Amenability, Nuclearity, and Injectivity

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March 2022

A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of Master of Science in Mathematics and Statistics.

Abstract

In this thesis, we study amenability on various objects. We begin by studying amenable groups: we prove a few useful characterizations of amenability and demonstrate the power of the notion by proving that it yields several useful results. Of particular interest are the stability theorems for approximate representations and the unitarizability theorem that we prove: the existence of similar stability and unitarizability results for amenable objects is a common theme throughout this thesis. We then consider the case of amenable Banach algebras and prove that they admit a stability theorem for approximate representations. In the next section, we prove that nuclear C*-algebras are amenable as Banach algebras and use this to obtain a unitarizability result for nuclear C*-algebras. We also consider the notion of strong amenability for C*-algebras. Finally, we introduce the notion of injectivity for von Neumann algebras and show that the amenability of a group is characterized by the injectivity of the group von Neumann algebra.

Résumé

Dans cette thèse, nous étudions le propriété de la moyennabilité sur divers objets. Nous commençons par étudier les groupes moyennables : nous prouvons quelques caractérisations utiles de la moyennabilité et démontrons la puissance du concept en prouvant qu'il donne plusieurs résultats utiles. Nous sommes particulièrement intéressés par les théorèmes de stabilité pour les répresentations approximatives et par le théorème d'unitarisation que nous démontrons : l'existence de résultats similaires de stabilité et d'unitarisation pour des objets moyennables est un théme commun dans cette thèse. Nous considérons ensuite le cas des algèbres de Banach moyennables et démontrons qu'elles admettent un théorème de stabilité pour les répresentations approximatives. Dans la section suivante, nous démontrons que les C*-algèbres nucléaires sont moyennables en tant ce que algèbres de Banach et utilisons ce fait pour obtenir un résultat d'unitarisabilité pour les C*-algèbres nucléaires. Nous considérons aussi le concept de moyennabilité forte pour les C*-algèbres. Enfin, nous introduisons le concept d'injectivité pour les algèbres de von Neumann et prouvons que la moyennabilité d'un groupe est caracterisée par l'injectivité de l'algèbre de von Neumann du groupe.

Acknowledgements

I would like to thank my supervisor, Mikaël Pichot, for his patient supervision, much helpful advice, and insightful remarks far too numerous for me to count.

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Chapter 1

Introduction

The concept of amenability originated in 1929 when J. von Neumann observed that the existence of a paradoxical decomposition of the unit ball in $\mathbb{R}^3$ and non-existence of such a decomposition of the unit disk in $\mathbb{R}^2$ could be attributed to the fact that $E(2)$ admits a finitely additive left invariant probability measure, while $E(3)$ does not. This led him to define a discrete group as *amenable* if it admits a finitely additive left invariant probability measure. The concept was later extended to Banach algebras by B. E. Johnson in 1972 (see [Joh72]) and has since grown into a deep and fascinating area of research with applications in diverse fields, such as abstract harmonic analysis, operator algebras, and ergodic theory.

One of the many applications of amenability is to the question of the unitarizability of bounded representations. In 1950, M. M. Day and J. Dixmier proved (independently) that uniformly bounded representations of amenable groups on Hilbert spaces are equivalent to unitary representations (see [Day50] and [Dix50]). An analogous result for strongly amenable C*-algebras was obtained by J. Bunce in 1972 (see [Bun72a], [Bun72b]), and U. Haagerup's proof that nuclear C*-algebras admit virtual diagonals belonging to a certain closed convex hull (see [Haa83]) later permitted the extension of the result to nuclear C*-algebras.

Another useful application of amenability is to the question of the stability of certain "approximate representations". In 1982, D. Kazhdan proved that if $\varphi : G \to U(\mathcal{H})$ is a map of a discrete amenable group G into the unitary group $U(\mathcal{H})$ of some Hilbert space $\mathcal{H}$ which is "approximately" a representation in the sense that the operator norm of $\varphi(xy) - \varphi(x)\,\varphi(y)$ is small for all $x, y \in G$, then there exists some genuine unitary representation $\pi : G \to U(\mathcal{H})$

such that $\varphi(x)$ is close in operator norm to $\pi(x)$ for all $x \in G$ (see [Kaz82]). A few years later, similar results were obtained for amenable Banach algebras by B. E. Johnson (see [Joh88]). In the previous decade, there appears to have been a surge of renewed interest in such stability theorems for approximate representations, with results of this type appearing in (among others) [BOT13], [Sht13], [GH17], and [dCOT19].

In this thesis, we aim to introduce various notions of amenability for groups, Banach algebras, C*-algebras, and von Neumann algebras and to prove some of the unitarizability and stability results we have mentioned. We assume that the reader is familiar with the basic theory of Banach algebras, C*-algebras, and von Neumann algebras and freely use results from these areas without supplying proof or reference.

Chapter 2

Amenable Groups

The primary references for this chapter are [Pie84] and [BO08].

2.1 Amenability

If $\mathcal{H}$ is a Hilbert space, then we let $B(\mathcal{H})$ denote the *-algebra of bounded operators on $\mathcal{H}$.

Definition 2.1.1. Let G be a group and $\mathcal{H}$ be a Hilbert space. A map $\varphi : G \to B(\mathcal{H})$ is *uniformly bounded* with respect to the operator norm on $B(\mathcal{H})$ if

$$\|\varphi\| = \sup\{\|\varphi(x)\|_{\text{op}} : x \in G\} < \infty$$

If G is a group, $\mathcal{H}$ is a Hilbert space, and $\mathcal{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra, then we let $\ell^\infty(G, \mathcal{M})$ denote the space of uniformly bounded maps from G to $\mathcal{M}$; when $\mathcal{M} = \mathbb{C}$, we usually write $\ell^\infty(G)$ rather than $\ell^\infty(G, \mathbb{C})$. For every bounded operator $T \in B(\mathcal{H})$, we let $\kappa_T : G \to B(\mathcal{H})$ denote the map defined by $\kappa_T(x) = T$. If $\Psi : \ell^\infty(G, \mathcal{M}) \to \mathcal{M}$ is a linear map, then we often employ the notation $\Psi_x\, \varphi(x) = \Psi(\varphi)$.

Definition 2.1.2. Let G be a discrete group, $\mathcal{H}$ be a Hilbert space, and $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra. An *invariant mean* on $\ell^\infty(G, \mathcal{M})$ is a linear map $\mathbb{E} : \ell^\infty(G, \mathcal{M}) \to \mathcal{M}$ such that $\|\mathbb{E}\|_{\text{op}} = 1$, $\mathbb{E}(\kappa_T) = T$ for all $T \in \mathcal{M}$, and

$$\mathbb{E}_x\, \varphi(sx) = \mathbb{E}_x\, \varphi(x)$$

for all $\varphi \in \ell^\infty(G, \mathcal{M})$ and $s \in G$.

Definition 2.1.3. A discrete group G is *amenable* if $\ell^\infty(G)$ admits an invariant mean.

The following result is due to Tomiyama (see Theorem 1 in [Tom57]).

Proposition 2.1.4. *Let $\mathcal{A}$ be a C*-algebra and $\mathcal{B} \subseteq \mathcal{A}$ be a C*-subalgebra of $\mathcal{A}$. If $\pi : \mathcal{A} \to \mathcal{B}$ is a linear map such that $\|\pi\|_{\mathrm{op}} = 1$ and $\pi(x) = x$ for all $x \in \mathcal{B}$, then π is positive and*

$$\pi(axb) = a\,\pi(x)\,b$$

for all $x \in \mathcal{A}$ and $a, b \in \mathcal{B}$.

Proof. By replacing π with $\pi^{**} : \mathcal{A}^{**} \to \mathcal{B}^{**}$ if necessary, it may be assumed without loss of generality that $\mathcal{A}$ is unital and $\mathcal{B}$ is a W*-algebra. Let $p \in \mathcal{B}$ be a projection; then $px \in \mathcal{B}$ and $(1-p)\,x \in \mathcal{B}$ for all $x \in \mathcal{B}$, and thus $\pi(p\,\pi(x)) = p\,\pi(x)$ and $\pi((1-p)\,\pi(x)) = (1-p)\,\pi(x)$ for all $x \in \mathcal{A}$. It follows that

$$\begin{aligned}(1+2n+n^2)\,\|(1-p)\,\pi(px)\|^2 &= \|(1-p)\,\pi(px + n\,(1-p)\,\pi(px))\|^2 \\ &\le \|1-p\|^2\,\|\pi\|_{\mathrm{op}}^2\,\|px + n\,(1-p)\,\pi(px)\|^2 \\ &\le \|px + n\,(1-p)\,\pi(px)\|^2 \\ &\le \|px\|^2 + n^2\,\|(1-p)\,\pi(px)\|^2\end{aligned}$$

for all $n \in \mathbb{N}$, so $\|(1-p)\,\pi(px)\| = 0$, which implies that $(1-p)\,\pi(px) = 0$. Let $q = 1-p$; then q is a projection, so $p\,\pi((1-p)\,x) = (1-q)\,\pi(qx) = 0$, and thus it follows that

$$\pi(px) = p\,\pi(px) = p\,\pi(x)$$

for all $x \in \mathcal{A}$ and every projection $p \in \mathcal{B}$. As $\mathcal{B}$ is a W*-algebra, it is the closed linear span of its projections, so $\pi(ax) = a\,\pi(x)$ for all $x \in \mathcal{A}$ and $a \in \mathcal{B}$. As π is unital and $\|\pi\|_{\mathrm{op}} \le 1$, it follows that π is a positive map, so $\pi(x^*) = \pi(x)^*$ for all $x \in \mathcal{A}$, which in turn implies that $\pi(xb) = \pi(x)\,b$ for all $x \in \mathcal{A}$ and $b \in \mathcal{B}$. ∎

Corollary 2.1.5. *Let G be a discrete group, $\mathcal{H}$ be a Hilbert space, and $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra. If $\mathbb{E} : \ell^\infty(G, \mathcal{M}) \to \mathcal{M}$ is an invariant mean, then $\mathbb{E}$ is positive and*

$$\mathbb{E}(\kappa_S \varphi \kappa_T) = S\,\mathbb{E}(\varphi)\,T$$

for all $S, T \in \mathcal{M}$ and $\varphi \in \ell^\infty(G, \mathcal{M})$.

Proof. Let $\iota : \mathcal{M} \hookrightarrow \ell^\infty(G, \mathcal{M})$ be the canonical inclusion map; then clearly $\|\iota \circ \mathbb{E}\|_{\text{op}} = 1$ and $(\iota \circ \mathbb{E})(\kappa_T) = \iota(T) = \kappa_T$ for all $T \in B(\mathcal{H})$, so it follows by Proposition 2.1.4 that

$$(\iota \circ \mathbb{E})(\kappa_S \varphi \kappa_T) = \kappa_S\, (\iota \circ \mathbb{E})(\varphi)\, \kappa_T = \iota(S)\, (\iota \circ \mathbb{E})(\varphi)\, \iota(T) = \iota(S\, \mathbb{E}(\varphi)\, T)$$

for all $S, T \in \mathcal{M}$ and $\varphi \in \ell^\infty(G, \mathcal{M})$; as ι is an embedding, this implies that

$$\mathbb{E}(\kappa_S \varphi \kappa_T) = S\, \mathbb{E}(\varphi)\, T$$

for all $S, T \in \mathcal{M}$ and $\varphi \in \ell^\infty(G, \mathcal{M})$. Furthermore, as $\iota \circ \mathbb{E}$ is positive and ι is an isometric *-homomorphism, it follows that $\mathbb{E}$ is positive. □

If G is a discrete group, $\mathcal{H}$ is a Hilbert space, $\mathcal{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra, and $\mathbb{E}$ is an invariant mean on $\ell^\infty(G, \mathcal{M})$, then the bounded linear map $\Psi : \ell^\infty(G, \mathcal{M}) \to \mathcal{M}$ defined by

$$\Psi(\varphi) = \tfrac{1}{2}\, \mathbb{E}_x\, \mathbb{E}_y\, (\varphi(xy^{-1}) + \varphi(yx^{-1}))$$

is an invariant mean on $\ell^\infty(G, \mathcal{M})$ such that

$$\Psi_x\, \varphi(sx) = \Psi_x\, \varphi(xs) = \Psi_x\, \varphi(x^{-1}) = \Psi_x\, \varphi(x)$$

for all $\varphi \in \ell^\infty(G, \mathcal{M})$ and $s \in G$. Henceforth, we assume whenever it is convenient to do so that the invariant mean we are working with satisfies this property.

We now demonstrate a simple, yet invaluable construction that allows us to extend an invariant mean on $\ell^\infty(G)$ to an invariant mean on $\ell^\infty(G, \mathcal{M})$, for any Hilbert space $\mathcal{H}$ and von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$.

Proposition 2.1.6. *Let G be a discrete amenable group and $\mathcal{H}$ be a Hilbert space. If $\mathcal{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra, then there exists an invariant mean on $\ell^\infty(G, \mathcal{M})$.*

Proof. As G is amenable, there exists an invariant mean $\mathbb{E} \in \ell^\infty(G)^*$. For every map $\varphi \in \ell^\infty(G, B(\mathcal{H}))$ and vector $\eta \in \mathcal{H}$, define a bounded linear functional $\psi_{\varphi,\eta} \in \mathcal{H}^*$ by

$$\psi_{\varphi,\eta}(\xi) = \mathbb{E}_x\, \langle \xi, \varphi(x)\, \eta \rangle$$

It follows by the Riesz representation theorem that a map $\mathbb{E} : \ell^\infty(G, B(\mathcal{H})) \to B(\mathcal{H})$ can be defined by

$$\psi_{\varphi,\eta}(\xi) = \langle \xi, \mathbb{E}(\varphi)(\eta) \rangle$$

It is easily verified that $\mathbb{E}$ defines an invariant mean on $\ell^\infty(G, B(\mathcal{H}))$. Let $\varphi \in \ell^\infty(G, \mathcal{M})$; then it follows by Corollary 2.1.5 that

$$T\,\mathbb{E}(\varphi) = \mathbb{E}(\kappa_T \varphi) = \mathbb{E}(\varphi \kappa_T) = \mathbb{E}(\varphi)\,T$$

for all $T \in \mathcal{M}'$, which implies that $\mathbb{E}(\varphi) \in \mathcal{M}''$. Furthermore, as $\mathcal{M}$ is a von Neumann algebra, it follows by the von Neumann bicommutant theorem that $\mathcal{M}'' = \mathcal{M}$, so $\mathbb{E}(\varphi) \in \mathcal{M}$ for all $\varphi \in \ell^\infty(G, \mathcal{M})$ and thus $\mathbb{E}|_{\ell^\infty(G,\mathcal{M})}$ is an invariant mean on $\ell^\infty(G, \mathcal{M})$. □

Henceforth, every amenable group G will implicitly be equipped with an invariant mean on $\ell^\infty(G, \mathcal{M})$ for every Hilbert space $\mathcal{H}$ and von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$; abusing notation, we will write $\mathbb{E}$ for all such invariant means.

Remark 2.1.7. It follows by construction that the invariant mean $\mathbb{E}$ defined in Proposition 2.1.6 satisfies the following properties:

- $\mathbb{E}_x \langle \xi, \varphi(x)\,\eta \rangle = \langle \xi, \mathbb{E}_x\, \varphi(x)\,\eta \rangle$
- $\mathbb{E}_x \langle \varphi(x)\,\xi, \eta \rangle = \langle \mathbb{E}_x\, \varphi(x)\,\xi, \eta \rangle$

for all $\varphi \in \ell^\infty(G, \mathcal{M})$ and $\xi, \eta \in \mathcal{H}$; henceforth, we assume whenever it is convenient to do so that the invariant mean we are working with satisfies this property.

The following result is a slight variation on Proposition 2.2 in [dCOT19].

Proposition 2.1.8. *Let G be a discrete amenable group and $\mathcal{H}$ be a Hilbert space. If $\varphi : G \to B(\mathcal{H})$ is a uniformly bounded map such that $\|\varphi\| \leq 1$, then the map $\psi : G \to B(\mathcal{H})$ defined by*

$$\psi(x) = \mathbb{E}_y\, \varphi(y)^*\, \varphi(yx)$$

is positive definite and $\|\psi\| \leq 1$. Furthermore, if φ is a unitary map, then ψ is unital.

Proof. It follows by the right invariance, additivity, and positivity of $\mathbb{E}$ that

$$\begin{aligned}\sum_{x,y\in F} \langle \psi(x^{-1}y)\,\xi_y, \xi_x\rangle &= \sum_{x,y\in F} \mathbb{E}_z\,\langle \varphi(z)^*\,\varphi(zx^{-1}y)\,\xi_y, \xi_x\rangle \\ &= \sum_{x,y\in F} \mathbb{E}_z\,\langle \varphi(zx)^*\,\varphi(zy)\,\xi_y, \xi_x\rangle \\ &= \sum_{x,y\in F} \mathbb{E}_z\,\langle \varphi(zy)\,\xi_y, \varphi(zx)\,\xi_x\rangle \\ &= \mathbb{E}_z\,\langle \sum_{y\in F} \varphi(zy)\,\xi_y, \sum_{x\in F} \varphi(zx)\,\xi_x\rangle \geq 0\end{aligned}$$

for every finite subset $F \subseteq G$ and all $\xi_x, \xi_y \in \mathcal{H}$. □

2.2 The Følner and Reiter Conditions

In this section we prove a couple of useful characterizations of amenability.

Definition 2.2.1. Let G be a group. The *left translation action* of G on $\ell^\infty(G)$ is the action defined by

$$(s \cdot f)(x) = f(s^{-1}x)$$

We typically write sf rather than $s \cdot f$, except where it might lead to confusion.

Definition 2.2.2. A countable discrete group G satisfies the *Reiter condition* if there exists a net (μ_α) of nonnegative functions in $\ell^1(G)$ with unit norm such that $\|s\mu_\alpha - \mu_\alpha\|_1 \to 0$ for all $s \in G$.

If X is a set, then we let $|X|$ denote the cardinality of X.

Definition 2.2.3. A discrete group G satisfies the *Følner condition* if there exists a sequence (F_n) of finite subsets of G such that

$$\frac{|sF_n \triangle F_n|}{|F_n|} \to 0$$

for all $s \in G$.

If X is a nonempty countable set, then every absolutely summable function $\mu \in \ell^1(X)$ induces a bounded linear functional $\tau_\mu \in \ell^\infty(X)^*$ under the definition

$$\tau_\mu(f) = \sum_{x \in X} \mu(x)\, f(x)$$

If $\mathcal{A}$ is a C*-algebra, then we let $\mathrm{St}(\mathcal{A})$ denote the set of states on $\mathcal{A}$.

Theorem 2.2.4. *Let X be a nonempty countable set. The set $W \subseteq \mathrm{St}(\ell^\infty(X))$ defined by*

$$W = \{\tau_\mu \in \ell^\infty(X)^* : \mu \geq 0,\ \|\mu\|_1 = 1\}$$

is dense in $\mathrm{St}(\ell^\infty(X))$ with respect to the weak- topology.*

Proof. Suppose that W is not dense in $\mathrm{St}(\ell^\infty(X))$ with respect to the weak-* topology; then there exists a state $\psi \in \mathrm{St}(\ell^\infty(X)) \setminus \overline{W}$, where $\overline{W}$ denotes the closure of W with respect to the weak-* topology, and it follows by the Hahn-Banach separation theorem that there exists a function $f_0 \in \ell^\infty(X)$ and a real numbers $\alpha \in \mathbb{R}$ such that

$$\operatorname{Re} \tau(f_0) < \alpha < \operatorname{Re} \psi(f_0)$$

for all $\tau \in \overline{W}$. Let $f = f_0 + f_0^* + \|f_0 + f_0^*\|_\infty\, \chi_X$ and $\beta = 2\alpha + \|f_0 + f_0^*\|_\infty$; then

$$\tau(f) < \beta < \psi(f)$$

for all $\tau \in \overline{W}$ and f is nonnegative. As f is nonnegative, there exists a sequence (x_n) of elements in X such that $f(x_n) \to \|f\|_\infty$. Let (μ_n) be the sequence of nonnegative functions in $\ell^1(X)$ defined by $\mu_n = \chi_{\{x_n\}}$. Then $\tau_{\mu_n}(f) \to \|f\|_\infty$ and $\tau(f) \leq \|f\|_\infty$ for all $\tau \in W$, so $\|f\|_\infty = \sup\{\tau(f) : \tau \in W\}$. However, this is a contradiction, as

$$\|f\|_\infty = \sup\{\tau(f) : \tau \in W\} \leq \beta < \psi(f) \leq |\psi(f)| \leq \|\psi\|_{\mathrm{op}}\, \|f\|_\infty = \|f\|_\infty$$

It thus follows that W is dense in $\mathrm{St}(\ell^\infty(X))$ with respect to the weak-* topology. □

Lemma 2.2.5. *Let X be a nonempty countable set. If $\mu \in \ell^1(X)$ is an absolutely summable function, then*

$$\psi(\mu) = \tau_\mu(\psi \circ \chi_{\{\cdot\}})$$

for all $\psi \in \ell^1(X)^*$.

Proof. As X is a countable set, it is the limit of an ascending chain $F_1 \subseteq F_2 \subseteq \cdots \subseteq X$ of finite subsets. Let (μ_n) be the sequence of finitely supported functions in $\ell^1(X)$ defined by $\mu_n = \mu\chi_{F_n}$. As μ_n is finitely supported for all $n \in \mathbb{N}$, it follows by linearity that

$$\psi(\mu_n) = \sum_{x \in X} \mu_n(x)\, \psi(\chi_{\{x\}}) = \tau_{\mu_n}(\psi \circ \chi_{\{\cdot\}})$$

for all $\psi \in \ell^1(X)^*$ and $n \in \mathbb{N}$; as $\|\mu_n - \mu\|_1 \to 0$, it then follows by the continuity of ψ that $\psi(\mu) = \tau_\mu(\psi \circ \chi_{\{\cdot\}})$ for all $\psi \in \ell^1(X)^*$. ☐

Proposition 2.2.6. *Let X be a nonempty countable set. If (μ_α) is a net in $\ell^1(X)$ such that $(\tau_{\mu_\alpha}) \to \tau_\mu$ in the weak-* topology, then $(\mu_\alpha) \to \mu$ in the weak topology.*

Proof. It follows by Lemma 2.2.5 that

$$\psi(\mu_\alpha) = \tau_{\mu_\alpha}(\psi \circ \chi_{\{\cdot\}}) \to \tau_\mu(\psi \circ \chi_{\{\cdot\}}) = \psi(\mu)$$

for all $\psi \in \ell^1(X)^*$, and thus $(\mu_\alpha) \to \mu$ in the weak topology. ☐

Lemma 2.2.7. *Let X be a countable set. If $\mu \in \ell^1(X)$ and $\nu \in \ell^1(X)$ are nonnegative absolutely summable functions, then*

$$\|\mu - \nu\|_1 = \int_0^\infty |\{x \in X : \mu(x) > t\} \triangle \{x \in X : \nu(x) > t\}|\, dt$$

Proof. As

$$|\mu(x) - \nu(x)| = \int_0^\infty |\chi_{(0,\mu(x))}(t) - \chi_{(0,\nu(x))}(t)|\, dt$$

for all $x \in X$ and

$$\sum_{x \in X} |\chi_{(0,\mu(x))}(t) - \chi_{(0,\nu(x))}(t)| = |\{x \in X : \mu(x) > t\} \triangle \{x \in X : \nu(x) > t\}|$$

for all $t \in [0, \infty)$, it follows by Tonelli's theorem that

$$\|\mu - \nu\|_1 = \sum_{x \in X} |\mu(x) - \nu(x)| = \int_0^\infty |\{x \in X : \mu(x) > t\} \triangle \{x \in X : \nu(x) > t\}|\, dt$$ ☐

Theorem 2.2.8. *Let G be a countable discrete group. The following are equivalent:*

1. *G is amenable*

2. G satisfies the Reiter condition

3. G satisfies the Følner condition

Proof. Suppose that G is amenable and let $\mathbb{E} \in \ell^\infty(G)^*$ be an invariant mean. By Theorem 2.2.4, there exists a net (μ_α) of nonnegative functions in $\ell^1(G)$ with unit norm such that $(\tau_{\mu_\alpha}) \to \mathbb{E}$ in the weak-* topology. Let $K = \{s_1, \ldots, s_k\} \subseteq G$ be any finite subset of G. As

$$\tau_{s\mu_\alpha - \mu}(f) = \tau_{s\mu_\alpha}(f) - \tau_\mu(f) = \tau_{\mu_\alpha}(s^{-1}f) - \tau_\mu(f) \to \mathbb{E}(s^{-1}f) - \mathbb{E}(f) = 0$$

for all $f \in \ell^\infty(G)$ and $s \in G$, it follows by Proposition 2.2.6 that $(s\mu_\alpha - \mu) \to 0$ in the weak topology for all $s \in G$, and thus in turn $(s_1\mu_\alpha - \mu_\alpha, \ldots, s_k\mu_\alpha - \mu_\alpha) \to (0, \ldots, 0)$ in the weak topology on $\ell^1(G)^{\oplus|K|}$. By Mazur's lemma, there exists a sequence (ν_n) of finite convex combinations of elements in $\{\mu_\alpha\}$ such that

$$\sum_{s \in K} \|s\nu_n - \nu_n\|_1 = \|(s_1\nu_n, \ldots, s_k\nu_n)\|_{\ell^1(G)^{\oplus|K|}} \to 0$$

This implies that $\|s\nu_n - \nu_n\|_1 \to 0$ for all $s \in K$, and thus for every $\varepsilon > 0$ and every finite subset $K \subseteq G$ there exists a nonnegative function $\nu_{\varepsilon,K} \in \ell^1(G)$ with unit norm such that $\|s\nu_{\varepsilon,K} - \nu_{\varepsilon,K}\|_1 < \varepsilon$ for all $s \in K$. As G is countable, it is the limit of an ascending chain $F_1 \subseteq F_2 \subseteq \cdots \subseteq G$ of finite subsets; it then follows that $\|s\nu_{n^{-1},F_n} - \nu_{n^{-1},F_n}\|_1 \to 0$ for all $s \in G$, and so the implication (1) $\Rightarrow$ (2) holds. Suppose that G satisfies the Reiter condition, let $K \subseteq G$ be any finite subset of G, let $\varepsilon > 0$, and let $\mu \in \ell^1(G)$ be a nonnegative function such that $\|\mu\|_1 = 1$ and $\|s\mu - \mu\|_1 < \varepsilon$ for all $s \in K$. For every real number $t \in [0, 1)$, let $F_t = \{x \in G : \mu(x) > t\}$; then it follows by Lemma 2.2.7 that

$$\int_0^1 |sF_t \triangle F_t|\, dt = \|s\mu - \mu\|_1 < \varepsilon \|\mu\|_1 = \varepsilon \int_0^1 |F_t|\, dt$$

so there exists some $t \in [0, 1)$ such that $|sF_t \triangle F_t| < \varepsilon |F_t|$. As this holds for every finite subset $K \subseteq G$ and all $\varepsilon > 0$, it follows that G satisfies the Følner condition, and thus the implication (2) $\Rightarrow$ (3) holds. Suppose now that G satisfies the Følner condition, let (F_n) be a sequence of finite subsets of G witnessing the Følner condition, and let (μ_n) be the sequence of nonnegative functions in $\ell^1(G)$ defined by $\mu_n = \frac{1}{|F_n|}\chi_{F_n}$. Then

$$\|s\mu_n - \mu_n\|_1 = \frac{1}{|F_n|} \sum_{x \in G} |\chi_{sF_n}(x) - \chi_{F_n}(x)| = \frac{|sF_n \triangle F_n|}{|F_n|} \to 0$$

for all $s \in G$, and so the implication (3) $\Rightarrow$ (2) holds. Suppose now that G satisfies the Reiter condition and let (μ_α) be a net in $\ell^1(G)$ indexed by a directed set $\mathcal{I}$ witnessing the Reiter condition. As $\|\tau_{\mu_\alpha}\|_{\text{op}} \le \|\mu_\alpha\|_1 = 1$ for all $\alpha \in \mathcal{I}$, it follows by the Banach-Alaoglu theorem that (τ_{μ_α}) has a subnet (τ_{μ_β}) converging to some bounded linear functional $\tau \in \ell^\infty(G)^*$ such that $\|\tau\|_{\text{op}} \le 1$ with respect to the weak-* topology. As

$$\tau_{\mu_\alpha}(\chi_G) = \sum_{x\in G} \mu_\alpha(x)\,\chi_G(x) = \sum_{x\in G} \mu_\alpha(x) = \sum_{x\in G} |\mu_\alpha(x)| = \|\mu_\alpha\|_1 = 1$$

for all $\alpha \in \mathcal{I}$ and $\tau_{\mu_\beta}(\chi_G) \to \tau(\chi_G)$, in turn $\tau(\chi_G) = 1$, and thus τ is a state on $\ell^\infty(G)$. Furthermore, as $|\tau_{\mu_\beta}(s^{-1}f) - \tau_{\mu_\beta}(f)| \to |\tau(s^{-1}f) - \tau(f)|$ and

$$|\tau_{\mu_\alpha}(s^{-1}f) - \tau_{\mu_\alpha}(f)| = |\tau_{s\mu_\alpha - \mu_\alpha}(f)| \le \|\tau_{s\mu_\alpha-\mu_\alpha}\|_{\text{op}}\,\|f\|_\infty \le \|s\mu_\alpha - \mu_\alpha\|_1\,\|f\|_\infty \to 0$$

for all $f \in \ell^\infty(G)$ and $s \in G$, it follows that $|\tau(s^{-1}f) - \tau(f)| = 0$ for all $f \in \ell^\infty(G)$ and $s \in G$, so τ is an invariant mean on $\ell^\infty(G)$ and the implication (2) $\Rightarrow$ (1) holds. □

2.3 Examples of Amenable Groups

Proposition 2.3.1. *Let $H \hookrightarrow G \twoheadrightarrow K$ be a short exact sequence of discrete groups. If H and K are amenable, then G is amenable.*

Proof. As $H \hookrightarrow G \twoheadrightarrow K$ is a short exact sequence, there exists a normal subgroup $N \trianglelefteq G$ such that $H \cong N$ and $K \cong G/N$; it thus suffices to show that if $N \trianglelefteq G$ is a normal subgroup such that N and G/N are amenable, then G is amenable. Let τ_N and $\tau_{G/N}$ be invariant means on $\ell^\infty(N)$ and $\ell^\infty(G/N)$, respectively. For every uniformly bounded function $f \in \ell^\infty(G)$, let $g_f \in \ell^\infty(G/N)$ be the uniformly bounded function defined by

$$g_f(xN) = \tau_N(x^{-1}f)$$

As τ_N is an invariant mean, this is clearly well-defined. Now let $\tau \in \text{St}(\ell^\infty(G))$ be the state defined by

$$\tau(f) = \tau_{G/N}(g_f)$$

As

$$g_{sf}(xN) = \tau_N(x^{-1}sf) = \tau_N((s^{-1}x)^{-1}f) = g_f(s^{-1}xN)$$

for all $s, x \in G$, it follows by the left invariance of $\tau_{G/N}$ that τ is an invariant mean on $\ell^\infty(G)$, and thus G is amenable. □

Proposition 2.3.2. *If G is a finite discrete group, then G is amenable.*

Proof. It suffices to let $\mathbb{E}$ be the usual average defined by $\mathbb{E}(f) = |G|^{-1} \sum_{x \in G} f(x)$. □

Proposition 2.3.3. *The discrete group $(\mathbb{Z}, +)$ is amenable.*

Proof. For every $n \in \mathbb{N}$, let $F_n = \{1, \ldots, n\}$; then

$$\frac{|(m + F_n) \triangle F_n|}{|F_n|} \leq \frac{2m}{n} \to 0$$

for all $m \in \mathbb{Z}$, so (F_n) is a sequence of finite subsets witnessing the Følner condition for $(\mathbb{Z}, +)$. It then follows by Theorem 2.2.8 that $(\mathbb{Z}, +)$ is amenable. □

Proposition 2.3.4. *Let G be a countable discrete group. If G is the limit of an ascending chain $F_1 \subseteq F_2 \subseteq \cdots \subseteq G$ of discrete amenable subgroups, then G is amenable.*

Proof. Let $K \subseteq G$ be a finite subset and $\varepsilon > 0$; as K is finite, there exists some $n \in \mathbb{N}$ such that $K \subseteq F_n$. As F_n is amenable, it follows by Theorem 2.2.8 that there exists a finite subset $F \subseteq G$ such that $|sF \triangle F| < \varepsilon |F|$ for all $s \in K$; as there exists such a subset for every finite subset $K \subseteq G$ and all $\varepsilon > 0$, it follows that G satisfies the Følner condition, and thus Theorem 2.2.8 implies that G is amenable. □

Proposition 2.3.5. *If G is a countable discrete abelian group, then G is amenable.*

Proof. As G is countable and abelian, it is the limit of an ascending chain of finitely generated abelian subgroups. It follows by Proposition 2.3.1, Proposition 2.3.2, and Proposition 2.3.3 that every finitely generated discrete abelian group is amenable, and thus Proposition 2.3.4 implies that G is amenable. □

2.4 Unitarizability of Bounded Representations

The following unitarizability result is due to Day (see [Day50]) and Dixmier (see [Dix50]).

Theorem 2.4.1 (Day-Dixmier Theorem). *Let G be a discrete amenable group and $\mathcal{H}$ be a Hilbert space. If $\pi : G \to B(\mathcal{H})$ is a uniformly bounded representation, then there exists an invertible operator $V \in B(\mathcal{H})$ such that $\|V\|_{\mathrm{op}}, \|V^{-1}\|_{\mathrm{op}} \leq \|\pi\|$ and such that $V^{-1}\,\pi(\,\cdot\,)\,V$ is a unitary representation on $\mathcal{H}$.*

Proof. Define a new inner product $\langle \cdot, \cdot \rangle_U$ on $\mathcal{H}$ by

$$\langle \xi, \eta \rangle_U = \mathbb{E}_y \, \langle \pi(y^{-1})\, \xi, \pi(y^{-1})\, \eta \rangle$$

Then $\|\xi\|_U \leq \|\pi\|\,\|\xi\|$ and $\|\xi\| \leq \|\pi\|\,\|\xi\|_U$ for all $\xi \in \mathcal{H}$, so $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_U$ are equivalent inner products, and thus $(\mathcal{H}, \langle \cdot, \cdot \rangle_U)$ is also a Hilbert space. It follows by the left invariance of the mean that

$$\begin{aligned}
\langle \pi(x)\, \xi, \eta \rangle_U &= \mathbb{E}_y \, \langle \pi(y^{-1})\, \pi(x)\, \xi, \pi(y^{-1})\, \eta \rangle \\
&= \mathbb{E}_y \, \langle \pi(y^{-1}x)\, \xi, \pi(y^{-1})\, \eta \rangle \\
&= \mathbb{E}_y \, \langle \pi(y^{-1})\, \xi, \pi(y^{-1}x^{-1})\, \eta \rangle \\
&= \mathbb{E}_y \, \langle \pi(y^{-1})\, \xi, \pi(y^{-1})\, \pi(x^{-1})\, \eta \rangle \\
&= \langle \xi, \pi(x^{-1})\, \eta \rangle_U
\end{aligned}$$

for all $x \in G$ and $\xi, \eta \in \mathcal{H}$, so it follows that π is a unitary representation on $(\mathcal{H}, \langle \cdot, \cdot \rangle_U)$. Let $T : (\mathcal{H}, \langle \cdot, \cdot \rangle) \to (\mathcal{H}, \langle \cdot, \cdot \rangle_U)$ be the identity operator defined by $T(\xi) = \xi$. Let $T = V|T|$ be the polar decomposition of T; as T is invertible, V can be taken to be unitary. Then $V^{-1}\,\pi(\,\cdot\,)\,V$ is a unitary representation on $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, and furthermore

$$\|V\xi\| \leq \|\pi\|\,\|V\xi\|_U = \|\pi\|\,\|\xi\|$$

and

$$\|V^{-1}\xi\| = \|\xi\|_U \leq \|\pi\|\,\|\xi\|$$

for all $\xi \in \mathcal{H}$, so $\|V\|_{\mathrm{op}}, \|V^{-1}\|_{\mathrm{op}} \leq \|\pi\|$. □

It is an open question whether or not unitarizability of uniformly bounded representations is equivalent to amenability for discrete groups.

2.5 Unitarily Invariant Norms

If $\mathcal{H}$ is a Hilbert space, then we let $U(\mathcal{H})$ denote the group of unitary operators on $\mathcal{H}$; similarly, if $\mathcal{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra, then we let $U(\mathcal{M})$ denote the group of unitary operators in $\mathcal{M}$. Throughout the remainder of this chapter, we allow semi-norms to take values in $[0, \infty]$.

Definition 2.5.1. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra. A semi-norm $\|\cdot\|$ on $\mathcal{M}$ is *unitarily invariant* if

$$\|UTV\| = \|T\|$$

for all $T \in \mathcal{M}$ and $U, V \in U(\mathcal{M})$.

Examples of unitarily invariant semi-norms include the operator norm and, if $\mathcal{M}$ is equipped with a tracial state τ, the Schatten p-norms arising from τ.

Proposition 2.5.2. *Let $\mathcal{H}$ be a Hilbert space. If $\|\cdot\|$ is a unitarily invariant semi-norm on $B(\mathcal{H})$, then the following relations hold:*

$$\|RTS\| \leq \|R\|_{\text{op}} \, \|T\| \, \|S\|_{\text{op}}$$

$$\|T\| = \|T^*\| = \||T|\|$$

$$\|T^*T\| = \|TT^*\|$$

$$0 \leq R \leq S \implies \|R\| \leq \|S\|$$

for all $R, S, T \in B(\mathcal{H})$.

Proof. Suppose that $\|R\|_{\text{op}}, \|S\|_{\text{op}} < 1$; then by the Russo-Dye theorem there exist unitaries $U_1, \ldots, U_n, V_1, \ldots, V_m \in U(\mathcal{H})$ and $\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_m \in [0, 1]$ such that $R = \sum_{i=1}^n \mu_i U_i$, $S = \sum_{j=1}^m \nu_j V_j$, and $\sum_{i=1}^n \mu_i = \sum_{j=1}^m \nu_j = 1$. It thus follows by the triangle inequality and the unitary invariance of the norm that

$$\|RTS\| \leq \sum_{i=1}^n \sum_{j=1}^m \mu_i \nu_j \|U_i T V_j\| = \sum_{i=1}^n \mu_i \sum_{j=1}^m \nu_i \|T\| = \|T\|$$

Now let R and S be arbitrary and let $R_n = (\|R\|_{\text{op}} + \frac{1}{n})^{-1} R$ and $S_n = (\|S\|_{\text{op}} + \frac{1}{n})^{-1} S$ for every $n \in \mathbb{N}$. Then $\|R_n\|_{\text{op}}, \|S_n\|_{\text{op}} < 1$, so it follows that

$$\begin{aligned}\|RTS\| &= (\|R\|_{\text{op}} + \tfrac{1}{n})\,(\|S\|_{\text{op}} + \tfrac{1}{n})\,\|R_n T S_n\| \\ &\leq (\|R\|_{\text{op}} + \tfrac{1}{n})\,(\|S\|_{\text{op}} + \tfrac{1}{n})\,\|T\| \\ &\to \|R\|_{\text{op}}\,\|T\|\,\|S\|_{\text{op}}\end{aligned}$$

and thus the first relation holds. Let $T = U|T|$ be the polar decomposition of T; then

$$\|T\| = \|U|T|\| \leq \|U\|_{\text{op}}\,\||T|\| \leq \||T|\| = \|U^*T\| \leq \|U^*\|_{\text{op}}\,\|T\| \leq \|T\|$$

so $\|T\| = \||T|\|$, and

$$\|T^*\| = \||T|U^*\| \leq \||T|\|\,\|U^*\|_{\text{op}} \leq \||T|\| = \||T|^*\| = \|T^*U\| \leq \|T^*\|\,\|U\|_{\text{op}} \leq \|T^*\|$$

so $\|T^*\| = \||T|\|$. Furthermore,

$$\|T^*T\| = \||T|^2\| = \||T||T|^*\| = \|U^*TT^*U\| \leq \|TT^*\| = \|U|T||T|U^*\| \leq \||T||T|\| = \|T^*T\|$$

so $\|T^*T\| = \|TT^*\|$. For the final relation, R and S are positive operators, so $R^{1/2}$ and $S^{1/2}$ are well-defined, and thus it follows by Douglas' lemma that there exists a bounded operator $T \in B(\mathcal{H})$ such that $\|T\|_{\text{op}} \leq 1$ and $R^{1/2} = T^* S^{1/2}$. Then $R^{1/2} = (R^{1/2})^* = S^{1/2} T$, so $R = R^{1/2} R^{1/2} = T^* S T$ and

$$\|R\| = \|T^*ST\| \leq \|T^*\|_{\text{op}}\,\|S\|\,\|T\|_{\text{op}} = \|T\|_{\text{op}}^2\,\|S\| \leq \|S\| \qquad \square$$

Lemma 2.5.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $\|\cdot\|$ be a unitarily invariant semi-norm on $B(\mathcal{H})$. If $S \in B(\mathcal{K})$ is a normal operator, $T : \mathcal{H} \to \mathcal{K}$ is a bounded operator, and $f \in \mathcal{B}(\sigma(S))$ and $g \in \mathcal{B}(\sigma(S))$ are Borel functions such that $0 \leq f(t) \leq g(t)$ for all $t \in \sigma(S)$, then*

$$\|T^*\, f(S)\, T\| \leq \|T^*\, g(S)\, T\|$$

Proof. As $0 \leq f(t) \leq g(t)$ for all $t \in \sigma(S)$, it follows that $0 \leq f(S) \leq g(S)$. Furthermore, if $R \in B(\mathcal{K})$ is a positive operator, then $R^{1/2}$ is well-defined, so $T^*RT = T^*R^{1/2}R^{1/2}T$ is also positive; thus $0 \leq T^*\, f(S)\, T \leq T^*\, g(S)\, T$, and it follows by Proposition 2.5.2 that

$$\|T^*\, f(S)\, T\| \leq \|T^*\, g(S)\, T\| \qquad \square$$

We will require a number of inequalities.

Proposition 2.5.4. *Let $\mathcal{H}$ be a Hilbert space and $\|\cdot\|$ be a unitarily invariant semi-norm on $B(\mathcal{H})$. If $T \in B(\mathcal{H})$ is a positive operator, then*

$$\|I_{\mathcal{H}} - T\| \le \|I_{\mathcal{H}} - T^2\|$$

Proof. As $|1-t| \le |1-t^2|$ for all $t \in [0,\infty)$, it follows by Proposition 2.5.2 and Lemma 2.5.3 that

$$\|I_{\mathcal{H}} - T\| = \||I_{\mathcal{H}} - T|\| \le \||I_{\mathcal{H}} - T^2|\| = \|I_{\mathcal{H}} - T^2\| \qquad \square$$

Proposition 2.5.5. *Let $0 < \delta \le 1$, $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and $\|\cdot\|$ be a unitarily invariant semi-norm on $B(\mathcal{H})$. If $S \in B(\mathcal{K})$ is a positive operator and $T : \mathcal{H} \to \mathcal{K}$ is a bounded operator, then*

$$\|T^*\, \chi_{[\delta,1]}(S)\, T\| \le \tfrac{1}{\delta}\, \|T^* S T\|$$

Proof. As $0 \le \chi_{[\delta,1]}(t) \le \frac{1}{\delta}\, t$ for all $t \in [0,\infty)$, the proposition follows by Lemma 2.5.3. $\square$

Proposition 2.5.6. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $\|\cdot\|$ be a unitarily invariant semi-norm on $B(\mathcal{H})$. If $S \in B(\mathcal{K})$ is a positive contraction and $T : \mathcal{H} \to \mathcal{K}$ is a contraction, then*

$$\|T^*\,(I_{\mathcal{K}} - S)\, T\| \le \|I_{\mathcal{H}} - T^* S T\|$$

Proof. As S and T^*T are positive contractions, in turn $I_{\mathcal{K}} - S$ and $I_{\mathcal{H}} - T^*T$ are positive operators, so $0 \le T^*\,(I_{\mathcal{K}} - S)\, T \le I_{\mathcal{H}} - T^* S T$. The proposition then follows immediately by Proposition 2.5.2. $\square$

Proposition 2.5.7. *Let $0 \le \delta < 1$, $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and $\|\cdot\|$ be a unitarily invariant semi-norm on $B(\mathcal{H})$. If $S \in B(\mathcal{K})$ is a positive contraction and $T : \mathcal{H} \to \mathcal{K}$ is a contraction, then*

$$\|I_{\mathcal{H}} - T^*\, \chi_{[\delta,1]}(S)\, T\| \le \|I_{\mathcal{H}} - T^*T\| + \tfrac{1}{1-\delta}\, \|I_{\mathcal{H}} - T^* S T\|$$

Proof. As $0 \leq 1 - \chi_{[\delta,1]}(t) \leq \frac{1}{1-\delta}(1-t)$ for all $t \in [0,1]$, it follows by Lemma 2.5.3 and Proposition 2.5.6 that

$$\begin{aligned}\|T^*T - T^*\,\chi_{[\delta,1]}(S)\,T\| &= \|T^*\,(I_{\mathcal{K}} - \chi_{[\delta,1]}(S))\,T\| \\ &\leq \tfrac{1}{1-\delta}\,\|T^*\,(I_{\mathcal{K}} - S)\,T\| \\ &\leq \tfrac{1}{1-\delta}\,\|I_{\mathcal{H}} - T^*ST\|\end{aligned}$$

It then follows by the triangle inequality that

$$\begin{aligned}\|I_{\mathcal{H}} - T^*\,\chi_{[\delta,1]}(S)\,T\| &\leq \|I_{\mathcal{H}} - T^*T\| + \|T^*T - \chi_{[\delta,1]}(S)\,T\| \\ &\leq \|I_{\mathcal{H}} - T^*T\| + \tfrac{1}{1-\delta}\,\|I_{\mathcal{H}} - T^*ST\|\end{aligned}$$

□

Proposition 2.5.8. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $\|\cdot\|$ be a unitarily invariant semi-norm on $B(\mathcal{H})$. If $S \in B(\mathcal{K})$ is a contraction and $T : \mathcal{H} \to \mathcal{K}$ is a bounded operator, then*

$$\|T^*ST\| \leq \|T^*T\|$$

Proof. As S^*S is positive and $\|S^*S\|_{\text{op}} = \|S\|_{\text{op}}^2 \leq 1$, it follows that $I_{\mathcal{K}} - S^*S$ is positive, so $T^*T - T^*S^*ST$ is a positive operator. Let $T^*ST = U|T^*ST|$ be the polar decomposition of T^*ST; then

$$0 \leq (TU - ST)^*(TU - ST) = U^*T^*TU - 2\,|T^*ST| + T^*S^*ST$$

It follows that $0 \leq |T^*ST| \leq \frac{1}{2}\,(U^*T^*TU + T^*S^*ST) \leq \frac{1}{2}\,(U^*T^*TU + T^*T)$, and thus

$$\begin{aligned}\|T^*ST\| &= \||T^*ST|\| \\ &\leq \tfrac{1}{2}\,\|U^*T^*TU + T^*T\| \\ &\leq \tfrac{1}{2}\left(\|U^*T^*TU\| + \|T^*T\|\right) \\ &\leq \|T^*T\|\end{aligned}$$

□

Proposition 2.5.9. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $\|\cdot\|$ be a unitarily invariant semi-norm on $B(\mathcal{H})$. If $S \in B(\mathcal{K})$ is a positive contraction and $T : \mathcal{H} \to \mathcal{K}$ is a contraction, then*

$$\|I_{\mathcal{H}} - T^*T\| \leq \|I_{\mathcal{H}} - T^*ST\|$$

Proof. As S and T^*T are positive contractions, $I_{\mathcal{K}} - S$ and $I_{\mathcal{H}} - T^*T$ are positive operators, which implies in turn that $I_{\mathcal{H}} - T^*ST - (I_{\mathcal{H}} - T^*T) = T^*(I_{\mathcal{K}} - S)T$ is a positive operator and $0 \leq I_{\mathcal{H}} - T^*T \leq I_{\mathcal{H}} - T^*ST$. The proposition then follows by Proposition 2.5.2. □

If $\mathcal{H}$ is a separable Hilbert space, then every unitarily invariant semi-norm $\|\cdot\|$ on $B(\ell^2(\mathbb{N}))$ induces a unitarily invariant semi-norm $\|\cdot\|_{B(\mathcal{H})}$ on $B(\mathcal{H})$ in the following way: fixing an isometry $U : \mathcal{H} \to \ell^2(\mathbb{N})$, let

$$\|T\|_{B(\mathcal{H})} = \|UTU^*\|$$

It is easily verified that this is indeed a unitarily invariant semi-norm. Let $U : \mathcal{H} \to \ell^2(\mathbb{N})$ and $V : \mathcal{H} \to \ell^2(\mathbb{N})$ be isometries; then it follows by Theorem 2.5.2 that

$$\|UTU^*\| = \|UV^*VTV^*VU^*\| \leq \|UV^*\|_{\mathrm{op}} \|VTV^*\| \|VU^*\|_{\mathrm{op}} = \|VTV^*\|$$

for all $T \in B(\mathcal{H})$, and swapping U and V shows that $\|UTU^*\| = \|VTV^*\|$ for all $T \in B(\mathcal{H})$; thus $\|\cdot\|_{B(\mathcal{H})}$ does not depend on the choice of isometry $U : \mathcal{H} \to \ell^2(\mathbb{N})$. Moreover, it can easily be shown that if $\|\cdot\|$ is the operator norm on $B(\ell^2(\mathbb{N}))$, then $\|\cdot\|_{B(\mathcal{H})}$ is the operator norm on $B(\mathcal{H})$, and that if $\mathcal{K}$ is another separable Hilbert space, then

$$\|T^*T\|_{B(\mathcal{H})} = \|TT^*\|_{B(\mathcal{K})}$$

for every bounded operator $T : \mathcal{H} \to \mathcal{K}$.

The following result is a slight variation on Lemma 2.10 from [dCOT19].

Proposition 2.5.10. *Let G be a countable discrete amenable group, $\mathcal{H}$ be a Hilbert space, and $\|\cdot\|$ be a unitarily invariant semi-norm on $B(\mathcal{H})$ which is lower semi-continuous in the weak operator topology. If $\varphi : G \to B(\mathcal{H})$ is a uniformly bounded map, then*

$$\|\mathbb{E}_x \varphi(x)\| \leq \mathbb{E}_x \|\varphi(x)\|$$

Proof. Let $\Phi : \ell^1(G) \to B(\mathcal{H})$ be the bounded linear map defined by

$$\Phi(\mu) = \sum_{x \in G} \mu(x)\, \varphi(x)$$

As this sum converges in operator norm, it also converges in the weak operator topology. For any nonnegative function $\mu \in \ell^1(G)$ and finite subset $F \subseteq G$, it follows by the triangle inequality that

$$\|\sum_{x \in F} \mu(x)\,\varphi(x)\| \leq \sum_{x \in F} \mu(x)\,\|\varphi(x)\| \leq \sum_{x \in G} \mu(x)\,\|\varphi(x)\| = \tau_\mu(\|\varphi(\,\cdot\,)\|)$$

As G is countable, it is the limit of an ascending chain $F_1 \subseteq F_2 \subseteq \cdots \subseteq G$ of finite subsets. As $\|\cdot\|$ is lower semi-continuous with respect to the weak operator topology, it follows that

$$\|\Phi(\mu)\| = \|\sum_{x \in G} \mu(x)\,\varphi(x)\| \leq \liminf_{n \to \infty} \|\sum_{x \in F_n} \mu(x)\,\varphi(x)\| \leq \tau_\mu(\|\varphi(\,\cdot\,)\|)$$

By Theorem 2.2.4, there exists a net (μ_α) of nonnegative functions in $\ell^1(G)$ with unit norm such that (τ_{μ_α}) converges to $\mathbb{E} \in \ell^\infty(G)^*$ in the weak-* topology on $\ell^\infty(G)^*$. It follows that

$$\begin{aligned}
\langle \Phi(\mu_\alpha)\,\xi, \eta\rangle &= \langle \sum_{x \in G} \mu_\alpha(x)\,\varphi(x)\,\xi, \eta\rangle \\
&= \sum_{x \in G} \mu_\alpha(x)\,\langle \varphi(x)\,\xi, \eta\rangle \\
&= \tau_{\mu_\alpha}(\langle \varphi(\,\cdot\,)\,\xi, \eta\rangle) \\
&\to \mathbb{E}_x\,\langle \varphi(x)\,\xi, \eta\rangle
\end{aligned}$$

for all $\xi, \eta \in \mathcal{H}$, so $(\Phi(\mu_\alpha)) \to \mathbb{E}(\varphi)$ in the weak operator topology, and thus

$$\|\mathbb{E}(\varphi)\| \leq \liminf_\alpha \|\Phi(\mu_\alpha)\| \leq \liminf_\alpha \tau_{\mu_\alpha}(\|\varphi(\,\cdot\,)\|) = \mathbb{E}_x\,\|\varphi(x)\| \qquad \square$$

Remark 2.5.11. The proof of the above proposition implies that $\mathbb{E}(\varphi) \in \overline{\operatorname{conv}\{\varphi(x) : x \in G\}}$, where the closure is with respect to the weak operator topology.

2.6 Stability of Approximate Representations

In this section, we will prove that discrete amenable groups admit certain stability results for approximate representations. We begin by specifying what we mean by an "approximate representation".

Definition 2.6.1. Let G be a discrete group and $\mathcal{H}$ be a Hilbert space. A map $\varphi : G \to B(\mathcal{H})$ is an *ε-representation* with respect to a semi-norm $\|\cdot\|$ on $B(\mathcal{H})$ if

$$\|\varphi(xy) - \varphi(x)\,\varphi(y)\| \leq \varepsilon$$

for all $x, y \in G$.

In order to prove the promised stability results, we will require the following version of Stinespring's dilation theorem.

Theorem 2.6.2 (Stinespring's Dilation Theorem). *Let G be a discrete group and $\mathcal{H}$ be a Hilbert space. If $\psi : G \to B(\mathcal{H})$ is a positive definite map, then there exists a Hilbert space $\mathcal{K}$, a bounded operator $U : \mathcal{H} \to \mathcal{K}$, and a unitary representation $\pi : G \to U(\mathcal{K})$ such that*

$$\psi(x) = U^*\,\pi(x)\,U$$

for all $x \in G$. Furthermore, if G is countable and $\mathcal{H}$ is separable, then $\mathcal{K}$ can be taken to be separable.

Proof. Let $C_{\text{fin}}(G, \mathcal{H})$ denote the vector space of finitely supported functions from G to $\mathcal{H}$ and define a sesquilinear form on $C_{\text{fin}}(G, \mathcal{H})$ by

$$(f_1, f_2)_\psi = \sum_{x,y\,\in G} \langle \psi(x^{-1}y)\, f_1(y), f_2(x)\rangle_{\mathcal{H}}$$

As ψ is positive definite, $(\cdot, \cdot)_\psi$ is positive semidefinite, so it follows by the Cauchy-Schwarz inequality that $\mathcal{N} = \{f : (f, f)_\psi = 0\}$ is a subspace of $C_{\text{fin}}(G, \mathcal{H})$. Define an inner product on $C_{\text{fin}}(G, \mathcal{H})/\mathcal{N}$ by

$$\langle [f_1], [f_2]\rangle_{\mathcal{K}} = (f_1, f_2)_\psi$$

where $[f]$ denotes the equivalence class of f in $C_{\text{fin}}(G, \mathcal{H})/\mathcal{N}$. Let $\mathcal{K}$ be the completion of $C_{\text{fin}}(G, \mathcal{H})/\mathcal{N}$ with respect to the inner product, let $U : \mathcal{H} \to \mathcal{K}$ be the linear operator defined by

$$U(\xi) = [\delta_e\, \xi]$$

and let $\pi : G \to U(\mathcal{K})$ be the left regular representation on $C_{\text{fin}}(G, \mathcal{H})/\mathcal{N}$, where

$$(\delta_x\, \xi)(y) = \begin{cases} \xi, & y = x \\ 0, & y \neq x \end{cases}$$

Then

$$\begin{aligned} \langle U^* \,\pi(x)\, U\xi, \eta\rangle_{\mathcal{H}} &= \langle \pi(x)\, U\xi, U\eta\rangle_{\mathcal{K}} \\ &= \langle \pi(x)\, [\delta_e\, \xi], [\delta_e\, \eta]\rangle_{\mathcal{K}} \\ &= \langle [\delta_x\, \xi], [\delta_e\, \eta]\rangle_{\mathcal{K}} \\ &= \sum_{y,z\,\in G} \langle \psi(y^{-1}z)\, (\delta_x\, \xi)(z), (\delta_e\, \eta)(y)\rangle_{\mathcal{H}} \\ &= \langle \psi(x)\, \xi, \eta\rangle_{\mathcal{H}} \end{aligned}$$

for all $\xi, \eta \in \mathcal{H}$, so $\psi(x) = U^* \,\pi(x)\, U$ for all $x \in G$. □

Remark 2.6.3. As $U^*U = U^* \,\pi(e)\, U = \psi(e)$, it follows that $\|U\|_{\text{op}}^2 = \|\psi(e)\|_{\text{op}}$; furthermore, if ψ is unital, then U is an isometry.

Corollary 2.6.4. *Let G be a discrete group and $\mathcal{H}$ be a Hilbert space. If $\psi : G \to B(\mathcal{H})$ is a positive definite map, then $\psi(x^{-1}) = \psi(x)^*$ for all $x \in G$.*

The proof of the following theorem is based on that of Theorem 3.1 in [dCOT19].

Theorem 2.6.5. *Let $\varepsilon \geq 0$, G be a countable discrete amenable group, $\mathcal{H}$ be a separable Hilbert space, and $\|\cdot\|$ be a unitarily invariant semi-norm on $B(\ell^2(\mathbb{N}))$ which is lower semi-continuous in the weak operator topology. If $\psi : G \to B(\mathcal{H})$ is a positive definite map such that $\|\psi\| \leq 1$ and $\mathbb{E}_x \|I_{\mathcal{H}} - \psi(x)^*\, \psi(x)\| \leq \varepsilon$, then there exists a separable Hilbert space $\mathcal{K}$, a partial isometry $U : \mathcal{H} \to \mathcal{K}$, and a unitary representation $\pi : G \to U(\mathcal{K})$ such that*

$$\|\psi(x) - U^* \,\pi(x)\, U\| \leq 3\, \|I_{\mathcal{H}} - \psi(e)\| + 4\varepsilon$$

for all $x \in G$ and

$$\|I_{\mathcal{H}} - U^*U\| \leq \|I_{\mathcal{H}} - \psi(e)\| + \tfrac{4}{3}\varepsilon, \qquad \|I_{\mathcal{K}} - UU^*\| \leq 4\varepsilon$$

where $\|\cdot\|$ is defined on $B(\mathcal{H})$ and $B(\mathcal{K})$ by fixing isometries $\mathcal{H} \hookrightarrow \ell^2(\mathbb{N})$ and $\mathcal{K} \hookrightarrow \ell^2(\mathbb{N})$.

Proof. As ψ is positive definite and $\|\psi\| \leq 1$, it follows by Stinespring's dilation theorem that there exists a separable Hilbert space $\mathcal{K}$, a bounded operator $U : \mathcal{H} \to \mathcal{K}$, and a unitary representation $\pi : G \to U(\mathcal{K})$ such that $\psi(x) = U^* \pi(x) U$. Let $T = \mathbb{E}_x \pi(x) UU^* \pi(x)^*$ and $P = \chi_{[1/4,1]}(T)$. As $\|U\|_{\mathrm{op}}^2 = \|\psi(e)\|_{\mathrm{op}} \leq 1$, it follows by Proposition 2.5.7 and Proposition 2.5.10 that

$$\begin{aligned}
\|I_{\mathcal{H}} - U^*PU\| &\leq \tfrac{4}{3} \|I_{\mathcal{H}} - U^*TU\| + \|I_{\mathcal{H}} - U^*U\| \\
&\leq \tfrac{4}{3} \mathbb{E}_x \|I_{\mathcal{H}} - U^* \pi(x) UU^* \pi(x)^* U\| + \|I_{\mathcal{H}} - U^*U\| \\
&= \tfrac{4}{3} \mathbb{E}_x \|I_{\mathcal{H}} - \psi(x)^* \psi(x)\| + \|I_{\mathcal{H}} - \psi(e)\| \\
&\leq \|I_{\mathcal{H}} - \psi(e)\| + \tfrac{4}{3}\varepsilon
\end{aligned}$$

Let $S = PU$ and let $S = V|S|$ be the polar decomposition of S. Then

$$\|I_{\mathcal{H}} - |S|^2\| = \|I_{\mathcal{H}} - S^*S\| = \|I_{\mathcal{H}} - U^*PU\| \leq \|I_{\mathcal{H}} - \psi(e)\| + \tfrac{4}{3}\varepsilon$$

and as SS^* is a positive operator and $\|SS^*\|_{\mathrm{op}}, \|V\|_{\mathrm{op}} \leq 1$, it follows by Proposition 2.5.9 that

$$\|I_{\mathcal{H}} - V^*V\| \leq \|I_{\mathcal{H}} - V^*SS^*V\| = \|I_{\mathcal{H}} - |S||S|^*\| = \|I_{\mathcal{H}} - |S|^2\| \leq \|I_{\mathcal{H}} - \psi(e)\| + \tfrac{4}{3}\varepsilon$$

As P is an orthogonal projection, $P\mathcal{K}$ is a closed subspace of $\mathcal{K}$, therefore a Hilbert space. As $T \in \pi(G)'$, in turn $P \in \pi(G)'$ as well; as P is an orthogonal projection, it is the identity on $P\mathcal{K}$. It follows that the map $\pi_0 : G \to U(P\mathcal{K})$ defined by $\pi_0(x) = P\,\pi(x)\,P$ is a unitary representation on $P\mathcal{K}$, and thus it follows by Proposition 2.5.8 and Proposition 2.5.6 that

$$\begin{aligned}
\|\psi(x) - S^* \pi_0(x)\, S\| &= \|\psi(x) - U^* \pi_0(x)\, U\| \\
&= \|U^* \pi(x)\, U - U^* \pi_0(x)\, U\| \\
&= \|U^* P^\perp \pi(x)\, P^\perp U\| \\
&\leq \|U^* P^\perp P^\perp U\| \\
&= \|U^* P^\perp U\| \\
&\leq \|I_{\mathcal{H}} - U^*PU\| \\
&\leq \|I_{\mathcal{H}} - \psi(e)\| + \tfrac{4}{3}\varepsilon
\end{aligned}$$

for all $x \in G$, which implies by Proposition 2.5.2 and Proposition 2.5.4 that

$$\begin{aligned}
\|\psi(x) - V^* \pi_0(x) V\| &\le \|\psi(x) - S^* \pi_0(x) S\| + \|S^* \pi_0(x) S - S^* \pi_0(x) V\| \\
&\quad + \|S^* \pi_0(x) V - V^* \pi_0(s) V\| \\
&\le \|\psi(x) - S^* \pi_0(x) S\| + \|S^* \pi_0(x) V\|_{\mathrm{op}} \||S| - I_{\mathcal{H}}\| \\
&\quad + \||S| - I_{\mathcal{H}}\| \|V^* \pi_0(x) V\|_{\mathrm{op}} \\
&\le \|\psi(x) - S^* \pi_0(x) S\| + 2 \|I_{\mathcal{H}} - |S|\| \\
&\le \|\psi(x) - S^* \pi_0(x) S\| + 2 \|I_{\mathcal{H}} - |S|^2\| \\
&\le \|\psi(x) - S^* \pi_0(x) S\| + 2 \|I_{\mathcal{H}} - \psi(e)\| + \tfrac{8}{3}\varepsilon \\
&\le 3 \|I_{\mathcal{H}} - \psi(e)\| + 4\varepsilon
\end{aligned}$$

for all $x \in G$. Furthermore, as $|S||S|$ is a positive operator and $\||S||S|\|_{\mathrm{op}}, \|V\|_{\mathrm{op}} \le 1$, it follows by Proposition 2.5.9, Proposition 2.5.2, Proposition 2.5.5, Proposition 2.5.10, and Proposition 2.5.6 that

$$\begin{aligned}
\|P - VV^*\| &\le \|P - V|S||S|V^*\| \\
&= \|P - SS^*\| \\
&= \|P\,(I_{\mathcal{K}} - UU^*)\,P\| \\
&= \|P\,(I_{\mathcal{K}} - UU^*)^{1/2}\,(I_{\mathcal{K}} - UU^*)^{1/2}\,P\| \\
&= \|(I_{\mathcal{K}} - UU^*)^{1/2}\,P\,(I_{\mathcal{K}} - UU^*)^{1/2}\| \\
&\le 4\,\|(I_{\mathcal{K}} - UU^*)^{1/2}\,T\,(I_{\mathcal{K}} - UU^*)^{1/2}\| \\
&\le 4\,\mathbb{E}_x \|(I_{\mathcal{K}} - UU^*)^{1/2}\,\pi(x)\,UU^*\,\pi(x)^*\,(I_{\mathcal{K}} - UU^*)^{1/2}\| \\
&= 4\,\mathbb{E}_x \|U^*\,\pi(x)^*\,(I_{\mathcal{K}} - UU^*)\,\pi(x)\,U\| \\
&\le 4\,\mathbb{E}_x \|I_{\mathcal{H}} - U^*\,\pi(x)^*\,UU^*\,\pi(x)\,U\| \\
&= 4\,\mathbb{E}_x \|I_{\mathcal{H}} - \psi(x)^*\,\psi(x)\| \\
&\le 4\varepsilon
\end{aligned}$$

The proof is then complete upon renaming V to U, $P\mathcal{K}$ to $\mathcal{K}$, and π_0 to π. □

The following result is a slight variation on Theorem 5.2 in [dCOT19]; it is our first example of a stability theorem for approximate representations of amenable groups.

Theorem 2.6.6. *Let $\varepsilon \geq 0$, G be a countable discrete amenable group, $\mathcal{H}$ be a separable Hilbert space, and $\|\cdot\|$ be a unitarily invariant semi-norm on $B(\ell^2(\mathbb{N}))$ which is lower semi-continuous in the weak operator topology. If $\varphi : G \to U(\mathcal{H})$ is an ε-representation with respect to $\|\cdot\|$, then there exists a separable Hilbert space $\mathcal{K}$, a partial isometry $U : \mathcal{H} \to \mathcal{K}$, and a unitary representation $\pi : G \to U(\mathcal{K})$ such that*

$$\|\varphi(x) - U^* \pi(x) U\| \leq 9\varepsilon$$

for all $x \in G$ and

$$\|I_{\mathcal{H}} - U^*U\| \leq \tfrac{8}{3}\varepsilon, \qquad \|I_{\mathcal{K}} - UU^*\| \leq 8\varepsilon$$

where $\|\cdot\|$ is defined on $B(\mathcal{H})$ and $B(\mathcal{K})$ by fixing isometries $\mathcal{H} \hookrightarrow \ell^2(\mathbb{N})$ and $\mathcal{K} \hookrightarrow \ell^2(\mathbb{N})$.

Proof. Let $\psi : G \to B(\mathcal{H})$ be the unital positive definite map defined by

$$\psi(x) = \mathbb{E}_y \, \varphi(y)^* \, (yx)$$

It then follows by Proposition 2.5.10 and the unitary invariance of $\|\cdot\|$ that

$$\begin{aligned}
\|\varphi(x) - \psi(x)\| &= \|\varphi(x) - \mathbb{E}_y \, \varphi(y)^* \, \varphi(yx)\| \\
&\leq \mathbb{E}_y \, \|\varphi(x) - \varphi(y)^* \, \varphi(yx)\| \\
&= \mathbb{E}_y \, \|\varphi(y) \, \varphi(x) - \varphi(yx)\| \leq \varepsilon
\end{aligned}$$

for all $x \in G$. As $\psi(x)^* \, \psi(x)$ is a positive operator and $\|\psi(x)^* \, \psi(x)\|_{\mathrm{op}} \leq \|\psi\|^2 = 1$ for all $x \in G$, it follows that $I_{\mathcal{H}} - \psi(x)^* \, \psi(x)$ is a positive operator for all $x \in G$. Furthermore, $|\varphi(x) - \psi(x)|^2$ is also positive for all $x \in G$; as

$$I_{\mathcal{H}} - \psi(x)^* \, \psi(x) + |\varphi(x) - \psi(x)|^2 = \mathbb{E}_y \, |\varphi(yx) - \varphi(y) \, \varphi(x)|^2$$

for all $x \in G$, it follows that

$$0 \leq I_{\mathcal{H}} - \psi(x)^* \, \psi(x) \leq \mathbb{E}_y \, |\varphi(yx) - \varphi(y) \, \varphi(x)|^2$$

for all $x \in G$. This then implies by Proposition 2.5.2 and Proposition 2.5.10 that

$$\begin{aligned}\|I_{\mathcal{H}} - \psi(x)^* \psi(x)\| &\leq \|\mathbb{E}_y |\varphi(yx) - \varphi(y)\varphi(x)|^2\| \\ &\leq \mathbb{E}_y \||\varphi(yx) - \varphi(y)\varphi(x)|^2\| \\ &\leq \mathbb{E}_y \|\varphi(yx) - \varphi(y)\varphi(x)\| \|\varphi(yx) - \varphi(y)\varphi(x)\|_{\mathrm{op}} \\ &\leq \mathbb{E}_y \|\varphi(yx) - \varphi(y)\varphi(x)\| (\|\varphi(yx)\|_{\mathrm{op}} + \|\varphi(y)\varphi(x)\|_{\mathrm{op}}) \\ &= 2\,\mathbb{E}_y \|\varphi(yx) - \varphi(y)\varphi(x)\| \leq 2\varepsilon\end{aligned}$$

for all $x \in G$; the proposition then follows by Theorem 2.6.5. □

Remark 2.6.7. If $\|\cdot\|$ is submultiplicative, then

$$\|I_{\mathcal{H}} - \psi(x)^* \psi(x)\| \leq \mathbb{E}_y \|\varphi(yx) - \varphi(y)\varphi(x)\|^2 \leq \varepsilon^2$$

for all $x \in G$, and the bounds can be improved to

$$\|\varphi(x) - U^* \pi(x) U\| \leq \varepsilon + 4\varepsilon^2, \qquad \|I_{\mathcal{H}} - U^*U\| \leq \tfrac{4}{3}\varepsilon^2, \qquad \|I_{\mathcal{K}} - UU^*\| \leq 4\varepsilon^2$$

The following stability result is due to Kazhdan (see Theorem 1 in [Kaz82]), though his original proof was very different from the one we have supplied.

Theorem 2.6.8. *Let $0 < \varepsilon < \frac{1}{4}$, G be a countable discrete amenable group, and $\mathcal{H}$ be a separable Hilbert space. If $\varphi : G \to U(\mathcal{H})$ is an ε-representation with respect to the operator norm, then there exists a unitary representation $\pi : G \to U(\mathcal{H})$ such that*

$$\|\varphi(x) - \pi(x)\|_{\mathrm{op}} < 2\varepsilon$$

for all $x \in G$.

Proof. It follows by Theorem 2.6.6 and Remark 2.6.7 that there exists a separable Hilbert space $\mathcal{K}$, a partial isometry $U : \mathcal{H} \to \mathcal{K}$, and a unitary representation $\pi : G \to U(\mathcal{K})$ such that

$$\|\varphi(x) - U^* \pi(x) U\|_{\mathrm{op}} < 2\varepsilon$$

for all $x \in G$ and

$$\|I_{\mathcal{H}} - U^*U\|_{\mathrm{op}} < 1, \qquad \|I_{\mathcal{K}} - UU^*\|_{\mathrm{op}} < 1$$

It then follows that U is unitary, so $U^* \pi(\cdot) U$ is a unitary representation on $\mathcal{H}$. □

Chapter 3

Amenable Banach Algebras

The primary references for this chapter are [Joh72] and [Run02].

3.1 Amenability

Definition 3.1.1. Let $\mathcal{A}$ be a Banach algebra. A Banach space X is a *Banach $\mathcal{A}$-module* if it is a two-sided $\mathcal{A}$-module and there exists a real number $M > 0$ such that

$$\|a \cdot x\| \leq M \|a\| \|x\|, \qquad \|x \cdot a\| \leq M \|a\| \|x\|$$

for all $a \in \mathcal{A}$ and $x \in X$.

If $\mathcal{A}$ is a Banach algebra and X is a Banach $\mathcal{A}$-module, then the dual space X^* can be made into a Banach $\mathcal{A}$-module by equipping it with the actions defined by

$$(a \cdot f)(x) = f(x \cdot a), \qquad (f \cdot a)(x) = f(a \cdot x)$$

where $a \in \mathcal{A}$, $x \in X$, and $f \in X^*$.

Definition 3.1.2. Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-module. A bounded linear map $D : \mathcal{A} \to X$ is a *derivation* if

$$D(xy) = x \cdot D(y) + D(x) \cdot y$$

for all $x, y \in \mathcal{A}$.

Definition 3.1.3. Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-module. The *inner derivation* induced by a bounded linear functional $f \in X^*$ is the derivation $\delta(f) : \mathcal{A} \to X^*$ defined by

$$\delta(f)(x) = x \cdot f - f \cdot x$$

Definition 3.1.4. A Banach algebra $\mathcal{A}$ is *amenable* if for every Banach $\mathcal{A}$-module X and every derivation $D : \mathcal{A} \to X^*$ there exists a bounded linear functional $f \in X^*$ such that $D = -\delta(f)$.

If X and Y are Banach spaces, then we let $X \widehat{\otimes} Y$ denote the projective tensor product of X and Y.

Definition 3.1.5. Let $\mathcal{A}$ be a Banach algebra. The *multiplication map* on $\mathcal{A}$ is the bounded linear map $\pi : \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$ defined by letting

$$\pi(x \otimes y) = xy$$

for $x, y \in \mathcal{A}$ and extending using the universal property.

We denote the natural embedding of a normed vector space X into its double dual by $\iota_X : X \hookrightarrow X^{**}$.

Definition 3.1.6. Let $\mathcal{A}$ be a Banach algebra and $\pi : \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$ be the multiplication map on $\mathcal{A}$. A *virtual diagonal* for $\mathcal{A}$ is a bounded linear functional $M \in (\mathcal{A} \widehat{\otimes} \mathcal{A})^{**}$ such that

1. $x \cdot M = M \cdot x$
2. $x \cdot \pi^{**}(M) = \iota_{\mathcal{A}}(x)$

for all $x \in \mathcal{A}$.

Definition 3.1.7. Let $\mathcal{A}$ be a Banach algebra and $\pi : \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$ be the multiplication map on $\mathcal{A}$. An *approximate diagonal* for $\mathcal{A}$ is a bounded net (m_α) in $\mathcal{A} \widehat{\otimes} \mathcal{A}$ such that

1. $(x \cdot m_\alpha - m_\alpha \cdot x) \to 0$ in the norm topology on $\mathcal{A} \widehat{\otimes} \mathcal{A}$
2. $(x \cdot \pi(m_\alpha)) \to x$ in the norm topology on $\mathcal{A}$

for all $x \in \mathcal{A}$.

Theorem 3.1.8. *Let $\mathcal{A}$ be a unital Banach algebra. The following are equivalent:*

1. *$\mathcal{A}$ is amenable*

2. *$\mathcal{A}$ has a virtual diagonal*

3. *$\mathcal{A}$ has an approximate diagonal*

Proof. Suppose that $\mathcal{A}$ is amenable; as $\delta(\iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(1_{\mathcal{A} \widehat{\otimes} \mathcal{A}}))$ is a derivation and

$$\begin{aligned}
\pi^{**}(\delta(\iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(1_{\mathcal{A} \widehat{\otimes} \mathcal{A}}))(x))(f) &= \delta(\iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(1_{\mathcal{A} \widehat{\otimes} \mathcal{A}}))(x)(\pi^*(f)) \\
&= (x \cdot \iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(1_{\mathcal{A} \widehat{\otimes} \mathcal{A}}) - \iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(1_{\mathcal{A} \widehat{\otimes} \mathcal{A}}) \cdot x)(\pi^*(f)) \\
&= \iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(1_{\mathcal{A} \widehat{\otimes} \mathcal{A}})(\pi^*(f) \cdot x - x \cdot \pi^*(f)) \\
&= (\pi^*(f) \cdot x - x \cdot \pi^*(f))(1_{\mathcal{A} \widehat{\otimes} \mathcal{A}}) \\
&= \pi^*(f)(x) - \pi^*(f)(x)
\end{aligned}$$

for all $x \in \mathcal{A} \widehat{\otimes} \mathcal{A}$ and $f \in \mathcal{A}^*$, it follows that there exists a bounded linear functional $h \in \ker \pi^{**}$ such that $\delta(\iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(1_{\mathcal{A} \widehat{\otimes} \mathcal{A}})) = -\delta(h)$. Let $M = \iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(1_{\mathcal{A} \widehat{\otimes} \mathcal{A}}) + h$; then

$$x \cdot M - M \cdot x = \delta(M)(x) = \delta(\iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(1_{\mathcal{A} \widehat{\otimes} \mathcal{A}}) + h)(x) = \delta(\iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(1_{\mathcal{A} \widehat{\otimes} \mathcal{A}}))(x) + \delta(h)(x) = 0$$

for all $x \in \mathcal{A}$ and

$$\begin{aligned}
(x \cdot \pi^{**}(M))(f) &= \pi^{**}(M)(f \cdot x) \\
&= \pi^{**}(\iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(1_{\mathcal{A} \widehat{\otimes} \mathcal{A}}))(f \cdot x) \\
&= \iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(1_{\mathcal{A} \widehat{\otimes} \mathcal{A}})(\pi^*(f \cdot x)) \\
&= \pi^*(f \cdot x)(1_{\mathcal{A} \widehat{\otimes} \mathcal{A}}) \\
&= (f \cdot x)(\pi(1_{\mathcal{A} \widehat{\otimes} \mathcal{A}})) \\
&= (f \cdot x)(1_{\mathcal{A}}) \\
&= f(x) \\
&= \iota_{\mathcal{A}}(x)(f)
\end{aligned}$$

for all $x \in \mathcal{A}$ and $f \in \mathcal{A}^*$, so $x \cdot \pi^{**}(M) = \iota_{\mathcal{A}}(x)$ for all $x \in \mathcal{A}$, which proves the implication (1) $\Rightarrow$ (2). Suppose now that $\mathcal{A}$ has a virtual diagonal $M \in (\mathcal{A} \widehat{\otimes} \mathcal{A})^{**}$. As $\iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(\mathcal{A} \widehat{\otimes} \mathcal{A})$ is dense in $(\mathcal{A} \widehat{\otimes} \mathcal{A})^{**}$ with respect to the weak-* topology, there exists a bounded net (m_α) in $\mathcal{A} \widehat{\otimes} \mathcal{A}$ such that $(\iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(m_\alpha)) \to M$ in the weak-* topology. In particular, this implies that

$$
\begin{aligned}
f(x \cdot m_\alpha - m_\alpha \cdot x) &= (f \cdot x - x \cdot f)(m_\alpha) \\
&= \iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(m_\alpha)(f \cdot x - x \cdot f) \\
&\to M(f \cdot x - x \cdot f) \\
&= (x \cdot M - M \cdot x)(f)
\end{aligned}
$$

for all $x \in \mathcal{A}$ and $f \in (\mathcal{A} \widehat{\otimes} \mathcal{A})^*$, and thus $(x \cdot m_\alpha - m_\alpha \cdot x) \to 0$ in the weak topology for all $x \in \mathcal{A}$. Furthermore, as

$$
\begin{aligned}
f(x \cdot \pi(m_\alpha)) &= (f \cdot x)(\pi(m_\alpha)) \\
&= \pi^*(f \cdot x)(m_\alpha) \\
&= \iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(m_\alpha)(\pi^*(f \cdot x)) \\
&\to M(\pi^*(f \cdot x)) \\
&= \pi^{**}(M)(f \cdot x) \\
&= (x \cdot \pi^{**}(M))(f) \\
&= \iota_{\mathcal{A}}(x)(f) \\
&= f(x)
\end{aligned}
$$

for all $x \in \mathcal{A}$ and $f \in \mathcal{A}^*$, it follows that $(x \cdot \pi(m_\alpha)) \to x$ in the weak topology for all $x \in \mathcal{A}$. Passing to the product topology (which coincides with the weak topology on the product space) and applying Mazur's lemma then yields a bounded net (m_β) such that $(x \cdot m_\beta - m_\beta \cdot x) \to 0$ in the norm topology on $\mathcal{A} \widehat{\otimes} \mathcal{A}$ and $(x \cdot \pi(m_\beta)) \to x$ in the norm topology on $\mathcal{A}$ for all $x \in \mathcal{A}$; thus the implication (2) $\Rightarrow$ (3) holds. Suppose now that $\mathcal{A}$ has an approximate diagonal (m_α) indexed by a directed set $\mathcal{I}$ and let

$$
m_\alpha = \sum_{n=1}^{\infty} x_{n,\alpha} \otimes y_{n,\alpha}
$$

where $\sum_{n=1}^{\infty}\|x_{n,\alpha}\|\,\|y_{n,\alpha}\| < \infty$ for all $\alpha \in \mathcal{I}$. Let X be an arbitrary Banach $\mathcal{A}$-module and let $D : \mathcal{A} \to X^*$ be an arbitrary derivation. Let (f_α) be the net in X^* defined by

$$f_\alpha = -\sum_{n=1}^{\infty} x_{n,\alpha} \cdot D(y_{n,\alpha})$$

As (f_α) is bounded, it follows by the Banach-Alaoglu theorem that it has a subnet (f_β) converging to some bounded linear functional $f \in X^*$ in the weak-* topology. Then

$$\begin{aligned}
D(a)(x \cdot \pi(m_\beta)) &= D(a)\Big(x \cdot \sum_{n=1}^{\infty} x_{n,\beta}\, y_{n,\beta}\Big) \\
&= \sum_{n=1}^{\infty} (x_{n,\beta}\, y_{n,\beta} \cdot D(a))(x) \\
&= \sum_{n=1}^{\infty} (x_{n,\beta} \cdot D(y_{n,\beta}\, a) - x_{n,\beta} \cdot D(y_{n,\beta}) \cdot a)(x) \\
&= -\sum_{n=1}^{\infty} ((x_{n,\beta} \cdot D(y_{n,\beta})) \cdot a - a \cdot (x_{n,\beta} \cdot D(y_{n,\beta})))(x) \\
&= -\sum_{n=1}^{\infty} (x_{n,\beta} \cdot D(y_{n,\beta}))(a \cdot x - x \cdot a) \\
&= f_\beta(a \cdot x - x \cdot a) \\
&\to f(a \cdot x - x \cdot a) \\
&= (f \cdot a - a \cdot f)(x) \\
&= -\delta(f)(a)(x)
\end{aligned}$$

for all $a \in \mathcal{A}$, $x \in X$, and $f \in X^*$; as (m_α) is an approximate diagonal, $(x \cdot \pi(m_\beta)) \to x$ in the norm topology for all $x \in X$, so in turn $D(a)(x \cdot \pi(m_\beta)) \to D(a)(x)$ for all $a \in \mathcal{A}$ and $x \in X$, and thus $D = -\delta(f)$, which proves the implication (3) ⇒ (1). □

3.2 Stability of Approximate Representations

As with groups, amenability for Banach algebras allows us to obtain stability theorems for approximate representations. We begin by specifying what we mean by "ε-homomorphism" in the case of Banach algebras.

Definition 3.2.1. Let $\varepsilon \geq 0$ and $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is an *ε-homomorphism* if

$$\|\varphi(xy) - \varphi(x)\,\varphi(y)\| \leq \varepsilon\,\|x\|\,\|y\|$$

for all $x, y \in \mathcal{A}$.

The following result is due to Johnson (see Theorem 3.1 in [Joh88]).

Theorem 3.2.2. *Let $\mathcal{A}$ be an amenable Banach algebra and $\mathcal{B}$ be a Banach algebra with a predual $\mathcal{B}_*$. For every real number $M > 0$, there exist real numbers $\delta, \kappa > 0$ such that if $\varepsilon \in (0, \delta)$ and $\varphi : \mathcal{A} \to \mathcal{B}$ is a bounded ε-homomorphism with $\|\varphi\|_{\mathrm{op}} \leq M$, then there exists an algebra homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ such that*

$$\|\varphi - \pi\|_{\mathrm{op}} \leq \kappa\varepsilon$$

Proof. It may be assumed without loss of generality that $\mathcal{A}$, $\mathcal{B}$, and φ are unital. Let (m_α) be an approximate diagonal for $\mathcal{A}$ indexed by a directed set $\mathcal{I}$ and let

$$m_\alpha = \sum_{j=1}^{\infty} x_{j,\alpha} \otimes y_{j,\alpha}$$

where $\kappa_0 = \sup\{\sum_{j=1}^{\infty} \|x_{j,\alpha}\|\,\|y_{j,\alpha}\| : \alpha \in \mathcal{I}\} < \infty$. Let $\delta = \frac{1}{4}\,(\kappa_0 + M^2\kappa_0^2)^{-1}$ and $\kappa = \frac{5}{3} M\kappa_0$. Let $\psi_0 = \varphi$ and for every $n \in \mathbb{N}$, let $\psi_n : \mathcal{A} \to \mathcal{B}$ be the unital linear map defined by

$$\psi_n(x) = \psi_{n-1}(x) + \sigma_n(x)$$

where the sequence (σ_n) is as defined below. For every $n \in \mathbb{N}$ and $\alpha \in \mathcal{I}$, let $\sigma_{n,\alpha} : \mathcal{A} \to \mathcal{B}$ be the linear map defined by

$$\sigma_{n,\alpha}(x) = \sum_{j=1}^{\infty} \psi_{n-1}(x_{j,\alpha})\,(\psi_{n-1}(y_{j,\alpha} x) - \psi_{n-1}(y_{j,\alpha})\,\psi_{n-1}(x))$$

Fix an isometric isomorphism $\Phi : \mathcal{B} \to (\mathcal{B}_*)^*$. It follows by the Banach-Alaoglu theorem and Tychonoff's theorem that $(\Phi \circ \sigma_{n,\alpha})$ has a subnet $(\Phi \circ \sigma_{n,\beta})$ that converges in the product topology induced by the weak-* topology on $(\mathcal{B}_*)^*$ to some bounded linear map $\omega_n : \mathcal{A} \to (\mathcal{B}_*)^*$. For every $n \in \mathbb{N}$, let $\sigma_n : \mathcal{A} \to \mathcal{B}$ be the linear map defined by

$$\sigma_n(x) = \Phi^{-1}(\omega_n(x))$$

Let $\varepsilon_0 = \varepsilon$ and for every $n \in \mathbb{N}$, let

$$\varepsilon_n = \sup\{\|\psi_n(xy) - \psi_n(x)\,\psi_n(y)\| : \|x\| \le 1,\ \|y\| \le 1\}$$

For every $n \in \mathbb{N}$ and $\alpha \in \mathcal{I}$, let $\Psi_{1,n,\alpha}, \Psi_{2,n,\alpha}, \Psi_{3,n,\alpha}, \Psi_{4,n,\alpha}, \Psi_{5,n,\alpha} : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ be the maps defined by

$$\Psi_{1,n,\alpha}(x,y) = \Big(1_{\mathcal{B}} - \sum_{j=1}^{\infty} \psi_n(x_{j,\alpha}\, y_{j,\alpha})\Big)(\psi_n(xy) - \psi_n(x)\,\psi_n(y))$$

$$\Psi_{2,n,\alpha}(x,y) = \Big(\sum_{j=1}^{\infty} (\psi_n(x_{j,\alpha}\, y_{j,\alpha}) - \psi_n(x_{j,\alpha})\,\psi_n(y_{j,\alpha}))\Big)(\psi_n(xy) - \psi_n(x)\,\psi_n(y))$$

$$\Psi_{3,n,\alpha}(x,y) = \sum_{j=1}^{\infty} (\psi_n(x x_{j,\alpha}) - \psi_n(x)\,\psi_n(x_{j,\alpha}))(\psi_n(y_{j,\alpha}\, y) - \psi_n(y_{j,\alpha})\,\psi_n(y))$$

$$\Psi_{4,n,\alpha}(x,y) = \sum_{j=1}^{\infty} (\psi_n(x_{j,\alpha})\,\psi_n(y_{j,\alpha}\, xy) - \psi_n(x x_{j,\alpha})\,\psi_n(y_{j,\alpha}\, y))$$

$$\Psi_{5,n,\alpha}(x,y) = \sum_{j=1}^{\infty} (\psi_n(x x_{j,\alpha})\,\psi_n(y_{j,\alpha}) - \psi_n(x_{j,\alpha})\,\psi_n(y_{j,\alpha}\, x))\,\psi_n(y)$$

Let $\pi : \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A} \to \mathcal{A}$ be the multiplication map; as $\mathcal{A}$ is unital and (m_α) is an approximate diagonal, it is clear that $\|1_{\mathcal{A}} - \pi(m_\alpha)\| \to 0$, and thus

$$\|\Psi_{1,n,\alpha}(x,y)\| \le \varepsilon_n\, \|x\|\, \|y\|\, \|\psi_n\|_{\mathrm{op}}\, \|1_{\mathcal{A}} - \pi(m_\alpha)\| \to 0$$

for all $x, y \in \mathcal{A}$ and $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, let $\tau_n : \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A} \to \mathcal{B}$ be the linear map defined by letting

$$\tau_n(x \otimes y) = \psi_n(x)\,\psi_n(y)$$

for all $x, y \in \mathcal{A}$ and extending using the universal property; then

$$\begin{aligned}\|\tau_n(m_\alpha \cdot xy - x \cdot m_\alpha \cdot y)\| &= \|(y \cdot \tau_n)(m_\alpha \cdot x - x \cdot m_\alpha)\| \\ &\le \|y \cdot \tau_n\|_{\mathrm{op}}\, \|m_\alpha \cdot x - x \cdot m_\alpha\| \to 0\end{aligned}$$

for all $x, y \in \mathcal{A}$ and $n \in \mathbb{N}$, so

$$\|\Psi_{4,n,\alpha}(x,y)\| = \|\tau_n(m_\alpha \cdot xy - x \cdot m_\alpha \cdot y)\| \to 0$$

for all $x, y \in \mathcal{A}$ and $n \in \mathbb{N}$, and

$$\|\tau_n(x \cdot m_\alpha - m_\alpha \cdot x)\| \leq \|\tau_n\|_{\mathrm{op}} \, \|x \cdot m_\alpha - m_\alpha \cdot x\| \to 0$$

for all $x \in \mathcal{A}$ and $n \in \mathbb{N}$, so

$$\|\Psi_{5,n,\alpha}(x, y)\| \leq \|\tau_n(x \cdot m_\alpha - m_\alpha \cdot x)\| \, \|\psi_n(y)\| \to 0$$

for all $x, y \in \mathcal{A}$ and $n \in \mathbb{N}$. Furthermore, it follows that

$$\|\sigma_{n,\alpha}(x)\| \leq \sum_{j=1}^{\infty} \|\psi_{n-1}(x_{j,\alpha})\| \, \|\psi_{n-1}(y_{j,\alpha} x) - \psi_{n-1}(y_{j,\alpha}) \, \psi_{n-1}(x)\| \leq \kappa_0 \, \|\psi_{n-1}\|_{\mathrm{op}} \, \varepsilon_{n-1} \, \|x\|$$

for all $x \in \mathcal{A}$, $n \in \mathbb{N}$, and $\alpha \in \mathcal{I}$, so

$$\|\sigma_n(x)\| \leq \liminf_{\beta} \|\sigma_{n,\beta}(x)\| \leq \kappa_0 \, \|\psi_{n-1}\|_{\mathrm{op}} \, \varepsilon_{n-1} \, \|x\|$$

for all $x \in \mathcal{A}$ and $n \in \mathbb{N}$, which implies that

$$\|\sigma_n(x) \, \sigma_n(y)\| \leq \|\sigma_n(x)\| \, \|\sigma_n(y)\| \leq \kappa_0^2 \, \|\psi_{n-1}\|_{\mathrm{op}}^2 \, \varepsilon_{n-1}^2 \, \|x\| \, \|y\|$$

for all $x, y \in \mathcal{A}$ and $n \in \mathbb{N}$. Moreover, as

$$\|\psi_n(x_{j,\alpha} y_{j,\alpha}) - \psi_n(x_{j,\alpha}) \, \psi_n(y_{j,\alpha})\| \leq \varepsilon_n \, \|x_{j,\alpha}\| \, \|y_{j,\alpha}\|$$

for all $n \in \mathbb{N}$ and $\alpha \in \mathcal{I}$, it follows that

$$\|\Psi_{2,n,\alpha}(x, y)\| \leq \kappa_0 \, \varepsilon_n^2 \, \|x\| \, \|y\|$$

for all $x, y \in \mathcal{A}$, $n \in \mathbb{N}$, and $\alpha \in \mathcal{I}$, and as

$$\|(\psi_n(x x_{j,\alpha}) - \psi_n(x) \, \psi_n(x_{j,\alpha}))(\psi_n(y_{j,\alpha} y) - \psi_n(y_{j,\alpha}) \, \psi_n(y))\| \leq \varepsilon_n^2 \, \|x\| \, \|y\| \, \|x_{j,\alpha}\| \, \|y_{j,\alpha}\|$$

for all $x, y \in \mathcal{A}$, $n \in \mathbb{N}$, and $\alpha \in \mathcal{I}$, it follows that

$$\|\Psi_{3,n,\alpha}(x, y)\| \leq \kappa_0 \, \varepsilon_n^2 \, \|x\| \, \|y\|$$

for all $x, y \in \mathcal{A}$, $n \in \mathbb{N}$, and $\alpha \in \mathcal{I}$. As

$$\begin{aligned} \sum_{j=1}^{5} \Psi_{j,n-1,\alpha}(x, y) = \psi_{n-1}(xy) - \psi_{n-1}(x) \, \psi_{n-1}(y) + \sigma_{n,\alpha}(xy) \\ - \sigma_{n,\alpha}(x) \, \psi_{n-1}(y) - \psi_{n-1}(x) \, \sigma_{n,\alpha}(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$, $n \in \mathbb{N}$, and $\alpha \in \mathcal{I}$, it follows that

$$\|\psi_n(xy) - \psi_n(x)\,\psi_n(y)\| \leq \|\sigma_n(x)\,\sigma_n(y)\| + \liminf_{\beta} \sum_{j=1}^{5} \|\Psi_{j,n-1,\beta}(x,y)\|$$

for all $x, y \in \mathcal{A}$ and $n \in \mathbb{N}$. As

$$\|\psi_n(x)\| \leq \|\psi_{n-1}(x)\| + \|\sigma_n(x)\| \leq (1 + \kappa_0\,\varepsilon_{n-1})\,\|\psi_{n-1}\|_{\mathrm{op}}\,\|x\|$$

for all $x \in \mathcal{A}$ and $n \in \mathbb{N}$, it follows that $\|\psi_n\|_{\mathrm{op}} \leq (1 + \kappa_0\,\varepsilon_{n-1})\,\|\psi_{n-1}\|_{\mathrm{op}}$ for all $n \in \mathbb{N}$, and furthermore

$$\varepsilon_n = \sup\{\|\psi_n(xy) - \psi_n(x)\,\psi_n(y)\| : \|x\| \leq 1,\ \|y\| \leq 1\} \leq (2\kappa_0 + \kappa_0^2\,\|\psi_{n-1}\|_{\mathrm{op}}^2)\,\varepsilon_{n-1}^2$$

for all $n \in \mathbb{N}$. It follows by induction that $\|\psi_n\|_{\mathrm{op}} \leq (2 - 2^{-n})\,\|\varphi\|_{\mathrm{op}}$ and $\varepsilon_n \leq 2^{-n}\,\varepsilon$ for all $n \in \mathbb{N}$, so $(\varepsilon_n) \to 0$. As

$$\|\psi_n(x) - \psi_{n-1}(x)\| = \|\sigma_n(x)\| \leq 2^{-n}\,(2 - 2^{-n})\,\kappa_0\,\varepsilon\,\|\varphi\|_{\mathrm{op}}\,\|x\|$$

for all $x \in \mathcal{A}$ and $n \in \mathbb{N}$, it follows that $(\psi_n(x))$ converges in norm for all $x \in \mathcal{A}$. Let $\pi : \mathcal{A} \to \mathcal{B}$ be the bounded linear map defined by

$$\pi(x) = \lim_{n \to \infty} \psi_n(x)$$

Then π is an algebra homomorphism and

$$\begin{aligned}
\|\varphi(x) - \pi(x)\| &\leq \sum_{n=1}^{\infty} \|\psi_n(x) - \psi_{n-1}(x)\| \\
&\leq \kappa_0\,\varepsilon\,\|\varphi\|_{\mathrm{op}}\,\|x\| \sum_{n=1}^{\infty} 2^{-n}\,(2 - 2^{-n}) \\
&= \tfrac{5}{3}\,\kappa_0\,\varepsilon\,\|\varphi\|_{\mathrm{op}}\,\|x\| \\
&\leq \tfrac{5}{3}\,M\kappa_0\,\varepsilon\,\|x\|
\end{aligned}$$

for all $x \in \mathcal{A}$. □

Remark 3.2.3. In particular, every von Neumann algebra admits a unique predual, so the above theorem holds whenever $\mathcal{B}$ is a von Neumann algebra.

Chapter 4

Nuclear and Strongly Amenable C*-Algebras

Throughout this chapter we will make frequent reference to the notion of *injectivity* for von Neumann algebras; we refer the reader to Chapter 5 for the definition.

4.1 Nuclearity

Definition 4.1.1. A C*-algebra $\mathcal{A}$ is *nuclear* if for every C*-algebra $\mathcal{B}$ there exists a unique norm on $A \otimes B$ such that the completion of the tensor product with respect to the norm is a C*-algebra.

The following inequality is due to Haagerup (see Theorem 1.1 in [Haa85]). The proof is rather long (exceeding the total length of the remainder of this chapter), so it is with some regret that we choose to omit it for the sake of keeping this thesis at a reasonable length.

Theorem 4.1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. If $V : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ is a bounded bilinear form, then there exist states $\varphi_1, \varphi_2 \in \mathrm{St}(\mathcal{A})$ and states $\psi_1, \psi_2 \in \mathrm{St}(\mathcal{B})$ such that*

$$|V(x, y)| \leq \|V\| \left(\varphi_1(x^*x) + \varphi_2(xx^*)\right)^{1/2} \left(\psi_1(y^*y) + \psi_2(yy^*)\right)^{1/2}$$

for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$. Furthermore, if $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras and V is separately ultraweakly continuous, then the states can be taken to be normal.

The following theorems are due to various results of Choi, Connes, Effros, Elliott, and Lance (see Section 7 in [Con76], Corollary 3.2 in [CE76], Theorem 3 in [CE77], Theorem 6.4 in [EL77], and Theorem 4 in [Ell78]). The proofs are again quite long, so we omit them for the same reason. Fortunately, these will be the last such omissions.

Theorem 4.1.3. *Let $\mathcal{H}$ be a Hilbert space. A separable von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ is injective if and only if there exists an ascending chain $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \cdots \subseteq \mathcal{M}$ of finite dimensional *-subalgebras of $\mathcal{M}$ such that $\mathcal{M} = \overline{\bigcup_{n=1}^{\infty} \mathcal{M}_n}$, where the closure is with respect to the ultraweak topology.*

Theorem 4.1.4. *Let $\mathcal{H}$ be a Hilbert space. A von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ is injective if and only if it is generated by an increasing net of injective countably generated von Neumann subalgebras of $\mathcal{M}$.*

Theorem 4.1.5. *A C*-algebra $\mathcal{A}$ is nuclear if and only if $\mathcal{A}^{**}$ is injective.*

We take this opportunity to fix the notation that we will require throughout the remainder of this section. If $\mathcal{A}$ is a C*-algebra, then we let $\mathrm{Bil}(\mathcal{A})$ denote the set of bounded bilinear forms on $\mathcal{A}$; furthermore, we let $\Phi : \mathcal{A} \times \mathrm{Bil}(\mathcal{A}) \to \ell^\infty(\mathcal{A})$ and $\Psi : \mathcal{A} \times \mathrm{Bil}(\mathcal{A}) \to \ell^\infty(\mathcal{A})$ be the maps defined by

$$\Phi(a, V)(x) = V(ax^*, x), \qquad \Psi(a, V)(x) = V(x^*, xa)$$

and let $\Theta : \mathrm{Bil}(\mathcal{A}) \to \ell^\infty(\mathcal{A})$ be the map defined by

$$\Theta(V)(x) = V(x^*, x)$$

We also let $\kappa : (\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})^* \to \mathrm{Bil}(\mathcal{A})$ and $\kappa_{\mathrm{op}} : (\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})^* \to \mathrm{Bil}(\mathcal{A})$ be the maps defined by

$$\kappa(f)(x, y) = f(x \otimes y), \qquad \kappa_{\mathrm{op}}(f)(x, y) = f(y \otimes x)$$

Finally, if $\mathcal{M}$ is a von Neumann algebra, then we let $\mathrm{Bil}_\sigma(\mathcal{M})$ denote the set of separately ultraweakly continuous bilinear forms on $\mathcal{M}$, let $\mathcal{Z}(\mathcal{M})$ denote the center of $\mathcal{M}$, and let $I(\mathcal{M})$ denote the semigroup of isometries in $\mathcal{M}$.

The following result is due to Haagerup (see Theorem 2.1 in [Haa83]).

Theorem 4.1.6. *Let $\mathcal{H}$ be a Hilbert space. If $\mathcal{M} \subseteq B(\mathcal{H})$ is an injective von Neumann algebra, then there exists a state $m \in \mathrm{St}(\ell^\infty(I(\mathcal{M})))$ such that*

$$m(\Phi(a,V)|_{I(\mathcal{M})}) = m(\Psi(a,V)|_{I(\mathcal{M})})$$

for all $V \in \mathrm{Bil}_\sigma(\mathcal{M})$ and $a \in \mathcal{M}$.

We will require a few lemmas in order to prove the theorem; the following remark will also be helpful.

Remark 4.1.7. If $\mathcal{A}$ is a C*-algebra and $\pi : \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$ is the multiplication map, then

$$\kappa(\pi^*(\varphi))(x,y) = \pi^*(\varphi)(x \otimes y) = \varphi(\pi(x \otimes y)) = \varphi(xy)$$

for all $x, y \in \mathcal{A}$ and $\varphi \in \mathcal{A}^*$.

Lemma 4.1.8. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra, $p \in \mathcal{M}$ be the largest finite projection in $\mathcal{Z}(\mathcal{M})$, and $\pi : \mathcal{M} \widehat{\otimes} \mathcal{M} \to \mathcal{M}$ be the multiplication map. If $m \in \mathrm{St}(\ell^\infty(I(\mathcal{M})))$ is a state such that*

$$m(\Theta(\kappa_{\mathrm{op}}(\pi^*(\varphi)))|_{I(\mathcal{M})}) = \varphi(p)$$

for every positive normal linear functional $\varphi \in \mathcal{M}^*$, then the linear functionals $\omega_1, \omega_2 \in \mathcal{M}^*$ defined by

$$\omega_1(a) = m(\Phi(a,V)|_{I(\mathcal{M})}), \qquad \omega_2(a) = m(\Psi(a,V)|_{I(\mathcal{M})})$$

are ultraweakly continuous for every $V \in \mathrm{Bil}_\sigma(\mathcal{M})$.

Proof. Let $V \in \mathrm{Bil}_\sigma(\mathcal{M})$ be a bilinear form; then it follows by Theorem 4.1.2 that there exist normal states $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \mathrm{St}(\mathcal{M})$ such that

$$|V(x,y)| \leq \|V\| \left(\varphi_1(x^*x) + \varphi_2(xx^*)\right)^{1/2} \left(\varphi_3(y^*y) + \varphi_4(yy^*)\right)^{1/2}$$

for all $x, y \in \mathcal{M}$. For $n = 1, \ldots, 4$, let $\psi_n \in \mathcal{M}^*$ be the bounded linear functional defined by

$$\psi_n(a) = m(\Phi(a, \kappa_{\mathrm{op}}(\pi^*(\varphi_n)))|_{I(\mathcal{M})})$$

It follows that $\psi_n|_{p\mathcal{M}}$ is normal for $n = 1, \ldots, 4$, and Remark 4.1.7 implies that

$$\kappa_{\mathrm{op}}(\pi^*(\varphi))(x, y) = \kappa(\pi^*(\varphi))(y, x) = \varphi(yx)$$

for all $x, y \in \mathcal{M}$ and $\varphi \in \mathcal{M}^*$, so

$$\begin{aligned}
\Phi(a, \kappa_{\mathrm{op}}(\pi^*(\varphi_n)))(x) &= \kappa_{\mathrm{op}}(\pi^*(\varphi_n))(ax^*, x) \\
&= \varphi_n(xax^*) \\
&= \varphi_n(axx^*) \\
&= (\varphi_n \cdot a)(xx^*) \\
&= \kappa_{\mathrm{op}}(\pi^*(\varphi_n \cdot a))(x^*, x) \\
&= \Theta(\kappa_{\mathrm{op}}(\pi^*(\varphi_n \cdot a)))(x)
\end{aligned}$$

for all $x \in \mathcal{M}$, $a \in \mathcal{Z}(\mathcal{M})$, and $n = 1, \ldots, 4$, and thus

$$\begin{aligned}
\psi_n(1 - p) &= m(\Phi(1 - p, \kappa_{\mathrm{op}}(\pi^*(\varphi)))|_{I(\mathcal{M})}) \\
&= m(\Theta(\kappa_{\mathrm{op}}(\pi^*(\varphi_n \cdot (1 - p))))|_{I(\mathcal{M})}) \\
&= (\varphi_n \cdot (1 - p))(p) \\
&= \varphi_n(p - p^2) \\
&= 0
\end{aligned}$$

for $n = 1, \ldots, 4$. As ψ_n vanishes on the properly infinite part of $\mathcal{M}$, it follows that ψ_n is normal for $n = 1, \ldots, 4$. As

$$\begin{aligned}
|V(au^*, u)| &\leq \|V\| \left(\varphi_1(ua^*au^*) + \varphi_2(au^*ua^*)\right)^{1/2} \left(\varphi_3(u^*u) + \varphi_4(uu^*)\right)^{1/2} \\
&\leq \sqrt{2}\, \|V\| \left(\varphi_1(ua^*au^*) + \varphi_2(aa^*)\right)^{1/2}
\end{aligned}$$

for all $a \in \mathcal{M}$ and $u \in I(\mathcal{M})$ and

$$\begin{aligned}
|V(u^*, ua)| &\leq \|V\| \left(\varphi_1(uu^*) + \varphi_2(u^*u)\right)^{1/2} \left(\varphi_3(a^*u^*ua) + \varphi_4(uaa^*u^*)\right)^{1/2} \\
&\leq \sqrt{2}\, \|V\| \left(\varphi_3(a^*a) + \varphi_4(uaa^*u^*)\right)^{1/2}
\end{aligned}$$

for all $a \in \mathcal{M}$ and $u \in I(\mathcal{M})$, it follows that

$$\begin{aligned}
|\omega_1(a)| &\leq \sqrt{2}\, \|V\|\, m((\Phi(a^*a, \kappa_{\mathrm{op}}(\pi^*(\varphi_1))) + \varphi_2(aa^*))^{1/2}|_{I(\mathcal{M})}) \\
&\leq \sqrt{2}\, \|V\| \left(\psi_1(a^*a) + \varphi_2(aa^*)\right)^{1/2}
\end{aligned}$$

for all $a \in \mathcal{M}$ and

$$|\omega_2(a)| \leq \sqrt{2}\,\|V\|\, m((\varphi_3(a^*a) + \Phi(aa^*, \kappa_{\mathrm{op}}(\pi^*(\varphi_4))))^{1/2}|_{I(\mathcal{M})})$$
$$\leq \sqrt{2}\,\|V\|\,(\varphi_3(a^*a) + \psi_4(aa^*))^{1/2}$$

for all $a \in \mathcal{M}$, so ω_1 and ω_2 are ultraweakly continuous. □

Lemma 4.1.9. *Let $\mathcal{H}$ be a Hilbert space, $(\mathcal{M}_\alpha)$ be an increasing net of von Neumann algebras acting on $\mathcal{H}$ indexed by a directed set $\mathcal{I}$, and $\mathcal{M} = \overline{\bigcup_{\alpha \in \mathcal{I}} \mathcal{M}_\alpha}$, where the closure is with respect to the ultraweak topology. If for every $\alpha \in \mathcal{I}$ there exists a state $m_\alpha \in \mathrm{St}(\ell^\infty(I(\mathcal{M}_\alpha)))$ witnessing the equality in Theorem 4.1.6 for $\mathcal{M}_\alpha$, then there exists a state $m \in \mathrm{St}(\ell^\infty(I(\mathcal{M})))$ witnessing the equality in Theorem 4.1.6 for $\mathcal{M}$.*

Proof. Let $V \in \mathrm{Bil}_\sigma(\mathcal{M})$ be a bilinear form; then $V|_{\mathcal{M}_\alpha \times \mathcal{M}_\alpha} \in \mathrm{Bil}_\sigma(\mathcal{M}_\alpha)$ for all $\alpha \in \mathcal{I}$. For every $\alpha \in \mathcal{I}$, let $\tau_\alpha \in \mathrm{St}(\ell^\infty(I(\mathcal{M})))$ be the state defined by

$$\tau_\alpha(f) = m_\alpha(f|_{I(\mathcal{M}_\alpha)})$$

It follows by the Banach-Alaoglu theorem that there exists a subnet (τ_β) of (τ_α) converging to some state $\tau \in \mathrm{St}(\ell^\infty(I(\mathcal{M})))$ in the weak-* topology, and it is clear that

$$\tau(\Phi(a, V)|_{I(\mathcal{M})}) = \tau(\Psi(a, V)|_{I(\mathcal{M})})$$

for all $a \in \bigcup_{\alpha \in \mathcal{I}} \mathcal{M}_\alpha$. Let $\pi : \mathcal{M} \mathbin{\widehat{\otimes}} \mathcal{M} \to \mathcal{M}$ be the multiplication map and suppose that $\mathcal{M}$ is a finite von Neumann algebra; then it follows by Remark 4.1.7 that

$$\Theta(\kappa_{\mathrm{op}}(\pi^*(\varphi)))(u) = \kappa_{\mathrm{op}}(\pi^*(\varphi))(u^*, u) = \kappa(\pi^*(\varphi))(u, u^*) = \varphi(uu^*) = \varphi(1_{\mathcal{M}})$$

for all $u \in I(\mathcal{M})$ and $\varphi \in \mathcal{M}^*$, so

$$\tau(\Theta(\kappa_{\mathrm{op}}(\pi^*(\varphi)))|_{I(\mathcal{M})}) = \varphi(1_{\mathcal{M}})$$

for every positive normal linear functional $\varphi \in \mathcal{M}^*$ and it follows by Lemma 4.1.8 that

$$\tau(\Phi(a, V)|_{I(\mathcal{M})}) = \tau(\Psi(a, V)|_{I(\mathcal{M})})$$

for all $a \in \mathcal{M}$, so it suffices to let $m = \tau$. Suppose now that $\mathcal{M}$ is not a finite von Neumann algebra and let $p \in \mathcal{M}$ be the largest finite projection in $\mathcal{Z}(\mathcal{M})$. As $(1 - p)\,\mathcal{M}$ is properly infinite, there exists a sequence (u_n) of isometries in $(1-p)\mathcal{M}$ such that $(u_n^*) \to 0$ in the ultrastrong topology. Let (v_n) be the sequence of isometries in $I(\mathcal{M})$ defined by

$$v_n = p + u_n$$

For every $n \in \mathbb{N}$, let $\psi_n \in \mathrm{St}(I(\mathcal{M}))$ be the state defined by

$$\psi_n(f) = \tau((v_n \cdot f)|_{I(\mathcal{M})})$$

where $I(\mathcal{M})$ acts on $\ell^\infty(I(\mathcal{M}))$ by left translation. It follows by the Banach-Alaoglu theorem that (ψ_n) has a subnet (ψ_α) converging in the weak-* topology to some state $m \in \mathrm{St}(I(\mathcal{M}))$. For every $n \in \mathbb{N}$, let $V_n \in B_\sigma(\mathcal{M})$ be the bilinear form defined by

$$V_n(x, y) = V(xv_n^*, v_n y)$$

It follows from the first part of the proof that

$$\psi_n(\Phi(a, V)|_{I(\mathcal{M})}) = \tau(\Phi(a, V_n)|_{I(\mathcal{M})}) = \tau(\Psi(a, V_n)|_{I(\mathcal{M})}) = \psi_n(\Psi(a, V)|_{I(\mathcal{M})})$$

for all $a \in \bigcup_{\alpha \in \mathcal{I}} \mathcal{M}_\alpha$ and $n \in \mathbb{N}$, which implies that

$$m(\Phi(a, V)|_{I(\mathcal{M})}) = m(\Psi(a, V)|_{I(\mathcal{M})})$$

for all $a \in \bigcup_{\alpha \in \mathcal{I}} \mathcal{M}_\alpha$. As $p \le uu^*$ for all $u \in I(\mathcal{M})$, it follows that

$$\varphi(p) \le m(\Theta(\kappa_{\mathrm{op}}(\pi^*(\varphi)))|_{I(\mathcal{M})})$$

for every positive normal linear functional $\varphi \in \mathcal{M}^*$; as $(v_n\, v_n^*) \to p$ in the ultraweak topology, it follows that

$$m(\Theta(\kappa_{\mathrm{op}}(\pi^*(\varphi)))|_{I(\mathcal{M})}) \le \limsup_\alpha \, \tau(v_\alpha \cdot \Theta(\kappa_{\mathrm{op}}(\pi^*(\varphi)))|_{I(\mathcal{M})}) \le \limsup_\alpha \, \varphi(v_\alpha\, v_\alpha^*) = \varphi(p)$$

for every positive normal linear functional $\varphi \in \mathcal{M}^*$, which then implies by Lemma 4.1.8 that

$$m(\Phi(a, V)|_{I(\mathcal{M})}) = m(\Psi(a, V)|_{I(\mathcal{M})})$$

for all $a \in \mathcal{M}$. □

We can now prove Theorem 4.1.6.

Proof of Theorem 4.1.6. Suppose that $\mathcal{M}$ is finite dimensional; then the space of continuous functions $C(U(\mathcal{M}))$ on the unitary group $U(\mathcal{M})$ admits a normalized Haar measure ν. Let $\psi_\nu \in C(U(\mathcal{M}))^*$ be the bounded linear functional defined by

$$\psi_\nu(f) = \int_{U(\mathcal{M})} f(u)\, d\nu(u)$$

It follows by the Hahn-Banach extension theorem that there exists a state $m \in \mathrm{St}(\ell^\infty(U(\mathcal{M})))$ extending ψ_ν. As m is right invariant on $C(U(\mathcal{M}))$, it follows that

$$m(\Phi(a,V)|_{U(\mathcal{M})}) = m(\Psi(a,V)|_{U(\mathcal{M})})$$

for all $a \in U(\mathcal{M})$. As $\mathcal{M}$ is finite dimensional, it follows that $U(\mathcal{M}) = I(\mathcal{M})$, and moreover $\mathcal{M} = \mathrm{Span}(U(\mathcal{M}))$, which proves the result when $\mathcal{M}$ is finite dimensional; it then follows by Thereom 4.1.3 and Lemma 4.1.9 that the result holds when $\mathcal{M}$ is separable. If $\mathcal{M}$ is countably generated, then it is a direct sum of injective separable von Neumann algebras, and the result follows for the general case from Theorem 4.1.4. □

The following result is due to Haagerup (see Theorem 3.1 in [Haa83]).

Theorem 4.1.10. *Let $\mathcal{A}$ be a C^*-algebra. If $\mathcal{A}$ is nuclear, then it has a virtual diagonal belonging to $\overline{\mathrm{conv}\{\iota_{\mathcal{A}\widehat{\otimes}\mathcal{A}}(x^*\otimes x) : \|x\| \leq 1\}}$, where the closure is with respect to the weak-* topology; in particular, it is amenable as a Banach algebra.*

Proof. As $\mathcal{A}$ is nuclear, it follows by Theorem 4.1.5 that $\mathcal{A}^{**}$ is injective, so Theorem 4.1.6 implies that there exists a state $m \in \mathrm{St}(\ell^\infty(I(\mathcal{A}^{**})))$ such that

$$m(\Phi(a,V)|_{I(\mathcal{A}^{**})}) = m(\Psi(a,V)|_{I(\mathcal{A}^{**})})$$

for all $V \in \mathrm{Bil}_\sigma(\mathcal{A}^{**})$ and $a \in \mathcal{A}^{**}$. If $V \in \mathrm{Bil}(\mathcal{A})$ is a bilinear form, then it has a unique extension to a separately ultraweakly continuous bilinear form $V_\sigma \in \mathrm{Bil}_\sigma(\mathcal{A}^{**})$. Now let $M \in (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ be the bounded linear functional defined by

$$M(f) = m(\Theta(\kappa(f)_\sigma)|_{I(\mathcal{A}^{**})})$$

As

$$\kappa(a\cdot f)(x,y) = (a\cdot f)(x\otimes y) = f(x\otimes ya) = \kappa(f)(x,ya)$$

and

$$\kappa(f \cdot a)(x, y) = (f \cdot a)(x \otimes y) = f(ax \otimes y) = \kappa(f)(ax, y)$$

for all $a, x, y \in \mathcal{A}$ and $f \in (\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})^*$, it follows that

$$\begin{aligned}
(a \cdot M)(f) &= M(f \cdot a) \\
&= m(\Theta(\kappa(f \cdot a)_\sigma)|_{I(\mathcal{A}^{**})}) \\
&= m(\Phi(a, \kappa(f)_\sigma)|_{I(\mathcal{A}^{**})}) \\
&= m(\Psi(a, \kappa(f)_\sigma)|_{I(\mathcal{A}^{**})}) \\
&= m(\Theta(\kappa(a \cdot f)_\sigma|_{I(\mathcal{A}^{**})}) \\
&= M(a \cdot f) \\
&= (M \cdot a)(f)
\end{aligned}$$

for all $a \in \mathcal{A}$ and $f \in (\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})^*$, so $a \cdot M = M \cdot a$ for all $a \in \mathcal{A}$. Let $\pi : \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A} \to \mathcal{A}$ be the multiplication map; then it follows by Remark 4.1.7 that

$$\kappa(\pi^*(f \cdot a))(x, y) = (f \cdot a)(xy) = f(axy)$$

for all $a, x, y \in \mathcal{A}$ and $f \in \mathcal{A}^*$, so

$$\begin{aligned}
(a \cdot \pi^{**}(M))(f) &= \pi^{**}(M)(f \cdot a) \\
&= M(\pi^*(f \cdot a)) \\
&= m(\Theta(\kappa(\pi^*(f \cdot a))_\sigma)|_{I(\mathcal{A}^{**})}) \\
&= f(a) \\
&= \iota_{\mathcal{A}}(a)(f)
\end{aligned}$$

for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$, which implies that M is a virtual diagonal for $\mathcal{A}$. Suppose that $M \notin \overline{\mathrm{conv}\{\iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(x^* \otimes x) : \|x\| \leq 1\}}$; then it follows by the Hahn-Banach separation theorem that there exists a linear functional $f \in (\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})^*$ and a real number $\beta \in \mathbb{R}$ such that

$$\mathrm{Re}\, \kappa(f)(x^*, x) = \mathrm{Re}\, f(x^* \otimes x) = \mathrm{Re}\, \iota_{\mathcal{A} \widehat{\otimes} \mathcal{A}}(x^* \otimes x)(f) < \beta < \mathrm{Re}\, M(f)$$

for all $x \in \mathcal{A}$ such that $\|x\| \leq 1$. Let $u \in I(\mathcal{A}^{**})$ be an arbitrary isometry; then it follows by Kaplansky's density theorem that there exists a net (x_α) in the unit ball of $\mathcal{A}$ such that $(\iota_{\mathcal{A}}(x_\alpha)) \to u$ in the ultrastrong* topology. However, $\kappa(f)_\sigma$ is separately ultrastrong* continuous, so this implies that

$$\operatorname{Re} \kappa(f)_\sigma(u^*, u) \leq \beta < \operatorname{Re} M(f) = \operatorname{Re} m(\Theta(\kappa(f)_\sigma)|_{I(\mathcal{A}^{**})})$$

for all $u \in I(\mathcal{A}^{**})$, which is a contradiction. $\square$

Remark 4.1.11. The above proof implies that $M \in \overline{\operatorname{conv}\{f \to \kappa(f)_\sigma(u^*, u) : u \in I(\mathcal{A}^{**})\}}$, where the closure is with respect to the weak-* topology.

4.2 Unitarizability of Bounded Representations

In Chapter 2, we proved that every uniformly bounded representation of an amenable group on a Hilbert space is equivalent to a unitary representation; in this section, we prove an analogous result for bounded representations of nuclear C*-algebras.

Lemma 4.2.1. *Let $\mathcal{A}$ be a unital C*-algebra, $\mathcal{H}$ be a Hilbert space, and $\pi : \mathcal{A} \to B(\mathcal{H})$ be a bounded unital representation. If there exists a positive invertible operator $T \in B(\mathcal{H})$ such that $T\,\pi(x) = \pi(x^*)^*\,T$ for all $x \in \mathcal{A}$, then there exists a positive invertible operator $V \in B(\mathcal{H})$ such that $V^{-1}\,\pi(\,\cdot\,)\,V$ is a unital *-representation.*

Proof. As T is positive, the operator $T^{1/2}$ is well-defined; as T is invertible, $T^{1/2}$ is as well. Let $V = T^{-1/2}$; then

$$V^{-1}\,\pi(x)\,V = VT\,\pi(x)\,V = V\,\pi(x^*)^*\,TV = V\,\pi(x^*)^*\,V^{-1} = (V^{-1}\,\pi(x^*)\,V)^*$$

for all $x \in \mathcal{A}$, so it follows that $V^{-1}\,\pi(\,\cdot\,)\,V$ is a *-representation. $\square$

Theorem 4.2.2. *Let $\mathcal{A}$ be a nuclear C*-algebra and $\mathcal{H}$ be a Hilbert space. If $\pi : \mathcal{A} \to B(\mathcal{H})$ is a bounded unital representation, then there exists an invertible operator $V \in B(\mathcal{H})$ such that $V^{-1}\,\pi(\,\cdot\,)\,V$ is a unital *-representation.*

Proof. As $\mathcal{A}$ is nuclear, it follows by Theorem 4.1.10 that it is amenable as a Banach algebra and admits a virtual diagonal $M \in \overline{\operatorname{conv}\{\iota_{\mathcal{A}\,\widehat{\otimes}\,\mathcal{A}}(x^* \otimes x) : \|x\| \leq 1\}}$, where the closure is with respect to the weak-* topology. For each $\xi, \eta \in \mathcal{H}$, let $\varphi_{\xi,\eta} \in (\mathcal{A} \widehat{\otimes} \mathcal{A})^*$ be the bounded linear functional defined by letting

$$\varphi_{\xi,\eta}(x \otimes y) = \langle \pi(y)\, \xi, \pi(x^*)\, \eta \rangle$$

for $x, y \in \mathcal{A}$ and extending using the universal property. Let $T \in B(\mathcal{H})$ be the bounded operator defined by

$$\langle T\xi, \eta \rangle = M(\varphi_{\xi,\eta})$$

As

$$\begin{aligned} \varphi_{\pi(x)\,\xi,\eta}(y \otimes z) &= \langle \pi(z)\, \pi(x)\, \xi, \pi(y^*)\, \eta \rangle \\ &= \langle \pi(zx)\, \xi, \pi(y^*)\, \eta \rangle \\ &= \varphi_{\xi,\eta}(y \otimes zx) \\ &= (x \cdot \varphi_{\xi,\eta})(y \otimes z) \end{aligned}$$

for all $x, y, z \in \mathcal{A}$ and $\xi, \eta \in \mathcal{H}$, it follows that $\varphi_{\pi(x)\,\xi,\eta} = x \cdot \varphi_{\xi,\eta}$ for all $x \in \mathcal{A}$ and $\xi, \eta \in \mathcal{H}$, and similarly $\varphi_{\xi,\pi(x^*)\,\eta} = \varphi_{\xi,\eta} \cdot x$ for all $x \in \mathcal{A}$ and $\xi, \eta \in \mathcal{H}$. This implies that

$$\begin{aligned} \langle T\, \pi(x)\, \xi, \eta \rangle &= M(\varphi_{\pi(x)\,\xi,\eta}) \\ &= M(x \cdot \varphi_{\xi,\eta}) \\ &= (M \cdot x)(\varphi_{\xi,\eta}) \\ &= (x \cdot M)(\varphi_{\xi,\eta}) \\ &= M(\varphi_{\xi,\eta} \cdot x) \\ &= M(\varphi_{\xi,\pi(x^*)\,\eta}) \\ &= \langle T\xi, \pi(x^*)\, \eta \rangle \\ &= \langle \pi(x^*)^*\, T\xi, \eta \rangle \end{aligned}$$

for all $x \in \mathcal{A}$ and $\xi, \eta \in \mathcal{H}$, and it follows that $T\, \pi(x) = \pi(x^*)^*\, T$ for all $x \in \mathcal{A}$. If $V \in \mathrm{Bil}(\mathcal{A})$ is a bilinear form, then it extends uniquely to a separately ultraweakly continuous bilinear form $V_\sigma \in \mathrm{Bil}_\sigma(\mathcal{A}^{**})$; similarly, if $\psi : \mathcal{A} \to B(\mathcal{H})$ is a bounded representation, then it extends to an ultraweakly continuous bounded representation $\psi_\sigma : \mathcal{A}^{**} \to B(\mathcal{H})$. Moreover, it follows by Remark 4.1.11 that $M \in \overline{\text{conv}\{f \to \kappa(f)_\sigma(u^*, u) : u \in I(\mathcal{A}^{**})\}}$, where the closure is with respect to the weak-* topology, so in turn $\langle T\xi, \eta\rangle \in \overline{\text{conv}\{\langle \pi_\sigma(u)^* \, \pi_\sigma(u)\, \xi, \eta\rangle : u \in I(\mathcal{A}^{**})\}}$ for all $\xi, \eta \in \mathcal{H}$. In particular, this implies that $\langle T\xi, \xi\rangle \in \overline{\text{conv}\{\langle \pi_\sigma(u)^* \, \pi_\sigma(u)\, \xi, \xi\rangle : u \in I(\mathcal{A}^{**})\}}$ for all $\xi \in \mathcal{H}$, and thus T is a positive operator. Furthermore, as

$$\|\xi\| = \|\pi_\sigma(u^*u)\, \xi\| = \|\pi_\sigma(u^*)\, \pi_\sigma(u)\, \xi\| \leq \|\pi_\sigma\|_{\text{op}}\, \|\pi_\sigma(u)\, \xi\|$$

for all $u \in I(\mathcal{A}^{**})$ and $\xi \in \mathcal{H}$, in turn

$$\langle (\pi_\sigma(u)^* \, \pi_\sigma(u) - \|\pi_\sigma\|_{\text{op}}^{-2}\, I_{\mathcal{H}})\, \xi, \xi\rangle = \|\pi_\sigma(u)\, \xi\|^2 - \|\pi_\sigma\|_{\text{op}}^{-2}\, \|\xi\|^2 \geq 0$$

for all $u \in I(\mathcal{A}^{**})$ and $\xi \in \mathcal{H}$, so $\|\pi_\sigma\|_{\text{op}}^{-2}\, I_{\mathcal{H}} \leq \pi_\sigma(u)^* \, \pi_\sigma(u)$ for all $u \in I(\mathcal{A}^{**})$; this implies that T is invertible, and thus the proposition follows by applying Lemma 4.2.1. □

If $\mathcal{A}$ is a unital C*-algebra, then we let $U(\mathcal{A})$ denote the group of unitary elements in $\mathcal{A}$.

Definition 4.2.3. A unital C*-algebra $\mathcal{A}$ is *strongly amenable* if for every unital Banach $\mathcal{A}$-module X and every derivation $D : \mathcal{A} \to X^*$ there exists a bounded linear functional $f \in \overline{\text{conv}\{D(u) \cdot u^* : u \in U(\mathcal{A})\}}$ such that $D = -\delta(f)$, where the closure is with respect to the weak-* topology.

The next result follows immediately from the above definition.

Proposition 4.2.4. *Let $\mathcal{A}$ be a unital C*-algebra. If $\mathcal{A}$ is strongly amenable, then $\mathcal{A}$ is amenable as a Banach algebra.*

When $\mathcal{A}$ is strongly amenable and not merely nuclear, there exist more elementary proofs of Theorem 4.2.2 (see [Bun72a], [Bun72b]) that do not rely on Theorem 4.1.10. The proof of the following theorem is essentially the method of [Bun72b].

If $\mathcal{H}$ is a Hilbert space, then we let $\mathcal{C}_1(\mathcal{H})$ denote the ideal of trace class operators in $B(\mathcal{H})$ and let $\Phi_{\text{Tr}} : B(\mathcal{H}) \to \mathcal{C}_1(\mathcal{H})^*$ denote the isometric isomorphism defined by

$$\Phi_{\text{Tr}}(S)(T) = \text{Tr}(ST)$$

Theorem 4.2.5. *Let $\mathcal{A}$ be a strongly amenable unital C*-algebra and $\mathcal{H}$ be a Hilbert space. If $\pi : \mathcal{A} \to B(\mathcal{H})$ is a bounded unital representation, then there exists a positive invertible operator $V \in B(\mathcal{H})$ such that $V^{-1}\, \pi(\,\cdot\,)\, V$ is a unital *-representation.*

Proof. Let $C_1(\mathcal{H})$ be equipped with the left and right actions defined by

$$x \cdot T = \pi(x)\, T, \qquad\qquad T \cdot x = T\, \pi(x^*)^*$$

for $x \in \mathcal{A}$ and $T \in \mathcal{C}_1(\mathcal{H})$; then $\mathcal{C}_1(\mathcal{H})$ is a Banach $\mathcal{A}$-module. Let $D : \mathcal{A} \to \mathcal{C}_1(\mathcal{H})^*$ be the bounded linear map defined by

$$D(x)(T) = \mathrm{Tr}(x \cdot T - T \cdot x)$$

Then

$$\begin{aligned} D(xy)(T) &= \mathrm{Tr}(xy \cdot T - T \cdot xy) \\ &= \mathrm{Tr}(y \cdot (T \cdot x) - (T \cdot x) \cdot y) + \mathrm{Tr}(x \cdot (y \cdot T) - (y \cdot T) \cdot x) \\ &= D(y)(T \cdot x) + D(x)(y \cdot T) \\ &= (x \cdot D(y))(T) + (D(x) \cdot y)(T) \end{aligned}$$

for all $x, y \in \mathcal{A}$ and $T \in \mathcal{C}_1(\mathcal{H})$, and thus D is a derivation. As $\mathcal{A}$ is strongly amenable, it follows that there exists some bounded linear functional $f \in \overline{\mathrm{conv}\{D(u) \cdot u^* : u \in U(\mathcal{A})\}}$ such that $D = -\delta(f)$, where the closure is with respect to the weak-* topology on $\mathcal{C}_1(\mathcal{H})^*$. Let $S_f = \Phi_{\mathrm{Tr}}^{-1}(f)$. As

$$\begin{aligned} \mathrm{Tr}(\pi(x)\, T - \pi(x^*)^*\, T) &= \mathrm{Tr}(\pi(x)\, T) - \mathrm{Tr}(\pi(x^*)^*\, T) \\ &= \mathrm{Tr}(\pi(x)\, T) - \mathrm{Tr}(T\, \pi(x^*)^*) \\ &= \mathrm{Tr}(x \cdot T) - \mathrm{Tr}(T \cdot x) \\ &= \mathrm{Tr}(x \cdot T - T \cdot x) \\ &= D(x)(T) \end{aligned}$$

for all $x \in \mathcal{A}$ and $T \in \mathcal{C}_1(\mathcal{H})$ and

$$\begin{aligned} \mathrm{Tr}(\pi(x^*)^*\, S_f T) - \mathrm{Tr}(S_f\, \pi(x)\, T) &= \mathrm{Tr}(S_f T\, \pi(x^*)^*) - \mathrm{Tr}(S_f\, \pi(x)\, T) \\ &= \Phi_{\mathrm{Tr}}(S_f)(T \cdot x) - \Phi_{\mathrm{Tr}}(S_f)(x \cdot T) \\ &= (x \cdot \Phi_{\mathrm{Tr}}(S_f) - \Phi_{\mathrm{Tr}}(S_f) \cdot x)(T) \\ &= (x \cdot f - f \cdot x)(T) \\ &= \delta(f)(x)(T) \end{aligned}$$

for all $x \in \mathcal{A}$ and $T \in \mathcal{C}_1(\mathcal{H})$, in turn

$$\begin{aligned}\Phi_{\mathrm{Tr}}((I_{\mathcal{H}} - S_f)\,\pi(x) - \pi(x^*)^*\,(I_{\mathcal{H}} - S_f))(T) &= \mathrm{Tr}((I_{\mathcal{H}} - S_f)\,\pi(x)\,T - \pi(x^*)^*\,(I_{\mathcal{H}} - S_f)\,T) \\ &= D(x)(T) + \delta(f)(x)(T)\end{aligned}$$

for all $x \in \mathcal{A}$ and $T \in \mathcal{C}_1(\mathcal{H})$. It follows that $(I_{\mathcal{H}} - S_f)\,\pi(x) - \pi(x^*)^*\,(I_{\mathcal{H}} - S_f) \in \ker \Phi_{\mathrm{Tr}}$ for all $x \in \mathcal{A}$; as Φ_{Tr} is an embedding, this implies that $(I_{\mathcal{H}} - S_f)\,\pi(x) = \pi(x^*)^*\,(I_{\mathcal{H}} - S_f)$ for all $x \in \mathcal{A}$. As

$$\begin{aligned}(D(u)\cdot u^*)(T) &= D(u)(u^*\cdot T) \\ &= \mathrm{Tr}(uu^*\cdot T - u^*\cdot T\cdot u) \\ &= \mathrm{Tr}(T - u^*\cdot T\cdot u) \\ &= \mathrm{Tr}(T - \pi(u^*)\,T\,\pi(u^*)^*) \\ &= \mathrm{Tr}(T - \pi(u^*)^*\,\pi(u^*)\,T) \\ &= \Phi_{\mathrm{Tr}}(I_{\mathcal{H}} - \pi(u^*)^*\,\pi(u^*))(T)\end{aligned}$$

for all $u \in U(\mathcal{A})$ and $T \in \mathcal{C}_1(\mathcal{H})$, it follows that $f \in \overline{\mathrm{conv}\{\Phi_{\mathrm{Tr}}(I_{\mathcal{H}} - \pi(u)^*\,\pi(u)) : u \in U(\mathcal{A})\}}$, where the closure is with respect to the weak-* topology on $\mathcal{C}_1(\mathcal{H})^*$, and thus in turn $I_{\mathcal{H}} - S_f \in \overline{\mathrm{conv}\{\pi(u)^*\,\pi(u) : u \in U(\mathcal{A})\}}$, where the closure is with respect to the ultraweak topology on $B(\mathcal{H})$. This implies that $I_{\mathcal{H}} - S_f$ is a positive operator. Furthermore, as

$$\|\xi\| = \|\pi(u^*)\,\pi(u)\,\xi\| \le \|\pi\|_{\mathrm{op}}\,\|\pi(u)\,\xi\|$$

for all $u \in U(\mathcal{A})$ and $\xi \in \mathcal{H}$, in turn

$$\langle(\pi(u)^*\,\pi(u) - \|\pi\|_{\mathrm{op}}^{-2}\,I_{\mathcal{H}})\,\xi, \xi\rangle = \|\pi(u)\,\xi\|^2 - \|\pi\|_{\mathrm{op}}^{-2}\,\|\xi\|^2 \ge 0$$

for all $u \in U(\mathcal{A})$ and $\xi \in \mathcal{H}$, so $\|\pi\|_{\mathrm{op}}^{-2}\,I_{\mathcal{H}} \le \pi(u)^*\,\pi(u)$ for all $u \in U(\mathcal{A})$; this then implies that $I_{\mathcal{H}} - S_f$ is invertible, and thus the proposition follows by applying Lemma 4.2.1 with $T = I_{\mathcal{H}} - S_f$. □

Chapter 5

Injective von Neumann Algebras

5.1 Injectivity

Definition 5.1.1. Let $\mathcal{H}$ be a Hilbert space. A von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ is *injective* if every unital completely positive linear map from a closed unital *-subalgebra of a unital C*-algebra $\mathcal{A}$ to $\mathcal{M}$ extends to a unital completely positive linear map from $\mathcal{A}$ to $\mathcal{M}$.

Definition 5.1.2. Let $\mathcal{H}$ be a Hilbert space. A von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ has the *extension property* if there exists a bounded linear map $E : B(\mathcal{H}) \to \mathcal{M}$ such that $\|E\|_{\text{op}} = 1$ and $E|_{\mathcal{M}}$ is the identity map.

Loebl attributes the following result to Tomiyama (see [Loe74]).

Proposition 5.1.3. *Let $\mathcal{H}$ be a Hilbert space. A von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ has the extension property if and only if $\mathcal{M}'$ has the extension property.*

Proof. Suppose first that $\mathcal{M}$ has the extension property; then there exists a bounded linear map $E : B(\mathcal{H}) \to \mathcal{M}$ such that $\|E\|_{\text{op}} = 1$ and $E|_{\mathcal{M}}$ is the identity map. Let $(\mathcal{M}, \mathcal{H}, J, P)$ be a standard form of $\mathcal{M}$ and let $Q : B(\mathcal{H}) \to \mathcal{M}'$ be the linear map defined by

$$Q(S) = J\,E(JSJ)\,J$$

Then

$$Q(Q(S)) = J\,E(J^2\,E(JSJ)\,J^2)\,J = J\,E(E(JSJ))\,J = J\,E(JSJ)\,J = Q(S)$$

for all $S \in B(\mathcal{H})$ and $\|Q\|_{\text{op}} = 1$, so $\mathcal{M}'$ has the extension property. To show the converse, suppose now that $\mathcal{M}'$ has the extension property; then it follows by the first part of the proof that $\mathcal{M}'' = \mathcal{M}$ has the extension property. $\square$

The following theorem is due to Loebl (see [Loe74]).

Proposition 5.1.4. *Let $\mathcal{H}$ be a Hilbert space. A von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ has the extension property if and only if it is injective.*

Proof. Suppose that $\mathcal{M}$ has the extension property. Let $\mathcal{A}$ be a unital C*-algebra, $\mathcal{B} \subseteq \mathcal{A}$ be a closed unital *-subalgebra of $\mathcal{A}$, and $\varphi : \mathcal{B} \to \mathcal{M}$ be a unital completely positive linear map. Let $\iota : \mathcal{M} \hookrightarrow B(\mathcal{H})$ be the canonical inclusion map; then ι is unital and completely positive. Let $\psi = \iota \circ \varphi$; then ψ is a composition of unital completely positive maps, therefore itself unital and completely positive, and it follows by Arveson's extension theorem that there exists a unital completely positive map $\Psi : \mathcal{A} \to B(\mathcal{H})$ such that $\Psi|_{\mathcal{B}} = \psi$. As $\mathcal{M}$ has the extension property, there exists a bounded linear map $E : \mathcal{B}(\mathcal{H}) \to \mathcal{M}$ such that $\|E\|_{\text{op}} = 1$ and $E|_{\mathcal{M}}$ is the identity map; then E is completely positive, so in turn $E \circ \Psi$ is a unital completely positive linear map from $\mathcal{A}$ to $\mathcal{M}$. As

$$(E \circ \Psi)|_{\mathcal{B}} = E \circ \psi = E \circ \iota \circ \varphi = \varphi$$

it then follows that $\mathcal{M}$ is injective. Conversely, suppose that $\mathcal{M}$ is injective. As $\mathcal{M}$ is closed in the weak operator topology, it is closed in the norm topology; as the identity map on $\mathcal{M}$ is a unital completely positive linear map, it then follows by injectivity that there exists a unital completely positive map $E : B(\mathcal{H}) \to \mathcal{M}$ extending the identity map on $\mathcal{M}$. Furthermore, as E is unital and completely positive, it follows that $\|E\|_{\text{op}} = 1$. $\square$

Definition 5.1.5. Let $\mathcal{H}$ be a Hilbert space. A von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ has Schwartz's *property* (P) if

$$\mathcal{M}' \cap \overline{\operatorname{conv}\{uTu^* : u \in U(\mathcal{M})\}} \neq \emptyset$$

for all $T \in B(\mathcal{H})$, where the closure is with respect to the weak operator topology.

The following result is due to Schwartz (see Lemma 5 in [Sch63]).

Proposition 5.1.6. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra. If $\mathcal{M}$ has property (P), then $\mathcal{M}'$ has the extension property.*

Proof. For every $T \in B(\mathcal{H})$, let $\mathcal{W}(T) = \overline{\mathrm{conv}\{uTu^* : u \in U(\mathcal{M})\}}$, where the closure is with respect to the weak operator topology. Let $\mathcal{I}$ be the set of linear maps $\psi : B(\mathcal{H}) \to B(\mathcal{H})$ such that $\|\psi\|_{\mathrm{op}} = 1$, $\psi(x) = x$ for all $x \in \mathcal{M}$, and $\psi(T) \in \mathcal{W}(T)$ for all $T \in B(\mathcal{H})$. As the identity map belongs to $\mathcal{I}$, it is nonempty. Define a partial order on $\mathcal{I}$ by letting $\varphi \leq \psi$ if $\mathcal{W}(\psi(T)) \subseteq \mathcal{W}(\varphi(T))$ for all $T \in B(\mathcal{H})$. Let $\{\psi_\alpha : \alpha \in \mathcal{J}\}$ be a chain in $\mathcal{I}$; then taking a Banach limit on $\mathcal{J}$ yields an upper bound for the chain. It follows by Zorn's lemma that $\mathcal{I}$ has a maximal element $\psi \in \mathcal{I}$. Suppose that there exists some $T \in B(\mathcal{H})$ such that $\mathcal{W}(\psi(T))$ contains at least two distinct elements. As $\mathcal{M}$ has property (P), there exists a net (μ_α) of finitely supported nonnegative functions on $U(\mathcal{M})$ indexed by a directed set $\mathcal{J}$ such that $\sum_{u \in U(\mathcal{M})} \mu_\alpha(u) = 1$ for all $\alpha \in \mathcal{J}$ and $(\sum_{u \in U(\mathcal{M})} \mu_\alpha(u)\, u\, \psi(T)\, u^*)$ converges to an element of $\mathcal{M}'$ in the weak operator topology; taking a Banach limit on $\mathcal{J}$ then yields a bounded linear map $\varphi \in \mathcal{I}$ such that $\psi \leq \varphi$ and $\varphi(T) \in \mathcal{M}'$. As $\varphi(T) \in \mathcal{M}'$, it follows that $\mathcal{W}(\varphi(T)) = \{\varphi(T)\}$, so $\mathcal{W}(\varphi(T))$ is a proper subset of $\mathcal{W}(\psi(T))$. However, this contradicts the maximality of ψ, and it thus follows that $\mathcal{W}(\psi(T))$ is a singleton for all $T \in B(\mathcal{H})$. As $\mathcal{M}' \cap \mathcal{W}(\psi(T)) \neq \emptyset$ for all $T \in B(\mathcal{H})$, this implies that $\psi(B(\mathcal{H})) \subseteq \mathcal{M}'$, and thus $\mathcal{M}'$ has the extension property. □

Corollary 5.1.7. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra. If $\mathcal{M}$ has property (P), then $\mathcal{M}$ is injective.*

Proof. This follows from Proposition 5.1.6, Proposition 5.1.3, and Proposition 5.1.4. □

5.2 Group von Neumann Algebras

Definition 5.2.1. Let G be a countable discrete group and $\lambda : G \to U(\ell^2(G))$ be the left regular representation. The *group von Neumann algebra* $\mathcal{L}(G)$ of G is the von Neumann algebra defined by

$$\mathcal{L}(G) = \lambda(G)''$$

Remark 5.2.2. There exists a canonical trace τ on $\mathcal{L}(G)$ defined by

$$\tau(x) = \langle x\chi_{\{e\}}, \chi_{\{e\}}\rangle_2$$

Theorem 5.2.3. *Let G be a discrete group. The following are equivalent:*

1. *G is amenable*

2. *$\mathcal{L}(G)$ is injective*

3. *$\mathcal{L}(G)$ has property (P)*

4. *$\pi(G)''$ has property (P) for every Hilbert space $\mathcal{H}$ and every unitary representation $\pi : G \to U(\mathcal{H})$*

Proof. The implication (3) ⇒ (2) follows from Corollary 5.1.7, so it suffices to show that the implications (1) ⇒ (4) ⇒ (3) and (2) ⇒ (1) hold. Suppose that G is amenable, let $\mathcal{H}$ be a Hilbert space, let $\pi : G \to U(\mathcal{H})$ be a unitary representation, and let $\mathcal{M} = \pi(G)''$. For every bounded operator $T \in B(\mathcal{H})$, let $S_T = \mathbb{E}_x\, \pi(x)\, T\, \pi(x)^*$; then

$$\pi(x)\, S_T = \mathbb{E}_y\, \pi(xy)\, T\, \pi(y)^* = \mathbb{E}_y\, \pi(y)\, T\, \pi(x^{-1}y)^* = S_T\, \pi(x^{-1})^* = S_T\, \pi(x)$$

for all $x \in G$. As $\pi(G)'$ is a von Neumann algebra, $\mathcal{M}' = \pi(G)''' = \pi(G)'$, and thus $S_T \in \mathcal{M}'$. Furthermore, as $\pi(G) \subseteq \mathcal{M}$, it follows by Remark 2.5.11 that $S_T \in \overline{\text{conv}\{uTu^* : u \in U(\mathcal{M})\}}$, where the closure is with respect to the weak operator topology; thus $\mathcal{M} = \pi(G)''$ has property (P). Suppose now that (4) holds, let $\mathcal{H} = \ell^2(G)$, and let $\pi : G \to U(\ell^2(G))$ be the left regular representation; then $\mathcal{L}(G) = \pi(G)''$ has property (P), so the implication (4) ⇒ (3) holds. Suppose now that $\mathcal{L}(G)$ is injective; then it follows by Proposition 5.1.4 that $\mathcal{L}(G)$ has the extension property, so by Proposition 2.1.4 there exists a bounded linear map $E : B(\ell^2(G)) \to \mathcal{L}(G)$ such that $\|E\|_{\text{op}} = 1$ and $E(xTy) = x\, E(T)\, y$ for all $x, y \in \mathcal{L}(G)$ and $T \in B(\ell^2(G))$. For every $f \in \ell^\infty(G)$, let $M_f \in B(\ell^2(G))$ be the multiplication operator associated to f; then

$$\lambda(s)\, M_f\, \lambda(s^{-1}) = M_{sf}$$

for all $s \in G$. Let $\tau \in \mathrm{St}(\mathcal{L}(G))$ be the canonical trace on $\mathcal{L}(G)$ and let $\psi \in \ell^\infty(G)^*$ be the bounded linear functional defined by

$$\psi(f) = (\tau \circ E)(M_f)$$

Then $\psi(\chi_G) = \tau(E(I_{\ell^2(G)})) = \tau(I_{\ell^2(G)}) = 1$ and $\|\psi\|_{\mathrm{op}} = 1$, and moreover

$$\begin{aligned}\psi(sf) &= (\tau \circ E)(M_{sf}) \\ &= (\tau \circ E)(\lambda(s)\, M_f\, \lambda(s^{-1})) \\ &= \tau(\lambda(s)\, E(M_f)\, \lambda(s^{-1})) \\ &= (\tau \circ E)(M_f) \\ &= \psi(f)\end{aligned}$$

for all $f \in \ell^\infty(G)$ and $s \in G$, so ψ is an invariant mean on $\ell^\infty(G)$. □

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