AN INEQUALITY IN THE GEOMETRY OF NUMBERS

by

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... ויקח מאבני המקום וישם מראשתיו...

וישכם יעקב בבקר ויקח את האבן

אשר שם מראשתיו...

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CONTENTS

INTRODUCTION

There appeared in 1883 an article by Barlow [1] in Nature in which the problem of the densest packing of spheres is discussed from the point of view of the physical chemist. While Thue [2,3] may be credited with the earliest mathematical formulation of a packing problem, namely that of the densest packing of equal circles in the plane it was not until H. Minkowski [4] had laid a firm foundation to the Geometry of Numbers that wider interest was stirred in problems of this type. The results of subsequent investigations up to 1953 have been assembled by L. Fejes Toth [5]. The particular question with which we deal in this thesis has already been investigated by L. Fejes Toth [6] and C. A. Rogers [7]. However our approach to the problem and its formulation, the methods we develop and the precise results we obtain are distinct from those of the latter.

The line of research which is continued here was initiated by H. Zassenhaus [8] and has been successfully applied by N. Smith [9] to the packing of the star-shaped domain $| xy | \leq 1$.

Motivated by the necessity of a compactness property we define the concept of a quasi-Jordan polygon in terms of which we introduce a more general definition of a packing, more general in that the usual definition is comprehended as a special case. Our main result, stated in Theorem 2, consists of an inequality involving the number of points of the packing and the 'area' and 'perimeter' of the quasi-Jordan polygon relating to it. The proof of the inequality proceeds by induction. It depends primarily on the possibility of decomposition of the packing (Theorems 3, 4) which in turn depends on the decomposability of a quasi-Jordan polygon as discussed in Theorem 1.

CHAPTER I

A convex Jordan curve, Γ, with a centre of symmetry, O, defines a function μ: $R^2 \times R^2 \to R$, $\mu(P_1,P_2) = \frac{|\overrightarrow{P_1P_2}|}{|\overrightarrow{OP}|}$ where $P \in \Gamma$ and $\overrightarrow{OP} = \lambda \overrightarrow{P_1P_2}$, $\lambda \in R$. μ has the properties of a distance function:

(a) $\mu(P_1,P_1) = 0$ since $|\overrightarrow{P_1P_1}| = 0$, $|\overrightarrow{OP}| \geqslant \varepsilon > 0$;

(b) $\mu(P_1,P_2) > 0$ for $P_1 \neq P_2$ since $|\overrightarrow{P_1P_2}| > 0$, $|\overrightarrow{OP}| \geqslant \varepsilon > 0$;

(c) $\mu(P_1,P_2) = \mu(P_2,P_1)$ by the central symmetry of Γ;

(d) $\mu(P_1,P_2) + \mu(P_2,P_3) \geqslant \mu(P_1,P_3)$;

(e) $\mu(P,Q) = \mu(R,S)$ if $\overrightarrow{PQ} = \overrightarrow{RS}$;

(f) $\mu(P,R) = \lambda\mu(P,Q)$ if $\overrightarrow{PR} = \lambda\overrightarrow{PQ}$ $(\lambda > 0)$.

A proof of (d) may be found in Bonnesen-Fenchel [10]. The function, μ, is generally termed the "Minkowski distance" or "radial distance" defined by Γ. A point set, E, in R is said to be admissible with respect to Γ if the μ-distance between any two points of E is ≥ 1.

<u>Definition 1:</u> A <u>quasi-Jordan polygon</u>, Π, is defined as the image of a Jordan polygon, K, with vertices $P_1, \dots, P_n$ [1)] under a unique mapping, θ, into the plane, of the domain K* bounded by K which carries the triangles, $T_1, \dots, T_{n-2}$, of a vertex triangulation of K* barycentrically into triangles, $\overline{T}_1, \dots, \overline{T}_{n-2}$, not necessarily proper, subject to the conditions:

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1) Hereafter subscripts shall denote least positive residues modulo n or modulo such other integer as will be clear from the text.

a) Θ preserves the orientation of the triangles $T_1, \cdots, T_{n-2}$;

b) The sum of the angles at ΘP_i in the triangles $\overline{T}_j$ for which T_j has P_i as a vertex is less than or equal to $360°$

It follows from the uniqueness of Θ and conditions a) and b) that if T_i and T_j have a common vertex $\overline{T}_i$ and $\overline{T}_j$ have no interior points in common.

In addition we define:

(i) $\overline{P}_i = \Theta P_i$ as the vertices of Π;

(ii) as the angle, $\angle\overline{P}_i$, at $\overline{P}_i$, the sum of the angles at $\overline{P}_i$ in the triangles $\overline{T}_j$ for which T_j has P_i as a vertex so that $0° \leq \angle\overline{P}_i \leq 360°$;

(iii) $\sum_{i=1}^{n-1} \mu(\overline{P}_i, \overline{P}_{i+1}) + \mu(\overline{P}_n, \overline{P}_1)$ as the μ-length of Π;

(iv) $\Pi^* = \bigcup_{i=1}^{n-2} \overline{T}_i$ as the quasi-Jordan domain bounded by Π;

(v) the sum of the areas of $\overline{T}_1, \ldots, \overline{T}_{n-2}$ as the area of Π^*;

(vi) as a (simple) path in Π^*, the image under Θ of a (simple) path in K^*.

From the definition of Θ it follows in accordance with (vi) that a polygonal path in Π^* is the image of a polygonal path in K^*. We shall, when referring to a qualified subset of Π^*, mean the image under Θ of a subset, so qualified, of K^* as for example in (i) and (vi) above.

<u>Theorem 1:</u> A simple path $\overline{\lambda} = \overline{P}_1\overline{Q}_1\ldots\overline{Q}_r\overline{P}_j$ in the quasi-Jordan domain bounded by $\overline{P}_1\ldots\overline{P}_n$ determines a pair of quasi-Jordan polygons, $\overline{P}_1\ldots\overline{P}_j\overline{Q}_r\overline{Q}_{r-1}\ldots\overline{Q}_1$ and $\overline{P}_1\overline{Q}_1\ldots\overline{Q}_r\overline{P}_j\ldots\overline{P}_n$.

Proof: The pre-image of $\bar{\lambda}$ in K*, a Jordan antecedent of Π under a mapping θ is a simple polygonal path with end points P_1 and P_j and vertices, $Q_1, \dots, Q_s$. Amongst the latter, $Q_{i_1}, \dots, Q_{i_r}$ satisfy $\theta Q_{i_k} = \bar{Q}_k$ $(k = 1, \dots, r)$ while the remainder, $Q_{i_{r+1}}, \dots, Q_{i_s}$ are contained in the sides of the triangles in the vertex triangulation of K*. The points $Q_1, \dots, Q_s$ determine a refinement of the triangulation of K* which is furthermore a vertex triangulation of K^*_1 and K^*_2, the domains into which λ divides K*. θK_1 and θK_2 are quasi-Jordan polygons with vertices $\bar{P}_1, \dots, \bar{P}_j, \theta Q_s, \theta Q_{s-1}, \dots, \theta Q_1$ and $\bar{P}_1, \theta Q_1, \dots, \theta Q_s, \bar{P}_j, \dots, \bar{P}_n$ respectively for the conditions of Definition 1 are satisfied. It remains to show that:

Lemma 1: If $\bar{P}_1 \dots \bar{P}_n$ is a quasi-Jordan polygon and $\bar{P}_{i-1}, \bar{P}_i, \bar{P}_{i+1}$ are collinear, $\bar{P}_1 \dots \hat{\bar{P}}_i \dots \bar{P}_n$ is a quasi-Jordan polygon.

Proof: Clearly we may assume that the Jordan antecedent K* of $P_1 \dots P_n$ is convex. In the triangulation of K* let $T_1', \dots, T_r'$ be all of those triangles which contain P_i. Let $P_{i-1} P_i P_{i+1} P_{j_1} \dots P_{j_{r-1}}$ be the boundary of $\cup T_i'$. We introduce a vertex triangulation of the domain K^*_i bounded by the convex polygon $P_{i-1} P_{i+1} P_{j_1} \dots P_{j_{r-1}}$ in the following manner. Let $\bar{P}_{j_k}$ be a vertex amongst $\bar{P}_{j_1}, \dots, \bar{P}_{j_{r-1}}$ which is nearest to $\bar{P}_{i-1} \bar{P}_{i+1}$ then $P_{i-1} P_{i+1} P_{j_k}$ shall be a triangle of the triangulation. In a similar way, if $k \neq r-1$, a triangle may be determined in the domain bounded by $P_{i-1} P_{j_k} \dots P_{j_{r-1}}$ having as one of its sides $P_{i-1} P_{j_k}$

and, if $k \neq 1$ in the domain bounded by $P_{i+1} P_{j_k} P_{j_{k-1}} \cdots P_{j_1}$, a triangle one of whose sides is $P_{i+1} P_{j_k}$. Continuing in this way there results a triangulation of K^*_i with the following property. There exists a mapping, θ', of K^*_i which coincides with θ on $P_{i-1}, P_{i+1}, P_{j_1}, \ldots, P_{j_{r-1}}$; which maps K^*_i onto $\cup \overline{T}_i'$ and which maps the triangles of the new triangulation of K^*_i onto triangles which satisfy the conditions of Definition 1. Let θ'' be a mapping which coincides with θ on $K^* - K^*_i$ and with θ' on K^*_i. The domain bounded by $P_1 \cdots \hat{P}_i \cdots P_n$ is a Jordan antecedent under the mapping θ'' defining the quasi-Jordan polygon $\bar{P}_1 \ldots \bar{P}_i \ldots \bar{P}_n$.

<u>Definition 2:</u> We define $\mathbb{P}$, a <u>packing with respect to Γ</u>, as a pair (E,Π) where E is a finite point set and Π is a quasi-Jordan polygon for which Π* contains E and whose vertices are contained in E subject to: If $P, Q \in E$ and the straight segment, PQ, is a path in Π* then $\mu(P,Q) \geq 1$ where μ is the Minkowski distance defined by Γ.

CHAPTER II

Our principal result is the following:

Theorem 2: Let Γ be a convex, centrally symmetric Jordan curve, μ the Minkowski distance defined thereby and $\mathbb{P} = (E, \Pi)$ a packing with respect to Γ. Then there holds the inequality:

$$(I_n) \qquad \mathcal{J}(\Pi) = \frac{A(\Pi^*)}{\Delta} + \frac{M(\Pi)}{2} + 1 \geqslant n$$

where $A(\Pi^*)$ is the area of Π^*, $M(\Pi)$ the μ-length of Π, Δ the determinant of the critical lattice with respect to Γ and n the number of points in E.

We first prove (I_3). To do so we shall require

Lemma 2: If T, a triangle with admissible vertices, P, Q, R, has a pair of sides of μ-length greater than 1 there exists a triangle T' with admissible vertices for which

$$\mathcal{J}(T') < \mathcal{J}(T)$$

Proof: Let $\mu(P,Q) > 1$, $\mu(P,R) > 1$. Since P lies outside $\Gamma(Q)$ and $\Gamma(R)$ there exists a neighbourhood of P with the same property and in particular a point P' for which $\overrightarrow{QP'} = \lambda \overrightarrow{QP}$ $(0<\lambda<1)$.

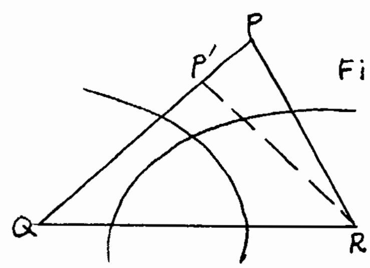

Fig. I.

Let T' be the triangle with vertices P', Q, R. We have $A(T'^*) < A(T^*)$ and since

$$\mu(P', R) \leq \mu(P', P) + \mu(P, R),$$

$$\mu(P', R) + \mu(P', Q) \leq \mu(P', P) + \mu(P', Q) + \mu(P, R)$$

and $\qquad M(T') \leq M(T).$

Hence $\qquad \mathcal{J}(T') < \mathcal{J}(T).$

At a later stage we shall require also the following lemma the proof of which is similar to the above.

<u>Lemma 3:</u> If K, a convex quadrilateral with vertices P, Q, R, S admissible, has a pair of opposite sides and both diagonals of μ-length greater than 1 there exists a quadrilateral K' with admissible vertices such that $\mathcal{J}(K') < \mathcal{J}(K)$.

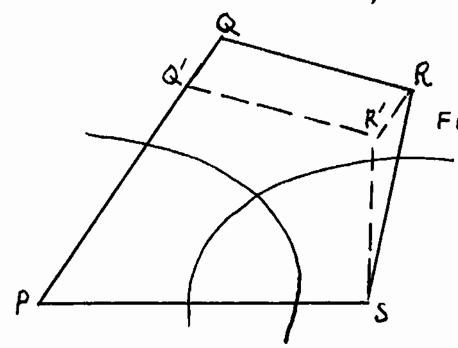

Fig. 2.

<u>Proof:</u> Let $\mu(P,Q) > 1$, $\mu(S,R) > 1$ and of $\angle P + \angle S$ and $\angle Q + \angle R$ let $\angle Q + \angle R \geqslant \pi$. Since Q and R each lie outside both Γ(P) and Γ(S) there exist neighbourhoods of Q and R with the same property and, in particular, points Q' and R' for which $\overrightarrow{PQ'} = \lambda \overrightarrow{PQ}$ $(0 < \lambda < 1)$ and $\overrightarrow{Q'R'} = \overrightarrow{QR}$. The quadrilateral, K', with vertices P, Q', R', S is again admissible while A(K'*) < A(K*). Moreover, since

$$\mu(R',S) \leqslant \mu(R,S) + \mu(R,R'),$$
$$\mu(R',S) + \mu(Q',P) \leq \mu(R,S) + \mu(Q,Q') + \mu(Q',P);$$

so that $\mu(Q',P) + \mu(R',S) + \mu(Q',R') + \mu(P,S) \leq \mu(Q,P) + \mu(R,S) + \mu(Q,R) + \mu(P,S)$

and $\qquad M(K') \leqslant M(K).$

Hence $\qquad \mathcal{J}(K') < \mathcal{J}(K).$

We shall speak of K under these circumstances as being reducible. More generally, if $\mathbb{P} = (E, \pi)$ is a packing in which E contains n points and there exists a packing $\mathbb{P}' = (E', \pi')$

in which E' contains n points and $\mathcal{J}(\Pi') < \mathcal{J}(\Pi)$ we shall say $\mathbb{P}$ is reducible. Two packings (E,Π) and (E',Π') will be called equivalent if $\mathcal{J}(\Pi) = \mathcal{J}(\Pi')$, E and E' containing the same number of points.

Returning to the proof of (I_3) let T be a triangle with admissible vertices, P, Q, R. By Lemma 2 a necessary condition that T be irreducible is that at most one side of T is of μ-length greater than 1. Assume then that $\mu(P,Q) = \mu(P,R) = 1$. Referring to Figure 3,

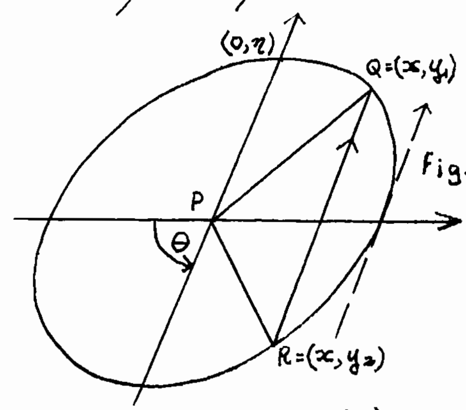

Fig.3.

$$\mathcal{J}(T(x)) = \frac{\sin\theta}{2\Delta}x(y_1-y_2) + \frac{1}{2\eta}(y_1-y_2)+2.$$

The second derivative with respect to x[2],

$$\mathcal{J}''(x) = \frac{\sin\theta}{2\Delta}[x(y_1''-y_2'')+2(y_1'-y_2')] + \frac{1}{2\eta}(y_1'-y_2').$$

Since $y_1', y_1'' < 0$ and $y_2', y_2'' > 0$ it follows that $\mathcal{J}''(x) < 0$. Under the circumstances if $T(x)$ is irreducible either x = 0 or $\mu(Q,R) = 1$. If x = 0 then $\mu(P,Q) = 2$ and (I_3) is satisfied with equality. If $\mu(Q,R) = 1$ then (I_3) will be established with the proof of

<u>Lemma 4:</u> The lattice, Λ, generated by $\vec{OP}$, $\vec{OQ}$ for which $\mu(O,P) = \mu(O,Q) = \mu(P,Q) = 1$ is admissible.

<u>Proof:</u> Let P_1 and P_2 be distinct points in Λ. We

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2) We are assuming throughout that Γ is twice differentiable. The possibility of removing this restriction may be made to depend upon that of smoothing the vertices of a convex polygon in a suitable manner and the application of the "Auswahlsatz" of Blaschke [11].

must show that $\mu(P_1, P_2) \geq 1$. There exists $P_3 \in \Lambda$ such that $\overrightarrow{P_1P_3} = \lambda_1\overrightarrow{OP}$, $\overrightarrow{P_2P_3} = \lambda_2\overrightarrow{OQ}$, λ_1 and λ_2 rational integers not both zero. If $\lambda_1 = 0$, $\overrightarrow{P_2P_1} = \lambda_2\overrightarrow{OQ}$, $\mu(P_1, P_2) = |\lambda_2|\mu(O,Q)$ so that $\mu(P_1, P_2) \geq 1$ since $|\lambda_2| \geq 1$. Similarly, if $\lambda_2 = 0$, $\mu(P_1, P_2) = |\lambda_1| \geq 1$.

Let λ_1 and λ_2 both be different from zero. If $|\lambda_1| \neq |\lambda_2|$ let $|\lambda_1| < |\lambda_2|$. Since

$$\mu(P_1, P_2) + \mu(P_1, P_3) \geqslant \mu(P_2, P_3),$$
$$\mu(P_1, P_2) + |\lambda_1| \geqslant |\lambda_2|,$$
$$\mu(P_1, P_2) \geqslant |\lambda_2| - |\lambda_1| \geqslant 1.$$

It remains to consider the case in which $|\lambda_1| = |\lambda_2|$. In this case $\overrightarrow{P_1P_2} = \lambda_1(\overrightarrow{OP} \pm \overrightarrow{OQ})$.

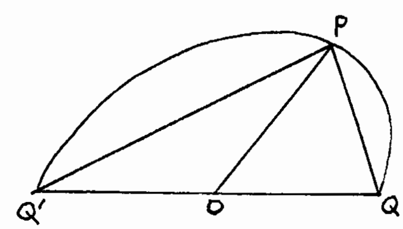

Fig. 4

Referring to Figure 4, $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$ while $\overrightarrow{Q'P} = \overrightarrow{Q'O} + \overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{OP}$. But $\mu(P,Q) = 1$ and

$\mu(P,Q') + \mu(P,Q) \geq \mu(Q',Q) = 2$.

Thus $\mu(P,Q') \geq 1$ and $\mu(P_1, P_2) \geq |\lambda_1|$. Hence $\mu(P_1, P_2) \geq 1$. This completes the proof of (I_3).

<u>Theorem 3:</u> If there exists a polygonal path, $\lambda = Q_1Q_2 \cdots Q_r$ in Π^* for which:

(a) the interior points of the path are contained in the interior of Π,

(b) the vertices and end points are contained in E,

(c) the sides are of μ-length 1,

then there exists λ_o, a simple polygonal path in Π^* with end points Q_1 and Q_r and vertices a subset of those of λ satisfying conditions (a) – (c).

Proof: If λ is itself a simple path then $\lambda_0 = \lambda$. Otherwise it suffices to show that there exists a polygonal path, λ', in Π^* with end points Q_1 and Q_r, whose vertices are a proper subset of those of λ and which satisfies conditions (a) – (c).

Assume that λ is not a simple path. Then at least one of the following holds:

(i) A pair of vertices of λ , say Q_j and Q_k $(j \neq k)$ coincide.[3]

(ii) A pair of sides of λ , say Q_jQ_{j+1} and Q_kQ_{k+1} intersect.[4]

In case (i) certainly $|k-j| > 1$. Let $k > j$, then $Q_1 \cdots Q_j Q_k \cdots Q_r$ has the properties of λ'. In case (ii) let Q_jQ_{j+1} and Q_kQ_{k+1} intersect at X. Assume that Q_jQ_k is not a path in Π^* and $\mu(Q_j, Q_k) \leq 1$. Since Q_jXQ_k is a path in Π^* we deduce, from the triangles in the triangulation of Π^* which cover the path Q_jXQ_k that since they do not cover Q_jQ_k there exists a vertex, P_{s_1} , of Π in the interior of the triangle Q_jXQ_k.

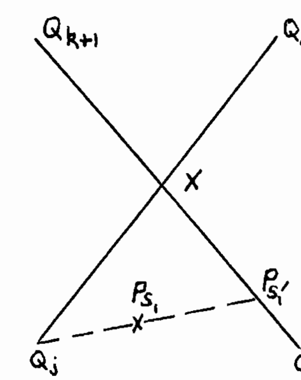

Fig. 5.

Since $\mu(Q_j, P_{s_1}) < 1$, $Q_jP_{s_1}$ is not a path and there exists a second vertex P_{s_2} of Π in the interior of the triangle

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3) That is, they have the same pre-image in the Jordan antecedent.

4) The paths in the Jordan antecedent, of which Q_jQ_{j+1} and Q_kQ_{k+1} are the images, intersect.

$Q_j X P_{s_1'}$ where $\overrightarrow{Q_j P_{s_1'}} = \lambda \overrightarrow{Q_j P_{s_1}}$, $P_{s_1'} \in XQ_k$. Continuing the argument we find a vertex P_s of Π for which $Q_j P_s$ is a path in Π^* but $\mu(Q_j, P_s) < 1$ which is a contradiction. Hence, either $\mu(Q_j, Q_k) = 1$ and $Q_j Q_k$ is a path in Π^* or $\mu(Q_j, Q_k) > 1$. But by the same argument $\mu(Q_{j+1}, Q_{k+1}) \geq 1$ while $\mu(Q_j, Q_k) + \mu(Q_{j+1}, Q_{k+1}) \leq \mu(Q_j, Q_{j+1}) + \mu(Q_k, Q_{k+1}) = 2$. Hence $\mu(Q_j, Q_k) = 1$ and $Q_j Q_k$ is a path in Π^*. In that $|k-j| > 1$, letting $k > j$, $Q_1 \cdots Q_j Q_k \cdots Q_r$ has the properties of λ' .

We shall say that a pair of points of E are linked if there exists a path in Π^* satisfying conditions (a) – (c) of Theorem 3 which has this pair of points as end points. A simple path in Π^* with the properties (a) – (c) of Theorem 3 will be called a linkage (of the end points). Of particular interest will be the set of points of E each of which is linked to a particular vertex. We shall denote the set linked to P_i by $\mathcal{L}(P_i)$.

Theorem 4: A necessary condition that (E, Π) be irreducible is that either

(a) there is a vertex of Π which is an interior point of a side of Π and whose pre-image in the Jordan antecedent, K^*, of Π^* is a simple polygonal path joining a vertex of K to an interior point of a side of K[5],

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5) A special case of this is realized, when the angle at a vertex, P, is zero, in the vertex preceding or following P.

(b) there is a linkage between a pair of vertices of Π,

or (c) (E,Π) is equivalent to a packing in which (a) or (b) holds.

Proof: Let us assume the contrary. The negation of (b) implies that $\mathcal{L}(P_i) \cap \mathcal{L}(P_j) = \emptyset$ $(i \neq j;\ i=1,\ldots,s;\ j=1,\ldots,s)$ where $P_1, \ldots P_s$ are the vertices of Π. Hence there exists for each point Q_{ir} of $\mathcal{L}(P_i)$ a neighbourhood $V(Q_{ir}, \varepsilon_{ir})$ in the interior of Π no point of which is linked to a point of $E - \mathcal{L}(P_i) - P_i$ and in particular a neighbourhood $V(P_i, \varepsilon_i)$ of P_i no point of which is linked to a point of $E - \mathcal{L}(P_i) - P_{i-1} - P_{i+1}$. Further, let $\delta_1 = d(E - \{P_i\}, \Pi)$, the distance between these two sets in the usual sense save that only straight paths in Π* between their respective points are to be considered. Let $\delta_2 = \min\{d(P_j, \Pi - P_{j-1}P_j - P_jP_{j+1}) \mid j=1,\ldots,s\}$. The negation of (a) implies $\delta_2 > 0$. Let $\bar{\varepsilon}_i = \min\{\varepsilon_{i1}, \varepsilon_{i2}, \ldots; \varepsilon_i, \frac{\delta_1}{2}, \frac{\delta_2}{2}\}$. A mapping, τ_i, of E and Π under which $E - \mathcal{L}(P_i) - P_i$ remains fixed $\tau_i P_i \in V(P_i, \bar{\varepsilon}_i)$ and $\overrightarrow{\tau_i P_i\, \tau_i Q_{ir}} = \overrightarrow{P_i Q_{ir}}$ and which maps Π into $\tau_i \Pi = P_1 \ldots P_{i-1} (\tau_i P_i) P_{i+1} \ldots P_s$ induces a transformation of (E,Π) into $(\tau_i E, \tau_i \Pi)$ which, if $\mu(P_{i-1}, \tau_i P_i) \geq 1$, $\mu(\tau_i P_i, P_{i+1}) \geq 1$, is again a packing provided $\tau_i \Pi$ is quasi-Jordan. Furthermore, neither (a) nor (b) holds in $\tau_i \Pi$. We note in particular that if P, Q in E, PQ not a path in Π, $\tau_i P\, \tau_i Q$ is a path in $(\tau_i \Pi)^*$ then clearly $\mu(\tau P, \tau Q) \geq 1$. For, under the circumstances there exists a vertex of Π a neighbourhood of which intersects PQ and no point of which is linked to both P and Q.

In a similar way, the negation of (a) and (b) leads to a mapping τ_{ij} of Π and E defined by local variation of the vertices P_i and P_j which, if $\tau_{ij}\Pi$ is quasi-Jordan, induces a transformation of (E,Π) which is again a packing and neither (a) nor (b) hold in $\tau_{ij}\Pi$.

The existence of a local variation of the above type under which Π remains quasi-Jordan will be ensured if $\angle P_i < 360^\circ$ $(i=1,\ldots,s)$. Let us suppose $\angle P_i = 360^\circ$. If $\mu(P_{i-1},P_i) > \mu(P_i,P_{i+1})$ rotation of P_i along $\mu(P_i,P_{i+1})\Gamma(P_{i+1})$ in that sense which decreases $\angle P_{i-1}$ (see Figure 6) clearly decreases both $A(\Pi)$ and $M(\Pi)$ hence also $\mathcal{F}(\Pi)$ which is a contradiction. Thus $\mu(P_{i-1},P_i) = \mu(P_i,P_{i+1})$, in which case $\mathcal{F}(\Pi)$ remains fixed under a rotation of P_i but continuance of such a variation leads to a packing, (E',Π'), in which (a) or (b) holds and for which $\mathcal{F}(\Pi') = \mathcal{F}(\Pi)$, i.e. to condition (c).

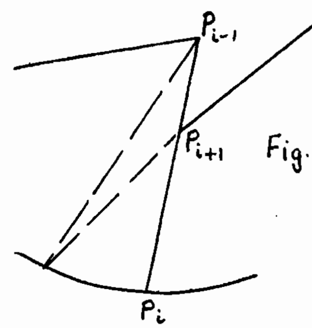


Fig. 6.

Assume then that $\angle P_i < 360^\circ$ $(i=1,\ldots,s)$. It remains for us to consider the case in which $0^\circ < \angle P_i < 360^\circ$ $(i=1,\ldots,s)$.

There exists amongst the triangles of the vertex triangulation, one of which two sides are contained in Π, say $P_{i-k}P_iP_{i+l}$ for which $\angle P_i < 180^\circ$. For there exists, certainly, a triangle with a pair of sides in Π say $P_{q-1}P_qP_{q+1}$. If $\angle P_q < 180^\circ$ our assertion is valid. Otherwise we may replace

Π, for the purpose of this argument, by $\Pi' = P_1, \ldots, P_{q-1}\ P_{q+1}, \ldots, P_s$ having the triangulation of Π with the triangle $P_{q-1}P_qP_{q+1}$ removed. Either a triangle of the type we seek exists in Π' or we modify it in the same way as Π. Continuing in this way we come to a polygon $\Pi'' = P_{i_1}P_{i_2} \ldots\ P_{i_{s'}}$, $1 \leq i_1 < i_2 < \ldots < i_{s'} \leq s$ having the triangulation of Π with all those of the type $P_{q-1}P_qP_{q+1}$ removed. There exists amongst the remainder one, say $P_{i_{r-1}}P_{i_r}P_{i_{r+1}}$, and $\angle P_{i_r} < 180^\circ$, which as a triangle in the triangulation of Π is of the asserted type.

This argument provides for the existence of a pair of consecutive sides, $P_{i-1}\ P_i$, $P_i P_{i+1}$ say, for which $\angle P_i < 180^\circ$ and neither $P_{i-2}\ P_{i-1}$ and $P_i P_{i+1}$ nor $P_{i-1}\ P_i$ and $P_{i+1}\ P_{i+2}$ intersect. At least one of $\mu(P_{i-1}, P_i) = 1$, $\mu(P_i, P_{i+1}) = 1$ holds. For otherwise application of Lemma 2 contradicts the irreducibility of (E, Π). Let $\mu(P_i, P_{i+1}) = 1$. We distinguish two cases according as:

A) $\angle P_{i+1} < \pi$

B) $\angle P_{i+1} \geq \pi$.

Case A). By application of Lemma 3, at least one of $\mu(P_{i-1}, P_i)$, $\mu(P_{i+1}, P_{i+2}) = 1$. Let $\mu(P_{i-1}, P_i) = 1$ and assume $\mu(P_{i+1}, P_{i+2}) > 1$. We study the variation in $\mathfrak{J}(\Pi)$ resulting from a variation of P_{i+1} along $P_{i+1}P_{i+2}$ under which $\mu(P_{i-1}, P_i)$ and $\mu(P_i, P_{i+1})$ each remain equal to one. Referring to Figure 7, we note that $\mathfrak{J}(\Pi)$

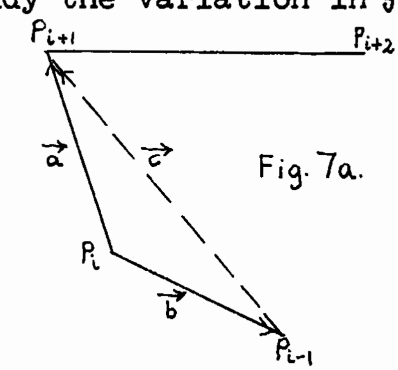


Fig. 7a.

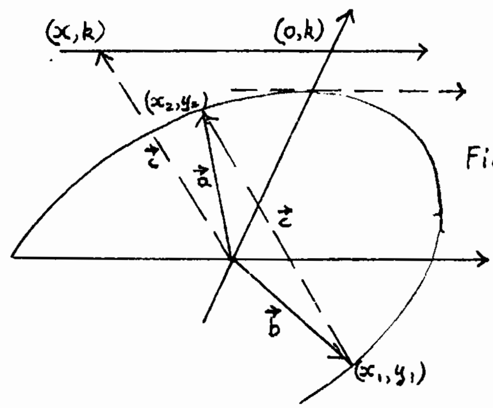


Fig. 7b.

is a linear function with positive coefficients of x and $f(x)$, $f(x) = x_1 y_2 - x_2 y_1$.

Noting that

$$(1)\quad \begin{cases} x_2 - x_1 = x \\ y_2 - y_1 = k \end{cases}$$

$$f(x) = x_1 k - x y_1 .$$

Denoting $\frac{dy_i}{dx_i}$, $\frac{d^2 y_i}{dx_i^2}$ by y_i', y_i'' respectively ($i = 1, 2$) we find:

$$f'(x) = -y_1 + (k - x y_1') \frac{dx_1}{dx} .$$

From equations (1), $\frac{dx_1}{dx} = y_2'(y_1' - y_2')^{-1}$; $\frac{dx_2}{dx} = y_1'(y_1' - y_2')^{-1}$

and $\frac{d^2 x_1}{dx^2} = \left[(y_1')^2 y_2'' - (y_2')^2 y_1''\right](y_1' - y_2')^{-3}$.

$$f''(x) = -2y_1' \frac{dx_1}{dx} - x\left(\frac{dx_1}{dx}\right)^2 y_1'' + (k - x y_1') \frac{d^2 x_1}{dx^2}$$

$$= -\frac{2y_1' y_2'}{y_1' - y_2'} + \frac{k - x y_1'}{(y_1' - y_2')^3}(y_1')^2 y_2'' - \frac{k - x y_2'}{(y_1' - y_2')^3}(y_2')^2 y_1'' .$$

Since, if $x_2 \geq 0$ $\mathcal{J}(\Pi)$ would be reducible by translation of P_{i+1} towards P_{i+2} holding P_i fixed, we may assume $x_2 < 0$ and hence $y_2' > 0$. Returning to the expression for $f''(x)$:

if $y_1' \geq 0$, $y_1' - y_2' > 0$; if $y_1' < 0$, $y_1' - y_2' < 0$,
thus $-\frac{2y_1' y_2'}{y_1' - y_2'} \leq 0$;
if $y_1' \geq 0$, $k - x y_1' > 0$, $(y_1' - y_2')^3 > 0$;
if $y_1' < 0$, $y_1' < \frac{y_2 - y_1}{x_2 - x_1} = \frac{k}{x}$, $(x_2 < 0 \Rightarrow x < 0)$ $x y_1' > k$, $k - x y_1' < 0$;
also $(y_1' - y_2')^3 < 0$; thus, since $y_2'' \leq 0$, $\frac{k - x y_1'}{(y_1' - y_2')^3}(y_1')^2 y_2'' \leq 0$;
if $y_1' \geq 0$, $(y_1' - y_2')^3 > 0$, $y_1'' \geq 0$; if $y_1' < 0$, $(y_1' - y_2')^3 < 0$, $y_1'' \leq 0$;
and, since $k - x y_2' > 0$, $-\frac{k - x y_2'}{(y_1' - y_2')^3}(y_2')^2 y_1'' \leq 0$.

Thus $f''(x) \leq 0$, and, by a suitable choice of dx , $\mathcal{J}(\Pi)$ can be decreased. Let us assume, then, that $\mu(P_{i+1}, P_{i+2}) = 1$.

We consider now the variation in $\mathcal{J}(\Pi)$ resulting from a variation of P_i and P_{i+1} under which $\mu(P_{i-1}, P_i)$, $\mu(P_i, P_{i+1})$, $\mu(P_{i+1}, P_{i+2})$ remain equal to 1.

$\mathcal{J}(\Pi)$ is an increasing linear function of φ where (see Figure 8) $\varphi = ry + x_1 y_2 - x_2 y_1$.

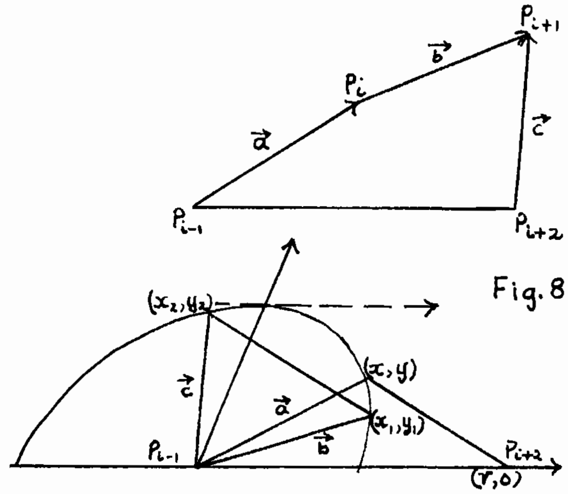


Fig. 8.

Noting that

(2) $\qquad x_2 - x_1 = x - r$

(3) $\qquad y_2 - y_1 = y$,

(4) $\qquad \varphi = (r + x_1)y + (r - x)y_1$.

Subject to the constraints (2), (3) and

$$(x, y),\ (x_1, y_1),\ (x_2, y_2) \in \Gamma(P_{i-1})$$

φ has one degree of freedom. Choosing x as independent variable and with y_i' , y_i'' $(i = 1, 2)$ as before we find:

$$\varphi''(x) = (r + x_1)y'' + 2(y' - y_1')\frac{dx_1}{dx} + (r - x)y_1''\left(\frac{dx_1}{dx}\right)^2 + \left[(r - x)y_1' + y\right]\frac{d^2x_1}{dx^2}.$$

In virtue of (2) and (3),

$$\frac{dx_1}{dx} = \frac{y' - y_2'}{y_2' - y_1'}\ ; \qquad \frac{d^2x_1}{dx^2} = (y_2' - y_1')^{-3}\left[(y_2' - y_1')^2 y'' + (y' - y_2')^2 y_1'' - (y' - y_1')^2 y_2''\right]$$

so that

(5)
$$\begin{aligned} \varphi''(x) \;=\; & 2(y' - y_1')(y' - y_2')(y_2' - y_1')^{-1} \\ & + \left[y + (r + x_1)y_2' - (x + x_1)y_1'\right](y_2' - y_1')^{-1} y'' \\ & + \left[(r - x)y_2' + y\right](y' - y_2')^2 (y_2' - y_1')^{-3} y_1'' \\ & - \left[(r - x)y_1' + y\right](y' - y_1')^2 (y_2' - y_1')^{-3} y_2'' . \end{aligned}$$

It suffices to assume that $y_1 \geq 0$, i.e. $\angle P_{i+2}\, P_{i-1}\, P_i + \angle P_i \geq \pi$ for otherwise $\angle P_{i-1}\, P_{i+2}\, P_{i+1} + \angle P_{i+1} \geq \pi$ and we may interchange the roles of P_{i-1} and P_{i+2} and so on. Assume then that $y_1 \geq 0$.

By an argument similar to that used in the proof of Theorem 3 we find that $\mu(P_{i-1}, P_{i+2}) \geq 1$. For otherwise there is a point of E in the interior of $P_{i-1}\, P_i\, P_{i+1}\, P_{i+2}$. The line parallel to $P_i P_{i+1}$ through such a point, say Q, which is nearest to $P_i P_{i+1}$ cuts at least one of $P_{i-1} P_i$, $P_{i+1} P_{i+2}$ say $P_{i+1} P_{i+2}$ at X. Clearly QP_i is a path in Π* and, since $\Gamma(P_i)$, $\Gamma(P_{i+2})$ intersect at P_{i+1} and a point outside $P_{i-1}\, P_i\, P_{i+1}\, P_{i+2}$, $\mu(Q, P_{i+2}) < 1$. Hence there is a point Q' in the triangle $P_{i+2}QX$ for which in turn $\mu(Q', P_{i+2}) < 1$. Continuing in this way we arrive at a contradiction.

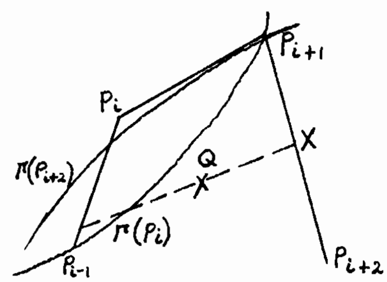


Fig. 9.

Returning to (5), since $\angle P_{i-1}\, P_i\, P_{i+1} < 180°$, $x_1 > x$ and since $\mu(P_{i-1}, P_{i+2}) \geq 1$, $r > x$ and, by (2), $x_1 > x_2$.

Hence

(6) $\quad y_2' > y' > y_1'$

and $\quad 2(y' - y_1')(y' - y_2')(y_2' - y_1')^{-1} < 0.$

As to the coefficients of y'', y_1'' and y_2'' we find:

since $\quad \dfrac{y}{r-x} = \dfrac{y_2 - y_1}{x_1 - x_2} > -y_2'$, $\; r - x > 0$,

(7) $\quad y + (r-x)y_2' > 0,$

$$y + (r+x_1)y_2' > (x+x_1)y_2' ,$$

$$\begin{aligned} y + (r+x_1)y_2' - (x+x_1)y_1' &> (x+x_1)(y_2'-y_1') \\ &\doteq (r+x_2)(y_2'-y_1') \\ &> 0 ; \end{aligned}$$

and from (6), (7)

$$[(r-x)y_2' + y](y'-y_2')^2(y_2'-y_1')^{-3} > 0 ;$$

also, since

$$\frac{y}{r-x} = \frac{y_2-y_1}{x_1-x_2} < -y_1' \quad , \quad (r-x)y_1' + y < 0$$

$$-[(r-x)y_1' + y](y'-y_1')^2(y_2'-y_1')^{-3} > 0 .$$

In that y'', y_1'', $y_2'' < 0$ we have established that

$$\varphi''(x) < 0$$

In particular for y_1 in the neighbourhood of zero we see that

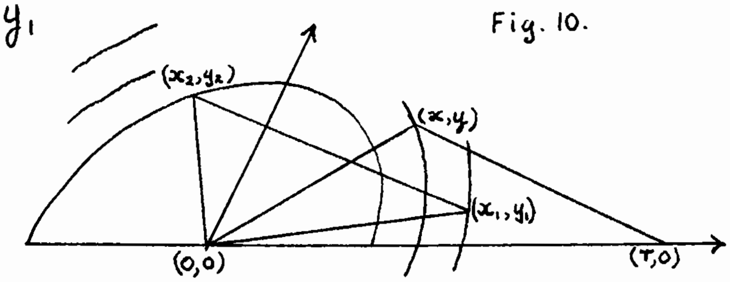
$$\lim_{y_1\to 0_+} \varphi'(x) = \lim_{y_1\to 0_-} \varphi'(x) = (r-x)y_2' + (r+x_2)y' \neq 0 .$$

Thus, by a suitable choice, as to sign, of dx, $d\varphi < 0$ and $\mathcal{J}(\Pi)$ may be decreased.

Case B). We again consider the variation in $\mathcal{J}(\Pi)$ resulting from a variation of P_i and P_{i+1} under which $\mu(P_{i-1}, P_i)$, $\mu(P_i, P_{i+1})$ and $\mu(P_{i+1}, P_{i+2})$ remain constant. In this case $\mathcal{J}(\Pi)$ is a linear function of ψ where (see Figure 10)

$$\begin{aligned} \psi(x) &= ry - (x_1y_2 - x_2y_1) \\ &= (r-x_1)y - (r-x)y_1 \end{aligned}$$

Fig. 10.



since

(2) $x_2 - x_1 = x - r$

(3) $y_2 - y_1 = y$.

$$\psi'(x) = \left[(r-x_1)y'+y_1\right] - \left[(r-x)y_1'+y\right]\frac{y'-y_2'}{y_2'-y_1'}$$

(i) $x<0,\ y>0 \Rightarrow y_2'>y'>0;\quad y>y_2>0;\quad x_2<0;$

hence $y_1<0;\quad x_1>0;\quad y_1'\geq y_2'>y';\ \frac{y'-y_2'}{y_2'-y_1'}>0$

and $\quad (r-x)y_1'+y>0;$

if $\ x\leq x_2\ ,\ r-x_1\leq 0,\ (r-x_1)y'+y_1<0;$

if $\ x>x_2,\ \frac{y-y_2}{x-x_2}>y',\ (r-x_1)y'+y_1<0;$

(ii) $x\geq 0, y\geq 0 \Rightarrow y_2'>y';\ y'<0;\ x>x_2,\ (r-x_1)>0;$

if $\ x_1<0, y_1<0,\quad y_1'<0;\quad y_1'\geq y_2';\quad \frac{y'-y_2'}{y_2'-y_1'}>0;\quad (r-x_1)y'+y_1<0;$

$x_2>x_1;\quad r-x<0;\quad (r-x)y_1'+y>0;$

if $\ x_1\geq 0,\ y_1<0,\quad y_1'>0;\quad y_1'\geq y_2';\quad \frac{y'-y_2'}{y_2'-y_1'}>0;\ (r-x_1)y'+y_1<0;$

if $\ r-x\geq 0,\ (r-x)y_1'+y\geq 0,$

if $\ r-x<0,\ \frac{y_2-y_1}{x_2-x_1}>y_1',\ (r-x)y_1'+y>0;$

if $\ x_1\geq 0,\ y_1\geq 0,\quad y_1'<0;\quad y_2'\geq y_1';\quad \frac{y'-y_2'}{y_2'-y_1'}<0;$

$y_1<y_2,\ x_1>x_2,\ r-x>0,\ \frac{y_2-y_1}{x_2-x_1}>y_1',\ (r-x)y_1'+y<0;$

$\frac{y_1}{x_1-r}=\frac{y_2-y}{x_2-x}>y',\ (r-x_1)y'+y_1<0;$

(iii) $x\geq 0, y<0 \Rightarrow y'>0;\quad y'>y_2';\quad y_1>0;\ y_2'\geq y_1';\ \frac{y'-y_2'}{y_2'-y_1'}>0;$

if $\ x_1\geq 0,\ x_2<x_1;\ r-x>0;\ y_1'<0\ ;\ (r-x)y_1'+y<0;$

if $\ x_1<0,\ y_1'>0;$ if $\ r-x<0,\ (r-x)y_1'+y<0;$

if $\ r-x>0,\ \frac{y_1-y_2}{x_1-x_2}>y_1',\ (r-x)y_1'+y<0;$

if $\ r-x_1\geq 0,\ (r-x_1)y'+y_1>0;$

if $\ r-x_1<0,\ \frac{y_2-y_1}{x_2-x_1}>y',\ (r-x_1)y'+y_1>0;$

(iv) $x<0,\ y<0 \Rightarrow y'<0;\quad y'>y_2';\ y_1>y_2;\quad x_1>x_2>x;$

$0>y_2'>y_1',\quad \frac{y'-y_2'}{y_2'-y_1'}>0;$

$r-x>0,\ (r-x)y_1'+y<0;$

$r-x_1<0,\ (r-x_1)y'+y_1>0\ .$

Thus $\psi'(x) < 0$ when $y > 0$ and $\psi'(x) > 0$ when $y < 0$. Hence ψ can be decreased by varying P_i so as to decrease $\angle P_i P_{i-1} P_{i+2}$. This completes the proof of Theorem 4.

Turning to the proof of (I_n), let the points of E in a packing $\mathbb{P} = (E,\Pi)$ be enumerated and further, since $\mathfrak{F}(\Pi)$ is invariant under a translation of $\mathbb{P}$ let the first point be the origin of a rectangular coordinate system with respect to which the remainder have coordinates (x_1, y_1), (x_2, y_2), $\ldots$, (x_{n-1}, y_{n-1}). This provides a correspondence between a packing and a point $\xi \equiv (x_1, y_1, x_2, y_2, \ldots, x_{n-1}, y_{n-1})$ in R^{2n-2}.

Since $M(\Pi) \geq 2\mu(P_i, P_j)$, $P_i, P_j \in E$, if $|\overrightarrow{P_i P_j}| \geqslant Kn$ where $K = \sup |OP|$, $P \in \Gamma(0)$, then $\mathfrak{F}(\Pi) > \frac{1}{2}M(\Pi) \geqslant n$. We may therefore confine our consideration to the hypercube, S: $|x_i| \leqslant Kn$, $|y_i| \leqslant Kn$ $(i = 1, \ldots, n-1)$.

The points, ξ , are further restricted by the conditions which define a packing. Since these are all expressible as weak inequalities to be satisfied by continuous functions of the coordinates of the points of E, ξ, when restricted to S, is contained in the union, T, of finitely many closed sets. Thus T is closed and, since furthermore it is bounded, it is compact.

Consider $\mathfrak{F}$ as a function of ξ . Certainly it is bounded below. It has therefore a greatest lower bound and indeed assumes an absolute minimum at $\xi_0 \in$ T. It suffices to prove I_n for the packing (E_0, Π_0) corresponding to ξ_0 .

Let us particularize I_n to $I_{(n,m)}$ corresponding to m, the number of points of E not in Π. We apply induction over the index set $\{(n,m) \mid 0 \leq m \leq n-3\}$ and assume that the inequalities $\{I_{(n',m')} \mid n' \leq n,\ m' < m;\ n' < n,\ m' \leq m\}$ are true.

By Theorem 4, Π satisfies one of the conditions (a), (b) or (c) of that theorem and it is sufficient to assume either (a) or (b) holds. The vertex of condition (a) or the linkage, λ, of (b) divides (E_0,Π_0) into two packings (E_1,Π_1) and (E_2,Π_2) for which in case (a)

$$M(\Pi_1) + M(\Pi_2) = M(\Pi_0)$$

$$n_1 + n_2 = n + 1 ,$$

in case (b)

$$M(\Pi_1) + M(\Pi_2) = M(\Pi_0) + 2M(\lambda)$$

$$n_1 + n_2 = n + n(\lambda) ,$$

and in both cases

$$A(\Pi_1) + A(\Pi_2) = A(\Pi_0) ,$$

where n is the number of points of E in E_i $(i = 1,2)$ and $n(\lambda)$ the number of points of E in λ. Clearly, $n(\lambda) = M(\lambda) + 1$.

Applying the inductional assumption to (E_1,Π_1), (E_2,Π_2):

$$\frac{A(\pi_1)}{\Delta} + \frac{M(\pi_1)}{2} + 1 \geq n_1$$

$$\frac{A(\pi_2)}{\Delta} + \frac{M(\pi_2)}{2} + 1 \geq n_2$$

Adding, we have in case (a):

$$\frac{A(\pi_0)}{\Delta} + \frac{M(\pi_0)}{2} + 2 \geq n+1$$

and in case (b):

$$\frac{A(\Pi_0)}{\Delta} + \frac{M(\Pi_0)}{2} + M(\lambda) + 2 \geqslant n + n(\lambda) = n + M(\lambda) + 1 .$$

Thus in both cases:

$$\frac{A(\Pi_0)}{\Delta} + \frac{M(\Pi_0)}{2} + 1 \geqslant n,$$

and $\left\{ I_{(n',m')} \mid n' \leq n,\ m' < m;\ n' < n,\ m' \leq m \right\}$ implies $I_{(n,m)}$. In particular, as a special case of the above argument we see that $\left\{ I_{(n,0)} \mid 3 \leq n' < n \right\}$ implies $I_{(n,0)}$. Having already established $I_{(3,0)}$ the proof of Theorem 2 is complete.

Corollary (Theorem 2): Let (E,Π) be a packing in which $\mathcal{J}(\Pi) = n$, the number of points of E, and Π is a Jordan polygon. There is a triangulation of Π^* by triangles with vertices the points of E, sides of μ-length 1 and area each $\frac{1}{2}\Delta$.

Proof: When $n = 3$ Π is a triangle which, from the proof of (I_3), has sides of μ-length 1 so that $\frac{A(\Pi)}{\Delta} = 3 - \frac{3}{2} - 1$ and $A(\Pi) = \frac{1}{2}\Delta$.

For $n > 3$, referring to Theorem 2 we find that the possibilities (a) and (c) of that theorem do not apply since Π is a Jordan polygon. Hence there is a linkage between a pair of vertices of Π which divides it into a pair of Jordan polygons Π_1 and Π_2 and the packing (E,Π) into packings (E_1,Π_1) and (E_2,Π_2). Letting ν be the μ-length of the linkage, n_1 and n_2 the number of points of E_1 and E_2 respectively we have:

$$\mathcal{J}(\Pi_1) + \mathcal{J}(\Pi_2) = \frac{A(\Pi)}{\Delta} + \frac{M(\Pi)}{2} + \nu + 2 ,$$

$$\mathfrak{J}(\pi_1) + \mathfrak{J}(\pi_2) = n + \nu + 1$$
$$= n_1 + n_2$$

Hence $\mathfrak{J}(\Pi_i) = n_i$ $(i = 1, 2)$ and the corollary is applicable to Π_1 and Π_2. If it is valid for Π_1 and Π_2 it is valid for Π so that applying induction over the same index set as in the proof of Theorem 2, the corollary is established.

REFERENCES

1. Barlow, W.: Probable Nature of the Internal Symmetry of Crystals. Nature vol. 29, 1883-4, pp. 186-8, 205-7, 404.

2. Thue, A.: Om Nogle Geometrisk Taltheoretiske Theoremer. Forhdl. Skand. Naturforsk. 14, 1892, pp. 352-3.

3. Thue, A.: Uber die dichteste Zusammenstellung von kongruenten Kreisen in einer Ebene. Christiania Vid. Selsk. Skr. 1, 1910, pp. 3-9.

4. Minkowski, H.: Geometrie der Zahlen, 2nd ed. Leipzig and Berlin, 1910.

5. Fejes Toth, L.: Lagerungen in der Ebene auf der Kugel und im Raum. Berlin, 1953.

6. Fejes Toth, L.: Some packing and covering theorems, Acta Sci. Math. (Szeged) 12 A, 1950, pp. 62-7.

7. Rogers, C.A.: The closest packing of convex two-dimensional domains. Acta Math. 86, 1951, pp. 309-321.

8. Zassenhaus, H.: Methoden und Probleme der Modernen Algebra. [An address before the Hamburg Math. Seminar, Aug. 1, 1947] Hamburg Univ. Press.

9. Smith, N.: A Statistical Problem in the Geometry of Numbers. McGill Thesis, 1951.

10. Bonnesen-Fenchel: Konvexe Korper. Berlin, 1934, pp. 21-3.

11. Blaschke, W.: Kreis und Kugel. Leipzig, 1916.