# Fractional Edge and Total Colouring

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To Amy.

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# PREFACE

The main contributions of this thesis are contained in the Chapters 4, 5, 6, and 7. An extended abstract describing the results of Chapter 4 appeared in the proceedings of *EuroCOMB 2009* [65] and is joint work with Bruce Reed and Conor Meagher. An extended abstract describing the results of Chapter 6 appeared in the proceedings of *LAGOS 2009* [55] and is joint work with Bruce Reed and Takehiro Ito. Chapters 5 and 7 are joint work with Bruce Reed. In each case, all authors contributed equally.

## ABSTRACT

Many problems which seek to schedule, sequence, or time-table a set of events subject to given constraints can be modelled as graph colouring problems. In this thesis, we study the edge and total colouring problems which, like all NP problems, can be formulated as integer programs and subjected to a two-pronged attack: we first solve the fractional relaxation and then use this solution to solve or obtain an approximation of the solution of the integer program. We focus on the complexity of solving the fractional relaxations of the integer programs for the edge and total colouring problems.

For each $\varepsilon > 0$, we give a linear time algorithm which determines the fractional chromatic index of a graph $G$ with maximum degree at least $\varepsilon|G|$. For graphs with large maximum degree, this improves on Padberg and Rao's polynomial time algorithm to determine the fractional chromatic index for general graphs. Both algorithms rely on a theorem of Edmonds showing that the fractional chromatic index of a graph is determined by its maximum degree and overfull subgraphs. Our algorithm exploits the fact that overfull subgraphs are related to small cuts in a graph and have simple intersection patterns when the maximum degree is large.

The complexity of determining the fractional total colouring number is currently unresolved. We focus on graphs with large maximum degree, applying the very successful techniques for fractional edge colouring to fractional total colouring. We characterize graphs with maximum degree $\Delta$ whose fractional total colouring number is $\Delta + 2$, sharpening a result of Kilakos and Reed who showed it is between $\Delta + 1$ and $\Delta + 2$. We show graphs whose fractional total colouring number is less than $\Delta + 2$ have a special fractional vertex colouring which extends to a fractional total colouring using less than $\Delta + 2$ colours. We

extend these ideas by giving necessary conditions a fractional vertex $\beta$-colouring must satisfy to be extendable to a fractional total $\beta$-colouring. We conjecture these conditions are sufficient when $G$ satisfies $\Delta > \frac{1}{2}|G|$. We verify a special case of this conjecture by giving a polynomial time algorithm which constructs an optimal fractional total colouring of a graph $G$ with maximum degree at least $\frac{3}{4}|G|$ and containing no overfull subgraphs.

# ABRÉGÉ

De nombreux problèmes qui consistent à programmer, ordonner, ou encore planifier un ensemble d'événements étant données des contraintes peuvent être modilisés par un probème de coloriage de graphe. Dans cette thèse, nous étudions les problèmes du coloriage d'arêtes et du coloriage total, qui comme tous les problèmes NP, peuvent être formulés sous forme de programmation entière, et traités avec une approche en deux temps: on résoud d'abord la relaxation fractionnaire du programme entier, puis on utilise la solution du programme relaché pour déterminer ou approximer la solution du programme entier. Nous nous concentrons sur la complexité de la relaxation fractionnaire pour les problèmes du coloriage d'arêtes et du coloriage total.

Pour tout $\varepsilon > 0$, nous donnons un algorithme linéaire qui détermine l'indice chromatique fractionnaire d'un graphe $G$ dont le degré maximal est au moins $\varepsilon|G|$. Pour les graphes dont le degré maximum est grand, ceci améliore l'algorithme polynomial de Padberg et Rao pour l'indice chromatique fractionnaire. Les deux algorithmes reposent sur un théorème dû à Edmonds qui montre que l'indice chromatique fractionnaire est déterminé par le degré maximum et les sous-graphes overfull. Notre algorithme exploite le fait que les sous-graphes overfull sont reliés aux petites coupes, et ont des motifs d'intersections simples lorsque le degré maximum est grand.

La complexité du problème de coloriage total fractionnaire est toujours inconnue. Nous nous concentrons sur les graphes dont le degré maximum est grand, et appliquons au coloriage total fractionnaire les le techniques qui se sont avérées fructueuses pour le problème de coloriage d'arêtes . Nous charactérisons les graphes dont le degré maximum est $\Delta$, et dont le nombre chromatique total fractionnaire est $\Delta + 2$, ce qui améliore un résultat de Kilakos et Reed qui ont montré qu'il est compris entre $\Delta + 1$ et $\Delta + 2$. Nous montrons que les graphes dont le nombre chromatique total fractionnaire est moins de $\Delta+2$ possèdent un coloriage fractionnaire des noeuds qui s'étend en un coloriage total fractionnaire utilisant moins de $\Delta+2$ couleurs. Nous généralisons ces idées en donnant des conditions nécéssaires que doit remplir un $\beta$-coloriage des noeuds pour pouvoir être étendu en un $\beta$-coloriage total fractionnaire. Nous conjecturons que ces conditions sont suffisantes lorsque $\Delta > \frac{1}{2}|G|$. Nous vérifions un cas particulier de cette conjecture en concevant un algorithme polynomial qui construit un coloriage total fractionnaire pour un graphe $G$ dont le degré maximum est au moins $\frac{3}{4}|G|$ et qui ne contient pas de sous-graphe overfull.

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# CHAPTER 1
# Introduction

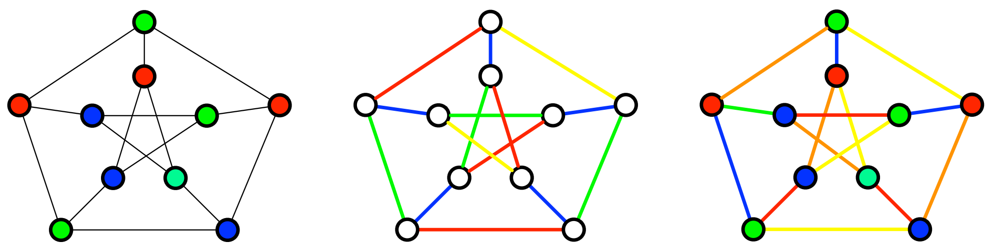
Combinatorial optimization[1] seeks to find a best solution in a set of feasible solutions. Since this set is typically very large, the naive method of enumerating each possible feasible solution is usually computationally prohibitive. Hence, one normally seeks efficient methods whose runtime is bounded by a polynomial in the size of the input. In this thesis, we focus on efficient algorithms for various graph colouring problems.

Graph colouring problems, and more generally combinatorial optimization problems, often arise in modelling the real world. For example, in 1852, Francis Guthrie sought to colour the regions of a map such that no two regions which share a common border receive the same colour (see Chapter 6 in [10]), and in 1949, Claude Shannon considered colour coding the wires in electrical panels which connect devices such as relays and switches [104]. In fact, many problems which seek to schedule, sequence, or time-table a set of events subject to given constraints can be modelled as graph colouring problems.

In this thesis, we focus on the vertex, edge, and total colouring problems. A *vertex $k$-colouring* of $G$ is an assignment of $k$ colours (i.e. the numbers $1, ..., k$) to the vertices of $G$ such that each vertex receives a colour and no two adjacent vertices receive the same colour. The *chromatic number*, denoted $\chi(G)$, is the smallest $k \geq 0$ such that $G$ has a vertex $k$-colouring. An *edge $k$-colouring* of $G$ is an assignment of the colours $\{1, ..., k\}$ to

[1] We use standard notation from graph theory and combinatorial optimization. We refer the reader to the books [12, 25, 101] for more information and the appendix for some notation.

the edges of $G$ such that each edge receives a colour and no two incident edges receive the same colour. The *chromatic index* of $G$, denoted $\chi'(G)$, is the smallest $k \geq 0$ such that $G$ has an edge $k$-colouring. A *total $k$-colouring* of $G$ is an assignment of the colours $\{1, ..., k\}$ to the vertices and edges of G such that each vertex and each edge receives a colour, no two adjacent vertices receive the same colour, no two incident edges receive the same colour, and no edge receives the same colour as one of its endpoints. The *total chromatic number* of $G$, denoted $\chi''(G)$, is the smallest $k \geq 0$ such that $G$ has a total $k$-colouring.



(a) A vertex 3-colouring (b) An edge 4-colouring (c) A total 5-colouring

Figure 1–1: A vertex, edge and total colouring of the Peterson graph.

Determining each of $\chi(G), \chi'(G)$ and $\chi''(G)$ is thought to be very difficult (i.e. NP-complete) [54, 63, 95]. Like all NP problems, they can be formulated as integer programs (IPs) and subjected to a two-pronged attack: we first solve the fractional relaxation and then use this solution to determine or obtain an approximation to the solution to the integer program. The first prong of this attack, namely, solving the fractional relaxation of the vertex, edge and total colouring IPs, is of central importance in this thesis. The complexity of the first two of these is known, while the third is currently unresolved.

In Section 1.1, we discuss the complexity of determining $\chi(G), \chi'(G)$ and $\chi''(G)$, establishing many key results which reappear throughout this thesis. We also discuss several results which yield simple bounds on each of these invariants. In Section 1.2, we expand on the two-pronged attack. In the remainder of this section, we outline the chapters of this thesis. The uninitiated reader may wish to read Sections 1.1 and 1.2 first.

In Chapter 2, we apply an iterative approach to vertex colouring. We discuss an algorithm which iteratively chooses a stable set containing the most uncoloured vertices. We show the number of uncoloured vertices decreases by a fixed proportion in each iteration. This algorithm will be used to show that the chromatic number of any $n$-vertex graph is at most $(1 + \ln n)$ times its fractional chromatic number. It will follow that it is NP-hard to closely approximate the fractional chromatic number of a graph. In contrast, we show that if a graph is perfect, then we can exploit this fact to efficiently find a stable set meeting all its maximum cliques. This allows us to iteratively optimally colour perfect graphs in polynomial time.

In Chapter 3, we discuss applying a similar iterative approach to edge colouring. Given a graph $G$, we pick a matching $M$ and repeat the approach on the graph $G - M$. Ideally, we want to choose a matching whose removal decreases the chromatic index by one. This approach works for bipartite graphs, since any such graph contains a matching which saturates each vertex of maximum degree. It follows that the chromatic index of any bipartite graph is equal to its maximum degree. As we discuss now, this is not the case for general graphs, and so, a more sophisticated approach is necessary.

Goldberg [40] and, independently, Seymour [103] conjectured that the chromatic index of any multigraph is at most 1 more than its fractional chromatic index. (Goldberg actually conjectured something slightly stronger, which we will discuss later.) More strongly, Seymour [103] speculated that a simple iterative approach should yield such an edge colouring. He conjectured that any graph $G$ whose fractional chromatic index $\beta$ is greater than 3 has a matching $M$ whose deletion leaves $G - M$ with fractional chromatic index at most $\lceil \beta \rceil - 1$. He proved that when $\beta \leq 6$, the chromatic index is at most $\lceil \beta \rceil + 1$, and so, this would imply the Goldberg and Seymour's conjecture. Unfortunately, Rizzi showed Seymour's stronger conjecture is false, giving for each integer $\beta > 3$ a construction of a $\beta$-regular graph $G$ whose fractional chromatic index is equal to $\beta$ and containing no such matching [94]. Each $\beta$-regular graph $G$ with fractional chromatic index equal to $\beta$ has a perfect matching $M$, and so, unlike iteratively edge colouring bipartite graphs, it is not enough to ensure the matching $M$ saturates each vertex of maximum degree. The key to understanding the additional properties $M$ must satisfy is Edmonds' linear description of the matching polytope [27] which yields that the fractional chromatic index of a graph $G$ is determined by its maximum degree $\Delta(G)$ and the following edge-dense subgraphs.

**Definition 1.1** (Overfull subgraph)**.** A subgraph $H$ of $G$ with $|H| > 1$ is *overfull* if $\frac{2|E(H)|}{|H|-1} > \Delta(G)$.

**Theorem 1.2.** *[Edmonds' Fractional Edge Colouring Theorem][27] For any graph G with maximum degree $\Delta(G)$, the fractional chromatic index is*

$$\chi'_f(G) = \max\left\{\Delta(G), \max_{H\subseteq G, |H|>1 \ odd} \frac{2|E(H)|}{|H|-1}\right\}.$$

Kahn [58] uses Edmonds' fractional edge colouring theorem to show Goldberg and Seymour's conjecture is asymptotically true. He avoids the problems which occur when removing a single matching by applying a more complicated iterative approach which removes a set of matchings from the graph. He shows that if a graph $G$ has large enough fractional chromatic index, then there exists a set of $N$ matchings ($N$ sufficiently large), such that by removing these matchings from $G$, the maximum degree is at most $\chi'_f(G)-(1-o(1))N$ and any odd subgraph $H$, $|H| > 1$, satisfies $\frac{2|E(H)|}{|H|-1} \leq \chi'_f(G) - (1-o(1))N$. These facts together with Theorem 1.2 imply the fractional chromatic index drops by $(1-o(1))N$.

The difficulty in dealing with overfull subgraphs iteratively is that you have to deal with them all at once. In Chapter 4, we turn to large degree graphs which are easier to handle because they do not have too many overfull subgraphs. We use this fact to describe a linear time algorithm to determine the fractional chromatic index of a graph with large maximum degree.

The difficulty in applying an iterative approach for graphs with large maximum degree is that eventually, we iterate out of the class (i.e. the maximum degree gets too small). One way to avoid this is by modifying the standard iterative approach to reduce to a different case. For example, Perkovic and Reed [89] give an algorithm which edge $\Delta(G)$-colours a subclass of $\Delta(G)$-regular graphs with large maximum degree by iteratively removing matchings until they reduce to a bipartite graph. The fact that any bipartite graph $H$ is edge $\Delta(H)$-colourable allows them to finish the colouring. Frieze, Jackson, McDiarmid, and Reed [33] give an algorithm which attempts to edge $\Delta(G)$-colour by iteratively removing matchings until they find a graph whose vertices of maximum degree induce a stable set. Fournier [32] showed that if the vertices of maximum degree in a graph $H$ induce an acyclic graph, then $H$ is edge $\Delta(H)$-colourable. This allows them to finish the colouring. In Chapter 7, we find fractional total colourings of graphs with large maximum degree and not containing overfull subgraphs by iteratively removing total stable sets until we reduce our problem to finding a fractional edge colouring of an auxiliary graph. We then finish the total colouring by applying fractional edge colouring techniques.

In Chapter 5, we discuss an alternate approach to total colouring. We first choose a vertex colouring and then choose an edge colouring which does not conflict it. It is believed that for any vertex $(\Delta(G)+3)$-colouring there exists an edge $(\Delta(G)+3)$-colouring which extends it and hence when combined with it yields a total $(\Delta(G)+3)$-colouring. A simple proof shows that for any vertex $(\Delta(G)+3)$-colouring there exists a fractional edge $(\Delta(G)+3)$-colouring which extends it. We discuss the Conformability Conjecture which gives necessary and sufficient conditions for a vertex $(\Delta(G)+1)$-colouring of a graph $G$

with large maximum degree $\Delta(G)$ to have an edge $(\Delta(G)+1)$-colouring which extends it. We make an analogous conjecture for fractional total colouring. This is related to the result of Chapter 7 which verifies a special case of this conjecture. In Chapter 6, we prove that a connected graph $G$ with maximum degree $\Delta(G)$ has fractional total colouring number equal to $\Delta(G)+2$ precisely when $G = K_{2n}$ or $G = K_{n,n}$ for some integer $n \geq 1$. To do so, we show that if $G$ is neither of these two graphs then a certain fractional vertex colouring can be extended to a fractional total colouring using less than $\Delta(G)+2$ colours.

In Chapter 8, we conclude by discussing directions for future research.

## 1.1 Determining and Bounding $\chi$, $\chi'$, and $\chi''$.

In this section, we discuss the complexity of determining the chromatic number, the chromatic index, and the total colouring number. Since determining each of these is thought to be very difficult, we then turn to simple bounds on each of these invariants. It is easy to prove that the chromatic number of a graph $G$ is at least the size of a largest clique in $G$, denoted $\omega(G)$, and at most one more than the maximum degree of $G$, denoted $\Delta(G)$. Neither of these bounds need be very tight, in fact, most $n$-vertex graphs $G$ satisfy $\omega(G) \leq 2\log n$, $\Delta(G) \geq \frac{n}{2}$ and have chromatic number approximately $\frac{n}{2\log n}$ (see, for example, [4]). In contrast, the chromatic index of any simple graph is always within one of its trivial lower bound, indeed, it is always at least $\Delta(G)$ and at most $\Delta(G)+1$. Similarly, the total colouring number is thought to be always within one of its trivial lower bound, specifically, it is known to be at least $\Delta(G)+1$, and it is conjectured to be at most $\Delta(G)+2$. In Section 1.2, we will further improve on these bounds by considering the two-pronged attack.

### 1.1.1 Vertex Colouring

A graph has chromatic number exactly zero precisely if it has no vertices, and chromatic number at most one precisely if it has no edges. A graph has chromatic number at most two precisely when it is bipartite. It is well-known that there exists an algorithm to determine if a graph is bipartite in linear time (see [12] for example). Progressing beyond this is thought to be difficult as Karp showed that for any $k \geq 3$, deciding if a graph is vertex k-colourable is NP-hard [63]. His proof implies that there does not exist any $\alpha$-approximation algorithm[2] for vertex 3-colouring with $\alpha < \frac{4}{3}$ unless P = NP. The theory of probabilistically checkable proofs (PCPs) is able to sharpen this result further. Indeed, Bellare, Goldreich, and Sudan [8], building upon the work of Lund and Yannakakis [76] and Fürer [35], showed that the PCP theorem implies:

**Theorem 1.3.** *[8] For any $\varepsilon > 0$, there does not exist any polynomial time $\alpha$-approximation for vertex colouring with $\alpha < n^{\frac{1}{7}-\varepsilon}$ unless P = NP.*

We remark that under the stronger assumption that NP $\neq$ ZPP, there does not exist any polynomial time $n^{1-\varepsilon}$-approximation for vertex colouring for any $\varepsilon > 0$ [66]. The best known positive result for approximating the chromatic number of a graph is by Halldórsson:

**Theorem 1.4.** *[47] There exists an algorithm which given any graph $G$ finds a vertex $\alpha \cdot \chi(G)$-colouring with $\alpha = O\left(\frac{n(\log\log n)^2}{\log^3 n}\right)$ in polynomial time.*

There are two very natural bounds on the chromatic number. For any clique $C$ and vertex colouring of a graph $G$, each vertex of $C$ must receive a unique colour and so $\chi(G) \geq |C|$. So, letting $\omega(G)$ be the size of the largest clique in a graph $G$, we have

**Observation 1.5.** *$\chi(G) \geq \omega(G)$.*

On the other hand,

[2] Specifically, for a class of minimization problems $\mathcal{P}$, such as vertex colouring, an $\alpha$-approximation algorithm for $\mathcal{P}$ is an algorithm which given $P \in \mathcal{P}$ returns a solution to $P$ with the value no more than $\alpha$ times the value of an optimal solution to $P$ in time polynomial in the size of $P$. The statement that there does not exist an $\alpha$-approximation algorithm for $\mathcal{P}$ unless P = NP should be interpreted as: it is NP-hard to determine whether given $P \in \mathcal{P}$ there exists a solution of value at most $\alpha$ times the optimum.

**Lemma 1.6** (Greedy colouring procedure)**.** *For each graph $G$ with maximum degree $\Delta(G)$, $\chi(G) \leq \Delta(G) + 1$.*

*Proof.* Order the vertices of $G$ arbitrarily as $v_1, ..., v_n$. We iteratively assign to a vertex $v_i$ in turn the smallest colour of $\{1, ..., \Delta(G)+1\}$ which is not assigned to any of its neighbours which appear earlier in this order. As $v_i$ has at most $\Delta(G)$ such neighbours, there always exists such an unassigned colour. □

Now by choosing the ordering of the vertices of $G$ carefully, one can show $\chi(G)$ achieves the upper bound of $\Delta(G) + 1$ in only two special cases.

**Theorem 1.7** (Brook's Theorem [13])**.** *For any graph $G$, $\chi(G) \leq \Delta(G)$ unless some component of $G$ is a clique on $\Delta(G) + 1$ vertices or $\Delta(G) = 2$ and some component of $G$ is a cycle with an odd number of vertices.*

### 1.1.2 Edge Colouring

Holyer showed that it is NP-complete to decide whether a graph is edge $\Delta(G)$-colourable for all $\Delta(G) \geq 3$ [54]. In fact, Hoyler's result implies for all $\varepsilon > 0$, there does not exist a $(\frac{4}{3} - \varepsilon)$-approximation algorithm for edge colouring unless P=NP. Though, as we discuss now, in contrast to the inapproximability of the chromatic number, the chromatic index is always within one of its trivial lower bound.

For each vertex $v$ and edge colouring of a graph $G$, the edges incident to $v$, denoted $\delta(v)$, must each receive a unique colour. Hence, if $G$ has maximum degree $\Delta(G)$ then $\chi'(G) \geq \Delta(G)$. Vizing's Theorem shows that this lower bound is never far from being correct:

**Theorem 1.8** (Vizing's Theorem)**.** *[110] For any graph $G$, $\chi'(G) \leq \Delta(G) + 1$.*

The proof of Vizing's Theorem applies an iterative approach with a similar flavour to that in the proof of Lemma 1.6. We iteratively assign colours to the edges of the graph. The difference is that we may need to change a few of the colours on already coloured edges to ensure each edge has an available colour. The proof yields an algorithm to construct an edge $(\Delta(G)+1)$-colouring of any simple $n$-vertex $m$-edge graph $G$ in $O(nm)$ time [32]. Gabow *et al.* have improved this runtime to $O(m\sqrt{n \log n})$ [37]. Relaxing the constraint that the algorithm must run in polynomial time, Beigel and Eppstein gave an $O(1.5039^n)$ time algorithm to determine if a graph has a edge 3-colouring [7]. Subsiquently, Eppstein has given an $O(2^{n/2})$ time algorithm for the same problem [29].

Polynomial time algorithms to find optimal edge colourings exist for specific, well structured classes of graphs. For example, we have already discussed an iterative approach for colouring bipartite graph. Alon gave an $O(m \log m)$ time algorithm to find such a colouring [3]. For any planar graph of maximum degree $\Delta(G) \geq 7$ its chromatic index is equal to $\Delta(G)$ [97, 109, 110], and Cole and Kowalik give a linear time algorithm for colouring planar graphs of maximum degree at least 9 [26].

### 1.1.3 Total Colouring

The total colouring number was introduced independently by Behzad [6] and Vizing [110]. As was the case with edge colouring, deciding whether or not $\chi''(G)$ is equal to $\Delta(G)+1$ is NP-hard, for $G$ with maximum degree $\Delta(G) \geq 3$. Interestingly, it is NP-hard even for $k$-regular bipartite graphs, $k \geq 3$ [95, 79].

For any vertex $v$ and total colouring of a graph $G$, the vertex $v$ and the edges incident to $v$ each receive a unique colour, and so, if $G$ has maximum degree $\Delta(G)$, then $\chi''(G) \geq \Delta(G)+1$. Analogous to Vizing's Theorem, the well-known Total Colouring Conjecture of Behzad [6] and Vizing [110] states that $\chi''(G)$ is one of two values:

**Conjecture 1.9** (Total Colouring Conjecture)**.** *For any graph $G$ of maximum degree $\Delta(G)$, $\chi''(G) \leq \Delta(G)+2$.*

As any bipartite graph can be edge $\Delta(G)$-coloured and vertex 2-coloured, it trivially follows that any bipartite graph has a total $(\Delta(G)+2)$-colouring. The Total Colouring

Conjecture is also known to hold for $r$-partite graphs, interval graphs, multigraphs of maximum degree at most 5, and planar graphs with maximum degree at least 7 (see, for example, [56]). Hilton and Hind showed the conjecture holds for graphs with sufficiently high maximum degree:

**Theorem 1.10.** *[49] If a graph $G$ has maximum degree $\Delta(G) \geq \frac{3}{4}|G|$, then $\chi''(G) \leq \Delta(G)+2$.*

McDiarmid and Reed [78] show there exists a constant $c < \sqrt{2}$ so that the number of simple graphs on $n$ vertices satisfying $\chi''(G) > \Delta(G)+2$ is at most $o(c^{n^2})$. Hence, almost all simple graphs satisfy $\chi''(G) \leq \Delta(G)+2$.

Kostochka showed for any multigraph with maximum degree $\Delta(G)$, $\chi''(G) \leq \left\lfloor \frac{3}{2}\Delta(G) \right\rfloor$ [69, 70, 71]. In a similar vien, Hind proved $\chi''(G) \leq \chi'(G) + 2\left\lceil \sqrt{\chi(G)} \right\rceil$ [51, 52] and Sánchez-Arroyo proved $\chi''(G) \leq \chi'(G) + \left\lfloor \frac{1}{3}\chi(G) \right\rfloor + 2$ [96]. Under the assumption that $t$ satisfies $t! > n$, McDiarmid and Reed [78] showed that $\chi''(G) \leq \chi'(G) + t + 1$. Hind [53] proved that $\chi''(G) \leq \Delta(G) + 2\left\lceil n/\Delta(G) \right\rceil + 1$. Chetwind and Häggkvist [16] proved that $\chi''(G) \leq \Delta(G) + 18\Delta(G)^{1/3}\log(3\Delta(G))$. Perhaps the strongest piece of evidence for the truth of the Total Colouring Conjecture is due to Molloy and Reed who proved:

**Theorem 1.11.** *[81] There exists an absolute constant $C$ such that for any graph $G$ of maximum degree $\Delta(G)$, $\chi''(G) \leq \Delta(G) + C$.*

## 1.2 The Two-Pronged Attack

Our main focus in this thesis is solving or obtaining an approximation to the solution of the fractional relaxations of the vertex, edge and total colouring integer programs. Though solving each of these is interesting in its own right, it is in combination with the second prong of the attack from which some of the most beautiful conjectures and results follow. In this section, we discuss applying this approach to the vertex, edge and total colouring problems. Whilst doing so, we sharpen the bounds given in Section 1.1 and overview the complexity of solving the fractional relaxations of the integer programs corresponding to each of these problems.

### 1.2.1 Fractional Vertex Colouring

A *stable set* of a graph $G$ is a subset $S$ of $G$ for which each pair of vertices in $S$ are nonadjacent. It is easy to see that in any vertex colouring a set of vertices receiving the same colour, or *colour class*, is a stable set of $G$. So, a vertex $k$-colouring of $G$ is a covering of the vertices of $G$ by $k$ stable sets, and can be formulated as the following IP:

$$\begin{array}{lll} \min & \mathbf{1}^T x & \\ \text{s.t.} & \sum_{S \ni v} x_S \geq 1 & \forall v \in G \\ & x \geq 0 & \\ & x \in \mathbb{Z}^{\mathcal{S}(G)}, & \end{array} \tag{1.1}$$

where $\mathcal{S}(G)$ is the collection of all stable sets of $G$. The chromatic number of $G$ is the optimal value of (1.1).

The *fractional chromatic number* is the optimal value to the following linear program (LP):[3]

$$\begin{array}{lll} \min & \mathbf{1}^T x & \\ \text{s.t.} & \sum_{S \ni v} x_S \geq 1 & \forall v \in G \\ & x \geq 0 & \\ & x \in \mathbb{R}^{\mathcal{S}(G)}. & \end{array} \tag{1.2}$$

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[3] The reader may wish to refer to [100] for equivalent definitions of the fractional chromatic number and other fractional invariants.

As is the case for the chromatic number, we have the following easily derived bounds on $\chi_f(G)$. Calling the linear program (1.2) the *primal*, its *dual* is

$$
\begin{array}{lll}
\max & \mathbf{1}^T y & \\
\text{s.t.} & \sum_{v \in S} y_v \leq 1 & \forall S \in \mathcal{S}(G) \\
& y \geq 0 & y \in \mathbb{R}^{V(G)}.
\end{array}
\tag{1.3}
$$

Now, for any primal feasible solution $x$ and dual feasible solution $y$, (1.2) and (1.3) imply: $\mathbf{1}^T x \geq \mathbf{1}^T y$. (This statement is normally referred as *weak duality.* The *(strong) duality theorem of linear programming* (see, for example, [101]) yields that if one of (1.2) and (1.3) is feasible and has a finite optimum then so does the other, where in fact, the two optima are equal).

Now, $z = (\frac{1}{\alpha(G)}, ..., \frac{1}{\alpha(G)})$ is a feasible solution to (1.3), and so weak duality implies $\chi_f(G) \geq \mathbf{1}^T z = \frac{|V(G)|}{\alpha(G)}$. Moreover, as this also holds for any subgraph of $G$, we have

**Observation 1.12.** $\chi_f(G) \geq \max_{H \subseteq G} \frac{|V(H)|}{\alpha(H)}$.

In particular, letting $\omega(G)$ be the size of a largest clique in $G$ and letting $H$ be a maximum size clique, we have

**Observation 1.13.** $\chi_f(G) \geq \omega(G)$.

Now, each feasible solution of the vertex colouring IP is feasible for the fractional vertex colouring LP, and so we have:

**Observation 1.14.** $\chi_f(G) \leq \chi(G)$.

This bound does not always hold with equality. To see this consider the cycle on five vertices $C_5$. It is easily seen that $\chi(C_5) = 3$. We find a fractional vertex $\frac{5}{2}$-colouring by assigning weight $x_S = \frac{1}{2}$ to each stable set $S$ of size two in $C_5$ and weight $x_{S'} = 0$ for each other stable set $S'$. As $\alpha(C_5) = 2$, Observation 1.12 implies $\chi_f(C_5) = \frac{5}{2}$.

As discussed in the introduction, Johnson [57], Lovász [73], and Chvátal [24] applied an iterative approach to show $\chi(G) \leq (1+\ln n)\chi_f(G)$. Unfortunately this is a double-edged sword, as this result together with the above inapproximability result for computing the chromatic number implies that it is also NP-hard to approximate the fractional chromatic number within a multiplicative factor of $n^{\frac{1}{7}-\varepsilon}$ for each $\varepsilon > 0$ [76]. We discuss these results in detail in Chapter 2.

The well-studied class of perfect graphs (see [2]) focuses on graphs with $\chi(G) = \chi_f(G) = \omega(G)$. A graph is *perfect* if for each induced subgraph $H$ of $G$, $\chi(H) = \omega(H)$. Motivated by the Shannon zero-error capacity of a graph [105], Claude Berge (see [1]) introduced the class of perfect graphs and posed two conjectures. The first was resolved by Lovász in 1972:

**Theorem 1.15** (Weak perfect graph theorem)**.** *[72] A graph $G$ is perfect if and only if $\overline{G}$ is perfect.*

The second Berge conjecture was proven true by Chudnovsky, Robertson, Seymour, and Thomas in 2002:

**Theorem 1.16** (Strong perfect graph theorem)**.** *[22] A graph is perfect if and only if it contains neither an odd chordless cycle of length at least five or the complement of such a cycle as an induced subgraph.*

Theorem 1.15 is the first step in showing there exists an algorithm to find an optimal vertex colouring of a perfect graph in polynomial time. Grötschel, Lovász and Schrijver [44] use an iterative approach together with semidefinite programming techniques and the ellipsoid method to do exactly this. We discuss their result in detail in Section 2.2. We remark that subsequently, Chudnovsky, Cornuéjols, Liu, Seymour, and Vušković [21] showed how to recognize perfect graphs in polynomial time.

### 1.2.2 Fractional Edge Colouring

In stark contrast to inapproximability of fractional vertex colouring, we can solve the fractional relaxation of the edge colouring IP in polynomial time. A *matching* of a graph

$G$ is a subset $M$ of $E(G)$ such that no two edges in $M$ share an endpoint. In any edge colouring, a set of edges receiving the same colour is a matching. So, an edge $k$-colouring of a graph $G = (V, E)$ is a covering of the edge set $E$ of $G$ with $k$ matchings of $G$. Hence, the chromatic index is the optimal value to the following IP:

$$\begin{array}{lll} \min & \mathbf{1}^T y & \\ \text{s.t.} & \sum_{M \ni e} y_M \geq 1 & \forall e \in E(G) \\ & y \geq 0 & \\ & y \in \mathbb{Z}^{\mathcal{M}(G)}, & \end{array} \tag{1.4}$$

where $\mathcal{M}(G)$ is the set of all matchings of $G$.

The *fractional chromatic index* of a graph $G$, denoted $\chi'_f(G)$, is the optimal value to the following LP:

$$\begin{array}{lll} \min & \mathbf{1}^T y & \\ \text{s.t.} & \sum_{M \ni e} y_M \geq 1 & \forall e \in E(G) \\ & y \geq 0 & \\ & y \in \mathbb{R}^{\mathcal{M}(G)}. & \end{array} \tag{1.5}$$

For any graph $G$, we can determine $\chi'_f(G)$ in polynomial time. Padberg and Rao [88] give a polynomial time algorithm to find a most odd overfull subgraph. This together with Edmonds' fractional edge colouring theorem yields a polynomial algorithm to determine the fractional chromatic index. Further, Edmonds' polynomial time algorithm to solve the maximum weight matching problem [28], together with the ellipsoid method [44], implies one can optimally fractionally edge colour a graph in polynomial time.

Trivially, $\chi'(G) \geq \chi'_f(G)$. Hilton conjectured that if the maximum degree is large then the fractional chromatic index should determine the chromatic index. His *Overfull Conjecture* states that for any simple graph $G$ with maximum degree $\Delta(G) > \frac{1}{3}|G|$, we have $\chi'(G) = \left\lceil \chi'_f(G) \right\rceil$ [18, 20]. If this conjecture were true, then for any such simple graph $G$, the algorithm of Chapter 4 would determine the chromatic index in linear time.

Goldberg conjectured that the chromatic index of any multigraph is at most the maximum of $\Delta(G)+1$ and $\left\lceil \chi'_f(G) \right\rceil$. This is often referred to as the *Goldberg-Seymour Conjecture*. If true, then the fractional chromatic index would approximate the chromatic index within an additive factor of 1. This has been attacked from many different angles. A long series of results [104, 5, 40, 41, 87, 107, 31, 99], has approached this conjecture by studying the weaker proposition that for each odd integer $m \geq 3$, every multigraph $G$ with $\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1}$ satisfies $\chi'(G) = \left\lceil \chi'_f(G) \right\rceil$. This starts with the classical result of Shannon [104] who showed $\chi'(G) \leq \frac{3}{2}\Delta(G)$. More recently Schiede [99] showed this was true for $m = 15$.

Approaching this conjecture from another angle, Plantholt showed if $G$ is an $n$-vertex multigraph then $\chi'(G) \leq \left\lceil \chi'_f(G) \right\rceil + \lceil n/8 \rceil - 1$ [90]. Plantholt showed if $G$ is an $n$-vertex multigraph, $n$ is even and $n \geq 572$, then $\chi'(G) \leq \left\lceil \chi'_f(G) \right\rceil + 1 + \sqrt{n \ln n/10}$ [91]. In the same article, he proved for any $\varepsilon > 0$, there exists an $N(\varepsilon) > 0$ such that $\chi'(G) \leq \left\lceil \chi'_f(G) \right\rceil + \varepsilon n$ whenever $n > N(\varepsilon)$ [91]. Sanders and Steurer proved that in time polynomial in the input size, they can find an edge $\beta$-colouring, where $\beta = \left(1 + \sqrt{\frac{4.5}{\chi'_f(G)}}\right) \chi'_f(G)$ [98]. Recently, Chen, Yu, and Zang [15] showed $\chi'(G) \leq \max\{\Delta(G) + \sqrt{\Delta(G)/2}, \left\lceil \chi'_f(G) \right\rceil\}$. Very recently, Plantholt [92] uses a complicated iterative approach in a preprint claiming $\chi'(G) \leq \left\lceil \chi'_f(G) \right\rceil + \log_{3/2}(\min\{(|G|+1)/3, \left\lceil \chi'_f(G) \right\rceil\})$.

### 1.2.3 Fractional Total Colouring

Unlike vertex colouring and edge colouring, the complexity of solving the fractional relaxation of the total colouring IP is still unresolved. A *total stable set* $T$ is a set of edges and vertices of $G$ such that $V(G) \cap T$ induces a stable set in $G$ and $E(G) \cap T$ induces a matching in $G - (V(G) \cap T)$. It is easy to see that each colour class of a total $k$-colouring is a total stable set, and so, a total $k$-colouring is a covering of $V(G) \cup E(G)$ by $k$ total stable sets. Hence, the total colouring number of a graph $G$ is the optimal value to the following IP:

$$\begin{array}{lll} \min & \mathbf{1}^T z & \\ \text{s.t.} & \sum_{T \ni u} z_T \geq 1 & \forall u \in V(G) \cup E(G) \\ & z \geq 0 & \\ & z \in \mathbb{R}^{\mathcal{T}(G)}, & \end{array} \tag{1.6}$$

where $\mathcal{T}(G)$ is the set of all total stable sets of $G$.

The *fractional total colouring number* of a graph $G$, denoted $\chi''_f(G)$, is the optimal value to the following LP:

$$\begin{array}{lll} \min & \mathbf{1}^T z & \\ \text{s.t.} & \sum_{T \ni u} z_T \geq 1 & \forall u \in V(G) \cup E(G) \\ & z \geq 0 & \\ & z \in \mathbb{R}^{\mathcal{T}(G)}, & \end{array} \tag{1.7}$$

where $\mathcal{T}(G)$ is the set of all total stable sets of $G$.

If one could optimize any linear weight function over the total stable set polytope $TP(G)$ (the convex combination of all incidence vectors of total stable sets of a graph), then one could determine the fractional total colouring number in polynomial time. This is not likely to be the case, since letting $w \in \mathbb{R}^{V \cup E}$ where $w_x = 1$ whenever $x \in V$ and $w_x = 0$ otherwise, we have that $\max\{w^T t : t \in TP(G)\}$ is equal to the size of a maximum stable set in $G$. It is known that approximating the size of a maximum stable set of an $n$-vertex graph to within a factor of $n^{\frac{1}{2}-\varepsilon}$, $\varepsilon > 0$, is NP-hard [106]. So, in contrast to the polyhedral approach for fractional edge colouring, we must consider a different approach.

On the other hand, we can approximate the fractional total colouring number of any multigraph within 2, because it exceeds the fractional chromatic index by at most that amount. Kilakos and Reed strengthened this result by showing that simple graphs have fractional total colouring number at most $\Delta(G) + 2$ [67], thereby proving a fractional analogue of the Total Colouring Conjecture. In Chapter 6, we sharpen their result, characterizing exactly those graphs which satisfy $\chi''_f(G) = \Delta(G) + 2$. In Chapter 7, we show graphs $G$ with $|G| \geq 320$, maximum degree $\Delta(G) \geq \frac{3}{4}|G|$, and containing no overfull subgraphs satisfy $\chi''_f(G) = \Delta(G)+1$. In fact, if the Overfull Conjecture is true, then the proof implies each such graph satisfies $\chi''(G) = \Delta(G) + 1$.

The Overfull Conjecture aims to describe graphs with large maximum degree whose chromatic index is $\Delta + 1$. An analogous conjecture for total colouring would state that if the maximum degree of a graph $G$ is large enough, then $\chi''(G) = \left\lceil \chi''_f(G) \right\rceil$. In Chapter 5, we disprove this conjecture showing there does not exist an $\varepsilon > 0$ such that each graph $G$ with $\Delta(G) \geq \varepsilon|G|$, satisfies $\chi''(G) = \left\lceil \chi''_f(G) \right\rceil$.

# CHAPTER 2
# An Iterative Approach to Vertex Colouring

In this chapter, we discuss an iterative approach to vertex colouring. Given a graph $G$, each of the methods we discuss chooses one stable set $S$ per iteration and then repeats the method on the graph $G - S$. We upper bound the total number of stable sets needed to cover the vertices of $G$ by ensuring a certain parameter is reduced in each iteration.

In Section 2.1, we discuss an algorithm which iteratively chooses a stable set containing the most uncoloured vertices. We show the number of uncoloured vertices decreases by a fixed proportion in each iteration. This algorithm will be used to show that the optimal value of the vertex colouring integer program is no more than $(1 + \ln n)$ times the value of its fractional relaxation. This small integrality gap together with the inapproximability of vertex colouring implies the inapproximability of the fractional relaxation of the vertex colouring IP.

In Section 2.2, we discuss colouring perfect graphs. In each iteration, we choose a stable set whose removal decreases the clique number by one. This leads to a polynomial time algorithm for colouring perfect graphs.

## 2.1 A Greedy Approach

Letting $G$ be an $n$-vertex graph, we apply the following iterative approach. For each iteration $i \geq 1$, let $U_i$ be the uncoloured vertices at the beginning of iteration $i$. (Initially, $U_1 = V(G)$.) We choose a stable set $S_i$ of $G$ to add to our colouring. We update $U_{i+1} = U_i - S_i$, and repeat.

We show that there exists a certain natural probability distribution which ensures that if $S$ is chosen from this distribution, then for any iteration $i$, $\mathbf{E}\{|U_i - S|\} \leq |U_i|\left(1 - \frac{1}{\chi_f(G)}\right)$. Hence, there exists a stable set $S'$ such that $|U_{i+1}| = |U_i - S'| \leq |U_i|\left(1 - \frac{1}{\chi_f(G)}\right)$ and we let $S_i = S'$. By choosing the stable sets $S_1, ..., S_i$ this way, it follows that the number of uncoloured vertices at the end of iteration $i$ is

$$|U_{i+1}| \quad \leq \quad |U_i|\left(1 - \frac{1}{\chi_f(G)}\right) \leq n\left(1 - \frac{1}{\chi_f(G)}\right)^i.$$

Hence, after $i^* = \lceil \ln n \cdot \chi_f(G) \rceil$ iterations, the stable sets $S_1, ..., S_{i^*}$ form a vertex colouring since the number of uncoloured vertices is at most

$$\begin{aligned} n\left(1 - \frac{1}{\chi_f(G)}\right)^{\lceil \ln n \cdot \chi_f(G) \rceil} &= n\left(\left(1 - \frac{1}{\chi_f(G)}\right)^{\chi_f(G)}\right)^{\frac{\lceil \ln n \cdot \chi_f(G) \rceil}{\chi_f(G)}} \\ &< ne^{-\ln n} = 1. \end{aligned}$$

This proves the following theorem.

**Theorem 2.1.** *[24, 57, 73] For any graph $G$, $\chi(G) \leq \lceil \ln n \cdot \chi_f(G) \rceil$.*

The key to defining this probability distribution is the following definitions and lemma.

**Definition 2.2.** For any set $S \subseteq V(G)$ define its *incident vector* $\chi^S \in \{0,1\}^{V(G)}$ by

$$\chi_v^S = \begin{cases} 1 & v \in S \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 2.3.** The *stable set polytope* of a graph $G$ is the convex hull of the incidence vectors of stable sets of $G$:

$$\mathrm{STAB}(G) = conv(\{\chi^S : S \in \mathcal{S}(G)\}).$$

**Lemma 2.4.** *Any graph $G$ has a fractional vertex $\beta$-colouring if and only if $\left(\frac{1}{\beta}, ..., \frac{1}{\beta}\right) \in STAB(G)$.*

*Proof.* $G$ has a fractional vertex $\beta$-colouring precisely when there exists a nonnegative weighting $w \in \mathbb{R}^{\mathcal{S}(G)}$ of stable sets such that $\sum_{S\in\mathcal{S}(G)} w_S = \beta$ and $\sum_{S\ni v} w_S = 1$ for all $v \in V(G)$. This is true precisely when $w$ satisfies $\sum_{S\in\mathcal{S}(G)} \frac{w_S}{\beta} = 1$, and $\sum_{S\in\mathcal{S}(G)} \frac{w_S}{\beta}\chi^S = \left(\frac{1}{\beta}, ..., \frac{1}{\beta}\right)$, that is when $\left(\frac{1}{\beta}, ..., \frac{1}{\beta}\right) \in$STAB$(G)$. $\square$

This argument shows that if $w$ is a fractional vertex $\chi_f(G)$-colouring, then $\frac{1}{\chi_f(G)}w$ is a probability distribution on the stable sets of $G$ such that if $S$ is a stable set drawn at random from this distribution, i.e. the probability we choose $S$ is $\frac{1}{\chi_f(G)}w_S$, then for each $v \in V(G)$,

$$\mathbf{P}\{v \in S\} = \sum_{S\ni v} \frac{1}{\chi_f(G)} w_S = \frac{1}{\chi_f(G)} \sum_{S\ni v} w_S \geq \frac{1}{\chi_f(G)}.$$

Hence, by linearity of expectation, the expected size of a stable set $S$ drawn at random from this distribution is

$$\mathbf{E}\{|S|\} = \sum_{v\in G} \mathbf{P}\{v \in S\} \geq \frac{|G|}{\chi_f(G)}.$$

More strongly, for any set $X \subseteq V(G)$, if $S$ is a stable set drawn at random from this distribution, then the expected number of vertices of $X$ in $S$ is

$$\mathbf{E}\{|X \cap S|\} = \sum_{v\in X} \mathbf{P}\{v \in S\} \geq \frac{|X|}{\chi_f(G)}.$$

In our iterative algorithm, $i \geq 0$, we choose the stable set $S_i$ from this distribution. Since $|U_{i+1}| = |U_i - S_i| = |U_i| - |U_i \cap S_i|$, it follows that

$$\mathbf{E}\{|U_{i+1}|\} \quad = \quad |U_i| - \mathbf{E}\{|U_i \cap S_i|\} \leq |U_i| \left(1 - \frac{1}{\chi_f(G)}\right),$$

as desired.

### 2.1.1 The Inapproximability of the Fractional Chromatic Number

Given $\chi_f(G)$ for a graph $G$, Theorem 2.1 allows us to approximately determine $\chi(G)$. More strongly, if there exists a polynomial time algorithm which approximately determines the fractional chromatic number within a multiplicative factor of $\alpha$, then Theorem 2.1 implies there exists a polynomial time which approximately determines the chromatic number within a multiplicative factor of $\lceil \ln n \cdot \alpha \rceil$. So, Theorem 1.3 implies that for all $\varepsilon > 0$, $\lceil \ln n \cdot \alpha \rceil > n^{\frac{1}{7}-\varepsilon}$. Since for all $\varepsilon > 0$ and $n$ large enough, $\left\lceil \ln n \cdot n^{\frac{1}{7}-2\varepsilon} \right\rceil < n^{\frac{1}{7}-\varepsilon}$, we have

**Theorem 2.5.** *[8] For each $\delta > 0$ there does not exist any polynomial time $\alpha$-approximation for fractional vertex colouring with $\alpha < n^{\frac{1}{7}-\delta}$ unless* $\mathrm{P} = \mathrm{NP}$.

Now, if we have a polynomial time algorithm to find a fractional vertex $\alpha$-colouring of $G$, then we can apply the iterative approach of the previous section to find a vertex $\lceil \ln n \cdot \alpha \rceil$-colouring of $G$ in polynomial time. We first apply the given algorithm to find a fractional vertex $\alpha$-colouring $x$. Since it runs in polynomial time, $x$ gives positive weight to at most a polynomial number of stable sets. We apply the iterative approach, picking a 'best' stable set deterministically in the following way. If $S_1, ..., S_{i-1}$ are the stable sets chosen so far, then we choose $S_i$ such that $|U_i - S_i| \leq |U_i| \left(1 - \frac{1}{\alpha}\right)$. The proof in the previous section can be easily modified to show that there always exists such a stable set given positive weight by $x$. Clearly, it is enough to choose $S_i$ to be a stable set with $x_{S_i} > 0$ and containing the most uncoloured vertices. So, we can choose $S_i$ in polynomial time by simply checking each stable set given positive weight.

Johnson [57] and Lovász [73] prove the following stronger result. Given a set of elements $\{1, ..., n\}$ and collection $\mathcal{C}$ of subsets of $\{1, ..., n\}$, a *set covering* is a subset $\mathcal{S} \subseteq \mathcal{C}$ such that each element is in some subset of $\mathcal{S}$. They show the total number of stable sets returned by the greedy heuristic, where at each iteration we choose a stable set of largest size, is at most $(1 + \ln \mu)$ times the minimum size of a set covering, where $\mu$ is the size of a largest subset. Chvátal [24] considered weighted set covering, where each set is given a nonnegative weight, and a greedy heuristic which at each iteration chooses a set with largest total weight. He proves that the total weight of stable sets returned by the (modified) greedy heuristic is at most $(1 + \ln \mu)$ times the minimum weight of a set covering.

## 2.2 A Perfect Approach

We remind the reader that a graph is *perfect* if for each induced subgraph $H$ of $G$, $\chi(H) = \omega(H)$, where $\omega(H)$ is the size of a largest clique in $H$. We note that Observations 1.13 and 1.14 imply $\omega(G) = \chi_f(G) = \chi(G)$. The main focus of the remainder of this chapter is the following result of Grötschel, Lovász and Schrijver.

**Theorem 2.6.** *[44] There exists a polynomial algorithm to find an optimal vertex colouring of a perfect graph.*

In proving Theorem 2.6 they apply the iterative approach. Letting $G$ be a perfect graph, we find a stable set $S$ of $G$ which intersects each maximum clique in $G$ in polynomial time. Clearly, $G-S$ is perfect. Moreover, $\omega(G-S) = \omega(G)-1$, and so, $\chi(G-S) = \chi(G)-1$. Hence, we can recursively apply the algorithm to find a vertex $(\omega(G)-1)$-colouring of $G-S$. Combining this colouring with $S$ yields a vertex $\omega(G)$-colouring of $G$. It follows easily that this algorithm runs in polynomial time.

In describing how to find such a stable set, we need the following consequences of the ellipsoid method which can be found in Grötschel, Lovász and Schrijver [44].

**Definition 2.7.** A *separation oracle* for a convex region $K \subseteq \mathbb{R}^n$ is a subroutine which given $y \in \mathbb{R}^n$ either decides that $y \in K$ or finds a vector $a \in \mathbb{R}^n$ and scalar $b \in \mathbb{R}$ such that for each $x \in K, a^T x \leq b$, but $a^T y > b$. Here $\{x \in \mathbb{R}^n : a^T x = b\}$ is called a *separating hyperplane.*

**Definition 2.8.** Let $\varphi$ be a positive integer. A *well-described polyhedron* is a triple $(P; n, \varphi)$ where $P \subseteq \mathbb{R}^n$ is a polyhedron such that there exists a system of inequalities with rational coefficients that has solution set $P$, i.e. $P = \{x : Ax \leq b\}$, and such that the encoding length of each inequality of the system is at most $\varphi$.

We consider a class $\mathcal{P}$ of well-described polyhedra for which we have a separation oracle that runs in time polynomial bounded in $n$ and $\varphi$. Grötschel *et al.* show there exists another polynomial time algorithm which given $P \in \mathcal{P}$ and $x^* \in P$ decomposes it as $x^* = \sum_{i=1}^{t} \lambda_i z_i$, where $z_1, ..., z_t$ are extreme points of $P$ and $\lambda_1, ..., \lambda_t$ are positive scalars with $\sum_{i=1}^{t} \lambda_i = 1$. Furthermore, Grötschel, Lovász and Schrijver show the equivalence of separation and optimization [43]. For our purposes, we need the following:

**Theorem 2.9.** *[44] Suppose we have a class $\mathcal{P}$ of well-described polyhedra for which we have a separation oracle that runs in time polynomially time. Then there exists an algorithm such that for any $c \in \mathbb{Z}^n$ and polyhedron $P \in \mathcal{P}$ finds an optimum extreme point solution of $P$ in a polynomial time.*

**Remark 2.10.** By polynomial time in Theorem 2.9, we mean time polynomially bounded in $n$ and $\varphi$.

We can now describe how to find a stable set of a perfect graph $G$ which intersects every maximum clique in $G$. We first determine $\omega(G)$:

**Theorem 2.11.** *[44] There exists a polynomial time algorithm to determine $\omega(G)$ for any perfect graph $G$.*

The first proof of this theorem uses semidefinite programming techniques to optimize over the *theta body* of a graph $H$, denoted $\mathrm{TH}(H)$. The theta body of a graph $H$, which was introduced by Lovász [74], is a convex body containing the stable set polytope $\mathrm{STAB}(H)$

and contained in QSTAB($G$), the *fractional stable set polytope*:

$$\text{QSTAB}(G) := \left\{ z \in R^{V(G)} : z \geq 0; \forall C \in \mathcal{C}(G), \sum_{v \in C} z_v \leq 1 \right\},$$

where $\mathcal{C}(G)$ is the collection of all cliques in $G$. Fulkerson [34], and later Chvátal [23], showed that for a perfect graph $H$, STAB($H$) = QSTAB($H$). Furthermore, Grötschel, Lovász and Schrijver [44] described a polynomial time separation oracle for a theta body. These remarks taken together imply we can optimize over STAB($H$) of a perfect graph $H$ in polynomial time. Now by Theorem 1.15, $\overline{G}$ is perfect and so we can find a maximum stable set of STAB($\overline{G}$), which is a maximum clique in $G$. This establishes Theorem 2.11.

Since $\chi(G) = \omega(G)$, Lemma 2.4 implies $\left(\frac{1}{\omega(G)}, ..., \frac{1}{\omega(G)}\right) \in$ STAB($G$). Hence, the Grötschel *et al.* algorithms above imply there exists a polynomial time algorithm which finds stable sets $S_1, ..., S_t$ and corresponding positive weights $\lambda_1, ..., \lambda_t$ such that $\left(\frac{1}{\omega(G)}, ..., \frac{1}{\omega(G)}\right) = \sum_{i=1}^{t} \lambda_i \chi^{S_i}$. Evidently, $\omega(G)\lambda$ is a fractional vertex $\omega(G)$-colouring of $G$. Furthermore, since the total weight of stable sets intersecting each maximum clique is exactly $\omega(G)$, each stable set $S_i$ such that $\lambda_{S_i} > 0$ must intersect each maximum clique of $G$. So, we can return any such $S_i$.

To finish this section, we sketch the proof of Theorem 2.11 in a way different from outlined above. Though we still use semidefinite programming, the algorithm we present here is due to Karger, Motwani and Sundan [62]. We follow the exposition of Reed in [93]. The key idea is that a perfect graph $G$ with $n$ vertices has a vertex $k$-colouring precisely when there exists a certain type of embedding of $G$ in $\mathbb{R}^n$. To explain this, we describe a relationship between vertex 4-colourings of a graph and embeddings of its vertex set in $\mathbb{R}^2$.

Given a graph $G$ with a vertex 4-colouring $S_1, S_2, S_3, S_4$, associate with each colour class exactly one vector from $(1,0), (-1,0), (0,1), (0,-1)$, and assign to each vertex $v_i \in V(G)$ the vector $x^i$ associated with its colour class. Notice, we have $x^i \cdot x^j \leq 0$ for each edge $v_iv_j \in E(G)$, and $x^i \cdot x^i = 1$ for each $v_i \in V(G)$.[1] Conversely, suppose there exists a set of vectors $x^1, .., x^n$ one for each vertex and all in $\mathbb{R}^2$ such that $x^i \cdot x^j \leq 0$ for each edge $v_iv_j \in E(G)$, and $x^i \cdot x^i = 1$ for each $v_i \in V(G)$. We first rotate this set of (finite) vectors so that no vector has a zero coordinate. (We note that such a rotation preserves the dot product of any two unit vectors.) It follows that any two vectors $x^i$ and $x^j$ in the same quadrant satisfy $x^i \cdot x^j > 0$. Since $x^i \cdot x^j \leq 0$ for any $v_iv_j \in E(G)$, it must be the case that $x^i$ and $x^j$ are in different quadrants. So, the vertices corresponding to vectors in one quadrant form a stable set. It follows that we can vertex 4-colour $G$.

As computing a vertex 4-colouring is NP-hard, the above approach is unlikely to produce an efficient algorithm to vertex 4-colour a graph. So, we consider the following *vector program* which is an approximate version of the problem. Define $\mu(G)$ to be the optimal value of

$$\begin{array}{lll} \min & \mu & \\ \text{s.t.} & x^i \cdot x^j \leq \mu & \forall i, j \text{ s.t. } v_iv_j \in E(G) \\ & x^i \cdot x^i = 1 & \forall i \\ & x^i \in \mathbb{R}^n & \forall i \end{array} \tag{2.1}$$

We will discuss in a moment that as a consequence of the ellipsoid method, we can approximate this vector program within an additive error of $\varepsilon$ in time polynomial in $\log(1/\varepsilon)$ and the number of constraints. From this we will be able to determine $\omega(G)$ since for perfect graphs we have:

**Lemma 2.12.** *[62] If $G$ is perfect then $\mu(G) = -\frac{1}{\omega(G)-1}$.*

[1] Here $x^i \cdot x^j$ denotes the dot product of the vectors $x^i = (x^i_1, ..., x^i_m)$ and $x^j = (x^j_1, ..., x^j_m)$ which equals $\sum_{k=1}^m x^i_k x^j_k$.

As $\omega(G)$ is an integer between 1 and $n$, the possible values for $\mu(G)$ are $-1, -\frac{1}{2}, ..., -\frac{1}{n-1}$, and so no two possible values are within $\frac{1}{n(n-1)}$. Hence, if we approximate this vector program to within an error of $\frac{1}{2n^2}$ then we can determine $\mu(G)$, and hence $\omega(G)$, exactly in polynomial time. So, we need only prove the lemma and discuss how to approximate (2.1). We start with the latter.

The same ideas as in Theorem 2.9 apply in a more general setting which allows us to solve (2.1). A matrix $A = \{a_{ij}\}$ is *symmetric positive semidefinite* if there exists vectors $v^1, ..., v^n$ each in $\mathbb{R}^n$ such that the entry in the $i$th row and $j$th column is given by $a_{ij} = v^i \cdot v^j$. Given a solution to (2.1) $x^1, ..., x^n$, we consider the $n \times n$ symmetric positive semidefinite matrix $M = \{m_{ij}\}$ where the entry in the $i$th row and $j$th column is given by $m_{ij} = x^i \cdot x^j$. So, $M$ is a feasible solution of the same value to the following *semidefinite program*:

$$\begin{array}{rll} \text{min} & \mu & \\ \text{where} & M = \{m_{ij}\} \text{ is positive semidefinite} & \\ \text{s.t.} & m_{ij} \leq \mu & \forall i, j \\ & m_{ij} = m_{ji} & \forall i, j \\ & m_{ii} = 1 & \forall i \end{array} \tag{2.2}$$

Conversely, given any symmetric positive semidefinite matrix $M$ which is a solution to (2.2), there exists an $n \times n$ matrix $U$ such that $U^T U = M$; this is called a *Cholesky decomposition of* $M$ (see [42]). So, if $M$ is a solution to (2.2) then the column vectors of $U$, $x^1, ..., x^n$, form a solution to (2.1) of the same value. It follows that a graph has a solution of value $k$ to (2.1) precisely when it has a solution of value $k$ to (2.2), and so we can focus on solving (2.2).

It turns out that if we have a polynomial time separation oracle for the feasible region $K$ of (2.2), then we can apply the ellipsoid method to approximate (2.2) within an additive error of $\varepsilon$ in time polynomial in $\log(1/\varepsilon)$ and $n$. The separation oracle is straightforward for the constraints $m_{ij} \leq \mu$, $m_{ij} = m_{ji}$, and $m_{ii} = 1$, as there are only a polynomial number of them. Grötschel, Lovász and Schrijver complete the separation oracle for $K$ by showing how to test if a matrix is positive semidefinite in polynomial time [44].

So given an approximate solution $M$ to (2.2), it remains to find an approximate solution to (2.1). For any symmetric positive semidefinite matrix $M$ one can find the Cholesky decomposition in $n^3/3$ floating point operations (see for example [42]). Unfortunately, this does not imply a polynomial time algorithm since the entries of $U$ may be irrational. On the other hand, for any $\delta > 0$ and symmetric positive semidefinite matrix $M$ there exists an algorithm which finds an $n \times n$ matrix $U$ such that $||U^T U - M||_\infty < \delta$ in time polynomial in $n$ and $\log(1/\delta)$ [42]. Since we are only interested in approximating (2.1) this is sufficient. We omit further details.

We now finish the details of Lemma 2.12.

*Proof of Lemma 2.12.* We prove the lemma in two steps:

1. $\left(\mu(G) \geq -\frac{1}{\omega(G)-1}\right)$. We show that for $n \geq k$ and any set $S$ of $k$ unit vectors in $\mathbb{R}^n$ there exists distinct $y^i$ and $y^j$ in $S$ such that $y^i \cdot y^j \geq -\frac{1}{k-1}$. In particular, if $\{x^1, ..., x^{\omega(G)}\}$ is a set of unit vectors corresponding to some $\omega(G)$ clique, then there exists distinct $x^i$ and $x^j$ in $S$ such that $x^i \cdot x^j \geq -\frac{1}{\omega(G)-1}$. Hence, $\mu(G) \geq -\frac{1}{\omega(G)-1}$. To prove the claim, we note that

$$0 \leq \left(\sum_i y^i\right)^2 = \sum_i y^i \cdot y^i + \sum_{i<j} 2y^i \cdot y^j = k + \sum_{i<j} 2y^i \cdot y^j,$$

   and so $-\frac{k}{2} \leq \sum_{i<j} y^i \cdot y^j$. Hence, we have the average $\binom{k}{2}^{-1} \sum_{i<j} y^i \cdot y^j \geq -\frac{1}{k-1}$. It follows there must exist a pair of vectors $y^i$ and $y^j$ which achieve as much as this average.

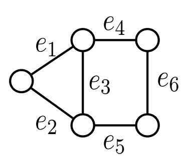
2. $\left(\mu(G) \leq -\frac{1}{\omega(G)-1}\right)$. We show for $n \geq k$ there are $k$ unit vectors $y^1, ...y^k$ in $\mathbb{R}^n$ such that for all $i \neq j$, $y^i \cdot y^j = -\frac{1}{k-1}$. Hence for $k = \chi(G)$, letting $S_1, ..., S_{\omega(G)}$ be an

optimal colouring of $G$, and assigning each vertex $v_i \in S_j$ the vector $x^i = y^i$, we have a feasible solution to (2.1) of value $-\frac{1}{\chi(G)-1}$. Hence, we have $\mu(G) \leq -\frac{1}{\chi(G)-1} = -\frac{1}{\omega(G)-1}$.

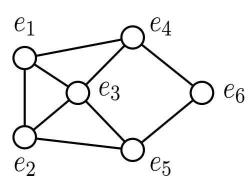
We define $y^1, ..., y^k$ in $\mathbb{R}^n$ as follows: for each $i$ between 1 and $k$, let $y^i$ have the first $i-1$ coordinates set to $\frac{1}{\sqrt{k(k-1)}}$, the $i$th coordinate set to $-\sqrt{\frac{k-1}{k}}$, the coordinates $i+1$ to $k$ set to $\frac{1}{\sqrt{k(k-1)}}$, and the remaining coordinates set to 0. It is straightforward to verify that these satisfy the desired condition.

This completes the proof of of Lemma 2.12. □

To finish this chapter, we remark that there is no direct combinatorial algorithm to vertex colour perfect graphs. For this reason, much effort has been expended in finding efficient combinatorial algorithms for subclasses of perfect graphs. For example, linear time algorithms exist to find optimal vertex colourings of bipartite graphs, interval graphs, and chordal graphs. Another class of perfect graphs, which is related to our discussion in Chapter 3, is the class of line-graphs of bipartite graphs. For a graph $G$, the *line graph*, $L(G)$, of $G$ is the graph whose vertex set corresponds to the edge set of $G$, two vertices of which are adjacent precisely if the corresponding edges are incident in $G$ (see Fig. 2–1).




(a) $G$




(b) $L(G)$

Figure 2–1: A graph $G$ and its line graph $L(G)$.

**Observation 2.13.** *The set of edge $k$-colourings of $G$ are in one-to-one correspondence with the set of vertex $k$-colourings of $L(G)$ and so $\chi'(G) = \chi(L(G))$.*

**Observation 2.14.** *If $G$ is bipartite, then $\omega(L(G)) = \Delta(G)$.*

As we will see in Chapter 3, König's edge colouring theorem states that if $G$ is bipartite then $\chi'(G) = \Delta(G)$. As any induced subgraph of a bipartite graph is bipartite, it follows easily that $G$ is perfect. Furthermore, we describe a polynomial time algorithm to find an edge $\Delta(G)$-colouring of $G$ from which an optimal vertex colouring of $L(G)$ is easily derived. For more information about colouring perfect graphs we refer the interested reader to [2, 77].

# CHAPTER 3
## Matchings and Edge colouring

In this chapter, we apply an iterative approach to the edge colouring problem. Given a graph $G$, we pick a matching $M$ and repeat the approach on the graph $G - M$. Ideally, we want to choose $M$ so that $\chi'(G - M) = \chi'(G) - 1$.

This approach works for any bipartite graph, since in any such graph there always exists a matching which saturates each vertex of maximum degree. This result relies on Hall's Theorem which describes the structure of bipartite graphs without large matchings. We discuss this in Section 3.1. Before turning to the general case, we extend this to describe the structure of graphs without large matchings in Section 3.2. We discuss a result, which will be used in later chapters, describing when there exists a matching saturating a subset of vertices in a graph.

As discussed in the introduction, the *Goldberg-Seymour Conjecture* states that for any multigraph $G$, $\chi'(G) \leq \max\{\Delta(G) + 1, \left\lceil \chi'_f(G) \right\rceil\}$. Seymour [103] conjectured that for any graph $G$, $\chi'(G) \leq 1 + \left\lceil \chi'_f(G) \right\rceil$ and proved that this is true when $\left\lceil \chi'_f(G) \right\rceil \leq 6$. He further conjectured that any graph $G$ with $\left\lceil \chi'_f(G) \right\rceil \geq 4$ has a matching $M$ whose deletion leaves $G - M$ satisfying $\left\lceil \chi'_f(G) \right\rceil = \left\lceil \chi'_f(G - M) \right\rceil + 1$ [103]. Unfortunately, Rizzi disproved this second conjecture giving for each integer $r > 3$ a construction of an $r$-regular graph $G$ with $\chi'_f(G) = r$ and not containing such a matching [94]. Any such graph has a perfect matching, and so unlike iteratively edge colouring bipartite graphs, it is not enough to saturate the vertices of maximum degree in $G$. Edmonds described the additional properties a matching must satisfy. The key is his characterization of the matching polytope, which we describe and prove in Section 3.3. He used this characterization to show that the fractional chromatic index is determined by a graph's maximum degree and odd overfull subgraphs. We remind the reader that a subgraph $H$ of $G$ is *odd overfull* if $H$ is odd and $\frac{2|E(H)|}{|H|-1} > \Delta$.

**Remark 3.1.** Most of the literature refers to these subgraphs as *overfull.* Odd overfull subgraphs are an essential tool for fractional edge colouring. We warn the reader that for fractional total colouring it will be important to consider both even and odd overfull subgraphs.

Edmonds' fractional edge colouring theorem [27] (see Theorem 1.2) states that for any graph $G$,

$$\chi'_f(G) = \max\left\{\Delta, \max_{H \subseteq G, |H|>1 \text{ odd}} \frac{2|E(H)|}{|H|-1}\right\}.$$

In Section 3.4, we discuss why Edmonds' maximum matching algorithm together with the ellipsoid method yields an algorithm to determine the fractional chromatic index in polynomial time.

We remind the reader of the iterative approach for colouring perfect graphs described in Section 2.2. Here $\chi(G)$ is equal to the size of a maximum clique $G$ and the colouring algorithm chooses a stable set $S$ which intersects each maximum clique of $G$. Hence, the removal of $S$ decreases the chromatic number by 1. In a similar way, the Goldberg-Seymour Conjecture together with Edmonds' fractional edge colouring theorem describes the properties a matching $M$ would need to satisfy to ensure that the upper bound in the Goldberg-Seymour Conjecture drops by 1.

In Section 3.5, we show, following Kahn, that the iterative approach asymptotically works. He used it to prove:

**Theorem 3.2.** *[58] For any multigraph $G$, $\chi'(G) \leq (1 + o(1))\chi'_f(G)$.*

Kahn applies a more complicated iterative procedure which selects a set of $N$ matchings to remove from the graph. Kahn ensures that by doing so the fractional chromatic index drops by $(1 - o(1))N$. To do so, he uses Edmonds' fractional edge colouring theorem and focuses on both the maximum degree and the value of $\frac{2|E(H)|}{|H|-1}$ for each odd overfull subgraph $H$.

We finish this section with some standard definitions. A *matching* $M$ of a graph $G$ is a subset of the edge set $E$ such that no two edges in $M$ share an endpoint. A matching $M$ *saturates* a subset $X$ of $V$ if $X \subseteq V(M)$. If a vertex $v$ is not an endpoint of some edge in a matching $M$, we say $M$ *misses* $v$. A matching is *perfect* if it saturates $V(G)$, and *near perfect* if it saturates $V(G) - v$ for some vertex $v \in V(G)$. $G$ is *factor-critical* if for each vertex $v \in G$, $G - v$ has a perfect matching. A *vertex cover* is a subset of vertices $S$ such that each edge of $G$ has at least one endpoint in $S$. We refer the reader to Lovász and Plummer's book *Matching Theory* [75] for further definitions. We note that all the results in this chapter hold for both graphs and multigraphs.

### 3.1 Hall's Theorem and Edge Colouring Bipartite Graphs

In this section, we show that for any bipartite graph $G$ of maximum degree $\Delta$, we have $\chi'(G) = \Delta$ and that we can find an edge $\Delta$-colouring in polynomial time. We apply the iterative approach. We show $G$ contains a matching $M_\Delta$ saturating each vertex of maximum degree in $G$, and then iterate on the bipartite graph $G - M$. We start by developing some classical tools.

König's minimax theorem states that for any bipartite graph $G$, the size of a maximum matching in $G$ is equal to the minimum size of a vertex cover in $G$ (see for example [75]). A consequence of this theorem is Hall's Theorem for bipartite matching.

**Theorem 3.3** (Hall's Theorem [46])**.** *A bipartite graph $G$ with bipartition $(A, B)$ has a matching saturating $A$ precisely if $|N(X)| \geq |X|$ for each $X \subseteq A$.*

*Proof.* If there exists $X \subseteq A$ such that $|N(X)| < |X|$ clearly no matching can saturate $X \subseteq A$. Conversely, if no matching saturates $A$, then there exists a vertex cover $S$ of size less than $|A|$. Evidently, $N(A - S) \subseteq S \cap B$, and so, $|N(A - S)| \leq |S \cap B| < |A - S|$. $\square$

We will repeatedly use the following consequence of Hall's Theorem and the lemma which flows from it.

**Corollary 3.4.** *Let $G$ be a bipartite graph. Then $G$ has a matching saturating each vertex of maximum degree $\Delta$.*

**Lemma 3.5.** *Let $G$ be a bipartite graph. Then there exists a polynomial time algorithm to find a matching saturating each vertex of maximum degree.*

We will prove these in a moment, we first show how to use them to describe the algorithm to edge $\Delta$-colour a bipartite graph.

**Theorem 3.6.** *There exists an algorithm which given a bipartite graph $G$ of maximum degree $\Delta$ finds an edge $\Delta$-colouring in polynomial time.*

*Proof.* We apply the iterative approach. If $E(G) = \emptyset$ return $\emptyset$. Otherwise, let $M_\Delta$ be a matching of $G$ saturating each vertex of maximum degree in $G$. Recursively apply the algorithm to find an edge $(\Delta - 1)$-colouring $M_1, ..., M_{(\Delta-1)}$ of $G - M_\Delta$. Return the edge colouring $M_1, ..., M_{(\Delta-1)}, M_\Delta$.

Clearly, this algorithm recursively calls itself $\Delta$ times. By Lemma 3.5, if $G$ is simple, then $\Delta \leq |G|$, and so, this algorithm runs in polynomial time. If $G$ is a multigraph, then assume $G$ is given as a simple graph with edge multiplicities $\mu$. In each iteration, for the matching $M_\Delta$, we let $r$ be the minimum multiplicity of any edge in $M_\Delta$ and remove $r$ copies of it. Since every copy of some edge is removed, this algorithm recursively calls itself at most $|G|^2$ times. □

A corollary of Theorem 3.6 is the following well-known theorem.

**Corollary 3.7** (König's Edge Colouring Theorem)**.** *[68] For any bipartite graph $G$ of maximum degree $\Delta$, $\chi'(G) = \Delta$.*

We now prove Corollary 3.4 and Lemma 3.5.

*Proof of Corollary 3.4.* Let $G$ have bipartition $(A, B)$. For any subset $X$ of $A$ containing only vertices of degree $\Delta$, $|N(X)| \geq |X|$, otherwise some vertex in $N(X)$ has degree greater than $\Delta$. Hence, if $A_\Delta$ is the vertices of degree $\Delta$ in $A$, then Hall's Theorem implies the bipartite graph induced by $A_\Delta \cup B$ has a matching saturating each vertex of $A_\Delta$. Hence, $G$ has a matching $M_A$ such that $V(M_A) \cap A$ is exactly the vertices of degree $\Delta$ in $A$. Similarly, $G$ has a matching $M_B$ such that $V(M_B) \cap B$ is exactly the vertices of degree $\Delta$ in $B$. Each component of the graph induced by the edges of $M_A \cup M_B$ is either a path or an even cycle. We claim that for each component we can choose a matching saturating all vertices of degree $\Delta$ in that component. The union of all such matchings is the desired matching.

Let $C$ be some component. If $C$ is an even cycle, then $M_A \cap E(C)$ saturates each vertex of $V(C)$. Otherwise, if $C$ is a path, then by the way we chose $M_A$ and $M_B$, each vertex which is a internal vertex of this path has degree $\Delta$ in $G$. Since the edges in a path are alternately in $M_A$ and $M_B$, and paths containing an even number of edges start and end in the same side of the partition, it follows that if $C$ is a path starts and ends on a vertex of degree $\Delta$, then it contains an odd number of edges. Hence, for any component $C$ which is a path, one of $M_A \cap E(C)$ and $M_B \cap E(C)$ saturates each vertex degree $\Delta$ in $C$. $\square$

*Proof of Lemma 3.5.* We assign to each edge $e$ weight $w_e \in \{0, 1, 2\}$ equal to the number of endpoints of $e$ which have degree $\Delta$ in $G$. We find a maximum weight matching $M$ in $G$ with respect to these edge weights. Clearly, $M$ saturates the maximum number of vertices of degree $\Delta$ in $G$, and so by Corollary 3.4, $M$ is the desired matching. Since it is well known that one can find a maximum weight matching of a graph in polynomial time (see for example [102]), this completes the proof. $\square$

As a final remark, we note that if $G$ has bipartition $(A, B)$ and there exists a set $X \subseteq A$ such that $|N(X)| < |X|$, then any matching in $G$ can saturate at most $|A| - (|X| - |N(X)|)$ vertices of $A$. In fact, by choosing $X = A - S$ for a minimum vertex cover $S$, König's minimax theorem implies:

**Corollary 3.8.** *The number of edges in a maximum matching of a bipartite graph $G$ with bipartition $(A, B)$ is* $\min_{X \subseteq A}(|A| - (|X| - |N(X)|))$.

In the next section, we discuss the Tutte-Berge formula which is an extension of this corollary to general graphs.

## 3.2 The Tutte-Berge Formula

Analogous to the proof of Theorem 3.6, we present an iterative method for total colouring in Chapter 7. Both of these methods require finding matchings with 'good' properties. In this section, we discuss our main tool for finding these matchings, the Tutte-Berge formula.

For any graph $H$, let $\mathrm{oc}(H)$ be the number of odd components in $H$.

**Theorem 3.9** (Tutte-Berge formula)**.** *[9, 108] The number of edges in a maximum matching of a graph $G = (V, E)$ is*

$$\frac{1}{2} \min_{K \subseteq V}(|G| - (\mathrm{oc}(G - K) - |K|)).$$

The Gallai-Edmonds structure theorem (see for example [75]) is an extension of the Tutte-Berge formula, characterizing a canonical decomposition with respect to a maximum matching. We omit a description of this decomposition. We instead turn to describing the weaker properties which we will use in our iterative methods.

It is clear that the Tutte-Berge formula extends Hall's Theorem, indeed, we often use the following special case of the Tutte-Berge formula:

**Theorem 3.10** (Tutte's theorem)**.** *[108] If a graph $G$ has no perfect matching, then there exists a set $K \subseteq G$ such that $G - K$ has more than $|K|$ odd components.*

If $K$ is chosen to be maximal, that is, there does not exist $K'$ with $K \subset K'$ such that $G - K'$ has more than $|K'|$ odd components, then we have the following stronger statement:

**Corollary 3.11.** *If a graph $G$ has no perfect matching, then there exists a set $K \subseteq V(G)$ such that $G - K$ has $r \geq |K| + 1$ odd components $O_1, ..., O_r$ each of which is factor-critical and, in addition, $G = K \cup \bigcup_{i=1}^{r} O_i$.*

*Proof.* Let $K$ as in the corollary statement be chosen to be maximal, and let $O_1, ..., O_r$ be the odd components of $G - K$, and $E_1, ..., E_t$ be the even components of $G - K$. Now, if $E_j$ is an even component of $G - K$, then for any vertex $v \in E_j$, $E_j - v$ contains an odd component $O$. So, $G - (K \cup \{v\})$ has at least $|K \cup \{v\}| + 1$ odd components $O_1, ..., O_r, O$. Since this contradicts the maximality of $K$, each component is odd.

If some $O_j$ is not factor-critical, then there exists a vertex $v \in O_j$ satisfying $O_j - v$ does not have a perfect matching. Hence, the Tutte-Berge Formula implies the subgraph induced by $O_j - v$ contains a set $K'$ such that $(O_j - v) - K'$ has $\ell \geq |K'| + 1$ odd components $O'_1, ..., O'_\ell$. Since $|O_j - v|$ is even, a simple parity count implies that $\ell \geq |K'| + 2$. So, $K' \cup \{v\}$ is a subset of $O_j$ such that $O_j - (K' \cup \{v\})$ has at least $|K' \cup \{v\}| + 1$ odd components. We now have that $K \cup (K' \cup \{v\})$ contradicts the maximality of $K$. Hence, $O_j$ is factor-critical. This completes the proof of the corollary. $\square$

Similar to finding $M_A$ in the proof of Corollary 3.4, we also need a tool for finding a matching which saturates a given set. We use the following corollary:

**Corollary 3.12** (see Exercise 3.1.8 in [75])**.** *For any graph $G$ and $Z \subseteq V(G)$ either $G$ has a matching saturating $Z$ or there exists $K \subseteq V(G)$ such that $G - K$ has at least $|K| + 1$ odd components completely contained in $Z$.*

### 3.3 The Matching Polytope and Edmonds' Characterization

We now turn to a polyhedral approach to edge colouring. The main tool is Edmonds' linear description of the matching polytope, which we will introduce in a moment. We apply this characterization in Section 3.4 to fractional edge colouring.

Given any nonnegative weight vector $w \in \mathbb{R}^E$ of a graph $G = (V, E)$, a *maximum weight matching* is a matching $M$ maximizing $\sum_{e \in M} w_e$. Edmonds described the first polynomial time algorithm for finding a maximum weight matching in a graph [28]. This, his famous blossom algorithm, was implemented by Gabow with runtime $O(n(m+n \log n))$ [36].

Edmonds used his algorithm to prove the following.

**Definition 3.13.** The *matching polytope* of a graph $G$ is the convex hull of the incidence vectors of matchings of $G$:

$$\mathrm{MP}(G) = conv(\chi^M : M \in \mathcal{M}(G)),$$

where $\mathcal{M}(G)$ is the set of all matchings of $G$.

**Theorem 3.14** ([27])**.** *For any graph $G$, the matching polytope is equal to:*

$$\left\{ x \in \mathbb{R}^{E(G)} \quad s.t. \quad \begin{array}{cc} \sum_{e \in \delta(v)} x_e \leq 1 & \forall v \in G \\ \sum_{e \in E(H)} x_e \leq \frac{|H|-1}{2} & \forall H \subseteq G, |H| \ odd \\ x_e \geq 0 & \forall e \in E(G). \end{array} \right\}. \tag{3.1}$$

We prove this theorem now, following the approach of [101]. We first consider the perfect matching polytope

$$\mathrm{P}(G) = conv(\chi^M : M \in \mathcal{M}(G), M \text{ perfect}).$$

Edmonds gave the following characterization:

**Theorem 3.15** ([27]). *For any graph $G$, the perfect matching polytope is equal to:*

$$\left\{ x \in \mathbb{R}^{E(G)} \quad s.t. \quad \begin{array}{cc} \sum_{e \in \delta(v)} x_e = 1 & \forall v \in V \\ \sum_{e \in \delta(H)} x_e \geq 1 & \forall H \subseteq G, |H| \ odd, |H| > 1 \\ x_e \geq 0 & \forall e \in E(G). \end{array} \right\}. \tag{3.2}$$

This theorem yields Theorem 3.14, as we prove later.

*Proof.* It is easy to see that the incidence vector of any perfect matching satisfies the inequalities of (3.2), hence so does any convex combination. Proving that for any $x \in \mathbb{R}^{E(G)}$ satisfying the inequalities of (3.2) we have $x \in \mathrm{P}(G)$ is much more interesting.

Assume that $G$ is a minimum counter-example to the statement. Specifically, there must exist an extreme point $x$ of the polytope (3.2) such that $x \notin \mathrm{P}(G)$. Out of all such counter-examples choose $G$ minimizing $|G| + |E(G)|$. By minimality, we have that $G$ is connected and $x_e \in (0, 1)$ for each edge $e \in E(G)$. Since, if $G$ is not connected, then one of its components is a counter-example. If $x_e = 0$ for some edge $e \in E(G)$, then $G - e$ is a counter-example, and if $x_{uv} = 1$, where $uv \in E(G)$, then $G - u - v$ is a counter-example. Furthermore, $|E(G)| \geq |G| + 1$. Since $G$ is connected and so if $|E(G)| \leq |G|$, then $G$ either contains a vertex of degree 1 or is an even cycle. If it contains a vertex $v$ of degree 1, then necessarily $\sum_{e \in \delta(v)} x_e = 1$ and so the unique edge $e \in \delta(v)$ satisfies $x_e = 1$, a contradiction. If $G$ is an even cycle, then there exists an $\alpha \in (0, 1)$ such that $x = \alpha\chi^{M_1} + (1 - \alpha)\chi^{M_2}$ for the two perfect matchings $M_1$ and $M_2$ in $G$, contradicting the fact that $x$ was an extreme point of (3.2).

Since $x$ is an extreme point of (3.2), it satisfies $|E(G)| \geq |G| + 1$ of the constraints with equality. So for at least one odd $H$, $|H| > 1$, we have $\sum_{e \in \delta(H)} x_e = 1$. This allows us to split $G$ and $x$ into two smaller examples using the following contraction operation:

**Definition 3.16.** Define $G/H$ to be the graph on vertex set $V(G-H) \cup \{h\}$ such that for each edge $uv \in E(G)$ such that $u, v \in G - H$ there exists an edge $uv \in E(G/H)$ and for each edge $yz \in E(G)$ such that $y \in H$ and $z \in G - H$ there exists an edge $hz \in E(G/H)$.

Consider the graph $G_1 = G/H$ and let $x^1 \in \mathbb{R}^{E(G_1)}$ be the projection of $x$ onto $G_1$, that is, for each edge $uv \in E_G(G-H)$, $x^1_{uv} = x_{uv}$, and for each edge $yz \in \delta_G(H)$, with $y \in H$ and $z \in G - H$, $x^1_{hy} = x_{yz}$. Since $\sum_{e \in \delta(h)} x_e = \sum_{e \in \delta(H)} x_e = 1$, it is easily checked that $x^1$ satisfies (3.2) for $G_1$, and so since $G$ was a minimum counter-example, $x^1$ is a convex combination of perfect matchings in $G_1$. In a similar way, define $G_2 = G/(G-H)$ and $x^2 \in \mathbb{R}^{E(G_2)}$ to be the projection of $x$ onto $G_2$. For the same reasons, $x^2$ is a convex combination of perfect matchings in $G_2$. It is now a routine matter to combine these two convex combinations to show that $x$ must be a convex combination of perfect matchings in $G$. As similar ideas will show up in the next chapter, we complete the details.

As $x^1 \in \mathrm{P}(G_1)$, there exists a list of perfect matchings $M^1_1, ..., M^1_s$ of $G_1$ and corresponding rational constants $\lambda^1_1, ..., \lambda^1_s$, such that $x^1 = \sum_{i=1}^{s} \lambda^1_i \chi^{M^1_i}$. Similarly as $x^2 \in \mathrm{P}(G_2)$, we have $x^2 = \sum_{j=1}^{t} \lambda^2_j \chi^{M^2_j}$ for perfect matchings $M^2_1, ..., M^2_t$ of $G_2$ and corresponding rational constants $\lambda^2_1, ..., \lambda^2_t$. Now, by allowing these two lists of perfect matchings $M^1_1, ..., M^1_s$ and $M^2_1, ..., M^2_t$ to have repetitions, we can assume that these lists have the same size, i.e. $s = t = K$, and $\lambda^1_1 = ... = \lambda^1_s = \lambda^2_1 = ... = \lambda^2_t = \frac{1}{K}$.

Let $e$ be an edge of $\delta(H)$. If $M^1$ is a perfect matching of $G_1$ containing $e^1$, the image of $e$ in $E(G_1)$, and $M^2$ is a perfect matching of $G_2$ containing $e^2$, the image of $e$ of $E(G_2)$, then $M^1 \cup M^2$ is a perfect matching of $G$. Now, since $\sum_{M^1_i \ni e^1} \lambda^1_i = \sum_{M^2_j \ni e^2} \lambda^2_j = x_e$, there exists exactly $Kx_e$ matchings $M^1_i$ containing the edge $e^1$ and $Kx_e$ matchings $M^2_j$ containing the edge $e^2$. Hence, by relabelling if necessary, we have that $M^1_1 \cup M^2_1, ..., M^1_K \cup M^2_K$ are perfect matchings in $G$. Furthermore, $x = \sum_{i=1}^{K} \frac{1}{K} \chi^{M^1_i \cup M^2_i}$, which implies that $x \in \mathrm{P}(G)$. This contradiction finishes the proof. $\square$

*Proof of Theorem 3.14.* Again proving that any $x \in \mathrm{MP}(G)$ satisfies these inequalities is easy as they are satisfied for the incidence vector of any matching. The interesting part is the other direction.

Let $x \in \mathbb{R}^{E(G)}$ satisfy (3.1). Construct the auxiliary graph $G'$ by taking two copies of $G$, denoted by $G_1$ and $G_2$, where for each vertex $v \in G$, its two copies $v_1$ and $v_2$ are adjacent in $G'$. We define a new vector $x' \in \mathbb{R}^{E(G')}$ and show that it satisfies (3.2) for $G'$. For each edge $e \in E(G)$ let its two copies be $e_1$ and $e_2$, and set $x'_{e_1} = x'_{e_2} = x_e$. For each vertex $v \in G$, let $x'_{v_1v_2} = 1 - x(\delta(v))$. Clearly, for each vertex $v \in V(G')$, $\sum_{e\in\delta(v)} x'_e = 1$. Moreover, as $x$ satisfies (3.1), we have that $x' \geq 0$. To show that $x'$ satisfies (3.2) in $G'$, it remains to show that for $H \subseteq G'$, $|H|$ is odd and $|H| > 1$, we have $\sum_{e\in\delta(H)} x'_e \geq 1$.

Fix such a subgraph $H$. Let $H_1$ be the set of vertices $v_1$ of $V(H) \cap V(G_1)$ such that $v_2 \notin V(H)$, and let $H_2$ be the set of vertices $v_2$ of $V(H) \cap V(G_2)$ such that $v_1 \notin V(H)$. Since one of $|H_1|$ or $|H_2|$ is odd, assume without loss of generality it is $|H_1|$. We have

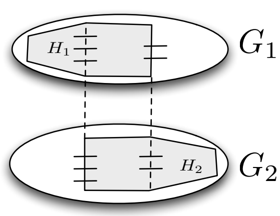



Figure 3–1: The graph $G'$ with subgraphs $H$, $H_1$, and $H_2$.

$\sum_{e\in\delta(v)} x'_e = 1$, and so,

$$\sum_{e\in\delta(H_1)} x'_e = \sum_{v\in V(H_1)} \left( \sum_{e\in\delta(v)} x'_e - \sum_{e\in\delta(v)\cap E(H_1)} x'_e \right) = |H_1| - 2 \sum_{e\in E(H_1)} x'_e.$$

Hence, as $x$ satisfies (3.1), $\sum_{e\in\delta(H_1)} x'_e = |H_1| - 2\sum_{e\in E(H_1)} x'_e \geq 1$. Furthermore, since each edge of $\delta(H_1)$ is either in $\delta(H)$ or it is $e_1 \in E(H \cap G_1)$ and its copy $e_2$ is in $\delta(H)$, we have $\sum_{e\in\delta(H)} x'_e \geq \sum_{e\in\delta(H_1)} x'_e \geq 1$. It follows that $x'$ satisfies (3.2) in $G'$, and so by Theorem 3.15, $x' \in \mathrm{P}(G')$. So, there exists $\lambda \geq 0$ such that $\mathbf{1}^T\lambda = 1$ and $x' = \sum_{M \in \mathcal{M}(G'), M \text{ perfect}} \lambda_M \chi^M$. Now, since for each perfect matching $M$ in $G'$ we have a matching in $G_1$, given by $M \cap E(G_1)$, it follows that $x = \sum_{M \in \mathcal{M}(G'), M \text{ perfect}} \lambda_M \chi^{M \cap E(G_1)}$. Hence, $x \in \mathrm{MP}(G_1)$, and since $G_1 = G$ we have $x \in \mathrm{MP}(G)$. □

## 3.4 The Fractional Chromatic Index

We remind the reader that a *fractional edge $\beta$-colouring* of a graph $G = (V, E)$ is a solution of value $\beta$ to the following LP:

$$\begin{array}{lll} \min & \mathbf{1}^T y & \\ \text{s.t.} & \sum_{M \ni e} y_M \geq 1 & \forall e \in E \\ & y \geq 0, & \\ & y \in \mathbb{R}^{\mathcal{M}(G)}. & \end{array} \tag{3.3}$$

The *fractional chromatic index of* $G$, $\chi'_f(G)$, is the smallest $\beta \geq 0$ such that there exists a fractional edge $\beta$-colouring of $G$. Clearly, as every integral edge $\beta$-colouring is a fractional edge $\beta$-colouring we have $\chi'_f(G) \leq \chi'(G)$.

The dual of (3.3) is:

$$\begin{array}{lll} \max & \mathbf{1}^T z & \\ \text{s.t.} & \sum_{e \in M} z_e \leq 1 & \forall M \in \mathcal{M}(G) \\ & z \geq 0 & \\ & z \in \mathbb{R}^{E}. & \end{array} \tag{3.4}$$

Weak duality immediately yields the following lower bounds:

**Observation 3.17.** *For any graph $G$ of maximum degree $\Delta$, $\chi'_f(G) \geq \Delta$.*

*Proof.* For any vertex $v \in V$, let $z^v_e = 1$ if $e \in \delta(v)$ and $z^v_e = 0$ otherwise. Since, $z^v$ is dual feasible, $\chi'_f(G) \geq \max_{v \in G} \mathbf{1}^T z^v = \max_{v \in G} |\delta(v)| = \Delta$. □

**Observation 3.18.** *For any graph $G$, $\chi'_f(G) \geq \frac{|E(G)|}{\lfloor |G|/2 \rfloor}$.*

*Proof.* This follows since any matching contains at most $\lfloor |G|/2 \rfloor$ edges, $z = \left( \frac{1}{\lfloor |G|/2 \rfloor}, ..., \frac{1}{\lfloor |G|/2 \rfloor} \right)$ is dual feasible. $\square$

Furthermore, since $\chi'_f(G) \geq \chi'_f(H)$ for any subgraph $H$ of $G$:

**Observation 3.19.** *For any graph $G$,*

$$\chi'_f(G) \geq \max_{H \subseteq G, |H|>1} \frac{|E(H)|}{\lfloor |H|/2 \rfloor}.$$

Now, if $|H|$ is even, then $\frac{|E(H)|}{\lfloor |H|/2 \rfloor} = \frac{\sum_{v \in V} |\delta(v)|}{|H|} \leq \Delta$ and we arrive at the following lower bound:

**Observation 3.20.**

$$\chi'_f(G) \geq \max \left\{ \Delta, \max_{H \subseteq G, |H|>1 \ odd} \frac{2|E(H)|}{|H|-1} \right\}.$$

Edmonds' characterization of the matching polytope implies that this lower bound is exactly the right value for the fractional chromatic index. To see this, we have as in Lemma 2.4:

**Lemma 3.21.** *There exists a fractional edge $\beta$-colouring precisely when $\left( \frac{1}{\beta}, ..., \frac{1}{\beta} \right)$ is in the matching polytope.*

With this in hand, we are able to prove Edmonds' fractional edge colouring theorem.

*Proof of Theorem 1.2.* By Lemma 3.21, $\chi'_f(G)$ is the smallest positive $\beta$ such that $\left( \frac{1}{\beta}, ..., \frac{1}{\beta} \right) \in$ MP$(G)$. Reformulating using Theorem 3.14, $\chi'_f(G)$ is the smallest $\beta > 0$ such that

1) $\beta^{-1}|\delta(v)| \leq 1$, $\forall v \in G$, and
2) $\beta^{-1}|E(H)| \leq \frac{|H|-1}{2}$, $\forall H \subseteq G, |H|$ odd, $|H| > 1$.

Clearly, 1) is satisfied precisely when $\beta \geq \Delta$ and 2) is satisfied precisely when for each odd $H$, $|H| > 1$, $\beta \geq \frac{2|E(H)|}{|H|-1}$. Hence, the lemma follows. $\square$

We now turn to applying these results to determine $\chi'_f(G)$ and find an optimal edge colouring. One way to compute $\chi'_f(G)$ is to computing $\Delta$ and an odd overfull subgraph $H$ maximizing $\frac{2|E(H)|}{|H|-1}$. Computing $\Delta$ is an easy matter. On the other hand, as there is an exponential number of odd subgraphs, determining the latter value is more involved. Padberg-Rao give a method for doing this [88].

We finish off this section by showing how to fractionally edge colour in polynomial time by applying the ellipsoid method. On one hand, this method seems less desirable as it leaves us with little combinatorial insight into the difficulty of constructing an optimal fractional edge colouring. On the other hand, this method emphasizes the importance of Edmonds' characterization of the matching polytope and the maximum matching algorithm.

We prove:

**Theorem 3.22.** *[84] There exists an algorithm such that given any graph $G$ determines the fractional chromatic index and an optimal fractional edge colouring in polynomial time.*

The theorem relies on the following consequence of the ellipsoid method:

**Theorem 3.23.** *[44] There exists an algorithm that, for any $c \in \mathbb{Z}^n$ and well-described polyhedron $(P; n, \varphi)$ for which we have a polynomial time separation oracle, either:*

*i) finds an optimum dual solution which is an extreme point of the dual polytope to $P$, or*

*ii) asserts that the dual problem is unbounded or has no solution,*

*in a polynomial time.*

*Proof of Theorem 3.22.* It is enough to show we can separate over a polytope of the form (3.4) in polynomial time. Let $z$ be a vector in $\mathbb{R}^{E(G)}$. We can clearly check if $z \geq 0$ in polynomial time. Hence, we need only check if for each matching $M \in \mathcal{M}(G)$, we have $\sum_{e \in M} z_e \leq 1$. By Edmonds' blossom algorithm [28], we can find a maximum weight matching $M'$ with edge weights $z$ in polynomial time. Now, if $\sum_{e \in M'} z_e \leq 1$, then $z$ is in the polytope of (3.4); otherwise, $\sum_{e \in M'} z_e > 1$ and so $\sum_{e \in M'} z_e \leq 1$ is our violated constraint. $\square$

### 3.5 Approximating the Chromatic Index

A remarkable conjecture of Goldberg [40] and independently Seymour [103] states that for any multigraph $G$,

$$\chi'(G) \leq \max\{\Delta(G) + 1, \lceil \chi'_f(G) \rceil\}.$$

In this section, we discuss applying the iterative approach to prove this conjecture. Very recently, Plantholt [92] uses a complicated variant of this approach in a preprint claiming $\chi'(G) \leq \left\lceil \chi'_f(G) \right\rceil + \log_{3/2}(\min\{(|G|+1)/3, \left\lceil \chi'_f(G) \right\rceil\})$. As discussed in the introduction, it is difficult to ensure that right-hand side of the upper bound in the Goldberg-Seymour conjecture drops by 1. In particular, a graph $G$ may not contains a matching $M$ for which the chromatic index of $G - M$ is one less than the chromatic index of $G$.

Kahn's proof of Theorem 3.2 also uses a sophisticated variant of the iterative approach together with Edmonds' theorem for edge colouring. Kahn gets around the problem encountered when choosing a single matching by choosing a large set of matchings to remove via the probabilistic method. In particular, he finds a set of $N$ matchings of the graph such that when we remove this set of matchings the fractional chromatic index drops by about $N$. We now turn to a sketch of Kahn's result, following the approach of Molloy and Reed [82].

Kahn proves:

**Lemma 3.24.** *$\forall \varepsilon > 0$, $\exists C(\varepsilon) > 0$ such that if $\chi'_f(G) > C(\varepsilon)$ then we can find $N = \left\lfloor \chi'_f(G)^{\frac{3}{4}} \right\rfloor$ matchings $M_1, ..., M_N$ each in $\mathcal{M}(G)$ and such that $\chi'_f(G - \bigcup_{i=1}^{N} M_i) \leq \chi'_f(G) - (1+\varepsilon)^{-1} N$.*

The theorem follows easily from this lemma.

*Proof of Theorem 3.2.* Fixing $\varepsilon > 0$, we claim that for the $C(\varepsilon)$ of Lemma 3.24, that $\chi'(G) \leq (1+\varepsilon)\chi'_f(G) + C(\varepsilon)$. As this is true for all $\varepsilon > 0$, the theorem follows.

If $\chi'_f(G) \leq C(\varepsilon)$ then by Shannon's Theorem above and the fact that $\chi'_f(G) \geq \Delta$, $\chi'(G) \leq \frac{3}{2}\Delta \leq 2\chi'_f(G) \leq \chi'_f(G) + C(\varepsilon)$. So, we assume that each multigraph $H$ with $\chi'_f(H) < B$, $B > C(\varepsilon)$ we have $\chi'(H) \leq (1+\varepsilon)\chi'_f(H) + C(\varepsilon)$ and prove the claim by induction on $B$. Now, if $\chi'_f(G) \geq B$ then by Lemma 3.24, there exists a set of matchings $M_1, ..., M_N$ such that $\chi'_f(G - \bigcup_{i=1}^N M_i) \leq \chi'_f(G) - (1+\varepsilon)^{-1}N$. Hence, by induction

$$\begin{aligned}\chi'(G) &\leq \chi'(G - \bigcup_{i=1}^{N} M_i) + N \\ &\leq (1+\varepsilon)\chi'_f\left(G - \bigcup_{i=1}^{N} M_i\right) + N + C(\varepsilon) \\ &\leq (1+\varepsilon)\chi'_f(G) + C(\varepsilon).\end{aligned}$$

The claim now follows. □

We now briefly outline the proof of Lemma 3.24. Fix $\varepsilon > 0$, and let $c^* = \chi'_f(G) - (1+\varepsilon)^{-1}N$. Theorem 3.3 implies that a set of matchings $M_1, ..., M_N$ satisfies $G' = G - \bigcup_{i=1}^N M_i$ has chromatic index at most $c^*$ precisely when

(A) $\forall v \in G'$, $d_{G'}(v) \leq c^*$

and

(B) $\forall H \subseteq G'$ with $|H|$ odd, $|E(H)| \leq \left(\frac{|H|-1}{2}\right) c^*$.

Kahn actually insists on a strengthening of (A) which allows a significant weakening of (B). Specifically, he ensures the following holds:

(C) $\forall v \in G'$, $d_{G'}(v) \leq c^* - N\varepsilon/4$.

The advantage to this strengthening is that if (C) holds then any odd subgraph failing to satisfy (B) has at most $\frac{\Delta}{N\varepsilon/4}$ vertices. Furthermore, it is easy to see that if a subgraph violates (B), then it has a connected subgraph which also violates (B). So given (C), to ensure (B) we need only ensure the following holds:

(D) $\forall$ odd connected $H \subseteq G'$ with $|H| \leq \frac{\Delta}{N\varepsilon/4}$: $|E(H)| \leq \left(\frac{|H|-1}{2}\right) c^*$.

Kahn's approach to showing that (C) and (D) hold is probabilistic. If one considers an optimal fractional colouring, and associated probability distribution $\lambda$ as in Lemma 3.21, then an easy expected value computation shows that every vertex v satisfies the bound of (C) with positive probability, and every H satisfies the bound of (D) with positive probability. The difficulty in the proof is that we need to handle all the vertices and small odd subsets of vertices at once. Part of Kahn's approach to doing so, is replacing (B) by (D), which greatly reduces the number of overfull subgraphs we have to consider. He also shows that if $\lambda$ arises from an optimal or near-optimal fractional colouring then each vertex satisfies (C) with probability which is close to 1 (not just greater than 0) and that every $H$ satisfies D with probability which is close to 1 (not just greater than 0). Finally, he considers $\lambda$ arising from a nearly optimal fractional colouring which has a special structure (it is a hardcore distribution). This ensures that the event that a particular edge is in a random matching is nearly independent of our choice of the matching in far apart parts of the graph. This allows him to combine the Lovász Local Lemma with the fact that (C) is very likely to hold for a specific $v$ and (D) is very likely to hold for a specific $H$ to show that with positive probability (C) holds for all $v$ and (D) holds for all $H$. Summarizing, each vertex and overfull subgraph is easy to handle, and the key to Kahn's proof is exploiting their properties to handle all of them at once.

# CHAPTER 4
# Determining the Fractional Chromatic Index

In this chapter, we discuss two methods for determining the fractional chromatic index. The first method follows from the work of Padberg and Rao.

**Theorem 4.1.** *[88] There exists an algorithm which given a multigraph $G$ with maximum degree $\Delta(G)$, determines $\chi'_f(G)$ in $O(n^4(\log n + \log \Delta(G))^2)$ time.*

The key to the proof of Theorem 4.1 is a method for determining if a graph contains an odd overfull subgraph, and the key to this method is the following observation.

**Observation 4.2.** *[Cut conditions for overfull subgraphs] A subgraph $F$ of $G$ is overfull precisely when*

$$|\delta(F)| + \sum_{v \in F} (\Delta(G) - d(v)) < \Delta(G).$$

To illustrate this, we consider finding an odd overfull subgraph in the special case when $G$ is $\Delta(G)$-regular, that is, each vertex $v \in G$ satisfies $|\delta(v)| = \Delta(G)$. For any subgraph $H$ of $G$, the summation in Observation 4.2 is exactly 0, and so,

**Observation 4.3.** *A subgraph $H$ of $G$ is odd overfull precisely when $|H|$ is odd and $|\delta(H)| < \Delta(G)$.*

We focus on determining if $G$ contains an odd subgraph $H$ satisfying $|\delta(H)| < \Delta(G)$. We will prove that if $F$ is a subgraph of $G$ minimizing $|\delta(F)|$ and $H$ is an odd subgraph of $G$ satisfying $|\delta(H)| < \Delta(G)$, then there exists some odd subgraph $H'$ with $|\delta(H')| < \Delta(G)$ completely contained either in $F$ or $G - F$. So, in searching for an odd overfull subgraph, we first find a subgraph $F$ minimizing $|\delta(F)|$. If $|F|$ is even then recursively check $F$ and

$G - F$. If $|F|$ is odd then $F$ is odd overfull whenever $|\delta(F)| < \Delta(G)$ (since $G$ is $\Delta(G)$-regular and so $|F| > 1$), otherwise, $G$ contains no odd overfull subgraph. As finding $F$ minimizing $|\delta(F)|$ is the polynomial solvable minimum cut problem this can be done in polynomial time. We describe the Padberg-Rao method in Section 4.2. In Section 4.2.2, we show how to apply their method to prove Theorem 4.1.

We now turn to the second method for determining the fractional chromatic index. We show that by considering the cut condition and the intersection patterns of overfull subgraphs, we can determine the chromatic index for (simple) graphs with large maximum degree in linear time. We prove:

**Theorem 4.4.** *For all $\gamma > 0$, there exists an algorithm which given a graph $G$ with maximum degree $\Delta$ satisfying $\Delta \geq \gamma|G|$ determines $\chi'_f(G)$ in $O(|G| + |E(G)|)$ time.*

The cut condition ensures that in any odd overfull subgraph $H$, most vertices of $H$ have many neighbours in $H$ while most vertices outside of $H$ have few neighbours in $H$. This allows us to show that if $G$ has high maximum degree $\Delta$ then it has very few odd overfull subgraphs. In particular, if $\Delta \geq \frac{1}{2}|G|$, then there is only one, while if $\Delta \geq \gamma|G|$, then there are at most $O\left((3\left\lceil\gamma^{-1}\right\rceil^2)^{\left\lceil\gamma^{-1}\right\rceil}\right)$. We prove Theorem 4.4 by finding all odd overfull subgraphs $H$ of $G$. Since it is trivial to calculate $\Delta$ in linear time, it follows that given the set of all odd overfull subgraphs for $G$, we can apply Theorem 1.2 to calculate $\chi'_f(G)$ in linear time.

We henceforth assume that $G$ is an $n$-vertex $m$-edge graph of maximum degree $\Delta$. In Section 4.3, we start by presenting some key observations used in our algorithm. We illustrate how to use these observations in Section 4.3.1 by describing a linear time algorithm to determine $\chi'_f(G)$ given a graph $G$ of maximum degree $\Delta \geq \frac{n}{2}$. In Section 4.3.2, we prove Theorem 4.4 in its full generality. In the remainder of this section, we overview some results and conjectures related to Theorem 4.4.

In related work, Niessen [85, 86] showed that when $\Delta \geq \frac{n}{2}$, $G$ contains at most 1 odd overfull subgraph and gave a linear time algorithm to find it if it exists. In addition, when $\Delta > \frac{n}{3}$ he showed that $G$ contains at most 3 odd overfull subgraphs and gave an $O(n^{5/3}m)$ time algorithm to find them if they exist.

The following conjecture of Hilton implies that for any high degree simple graph, Theorem 4.4 computes the chromatic index in linear time.

**Conjecture 4.5** (Hilton's Overfull Conjecture)**.** *[18, 20] Any simple graph $G$ with maximum degree $\Delta > \frac{1}{3}|G|$ satisfies $\chi'(G) = \left\lceil \chi'_f(G) \right\rceil$.*

We would like to apply an iterative approach to prove Hilton's conjecture and actually find an optimal colouring in high degree graphs. The difficulty is that eventually the maximum degree will drop below $\frac{1}{3}|G|$ and the conjecture no longer holds. The smallest such example is the Peterson graph minus one vertex $P'$ (depicted in Figure 4–1). (It is

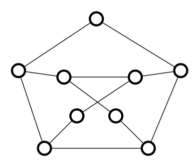


Figure 4–1: The Peterson graph minus one vertex.

easy to verify that $P'$ contains no odd overfull subgraph, but has chromatic index equal to 4. Since $\Delta = 3 = \frac{1}{3}|P'|$, this example implies the bound on $\Delta$ in Hilton's conjecture cannot be lowered further.) So, if we are to apply an iterative approach to prove Hilton's conjecture, then it will need to be modified in order to avoid this problematic situation. One way to do so is to iteratively reduce the input graph to a base case which is easily handled. We now turn to two related results which do exactly this.

The first is Perkovic and Reed's attack on the following special case of Hilton's Overfull Conjecture [89].

**Conjecture 4.6.** *(see [17]) If $G$ is a $\Delta$-regular simple graph with an even number of vertices at most $2\Delta$, then $G$ is edge $\Delta$-colourable.*

It is easily checked that a graph $G$ as in the conjecture can contain no odd overfull subgraphs. Perkovic and Reed show that if a graph $G$ satisfies the conditions of the conjecture, has large enough maximum degree, and contains no

(i) bipartite subgraph $H$ such that each vertex $v \in H$ satisfies $d_H(v) \geq \Delta - \Delta^{39/40}$, or

(ii) subgraph $H$ such that $|G| - |H| \geq \Delta - \Delta^{39/40}$ and each vertex $v \in H$ satisfies $d_H(v) \geq \Delta - \Delta^{39/40}$,

then there exists a set of matchings $M_1, ..., M_k$ such that $G - \bigcup_{i=1}^{k} M_i$ is a bipartite graph of maximum degree $\Delta - k$. Here we can handle the base case, since by König's Theorem, $\chi'(G - \bigcup_{i=1}^{k} M_i) = \Delta(G - \bigcup_{i=1}^{k} M_i) = \Delta - k$. It follows that the conjecture holds in this case.

In the introduction, we mentioned that Frieze, Jackson, McDiarmid, and Reed [33] showed that the proportion of $n$-vertex graphs with chromatic index $\Delta + 1$, denoted $p_n$, satisfies $n^{-(1/2+\varepsilon)n} < p_n < n^{-(1/8-\varepsilon)n}$. To do so, they give a polynomial time algorithm which attempts to edge $\Delta$-colour a graph following a similar approach. Letting $Z$ be the set of vertices of maximum degree in a graph $G$, the algorithm tries to remove a set of matchings $M_1, ..., M_k$ such that $Z$ is a stable set in $G' = G - \bigcup_{i=1}^{k} M_i$, $\Delta(G') = \Delta - k$, and the only vertices of maximum degree in $G'$ are $Z$. Since Fournier [32] showed that a graph $G'$ whose vertices of maximum degree form a stable set is edge $\Delta(G')$-colourable. It follows that if $M_1, ..., M_k$ exist, then $G$ is edge $\Delta$-colourable. Frieze *et al.* show that the probability this approach works on a random graph $G_{n,p}$ is at least $1 - O(\exp\{-\frac{1}{2}cn \log n\})$ where $c < \min\{p/2, 1/3\}$. This improved on a result of Erdös and Wilson [30] who showed almost every graph has chromatic index $\Delta$.

## 4.1 Proof of Theorem 4.1

We describe a polynomial time algorithm to determine the fractional chromatic index of any mutligraph. We assume that we are given a multigraph $G$ where the number of parallel edges between any two vertices is encoded as an edge multiplicity vector $\mu \in \mathbb{Z}^{E(G)}$. Since for any edge $e \in E(G)$, $\mu_e \leq \Delta(G)$, one can determine $\Delta(G)$ in $O((n+m)\log \Delta(G))$ time by summing up the multiplicities of the edges adjacent to each vertex. Hence, by Theorem 1.2 the following lemma allows us to determine $\chi'_f(G)$ in $O(n^4(\log n + \log \Delta(G))^2)$ time.

**Lemma 4.7.** *[88] There exists an algorithm which given a multigraph $G$ with maximum degree $\Delta(G)$, either returns an odd overfull subgraph $F$ satisfying $\frac{2|E(F)|}{|F|-1} = \chi'_f(G)$, or asserts that $G$ contains no odd overfull subgraph in $O(n^4(\log n + \log \Delta(G))^2)$ time.*

As outlined in the introduction, the key to the proof of Lemma 4.7 is Padberg and Rao's method for determining if a graph contains an odd overfull subgraph. We now turn to a description of their method.

## 4.2 The Padberg-Rao Method

In this section, we describe Padberg and Rao's method for finding an odd overfull subgraph in polynomial time.

**Lemma 4.8.** *[88] There exists an algorithm, denoted* $\mathrm{OF}(G)$*, which given a multigraph $G$ with maximum degree $\Delta(G)$, either returns an odd overfull subgraph or asserts that no such subgraph exists in $O(n^4 \log \Delta(G))$ time.*

We mimic the $\Delta(G)$-regular case sketched in the introductory section. We construct an auxiliary graph which is nearly $\Delta(G)$-regular. Let $G'$ be the graph built by taking a copy of $G$ plus a vertex $v'$ such that for each $u \in G$, there are $\Delta(G) - d(u)$ edges between $u$ and $v'$ (see Figure 4–2).

**Observation 4.9.** *Each vertex $v \in G' - v'$ has degree $\Delta(G)$ in $G'$.*

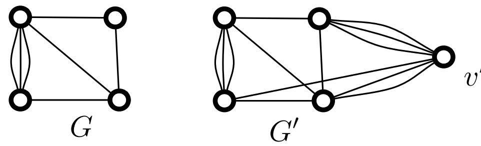



Figure 4–2: A multigraph $G$ and the multigraph $G'$ built from it.

**Observation 4.10.** *Suppose $H'$ is a subgraph of $G'$ not containing $v'$ and $H$ is the subgraph of $G$ induced on the vertices of $H'$. Then*

$$|\delta_{G'}(H')| = |\delta_G(H)| + \sum_{v \in H} (\Delta(G) - d_G(v)).$$

Now similar to the $\Delta(G)$-regular case, in order to determine if $G$ contains an odd overfull subgraph it is enough to look for an odd subgraph $H'$ of $G' - v'$ with $|\delta_{G'}(H')| < \Delta(G)$:

**Observation 4.11.** *$G$ contains an odd overfull subgraph precisely when $G'$ contains an odd subgraph $H'$ not containing $v'$ and satisfying $|\delta_{G'}(H')| < \Delta(G)$.*

*Proof.* By Observation 4.2, $H$ is an odd overfull subgraph of $G$ precisely when $|\delta_G(H)| + \sum_{v \in H}(\Delta(G) - d_G(v)) < \Delta(G)$. Letting $H'$ be the subgraph of $G'$ induced on the vertex set of $H$, Observation 4.10 implies this latter statement is equivalent to $|\delta_{G'}(H')| < \Delta(G)$. $\square$

In solving this problem, we solve the following more general problem. Let $p \in \{0,1\}^{V(G)}$ be such that $\sum_{v \in G} p_v \equiv 0 \mod 2$; we call such a $p$ a *parity function*. A $k$ *odd cut-set* of $G$ with respect to $p$ is a set $S$ such that $\sum_{v \in S} p_v$ is odd and $|\delta(S)| \leq k$.

**Lemma 4.12** (Padberg-Rao algorithm)**.** *[88] There exists an algorithm which given a multigraph $H$, a parity function $p$, and an integer $k$ either returns a $k$ odd cut-set of $H$ or asserts that no $k$ odd cut-set exists in $O(n^4 \log \Delta(G))$ time.*

We prove this lemma in Section 4.2.1. Before doing so, we describe how to apply it to finish the proof Lemma 4.8.

For the graph $G'$ above, we define the following parity function: $p_u = 1$, $\forall u \in G' - v'$, $p_{v'} = 1$ whenever $|G|$ is odd and $p_{v'} = 0$ whenever $|G|$ is even.

**Observation 4.13.** *$G'$ contains an odd subgraph $H'$ not containing $v'$ and satisfying $|\delta_{G'}(H')| < \Delta(G)$ precisely when $G'$ has a $(\Delta(G) - 1)$ odd cut-set with respect to $p$.*

*Proof.* Suppose $G'$ contains an odd subgraph $H'$ not containing $v'$ and satisfying $|\delta_{G'}(H')| < \Delta(G)$. Since for each vertex $v$ of $G' - v'$, $p_v = 1$, we have $\sum_{v \in H'} p_v$ is odd. Hence, $V(H')$ is a $(\Delta(G) - 1)$ odd cut-set with respect to $p$.

Conversely, if $S$ is a $(\Delta(G)-1)$ odd cut-set of $G'$ then as both $\sum_{v \in S} p_v$ and $\sum_{v \in G'-S} p_v$ are odd, assume without loss of generality that $S$ does not contain $v'$. Since for each vertex $v$ of $G - v'$, $p_v = 1$, we have that $|S|$ is odd. Hence, as $|\delta_G(S)| < \Delta(G)$, $S$ induces the desired subgraph. $\square$

Observation 4.13 together with Obervation 4.11 implies that $G$ contains an odd overfull subgraph precisely when $G'$ has a $(\Delta(G) - 1)$ odd cut-set with respect to $p$. We apply the Padberg-Rao algortihm to $G'$, $p$ and $k = \Delta(G) - 1$. If no $k$ odd cut-set exists then $G$ contains no odd overfull subgraph. Otherwise, if $S$ is a $k$ odd cut-set then our above remarks imply either $S$ of $G' - S$ induces an odd overfull subgraph in $G$. It remains to bound the running time.

Trivially, we can construct $G'$ from $G$ in $O(n + m)$ time. Moreover, we only apply the Padberg-Rao algorithm once to $G'$ which satisfies $|G'| = n + 1$. Hence we have a $O(n^4 \log \Delta(G))$ time algorithm to determine if $G$ contains an odd overfull subgraph.

### 4.2.1 Finding a $k$ Odd Cut-Set

In this section, we prove Lemma 4.12. The algorithm is recursive and takes as input the tuple $(H, p, k)$, where $H$ is a multigraph, $p \in \{0, 1\}^{V(H)}$ is a parity function, and $k$ is an integer. As output it either returns a $k$ odd cut-set $S$ or asserts that no $k$ odd cut-set exists. If every vertex of $H$ has parity 0 then we return no $k$ odd cut-set exists. Otherwise, let $s$ and $t$ be two vertices of $H$ such that $p_s = p_t = 1$. Let $S$ be a subset of $H$ satisfying $s \in S$, $t \notin S$, and minimizing $|\delta(S)|$, that is, a *minimum* $s-t$*-cut.*

If $|\delta(S)| > k$ then any $k$ odd cut-set $S'$ either has both $s$ and $t$ in $S'$ or both $s$ and $t$ in $H - S'$. We consider the graph $H' = H/\{s,t\}$ where for each vertex $v$ of $H - \{s,t\}$ we have parity $p'_v = p_v$ and for the (added) vertex $v^* \in H' - H$ we set $p'_{v^*} = 0$.

**Observation 4.14.** *$H'$ with parities $p'$ has a $k$ odd cut-set precisely when $H$ with parities $p$ does.*

*Proof.* Let $S$ be a subset of $H'$. We can assume that $v^* \notin S$, since a $S$ is a $k$ odd cut-set of $H'$ precisely when $H' - S$ is. Now, $\sum_{v\in S} p_v = \sum_{v\in S} p'_v$ and the value of $|\delta(S)|$ remains unchanged, and so, $S$ is a $k$ odd cut-set of $H'$ with parities $p'$ precisely when $S$ is a $k$ odd cut-set of $H$ with parities $p$. $\square$

Hence, we can recursively apply our algorithm to the instance $(H', p', k)$. We note that $|H'| = |H| - 1$.

If $|\delta(S)| \leq k$ then we consider two cases: $\sum_{v\in S} p_v$ is odd and $\sum_{v\in S} p_v$ is even. If $\sum_{v\in S} p_v$ is odd, then $S$ is a $k$ odd cut-set and we return it. Suppose $\sum_{v\in S} p_v$ is even. By the following lemma we can split our problem into two smaller instances:

**Lemma 4.15** (The uncrossing lemma)**.** *Let $S$ be a minimum $s-t$-cut. For any integer $k \geq |\delta(S)|$, if there exists a $k$ odd cut-set then there exists a $k$ odd cut-set $S^o$ such that either $S^o \subseteq S$ or $S^o \subseteq H - S$.*

The proof is a simple counting argument, which we defer for the moment. For now, we focus on how to use it to complete the algorithm.

The uncrossing lemma yields that it is enough to determine whether either of the following two smaller instances contains a $k$ odd cut-set:

1. $(H_1, p^1, k)$, where $H_1 = H/S$ and $p^1$ is the parity vector such that for each vertex $v$ in $H_1 \cap H$, $p^1_v = p_v$ and for the (added) vertex $v^* \in H_1 - H$, $p^1_{v^*} = 0$.

2. $(H_2, p^2, k)$, where $H_2 = H/(H-S)$ and $p^2$ is the parity vector such that for each vertex $v$ in $H_2 \cap H$, $p^2_v = p_v$ and for the (added) vertex $v^* \in H_2 - H$, $p^2_{v^*} = 0$.

The uncrossing lemma directly yields the following observation:

**Observation 4.16.** *$H$ with parities $p$ contains a $k$ odd cut-set precisely when one of $H_1$ with parities $p^1$ and $H_2$ with parities $p^2$ does.*

Hence, we can recursively apply our algorithm to the two tuples $(H_1, p^1, k)$ and $(H_2, p^2, k)$. We note that since $1 < |S| < |H|$ and $\sum_{v \in S} p_v$ is even, we have that both $|H_1| < |H|$ and $|H_2| < |H|$.

At each iteration, the algorithm either terminates, finds the desired cut, or our instance becomes smaller. It follows that the above algorithm will either find a $k$ odd cut-set cut or terminate. Hence, we need only prove the run time of our algorithm and the uncrossing lemma. We start with the latter.

*Proof of Lemma 4.15.* We can assume that $\sum_{v \in S} p_v$ is even, since otherwise $S^o = S$ satisfies the lemma. Let $T = H - S$. Let $S'$ be a $k$ odd cut-set and $T' = H - S'$. It follows that the total parities of exactly one of $S' \cap S$ and $S' \cap T$ is also odd; by relabelling, if necessary, we assume that $\sum_{v \in S' \cap S} p_v$ is odd. It follows that $\sum_{v \in T' \cap S} p_v$ is also odd (see

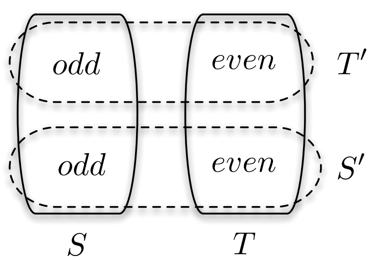



Figure 4–3: $S$ even cut-set, $S'$ odd cut-set.

Fig. 4–3.) Now some simple counting implies $|\delta(S)| + |\delta(S')| \geq |\delta(S' \cap T)| + |\delta(T' \cap S)|$ and $|\delta(S)| + |\delta(S')| \geq |\delta(S' \cap S)| + |\delta(T' \cap T)|$. So, if $t \in S' \cap T$, then since $S$ minimized

$|\delta(S)|$ over all $s-t$-cuts, we have $|\delta(S'\cap T)| \geq |\delta(S)|$. Hence,

$$|\delta(S)| + |\delta(S')| \geq |\delta(S'\cap T)| + |\delta(T'\cap S)| \geq |\delta(S)| + |\delta(T'\cap S)|.$$

This implies $|\delta(S')| \geq |\delta(T'\cap S)|$. Letting $S^o = T'\cap S$ we have $S^o$ is a $k$ odd cut-set for which $S^o \subseteq S$. Otherwise, $t \in T'\cap T$, and so, $|\delta(T'\cap T)| \geq |\delta(S)|$. It follows that $|\delta(S') \geq |\delta(S'\cap S)|$, and so, letting $S^o = S'\cap S$ we have $S^o$ is a $k$ odd cut-set for which $S^o \subseteq S$. □

It remains to bound the run time. In each iteration, the algorithm finds a minimum $s-t$-cut, and then either asserts no $k$ odd cut-set exists, returns $S$ as a $k$ odd cut-set, or recursively applies the algorithm to smaller instances. Clearly, constructing the smaller instances can be done in linear time. Moreover, given a $k$ odd cut-set in one of these smaller instances, our above remarks imply there exists a linear time algorithm to compute a $k$ odd cut-set of $H$. Hence, if $H$ has $n$ vertices and $m$ edges, the above algorithm runs in time

$$\tau(H,p,k)\cdot(MC(n,m) + O(n+m)),$$

where $\tau(H,p,k)$ is the total number of instance of minimum cut which must be solved given a particular input tuple $(H,p,k)$ and $MC(n,m)$ is the complexity of finding a minimum $s-t$ cut in the graph $H$.

Given a simple graph $G$ where each edge $e$ has capacity $c_e$, we can find a max-flow min-cut in $O(|G|^3)$ elementary arithmetic and other operations [64]. In the Padberg-Rao Method, we encode each multigraph $H$ as a simple graph with edge multiplicity function $\mu \in \mathbb{Z}^{E(H)}$. Since each edge $e$ has multiplicity $\mu_e \leq \Delta(H)$, we have that each operation takes at most $\log\Delta(H)$ bit operations. Hence, this algorithm takes $O(n^3\log\Delta(H))$ bit operations. Since this is an upper bound on the running time of this algorithm, we can find a max-flow min-cut in $O(n^3\log\Delta(H))$ time. We remark that this is not the fastest max-flow min-cut algorithm. For example, Goldberg and Rao [39] give an algorithm which takes in $O(m^{3/2}\log(n^2/m)\log U)$ operations, where $U$ is the maximum capacity on any edge.

To finish the proof the runtime of the Padberg-Rao algorithm, it is enough to prove for each instance $\tau(H,p,k) \leq n$. This follows from the next claim.

**Claim 4.17.** $\tau(H,p,k) \leq \max\{0, (\sum_{v\in H} p_v) - 1\}$.

*Proof.* The proof is by induction on $\sum_{v\in H} p_v$. The base case when $\sum_{v\in H} p_v = 0$ is trivial as we solve no instances of minimum cut. Assume the claim holds for all graphs $H'$ with $\sum_{v\in H'} p_v < 2\ell$, $\ell \geq 1$ and that $(H,p)$ satisfies $\sum_{v\in H} p_v = 2\ell$. Let $s$ and $t$ be two vertices with nonzero parity. Our algorithm first solves one instance of minimum cut and then considers three options. Let $S$ be a cut-set with $s \in S$, $t \notin T$, and minimizing $|\delta(S)|$. If $\sum_{v\in S} p_v$ is odd then our algorithm terminates, and so as $\tau(H,p,K) = 1$, the claim holds. If $\sum_{v\in S} p_v$ is even and $|\delta(S)| \leq k$ then our algorithm recurses on the two instances $(H_1, p^1, k)$ and $(H_2, p^2, k)$. Note that by construction

$$\sum_{v\in H_1} p_v^1 + \sum_{v\in H_2} p_v^2 = \sum_{v\in H} p_v.$$

Hence, including the 1 instance we already solved, the induction hypothesis yields:

$$\tau(H,p,k) \leq 1 + \sum_{v\in H_1} p_v^1 - 1 + \sum_{v\in H_2} p_v^2 - 1 = \sum_{v\in H} p_v - 1.$$

Finally, if $\sum_{v\in S} p_v$ is even and $|\delta(S)| > k$ then our algorithm recurses on the instance $(H', p', k)$. Since $\sum_{v\in H'} p'_v = \sum_{v\in H} p_v - 2$, the induction hypothesis yields:

$$\tau(H,p,k) \leq 1 + \sum_{v\in H'} p'_v - 1 = \sum_{v\in H} p_v - 2.$$

Hence, the claim holds. $\square$

This completes the proof of Lemma 4.12.

### 4.2.2 Determining the Fractional Chromatic Index Revisited

In this section, we prove Lemma 4.7. Let $\Lambda(G) = \max_{H \subseteq G, |H|>1 \text{ odd}} \frac{2|E(H)|}{|H|-1}$. The core of the algorithm in Lemma 4.7 is the following subroutine:

**Lemma 4.18.** *There exists an algorithm, denoted* ALPHA_OF$(G, \alpha)$*, which given a multigraph $G$ and a rational number $\alpha = \frac{r}{k}$ where $\alpha \geq \Delta(G)$ and $r, k \in \mathbb{Z}$, either returns an odd subgraph $H$, $|H| > 1$ and satisfying $\frac{|E(H)|}{|H|-1} > \alpha$, or asserts that $\Lambda(G) \leq \alpha$ in $O(n^4 \log r)$ time.*

Lemma 4.18, which we prove at the end of the section, uses algorithm OF$(G)$ (Lemma 4.8) as a subroutine. We first prove Lemma 4.7 using Lemma 4.18.

*Proof of Lemma 4.7.* For any subgraph $H$ of $G$, we have $|E(H)| \leq \frac{n\Delta(G)}{2}$, and so trivially, $\Lambda(G) \in (0, n\Delta(G)]$. We first apply $OF(G)$ to determine whether $G$ contains an odd overfull subgraph. If no, then we assert that no such subgraph exists. Otherwise, we find an odd overfull subgraph $F$ with $\frac{2|E(F)|}{|F|-1} = \Lambda(G) \in (\Delta(G), n\Delta(G)]$. The following observation allows us to use binary search to find $\Lambda(G)$ and such a subgraph $F$.

**Observation 4.19.** *Suppose $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers satisfying $a, c \in \mathbb{Z}$ and $b, d \in \{1, ..., n-1\}$. Then $|\frac{a}{b} - \frac{c}{d}|$ is either $0$ or greater than $\frac{1}{n^2}$.*

*Proof.* $|\frac{a}{b} - \frac{c}{d}| = |\frac{ad-bc}{bd}|$. Hence, if $ad - bc \neq 0$ then $|\frac{a}{b} - \frac{c}{d}| \geq \frac{1}{bd} > \frac{1}{n^2}$. □

Our initial interval is $(\Delta(G), n\Delta(G)]$. Given any interval $(\ell', r']$ containing $\Lambda(G)$, we need to determine whether $\Lambda(G) \in (\ell', \frac{\ell'+r'}{2}]$ or $\Lambda(G) \in (\frac{\ell'+r'}{2}, t']$. For $\alpha' = \frac{\ell'+t'}{2}$, we apply ALPHA_OF$(G, \alpha')$ (Lemma 4.18) to test whether $\Lambda(G) \leq \alpha'$. If $\Lambda(G) > \alpha'$, then $\Lambda(G) \in (\frac{\ell'+r'}{2}, t']$; otherwise, $\Lambda(G) \in (\ell', \frac{\ell'+r'}{2}]$.

We binary search until the size of the interval $(\ell, r]$ is less than $\frac{1}{n^2}$. We can do so by testing at most $3 \log n + \log \Delta(G)$ values of $\alpha$. Since $G$ contains an odd overfull subgraph, Observation 4.19 implies any subgraph $F$ with $\frac{2|E(F)|}{|F|-1} > \ell$ is odd overfull and $\frac{2|E(F)|}{|F|-1} = \Lambda(G)$. Hence, we return the subgraph returned by $\textsc{alpha\_OF}(G, \ell)$.

By renormalizing, we can always assume that for each interval $(\ell', r']$ in iteration $i$, we have $\ell' = \frac{a}{2^i}$, $r' = \frac{b}{2^i}$, where $a, b \in \mathbb{Z}$ such that $a \leq b \leq n\Delta(G)2^i$. Since the number of iterations is at most $3 \log n + \log \Delta(G)$, for each call to $\textsc{alpha\_OF}(G, \alpha)$, we have $\alpha = \frac{\ell' + r'}{2} = \frac{r}{k}$ for $r, k \in \mathbb{Z}$ satisfying $r \leq n^4 \Delta(G)^2$. It follows that the total runtime is $O(n^4(\log n + \log \Delta(G))^2)$. □

*Proof of Lemma 4.18.* We reduce determining if $G$ contains an odd subgraph $H$, $|H| > 1$ and satisfying $\frac{|E(H)|}{|H|-1} > \alpha$ to the question of determining if an auxiliary graph $G_{r,k}$ contains an odd overfull subgraph and apply Lemma 4.8. The graph $G_{r,k}$ is constructed by taking the disjoint union of two vertices $u^*$ and $v^*$ for which there are $r$ copies of the edge $u^*v^*$, and a copy of $G$, where each edge $e \in E(G)$ has $k$ copies.

**Claim 4.20.** *$\Delta(G_{r,k}) = r$.*

*Proof.* Each vertex $v$ of $V(G)$ in $G_{r,k}$ has degree $d_{G_{r,k}}(v) = kd_G(v) \leq k\Delta(G) \leq k\alpha \leq r$, and both $u^*$ and $v^*$ have degree $r$. □

**Claim 4.21.** *If $G_{r,k}$ contains an odd overfull subgraph $H_{r,k}$, then $H_{r,k} \cap V(G)$ is also an odd overfull subgraph of $G_{r,k}$.*

*Proof.* Observation 4.2 implies $|\delta(H_{r,k})| < \Delta(G_{r,k})$, and so $u^*v^* \notin \delta(H_{r,k})$, since the number of copies of $u^*v^*$ is $\Delta(G_{r,k})$. Furthermore, if $u^*v^*$ is contained in $H_{r,k}$, then the vertices of $u^*v^*$ contribute 0 to the left-hand side of the inequality of Observation 4.2. Hence, the value of left-hand side of this inequality is the same for both $H_{r,k}$ and $H_{r,k} \cap V(G)$. Hence, $H_{r,k} \cap V(G)$ is also an odd overfull subgraph of $(G_{r,k})$. □

**Claim 4.22.** *$G_{r,k}$ contains an odd overfull subgraph $H_{r,k}$ precisely when $\Lambda(G) > \alpha$.*

*Proof.* By Claim 4.21, we can assume $H_{r,k}$ does not contain $u^*v^*$. Letting $H$ be the subgraph of $G$ induced on the vertices $H_{r,k}$, we have

$$\Delta(G_{r,k}) < \frac{2|E(H_{r,k})|}{|H_{r,k}|-1} = k\frac{2|E(H)|}{|H|-1}.$$

Hence,

$$\frac{2|E(H)|}{|H|-1} > \frac{1}{k}\Delta(G_{r,k}) = \frac{r}{k} = \alpha.$$

The claim now follows. □

We determine whether $G_{r,k}$ contains an odd overfull subgraph. By Claim 4.20, $\Delta(G_{r,k}) = r$. Since $|V(G_{r,k})| = n+2$ and $|E(G_{r,k})| = m+1$, we can do so in $O(n^4 \log r)$ time by applying $\mathrm{OF}(G_{r,k})$. If $H_{r,k}$ is the returned subgraph, then by Observation 4.22, $H_{r,k} \cap V(G)$ is odd overfull in $G_{r,k}$. Letting $H$ be the subgraph of $G$ induced on the vertices $H_{r,k} \cap V(G)$, we return $H$. Otherwise, we assert that $\Lambda(G) \leq \alpha$. □

### 4.3 Determining the Fractional Chromatic Index for Graphs of Large Degree

In this Section, we prove Theorem 4.4. The cut condition of overfull subgraphs allows us to check if a particular subgraph $H$ of $G$ is odd overfull in linear time via one scan of the edge set. In this section, we will use the following stronger version.

**Definition 4.23.** The *deficiency* of a vertex $v$ in $G$ is $\mathrm{def}(v) := \Delta - d(v)$. The *deficiency* of a subgraph $H$ of $G$ is $\mathrm{def}(H) := \sum_{v \in H} \mathrm{def}(v)$.

**Fact 4.24.** *A subgraph $H$ of $G$ is odd overfull precisely if $|H|$ is odd and* $\mathrm{def}(H) + |\delta(H)| \leq \Delta - 2$.

*Proof.* As $|H|$ is odd, $\mathrm{def}(H) + |\delta(H)| = \Delta|H| - 2|E(H)|$ has the same parity as $\Delta$. Hence, it cannot be the case that $\mathrm{def}(H) + |\delta(H)| = \Delta - 1$. □

The cut condition also yields the following observations which are key to our algorithm for finding a set of candidates containing all the odd overfull subgraphs of $G$ in linear time.

**Definition 4.25.** A vertex $v \in G$ is *$\varepsilon$-special* for an odd overfull subgraph $H$ of $G$ if it is incident to more than $\varepsilon\Delta$ edges of $\delta(H)$, or it is in $H$ and has deficiency at least $\varepsilon\Delta$.

**Observation 4.26.** *$\forall \varepsilon > 0$ and any odd overfull subgraph $H$ of $G$, the number of vertices which are $\varepsilon$-special for $H$ is at most $\varepsilon^{-1}$.*

**Observation 4.27.** *$\forall \varepsilon > 0$ and $v, w \in G$ with at least $3\varepsilon\Delta$ common neighbours either at least one of $w$ or $v$ is $\varepsilon$-special for $H$, both are in $H$, or neither are in $H$.*

The following two observations are the key to understanding the intersection patterns of odd overfull subgraphs.

**Observation 4.28.** *For any $X \subseteq V(H)$, $\sum_{v\in X}(\Delta - d_X(v)) \geq |X|(\Delta + 1 - |X|)$.*

**Observation 4.29.** *For any two subgraphs $A_1$ and $A_2$,*

$$\sum_{v\in(A_1\oplus A_2)} (\Delta - d_{(A_1\oplus A_2)}(v)) \leq \sum_{v\in A_1} (\Delta - d_{A_1}(v)) + \sum_{v\in A_2} (\Delta - d_{A_2}(v)).$$

*Proof.* The observation follows easily from the facts that $\delta(A_1 \oplus A_2) \subseteq \delta(A_1) \cup \delta(A_2)$ and $\sum_{v\in(A_1\oplus A_2)}(\Delta - d(v)) \leq \sum_{v\in A_1}(\Delta - d(v)) + \sum_{v\in A_2}(\Delta - d(v))$. □

Observations 4.28 yields that odd overfull subgraphs contain many vertices:

**Observation 4.30.** *Any overfull subgraph $A$ satisfies $|A| \geq \Delta + 1$.*

*Proof.* Observation 4.28 yields, $|A|(\Delta + 1 - |A|) \leq \Delta - 2$. It follows that $|A| \geq \Delta + 1$. □

Together Observations 4.28 and 4.29 yield the following important lemma.

**Lemma 4.31.** *(e.g. [85]) For two distinct odd overfull subgraphs $H_1$ and $H_2$ of $G$, the number of vertices in the symmetric difference of their vertex set $H_1 \oplus H_2$ is at least $\Delta$.*

*Proof.* Letting $X = H_1 \oplus H_2$, Observation 4.29 implies $\sum_{v\in X}(\Delta - d_X(v)) \leq 2\Delta - 4$, and so by Observation 4.28, $|X|(\Delta - |X| + 1) \leq 2\Delta - 4$. It follows that either $|X| \leq 1$ or $|X| \geq \Delta$. In the former case, $|H_2 - H_1| + |H_1 - H_2| \leq 1$ and we have contradiction to the fact that

$V(H_1)$ and $V(H_2)$ are distinct odd sets. Hence, we must have the latter case, from which the lemma follows. $\square$

To finish this section, we prove that for the special case handled in the next section (i.e. $\Delta \geq \frac{n}{2}$), $G$ contains at most one odd overfull subgraph. We need the following observation whose proof is similar to that of Observation 4.29.

**Observation 4.32.** *For any two subgraphs $A_1$ and $A_2$,*

$$\sum_{v \in (A_1 - A_2)} (\Delta - d_{(A_1 - A_2)}(v)) \leq \sum_{v \in A_1} (\Delta - d_{A_1}(v)) + \sum_{v \in A_2} (\Delta - d_{A_2}(v)).$$

**Lemma 4.33.** *If $G$ has maximum degree $\Delta \geq \frac{n}{2}$, then $G$ contains at most one odd overfull subgraph.*

*Proof.* By Lemma 4.31, if $G$ contains two distinct odd overfull subgraphs $H_1$ and $H_2$, then $|H_1 \oplus H_2| \geq \Delta$. Assume with out loss of generality that $|H_1 - H_2| \geq \frac{1}{2}\Delta$. As in the proof of Lemma 4.31, letting $X = H_1 - H_2$, Observation 4.32 implies $\sum_{v \in X}(\Delta - d_X(v)) \leq 2\Delta - 4$, and so by Observation 4.28, $|X|(\Delta - |X| + 1) \leq 2\Delta - 4$. It follows that $|H_1 - H_2| = |X| \geq \Delta$. By Observation 4.30, $|H_2| \geq \Delta + 1$, and so, we have a contradictions as $n \geq |H_2| + |H_1 - H_2| \geq 2\Delta + 1$. $\square$

### 4.3.1 The Case $\Delta \geq \frac{n}{2}$

We now present a linear time algorithm which determines $\chi'_f(G)$ for any graph $G$ satisfying $\Delta \geq \frac{n}{2}$. We can assume that $n > 360$, as otherwise we can decide if $G$ contains an odd overfull subgraph in constant time by checking all subgraphs of size at least $\Delta + 1$.

**Definition 4.34.** $\forall \varepsilon > 0$, define $L_\varepsilon := \{w \in G : d(w) \geq (1 - \varepsilon)\Delta\}$.

For any odd overfull subgraph $H$ of $G$, every vertex of $H - L_\varepsilon$ is $\varepsilon$-special, and so by Observation 4.26, if $|L_\varepsilon| \leq \varepsilon^{-1}$ then $H$ has at most $2\varepsilon^{-1}$ vertices. It follows that $H$ contains at most $2\varepsilon^{-1}|H|$ edges and is not odd overfull. So, we can assume that $|L_\varepsilon| > \varepsilon^{-1}$.

Set $\varepsilon = \frac{1}{30}$. We describe a linear time subroutine which given a vertex $v$ of $L_\varepsilon$ determines if there is an odd overfull subgraph $H$ for which $v$ is not an $\varepsilon$-special vertex and if so finds the unique such $H$. By Observation 4.26, applying this to each of 31 different vertices in turn we either find the unique odd overfull subgraph $H$ or determine no such subgraph exists.

Given $v$, our first step is to determine a set $\mathcal{C}$ of three candidate subgraphs. We ensure that if $H$ is an odd overfull subgraph $H$ for which $v$ is not an $\varepsilon$-special vertex, then there exists a candidate subgraph $C \in \mathcal{C}$ such that $H - S_H = C - S_H$, where $S_H$ is the set of $\varepsilon$-special vertices for $H$. In doing so, we focus on the intersection of the neighbourhood of each vertex with the neighbourhood of $v$.

**Definition 4.35.** *$\forall v \in L_\varepsilon$, define*

$T_\varepsilon(v) := \{w \in L_\varepsilon : |N(w) \cap N(v)| \geq 3\varepsilon\Delta\}$, *and*

$S_\varepsilon(v) := \{w \in L_\varepsilon : |N(w) \cap N(v)| \leq 3\varepsilon\Delta\}$.

We exploit the following two corollaries of Observation 4.27.

**Observation 4.36.** *If $v \in H$ then $(T_\varepsilon(v) - S_H) \subseteq H$. Otherwise, $(T_\varepsilon(v) - S_H) \cap H = \emptyset$.*

**Observation 4.37.** *Either $(S_\varepsilon(v) - S_H) \subseteq H$ or $(S_\varepsilon(v) - S_H) \cap H = \emptyset$.*

*Proof.* Since every vertex in $L_\varepsilon$ has minimum degree $(1-\varepsilon)\Delta$, every vertex of $S_\varepsilon(v)$ has at least $\frac{26}{30}\Delta$ neighbours in $G - N(v)$. Since $|G - N(v)| \leq \frac{31}{30}\Delta$, any pair of vertices in $S_\varepsilon(v)$ have $\frac{21}{30}\Delta$ common neighbours. By Observation 4.27, if there exist two vertices $x$ and $y$ of the set $S_\varepsilon(v)$ such that $x \in H$ and $y \notin H$ then one of them is an $\varepsilon$-special vertex for $H$. $\square$

Combining these we obtain:

**Observation 4.38.** *$H - S_H$ is one of $S_\varepsilon(v) - S_H$, $T_\varepsilon(v) - S_H$, or $L_\varepsilon - S_H$.*

So, we need only find for each $A \in \{L_\varepsilon, S_\varepsilon(v), T_\varepsilon(v)\}$ all the odd overfull subgraphs $H$ satisfying $A - S_H = H - S_H$. The first step is to determine $S_\varepsilon(v)$, $T_\varepsilon(v)$, and $L_\varepsilon$ which can be done by scanning through the edge set once. We then apply the following subroutine to each of the three choices for $A$, which we can do since $\Delta \geq \frac{n}{2} > 180 = 6\varepsilon^{-1}$.

FIND-OVERFULL

INPUT: $\varepsilon > 0$, $G$ satisfying $\Delta > 6\varepsilon^{-1}$, $A \subseteq G$

OUTPUT: returns odd overfull subgraph $H$ of $G$ with $H - S_H = A - S_H$, or asserts no such subgraph exists.

We let $J := \{v \in G \mid |N(v) \cap A| \geq \frac{\Delta}{2}\}$. If there exists an odd overfull subgraph $H$, then it is the unique odd overfull subgraph. By Observation 4.26, vertices in $B := (J \setminus V(H)) \cup (V(H) \setminus J)$ contribute at least $\frac{\Delta}{2} - \varepsilon^{-1}$ to $\text{def}(H) + |\delta(H)|$. Thus, $|B| \leq 2$. If $J$ is even then $B$ must be a single vertex, and since $H$ is the unique odd overfull subgraph either $H = J \cup B$ or $H = J \setminus B$. It is easy to see that this vertex must be either the vertex $m$ of $V(G) \setminus J$ with the **m**ost neighbours in $J$, or the vertex $f$ of $J$ with the **f**ewest number of neighbours in $J$. So, we check if either $J \cup \{m\}$ or $J \setminus \{f\}$ induce an odd overfull subgraph. In the same vein, if $J$ is odd then $B$ is contained in the union of the two vertices of $V(G) \setminus J$ with the most neighbours in $J$ and the two vertex of $J$ with the fewest neighbours in $J$. So, there are at most $2^4$ choices for $H$ given $J$ and we check them all.

As determining if a particular subgraph is odd overfull takes linear time, this subroutine also takes linear time. This completes the proof when $\Delta \geq \frac{n}{2}$.

### 4.3.2 General Case

Let $\gamma > 0$ and $G$ satisfy $\Delta \geq \gamma n$. As in the previous section, to find each odd overfull subgraph $H$, we focus on finding vertices which are not $\varepsilon$-special for $H$. Unlike the previous section, there may be many odd overfull subgraphs and a vertex may be not $\varepsilon$-special for many odd overfull subgraphs. We instead find a set $S$ of vertices contained in $H$ which are not $\varepsilon$-special for $H$ and such that any pair of vertices in $S$ has very few common neighbours. By choosing this set $S$ as large as possible it will follow that, up to the $\varepsilon$-special vertices, $H$ has vertex set $\bigcup_{v \in S} T_\varepsilon(v)$. Given $\bigcup_{v \in S} T_\varepsilon(v)$, we can then apply FIND-OVERFULL to find $H$.

Set $\varepsilon = \frac{2}{2(\lceil \gamma^{-1} \rceil + 1) + 3(\lceil \gamma^{-1} \rceil + 1)^2}$. We can assume $n > 6\gamma^{-1}\varepsilon^{-1}$, as otherwise we can find all odd overfull subgraphs of $G$ in constant time. We break our algorithm into two steps:

**Lemma 4.39.** *In linear time we can find a list $\mathcal{S}$ of at most $(\varepsilon^{-1} + 2)^{\lceil \gamma^{-1} \rceil}$ sets each of at most $\lceil \gamma^{-1} \rceil$ vertices of $L_\varepsilon$ such that for each odd overfull subgraph $H$ of $G$ there exists an $S \in \mathcal{S}$ satisfying:*

*(a) $S$ contains no $\varepsilon$-special vertex for $H$, and*

*(b) every vertex of $L_\varepsilon$ which is not $\varepsilon$-special for $H$ shares at least $3\varepsilon\Delta$ neighbours with some $v \in S$.*

**Lemma 4.40.** *Given a list $S$ of at most $\lceil \gamma^{-1} \rceil$ vertices of $L_\varepsilon$ we can compute all odd overfull subgraphs $H$ of $G$ satisfying (a) and (b) with respect to $S$ in linear time.*

*Proof of Lemma 4.39.* We describe a subroutine that constructs lists $\mathcal{S}_0, \mathcal{S}_1, ..., \mathcal{S}_c$ of sets such that $c = \lceil \gamma^{-1} \rceil$ and $\emptyset = \mathcal{S}_0 \subseteq ... \subseteq \mathcal{S}_c = \mathcal{S}$ satisfying the lemma. To do so, we ensure that for each $i$ between 0 and $c$:

(i) for each set $S$ in list $\mathcal{S}_i$, every pair of different vertices $x, y$ from $S$ satisfy $|N(x) \cap N(y)| < 3\varepsilon\Delta$.

(ii) for each odd overfull subgraph $H$ of $G$, there exists some set $S \in \mathcal{S}_i$ satisfying (a) such that either $S$ also satisfies (b) or $|S| = i$.

We show below that any set $S$ satisfying (i) has at most $c$ elements. Thus, (i) and (ii) imply that upon termination (a) and (b) hold.

**Constructing $\mathcal{S}_{i+1}$ from $\mathcal{S}_i$:** We show how to construct $\mathcal{S}_{i+1}$ satisfying (i) and (ii) for $i+1$ from $\mathcal{S}_i$ satisfying (i) and (ii) for $i$. For each set $S \in \mathcal{S}_i$ we do the following. First we add $S$ to $S_{i+1}$. Set $P'_i(S) = \{w \in L_\varepsilon : |N(w) \cap N(u)| < 3\varepsilon\Delta \ \forall u \in S\}$. Choose a set $P_i(S)$ of $\min\{|P'_i(S)|, \varepsilon^{-1}+1\}$ vertices from $P'_i(S)$. For each $w \in P_i(S)$ we add $S \cup \{w\}$ to $\mathcal{S}_{i+1}$.

Clearly $S_0 = \emptyset$ satisfies (i) and (ii). By induction assume that $S_i$ satisfies (i), and so, by the way we chose $P'_i(S)$, $\mathcal{S}_{i+1}$ also satisfies (i). Consider now any odd overfull subgraph $H$ of $G$. By induction, there is an $S \in \mathcal{S}_i$ such that (ii) is satisfied for $H$ with respect to $i$. If $H$ and $S$ satisfy (b) then, since $S \in \mathcal{S}_{i+1}$, (ii) is satisfied for $H$ with respect to $i+1$. Otherwise, $P'_i(S)$ contains a vertex which is not $\varepsilon$-special for $H$. Since $P_i(S)$ either is $P'_i(S)$ or contains more than $\varepsilon^{-1}$ vertices, Observation 4.26 implies that $P_i(S)$ also contains a vertex $u$ which is not $\varepsilon$-special for $H$. Thus, $S \cup \{u\} \in \mathcal{S}_{i+1}$ satisfies (ii) for $H$ with respect to $i+1$.

To conclude we bound the size of a set $S$ satisfying (i). We assume for contradiction that there exists a set $S$ such that $S$ satisfies (i) and $|S| = c+1$. Since $S \subseteq L_\varepsilon$ and each pair of distinct vertices of $S$ share at most $3\varepsilon\Delta$ common neighbours,

$$\begin{aligned} \left|\bigcup_{i=1}^{|S|} N(v_i)\right| &\geq |S|(1-\varepsilon)\Delta - \binom{|S|}{2} 3\varepsilon\Delta \\ &> |S|\Delta - \left(\frac{|S|}{3} + \frac{|S|^2}{2}\right) 3\varepsilon\Delta. \end{aligned}$$

Since, $\varepsilon = \frac{2}{2(\lceil\gamma^{-1}\rceil+1)+3(\lceil\gamma^{-1}\rceil+1)^2}$ and $|S| = c+1 = \lceil\gamma^{-1}\rceil + 1$ we have

$$\left|\bigcup_{i=1}^{|S|} N(v_i)\right| > (\lceil\gamma^{-1}\rceil + 1)\Delta - \Delta = \lceil\gamma^{-1}\rceil \Delta = n,$$

a contradiction. It follows that any $S$ satisfying (i) has at most $\lceil\gamma^{-1}\rceil = c$ elements. Now, $|S_1| \leq \varepsilon^{-1}+2$ and for each $i$ between 2 and $c$, $|S_{i+1}| \leq (\varepsilon^{-1}+2)|S_i|$. Hence, there can be at most $(\varepsilon^{-1}+2)^{\lceil \gamma^{-1} \rceil}$ sets in $\mathcal{S}$. For each set $S$ in list $\mathcal{S}_i$, by scanning the edge set we can construct the set $P_i(S)$ in linear time. Based on the size of $P_i(S)$, the subroutine adds a constant number of sets to $\mathcal{S}_{i+1}$ and so the subroutine takes linear time. □

*Proof of Lemma 4.40.* We break our proof into two parts:

(I) we describe a linear time subroutine which given a subset $A$ of $S$ determines if there is an odd overfull subgraph $H$ of $G$ satisfying (a) and (b) with respect to $S$ where $A = S \cap H$, and

(II) we show that for each subset $A$ of $S$ at most one odd overfull subgraph of $G$ satisfies $A = S \cap H$ and both (a) and (b) with respect to $S$.

Thus, if we check each of the $2^{|S|} \leq 2^{\lceil \gamma^{-1} \rceil}$ subsets of $S$, we are guaranteed to find each $H$ satisfying (a) and (b) with respect to $S$.

**(I)** Observation 4.27 implies:

**Definition 4.41.** $\forall \varepsilon > 0$, and $A \subseteq L_\varepsilon$, we use $T_\varepsilon(A)$ to denote $\{w \in L_\varepsilon : \exists v \in A \text{ s.t. } |N(w) \cap N(v)| \geq 3\varepsilon\Delta\}$.

**Observation 4.42.** *If $A = S \cap H$ for some odd overfull subgraph $H$ of $G$ satisfying (a) and (b) with respect to $S$, then $T_\varepsilon(A) - S_H \subseteq H - S_H$.*

In fact, as $S$ satisfies (b) with respect to $H$, if a vertex $u \in L_\varepsilon - T_\varepsilon(A)$ is not an $\varepsilon$-special vertex for $H$ then for some vertex in $v$ of $S - A$ we have $|N(u) \cap N(v)| \geq 3\varepsilon\Delta$. Thus by Observation 4.27, $u \in G - H$. Hence,

**Observation 4.43.** *If $A = S \cap H$ for some odd overfull subgraph $H$ of $G$ satisfying (a) and (b) with respect to $S$, then $T_\varepsilon(A) - S_H = H - S_H$.*

Therefore, as $n > 6\gamma^{-1}\varepsilon^{-1}$, we have $\Delta > 6\varepsilon^{-1}$, and so, we can use Subroutine Find-Overfull (Section 4.3.1) with $A = T_\varepsilon(A)$ to return one such $H$. As constructing $T_\varepsilon(A)$

can be done by scanning through the edge set once and Subroutine Find-Overfull takes linear time, the subroutine takes linear time.

**(II)** Suppose $H_1$ is an odd overfull subgraph $H_1$ of $G$ satisfying $A = S \cap H_1$ and both (a) and (b) with respect to $S$. Assume for contradiction that $H_2$ is a different odd overfull subgraph of $G$ also satisfying $A = S \cap H_2$ and both (a) and (b) with respect to $S$. As in (I), $T_\varepsilon(A) - S_{H_1} = H_1 - S_{H_1}$ and $T_\varepsilon(A) - S_{H_2} = H_2 - S_{H_2}$. So, $H_1 - (S_{H_1} \cup S_{H_2}) = H_2 - (S_{H_1} \cup S_{H_2})$. By Observation 4.26, $|S_{H_1}| \le \varepsilon^{-1}$ and $|S_{H_2}| \le \varepsilon^{-1}$, and so $|H_1 \oplus H_2| \le 2\varepsilon^{-1}$. We now have a contradiction by Lemma 4.31, since $|H_1 \oplus H_2| \ge \Delta > 6\varepsilon^{-1}$.

This completes the proof of Lemma 4.40. $\square$

We remark that since $\mathcal{S}$ satisfies $|\mathcal{S}| \le (\varepsilon^{-1}+2)^{\lceil \gamma^{-1} \rceil}$ and $\varepsilon^{-1} = \frac{3}{2}\left\lceil \gamma^{-1} \right\rceil^2 + 4\left\lceil \gamma^{-1} \right\rceil + \frac{5}{2}$, we have at most $|\mathcal{S}| \le (\frac{3}{2}\left\lceil \gamma^{-1} \right\rceil^2 + 4\left\lceil \gamma^{-1} \right\rceil + \frac{7}{2})^{\lceil \gamma^{-1} \rceil}$. Since for each $S \in \mathcal{S}$ we need only check each of the $2^{|S|} \le 2^{\lceil \gamma^{-1} \rceil}$ subsets of $S$ and each subset corresponds to at most one odd overfull subgraph, the total number of odd overfull subgraphs is at most $(3\left\lceil \gamma^{-1} \right\rceil^2 + 8\left\lceil \gamma^{-1} \right\rceil + 7)^{\lceil \gamma^{-1} \rceil}$.

# CHAPTER 5
## Total Colouring

Recall that a total colouring is an assignment of colours to the vertices and edges of a graph such that:

1. no two adjacent vertices share the same colour,
2. no two incident edges share the same colour, and
3. no edge shares a colour with one of its endpoints.

Recall further that the total colouring number of a graph $G$, denoted $\chi''(G)$, is the smallest $k \geq 0$ such that $G$ has a total colouring using $k$ colours. Analogous to Vizing's Theorem, the well-known Total Colouring Conjecture of Behzad [6] and Vizing [110] states that for each simple graph of maximum degree $\Delta$, $\Delta + 1 \leq \chi''(G) \leq \Delta + 2$. Now, the greedy colouring procedure (Lemma 1.6) yields that any simple graph has a vertex $(\Delta + 1)$-colouring and Vizing's Theorem (Theorem 1.8) implies that any simple graph has an edge $(\Delta + 1)$-colouring. So, the difficultly in finding total colourings is to choose a vertex colouring and an edge colouring which do not conflict. In this chapter, we focus on finding total colourings for simple graphs by choosing such a pair of colourings.

One approach to finding such a pair of colourings is to first choose an edge colouring and then try to extend it by choosing a nonconflicting vertex colouring. The difficulty with this approach is that even if we allow up to $(2\Delta - 1)$ colours for the edge colouring there may not exist a vertex $(2\Delta - 1)$-colouring which extends it. Molloy and Reed [82] give the following example. Let $G$ be the graph built by taking a clique on $n$-vertices and adding a pendant edge at each vertex (see Figure 5–1 for an example). It is easily seen that $\chi'(G) = \Delta$ and for any edge $\Delta$-colouring, each vertex of the clique is incident to an

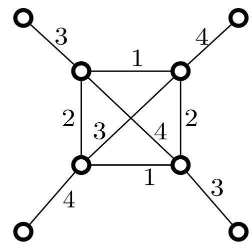



Figure 5–1: $G$ when $\Delta = 4$ and a bad edge colouring.

edge of each colour. Given such an edge $\Delta$-colouring, to colour the vertices of the clique so as to avoid conflicts, we need an additional $\Delta$ colours, and so, the best total colouring extending this edge colouring, uses at least $2\Delta$ colours.

McDiarmid and Reed [78] obtain a total $(\Delta + 2f(\Delta))$-colouring of any graph, where $f(\Delta) = O(\Delta^{\frac{2}{3}} \log \Delta)$, as follows. Given an edge $(\Delta + 1)$-colouring, they choose a vertex $(\Delta + f(\Delta) - 1)$-colouring which does not conflict the edge colouring too much. Specifically, they show that they can choose the vertex colouring such that if $R$ is the graph induced on the edges which are involved in some conflict, then $\Delta(R) \leq f(\Delta)$. By Vizing's Theorem, they then can recolour these edges at a cost of $f(\Delta) + 1$ further colours.

Fixing a vertex colouring and then choosing an edge colouring so as to avoid conflicts seems much more promising. An edge $c$-colouring $M_1, ..., M_c$ *extends* a vertex $c$-colouring $S_1, ..., S_c$ if for each $i = 1, ..., c$, $S_i \cup M_i$ is a total stable set. It is conjectured that for *any* vertex $(\Delta+3)$-colouring there exists an edge $(\Delta+3)$-colouring which extends it and hence when combined with it yields a total $(\Delta + 3)$-colouring. The key idea is that the vertex $(\Delta + 3)$-colouring places very few constraints on the edge colouring. We discuss this in Section 5.1.

In Section 5.2, we show that any vertex $(\Delta + 3)$-colouring can be combined with any edge $(\Delta + 1)$-colouring to construct a fractional total $(\Delta + 3)$-colouring. We discuss how Kilakos and Reed build on this idea to construct a fractional total $(\Delta+2)$-colouring of any graph. In Chapter 6, we will sharpen their approach, showing all but at most two graphs of maximum degree $\Delta$ have a fractional total $\alpha$-colouring with $\alpha < \Delta + 2$.

In Section 5.3, we discuss a condition proposed by Chetwynd and Hilton which is required for a vertex $(\Delta + 1)$-colouring to be extended to a total $(\Delta + 1)$-colouring. They conjecture that for graphs with large enough maximum degree and not containing a certain bad subgraph, if there exists a vertex colouring satisfying this condition, then there exists a total $(\Delta + 1)$-colouring. Now, not every vertex colouring satisfying their condition is extendable to a total $(\Delta+1)$-colouring. For example in Chapter 7, we show that one needs to ensure that for each colour class $S$, $G-S$ does not contain an odd component each vertex of which has degree $\Delta$ in $G$. It would be interesting to know if this is the only problem which prevents a vertex colouring satisfying their condition from being extended to a total $(\Delta+1)$-colouring of the graph. In Section 5.4, for each $\beta \in [\Delta+1, \Delta+2]$, we give a condition which is required for a fractional vertex $\beta$-colouring to be extended to a fractional total $\beta$-colouring. We propose, as a fractional version of Chetwynd and Hilton's Conjecture, that for graphs with large enough maximum degree and not containing a certain bad subgraph, if there exists a fractional vertex $\beta$-colouring satisfying this condition, then there exists a total $\beta$-colouring. In Chapter 7, we will verify the truth of this conjecture for graphs with large maximum degree $\Delta$ and containing no overfull subgraphs.

Hilton's Overfull Conjecture (Conjecture 4.5) aims to describe graphs with large maximum degree whose chromatic index is $\Delta + 1$. An analogous conjecture for total colouring would state that if the maximum degree of a graph $G$ is large enough, then $\chi''(G) = \left\lceil \chi''_f(G) \right\rceil$. If true, then the conjecture given in Section 5.4 would determine the total colouring number. In Section 5.5, we disprove this conjecture showing there does not exist an $\varepsilon > 0$ such that each graph $G$ with $\Delta(G) \geq \varepsilon|G|$, satisfies $\chi''(G) = \left\lceil \chi''_f(G) \right\rceil$.

## 5.1 The List Edge Chromatic Number

Determining if a vertex $c$-colouring extends to a total $c$-colouring is an instance of the list edge colouring problem, which we now define.

**Definition 5.1.** Given a list $L_e$ of colours for each edge $e$ in a graph $G$, an edge colouring of $G$ is *acceptable* if every edge is coloured with a colour from its list. The *list chromatic index*, denoted $\chi'_\ell(G)$, is the smallest $k \geq 0$ satisfying: if every list has size at least $k$ then there exists an acceptable edge colouring.

Given a vertex $c$-colouring, we simplely set for each edge $e$, $L_e$ to be the set obtained from $1, .., c$ by removing the colours of the endpoints of $c$. Clearly, there is an acceptable colouring for these lists precisely if the vertex colouring extends to a total $c$-colouring. This prompts our interest in list colouring and the list edge chromatic number. Now, if each edge $e$ is given the same list of size $t$, then a list edge colouring exists precisely when $t \geq \chi'(G)$. Hence,

**Observation 5.2.** *$\chi'_\ell(G) \geq \chi'(G)$.*

It is conjectured that the converse is true:

**Conjecture 5.3** (The list edge colouring conjecture)**.** *(see [56]) Every graph $G$ satisfies $\chi'_\ell(G) = \chi'(G)$.*

Galvin [38] proved that the conjecture is true for bipartite graphs, thus proving a famous conjecture of Dinitz on partial Latin squares (see [56] for more information). Bollobás and Harris showed that for $c > \frac{11}{6}$ and maximum degree $\Delta$ large enough, $\chi'_\ell(G) \leq c\Delta$ [11]. Kahn extended the ideas of the proof of Theorem 3.2 to show that for any multigraph $G$, $\chi'_\ell(G) \leq (1 + o(1))\chi'_f(G)$ [59]. This bound was improved to $\chi'_\ell(G) \leq \Delta + O(\Delta^{2/3}\sqrt{\log \Delta})$ by Häggkvist and Janssen [45], and sharpened by Molloy and Reed to $\chi'_\ell(G) \leq \Delta + O(\Delta^{1/2}(\log \Delta)^4)$ [83].

As described above the total colouring number and list edge colouring number are intimately related:

**Lemma 5.4.** *Every graph $G$ satisfies $\chi''(G) \leq \chi'_\ell(G) + 2$.*

*Proof.* Colour the vertices of $G$ with $\chi'_\ell(G) + 2 \geq \Delta + 2$ colours $S_1, ..., S_{\chi'_\ell(G)+2}$. For each edge $e \in E(G)$, let $L_e$ be the list of $\chi'_\ell(G)$ colours none of which are assigned to either endpoint of $e$. Let $M_1, ..., M_{\chi'_\ell(G)+2}$ be a list edge colouring of $G$ with respect to these lists. The way we chose the lists implies for each $i$, $S_i \cup M_i$ is a total stable set, and so $S_1 \cup M_1, ..., S_{\chi'_\ell(G)+2} \cup M_{\chi'_\ell(G)+2}$ is a total $(\chi'_\ell(G) + 2)$-colouring of $G$. $\square$

If the list edge colouring conjecture were true then this lemma together with Vizing's Theorem would imply that for any graph $G$, $\chi''(G) \leq \Delta + 3$, since $\chi''(G) \leq \chi'_\ell(G) + 2 = \chi'(G) + 2 \leq \Delta + 3$.

## 5.2 Fractional Total Colouring Revisited

In this section, we discuss bounding the fractional total colouring number. We show we can approximate it within an additive factor of 2. We remind the reader that a fractional total $\beta$-colouring of a graph $G$ is a solution of (not necessarily optimal) value $\beta$ to the linear program

$$\begin{array}{lll} \min & \mathbf{1}^T w & \\ \text{s.t.} & \sum_{T \ni x} w_T \geq 1 & \forall x \in V(G) \cup E(G) \\ & w \geq 0 & \\ & w \in \mathbb{R}^{\mathcal{T}(G)}. & \end{array} \tag{5.1}$$

The fractional total colouring number of $G$, denoted $\chi''_f(G)$, is the smallest $\beta \geq 0$ for which there exists a fractional total $\beta$-colouring. Unlike the closely related questions of determining the fractional chromatic number and fractional chromatic index the complexity of determining the fractional total colouring number is still unresolved.

The dual of (5.1) is:

$$
\begin{array}{lll}
\max & \mathbf{1}^T z & \\
\text{s.t.} & \sum_{x \in T} z_x \leq 1 & \forall T \in \mathcal{T}(G) \\
& z \geq 0 & \\
& z \in \mathbb{R}^{V(G) \cup E(G)}. &
\end{array}
\tag{5.2}
$$

As was the case for the fractional chromatic index, weak duality yields the following lower bound:

**Observation 5.5.** *For any graph $G$ of maximum degree $\Delta$, $\chi''_f(G) \geq \Delta + 1$.*

*Proof.* For any vertex $v \in V$, let $z^v_x = 1$ if $x \in \{v\} \cup \delta(v)$ and $z^v_x = 0$ otherwise. Since $z^v$ is dual feasible, $\chi''_f(G) \geq \max_{v \in G} \mathbf{1}^T z^v = \max_{v \in G} |\delta(v)| + 1 = \Delta + 1$. □

Letting $\alpha_t(G)$ be the size of a largest total stable set in $G$, we have:

**Observation 5.6.** *For any graph $G$ of maximum degree $\Delta$, $\chi''_f(G) \geq \frac{|G|+|E(G)|}{\alpha_t(G)}$.*

*Proof.* This follows since $z = \left(\frac{1}{\alpha_t(G)}, ..., \frac{1}{\alpha_t(G)}\right)$ is dual feasible. □

We discussed in Section 5.1, the conjecture that the total colouring number is within two of the chromatic index. We now give a short argument which yields that the fractional total colouring number of any graph is within two of the chromatic index.

**Lemma 5.7.** *[67] For any graph $G$, $\chi''_f(G) \leq \chi'(G) + 2$.*

*Proof.* The main idea is that it is possible to separately choose a vertex colouring and an edge colouring of $G$, such that we can match up the stable sets and matchings which receive positive weight to find the desired weighting of total stable sets.

We start by arbitrarily choosing a vertex $(\chi'(G) + 2)$-colouring of $G$. For each colour class $S_i$, $1 \leq i \leq \chi'(G) + 2$, the induced subgraph $G - S_i$ has an edge $\chi'(G)$-colouring $y^i$. For each matching $M_j \in \mathcal{M}(G - S_i)$ with $y^i_{M_j} = 1$, we let the total stable set $T_{i,j} = S_i \cup M_j$

have weight $w_{T_{i,j}} = 1/\chi'(G)$. For all other total stable sets $T \in \mathcal{T}(G)$, we let $w_T = 0$. As $y^i$ was an edge $\chi'(G)$-colouring of $G - S_i$, each $v$ in $S_i$ is in exactly $\chi'(G)$ of the total stable sets $T_{i,j}$. Each $e = xy$ is in $T_{i,j(e,i)}$ for some $j(e,i)$ for every $i$ between 1 and $\chi'(G)+2$ such that $x \notin S_i$ and $y \notin S_i$. Hence, for any vertex or edge, the total weight given to total stable sets which contain it is 1. Thus, $w$ is a fractional total colouring and it is immediate that the total weight is $\chi'(G)+2$. □

This together with Vizing's theorem immediately implies:

**Corollary 5.8.** *For any graph $G$ with maximum degree $\Delta$, $\chi''_f(G) \leq \Delta + 3$.*

The same approach yields a fractional total $(\Delta+2)$-colouring provided $G$ has a vertex colouring $S_1, ..., S_{\Delta+2}$ such that each $G - S_i$ has fractional edge $\Delta$-colouring $y^i$. For each stable set $S_i$ and matching $M_i \in \mathcal{M}(G - S_i)$, we assign weight $\frac{y^i_M}{\Delta(G)}$ to the total stable set $S_i \cup M_i$. As the weights $\frac{y^i_M}{\Delta(G)}$ define a convex combination of matchings in $G - S_i$, we have that each $S_i$ is in weight 1 of total stable sets and the total weight of all total stable sets is $\Delta(G)+2$. Furthermore, for each edge $e$ and stable set $S_i$ not containing an endpoint of $e$, the total weight given to total stable sets containing $e$ and whose intersection with $V(G)$ is exactly $S_i$ is $\frac{1}{\Delta(G)}$. As the endpoints of each edge are contained in exactly two stable sets, we have that each edge is in weight 1 of total stable sets.

Kilakos and Reed [67] strengthen this to show the following fractional analogue of the Total Colouring Conjecture:

**Theorem 5.9.** *[67] For any graph $G$, $\Delta + 1 \leq \chi''_f(G) \leq \Delta + 2$.*

They obtain a vertex $(\Delta+2)$-colouring $S_1, ..., S_{\Delta+2}$ such that $G - S_i$ has a fractional edge $\Delta$-colouring for each $i$ between 1 and $\Delta+1$ and such that $S_{\Delta+2}$ has some special properties which allow them to fractionally total $(\Delta+2)$-colour $G$, even though $G - S_{\Delta+2}$ may not be fractionally edge $\Delta$-colourable.

In finding such a vertex $(\Delta + 2)$-colouring, Kilakos and Reed focus on overfull subgraphs, in particular, they focus on the cut-condition and intersection properties of overfull subgraphs. We give more details in Chapter 6. We then sharpen the techniques in their proof to show that $\chi''_f(G) = \Delta + 2$ precisely if some component of $G$ is either an even clique on $\Delta + 1$ vertices $K_{\Delta+1}$ or the complete bipartite graph $K_{\Delta,\Delta}$.

## 5.3 The Conformability Conjecture

We now examine a condition proposed by Chetwynd and Hilton which is required for a vertex $(\Delta+1)$-colouring to be extended to a total $(\Delta+1)$-colouring [18]. We then discuss their *Conformability Conjecture* which states that these conditions are sufficient when the graph has large maximum degree and does not contain a certain problematic subgraph.

Suppose $T_1 = S_1 \cup M_1, T_2 = S_2 \cup M_2, ..., T_{\Delta+1} = S_{\Delta+1} \cup M_{\Delta+1}$ is a total $(\Delta + 1)$-colouring. (Thus for each $i$, $S_i$ is a stable set and $M_i$ is a matching of $G - S_i$.) We say vertex $v$ is *missed* by $T_i$ if $v \notin S_i \cup V(M_i)$. Clearly, each vertex $v$ is missed by $\Delta - d_G(v)$ of the $T_i$. Trivially, if $|G - S_i|$ is odd, then $T_i$ misses a vertex. This yields:

**Observation 5.10.** *If $T_1, ..., T_{\Delta+1}$ is a total $(\Delta + 1)$-colouring, then the number of total stable sets for which $|G - (V(G) \cap T_i)|$ is odd is at most $\sum_{v \in G}(\Delta - d(v))$.*

Furthermore, since for any subgraph $H$ of $G$, $\chi''(G) \geq \chi''(H)$, we have:

**Observation 5.11.** *If $T_1, ..., T_{\Delta+1}$ is a total $(\Delta + 1)$-colouring, then for any subgraph $H$ of $G$ the number of total stable sets for which $|H - (V(G) \cap T_i)|$ is odd is at most $\sum_{v \in H}(\Delta - d_H(v))$.*

This observation suggests the following definition.

**Definition 5.12.** A vertex $(\Delta + 1)$-colouring $S_1, ..., S_{\Delta+1}$ is *conformable* if for each subgraph $H$ of $G$ the number of $S_i$ for which $|H - S_i|$ is odd is at most $\sum_{v \in H}(\Delta - d_H(v))$, and *nonconformable* otherwise.

Observation 5.11 now implies:

**Lemma 5.13.** *[18, 19] If every vertex* $(\Delta+1)$*-colouring of* $G$ *is nonconformable, then* $\chi''(G) \geq \Delta + 2$.

We note that the converse to Lemma 5.13 does not hold in general. To see this, consider the complete bipartite graph $K_{\Delta,\Delta}$ with $\Delta$ even and bipartition $(A, B)$. A conformable colouring is given by $S_1 = A, S_2 = B$, and $S_i = \emptyset$ for each $i$ between 3 and $\Delta+1$. On the other hand, the largest size of a total stable set in $G$ is $\Delta$, and so, $\chi''(G) \geq \frac{|G|+|E(G)|}{\Delta} = \frac{2\Delta+\Delta^2}{\Delta} = \Delta + 2$.

Hilton and Chetwyn [19] conjectured that if $\Delta \geq \frac{|G|+1}{2}$ then the converse of Lemma 5.13 is also true. Unfortunately, Chen and Fu [14] noticed the graph $K^*_{2n+1}$ obtained by subdividing any edge of an odd clique on $2n+1$ is a counterexample to this conjecture. It has a conformable vertex colouring while $\chi''(K^*_{2n+1}) = \Delta(K^*_{2n+1}) + 2$. Hamilton, Hilton

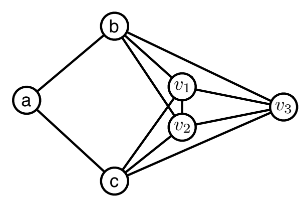



Figure 5–2: $K^*_5$ and a conformable colouring.

and Hind [48] proposed that $K^*_{2n+1}$ is the unique counterexample to the conjecture of Hilton and Chetwyn.

**Conjecture 5.14** (The Conformability Conjecture)**.** *[48] Suppose a graph* $G$ *has maximum degree* $\Delta \geq \frac{|V(G)|+1}{2}$ *and* $G$ *does not contain* $K^*_{2n+1}$ *as a subgraph whenever* $\Delta = 2n$. *Then* $\chi''(G) = \Delta + 1$ *precisely when there exists a conformable colouring of* $G$.

Hamilton *et al.* give the following reason why the believe that $K^*_{2n+1}$ is the only graph which needs to be excluded in Conjecture 5.14. Let $a$ be a vertex of minimum degree, $b$ and $c$ be the two vertices adjacent to $a$ and $v_1, ..., v_{2n-1}$ be the vertices in the non-neighbourhood of $a$. They show that up to relabelling the vertices $v_1, ..., v_{2n-1}$, $K^*_{2n+1}$ has exactly one conformable vertex colouring: $S_0 = \{a, v_1\}, S_1 = \{b, c\}, S_2 = \{v_2\}, ..., S_{2n-1} = \{v_{2n-1}\}, S_{2n} = \{\emptyset\}$. Hence by Observation 5.11, if $K^*_{2n+1}$ has a total $\Delta + 1$-colouring $T_1, ..., T_{\Delta+1}$ then up to relabelling it must be the case that the (vertex) stable sets of $T_1, ..., T_{\Delta+1}$ are exactly $S_0, ..., S_{2n}$. Now, this vertex colouring has exactly $2n - 2$ colour classes $S$ for which $|G - S|$ is odd and for the colour class $S_1$, $K^*_{2n+1} - S_1$ contains a singleton odd component. It follows that, letting $\mathrm{sc}(F)$ be the number of singletons in $F$, we have $H = K^*_{2n+1}$ satisfies:

$$\sum_{\ell=0}^{2n}([|H - S_\ell| - \mathrm{sc}(H - S_\ell)] \mod 2) + \sum_{\ell=0}^{2n} \mathrm{sc}(H - S_\ell) = 2n. \tag{5.3}$$

Since this is greater than $\sum_{v \in H}(\Delta - d(v)) = 2n - 2$, an argument similar to Observation 5.10 implies that no total $(\Delta + 1)$-colouring exists. Hamilton *et al.* [48] then prove that $K^*_{2n+1}$ is the unique graph $H$ for which any conformable colouring has the left-hand side of (5.3) greater than $\sum_{v \in H}(\Delta - d(v))$. We omit further details.

We note that a more natural weakening of Conjecture 5.14 which also accounts for the Chen and Fu counterexample is the following. For a vertex colouring $S_1, ..., S_{\Delta+1}$ of a graph $G$ to be extendable to a total $(\Delta + 1)$-colouring, the sum over $i = 1, ..., \Delta + 1$ of the number of vertices of $G - S_i$ missed by a maximum matching of $G - S_i$ is at most $\sum_{v \in G}(\Delta - d(v))$. More strictly, if $S_1, ..., S_{\Delta+1}$ is extendable then for each subgraph $H$ of $G$, the sum over $i = 1, ..., \Delta + 1$ of the number of vertices of $H - S_i$ missed by a maximum matching of $H - S_i$ is at most $\sum_{v \in H}(\Delta - d_H(v))$. We define such a vertex colouring to be *matching-conformable* and conjecture the following.

**Conjecture 5.15.** *If $G$ has a matching-conformable colouring, then $G$ has a total $(\Delta+1)$-colouring.*

There is some evidence to support Conjecture 5.14. For example, letting $\mathrm{def}(G) = \sum_{v\in G}(\Delta - d(v))$, Hamilton *et al.* showed the conjecture is true for odd order graphs $G$ satisfying $\Delta \geq (\sqrt{7}|G| + \mathrm{def}(G) + 1)/3$ and $\mathrm{def}(G) \leq |G| - \Delta - 1$ [48].

To conclude this section, we note that the Conformability Conjecture is aimed at identifying a simple reason why a graph has a total $(\Delta+1)$-colouring. Unfortunately, it is not clear if one can determine whether a graph is conformable in polynomial time. Letting $\mu(H)$ be the size of a maximum matching in a graph $H$, we have:

**Theorem 5.16.** *[111, 48] Suppose $G$ satisfies $|G| = 2n$ and $\Delta \geq 2n - 1$. Then $G$ is conformable precisely when $|E(\overline{G})| + \mu(\overline{G}) \leq n(2n - \Delta) - 1$.*

By Edmonds [28], we can determine $\mu(\overline{G})$ in polynomial time, and so, we can determine if a graph of even order $|G| = 2n$ and $\Delta \geq 2n-1$ is conformable in polynomial time. On the other hand, it is unknown whether there exists an algorithm to determine if a graph with odd order has a conformable colouring in polynomial time. We refer the interested reader to [48] for further discussion.

## 5.4 A Fractional Conformability Conjecture

In this section, we generalize the ideas of the Conformability Conjecture to fractional total colouring. We then discuss how the result in Chapter 7 proves a very natural special case of this conjecture.

Analogous to Observation 5.10 we have:

**Lemma 5.17.** *For any fractional total $\beta$-colouring $w$ of a graph $G$, we have*

$$\sum_{\substack{T \in \mathcal{T}(G) \\ |G - (V(G)\cap T)| \text{ is odd}}} w_T \leq \sum_{v\in G}(\beta - 1 - d(v)).$$

*Proof.* For each vertex $v$ of $G$, the total weight of total stable sets which contain an element of $v \cup \delta(v)$ is at least $d(v)+1$. Hence, letting $\mathcal{T}(v)$ be the set of total stable sets of $\mathcal{T}(G)$

which do not intersect $v \cup \delta(v)$, we have $\sum_{T \in \mathcal{T}(v)} w_T \leq \beta - 1 - d(v)$. For each total stable set $T$, let $m_T$ be the number of vertices which are not in $V(G) \cap T$ nor an endpoint of an edge in $E(G) \cap T$. On one hand, we have

$$\sum_{T \in \mathcal{T}(G)} w_T m_T = \sum_{v \in G} \sum_{T \in \mathcal{T}(v)} w_T \leq \sum_{v \in G} (\beta - 1 - d(v)). \tag{5.4}$$

On the other hand, if for the total stable set $T$ we have $|G - (T \cap V(G))|$ is odd, then $m_T \geq 1$, and so,

$$\sum_{\substack{T \in \mathcal{T}(G) \\ |G - (T \cap V(G))| \text{ is odd}}} w_T \leq \sum_{T \in \mathcal{T}(G)} w_T m_T. \tag{5.5}$$

The lemma follows by combining Equations 5.4 and 5.5. $\square$

Since for any subgraph $H$ of $G$, $\chi''_f(G) \geq \chi''_f(H)$, Lemma 5.17 suggests the following definition and implies the following theorem.

**Definition 5.18.** Let $\beta(G)$ be the minimum $\beta \geq \Delta + 1$ such that there exists a fractional vertex $\beta$-colouring $x$ satisfying for each $H \subseteq G$,

$$\sum_{\substack{S \in \mathcal{S}(G) \\ |H - S| \text{ odd}}} x_S \leq \sum_{v \in H} (\beta - 1 - d_H(v)). \tag{5.6}$$

**Theorem 5.19.** *For any graph $G$ of maximum degree $\Delta$, $\chi''_f(G) \geq \beta(G)$.*

We note that if a graph $G$ has a conformable colouring then $\beta(G) = \Delta + 1$. In particular, $\beta(K^*_{2n+1}) = \Delta(K^*_{2n+1}) + 1$. So, the bound of Theorem 5.19 is not always tight, since $\chi''_f(K^*_{2n+1}) > \Delta(K^*_{2n+1}) + 1$. Indeed, since $K^*_3 = K_{2,2}$ and $\alpha_t(K_{2,2}) = 2$, Observation 5.6 implies $\chi''_f(K_{2,2}) \geq \frac{|K_{2,2}| + |E(K_{2,2})|}{\alpha_t(K_{2,2})} = 4 = \Delta(K_{2,2}) + 2$. For larger $n$ a simple duality argument implies:

**Lemma 5.20.** *For each $n \geq 2$, $\chi''_f(K^*_{2n+1}) \geq 2n + 1 + \frac{1}{n+1}$.*

*Proof.* Let $a$ be the unique vertex of minimum degree. It is easy to verify that the largest colour class containing $a$ has size $n+2$, and any colour class not containing $a$ has size at most $n+1$. So, we can construct a feasible solution $z \in \mathbb{R}^{V(G)\cup E(G)}$ to (5.2) the dual to the fractional edge colouring LP by letting each $x \in V(G) \cup E(G) - \{a\}$ have weight $z_x = \frac{1}{n+1}$ and $z_a = 0$. Since $|V(G)| = 2n+2$ and $|E(G)| = (2n+1)n+1$, we have $\mathbf{1}^T z = \frac{1}{n+1}(2n+1+(2n+1)n+1) = 2n+1+\frac{1}{n+1}$. Hence, the lemma follows by weak duality. $\square$

As a fractional variant of the Conformability Conjecture we propose:

**Conjecture 5.21** (Fractional Conformability Conjecture)**.** *Suppose $G$ is a graph with maximum degree greater than $\frac{1}{2}|V(G)|$ and not containing $K^*_{2n+1}$, whenever $\Delta = 2n$. Then $\chi''_f(G) = \beta(G)$.*

## 5.5 No Analogue of the Overfull Conjecture for Total Colouring.

In this section, we discuss a conjecture which, similar to the Conformability Conjecture, seeks to describe the graphs with large maximum degree whose total chromatic number is $\Delta + 1$. Analogous to the Overfull Conjecture, it conjectures that there exists an $\varepsilon > 0$ such that each graph $G$ with $\Delta(G) \geq \varepsilon|G|$, satisfies $\chi''(G) = \left\lceil \chi''_f(G) \right\rceil$. We now give an example showing that this conjecture is false.

Consider $H^{2k}$ the graph built by taking $2k$ disjoint copies of $C_5$, where any two vertices not in the same $C_5$ are adjacent, i.e. $\overline{H^{2k}}$ is the disjoint union of $2k$ disjoint copies of $C_5$ (see Figure 5–3 for an example when $k = 1$). In this section, we prove the following lemmas.

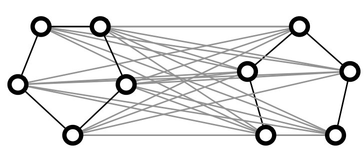


Figure 5–3: The graph $H^2$

**Lemma 5.22.** *For each* $k \geq 1$, $\chi''(H^{2k}) > \Delta(H^{2k}) + 1$.

**Lemma 5.23.** *For each* $k \geq 1$, $\chi''_f(H^{2k}) = \Delta(H^{2k}) + 1$.

Since $\Delta(H^{2k}) = 10k - 3$ and $|H^{2k}| = 10k$, it follows that there does not exist an $\varepsilon > 0$ such that each graph $G$ with $\Delta(G) \geq \varepsilon|G|$, satisfies $\chi''(G) = \left\lceil \chi''_f(G) \right\rceil$.

*Proof of Lemma 5.22.* Each vertex in $H^{2k}$ has degree $10k - 3$, and so, $\sum_{v \in H^{2k}} \Delta(H^{2k}) - d(v) = 0$. On the other hand, any vertex colouring of $H^{2k}$ must use at least $2k$ singleton colour classes, i.e., one per $C_5$. Together these two facts yield that every vertex $(\Delta + 1)$-colouring of $H^{2k}$ is nonconformable. By Lemma 5.13, this implies the result. □

*Proof of Lemma 5.23.* We construct a fractional total $(\Delta(H^{2k}) + 1)$-colouring of $H^{2k}$ in two steps. First, we assign weight to total stable sets whose intersection with $V(G)$ has size two, and second, we assign weight to total stable sets whose intersection with $V(G)$ has size zero.

In assigning weight to the total stable sets whose intersection with $V(G)$ has size zero, we need the following. Letting $H$ be a graph and $c \in \mathbb{R}^{E(H)}$, a *(fractional) weighted edge colouring (WEC)* is a solution to the following linear program.

$$\begin{array}{lll} \min & \mathbf{1}^T y & \\ \text{s.t.} & \sum_{M \ni e} y_M \geq c_e & \forall e \in E(H) \\ & y \geq 0, & \\ & y \in \mathbb{R}^{\mathcal{M}(H)}. & \end{array}$$

Edmonds' characterization of the matching polytope yields the optimal value to this linear program is $OPT(H, c) = \max\{\Delta(H, c), \Lambda(H, c)\}$, where $\Delta(H, c) := \max_{v \in H} c(\delta(v))$, and $\Lambda(H, c) := \max_{F \subseteq H, |F| > 1 \text{ odd}} \frac{2c(E(F))}{|F| - 1}$.

Label the stable sets of size two as $S_1, ..., S_{10k}$. For each $S_i$, let $e_i$ be the edge of the $C_5$ containing $S_i$ which is not incident with either vertex of $S_i$. We have $\Delta(H^{2k} - S_i - e_i) =$

$\Delta(H^{2k})-2$ and $H^{2k}-S_i-e_i$ contains no odd overfull subgraph. So, $\chi'_f(H^{2k}-S_i-e_i) = \Delta(H^{2k})-2 = 10k-5$. Let $y^i$ be an optimal fractional edge colouring of $H^{2k}-S_i-e_i$. For each matching $M_j \in \mathcal{M}(H^{2k}-S_i-e_i)$, let $T_{ij} = S_i \cup M_j$ have weight $w_{T_{ij}} = \frac{1/2}{10k-5} y^i_{M_j}$.

Now, consider the edge weights on $H^{2k}$ where each edge $e_i$ has weight $c_{e_i} = \frac{1}{2}$ and every other edge $e$ has weight $c_e = \frac{1}{2} - \frac{1/2}{10k-5}$. It is easily checked that $\Delta(H,c) = 5k-2$ and $\Lambda(H,c) = 5k-2$. So by Equation 6.3, $H$ with edge weights $c$ has a weighted fractional edge $(5k-2)$-colouring $z$. For each matching $M_k \in \mathcal{M}(H^{2k})$, let $T_k = M_k$ have weight $w_{T_k} = z_{M_k}$.

We claim that $w$ is a fractional total $(10k-2)$-colouring of $H^{2k}$. Each vertex $v$ is in exactly two stable sets of size two $S_i$ and $S_j$, and so,

$$\begin{aligned}
\sum_{T \ni v} w_T &= \sum_{T \ni S_i} w_T + \sum_{T \ni S_j} w_T \\
&= \sum_{M \ni \mathcal{M}(H^{2k}-S_i-e_i)} \frac{1/2}{10k-5} y^i_M + \sum_{M \ni \mathcal{M}(H^{2k}-S_j-e_j)} \frac{1/2}{10k-5} y^j_M \\
&= \frac{1}{2} + \frac{1}{2} = 1.
\end{aligned}$$

For each edge $e_i$, each of its endpoints is contained in exactly two stable sets of size two and $e_i$ is not contained in any total stable set $T$ with $T \cap V(H^{2k}) = S_i$. Hence, it is an edge of $H^{2k}-S_j-e_j$ for $10k-5$ values of $j$, and so,

$$\begin{aligned}
\sum_{T \ni e_i} w_T &= \sum_{j=1}^{10k} \sum_{\substack{M \in H^{2k}-S_j-e_j \\ M \ni e_i}} y^i_M \frac{1/2}{10k-5} + \sum_{\substack{M \in \mathcal{M}(H^{2k}) \\ M \ni e_i}} z_{M_k} \\
&= (10k-5)\frac{1/2}{10k-5} + \frac{1}{2} = 1.
\end{aligned}$$

For each remaining edge $e$, each of its endpoints is contained in exactly two stable sets of size two, and so, it is an edge of $H^{2k} - S_j - e_j$ for $10k - 4$ values of $j$. Hence,

$$\begin{aligned}\sum_{T \ni e} w_T &= \sum_{j=1}^{10k} \sum_{\substack{M \in H^{2k} - S_j - e_j \\ M \ni e}} y_M^i \frac{1/2}{10k-5} + \sum_{\substack{M \in \mathcal{M}(H^{2k}) \\ M \ni e}} z_{M_k} \\ &= (10k-4)\frac{1/2}{10k-5} + \frac{1}{2} - \frac{1/2}{10k-5} = 1.\end{aligned}$$

Finally, the total weight of this colouring is $\frac{1}{2}(10k) + 5k - 2 = 10k - 2$, as desired. $\square$

As a corollary to Lemma 5.23, we have that $\beta(H^{2k}) = \Delta(H^{2k}) + 1$. To see this, consider the fractional vertex $(\Delta(H^{2k}) + 1)$-colouring $x$, where for each stable set $S$ of size 2, we let $x_S = \frac{1}{2}$, we let $x_\emptyset = 5k - 2$, and all other stable set receive weight 0. Since each vertex is in exactly 2 stable sets of size 2, we have that $x$ is a fractional vertex colouring of total weight $10k\frac{1}{2} + 5k - 2 = 10k - 2 = \Delta(H^{2k}) + 1$, as desired. Since the fractional total $(\Delta(H^{2k}) + 1)$-colouring of Lemma 5.23 extends this colouring, by Theorem 5.19, $\beta(H^{2k}) = \Delta(H^{2k}) + 1$.

# CHAPTER 6
## A Characterization of Graphs with Fractional Total Colouring Number Equal to $\Delta + 2$

In this chapter, we characterize exactly those graphs with maximum degree $\Delta$ whose fractional total colouring number is $\Delta + 2$. This yields a simple linear-time algorithm to determine whether a given graph has fractional total colouring number $\Delta + 2$. We prove,

**Theorem 6.1.** *The fractional total colouring number of a connected graph $G$ of maximum degree $\Delta$ is $\Delta + 2$ precisely when $G = K_{2n}$ or $G = K_{n,n}$ for some integer $n \geq 1$.*

The easy direction of Theorem 6.1 is showing that $K_{2n}$, $n \geq 1$, has fractional total colouring number $2n+1$ and $K_{n,n}$, $n \geq 1$, has fractional total colouring number $n+2$. We do this in Section 6.1. In Section 6.2, we deal separately with two easy cases. We directly show that Theorem 6.1 holds in the case that $G$ has maximum degree $\Delta \leq 2$ and in the case that $G$ is an odd clique on $\Delta + 1$ vertices. For the other cases, we prove

**Lemma 6.2.** *Suppose $G$ is a connected non-empty graph satisfying $\Delta \geq 3$, $G \neq K_{\Delta+1}$ and $G \neq K_{\Delta,\Delta}$. Then for each edge $e \in E$ there exists a fractional total $(\Delta + 2)$-colouring $w^e$ of $G$ such that the weight of the stable sets containing $e$ is strictly greater than $1$.*

As discussed in Section 6.1, combining this result with LP duality and the easy fact that every graph has chromatic number at most $\Delta + 1$, easily yields the hard direction of Theorem 6.1. Thus, the key to Theorem 6.1 is Lemma 6.2. Section 6.3 gives its proof. In Section 6.4, we discuss a conjecture about graphs whose fractional total colouring number is close to $\Delta + 2$. We close this introductory section by sketching the proof of Lemma 6.2.

In proving Lemma 6.2, we sharpen the Kilakos and Reed [67] proof that for any graph $G$, $\Delta + 1 \leq \chi''_f(G) \leq \Delta + 2$. The crux of their proof is choosing a vertex colouring which can be extended to a fractional total $(\Delta + 2)$-colouring, by extending the approach used in the proof of Lemma 5.7. They obtain a vertex $(\Delta + 2)$-colouring $S_1, ..., S_{\Delta+2}$ such that $G - S_i$ has a fractional edge $\Delta$-colouring for each $i$ between 1 and $\Delta + 1$ and such that $S_{\Delta+2}$ has some special properties which allow them to fractionally total $(\Delta + 2)$-colour $G$, even though $G - S_{\Delta+2}$ may not be fractionally edge $\Delta$-colourable.

The key to finding their vertex colouring is the cut-condition (Observation 4.2) and the intersection properties of overfull subgraphs, which we now discuss. The cut-condition ensures that if all overfull subgraphs are vertex disjoint, then we can iteratively vertex colour each overfull subgraph separately because there are few edges from each such subgraph to the rest of the graph. Although the overfull subgraphs need not be disjoint they have the useful intersection property, that for any two overfull subgraphs $H_1$ and $H_2$, at least one of $H_1 - H_2$ and $H_2 - H_1$ is overfull. Letting $U$ be an overfull subgraph of a graph $G$ it follows that every minimal overfull subgraph is contained in $U$ or $G - U$. From this it follows that *minimal* overfull subgraphs are vertex disjoint.

We remark that it is crucial that we consider both even and odd overfull subgraphs, since minimal odd overfull subgraphs need not be vertex disjoint (an example is given in Figure 6–1).

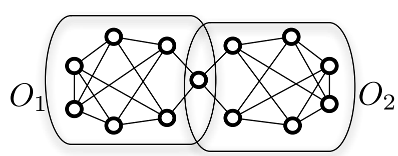



Figure 6–1: $O_1$ and $O_2$ are the only two minimal odd overfull subgraphs. They are not vertex disjoint. Notice $O_1 - O_2$ and $O_2 - O_1$ are minimal overfull subgraphs.

We note that given any overfull subgraph $U$, we can find all minimal overfull subgraphs by recursively checking the parts $U$ and $G - U$. An easy modification of Padberg and Rao's algorithm for finding odd overfull subgraphs, exploiting this fact, yields a polynomial time algorithm for determining if a graph $G$ contains an overfull subgraph. Thus in polynomial time, we can partition the vertex set of $G$ into $V(O_1), V(O_2), ..., V(O_k)$ and the rest $G - \bigcup_{j=1}^{k} V(O_j)$, where $O_1, O_2, ..., O_k$ are minimal overfull subgraphs and $G - \bigcup_{j=1}^{k} O_j$ contains no overfull subgraphs. Henceforth, we will assume that we are given this decomposition with the additional property that $|O_i| \leq |O_{i+1}|$ for $i$ between 1 and $k-1$.

Now, by using this partition we can find the vertex colouring as in Kilakos and Reed. In particular, they find a vertex colouring $S_1, S_2, ..., S_{\Delta+2}$ such that the following property holds:

(P) Each $O_j$ contains a vertex in colour class $S_i$ for each $i$ between 1 and $\Delta+1$, and each $O_j$ satisfying $|O_j| \geq \Delta + 2$ contains a vertex in colour class $S_{\Delta+2}$.

Thus, as required, $G - S_i$ has a fractional edge $\Delta$-colouring for each $i$ between 1 and $\Delta+1$ because $G - S_i$ contains no overfull subgraph. Moreover, Property (P) has the special properties alluded to before which allow Kilakos and Reed to complete the fractional total $(\Delta+2)$-colouring.

To prove Lemma 6.2, for each edge $e$, we will obtain a vertex colouring satisfying the conditions of Kilakos and Reed, and one extra condition. Specifically we insist that $e$ is in $G - S_1$ and that $G - S_1$ has a fractional edge $\Delta$-colouring in which the total weight of the matchings containing $e$ exceeds 1. We can then prove Lemma 6.2 by mimicking the approach of Kilakos and Reed. It remains then to discuss how to find the desired vertex colouring.

If $e$ is incident to a vertex $v$ of degree $\Delta$ in $G - S_1$ then our extra condition cannot be satisfied because the weight of the matchings in the colouring using the other edges incident to $v$ must be at least $\Delta - 1$. In the same vein, if $e$ is in an odd subgraph $H$ such that $|E(H)| = \Delta\frac{|H|-1}{2}$ then again simple counting shows that this is impossible. It turns out that Edmonds' characterization of the matching polytope easily implies that our desired special fractional edge-colouring of $G - S_1$ exists precisely if neither of these two possibilities occur.

We actually impose a slightly stronger condition.

**Definition 6.3.** A subgraph $F$ of $G$ is $\Delta$-*full* if $|E(F)| = \Delta\frac{|F|-1}{2}$ and $|F| > 1$.

Let $H_i = G - S_i$ for each $i$ between 1 and $\Delta + 2$. We call a vertex colouring $S_1, S_2, \ldots, S_{\Delta+2}$ satisfying Property (P) a *better colouring* for an edge $e = yz$ if

(i) $e \in E(H_1)$, $|\delta_{H_1}(y)| < \Delta$, and $|\delta_{H_1}(z)| < \Delta$, and

(ii) $H_1$ has no $\Delta$-full subgraph of $G$ which contains both $y$ and $z$.

Our remarks above imply the following result which we prove formally in Section 6.3.1:

**Lemma 6.4.** *If $G = (V, E)$ has a better colouring for an edge $e \in E$ then $G$ has a fractional total $(\Delta + 2)$-colouring $w^e$ such that $\sum_{T \ni e} w^e_T > 1$.*

**Remark 6.5.** It is easy to see that no such vertex colouring can exist for $K_{2n}$ nor $K_{n,n}$. For $K_{2n}$ every colour class of any vertex colouring has size at most 1 and the subgraph $K_{2n-1}$ is $(2n-1)$-full, hence for any edge Property (ii) can never be satisfied. For $K_{n,n}$ both endpoints of any edge $e = yz$ have degree $\Delta$ and every colour class is either completely contained in $N(y)$ or $N(z)$, hence Property (i) can never be satisfied.

To complete the proof of Lemma 6.2, it is enough to prove the following result which we sketch here and prove in Section 6.3.2:

**Lemma 6.6.** *Suppose $G = (V, E)$ is a connected non-empty graph satisfying $\Delta \geq 3$, $G \neq K_{\Delta+1}$ and $G \neq K_{\Delta,\Delta}$. Then for each edge $e \in E$, there exists a better colour for $e$.*

The main step in our proof of Lemma 6.6 is to first pick colour class $S_1$ so that Properties (i) and (ii) of better colourings hold, and so that we can extend the colouring to satisfy in addition Property (P). It will follow that in order to extend the colour class $S_1$ to a better colouring for $e$ it is enough to ensure that for each $O_i$, $|O_i - S_1| \geq \Delta$ whenever $|O_i| = \Delta + 1$ and $|O_i - S_1| \geq \Delta + 1$ otherwise.

For Property (i), we start by picking at most two vertices to be in $S_1$ so that both $|(N(y) - z) - S_1| < \Delta - 1$ and $|(N(z) - y) - S_1| < \Delta - 1$. This is mostly straightforward, except that we need to be careful not to pick two vertices out of any overfull subgraph of size at most $\Delta + 2$.

For Property (ii), we exploit the easy fact that if a subgraph is $\Delta$-full and contains no vertices of degree $\Delta$ then it is $K_\Delta$. We extend $S_1$ twice. First, so that if $e$ is contained in some minimal overfull subgraph $O_i$ then either $|O_i - S_1| \leq \Delta + 1$ or $V(O_i) - S_1$ does not contain any vertex $v$ satisfying $|N(v) \cap (V(O_i) - S_1)| = \Delta$. Second, so that no vertex $v$ of $(G - S_1) - \cup_{i=1}^{k} O_k$ satisfies $|N(v) \cap (G - S_1)| = \Delta$. Together these imply that if $e$ is contained in some $\Delta$-full subgraph $F$, then either $F = K_\Delta$ or, as shown in the Section 6.3.2, $G$ is some easily dealt with special cases. For the latter special cases, we separately show in Section 6.2 how to find the fractional total $(\Delta + 2)$-colouring $w^e$ such that for each edge $e$, $\sum_{T \ni e} w_T^e > 1$. For the former case, we show that by picking the first few vertices (used to establish Property (i)) carefully, we are able to ensure that $e$ is never contained in a $K_\Delta$.

## 6.1 Proving Theorem 6.1

We start by proving the easy direction of Theorem 6.1:

**Lemma 6.7.** *$\chi''_f(G) = \Delta + 2$ whenever $G = K_{2n}$ or $G = K_{n,n}$ for some integer $n \geq 1$.*

*Proof.* We remind the reader that $\alpha_t(G)$ is the size of a largest total stable set in $G$, and Observation 5.6 showed $\chi''_f(G) \geq (|G| + |E(G)|)/\alpha_t(G)$. If $G = K_{2n}$ then $\Delta = 2n - 1$ and we have

$$\chi''_f(G) \geq \frac{|G| + |E(G)|}{\alpha_t(K_{2n})} = \frac{2n + n(2n-1)}{n} = 2n + 1 = \Delta + 2.$$

If $G = K_{n,n}$ then $\Delta = n$ and we have

$$\chi''_f(G) \geq \frac{|G| + |E(G)|}{\alpha_t(K_{n,n})} = \frac{2n + n^2}{n} = n + 2 = \Delta + 2.$$

As $\chi''_f(G) \leq \Delta + 2$ for all graphs $G$, we have shown $\chi''_f(G) = \Delta + 2$ whenever $G = K_{2n}$ or $G = K_{n,n}$. □

Given Lemma 6.2 to finish the proof of Theorem 6.1 it is enough to show:

**Lemma 6.8.** *Suppose $G = (V, E)$ is a graph such that for every edge $e \in E$ there exists a fractional total $(\Delta + 2)$-colouring $w^e$ of $G$ such that $\sum_{T \ni e} w^e_T > 1$ for the edge $e$. Then, $\chi''_f(G) < \Delta + 2$.*

*Proof.* We assume for contradiction that $\chi''_f(G) = \Delta + 2$. We start by obtaining a new fractional total $(\Delta + 2)$-colouring $w^*$ of $G$, such that $\sum_{T \ni e} w^*_T > 1$ for every edge $e \in E$ by defining for each total stable set $T \in \mathcal{T}(G)$

$$w^*_T = \frac{1}{|E|} \sum_{e \in E} w^e_T.$$

We point out that $w^*$ is an optimal fractional total colouring, so we know by duality that there exists some dual solution with dual optimal value $\Delta + 2$. Consider the dual to the fractional total colouring linear program:

$$\begin{array}{lll} \max & \sum_{u \in V \cup E} \gamma_u & \\ \text{s.t.} & \sum_{u \in T} \gamma_u \leq 1 & \forall T \in \mathcal{T}(G) \\ & \gamma \geq 0 & \\ & \gamma \in \mathbb{R}^{V \cup E}. & \end{array} \tag{6.1}$$

Let $\gamma^*$ be some dual optimal solution. Since $\sum_{T \ni e} w^*_T > 1$ for every edge $e \in E$, complementary slackness implies that $\gamma^*_e = 0$ for every edge $e \in E$. Therefore, the maximum value to (6.1) is equal to the maximum value of the following linear program:

$$\begin{aligned} \max \quad & \textstyle\sum_{u \in V} \gamma'_u \\ \text{s.t.} \quad & \textstyle\sum_{u \in S} \gamma'_u \leq 1 \qquad \forall S \in \mathcal{S}(G) \\ & \gamma' \geq 0 \\ & \gamma' \in \mathbb{R}^V. \end{aligned} \tag{6.2}$$

Eq. (6.2) defines the well studied *fractional clique number* of a graph, which is dual to the fractional chromatic number:

$$\begin{aligned} \min \quad & \textstyle\sum_{S \in \mathcal{S}(G)} w'_S \\ \text{s.t.} \quad & \textstyle\sum_{S \ni u} w'_S \geq 1 \qquad \forall u \in V \\ & w' \geq 0 \\ & w' \in \mathbb{R}^{\mathcal{S}(G)}. \end{aligned}$$

Since every graph $G$ has a vertex colouring with at most $\Delta+1$ colours, duality implies that the maximum value to the LP (6.2) is at most $\Delta + 1$. But, this contradicts the maximum value to (6.1) is $\Delta + 2$, and hence either $G$ has a fractional total colouring whose total weight is strictly less than $\Delta+2$ or $G$ does not have a colouring satisfying Lemma 6.2. $\square$

## 6.2 Special Cases

In this section, we directly show that Theorem 6.1 is satisfied when the maximum degree $\Delta \leq 2$ or $G$ is an odd clique on $\Delta + 1$ vertices.

**Lemma 6.9.** *For each $n \geq 1$, $\chi''_f(K_{2n+1}) = \Delta(K_{2n+1}) + 1 = 2n + 1$.*

*Proof.* We show $\chi''(K_{2n+1}) = 2n+1$ with an easy reduction to an edge colouring problem. Let $G + v$ be the graph built by taking a copy of $G$ plus a universal vertex $v$. So, $G + v = K_{2n+2}$, $\Delta(G + v) = 2n + 1$. Even cliques $C$ are well known to be edge $\Delta(C)$-colourable, and so let $M_1, ..., M_{2n+1}$ be a $(2n + 1)$-edge colouring of $G + v$. Since $|\delta(v)| = 2n + 1$, for each $j$ between 1 and $2n + 1$, $M_j \cap \delta(v) \neq \emptyset$. For each matching $M_j$, if $M_j$ contains the edge $uv$, $u \in G$, then let $T_j = M_j - uv + u$. Now, it is easy to check that each $T_j$ is a total stable set of $G$. Moreover, each vertex and edge is contained in exactly 1 total stable set $T_j$. Hence, $w$ is a total $(2n+1)$-colouring of $G$. $\square$

Let $G$ be a connected graph of maximum degree $\Delta$. Trivially, if $\Delta = 0$ then $G$ is a single vertex and hence $\chi''_f(G) = \chi''(G) = 1$. Furthermore, if $\Delta = 1$ then $G = K_2$ and $\chi''_f(G) = \chi''(G) = 3$.

**Lemma 6.10.** *Suppose $G$ is a connected graph satisfying $\Delta = 2$.*

1. *If $G$ is a path on $n \geq 3$ vertices $P_n$ then $\chi''_f(P_n) = 3$.*
2. *If $G$ is a cycle $C_k$ then*

$$\chi''_f(C_k) = \begin{cases} 3 & k = 3n, n \geq 1 \\ 3 + 1/n & k = 3n+1, n \geq 1 \\ 3 + 1/(2n+1) & k = 3n+2, n \geq 1. \end{cases}$$

*Proof.* If $G$ is a path $P_n,\ n \geq 3$ or a cycle $C_{3n},\ n \geq 1$, then $\chi''_f(G) = \chi''(G) = 3 = \Delta + 1$. The only non-trivial cases are when $G$ is one of the cycles $C_{3n+1}$ or $C_{3n+2}$ for $n \geq 1$.

Let $G = C_{3n+1} = (v_0, v_2, ..., v_{3n})$ with $n \geq 1$. For each $i$, $0 \leq i \leq 3n$, define the total stable set $T_i = S_i \cup M_i$ where

$$S_i = \{v_j \mid j = i + 3k \mod (3n+1),\ k = 0, 1, ..., n-1\},$$

and

$$M_i = \{v_j v_{j+1} \mid j = i + 3k + 1 \mod (3n+1),\ k = 0, 1, ..., n-1\}.$$

Give $T_i$ weight $w_{T_i} = 1/n$. For all other total stable sets $T \in \mathcal{T}(G)$, give $T$ weight $w_T = 0$. Since each vertex and edge is in exactly $n$ of the $3n+1$ total stable sets $T_1, ..., T_{3n+1}$, $w$ is a fractional total $(3 + 1/n)$-colouring of $G$. We note that $\alpha_t(C_{3n+1}) = 2n$, and so $\chi''_f(C_{3n+1}) \geq 2(3n+1)/2n = 3 + 1/n$. Hence, $\chi''_f(C_{3n+1}) = 3 + 1/n$.

Let $G = C_{3n+2} = (v_0, v_2, ..., v_{3n+1})$ with $n \geq 1$. For each $i$, $0 \leq i \leq 3n+1$, define the total stable sets $T_i^1 = S_i \cup M_i \cup \{v_l\}$ and $T_i^2 = S_i \cup M_i \cup \{v_l v_{l+1}\}$, where $l = i + 3n$,

$$S_i = \{v_j \mid j = i + 3k \mod (3n+2),\ k = 0, 1, ..., n-1\},$$

and

$$M_i = \{v_j v_{j+1} \mid j = i + 3k + 1 \mod (3n+2),\ k = 0, 1, ..., n-1\},$$

Assign $w_{T_i^1} = 1/(2n+1)$, $w_{T_i^2} = 1/(2n+1)$, and $w_T = 0$ for all other total stable sets $T \in \mathcal{T}(G)$. Each vertex is in exactly $n+1$ of the total stable sets $T_0^1, ..., T_{3n+1}^1$ and $n$ of the total stable sets $T_0^2, ..., T_{3n+1}^2$. Hence, the total weight of stable sets containing this vertex is 1. Similarly, each the total weight of stable sets containing any edge is 1. Hence, $w$ is a fractional total $(3 + 1/(2n+1))$-colouring of $G$. Since $\alpha_t(C_{3n+2}) = 2n+1$, we have $\chi_f''(C_{3n+2}) \geq 2(3n+2)/(2n+1) = 3 + 1/(2n+1)$. Hence, $\chi_f''(C_{3n+2}) = 3 + 1/(2n+1)$. $\square$

### 6.3 Proving Lemma 6.2

As noted above Lemma 6.2 follows directly from Lemmas 6.4 and 6.6.

#### 6.3.1 Proof of Lemma 6.4

Suppose that a graph $G$ has a better colouring $S_1, S_2, \ldots, S_{\Delta+2}$ for an edge $e = yz \in E$. In this section, we show that this implies that $G$ has a fractional total $(\Delta+2)$-colouring $w^e$ such that $\sum_{T \ni e} w_T^e > 1$.

Let $p$ be the number of minimal overfull subgraphs $O_1, ..., O_p$ of size $\Delta + 1$. By the cut condition, any overfull subgraph $O$ satisfies $|\delta(O)| < \Delta$. Hence, for each $j = 1, ..., p$, $O_j$ contains at least two vertices all of whose neighbours are in $O$. Since each vertex in $O_j$ receives a unique colour, one such vertex is not in $S_1$. Select such a vertex $w_j$ from each $O_j$ and let $A$ be the set of the chosen vertices. Let $A_i = A \cap S_i$ for each $i$, $2 \leq i \leq \Delta + 1$.

Letting $H$ be a graph and $c \in \mathbb{R}^{E(H)}$, recall that a *weighted edge $\beta$-colouring* with respect to edge weights $c \in \mathbb{R}^{E(G)}$ is a (not necessarily optimal) solution to the following

LP of value $\beta$:

$$\begin{array}{lll} \min & \mathbf{1}^T w & \\ \text{s.t.} & \sum_{M \ni e} w_M \geq c_e & \forall e \in E(G) \\ & w \geq 0, & \\ & w \in \mathbb{R}^{\mathcal{M}(G)}. & \end{array} \tag{6.3}$$

Recall further that Edmonds' characterization of the matching polytope yields that the optimal value of this LP is

$$\max \left\{ \max_{v \in H} c(\delta(v)), \max_{F \subseteq H, |F|>1 \text{ odd}} \frac{2c(E(F))}{|F|-1} \right\}.$$

Let $\Lambda_{\max} = \max\left\{ \frac{2|E(U)|}{|U|-1} : U \subseteq H_1 \text{ and } y, z \in U \right\}$ and $\varepsilon = \min\{1, \Delta - \Lambda_{\max}\}$. We construct a weighted edge colouring of $H_1 := G - S_1$ where $c_e = 1 + \varepsilon$ for the edge $e$ and $c_f = 1$ for each edge $f \in E(H_1) - \{e\}$. Since $\{S_1, S_2, \ldots, S_{\Delta+2}\}$ is a better colouring for $e = yz$, we have $\Lambda_{\max} < \Delta$ and hence $0 < \varepsilon \leq 1$. It follows that since $|\delta_{H_1}(y)| \leq \Delta - 1$ and $|\delta_{H_1}(z)| \leq \Delta - 1$, we have for all $u \in H_1$ that $\max_{v \in H_1} c(\delta_{H_1}(v)) \leq \Delta$. Moreover, for all $U \subseteq H_1$ not containing both $y$ and $z$ we have $\frac{2c(E(U))}{|U|-1} = \frac{2|E(U)|}{|U|-1} \leq \Delta$, and for all $U$ with $|U| \geq 3$, odd, and containing both $y$ and $z$ we have

$$\frac{c(E(U))}{\left\lfloor \frac{1}{2}(|U|-1) \right\rfloor} \leq \Lambda_{\max} + \frac{\varepsilon}{\left\lfloor \frac{1}{2}(|U|-1) \right\rfloor} \leq \Delta.$$

Hence, there exists a weighted edge $\Delta$-colouring $y^1$ of $H_1$.

For each $i$, $2 \leq i \leq \Delta + 1$, we obtain a weighted edge $\Delta$-colouring $y^i$ of the graph $H_i := G - (S_i - A_i)$ with edge weights $c_f = 1/2$ for each edge $f \in \delta(A_i)$ and $c_f = 1$ otherwise. Such a weighted edge colouring $y^i$ exists: for any vertex $v$, $c(\delta(v)) \leq |\delta(v)| \leq \Delta$, no subgraph of $G - S_i$ is overfull and each subgraph $U$ of $H_i$ containing some vertex of

$v \in A$ satisfies

$$\frac{2c(E(U))}{|U|-1} \le \frac{\Delta(|U|-1) - \frac{1}{2}\delta(v) + \frac{1}{2}\delta(v)}{|U|-1} = \Delta.$$

Similarly, for $H_{\Delta+2} := G - S_{\Delta+2} + A$ with edge weights $c_f = 1/2$ for each edge $f \in \delta(A)$ and $c_f = 1$ otherwise, let $y^{\Delta+2}$ be a weighted edge $\Delta$-colouring.

We are now ready to construct a fractional total colouring $w^e$ of $G$ satisfying Lemma 6.2. For each $i$, $1 \le i \le \Delta+2$, let

$$T_{i,M} = \begin{cases} M \cup \big(S_i - \{w : w \in A_i, \delta(w) \cap M \neq \emptyset\}\big) & \text{for } 1 \le i \le \Delta+1 \\ M \cup S_{\Delta+2} \cup \{w : w \in A_i, \delta(w) \cap M = \emptyset\} & \text{for } i = \Delta+2 \end{cases}$$

and $w^e_{T_{i,M}} = y^i_M/\Delta$ for all $M \in \mathcal{M}(H_i)$ and for all other total stable sets $T \in \mathcal{T}(G)$ of $G$, let $w^e_T = 0$. Clearly, each $T_{i,M}$ is a total stable set of $G$. Moreover, for each vertex $u \in V - A$, if $u \in S_i$ then

$$\sum_{T \ni u} w^e_T = \sum_{M \in \mathcal{M}(H_i)} \frac{y^i_M}{\Delta} = \frac{\Delta}{\Delta} = 1.$$

For each vertex $v \in A_i$, $2 \le i \le \Delta+1$,

$$\sum_{T \ni v} w^e_T = \sum_{\substack{M \,\in\, \mathcal{M}(H_i) \\ \delta(v) \cap M = \emptyset}} \frac{y^i_M}{\Delta} + \sum_{\substack{M \,\in\, \mathcal{M}(H_{\Delta+2}) \\ \delta(v) \cap M = \emptyset}} \frac{y^{\Delta+2}_M}{\Delta} \ge \frac{\Delta}{2\Delta} + \frac{\Delta}{2\Delta} = 1.$$

For each edge $f \notin \delta(A) \cup \{e\}$ with one end in $S_k$ and one end in $S_m$,

$$\sum_{T \ni f} w^e_T = \sum_{\substack{i=1 \\ i \neq \{k,m\}}}^{\Delta+2} \sum_{\substack{M \in \mathcal{M}(H_i) \\ M \ni f}} \frac{y^i_M}{\Delta} \ge \sum_{\substack{i=1 \\ i \neq \{k,m\}}}^{\Delta+2} \frac{1}{\Delta} = 1.$$

For each edge $f \in \delta(A)$, with one end in $A \cap S_k$ and one end in $S_m$,

$$\begin{aligned}
\sum_{T \ni f} w_T^e &= \sum_{\substack{i=1 \\ i \neq m}}^{\Delta+2} \sum_{\substack{M \in \mathcal{M}(H_i) \\ M \ni f}} \frac{y_M^i}{\Delta} \\
&= \sum_{\substack{i=1 \\ i \neq \{k,m\}}}^{\Delta+1} \left[ \sum_{\substack{M \in \mathcal{M}(H_i) \\ M \ni f}} \frac{y_M^i}{\Delta} \right] + \sum_{\substack{M \in \mathcal{M}(H_k) \\ M \ni f}} \frac{y_M^k}{\Delta} + \sum_{\substack{M \in \mathcal{M}(H_{\Delta+2}) \\ M \ni f}} \frac{y_M^{\Delta+2}}{\Delta} \\
&\geq \frac{\Delta - 1}{\Delta} + \frac{1}{2\Delta} + \frac{1}{2\Delta} = 1.
\end{aligned}$$

Furthermore, for the edge $e = yz$ with $y \in S_l$ and $z \in S_m$, we have

$$\sum_{T \ni e} w_T^e = \sum_{\substack{i=1 \\ i \notin \{l,m\}}}^{\Delta+2} \sum_{\substack{M \in \mathcal{M}(H_i) \\ M \ni e}} \frac{y_M^i}{\Delta} \geq \frac{1+\varepsilon}{\Delta} + \sum_{\substack{i=2 \\ i \notin \{l,m\}}}^{\Delta+2} \frac{1}{\Delta} = \frac{1+\varepsilon}{\Delta} + \frac{\Delta - 1}{\Delta} > 1.$$

Finally, the total weight of this fractional total colouring is

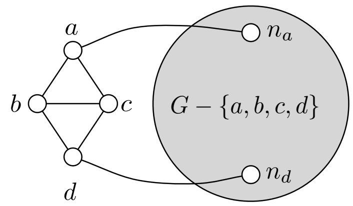
$$\sum_{T \in \mathcal{T}(G)} w_T^e = \sum_{i=1}^{\Delta+2} \sum_{M \in \mathcal{M}(H_i)} \frac{y_M^i}{\Delta} \leq \sum_{i=1}^{\Delta+2} 1 = \Delta + 2.$$

### 6.3.2 Proof of Lemma 6.6

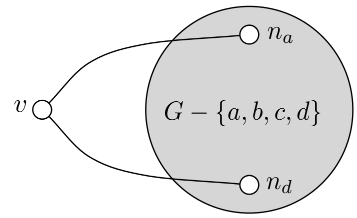
We now show how to construct a better colouring for an edge $e$. We start by dealing with the two special cases.

**Lemma 6.11.** *Suppose $G$ is a graph of maximum degree $\Delta = 3$, and edge $e = bc \in E(G)$ is contained in a subgraph $H$ which is the intersection of two $K_3$ subgraphs defined by $\{a, b, c, d\}$ with nonedge $ad$ (see Fig. 6–2(a)). Then $G$ has a fractional total $(\Delta + 2)$-colouring $w^e$ such that $\sum_{T \ni e} w_T^e > 1$.*

*Proof.* For ease of exposition, we can assume that $\delta(a) = \delta(d) = 3$, as otherwise, if $\delta(a) < 3$ then add pendant vertex $n_a$ adjacent to $a$ and if $\delta(d) < 3$ then add pendant vertex $n_d$ adjacent to $d$. As adding vertices and edge can only increase the fractional total colouring number we have $\chi''_f(G) \leq \chi''_f(G \cup \{n_a, n_d\})$. Hence, we can assume $N(a) = \{b, c, n_a\}$ and $N(d) = \{b, c, n_d\}$; notice it could be the case that $n_a = n_d$.




(a) $G$.




(b) $G/H$.

Figure 6–2: Special case 1.

To colour $G$, consider $G/H$, the graph obtained by contracting $H$. Specifically, $G/H$ is built by taking a copy of $G - H$ and a vertex $v$ adjacent to $n_a$ and $n_b$. We separately colour $H$ and $G/H$ and then combine the two colourings.

We start with $G/H$. Now, if $n_a \neq n_d$ then since $|\delta(H)| = 2$, we have $\Delta(G/H) \leq 3$ and $G/H$ is simple. We can obtain a fractional total colouring $w'$ of $G/H$ with total weight 5 by using the Kilakos-Reed algorithm (Theorem 5.9). If $n_a = n_d$ then the edge between $v$ and $n_a$ has multiplicity 2, and we obtain a fractional total colouring $w''$ of $G - H$ with total weight 5 by using the Kilakos-Reed algorithm. As $|\delta(n_a) \cap E(G - H)| = 1$, it is then trivial to find a fractional total colouring $w'$ of $G/H$ with total weight 5.

Now, $H$ has a fractional total 5-colouring $y$ where $e = bc$ is in weight $\frac{5}{3}$ of total stable set: $y_{\{a,d,bc\}} = 1$, $y_{\{b,cd\}} = y_{\{c,bd\}} = y_{\{b,ca\}} = y_{\{c,ba\}} = \frac{1}{2}$, and $y_{\{bc\}} = y_{\{ac,bd\}} = y_{\{ab,cd\}} = \frac{2}{3}$.

We find the desired colouring of $G$ by combining this colouring of $H$ with the fractional total 5-colouring of $G/H$. For each total set $T$ of $G$ we define our fractional total colouring

$w$ of $G$ as follows:

$$w_T = \begin{cases} w'_{(T-A)\cup\{v\}} & \text{if } A := T \cap V(H) = \{a, d, bc\} \\ \frac{1}{2}w'_{(T-A)\cup\{vn_a\}} & \text{if } A := T \cap V(H) \in \{\{an_a, b, cd\}, \{an_a, c, bd\}\} \\ \frac{1}{2}w'_{(T-A)\cup\{vn_d\}} & \text{if } A := T \cap V(H) \in \{\{dn_d, b, ca\}, \{dn_d, c, ba\}\} \\ \frac{1}{3}w'_{T-A} & \text{if } A := T \cap V(H) \in \{\{bc\}, \{ac, bd\}, \{ab, cd\}\} \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that for any $x \in V(G) \cup E(G)$, the total weight given to $x$ by $w$, i.e. $\sum_{T \ni x} w_T$, is at least the weight given to its image in $G/H$ or $H$ by the colourings $w'$ and $y$. Hence, it follows from our choice of $w'$ and $y$ that this is the desired fractional total $(\Delta + 2)$-colouring of $G$. □

**Lemma 6.12.** *Suppose that $G$ is the graph obtained by deleting a perfect matching from a clique on $k$ vertices, $k \geq 6$ and $k$ even, and $e \in E(G)$. Then, $G$ has a fractional total $k$-colouring $w^e$ such that $\sum_{T \ni e} w^e_T > 1$.*

*Proof.* We note that every edge is identical up to isomorphism. Furthermore, the fractional chromatic index of $G$ is its maximum degree $\Delta = k - 2$:

**Observation 6.13.** $\chi'_f(G) = \Delta$.

*Proof.* For any vertex $v \in V$, $\frac{2|E(G-\{v\})|}{|G-v|-1} = \frac{\Delta+\Delta(\Delta-1)}{\Delta} = \Delta$. Hence, since any odd overfull subgraph of $G$ contains at least $\Delta + 1 = k - 1$ vertices, we have $\chi'_f(G) = \Delta$. □

Let $a, b$ be two nonadjacent vertices of $G$ which are not endpoints of $e$. For every vertex $v \in V - \{a, b\}$, we find a fractional edge $\Delta$-colouring $y^v$ of $G - \{v\}$. Since $G - \{a, b\}$ is the graph obtained by deleting a perfect matching from a clique on $k - 2$ vertices,

$\chi'_f(G) \leq \Delta - 2$. We find a fractional edge colouring $y^{a,b}$ of $G - \{a, b\}$ with total weight $\Delta - 2$. Finally, find a fractional edge colouring $y$ of $G$ with total weight $\Delta$.

We construct the desired fractional total colouring $w^e$ of $G$ as follows. For each $v \in V - \{a, b\}$ and matching $M_j \in \mathcal{M}(G - \{v\})$, let $T_{v,M_j} = \{v\} \cup M_j$ and $w_{T_{v,M_j}} = y^v_{M_j}/\Delta$ For each matching $M_j \in \mathcal{M}(G - \{a, b\})$, let $T_{a,b,M_j} = \{a, b\} \cup M_j$ and $w_{T_{a,b,M_j}} = y^{a,b}_{M_j}/(\Delta - 2)$. For each matching $M_j \in \mathcal{M}(G)$, let $T_{M_j} = M_j$ and $w_{T_{M_j}} = y_{M_j}/\Delta$. All other total stable sets $T$ receive weight $w_T = 0$.

It remains only to show this is the desired colouring. Every vertex has weight 1. Every edge not adjacent to $a$ or $b$, in particular the edge $e$, has weight

$$\frac{\Delta - 2}{\Delta} + \frac{1}{\Delta - 2} + \frac{1}{\Delta} = \frac{\Delta^2 - 2\Delta + 2}{\Delta(\Delta - 2)} > 1.$$

Every other edge is adjacent to exactly one of $a$ or $b$ and therefore has weight

$$\frac{\Delta - 1}{\Delta} + \frac{1}{\Delta} = 1.$$

Finally the total weight of this total colouring is $\Delta + 2$. $\square$

We assume that $G$ and $e$ do not satisfies either of the conditions of Lemma 6.11 nor the conditions of Lemma 6.12. Call a stable $S_1$ *extendable for the edge* $e$ if

(I) $e \in E(G - S_1)$, $|\delta_{G-S_1}(y)| < \Delta$, and $|\delta_{G-S_1}(z)| < \Delta$,

(II) $G - S_1$ has no $\Delta$-full subgraph of $G$ which contains both $y$ and $z$, and

(III) for each $i$ between 1 and $k$ if $|O_i| = \Delta + 1$ then $|O_i - S_1| \geq \Delta$, otherwise $|O_i - S_1| \geq \Delta + 1$.

The key to the proof of Lemma 6.6 is to show that we can choose a stable set $S_1$ which is extendable for the edge $e$. The proof of Lemma 6.6 then follows from the next two lemmas.

**Lemma 6.14.** *If $S_1$ is an extendable stable set for the edge $e$ of a graph $G$, then there exists a better colouring of $G$ for $e$.*

**Lemma 6.15.** *Suppose $G$ is a graph and $e \in E(G)$. If $G$ has maximum degree $\Delta \geq 3$, $G \neq K_{\Delta+1}$, $G \neq K_{\Delta,\Delta}$ and $G$ and $e$ do not satisfy the conditions of Lemma 6.11 nor the conditions of Lemma 6.12, then there exists an extendable stable set $S_1$ for the edge $e$.*

The easier of the two lemmas in Lemma 6.14, and we start with it.

*Proof of Lemma 6.14.* It is enough to find a vertex $(\Delta+2)$-colouring $S'_1, ..., S'_{\Delta+2}$ satisfying Property (P) such that $S_1 \subseteq S'_1$ and $y$ and $z$ are not in $S_1$. To ensure that $y$ and $z$ are not in $S_1$, we use the following auxiliary graph $G'$: if $N(y)$ contains no vertex in $S_1$, then $\delta(y) < \Delta$, and we add a temporary vertex $y'$ adjacent to $y$ in $G$ and let $y'$ be in $S_1$, and if $N(z)$ contains no vertex in $S_1$, then we add a temporary vertex $z'$ adjacent to $z$ in $G$ and let $z'$ be in $S_1$. Since $\Delta \geq 3$, the minimal overfull subgraphs $O_1, ..., O_k$ of $G'$ are the same as those in $G$. Hence, it is enough to find a $\Delta + 2$ vertex colouring $S'_1, ..., S'_{\Delta+2}$ of $G'$ satisfying Property (P) and such that $S_1 \subseteq S'_1$.

We first colour each minimal overfull subgraph in order and then greedily colour the remaining vertices of $G - \bigcup_{i=1}^{k} O_i$. For each $i$ between 1 and $k$, we consider separately the cases when $O_i \cap S_1 = \emptyset$ and $O_i \cap S_1 \neq \emptyset$. We apply a simple modification of the greedy colouring procedure (Lemma 1.6) together with the fact that for overfull subgraphs $O$, the cut condition implies $|\delta(O)| \leq \Delta - 1$.

For the former case, let $t = \min\{\Delta + 2, |O_i|\}$ and pick a set of $U \subseteq O_i$ of $t$ vertices such that $U$ contains all the vertices of $O_i$ which are endpoints of edges in $\delta(O_i)$. Label the vertices of $U = \{u_1, ..., u_t\}$ such that if $\delta(u_l) \cap \delta(O_i) \neq \emptyset$ and $j < l$ then $\delta(u_j) \cap \delta(O_i) \neq \emptyset$. We assign to each vertex $u_j$ of $U$ the smallest colour not assigned to any vertex of $N(u_j) \cup \bigcup_{l=1}^{j-1} u_l$. Since $|\delta(O_i)| \leq \Delta - 1$, $|\delta(u_j) \cap \delta(O_i)| + |\bigcup_{l=1}^{j-1} u_l| \leq t - 1$. Hence, if $|O_i| = \Delta + 1$ then there always exists an available colour of $\{1, ..., \Delta+1\}$ and if $|O_i| \geq \Delta + 2$ then there always exists an available colour of $\{1, ..., \Delta + 2\}$. To colour the remaining vertices of $O_i$, we assign to each vertex $v \in O_i - U$ the smallest colour not assigned to any vertex of $N(v)$; trivially, there always exists two available colours of $\{1, ..., \Delta + 2\}$.

For the latter case, let $t = \min\{\Delta+1, |O_i|-1\}$ and pick a set of $U \subseteq O_i - S_1$ of $t$ vertices such that $U$ contains all the vertices of $O_i - S_1$ which are endpoints of edges in $\delta(O_i)$, and labelled $\{u_1, ..., u_t\}$ such that if $\delta(u_l) \cap \delta(O_i) \neq \emptyset$ and $j < l$ then $\delta(u_j) \cap \delta(O_i) \neq \emptyset$. We assign to each vertex $u_j$ of $U$ the smallest colour of $\{2, ..., \Delta+2\}$ not assigned to any vertex of $N(u_j) \cup \bigcup_{l=1}^{j-1} u_l$. Again we have $|\delta(u_j) \cap \delta(O_i)| + |\bigcup_{l=1}^{j-1} u_l| \leq t - 1$ and so there always exists an available colour. Again we colour each vertex $v \in O_i - U$ greedily.

It follows that we can find the desired better colouring of $G$ for $e$. $\square$

*Proof of Lemma 6.15.* We apply the following algorithm to find $S_1$:

1. Pick one or two vertices to be in $S_1$ based on the following three cases. Let $e = yz$.
   - A: If $N(y) \cap N(z) \neq \emptyset$, then add a vertex $w \in N(y) \cap N(z)$ to $S_1$ such that $w$ maximizes $|N(w) \cap (N(y) \cup N(z))|$ for all choices for $w$.
   - B: If Case A does not apply, $|\delta(y)| < \Delta$ and $|\delta(z)| = \Delta$, then add $z'$ to $S_1$, where $z'$ is a arbitrarily chosen vertex of $N(z) - y$. If Case A does not apply, $|\delta(y)| = \Delta$ and $|\delta(z)| < \Delta$, then add $y'$ to $S_1$, where $y'$ is a arbitrarily chosen vertex of $N(y) - z$.
   - C: If Case A does not apply, $|\delta(y)| = \Delta$ and $|\delta(z)| = \Delta$, then add nonadjacent vertices $y' \in N(y) - z$ and $z' \in N(z) - y$ to $S_1$ where there does not exist a minimal overfull graph $O_i$ of size at most $\Delta + 2$ containing both $y'$ and $z'$.
2. If $e$ is contained in exactly one minimal overfull subgraph $O_i$ of $G$ then we apply the following algorithm: If $|O_i - S_1| \leq \Delta + 1$ or $V(O_i) - S_1$ does not contain any vertex $v$ satisfying $|N(v) \cap (V(O_i) - S_1)| = \Delta$, then we stop. If we do not stop then we add such a vertex $v$ to $S_1$ and repeat Step 2.

3. If there exists a vertex $v$ of $(G - S_1) - \cup_{i=1}^{k} O_k$ such that $|N(v) \cap (G - S_1)| = \Delta$, then we add $v$ to $S_1$. We repeat Step 3 until no such vertex exits.

By design, if $S_1$ exists then it is a stable set. We claim that $S_1$ always exists and that it is extendable. To show that $S_1$ always exists, we show that the above algorithm is always successful. In particular, we need to ensure that when Case C applies, the vertices $y'$ and $z'$ exist.

**Lemma 6.16.** *If $N(y) \cap N(z) = \emptyset$, $|\delta(y)| = \Delta$ and $|\delta(z)| = \Delta$, then there exists nonadjacent vertices $y' \in N(y) - z$ and $z' \in N(z) - y$ such that no minimal overfull graph $O_i$ of size at most $\Delta + 2$ contains both $y'$ and $z'$.*

*Proof.* If every edge exists between $N(y) - z$ and $N(z) - y$ then the graph induced on $N(y) \cup N(z)$ is a $K_{\Delta,\Delta}$. Thus, there exist $y' \in N(y) - z$ and $z' \in N(z) - y$ such that $y'z' \notin E(G)$. If $y'$ and $z'$ are not contained in some minimal $\Delta$-full subgraph of size $\Delta + 1$ or $\Delta + 2$ then we are done. Otherwise, let $H$ be a minimal overfull subgraph containing $y'$ and $z'$. Partition the set $Y = N(y) - z$ into sets $Y_I = Y \cap H$ and $Y_O = Y - H$, and the set $Z = N(z) - y$ into sets $Z_I = Z \cap H$ and $Z_O = Z - H$. We show that there always exists an nonedge between either $Z_I$ and $Y_O$ or $Z_O$ and $Y_I$. We note that $y' \in Y_I$ and $z' \in Z_I$, and so $|Y_I| \geq 1$ and $|Z_I| \geq 1$.

First, assume $y, z \in H$. As $|H| \leq \Delta + 2$ it follows that, since $\Delta \geq 3$, $|Z_O| + |Y_O| \geq \Delta - 2 \geq 1$. Hence, as $|\delta(H)| \leq \Delta - 1$, either the desired nonedge exists or there is at most one edge between $Z_I$ and $Y_O$ and $Z_O$ and $Y_I$. Since the former case satisfies the lemma, assume we are in the latter. Now, if $|Z_O| = 0$, then $|Y_O| \geq \Delta - 2 \geq 1$ and $|Z_I| = \Delta - 1 \geq 2$, and so there exists a nonedge between $Y_O$ and $Z_I$. So, suppose $|Z_O| \geq 1$. Similarly, if $|Y_O| = 0$, then there exists a nonedge between $Y_I$ and $Z_O$, and so, suppose $|Y_O| \geq 1$. We now have $Z_I$ and $Y_O$ and $Z_O$ and $Y_I$ are all nonempty, and so, the desired nonedge exists.

Second, assume $y, z \notin H$. Since $yy' \in \delta(H)$ and $zz' \in \delta(H)$, either we have the desired nonedge or (a) $|Y_I||Z_O| + |Y_O||Z_I| \leq |\delta(H)| - 2 \leq \Delta - 3$. Since the former case satisfies the lemma, assume we are in the latter. Since $|Z_I| \geq 1$ and $|Y_I| \geq 1$, (a) implies $|Y_O| + |Z_O| \leq \Delta - 3$. It follows that $|Y_I| + |Z_I| \geq \Delta + 1$. Without loss of generality assume $|Y_I| \geq |Z_I|$, and so $|Y_I| \geq \frac{1}{2}(\Delta + 1)$. This with (a) implies $|Z_O| \leq 1$, and so $|Z_I| \geq \Delta - 2$. This with (a) implies that $|Y_O| \leq 1$, and so $|Y_I| \geq \max\{\frac{1}{2}(\Delta + 1), \Delta - 2\}$. Since, $\Delta \geq 3$, $|Y_I| \geq 2$. We now have $|\delta(H)| \geq |Z_I| + |Y_I| \geq \Delta$. This contradiction implies the desired nonedge exists.

Finally, assume $yz \in \delta(H)$, without loss of generality assume $y \in H$, $z \notin H$. Since $|Z_I| \geq 1$, either we have the desired nonedge or (b) $|Y_I||Z_O| + |Y_O||Z_I| \leq |\delta(H)| - 1 \leq \Delta - 2$. Since the former case satisfies the lemma, assume we are in the latter. Since $H$ is a minimal overfull subgraph, each vertex of $H$ has at least $\frac{1}{2}\Delta$ neighbours in $H$. Hence, $|Y_I| \geq \frac{1}{2}\Delta$. This with (b) implies $|Z_O| \leq 1$, and so, $|Z_I| \geq \Delta - 2$. Since $|\delta(H)| \geq |Z_I| + |Y_O| + 1$, this implies that $|Y_O| = 0$, and so, $|Y_I| = \Delta$. Hence, $|\delta(H)| \geq 2\Delta - 2 > \Delta$. This contradiction implies the desired nonedge exists. □

In order to complete the proof of Lemma 6.15, we need only prove that $S_1$ is extendable. As in Step 1 we never pick $y$ or $z$ to be in $S_1$ and as after these steps $|\delta_{H_1}(y)| < \Delta$ and $|\delta_{H_1}(z)| < \Delta$, it follows that Property (I) holds. Moreover, we ensure that if $|O_i| = \Delta + 1$ then $|O_i - S_1| \geq \Delta$, otherwise $|O_i - S_1| \geq \Delta + 1$ for $i$ between 1 and $k$. Hence, Property (III) holds. So, we need only prove that Property (II) holds, that is, $H_1 = G - S_1$ has no $\Delta$-full subgraph of $G$ which contains both $y$ and $z$. To do so, we show that if $H_1$ has such a $\Delta$-full subgraph $F$ then $F = G$, and $G$ and $e$ satisfy the conditions of Lemma 6.12. Since we assumed this was not the case, from this the lemma follows.

We first need some structural information about $\Delta$-full subgraphs in $H_1$.

**Lemma 6.17.** *No endpoint of $e = yz$ is contained in a $K_\Delta$ in $H_1$*

*Proof.* As the degree of the endpoints of $e$ is less than $\Delta$, no endpoint of $e$ is contained in the intersection of two $\Delta$-cliques and $e$ can not have only one endpoint inside a $K_\Delta$. Thus, if $e$ is contained in a $K_\Delta$ then both of its endpoints are contained in exactly one. Since $\Delta \geq 3$, $N(y) \cap N(z)$ contains at least one vertex, and so $S_1$ contains a vertex $v \in N(y) \cap N(z)$. This vertex must of have seen the most vertices of $N(y) \cup N(z)$ out of vertices in $N(y) \cap N(z)$. Thus, $v$ has at least $\Delta - 1$ neighbours inside the $K_\Delta$. It cannot be the case that $v$ has exactly $\Delta - 1$ neighbours inside the $K_\Delta$, as then $e$ is in the intersection of two $\Delta$-cliques, nor can it be the case that $v$ has $\Delta$ neighbours inside the $K_\Delta$, as then $G$ contains a $K_{\Delta+1}$. The lemma now follows. □

**Lemma 6.18.** *A subgraph $F$ is $\Delta$-full and satisfies $\Delta(F) < \Delta$ precisely if $F = K_\Delta$.*

*Proof.* Notice that

$$\frac{2|E(F)|}{|F|-1} \leq \frac{|F|(\Delta-1)}{|F|-1} \leq \frac{\Delta(|F|-1)}{|F|-1} = \Delta,$$

where the first inequality is strict precisely when $F$ is not $\Delta$-regular and the second is strict precisely when $|F| > \Delta$. In other words, $F$ is $\Delta$-full and satisfies $\Delta(F) < \Delta$ precisely when $F = K_\Delta$. □

**Lemma 6.19.** [67] *Let $H_1$ be a minimal overfull subgraph and $H_2$ be any other subgraph graph of $G$. If $|E(H_1 \cap H_2, H_1 - H_2)| \leq |E(H_1 \cap H_2, H_2 - H_1)|$, then $H_1 - H_2$ is overfull.*

*Proof.* We have

$$\begin{aligned} |E(H_1 \cap H_2)| \quad + \quad & |E(H_1 \cap H_2, H_1 - H_2)| \\ & \leq (|H_1 \cap H_2|\Delta - |E(H_1 \cap H_2, G - (H_1 \cap H_2))|)/2 \\ & \qquad\qquad\qquad\qquad + |E(H_1 \cap H_2, H_1 - H_2)| \\ & \leq \frac{|H_1 \cap H_2|}{2}\Delta \end{aligned}$$

This implies that $|H_1 - H_2| > 1$ (as otherwise we have

$$|E(H_1)| = |E(H_1 \cap H_2)| + |E(H_1 \cap H_2, H_1 - H_2)| \leq \frac{|H_1| - 1}{2}\Delta$$

which contradicts $H_1$ being overfull), thus we have

$$\begin{aligned} |E(H_1 - H_2)| &= |E(H_1)| - |E(H_1 \cap H_2, H_1 - H_2)| - |E(H_1 \cap H_2)| \\ &> \frac{|H_1| - 1}{2}\Delta - \frac{|H_1 \cap H_2|}{2}\Delta = \frac{|H_1 - H_2| - 1}{2}\Delta. \end{aligned}$$

Hence, $H_1 - H_2$ is overfull. □

**Lemma 6.20.** *If $F$ is a $\Delta$-full subgraph of $H_1$ containing the edge $e$, there exists an overfull subgraph $O_i$ of $G$ such that $F \subseteq O_i$.*

*Proof.* Lemma 6.17 implies that $F$ is not a clique on $\Delta$ vertices and so Lemma 6.18 together with the fact that the maximum degree of any vertex in $H_1 - \bigcup_{i=1}^{k} O_i$ is less than $\Delta$ implies that $F$ must intersect some $O_i$. If $|E(F \cap O_i, O_i - F)| \leq |E(F \cap O_i, F - O_i)|$ then Lemma 6.19 yields a contradiction to minimality of $O_i$. We must have $|E(F \cap O_i, O_i - F)| > |E(F \cap O_i, F - O_i)|$, and so either $F \subseteq O_i$ or

$$\begin{aligned} |E(F - O_i)| &= |E(F)| - (|E(F \cap O_i)| + |E(F \cap O_i, F - O_i)|) \\ &\geq \frac{1}{2}\Delta(|F| - 1) - \frac{1}{2}(\Delta|F \cap O_i| \\ &\quad -|E(F \cap O_i, G - (F \cap O_i))|) - |E(F \cap O_i, F - O_i)| \\ &> \frac{1}{2}\Delta(|F - O_i| - 1), \end{aligned}$$

thus $F - O_i$ is overfull in $G$. Since $F - O_i \subseteq H_1$, it is overfull in $H_1$. This contradiction finishes the proof. □

We can now prove that Property (II) of better colourings holds. By Lemma 6.20, if $e$ is contained in some $\Delta$-full subgraph $F$ in $H_1$, then $e \subseteq F \subseteq O_i$ for some $O_i$. Moreover, since $F$ is not a $K_\Delta$, we must have $|F| \geq \Delta + 1$ and so $|O_i| \geq \Delta + 2$. As $F$ contains some vertex of degree $\Delta$, by Step 2 of our algorithm for choosing the set $S_1$, we have $|O_i - S_1| = \Delta + 1$, $O_i - S_1$ contains some vertex $w$ satisfying $|\delta(w) \cap E(O_i - S_1)| = \Delta$, and each vertex $v$ of $O_i \cap S_1$ must also have $\Delta$ neighbours in $O_i$. Since each of these vertices has exactly $\Delta$ neighbours in $F$, it follows that $|S_i \cap O_i| = 1$. Hence, as $|F| \geq \Delta + 1$, it follows that $F = O_i - S_1$ and $|O_i| = \Delta + 2$. Moreover, as $|\delta(F)| + \sum_{v \in F}(\Delta - |\delta(v)|) = \Delta$ and, for $v \in O_i \cap S_1$, $|\delta(v) \cap \delta(F)| = \Delta$, we have that every vertex in $F$ has degree $\Delta$ in $G$. So, $O_i$ has exactly $\Delta + 2$ vertices and is $\Delta$ regular. Therefore, $O_i$ is exactly $K_{\Delta+2} - M$ for some perfect matching $M$. Since $G$ is connected, it must be the case that $G = O_1$. Hence, $G$ and $e$ satisfy the conditions of Lemma 6.12. This completes the proof of Lemma 6.8.

## 6.4 Small Overfull Subgraphs and the Fractional Total Colouring Number

A simple corollary of Edmonds' fractional edge colouring theorem yields that if a graph $G$ has fractional chromatic index close to $\Delta(G) + 1$, then it must have a small odd overfull subgraph.

**Corollary 6.21.** *For any graph $G$, if the smallest odd overfull subgraph has size at least $k+1$, then $\chi'_f(G) \leq \Delta(G) + \frac{\Delta(G)}{k}$.*

*Proof.* For each subgraph $H$ which is odd and contains at least three vertices, we have

$$\frac{2|E(H)|}{|H|-1} \leq \frac{\Delta(G)|H|}{|H|-1} = \Delta(G) + \frac{\Delta(G)}{|H|-1} \leq \Delta(G) + \frac{\Delta(G)}{k}.$$

The result is now immediate from Theorem 1.2. □

Now for a simple graph $G$, if $\chi'_f(G) = \Delta(G) + 1$ then $G$ contains an odd overfull subgraph $H$ satisfying $|H| \leq \Delta(G) + 1$. As any odd overfull subgraph contains at least

$\Delta(G)+1$ vertices, it follows that $\chi'_f(G) = \Delta(G)+1$ precisely when $G$ contains $K_{\Delta(G)+1}$, $\Delta(G)$ even, as a subgraph.

Theorem 6.1 give the analogous result for fractional total colouring. We note that both $K_{2l}$ and $K_{l,l}$ are overfull subgraphs and have $2l$ vertices. Reed conjectured that if $G$ has fractional total colouring number close to $\Delta(G)+2$ then $G$ contains a small overfull subgraph. In this section, we explain and strengthen this conjecture.

As an example, consider graphs of girth $g \geq 5$, that is, the shortest induced cycle has length at least $g$. Clearly, $G$ can contain neither $K_{2l}$ nor $K_{l,l}$ for $l \geq 2$. More strongly, graphs with large girth do not contain small overfull subgraphs:

**Observation 6.22.** *If $G$ has girth $g$ then each overfull subgraph $H$ satisfies $|H| > \frac{g-1}{2}\Delta(G)$.*

*Proof.* $H$ has at most

$$\frac{\binom{|H|}{g}g}{\binom{|H|-2}{g-2}} = \frac{(|H|)(|H|-1)}{g-1}$$

edges. If $H$ is overfull then

$$\Delta(G) < \frac{2|E(H)|}{|H|-1} \leq \frac{2|H|}{g-1}.$$

□

Hence, by the above conjecture of Reed, we would expect their fractional total colour number to be far away from $\Delta(G)+2$. A first step in this direction is Theorem 6.1 which yields the following:

**Corollary 6.23.** *If $G$ has girth $g \geq 5$ then $\chi''_f(G) < \Delta(G)+2$.*

This was strengthened by Kaiser, King, and Král [60] and Kardos, Kral, and Sereni [61] as follows:

**Theorem 6.24.** *For all $\epsilon > 0$, there exists a girth $g(\epsilon)$ such that if a graph $G$ has girth at least $g(\epsilon)$ then $\chi''_f(G) < \Delta(G)+1+\epsilon$.*

We conjecture that this phenomena should occur immediately.

**Conjecture 6.25.** *For each $k \geq 1$ and graph $G$ with girth $g \geq 3k+1$, $\chi''_f(G) \leq \Delta(G)+1+\frac{1}{k}$.*

We note that $\chi''_f(C_{3k+1}) = 3 + \frac{1}{k}$, and so this cannot be improve further. By Observation 6.22 this is a special case of the following stronger conjecture.

**Conjecture 6.26.** *For each $k \geq 1$, if each overfull subgraph $H$ of $G$ satisfies $|H| > k$ then $\chi''_f(G) \leq \Delta(G) + 1 + \frac{\Delta(G)}{k-\Delta(G)}$.*

# CHAPTER 7
## Fractional Total Colouring without Overfull Subgraphs

In this chapter, we prove the following theorem:

**Theorem 7.1.** *There exists an algorithm which given a graph $G$ with maximum degree $\Delta \geq \frac{3}{4}|G|$, either returns $\chi''_f(G)$ or an overfull subgraph in $O(|G| + |E(G)|)$ time.*

An easy extension of Theorem 4.4 yields that since $\Delta \geq \frac{3}{4}|G|$, there exists a linear time algorithm to find all overfull subgraphs of $G$. (Indeed, recall that given a subgraph $A$ of $G$, the Subroutine FIND-OVERFULL of Section 4.3.1 returns any odd subgraph $H$ of $G$ with $H - S_H = A - S_H$, where $S_H$ is the set of $\varepsilon$-special vertices for $H$. To do so, this subroutine sets $J := \{v \in G \mid |N(v) \cap A| \geq \frac{\Delta}{2}\}$ and checks whether any *odd* subgraph $H$ satisfying $|J \oplus H| \leq 2$ is overfull. We need only modify this subroutine to check *all* subgraphs $H$ satisfying $|J \oplus H| \leq 2$.) If $|G| < 320$, then we can determine the fractional total colouring number in constant time by applying the simplex algorithm. Since we can determine $\Delta$ in linear time, to prove Theorem 7.1 it is enough to prove the following lemma.

**Lemma 7.2.** *Suppose $G$ is a graph satisfying $|G| \geq 320$, with maximum degree $\Delta \geq \frac{3}{4}|G|$ and containing no overfull subgraph. Then $\chi''_f(G) = \Delta + 1$.*

We prove Lemma 7.2 by iteratively choosing total stable sets $T_1, ..., T_k$ until we have reduced our problem to finding an edge colouring of an auxiliary graph $G^*$. This auxiliary graph has maximum degree $\Delta(G^*) = \Delta - k \geq \frac{1}{2}|G|$ and contains no overfull subgraphs. By combining a fractional edge $\Delta(G^*)$-colouring of $G^*$ together with the stable sets $T_1, ..., T_k$ we will find the desired fractional total $(\Delta + 1)$-colouring of $G$. More strongly, if $G^*$ has an (integral) edge $\Delta(G^*)$-colouring then our method will yield an (integral) total $(\Delta+1)$-colouring of $G$. Since $G^*$ has large enough maximum degree, this leads to the following corollary.

**Corollary 7.3.** *Suppose that Hilton's Overfull Conjecture is true. Let $G$ be a graph satisfying $|G| \geq 320$, with maximum degree $\Delta \geq \frac{3}{4}|G|$ and containing no overfull subgraph. Then $\chi''(G) = \Delta + 1$.*

Hilton, Holroyd, and Zhao [50] prove the following related result.

**Theorem 7.4.** *[50] Suppose that the Hilton's Overfull Conjecture is true. Let $G$ be a graph of maximum degree $\Delta(G)$ and minimum degree $m(G)$.*

1. *If $G$ has order $2n$, $\Delta(G) \leq 2n-2$, $m(G) + \Delta(G) \geq \frac{3}{2}(2n) - 1$ and $\sum_{v \in G}(\Delta(G) - d(v)) \geq (\Delta(G) - m(G)) + n$, then $\chi''(G) = \Delta(G) + 1$.*
2. *If $G$ has order $2n+1$, $m(G) + \Delta(G) \geq \frac{3}{4}(2n+1) + \frac{1}{4}\sum_{v \in G}(\Delta(G) - d(v))$ and $\sum_{v \in G}(\Delta(G) - d(v)) \geq 2n + \Delta(G) - 2m(G)$, then $\chi''(G) = \Delta(G) + 1$.*

Restated in terms of fractional total colouring, Hilton *et al.* show that if $G$ satisfies either Condition 1. or Condition 2., then $G$ also satisfies $\chi''_f(G) = \Delta(G) + 1$.

Our proof of Lemma 7.2 is algorithmic and we will briefly discuss in Section 7.5 how to turn it into a polynomial time algorithm to find a fractional total $(\Delta+1)$-colouring of any graph satisfying its conditions. In Section 7.1, we start with a sketch of the proof of Lemma 7.2. Sections 7.2, 7.3, and 7.4 contain the details. We close this introductory section by discussing the relationship between Lemma 7.2 and the Fractional Conformability Conjecture (Conjecture 5.21).

We will find a vertex $(\Delta+1)$-colouring $S_1, ..., S_{\Delta+1}$ and then find a fractional edge $(\Delta+1)$-colouring which extends it. We choose our vertex colouring such that setting $k = |G| - \Delta - 1$, we have that $S_1, ..., S_k$ have size 2 and $S_{k+1}, ..., S_{\Delta+1}$ are singletons. Since $G$ contains no overfull subgraph, any such vertex colouring is conformable.

**Lemma 7.5.** *If $G$ has maximum degree $\Delta$ and contains no overfull subgraph then any vertex $(\Delta+1)$-colouring $S_1, ..., S_{\Delta+1}$ of $G$ such that for $i \leq |G|-\Delta-1$, $|S_i| = 2$ and for $i > |G|-\Delta-1$, $|S_i| = 1$ is conformable.*

*Proof.* Let $F$ be a subgraph of $G$. Trivially, if $F$ satisfies $\sum_{v\in F}(\Delta - d_F(v)) \geq \Delta+1$ then any vertex $(\Delta+1)$-colouring is conformable with respect to $F$. Otherwise, since $G$ contains no overfull subgraphs, $\sum_{v\in F}(\Delta - d_F(v)) = \Delta$. It is enough to show that there exists a stable set $S_j$ such that $|F - S_j|$ is even. If $F$ is odd then as $|F| = \sum_{i=1}^{\Delta+1} |S_i \cap F|$, we have for at least one $S_i$, $|S_i \cap F|$ is odd. If $F$ is even and $|S_i \cap F| = 1$ for each $i$ then $|F| = \Delta+1$ and $\Delta$ is odd. On the other hand, we have $2|E(F)| = \Delta(|V(F)|-1) = \Delta^2$, a contradiction as $\Delta$ is odd. This completes the proof. $\square$

We will prove below that since $\Delta(G) \geq \frac{3}{4}|G|$, we can find such a vertex $(\Delta+1)$-colouring. Hence, the graphs we consider satisfy $\beta(G) = \Delta+1$, and so, Lemma 7.2 verifies the Fractional Conformability Conjecture when $\Delta(G) \geq \frac{3}{4}|G|$ and $G$ contains no overfull subgraph.

Not all vertex colourings as in Lemma 7.5 are extendable. In particular, we will need to ensure that for each $i$, $G - S_i$ does not contain an odd component each vertex of which has degree $\Delta$ in $G$. For, a simple parity argument yields that any matching of $G - S_i$ misses at least one vertex of this odd component. This is problematic since no vertex of maximum degree is missed by any total stable set which receives positive weight in any fractional total $(\Delta+1)$-colouring. It would be interesting to know if this is the only problem which prevents a conformable colouring of a graph with large enough maximum degree from being extended to a total $(\Delta+1)$-colouring of the graph. We remark that this is a subtlety of the Conformability Conjecture. It only aims to characterize graphs with total $(\Delta+1)$-colourings and does not describe the conformable vertex colourings which can be extended to total $(\Delta+1)$-colourings.

### 7.1 Sketching the Proof of Lemma 7.2

To prove Lemma 7.2 we first vertex colour $G$ using $\Delta + 1$ colour classes $S_1, ..., S_{\Delta+1}$ each of size 1 or 2. Specifically, setting $k = |G| - \Delta - 1$, we have that $S_1, ..., S_k$ have size 2 and $S_{k+1}, ..., S_{\Delta+1}$ are singletons. We iteratively choose disjoint matchings $M_1, ..., M_k$ such that $M_i$ is disjoint from $S_i$. We will assign the total stable set $T_i = S_i \cup M_i$ weight 1 in our fractional total colouring. To complete our colouring we will need to find a nonnegative weighting $w$ on the total stable sets of total weight $\Delta + 1 - k$ so that:

(A) $\forall x \in \left(V(G) - \bigcup_{i=1}^{k} S_i\right) \cup \left(E(G) - \bigcup_{i=1}^{k} M_i\right), \sum_{x \in T} w_T \geq 1.$

We will further insist that:

(B) If $w_T > 0$, then $T \cap V(G) \in \{S_{k+1}, ..., S_{\Delta+1}\}$.

Guiding our approach is the following:

**Observation 7.6.** *Weightings satisfying (A) and (B) are equivalent to fractional edge $(\Delta + 1 - k)$-colourings of the graph $G^*$ defined as follows:*

**Definition 7.7.** Define $G^*$ to be the graph formed by taking a copy of $G - \bigcup_{j \leq k} M_j$ and adding vertex $v^*$ adjacent to every vertex in $\bigcup_{j=k+1}^{\Delta+1} S_j$.

By Edmonds' Fractional Edge Colouring Theorem (Theorem 1.2), $G^*$ has a fractional edge $(\Delta + 1 - k)$-colouring provided:

(C) $\Delta(G^*) = \Delta + 1 - k$, and

(D) every odd subgraph $H$ of $G^*$ satisfies $2|E(H)| \leq \Delta(G^*)(|H| - 1)$.

Since $|H|$ is odd, $\Delta(G^*)|H| - 2|E(H)|$ has the same parity as $\Delta(G^*)$, and so (D) is equivalent to:

(D$'$) $\forall$ odd $H \subseteq G^*, |H| > 1, |\delta(H)| + \sum_{v \in H}(\Delta(G^*) - d(v)) \geq \Delta(G^*) - 1.$

We will find each $M_i \in G - S_i$ in turn. Whilst doing so, we need to ensure that properties like (C) and (D′) are maintained throughout. We let

$$G_i := G - \bigcup_{j<i} M_j,$$

and insist that:

(C$_i$) $\forall v \in \bigcup_{j<i} S_j$, $d_{G_i}(v) \leq \Delta + 2 - i$, and $\forall v \in V(G) - \bigcup_{j<i} S_j$, $d_{G_i}(v) \leq \Delta + 1 - i$.

(D′$_i$) $\forall A \subseteq G_i$, $|A| > 1$, we have in $G_i$ that:

$$\sum_{v \in V(A) \cap \left(\bigcup_{j<i} S_j\right)} (\Delta + 2 - i - d_A(v)) + \sum_{v \in V(A) - \left(\bigcup_{j<i} S_j\right)} (\Delta + 1 - i - d_A(v)) \geq \Delta + 1 - i.$$

**Remark 7.8.** The more stringent condition on $d_{G_i}(v)$ for $v$ not yet coloured in Condition (C$_i$) is due to the fact that each of these vertices is necessarily missed by later matchings of total weight 1. The differing treatment of the uncoloured and coloured vertices of $A$ in (D′$_i$) is similarly motivated. We will prove below that in order to ensure that there are no odd overfull subgraphs in $G^*$, we need to ensure that (D′$_{k+1}$) holds for both odd and even subgraphs of $G_{k+1}$. This explains why (D′$_i$) is not restricted to odd subgraphs.

Finding a sequence of matchings such that (C$_i$) holds for all $i$ is not possible for all choices of $S_1, ..., S_{\Delta+1}$. Indeed, if $G - S_1$ has an odd component $H$ such that every vertex of $H$ has degree $\Delta$ in $G$, then we cannot even ensure (C) holds. This is because some vertex $v$ of $H$ is missed by $M_1$ and hence has degree greater than $\Delta + 1 - k$ in $G^*$. (Since if $v$ is hit by $M_2, ..., M_k$ it is adjacent to $v^*$ in $G^*$.) So, to ensure (C) holds we must choose $S_1$ so that no such $H$ exists. In the same vein, for each $i$, if (C$_{i+1}$) is to hold, then we must ensure that there does not exist an odd component $H$ of $G_i - S_i$ such that every vertex of $H$ satisfies (C$_i$) with equality. Doing so allows us to choose the matchings so that (C$_i$) holds for all $i$ by simply applying the Tutte-Berge Formula.

We turn now to ensuring that ($\mathrm{D}'_i$) holds for all $i$. In describing our approach to do so, we use the following observations.

**Observation 7.9.** *The left-hand side of ($D'_{i+1}$) is the left-hand side of ($D'_i$) minus* $|A| - |S_i \cap V(A)| - 2|M_i \cap E(A)|$*. The right-hand side of ($D'_{i+1}$) is the right-hand side of ($D'_i$) minus 1.*

**Corollary 7.10.** *If ($D'_i$) is tight for $A$, then in order for ($D'_{i+1}$) to hold, $|M_i \cap E(A)|$ must be a perfect or near perfect matching of $A - S_i$.*

We recall that Definition 6.3 defined a subgraph $F$ of $G$ to be $\Delta$-full if $|F| > 1$ and $\sum_{v \in F}(\Delta - d_F(v)) = \Delta$.

**Observation 7.11.** *($D_1$) is tight for $A$ precisely if $A$ is $\Delta$-full.*

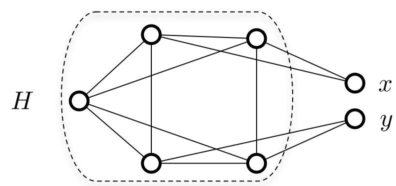



Figure 7–1: A set of four $\Delta$-full subgraphs.

Assume $G$ contains a subgraph $H$ and $G - H$ contains two nonadjacent vertices $x$ and $y$ such that each subgraph $F \in \{H, H + x, H + y, H + x + y\}$ is $\Delta$-full (for an example of such a graph see Fig. 7–1). By the above observations, in order for ($\mathrm{D}'_2$) to hold for a matching $M_1 \in G - S_1$, we need that for each such $F$, $M_1 \cap E(F)$ is either a perfect or near-perfect matching of $F - S_1$. If $\{x, y\} \cap S_1 = \emptyset$, then this is impossible, since if $H - S_1$ is odd, then no matching is perfect in both $(H + x) - S_1$ and $(H + y) - S_1$, whereas if $H - S_1$ is even, then as $x$ and $y$ are nonadjacent, no matching is perfect in both $H - S_1$ and $(H + x + y) - S_1$. So, we must choose $S_1$ so that no such triple $H, x, y$ exists. We actually insist that something slightly stronger holds:

($E_1$) If $G$ contains a subgraph $H$ and vertices $x, y$ such that all of $H, H+x, H+y$, and $H+x+y$ are $\Delta$-full, then $S_1$ contains exactly one of $\{x, y\}$.

In a similar vein, we ensure that for each $i \geq 2$, the following holds:

($E_i$) $G_i$ does not contain a subgraph $H$ and vertices $x, y$ such that all of $H, H+x, H+y$, and $H+x+y$ satisfy the bound of ($D_i'$) with equality.

We remark that ensuring that ($E_i$) holds actually prevents there from being an odd component of $G_i - S_i$ all of whose vertices satisfy ($C_i$) with equality. For, if $H$ is such an odd component in $G_i - S_i$, then for $\{x, y\} = S_i$, the triple $H, x, y$ fails the condition ($E_i$). Indeed, since ($D_i'$) holds, there are at least $\Delta+1-i$ edges of $E(G)$ from $H$ to $\{x, y\}$. Since $H+x+y$ satisfies the condition of ($D_i'$), we see that there are in fact exactly $\Delta+1-i$ such edges and $xy$ cannot be an edge of $G$. Thus, $H$ and $H+x+y$ both satisfy the bound of ($D_i'$) with equality. In the same vein, the fact that ($D_i'$) holds implies that each of $H+x$ and $H+y$ satisfies the bound of ($D_i'$) with equality, i.e. there are exactly $\frac{1}{2}(\Delta+1-i)$ edges from each of $x$ and $y$ to $H$.

To complete the proof of the Lemma 7.2 it is enough to prove the following lemmas:

**Lemma 7.12.** *Any graph $G$ of maximum degree $\Delta \geq \frac{3}{4}|G|$ has a vertex $(\Delta+1)$-colouring $S_1, ..., S_{\Delta+1}$ such that $|S_i| = 2$ for each $i$ between 1 and $k$, $|S_i| = 1$ for each $i$ between $k+1$ and $\Delta+1$ and such that $S_1$ satisfies ($E_1$).*

**Lemma 7.13.** *Suppose $S_1, ..., S_{\Delta+1}$ is a vertex $(\Delta+1)$-colouring such that $|S_i| = 2$ for each $i = 1...k$, and $|S_i| = 1$ otherwise. Given $G_1$ such that ($C_1$) and ($D_1'$) hold, and $S_1$ satisfying ($E_1$), then we can choose $M_1 \in G_1 - S_1$ so that ($C_2$), ($D_2'$), and ($E_2$) hold.*

**Lemma 7.14.** *Suppose $S_1, ..., S_{\Delta+1}$ is a vertex $(\Delta+1)$-colouring such that $|S_i| = 2$ for each $i = 1...k$, and $|S_i| = 1$ otherwise. Given $G_i$ such that ($C_i$), ($D_i'$), and ($E_i$) hold for $i \geq 2$, then we can choose $M_i \in G_i - S_i$ so that ($C_{i+1}$), ($D_{i+1}'$), and ($E_{i+1}$) hold.*

**Lemma 7.15.** *Suppose $S_1, ..., S_{\Delta+1}$ is a vertex $(\Delta+1)$-colouring such that $|S_i| = 2$ for each $i = 1...k$, and $|S_i| = 1$ otherwise. If ($C_{k+1}$) and ($D'_{k+1}$) hold for $G_{k+1}$, then so do (C) and (D′) for $G^*$.*

The proof of Lemma 7.15 is straightforward. Indeed, since $d_{G^*}(v^*) = \Delta + 1 - k$, (C) is immediate from ($\mathrm{C}_{k+1}$) and the proof that (D′) follows from ($\mathrm{D}'_{k+1}$) is not much more difficult. We defer further details until the end of this section.

The preliminary discussion above suggests the structure of the proof of Lemma 7.14 (and the proof of Lemma 7.13, in fact, we state and prove a common generalization). Since ($\mathrm{C}_i$), ($\mathrm{D}'_i$) and ($\mathrm{E}_i$) hold, ($\mathrm{C}_{i+1}$) can be obtained as a simple consequence of the Tutte-Berge Formula, the real difficulty is ensuring that ($\mathrm{D}'_{i+1}$) and ($\mathrm{E}_{i+1}$) continue to hold. It turns out that in doing so, we need only concern ourselves with subgraphs for which ($\mathrm{D}'_i$) is close to tight. To handle these problematic subgraphs we analyze how they intersect. For the smallest such subgraph $A$, every other problematic subgraph $B$ either:

1. essentially has the same vertex set as $A$,
2. essentially has the same vertex set as $G_i - A$, or
3. essentially has the same vertex set as $G_i$.

We will sketch how this allows us to find our matching $M_i$ below, after we have stated the common generalization of Lemmas 7.13 and 7.14. First, we give the proof of Lemma 7.12 which also combines an application of the Tutte-Berge Formula and an analysis of the intersection properties of subgraphs for which ($\mathrm{D}'_i$) (in this case ($\mathrm{D}'_1$)) is close to tight.

*Proof of Lemma 7.12.* To prove this lemma we combine a straightforward consequence of the Tutte-Berge Formula with a simple but powerful observation about the intersection patterns of $\Delta$-full subgraphs.

**Lemma 7.16.** *If $F$ is a graph of maximum degree $\Delta$ and satisfying $\frac{1}{2}|F| < \Delta < |F| - 1$, then $\overline{F}$ has a matching $M$ with at least $|F| - \Delta - 1$ edges. Furthermore, if $u, v \in F$ satisfy $d(u) < \Delta$ and $d(v) < \Delta$, then we can choose $M$ so that $uv \notin M$ and $\{u, v\} \cap V(M) \neq \emptyset$.*

*Proof.* If the first statement of the lemma fails to hold, then letting $r > 2\Delta + 2 - |F| > 2$ be the number of vertices missed by a maximum matching of $\overline{F}$, the Tutte-Berge Formula implies that there is a set $K$ such that $\overline{F} - K$ contains $|K| + r$ odd components. Since the $|K| + r - 2$ biggest such components contain at least $|K| + r - 2$ vertices, the smallest odd component of $\overline{F} - K$ has at most $\frac{1}{2}((|F| - |K|) - |K| - r + 2)$ vertices. Hence each of its vertices has at least $\frac{1}{2}(|F| - |K|) + \frac{1}{2}|K| + \frac{1}{2}r - 1$ neighbours in $G$ which lie in the other components of $\overline{F} - K$. This is a contradiction as $\frac{1}{2}(|F| + r) - 1 > \Delta$, and so $\overline{F}$ has the desired matching.

Assume $u$ and $v$ are vertices as in the statement of the lemma. Let $F'$ be the graph obtained from $F$ by adding the edge $uv$ if it is not in $F$. Since $\Delta(F') = \Delta$, a maximum matching $N$ of $\overline{F}'$ has at least $|F'| - \Delta - 1$ edges. Clearly, $N$ is also a matching of $\overline{F}$. Now, if $N$ does not satisfy the lemma, then $\{u, v\} \cap V(N) = \emptyset$. Since $\Delta < |F| - 1$ and the vertices of $F - V(N)$ induce a clique $C$ in $F$, $u$ is adjacent in $\overline{F}$ to some vertex $w$ in edge $f \in N$, and so $M = N - f + uw$ satisfies the lemma. $\square$

**Lemma 7.17.** *Suppose $\Delta \geq \frac{3}{4}|G|$ and $|G| > 4$. If $A_1$ and $A_2$ are both $\Delta$-full subgraphs, then $|A_1 \oplus A_2| \leq 2$. Hence, there are at most four $\Delta$-full subgraphs.*

*Proof.* The proof follows easily from the following simple observations:

**Observation 7.18.** *For any $X \subseteq V(H)$, $\sum_{v \in X}(\Delta - d_X(v)) \geq |X|(\Delta + 1 - |X|)$.*

**Observation 7.19.** *Any $\Delta$-full subgraph $A$ satisfies $|A| \geq \Delta$.*

*Proof.* If $A$ is $\Delta$-full, then by definition $\sum_{v\in A}(\Delta - d_A(v)) = \Delta$, and so, Observation 7.18 yields, $|A|(\Delta+1-|A|) \leq \Delta$. It follows that either $|A| \leq 1$ or $|A| \geq \Delta$. The first contradicts the definition of $\Delta$-full, and so, $|A| \geq \Delta$. $\square$

**Observation 7.20.** *For any two subgraphs $A_1$ and $A_2$,*

$$\sum_{v\in(A_1\oplus A_2)} (\Delta - d_{(A_1\oplus A_2)}(v)) \leq \sum_{v\in A_1}(\Delta - d_{A_1}(v)) + \sum_{v\in A_2}(\Delta - d_{A_2}(v)).$$

*Proof.* Observe that $\delta(A_1 \oplus A_2) \subseteq \delta(A_1) \cup \delta(A_2)$ and $\sum_{v\in(A_1\oplus A_2)}(\Delta - d(v)) \leq \sum_{v\in A_1}(\Delta - d(v)) + \sum_{v\in A_2}(\Delta - d(v))$. Hence,

$$\begin{aligned}
\sum_{v\in(A_1\oplus A_2)} (\Delta - d_{(A_1\oplus A_2)}(v)) &= |\delta(A_1 \oplus A_2)| + \sum_{v\in(A_1\oplus A_2)} (\Delta - d(v)) \\
&\leq |\delta(A_1)| + |\delta(A_2)| + \sum_{v\in A_1}(\Delta - d(v)) + \sum_{v\in A_2}(\Delta - d(v)) \\
&= \sum_{v\in A_1}(\Delta - d_{A_1}(v)) + \sum_{v\in A_2}(\Delta - d_{A_2}(v)),
\end{aligned}$$

as desired. $\square$

**Observation 7.21.** *Suppose $\Delta > 4$. For any two $\Delta$-full subgraphs $A_1$ and $A_2$, we have $|A_1 \oplus A_2|$ is either at most 2 or at least $\Delta - 1$.*

*Proof.* Letting $X = A_1 \oplus A_2$, Observation 7.20 implies $\sum_{v\in X}(\Delta - d_X(v)) \leq 2\Delta$, and so by Observation 7.18, $|X|(\Delta - |X| + 1) \leq 2\Delta$. Rearranging, we arrive at $(|X| - 2)\Delta \leq |X|(|X| - 1)$. If $|X| \leq 2$, then the observation follows and so assuming that $|X| > 2$, we find that

$$\Delta \leq \frac{|X|(|X| - 1)}{|X| - 2} \leq |X| + 1 + \frac{2}{|X| - 2}.$$

Hence, we have $|X| \geq \Delta - 1 - \frac{2}{|X|-2}$. Since, $\Delta \geq 5$ this implies $|X| \geq \Delta - 1$ as desired. $\square$

Now, if $A_1$ and $A_2$ fail to satisfy Lemma 7.17, then one of $|A_2 - A_1| \geq \frac{1}{2}(\Delta - 1)$ or $|A_1 - A_2| \geq \frac{1}{2}(\Delta - 1)$, and so by Observation 7.19, $|G| \geq \frac{3}{2}\Delta - \frac{1}{2} > |G|$, a contradiction since by assumption $|G| > 4$. □

With Lemmas 7.16 and 7.17 in hand, we can finish the proof of Lemma 7.12. We can assume $\Delta \leq |G| - 2$, as otherwise the lemma is trivial. If $G$ contains a triple $H, x, y$ such that all of $H, H + x, H + y$, and $H + x + y$ are $\Delta$-full, then both $x$ and $y$ have at most $\frac{1}{2}\Delta$ neighbours in $H$. Therefore, since $|H| \geq \Delta \geq \frac{3}{4}|G|$, we have $|G-(H+x+y)| < \frac{1}{4}|G| \leq \frac{1}{3}\Delta$, and so, $d(x) < \Delta$ and $d(y) < \Delta$. We let $u = x$ and $v = y$, choose a matching $M$ of $\overline{G}$ as in Lemma 7.16, and let $e_1, ..., e_k$ be edges of $M$ such that $e_1$ contains either $u$ or $v$. Otherwise, $G$ contains no such triple and we choose a matching $M$ of $\overline{G}$ with at least $|G| - \Delta - 1$ edges as in Lemma 7.16, and let $e_1, ..., e_k$ be edges of $M$. In either case, we let $S_i = e_i$, for $i = 1, ..., k$. Since there are exactly $2(\Delta + 1) - |G|$ vertices not in $\bigcup_{i=1}^{k} S_i$, we assign each vertex of $G - \bigcup_{i=1}^{k} S_i$ to a unique stable set $S_{k+1}, ..., S_{\Delta+1}$.

If $G$ contains a triple $H, x, y$ such that all of $H, H+x, H+y$, and $H+x+y$ are $\Delta$-full, then by Lemma 7.17, $G$ contains no other $\Delta$-full subgraph. Hence, since $|\{x, y\} \cap S_1| = 1$, ($E_1$) holds. Otherwise, if no such triple exists, then ($E_1$) holds for any choice of $S_1$. In either case, $S_1, ..., S_{\Delta+1}$ satisfies the lemma. □

We prove Lemmas 7.13 and 7.14 together using the following common generalization.

**Lemma 7.22.** *Let $H$ be a graph with $|H| \geq 320$, $D$ an integer with $D \geq \frac{|H|}{2} + 3$, $U$ be a subset of $V(H)$ with $|U| \leq 2D - \frac{|H|}{2}$ and $\{x, y\} \subseteq U$ such that:*

*(a) $\forall v \in U$, $d(v) \leq D - 1$ and $\forall v \in V(H) - U$, $d(v) \leq D$,*

*(b) $\forall A \subseteq H$ with $|A| > 1$, we have $\sum_{v \in A}(D - d_A(v)) - |V(A) \cap U| \geq D - 1$, and*

*(c) $\forall A_1, A_2 \subseteq H$ both satisfying the bound of (b) with equality we have either $|A_1 \oplus A_2| \leq 1$ or $|(A_1 \oplus A_2) \cap \{x, y\}| = 1$.*

*Then we can find a matching $M$ in $H - \{x, y\}$ such that:*

*(i)* $M$ *saturates the set* $\mathcal{X}$ *of vertices in* $H - \{x, y\}$ *for which the bound of (a) holds with equality,*

*(ii)* $\forall A \subseteq H$ *with* $|A| > 1$*, we have*

$$\sum_{v \in A} (D - 1 - d_A(v)) - |V(A) \cap (U - \{x, y\})| + 2|M \cap E(A)| \geq D - 2,$$

*and*

*(iii)* $\forall A_1, A_2 \subseteq H$ *both satisfying the bound of (ii) with equality we have* $|A_1 \oplus A_2| \leq 1$*.*

The proofs of Lemma 7.13 and 7.14, follow easily from Lemma 7.22. For both we set $H = G_i$, $U = \bigcup_{j \geq i} S_j$ and $D = \Delta + 2 - i$. Since $i \leq k = |G| - \Delta - 1$, we have $D \geq \frac{|H|}{2} + 3$ and $|U| \leq 2D - \frac{|H|}{2}$. Moreover, ($\mathrm{C}_i$) implies that (a) holds, ($\mathrm{D}'_i$) implies that (b) holds and ($\mathrm{E}_i$) implies that (c) holds. So, Lemma 7.22 guarantees that there exists a matching $M_i \in G_i - S_i$ such that (i), (ii), and (iii) each hold. It is immediate from (i) together with (a) that ($\mathrm{C}_{i+1}$) holds, from Observation 7.9 and (ii) that ($\mathrm{D}'_{i+1}$) holds and from (iii) that ($\mathrm{E}_{i+1}$) holds.

We now outline the proof of Lemma 7.22. Its details take up the remaining sections of this chapter. As discussed above, in proving this lemma, we must focus on ensuring that (ii) holds for subgraphs for which (b) is nearly tight. We now make explicit what we mean by nearly tight and how our matching must intersect these nearly tight subgraphs if (ii) and (iii) are to hold.

In discussing this, we find it convenient to keep track of the slack in the constraint of (b) using the following notation:

**Definition 7.23.** For a subgraph $A$ of $H$ with $|A| > 1$, the *slack of $A$* is

$$\mathrm{slack}(A) := \left( \sum_{v \in A} (D - d_A(v)) - |V(A) \cap U| \right) - (D - 1).$$

The following definition and observation helps us to understand the condition imposed by (ii):

**Definition 7.24.** The *cost of a matching $M$ for $A$ with respect to* $\{x,y\}$ is

$$\begin{aligned} \text{cost}(M, A) &:= |A| - |\{x, y\} \cap V(A)| - 2|M \cap E(A)| \\ &= |A - \{x, y\} - V(M)| + |M \cap \delta(A)|. \end{aligned}$$

**Observation 7.25.** *For a subgraph $A$ of $H$ and matching $M$ in $H - \{x, y\}$, the left-hand side of (ii) minus the right-hand side of (ii) is equal to slack$(A)$ − cost$(M, A) + 1$.*

Observation 7.25 implies for (ii) or (iii) to fail, we need a subgraph with $\text{slack}(A) - \text{cost}(M, A) + 1 \leq 0$. This motivates the following definition and yields the following corollary.

**Definition 7.26.** A subgraph $A \subseteq H$ is *nearly overfull* if $|A| \geq 4$ and $\text{slack}(A) \leq |A|$.

**Corollary 7.27.** *To ensure (ii) and (iii) hold it is enough to ensure that $M$ satisfies:*

*(iv) for each nearly overfull subgraph $A$ in $H$, slack$(A)$ − cost$(M, A) + 1 \geq 0$, and*

*(v) for any two nearly overfull subgraphs $A_1$ and $A_2$ in $H$ with slack$(A_1)$ − cost$(M, A_1) + 1 = 0$ and slack$(A_2)$ − cost$(M, A_2) + 1 = 0$, $|A_1 \oplus A_2| \leq 1$.*

Corollary 7.27 allows us to focus our attention on nearly overfull subgraphs. We will prove in Section 7.2 that for a smallest nearly overfull subgraph $A$, every nearly overfull subgraph of $H$ shares all but a constant number of vertices with $A$, $H - A$, or $H$. We find the desired matching by choosing an appropriate such subgraph $A$ and considering $A$ and $H - A$ separately.

We discuss the intersection properties of nearly overfull subgraphs in Section 7.2. In Section 7.3, we describe the matching lemmas needed to ensure (i), (iv), and (v) hold. The details of how we combine the pieces to prove Lemma 7.22 can be found in Section 7.4. We finish this section with a proof of Lemma 7.15.

*Proof of Lemma 7.15.* We start with (C). Each vertex $v$ in $G_{k+1}$ satisfies $d_{G_{k+1}}(v) \leq \Delta + 1 - k$ and each vertex $u \in \bigcup_{j=k+1}^{\Delta+1} S_j$ satisfies $d_{G_{k+1}}(u) \leq \Delta - k$. Hence, since each vertex either appears exactly once in some colour class $S_1, ..., S_k$ or is adjacent to $v^*$ in $G^*$, for each $v \in G^* - v^*$, $d_{G^*}(v) \leq \Delta - k + 1$ in $G^*$. Finally, $v^*$ is adjacent to each vertex in $\bigcup_{j=k+1}^{\Delta+1} S_j$ and so $d_{G^*}(v^*) = \Delta + 1 - k$. It remains to show (D$'$) holds.

(D$'_{k+1}$) yields that for each subgraph $A \subseteq G_{k+1}$, $|A| > 1$,

$$\sum_{v \in A} (\Delta + 1 - k - d_A(v)) - \left| A \cap \bigcup_{j>k} S_j \right| \geq \Delta - k = \Delta(G^*) - 1.$$

If $H$ is an odd subgraph of $G^*$ not containing $v^*$, then we have

$$|\delta_{G^*}(H)| + \sum_{u \in H} (\Delta(G^*) - d_{G^*}(u)) \;\;=\;\; \sum_{u \in H} (\Delta + 1 - k - d_H(u)) \geq \Delta(G^*) - 1.$$

If $H$ is an odd subgraph of $G^*$ containing $v^*$, then we have

$$\begin{aligned} |\delta_{G^*}(H)| + \sum_{u \in H} (\Delta(G^*) - d_{G^*}(u)) &= \sum_{u \in H - v^*} (\Delta + 1 - k - d_{(H-v^*)}(u)) \\ &\quad - |N(v^*) \cap H| + |N(v^*) - H| \\ &\geq \Delta - k + \left| (H - v^*) \cap \bigcup_{j>k} S_j \right| \\ &\quad - |N(v^*) \cap H| + |N(v^*) - H| \\ &= \Delta - k + |N(v^*) - H| \geq \Delta(G^*) - 1. \end{aligned}$$

Hence, (D$'$) holds. $\square$

## 7.2 Nearly Overfull Subgraphs and their Intersection Patterns

In this section, we develop further the intersection properties of nearly overfull subgraphs. We prove that for a smallest nearly overfull subgraph $A$, every nearly overfull subgraph of $H$ shares all but a constant number of vertices with $A$, $H - A$, or $H$. We then turn to nearly overfull subgraphs whose slack is at most 20 and describe their stronger intersection properties. Similar to Lemma 7.17, there are at most four of these subgraphs, a fact on which our proof of Lemma 7.22 relies.

We prove the following three lemmas:

**Lemma 7.28.** *Suppose $|H| \geq 65$, $D \geq \frac{1}{2}|H| + 3$, and $T$ is a nearly overfull subgraph with $|T| < |H| - 6$. Then each nearly overfull subgraph is within 6 of $T$, within 6 of $H$, or within 8 of $H - T$.*

**Lemma 7.29.** *Suppose $|H| \geq 320$ and $D \geq \frac{1}{2}|H| + 3$. If $A$ and $B$ are nearly overfull subgraphs satisfying slack$(A) \leq 20$ and slack$(B) \leq 20$, then $|A \oplus B| \leq 2$. Furthermore, if $|A \oplus B| = 2$ where $(A \oplus B) = \{x, y\}$ and each nearly overfull subgraph has nonnegative slack, then the subgraphs $(A \cap B), (A \cap B) + x, (A \cap B) + y$ and $(A \cap B) + x + y$ have slack less than each other nearly overfull subgraph $C$ satisfying $|A \oplus C| \leq 3$.*

**Lemma 7.30.** *Suppose $|H| \geq 65$, $D \geq \frac{1}{2}|H| + 3$, and $T^*$ is a nearly overfull subgraph with slack$(T^*) \leq 20$. Then each nearly overfull subgraph of $H$ is either within 6 of $T^*$, within 6 of $H$, or within 8 of $H - T^*$.*

Lemma 7.29 will be used to show that if $A$ and $B$ are any two nearly overfull subgraphs both with slack at most 20, then they also satisfy $|A \oplus B| \leq 2$. Using Lemmas 7.28 and 7.30, we will the prove the following lemma below.

**Definition 7.31.** A nearly overfull subgraph $T$ is *good* if:

(1) Each nearly overfull subgraph is within 6 of $T$, 6 of $H$, or 8 of $H - T$.

(2) Exactly one of the following holds:

  (2a) Each nearly overfull subgraph has slack at least 21 and slack$(T) = \min\{\text{slack}(B) : B$ nearly overfull and $|T \oplus B| \leq 6\}$.

  (2b) slack$(T) \leq 20$ and slack$(T) = \min\{\text{slack}(B) : B$ nearly overfull$\}$.

**Lemma 7.32.** *Suppose $|H| \geq 65$ and $D \geq \frac{1}{2}|H| + 3$. Then either each nearly overfull subgraph contains at least $|H| - 6$ vertices and has slack at least 21, or $H$ contains a good subgraph.*

In proving Lemmas 7.28, 7.29 and 7.30 we exploit the fact that $D \geq \frac{1}{2}|H|+3$. The basis to understanding how nearly overfull subgraphs can intersect are the following observations. As $|H| \leq 2D - 6$, we have,

**Observation 7.33.** *If $A$ is a nearly overfull subgraph of $H$, then slack$(A) \leq 2D - 6$.*

**Observation 7.34.** *For any two subgraphs $A$ and $B$, each of the following hold:*

1. $|\delta(A \cap B)| + \sum_{v \in A \cap B}(D - d(v)) - |(A \cap B) \cap U| \leq 2D - 2 + \mathit{slack}(A) + \mathit{slack}(B)$
2. $|\delta(A - B)| + \sum_{v \in A - B}(D - d(v)) - |(A - B) \cap U| \leq 2D - 2 + \mathit{slack}(A) + \mathit{slack}(B)$
3. $|\delta(B - A)| + \sum_{v \in B - A}(D - d(v)) - |(B - A) \cap U| \leq 2D - 2 + \mathit{slack}(A) + \mathit{slack}(B)$
4. $|\delta(A \oplus B)| + \sum_{v \in A \oplus B}(D - d(v)) - |(A \oplus B) \cap U| \leq 2D - 2 + \mathit{slack}(A) + \mathit{slack}(B)$

*Proof.* We have $\delta(A \cap B) \subseteq \delta(A) \cup \delta(B)$, and, as $D \geq d(v) + |\{v\} \cap U|$ for each vertex $v$,

$$\left(\sum_{v \in A \cap B}(D - d(v))\right) - |(A \cap B) \cap U| \leq \left(\sum_{v \in A}(D - d(v))\right) - |A \cap U| + \left(\sum_{v \in B}(D - d(v))\right) - |B \cap U|.$$

So the first result follows. The remaining results follow similarly. $\square$

Combining this observation with Observation 7.18 yields the following.

**Observation 7.35.** *If $|H| > 64$, then for any two nearly overfull subgraphs $A$ and $B$ of $H$, each of $A \cap B$, $A - B$, $B - A$, and $A \oplus B$ contains either at most 6 or at least $D - 6$ vertices.*

*Proof.* By Observation 7.33, slack$(A)$+slack$(B) \leq 4D - 12$, and so Observation 7.34 yields:

$$|\delta(A \cap B)| + \sum_{v \in A \cap B}(D - d(v)) - |(A \cap B) \cap U| \leq 6D - 14.$$

By Observation 7.18,

$$|A \cap B|(D + 1 - |A \cap B|) - |(A \cap B) \cap U| \leq 6D - 14.$$

Hence, $|A \cap B|(D - |A \cap B|) \leq 6D - 14$. A contradiction follows when $7 \leq |A \cap B| \leq D - 7$ as $D \geq \frac{1}{2}|H| + 3 > 35$. The remaining results follow similarly. $\square$

If $|H| > 64$ and $H$ contains some nearly overfull subgraph $F$, then Observation 7.35 allows us to classify each nearly overfull subgraph $A$ of $H$ into two types with respect to $F$:

**Close to $F$:** $A$ is within 6 vertices of $F$, i.e. $|F \oplus A| \leq 6$.

**Far from $F$:** at least one of $F - A$ and $A - F$ has size at least $D - 6$.

Since $D \geq \frac{n}{2} + 3$, it follows that closeness is an equivalence relation. In fact, there are at most three equivalence class. To see this, we start by noting that for each nearly overfull subgraph $A$ of $H$ we have $|\delta(A)| + \sum_{v \in A}(D - d(v)) \leq D + 2|A|$ and so Observation 7.18 implies:

**Observation 7.36.** *Every nearly overfull subgraph in $H$ has at least $D - 2 \geq \frac{1}{2}|H| + 1$ vertices.*

With these observations in hand, we can now prove our first structural lemma:

*Proof of Lemma 7.28.* Consider nearly overfull $B$ which is far from $T$. If $|B - T| \leq 6$, then $|T - B| \geq D - 6$ and $|B \cap T| \geq D - 6$ implying that $|T| \geq 2D - 12 \geq |H| - 6$, a contradiction. Suppose $|B - T| \geq D - 6$ and consider two cases: $|T \cap B| \geq D - 6$ and $|T \cap B| \leq 6$. In the former case, $|B| = |B - T| + |T \cap B| \geq 2(D - 6) \geq |H| - 6$ and so $B$ is within 6 of $H$. In the latter case, since $|H - T| \leq \frac{|H|}{2} - 1$ and $|B - T| \geq \frac{|H|}{2} - 3$, we have $|H - (T \cup B)| \leq 2$, and so, $B$ is within 8 of $H - T$. $\square$

We now turn to the intersection properties of nearly overfull subgraphs whose slack is at most 20. We start by improving Observations 7.35 and 7.36 for this case.

**Observation 7.37.** *Suppose $|H| > 40$. If nearly overfull $T$ has slack$(T) \leq 20$, then $|T| \geq D-1$.*

*Proof.* We have $\sum_{v \in T}(D - d_T(v)) \leq D + 19 + |T \cap U| \leq D + 19 + |T|$, and so, Observation 7.18 implies $|T|(D - |T|) \leq D + 19$. Since $D \geq \frac{1}{2}|H| + 3 > 23$, this implies either $|T| \leq 2$ or $|T| \geq D - 1$. Since $T$ is nearly overfull we have $|T| \geq 4$, and so, it must be the latter. $\square$

Analogous to Observation 7.35, if we assume $|H| \geq 109$, then we have the following.

**Observation 7.38.** *Suppose $|H| \geq 109$. Then for any two subgraphs $T_1$ and $T_2$ of $H$ with slack$(T_1) \leq 20$ and slack$(T_2) \leq 20$, each of $T_1 - T_2$, $T_2 - T_1$, and $T_1 \oplus T_2$ contains either at most 2 or at least $D - 2$ vertices.*

We need the following observation which describes the change in slack when we add or remove a vertex from a subgraph, and follows from the definition of slack.

**Observation 7.39.** *For any subgraph $F$ of $H$ and $u \notin F$,*

$$\textit{slack}(F + u) = \textit{slack}(F) + (D - 2|\delta(u) \cap \delta(F)| - |\{u\} \cap U|)\,.$$

*Proof.* By definition,

$$\begin{aligned}
\text{slack}(F+u) &= |\delta(F+u)| + \left( \sum_{v \in F-u} (D - d(v)) - |V(F+u) \cap U| \right) - (D-1) \\
&= |\delta(F)| + |\delta(u)| - 2|\delta(u) \cap \delta(F)| + \sum_{v \in F} (D - d(v)) + (D - d(u)) \\
&\qquad - |V(F) \cap U| - |\{u\} \cap U| - (D-1) \\
&= \text{slack}(F) + (D - 2|\delta(u) \cap \delta(F)| - |\{u\} \cap U|)\,,
\end{aligned}$$

as desired. $\square$

The proofs of Lemmas 7.29 and 7.30 are now easy.

*Proof of Lemma 7.29.* By Observations 7.38, if $|A \oplus B| > 2$, then $|A \oplus B| \geq D - 2$, and so one of $|A - B| \geq D - 2$ and $|B - A| \geq D - 2$ holds. Hence, Observation 7.37 implies $|H| \geq 2D - 3$, a contradiction as $D \geq \frac{1}{2}|H| + 3$.

Suppose $A \oplus B = \{x, y\}$. We claim

(1) each of $(A \cap B), (A \cap B) + x, (A \cap B) + y$ and $(A \cap B) + x + y$ has slack at most 44, and

(2) each other nearly overfull subgraph $C$ satisfying $|A \oplus C| \leq 3$ has slack greater than 44.

From this the lemma follows.

To prove Claim (1), it is enough to show that if $F$ is a nearly overfull subgraph satisfying $|A \oplus F| = |B \oplus F| = 1$, then $\text{slack}(F) \leq 44$. Since $\text{slack}(A) \leq 20$ and each nearly overfull subgraph has nonnegative slack, Observation 7.39 implies each vertex $u \in A$ has $d_A(u) \geq \frac{1}{2}D - 11$ and each vertex $u \notin A$ has $|\delta(u) \cap \delta(A)| \leq \frac{1}{2}D + 10$. Similarly, each vertex $u \in B$ has $d_B(u) \geq \frac{1}{2}D - 11$ and each vertex $u \notin B$ has $|\delta(u) \cap \delta(B)| \leq \frac{1}{2}D + 10$. These two facts imply each vertex in $A \oplus B$ has between $\frac{1}{2}D - 12$ and $\frac{1}{2}D + 12$ neighbours in $A$. Since $F$ satisfies $|A \oplus F| = |B \oplus F| = 1$, there exists a $v \in A \oplus B$ such that either $F = A + v$ or $F = A - v$, and so, Observation 7.39 implies $\text{slack}(C) \leq 44$. Claim (1) now follows.

We now prove Claim (2). We have $|\delta(A)| + \sum_{v \in A}(D - d(v)) - |U \cap A| \leq D + 19$. Since each vertex of $A \oplus B$ contributes at least $\frac{1}{2}D - 12$ to the left-hand side of this sumation, we have $\sum_{v \in (A \cap B)}(D - d(v)) - |U \cap (A \cap B)| \leq D + 19 - 2(\frac{1}{2}D - 12) \leq 43$. Hence, any vertex not in $A \cup B$ has at most 45 neighbours inside $A$ and each vertex in $(A \cap B)$ has at least $D - 45$ neighbours inside $A$. Observation 7.39 now implies that each nearly overfull subgraph $C$ satisfying $|A \oplus C| \leq 3$ and not equal to $(A \cap B), (A \cap B) + x, (A \cap B) + y$, and $(A \cap B) + x + y$ has slack at least $D - 100 > 44$ as required. Claim (2) now follows. $\square$

*Proof of Lemma 7.30.* By Lemma 7.28, it is enough to prove that if $|T^*| \geq |H| - 6$, then each nearly overfull subgraph is close to $T^*$. Assume otherwise, that there exists nearly overfull $F$ satisfying $|T^* \oplus F| > 6$. Observation 7.35 implies $|T^* \oplus F| \geq D - 6$, and so, one of $|T^* - F| \geq D - 6$ and $|F - T^*| \geq D - 6$ holds. The latter case is impossible since $|T^*| \geq |H| - 6$. In the former case, the fact that $|H| \leq 2D - 6$, implies $|F| \leq D$ and so $\text{slack}(F) \leq |F| \leq D$. By Observation 7.34, $|\delta(T^* - F)| + \sum_{v \in T^* - F}(D - d(v)) - |(T^* - F) \cap U| \leq 3D + 18$. It follows that $|T^* - F|(D - |T^* - F|) \leq 3D + 18$, from which we conclude $|T^* - F| \geq D - 3$. This is a contradiction as $|H| \geq |F| + |T^* - F| \geq D - 2 + D - 3 = 2D - 5$. □

We finish this section by proving Lemma 7.32.

*Proof of Lemma 7.32.* Assume that there exists a nearly overfull subgraph which either has less than $|H| - 6$ vertices or slack less than 20. In the latter case, letting $T^*$ be a subgraph of minimum slack, Lemma 7.30 implies $T^*$ is good. So, assume each nearly overfull subgraph has slack at least 21 and let $T$ be a nearly overfull subgraph containing less than $|H| - 6$ vertices. Let $T^*$ be a subgraph close to $T$ which minimizes slack. Since closeness is a equivalence relation, $T^*$ satisfies $\text{slack}(T^*) = \min\{\text{slack}(B) : B \text{ nearly overfull and } |T^* \oplus B| \leq 6\}$. We claim that $T^*$ also satisfies Property (1), which implies $T^*$ is good.

If $|T^*| < |H| - 6$, then the claim follows from Lemma 7.28. Otherwise, since $|T \oplus T^*| \leq 6$, we have $|H| - 12 \leq |T| < |H| - 6$. Let $A$ be nearly overfull subgraph far from $T$, we have either $|T - A| \geq D - 6$ or $|A - T| \geq D - 6$. In the latter case, we have $|H| \geq |T| + |A - T| \geq |H| - 12 + D - 6$, a contradiction. Assume the former case occurs but not the latter. Since $|A| \geq D - 2$, we have $|A \cap T| \geq D - 6$, and so, $|H| = |H - T| + |T \cap A| + |T - A| > 2D - 6 \geq |H|$, a contradiction. It follows that each nearly overfull subgraph is close to $T$. The claim now follows, since closeness is an equivalence relation, and so, each nearly overfull subgraph is close to $T^*$. □

## 7.3 The Matching Toolkit

In proving Lemma 7.22, we need to saturate the high degree vertices in $\mathcal{X}$ so as to ensure (i) holds, and choose matchings with enough edges to ensure (iv) and (v) (and hence (ii) and (iii)) hold. We present some tools for doing each of these separately and then discuss doing them both at once.

### 7.3.1 Saturating High Degree Vertices

We start by proving that if every vertex of a set $Z$ has high degree and does not induce an odd component, then we can find a matching saturating $Z$.

**Lemma 7.40.** *Suppose $Z$ is a set of vertices of $G$, every vertex of which has degree at least $\frac{1}{2}|G|$, and which is not the vertex set of an odd component of $G$. Then, there is a matching saturating $Z$.*

For $|G|$ at least 15, we then obtain the following stronger result.

**Lemma 7.41.** *Suppose $|G| \geq 15$, and $Z$ is a subset of $V(G)$, contain no vertices of degree less than $\frac{1}{2}|G|-2$ and containing at most $\frac{1}{2}|G|-5$ vertices of degree less than $\frac{1}{2}|G|$. If there is no matching saturating $Z$, then some subset of $Z$ is the vertex set of an odd component of $G$.*

We then apply Lemma 7.40 to prove the following:

**Lemma 7.42.** *If $H$ satisfies the conditions of Lemma 7.22, then $H-\{x,y\}$ has a matching satisfying (i).*

*Proof of Lemma 7.40.* If the lemma fails, then by Corollary 3.12 there exists a set $K \subseteq V(G)$ such that $G-K$ contains $t \geq |K|+1$ odd components $O_1, ..., O_t$ completely contained in $Z$. Letting $r = \min_j |O_j|$ we have:

$$|G| \geq r(|K|+1) + |K| = (r+1)(|K|+1) - 1.$$

Considering a vertex $v$ in an $O_j$ of size $r$ we have:

$$|K| \geq |N(v) \cap K| \geq |N(v)| - r + 1 \geq \frac{1}{2}|G| + 1 - r.$$

Combining these two inequalities yields:

$$|G| \geq (r+1)\left(\frac{1}{2}|G| + 2 - r\right) - 1.$$

This implies that either $r < 1$ or $r > \frac{1}{2}|G|$. The former is impossible. The latter implies $t = 1$, $K = \emptyset$, and hence $O_j \subseteq Z$ is the vertex set of an odd component of $G$. Since $Z$ has minimum degree $\frac{1}{2}|G|$, it intersects at most one component of $G$, and hence, $Z$ is the vertex set of an odd component of $G$, a contradiction. So, there must exist a matching saturating $Z$ as desired. $\square$

*Proof of Lemma 7.41.* As in the proof of Lemma 7.40, if there is no matching saturating $Z$, then there exists $K \subseteq V(G)$ such that $G - K$ contains $t \geq |K| + 1$ odd components $O_1, ..., O_t$ completely contained in $Z$, and for $r = \min_j |O_j|$ we have $|G| \geq (r+1)(|K|+1) - 1$, and $|K| \geq \frac{1}{2}|G| - 2 - r + 1$, which yields:

$$|G| \geq (r+1)\left(\frac{1}{2}|G| - r\right) - 1.$$

As $|G| \geq 15$, this implies either $r \leq 1$ or $r \geq \frac{1}{2}|G| - 2$.

Now, if $r \leq 1$, then $|K| \geq \frac{1}{2}|G| - 2$, $t \geq \frac{1}{2}|G| - 1$, and, since $|G| \geq |K| + t + 2|\{O_j : |O_j| > 1, j = 1...t\}|$, we see that there are at least $\frac{1}{2}|G| - 2$ odd components $O_j$, which are a single vertex of $Z$. By hypothesis, one of these vertices has at least $\frac{1}{2}|G|$ neighbours and hence $|K| \geq \frac{1}{2}|G|$, a contradiction.

If $r \geq \frac{1}{2}|G| - 2$, then either $|K| = 0$ and $t \geq 1$, or $|K| = 1$ and $t = 2$. In the former case, each odd component of $G - K$ is an odd component of $G$ contained in $Z$. In the latter case, the smaller of these two odd components has size between $\frac{1}{2}|G| - 2$ and $\frac{1}{2}|G| - \frac{1}{2}$. Hence, each of its vertices has degree at most $\frac{1}{2}|G| - \frac{1}{2}$, contradicting our degree assumptions. The lemma now follows. $\square$

*Proof of Lemma 7.42.* The key is the following observation, which we will use again in Section 7.4.

**Observation 7.43.** *No subset of $\mathcal{X}$ induces an odd component in $H - \{x, y\}$.*

*Proof.* Assume $Z \subseteq \mathcal{X}$ induces an odd component in $H - \{x, y\}$. Since $Z$ satisfies (b), there are at least $D - 1$ edges between $Z$ and $H - Z$ which must all go to $\{x, y\}$. Since $Z + x + y$ also satisfies (b) and $\{x, y\} \subseteq U$, $D - |N(x) \cap Z| + D - |N(y) \cap Z| - 2$ is also at least $D - 1$. So, both $Z$ and $Z + x + y$ satisfy (b) with equality, contradicting (c). $\square$

Each vertex in $\mathcal{X}$ has at least $D - 3 \geq \frac{1}{2}|H| \geq \frac{1}{2}|H - \{x, y\}|$ neighbours in $H - \{x, y\}$ and $\mathcal{X}$ does not induce an odd component in $H$. Hence to find $M$, we apply Lemma 7.40 to $G = H - \{x, y\}$ and $Z = \mathcal{X}$. $\square$

### 7.3.2 Finding Large Matchings

We start by showing that if $G$ is large enough and satisfies certain density properties (similar to that of being nearly overfull), then there exists a matching saturating nearly every vertex of $G$.

**Lemma 7.44.** *Let $G$ be a graph with maximum degree at most $D$ and such that for each subgraph $H$ of $G$ satisfying $D \leq |H| < |G| - 2$, we have $\sum_{v \in H}(D - d_H(v)) \geq D$. If $r \geq 2$ and $\sum_{v \in G}(D - d(v)) \leq rD$, then $G$ has a matching missing at most $r$ vertices.*

*Proof.* If the lemma fails to holds, then the Tutte-Berge formula (Theorem 3.9) yields that $G$ contains a set $K$ such that $G - K$ has $t \geq |K| + r + 1$ odd components $O_1, ..., O_t$. Since $r \geq 2$, this implies there are at least three odd components. For any odd component $O_j$, each edge of $\delta(O_j)$ has one endpoint in $O_j$ and the other in $K$. Hence, $\sum_{j=1}^{t} |\delta(O_j)| \leq |K|D$. Moreover, as there are at least three odd components, each odd component $O_i$ satisfying

$|O_i| \geq D$ also satisfies $|O_i| < |G| - 2$, and so, $|\delta(O_i)| + \sum_{v \in O_i}(D - d(v)) \geq D$. Since $(\ell - 1)(D - \ell) \geq 0$ for $1 \leq \ell \leq D$, we have for each odd component $O_i$ satisfying $|O_i| \leq D$ that $|O_j|(D - |O_j| + 1) \geq D$. By Observation 7.18,

$$\begin{aligned}
\sum_{j=1}^{t} D &\leq \sum_{j:|O_j| \geq D} D + \sum_{j:|O_j|<D} |O_j|(D - |O_j| + 1) \\
&\leq \sum_{j:|O_j| \geq D} \left( |\delta(O_j)| + \sum_{v \in O_j} (D - d(v)) \right) + \sum_{j:|O_j|<D} \left( |\delta(O_j)| + \sum_{v \in O_j} (D - d(v)) \right) \\
&\leq \sum_{v \in G} (D - d(v)) + \sum_{j=1}^{t} |\delta(O_j)| \\
&\leq (|K| + r)D.
\end{aligned}$$

Hence, $t \leq |K| + r$. This contradiction completes the proof. $\square$

The following is an easy corollary of Lemma 7.44.

**Lemma 7.45.** *Suppose $H$ satisfies the conditions of Lemma 7.22, and $T$ is a nearly overfull subgraph of minimum slack of $H$. Then $T - \{x, y\}$ has a matching $N$ with cost$(N, T) \leq 7$.*

*Proof.* Since $T$ is nearly overfull, $\sum_{v \in T} (D - d_T(v)) \leq 5D$, and so

$$\sum_{v \in T - \{x,y\}} \big(D - d_{(T - \{x,y\})}(v)\big) \leq 7D.$$

By minimality and Lemma 7.29, each subgraph $F \subseteq T - \{x, y\}$ with $|F| < |T - \{x, y\}| - 2$ satisfies $\sum_{v \in F}(D - d_F(v)) \geq D$. Hence, Lemma 7.44 yields that the cost of a maximum matching of $T - \{x, y\}$ is at most 7. $\square$

Now, if we put additional constraints on $r$, $D$ and $|G|$, then we are able to strengthen Lemma 7.44 as follows.

**Lemma 7.46.** *Suppose $r \in \mathbb{Z}$ satisfies $1 \leq r \leq 8$ and $|G| \geq 320$. Let $G$ be a graph with maximum degree at most $D$, minimum degree at least $2r+1$, and such that for each subgraph $H$ of $G$ satisfying $D \leq |H| < |G|-2$, we have $\sum_{v \in H}(D - d_H(v)) \geq D$.*

*If $D \geq \frac{1}{2}|G| + 3$ and $\sum_{v \in G}(D - d(v)) \leq rD$, then one of the following holds:*

1. *$G$ has a perfect or near-perfect matching missing any vertex,*
2. *there exists a set $K$ satisfying $|K| \leq 1$ and $G - K$ has exactly two odd components both of size at least $D - r + 1$, or*
3. *there exists a set $K$ satisfying $|K| \geq D - 2r$ and $G - K$ has at least $|K| + 1$ odd components, each of size at most $r$.*

*Proof.* If the lemma fails to hold, then there exists $K \subseteq V(G)$ such that $G - K$ has $t \geq |K| + 1$ odd components $O_1, ..., O_t$. We assume that $|K|$ is maximal and so each $O_i$ has a near perfect matching missing every vertex and every vertex of $G$ is in $K$ or some odd component. Furthermore, if there is exactly one odd component $O_1$, then $K = \emptyset$ and $O_1 = G$ has a near perfect matching missing any vertex, and so Case 1 applies. So, we can assume $t \geq 2$.

Now, if an odd component $O_j$ satisfies $|O_j| \geq |G| - 2$, then the minimum degree condition ensures that $O_j = G$. Hence, if $|O_j| \geq D$ then $|\delta(O_j)| \geq D$. As in the proof of Lemma 7.44, we have

$$\sum_{j:|O_j| \geq D} D + \sum_{j:|O_j|<D} |O_j|(D - |O_j| + 1) \leq (|K| + r)D.$$

Each odd component adds at least $D$ to this sum. Moreover, as $D \geq 163$ and $r \leq 8$, each odd component of size between $r+1$ and $D - r$ adds more than $rD$ to this sum, and so, each $O_j$ has size at most $r$ or at least $D - r + 1$. Now, as $|G| \leq 2D - 6$, there are at most two components of size at least $D - r + 1$, and thus at least $|K| - 1$ odd components of size at most $r$. Each vertex in a component of size at most r adds at least $D - |K| - r + 1$

to $\sum_{v\in G}(D - d(v))$, and so,

$$(|K| - 1)(D - |K| - r + 1) \leq \sum_{v\in G}(D - d(v)) \leq rD.$$

As $D \geq 163$ and $r \leq 8$, it follows that either $|K| \leq r + 1$ or $|K| \geq D - 2r$. In the former case, each vertex in an odd component of size at most $r$ has degree at most $2r$, and so by our degree assumptions, there are no such components; hence, there are exactly two components of size at least $D - r + 1$ satisfying Case 2. In the latter case, as $|G| \leq 2D - 6$ and $|G| \geq 2|K| + 1$, each odd component has size at most $r$; hence Case 3 applies. $\square$

**Corollary 7.47.** *Suppose $r \in \mathbb{Z}$ satisfies $1 \leq r \leq 8$ and $|G| \geq 320$. Let $G$ be a graph with maximum degree at most $D$, minimum degree at least $2r + 1$, and such that for each subgraph $H$ of $G$ satisfying $D \leq |H| < |G| - 2$, we have $\sum_{v\in H}(D - d_H(v)) \geq D$.*

*If either of the following two conditions hold, then $G$ has a perfect or near-perfect matching missing any vertex:*

1. $D \geq \frac{1}{2}|G| + 2r$ *and* $\sum_{v\in G}(D - d(v)) \leq rD$.
2. $D \geq \frac{1}{2}|G| + 4$ *and* $\sum_{v\in G}(D - d(v)) \leq \frac{9}{2}D$.

*Proof of 1.* By Lemma 7.46, if the corollary fails to hold, then there exists $K \subseteq V(G)$ such that either $|K| \leq 1$ and $G - K$ has two odd components both of size at least $D - (r - 1)$, or there exists a set $K$ satisfying $|K| \geq D - 2r$ and $G - K$ has $t \geq |K| + 1$ odd components $O_1, ..., O_t$, each of size at most $r$. The first of these conditions implies $|G| \geq 2D - 2(r - 1)$ and the second implies $|G| \geq 2D - 4r + 1$. In either case we contradict the fact that $|G| \leq 2D - 4r$. The desired result follows.

*Proof of 2.* Letting $r = 5$, Lemma 7.46 implies that if the corollary fails to hold, then there exists $K \subseteq V(G)$ such that either $|K| \leq 1$ and $G - K$ has two odd components both of size at least $D-4$, or there exists a set $K$ satisfying $|K| \geq D-10$ and $G-K$ has $t \geq |K|+1$ odd components $O_1, ..., O_t$, each of size at most 5.

The former implies $|G| \geq 2D-8$ and so both odd components have size exactly $D-4$. Observation 7.18 implies each odd component adds at least $5(D-4)$ to $\sum_{v \in G}(D-d(v))$. This is a contradiction as $\sum_{v \in G}(D-d(v)) \geq 10(D-4) > \frac{9}{2}D$.

In the latter case, we tighten our analysis of Lemma 7.46. We note that each non-singleton odd component adds at least $3D-6$ to $\sum_{j=1}^{t} |O_j|(D-|O_j|+1) \leq (|K|+\frac{9}{2})D$, and so if $s$ is the number of such components, we have $s \leq 2$. Now, each non-singleton odd component adds at least $3(D-|K|-4)$ to $\sum_{v \in G}(D-d(v))$. Moreover, there are $|K|+1-s$ singleton components, each adding $D-|K|$ to $\sum_{v \in G}(D-d(v))$, and so

$$(|K|+1-s)(D-|K|)+3(D-|K|-4)s \quad \leq \quad \sum_{v \in G}(D-d(v)) \leq \frac{9}{2}D.$$

It follows that either $|K| \leq 3$ or $|K| \geq D-4$. Since $|K| \geq D-6$, it must the latter case, and so, we have $|G| \geq 2|K|+1 \geq 2D-7$, a contradiction.

This completes the proof of the corollary. □

### 7.3.3 Combining the Tools

The following lemma allows us to combine the tools of Section 7.3.1 and 7.3.2.

**Lemma 7.48.** *If a graph $F$ has a matching $N$ saturating each vertex in some subset $X \subseteq V(F)$, then $F$ has a maximum matching $N^*$ saturating each vertex in $X$.*

*Proof.* Let $M$ be a maximum matching of $F$. Each component of the graph with vertex set $V(F)$ and edge set $N \cup M$ is either an even cycle or a path. For each component $C$, we will pick $N^*_C$ to be either the edges of $N \cap E(C)$ or $M \cap E(C)$ and then set $N^* = \bigcup_C N^*_C$.

For each component $C$ which is an even cycle, let $N^*_C$ be the edges of $N \cap E(C)$. For each component $C$ which is a path with an odd number of edges, the maximality of

$M$ implies that the first and last edges of the path are in $M$; we let $N^*_C$ be the edges of $M \cap E(C)$. For each component $C$ which is a path with an even number of edges, let $N^*_C$ be the edges of $N \cap E(C)$.

It is immediate that $|N^*| = |M|$ and that each endpoint of an edge in $N$ is the endpoint of some edge in $N^*$. Hence, $N^*$ is the desired matching. $\square$

### 7.4 Putting It All Together

We now show how to put the pieces together to prove Lemma 7.22. By Corollary 7.27, it is enough to choose a matching $M$ for which (i), (iv) and (v) hold. When $H$ contains no nearly overfull subgraph, (iv) and (v) hold automatically for any matching $M$. That we can choose a matching $M$ such that (i) holds follows trivially from Lemma 7.42. Thus we can assume that $H$ contains a nearly overfull subgraph.

By Lemma 7.32, we need to deal with the following two cases:

(1) each nearly overfull subgraph contains at least $|H| - 6$ vertices and has slack at least 21, or

(2) $H$ contains a good subgraph.

In either case, we start by choosing a nearly overfull subgraph $T$ of $H$ to focus on. In Case 1, we choose $T$ to be a nearly overfull subgraph minimize slack and in Case 2, we choose $T$ to be a good subgraph. We find a matching $M^1 \in T - \{x, y\}$ and at most a constant number of edges $E^*$ between $T - \{x, y\} - V(M^1)$ and $H - T - \{x, y\}$. In doing so, we ensure that $\mathcal{X} \cap T \subseteq V(M^1 \cup E^*)$ and that for the matching $M = M^1 \cup E^*$, the conditions of (iv) and (v) hold for each nearly overfull subgraph close to $T$. We then find a matching $M^2 \in H - T - \{x, y\} - V(E^*)$. We ensure that $\mathcal{X} - T - V(E^*) \subseteq V(M^2)$ and the matching $M = M^1 \cup M^2 \cup E^*$ satisfies (iv) and (v). It follows that $M = M^1 \cup M^2 \cup E^*$ satisfies the lemma.

In proving that (iv) and (v) hold, we use the following observation which allows us to bound the cost of the matching $M$ for each nearly overfull subgraph by comparing it to the cost of $M$ for $T$, $H-T$ and $H$.

**Observation 7.49.** *For any two subgraphs $A$ and $B$ of $H$ and matching $N$ of $H-\{x,y\}$, cost$(N,A) \leq$ cost$(N,B)+|(A \oplus B)-\{x,y\}|$.*

*Proof.* By definition, $\text{cost}(N,A)-\text{cost}(N,B)$ is equal to

$$|A-\{x,y\}-V(N)|+|N\cap\delta(A)|-|B-\{x,y\}-V(N)|-|N\cap\delta(B)|.$$

We have $|A-\{x,y\}-V(N)|-|B-\{x,y\}-V(N)| \leq |(A-B)-\{x,y\}-V(N)|$ and

$$\begin{aligned} |N\cap\delta(A)|-|N\cap\delta(B)| &\leq |N\cap E(A-B,G-(A\cup B))|+|N\cap E(A,B-A)| \\ &\leq |((A-B)-\{x,y\})\cap V(N)|+|(B-A)-\{x,y\}|. \end{aligned}$$

It follows that $\text{cost}(N,A)-\text{cost}(N,B) \leq |A-B-\{x,y\}|+|B-A-\{x,y\}|$, as desired. □

To choose the matchings $M^1, E^*$ and $M^2$, we use Lemmas 7.42, 7.45 and 7.48 and the following four lemmas.

**Lemma 7.50.** *Suppose $H$ satisfies the conditions of Lemma 7.22 and $A$ is a good subgraph satisfying slack$(A) \geq 21$ and $|A| > \frac{3}{4}|H|$. Then there exists a matching $M^1$ of $A-\{x,y\}$ and a subset $E^*$ of $\delta(A)$ such that $M^1 \cup E^*$ is a matching of $H-\{x,y\}$ satisfying cost$(M^1 \cup E^*, A) \leq 7$ and $\mathcal{X} \subseteq V(M^1\cup E^*)$.*

**Definition 7.51.** For any subgraph $A$ of $H$, let $A_e$ be a subgraph satisfying $|(A\oplus A_e)-\{x,y\}|=1$ and minimizing slack out of all such subgraphs.

**Lemma 7.52.** *Suppose $H$ satisfies the conditions of Lemma 7.22 and $A$ is a good subgraph satisfying slack$(A) \leq 20$ or $|A| \leq \frac{3}{4}|H|$. Then there exists a matching $M^1$ of $A-\{x,y\}$ and a subset $E^*$ of $\delta(A)$ such that $M^1\cup E^*$ is a matching of $H-\{x,y\}$ satisfying cost$(M^1, A) \leq 1$, $\mathcal{X}\cap A \subseteq V(M^1\cup E^*)$, and if $|A-\{x,y\}|$ is odd then, cost$(M^1\cup E^*, A_e)=0$.*

**Lemma 7.53.** *Suppose $H$ satisfies the conditions of Lemma 7.22 and $A$ is a good subgraph of $H$. For any set $S$ of at most three vertices, there exists a matching $M'$ of $(H - A) - S$ saturating each vertex of $(\mathcal{X} - A) - S$.*

**Lemma 7.54.** *Suppose $H$ satisfies the conditions of Lemma 7.22 and $A$ is a good subgraph of $H$. If there exists some nearly overfull subgraph far from $A$, then for any set $S$ of at most three vertices, there exists a matching $M''$ of $(H-A)-S$ satisfying cost$(M'', (H-A)-S) \leq 12$.*

We now deal with the two cases outlined above.

**Case 1.** Let $T$ be a nearly overfull subgraph of minimum slack. Lemma 7.45 implies $T - \{x, y\}$ has a matching $M'$ with $\text{cost}(M', T) \leq 7$. Since $|T| \geq |H| - 6$, Observation 7.49 implies $\text{cost}(M', H) \leq 13$. By Lemma 7.42, $H - \{x, y\}$ has a matching $M''$ satisfying (i). Hence by Lemma 7.48, $H - \{x, y\}$ has a matching $M$ satisfying (i) and such that $\text{cost}(M, H) \leq 13$. Since each nearly overfull subgraph $A$ is within 6 of $H$, Observation 7.49 implies $\text{cost}(M, A) \leq 19$. This fact together with the assumption that each nearly overfull subgraph has slack at least 21 implies (iv) and (v) also hold for this choice of $M$. (To satisfy the sketch above, set $M^1 = M \cap E(H)$, $E^* = M \cap \delta(H)$, and $M^2 = M - M^1 - E^*$.)

**Case 2.** Let $T$ be a good subgraph. First suppose $\text{slack}(T) \geq 21$ and $|T| > \frac{3}{4}|H|$. Choosing $M^1$ and $E^*$ as in Lemma 7.50, it is immediate that $M = M^1 \cup E^*$ satisfies (i) and $\text{cost}(M, T) \leq 7$. We claim each nearly overfull subgraph $A$ is within 6 of $T$, and so by Observation 7.49, $\text{cost}(M, A) \leq 13$. Since $\text{slack}(A) \geq 21$ for each such $A$, we see that (iv) and (v) hold for $M$. Thus, to complete the proof of this case, it remains to prove the claim. To this end, we note that if there exists a nearly overfull subgraph far from $T$, then either $|T - H| \geq D - 6$ or $|H - T| \geq D - 6$. The former case contradicts that $T$ is good. The latter case is also impossible, since $|T| \geq \frac{3}{4}|H|$, $|H| \leq 2D - 6$ and $D \geq 163$, and so $|H - T| \leq \frac{1}{4}|H| < D - 6$.

Second suppose $\text{slack}(T) \leq 20$ or $|T| \leq \frac{3}{4}|H|$. Choose $M^1$ and $E^*$ as in Lemma 7.52 and set $S = \{x, y\} \cup (V(E^*) - T)$. We note that since $\text{cost}(M^1, T) \leq 1$, it follows that $|E^*| \leq 1$, and so, $|S| \leq 3$. In finding the matching $M^2$, we consider the follow two cases: (A) each nearly overfull subgraph is close to $T$, and (B) there exists a nearly overfull subgraph far from $T$.

**Case A.** By Lemma 7.53, there is a matching $M^2$ of $(H - T) - S$ saturating each vertex of $(\mathcal{X} - T) - S$. Let $M = M^1 \cup E^* \cup M^2$. Clearly, (i) holds with respect to $M$. We now prove that (iv) and (v) also hold.

For any nearly overfull subgraph $A$, the fact that $\text{cost}(M, T) \leq 1$ combined with Observation 7.49 implies $\text{cost}(M, A) \leq 7$. Hence, if $\text{slack}(A) \geq 7$ then $\text{slack}(A) - \text{cost}(M, A) + 1 \geq 1$. More strongly, if $A$ fails the condition of (iv) then $\text{slack}(A) \leq 5$, and if $A_1$ and $A_2$ are nearly overfull subgraphs which fail the condition of (v) then $\text{slack}(A_1) \leq 6$ and $\text{slack}(A_2) \leq 6$. So, to ensure the conditions of (iv) and (v) hold, we can restrict our attention to nearly overfull subgraphs with slack at most 6.

Lemma 7.29 together with our choice of $T$ and $T_e$ implies the following.

**Observation 7.55.** *Suppose $A$ is a nearly overfull subgraph with slack$(A) \leq 20$.*

1. *If $|T - \{x, y\}|$ is even, then cost$(M^1 \cup E^*, A) \leq |(A \oplus T) - \{x, y\}| \leq 2$.*
2. *If $|T - \{x, y\}|$ is odd, then cost$(M^1 \cup E^*, A) \leq |(A \oplus T_e) - \{x, y\}| \leq 2$.*

*Proof.* By Lemma 7.29, each nearly overfull subgraph $A$ with $\text{slack}(A) \leq 20$ satisfies $|A \oplus T| \leq 2$. If $|T - \{x, y\}|$ is even, then since $\text{cost}(M^1 \cup E^*, T) = 0$, Part 1 follows immediately from Observation 7.49. Otherwise, $|T_e - \{x, y\}|$ is even and $\text{cost}(M^1 \cup E^*, T_e) = 0$. If $A$ is a nearly overfull subgraph with $\text{slack}(A) \leq 20$ and $|(T \oplus A) - \{x, y\}| \leq 1$, then since $|(T \oplus T_e) - \{x, y\}| \leq 1$ we have $|(T_e \oplus A) - \{x, y\}| \leq 2$, and so, the result follows from Observation 7.49. So, assume $A$ has $\text{slack}(A) \leq 20$ and $|(A \oplus T) - \{x, y\}| = 2$. By Lemma 7.29, each nearly overfull subgraph with slack at most 20 is contained in the set

$\{(A\cap T),(A\cap T)+u,(A\cap T)+v,(A\cap T)+u+v\}$, where $\{u,v\}=(A\oplus T)-\{x,y\}=(A\oplus T)$. It follows that $T_e$ is one of these subgraphs, and so, we have $|T_e\oplus A|\leq 2$. The result now follows from Observation 7.49. $\square$

Observation 7.55 together with Property (c) implies the following.

**Observation 7.56.** *For each nearly overfull subgraph $A\subseteq H$ with slack$(A)\leq 20$ we have slack$(A)-$ cost$(M^1\cup E^*,A)+1\geq 0$.*

*Proof.* Let $A$ be a nearly overfull subgraph with slack$(A)\leq 20$ and slack$(A)-\text{cost}(M^1\cup E^*,A)+1<0$. If $|T-\{x,y\}|$ is even, then Observation 7.55 implies slack$(A)=0$ and $|(T\oplus A)-\{x,y\}|=2$, a contradiction by (c). If $|T-\{x,y\}|$ is odd, then Observation 7.55 implies slack$(A)=0$ and $|(T_e\oplus A)-\{x,y\}|=2$. By our choice of $T_e$, slack$(T_e)=0$ which contradicts (c). $\square$

**Observation 7.57.** *For no pair of subgraphs $A_1$ and $A_2$ both with slack at most 20 and $|A_1\oplus A_2|=2$ do we have slack$(A_1)-$ cost$(M^1\cup E^*,A_1)+1=0$ and slack$(A_2)-$ cost$(M^1\cup E^*,A_2)+1=0$.*

*Proof.* If $|T-\{x,y\}|$ is even, then let $T'=T$; otherwise, let $T'=T_e$. By Observation 7.55, a nearly overfull subgraph $A$ with slack$(A)\leq 20$ satisfies slack$(A)-\text{cost}(M^1\cup E^*,A)+1=0$ only when 1) $|A-\{x,y\}|$ is even, $|(T'\oplus A)-\{x,y\}|=2$, and slack$(A)=1$, or 2) $|A-\{x,y\}|$ is odd, $|(T'\oplus A)-\{x,y\}|=1$, and slack$(A)=0$. By Lemma 7.29, at most one subgraph satisfies 1).

If $A$ satisfies 1), then by Lemma 7.29, each nearly overfull subgraph with slack at most 20 is contained in the set $\{(A\cap T'),(A\cap T')+u,(A\cap T')+v,(A\cap T')+u+v\}$, where $\{u,v\}=(A\oplus T')-\{x,y\}=(A\oplus T')$. Hence by (c), any two subgraphs $A_1$ and $A_2$ satisfying slack$(A_1)=$ slack$(A_2)=0$ also satisfy $|A_1\oplus A_2|\leq 1$. So, at most one nearly overfull subgraph $B$ satisfies 2), and $|A\oplus B|\leq 1$.

Assume no subgraph satisfies 1) and subgraphs $A_1$ and $A_2$ satisfy 2). If $|A_1 \oplus A_2| = 2$ then by (c), $|(A_1 \oplus A_2) \cap \{x, y\}| = 1$, and so, either $|(T' \oplus A_1) - \{x, y\}| = 0$ or $|(T' \oplus A_2) - \{x, y\}| = 0$. As this contradicts that $A_1$ and $A_2$ satisfy 2), we have $|A_1 \oplus A_2| \leq 1$. $\square$

Since for any nearly overfull subgraph $A$, $\text{cost}(M, A) \leq \text{cost}(M^1 \cup E^*, A)$, it now follows that $M$ satisfies (i), (iv), and (v).

**Case B.** By Lemma 7.53, there is a matching $M'$ of $(H - T) - S$ saturating each vertex of $(\mathcal{X} - T) - S$. By Lemma 7.54, there is a matching $M''$ of $(H - T) - S$ satisfying $\text{cost}(M'', (H - T) - S) \leq 12$. By Lemma 7.48, there is a matching $M^2$ of $(H - T) - S$ saturating each vertex of $(\mathcal{X} - T) - S$ and with $\text{cost}(M^2, (H - T) - S) \leq 12$. We let $M = M^1 \cup E^* \cup M^2$. It is immediate that (i) holds with respect to $M$. We now prove that (iv) and (v) also hold.

Since $T$ is good, each nearly overfull subgraphs is either with 6 of $T$, 6 of $H$ or 8 of $H - T$. For any nearly overfull subgraph $A$ within 6 of $T$, the fact that $\text{cost}(M, T) \leq 1$ combined with Observation 7.49 implies $\text{cost}(M, A) \leq 7$. We have $\text{cost}(M, H - T) \leq 13$, and so, $\text{cost}(M, H) \leq 14$. Hence, for any nearly overfull subgraph $A$ within 6 of $H$ or 8 of $H - T$, Observation 7.49 implies the cost for $A$ is at most 21. Hence, if $\text{slack}(A) \geq 21$ then $\text{slack}(A) - \text{cost}(M, A) + 1 \geq 1$. More strongly, if $A$ fails the condition of (iv) then $\text{slack}(A) \leq 19$, and if $A_1$ and $A_2$ are nearly overfull subgraphs which fail the condition of (v) then $\text{slack}(A_1) \leq 20$ and $\text{slack}(A_2) \leq 20$. So, to check that the conditions of (iv) and (v) hold, we can restrict our attention to nearly overfull subgraphs with slack at most 20. Furthermore, since $T$ is a good subgraph, if there exists a subgraph with slack at most 20, then $\text{slack}(T) \leq 20$ and so by Lemma 7.29 each nearly overfull subgraph with slack at most 20 is within 2 of $T$. Hence, Observations 7.56 and 7.57 imply that (iv) and (v) hold.

It remains only to prove Lemmas 7.50, 7.52, 7.53 and 7.54.

*Proof of Lemma 7.50.* By Lemma 7.45, $A - \{x, y\}$ has a maximum matching $M'$ with $\mathrm{cost}(M', A) \leq 7$. By Lemma 7.42, $H - \{x, y\}$ has a matching saturating each vertex in $\mathcal{X}$. By our choice of $A$, each vertex not in $A$ has at most $\frac{1}{2}D$ neighbours in $A$. Since $|A| > \frac{3}{4}|H|$, this implies each vertex of $H - A$ has degree less than $\frac{1}{2}D + \frac{|H|}{4} < D - 1$, and so we have $\mathcal{X} \subseteq A$. These facts imply that $H - \{x, y\}$ has a matching saturating each vertex of $\mathcal{X}$ using only edges from $\delta(A - \{x, y\}) \cup E(A - \{x, y\})$. Let $M''$ be such a matching with the maximum number of edges. To complete the proof, we will construct a matching $M$ satisfying $\mathcal{X} \subseteq V(M)$ and such that $\mathrm{cost}(M, A) \leq 7$, by choosing from the edges of $M' \cup M''$. The lemma then follows by letting $M^1 = M \cap E(A)$ and $E^* = M \cap \delta(A)$.

Each component of the graph with vertex set $H - \{x, y\}$ and edge set $M' \cup M''$ is either an even cycle contained completely in $A - \{x, y\}$ or a path completely contained in $A - \{x, y\}$ except for possibly the first and last vertex. For each component $C$, we will pick $M_C$ to be either the edges of $M' \cap E(C)$ or $M'' \cap E(C)$ and then set $M = \bigcup_C M_C$.

By the maximality of both $M'$ and $M''$, each component completely contained in $A - \{x, y\}$ which is a path with an odd number of edges is an edge of both $M'$ and $M''$. For each component $C$ which is either an even cycle or a path completely contained in $A - \{x, y\}$, we let $M_C = M'' \cap E(C)$. For each component $C$ which is a path starting in $A - \{x, y\}$ and ending in $H - A - \{x, y\}$ with an even number of edges, we let $M_C = M' \cap E(C)$, and with an odd number of edges we let $M_C = M'' \cap E(C)$. Each component $C$ which is a path starting and ending in $H - A - \{x, y\}$ must have an odd number of edges, and we let $M_C = M' \cap E(C)$.

As $\mathcal{X} \subseteq V(M'') \cap A \subseteq V(M) \cap A$, we have $\mathcal{X} \subseteq V(M)$. Moreover, each vertex of $(A - \{x, y\}) - V(M')$ corresponds uniquely to either a vertex of $(A - \{x, y\}) - V(M)$ or an edge of $\delta(A - \{x, y\}) \cap M$. Hence, $\mathrm{cost}(M, A) = \mathrm{cost}(M', A) \leq 7$. $\square$

*Proof of Lemma 7.52.* We first prove that $A-\{x,y\}$ has a perfect or near perfect matching missing any vertex. We then use this to pick $M^1$ and $E^*$.

Since $A$ is good, Lemma 7.29 implies each subgraph $A$ of $A-\{x,y\}$ with $|A| < |A-\{x,y\}|-2$ satisfies $\sum_{v\in A}(D-d_A(v)) \geq D$, and each vertex in $A-\{x,y\}$ has at least $\frac{1}{2}(D-1)-2 \geq 15$ neighbours in $A-\{x,y\}$. We consider two cases: 1) $|A-\{x,y\}| \leq 2D-28$ and 2) $|A-\{x,y\}| > 2D-28$.

**Case 1.** Since $A$ is nearly overfull, $\sum_{v\in A-\{x,y\}}\left(D-d_{(A-\{x,y\})}(v)\right) \leq 7D$. Hence by Corollary 7.47, $A-\{x,y\}$ has the desired matching.

**Case 2.** Since $D \geq \frac{1}{2}|H|+3$, we have $|A-\{x,y\}| > |H|-22$. As $|H| \geq 320$, we have $|A| > \frac{3}{4}|H|$ and so, $\text{slack}(A) \leq 20$. By assumption, $|U| \leq 2D-\frac{|H|}{2}$ and $|H| > |A-\{x,y\}| > 2D-28$, which yield $|U| < D+14$. It follows that $|\delta(A-\{x,y\})|+\sum_{v\in A-\{x,y\}}(D-d(v)) \leq 3D+20+|U| \leq \frac{9}{2}D$. Corollary 7.47 now implies $A-\{x,y\}$ has the desired matching.

We now pick $M^1$ and $E^*$. If $|A-\{x,y\}|$ is even, then letting $M^1$ be a perfect matching of $A-\{x,y\}$ and return $M^1$ and $E^*=\emptyset$. Since $\text{cost}(M^1\cup E^*)=0$, the lemma follows in this case. If $|A-\{x,y\}|$ is odd, then since $|(A\oplus A_e)-\{x,y\}|=1$ either $\exists v \in (A-A_e)-\{x,y\}$ or $\exists w \in (A_e-A)-\{x,y\}$. In the latter case by Observation 7.43 and our choice of $A_e$, $w$ has a neighbour $u$ in $A-\{x,y\}$; let $M^1$ be a perfect matching of $A-\{x,y,u\}$, we return $M^1$ and $E^*=\{uw\}$. In the former case, let $M^1$ be a perfect matching of $A_e-\{x,y\}=A-\{x,y,v\}$. By Observation 7.43 and our choice of $A_e$, either $v$ has a neighbour $u$ in $(H-A)-\{x,y\}$, or $v \notin \mathcal{X}$. In the former case, we return $M^1$ and $E^*=\{vu\}$, and in the latter, we return $M^1$ and $E^*=\emptyset$. In either case, $\mathcal{X}\cap A \subseteq V(M^1\cup E^*)$, $\text{cost}(M^1\cup E^*, A)=1$ and $\text{cost}(M^1\cup E^*, A_e)=0$, and so, the lemma now follows. □

*Proof of Lemma 7.53.* If $(H-A)-S$ contains no vertices of $\mathcal{X}$, then $M'=\emptyset$ will do. Otherwise, we claim that $G=(H-A)-S$ and $Z=(\mathcal{X}-A)-S$ satisfy the conditions of Lemma 7.41, and that no subset of $Z$ is the vertex set of an odd component. Hence, $(H-A)-S$ has a matching saturating $(\mathcal{X}-A)-S$, as desired.

By the choice of $A$, each vertex of $Z$ has at most $\frac{1}{2}D$ neighbours in $A$, and so at least $\frac{1}{2}(D-2)-|S-A|$ neighbours in $G$. Since $|S|\leq 3$ and $D\geq 163$, $|G|\geq 21$. By Observation 7.36, $|A|\geq \frac{|H|}{2}+1$, and so, $|G|\leq \frac{|H|}{2}-1-|S-A|\leq D-4-|S-A|$. It follows that each vertex of $Z$ has at least $\frac{1}{2}|G|-2$ neighbours in $G$. Each vertex of $Z$ with less than $\frac{1}{2}|G|\leq \frac{1}{2}D-2$ neighbours in $G$ has at least $\frac{1}{2}D-2$ neighbours in $A-S$. Since $A$ is nearly overfull, $|\delta(A)|\leq 5D$ and so $|\delta(A-S)|\leq 8D$. These two facts implies $Z$ contains at most $17\leq \frac{1}{2}|G|-5$ such vertices.

It remains to show that no subset of $Z$ is the vertex set of an odd component. If $O$ is such an odd component, then each edge of $\delta(O)$ has one endpoint in $O$ and the other in $S\cup A$. Hence, $|\delta(S\cup A)|=|\delta(O)|\geq |O|(D-1-(|O|-1))$. Since $A$ is nearly overfull, $|\delta(A\cup S)|\leq 8D$, and so, $|O|(D-|O|)\leq 8D$. It follows that either $|O|\leq 8$ or $|O|\geq D-8$. The former is impossible, since each vertex of $Z$ has at least $\frac{1}{2}D>8$ neighbours in $G$. The latter implies $|A|\leq |G|-|O|\leq D+2$, and so the definition of nearly overfull implies, $|\delta(A)|\leq |\delta(A)|+\sum_{v\in A}(D-d(v))\leq D-1+2|A|<3D+4$. Hence, $|\delta(A\cup S)|<3D+4+|S-A|D$ and so $|O|(D-|O|)<3D+4+|S-A|D$. It follows that either $|O|\leq 6$ or $|O|\geq D-3-|S-A|$. The former case is impossible. In the latter case, by Observation 7.36 and since $|A|\geq D-2$, it follows that $|H|\geq |A|+|O|+|S-A|\geq 2D-5$, a contradiction. □

*Proof of Lemma 7.54.* Let $B$ be a nearly overfull subgraph far from $A$. Since $A$ is good, $|B-A|\geq D-6\geq \frac{1}{2}|H|-3$. Since $|A|\geq \frac{1}{2}|H|+1$, we have $|H-(B\cup A)|\leq 2$. Hence, if we show $(B-A)-S$ has a matching missing at most 10 vertices, then the desired matching clearly exists. Since $|A|\geq D-2$ and $|G|\leq 2D-6$, it follows that $|B-A|\leq |G-A|\leq D-4$. So by Observation 7.34, $|\delta(B-A)|+\sum_{v\in B-A}(D-d(v))\leq 6D-14+|(B-A)\cap U|\leq 7D$.

Hence, $|\delta((B-A)-S)| + \sum_{v\in(B-A)-S}(D - d(v)) \leq 10D$, and so, Lemma 7.44 implies $(B-A)-S$ has a matching missing at most 10 vertices. $\square$

## 7.5 Algorithmic Considerations

To finish this chapter, we briefly discuss how to turn the proof of Lemma 7.2 into a polynomial time algorithm to find an optimal fractional total colouring of a graph $G$ with maximum degree $\Delta \geq \frac{3}{4}|G|$ and containing no overfull subgraph. The core of this algorithm is an extension of the linear time algorithm given in Chapter 4 to find all odd overfull subgraphs. We first describe this extension before sketching the details of the algorithm.

### 7.5.1 Finding All Nearly Overfull Subgraphs

Given a graph $F$ with maximum degree $\Delta \geq \frac{n}{2}$, we sketch a polynomial time algorithm to find all nearly overfull subgraphs. We mimic the algorithm given in Section 4.3.1, using the same notation.

Since a nearly overfull subgraph $A$ satisfies $\mathrm{def}(A) + |\delta(A)| \leq 5\Delta$, for any $\varepsilon > 0$, there can be at most $5\varepsilon^{-1}$ $\varepsilon$-special vertices for $A$. Recall that $\forall\varepsilon > 0$, we defined $L_\varepsilon$ to be $\{w \in G : d(w) \geq (1-\varepsilon)\Delta\}$. We describe a polynomial time subroutine which, for an appropriately chosen $\varepsilon > 0$, and given a vertex $v$ of $L_\varepsilon$, determines each nearly overfull subgraph $A$ for which $v$ is not an $\varepsilon$-special vertex. Applying this to each of $5\varepsilon^{-1}+1$ different vertices in turn we find all nearly overfull subgraphs.

Given a vertex $v$, our first step is to compute a set $\mathcal{C}$ of three candidate subgraphs. We ensure that if $A$ is nearly overfull and $v$ is not $\varepsilon$-special for $A$, then there exists a $C \in \mathcal{C}$ such that $A - S_A = C - S_A$, where $S_A$ is the set of $\varepsilon$-special vertices for $A$. It is straightforward to verify that as in Observation 4.38 we have

**Observation 7.58.** *If $v$ is not $\varepsilon$-special for nearly overfull $A$, then $A - S_A$ is one of $S_\varepsilon(v) - S_A$, $T_\varepsilon(v) - S_A$, or $L_\varepsilon - S_A$.*

So, we need only find for each $B \in \{L_\varepsilon, S_\varepsilon(v), T_\varepsilon(v)\}$ all the nearly overfull subgraphs $A$ satisfying $B - S_A = A - S_A$. The first step is to determine $S_\varepsilon(v)$, $T_\varepsilon(v)$, and $L_\varepsilon$ which can be done by scanning through the edge set once. Since, for any nearly overfull subgraph $A$, $|S_A| \le 5\varepsilon^{-1}$, to find all nearly overfull subgraphs close to $A$, we simply test each subgraph $A'$ for which $|A \oplus A'| \le 5\varepsilon^{-1}$. Since testing if a subgraph is nearly overfull can be done in polynomial time, it follows that this takes polynomial time.

### 7.5.2 The Algorithm

If $|G| < 320$ then we can find an optimal fractional total colouring of a graph $G$ in constant time by applying the simplex algorithm. Otherwise, to show the algorithm runs in polynomial time it is enough to show in polynomial time we can i) find a vertex colouring as in Lemma 7.12 , ii) find a matching as in Lemma 7.22, and iii) finish the fractional total colouring using the edge colouring of the auxiliary graph $G^*$.

**i.** We find a vertex $(\Delta + 1)$-colouring $S_1, ..., S_{\Delta+1}$ as in Lemma 7.12 in polynomial time. By Edmonds [28], we can find a maximum matching in $\overline{G}$ in polynomial time. Since each $\Delta$-full subgraph is nearly-overfull, we can determine each $\Delta$-full subgraph by testing each nearly overfull subgraph. Hence, the proof of Lemma 7.12 together with the algorithm given in Section 7.5.1 yields a polynomial time algorithm to find $S_1, ..., S_{\Delta+1}$.

**ii.** For $i = 1, ..., k$, we use Lemma 7.22 to find the matching $M_i$ of $G_i$. The first step is to find the set of all nearly overfull subgraphs, from which we can easily find our two-part partition. For each part, we need only find the matchings guaranteed by our matching tool kit of Section 7.3. That we can do so, follows from Edmonds' maximum matching algorithm, and that given that there exists a matching saturating a set $Z \subseteq V(H)$, there exists a polynomial time algorithm to find such a matching. The latter algorithm is as follows. We find a maximum weight matching $M$ in the auxiliary graph $H'$ built by taking a copy of $H$ where each edge $e$ has weight $w_e \in \{0, 1, 2\}$ equal to the number of endpoints of $e$ which are in $Z$. A maximum weight matching in $H'$ saturates the maximum number of vertices in $Z$, and so, $M$ is the desired matching. Hence, by combining these matchings as in Section 7.3.3, it follows that we can find the matching $M$ of Lemma 7.22 in polynomial time.

**iii.** Given the matchings $M_1, ..., M_k$, we construct the auxiliary graph $G^*$, and find a fractional edge $\Delta(G^*)$-colouring of $G^*$ in polynomial time. As described in Section 7.1, it is then straightforward to find the desired fractional total $(\Delta + 1)$-total colouring of $G$.

# CHAPTER 8
# Concluding Remarks

We conclude with several directions for future research. A fundamental question left unanswered is the complexity of fractional total colouring. Since resolving this currently seems out of reach, we present several promising and hopefully easier directions. We also discuss the connections between techniques presented in this thesis and Hilton's Overfull Conjecture. We finish by discussing a combinatorial algorithm for fractional edge colouring.

## 8.1 The Fractional Conformability Conjecture

We believe an important step to resolving the complexity of fractional total colouring is the Fractional Conformability Conjecture (Conjecture 5.21). An affirmative answer would reduce determining $\chi''_f(G)$ for graphs with maximum degree $\Delta(G) > \frac{1}{2}|G|$, to determining $\beta(G)$. Currently, the difficultly of determining $\beta(G)$ is also unresolved. In determining $\beta(G)$, it may help to use the fact that if a graph $G$ satisfies $\Delta > \frac{1}{2}|G|$, then we need only focus on graphs with overfull subgraphs. Indeed, if $G$ contains no overfull subgraphs, then $\beta(G) = \Delta + 1$ and we can find a $\beta(G)$ colouring in polynomial time by finding a maximum matching in $\overline{G}$.

We believe we can strengthen Lemma 7.2, by dropping the condition that $G$ must contain no overfull subgraphs. The intersection patterns of overfull subgraphs are simple when $G$ has maximum degree $\Delta \geq \frac{3}{4}|G|$, indeed, $G$ can contain at most two and any two overfull subgraphs $A$ and $B$ satisfy $|A \oplus B| \leq 1$. Furthermore, the cut-condition easily implies that if there are two overfull subgraphs, then for one $H$, $\sum_{v \in G}(\Delta - d_H(v)) \geq \frac{\Delta}{2}$.

This allows us to essentially focus on one overfull subgraph. We highlight two difficulties in extending our iterative approach to prove this strengthening.

In proving Lemma 7.2, we ensured that $G^*$ contains no overfull subgraph. This can be thought of as ensuring that $G^*$ contains no 'clique-like' subgraph. Theorem 6.1 implies that we need also be wary of 'complete bipartite-like' subgraphs. Indeed, in proving this strengthening we need to also ensure that no subgraph of $G^*$ is a nearly complete bipartite subgraph. We remark that this did not become an issue in the proof of Lemma 7.2 since the graphs we considered had large enough maximum degree so that no nearly complete bipartite subgraph could exist.

The second main issue results from the fact that we need to consider graphs $G$ whose maximum degree is as low as $\frac{|G|}{4}$. For such graphs, the intersection patterns of nearly overfull subgraphs are more complicated. Indeed, $G$ can now contain four vertex disjoint nearly overfull subgraphs. We need to consider a more complicated decomposition of our graph. Similar to Observation 7.35, we have that for any two nearly overfull subgraphs $A$ and $B$ of $H$, each of $A \cap B$, $A - B$, $B - A$, and $A \oplus B$ contains either at most 10 or at least $D - 10$ vertices. From this it follows that we can partition the vertex set into $t \leq 4$ parts $P_1, ..., P_t$ such that for each nearly overfull subgraph $A$ and $i = 1, .., t$ either $|A \oplus P_i| \leq 10$ or $|A \cap P_i| \leq 10$. This allows us to focus on finding the desired matchings in the subgraphs induced on each of the parts separately. The difficultly is in combining these matchings.

### 8.2 Total Colouring Graphs with Bounded Stability Number

We remind the reader that $\alpha(G)$ is the size of a largest stable set in a graph $G$. As we now show, an approach along the lines of the proof of Theorem 3.22 allows us to determine the fractional total colouring number of a graph $G$ with $\alpha(G)$ bounded by a constant in polynomial time.

**Theorem 8.1.** *For each $t \geq 0$, there exists an algorithm such that given any graph $G$ satisfying $\alpha(G) \leq t$, determines the fractional total colouring number and an optimal fractional total colouring in polynomial time.*

*Proof.* By Theorem 3.23, it is enough to show we can separate over a polytope of the form (5.2) in polynomial time. Let $z$ be a vector in $\mathbb{R}^{V(G)\cup E(G)}$. We can clearly check if $z \geq 0$ in polynomial time. For each stable set $S \in \mathcal{S}(G)$, it is enough to check if for each matching $M \in \mathcal{M}(G-S)$, we have $\sum_{e\in M} z_e \leq 1 - \sum_{v\in S} z_v$. Indeed, if each matching satisfies this inequality, then for each total stable set $T$ such that $T \cap V(G) = S$, we have $\sum_{x\in T} z_x \leq 1$. Hence, if this holds for each $S \in \mathcal{S}(G)$, then for each total stable set $T$, we have $\sum_{x\in T} z_x \leq 1$. Otherwise, there exists $S \in \mathcal{S}(G)$ and $M \in \mathcal{M}(G-S)$ such that $\sum_{x\in S\cup M} z_x > 1$ and so $\sum_{x\in S\cup M} z_x \leq 1$ is our violated constraint.

By Edmonds' blossom algorithm [28], for each $S \in \mathcal{S}(G)$, we can find a maximum weight matching $M \in \mathcal{M}(G-S)$ with edge weights $z$ in polynomial time. Since $\alpha(G) \leq t$, we can enumerate all such stable sets by checking each of the at most $|G|^t$ sets of at most $t$ vertices. These facts imply the desired polynomial time separation oracle for a polytope of the form (5.2) exists. $\square$

It would be interesting to develop a combinatorial algorithm which finds an optimal fractional total colouring of graphs with bounded stability number in polynomial time.

## 8.3 Hilton's Overfull Conjecture and Nearly Overfull Subgraphs

Nearly overfull subgraphs turn up very naturally in our iterative approach to determine the fractional total colouring number. We propose that subgraphs which are close to failing the cut condition can also be used to understand the chromatic index. Call a subgraph $H$ of $G$ *nearly overfull* with respect to edge colouring if $|H| > 4$ and $|\delta(H)| + \sum_{v\in V(H)}(\Delta(G) - |\delta(v)|) < \Delta + |H|$. We propose that if the intersection properties of nearly overfull subgraphs are simple then determining the chromatic index should be easier.

As an example, consider Hilton's Overfull Conjecture [18, 20]. We remind the reader that this states that if $\Delta > \frac{1}{3}|G|$ then $\chi'(G) = \Delta + 1$ precisely when $G$ contains an odd overfull subgraph. When $\Delta > \frac{1}{3}|G|$, $G$ can contain at most 3 overfull subgraphs. In fact, if $|G|$ is large enough, then $G$ can contain at most 3 nearly overfull subgraphs. The techniques introduced in Chapter 7 allow us to exploit this fact. Indeed, we can show that if $G$ has maximum degree $\Delta \geq \frac{1}{2}|G| + 3$ and contains no overfull subgraph, then there exists a matching $M$ saturating each vertex of maximum degree and such that $G - M$ contains no overfull subgraph (see Lemma 7.22). Hence, $\chi'_f(G-M) = \chi'_f(G)-1$. As discuss in Chapter 4, the difficultly with any iterative approach is that eventually the maximum degree will drop below $\frac{1}{3}|G|$ and the conjecture no longer holds. So, if we are to apply an iterative approach to prove Hilton's conjecture, then it will need to be modified in order to avoid this problematic situation. For example, the approaches of Perkovic and Reed [89] and Frieze, Jackson, McDiarmid, and Reed [33] iteratively reduced the input graph to a base case which is easily handled. It would be interesting to know if one can use the structure of nearly overfull subgraphs to iteratively find disjoint matchings $M_1, ..., M_k$ of a graph $G$ with sufficiently large maximum degree such that $G' = G - \bigcup_{i=1}^{k} M_i$ belongs to a class of graphs for which the chromatic index is $\Delta(G) - k$.

### 8.4 A Combinatorial Algorithm for Fractional Edge Colouring

We remark that one can use an iterative approach to construct an optimal fractional edge colouring of any multigraph in polynomial time. (To the author's knowledge, this algorithm has not previously appeared in print, though Meagher [80] notes in his M.Sc. thesis that such an algorithm exists.) As in Kahn's proof that the Goldberg-Seymour Conjecture is asymptotically true (see Theorem 3.2), this is more complicated than edge colouring bipartite graphs since we need to worry about reducing both the maximum degree and the edge-density of any odd overfull subgraph. We modify the simple iterative approach and apply the following two reductions. We show that if there exists a subgraph $H$ satisfying $1 < |H| < |G|$ and $\frac{2|E(H)|}{|H|-1} = \chi'_f(G)$, then we can reduce our problem to edge colouring $H$ and $G/H$, the graph obtained by contracting the vertices of $H$ into a single vertex. In doing so, we exploit the fact that $|\delta(H)| \leq \Delta(G)$, from which it follows that $\chi'_f(G) = \max\{\chi'_f(H), \chi'_f(G/H)\}$. If no such subgraph exists, then we can find a matching $M$ and scalar $\varepsilon > 0$ such that when we remove weight $\varepsilon$ of $M$ the fractional chromatic index drops by $\varepsilon$. We remove weight $\varepsilon$ of $M$ and repeat the procedure on the reduced graph.

We stress that this approach does not work for finding integral edge colourings because it may not be possible to set $\varepsilon = 1$, i.e. remove a whole matching. Indeed, there may exist some odd subgraph $H$ for which the value $\frac{2|E(H)|}{|H|-1}$ is not reduced enough. Though, it is possible to use this approach to bound the chromatic index. For example, Plantholt [92] uses a complicated variant of this approach in a preprint claiming $\chi'(G) \leq \left\lceil \chi'_f(G) \right\rceil + \log_{3/2}(\min\{(|G|+1)/3, \left\lceil \chi'_f(G) \right\rceil\})$. It would be interesting to know how far one can push such an approach.

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## Notation

A *graph* $G = (V, E)$ is a set $V$ of vertices and a set $E$ of unordered pairs of $V$. A *multigraph* $G = (V, E)$ a set $V$ of vertices and a multiset $E$ of unordered pairs of $V$. Letting $G$ be a graph or multigraph and $S$ and $T$ be a subsets of $V$, we define the following.

**Basic graph terminology**

| Term | Symbol | Description |
|---|---|---|
| vertex set | $V(G)$ | the set of vertices of $G$ |
| edge set | $E(G)$ | the set of edges of $G$ |
| size of $G$ | $\|G\| = \|V(G)\|$ | number of vertices of $G$ |
| clique | | a set of pairwise adjacent vertices |
| stable set | | a set of pairwise nonadjacent vertices |
| matching | | a set of edges no two of which share an endpoint |
| total stable set | | a stable set $S$ and matching $M$ such that no |
| | | vertex $v$ in $S$ is an endpoint of edge in $M$ |
| neighbour of $v$ | | a vertex $w$ adjacent to $v$ |
| neighbourhood of $v$ | $N(v)$ | set of neighbours of $v$ |
| edges incident to $v$ | $\delta(v)$ | set of edges which contain $v$ as endpoint |
| cut | $C = (S, T)$ | a two part partition of $V(G)$ |
| cut-set | $\delta(S)$ | set of edges $\{uv : u \in S, v \notin S\}$ |
| subgraph of $G$ induced on $S$ | $G[S]$ | $V(G[S]) = S$, |
| | | $E(G[S]) = \{uv \in E(G) : u, v \in S\}$ |
| complement of $G$ | $\overline{G}$ | $V(\overline{G}) = V(G)$, |
| | | $E(\overline{G}) = \{uv \in E(\overline{G}) : uv \notin E(G)\}$ |

*N.B. In general, we do not distinguish between a graph and its vertex set.*

**Graph invariants**

| Term | Symbol | Description |
|---|---|---|
| chromatic number of $G$ | $\chi(G)$ | See Chapter 2 |
| fractional chromatic number of $G$ | $\chi_f(G)$ | See Chapter 2 |
| chromatic index of $G$ | $\chi'(G)$ | See Chapter 3 |
| fractional chromatic index of $G$ | $\chi'_f(G)$ | See Chapter 3 |
| total colouring number of $G$ | $\chi''(G)$ | See Chapter 5 |
| fractional total colouring number of $G$ | $\chi''_f(G)$ | See Chapter 5 |
| clique number of $G$ | $\omega(G)$ | size of largest clique in $G$ |
| stability number of $G$ | $\alpha(G)$ | size of largest stable set in $G$ |
| total stability number of $G$ | $\alpha_t(G)$ | size of largest total stable set in $G$ |
| degree of $v \in G$ | $d(v)$ | size of $N(v)$ |
| maximum degree of $G$ | $\Delta(G)$ | $\max_{v \in G} d(v)$ |