On the Metrizability Problem for Projective Structures on Surfaces

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Abstract

In this thesis, we will follow a paper by Bryant *et al.* in finding the necessary and sufficient conditions for the existence of a Levi-Civita connection within a given projective structure $[\Gamma]$ on a surface. We will give an explicit formulation of a necessary condition, as an obstruction of order five in the Christoffel symbols of an arbitrary element of $[\Gamma]$. Our approach in finding this obstruction allows us to find sufficient conditions in the real analytic case as well. In terms of the Christoffel symbols of an arbitrary element of $[\Gamma]$, the explicit forms of the sufficient conditions are of order six in generic case, and of order eight in non-generic case. All the formulations will be projectively invariant and will be expressed in point invariants of the second-order ODE whose integral curves are geodesics of $[\Gamma]$. We will use a machinery developed by Hitchin and LeBrun, that we call minitwistor theory, to find the moduli space of these integral curves. Using this machinery and the notion of densities on a manifold, we give geometric interpretations of the formulation of the problem and derive a projective property of the space of metrics whose Levi-Civita connections belong to $[\Gamma]$.

Abstrait

Dans cette thèse, nous analysons un travail de Bryant *et al.* portant sur l'obtention de conditions nécessaires et suffisantes pour l'existence d'une connexion de Levi-Civita au sein d'une structure projective $[\Gamma]$ sur une surface. Nous donnons une formulation explicite d'une condition nécessaire sous la forme d'une obstruction d'ordre cinq sur les coefficients de Christoffel d'un élément arbitraire de $[\Gamma]$. Dans le cas analytique, notre approche nous permet d'obtenir des conditions suffisantes. En termes des coefficients de Christoffel d'un élément arbitraire de $[\Gamma]$, ces conditions suffisantes sont d'ordre six dans le cas générique et d'ordre huit dans le cas non-générique. Toutes les formulations obtenues sont projectivement invariantes et exprimées en termes d'invariants ponctuels des equations différentielles ordinaires d'ordre deux dont les solutions les géodésiques de $[\Gamma]$. Nous utlisons une technique développée par Hitchin et LeBrun, appelée théorie des mini-twisteurs, pour trouver l'espace des modules de ces courbes intégrales. En utlisant cette technique et la notion de densité sur une variété, nous donnons des interprétations géométriques de la formulation du problème et dérivons une propriété projectivement invariante de l'espace des métriques dont la connexion de Levi-Civita appartient a $[\Gamma]$.

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Contents

CHAPTER 1

Introduction

We being with a brief history motivating the problem that we will be dealing with in this thesis. After this we will give a formal formulation of the problem and an overview of the important results discussed in this thesis.

1.1 History and Motivation

The study of geometric structures on differentiable manifolds and fiber bundles, in particular jet spaces, goes back to the work of Riemann and Christoffel and is one of the main objectives of differential geometry and differential topology. There are various ways to define a geometric structure on a manifold. One important method is to define a geometric structure by a system of differential equations. The local and global study of manifolds or of fibred spaces with structures that are generated by a system of differential equations is what is called the geometry of differential equations. This subject has a very long and distinguished history whose progress is closely tied to development in Riemannian geometry, Finsler geometry, and Cartan spaces combined with the various theories of connections in fibred spaces. The survey [1] contains a great number of works on this topic and gives a good historical background of the subject.

The origin of the geometry of differential equations dates back to the earliest systematic studies of differential equations and the work of Lie on the invariants of differential equations. He investigated certain classes of equations relative to a specified group and provided an effective geometric interpretation of his results. What we mean by a group property of a system of differential equations is a property of the system which remains unchanged when the dependent or independent variables in the equation are subjected to a transformation belonging to a transformation group G. A system of differential equations is said to admit a group G if such a group property holds for that system. One very important and commonly investigated group property is the is the invariance of the solution of a differential equation so that if a transformation $g \in G$ acts on a solution of this system it will give another solution of the same system. Such group properties can be used to give a group classification of systems of differential equations, a subject first addressed by Lie. The investigation of group properties of system of differential equations was done in more generality by Cartan.

In this thesis we will be dealing with the geometrical study of a specific class of second-order differential equations

$$\frac{d^2y}{dx^2} = \Lambda(x, y, \frac{dy}{dx}), \tag{1.1}$$

in two variables and discuss the geometric structures that can be associated to it. The first geometric structure that we will associate to this class of ODEs consists of the geodesic lines on surfaces as one of the most important geometric structures that contains many aspects of the local classical theory of the surfaces as well as new approaches that are brought due to twistor theory. The notion of geodesic on a surface can be locally defined using a second-order differential equation. On a surface we can consider the class of all such second-order equations which result in the same geodesic curves on the surface as unparametrized curves. Such a class is called a projective structure (or projective connection) of the surface and can be formulated as a second-order differential equation $\frac{d^2y}{dx^2} = \Lambda(x, y, \frac{dy}{dx})$, where Λ is cubic in y'. This is the form of the second-order ODEs that we consider in this thesis.

Extensive geometric study of second-order differential equations was conducted by Cartan, Eisenhart and Douglas and others in the 1920s. Eisenhart was primarily interested in the case of equations that could be interpreted as describing the geodesics of an affine connection of a manifold. A systematic investigation of such single scalar second-order ODEs has been completed in the language of modern differential geometry and exterior differential systems in [18] and also in [3]. Cartan realized that the second-order ODEs of the form (1.1) induce a natural projective structure on their two dimensional solution space. In his paper [5] on projective connections, Cartan used his own approach (which extends the idea of affine connection as formulated by Levi-Civita and Weyl to a more general situation involving projective frames) to study the theory of connection and equivalence of geometric structures by means of G-structures. A modern interpretation of Cartan's ideas can be found in [27].

Around the time that Cartan published his paper on projective connections [5], a somewhat different way of investigating the same problem was conducted by several other authors, including Thomas [29] and Whitehead [35], using the language that we will use in our formulation of projective structures. A comparison of the two approaches is done in [7]. We use Thomas's approach in the statement of the problem. The generalization of the projective differential geometry of affine connections was studied by Douglas [8] under the name of general geometry of paths.

A projective structure on an n-dimensional manifold M, in the sense explained earlier, can be thought of as a local identification of M with $\mathbb{RP}^n$. There are many interesting examples of manifold that carry a projective structure, however, the existence and classification of projective structures on n-dimensional manifolds is an open problem for $n \geq 3$. It has been conjectured that every three-dimensional manifold can be equipped with a projective structure (c.f. [28]). This is a very difficult problem and its positive solution would imply, in particular, the Poincaré conjecture. A research report of the problem on compact surfaces can be found in [6].

Defining the projective equivalence relation between affine connections as above results in equivalent classes of affine connections which have the same geodesics as unparametrized curves. One can ask if there is a Levi-Civita connection in a given equivalence class and, if so, one can try to determine the number of projectively equivalent Levi-Civita connections as well as necessary and sufficient conditions for the existence of metrics giving rise to those Levi-Civita connections. If such a metric exists then the projective structure is called metrizable. The earliest attempt to solve this problem of finding necessary and sufficient conditions for the metrisability of a projective structure on a surface was first carried out by Roger Liouville [24].

In this thesis we will provide the necessary and sufficient conditions to this problem and will discuss two geometric treatments of the problems using twistor theory and density bundles as is done in [2].

1.2 Formulation and Results

In this section, we will give the formulation of the problem and state our results.

Suppose $g = (g_{ij})$ is a Riemannian metric on an open set $U \subseteq M$ with

$$\begin{aligned} g &= g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + g_{22}dx_2^2 \\ &= E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2, \end{aligned} \tag{1.2}$$

where (x, y) and (x_1, x_2) denote the same coordinates. We switch between the two notations as is convenient in order to keep expressions as simple as possible. We do the same for the components of the metric by letting $g_{11} = E,\ g_{12} = F,\ g_{22} = G$. The Christoffel symbols of the associated Levi-Civita connection is

$$\Gamma^a_{cb} = \frac{1}{2}g^{ad}\big(\frac{\partial g_{db}}{\partial x^c} + \frac{\partial g_{dc}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^d}\big), \tag{1.3}$$

Considering a projective structure $[\Gamma]$, pick two elements $\widetilde{\Gamma}^i{}_{jk},\ \Gamma^i{}_{jk} \in [\Gamma]$. As will be shown in Section 3.1, we should have

$$\widetilde{\Gamma}^i{}_{jk} = \Gamma^i{}_{jk} + \Upsilon_j\delta^i_k + \Upsilon_k\delta^i_j. \tag{1.4}$$

If there is a Levi-Civita connection in $[\Gamma]$ then it is projectively equivalent to $\Gamma^i{}_{jk}$. Therefore, the following system has to be satisfied for some g_{ij}'s and Υ^k's

$$\Gamma^i{}_{jk} + \Upsilon_j\delta^i_k + \Upsilon_k\delta^i_j = \frac{1}{2}g^{ad}\big(\frac{\partial g_{db}}{\partial x^c} + \frac{\partial g_{dc}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^d}\big).$$

Because we deal with torsion-free affine connections, there are six Christoffel symbols $\Gamma^i{}_{jk}$, so the system above consists of six equations. However there are three metric components g_{ij}, as well as a one-form $\Upsilon = \Upsilon_a dx^a$, that are to be determined. Therefore, the system is overdetermined. Another way of addressing the overdetermined nature of the problem is by deriving the associated second-order ODE of the projective structure as we mentioned in the previous section. In Section 3.1 we will show that it has the form

$$\frac{d^2y}{dx^2} = A_3(\frac{dy}{dx})^3 + A_2(\frac{dy}{dx})^2 + A_1\frac{dy}{dx} + A_0,$$

where

$$A_3 = \Gamma^1{}_{22},\ A_2 = 2\Gamma^1{}_{12} - \Gamma^2{}_{22},\ A_1 = \Gamma^1{}_{11} - 2\Gamma^2{}_{12},\ A_0 = -\Gamma^2{}_{11}, \tag{1.5}$$

where $\Gamma^i{}_{jk}$ is an arbitrary element of $[\Gamma]$. The second-order ODEs of this form constitute the class of ODEs that we are considering in our geometrical study of differential equations. It can be easily checked that the A_i's are invariant under the projective equivalence relation (1.4) meaning that replacing the Christoffel symbols $\Gamma^i{}_{jk}$ in equation (1.5) with $\widetilde{\Gamma}^i{}_{jk}$ such that relation (1.4) holds, does not change the A_i's. Therefore, in a given local coordinate, a projective structure can be represented by the A_i's. Due to a result by Cartan in [5], any choice of the A_i's in the equation above also gives rise to a projective structure. As a result, in a given local coordinate there is a one-to-one correspondence between the functions $A_0, ..., A_3$ and projective structures. If a projective structure is metrizable then we can substitute expression (1.3) for the Christoffel symbols in equation (1.5). It is straightforward to obtain the system

$$\begin{aligned}
A_0 &= \frac{1}{2}\frac{E\partial_y F - 2E\partial_x F + F\partial_x E}{EG - F^2},\\
A_1 &= \frac{1}{2}\frac{3F\partial_y E + G\partial_x E - 2F\partial_x F - 2E\partial_x G}{EG - F^2},\\
A_2 &= \frac{1}{2}\frac{2F\partial_y F + 2G\partial_y E - 3F\partial_x G - E\partial_y G}{EG - F^2},\\
A_3 &= \frac{1}{2}\frac{2G\partial_y F - G\partial_x G - F\partial_y G}{EG - F^2}.
\end{aligned} \tag{1.6}$$

In this system, we are given four functions $A_0, ..., A_3$ and want to find three functions $E,\ F,\ G$ such that (1.6) holds.

In order to state the first obstruction for metrizability of a projective structure, we associate to a projective structure $[\Gamma]$ a 6×6 matrix $\mathcal{M}[\Gamma]$ defined as

$$\mathcal{M}[\Gamma] := \big(\mathcal{V}, D_a\mathbf{V}, D_{(a}D_{b)}\mathbf{V}\big) \tag{1.7}$$

where the vector field $\mathbf{V}$ is given in (3.25) and depends on the A_i's up to their second derivatives. We have made use of abstract index notation in writing $\mathcal{M}[\Gamma]$ where $D_{(a}D_{b)}\mathbf{V} := \frac{1}{2}\big(D_{(a}D_{b)}\mathbf{V} + D_{(b}D_{a)}\mathbf{V}\big)$ and $D_a\mathbf{V} = \partial_a\mathbf{V} - V\Omega_a$ as will be explained in Section 2.3 and Ω_a's are given by (C.1). We define $D_aD_bD_c \cdots D_d\mathbf{V}$ recursively to be $\partial_a(D_bD_c\cdots D_d\mathbf{V}) - (D_bD_c\cdots D_d\mathbf{V})\mathbf{\Omega}_a$.

Theorem 1.2.1. *If a projective structure* $[\Gamma]$ *is metrizable then*

$$\det(\mathcal{M}[\Gamma]) = 0. \tag{1.8}$$

The proof will be given in Section 3.4.

We will see that investigating the metrizability of a projective structure leads to a first-order system of PDEs. Using the machinery of Section 2.3, we find the obstructions to the existence of a Levi-Civita connection in a projective structure $[\Gamma]$. In this way we can also find the dimension of the space of metrics over a sufficiently small open set of the surface of which the Levi-Civita connections belong to $[\Gamma]$. The first obstruction was given in Theorem (1.2.1). We will translate our problem to finding the dimension of the space of the parallel sections of a connection over a rank six vector bundle on a surface. In this manner, the matrix $\mathcal{M}[\Gamma]$ will be equivalent to a matrix consisting of the curvature matrix and its covariant derivatives up to degree two which is what we will also denote by $\mathcal{F}_2$ in our general treatment of first-order PDEs in Section 2.3 and is given by (2.25).

In Section 3.5, we first provide a sufficient condition for metrizability of a special class of projective structures, called generic, in sufficiently small neighborhood of certain points as follows:

Theorem 1.2.2. *Given a real analytic projective structure $[\Gamma]$ over a surface with associated coefficients A_i's and $\det(\mathcal{M}[\Gamma]) = 0$ such that $\mathrm{rank}(\mathcal{M}[\Gamma]) = 5$ and $\ker(\mathcal{M}[\Gamma]) =$ span$\{\mathbf{u}\}$ with $\mathbf{u} = (\mathbf{u}_1, ..., \mathbf{u}_6)$ and $\mathbf{u}_1\mathbf{u}_3 - (\mathbf{u}_2)^2 \neq 0$ at a point p, a sufficient condition for the metrizability of $[\Gamma]$ in a sufficiently small neighborhood of p is that the the rank of the 10×6 matrix with the rows*

$$(\mathbf{V}, D_a\mathbf{V}, D_{(a}D_{b)}\mathbf{V}, D_{(a}D_bD_{c)}\mathbf{V})$$

is equal to five. This condition holds if and only if two invariants E_1, E_2 defined in terms of the A_i's up to their sixth derivatives vanish. The expressions for these invariants depend on the given projective structure.

The construction of E_1 and E_2 will be explained in Section 3.5. We will show that for this specific choice of projective structures the fifth order equation (1.8) and these two sixth order invariants guarantee the involutivity of the system of first-order PDEs that constitute our problem and the general solution depends on three functions of two variables.

If a real analytic projective structure $[\Gamma]$ satisfies the non-degeneracy condition but not the other properties stated in Theorem (1.2.2), then we would obtain obstructions in terms of higher derivatives of the A_i's that ensure the metrizability of $[\Gamma]$. We will prove the theorem below in Section 3.5.

Theorem 1.2.3. *A real analytic projective structure* $[\Gamma]$ *over a surface* U *is metrizable in a sufficiently small neighborhood of a point* $p \in U$ *if and only if the rank of the* 6×21 *matrix with the rows*

$$\mathcal{M}_{max} := (\mathbf{V}, D_a\mathbf{V}, D_{(a}D_{b)}\mathbf{V}, D_{(a}D_bD_{c)}\mathbf{V}, D_{(a}D_bD_cD_{d)}\mathbf{V}, D_{(a}D_bD_cD_dD_{e)}\mathbf{V})$$

is smaller than six and there exists a vector $\mathbf{u} \in \ker(\mathcal{M}_{max})$ *such that* $\mathbf{u}_1\mathbf{u}_3 - (\mathbf{u}_2)^2$ *does not vanish at* p*.*

The sufficient condition we derived for the metrizability of a projective structure $[\Gamma]$ on a surface, may only hold in a sufficiently small open set of the surface. That is, even if $[\Gamma]$ is metrizable around any point of the surface, a global metric may fail to exist over the whole surface.

In Chapter 4 we will discuss a twistor version of the problem which we call the minitwistor theory following [4]. This approach will relate a real analytic projective structure on a surface to a complex surface with a family of rational curves with normal bundle of degree one, that we call minitwistor lines. This construction will be used as a geometric interpretation for the linearisation of an ODE associated to a projective structure to a system of linear first-order PDEs. Moreover, it allows us to construct the moduli space of a projective structure. In other words, we can construct the moduli space of the solutions of the second-order ODEs associated to a projective structure. Finally, we will show some properties of the space of metrics over a surface of which the Levi-Civita connections belongs to a given metrizable projective structure using densities on manifold. We mention in closing that the correspondence between local differential invariants of second-order ODEs defining a projective connection and formal neighborhoods of rational curves in twistor space has been explored in [16].

CHAPTER 2

Preliminaries

2.1 Density Bundles

In this chapter, we let M be an m-dimensional real manifold and V be a real vector space. The complex version of the definitions and theorems are straightforward to state. The material covered in this section is discussed in greater detail in [10], [25], and [26].

Definition 2.1.1. We call $f : \Lambda^m V^* \longrightarrow \mathbb{R}$ an r-density over V if $f(\lambda u) = |\lambda|^r f(u)$ for all $\lambda \in \mathbb{R}$. The linear space of r-densities on V is denoted by $|\Lambda|^r(V)$. The bundle of r-densities on a manifold M is obtained from TM by replacing each tangent space T_xM by the space of $|\Lambda|^r(T_xM)$ and is denoted by $|\Lambda|^r_M$. For an r-density we call r the weight of the density.

It is easy to show that the fibers of an r-density bundle are vector spaces. Since we will also consider tensor densities, we may refer to a r-density defined in the sense above, as a scalar r-density to avoid any confusion. Note that a scalar density on M is a line bundle.

Let Q and $\widetilde{Q}$ represent a scalar density of weight r in two coordinate systems $(x^i)_i$ and $(\widetilde{x}^i)_i$ on M. As we know, if an m-form u over $\Lambda^m(T^*M)$ transforms to $\widetilde{u}$ via the change of coordinates $(x^i) \to (\widetilde{x}^i(x))$ then $\widetilde{u} = Ju$ where $J = \det(\frac{\partial \widetilde{x}_i}{\partial x_j})$.

Thus, according to the definition we have

$$Q_x(u) = \widetilde{Q}_x\left(\frac{\widetilde{u}}{J}\right) = \frac{\widetilde{Q}_x}{|J^r|}(u) \Longrightarrow \widetilde{Q}_x = |J^r|Q_x. \tag{2.1}$$

As an example, consider an open set $U \subseteq M$ such that the canonical bundle (i.e. the highest exterior power of the cotangent bundle) is trivialisable over it. We can obtain a 1-density by first choosing a nonvanishing section $\mathbf{u}(x) \in \Gamma(\Lambda^m(T^*M))$ as a normalization and define

$$\begin{aligned} \omega : \Lambda^m(T^*M) &\longrightarrow \mathbb{R} \\ \mathbf{v}(x) &\longmapsto |g(x)|. \end{aligned} \tag{2.2}$$

where $\mathbf{v}(x) = g(x)\mathbf{u}(x)$. We can easily verify that ω is a 1-density. So on a sufficiently small open set of M, the density bundle we constructed above is identical to the canonical bundle restricted to the set of charts on U with positive Jacobian. We call a section $\eta \in |\Lambda|^1_M$ a volume form.

On any sufficiently small open set we can construct a positive 1-density, and using the fact that 1-densities form a vector space at each point, with a choice of partition of unity we obtain a 1-density over M. Having a 1-density over M at our disposal, we can raise any of its sections to the power r to construct a section in the r-density bundle as is clear from the definition. Therefore, it is clear that there is a one-to-one correspondence between the sections in the 1-density bundle and the r-density bundle. We can easily define the dual of an r-density of which the elements act on an element $f \in |\Lambda|^r_M$ on each fiber and give a scalar that would be a zero-density over M (the tensor product of any line bundle with its dual always has weight zero), thus the weight of the dual of an r-density is $-r$.

We can easily generalize this definition to tensor bundles.

Definition 2.1.2. For tensor bundle $\mathcal{T}M$ on a manifold M, the associated tensor density bundle of weight r has $\mathcal{T}M|_x \otimes |\Lambda|^r(T_xM)$ as its fibers at a point x and is denoted by $\mathcal{T}M \otimes |\Lambda|^r_M$

As an example, it is easy to see that the change in the representation of a $(1,2)$-tensor density with a change of coordinate $(x^i)_i \to (\widetilde{x}^i)_i$ is

$$\widetilde{T}^a_{\ bc} = J^3 \frac{\partial \widetilde{x}^a}{\partial x^i}\frac{\partial \widetilde{x}^j}{\partial x^b}\frac{\partial \widetilde{x}^k}{\partial x^c} T^i_{\ jk}.$$

The notion of covariant derivative can be extended to tensor densities as is done in [26]. The covariant derivative of a scalar density Q of weight r on an open set U, with respect to a connection with Christoffel symbols ${\Gamma^i}_{jk}$, is

$$\nabla_a Q = \frac{\partial}{\partial x^a} - r{\Gamma^b}_{ba}Q. \tag{2.3}$$

It is clear that the formulation for the covariant derivative of a 1-density is identical to the covariant derivative of an n-form as expected by our construction in (2.2). As a result, on an open set U over which our construction (2.2) of 1-densities hold, we sometimes denote the volume form η by $\eta_{[ab\cdots c]}$ which is the abstract index notation for differential forms of highest degree.

This easily gives the covariant derivative of a density tensor $\hat{T}$ with weight r and of order (p,q). In order to write the formula we do it for $(p,q) = (1,2)$ and the general case can be treated analogously. Notice that from the definition we have $\hat{T} = T \otimes Q$ were T is a $(1,2)$ tensor and Q is a scalar density of weight r. Therefore, we have

$$\begin{aligned}
\nabla_a \hat{T}^i_{jk} &= \nabla_a (T^i_{jk} Q) = (\nabla_a T^i_{jk}) Q + T^i_{jk} \nabla_a Q \\
&= \frac{\partial}{\partial x^a} \hat{T}^i_{jk} + (\Gamma^i_{ar} T^r_{jk} - \Gamma^r_{aj} T^r_{rk} - \Gamma^r_{ak} T^r_{jr}) Q - T^i_{jk} (r \Gamma^r_{ar} Q) \\
&= \frac{\partial}{\partial x^a} \hat{T}^i_{jk} + \Gamma^i_{ar} \hat{T}^r_{jk} - \Gamma^r_{aj} \hat{T}^r_{rk} - \Gamma^r_{ak} \hat{T}^r_{jr} - r \Gamma^r_{ar} \hat{T}^i_{jk}
\end{aligned} \tag{2.4}$$

It is clear that the only difference between the covariant derivatives of $\hat{T}$ and T is the last term which involves $-r\Gamma^r_{ar}$.

2.2 Jet Bundles

The material of this section can be found in more detail in [9] and [19]. Throughout the text we will make use of the abstract index notation in writing formulae and multi-index notation for taking derivatives, unless otherwise stated.

Whenever we have a system of differential equations we can regard all derivatives up to the highest degree appearing in the system as new variables and think of it as a system of constraints over a jet space. This means that we regard these derivatives as the local coordinates of the set of points in a fiber bundle called a jet bundle. In this way a system of differential equations is a set of constraints that define a submanifold of the jet bundle. In order to state the definition in a general form, we consider a fiber bundle (E, M, π) having M as its m-dimensional base manifold with N-dimensional fibers and π as the projection map. Let (x^i, u^a) with

$0 \le i \le m, 0 \le a \le N$, be a local trivialization for $E|_{\pi^{-1}(V)}, V \subseteq M$ and consider two local sections $s:\ V \longrightarrow E\ ,\ s':\ V' \longrightarrow E$ with $V \cap V' \neq \varnothing$. We define the equivalence relation j^p over $V \cap V'$ to be

$$j^p(s) = j^p(s') \iff \frac{\partial^{|I|} s^a}{\partial x^I}(x) = \frac{\partial^{|I|} s'^a}{\partial x^I}(x)\ ,\ |I| = p\ ,\ \forall x \in V \cap V' \tag{2.5}$$

where I is used for the multi-index notation.

We can define the vector space $J^p_x(E)$ to be the space of local sections of E equivalent with respect to the relation j^p. In other words, this space only contains the derivatives of sections of E up to degree p. Therefore, each element of this space is defined by assigning $N\binom{m+p}{p}$ coordinates which is the number of derivatives of a section up to order p, given the commutativity of partial derivatives. Define the pth-jet bundle to be $J^p(E) := \bigsqcup_{x \in M} J^p_x(E)/[j^p]$ with the equivalence relation j^p defined as above. Sometimes $J^p(M, E)$ is used instead of $J^p(E)$ to make clear what the base manifold is.

It is clear that $J^0(E) = E$ and that $J^p(E)$ has M as its base manifold with an obvious projection $\pi^p_0 : J^p(E) \longrightarrow J^0(E) := E$. There is also a natural projection $\pi^p_q :\ J^p(E) \longrightarrow J^q(E)$ for $p \ge q$ which ignores the derivatives of degree more than q. Any trivialization of E induces a trivialization of $J^p(E)$.

Let $\phi : \widetilde{V} := \pi^{-1}(V) \to \mathbb{R}^{(m+N)}$ be a trivialization of E such that $\phi(\widetilde{x}) = (x^i, u^a)$ at $\widetilde{x} = \pi^{-1}(x)$. Then the induced chart on $J^p(E)$ is:

$$\phi^p\ :\ \widetilde{V}^p := (\pi^p_0)^{-1}(\widetilde{V}) \longrightarrow \mathbb{R}^{m+N\binom{m+p}{p}}$$
$$\phi^p(j^p s(x)) = \left(x^i, \frac{\partial^{|I_q|} u^a}{\partial x^{I_q}}(x) \right)$$

where $1 \le i \le m,\ 1 \le q \le p,\ |I_q| = q$. We will denote $\frac{\partial^{|I|}}{\partial x^I} u^a(x)$ by $u^a_I(x)$.

Suppose the system F consists of r PDEs $F_s(x^i, u^a_I) = 0$, with $1 \le s \le r$, $|I| \le p$, that are of order p for N real-valued functions $(u^1, ..., u^N)$ over $\mathbb{R}^m$. The procedure explained above transforms F into a system of constraints in $J^p(\mathbb{R}^n, \mathbb{R}^m)$. This way all the derivatives in the equation become coordinates in $J^p(\mathbb{R}^n, \mathbb{R}^m)$. Comparing this example with the terminology of our discussion above, the fiber bundle E is the trivial $\mathbb{R}^N$ bundle over $M = \mathbb{R}^m$. This system defines a submanifold $\mathcal{M}^{(0)} \subseteq J^p(E)$ of codimension r as its zero locus called the equation manifold. Let $s(x) : M \to M \times U \subseteq E$ be a local smooth solution of the system F then $j^p(s) \in \mathcal{M}^{(0)}$ where j^p is defined as in (2.5). We define $\mathcal{S}^{(p)} \subseteq \mathcal{M}^{(0)}$ to be a manifold generated by the p-jets of smooth local solutions $s(x)$ of F for all choices of initial conditions (x_0^i, u_0^a). If $\mathcal{S}^{(p)} = \mathcal{M}^{(0)}$ then the system is called locally solvable. The manifold $\mathcal{M}^{(1)} \subseteq J^{(p+1)}$, called the first prolongation of $\mathcal{M}^{(0)}$, is defined by both the original system F and its first derivatives which are total derivatives with respect to the x^i's and contains $\mathcal{S}^{(p+1)}$ which is the $(p+1)$-jets of the smooth solution sections of F. So $\mathcal{M}^{(p+1)}$ is the zero locus of the system of PDEs $F^{(1)}$ in $J^{(p+1)}(E)$ defined as

$$F_s(x^i, u^a) = 0, \ \frac{DF_s}{Dx^i} = 0, \tag{2.6}$$

where $1 \leq i \leq m, \ 1 \leq a \leq N, \ 1 \leq s \leq r$. Because the prolonged system satisfies the original system as well, we have $\pi_p^{p+1}\mathcal{M}^1 \subseteq \mathcal{M}^{(0)}$ as a submanifold. It is important to note that we may have $\pi_p^{p+1}\mathcal{M}^{(1)}$ as a submanifold of $\mathcal{M}^{(0)}$ with nonzero codimension. This is because differentiating F and mixing the new equations with the old ones may lead to cancelation of derivatives of order p, and by projecting the result to $J^p(E)$ we obtain a non-regular equation which does not surject onto $\mathcal{M}^{(0)}$ but instead gives a submanifold of nonzero codimension. Also it is easy to see that $\pi_p^{p+1}\mathcal{S}^{(p+1)} = \mathcal{S}^p$. For instance over $U \subseteq J^1(\mathbb{R}^2, \mathbb{R})$ with coordinate (x_1, x_2, u, u_1, u_2) consider the system of constraints F which determines $\mathcal{M}^{(1)}$ defined as

$$F(x_i, u, u_i) = \begin{cases} u_1 = u + x_1, \\ u_2 = u^2 + x_2, \end{cases} \tag{2.7}$$

Therefore the first prolongation manifold $\mathcal{M}^{(1)}$, is determined by

$$F(x_i, u, u_i) \quad \text{and} \quad \frac{D}{Dx_i}F(x_i, u, u_i) = \begin{cases} u_{11} = u_1 + 1, \\ u_{12} = u_2, \\ u_{21} = 2uu_1, \\ u_{22} = 2uu_2 + 1. \end{cases} \tag{2.8}$$

Because $u_{12} = u_{21}$, the second and third equation in (2.8) gives a new constrain on the projection of $\mathcal{M}^{(1)}$ into $\mathcal{M}^{(0)} \subseteq J^1(\mathbb{R}^2, \mathbb{R})$, namely $u_2 = 2uu_1$, and is independent of the equations (2.7); therefore $\mathcal{M}^{(1)}$ does not surject to $\mathcal{M}^{(0)}$.

As a result it is possible to have a system of p-th order PDEs such that the solutions do not have a $(p+r)$-jet for some $r \geq 1$ because the constraints introduced by the prolongations may accumulate in such a way that the projection of some $\mathcal{M}^{(r)}$ into $\mathcal{M}^{(0)}$ becomes empty. To put it differently, if we note that

$$\mathcal{S}^{(p)} \subseteq \mathcal{M}^{(0)}, \ \mathcal{S}^{(p+1)} \subseteq \mathcal{M}^{(1)}, \ \pi_p^{p+1}\mathcal{M}^{(1)} \subseteq \mathcal{M}^{(0)}, \ \pi_p^{(p+1)}\mathcal{S}^{(p+1)} = \mathcal{S}^{(p)} \tag{2.9}$$

then we can say that projecting the prolongation manifolds $\mathcal{M}^{(r)}$ to $\mathcal{M}^{(0)}$ as we increase r takes us closer to $\mathcal{S}^{(p)} \subseteq \mathcal{M}^{(0)}$ and we expect that if $\mathcal{S}^{(p)}$ is empty there for some r the projection of $\mathcal{M}^{(r)}$ to $\mathcal{M}^{(0)}$ becomes empty for some $r \geq 1$.

A natural question is to ask whether a system necessarily has a smooth solution if differentiating it q times for some integer q, its zero locus (the prolongation manifolds $\mathcal{M}^{(q)}$ of $\mathcal{M}^{(0)}$) is observed to be not zero-dimensional. The general answer to this question makes use of Cartan's test which we will discuss briefly in the the appendix. We will however, discuss the answer in the case where F is a first-order system of PDEs in details in next section.

2.3 Involutivity of First-order PDEs

In this section, we state an important theorem concerning the integrability of a system of first-order PDEs and the dimension of the solution space with geometric interpretation of a system of PDEs that connects the system to the parallel sections of a connection on a vector bundle. As we will define the notion of holonomy for a connection, we will realize that the existence of a parallel section gives rise to a restriction on the holonomy.

We assume that the system F has the following form over the fiber bundle E,

$$F_a(x^i, u^a, u^a_i) = \frac{\partial u^a}{\partial x^i} - \psi^a_i(x,u) = 0, \tag{2.10}$$

where $1 \leq i \leq m,\ 1 \leq a \leq N$.

Definition 2.3.1. An r-dimensional distribution $\mathcal{D}$ on M is an assignment of an r-dimensional subspace $\mathcal{D}_p \subseteq T_pM$ at each point $p \in M$ such that $\mathcal{D}_p$ varies smoothly with respect to p, meaning that there exists a neighborhood $U \subseteq M$ of each point $p \in M$ such that $\mathcal{D}_q = \text{span}\{X_1(q), ..., X_r(q)\}\ \forall q \in U$, for a set of r vector fields $\{X_1, ..., X_r\}$ defined over U. A submanifold $N \subseteq M$is called an integral manifold of $\mathcal{D}$ if $T_qN = \mathcal{D}_q\ \forall q \in N$. The distribution $\mathcal{D}$ is called completely integrable if there exists a unique maximal integral manifold of $\mathcal{D}$ at every point of M .

Assuming that the the equation manifold of the system (2.10) that we denote by $\mathcal{M}^{(0)} \in J^1(\mathbb{R}^m, \mathbb{R}^N)$ exists and is r dimensional, it is easy to find an r-dimensional distribution $\mathcal{D}^r$ on E generating $\mathcal{S}^{(0)} \subseteq E$ as its integral manifold. Recall that $\mathcal{S}^{(0)}$ is a submanifold of E that is generated by sections $s(x) \subseteq E$ for which $F(s) = 0$

hold. Using chain rule, the form of a vector field X tangent to $\mathcal{M}$,with coordinates (x^i, u^a), can be written as

$$X = A^i(\frac{\partial}{\partial x^i} + \frac{\partial u^a}{\partial x^i}\frac{\partial}{\partial u^a}) \tag{2.11}$$

Therefore the r-dimensional distribution $\mathcal{D}$ generating $\mathcal{S}^{(0)}$ is generated by the vectors

$$X_i = \partial_{x^i} + \psi_i^a \partial_{u^a},\ 1 \le i \le m,\ 1 \le a \le N. \tag{2.12}$$

Now we state the classic version of Frobenius theorem which involves the existence of an integral manifold for a distribution. The proof can be found in [21].

Theorem 2.3.2 (Frobenius Theorem)**.** *Let $\mathcal{D}$ be an r-dimensional distribution over on E such that*

$$[X, Y] \in \mathcal{D},\ \ \forall X, Y \in \mathcal{D}.$$

Then $\mathcal{D}$ is completely integrable.

To prove the existence of $\mathcal{S}^{(0)}$ for the system (2.10), we look for the integral manifold of the r-dimensional distribution $\mathcal{D}$ we found in (2.12). It turns out that in order to have $[X_i, X_j] \in \mathcal{D}$ for the vector fields X_i in (2.12), the ψ_i^a's have to satisfy specific relations. Forming the commutator $[X_i, X_j]$, the only way that it can be written as a linear combination of the X_i's is when $[X_i, X_j] = 0,\ \forall i, j$ since any linear combination of the X_i's gives a nonzero coefficient for at least one vector ∂_{x^i} for some i while the coefficients of the ∂_{x^i}'s for the commutators $[X_i, X_j]$ are zero. This equality describes the equations that have to be satisfied by the functions ψ_i^a's in order to guarantee the complete integrability of $\mathcal{D}$. It is straightforward to derive these equations which would be as stated in the following theorem.

Theorem 2.3.3. *The system F defined in* (2.10) *has a unique smooth solution $u(x) = (u^1(x), ..., u^a(x))$ for any choice of initial value $u(x_0) = u_0$ if and only if the ψ_i^a's satisfy*

$$\frac{\partial \psi_i^a}{\partial x^i} - \frac{\partial \psi_j^a}{\partial x^j} + \frac{\partial \psi_i^a}{\partial u^b}\psi_j^b - \frac{\partial \psi_j^a}{\partial u^b}\psi_i^b = 0 \tag{2.13}$$

This theorem says that with the equalities above satisfied, the general solution of the system F in (2.10) only depends on the initial value u_0 meaning that it depends on N arbitrary constants specifying $u_0 = (u_0^a)_a$.

What if the relations in (2.13) are not satisfied? This means that the system F may not have a solution for arbitrary initial values. But we can still try to find those initial values for which a solution of F does exist. In other words, we are trying to find the integral manifolds of the distribution $\mathcal{D}$ generated by vector fields (2.12), whose coordinate functions satisfy equation (2.13). If such manifolds does not exist then it means that there is no sub-distribution of $\mathcal{D}$ that is integrable and as a result, for no choice of initial condition do we get a solution. Using $\frac{\partial u^a}{\partial x^i} = \psi_i^a(x,u)$ to substitute derivatives of the u^a's with the ψ_i^a's, we can look at the relations (2.13) as a set of equations

$$F_1(x,u) = 0 \tag{2.14}$$

involving the functions $\psi_i^a(x,u)$. These equations are satisfied over $\mathcal{S}^{(0)} \subseteq E$, as they express locus of points satisfying F and the integrability condition (2.13) simultaneously. We denote this locus by $\mathcal{M}_1$. We can differentiate equations $F_1(x,u) = 0$ similarly and get new equations that form a new set of equations

$$F_2(x,u) = 0.$$

If $\mathcal{M}_1$ is the equation manifold for the system F, namely $\mathcal{S}^{(0)} = \mathcal{M}_1$ then the system $F_2 = 0$ has to be satisfied over $\mathcal{M}_1$ as they ensure that the locus of points $\mathcal{S}^{(0)}$ is an integral manifold. In other words,they say whether the distribution of $\mathcal{D}$ generating $\mathcal{M}_1$ and found from the system $F_1 = 0$, is closed under the Lie bracket. If it is closed then we have found an integral manifold and we are done. If not then we consider the new locus $\mathcal{M}_2$ obtained from $F_2 = 0$ and follow a similar procedure by differentiating F_2. In this way we get other sets of relations $F_k = 0$ with a geometric interpretation similar to what we gave for $F_1 \text{ and } F_2$. To summarize what was done above, we can say that our first step was to find the locus of points satisfying $F_1 = 0$, denoted by $\mathcal{M}_1$, which is the intersection locus of the system F and integrability condition (2.13). Then the second step checks whether the locus is an integral manifold of F or not. If not we follow the same procedure for the locus of points satisfying $F_2 = 0$ denoted by $\mathcal{M}_2$ which is a subset of $\mathcal{M}_1$. This procedure can be continued for other F_k's.

Now we are in a situation similar to the question we asked before. Recall that we want to find initial values for which a solution $u(x)$ exists. Therefore, if we obtain more than N independent sets of equations $F_i(x,u)$ then we have too many constraints on the u^a's and no manifold $\mathcal{S}^{(1)} \in J^1(E)$ generated by 1-jets of smooth solution can exist. Thus, there exist $K \leq N$ such that F_{K+1} is a consequence of the sets $F_i = 0,\ 1 \leq i \leq K$. We have the following theorem from [30].

Theorem 2.3.4. *The system F in* (2.10) *admits solution if and only if there exists an integer $K \leq N$ such that the set of equations*

$$F_1 = F_2 = \cdots = F_K = 0$$

is compatible for all $x \in U \subseteq \mathbb{R}^N$ and the set of equations $F_{K+1} = 0$ is satisfied in the region in which $F_1, ..., F_K$ vanish. If q is the number of independent equations in the first K sets of equations, then the general solution of the system depends on $N - q$ arbitrary constants.

Notice that these arbitrary constants specify the locus of the initial values that give rise to a solution of the system. In other words,the solution space is an q-codimensional submanifold of E. Our discussion above proves the forward implication of the theorem.

Proof. Assume that the first K independent sets impose $q \leq N$ independent conditions

$$G_s(u, x) = 0,\ s = 1, ..., q. \tag{2.15}$$

Therefore $\mathrm{rank}(\frac{\partial G_s}{\partial u^a}) = q$, and by the implicit function theorem the relations (2.15) can be solved for say the first q functions $u^1, ..., u^q$. Therefore, we can write $u^\alpha = \phi^\alpha(u^{q+1}, ..., u^N, x)$, $1 \leq \alpha \leq q$. Differentiating ϕ^α's and using (2.10) to eliminate the derivatives $\partial_{x_i} u^\alpha$, we obtain

$$\psi_i^\alpha - \frac{\partial \phi^\alpha}{\partial u^\beta}\psi_i^\beta - \frac{\partial \phi^\alpha}{\partial x^i} = 0. \tag{2.16}$$

where $q \leq \beta \leq N$. These equations belong to the set $F_{K+1} = 0$ so they hold by assumption. By substituting $\psi_i^\alpha = \partial_{x_i} u^\alpha$ in the equation (2.16) and subtracting the result from (2.16) we get

$$\frac{\partial u^\alpha}{\partial x^i} - \psi_i^\alpha - \frac{\partial \psi^\alpha}{\partial u^\beta}\left(\frac{\partial u^\beta}{\partial x^i} - \psi_i^\beta\right) = 0, \tag{2.17}$$

So

$$\frac{\partial u^\beta}{\partial x^i} = \bar{\psi}_i^\beta(u^{q+1}, ..., u^N, x), \tag{2.18}$$

where $\beta = q + 1, ..., N$ and $\bar{\psi}_i^\beta = \psi_i^\beta|_{u^\alpha = \phi^\alpha(u^{q+1}, ..., u^N, x)}$. The system (2.18) is completely integrable as the consistency belongs to the set

$$F_1 = \cdots = F_K = 0.$$

Therefore, by the Frobenius Theorem (2.3.3) there is a solution which involves $(N - q)$ constants. □

In our problem, the functions ψ^a_i's in the system (2.10) are linear in u^a. So we have

$$\frac{\partial u^a}{\partial x^i} = \psi^a_{b,i}(x)u^b.$$

We can write this as

$$D\mathbf{u} := d\mathbf{u} + \mathbf{\Omega u} = 0, \tag{2.19}$$

where $\mathbf{u} = (u^1, ..., u^N)^T$ is a column vector and

$$\mathbf{\Omega} = \mathbf{\Omega}_1 dx^1 + \mathbf{\Omega}_2 dx^2 = -\psi^a_{b,i}(x)dx^i \tag{2.20}$$

is an $N \times N$ matrix-valued one-form on U. In its rewritten form, the problem becomes one of finding parallel sections of the connection $D = d + \mathbf{\Omega}$ over a vector bundle with N-dimensional fibers over U. Note that, as we assumed earlier, the total space is $\mathbb{R}^{m+N}$ over $U \subseteq \mathbb{R}^m$. Since our problem, explained in the next chapter, involves a two-dimensional base manifold, we consider the case $m = 2$. The same holds for other cases. Taking the exterior derivative of equation (2.19), we obtain

$$d(d\mathbf{u} + \mathbf{\Omega u}) = d\mathbf{\Omega u} + \mathbf{\Omega} \wedge d\mathbf{u} = (d\mathbf{\Omega} + \mathbf{\Omega} \wedge \mathbf{\Omega})\mathbf{u} = \mathbf{Fu} = 0, \tag{2.21}$$

where $\mathbf{F}$ is the curvature of the connection D. We have

$$\mathbf{F} = d\mathbf{\Omega} + \mathbf{\Omega} \wedge \mathbf{\Omega} = (\partial_1 \mathbf{\Omega}_2 - \partial_2 \mathbf{\Omega}_1 + [\mathbf{\Omega}_1, \mathbf{\Omega}_2])dx^1 \wedge dx^2 = F dx^1 \wedge dx^2 \tag{2.22}$$

where F is an $N \times N$ matrix. Note that this last equation has to be satisfied by a solution of our system. This is actually $F_1 = 0$, the first set of conditions we mentioned in Theorem (2.3.4). In our case they are linear homogeneous equations. If $F = 0$ then the connection is flat and the condition (2.13) in Theorem (2.3.3) is satisfied and N independent parallel sections of the connection exist which is equivalent to saying that the general solution of the system (2.19) depends on N arbitrary constants. On the other hand, if $\det F \neq 0$ then the only parallel section is the zero section.

To determine the dimension of the space of parallel sections of the connection D, we follow the procedure explained earlier: differentiating the equation (2.22). We obtain

$$0 = d(F\mathbf{u}) = dF\mathbf{u} - F\mathbf{\Omega u} = [(\partial_i F - F\Omega_i)\mathbf{u}]dx^i. \tag{2.23}$$

Using $F\mathbf{u} = 0$ we get

$$(D_i F)\mathbf{u} = 0,$$

where $D_i F = \partial_i F + [\Omega_i, F]$. We keep differentiating to obtain new sets of equation as constraints on the equation manifold. Expressed in matrix form, these sets of equations are denoted as

$$F\mathbf{u} = 0, \quad (D_i F)\mathbf{u} = 0, \quad (D_i D_j F)\mathbf{u} = 0, \quad \dots \tag{2.24}$$

The set of equations $F_1 = F_2 = \dots = 0$ stated in Theorem (2.3.4) are are identical to matrix equations above. After K times differentiating we obtain

$$\mathcal{F}_K \mathbf{u} = 0$$

where $\mathcal{F}_K$ is a $2^K \times N$ matrix is the matrix consisting of the sub-matrices

$$\mathcal{F}_K = \big(F, (D_i F), \dots, (D_i \cdots D_j F)\big) \tag{2.25}$$

with $\mathcal{F}_0 = F$. Theorem (2.3.4) tells us that this process eventually stops.

Theorem 2.3.5. *Assume that the ranks of the matrices $\mathcal{F}_K,\ K = 0,\ 1,\ 2, \dots,$ are maximal and constant. Let K_0 be the smallest natural number such that*

$$\operatorname{rank}(\mathcal{F}_{K_0}) = \operatorname{rank}(\mathcal{F}_{K_0+1}).$$

If K_0 exists then $\operatorname{rank}(\mathcal{F}_{K_0}) = \operatorname{rank}(\mathcal{F}_{K_0+k})\ \forall k \in \mathbb{N}$ and the space of parallel sections of the connection $D = d + \mathbf{\Omega}$ has dimension $N - \operatorname{rank}(\mathcal{F}_{K_0})$.

Note that in the theorem above, we asked for the local maximality of the rank of a finite number of matrices in the sense that the rank of the matrices do not decrease in an open set. This is always true for at least one sufficiently small open neighborhood of a point in the domain essentially because of the continuity of the determinant function on the space of matrices.

Our discussion shows that the existence of a non-trivial parallel section of the connection ∇ over the rank six vector bundle E, induces a restriction on the what is called the holonomy of ∇. In order to define the holonomy of a connection ∇ over E at point $x \in M$, denoted by $\operatorname{Hol}_x(\nabla)$, we use the parallel transport map

$$P_\gamma : E_x \to E_y$$

where $\gamma : [0,1] \to E]$ and $\gamma(0) \in E_x$ and $\gamma(1) \in E_y$. It is clear that for such choice of γ we have $P_\gamma \in \mathrm{Hom}(E_x, E_y)$. We define

$$\mathrm{Hol}_x(\nabla) \coloneqq \{P_\gamma \ : \ \gamma \text{ a loop based at } x\} \subseteq GL(E_x).$$

It is not hard to see that if M is connected then the holonomy groups at different points of M are isomorphic by conjugation, so we can omit the label x and therefore the holonomy group is a global invariant of the connection with $\mathrm{Hol}(\nabla) \in GL(N,\mathbb{R})$. When the surface M is not simply connected it is easier to consider the restricted holonomy group defined as

$$\mathrm{Hol}^0_x(\nabla) \coloneqq \{P_\gamma \ : \ \gamma \text{ a null-homotopic loop based at } x\} \subseteq GL(E_x).$$

Similarly to our argument above, we can omit the label x and obtain that the holonomy group $\mathrm{Hol}^0(\nabla)$ is a subgroup of $GL(N,\mathbb{R})$. Thus, we can define the holonomy algebra $\mathfrak{hol}(\nabla)$ to be the Lie algebra of $\mathrm{Hol}^0(\nabla)$. It is a Lie subalgebra of $\mathfrak{gl}(N,\mathbb{R})$ defined up to the adjoint action of $GL(N,\mathbb{R})$. Similarly, we have $\mathfrak{hol}_x(\nabla)$ which is the Lie algebra of $\mathrm{Hol}^0_x(\nabla)$.

In order to state the Ambrose-Singer Holonomy Theorem (c.f. [21]) we recall that for a affine connection we have its curvature $\mathcal{R} \in \Gamma(\Lambda^2 T^*M \otimes End(E))$ defined as

$$\mathcal{R}(X,Y)s \coloneqq (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})s.$$

where $X, Y \in TM$ and s is a section of E.

Theorem 2.3.6. *Let M be a manifold, E a vector bundle over it with a choice of connection, say ∇. Fix $x \in M$ so that $\mathfrak{hol}_x(\nabla)$ is a Lie subgroup of $\mathrm{End}(E_x)$. Then $\mathfrak{hol}_x(\nabla)$ is the vector space of all elements of $End(E_x)$ of the form $P_\gamma^{-1}\mathcal{R}(X,Y)P_\gamma$, where $x \in M$ and $\gamma : [0,1] \to M$ is a piecewise smooth curve with $\gamma(0) = x$ and $\gamma(1) = y$, $P_\gamma : E_x \to E_y$ is the parallel translation map and $X, Y \in T_yM$.*

This shows that the curvature $\mathcal{R}$ determines $\mathfrak{hol}(\nabla)$ and hence $\mathrm{Hol}^0(\nabla)$. For instance, if the connection is flat, so that $\mathcal{R} = 0$, then $\mathfrak{hol}(\nabla) = 0$ and therefore $\mathrm{Hol}^0(\nabla)$ is trivial.

Now according to our discussion following equation (2.22), we showed that the existence of a non-trivial solution to a system of first-order PDEs at a point x, forces the curvature tensor not to have a full-rank in a neighborhood of x, namely $\det(F) = 0$. Therefore, if the curvature of (E, D), defined in (2.22), does not vanish, a non-trivial solution to the system of linear PDEs (2.10) exists, if the holonomy of D lies in some proper subgroup of $GL(N, \mathbb{R})$. This is clear from the Ambrose-Singer Theorem, as the holonomy would be generated by

$$P_\gamma^{-1}\mathbf{F}(X,Y)P_\gamma = (P_\gamma^{-1}FP_\gamma)(dx \wedge dy(X,Y)),$$

where $X, Y \in TM$ and F is defined in (2.22).

CHAPTER 3

Necessary and Sufficient Conditions for Metrizability

In this chapter, we want to find the necessary and sufficient conditions for the problem posed in the introduction. Our treatment of the problem is the same as that in [2] and uses the material discussed in Section 2.3 and (2.2).

3.1 Projective Equivalence of Torsion-free Affine Connections

A connection allows us to associate to any pair consisting of a point and a direction in the tangent space of that point a unique maximal curve that is called a geodesic. To describe it explicitly, we fix a point p in an m-dimensional manifold M and a direction in T_pM and pick any vector in that direction, say $\mathbf{v} \in T_pM$. Then finding a curve such that the parallel transport of $\mathbf{v}$ along the curve remains tangent to it amounts to solving $\nabla_{\mathbf{v}}\mathbf{v} = 0$. It is clear that scaling the vector by a scalar does not affect the equation and the locus only depends on the direction. In local coordinates $(x^1, ..., x^m)$, if ${\Gamma^i}_{jk}$ are the Christoffel symbols for ∇, then $\nabla_{\mathbf{v}(t)}\mathbf{v}(t) = 0$ gives

$$\frac{d^2x^i}{dt^2} - {\Gamma^i}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \tag{3.1}$$

along the geodesic $\gamma(t) = (x^1(t), ..., x^m(t))$ where $\mathbf{v}(t) = \dot{\gamma}(t) = (\frac{dx^1}{dt}(t), ..., \frac{dx^m}{dt}(t))$.

Here we assume M is a surface with a local trivialization (x, y), which we sometimes denote by (x^1, x^2), over an open set U. To eliminate the parameter t in (3.1), assuming $\frac{dy}{dx} \neq 0$, we obtain a second-order ODE

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}(\frac{\dot{y}}{\dot{x}}) = \frac{d}{dt}(\frac{\dot{y}}{\dot{x}})\frac{1}{\dot{x}} = \frac{\ddot{y}}{\dot{x}^2} - \frac{\dot{y}\ddot{x}}{\dot{x}^3} \\ &= -\Gamma^2{}_{11} - 2\Gamma^2{}_{12}\frac{\dot{y}}{\dot{x}} - \Gamma^2{}_{22}(\frac{\dot{y}}{\dot{x}})^2 + \Gamma^1{}_{11}\frac{\dot{y}}{\dot{x}} + 2\Gamma^1{}_{12}(\frac{\dot{y}}{\dot{x}})^2 + \Gamma^1{}_{22}(\frac{\dot{y}}{\dot{x}})^3 \\ &= \Gamma^1{}_{22}(\frac{dy}{dx})^3 + (2\Gamma^1{}_{12} - \Gamma^2{}_{22})(\frac{dy}{dx})^2 + (\Gamma^1{}_{11} - 2\Gamma^2{}_{12})\frac{dy}{dx} - \Gamma^2{}_{11}\end{aligned} \tag{3.2}$$

We can write this as

$$\frac{d^2y}{dx^2} = A_3(\frac{dy}{dx})^3 + A_2(\frac{dy}{dx})^2 + A_1\frac{dy}{dx} + A_0 \tag{3.3}$$

where

$$A_3 = \Gamma^1{}_{22},\ A_2 = 2\Gamma^1{}_{12} - \Gamma^2{}_{22},\ A_1 = \Gamma^1{}_{11} - 2\Gamma^2{}_{12},\ A_0 = -\Gamma^2{}_{11}. \tag{3.4}$$

In this way we can associate to any affine connection, a second-order ODE.

We can look at this ODE in another way. A pair of a point and a direction in its tangent space, correspond to only one maximal geodesic. Therefore, we can think of the initial values of the system of ODEs in (3.1) to be a point in $\mathbb{P}(TM)$. For a manifold M, $\mathbb{P}(TM)$ is simply the fiber bundle obtained from TM by taking the quotient of each tangent space by scalar multiplication in order to obtain projective spaces as the fibers. Due to Picard's theorem, the solutions of an ODE depend smoothly on the initial conditions. Knowing that there is unique lift of a curve in M to $\mathbb{P}(TM)$, we expect that the lift of geodesics, as the solutions of (3.1), give a foliation of $\mathbb{P}(TM)$. In order to determine this foliation, we convert the second-order system (3.1) into a system of first-order ODEs by introducing new variables (y^1, y^2) as

$$\begin{aligned}y^i(t) &= \frac{dx^i}{dt}(t) \\ \frac{dy^i}{dt}(t) &= -\Gamma^i{}_{jk}y^j(t)y^k(t).\end{aligned} \tag{3.5}$$

Now we find the vector field that is tangent to the integral curves of (3.5). This vector field usually called the characteristic vector field of the system. An integral curve of (3.5), say $\gamma(t) = (x^i(t), y^i(t))$, lies in TM. Its tangent vector has the form

$$\begin{aligned}\dot{\gamma}(t) &= \dot{x}^i(t)\frac{\partial}{\partial x^i} + \dot{y}^i(t)\frac{\partial}{\partial y^i} \\ &= y^i\frac{\partial}{\partial x^i} - \Gamma^i{}_{jk}y^jy^k\frac{\partial}{\partial y^i}.\end{aligned} \tag{3.6}$$

This vector field is also called the geodesic flow or the geodesic spray as its integral curves are geodesics. Because the geodesic equation is invariant under scalar multiplication, the characteristic vector field is homogeneous of order one with respect to coordinates of the tangent bundle, so it also defines a vector field over $\mathbb{P}(TM)$.

Definition 3.1.1. Two connections Γ and $\widetilde{\Gamma}$ are projectively equivalent if they have identical geodesics as unparametrized curves. The set of all connections projectively equivalent to Γ is called the projective structure defined by Γ and is denoted by $[\Gamma]$.

One way to derive an expression for two projectively equivalent connections is to make use of geodesic spray as is done in [14]. Two alternative procedures ways are discussed in [10] and [12]. The fact that any point in $\mathbb{P}(TM)$ determines only one maximal geodesic suggests that if we have two projectively equivalent connections with the same geodesics as unparametrized curves, the foliation of $\mathbb{P}(TM)$ by the integral curves of their geodesic spray is identical. This is essentially due to the fact that projecting the geodesic flows to $\mathbb{P}(TM)$ normalizes the tangent vectors of the geodesics. Let V and $\widetilde{V} \in \Gamma(TTM)$ be geodesic sprays that correspond to projectively equivalent connections $\Gamma \,\mathrm{and}\, \widetilde{\Gamma}$. Using the map $\pi : TM \to \mathbb{P}(TM)$ and (3.6), at a point p we easily obtain

$$\begin{aligned}\pi_*(V) &= \pi_*(y^i \frac{\partial}{\partial x^i} - \Gamma^i{}_{jk} y^j y^k \frac{\partial}{\partial y^i}) \\ &= \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial y} + \left(A_0 + A_1\zeta + A_2\zeta^2 + A_3\zeta^3\right)\frac{\partial}{\partial \zeta}\end{aligned} \tag{3.7}$$

where $\zeta = \frac{y_2}{y_1}$ is an affine coordinate for the fiber $\mathbb{P}(T_pM)$ and A_i's are given in (3.4). Now if the integral curves of $\pi_*(V)$ and $\pi(\widetilde{V})$ were identical, we would have

$$\pi_*(V(p)) = c \cdot \pi_*(\widetilde{V}(p))$$

for some number c. Comparing the expressions using (3.7), we get $c = 1$. This indicates that V and $\widetilde{V}$ differ in the radial part meaning as follows. The projection $\pi_* : TTM \to T(\mathbb{P}(TM))$ is an $\mathbb{R}^*$ action. We can find the vector field over TM that generates this action which is not hard to see is $y_i \frac{\partial}{\partial y_i}$. This implies there exists a function $f \in C^\infty(TM)$ linear in y^i's, such that

$$V(x,y) - \widetilde{V}(x,y) = f(x,y) y_i \frac{\partial}{\partial y^i}. \tag{3.8}$$

Comparing the coefficients of $\frac{\partial}{\partial y^i}$, we obtain

$$\widetilde{\Gamma}^i{}_{jk} y^j y^k - \Gamma^i{}_{jk} y^j y^k = f y^i \Longrightarrow g^i_k y^k = f y^i,$$

where $g^i_k(x,y) = (\widetilde{\Gamma}^i{}_{jk} - \Gamma^i{}_{jk}) y^j$. Therefore, if $i \neq k$ then $g^i_k = 0$ and

$$g^i_k(x,y) = \delta^i_k \Upsilon_j y^j = (\widetilde{\Gamma}^i{}_{ji} - \Gamma^i{}_{ji}) y^j.$$

Since we are dealing with torsion-free connections we have $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$ and we similarly obtain

$$\delta^i_j \Upsilon_k y^k = (\widetilde{\Gamma}^i{}_{ji} - \Gamma^i{}_{ji}) y^k.$$

Thus the formulation for projectively equivalence of two connections is

$$\widetilde{\Gamma}^i{}_{jk} = \Gamma^i{}_{jk} + \Upsilon_j \delta^i_k + \Upsilon_k \delta^i_j. \tag{3.9}$$

3.2 Linearizing the System

Let $g = (g_{ij})$ be a Riemannian metric over an open set $U \subseteq M$ of a surface. We use the notation given by (1.2), and denote the inverse matrix of g_{ij} by g^{ij} . Assume that the Levi-Civita connection of this metric belongs to a projective structures $[\Gamma]$ with $A_0, ..., A_3$ as the coefficients of its associated second-order equation (3.2). From equation (3.4) where the A_i's are expressed as linear combination of Christoffel symbols, we realize that the A_i's form an affine space at each point of the surface. On the other hand, as we said in the introduction, any choice of smooth coefficients in equation (3.2) results in a projective structure, a result which is due to Cartan [5]. Thus, the set of all possible values of the A_i's at a point is an affine space which forms the fibers of a rank 4 vector bundle over U and a section of this vector bundle represent A_i's over U. We denote this vector bundle by $Pr(U)$.

As we saw earlier, in equation (3.2) the A_i's are expressed invariantly in terms of Christoffel symbols of any element of the projective class $[\Gamma]$. According to (1.3) Christoffel symbols are functions of the metric components and their first derivatives and thus the same holds for A_i's as linear functions of Christoffel symbols. Therefore, from (1.6) we have an operator

$$\sigma^0 : J^1(S^2(T^*U)) \longrightarrow J^0(Pr(U)) \tag{3.10}$$

where $S^2(T^*U)$ is the space of symmetric bilinear forms, which contains the metric g. Equation (1.6) indicates that σ^0 is homogeneous of degree zero because multiplying by any nonzero constant does not affect the A_i's.

We want to use equation (1.6) to express the g_{ij}'s in terms of the A_i's. The non-degeneracy condition on the metric, forces the condition $EG - F^2 \neq 0$. In order to include this condition, when we are expressing g_{ij} in terms of A_i's, we express the g_{ij}'s in the form

$$g_{ij} = \frac{\sigma_{ij}}{\Delta^2} \quad \text{and} \quad \Delta = \det(\sigma), \tag{3.11}$$

with σ a symmetric 2×2 matrix. We will also denote σ_{11}, σ_{12}, and σ_{22} by $\psi_1,\ \psi_2$, and ψ_3 respectively. By the above substitution of variables, it is straightforward to obtain the following set of equations from (1.6).

$$\begin{aligned} \frac{\partial \psi_1}{\partial x} &= \frac{2}{3}A_1\psi_1 - 2A_0\psi_2, \\ \frac{\partial \psi_3}{\partial y} &= 2A_3\psi_2 - \frac{2}{3}A_2\psi_3, \\ \frac{\partial \psi_1}{\partial y} + 2\frac{\partial \psi_2}{\partial x} &= \frac{4}{3}A_2\psi_1 - \frac{2}{3}A_1\psi_2 - 2A_0\psi_3, \\ \frac{\partial \psi_3}{\partial x} + 2\frac{\partial \psi_2}{\partial y} &= 2A_3\psi_1 - \frac{4}{3}A_1\psi_3 + \frac{2}{3}A_2\psi_2. \end{aligned} \tag{3.12}$$

In the substitution, we introduced a new metric tensor σ_{ij} which satisfies (3.12). The equations above are equivalent to

$$\nabla^{\Pi}_{(a}\sigma_{bc)} = 0, \tag{3.13}$$

where we have used abstract index notation and ∇^{Π} is an affine connection with ${\Pi^a}_{bc}$ as its Christoffel symbols and

$$\begin{aligned} {\Pi^1}_{11} &= \frac{1}{3}A_1, & {\Pi^2}_{11} &= -A_0, & {\Pi^1}_{12} &= \frac{1}{3}A_2, \\ {\Pi^2}_{12} &= -\frac{1}{3}A_1, & {\Pi^1}_{22} &= A_3, & {\Pi^2}_{22} &= -\frac{1}{3}A_2. \end{aligned} \tag{3.14}$$

Recall that σ is a symmetric $(2,0)$-tensor. As an example, we have

$$\begin{aligned} \nabla^{\Pi}_1\sigma_{12} &= \partial_1\sigma_{12} - {\Pi^a}_{11}\sigma_{a2} - {\Pi^a}_{12}\sigma_{1a} \\ \Rightarrow \nabla^{\Pi}_{(1}\sigma_{12)} &= \frac{1}{6}(2\nabla^{\Pi}_1\sigma_{12} + 2\nabla^{\Pi}_1\sigma_{21} + 2\nabla^{\Pi}_2\sigma_{11}) \\ &= \frac{1}{3}[2(\partial_1\psi_2 + ({\Pi^1}_{11} - {\Pi^2}_{12})\psi_2 - {\Pi^2}_{11}\psi_3 - {\Pi^1}_{12}\psi_1) \\ &\quad + (\partial_2\psi_1 - 2{\Pi^1}_{21}\psi_1 - 2{\Pi^2}_{11}\psi_2)] \\ &= \frac{1}{3}(2\partial_1\psi_2 + \partial_2\psi_1 - 4{\Pi^1}_{21}\psi_1 - (4{\Pi^2}_{21} + 2{\Pi^1}_{11})\psi_2 - 2{\Pi^2}_{11}\psi_3), \end{aligned}$$

which gives the third equation in the system (3.12).

The connection ∇^{Π} also belongs to the projective structure that is represented by the A_i's. The relation between $\Pi^a{}_{bc}$ and an element $\Gamma^a{}_{bc}$ of the projective structure $[\Gamma]$, satisfying (3.4), can be easily derived to be

$$\Pi^a{}_{bc} = \Gamma^a{}_{bc} - \frac{1}{3}\Gamma^d{}_{dc}\delta^a_b - \frac{1}{3}\Gamma^d{}_{db}\delta^a_c. \tag{3.15}$$

It is clear from above that $\Pi^a{}_{bc}$ and $\Gamma^a{}_{bc}$ satisfy equation (3.9) and therefore are projectively equivalent. The connection ∇^{Π} with the Christoffel symbols Π as defined above, is called the normal projective connection of a projective structure represented by A_i's. As is discussed in [20] these Christoffel symbols were first defined by T. Y. Thomas in [29] and can be obtained via Cartan's approach using G-structures.

We can summarize the above discussion in the following theorem, originally due to Roger Liouville.

Theorem 3.2.1 (Liouville [24])**.** *A projective structure* $[\Gamma]$ *corresponding to the second-order ODE* (3.3) *is metrizable on a neighborhood of a point* $p \in U$*, if and only if there exists functions* $\psi_i(x,y), 1 = 1,2,3$ *defined on a neighborhood of* p *such that* $\psi_1\psi_3 - \psi_2^2$ *does not vanish at* p *and the equations* (3.12) *hold on their domain of definition.*

3.2.1 Diffeomorphism Invariant Conditions of Projective Structures

The material of this section can be found in detail in [18]. Let U be an open set equipped with local coordinates (x,y) and projective structure $[\Gamma]$. Any local diffeomorphism of an open set U is represented by a change of local coordinates

$$(x,y) \longrightarrow (\widetilde{x}(x,y)), \widetilde{y}(x,y)).$$

A local invariant on such an open set is called a diffeomorphism invariant condition if changing the local coordinates does not affect its vanishing points.

As an example, recall that the correspondence between projective structures and second-order ODEs $y'' = \Lambda(x,y,y')$ with Λ being at most cubic in y', was explained in Section 3.1. It gives a diffeomorphism invariant condition for a second-order ODE that is associated to a projective structure as follows. Let us define

$$I_0(\Lambda) = \frac{\partial^4\Lambda}{(\partial y')^4}.$$

If it vanishes for an ODE $y'' = \Lambda(x,y,y')$, then Λ is at most cubic in y' and therefore the ODE is associated to a projective structure. It is clear from the definition of projective structure that acting a diffeomorphism Ψ on U gives rise to a new projective structure which we denote by $\Psi[\Gamma]$. In other words, the space of projective structures on an open set U is closed under the action of diffeomorphisms on U. Therefore, if we make a change of coordinates Ψ in an ODE that is associated to the projective structure $[\Gamma]$, the resulted ODE is associated to the projective class $\Psi[\Gamma]$. Therefore, the property of being cubic in y' for ODEs of the form $y'' = \Lambda(x,y,y')$ remains invariant under the action of diffeomorphisms. As a result, the vanishing points of I_0, as points in the space of functions in y', y, x, are invariant with respect to diffeomorphisms of U, $(x,y) \to (\widetilde{x}(x,y)), \widetilde{y}(x,y))$. Thus, we say the vanishing points of I_0 are diffeomorphism invariant. It is not hard to show directly that the condition $I_0 = 0$ is invariant under coordinate transformation

$$(x,y) \longrightarrow (\widetilde{x}(x,y)), \widetilde{y}(x,y)).$$

Note that the projective structure $[\Gamma]$ in a local coordinate (x,y) is represented by unique set of A_i's as was explained in Section 3.1. It is clear that changing the local coordinates gives rise to a different set of A_i's representing $[\Gamma]$. Thus, one can look for the suitable local coordinate in which some of the coefficients A_i vanish. For instance if curves with constant x are among the geodesics then $A_3 = 0$. Also A_2 can be made to vanish by choosing polar coordinates as geodesics. In this way the ODE (3.2) becomes second-order linear; however, since we are interested in deriving metrizability conditions that are diffeomorphism invariant, we will not make use of this freedom.

We have the following result, which will be useful later on, due to both Liouville and Tresse.

Theorem 3.2.2 (Liouville [23], Tresse [31])**.** *The ODE $y'' = \Lambda(y', y, x)$ is equivalent to $y'' = 0$ by point transformation if and only if $I_0 = I_1 = 0$, where*

$$\begin{aligned} I_0(\Lambda) &= \frac{\partial^4 \Lambda}{(\partial y')^4} \\ I_1(\Lambda) &= D_x^2 \Lambda_{11} - 4 D_x \Lambda_{01} - \Lambda_1 D_x \Lambda_{11} + 4\Lambda_1 \Lambda_{01} - 3\Lambda_0 \Lambda_{11} - 6 A_{00}, \end{aligned} \tag{3.16}$$

and

$$\Lambda_0 = \frac{\partial \Lambda}{\partial y}, \quad \Lambda_1 = \frac{\partial \Lambda}{\partial y'}, \quad D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + \Lambda \frac{\partial}{\partial y'}.$$

Note that if $I_0(\Lambda) = 0$ for a function $\Lambda(y', y, x)$ then Λ is cubic in y' and has the form

$$\Lambda(y', y, x) = A_3(x, y)(\frac{dy}{dx})^3 + A_2(x, y)(\frac{dy}{dx})^2 + A_1(x, y)\frac{dy}{dx} + A_0(x, y).$$

As a result of Tresse, if $I_0(\Lambda) = 0$, then I_1 is linear in y' and, by Liouville, takes the form $I_1 = -6L_1 - 6L_2 y'$ where

$$\begin{aligned} L_1 &= \frac{2}{3}\frac{\partial^2 A_2}{\partial x \partial y} - \frac{1}{3}\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial y^2} + A_0\frac{\partial A_2}{\partial y} + A_2\frac{\partial A_0}{\partial y}, \\ &- A_3\frac{\partial A_0}{\partial x} - 2A_0\frac{\partial A_3}{\partial x} - \frac{2}{3}A_1\frac{\partial A_1}{\partial y} + \frac{1}{3}A_1\frac{\partial A_2}{\partial x}, \\ L_2 &= \frac{2}{3}\frac{\partial^2 A_2}{\partial x \partial y} - \frac{1}{3}\frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_3}{\partial x^2} - A_3\frac{\partial A_1}{\partial x} - A_1\frac{\partial A_3}{\partial x}, \\ &+ A_0\frac{\partial A_3}{\partial y} + 2A_3\frac{\partial A_0}{\partial y} + \frac{2}{3}A_2\frac{\partial A_2}{\partial x} - \frac{1}{3}A_2\frac{\partial A_1}{\partial y}. \end{aligned} \tag{3.17}$$

We can give a geometric interpretation of the vanishing points of both I_0 and I_1, as functions in y', y, x, which uses the result of the proof of the sufficiency conditions of our problem in Section 3.5. We discussed earlier that if for a function $\Lambda(y', y, x)$ we obtain $I_0(\Lambda) = 0$ then the ODE $y'' = \Lambda(y', y, x)$ is associated to a projective structure $[\Gamma]$. Moreover, if for such a function, Λ, we have $I_1(\Lambda) = 0$ then the space of metrics whose Levi-Civita connections belong to $[\Gamma]$ is six-dimensional. This will be proved in Section 3.5. Now if a diffeomorphism Ψ acts on U then the projective structure $[\Gamma]$ is sent to another projective structure $\Psi[\Gamma]$. Also each metric g in the six-dimensional space of the metrics is sent to another metric Ψg. It is easy to check that if the Levi-Civita connection of g belongs to $[\Gamma]$ then the Levi-Civita connection of Ψg belongs to $\Psi[\Gamma]$. Therefore, the space of the metrics of which the Levi-Civita connections belong to $\Psi[\Gamma]$ is six-dimensional. As a result, the subspace of set of the functions $\Lambda(y', y, x)$ with $I_0(\Lambda) = I_1(\Lambda) = 0$ is closed under diffeomorphisms of U. Note that whenever $I_0(\Lambda) = 0$, we have $I_1(\Lambda) = 0$ if $L_1(\Lambda) = L_2(\Lambda) = 0$.

3.3 Prolonging the Linear System

As we explained in the introduction, our problem is overdetermined. In particular the last system of PDEs (3.12) is overdetermined. Since the formulation of the problem in the form of the system (3.12) is a first-order system of PDEs, we may think that we can follow the procedure stated in section (2.3). The problem is that not all of the first derivatives of the variables ψ_i are determined in the system. Therefore, we introduce the undetermined derivatives as new variables and introduce a procedure that will be called prolongation. It will turn out the that the second prolongation of the system gives us a system of the form used in Theorem (2.3.4).

Suppose that the system (3.12) has a solution which would be a graph $\mathcal{S} \subseteq \mathbb{R}^5$ given by

$$(x, y) \mapsto (x, y, \psi_1, \psi_2, \psi_3). \tag{3.18}$$

In order to find the tangent space at each point of this solution surface (if one exists), we have to determine the first derivatives of the functions ψ_1, ψ_2, ψ_3, introduced in equation (3.11), at each point; Namely $\partial_x\psi_i,\ \partial_y\psi_i, 1 \leq i \leq 3$. From equation (3.12) we can obtain at most four of them. The difference between the dimension of the tangent space at each point of the surface and the number of equations is $2 \cdot 3 - 4 = 2$. Let $\frac{\partial\psi_1}{\partial x}$ and $\frac{\partial\psi_2}{\partial x}$ be the two that are not determined. We introduce two new variables as follows:

$$\frac{1}{2}\mu(x, y) := \frac{\partial\psi_1}{\partial x}(x, y), \quad \frac{1}{2}\nu(x, y) := \frac{\partial\psi_1}{\partial y}(x, y). \tag{3.19}$$

The integrability condition $\partial_x\partial_y\psi_i = \partial_y\partial_x\psi_i$ gives three more equations as follows

$$\frac{\partial\mu}{\partial x} = P, \quad \frac{\partial\nu}{\partial y} = Q, \quad \frac{\partial\mu}{\partial y} - \frac{\partial\nu}{\partial x} = 0, \tag{3.20}$$

where $P \text{ and } Q$ can be obtained from (3.12) and are linear in the functions ψ_i, $\mu, \text{ and } \nu$ with coefficients depending on the A_i's and their derivatives up to the first degree. Their exact expression is given in (C.3).

Introducing these two variables can be thought of as prolonging the surface $\mathcal{S}$, which we assumed to be in $\mathbb{R}^5$, to another surface $\mathcal{S}^{(1)} \subseteq \mathbb{R}^7$ given by

$$(x, y) \mapsto (x, y, \psi_1, \psi_2, \psi_3, \mu, \nu).$$

This is what we referred to in Section 2.3 as prolongation of the equation manifold.

Similar to what we did for $\mathcal{S}$, we observe that while the system of nine equations from combining the systems (3.12), (3.19), and (3.20) is overdetermined for the new set of variables $\psi_1, \psi_2, \psi_3, \mu, \nu$, the difference between the number of first derivatives which span the tangent space at every point of the surface $\mathcal{S}^{(1)}$, and the number of equations is $2 \cdot 5 - 9 = 1$, which is less than what we had before prolonging the system. Therefore, it is likely that one more prolongation will result in enough equations to determine the tangent space at each point of the second prolongation manifold $\mathcal{S}^{(2)}$. If we let ρ to be

$$\rho(x,y) := \frac{\partial \mu}{\partial y}. \tag{3.21}$$

then from equations (3.21) and (3.20) we obtain

$$\frac{\partial \rho}{\partial x} = R, \quad \frac{\partial \rho}{\partial y} = S, \tag{3.22}$$

where R and S are given by (C.3). As functions of $(x, y, \psi_i, \mu, \nu, \rho)$, R and S are linear in terms of (ψ_i, μ, ν, ρ) because they form a first-order system of PDEs with all of their derivatives being determined. Now we have six variables having twelve first derivatives spanning the tangent space at each point of $\mathcal{S}^{(2)} \subseteq \mathbb{R}^8$, the second prolongation of the surface $\mathcal{S}$, which is defined by

$$(x,y) \mapsto (x,\ y,\ \psi_1,\ \psi_2,\ \psi_3,\ \mu,\ \nu,\ \rho).$$

All of the first derivatives of our variables are determined by the twelve equations in (3.12), (3.19), (3.20), (3.21) and (3.22). The integrability condition, which is $\partial_x\partial_y\rho = \partial_y\partial_x\rho$, adds an extra linear equation among the first derivatives of our variables. This is due to the fact that all the first derivatives of ψ_i, μ, ν, ρ are linear in terms of ψ_i, μ, ν, ρ. Thus, $\partial_x\rho$ is linear in the variables ψ_i, μ, ν, ρ and so is $\partial_y\partial_x\rho$. The same is true for $\partial_x\partial_y\rho$. Therefore, we have

$$\partial_x\partial_y\rho - \partial_y\partial_x\rho = \frac{\partial R}{\partial y} - \frac{\partial S}{\partial x} + S\frac{\partial R}{\partial \rho} - R\frac{\partial S}{\partial \rho} = 0. \tag{3.23}$$

As this equation is linear in (ψ_i, μ, ν, ρ) we get

$$\mathbf{V}\mathbf{u} = 0, \tag{3.24}$$

where $\mathbf{u} := (\psi_1, \psi_2, \psi_3, \mu, \nu, \rho)^T \in \mathbb{R}^6$ and $\mathbf{V} = (V_1, \dots, V_6)$ is given by

$$V_1 = 2\frac{\partial L_2}{\partial y} + 4A_2L_2 + 8A_3L_1, \quad V_2 = -2\frac{\partial L_1}{\partial x} - 2\frac{\partial L_2}{\partial x} - \frac{4}{3}A_1L_2 + \frac{4}{3}A_2L_1,$$
$$V_3 = 2\frac{\partial L_1}{\partial x} - 8A_0L_2 - 4A_1L_1, \quad V_4 = -5L_2, \quad V_5 = -5L_1, \quad V_0 = 0.$$

Here L_1, L_2 are given by equations (3.2.2). This integrability condition determines whether the solution obtained by prolongation exists or not. This is actually the first obstruction F_1 in (2.14) following the procedure we explained in Section 2.3. According to our construction in Section 2.3, $\mathbf{V}$ has to be the only non-zero row of the curvature $\mathbf{F}$ of the connection associated to our prolonged system and is therefore equivalent to equation (2.21).

3.4 Necessary Conditions

According to Section 2.3, all the first-order relations that we obtained throughout our prolongation should lead to a specific choice of connection on a rank six vector bundle. The vector $\mathbf{u}$, as defined above, is obtained from a trivialization of that vector bundle over the surface $\mathcal{S}$ (if it exists). Hence a solution $\mathbf{u}$ is a parallel section of the vector bundle and we would have

$$d\mathbf{u} + \mathbf{\Omega}\mathbf{u} = 0. \tag{3.25}$$

As we stated in Theorem (2.3.5), the obstructions to have a nonzero parallel section of this bundle involves the rank of the covariant derivatives of the curvature. For instance

$$\mathrm{rank}(\mathcal{F}_6) \leq 5,$$

where $\mathcal{F}_6$ consists of 6×6 sub-matrices F

$$(F,\ D_a F \ldots, (D_{a_1} \ldots D_{a_6})\, F), \tag{3.26}$$

with F being defined in (2.22) and is given by (C.2). Because the only non-zero row of the curvature is $\mathbf{V}$, we can replace F in the expression above with $\mathbf{V}$. Clearly, $\mathcal{F}_2$ is a submatrix of $\mathcal{F}_6$ and cannot have full rank. It is equivalent to the matrix consisting of the rows $\mathbf{V},\ D_a\mathbf{V}$ and $D_a D_b \mathbf{V}$. We note that

$$\begin{aligned} D_a D_b \mathbf{V} &:= D_a(\partial_a \mathbf{V} - \mathbf{V}\mathbf{\Omega}_a), \\ &= \partial_a\partial_b \mathbf{V} - (\partial_b \mathbf{V})\mathbf{\Omega}_a - (\partial_a \mathbf{V})\mathbf{\Omega}_b - \mathbf{V}(\partial_b \mathbf{\Omega}_a - \mathbf{\Omega}_a\mathbf{\Omega}_b). \end{aligned} \tag{3.27}$$

Also from (2.22) we have

$$\begin{aligned} F\mathbf{u} = 0 \Rightarrow \quad & (\partial_a \mathbf{\Omega}_b - \partial_b \mathbf{\Omega}_a + [\mathbf{\Omega}_a, \mathbf{\Omega}_b])\mathbf{u} \\ &= (\partial_a \mathbf{\Omega}_b - \partial_b \mathbf{\Omega}_a + \mathbf{\Omega}_a\mathbf{\Omega}_b - \mathbf{\Omega}_b\mathbf{\Omega}_a)\mathbf{u} = 0 \\ \Rightarrow \quad & (\partial_a \mathbf{\Omega}_b - \mathbf{\Omega}_b\mathbf{\Omega}_a)\mathbf{u} = (\partial_b \mathbf{\Omega}_a + \mathbf{\Omega}_a\mathbf{\Omega}_b)\mathbf{u}. \end{aligned}$$

Using the equation obtained above, we realize that the expression for $D_a D_b \mathbf{V}$ is symmetric in its indices, and therefore $D_a D_b \mathbf{V} = D_b D_a \mathbf{V}$. As a result, $\mathcal{F}_2$ is equivalent to a 6×6 matrix consisting of $\mathbf{V}$, $D_a \mathbf{V}$ and $D_{(a}D_{b)}\mathbf{V}$ which is $\mathcal{M}([\Gamma])$. Therefore $\det(\mathcal{M}[\Gamma]) = 0$ is the necessary condition stated in Theorem (1.2.1). We can also obtain the equation $\det(\mathcal{M}[\Gamma]) = 0$ directly by taking the covariant derivative of (3.24) and realizing the symmetry of second covariant derivatives of $\mathbf{V}$ which results in $(\mathcal{M}[\Gamma])\mathbf{u} = 0$. This completes the proof of Theorem (1.2.1).

As the V_i's are expressed in terms of the third derivatives of A_i's, the expression (1.8) involves the A_i's up to their fifth derivatives. It does not vanish on a projective structure in general, but vanishes on metrizable ones.

3.5 Sufficient Conditions

In this section, we apply Theorem (2.3.5) to the connection D, we defined in (3.25), in order to obtain the sufficient conditions of our problem. Motivated by (3.27), for this connection we can derive a symmetric property of covariant derivatives of any given degree $d \geq 2$ modulo lower degrees. Given $\mathbf{W} : U \longrightarrow \mathbb{R}^6$, similar to what we did in the calculation to obtain (3.27), we have

$$[D_i, D_j]\mathbf{W} = \mathbf{W}(\partial_i \mathbf{\Omega}_j - \partial_j \mathbf{\Omega}_i + \mathbf{\Omega}_i \mathbf{\Omega}_j - \mathbf{\Omega}_j \mathbf{\Omega}_i) = (\mathbf{W}F)\eta_{ij} = W_6 \mathbf{V} \eta_{ij}, \quad (3.28)$$

where $\eta_{11} = \eta_{00} = 0,\ \eta_{10} = -\eta_{01} = 1$, and F is given by the curvature $\mathbf{F} = F dx \wedge dy$, as in (2.22). The last equality holds since the only nonzero row of F is its sixth row which is equal to $\mathbf{V}$. Recalling that $V_6 = 0$ from (3.25), we have:

$$\begin{aligned}
D_i D_j \mathbf{V} &= D_{(i} D_{j)} \mathbf{V} + D_{[i} D_{j]} \mathbf{V} = D_{(i} D_{j)} \mathbf{V}, \\
D_i D_j D_k \mathbf{V} &= D_i D_{(j} D_{k)} \mathbf{V} + (D_j D_{(i} D_{k)} \mathbf{V} - D_j D_i D_k \mathbf{V}) \\
&\quad + (D_k D_{(i} D_{j)} \mathbf{V} - D_k D_i D_j \mathbf{V}) \\
&= D_i D_{(j} D_{k)} \mathbf{V} + (D_j D_{(i} D_{k)} \mathbf{V} - D_i D_j D_k \mathbf{V} + 2 D_{[i} D_{j]} D_k \mathbf{V}) \\
&\quad + (D_k D_{(i} D_{j)} \mathbf{V} - D_i D_k D_j \mathbf{V} + 2 D_{[i} D_{k]} D_j \mathbf{V}) \\
&= D_i D_{(j} D_{k)} \mathbf{V} + (D_j D_{(i} D_{k)} \mathbf{V} + 2 D_{[i} D_{j]} D_k \mathbf{V} \\
&\quad + (D_k D_{(i} D_{j)} \mathbf{V} + 2 D_{[i} D_{k]} D_j \mathbf{V}) \\
&\quad - 2 D_i D_j D_k \mathbf{V} \\
&= 3 D_{(i} D_j D_{k)} \mathbf{V} + 2(D_{[i} D_{j]} D_k \mathbf{V} + D_{[i} D_{k]} D_j \mathbf{V}) - 2 D_i D_j D_k \mathbf{V} \\
\Rightarrow D_i D_j D_k \mathbf{V} &= D_{(i} D_j D_{k)} \mathbf{V} + \frac{2}{3}(D_{[i} D_{j]} D_k \mathbf{V} + D_{[i} D_{k]} D_j \mathbf{V}) \\
&= D_{(i} D_j D_{k)} \mathbf{V} + \frac{2}{3}((D_k \mathbf{V})_6 \eta_{ij} + (D_j \mathbf{V})_6 \eta_{ik}) \mathbf{V}
\end{aligned}$$

In general we obtain

$$D_{a_1} D_{a_2} \ldots D_{a_k} \mathbf{V} = D_{(a_1} D_{a_2} \ldots D_{a_k)} \mathbf{V} + O(k-2) \quad (3.29)$$

where $O(k-2)$ involves derivatives of $\mathbf{V}$ up to order $k-2$.

This symmetry tells us that there are at most $n(K) = 1 + 2 + ... + K$ linearly independent covariant derivatives of F of order less than K, namely the symmetrized derivatives of the only nonzero column $\mathbf{V}$. Therefore, as we did at the end of last section, we replace the matrix F with $\mathbf{V}$ in our calculations.

Let $\mathcal{F}_K$ denote the matrix consisting of covariant derivatives of F of order less than or equal to K (see equation (2.25)). It is always possible to find a point $p \in U$, such that restricting F to a sufficiently small neighborhood around it, one can guarantee that the rank of the matrix $\mathcal{F}_K$ does not decrease for all $K = 0, 1, ..., 6$. The Frobenius Theorem (2.3.5) tells us the dimension of the space of parallel sections in the rank six bundle over that neighborhood. As we saw earlier $\mathcal{F}_2$ provides the necessary condition of Theorem (1.2.1), which is that $\det(\mathcal{M}[\Gamma]) = 0$ and $\mathbf{u}_1\mathbf{u}_3 - (\mathbf{u}_2)^2 \neq 0$. This is equivalent to saying that $\text{rank}(\mathcal{F}_2) \leq 5$ and $\det(\sigma_{ij}) \neq 0$.

Assuming the necessary conditions hold, we have at least one vector

$$\mathbf{u} = (\mathbf{u}_1, ..., \mathbf{u}_6) \in \ker(\mathcal{M}([\Gamma])),$$

with the property that $\mathbf{u}_1\mathbf{u}_3 - (\mathbf{u}_2)^2 \neq 0$. This property implies nondegeneracy of the form σ in (3.13). The first case we consider in order to derive the sufficient conditions is when $\text{rank}(\mathcal{F}_2) = 5$, that is the case for a generic projective structure.

Definition 3.5.1. We call a projective structure $[\Gamma]$ with $\det(\mathcal{M}[\Gamma]) = 0$ generic in a neighborhood U of a point p, if $\text{rank}(\mathcal{M}[\Gamma]) = 5$ in U and $\mathbf{u}_1\mathbf{u}_3 - \mathbf{u}_2^2 \neq 0$ where $\mathbf{u} \in \ker(\mathcal{M}([\Gamma]))$.

By the construction of Theorem (2.3.5), for a generic projective structure we have $6 \geq \text{rank}(\mathcal{F}_3) \geq \text{rank}(\mathcal{F}_2) = 5$. If $\text{rank}(\mathcal{F}_3) = 6$ then, according to Theorem (2.3.5), the number of parallel sections is zero. Therefore, in order to have a nonzero solution we need to have $\text{rank}(\mathcal{F}_3) = 5$ and there will be only a one-dimensional space of parallel sections. We want to find the differential relations in terms of the A_i's that will guarantee the condition $\text{rank}(\mathcal{F}_3) = 5$.

As for a generic projective structure $\text{rank}(\mathcal{M}[\Gamma]) = 5$ over U, it is obvious that for a generic projective structure the vectors $\mathbf{V}, D_a\mathbf{V}$ must be linearly independent, as otherwise $\text{rank}(\mathcal{M}[\Gamma]) \leq 3$. Since $\mathcal{M}[\Gamma]$ is generic, it forces exactly two of three vectors $D_{(a}D_{b)}\mathbf{V}$ to be independent. Similarly to what we did in (2.3.5), we take the covariant derivative of $D_{(a}D_{b)}\mathbf{V}$ one more time to obtain the system of four equations $(D_{(a}D_bD_{c)}\mathbf{V})\mathbf{u} = 0$. Since we want $\text{rank}\,\mathcal{F}_3$ not to increase, these new equations have to be identically satisfied. Note that all the entries of $\mathbf{V}$ and Ω are in

k	$\mathrm{rank}(J^{k+1}(S^2(T^*U)))$	$\mathrm{rank}(J^k(Pr(U)))$	$\mathrm{rank}(\ker(\sigma^k))$	$\mathrm{rank}(\mathrm{coker}(\sigma^k))$
-1	3	-	-	-
0	9	4	5	0
1	18	12	6	0
2	30	24	6	0
3	45	40	5	0
4	63	60	3	0
5	84	84	1	1=**1**
6	108	112	1	5=3+**2**
7	135	144	1	10=6+6-**2**

terms of the A_i's up to their third derivatives. Therefore, asking for these four new equations, which are the third covariant derivatives of $\mathbf{V}$, to vanish is equivalent to having four sixth order equations among the A_i's. But, if we look more closely we realize that not all of these four equations are new. Two of them are merely first derivatives of the assumptions (1.8). In order to realize this fact, we make a counting argument based on the dimensions of jet spaces.

Proof of Theorem (1.2.2). Prolong $\sigma^0 : J^1(S^2(T^*U)) \to J^0(Pr(U))$ in (3.10) to $\sigma^k : J^{k+1}(S^2(T^*U)) \to J^k(Pr(U))$ by differentiating the relations (1.6). The operator σ^k is a homogeneous bundle map from $(k+1)$-jets of metrics to k-jets of a rank four vector bundle - the space of projective structures. It differentiates the system (3.12) and gives the derivatives of the A_i's in terms of derivatives of the metric. Due to the homogeneity of the operator, its kernel contains at least a one-dimensional fiber at each point i.e., $\dim(\ker(\sigma^k)) \geq 1, \ \forall k$.

In the following table the bold numbers shows the number of obstructions which have to be satisfied to ensure the existence of a Levi-Civita connection in $[\Gamma]$ as a generic projective structure.

For $k \leq 4$ there is no obstruction and the σ^k's are surjective, since the highest order derivatives of the metric components always show up in the numerator of the expressions for all the derivatives of the A_i's, and never cancel each other out. At $k = 5$ we have $\mathrm{rank}(\ker(\sigma^5)) = 1$. This is because $\mathrm{rank}(\ker(\sigma^k)) \geq 1$ and none of the 6th derivatives of the metrics is eliminated by σ^5 as we discussed earlier in the proof. Since all the σ^k's with $k \leq 4$ were surjective, we get $\mathrm{rank}(\ker(\sigma^5)) = 1$. This fact and that $\dim(J^6(S^2(T^*U)) = \dim(J^5(Pr(U))) = 84$ implies that the image of the map σ^5 is 83-dimensional. Also the projective structures that correspond to a metric, have to satisfy equation (1.8), meaning that they are on the zero level set of $\det(\mathcal{M}[\Gamma])$. Since $\operatorname{rank}\mathcal{M}[\Gamma] = 5$ over U, this equation defines a smooth variety in $J^5(Pr(U))$ and thus the image of σ^5, defined by (1.8), is smooth and $\operatorname{rank}(\operatorname{coker}(\sigma^5)) = 1$. The argument we made to show $\operatorname{rank}(\ker(\sigma^5)) = 1$ implies that $\operatorname{rank}(\ker(\sigma^k)) = 1$ as long as the dimension of the domain is less than the dimension of the codomain.

Note that, as we said earlier, the entries of $D_a D_b \mathbf{V}$ are expressed in terms of the A_i's up to their fifth derivatives. The assumption that

$$\operatorname{rank}(\mathcal{M}[\Gamma]) = 5$$

makes the fifth order PDE $\det(\mathcal{M}[\Gamma]) = 0$ regular, meaning that it involves at least one fifth derivative of the A_i's. Therefore, it is a codimension one regular variety in J^5 and submerses onto lower jet spaces.

Considering the image of the map σ^6, we are exactly in the same situation as what we had in (2.6). The prolongations of the projective structures satisfying (1.8) will have their 6-jets in $J^6(Pr(U))$ constrained not only by equation (1.8), but also, by its first derivatives. In general, the k-jets of the regular solutions to (1.8) are constrained by $\det(\mathcal{M}(\Gamma)) = 0$ and all its derivatives up to order $(k-5)$. As a result, the first prolongation of the equation manifold of (1.8) is a regular smooth variety of codimension 3 in $J^6(Pr(U))$. Therefore, its dimension is $112 - 3 = 109$. Knowing that $\dim J^5(S^2(T^*U)) = 108$, we get $\operatorname{rank}(\operatorname{img}(\sigma^6)) = 108 - 1 = 107$. This indicates that the set of images of regular metric structures, meaning those with nonzero sixth order derivatives, has dimension 107 in $J^6(Pr(U))$ and is of codimension two in the space of projective structure that only satisfy the necessary condition (1.8). This implies the existence of two extra sixth order obstructions for a regular solution of (1.8) that corresponds to a metric structure. Let us denote these two obstructions by

$$E_1 = 0 \quad , \quad E_2 = 0. \tag{3.30}$$

Denoting the space of the 6-jets of projective structures coming from a metric structure by $\mathcal{N}$, we know that it is of codimension five in $J^6(Pr(U))$ with its constraints being equation (1.8) with its two first derivatives, and the constraints (3.30).

If we prove that there is no more constraint from the prolongation manifold $\mathcal{N}^{(1)}$ in $J^7(Pr(U))$ and that all the constraints are simply among the derivatives of obstructions we already have, then by Theorem (2.3.5) and using the fact that $\mathrm{rank}(\mathcal{F}_3) = \mathrm{rank}(\mathcal{F}_2) = \mathrm{rank}(\mathcal{F}_1) = 5$, we have a one-dimensional space of parallel sections of our connection. For $J^7(Pr(U))$, we can obtain twelve obstructions exactly in the same way. six obstruction are (1.8), its two first derivatives and its three second derivatives. The other six come from (3.30) and their first derivatives. On the other hand, $\mathrm{rank}(J^8(S^2(U))) = 135$ and $\mathrm{rank}(\ker(\sigma^7)) = 1$. Therefore, $\mathrm{rank}(\mathrm{img}(\sigma^7)) = 134$, and not all of the twelve obstructions that we found on the first prolongation of $\mathcal{N}$ are independent. There are two relations between the second derivatives of $\det(\mathcal{M}(\Gamma)) = 0$ and first derivatives of $E_1 = 0, E_2 = 0$, as these relations have appeared in the level of 7-jets. Thus, there cannot be any new obstruction on the first prolongation of $\mathcal{N}$. Theorem (2.3.5) applies and the system is involutive with a one-dimensional space of parallel sections. □

In order to obtain a better picture of E_1 and E_2, assume that the equation $\det(\mathcal{M}(\Gamma)) = 0$ gives $\mathbf{V}_{xy} \in \mathrm{span}\{\mathbf{V}, \mathbf{V}_x, \mathbf{V}_y, \mathbf{V}_{xx}, \mathbf{V}_{yy}\}$ so that

$$\mathbf{V}_{xy} = c_1\mathbf{V} + c_2\mathbf{V}_x + c_3\mathbf{V}_y + c_4\mathbf{V}_{xx} + c_5\mathbf{V}_{yy}$$

for some functions $c_1, \dots, c_5$ on U. Using (3.28), the equalities $\mathbf{V}_{xyy} = \mathbf{V}_{yyx}$ and $\mathbf{V}_{xyx} = \mathbf{V}_{xxy}$ hold modulo lower order terms. Therefore, both $\mathbf{V}_{xyy}$ and $\mathbf{V}_{xxy}$ belong to the span of $\{\mathbf{V}, \mathbf{V}_x, \mathbf{V}_y, \mathbf{V}_{xx}, \mathbf{V}_{yy}, \mathbf{V}_{xxx}, \mathbf{V}_{yyy}\}$. The vectors $\mathbf{V}_{xxx}, \mathbf{V}_{yyy}$ can be obtained once we know the sixth order obstructions which are the ones we found in our counting argument

$$E_1 := \det\begin{pmatrix}\mathbf{V}\\ \mathbf{V}_x\\ \mathbf{V}_y\\ \mathbf{V}_{xx}\\ \mathbf{V}_{yy}\\ \mathbf{V}_{xxx}\end{pmatrix}, \quad E_2 := \det\begin{pmatrix}\mathbf{V}\\ \mathbf{V}_x\\ \mathbf{V}_y\\ \mathbf{V}_{xx}\\ \mathbf{V}_{yy}\\ \mathbf{V}_{yyy}\end{pmatrix}.$$

Our problem for the non-generic cases is more complicated. Since whenever $\mathrm{rank}(\mathcal{M}[\Gamma]) \le 3$, the PDE $\det(\mathcal{M}(\Gamma)) = 0$ is not regular meaning that it no longer defines a smooth codimension one variety in $J^5(Pr(U))$, and the argument based on the dimensions of the jet bundles does not work.

For a generic projective structure, the non-degeneracy of the quadratic form σ, which is the solution of the system (3.12) obtained from a vector $\mathbf{u} \in \ker(\mathcal{M}([\Gamma]))$, follows directly from our assumptions that $\mathbf{u}_1\mathbf{u}_3 - (\mathbf{u}_2)^2 \neq 0$ and $\ker(\mathcal{M}([\Gamma]))$ spanned by $\mathbf{u}$. In the non-generic cases we have the following lemma in which $d([\Gamma])$ is the dimension of the vector space of solutions of $\det(\mathcal{M}([\Gamma]) = 0$.

Lemma 3.5.2. *If* $d([\Gamma]) \geq 2$ *then there are* $d([\Gamma])$ *independent non-degenerate quadratic forms among the solutions of* (1.8).

Proof. Let us assume that there is one degenerate quadratic form σ among solutions of (1.8). Then we can construct a vector field V such that at each point x it gives a null-vector of $\sigma(x)$. In flow-box coordinates of this vector field, we can diagonalize σ so that it takes the form $\sigma = \begin{pmatrix} \psi_1 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, in terms of ψ_1, the coefficients A_i's are determined from (3.12) to be

$$A_1 = \frac{3}{2}\frac{1}{\psi_1}\frac{\partial \psi_1}{\partial x}, \quad A_2 = \frac{3}{4}\frac{1}{\psi_1}\frac{\partial \psi_1}{\partial y}, \quad A_3 = 0,$$

with A_0 unspecified. Thus, in these local coordinates, there is no other quadratic form of the form $\sigma = \begin{pmatrix} \psi(x,y)\psi_1 & 0 \\ 0 & 0 \end{pmatrix}$ among the solutions of (1.8) and the only freedom is to rescale ψ_1 by a scalar which gives a one-dimensional subspace. In other words, no other possible degenerate solution of (1.8) independent of σ can have the same form. Therefore, in these local coordinates all other possible degenerate solutions have the form $\widetilde{\sigma} = \begin{pmatrix} \widetilde{\psi}_1 & \widetilde{\psi}_2 \\ \widetilde{\psi}_2 & \widetilde{\psi}_3 \end{pmatrix}$ with either $\widetilde{\psi}_3 \neq 0$ and $\widetilde{\psi}_1 = \widetilde{\psi}_2 = 0$ or $\widetilde{\psi}_i \neq 0$ for all i. However, in both cases the quadratic forms $a\sigma + b\widetilde{\sigma}$ are non-degenerate for almost all $a, b \in \mathbb{R}^2$. Hence the only case in which we cannot have as many non-degenerate quadratic forms as the dimension of the solution space of (3.12) is when $d([\Gamma]) \leq 1$. □

Proof of Theorem (1.2.3). We list different cases that can happen depending on $\operatorname{rank}(\mathcal{M}([\Gamma]))$.

- If $\operatorname{rank}(\mathcal{M}([\Gamma])) \leq 1$ then the second-order ODE (3.2) is equivalent to $y'' = 0$. When $\operatorname{rank}(\mathcal{M}([\Gamma])) = 0$ then $\mathbf{V} = 0$ and Theorem (2.3.5) implies that the connection D is flat and the system is completely integrable. Hence the space of parallel sections is 6-dimensional. This also implies that the invariants $L_1 = L_2 = 0$ and by Theorem (3.2.2) the system is equivalent to $y'' = 0$.

  If $\operatorname{rank}(\mathcal{M}([\Gamma])) = 1$ then $\partial_a \mathbf{V} - \mathbf{V}\mathbf{\Omega}_a = \mathbf{V}_a = \gamma_a \mathbf{V}$. According to the expression of $\mathbf{V}$, we obtain

$$\mathbf{V}\mathbf{\Omega}_a = (*, *, *, *, *, 5L_a) = 0.$$

As $V_6 = 0$, we obtain that the L_a's vanish and Theorem (3.2.2) applies proving that the system is equivalent to $y'' = 0$ and completely integrable.

- If $\mathrm{rank}(\mathcal{M}([\Gamma])) = 2$ then

$$\mathbf{V} + c_1\mathbf{V}_x + c_2\mathbf{V}_y = 0,$$

for functions $c_1,\ c_2$ at least one of which does not vanish. Based on our assumption and the symmetry of $\mathbf{V}_{ab}$, we have $\mathbf{V}_{ab} \in \mathrm{span}\{\mathbf{V},\ \mathbf{V}_a\}$. Thus, further differentiation does not introduce any new obstructions and the system is closed at the level of the first derivatives of $\mathbf{V}$. Using (2.3.5), the solution space has dimension four.

- If $\mathrm{rank}(\mathcal{M}([\Gamma])) = 3$, we have to consider two cases. The first one is when the set $\{\mathbf{V},\ \mathbf{V}_x,\ \mathbf{V}_y\}$ is linearly independent, in which case the argument is exactly the same as the previous case. We realize that the system is closed at that stage and the space of parallel sections is 3-dimensional. The second case is when either the set $\{\mathbf{V},\ \mathbf{V}_x,\ \mathbf{V}_{xx}\}$ or $\{\mathbf{V},\ \mathbf{V}_y,\ \mathbf{V}_{yy}\}$ is linearly independent. We will assume the first set is linearly independent and the same argument works for the second by interchanging x and y. It is obvious that y derivatives do not add any new obstructions and that an obstruction only involves x derivatives. Theorem (2.3.5) tells us that in order to have a solution space of dimension one, the sixth obstruction, which in this case is $\mathbf{V}_{xxxxx}$, must be in terms of lower order obstructions. So if $\mathbf{V}_{xxx}$ and $\mathbf{V}_{xxxx}$ are new obstruction then to have a nonzero solution space we must have

$$E := \det\begin{pmatrix} \mathbf{V} \\ \mathbf{V}_x \\ \mathbf{V}_{xx} \\ \mathbf{V}_{xxx} \\ \mathbf{V}_{xxxx} \\ \mathbf{V}_{xxxxx} \end{pmatrix} = 0.$$

If the matrix above has rank $3,\ 4$ or 5 then the solution space has $3,\ 2$ or 1 dimensions respectively.

- If $\mathrm{rank}(\mathcal{M}([\Gamma])) = 4$ then the vectors $\mathbf{V},\ \mathbf{V}_x,\ \mathbf{V}_y$ have to be linearly independent and the fourth vector has to be one of the second derivatives of $\mathbf{V}$, say $\mathbf{V}_{xx}$. Then the only higher order derivatives that may result in a new relation are higher order x derivatives of $\mathbf{V}$, for instance $\mathbf{V}_{xxx}$. If $\mathbf{V}_{xxx}$ gives a new relation then in order to obtain a nonzero solution from Theorem (2.3.5), $\mathbf{V}_{xxxx}$ must be expressed in terms of lower order relations and we would have

$$E := \det \begin{pmatrix} \mathbf{V} \\ \mathbf{V}_x \\ \mathbf{V}_y \\ \mathbf{V}_{xx} \\ \mathbf{V}_{xxx} \\ \mathbf{V}_{xxxx} \end{pmatrix} = 0,$$

and the solution space is one-dimensional. If $\mathbf{V}_{xxx}$ does not give any new relation then the solution space is two-dimensional. It is clear that in all cases the assumption in Theorem (1.2.3) together with previous lemma, ensure the existence of non-degenerate metrics in the projective structure.

□

As a result of the proof above, we have the following theorem.

Theorem 3.5.3. *The space of matrices compatible with a given projective structure can have dimensions* $0,\ 1,\ 2,\ 3,\ 4$ *or* 6*.*

CHAPTER 4

Minitwistor Theory of Projective Structures

In this chapter, we discuss the minitwistor theory which originates from the twistor theory developed by Roger Penrose. The twistorial treatment of our problem was originally done in [14] by Hitchin via Kodaira's deformation theory. In his work, Hitchin describes how we can associate to a surface Z with a family of rational curves with degree one normal bundle, the moduli space of its rational curves, denoted by X, which turns out to be a surface with a projective structure. The association motivates a notion of duality between X and Z. As is explained in [22], Z is the space of unparametrized geodesics of X. In other words, $\mathbb{P}(TX)$ has a double fiberation over Z and X and for any open set $U \subset X$ we have

$$\begin{array}{ccc} & \mathbb{P}(TU) & \\ {}^{\pi}\swarrow & & \searrow^{\tau} \\ U & \xrightarrow{\quad f \quad} & Z. \end{array} \tag{4.1}$$

Here, we will explain these construction and mention how the power of the twistor approach is used in them. Finally, we will see how this approach gives a geometric origin for the linearisation (3.12) of the non-linear system (3.3). We call an element of the family of rational curves a minitwistor line, following [4].

4.1 Minitwistor Correspondence

Suppose we have a rational curve Y in a complex surface Z with $N_Y \cong \mathcal{O}(d)$. Recall that $\dim(H^0(\mathbb{CP}^1, \mathcal{O}(1))) = d + 1$ and $H^1(\mathbb{CP}^1, \mathcal{O}(1)) = 0$, as we saw in Section A.We can apply the Kodaira Theorem (A.2.1) to get a complete family of rational curves $\{Y_x\}_x$ on Z containing Y with a complex surface X as its parametrization space, such that the isomorphism $T_xX \cong H^0(Y_x, N_{Y_x})$ holds canonically, so the $\dim(X) = d + 1$. The manifold X is called the moduli space of the family of minitwistor lines. Since any minitwistor line is represented by a point $x \in X$ and because $T_xX \cong H^0(Y_x, N_{Y_x})$, knowing $\dim(X) = d + 1$ implies that the degree of the normal bundle of a minitwistor line Y_x is d. Because we are dealing with Riemann spheres, this is sufficient to have $N_{Y_x} \cong \mathcal{O}(d), \ \forall x \in X$. In other words, the degree of the normal bundles of the minitwistor lines in the family are equal.

Suppose $d = 0$, meaning $N_Y \cong \mathcal{O}$. From the Kodaira Theorem, we obtain $\dim(X) = 1$. Also sections of the normal bundle of any minitwistor curve cannot vanish. Thus, the partial derivative of the minitwistor map with respect to x, introduced in Section A.2, cannot vanish since $\frac{\partial}{\partial x} f(y, x)$ with $y \in Y_x$ determines a section of N_{Y_x}. This means that the minitwistor lines never intersect. On the other hand, they form a complete family of curves and cover a neighborhood around any minitwistor line. Therefore, as an extension of the Kodaira Theorem, we can say that on a surface a neighborhood of a Riemann sphere with trivial normal bundle is biholomorphic to $\mathbb{CP}^1 \times \mathbb{C}$.

In the case when $d = 1$, we have $\deg(N_Y) = 1$. Before going any further, we explain a well-known construction used to lower the degree of the normal bundle. We do this in order to be able to use the case $d = 0$ that we already considered.

It is well-known that $\deg(N_Y) = Y \cdot Y$, where $Y \cdot Y$ is the self-intersection number of the curve Y(c.f. [13]). A standard procedure for lowering the self-intersection number of a curve, called blowing up a point of the curve, can thus be used to reduce the degree of the normal bundle of a curve. By blowing up a point $y \in Y$ in a surface Z, we lift Y to a curve $\widetilde{Y}$ in a surface $\widetilde{Z}$ such that $\widetilde{Y} \text{ and } \widetilde{Z}$ are diffeomorphic to $Y \text{ and } Z$ except at x . At the point x, we replace the point with the exceptional divisor $E := \mathbb{CP}^1$ in a consistent way. Another way of looking at this procedure is to take a neighborhood around x and map it to $\mathbb{C}^2$ such that x goes to the origin. Then using the natural projection $\pi : \mathcal{O}(-1) \to \mathbb{C}^2$, we can replace that neighborhood with the corresponding open set in $\mathcal{O}(-1)$. Suppose two curves Y and Y' intersect at p with intersection number one. Thus, they are transverse at p. If we blow up the point p, the blow up of Y and Y' no longer intersect as their only intersection point lifts to different points of E. The assumptions that the two curves were transverse at their intersection point and that the surface and its blow up are identical everywhere other than the point p were crucial. It is not difficult to derive properties of blowing up a point (c.f. [32]), including the decrease of the self-intersection number of a rational curve by one after blowing up one of its points. Therefore, blowing up any point of a curve lowers the degree of its normal bundle by one. In general, we can blow up a submanifold $S \subseteq M$, where $\operatorname{codim}(S) \geq 2$, by replacing it with the exceptional divisor $E := \mathbb{P}(N(S))$.

When $d = 1$, we have $N_Y \cong \mathcal{O}(1)$ with $H^0(Y, N_Y) = \mathbb{C}^2$ and $H^1(Y, N_Y) = 0$. Applying the Kodaira Theorem, we get a two-dimensional parametrizing space X. Here, we make use of the canonical isomorphism $T_x X \cong H^0(Y_x, N_{Y_x})$ in order to study the differential geometry of X. We fix a point $0 \in X$ representing the curve $Y_0 \subseteq Z$ and a one-dimensional subspace $l \in T_0 X$. This subspace corresponds to a one-dimensional subspace $\hat{l} \in H^0(Y_0, N_{Y_0})$. Remember that because $H^1(Y, N_Y) = 0$ then $H^0(Y, N_Y)$ coincides with the space of global sections of the normal bundle of Y. Since $N_Y \cong \mathcal{O}(1)$, we know from Section A that all the proper subspaces of this vector space are of the form $az_0 + bz_1$, where (z_0, z_1) are homogeneous coordinates of $\mathbb{CP}^1$. Therefore, the one-dimensional subspace $l \subseteq T_o X$ can be identified with the pair $(b, -a)$, the homogeneous coordinates of the vanishing point of its corresponding section $\hat{l}$. We can then blow up this point of the curve in order to lift Y to the curve $\widetilde{Y}$ which has a trivial normal bundle. The Kodaira Theorem applies to $\widetilde{Y} \subseteq \widetilde{Z}$. From the case we considered earlier we get a one parameter family of rational curves in the blown up surface $\widetilde{Z}$ parametrized by $\widetilde{X}$. We know that E and $\widetilde{Y}$ are transverse in $\widetilde{Z}$, therefore any curve in a neighborhood of $\widetilde{Y}$ is transverse to E. Thus, rational curves belonging to the one parameter family of minitwistor lines $\{Y_t\}_t$, which contains $\widetilde{Y}$ and covers a neighborhood of it, is transverse to E, as they cannot intersect each other. Projecting everything down to Z, we obtain the diagram

$$\begin{array}{ccc} T_t\widetilde{X} & \xrightarrow{\cong} & H^0(\widetilde{Y}_t, N_{\widetilde{Y}_t}) \\ \phi\downarrow & & \downarrow\pi_* \\ T_{x(t)}X & \xrightarrow{\cong} & H^0(Y_x, N_{Y_x}) \end{array}$$

where π_* is the induced map from blow up on the sheaf of sections of normal bundles (c.f. [13]). The map ϕ can be chosen uniquely such that the diagram commutes, because by the completeness part of the Kodaira Theorem the projections of all the deformations $\widetilde{Y}_t \subseteq \widetilde{Z}$ to $Y_{x(t)} \subseteq Z$ belong to the family containing Y_0. The image of $\widetilde{X}(t)$ under the map ϕ gives a curve $\gamma(t) \subseteq X$ such that $\gamma(0) = 0$ and $\dot{\gamma}(0) = l$ where l is the one-dimensional subspace that we fixed at the beginning. Also $\gamma(t)$ parametrizes the one-parameter family of curves $Y_t := \pi_*(\widetilde{Y}_t)$ containing $Y_0 = \pi_*(\widetilde{Y}_0)$. Since the exceptional divisor E maps to p and all $\widetilde{Y}_t$ are transverse to E, the curves Y_t intersect Y_0 at p.

Now we see that there is a notion of duality between the points in X and the curves in Y and vice versa, in the sense that any point $x \in X$ represents a curve $Y_x \subseteq Z$ and any point $y \in Y_x$ represents a curve passing through $x \in X$ which parametrizes the one-parameter subset of the minitwistor lines that intersect Y at y. In other words, we associate to a one-dimensional subspace $l \subseteq T_xX$ a point with homogeneous coordinates $[b : -a] \in Y \subseteq Z$ using the canonical isomorphism in the Kodaira Theorem. Therefore, there is a one-to-one correspondence between points of the fiber $\mathbb{P}(T_xX) \subseteq \mathbb{P}(TX)$ and points of the curve $Y_x \subseteq Z$. The minitwistor operator $\tau(y, x)$, defined in diagram (4.1), sends the fiber $\mathbb{P}(T_xX) \subseteq \mathbb{P}(TX)$ to the rational curve $Y_x \subseteq Z$.

Here we want to show that the curves in X, represented by points in Y, are complex geodesics and constitute a projective structure. In order to verify this claim, we make use of what we discussed earlier that for a projectively equivalent class of connections $[\Gamma]$ the form of the geodesic spray of each element ∇ with Christoffel symbols Γ^i_{jk} is given by (3.6) over TTX. The projection of the geodesic spray of projectively equivalent connections to $\mathbb{P}(TX)$ are identical and is given by (3.7). We also deduced that the geodesic spray of any two elements differ by $a(x,y)y_i\frac{\partial}{\partial y_i}$ with $a(x, y)$ linear in y. In general, the notion of spray in differential geometry defines a spray to be a vector field of the form $V = y_i\frac{\partial}{\partial x_i} - G_i(x, y)\frac{\partial}{\partial y_i}$ on the tangent bundle with y_i's as the fiber coordinates, where the functions $G_i(x, y)$ are homogenous of degree two in y. If $G_i(x, y)$ is quadratic in y then the spray defines an affine connection.

On X we have associated to any pair (x, l) such that $x \in X$ and l is a one-dimensional subspace of T_xX, a curve $\gamma(t)$ passing through x with $\dot{\gamma}(x) \in l$. Therefore, we have a well-defined lift of this foliation to $\mathbb{P}(TX)$. Following [22], we define a holomorphic projective structure similar to what we have in real case.

Definition 4.1.1. On an n-dimensional complex manifold X, we call a system $\mathcal{L}$ of complex curves which are inextensible immersed connected one-dimensional complex manifolds, a holomorphic projective structure if there is a unique association between curves in $\mathcal{L}$ and any holomorphic direction (i.e. to any element of the projectivized holomorphic tangent bundle $\mathbb{P}(TX)$) such that the curve varies holomorphically with the initial direction. In other words, the lift of the elements of the system $\mathcal{L}$, which is given by

$$\hat{l} := \{(x, T_x l) | x \in l,$$

for all $l \in \mathcal{L}$, foliates $\mathbb{P}(TX)$ holomorphically. We call the curves of $\mathcal{L}$ geodesics.

In the case $d = 1$, we have a holomorphic projective structure on the surface X. Comparing the definition above with our discussion in Section 3.1, the reason that the elements of $\mathcal{L}$ are called geodesics becomes apparent with the lemma below.

Theorem 4.1.2. *Let X be a complex n-dimensional manifold with a holomorphic projective structure $\mathcal{L}$. For a choice of coordinate chart over $U \subseteq X$, there exists a unique set of trace-free Christoffel symbols ${\Gamma^i}_{jk} \in \mathcal{O}_U$ (i.e ${\Gamma^i}_{jk} = 0$ for $j = i$) such that each element of the system $\mathcal{L}$ is a geodesic with respect to the affine connection defined by the Christoffel symbols over U.*

We adopt the proof given in [14] only for a surface (which can also be applied to the general form of the theorem).

Proof. The theorem follows if we find a vector field over TU that generates the curves in $\mathcal{L}$ and has the form of the geodesic spray of an affine connection, namely (3.6). Since we have a foliation of $\mathbb{P}(TU)$, we also have a vector field $\widetilde{V}$ over $\mathbb{P}(TU) \cong \mathbb{P}^1 \times U$ that generats the foliation. We can choose an open cover $\{W_a\}_a \subseteq \mathbb{P}^1$ such that $\widetilde{V}$ can be trivialized to $\widetilde{V}_a$ over $W_a \times U$. Using the map $\pi : TU \to \mathbb{P}(TU)$, we obtain their pull-backs $\pi^*(\widetilde{V}_a)$ which are a family of vector fields of the form

$$V_a(x,y) := y^i \frac{\partial}{\partial x^i} + f_a^i(x,y)\frac{\partial}{\partial y^i} + g_a(x,y) y^i \frac{\partial}{\partial y^i},$$

where the second term comes from the projection map π, as in (3.8). Because multiplying the tangent vector (y^1, y^2) by a scalar will not change the foliation of $\mathbb{P}(TU)$, we obtain $v(x, ry) = rv(x, r), r \in \mathbb{C}$. Therefore, the functions f_a^i are quadratic in the y^i's and g_a is linear in the y^i's, as we knew. Suppose $W_a \cap W_b \neq \varnothing$; then on the intersection for each i we have $f_a^i = f_b^i$, as they are determined by the vector field over $\mathbb{P}(TM)$. Therefore,

$$v_a - v_b = (g_a(x,y) - g_b(x,y))y^i \frac{\partial}{\partial y^i} = h_{ab}(x,y)y^i \frac{\partial}{\partial y^i},$$

where $h_{ab}(x,y)$ is linear in y. We can conclude that the v_a's can be glued together if and only if the transition maps h_{ab}'s do not impose any obstruction. From Section A.1, we know that $H^1(\mathbb{CP}^1, \mathcal{O}(1)) = 0$, meaning that there is no obstruction for the existence of a global sections in $\mathcal{O}(1)$ once the local trivializations are known. Therefore, for any choice of the functions g_a's we obtain a geodesic spray over $\mathbb{CP}^1 \times U$.

Among the sprays that we obtain from different choices of g_a's over each W_a, there is a special subset of the g_a's that gives rise to the connection with trace-free Christoffel symbols. As f_a is quadratic in y and g_a is linear, over any $W_a \times U$ we have

$$f_a^i(x,y)\frac{\partial}{\partial y^i} + g_a(x,y)y^i\frac{\partial}{\partial y^i} = \widetilde{\Gamma}^i{}_{jk}(x)y^k y^j \frac{\partial}{\partial y^i} + (\Lambda_j y^j(x))y^i \frac{\partial}{\partial y^i}.$$

Now we set $\Lambda_j = -\Gamma^i_{ij}$, where again the repeated indices do not denote summation. In this way we obtain

$$V_a = y^i \frac{\partial}{\partial x^i} + \Gamma^i{}_{jk}(x)y^k y^j \frac{\partial}{\partial y^i},$$

where $\Gamma^i{}_{jk} = 0$ for $j = i$. once we know trivializations of a spray, we can find the transition maps over subsets $(W_a \cap W_b) \times U$, which in this case are

$$V_a - V_b = h_{ab} y^i \frac{\partial}{\partial y^i}.$$

However, from our construction of the V_a's, we know that they have no component of the form $y^i \frac{\partial}{\partial y^i}$. Therefore, $h_{ab} = 0$ and we obtain the unique set of trace-free Christoffel symbols mentioned in the theorem, which are $\Gamma^i{}_{jk} = \widetilde{\Gamma}^i{}_{jk}$ for $j,k \neq i$ and zero otherwise. □

The procedure described above can be reversed in the sense that we can take the quotient of the projectivized tangent bundle of a surface over the one-dimensional foliation given by geodesics. It turns out that the quotient space is a two-dimensional complex manifold together with a family of rational curves with degree one normal bundle which are the projection of the fibers of the projectivized tangent bundle. The construction is discussed in generality and detail in [22].

When the canonical bundle of Z is restricted to a minitwistor line, the definition of normal bundle gives the exact sequence $0 \to T_Y(p) \to T_Z|_Y(p) \to N_Y(p) \to 0$ on each fiber, in which case the associated dual spaces satisfy the exact sequence $0 \to N_Y^*(p) \to \Omega_Z|_Y(p) \to \Omega_Y(p) \to 0$, where Ω_Z is the cotangent bundle of Z.

We know that if $0 \to V_1 \to V \to V_2 \to 0$ is an exact sequence of vector spaces $V_1,\ V$ and V_2 with dimensions r_1, r and r_2 then there is a natural isomorphism

$$\Lambda^r V \cong \Lambda^{r_1} V_1 \otimes \Lambda^{r_2} V_2.$$

Thus, we have

$$K_Z|_Y \cong \Omega_Y \otimes N_Y^*. \tag{4.2}$$

As we obtained in Section A.1, the dual of the tangent space of the Riemann sphere is of degree -2. We also have $N_Y \cong \mathcal{O}(1)$, and therefore $N^*(\mathbb{P}^1) \cong \mathcal{O}(-1)$. As a result, we have

$$K_Z|_Y \cong \Omega_Y \otimes N_Y^* \cong \mathcal{O}(-2) \otimes \mathcal{O}(-1) \cong \mathcal{O}(-3).$$

Knowing that the restriction of the canonical bundle of Z to a minitwistor line $K_Z|_Y$ is isomorphic to $\mathcal{O}(-3)$, we obtain $K_Z^{-2}|_Y \cong K_Z|_Y^{-2} \cong \mathcal{O}(6)$ and consequently $K_Z^{-\frac{2}{3}}|_Y \cong \mathcal{O}(2)$, where K_Z^{α} is a scalar density of weight α, as we defined in Section 2.1.

4.2 Alternative Derivations

According to the paper [4], if U is metrizable then the density $K_Z^{-2/3}$ admits a global section. Given a global section of this line bundle, say $s \in K_Z^{-2/3}$, its restriction to a minitwistor line Y can be expresses as a quadratic polynomial in homogeneous coordinates, say $s = \widetilde{\sigma}_{ij}\widetilde{z}^i\widetilde{z}^j$ with a symmetric matrix $\widetilde{\sigma}_{ij}$. Using the map $\tau : \mathbb{P}(TU) \to Z$ in diagram (4.1), we can pull-back the restriction of this section over a minitwistor line Y to the corresponding fiber of $\mathbb{P}(TU)$. Recall that $\mathbb{P}(TU)$ fibers over Z and the preimage of any minitwistor line is a fiber of $\mathbb{P}(TU)$ i.e. $\mathbb{P}(T_xU) = \tau^{-1}(Y_x)$. Therefore, $\mathcal{O}(2)(\mathbb{P}(T_xU)) \ni \tau^* s = \sigma_{ij}z^iz^j$ where coordinates $z^i = \tau^*(\widetilde{z}^i)$ are the homogeneous coordinates of the fiber $\mathbb{P}(T_xM)$ and $\sigma_{ij} = \tau^*(\widetilde{\sigma}_{ij})$

is a symmetric 2-tensor. Let $D \in \mathfrak{X}(\mathbb{P}(TU))$ be the geodesic spray of U defined by Christoffel symbols ${\Pi^i}_{jk}$ in (3.14), namely

$$D_x = z^a \frac{\partial}{\partial x^a} - {\Pi^a}_{bc} z^b z^c \frac{\partial}{\partial z^a}. \tag{4.3}$$

Recall that the quadratic form $\widetilde{\sigma}$ over Z is obtained due to the existence of a metric whose Levi-Civita connection is projectively equivalent to the affine connection with Christoffel symbols ${\Pi^i}_{jk}$.

Since the minitwistor map sends each geodesic to a point in Z, as explained in the construction of Z in last section, the push-forward of this vector field to Z at any point is zero and we have $D_x(\tau^*(s)) = \tau_*(D_x)s = 0$. Expanding this equation gives the linear system of ODEs (3.12):

$$\begin{aligned} D_x(\sigma_{ij} z^i z^j) &= z^a z^i z^j \frac{\partial}{\partial x^a} \sigma_{ij}(x) - {\Pi^a}_{bc} \sigma_{ij} z^b z^c \frac{\partial}{\partial z^a} z^i z^j \\ &= z^a z^i z^j \frac{\partial}{\partial x^a} \sigma_{ij}(x) - {\Pi^a}_{bc} \sigma_{ij} z^b z^c (\delta^i_a z^j + \delta^j_a z^i) \\ &= z^a z^i z^j \frac{\partial}{\partial x^a} \sigma_{ij}(x) - z^j z^b z^c {\Pi^i}_{bc} \sigma_{ij} - z^i z^b z^c {\Pi^j}_{bc} \sigma_{ij} \\ &= z^a z^i z^j \Big(\frac{\partial}{\partial x^a} \sigma_{ij}(x) - {\Pi^b}_{ij} \sigma_{ab} - {\Pi^b}_{ij} \sigma_{ab} \Big) \\ &= z^i z^j z^k \nabla^{\Pi}_{(i} \sigma_{jk)} = 0, \end{aligned} \tag{4.4}$$

where in the last term repeated indices again do not denote summation. This system is therefore equivalent to equation (3.13). Also we can use the projective coordinate $\zeta = \frac{z^1}{z^2}$ to get another form for the geodesic spray as in (3.7), which is

$$D = \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial y} + \big(A_0 + A_1 \zeta + A_2 \zeta^2 + A_3 \zeta^3\big) \frac{\partial}{\partial \zeta},$$

Using this form, we get

$$D_x(\sigma_{ij} z^i z^j) = D_x(\psi_1 + 2\psi_2 \zeta + \psi_3 \zeta^2) = 0.$$

which gives the system (3.12) directly.

This was a geometric interpretation of the way the nonlinear equation (3.2) could be written as a system of linear PDEs. The geodesic flow annihilates the pull-back of a global section in the anti-canonical divisor $K_Z^{-2/3}$ of the minitwistor space Z and it certainly results in a linear system of equations. The section could be constructed due to metrizability of U. Recall that we used the coefficients of the

Christoffel symbols of the projective connection to express the geodesic flow. As a result σ_{ab} satisfies assumptions in Liouville lemma (3.2.1) and $\frac{\sigma_{ab}}{(\det\sigma)^2}$ is the metric we were looking for. All our discussion in this section involved complex surfaces. Suppose that the maps in diagram (4.1) are invariant under an anti-holomorphic involution Ψ of Z with coordinates (z^1, z^2), meaning that $\frac{\partial\Psi}{\partial\overline{z}^i} = 0$. Then $f^{-1}\circ\Psi\circ f$ would be an anti-holomorphic involution of U, since

$$\frac{\partial}{\partial z^i}\left(f^{-1}\circ\Psi\circ f(z)\right) = \frac{\partial(f^{-1}\circ\Psi\circ f)}{\partial((\Psi\circ f)^k)}\cdot\frac{\partial(\Psi\circ f)^k}{\partial(f^j)}\cdot\frac{\partial f^j}{\partial z^i},$$

and because Ψ is invariant under f we get

$$\frac{\partial(\Psi\circ f)^k}{\partial f^j} = \frac{\partial\Psi^k}{\partial\overline{z}^j} = 0.$$

Consequently, we recover a real structure on U which would be $U\cap\Psi(U)$.

Note that in our construction we made use of global structure of $\mathbb{CP}^1$ to derive local properties of X, using the notion of duality explained between points and curves. The power of the twistorial approach is that the lines are constrained only by complex geometry and no differential geometry of Z is involved. The only information we needed to describe the differential geometry of X was the degree of the normal bundle of the rational curves.

We can use the notion of densities to derive the linear system (3.12) from another perspective and say more about its solutions. In a class of projectively equivalent torsion-free affine connections $[\Gamma]$ over a surface, if $\widetilde{\nabla},\ \nabla\in[\Gamma]$ satisfy (3.9) then using equation (2.3), for a section $h\in|\Lambda|^r_M$ we obtain

$$\widetilde{\nabla}_j h - \nabla_j h = (-r\widetilde{\Gamma}^i{}_{ij} + r\Gamma^i{}_{ij})h = -r3\Upsilon_j h. \tag{4.5}$$

We introduce the terminology projective density of weight ω for sections of the density bundle $|\Lambda|_M^{-\frac{\omega}{n+1}}$. Using equation (4.5), for a section of this bundle we have

$$\widetilde{\nabla} h = \nabla h + \omega\Upsilon h, \tag{4.6}$$

where the 1-form Υ is given by $\Upsilon = \Upsilon_j dx^j$. We denote the density bundles of projective weight ω by $\mathcal{E}(\omega)$ and we have $\mathcal{E}(\omega)\cong|\Lambda|_M^{-\frac{\omega}{n+1}}$. Therefore, a volume form has projective weight -3 as it is a 1-density so $r = 1$ in equation (4.6).

Let us restrict ourselves to a sufficiently small open set $U\subseteq M$ such that $\widetilde{\nabla}\eta = \theta\eta$ for a 1-form $\theta = \theta_i dx^i$ and $\eta\in|\Lambda|^1_M$. First, we change the projective representative from $\widetilde{\nabla}$ to ∇ so that

$$\Gamma^i{}_{jk} = \widetilde{\Gamma}^i{}_{jk} + \frac{\theta_j}{3}\delta^i_k + \frac{\theta_k}{3}\delta^i_j.$$

Then, using (4.6), we obtain

$$\nabla\eta = \widetilde{\nabla}\eta - 3\cdot\frac{\theta}{3}\eta = 0.$$

Therefore η is parallel with respect to ∇. If we changed the volume form by $\hat{\eta} = e^{3f}\eta$ then for a projective representative $\hat{\nabla}$ related to ∇ by $\Upsilon = \nabla f$ through equation (3.9), we would have

$$\hat{\nabla}\hat{\eta} = \nabla(e^{3f}\eta) - 3\nabla f\eta = 0. \tag{4.7}$$

This example provides a way of determining the projective weight of a scalar density over U. As we said earlier, the sections of any scalar density are in correspondence with volume forms in $\mathcal{E}(-3)$. For a volume form $\eta \in \mathcal{E}(-3)$, parallel with respect to ∇, the corresponding section $h \in \mathcal{E}(\omega)$ is also parallel with respect to ∇. This is because $h \in |\Lambda|_M^{-\frac{\omega}{3}}$ so that $h = \eta^{-\frac{\omega}{3}}$ and we have

$$\begin{aligned}\nabla_a h = \nabla_a \eta^{-\frac{\omega}{3}} &= -\frac{\omega}{3}\eta^{-1-\frac{\omega}{3}}\partial_a\eta + \frac{\omega}{3}(\Gamma^i{}_{ia})\eta^{-\frac{\omega}{3}} \\ &= -\frac{\omega}{3}\eta^{-1-\frac{\omega}{3}}(\partial_a\eta - \Gamma^i{}_{ia}\eta) \\ &= -\frac{\omega}{3}\eta^{-1-\frac{\omega}{3}}\nabla_a\eta = 0\end{aligned} \tag{4.8}$$

Suppose we have a line bundle with a choice of parallel section h, namely $\nabla h = 0$. Given a real valued function $f \in C^\infty(M)$, let $\hat{\nabla}$ be a projective representative related to ∇ via equation (3.9) and $\Upsilon = \nabla f$. If $\hat{h} = e^{-\omega f}h$ satisfies $\hat{\nabla}\hat{h} = 0$ then $h \in \mathcal{E}(\omega)$. In other words, if being parallel with respect to ∇ remains invariant under the transformation

$$h \mapsto \hat{h} = e^{-\omega f}h$$

for projective representative $\hat{\nabla}$ then the line bundle has projective weight ω and the line bundle is a scalar density bundle of weight $-\frac{\omega}{n+1}$. As the initial parallel section h comes from a parallel volume form, the transformation $h \mapsto \hat{h} = e^{-\omega f}$ is induced by the transformation of volume forms $\eta \mapsto \hat{\eta} = e^{3f}\eta$.

The solution for the system of ODEs (3.12) was equivalent to a quadratic form satisfying the equation $\nabla^{\Pi}_{(i}\sigma_{jk)} = 0$, where the Π's are the Christoffel symbols of the projective connection of $[\Gamma]$ in that local chart defined in (3.14). In this section, we want to relate this solution to the solution of $\nabla_{(i}\sigma_{jk)}$ for an arbitrary element of $[\Gamma]$ which is not necessarily Π. First let us find the projective weight of quadratic forms satisfying $D_x(\sigma) = 0$. Let $\hat{D}$ and D represent the geodesic spray of $\hat{\nabla}$ and ∇, with

Christoffel symbols $\hat{\Gamma}$ and Γ respectively, satisfying equation (3.9) with $\Upsilon = \nabla f$. As we did in (4.4) we can write this equation as

$$\begin{aligned}\hat{D}_x(\sigma_{jk}) - D_x(\sigma_{jk}) &= z^i z^j z^k[-(\hat{\Gamma}^r_{ij} - \Gamma^r_{ij})\hat{\sigma}_{rk} - (\hat{\Gamma}^r_{ik} - \Gamma^r_{ik})\hat{\sigma}_{jr}] \\ &= z^i z^j z^k[-(\Upsilon_j\delta^r_i + \Upsilon_i\delta^r_j)\hat{\sigma}_{rk} - (\Upsilon_i\delta^r_k + \Upsilon_k\delta^r_i)\hat{\sigma}_{jr}] \\ &= z^i z^j z^k[-(\Upsilon_j\hat{\sigma}_{ik} + \Upsilon_i\hat{\sigma}_{jk}) - (\Upsilon_i\hat{\sigma}_{jk} + \Upsilon_k\hat{\sigma}_{ji})] \\ &= z^i z^j z^k(-4\Upsilon_{(i}\sigma_{jk)})\end{aligned} \tag{4.9}$$

where in the last term, repeated indices do not account for summation. We the make the transformation

$$\sigma \longrightarrow \hat{\sigma} = e^{-\omega f}\sigma \tag{4.10}$$

We obtain

$$\begin{aligned}\hat{D}_x(\hat{\sigma}_{jk}) &= D_x(e^{-\omega f}\sigma_{jk}) + z^i z^j z^k(-4\Upsilon_{(i}e^{-\omega f}\sigma_{jk)}) \\ &= e^{-\omega f}D_x(\sigma_{jk}) + z^i z^j z^k(-\omega\sigma_{(jk}\nabla_{i)}fe^{-\omega f} - 4\Upsilon_{(i}\hat{\sigma}_{jk)}) \\ &= e^{-\omega f}D_x(\sigma_{jk}) + (-\omega - 4)z^i z^j z^k\Upsilon_{(i}\hat{\sigma}_{jk)}\end{aligned} \tag{4.11}$$

Assume that $D_x(\sigma_{jk}) = 0$, then in order to have $\hat{D}_x(\hat{\sigma}_{jk}) = 0$ with the transformation (4.10), we get $\omega = -4$. This implies that the linear operator

$$\sigma_{ij} \longrightarrow \nabla_{(i}\sigma_{jk)}$$

which is equivalent to $\sigma_{ij} \longrightarrow D_x(\sigma_{ij})$, is projectively invariant on symmetric two-tensors of projective weight $\omega = -4$.

Now suppose we have an kernel σ for this operator. Let η be the volume form parallel with respect to the connection ∇ as assumed earlier. Define $\sigma^{ij} = \eta^{ia}\eta^{jb}\sigma_{ab}$, which belongs to $S^2(TU) \otimes \mathcal{E}(2)$ since $\eta^{ab} \in \mathcal{E}(3)$ and $\sigma_{ab} \in \mathcal{E}(-4)$. In order to write down the covariant derivative of the σ^{ab}'s with respect to the connection ∇ for which equation (3.13) holds, we use the fact that the volume form η_{ab} and its dual η_{ab} are parallel and obtain

$$\begin{aligned}\nabla_{(1}\sigma_{12)} = 0 &= \frac{1}{3}(2\nabla_1\sigma_{12} + \nabla_2\sigma_{11}) \\ \Rightarrow 0 &= \eta^{21}\eta^{12}(2\nabla_1\sigma_{12} + \nabla_2\sigma_{11}) \\ &= (2\nabla_1(\eta^{21}\eta^{12}\sigma_{12}) + \nabla_2(\eta^{21}\eta^{12}\sigma_{11})) \\ &= 2\nabla_1\sigma^{21} - \nabla_2\sigma^{22} \\ &\Rightarrow \nabla_2\sigma_{22} = 2\nabla_1\sigma^{21}.\end{aligned} \tag{4.12}$$

We can similarly consider other cases and obtain

$$\nabla_a \sigma^{bc} = \delta_a^b \mu^c + \delta_a^c \mu^b. \tag{4.13}$$

where μ^a is some smooth function.

Now we want to see whether the projective structure containing ∇ with $\nabla_a(\sigma^{bc})$ having the form above, is metrizable or not. Note that in deriving the above equation we made use of the equation $\nabla_{(i}\sigma_{jk)} = 0$ and not the equation $\nabla^{\Pi}_{(i}\sigma_{jk)}$ from which the Liouville theorem gives the metrizability of the projective structure. The following theorem from [11] provides the answer.

Theorem 4.2.1. *Suppose $\nabla \in [\Gamma]$ admits a parallel volume form η and there is metric tensor such that*

$$\nabla_a \sigma^{bc} = \delta_a^b \mu^c + \delta_a^c \mu^b$$

for some function $\mu^b \in C^\infty(U)$, then $[\Gamma]$ is metrizable.

Proof. The projective equivalence relation (3.9) gives the following relation between the covariant derivative of a vector with respect to $\hat{\nabla}, \nabla \in [\Gamma]$:

$$\begin{aligned} \hat{\nabla}_a X^b = \partial_a X^b + \hat{\Gamma}_a^b i X^i &= \nabla_a X^b + (\Upsilon_a \delta_i^b + \Upsilon_i \delta_a^b) X^i \\ &= \nabla_a X^b + \Upsilon_a X^b + \delta_a^b \Upsilon_i X^i. \end{aligned} \tag{4.14}$$

where $\Upsilon = \nabla f$ for some function $f \in C^\infty(M)$. As we showed above, σ^{ab} has projective weight 2, thus the equation (4.13) remains unchanged if we change ∇ to $\hat{\nabla}$ such that relation (3.9) holds for $\Upsilon = \nabla f$ and also make the transformation $\hat{\sigma}^{ab} = e^{-2f}\sigma^{ab}$. This becomes more clear if we compute $\hat{\nabla}_a \hat{\sigma}^{bc})$ using (4.14), which gives

$$\begin{aligned} \hat{\nabla}_a(\hat{\sigma}^{bc}) &= \hat{\nabla}_a(e^{-2f}\sigma^{bc}) \\ &= e^{-2f}\left(-2\Upsilon_a\sigma^{bc} + \nabla_a\sigma^{bc} + 2\Upsilon_a\sigma^{bc} + \delta_a^b\Upsilon_d\sigma^{dc} + \delta_a^c\Upsilon_d\sigma^{bd}\right) \\ &= e^{-2f}\left(\delta_a^b\mu^c + \delta_a^c\mu^b + \delta_a^b\Upsilon_d\sigma^{dc} + \delta_a^c\Upsilon_d\sigma^{bd}\right) \end{aligned}$$

This can be written as

$$\hat{\nabla}_a\hat{\sigma}^{bc} = \delta_a^b\hat{\mu}^c + \delta_a^c\hat{\mu}^b \quad \text{where} \quad \hat{\mu}^a = e^{-2f}\left(\mu^a + \Upsilon_b\sigma^{ab}\right).$$

As we discussed earlier, the transformation $\hat{\sigma}^{ab} = e^{-2f}\sigma^{ab}$ induces a change of volume form as $\hat{\eta} = e^{3f}\eta$. Set η to be the parallel volume form with respect to ∇, which is assumed to exist, then $\hat{\eta}$ is a parallel volume form with respect to $\hat{\nabla}$ as we showed in (4.7). Define

$$\det(\sigma) = \eta_{ab}\eta_{cd}\sigma^{ac}\sigma^{bd}, \tag{4.15}$$

and we obtain,

$$\begin{aligned}\hat{\det}(\hat{\sigma}) &= \hat{\eta}_{ab}\hat{\eta}_{cd}\hat{\sigma}^{ac}\hat{\sigma}^{bd} \\ &= e^{6f}e^{-4f}\eta_{ab}\eta_{cd}\sigma^{ac}\sigma^{bd} = e^{2f}\det(\sigma).\end{aligned}$$

Now if we put $f = -\frac{1}{2}\log\det(\sigma)$, then $\hat{\det}(\hat{\sigma}) = 1$. This means that the volume form $\hat{\eta}$ which is parallel with respect to $\hat{\nabla}$, is the volume form of the metric tensor $\hat{\sigma}^{ab}$. Now the we can use lemma below which applies to this situation and conclude that $\hat{\nabla}$ is the Levi-Civita connection for the metric $\hat{\sigma}^{ab}$ Because ∇ is projectively equivalent to $\hat{\nabla}$, we conclude that the projective structure containing ∇ is metrizable and contains the Levi-Civita connection of the metric $\hat{\sigma^{ab}} = (\det\sigma)\sigma^{ab}$. Note that the Levi-Civita connection of σ^{ab} may not be contained in this projective structure. □

Lemma 4.2.2. *A torsion-free connection ∇ is the Levi-Civita connection of a metric g^{ab} if $\nabla_a g^{bc} = \delta^b_a\mu^c + \delta^c_a\mu^b$ and $\nabla\eta_{bc} = 0$ where η_{ab} is the volume for of the metric g^{ab}.*

Proof. Let D denote the Levi-Civita connection of the metric g^{ab}. As we are dealing with torsion-free affine connections we obtain

$$\nabla_a\omega_b = D_a\omega_b - \Gamma^c_{ab}\omega_c, \tag{4.16}$$

for any 1-form ω, with $\Gamma^a_{bc} = \Gamma^a_{cb}$.

As any metric is parallel with respect to its Levi-Civita connection we have $\nabla_a g^{bc} = \Gamma^b_{ad}g^{dc} + \Gamma^c_{ad}g^{bd}$. Therefore, according to the assumptions we have

$$\Gamma^b_{ad}g^{dc} + \Gamma^c_{ad}g^{bd} = \delta^b_a\mu^c + \delta^c_a\mu^b \tag{4.17}$$

We can contract both sides of the above equation by multiplying g_{bc} to get

$$\begin{aligned}\Gamma^b_{ad}g^{dc}g_{bc} + \Gamma^c_{ad}g^{bd}g_{bc} &= 2\Gamma^b_{ad}\delta^d_b + 2\Gamma^c_{ad}\delta^d_c = 4\Gamma^b_{ab} \\ &= 2\delta^b_a\mu_b + 2\delta^c_a\mu_c = 4\mu_a\end{aligned}$$

and thus $\Gamma^b_{ab} = \mu_a$ where $\mu_a = g_{ab}\mu^b$. Now using the assumption that the volume form η^{ab} is parallel with respect to ∇ we obtain

$$\nabla_a\eta_{bc} = -\Gamma^b_{ab}\eta_{bc} = 0.$$

Thus $\Gamma^b_{ab} = \mu_a = 0$ and $\nabla_a g^{bc} = 0$. Knowing this and $\nabla_a(g^{db}g_{bc}) = 0$, we obtain

$$0 = g^{db}\nabla_a g_{bc} = -g^{db}\left(\Gamma^e_{ab}g_{ec} + \Gamma^e_{ac}g_{be}\right) = -g^{db}\left(\Gamma_{abc} + \Gamma_{acb}\right)$$

where $\Gamma_{abc} = \Gamma^e_{ab}g_{ec}$. Contracting the equation by g_{fd} gives

$$\Gamma_{afc} + \Gamma_{acf} = 0. \tag{4.18}$$

Since the connection is torsion-free we have $\Gamma^a_{bc} = \Gamma^a_{cb}$ and therefore $\Gamma_{afc} = \Gamma_{fac}$. Combining this with (4.18), gives $\Gamma_{abc} = 0$. Thus, $\Gamma^a_{bc} = 0$ and from (4.16) we obtain $\nabla = D$. □

An easy corollary of what we discussed above is the as below.

Theorem 4.2.3. *There is a one-to-one correspondence between metric tensors satisfying $\nabla_a\sigma^{bc} = \delta^b_a\mu^c + \delta^c_a\mu^b$, for some function μ^a, and metric connections that are projectively equivalent to ∇.*

Another way of addressing this theorem is by using Theorem (3.2.1). The proof is easy if we recall that ∇ and ∇^{Π} are projectively equivalent and relation (3.9) holds for some 1-form Υ. If we restrict ourselves to a sufficiently small open set such that $\Upsilon = \nabla f$ then we obtain $\nabla^{\Pi}_a\hat{\sigma}^{bc} = \delta^b_a\hat{\mu}^c + \delta^c_a\hat{\mu}^b$, where $\hat{\sigma}^{bc} = e^{-2f}\sigma^{ab}$ and $\hat{\mu}^a = e^{-2f}\left(\mu^a + \Upsilon_b\sigma^{ab}\right)$. Recall that if a metric σ^{ab} satisfies $\nabla_a\sigma^{bc} = \delta^b_a\mu^c + \delta^c_a\mu^b$ then $\nabla_{(a}\sigma_{bc)} = 0$. Therefore, $\nabla^{\Pi}_{(a}\hat{\sigma}_{bc)} = 0$ and Theorem (3.2.1) guarantees the metrizability of the projective structure $[\Gamma]$ containing ∇^{Π}. As ∇^{Π} and ∇ are projectively equivalent, we have $\nabla \in [\Gamma]$. As we said the metrics would be $g^{ab} = \det(\sigma)\sigma^{ab}$. By the definition of $\det(\sigma)$ in (4.15), it has projective weight -2. Also, as we discussed earlier, σ^{ab} has projective weight 2. Therefore, the projective weight of the metric is g^{ab} is zero as is required. Note that the expression (4.13) is the first prolongation of the linear system (3.13).

APPENDIX A

Complex Geometry

In this section, we state some definitions and theorems in complex geometry that mainly deal with holomorphic line bundles which are used in the construction of minitwistors in Chapter 4. The material discussed here can be found in more detail in [15], [32] and [34].

A.1 Holomorphic Line Bundles

Let Z be a complex manifold. On any embedded submanifold $Y \subseteq Z$ we can associate a vector bundle called the normal bundle. The fibers of this bundle at each point $p \in Y$ are the quotient vector space obtained from the exact sequence

$$0 \longrightarrow T_Y(p) \longrightarrow T_Z|_Y(p) \longrightarrow N_Y(p) \longrightarrow 0, \tag{A.1}$$

where T_Y is the tangent bundle of Y and $T_Z|_Y$ is the restriction of the tangent bundle of Z to Y. It is clear that $\mathrm{rank}(N) = \mathrm{rank}(Z) - \mathrm{rank}(Y)$. We are interested in the case in which the submanifold Y is a rational curve, meaning that it is an embedding of the Riemann sphere i.e., $\mathbb{CP}^1$, and the manifold Z is a complex surface. In this case N_Y has rank one, and is thus a line bundle. Line bundles defined over a Riemann sphere have many interesting properties. In order to state them we first need to give some definitions and theorems. As the construction of vector bundles suggests, any vector bundle is defined by its transition function on the intersections of its domain of trivializations. For instance we can define a line bundle over a Riemann sphere by choosing two points $p \, \text{and} \, q \in \mathbb{CP}^1$ and considering open sets $U_0 = \mathbb{CP}^1 \backslash \{q\} \, \text{and} \, U_1 = \mathbb{CP}^1 \backslash \{p\}$. From our choice of U_0 and U_1 we obtain a chart for $\mathbb{CP}^1$ via stereographic projection to the complex plane. Suppose z and $\widetilde{z}$ are the local coordinates for U and $\widetilde{U}$ respectively with $z(p) = 0$. The line bundle L_p is defined if the transition function for trivializations over $U_0 \, \text{and} \, U_1$ is $g_{01} = z$ where z is the coordinate of U_0 and g_{01} is a map from U_1 to U_0. This bundle has a canonical section s_p such that $s_p|_{U_1} = 0$ and $s_p|_{U_0} = z$, and therefore $s_p(p) = 0$. It would appear that this construction of L_p depends on the choices of p and q, but it turns out for any choice of p and q of we get isomorphic line bundles as the lemma below implies.

Lemma A.1.1. *Given any two pair of points $(p, q), (p', q') \in \mathbb{CP}^1$ the exist a holomorphic diffeomorphism Ψ such that $\Psi(p) = p'$ and $\Psi(q) = q'$.*

The prove easily follows if we note that $\mathbb{CP}^1 \backslash \{p\} \cong \mathbb{C}$ and we can construct a holomorphic diffeomorphism Ψ_1 such that $\Psi_1(p) = p$ and $\Psi_1(q) = q'$, using stereographic projection and a translation. Similarly we have a holomorphic diffeomorphism Ψ_2 such that $\Psi_1(p) = p'$ and $\Psi_1(q') = q'$. Thus, $\Psi = \Psi_2 \circ \Psi_1$ is the map stated in the theorem. We denote the isomorphic class of line bundles L_p by $\mathcal{O}(1)$.

We can change the transition function to $g_{01} = z^n$, and proceed similarly to obtain an isomorphic class of line bundles which we denote by $\mathcal{O}(n)$. When we regard $\mathcal{O}(n)$ as a line bundle we mean an arbitrary line bundle that belongs to $\mathcal{O}(n)$. In this way, it is easy to define the notion of degree for these types of bundles. First we choose a section s of the line bundle $\mathcal{O}(n)$. Suppose the trivializations of the section over U_0 and U_1 is s_0 and s_1 respectively. Writing the power series of these two sections, we obtain

$$\sum a_m z^m \quad \text{and} \quad \sum \widetilde{a}_m \widetilde{z}^m.$$

This is due to the fact that the trivialized section, say s_0, can be regarded as a complex valued function over U_0, and as all functions are holomorphic we can write its power series in terms of z, the coordinate of the base open set U_0. Over $U_0 \cap U_1$ we have $z = \frac{1}{\widetilde{z}}$ and therefore

$$s_0 = z^n s_1 \Longrightarrow \sum a_m z^m = z^n \sum a_m z^{-m} \tag{A.2}$$

$$\Rightarrow \; a_m = 0 \; \forall \; m \geq n \; \text{and} \; \widetilde{a}_0 = a_{n-1}, \; \widetilde{a}_{n-1} = a_0, \; \cdots. \tag{A.3}$$

We call n the degree of $\mathcal{O}(n)$ since all the sections of this line bundle can be presented by polynomials of degree n. Thinking of z as the homogeneous coordinate arising from embedding of $\mathbb{CP}^1$ in $\mathbb{C}^2$, the global expression for s as a global section of $\mathcal{O}(n)$ would be $s = a_i z_1^{(n-i)} z_2^i$ with $0 \leq i \leq n$ and (z_i, z_2) the homogeneous coordinate for $\mathbb{CP}^1$ usually denoted by $[z_1 : z_2]$. It is obvious that any global section of $\mathcal{O}(n)$ vanishes at n points. We note that $\mathcal{O} \cong \mathcal{O}(0)$ where $\mathcal{O}$ denotes the trivial line bundle. this notation makes sense if we note that $\mathcal{O}$ is simply the sheaf of holomorphic functions over the Riemann sphere and so according to Theorem (A.1.2) stated below, its global sections are constant functions over $\mathbb{CP}^1$.

Theorem A.1.2. *On a connected and compact complex surface, the only holomorphic functions are the constants.*

The proof follows if we realize that the compactness of the manifold M implies that the modulus of any function f is maximum at some point $p \in M$. But then in a coordinate chart ϕ around p the module of the function $f \circ \phi^{-1}$ attains a maximum at an interior point of an open set of $\mathbb{C}$ which cannot be true unless f is constant. Using connectedness, f has to be constant over M. As sections of $\mathcal{O}$ are constants according to the theorem above, equation (A.3) implies that it is isomorphic to $\mathcal{O}$. It is easy to see that if a line bundle has a nonvanishing global section then the section itself gives an isomorphism between the line bundle and $\mathcal{O}$.

Recall that $H^p(M, V)$ is the pth cohomology space of the sheaf of germs of sections of the vector bundle V. For $p = 0$ it is the space of global sections of V if $H^1(M, V) = 0$. So we have $\dim(H^0(\mathbb{CP}^1, \mathcal{O}(n))) = n + 1$, because any global section of $\mathcal{O}(n)$ is identified by the coefficients a_i which are n-tuples as showed in (A.3). Here, we identify a line bundle with its sheaf of sections by abuse of notation. The sheaf $\mathcal{O}^*$ denotes the group sheaf of nonzero holomorphic functions under multiplication. The elements of $H^1(Z, \mathcal{O}^*)$ represent the isomorphism classes of holomorphic line bundles on Z (c.f. [13]). In other words,each element of this bundle represents a class of isomorphic line bundles and is also called the Picard group of Z, denoted by $Pic(Z)$. Now we can define the degree of any isomorphic class of line bundles by considering the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp(2\pi i f)} \mathcal{O}^* \longrightarrow 1, \tag{A.4}$$

The exactness of the sequence (A.4) is due to the fact that every nonzero holomorphic function can be expressed locally by the exponential of a holomorphic function.

Note that the image of the sheaf $\mathcal{O}$ under the exponential map is a presheaf and not necessarily a sheaf, and its associated sheaf is $\mathcal{O}^*$. The Mayer-Vietoris sequence gives the exact sequence below for corresponding cohomologies.

$$\begin{aligned} 0 &\to H^0(\mathbb{CP}^1,\mathbb{Z}) \to H^0(\mathbb{CP}^1,\mathcal{O}) \to H^0(\mathbb{CP}^1,\mathcal{O}^*) \\ &\to H^1(\mathbb{CP}^1,\mathbb{Z}) \to H^1(\mathbb{CP}^1,\mathcal{O}) \to H^1(\mathbb{CP}^1,\mathcal{O}^*) \\ &\xrightarrow{c_1([L])} H^2(\mathbb{CP}^1,\mathbb{Z}) \to H^2(\mathbb{CP}^1,\mathcal{O}) \to H^2(\mathbb{CP}^1,\mathcal{O}^*) \to 0. \end{aligned} \tag{A.5}$$

In order to simplify the exact sequence above, we need to state an important theorem in complex geometry.

Theorem A.1.3 (c.f. Voisin [32])**.** *If X is a Kähler manifold then $H^{p,q}(X)$ is canonically isomorphic to $H^q(X,\Omega^p_X)$.*

In the above theorem the term Kähler manifold refers to a complex manifold with a choice of Hermitian metric such that the imaginary part of the metric is a closed two-form. All complex projective spaces are Kähler. Because the isomorphism is canonical it is independent of the choice of the metric. The vector space $H^{p,q}(X)$ is the de Rham cohomology class which is represented by a closed form of type (p,q) on the complex manifold and $H^q(X,\Omega^p_X)$ is the qth Dolbeault cohomology of the vector bundle $\bigwedge^p \Omega_X$, where Ω_X is the cotangent bundle of X. Using this theorem we have

$$H^2(\mathbb{CP}^1,\mathcal{O}) \cong H^2(\mathbb{CP}^1,\Omega^0_X) \cong H^{(0,2)}(\mathbb{CP}^1) = 0. \tag{A.6}$$

Thus (A.5) becomes

$$0 \to \frac{H^1(\mathbb{CP}^1,\mathcal{O})}{H^1(\mathbb{CP}^1,\mathbb{Z})} \to H^1(\mathbb{CP}^1,\mathcal{O}^*) \xrightarrow{c_1([L])} H^2(\mathbb{CP}^1,\mathbb{Z}) \to 0 \tag{A.7}$$

We call $c_1([L]) \in H^2(\mathbb{CP}^1,\mathbb{Z})$, which appears in (A.7), the degree of the class of line bundles $[L]$. We know from topology that $H^2_{sing}(\mathbb{CP}^1) = \mathbb{Z}$ and due to the fact that all the cohomology theories with the same coefficients are isomorphic (c.f. [33]), we have the same result for cohomology of the constant sheaf of integers, that is $H^2(\mathbb{CP}^1,\mathbb{Z}) = \mathbb{Z}$. Therefore, $c_1([L]) \in \mathbb{Z}$.

The degree map c_1 is a homomorphism with respect to the tensor product of line bundles in $H^1(\mathbb{CP}^1,\mathcal{O}^*)$ since $\mathcal{O}^*$ is a multiplication group. As a result

$$\deg(L \otimes L') = \deg(L) + \deg(L') \text{ and } \deg(L^*) = -\deg(L). \tag{A.8}$$

where L^* is the dual line bundle of L. In the last equality we used the fact that the endomorphism $\mathrm{Hom}(L,L) = L \otimes L^*$ is canonically trivial since the only homomorphism from a one-dimensional vector space to itself is multiplication by scalars. Because of this L^* is usually denoted by L^{-1} and

$$0 = c_1(\mathrm{Hom}(L,L)) = c_1(L \otimes L^*) = c_1(L) - c_1(L^*).$$

Using this homomorphism we can prove that the degree of any line bundle over a $\mathbb{CP}^1$ is the number of zeros of an arbitrary section of the line bundle, which implies that all the sections have the same number of zeros. To do this, choose a section $s \in L$ that vanishes at p in which case the section $s \otimes s_p^* \in L \otimes L_p^*$ is nonvanishing where s_p is the canonical section of L_p. Therefore, we have $L \otimes L_p^* \cong \mathcal{O}$ because of existence of a nonvanishing section. If $c_1([L_p]) = 1$ then using (A.8), we get that $c_1([L]) = 1$. In general, for a section s in a line bundle $\widetilde{L}$ vanishing at m points $p_1, ..., p_m$ counting multiplicities, we can make similar argument for $s \otimes_{i=1}^m s_{p_i}^* \in \widetilde{L} \otimes_{i=1}^m L_{p_i}^*$ to show that $c_1([\widetilde{L}]) = m$. The choice of the degree of L_p is essentially a normalization. There is no line bundle with a positive degree less than L_p as it cannot vanish anywhere and is therefore trivial. An obvious corollary is that global sections of line bundles with negative degree never vanish. Actually there is no global section in a negative degree line bundle because if one were to exist it would have to be nonvanishing and thus would give an isomorphism to the trivial line bundle which is impossible as they have different degrees.

It is easy to see that the cotangent bundle of a Riemann sphere, also called its canonical bundle, denoted by K, is isomorphic to $\mathcal{O}(-2)$. In order to determine its degree, we only need to consider one section and find the degree of the transition function. As before, let us work in a charts $U_0,\ U_1$ with local coordinates $z, \text{ and } \widetilde{z}$. Since the coordinate on U_1 is $\widetilde{z} = \frac{1}{z}$, this section will have the form $d(\frac{1}{\widetilde{z}}) = -\widetilde{z}^{-2} d\widetilde{z}$ on U_1. Therefore,

$$\begin{aligned} dz = -\widetilde{z}^{-2} d\widetilde{z} &\Longrightarrow g_{01}(z) = -z^{-2} \\ &\Longrightarrow K_{\mathbb{CP}^1} := T^*\mathbb{CP}^1 \cong \mathcal{O}(-2). \end{aligned} \tag{A.9}$$

Another consequence of Theorem (A.1.3) is Serre duality, which is stated below.

Theorem A.1.4 (Serre Duality). *If E is a line bundle on an n-dimensional compact complex manifold Z, then*

$$H^p(Z,E) \cong H^{(n-p)}(Z, K_Z \otimes E^*)^*.$$

Recall that K_Z is the sheaf of sections of the canonical line bundle. Using Serre duality, we can simplify the exact sequence (A.7) in the case M is a Riemann sphere. For $n = 1$ we have

$$H^1(M,\mathcal{O}) \cong H^0(\mathbb{CP}^1,\mathcal{O}\otimes K_{\mathbb{CP}^1}) \cong H^0(\mathbb{CP}^1,\mathcal{O}\otimes\mathcal{O}(-2)) \cong H^0(\mathbb{CP}^1,\mathcal{O}(-2)).$$

Since $\mathcal{O}(-2)$ has no global section due to its negative degree, $H^1(M,\mathcal{O})$ must vanish. Therefore, the sequence (A.7) simplifies to

$$0 \to H^1(\mathbb{CP}^1,\mathcal{O}^*) \xrightarrow{c_1([L])} \mathbb{Z} \to 0. \tag{A.10}$$

As a result of the isomorphism between $H^1(\mathbb{CP}^1,\mathcal{O}^*)$ and $\mathbb{Z}$ the degree of a line bundle, identifies isomorphism class to which the line bundle belongs. Thus, all line bundles with the same degree are isomorphic. Using the relation (A.8) we conclude that $\mathcal{O}(1)$ generates all the line bundles with positive degree through tensor products; its dual, $\mathcal{O}(-1) := \mathcal{O}^*(1)$, generates all those with negative degree. For instance, $\mathcal{O}(n) = \otimes_{i=1}^n \mathcal{O}(1)$. Therefore, any line bundle L on a Riemann sphere is isomorphic to $\mathcal{O}(n)$ for $n = \deg(L)$. This is the special case of the Birkhoff-Grothendieck Theorem.

Theorem A.1.5. *A rank k-holomorphic vector bundle $E \longrightarrow \mathbb{CP}^1$ is isomorphic to a direct sum of line bundles $\mathcal{O}(m_1)\oplus\cdots\oplus\mathcal{O}(m_k)$ for some integers m_i.*

We can get a better geometric image of $\mathcal{O}(-1)$ by constructing an example of a degree minus one line bundle called the tautological line bundle.

Example (tautological line bundle)**.** Consider the Riemann sphere with the coordinate patches z and $\tilde{z}$ for $U_0,$ and U_1 that we defined previously. Then

$$\mathcal{O}(-1) \cong \{(p,(Z^0,Z^1)) \in \mathbb{CP}^1\times\mathbb{C}^2 | p = [Z^0 : Z^1]\} \tag{A.11}$$

According to (A.1.5) if we can show that the transition function is $g_{01}(z) = z^{-1}$ for the line bundle on the right hand side, then we are done. Consider the projection $\pi : \mathbb{C}^2 \to \mathbb{CP}^1$. The fiber above a point $[Z^0 : Z^1]$ is

$$\pi^{-1}([Z^0:Z^1]) = c(Z^0,Z^1) := c(Z^0,Z^1)|c\in\mathbb{C}.$$

Since Z^i does not vanish over U_i we can define the following trivializations:

$$\phi_0([Z^0:Z^1],c(Z^0,Z^1)) = (z,cZ^0) \subseteq U_0\times\mathbb{C} \text{ and } z = \frac{Z^1}{Z^0}$$
$$\phi_1([Z^0:Z^1],c(Z^0,Z^1)) = (\tilde{z},cZ^1) \subseteq U_1\times\mathbb{C} \text{ and } \tilde{z} = \frac{Z^0}{Z^1}$$

Thus we would have

$$g_{01}(z)cZ^1 = cZ^0 \Longrightarrow g_{01}(z) = \frac{Z^0}{Z^1} = z^{-1}.$$

A.2 Deformation Theory

Consider a complex surface Z which contains a family of rational curves, i.e. embeddings of the Riemann sphere. The normal bundle of a rational curve in this case is a line bundle over a Riemann sphere, so we only need to know the degree of the normal bundle in order to know its isomorphism class. Conversely, knowing the isomorphism class of the normal bundle is sufficient to know how the rational curve is embedded in Z. The following theorem is due to Kodaira.

Theorem A.2.1 (c.f. [14])**.** *If $Y \subseteq Z$ is a compact submanifold with $H^1(Y, N_Y) = 0$ then Y belongs to a locally complete family $\{Y_x : x \in X\}$ for some complex manifold X, and there is a canonical isomorphism between T_xX and $H^0(Y_x, N_{Y_x})$.*

In the statement of the theorem we are given global information about the complex surface Z as it contains a family of rational curve with degree one normal bundles. The theorem allows us to investigate the local geometry of the space parametrizing the family of rational curves. The parametrizing space and the canonical isomorphism can be geometrically interpreted. Any submanifold $Y \subseteq Z$ is represented by a point $x_0 \in X$. Let us denote it by Y_{x_0}. There exists a function $f(y, x) : Y_{x_0} \times X \longrightarrow Z$ that we call a minitwistor map such that $f(Y_{x_0}, x)$ is an embedding of $\mathbb{CP}^1$, for all $x \in X$. In other words,the vector field $\frac{\partial}{\partial x} f(y, x)|_{x_0}$ gives a global section of the normal bundle over Y_{x_0}. Also as will be explained later each normal bundle 'close enough' to the zero section, represents another curve close to Y_{x_0} and conversely, each curve 'close enough' to Y_{x_0} represents a section of the normal bundle. The completeness part of the theorem formally defined below, which may be found in [9], says in informal language that any rational curve close enough to Y_{x_0} is contained in the family.

Definition A.2.2. A pair $(\mathcal{F}, X)$ is called a d-dimensional complete analytic family of compact submanifolds of an $d + r$-dimensional complex manifold Z if

- $\mathcal{F}$ is a complex analytic submanifold of $Z \times X$ of codimension r with the property that for each $x \in X$ the intersection $Y_x \times x := \mathcal{F} \cap (Z \times x)$ is a compact submanifold of $Z \times x$ of dimension d.

- There exists an isomorphism

$$T_xX \cong H^0(Y_x, N_{Y_x})$$

where $Y_x \subseteq Z$.

APPENDIX B

Cartan's Test

In this appendix, we will discuss Cartan's test as a tool for finding the degree of generality of the set of solutions of an involutive analytic system of PDEs. The material covered in this section is discussed in greater detail in [9] and [17]. In order to state Cartan's test, we need to give some definitions as below in which M is a smooth m-dimensional manifold.

Definition B.0.3. A graded differential ideal $\mathcal{I} \subseteq \Omega^*(M)$ over M is a set of differential forms that are closed under exterior differentiation and wedge product, i.e. for all θ in $\mathcal{I}$

$$d\theta,\ \theta \wedge \eta \in \mathcal{I},$$

where η is an arbitrary differential form on M. We denoted the set of k-forms of $\mathcal{I}$ by $\mathcal{I}^k := \mathcal{I} \cap \Omega^k$.

A differential ideal can be defined by its generators. For instance we can write $\mathcal{I} =< \theta_1, ..., \theta^n >_{\text{diff}}$ which means that $\mathcal{I}$ consists of differential forms θ^i, $d\theta_i$ and their wedge product with all the differential forms in $\Omega^*(M)$. As we will see in an example, a system of PDEs can be represented with its associated differential ideal.

Definition B.0.4. An exterior differential system (EDS) is a pair $(M, \mathcal{I})$ where $\mathcal{I}$ is a graded differential ideal over M.

Definition B.0.5. An integral manifold of an EDS $(M,\mathcal{I})$ is a submanifold S of M such that $\mathcal{I}|_{TS} = 0$, i.e. for any integer $k \geq 1$,

$$\theta(v_1, ..., v_k) = 0,$$

where $\theta \in \mathcal{I}^k$ and $v_1, ..., v_k \in TS$.

Similarly we can define an integral element of an EDS.

Definition B.0.6. A k-dimensional subspace $E \subseteq T_xM$ is an integral element of $\mathcal{I}$ if $\theta(v_1, ..., v_k) = 0$, where $\theta \in \mathcal{I}^k$ and $v_1, ..., v_k \in E$. The set of all k-dimensional integral elements of $(M,\mathcal{I})$ is denoted by $V_k(\mathcal{I})$.

The set $V_k(\mathcal{I})$ is clearly a submanifold of the Grassmanian of all k-planes in TM, namely $Gr_k(TM)$.

A question similar to what we asked before would be whether for a set of k-dimensional integral elements of $\mathcal{I}$ there exists a k-dimensional integral manifold of $\mathcal{I}$ to which the integral elements are tangent.

The restriction of an integral element of $\mathcal{I}$ is an integral element meaning that is if G is a p-dimensional subspace of $E \in V_k(\mathcal{I})$ then $G \in V_p(\mathcal{I})$. The converse is not be true meaning that if $E \in V_k(\mathcal{I})$ and $G \in V_p(\mathcal{I})$ then $E \oplus G$ may not belong to $V_{p+k}(\mathcal{I})$.

Definition B.0.7. Let $E \in V_k(\mathcal{I}) \subseteq T_xM$ be spanned by $\{e_1, ..., e_k\}$. The polar space of E is

$$H(E) := \{v \in T_xM \mid \theta(v, e_1, ..., e_k) = 0, \forall \theta \in \mathcal{I}^{k+1}\} \subseteq T_xM.$$

It is clear that E is contained in $H(E)$. However, $H(E)$ is not necessarily an integral element. An integral element that is a one-dimensional extension of E can be obtained from $H(E)$ and is equal to $E \oplus \{v\}$ for some $v \in H(E)$ and $v \notin E$.

An integral element E is called regular if the dimension of $H(E)$ is constant in a neighborhood of E in $V_k(\mathcal{I})$. Moreover, E is called ordinary of the intersection of $V_k(\mathcal{I})$ with an open neighborhood of E is a smooth submanifold of $Gr_k(TM)$.

The dimension of the set of all $(k+1)$-dimensional integral elements that contain E would be $\dim(H(E)) - k$. Define

$$r(E) := \dim(H(E)) - k - 1.$$

If no such an extension exist then $r(E) = -1$.

An integral manifold of $\mathcal{I}$, say $S \subseteq M$, is called ordinary (regular) if all of its tangent spaces are ordinary (regular) integral elements. For a regular integral manifold we define $r(S) := r(T_xS)$ for some $x \in S$.

Theorem B.0.8 (Cartan-Kähler Theorem)**.** *Let $\Sigma \subseteq M$ be an n-dimensional analytic submanifold whose tangent spaces are regular integral elements for a real analytic EDS $(M, \mathcal{I})$, such that $\dim(H(T_x\Sigma)) = n+1$. Then there exists an open neighborhood of $x \in \Sigma$ and a unique $(n+1)$-dimensional integral manifold $S \subseteq M$ containing $\Sigma \cap U$.*

The fact that $\dim(H(E)) = n+1$ provides the uniqueness part of the theorem. Otherwise, if $\dim(H(E)) = n+r+1$ with $r \neq 0$, we consider a submanifold $R \subseteq M$, called restraining manifold, such that R is of codimension r in M and $\Sigma \subseteq R$ and at each point $p \in \Sigma$ the tangent space T_pR intersects $H(T_p\Sigma)$ transversally and as a result, the theorem above holds in R and we have a unique extension of Σ in R.

Using the Cartan-Kähler Theorem, we can successively construct the integral manifold of a differential ideal with a choice of restraining manifolds at each step. Therefore, for a given integral element $E \in V_n(\mathcal{I})$ at point p, we obtain an n-dimensional integral manifold N with $T_pN = E$. In order to address this construction, we define a regular flag of integral elements at a point $p \in M$ to be a set of n integral elements

$$(0) = E_0 \subset E_1 \subset \cdots \subset E_n = E \subset T_pM$$

such that $E_i \in V_i(\mathcal{I})$ and $E_1, ..., E_{n-1}$ are regular. Such a flag is called a regular flag for the integral element E. Given an integral element $E \in V_n(\mathcal{I})$ at point p, in order to use the Cartan-Kähler Theorem to investigate whether there exist an integral manifold $N \subseteq M$ such that $T_pN = E$, we only need to find a regular flag of E. Then by applying the Cartan-Kähler Theorem to each E_i in the flag we get the sufficient condition for the existence of N.

Practically speaking, our discussion above is not of great significance as finding a regular flag of E may not be straightforward. Also at point p, it is possible that the tangent space of not all of the integral manifolds have a regular flag.

Cartan's test helps to get around with this difficulty. For a flag $\mathcal{F}$ of E , not necessarily regular, set

$$c(E_i) := \dim(T_pM) - \dim(H(E_i)) \quad i = 0, ..., n.$$

where E_i's are the elements of the flag.

Theorem B.0.9 (Cartan's test)**.** *For an EDS* $(M,\mathcal{I})$ *let* $\mathcal{F} = (E_0, ..., E_n)$ *be an integral flag of* $E \in V_n(\mathcal{I})$ *and* $E \subseteq T_pM$. *Then*

$$\operatorname{codim}(V_n(\mathcal{I})) \geq c(\mathcal{F}) := c(E_0) + \cdots + c(E_{n-1})$$

in the Grassmannian $Gr_n(TM)$ *at* E. *Moreover,* $V_n(\mathcal{I})$ *is a smooth submanifold of* $Gr_n(TM)$ *at* E *of codimension* $c(\mathcal{F})$ *is and only if the flag* $\mathcal{F}$ *is regular.*

With Cartan's test at our disposal, for a quasi-linear system of PDEs with a given flag of an integral element, it is a matter of linear algebra to look for the existence of integral manifolds for it. If the equality holds for a flag of E then we say that the flag has passed the test and therefore is regular. As a result of our previous discussion, if we can find a flag of E that passes Cartan's test then E is an ordinary integral element and the Cartan-Kähler Theorem guarantees the existence of at least one real-analytic n-manifold manifold $N \subseteq M$ such that $T_pN = E$.

Example. The second-order PDEs of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + D + E(u_{xx}u_{yy} - u_{xy}^2) = 0,$$

where $A,\ B,\ C,\ D,\ E$ are functions of $x,\ y,\ u,\ u_x,\ u_y$ are called the Monge-Ampère equations. The 1-jet of solutions $u(x,y)$ define $\mathcal{S}^{(1)} \subseteq J^1(\mathbb{R}^2,\mathbb{R})$. We also want the variables x and y to be independent which gives the condition $dx \wedge dy \neq 0$ over $\mathcal{S}^{(1)}$. With this condition $\mathcal{S}^{(1)}$ will be a submanifold of the integral manifold of the 1-form $\theta^1 = du - pdx - qdy$ with $p = u_x$ and $q = u_y$. Thus, assuming that $dx \wedge dy \neq 0$ holds everywhere in $\mathcal{S}^{(1)}$, a function $u(x,y)$ satisfies the Monge-Ampère equation if and only if the first prolongation of the surface it spans, namely $\mathcal{S}^{(1)}$, is an integral manifold of θ^1 and

$$\theta^2 = Adp \wedge dy + B(dq \wedge dy - dp \wedge ddx) - Cdq \wedge dx + Ddx \wedge dy + Edp \wedge dq.$$

Knowing that $J^1(\mathbb{R}^2,\mathbb{R}) \cong \mathbb{R}^5$ with coordinates (x,y,u,p,q), we consider the EDS $(\mathbb{R}^5,\mathcal{I})$, where $\mathcal{I} =< \theta^1, \theta^2 >_{\text{diff}}$, to find $\mathcal{S}^{(1)}$ and the general solution to the Monge-Ampère equation.

We consider the case $A = B = C = 0$ and $D = E = 1$. It is clear that $V_1(\mathcal{I}) = \{\theta^1\}^{\perp}$ and thus it is spanned by the integral elements $\partial_p, \partial_q, \partial_x + p\partial_u$ and $\partial_y + q\partial_u$. Let $E_1 = \partial_p$ be the first integral element of the flag. The two-forms of the ideal are of the form $\theta^2,\ d\theta^1,\ \theta^1 \wedge \gamma$,where γ is an arbitrary one-form. Thus the vectors annihilating the one-forms $\partial_p \lrcorner \theta^2$, $\partial_p \lrcorner d\theta^1$ and θ^1 determine $H(E_1)$. Therefore, we have $H(E_1) = \{dq, dx.\theta^1\}^\perp = \{\partial_p, \partial_y + q\partial_u\}$. As E_1 has a one-dimensional extension, E_2 is uniquely determined and is equal to $H(E_1)$. It is easy to see that $H(E_2) = E_2$ and there is not extension of E_2. Now with our choices of E_1 and E_2 we have a flag $\mathcal{F}: \{0\} \subset E_1 \subset E_2 = E \subset T_x\mathbb{R}^5$. According to our discussion above, we have $c(E_0) = 5 - 4 = 1$ and $c(E_1) = 5 - 2 = 3$ and $c(E_2) = 5 - 2 = 3$ which gives $c(\mathcal{F}) = c(E_0) + c(E_1) = 4$. In order to see if this flag passes the Cartan's test we need to compute the codimension of $V_2(\mathcal{I})$ in $Gr_2(T\mathbb{R}^5)$. In a sufficiently small neighborhood of $E_2 \in V_2(\mathcal{I})$ the two-planes are spanned by the vectors

$$v_1 = \partial_p + a(\partial_x + p\partial_u) + b\partial_q + c\partial_u \quad \text{and} \quad v_2 = \partial_y + q\partial_u + d(\partial_x + p\partial_u) + e\partial_q + f\partial_u$$

for some $(a, b, ..., f)$. The conditions

$$\theta^1(v_1) = \theta^1(v_2) = 0, \quad d\theta^1(v_1, v_2) = \theta^2(v_1, v_2) = 0$$

give $c = f = e - a = b + d = 0$. Therefore, the fiber codimension of $V_2(I)$ at E_2 is four, as it is determined by knowing b and e, which is equal to $c(\mathcal{F})$. Therefore, the flag has passed the Cartan's test and the general solution of the Monge-Ampère equation depends on two function of one variable.

APPENDIX C

Long Formulae of Chapter 3

Recall that we considered the connection $D = d + \Omega_1 dx + \Omega_2 dy$ in (2.20). In our analysis in Section 3.4, we consider such a connection over the rank six we considered defined in (3.25). In this case the expressions for $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ are as below.

$$\begin{pmatrix} -\frac{2}{3}A_1 & 2A_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ -2A_3 & -\frac{2}{3}A_2 & \frac{4}{3}A_1 & 0 & 1 & 0 \\ (\Omega_1)_{41} & (\Omega_1)_{42} & (\Omega_1)_{43} & -\frac{1}{3}A_1 & -3A_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ (\Omega_1)_{61} & (\Omega_1)_{62} & (\Omega_1)_{63} & (\Omega_1)_{64} & (\Omega_1)_{65} & (\Omega_1)_{66} \end{pmatrix},$$
$$\begin{pmatrix} -\frac{2}{3}A_1 & 2A_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ -2A_3 & -\frac{2}{3}A_2 & \frac{4}{3}A_1 & 0 & 1 & 0 \\ (\Omega_1)_{41} & (\Omega_1)_{42} & (\Omega_1)_{43} & -\frac{1}{3}A_1 & -3A_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ (\Omega_1)_{61} & (\Omega_1)_{62} & (\Omega_1)_{63} & (\Omega_1)_{64} & (\Omega_1)_{65} & (\Omega_1)_{66} \end{pmatrix}, \tag{C.1}$$

with entries $(\Omega_i)_{ab}$ being as below:

$$
\begin{aligned}
(\Omega_1)_{41} &= -\frac{4}{3}\partial_x A_2 + 4A_0A_3 + \frac{2}{3}\partial_y A_1, \\
\Omega_1)_{42} &= -2\partial_y A_0 + \frac{2}{3}\partial_x A_1 + 4A_2A_0 - \frac{4}{9}(A_1)^2, \\
(\Omega_1)_{43} &= 2\partial_x A_0 - 4A_0A_1, \\
(\Omega_1)_{61} &= -\frac{4}{3}\partial_x\partial_y A_2 - \frac{20}{3}A_0A_2A_3 + \frac{2}{3}\partial_x^2 A_1 + 4A_3\partial_y A_0 - 2A_0\partial_y A_3 \\
&\quad - \frac{16}{9}A_2\partial_x A_2 + \frac{8}{9}A_2\partial_y A_1, \\
(\Omega_1)_{62} &= \frac{2}{3}\partial_x\partial_y A_1 - \frac{4}{3}A_1\partial_y A_1 + 2A_0\partial_y A_2 - 2\partial_y^2 A_0 + 4A_2\partial_y A_0 + 4A_3\partial_x A_0 \\
&\quad + 6A_0\partial_x A_3 + \frac{8}{9}A_1\partial_x A_2 + \frac{4}{3}A_0A_1A_3A_2 - \frac{4}{3}A_0(A_2)^2, \\
(\Omega_1)_{63} &= 2\partial_x\partial_y A_0 + \frac{2}{3}A_0\partial_x A_2 - \frac{4}{3}A_2\partial_x A_0 - 4A_1\partial_y A_0 - \frac{4}{3}A_0\partial_y A_1 \\
&\quad + \frac{8}{3}A_0A_1A_2 + 4A_3(A_0)^2, \\
(\Omega_1)_{64} &= \frac{4}{3}\partial_x A_2 - \partial_y A_1 + 5A_0A_3, \\
(\Omega_1)_{65} &= \frac{1}{3}\partial_x A_1 - 4\partial_y A_0 + 3A_2A_0 - \frac{2}{9}(A_1)^2, \\
(\Omega_1)_{66} &= \frac{1}{3}A_1, \\
(\Omega_2)_{51} &= -2\partial_y A_3 - 4A_3A_2, \\
(\Omega_2)_{52} &= 2\partial_x A_3 - \frac{2}{3}\partial_y A_2 + 4A_1A_3 - \frac{4}{9}(A_2)^2, \\
(\Omega_2)_{53} &= \frac{4}{3}\partial_y A_1 - \frac{2}{3}\partial_x A_2 + 4A_0A_3, \\
(\Omega_2)_{61} &= -2\partial_x\partial_y A_3 - 4A_2\partial_x A_3 - \frac{4}{3}A_3\partial_x A_2 + \frac{2}{3}A_3\partial_y A_1 + \frac{4}{3}A_1\partial_y A_3 \\
&\quad - 4A_0(A_3)^2 - \frac{8}{3}A_1A_2A_3, \\
(\Omega_2)_{62} &= 2\partial_x^2 A_3 - \frac{4}{3}A_2\partial_x A_2 - \frac{4}{3}A_0A_2A_3 - \frac{2}{3}\partial_x\partial_y A_2 + 4A_1\partial_x A_3 + 4A_0\partial_y A_3 \\
&\quad + 6A_3\partial_y A_0 + \frac{4}{3}A_3(A_1)^2 + 2A_3\partial_x A_1 + \frac{8}{9}A_2\partial_y A_1, \\
(\Omega_2)_{63} &= -\frac{2}{3}\partial_x^2 A_2 + \frac{8}{9}A_1\partial_x A_2 + 4A_0\partial_x A_3 - 2A_3\partial_x A_0 - \frac{16}{9}A_1\partial_y A_1 \\
&\quad + \frac{4}{3}\partial_x\partial_y A_1 + \frac{20}{3}A_0A_1A_3, \\
(\Omega_2)_{64} &= 4\partial_x A_3 - \frac{1}{3}\partial_y A_2 + 3A_1A_3 - \frac{2}{9}(A_2)^2, \\
(\Omega_2)_{65} &= \partial_x A_2 - \frac{4}{3}\partial_y A_1 + 5A_0A_3, \\
(\Omega_2)_{66} &= \frac{1}{3}A_2,
\end{aligned}
$$

The expression for the curvature **F** of this connection as was defined in (2.22) is

$$\mathbf{F} = d\mathbf{\Omega} + \mathbf{\Omega} \wedge \mathbf{\Omega} = F dx \wedge dy = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{pmatrix} dx \wedge dy, \tag{C.2}$$

where the $V_1, ..., V_6$ are given in (3.25).

In the prolongation process in Section 3.3 we defined the variables P, Q, R, S which have the following expressions.

$$\begin{aligned} P &= -(\Omega_1)_{41}\psi_1 - (\Omega_1)_{42}\psi_2 - (\Omega_1)_{43}\psi_3 - (\Omega_1)_{44}\mu - (\Omega_1)_{45}\nu, \\ Q &= -(\Omega_1)_{51}\psi_1 - (\Omega_1)_{52}\psi_2 - (\Omega_1)_{53}\psi_3 - (\Omega_1)_{54}\mu - (\Omega_1)_{55}\nu, \\ R &= -(\Omega_1)_{61}\psi_1 - (\Omega_1)_{62}\psi_2 - (\Omega_1)_{63}\psi_3 - (\Omega_1)_{64}\mu - (\Omega_1)_{65}\nu - (\Omega_1)_{66}\rho, \\ S &= -(\Omega_2)_{61}\psi_1 - (\Omega_2)_{62}\psi_2 - (\Omega_2)_{63}\psi_3 - (\Omega_1)_{64}\mu - (\Omega_1)_{65}\nu - (\Omega_1)_{66}\rho. \end{aligned} \tag{C.3}$$

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