t}} c^n\alpha^{n-1}\psi(\frac{\alpha}{b_n})\right)=O\left(\sum_{0<c\leq\sqrt{t}} c^n(\frac{t}{c}-c-\beta)^{n-1}\psi\left(\frac{t}{2cb_n}-\frac{c}{2b_n}-\frac{\beta}{2b_n}\right)\right)$$

If (k,l) is an exponent pair [GK], by [GK, Lemma 4.3, p. 39], if $f(x)$ satisfies the properties in the definition of exponent pairs, then

$$\sum_{m\in[a,b]}\psi(f(m))\ll t^{k/(k+1)}N^{((1-s)k+l)/(k+1)}+t^{-1}N^s. \tag{3.16}$$

We apply (3.16) to $f(x)=(tx^{-1}-x-\beta)/(2b_n)$. Using a dyadic decomposition we get

$$\sum_{m\in[2^{-j-1}u,2^{-j}u]}\psi(f(m))\ll t^{k/(k+1)}(2^{-j-1}u)^{(-k+l)/(k+1)}+t^{-1}(2^{-j-1}u)^2,$$

for $u\leq\sqrt{t}$. If $k<l$ the series $2^{-j(-k+l)/(k+1)}$ converges and we get the estimate

$$\sum_{m\leq u}\psi(f(m))\ll t^{k/(k+1)}u^{(l-k)/(k+1)}+t^{-1}u^2\ll t^{(k+l)/(2k+2)}.$$

This implies, using summation by parts, that

$$R(t) = O(t^{n-1/2+(k+l)/(2k+2)}). \tag{3.17}$$

The exponent pair $(11/30, 16/30)$, see [GK] gives the statement of theorem 1.2.1. The conjectural best exponent pairs $(\delta, 1/2+\delta)$ gives the conjecture 1.3.

3.3 Proof of Theorem 1.2.2

In theorem 1.2.2 we assume that at least one of the coefficients ${d_j}^2/{d_n}^2, 1 \le j \le n-1$, is irrational. Without loss of generality, we can assume that this happens for $j = n-1$. In fact, obtaining the formula (3.10) was based on an optional ordering for the summations over $k_1, k_2, \ldots, k_{n-1}$ in (3.4).

According to Vaaler's theorem [Va], see also [GK, p.116], for every positive integer J, there exist constants $\{\gamma_j; 1 \le |j| \le J\}$, satisfying the property $|\gamma_j| \ll 1/|j|$, such that for every real number ω

$$\psi(\omega) - \sum_{1 \le |j| \le J} \gamma_j e^{2\pi i(j\omega)} \ll \frac{1}{J}. \tag{3.18}$$

Therefore, by fixing J and taking $\omega = (\alpha - s_{n-1})/b_n$ in Vaaler's theorem, we have

$$\sum_{k_{n-1}} \psi\left(\frac{\alpha - s_{n-1}}{b_n}\right) \ll \sum_{1 \le |j| \le J} |\gamma_j| \left| \sum_{k_{n-1}} \exp(2\pi i j(\alpha - s_{n-1})/b_n)) \right| + \frac{\alpha - s_{n-2}}{J}. \tag{3.19}$$

To estimate the right-hand side of (3.19), we have

$$\left|\sum_{k_{n-1}} e^{2\pi i(j\frac{\alpha-s_{n-1}}{b_n})}\right| = \left|e^{2\pi i(j\frac{\alpha-s_{n-2}}{b_n})}\sum_{k_{n-1}} e^{-2\pi i(j\frac{b_{n-1}k_{n-1}}{b_n})}\right|$$

$$= \left|\frac{1-e^{-2\pi i(j\frac{b_{n-1}}{b_n})((\frac{\alpha-s_{n-2}}{b_{n-1}})+1)}}{1-e^{-2\pi i(j\frac{b_{n-1}}{b_n})}}\right| \leq \frac{1}{|\sin(\frac{\pi j b_{n-1}}{b_n})|} \leq \frac{1}{2\|\frac{jb_{n-1}}{b_n}\|}, \tag{3.20}$$

where, for every real number θ, $\|\theta\|$ is the distance between θ and the nearest integer.

By Diophantine approximation, the equation $\|j\theta\| < \frac{1}{j\log^2 j}$ has infinitely many integer solutions for almost no θ. In the other words, for almost all irrational θ there exists a constant K_θ such that, $\|j\theta\| \geq \frac{K_\theta}{j\log^2 j}$ for every positive integer j. Therefore, applying this approximation for the right-hand side of (3.20), we have

$$\left|\sum_{k_{n-1}} \exp(2\pi i j(\alpha - s_{n-1})/b_n)\right| \leq \frac{1}{2\|jb_{n-1}/b_n\|} \leq K|j|\log^2 j, \tag{3.21}$$

where K is a positive constant, dependent on b_{n-1}/b_n, and δ is an arbitrary positive real number.

Substituting (3.21) in (3.19), and noticing that $|\gamma_j| \ll 1/|j|$, we obtain

$$\sum_{k_{n-1}} \psi\left(\frac{\alpha - s_{n-1}}{b_n}\right) \ll J\log^2 J + \frac{\alpha - s_{n-2}}{J}. \tag{3.22}$$

Substituting (3.22) in (3.10), we have

$$R(t) \ll \sum_{0<c\leq\sqrt{t}} c^n \sum_{k_1}\sum_{k_2}\cdots\sum_{k_{n-2}}\left(J\log^2 J + \frac{\alpha - s_{n-2}}{J}\right). \tag{3.23}$$

For the last summation on the right-hand side of (3.23), we have

$$\sum_{k_{n-2}} \left(J\log^2 J + \frac{\alpha - s_{n-2}}{J} \right) = \sum_{k_{n-2}} \left(J\log^2 J + \frac{\alpha - s_{n-3} - b_{n-3}k_{n-3}}{J} \right)$$
$$\ll (\alpha - s_{n-3}) J\log^2 J + (\alpha - s_{n-3})^2 J^{-1}. \quad (3.24)$$

Therefore, by induction, we have

$$\sum_{k_1} \sum_{k_2} \cdots \sum_{k_{n-2}} \left(J\log^2 J + \frac{\alpha - s_{n-2}}{J} \right) \ll \alpha^{n-2} J\log^2 J + \alpha^{n-1} J^{-1}. \quad (3.25)$$

Substituting (3.25) in (3.23) and using (3.3), we see that

$$\begin{aligned} R(t) &\ll \sum_{0<c\leq\sqrt{t}} c^n \alpha^{n-2} J\log^2 J + \alpha^{n-1} J^{-1} \\ &\ll \sum_{0<c\leq\sqrt{t}} \left\{ c^n (\frac{t}{c})^{n-2} J\log^2 J + c^n (\frac{t}{c})^{n-1} \frac{1}{J} \right\} \\ &= t^{n-2} \sum_{0<c\leq\sqrt{t}} \left(c^2 J\log^2 J + tcJ^{-1} \right). \end{aligned} \quad (3.26)$$

Taking $J = c^\rho$ on the right-hand side of (3.26), we have

$$R(t) \ll t^{n-2} \sum_{0<c\leq\sqrt{t}} c^{2+\rho} \rho^2 \log^2 c + t^{n-1} \sum_{0<c\leq\sqrt{t}} c^{1-\rho} \ll \rho^2 t^{n+(-1+\rho)/2} \log^2 t + t^{n-\rho/2}. \quad (3.27)$$

So, to optimize the estimate on the right-hand side of (3.27), we choose $\rho = 1/2 - 2\log\log t/\log t$ and we are done: $R(t) \ll t^{n-1/4} \log t$.

CHAPTER 4
Cramér's Formula (L^2-Estimates)

4.1 Estimates for regularized spectral counting function

The idea of the proof of the theorem 1.2.3 is to use the Poisson summation formula to write the error term, corresponding to type II eigenvalues, in a form which can be estimated by the method of the stationary phase.

The spectral counting function is defined by

$$N(t) = N_T(t) + N_H(t), \tag{4.1}$$

where $N_T(t)$ is the spectral counting function of the torus, defined by:

$$N_T(t) = \#\{\lambda \in \Sigma_1; \lambda \leq t\},$$

and $N_H(t)$ is defined by

$$N_H(t) = \#\{\lambda \in \Sigma_2;\ \lambda \leq t\}.$$

The estimates for $N_T(t)$ are well-known. For example,

$$N_T(t) = \frac{t}{4\pi} + O(t^{\frac{1}{2}}), \tag{4.2}$$

will suffice for our purposes. This bound was known to Gauss. To evaluate $N_H(t)$, we write:

$$N_H(t) = \sum_{c(c+(2k+1))\leq t/2\pi} 2c. \tag{4.3}$$

To estimate (4.3) we split the sum, into two pieces: Define $A_t = \{(x,y); x > 0, y > 0, x(x+y) \leq t\}$ and $B_t = \{(x,y); x > 0, y > 0, x(x+2y) \leq t\}$. Then, we have

$$N_H(2\pi t) = N_A(2\pi t) - N_B(2\pi t), \tag{4.4}$$

where

$$N_A(2\pi t) = \sum_{(c,k)\in Z^2} (2c)\chi_{A_t}(c,k), \tag{4.5}$$

and

$$N_B(2\pi t) = \sum_{(c,k)\in Z^2} (2c)\chi_{B_t}(c,k). \tag{4.6}$$

In order to apply the Poisson summation formula for $N_A(2\pi t)$ and $N_B(2\pi t)$, we need to regularize the characteristic functions χ_{A_t} and χ_{B_t}. Take ρ to be a smooth symmetric positive function on R^2 with $\int_{R^2} \rho(x,y)dxdy = 1$ and $\text{supp}(\rho) \subseteq [-1,1]^2$. Let $\rho_\epsilon(x,y) = \epsilon^{-2}\rho(\frac{x}{\epsilon}, \frac{y}{\epsilon})$, where we make an explicit choice of $\epsilon > 0$ later on. Consider the mollified counting functions:

$$N_A^\epsilon(t) := \sum_{(c,k)\in Z^2} (2c)\chi_{A_t}(c,k) * \rho_\epsilon(c,k), \tag{4.7}$$

and

$$N_B^\epsilon(t) := \sum_{(c,k)\in Z^2} (2c)\chi_{B_t}(c,k) * \rho_\epsilon(c,k). \tag{4.8}$$

Lemma 4.1.1 *Let T be an arbitrarily large number and put $\epsilon = T^{-1}$. Then, for $1 < t < T$ and $C > 2$ we have,*

$$N_A^\epsilon(t - C) \leq N_A(2\pi t) \leq N_A^\epsilon(t + C),$$

and

$$N_B^\epsilon(t - C) \leq N_B(2\pi t) \leq N_B^\epsilon(t + C).$$

Proof. We prove the first series of inequalities in 4.1.1. The second series follows in the same way. Given $A_t = \{(x,y); x > 0, y > 0, x(x+y) \leq t\}$, let ∂A_t to be the hyperbola $x(x+y) = t$. If a point $X = (x,y) \in {Z_+}^2$ lies at a distance greater than $\sqrt{2}\epsilon$ from ∂A_t, then $\chi_{A_t} * \rho_\epsilon(X) = \chi_{A_t}(X)$. Therefore, by taking $\Omega_1 = \{(c,k) \in Z^2; \text{dist}((c,k), \partial A_{t+K\epsilon}) > \sqrt{2}\epsilon\}$, we have,

$$\begin{aligned} N_A^\epsilon(t + K\epsilon) &= \sum_{(c,k)\in Z^2} (2c)(\chi_{A_{t+K\epsilon}} * \rho_\epsilon)(c,k) \\ &= \sum_{(c,k)\in\Omega_1} (2c)\chi_{A_{t+K\epsilon}}(c,k) + \sum_{(c,k)\in Z^2\setminus\Omega_1} (2c)(\chi_{A_{t+K\epsilon}} * \rho_\epsilon)(c,k). \end{aligned}$$

On the other hand,

$$N_A(2\pi t) = \sum_{(c,k)\in Z^2} (2c)\chi_{A_t}(c,k).$$

So, to get $N_A^\epsilon(t+K\epsilon) \geq N_A(2\pi t)$, it suffices to choose ϵ and K so that $Z^2 \cap A_t \subseteq \Omega_1$. Since the closest point of $Z^2 \cap A_t$ to $\partial A_{t+K\epsilon}$ is $(1, [t-1])$, it suffices to require that:

$$\text{dist}((1,t), (\frac{-t+\sqrt{t^2+4t+4K\epsilon}}{2}, t)) > \sqrt{2}\epsilon. \tag{4.9}$$

Equation (4.9) is equivalent to $4K\epsilon > 4\epsilon^2 + 4 + 4\epsilon t + 8\epsilon$. So, it is enough to choose $K \geq 2T$ and $\epsilon \leq \frac{1}{T}$. The inequality $N_A^\epsilon(t-C) \leq N_A(2\pi t)$ can be proved in the same way and we are done.

Lemma 4.1.1 will help us to convert our results on $N_A^\epsilon(t)$ and $N_B^\epsilon(t)$ back to $N_H(t)$.

Remark 3

1. *Henceforth, we always assume $\epsilon = T^{-1}$ for a fixed large T and $t \in [1,T]$. Also we assume that δ is an arbitrary small positive number independent of T.*
2. *By the notation $f(x) \ll g(x)$, we mean that there exists a positive constant C such that $|f(x)| \leq C|g(x)|$ for every x.*

Proposition 4.1.2 *The following asymptotic expansion holds for N_A^ϵ:*

$$N_A^\epsilon(t) = \frac{4}{3}t^{\frac{3}{2}} - \frac{3}{2}t + R_A^\epsilon(t) + O(t^{\frac{1}{2}+\delta}), \tag{4.10}$$

where,

$$\begin{aligned} R_A^\epsilon(t) &= \frac{1}{\sqrt{2}\pi} \sum_{0<\nu\leq\mu} t^{\frac{3}{4}} \cos(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu) \\ &+ \frac{1}{\sqrt{2}\pi} \sum_{0<\nu<\mu} t^{\frac{3}{4}} \cos(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu). \end{aligned} \tag{4.11}$$

Proof. Applying the Poisson summation formula to $N_A^\epsilon(t)$ in (4.7) gives:

$$\begin{aligned} N_A^\epsilon(t) &= \sum_{\lambda,\nu} \widehat{2x\chi_A}(\lambda,\nu)\widehat{\rho_\epsilon}(\lambda,\nu) \\ &= \widehat{2x\chi_A}(0,0)\widehat{\rho_\epsilon}(0,0) + \sum_{\lambda\neq 0,\nu=0} \widehat{2x\chi_A}(\lambda,\nu)\widehat{\rho_\epsilon}(\lambda,\nu) \\ &+ \sum_{\lambda=0,\nu\neq 0} \widehat{2x\chi_A}(\lambda,\nu)\widehat{\rho_\epsilon}(\lambda,\nu) + \sum_{\lambda\neq 0,\nu\neq 0} \widehat{2x\chi_A}(\lambda,\nu)\widehat{\rho_\epsilon}(\lambda,\nu). \end{aligned} \tag{4.12}$$

We first estimate each term on the right-hand side of (4.12). For the first term, we get:

$$\widehat{2x\chi_A}(0,0) = \int\int_A 2xdydx = \int_0^{\sqrt{t}}\int_0^{\frac{t}{x}-x} 2xdydx = \frac{4}{3}t^{\frac{3}{2}}.$$

Since $\widehat{\rho_\epsilon}(0,0)=1$,

$$\widehat{2x\chi_A}(0,0).\widehat{\rho_\epsilon}(0,0) = \frac{4}{3}t^{\frac{3}{2}}. \tag{4.13}$$

To evaluate the second term in (4.12), we write

$$\begin{aligned} \widehat{2x\chi_A}(\lambda,0) &= \int\int_A 2xe^{2\pi i\lambda x}dydx = \int_0^{\sqrt{t}}\int_0^{\frac{t}{x}-x} 2xe^{2\pi i\lambda x}dydx \\ &= -\frac{t}{\pi i\lambda} + \frac{4}{(2\pi i\lambda)^3} + \left(\frac{4\sqrt{t}}{(2\pi i\lambda)^2} - \frac{4}{(2\pi i\lambda)^3}\right)e^{2\pi i\lambda\sqrt{t}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{\lambda\neq 0}\widehat{2x\chi_A}(\lambda,0).\widehat{\rho_\epsilon}(\lambda,0) &= \sum_{\lambda\neq 0}\left(\frac{-t}{\pi i\lambda} + \frac{4}{(2\pi i\lambda)^3} + \frac{8\sqrt{t}\pi i\lambda - 4}{(2\pi i\lambda)^3}e^{2\pi i\lambda\sqrt{t}}\right).\widehat{\rho_\epsilon}(\lambda,0) \\ &= \sum_{\lambda\neq 0}\frac{-t}{\pi i\lambda}\widehat{\rho_\epsilon}(\lambda,0) + O(\sqrt{t}). \end{aligned} \tag{4.14}$$

Without loss of generality, we can assume that $\rho(x,y) = \varrho(x)\varrho(y)$, where, $\varrho \in C_0^\infty((0,1))$ such that $\int \varrho(x)dx = 1$. Then,

$$\sum_{\lambda\neq 0} \frac{\widehat{\rho}(\epsilon\lambda, 0)}{\lambda} = \sum_{\lambda\neq 0} \frac{\widehat{\varrho}(\epsilon\lambda)}{\lambda} = \sum_{\lambda\neq 0} \frac{1}{\lambda}\int_R e^{2\pi i\epsilon\lambda x}\varrho(x)dx = \int_R \left(\sum_{\lambda\neq 0} \frac{e^{2\pi i\epsilon\lambda x}}{\lambda}\right)\varrho(x)dx,$$

where, for the last equality we have used the absolutely convergence of the summation $\sum_{\lambda\neq 0}\frac{1}{\lambda}\int_R e^{2\pi i\epsilon\lambda x}\varrho(x)dx$. This follows by noticing that $\frac{1}{\lambda}\int_R e^{2\pi i\epsilon\lambda x}\varrho(x)dx = -\frac{1}{2\pi i\epsilon\lambda^2}\int_R e^{2\pi i\epsilon\lambda x}\varrho'(x)dx$.

Using the formula $[z] - z + \frac{1}{2} = \sum_{n\neq 0}\frac{e^{2\pi inz}}{2\pi in}$, which holds for every $z \notin Z$, we get:

$$\sum_{\lambda\neq 0} \frac{\widehat{\rho}(\epsilon\lambda, 0)}{\lambda} = \int_R 2\pi i([\epsilon x] - \epsilon x + \frac{1}{2})\varrho(x)dx = \int_R 2\pi i(\frac{1}{2} - \epsilon x)\varrho(x)dx = \pi i + O(\epsilon) \tag{4.15}$$

since $[\epsilon x] = 0$, because $\varrho \in C_0^\infty((0,1))$.

Therefore, substituting (4.15) into (4.14) gives the following result for the second term on the right-hand side of (4.12):

$$\sum_{\lambda\neq 0} \widehat{2x\chi_A}(\lambda,0)\widehat{\rho}_\epsilon(\lambda,0) = -t + O(\epsilon t) + O(\sqrt{t}) = -t + O(\sqrt{t}), \tag{4.16}$$

since $\epsilon = T^{-1}$.

For the third term on the right-hand side of (4.12), we have:

$$\begin{aligned}
\widehat{2x\chi_A}(0,\nu) &= \int\int_A 2xe^{2\pi i\nu y}dydx \\
&= \int_0^{\sqrt{t}} \frac{2x}{2\pi i\nu}e^{2\pi i\nu(t/x-x)}dx - \int_0^{\sqrt{t}} \frac{2x}{2\pi i\nu}dx \\
&= \int_0^1 \frac{2tx}{2\pi i\nu}e^{2\pi i\sqrt{t}\nu(1/x-x)}dx - \int_0^1 \frac{2tx}{2\pi i\nu}dx.
\end{aligned} \tag{4.17}$$

We claim that the first integral on the right-hand side of (4.17) is $\ll \frac{\sqrt{t}}{\nu^2}$. To prove this, put $f(x) = \frac{1}{x} - x$. Since f has no critical point, we integrate by parts to get:

$$\int_0^1 xe^{2\pi i\sqrt{t}\nu f(x)}dx = \left[\frac{xe^{2\pi i\sqrt{t}\nu f(x)}}{\sqrt{t}\nu f'(x)}\right]_0^1 - \int_0^1 e^{2\pi i\sqrt{t}\nu f(x)}\frac{\sqrt{t}\nu f'(x) - x\sqrt{t}\nu f''(x)}{(\sqrt{t}\nu f'(x))^2}dx$$
$$\ll \frac{1}{\sqrt{t}\nu}.$$

Therefore,

$$\sum_{\nu\neq 0}\widehat{2x\chi_A}(0,\nu)\widehat{\rho}_\epsilon(0,\nu) = \sum_{\nu\neq 0} O\left(\frac{\sqrt{t}}{\nu^2}\right) - t\sum_{\nu\neq 0}\frac{\widehat{\rho}_\epsilon(0,\nu)}{2\pi i\nu} = O(\sqrt{t}) - \frac{t}{2} + O(\epsilon t)$$
$$= -\frac{t}{2} + O(\sqrt{t}), \tag{4.18}$$

since by symmetry $\sum_{\nu\neq 0}\frac{\widehat{\rho}_\epsilon(0,\nu)}{2\pi i\nu} = \sum_{\lambda\neq 0}\frac{\widehat{\rho}_\epsilon(\lambda,0)}{2\pi i\lambda} = \frac{1}{2} + O(\epsilon)$ (see (4.15)) and $\epsilon = T^{-1}$.

Finally, for the fourth term on the right-hand side of (4.12), we need the following proposition:

Proposition 4.1.3 *The sum*

$$\sum_{\lambda\neq 0}\sum_{\nu\neq 0}\widehat{2x\chi_A}(\lambda,\nu)\widehat{\rho}_\epsilon(\lambda,\nu) = R_A^\epsilon(t) + O(t^{\frac{1}{2}+\delta}), \tag{4.19}$$

where,

$$R_A^\epsilon(t) = \frac{1}{\sqrt{2}\pi}\sum_{0<\nu\leq\mu} t^{\frac{3}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu)$$
$$+ \frac{1}{\sqrt{2}\pi}\sum_{0<\nu<\mu} t^{\frac{3}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu). \tag{4.20}$$

Proof. See appendix A.

So, combining the results (4.13), (4.16), (4.18) and (4.19) for the four terms in (4.12) proves Proposition 4.1.2 and we are done.

Remark 4 *The following similar estimate holds for* N_B^ϵ*:*

$$N_B^\epsilon(t) = \frac{2}{3}t^{\frac{3}{2}} - t + R_B^\epsilon(t) + O(t^{\frac{1}{2}+\delta}), \tag{4.21}$$

where

$$\begin{aligned} R_B^\epsilon(t) &= \frac{1}{\pi}t^{\frac{3}{4}} \sum_{0<\nu\leq\mu} \cos(2\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon((\mu+\nu)/2,\nu) \\ &+ \frac{1}{\pi}t^{\frac{3}{4}} \sum_{0<\nu<\mu} \cos(2\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon((\mu+\nu)/2,\nu). \end{aligned} \tag{4.22}$$

4.2 Proof of theorem 1.2.3

Given the formulas for the regularized counting functions in Proposition 4.1.2 and Remark 4, we prove Theorem 1.2.3 in three steps: First, we find a new expression for R_A^ϵ in Proposition 4.2.1 so as to effectively estimate averages over short spectral intervals. Then, we evaluate L^2-estimates for $R_A^\epsilon(t)$ and $R_A^\epsilon(t+C) - R_B^\epsilon(t-C)$. Finally, using Lemma 4.1.1, we get rid of the mollifier ϵ and prove Theorem 1.2.3.

4.2.1 Step1: A new expression for R_A^ϵ:

Following an argument of Cramér [Cr], we claim:

Proposition 4.2.1 *One can rewrite* $R_A^\epsilon(t)$ *in the form:*

$$\begin{aligned} R_A^\epsilon(t) &= \frac{1}{2\sqrt{2}\pi^2} \sum_{0<\nu\leq\mu} \mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) \\ &+ \frac{1}{2\sqrt{2}\pi^2} \sum_{0<\nu<\mu} \mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) + O(t^{\frac{1}{2}+\delta}), \end{aligned} \tag{4.23}$$

where $\theta(f(x)) = f(x+1) - f(x)$.

Proof. Let $F_A^\epsilon(t)$ be the first summation on the right-hand side of (4.20), that is:

$$F_A^\epsilon(t) = \frac{1}{\sqrt{2}\pi}\sum_{0<\nu\leq\mu} t^{\frac{3}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu).$$

Then,

$$\begin{aligned}\int_0^t F_A^\epsilon(u)du &= \frac{t^{\frac{5}{4}}}{2\sqrt{2}\pi^2}\sum_{0<\nu\leq\mu}\mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\widehat{\rho}_\epsilon(\mu+\nu,\nu)\\ &+ \frac{5t^{\frac{3}{4}}}{16\sqrt{2}\pi^3}\sum_{0<\nu\leq\mu}\mu^{-\frac{9}{4}}\nu^{-\frac{5}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\widehat{\rho}_\epsilon(\mu+\nu,\nu) + O(t^{\frac{1}{4}}).\end{aligned}$$

Writing $\int_t^{t+1} F_A^\epsilon(u)du = \int_0^{t+1} F_A^\epsilon(u)du - \int_0^t F_A^\epsilon(u)du$, we get:

$$\int_t^{t+1} F_A^\epsilon(u)du = \frac{1}{2\sqrt{2}\pi^2}\sum_{0<\nu\leq\mu}\mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu)$$

$$+\frac{5}{16\sqrt{2}\pi^3}\sum_{0<\nu\leq\mu}\mu^{-\frac{9}{4}}\nu^{-\frac{5}{4}}\theta\left(t^{\frac{3}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) + O(t^{\frac{1}{4}}), \quad (4.24)$$

where $\theta(f(x)) = f(x+1) - f(x)$.

Next, we need:

Lemma 4.2.2 *The following holds,*

$$R_A^\epsilon(t) = \int_t^{t+1} R_A^\epsilon(u)du + O(t^{\frac{1}{2}+\delta}). \quad (4.25)$$

Proof. Write

$$\int_t^{t+1} R_A^\epsilon(u)du = R_A^\epsilon(t) + \int_t^{t+1}(R_A^\epsilon(u) - R_A^\epsilon(t))du. \quad (4.26)$$

For $t \leq u \leq t+1$,

$$R_A^\epsilon(u) - R_A^\epsilon(t) = N_A^\epsilon(u) - N_A^\epsilon(t) + O(\sqrt{t}).$$

So, using Lemma 4.1.1,

$$\begin{aligned} R_A^\epsilon(u) - R_A^\epsilon(t) &= O\left(N_A(2\pi(u+C)) - N_A\left(2\pi(t-C)\right)\right) + O(\sqrt{t}) \\ &= O(N_A(2\pi u) - N_A(2\pi t)) + O(\sqrt{t}). \end{aligned} \tag{4.27}$$

From the definition of $N_A(2\pi t)$ (see (4.5)),

$$N_A(2\pi u) - N_A(2\pi t) \leq \sum_{c|[t+1], c \leq \sqrt{[t+1]}} 2c = O_\delta(t^{\frac{1}{2}+\delta}), \tag{4.28}$$

for any $\delta > 0$. The lemma follows from (4.26), (4.27) and (4.28).

Thus, from (4.24) and Lemma 4.2.2, it follows that:

$$\begin{aligned} R_A^\epsilon(t) &= \frac{1}{2\sqrt{2}\pi^2} \sum_{0<\nu\leq\mu} \mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) \\ &+ \frac{5}{16\sqrt{2}\pi^3} \sum_{0<\nu\leq\mu} \mu^{-\frac{9}{4}}\nu^{-\frac{5}{4}}\theta\left(t^{\frac{3}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) \\ &+ \frac{1}{2\sqrt{2}\pi^2} \sum_{0<\nu<\mu} \mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) \\ &+ \frac{5}{16\sqrt{2}\pi^3} \sum_{0<\nu<\mu} \mu^{-\frac{9}{4}}\nu^{-\frac{5}{4}}\theta\left(t^{\frac{3}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) + O(t^{\frac{1}{2}+\delta}). \end{aligned} \tag{4.29}$$

We claim that the second and the fourth terms on the right-hand side of (4.29) are $O(t^{\frac{1}{4}})$. Indeed, to bound the second sum on the right-hand side of (4.29), use that

$\theta(f(t)) = \int_t^{t+1} f'(u)du$ to get

$$\sum_{0<\nu\leq\mu} \mu^{-\frac{9}{4}}\nu^{-\frac{5}{4}}\theta\left(t^{\frac{3}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) \ll \sum_{0<\nu\leq\mu} \mu^{-\frac{9}{4}}\nu^{-\frac{5}{4}}t^{\frac{1}{4}}\sqrt{\mu\nu} = O(t^{\frac{1}{4}}). \tag{4.30}$$

The estimate for the fourth sum on the right-hand side of (4.29) is the same as in (4.30).

Consequently, from (4.29) and (4.30), Proposition 4.2.1 follows.

4.2.2 Step2: L^2-estimate for R_A^ϵ:

We now show that for any $\delta > 0$,

$$\int_1^T |R_A^\epsilon(t)|^2 dt = c_1 T^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}),$$

where c_1 is a positive constant.

For simplicity, we do the computations for $E_A^\epsilon(t)$, which is the first summation on the right-hand side of (4.23) in Proposition 4.2.1; that is,

$$E_A^\epsilon(t) = \frac{1}{2\sqrt{2}\pi^2}\sum_{0<\nu\leq\mu} \mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu).$$

Then,

$$\int_1^T |E_A^\epsilon(t)|^2 dt = \frac{1}{8\pi^4}\sum_{\substack{0<\nu\leq\mu,\\ 0<\nu'\leq\mu'}} \int_1^T \theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu'\nu'}-\frac{\pi}{4})\right)dt$$
$$\times\mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\mu'^{-\frac{7}{4}}\nu'^{-\frac{3}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu)\overline{\widehat{\rho}_\epsilon(\mu'+\nu',\nu')}. \tag{4.31}$$

Let $n=\mu\nu$, $m=\mu'\nu'$, $\theta_n(t)=\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{n}-\frac{\pi}{4})\right)$ and $\theta_m(t)=\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{m}-\frac{\pi}{4})\right)$. It follows that,

$$\begin{aligned}|\int_1^T \theta_n(t)\theta_m(t)dt| &\ll |\int_1^T \theta\left(t^{\frac{5}{4}}e^{4\pi i\sqrt{t}\sqrt{n}}\right)\theta\left(t^{\frac{5}{4}}e^{4\pi i\sqrt{t}\sqrt{m}}\right)dt| \\ &+ |\int_1^T \theta\left(t^{\frac{5}{4}}e^{-4\pi i\sqrt{t}\sqrt{n}}\right)\theta\left(t^{\frac{5}{4}}e^{4\pi i\sqrt{t}\sqrt{m}}\right)dt|. \qquad (4.32)\end{aligned}$$

For $m>n$, both integrals on the right-hand side of (4.32) are bounded by:

$$\left|\int_1^T G(t)\ d\left(\frac{e^{4\pi i\sqrt{t}(\sqrt{m}-\sqrt{n})}}{\sqrt{m}-\sqrt{n}}\right)\right| < \frac{|G(T)+|G(1)|+\int_1^T|G'(t)|dt}{\sqrt{m}-\sqrt{n}}, \qquad (4.33)$$

where $G(t)=\frac{t^3}{2\pi i}\left((1+\frac{1}{t})^{\frac{5}{4}}e^{4\pi i\sqrt{m}(\sqrt{t+1}-\sqrt{t})}-1\right)\left((1+\frac{1}{t})^{\frac{5}{4}}e^{-4\pi i\sqrt{n}(\sqrt{t+1}-\sqrt{t})}-1\right)$.

By Taylor expansion, one can show that $G(t)\ll\min\{t^3,t^2\sqrt{mn}\}$ and $G'(t)\ll\min\{t^2+t^{\frac{3}{2}}m^{\frac{1}{2}},t\sqrt{mn}\}$. So,

$$|\int_1^T \theta_n(t)\theta_m(t)dt| \ll \frac{\min\{T^3+T^{\frac{5}{2}}m^{\frac{1}{2}},T^2m^{\frac{1}{2}}n^{\frac{1}{2}}\}}{\sqrt{m}-\sqrt{n}}. \qquad (4.34)$$

Next, we recall that:

$$\begin{aligned}&\sum_{\substack{0<\nu\leq\mu,\\0<\nu'\leq\mu',\\\mu'\nu'\neq\mu\nu}}\left(\int_1^T\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu'\nu'}-\frac{\pi}{4})\right)dt\right)\mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\mu'^{-\frac{7}{4}}\nu'^{-\frac{3}{4}} \\ &= 2\sum_{m>0}\sum_{\substack{\mu'|m,\\\mu'\geq\sqrt{m}}}\sum_{0<n<m}\sum_{\substack{\mu|n,\\\mu\geq\sqrt{n}}}\left(\int_1^T\theta_n(t)\theta_m(t)dt\right)n^{-\frac{3}{4}}\mu^{-1}m^{-\frac{3}{4}}\mu'^{-1} \\ &\ll \sum_{0<m}\sum_{0<n<m}\left(\int_1^T\theta_n(t)\theta_m(t)dt\right)n^{-\frac{5}{4}+\delta}m^{-\frac{5}{4}+\delta}. \qquad (4.35)\end{aligned}$$

Therefore, substituting the estimate (4.34) in (4.35), gives:

$$\sum_{\substack{0<\nu\le\mu,\\ 0<\nu'\le\mu',\\ \mu'\nu'\neq\mu\nu}} \left(\int_1^T \theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu'\nu'}-\frac{\pi}{4})\right)dt\right)\mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\mu'^{-\frac{7}{4}}\nu'^{-\frac{3}{4}}$$

$$\begin{aligned}
&\ll \sum_{0<m}\sum_{0<n<m}\left(\frac{\min\{T^3+T^{\frac{5}{2}}m^{\frac{1}{2}},T^2m^{\frac{1}{2}}n^{\frac{1}{2}}\}}{\sqrt{m}-\sqrt{n}}\right)n^{-\frac{5}{4}+\delta}m^{-\frac{5}{4}+\delta}\\
&\ll \sum_{\substack{0<m\le T,\\ 0<n<m}}\left(\frac{T^2m^{\frac{1}{2}}n^{\frac{1}{2}}}{\sqrt{m}-\sqrt{n}}\right)n^{-\frac{5}{4}+\delta}m^{-\frac{5}{4}+\delta}+\sum_{\substack{m>T\\ 0<n<m}}\left(\frac{T^{\frac{5}{2}}m^{\frac{1}{2}}+T^3}{\sqrt{m}-\sqrt{n}}\right)n^{-\frac{5}{4}+\delta}m^{-\frac{5}{4}+\delta}.\\
&= O(T^{2+3\delta})+O(T^{\frac{9}{4}+\delta})=O(T^{\frac{9}{4}+\delta}).
\end{aligned}\tag{4.36}$$

Thus, we are left with the case where $m=n$, that is $\mu\nu=\mu'\nu'$. This diagonal case will give the leading term in (4.31). We have,

$$(\theta_n(t))^2=\frac{i}{4}\theta^2\left(t^{\frac{5}{4}}e^{4\pi i\sqrt{t}\sqrt{n}}\right)+\frac{i}{4}\theta^2\left(t^{\frac{5}{4}}e^{-4\pi i\sqrt{t}\sqrt{n}}\right)+\frac{1}{2}\theta\left(t^{\frac{5}{4}}e^{4\pi i\sqrt{t}\sqrt{n}}\right)\theta\left(t^{\frac{5}{4}}e^{-4\pi i\sqrt{t}\sqrt{n}}\right).\tag{4.37}$$

The same argument used to prove (4.34) shows that:

$$\sum_{n>0}\int_1^T\theta^2\left(t^{\frac{5}{4}}e^{4\pi i\sqrt{t}\sqrt{n}}\right)\times n^{-\frac{5}{2}+\delta}\ll\sum_{n>0}\frac{T^2n^{\frac{1}{2}}n^{\frac{1}{2}}}{\sqrt{n}+\sqrt{n}}\times n^{-\frac{5}{2}+\delta}=O(T^2),$$

and the same estimate holds for $\theta^2\left(t^{\frac{5}{4}}e^{-4\pi i\sqrt{t}\sqrt{n}}\right)$.
So, we just continue with $\frac{1}{2}\theta\left(t^{\frac{5}{4}}e^{4\pi i\sqrt{t}\sqrt{n}}\right)\theta\left(t^{\frac{5}{4}}e^{-4\pi i\sqrt{t}\sqrt{n}}\right)$.

Now, for $n < T$, using the fact that $\theta(f(t)) = f(t+1) - f(t) = f'(t) + \int_t^{t+1} du \int_t^u f''(s)ds$, we have:

$$\begin{aligned}
&\theta\left(t^{\frac{5}{4}} e^{4\pi i\sqrt{t}\sqrt{n}}\right)\theta\left(t^{\frac{5}{4}} e^{-4\pi i\sqrt{t}\sqrt{n}}\right) \\
=\ & \left(2\pi i\sqrt{n}t^{\frac{3}{4}} e^{4\pi i\sqrt{t}\sqrt{n}} + O(nt^{\frac{1}{4}})\right)\left(-2\pi i\sqrt{n}t^{\frac{3}{4}} e^{-4\pi i\sqrt{t}\sqrt{n}} + O(nt^{\frac{1}{4}})\right) \\
=\ & 4\pi^2 n t^{\frac{3}{2}} + O(n^{\frac{3}{2}} t + n^2 t^{\frac{1}{2}}).
\end{aligned}$$

On the other hand, for $n \geq T$,

$$\theta\left(t^{\frac{5}{4}} e^{4\pi i\sqrt{t}\sqrt{n}}\right)\theta\left(t^{\frac{5}{4}} e^{-4\pi i\sqrt{t}\sqrt{n}}\right) = O(t^{\frac{5}{2}}).$$

Therefore,

$$\begin{aligned}
&\sum_{0<\nu\leq\mu}\ \sum_{\substack{0<\nu'\leq\mu',\\ \mu'\nu'=\mu\nu}} \left(\int_1^T \theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\right) \cdot \theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu'\nu'} - \frac{\pi}{4})\right) dt\right) \\
&\qquad \times\ \mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\mu'^{-\frac{7}{4}}\nu'^{-\frac{3}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu)\overline{\widehat{\rho}_\epsilon(\mu'+\nu',\nu')} \\
&= \sum_{0<n<T}\ \sum_{\mu|n,\mu\geq\sqrt{n}}\ \sum_{\mu'|n,\mu'\geq\sqrt{n}} \left(\frac{8\pi^2}{5} n T^{\frac{5}{2}} + O(n^{\frac{3}{2}}T^2 + n^2 T^{\frac{3}{2}})\right) n^{\frac{-3}{2}}\mu^{-1}\mu'^{-1} \\
&\times\ \widehat{\rho}_\epsilon(\mu+\frac{n}{\mu},\frac{n}{\mu})\overline{\widehat{\rho}_\epsilon(\mu'+\frac{n}{\mu'},\frac{n}{\mu'})} + \sum_{n\geq T}\ \sum_{\substack{\mu|n,\mu\geq\sqrt{n},\\ \mu'|n,\mu'\geq\sqrt{n}}} O(T^{\frac{5}{2}})n^{\frac{-3}{2}}\mu^{-1}\mu'^{-1} + O(T^2) \\
&= T^{\frac{5}{2}} \sum_{0<n<T}\ \sum_{\substack{\mu|n,\mu\geq\sqrt{n},\\ \mu'|n,\mu'\geq\sqrt{n}}} \frac{8\pi^2}{5} n^{\frac{-1}{2}}\mu^{-1}\mu'^{-1}\widehat{\rho}_\epsilon(\mu+\frac{n}{\mu},\frac{n}{\mu})\overline{\widehat{\rho}_\epsilon(\mu'+\frac{n}{\mu'},\frac{n}{\mu'})} + O(T^{2+\delta})
\end{aligned} \tag{4.38}$$

We split the sum in (4.38) into the pieces where $\mu \le T^{1/4}$ and $\mu > T^{1/4}$. We claim that the piece where $\mu > T^{1/4}$ is residual. To see this, note that:

$$T^{\frac{5}{2}} \sum_{\substack{0<\nu\le\mu,\\ \mu>T^{1/4}}} \sum_{\substack{0<\nu'\le\mu',\\ \mu'\nu'=\mu\nu}} \mu^{-1}\mu'^{-1}(\mu\nu)^{-\frac{1}{2}} = T^{\frac{5}{2}} \sum_{\mu>T^{1/4}} \mu^{-\frac{3}{2}} \sum_{0<\nu\le\mu} \nu^{-\frac{1}{2}} \sum_{\mu'|\mu\nu,\mu'\ge\sqrt{\mu\nu}} \mu'^{-1}$$

$$\begin{aligned} &\le T^{\frac{5}{2}} \sum_{\mu>T^{1/4}} \mu^{-\frac{3}{2}} \sum_{0<\nu\le\mu} \nu^{-\frac{1}{2}} \frac{1}{\sqrt{\mu\nu}} d(\mu\nu) = T^{\frac{5}{2}} \sum_{\mu>T^{1/4}} \mu^{-2} d(\mu) \sum_{0<\nu\le\mu} \nu^{-1} d(\nu) \\ &= T^{\frac{5}{2}} \sum_{\mu>T^{1/4}} \mu^{-2} d(\mu) \log^2(\mu) = O_\delta(T^{\frac{9}{4}+\delta}). \end{aligned} \tag{4.39}$$

So, if $\mu \le T^{1/4}$ then $\mu' \le \mu'\nu' = \mu\nu \le \mu^2 \le T^{\frac{1}{2}}$. Since $\epsilon = T^{-1}$, we have $\epsilon\nu \le \epsilon\mu \le T^{-\frac{3}{4}}$ and $\epsilon\nu' \le \epsilon\mu' \le T^{-\frac{1}{2}}$. Therefore, by Taylor expanding the functions $\widehat{\rho}(\epsilon\mu + \epsilon\nu, \epsilon\nu)$ and $\widehat{\rho}(\epsilon\mu' + \epsilon\nu', \epsilon\nu')$ around the point $(0,0)$ and using (4.39), we can evaluate the summation in (4.38) as follows:

$$\begin{aligned} &T^{\frac{5}{2}} \sum_{0<\nu\le\mu} \sum_{\substack{0<\nu'\le\mu',\\ \mu'\nu'=\mu\nu}} \mu^{-1}\mu'^{-1}(\mu\nu)^{-\frac{1}{2}} \widehat{\rho}_\epsilon(\mu+\nu,\nu)\overline{\widehat{\rho}_\epsilon(\mu'+\nu',\nu')} \\ &= T^{\frac{5}{2}} \sum_{0<\nu\le\mu} \sum_{\substack{0<\nu'\le\mu',\\ \mu'\nu'=\mu\nu}} \mu^{-1}\mu'^{-1}(\mu\nu)^{-\frac{1}{2}} + O_\delta(T^{\frac{9}{4}+\delta}). \end{aligned} \tag{4.40}$$

Therefore, substituting (4.40) in (4.38), we get:

$$\begin{aligned} \sum_{0<\nu\le\mu} \sum_{\substack{0<\nu'\le\mu',\\ \mu'\nu'=\mu\nu}} &\left(\int_1^T \theta\left(t^{\frac{5}{4}} \sin(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\right) \cdot \theta\left(t^{\frac{5}{4}} \sin(4\pi\sqrt{t}\sqrt{\mu'\nu'} - \frac{\pi}{4})\right) dt \right) \\ &\times \mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\mu'^{-\frac{7}{4}}\nu'^{-\frac{3}{4}} \widehat{\rho}_\epsilon(\mu+\nu,\nu)\overline{\widehat{\rho}_\epsilon(\mu'+\nu',\nu')} \end{aligned}$$

$$= T^{\frac{5}{2}} \sum_{0<n<T} \sum_{\mu|n,\mu\ge\sqrt{n}} \sum_{\mu'|n,\mu'\ge\sqrt{n}} \frac{8\pi^2}{5} n^{-\frac{1}{2}}\mu^{-1}\mu'^{-1} + O_\delta(T^{\frac{9}{4}+\delta}). \tag{4.41}$$

Finally, combining the results from (4.36) and (4.41), gives:

$$\int_1^T |E_A^\epsilon(t)|^2 dt = c_{11} T^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}),$$

where,

$$c_{11} := \frac{1}{10\pi^2} \sum_{\mu=1}^{\infty} \mu^{-\frac{3}{2}} \sum_{0<\nu\leq\mu} \nu^{-\frac{1}{2}} \sum_{\mu'|\mu\nu, \mu'\geq\sqrt{\mu\nu}} \mu'^{-1} = \frac{1}{10\pi^2} \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \sum_{\mu|n, \mu\geq\sqrt{n}} \mu^{-1} \sum_{\mu'|n, \mu'\geq\sqrt{n}} \mu'^{-1}.$$

The argument for $R_A^\epsilon(t)$ follows in the same way and one gets:

$$\int_1^T |R_A^\epsilon(t)|^2 dt = c_1 T^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}), \tag{4.42}$$

where,

$$\begin{aligned} c_1 &= \frac{1}{10\pi^2} \Big(\sum_{n=1}^{\infty} n^{-\frac{1}{2}} \sum_{\mu|n, \mu\geq\sqrt{n}} \mu^{-1} \sum_{\mu'|n, \mu'\geq\sqrt{n}} \mu'^{-1} + \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \sum_{\mu|n, \mu>\sqrt{n}} \mu^{-1} \sum_{\mu'|n, \mu'>\sqrt{n}} \mu'^{-1} \\ &+ 2 \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \sum_{\mu|n, \mu\geq\sqrt{n}} \mu^{-1} \sum_{\mu'|n, \mu'>\sqrt{n}} \mu'^{-1} \Big) \end{aligned}$$

Remark 5 *The argument for $R_B^\epsilon(t)$ is the same as for $R_A^\epsilon(t)$. The result is that:*

1.

$$\int_1^T |R_B^\epsilon(t)|^2 dt = c_2 T^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}), \tag{4.43}$$

where $c_2 = 2c_1$.

2.

$$\int_1^T |R_A^\epsilon(t+C) - R_B^\epsilon(t-C)|^2 dt = cT^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}), \tag{4.44}$$

for $c = c_1 + c_2 - 2c_3$, where c_3 is a positive constant defined by,

$$\begin{aligned} c_3 &= \frac{1}{10\pi^2}\Big(\sum_{n=1}^{\infty} n^{-\frac{1}{2}} \sum_{\mu|n,\mu\geq\sqrt{n}} \mu^{-1} \sum_{\mu'|4n,\mu'\geq 2\sqrt{n}} \mu'^{-1} + \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \sum_{\mu|n,\mu>\sqrt{n}} \mu^{-1} \sum_{\mu'|4n,\mu'>2\sqrt{n}} \mu'^{-1} \\ &+ \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \sum_{\mu|n,\mu\geq\sqrt{n}} \mu^{-1} \sum_{\mu'|4n,\mu'>2\sqrt{n}} \mu'^{-1} + \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \sum_{\mu|n,\mu>\sqrt{n}} \mu^{-1} \sum_{\mu'|4n,\mu'\geq 2\sqrt{n}} \mu'^{-1}\Big). \end{aligned}$$

Also the same result is true for $\int_1^T |R_A^\epsilon(t-C) - R_B^\epsilon(t+C)|^2 dt$.

Remark 6 *One can rewrite c as the following:*

$$\begin{aligned} c &= \frac{1}{5\pi^2}\sum_{n=1}^{\infty} n^{-\frac{5}{4}}\delta(n)\left(6\delta(n) - \delta(4n)\right) + \frac{2}{5\pi^2}\sum_{n=1}^{\infty} n^{-4}\delta(n^2) \\ &+ \frac{1}{10\pi^2}\sum_{n=1}^{\infty} n^{-4}\left(6\delta(n^2) - \delta(4n^2)\right) + \frac{1}{4\pi^2}\sum_{n=1}^{\infty} n^{-3}, \end{aligned}$$

where $\delta(n) = \sum_{d|n, d<\sqrt{n}} d$.

4.2.3 Step 3: Eliminating the mollification:

The last step in the proof of the Theorem 1.2.3 is to use Lemma 4.1.1 to get rid of the mollification in ϵ and prove the L^2-estimate for $R_H(t)$, which is the error term corresponding to type II eigenvalues. From Lemma 4.1.1, by choosing $\epsilon = T^{-1}$ and $t \in [1, T]$, we get,

$$(N_A^\epsilon(t-C) - N_B^\epsilon(t+C))^2 \leq (N_2(2\pi t))^2 \leq (N_A^\epsilon(t+C) - N_B^\epsilon(t-C))^2. \quad (4.45)$$

For simplicity we do the calculations for the second inequality in (4.45), the other should be proceeded similarly. Taking L^2-norms in (4.45) gives:

$$\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2} + R_H(t))^2 dt \leq \int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2} + O(t^{\frac{1}{2}+\delta}) + R_A^\epsilon(t+C) - R_B^\epsilon(t-C))^2 dt. \quad (4.46)$$

Thus, by expanding both sides in (4.46) we get:

$$\begin{aligned}
&\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2})^2 dt + \int_1^T (R_H(t))^2 dt + 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2}) R_H(t) dt \leq \\
&\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2})^2 dt + \int_1^T (R_A^\epsilon(t+C) - R_B^\epsilon(t-C) + O(t^{\frac{1}{2}+\delta}))^2 dt \\
+ \; & 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2} + O(t^{\frac{1}{2}+\delta}))(R_A^\epsilon(t+C) - R_B^\epsilon(t-C)) dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_1^T (R_H(t))^2 dt + 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2}) R_H(t) dt \;&\leq\; \int_1^T (R_A^\epsilon(t+C) - R_B^\epsilon(t-C))^2 dt \\
&+\; 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2} + O(t^{\frac{1}{2}+\delta}))(R_A^\epsilon(t+C) - R_B^\epsilon(t-C)) dt.
\end{aligned}$$

We claim that

$$\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2} + O(t^{\frac{1}{2}+\delta}))(R_A^\epsilon(t+C) - R_B^\epsilon(t-C)) dt = O(T^{\frac{9}{4}}) \qquad (4.47)$$

To see this, note that

$$\begin{aligned}
\int_1^T t^{\frac{3}{2}} E_A^\epsilon(t) dt \;&=\; \frac{1}{2\pi} \sum_{0<\nu\leq\mu\leq T^{1+\alpha}} (\int_1^T t^{\frac{9}{4}} e^{2\pi i\sqrt{t}\sqrt{\mu\nu}} dt) \mu^{-\frac{5}{4}} \nu^{-\frac{1}{4}} \widehat{\rho}_\epsilon(\mu+\nu, \nu) + O(T^{-\infty}) \\
&\ll\; \sum_{0<\nu\leq\mu\leq T^{1+\alpha}} T^{\frac{11}{4}} \mu^{-7/4} \nu^{-\frac{3}{4}} = O(T^{\frac{9}{4}}).
\end{aligned}$$

Similarly, we have $\int_1^T t^{\frac{3}{2}} R_A^\epsilon(t) dt = O(T^{\frac{9}{4}})$ and $\int_1^T t^{\frac{3}{2}} R_B^\epsilon(t) dt = O(T^{\frac{9}{4}})$, which proves our claim in (4.47).

Hence,

$$\int_1^T (R_H(t))^2 dt + 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2}) R_H(t) dt \leq \int_1^T (R_A^\epsilon(t+C) - R_B^\epsilon(t-C))^2 dt + O(T^{\frac{9}{4}}),$$

which implies that

$$\int_1^T (R_H(t))^2 dt + 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2})R_H(t)dt \le cT^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}).$$

On the other hand, from the leftmost inequality in (4.45), we also have

$$\int_1^T (R_H(t))^2 dt + 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2})R_H(t)dt \ge cT^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}).$$

Hence,

$$\int_1^T (R_H(t))^2 dt + 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2})R_H(t)dt = cT^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}). \tag{4.48}$$

Similarly, it is also true that

$$\int_1^T (R_H(t))^2 dt - 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2})R_H(t)dt = cT^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}), \tag{4.49}$$

since

$$R_B^\epsilon(t-C) - R_A^\epsilon(t+C) + O(t^{\frac{1}{2}+\delta}) \le -R_H(t) \le R_B^\epsilon(t+C) - R_A^\epsilon(t-C) + O(t^{\frac{1}{2}+\delta}).$$

Therefore, by adding a term $\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2}$ to both sides of this inequality and taking L^2-norms we are done.

Combining (4.48) and (4.49), proves that

$$\int_1^T (R_H(t))^2 dt = cT^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}).$$

Now $R_H(t)$ is the error term corresponding to $N_H(2\pi t)$ and we know that it differs with $R(2\pi t)$ which is the error term corresponding to $N(2\pi t)$ only by a term of order

$O(\sqrt{t})$. Therefore,

$$\int_1^T (R(t))^2 dt = c(2\pi)^{7/2} T^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}).$$

This proves Theorem 1.2.3.

4.3 Proof of theorem 1.2.4

Let $N(t)$ be the spectral counting function of the $(2n+1)$-dimensional Heisenberg manifold. Therefore,

$$N(t) = N_T(t) + N_H(t),$$

where $N_T(t)$ is the spectral counting function of $2n$-dimensional torus with the metric $h = I_{2n\times 2n}$ and $N_H(t)$ is defined by

$$N_H(t) = \#\{(c, k_1, k_2, ..., k_n); c > 0, k_j \geq 0, 2\pi c(c + 2k_1 + ... + 2k_n + n) \leq t\}.$$

For $N_T(t)$ we use the trivial estimate resulted from Hörmander's theorem:

$$N_T(t) = \frac{1}{n!2^n}\left(\frac{t}{2\pi}\right)^n + O\left(t^{n-\frac{1}{2}}\right),$$

and we continue with computing $N_H(t)$:

$$\begin{aligned} N_H(2\pi t) &= \sum_{c(c+2\sum k_j+n)\leq t} 2c^n = \sum_{c(c+2k+n)\leq t} 2c^n \sum_{k_1+...+k_n=k} 1 = \sum_{c(c+2k+n)\leq t} 2c^n \binom{k+n-1}{n-1} \\ &= \sum \frac{2}{(n-1)!} c^n k^{n-1} + \sum \frac{n}{(n-2)!} c^n k^{n-2} + O\left(t^{n-\frac{1}{2}}\right). \end{aligned}$$

Let $A_t = \{(x,y); x(x+2y+n) \le t, x>0, y>0\}$ and $\rho_\epsilon(x,y)$ be as defined in third section. We define the mollified counting function $N_\epsilon(t)$ as:

$$\begin{aligned} N_\epsilon(t) \quad &:= \quad \frac{2}{(n-1)!} \sum_{(c,k)\in Z^2} (c^n k^{n-1}) \chi_{A_t}(c,k) * \rho_\epsilon(c,k) \\ &+ \quad \frac{n}{(n-2)!} \sum_{(c,k)\in Z^2} (c^n k^{n-2}) \chi_{A_t}(c,k) * \rho_\epsilon(c,k). \end{aligned} \tag{4.50}$$

Proposition 4.3.1 *The following asymptotic expansion holds for $N_\epsilon(t)$:*

$$N_\epsilon(t) = \frac{2^{n+1} n!}{(2n+1)!} t^{n+\frac{1}{2}} - \frac{1}{n! 2^n} t^n + R_\epsilon(t) + O\left(t^{n-\frac{1}{2}+\delta}\right), \tag{4.51}$$

where,

$$R_\epsilon(t) = \frac{t^{n-\frac{1}{4}}}{2^{n-1}(n-1)!\pi} \sum_{0<\nu<\mu} (-1)^{\nu n} \cos(2\pi\sqrt{t\mu\nu} - \frac{\pi}{4}) \mu^{-\frac{5}{4}} \nu^{-\frac{1}{4}} (1-\frac{\nu}{\mu})^{n-1} \widehat{\rho_\epsilon}(\frac{\mu+\nu}{2}, \nu).$$

Proof. Applying the Poisson summation formula to the first sum in $N_\epsilon(t)$ (defined by (4.50)) gives:

$$\begin{aligned} &\sum_{(c,k)\in Z^2} (c^n k^{n-1}) \chi_{A_t}(c,k) * \rho_\epsilon(c,k) = \sum_{\lambda,\nu} \widehat{x^n y^{n-1} \chi_A}(\lambda,\nu) \widehat{\rho_\epsilon}(\lambda,\nu) \\ &= \widehat{x^n y^{n-1} \chi_A}(0,0) \widehat{\rho_\epsilon}(0,0) + \sum_{\lambda\ne 0,\nu=0} \widehat{x^n y^{n-1} \chi_A}(\lambda,\nu) \widehat{\rho_\epsilon}(\lambda,\nu) \\ &+ \sum_{\lambda=0,\nu\ne 0} \widehat{x^n y^{n-1} \chi_A}(\lambda,\nu) \widehat{\rho_\epsilon}(\lambda,\nu) + \sum_{\lambda\ne 0,\nu\ne 0} \widehat{x^n y^{n-1} \chi_A}(\lambda,\nu) \widehat{\rho_\epsilon}(\lambda,\nu). \end{aligned} \tag{4.52}$$

We first estimate each term on the right-hand side of (4.52). For the first term, we get:

$$\begin{aligned}
\widehat{x^n y^{n-1}\chi_A}(0,0) &= \int_0^{\sqrt{t}}\int_0^{\frac{t}{2x}-\frac{x}{2}-\frac{n}{2}} x^n y^{n-1} dy dx = \int_0^{\sqrt{t}} \frac{1}{n2^n}(t-x^2-nx)^n dx \\
&= \frac{(n!)^2 2^n}{(2n+1)!n} t^{n+\frac{1}{2}} - \frac{1}{2^{n+1}} t^n + O\left(t^{n-\frac{1}{2}}\right). \qquad (4.53)
\end{aligned}$$

Also similar computations like those we did in third section shows that:

$$\sum_{\lambda\neq 0} \widehat{x^n y^{n-1}\chi_A}(\lambda,0).\widehat{\rho_\epsilon}(\lambda,0) = \frac{-1}{n2^{n+1}} t^n + O\left(t^{n-\frac{1}{2}}\right), \qquad (4.54)$$

and

$$\sum_{\nu\neq 0} \widehat{x^n y^{n-1}\chi_A}(0,\nu)\widehat{\rho_\epsilon}(0,\nu) = O\left(t^{n-\frac{1}{2}}\right). \qquad (4.55)$$

For the fourth term on the right-hand side of (4.52),

$$\begin{aligned}
\widehat{x^n y^{n-1}\chi_A}(\lambda,\nu) &= \int_0^{\sqrt{t}}\int_0^{\frac{t}{2x}-\frac{x}{2}-\frac{n}{2}} x^n y^{n-1} e^{2\pi i(\lambda x+\nu y)} dy dx \\
&= \frac{(-1)^{\nu n}}{2^n \pi i\nu}\int_0^{\sqrt{t}} x^n(\frac{t}{x}-x-n)^{n-1} e^{\pi i((2\lambda-\nu)x+\frac{\nu t}{x})} dx \\
&- \frac{(-1)^{\nu n}(n-1)}{2^n(\pi i\nu)^2}\int_0^{\sqrt{t}} x^n(\frac{t}{x}-x-n)^{n-2} e^{\pi i((2\lambda-\nu)x+\frac{\nu t}{x})} dx \\
&= \frac{(-1)^{\nu n} t^n}{2^n \pi i\nu}\int_0^1 x(1-x^2-\frac{nx}{\sqrt{t}})^{n-1} e^{\pi i\sqrt{t}((2\lambda-\nu)x+\frac{\nu}{x})} dx \\
&- \frac{(-1)^{\nu n}(n-1)t^{n-1}}{2^n(\pi i\nu)^2}\int_0^1 x^n(1-x^2-\frac{nx}{\sqrt{t}})^{n-2} e^{\pi i\sqrt{t}((2\lambda-\nu)x+\frac{\nu}{x})} dx
\end{aligned}$$

Now using the method of the stationary phase and following the same argument as in the appendix, we will have:

$$\sum_{\lambda\neq 0,\nu\neq 0} \widehat{x^n y^{n-1}\chi_A}(\lambda,\nu)\widehat{\rho_\epsilon}(\lambda,\nu)$$
$$= \frac{t^{n-\frac{1}{4}}}{2^{n-1}\pi}\sum_{0<\nu<\mu}(-1)^{\nu n}\cos(2\pi\sqrt{t\mu\nu}-\frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}(1-\frac{\nu}{\mu})^{n-1}\widehat{\rho_\epsilon}(\frac{\mu+\nu}{2},\nu)+O\left(t^{n-\frac{1}{2}+\delta}\right) \tag{4.56}$$

Combining the results from (4.53),..., (4.56), we have proved that:

$$\frac{2}{(n-1)!}\sum_{(c,k)\in Z^2}(c^n k^{n-1})\chi_{A_t}(c,k)*\rho_\epsilon(c,k) = \frac{2^{n+1}n!}{(2n+1)!}t^{n+\frac{1}{2}} - \frac{t^n}{(n-1)!2^n} - \frac{t^n}{n!2^n} + R_\epsilon(t) + O\left(t^{n-\frac{1}{2}+\delta}\right)$$

Finally applying the Poisson summation formula to the second sum in $N_\epsilon(t)$ (defined by (4.50)) and using the same argument that we used for the first sum, we get:

$$\frac{n}{(n-2)!}\sum_{(c,k)\in Z^2}(c^n k^{n-2})\chi_{A_t}(c,k)*\rho_\epsilon(c,k) = \frac{1}{(n-1)!2^n}t^n + O\left(t^{n-\frac{1}{2}}\right)$$

This completes the proof of proposition 4.3.1.

Given the estimate (4.51) for $N_\epsilon(t)$, the rest of the proof for theorem 1.2.4 follows exactly like the proof of theorem 1.2.3.

APPENDIX A

4.4 Proof of Proposition 4.1.3:

After a simple integration, we get for $\nu \neq 0$:

$$\begin{aligned}\widehat{2x\chi_A}(\lambda,\nu) &= \int\int_A 2xe^{2\pi i(\lambda x+\nu y)}dydx = \int_0^{\sqrt{t}}\int_0^{t/x-x} 2xe^{2\pi i\lambda x}e^{2\pi i\nu y}dydx \\ &= \int_0^{\sqrt{t}} \frac{2x}{2\pi i\nu}e^{2\pi i\lambda x}e^{2\pi i\nu(t/x-x)}dx - \int_0^{\sqrt{t}}\frac{2x}{2\pi i\nu}e^{2\pi i\lambda x}dx. \qquad (4.57)\end{aligned}$$

The summation over the second integral in (4.57) leads to a term of order $O(t^{\frac{1}{2}+\delta})$ for every positive δ. To see this, for $\nu \neq 0$,

$$\int_0^{\sqrt{t}} \frac{x}{\nu}e^{2\pi i\lambda x}dx = \frac{1}{\nu}\left[\frac{xe^{2\pi i\lambda x}}{2\pi i\lambda} - \frac{e^{2\pi i\lambda x}}{2\pi i\lambda^2}\right]_0^{\sqrt{t}} \ll \frac{\sqrt{t}}{\nu\lambda}+\frac{1}{\nu\lambda^2}.$$

Therefore,

$$\begin{aligned}\sum_{\lambda\neq 0,\nu\neq 0}\left(\int_0^{\sqrt{t}}\frac{2x}{2\pi i\nu}e^{2\pi i\lambda x}dx\right)\widehat{\rho_\epsilon}(\lambda,\nu) &\ll \sum_{0<\lambda\leq t^{1+\alpha}}\sum_{0<\nu\leq t^{1+\alpha}}\left(\frac{\sqrt{t}}{\nu\lambda}+\frac{1}{\nu\lambda^2}\right)+O(t^{-\infty}) \\ &= O(t^{\frac{1}{2}}\ln^2(t)) = O(t^{\frac{1}{2}+\delta}), \qquad (4.58)\end{aligned}$$

where α and δ are arbitrarily small positive numbers.

To evaluate the first integral on the right-hand side of (4.57), make the change of variable $y = \frac{x}{\sqrt{t}}$,

$$\int_0^{\sqrt{t}}\frac{2x}{2\pi i\nu}e^{2\pi i\lambda x}e^{2\pi i\nu(t/x-x)}dx = \int_0^1 \frac{2ty}{2\pi i\nu}e^{2\pi i\sqrt{t}((\lambda-\nu)y+\frac{\nu}{y})}dy. \qquad (4.59)$$

It is convenient to introduce the new variable $\mu = \lambda-\nu$. Let $f(y) = \mu y + \frac{\nu}{y}$. Then, the phase, $f(y)$, has no critical point iff $\mu = 0$ or $\frac{\nu}{\mu} < 0$ or $\frac{\nu}{\mu} > 1$. We show that in any of these cases, the summation over the integral in (4.59) leads to a term of order $O(t^{\frac{1}{2}+\delta})$ for every positive δ. To see this, note that

$$\int_0^1 ye^{2\pi i\sqrt{t}(\mu y+\frac{\nu}{y})}dy = \left[\frac{ye^{2\pi i\sqrt{t}f(y)}}{\sqrt{t}f'(y)}\right]_0^1 - \frac{1}{\sqrt{t}}\int_0^1 \left(\frac{f'(y)-yf''(y)}{f'^2(y)}\right)e^{2\pi i\sqrt{t}f(y)}dy$$
$$\ll \frac{1}{\sqrt{t}|\mu-\nu|} + \frac{1}{\sqrt{t}}\max_{0\le y\le 1}\left|\frac{1}{f'(y)}\right| + \frac{1}{\sqrt{t}}\int_0^1 \left|\frac{f''(y)}{f'^2(y)}\right|dy = \frac{3}{\sqrt{t}|\mu-\nu|}.$$

Hence,

$$\sum_{\mu=0 \text{ or } 1<\frac{\nu}{\mu} \text{ or } \frac{\nu}{\mu}<0}\left(\int_0^1 \frac{ty}{\nu}e^{2\pi i\sqrt{t}(\mu y+\frac{\nu}{y})}dy\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu)$$

$$\ll \sqrt{t}\sum_{0<\nu}\frac{1}{\nu^2} + \sqrt{t}\sum_{0<\mu<\nu\le t^{1+\alpha}}\frac{1}{(\nu-\mu)\nu} + \sqrt{t}\sum_{\substack{\frac{\nu}{\mu}<0,0<|\mu|\le t^{1+\alpha},\\ 0<|\nu|\le t^{1+\alpha}}}\frac{1}{(\nu-\mu)\nu}$$
$$+O(t^{-\infty}) = O(t^{\frac{1}{2}+\delta}). \tag{4.60}$$

Therefore, combining the results form (4.57), (4.58), (4.59) and (4.60), we have

$$\sum_{\lambda\neq 0}\sum_{\nu\neq 0}\widehat{2x\chi_A}(\lambda,\nu)\widehat{\rho}_\epsilon(\lambda,\nu) = \sum_{0<\frac{\nu}{\mu}\le 1,\mu\neq 0}\left(\int_0^1 \frac{ty}{\pi i\nu}e^{2\pi i\sqrt{t}(\mu y+\frac{\nu}{y})}dy\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu)+O(t^{\frac{1}{2}+\delta}). \tag{4.61}$$

If $0<\frac{\nu}{\mu}\le 1$, then the phase has a critical point $\sqrt{\frac{\nu}{\mu}}$. Without loss of generality assume that $0<\nu\le\mu$. After making a change of variable $z=\sqrt{\frac{\mu}{\nu}}y$, we get:

$$\int_0^1 \frac{ty}{\nu}e^{2\pi i\sqrt{t}(\mu y+\frac{\nu}{y})}dy = \int_0^{\sqrt{\frac{\mu}{\nu}}} \frac{tz}{\mu}e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz$$
$$= \int_0^{\frac{1}{2}} \frac{tz}{\mu}e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz + \int_{\frac{1}{2}}^1 \frac{tz}{\mu}e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz + \int_1^{\sqrt{\frac{\mu}{\nu}}} \frac{tz}{\mu}e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz. \tag{4.62}$$

Using a standard integration by parts, one can see that $\int_0^{\frac{1}{2}} \frac{tz}{\mu} e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz = O(\frac{t}{\mu\sqrt{t\mu\nu}})$. Therefore the summation over this integral leads to a term of order $O(t^{\frac{1}{2}+\delta})$, that is:

$$\sum_{0<\nu\leq\mu}\left(\int_0^{\frac{1}{2}} \frac{tz}{\pi i\mu} e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) = O(t^{\frac{1}{2}+\delta}). \tag{4.63}$$

Consider the second integral on the right-hand side of (4.62). Applying the method of stationary phase(see [Cop]), we get:

$$\int_{\frac{1}{2}}^1 \frac{tz}{\mu} e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz = \frac{t}{2\sqrt{2}\mu\sqrt[4]{t\mu\nu}} e^{4\pi i\sqrt{t\mu\nu}+\frac{i\pi}{4}} + O(\frac{t}{\mu\sqrt{t\mu\nu}}), \tag{4.64}$$

and therefore, taking the summation we have:

$$\begin{aligned}&\sum_{0<\nu\leq\mu}\left(\int_{\frac{1}{2}}^1 \frac{tz}{\pi i\mu} e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu)\\ &= \frac{t^{\frac{3}{4}}}{2\pi\sqrt{2}}\sum_{0<\nu\leq\mu} e^{4\pi i\sqrt{t\mu\nu}-\frac{i\pi}{4}}\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu) + O(t^{\frac{1}{2}+\delta}).\end{aligned} \tag{4.65}$$

To evaluate the third integral on the right-hand side of (4.62), we use the following lemma(for proof, see [Cop] pages 29–33):

Lemma 4.4.1 *Suppose f and ϕ are analytic functions, regular in a simply connected open region D in the complex plane, containing the interval $[1,a]$ from the real axis. Also, suppose that f is real on the real axis and has exactly one stationary point $x=1$ in $[1,a]$ where $f''(1)>0$. Then,*

$$\int_1^a \phi(x)e^{isf(x)}dx = \sqrt{\frac{\pi}{2sf''(1)}}\phi(1)e^{isf(1)+\frac{i\pi}{4}} + O(\frac{1}{\varepsilon s}), \tag{4.66}$$

where $\varepsilon := \sqrt{f(a)-f(1)}$.

Therefore, from (4.66) we get that:

$$\int_1^{\sqrt{\frac{\mu}{\nu}}} \frac{tz}{\mu} e^{2\pi i\sqrt{t\mu\nu}(\frac{1}{z}+z)}dz = \frac{t}{2\sqrt{2}\mu\sqrt[4]{t\mu\nu}} e^{4\pi i\sqrt{t\mu\nu}+\frac{i\pi}{4}} + O(\frac{t}{\mu\varepsilon\sqrt{t\mu\nu}}), \tag{4.67}$$

where $\varepsilon = \frac{\sqrt{\mu}-\sqrt{\nu}}{\sqrt[4]{\mu\nu}}$. Hence, taking the summation gives:

$$\sum_{0<\nu\leq\mu} \left(\int_1^{\sqrt{\frac{\mu}{\nu}}} \frac{tz}{\pi i\mu} e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz \right) \widehat{\rho}_\epsilon(\mu+\nu,\nu)$$
$$= \frac{t^{\frac{3}{4}}}{2\pi\sqrt{2}} \sum_{0<\nu<\mu} e^{4\pi i\sqrt{t\mu\nu}-\frac{i\pi}{4}} \mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu) + O(t^{\frac{1}{2}+\delta}). \tag{4.68}$$

Combining (4.63), (4.65) and (4.68), we find that:

$$\sum_{0<\nu\leq\mu} \left(\int_0^{\sqrt{\frac{\mu}{\nu}}} \frac{tz}{\pi i\mu} e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz \right) \widehat{\rho}_\epsilon(\mu+\nu,\nu)$$
$$= \frac{t^{\frac{3}{4}}}{2\pi\sqrt{2}} \sum_{0<\nu\leq\mu} e^{4\pi i\sqrt{t\mu\nu}-\frac{i\pi}{4}} \mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu)$$
$$+\frac{t^{\frac{3}{4}}}{2\pi\sqrt{2}} \sum_{0<\nu<\mu} e^{4\pi i\sqrt{t\mu\nu}-\frac{i\pi}{4}} \mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu) + O(t^{\frac{1}{2}+\delta}). \tag{4.69}$$

Given a similar result as the one in (4.69) for the case $\mu \leq \nu < 0$, we have proved that:

$$\sum_{\lambda\neq 0}\sum_{\nu\neq 0} \widehat{2x\chi_A}(\lambda,\nu)\widehat{\rho}_\epsilon(\lambda,\nu) = \sum_{0<\frac{\nu}{\mu}\leq 1,\mu\neq 0} \left(\int_0^1 \frac{ty}{\pi i\nu} e^{2\pi i\sqrt{t}(\mu y+\frac{\nu}{y})}dy \right) \widehat{\rho}_\epsilon(\mu+\nu,\nu) + O(t^{\frac{1}{2}+\delta})$$
$$= \frac{1}{\sqrt{2}\pi} \sum_{0<\nu\leq\mu} t^{\frac{3}{4}} \cos(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu)$$
$$+\frac{1}{\sqrt{2}\pi} \sum_{0<\nu<\mu} t^{\frac{3}{4}} \cos(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu) + O(t^{\frac{1}{2}+\delta}),$$

which proves the proposition.

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