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# The axisymmetric and the plane jet in a coflow

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**Abstract**

The approximate variation of the mean properties in an axisymmetric jet and a two dimensional jet with an ambient coflow in an infinite duct have been determined using the excess momentum equation and an entrainment function. The turbulent velocity flux, consisting of a portion due to the excess jet velocity and a portion due to the turbulent coflow velocity, is carried by a velocity approximately equal to the top hat velocity. The entrainment into this flow is driven by the excess jet velocity and the entrainment function varies to allow for the change in entrainment from the strong jet to the weak jet. The entrainment constant is determined from the data of Nickels and Perry [14] for the axisymmetric case and the data of Bradbury and Riley [6] for the two dimensional case. All coflow experiments are in ducts and the effect of the duct is explored for the axisymmetric case. This paper is a prelude to the study of a buoyant jet in a coflow and a buoyant jet in a crossflow.

**Résumé**

Les variations approchées des propriétés moyennes d'un jet axisymétrique et d'un jet plan dans un courant de même direction en conduite infinie ont été déterminés en utilisant l'équation des quantités de mouvement supplémentaires et une fonction d'entrainement. Le flux de vitesse turbulente, composé en partie du supplément de vitesse dû au jet, en partie du courant turbulent, est transporté à une vitesse approximativement égale à celle du 'dessus de chapeau' (de l'axe du jet). L'entrainement dans cet écoulement est piloté par le supplement de vitesse du jet, et la fonction d'entrainement varie pour permettre de passer de l'entrainmement du jet fort à celui du jet faible. La constante d'entrainement est determinée à partir des données de Nickels et Perry [14] pour le cas axisymétrique, et celles de Bradbury et Riley [6] pour le jet plan. Toutes les expériences de courant portant sont faites en conduites et l'influence de la conduite est étudiée dans le cas axisymétrique. Cet article est un prélude à l'étude d'un jet flottant en co-courant et en

courant traversier.

**Introduction**

Buoyant jets in a still fluid and in a crossflow have been studied for a long time (Wood et al. [20]). For jets in a still fluid, the variation of the mean properties in the turbulent region of the jet are obtained using the momentum equation, the continuity equation and a closure equation in the form of an entrainment equation whose coefficient is proportional to the mean centre line velocity (Morton, Taylor and Turner [13]). Taylor [18] showed that, outside the turbulent region, the flow is irrotational and can be approximated with a line of sinks whose strengths are proportional to the entrainment velocity. A second approach to the closure equation is to use empirical measurements of the constant rate of spread of the turbulent region (Wood et.al. [20], Wright [21], Chu et.al [8]).

In most studies, the ratio of the crossflow velocity to the initial jet velocity is quite large (>0.05). This is appropriate for cases where the flow is in the atmosphere. When the flow is in the ocean, such as from a submerged sewage outfall, the normal crossflow is a very small percentage of the initial jet velocity. For this very small ratio of the crossflow velocity to the initial velocity, detailed entrainment velocity measurements have shown that the flow outside the turbulent region can be obtained by superimposing the irrotational crossflow with the sink appropriate to the jet entrainment (Gaskin [10]). This illustrated how, as the crossflow increases, the entrainment flow changes gradually from the normal entrainment flow into a sink to a forced entrainment flow. This is the motivation for exploring the use of the continuity and momentum equations combined with an entrainment function as a closure assumption.

The behaviour of a buoyant jet in any flow can be divided into a number of regimes in which particular physical processes dominate. In these regimes, simple dimensional analysis allows the form of the trajectory, the width and the dilution to be determined and any numerical model should, in the limit, satisfy all these regimes. One of the simpler of these regimes is a jet in a coflow without ambient turbulence and this is the case considered here.

For a plane or axisymmetic jet in a still fluid, it has been well established that the velocity and turbulence profiles are self preserving (Hussein et al. [11], Papanicolaou and List [15], Wood et.al. [20] and others). There have also been a number of studies of jets released parallel and in the direction of a flowing fluid (Forstall and Shapiro [9], Bradbury and Riley [6], Smith and Hughes [17], Antonia and Bilger [3], Nickels and Perry [14], and Chu et.al. [8]). These cases have been called plane or axisymmetric jets in a coflow and it has been shown that the excess mean velocities are approximately self-preserving in both the axisymmetric and the two dimensional flow, assuming a point momentum source. However, the product of the turbulence velocities are not self preserving (Townsend [19], Antonio & Bilger [3], Nickels and Perry [14]) and this suggests that the normal entrainment constant changes from that for a strong jet to that for a weak jet (see Hussein et al.[11] for a discussion on self preservation). The flow is completely self preserving only in the limit of a strong jet and a weak jet. However, a useful solution for engineering applications can be obtained if the assumption of self preservation of the mean excess velocities is used over the complete range from a strong jet to a weak jet and the form of the entrainment can be approximated.

In the first part of this paper, a new approach to an analysis of the mean velocities in a jet in a coflow without ambient turbulence and in an infinite duct is discussed. In the second portion, modifications for the case where the coflow is in a duct are discussed. Finally the method developed for the axisymmetric jet is applied to a plane jet in an infinite duct.

**The axisymmetric jet with a coflow in a large duct**

The jet flow is illustrated in Figure 1. The flow consists of a non-turbulent coflow outside the jet and a turbulent region inside the jet. This turbulent region consists of the jet flow ($u_{eg}$ region) and the turbulent portion of the coflow within the jet boundaries ($U_\infty$ region). When the jet occupies only a small portion of the duct, the flow is long and narrow and this allows the boundary layer assumption to be made. This implies that the time averaged excess velocity (subscript e) and tracer distributions are self similar and they are normally assumed to have a Gaussian velocity distribution (subscript g).

$$u_{eg} = U_{eg} e^{-\left(\frac{r}{b_g}\right)^2} \qquad (1)$$

This is illustrated in Figure 1, where $u_{eg}$ is the velocity at a radius r, $U_{eg}$ is the centre line velocity and $b_g$ is a characteristic radius (in this case it is the radius at which the value of $u_{eg}/U_{eg}$ equals 1/e). The continuity equation for the jet in an ambient flow in a frictionless duct of area A is,

$$\int_0^A u_{eg}\, 2\pi r dr + \int_0^A U_\infty\, 2\pi r dr = \frac{\pi}{4}\, d^2\, U_{eo} + A\, U_{\infty o} \tag{2}$$

Neglecting the friction on the sides of the duct and assuming the duct is large enough not to affect the coflow velocity, the coflow at the orifice, $U_{\infty o}$, is equal to the coflow downstream, $U_\infty$. The momentum equation yields,

$$\begin{aligned} &\int_0^A (u_{eg} + U_\infty)^2\, 2\pi r dr + \int_0^A \frac{p_x}{\rho}\, 2\pi r dr \\ &= \int_0^A (U_{eo} + U_\infty)^2\, 2\pi\, r dr + \int_0^A \frac{p_o}{\rho}\, 2\pi\, r dr \end{aligned} \tag{3}$$

where $p_o$ and $p_x$ are the pressures at the orifice and at a distance x downstream. Now, subtracting $U_\infty$ times the continuity equation, dividing by the maximum average velocity ($U_{eg}$) and the width ($b_g$) and assuming $p_o$ is zero, we get

$$\begin{aligned} &U_{eg}^2 b_g^2 \int_0^\infty \left[ \left( \frac{u_{eg}}{U_{eg}} \right)^2 + \frac{p_x}{\rho U_{eg}^2} \right] 2\pi \frac{r}{b_g} d\frac{r}{b_g} + U_\infty U_{eg} b_g^2 \int_0^\infty \frac{u_{eg}}{U_{eg}} 2\pi \frac{r}{b_g} d\frac{r}{b_g} \\ &= \frac{\pi}{4} d^2 \left( U_{eo}^2 + U_\infty U_{eo} \right) \end{aligned} \tag{4}$$

The first term on the left hand side of equation (4) is the momentum due to the jet excess velocity and the contribution of the streamwise pressure gradient and the second term is the momentum due to the coflow velocity.

For a jet in a still fluid (or zero ambient flow) Hussein et. al. [11] show that the first term can be replaced by the momentum integral to the second order,

$$\begin{aligned} &U_{eg}^2 b_g^2 \int_0^\infty \left[ \left( \frac{u_{eg}}{U_{eg}} \right)^2 + \frac{p_x}{\rho U_{eg}^2} \right] 2\pi \frac{r}{b_g} d\frac{r}{b_g} \\ &= U_{eg}^2 b_g^2 \int_0^\infty \left[ \left( \frac{u_{eg}}{U_{eg}} \right)^2 + \overline{u'^2} - \frac{1}{2} \left( \overline{v'^2} + \overline{w'^2} \right) \right] 2\pi \frac{r}{b_g} d\frac{r}{b_g} \end{aligned} \tag{5}$$

where u′, v′ and w′ are the dimensionless turbulence velocities. The first term on the right hand side, due to the time averaged jet excess velocities with assumed Gaussian distribution, can be integrated to give a shape constant for the momentum flux $I_m$ of π/2. Hussein et. al.'s [11] data suggests that the contribution of the turbulence velocities is about 10% of that of the mean velocities. We will combine the two contributions by increasing $I_m$ by 10% to 1.72. The second dimensionless term in equation (4) is the shape function for the volume flux,

$$I_q = \int_0^\infty \frac{u_{eg}}{U_{eg}} 2\pi \frac{r}{b_g} d\frac{r}{b_g} = \pi \qquad (6)$$

After substitution, the momentum equation becomes

$$I_m U_{eg}^{\ 2} b_g^{\ 2} + U_\infty I_q U_{eg} b_g^{\ 2} = \frac{\pi}{4}\left(U_{eo}^{\ 2} d^2 + U_\infty U_{eo} d^2\right) \qquad (7)$$

The value of the length scale for the strong jet (subscript J) to the weak jet (subscript WJ) transition is

$$l_{J,WJ} = \left[\frac{\pi}{4}\left(\frac{U_{eo}^2 d^2 + U_\infty U_{eo} d^2}{U_\infty^{\ 2}}\right)\right]^{\frac{1}{2}} \qquad (8)$$

Writing $b' = b_g/l_{J,WJ}$ and $U' = U_{eg}/U_\infty$ then the dimensionless form of equation (7) is,

$$I_m U'^2 b'^2 + I_q U' b'^2 = 1 = f + e \qquad (9)$$

where f, equal to $I_m U'^2 b'^2$, is the momentum due to the excess jet velocity and e, equal to $I_q U' b'^2$, is the momentum due to the coflow velocity. This can be rewritten as

$$\left(I_q U' b'^2 + \frac{I_q^2}{I_m} b'^2\right)\frac{I_m}{I_q} U' = q' \frac{I_m}{I_q} U' = 1 \qquad (10)$$

where q' is equal to $I_q U' b'^2 + I_q^2 b'^2/I_m$. The flow consists of a non turbulent volume flux outside the jet with a velocity of $U_4$ and a turbulent volume flux within the jet with a velocity of $U_\infty + u_{eg}$. The turbulent volume flux, q', can be decomposed into a term related to the excess velocity ($I_q U' b'^2 = \pi U' b'^2$ ) and a second term related to the turbulent portion of the coflow velocity ($I_q^2 b'^2/I_m = 5.81 b'^2$ ). It is also worth noting that the velocity carried by q' is the top hat velocity ($(I_m/I_q)U' = 0.54U'$ ) used by Morton et. al. [13] for jets in a still fluid and Chu et al. [8] for a jet in a coflow.

Using this definition for q' and noting that $U' = (I_q/I_m)(f/e)$ and $b' = (I_m^{0.5}/I_q)(e/f^{0.5})$, we get

$$q' = e + \frac{e^2}{1-e} = \frac{e}{1-e} \tag{11}$$

The entrainment into the turbulent flow, q', which consists of the excess velocity flux and the turbulent coflow flux (see Figure 1), is driven by the excess velocity, U', and this leads to

$$\frac{d}{ds'}q' = \frac{de}{dx'}\left(\frac{1}{1-e}\right)^2 = 2\pi b'\alpha_c U' = 2\pi\alpha_c \frac{(1-e)^{\frac{1}{2}}}{I_m^{\frac{1}{2}}} \tag{12}$$

where $\alpha_c$ is the entrainment constant for the coflowing jet and $x' = x/l_{J,WJ}$. When e (momentum due to the coflow) is small $\alpha_c$ will tend to the entrainment constant for a jet in a still fluid $\alpha_j$. Hussein et al. [11] get $\alpha_j$ equal to 0.057 and hence equations (10) and (12) give the solution for a jet in a still fluid. The solutions are

$$b' = 2\pi\frac{\alpha_j}{I_q}x' = 0.11x' \tag{13}$$

and

$$U'_{eg} = \frac{I_q}{(I_m)^{\frac{1}{2}}2\pi\,\alpha_j\,x'} = \frac{6.7}{x'} \tag{14}$$

For the strong jet in a coflow the data from Nickels and Perry [14] give a constant of 6.9 in equation (14).

In the weak jet the boundary between the turbulent fluid and the irrotational fluid is more convoluted than in the strong jet. Townsend [19] suggests that the variation in the entrainment might be explained by assuming, at least in plane jets and wakes that " (1) the basic entrainment is carried out by ordinary eddies of the turbulent motion, and (2) the additional folding is carried out by a distinct group of eddies, the entrainment eddies, which develop in intensity sufficient to produce large entrainment ratios in wakes and "weak" jets". To satisfy the additional entrainment caused by the convoluted boundary between the turbulent and the irrotational flow we write

$$\alpha_c = \alpha_j\ (1 + k_e e) \tag{15}$$

This satisfies the conditions when e tends to zero (jet in still fluid) and allows for the extra entrainment when e is large (weak jet). Hence

$$\frac{d}{ds'}q' = \frac{de}{dx'}\left(\frac{1}{1-e}\right)^2 = 2\pi b'\alpha_j(1 + k_e e)\frac{(1-e)^{\frac{1}{2}}}{I_m^{\frac{1}{2}}} \tag{16}$$

Substituting for e and letting f tend to zero, the solution gives the velocity decay as $x^{-2/3}$. This is the value obtained by simple dimensional analysis for a weak jet (Wood et al.[20]).

To obtain a complete solution, the initial conditions must be obtained from the zone of flow establishment.

**The zone of flow establishment**

To integrate the equations it is necessary to determine the value of f and x' at the end of the zone of flow establishment (ZFE). The dimensionless position is given by $x' = x/\,l_{J,WJ}$. Experiments show that the length of the ZFE depends on the relative magnitudes of the coflow and initial jet velocities (Abramovich [1], Rajaratnam [16]). For coflows less than the initial jet velocity, the length of the ZFE increases exponentially (from x/d = 7 for U' = 4 which is a still fluid, to x/d = 25 for U' = 1.33). For coflows greater than the initial jet velocity, the length of the ZFE decreases exponentially, returning to the still fluid value for U' = 0.33 and decreasing gradually for higher coflows. x' at the end of the zone of flow establishment is

$$x'^2 = \frac{(x/d)^2}{\pi/4} \frac{1}{U'^2_o + U'_o} \qquad (17)$$

where $U'_o$ is $U_{eo}/U_\infty$. Nickels and Perry [14] conducted the definitive experiments in a large wind tunnel for a jet in a coflow. The values of U' were 20, 10 and 2 and equation (17) yields x' values of 0.385, 0.97 and 6 respectively. These values at the end of the zone of flow establishment are in agreement with Nickels and Perry's results within an acceptable error. Equating the momentum flux at the orifice and the end of the zone of flow establishment and noting that the maximum velocity is constant, we obtain the value of $b'^2$ as $1/(I_m U'^2 + I_q U')$. This leads to values of b' of 0.038, 0.073 and 0.282 respectively. Then $f_o$ equal to $I_m U'^2/(I_m U'^2 + I_q U')$ is calculated, giving values of 0.909, 0.833 and 0.5 respectively.

Using these calculated initial conditions for U' of 20, the complete solutions were plotted for trial values of $k_e$ of 0, 0.5, 1.0 and 1.5, and the value of 1.0 was selected as the best-fit to Nickels & Perry's [14] data. The model using $k_e$ = 1.0 is compared to measured velocity data in Figure 2 and the agreement is satisfactory. In Figure 3 the model width is compared with the experimental data.

**The effects of a finite duct**

All experiments are carried out in finite ducts, which modify the jet behaviour from that found in infinite ducts. The duct confines the flow, causing the entrainment into the jet to be fed from a reversal in the coflow (Hussein et al. [11]) as shown in Figure 4. The flow reversal reduces the coflow velocity. The total momentum of the flow is split between the jet momentum and the momentum that contributes to a reduction in the coflow velocity resulting in a reduction of the jet momentum. A model for this effect is developed assuming a frictionless duct and accounting for the contributions due to jet turbulence and mean pressure variations.

If the value of the coflow velocity is $U_{\infty 0}$ at the end of the zone of flow establishment, the continuity equation is given by equation (2). The momentum equation is

$$\int_0^A (u_{eg}+U_{\infty x})^2+\left(\frac{p_x+p_v}{\rho}\right)2\pi r dr+A_\infty U_{\infty x}{}^2+\frac{p_{\infty x}}{\rho}A_\infty$$
$$=\int_0^A (U_{eo}+U_{\infty o})^2+\frac{p_o}{\rho}2r\pi dr+A_{\infty o}U_{\infty o}{}^2+\frac{p_\infty}{\rho}A_{\infty o} \qquad (18)$$

where $p_x$ is measured from $p_o$ and $p_v$ is the variable part of the pressure due to the duct influence. Subtracting $U_{\infty o}$ times the continuity equation from the momentum equation and setting $p_o$ to zero gives

$$\int_0^A (u_{eg}+U_{\infty x})^2 - U_{eg}U_{\infty o}2\pi r dr+U_{\infty x}\left(A_\infty U_{\infty x}-AU_{\infty 0}\right)$$
$$+\int_0^A\left(\frac{p_v+p_x}{\rho}\right)2\pi r dr+A_\infty\left(\frac{p_{\infty x}}{\rho}\right)=\frac{\pi}{4}d^2\left(U_{eo}{}^2+U_{eo}U_{\infty o}\right)+\left(\frac{p_\infty}{\rho}\right)A_{\infty o} \qquad (19)$$

Assuming that the jet maintains approximate self-similarity so that the shape constants for an infinite duct still apply, that the momentum flux due to the portion of the pressure in an infinite duct can still be represented by the shape factor $I_m$ increased by 10% (as per Hussein [11]) and that $A_\infty$ . A. Writing $p_{\infty x} = p_\infty + \Delta p_\infty$ and $U_{\infty x} = U_4 + \Delta U_{\infty 4}$ , we get

$$I_m U_{eg}{}^2 b_g{}^2+U_{eg}{}^2 b_g{}^2\int_0^A\left(\frac{p_v}{\rho U_{eg}{}^2}\right)2\pi\frac{r}{b_g}d\frac{r}{b_g}+\left(U_{\infty o}+2\Delta U_\infty\right)I_q U_{eg}b_g{}^2$$
$$+AU_{\infty o}\Delta U_\infty+A\Delta U_\infty{}^2+A_\infty\left(\frac{p_\infty+\Delta p_\infty}{\rho}\right)-A\frac{p_\infty}{\rho}=\frac{\pi}{4}d^2\left(U_{eo}{}^2+U_{eo}U_\infty\right) \qquad (20)$$

Using Bernoulli's equation along a streamline located in the coflow, we get

$$-\frac{\Delta p_\infty}{\rho} = U_{\infty o}\Delta U_\infty + \Delta U_\infty^{\,2} \tag{21}$$

Substituting (21) into (20) and rearranging

$$I_m U_{eg}^{\,2} b_g^{\,2} + U_{eg}^{\,2} b_g^{\,2} \int_0^A \left[\left(\frac{p_v}{\rho U_{eg}^{\,2}}\right) - \left(\frac{p_{\infty x}}{\rho U_{eg}^{\,2}}\right)\right] 2\pi \frac{r}{b_g} d\frac{r}{b_g} \tag{22}$$

$$+ \left(U_{\infty o} + 2\Delta U_\infty\right) I_q U_{eg} b_g^{\,2} = \frac{\pi}{4} d^2 \left(U_{eo}^{\,2} + U_{eo} U_{\infty o}\right)$$

On the right hand side of equation (22), the first term is the jet excess momentum for an infinite duct, the second term is the reducton in jet excess momentum due to the finite duct, the third term is the momentum due to the coflow in the infinite duct and the fourth term is the increase in the coflow momentum due to the finite duct. Let the transition length scale for strong jet to weak jet, $l_{J,WJ}$ be as before as given in equation (8) but replacing $U_\infty$ with $U_{\infty 0}$ and write $b' = b_g/l_{J,WJ}$, $U' = U_{eg}/U_{\infty 0}$, $U'_\infty = U_{\infty x}/U_{\infty 0}$ and $\Delta U'_\infty = \Delta U_\infty/U_{\infty 0}$. Assuming that the changing portion of the pressure, $p_v$, is approximately self similar, then the dimensionless value of the pressure difference can be described by a shape function, $I_p$, which is determined by fitting the curve to the experimental data as,

$$I_p = \int_0^A \left[\left(\frac{p_v}{\rho U_{eg}^{\,2}}\right) - \left(\frac{p_{\infty x}}{\rho U_{eg}^{\,2}}\right)\right] 2\pi \frac{r}{b_g} d\frac{r}{b_g} = 0.01 \tag{23}$$

Equation (22) can be written in dimensionless form as,

$$I_m U'^2 b'^2 + I_p U'^2 b'^2 + \left(1 + 2\Delta U'_\infty\right) I_q U' b'^2 = 1 \tag{24}$$

As before letting $f = I_m U'^2 b'^2$ and $e = I_q U' b'^2$, substitution gives

$$\left(1 + \frac{I_p}{I_m}\right) f + \left(1 + 2\Delta U'_\infty\right) e = 1 \tag{25}$$

We can write the momentum equation as a turbulent volume flux q' carried by a velocity of $(I_m/I_q)U'$ as,

$$\left[\left(1 + \frac{I_p}{I_m}\right) I_q U' b'^2 + \left(1 + 2\Delta U'_\infty\right) \frac{I_q^2}{I_m} b'^2\right] \frac{I_m}{I_q} U' = q' \frac{I_m}{I_q} U' = 1 \tag{26}$$

and q' can be rewritten assuming we can use the previous definitions of U' and b' for a jet in an infinite duct,

$$q' = \left(1+\frac{I_p}{I_m}\right) I_q U'b'^2 + \left(1+2\Delta U'_\infty\right)\frac{I_q^2}{I_m} b'^2$$
$$= \left(1+\frac{I_p}{I_m}\right)\left[\frac{e}{1-(1+2\Delta U'_\infty)e}\right] \qquad (27)$$

Differentiating q' and assuming that the equation for the entrainment into the turbulent region of the jet does not change (i.e. it is still driven by the jet excess velocity) we get

$$\left(1+\frac{I_p}{I_m}\right)\left[\frac{\frac{de}{dx} - 2e^2\frac{d\Delta U'_\infty}{dx}}{\left(1+\left[1+2\Delta U'_\infty\right]e\right)^2}\right] = 2\pi\,\alpha_j\left(1+k_e e\right)\left(\frac{1-e}{I_m^{\frac{1}{2}}}\right)^{\frac{1}{2}} \qquad (28)$$

Due to the confinement of the flow by the duct, the increase in the flow in the turbulent region must come from a decrease in the ambient coflow. If the ambient coflow velocity is still assumed constant over the area ( $U_\infty$' is only a function of x' ), then

$$\frac{de}{dx} = -A'_\infty\frac{d\Delta U'_\infty}{dx} \qquad (29)$$

With the initial conditions and equations (28) and (29), the effect of the finite duct was explored for all cases and the effect on the experiments of Nickels and Perry [14] was negligible. This is illustrated for the case of U' of 20 and A' of 5.12 in Figure 5. The effect of the duct area is not apparent until x' is well beyond the experimental data. The model indicates that the jet velocity and hence momentum begins to decrease once the jet occupies 15% of the duct, decreases very rapidly once the jet occupies 45% of the duct and at some point downstream will no longer be distinguishable from the coflow. Baturin [4] had similar observations in ventilation applications. The reduction in the coflow velocity is also illustrated.

**The asymptotic solution for the plane jet in a coflow**

The method described above can also be used for a plane jet in a coflow and, for the case of an infinite duct, will be summarized. With the assumed Gaussian velocity distribution, the momentum equation for this case is

$$I_m U_{eg}^2 b_g + U_\infty I_q U_{eg} b_g = U_{eo}^2 d + U_\infty U_{eo} d \qquad (30)$$

Where the shape constant for the discharge is

$$I_q = 2\int_0^\infty\left(\frac{u_{eg}}{U_{eg}}\right) d\,\frac{b}{b_g} = 1.77 \qquad (31)$$

In the calculations for $I_m$ the results of Bradbury [5] suggest that the turbulent terms balance the pressure distribution term, hence

$$I_m = 2\int_0^\infty \left(\frac{u_{eg}}{U_{eg}}\right)^2 d\,\frac{b}{b_g} = 1.25 \tag{32}$$

The length scale for the transition from the strong jet (J) to the weak jet (WJ) is again given by equation (8). Defining b' = $b_g/l_{J,WJ}$ and U' = $U_{eg}/U_\infty$ we get the dimensionless form of the momentum equation,

$$I_m U'^2 b' + I_q U'b' = f + e = 1 \tag{33}$$

This may be written as

$$\left(I_q U'b' + \frac{I_q^2}{I_m} b'\right)\frac{I_m}{I_q} U' = 1 \tag{34}$$

defining

$$q' = \left(I_q U'b' + \frac{I_q^2}{I_m} b'\right) = e + \frac{e^2}{f} = \frac{e}{1-e} \tag{35}$$

Then, assuming equation (14) as the function for the entrainment but with a new value of $k_e$,

$$\frac{dq'}{dx'} = \frac{de}{dx'}\frac{1}{(1-e)^2} = 2\alpha_j(1 + k_e e)\frac{I_q}{I_m}\frac{1-e}{e} \tag{36}$$

With a strong jet e tends to zero and hence from equation (36)

$$e^2 = \left(4\alpha_j \frac{I_q}{I_m}\right)x' \tag{37}$$

and thus

$$U'^2 = \left(\frac{I_q f}{I_m e}\right)^2 = \frac{I_q}{(4 I_m \alpha_j)x'} = \frac{0.353}{\alpha_j}\frac{1}{x'} \tag{38}$$

Bradbury and Riley get a coefficent of 6.25 and hence the entrainment constant $\alpha_j$ is 0.057. To obtain the complete solution we need the initial conditions. Bradbury and Riley's [6] experiments had a maximum velocity ratio of 6 and, to cover the experimental range, the initial conditions will be determined for this ratio. It is also worth noting that for the experiments the inlet velocity was not uniform and thus the excess momentum equation is written as

$$I_{mo} U_{eo}^2 d + U_\infty I_{qo} U_{eo} d = I_m U_{eg}^2 b + U_\infty I_q U_{eg} b \tag{39}$$

where $I_m$ and $I_q$ are the shape functions determined in the inlet. This gives a new value of

$$\frac{l_{J,WJ}}{d}=\frac{I_{mo}U_{eo}^2+U_\infty I_{qo}U_{eo}}{U_\infty^2} \qquad (40)$$

where U'$_o$ is $U_{eo}/U_\infty$ . Now for a strong jet x/d is approximately equal to 5 (Albertson et al [2]) hence

$$x'=\frac{x}{l_{J,WJ}}=\frac{x}{d}\frac{d}{l_{J,WJ}}=\frac{5}{I_{mo}U'^2_o+I_{qo}U'_o} \qquad (41)$$

Assuming that the maximum velocity does not change from the inlet to the end of the zone of flow establishment, using the excess momentum equation and dividing the velocities by $U_\infty$, we get

$$\frac{b}{d}=\frac{I_{mo}U'^2_o+I_{qo}U'_o}{I_mU'^2_o+I_qU'_o} \qquad (42)$$

and

$$f_o=I_mU'^2_ob'_o=\frac{I_mU'^2_o}{I_mU'^2_o+I_qU'_o} \qquad (43)$$

and

$$b'_o=\frac{b}{l_{J,WJ}}=\frac{1}{I_mU'^2_o+I_qU'_o} \qquad (44)$$

For U'$_o$ equalling 6, $b_o$' is 0.0179 and $f_o$ is 0.81. As pointed out by Bradbury and Riley [6], there still remains a problem with x' which depends on the shape factor in the inlet. If the velocity distribution is uniform, x' is 0.12 and if parabolic x' is 0.09. This is a minor change.
Accepting these initial conditions the complete solution is obtained by integrating equation (36) with a value of $k_e$ of 0.80. Figure 6 is the graph of $1/U'^2$ as a function of x' calculated in this manner, which is compared to the results obtained by Bradbury and Riley [6]. Similarly Figure 7 shows the width growth Δ (Δ is 0.832b') as a function of x'.

**The spread function**

In this paper the closure assumption is a modified form of the entrainment assumption. It is also possible to use the spread assumption below

$$\frac{db'}{dx'}=k_s\frac{cU'_{eg}}{U'_\infty+cU'_{eg}} \qquad (45)$$

For both the axisymmetric and the two dimensional flow the experimental value of $k_s$ is 0.11 and c is an empirical constant. It is noteworthy that Wood et al. [20] and Knudsen [12] used the data available at the time and empirically determined the value of c as 1. For the axisymmetric case, Chu, Lee and Chu [8] used Chu's [7] concept of dominant eddy hypothesis to obtain a value of 0.5 (approximately the top hat velocity $I_m/I_q$) and Wright [21] also suggests a value of 0.5. For the two dimensional case the data suggest that c equals 1.

**Conclusion**

A modified entrainment equation combined with the momentum equation can be used to obtain an approximate solution for the mean properties of jets in a coflow. The turbulent velocity flux, consisting of a portion due to the excess jet velocity and a portion due to the turbulent coflow velocity, is carried by a velocity approximately equal to the top hat velocity. The entrainment into this flow is driven by the jet excess velocity and the modified entrainment function allows for the change in entrainment from the strong jet to the weak jet. The same functional relationship for the entrainment is applied for an axisymmetric and a plane jet in a coflow. It is thought that this modification is due to clustering of eddies distorting the boundary between the turbulent fluid and the irrotational fluid. With the constant, which depends the additional entrainment in the weak jet case, the predicted mean properties agree with the experimental results. For the axisymmetric jet an allowance can be made for a finite duct size

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**Nomenclature / Notation**

$A$, $A_4$, $A_j$ duct area, coflow area, jet area

$b_g$ jet characteristic jet half-width where $u_{eg}/U_{eg} = 1/e$

d diameter of jet at origin or orifice diameter

e momentum flux due to coflow velocity

f momentum flux due to excess jet velocity

$I_q$, $I_m$, $I_p$ shape constant for volume flux , for momentum flux, for jet pressure difference

$k_e$ constant accounting for additional entrainment in a weak jet

$l_{J,WJ}$ length scale for transition from a strong jet (J) to a weak jet (WJ)

$M_{eo}$ initial jet momentum flux

$p_o$, $p_x$, $p_v$ pressure in jet at orifice, at x in an infinite duct, variation at x due to finite duct

$p_{4o}$, $p_4$ pressure in the coflow at orifice, at a distance x from origin in an infinite duct

$p_{4x}$, $\Delta p_4$ in a finite duct: pressure in the coflow at a distance x, variation from $p_4$

q' turbulent volume flux within jet

r jet radius

u', v', w' dimensionless turbulent velocity fluctuations

$u_e$, $u_{eg}$ local time averaged excess jet velocity, " with gaussian distribution

$U_{eo}$, $U_{eg}$ initial jet excess velocity, time averaged centreline jet excess velocity

$U_4$, $U_{40}$, $U_{4x}$ coflow velocity, initial coflow velocity, coflow velocity at a distance x

x downstream distance

$\alpha_j$, $\alpha_c$ entrainment constant for jet in a still fluid, for jet in a coflow

$\rho$ fluid density

' indicates a dimensionless value

**Figures for "The axisymmetric and a plane jet in a coflow"**

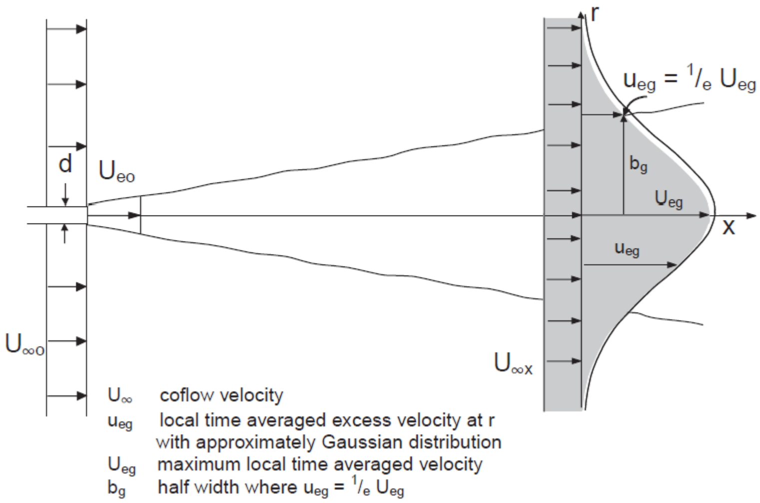


**Figure 1**. The nomenclature for an axisymmetric jet in a coflow. The shaded portion within the jet is turbulent.

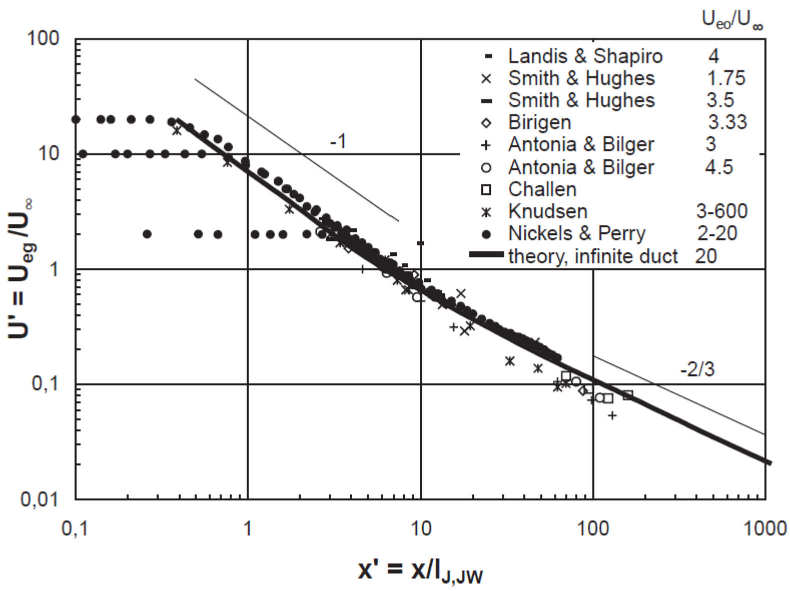


**Figure 2**. The variation of $U_{eg}/U_4$ as a function of $x/l_{J,WJ}$ for an axisymmetric jet in a non-turbulent coflow in an infinite duct. Experimental data is compared to the present integral theory.

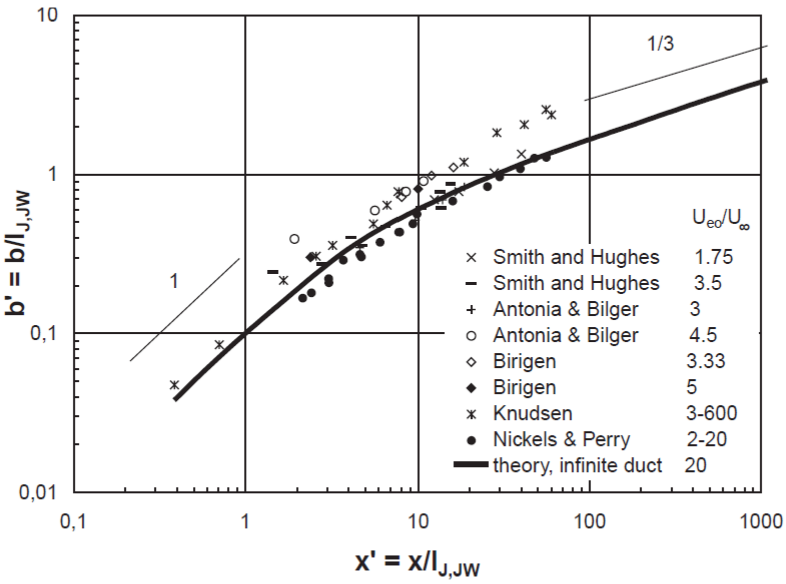


**Figure 3**. The value of the width of the jet ($b_g/1.414$) as a function of $x/l_{J,WJ}$ for an axisymmetric jet in a non-turbulent coflow in an infinite duct. Experimental data is compared to the present integral theory.

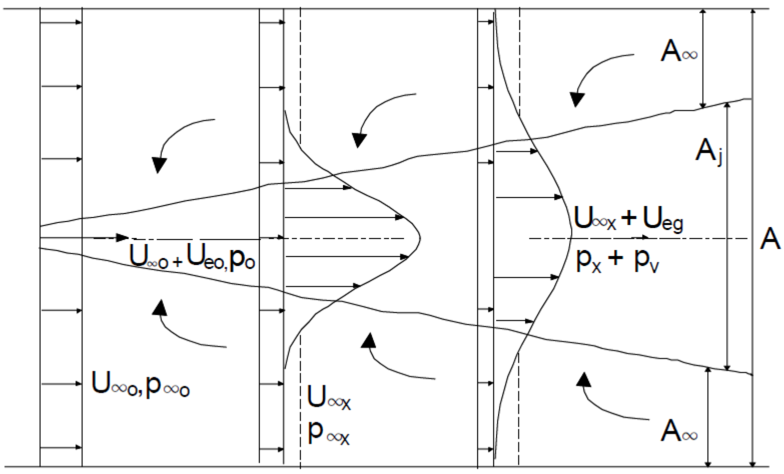


**Figure 4**. The nomenclature for the axisymmetric jet in a coflow in a finite duct illustrating the modifications due to the effect of the confinement of the flow.

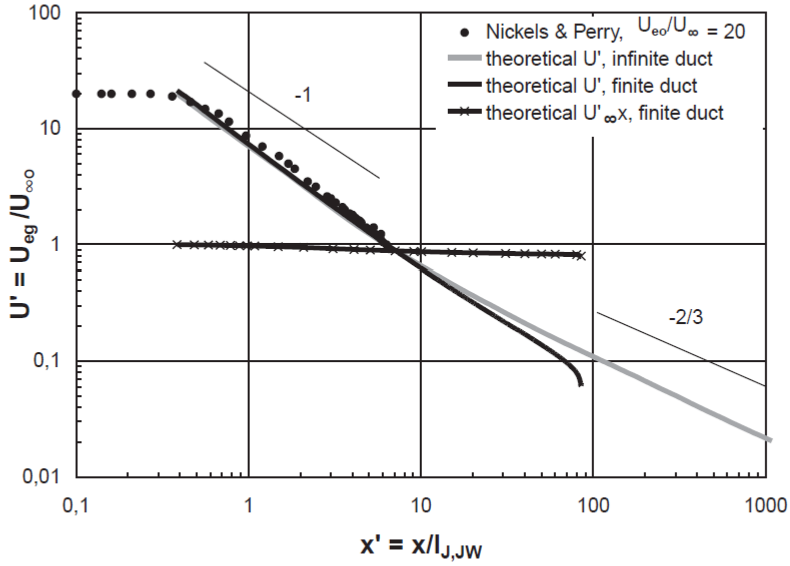


**Figure 5**. The decay of $U_{eg}/U_{4o}$ and $U_{4x}/U_{4o}$ as a function of $x/l_{J,WJ}$ due to the finite duct. The solid line is the empirical theory with a non turbulent coflow and the grey line allows for the correction for the duct size. The data of Nickels and Perry [14] for a velocity ratio of 20 is shown.

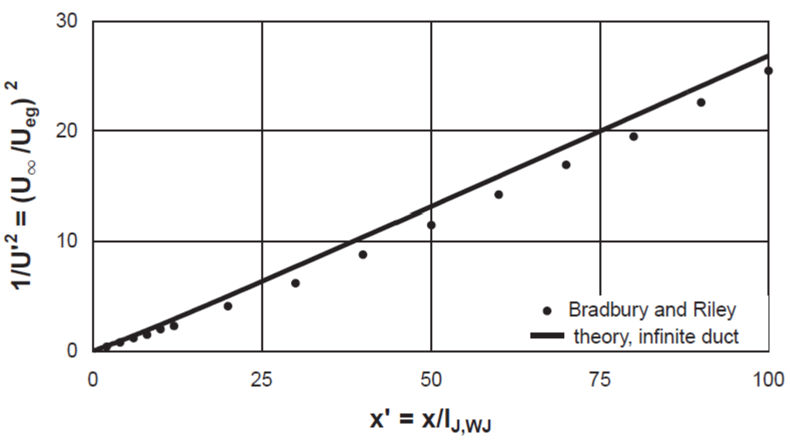


**Figure 6**. The variation $(U_4/U_{eg})^2$ as a function of $x/l_{J,WJ}$ for a plane jet in a coflow. The experiments were reported by Bradbury and Riley [6].

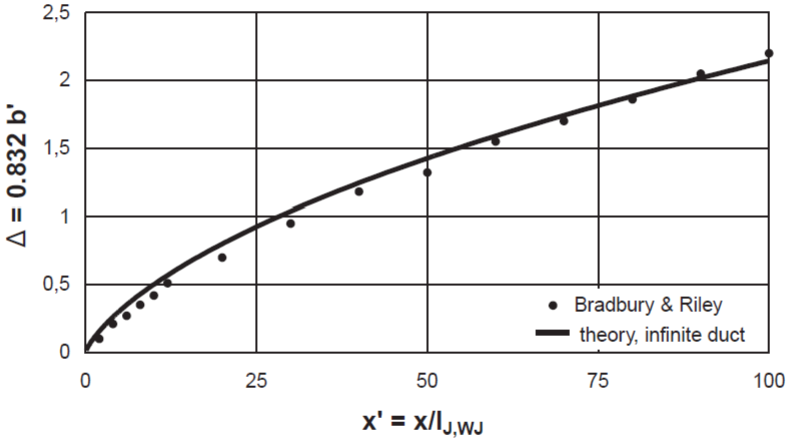


**Figure 7**. The variation of the width Δ (Δ = 0.832 b') as a function of $x/l_{J,WJ}$ for a plane jet in a coflow. The experiments were reported by Bradbury and Riley [6].