Modular forms, quaternion algebras, and special values of L-functions

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IN MEMORY OF

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Abstract

Let f be a cusp form of weight 2 and level N. Let K be an imaginary quadratic field of discriminant $-D$, and $\mathcal{A}$ an ideal class of K. We obtain precise formulas for the special values of the L-functions associated to the Rankin convolution of f and a theta series associated to the ideal class $\mathcal{A}$, in terms of the Petersson scalar product of f with the theta series associated to an Eichler order in a positive definite quaternion algebra. Our work is an extension of the work done by Gross [7]. The central tools used in this thesis are Rankin's method and a reformulation of Gross of work of Waldspurger concerning central critical values.

Résumé

Soit f une forme parabolique de poids 2 et de niveau N. Soit K un corps quadratique imaginaire de discriminant D, et $\mathcal{A}$ une classe d'idéaux de K. On donne une formule pour les valeurs spéciales de la fonction L associée à la convolution de Rankin de f et d'une série theta associée à la classe $\mathcal{A}$, en terme du produit scalaire de Petersson de f et d'une série theta associée à un ordre d'Eichler dans une algèbre de quaternions positive définie. Cette thèse est une extension d'un travail de Gross [7]. L'ingrédient essentiel y est la méthode de Rankin et les travaux de Waldspurger sur les valeurs centrales critiques de fonctions L.

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Contents

Introduction

In this thesis we study certain L-series of Rankin type. These L-series are of great significance in the study of elliptic curves. We will obtain the special values of these L-series in terms of theta series associated to some definite quaternion algebras. Here we review these L-series and related topics.

An elliptic curve E over a field F is a curve (one dimensional variety) of genus one, contained in $\mathbb{P}^2(F)$, the projective plane over F. In affine coordinates the defining equation of the curve E, defined over $\mathbb{Q}$, is an equation of the form

$$y^2 = x^3 - Ax - B, \tag{0.1}$$

where $A, B \in \mathbb{Q}$. For any number field F we let $E(F)$ be the set of solutions to the equation (0.1) in $\mathbb{P}^2(F)$. This set is indeed an abelian group with a natural composition law. We have:

Mordell-Weil Theorem [14, page220] For any number field F the group $E(F)$ is a finitely generated abelian group.

By the above theorem we have

$$E(F) \cong E_{tor}(F) \oplus \mathbb{Z}^r$$

for some non-negative integer r, which we call the (algebraic) rank of E over

F. The rank r turns out to be a mysterious number and there are quite a number of fascinating conjectures concerning that number.

The theory of elliptic curves over $\mathbb{Q}$, and in particular the rank, is closely related to the theory of modular forms through the Shimura-Taniyama-Weil conjecture [14, page 362] and the Birch and Swinnerton-Dyer conjecture [14, page 362]. If E is an elliptic curve over $\mathbb{Q}$, then

$$L_F(E, s) = \sum_{n \geq 0} c_F(n) n^{-s}$$

is an L-series which somehow records the number of elements in $E(F_p)$ for various primes p in its coefficients [14] . The Shimura-Taniyama-Weil conjecture, which after [16] can be called a theorem (in most cases), says that the inverse Mellin transform of $L_{\mathbb{Q}}(E, s)$ which is defined as

$$f_E(\tau) = \sum c_{\mathbb{Q}}(n) e^{2\pi i n \tau},$$

is a weight 2 cusp form for the congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$, where N is a positive integer called the conductor of E.

The Birch and Swinnerton-Dyer conjecture predicts that the algebraic rank r of E over F is indeed equal to the analytic rank of E over F which is defined to be the order of vanishing of $L_F(E, s)$ at 1. This conjecture also predicts a value for the quantity

$$\lim_{s \to 1} \frac{L_F(E, s)}{(s-1)^r}$$

in terms of some subtle algebraic invariants of E. There is a great deal of evidence for this conjecture. See for example [14] for a list of such evidences.

In chapter 1 we have given some background materials which will be used later on. In chapter 2 we review quaternion algebras, which are used in our main result in chapter 4. In chapter 3 in which we will follow the methods of [8], we define the Rankin L-series $L_{\mathcal{A}}(f,s)$ as

$$L_{\mathcal{A}}(f,s) = \sum_{\substack{m\geq 1 \\ (m,N)=1}} \frac{\epsilon(m)}{m^{2s-1}} \sum_{m\geq 1} \frac{a_m r_m}{m^s}.$$

Here

$$f(z) = \sum a_n e^{2\pi i n z}$$

is a cusp form in $S_2^{\text{new}}(\Gamma_0(N))$, $\mathcal{A}$ is an ideal class of the imaginary quadratic field K of discriminant $-D$, $r_m = r_{\mathcal{A}}(m)$ is the number of integral ideals of norm m in the class $\mathcal{A}$ and ϵ is the Dirichlet character associated to K. (See section 3.1).

The L-series $L_{\mathcal{A}}(f,s)$ extends analytically to an entire function of s and satisfies the functional equation

$$L^{\star}_{\mathcal{A}}(f,s) := \left(\frac{DN}{4\pi^2}\right)^2 \Gamma(s)^2 L_{\mathcal{A}}(f,s) = -\epsilon(N) L^{\star}_{\mathcal{A}}(f, 2-s)$$

(see Theorem 3.1). Our main result will give the value $L_{\mathcal{A}}(f,1)$ in terms of the Petersson scalar product of f and a theta series associated to Eichler orders in a definite quaternion algebra. See [2], [1]and [5] for some of the applications of this result. Note that for any character χ on $\text{Pic}(\mathcal{O})$

$$L_K(f,\chi,s) = \sum_{\mathcal{A}\in \text{Pic}(\mathcal{O})} \chi(\mathcal{A}) L_{\mathcal{A}}(f,s),$$

where $\mathcal{O}$ is the ring of integers of K. Using this fact we will be able to calculate $L_K(f, \chi, 1)$. We will use Rankin method to obtain a formula for $L_{\mathcal{A}}(f, 1)$ as Petersson scalar product (on $\Gamma_0(N)$) of f with a modular form $\Phi_{\mathcal{A}}$ (Theorem 3.8). We will conclude chapter 3 with explicitly calculating the coefficients of $\Phi_{\mathcal{A}}$ (Theorem 3.14). Our main result is proved in chapter 4 (Theorem 4.19), where we prove that the theta series $\Phi_{\mathcal{A}}$ is indeed a multiple of the theta series $\theta_{\mathcal{A}}$ associated to Eichler orders in a definite quaternion algebra (Proposition 4.18). Our main result (Theorem 4.19) was first proved by Gross in the special case where N and D are both prime [7]. A proof for the more general case, where D is not necessarily prime, has been suggested in [7] without any details.

Chapter 1

Preliminaries

1.1 Theta series of imaginary quadratic fields

Let K be an imaginary quadratic field of discriminant $-D$, and let $\mathcal{O} = \mathcal{O}_K$ be the ring of integers of K. We denote by $u = u(-D)$ the cardinality of $\mathcal{O}^\times/\langle\pm 1\rangle$, where $\mathcal{O}^\times$ is the group of units of $\mathcal{O}$. Then $u = 1$ except when $D = -3$ or $D = -4$, where $u = 3$ and 2 respectively. Let $h = h(-D)$ be the class number of K and $\mathcal{A}$ be a fixed ideal class of $\mathcal{O}$. For any ideal class $\mathcal{B}$ we define the theta series $E_{\mathcal{B}}(z)$ as

$$E_{\mathcal{B}}(z) = \frac{1}{2u}\sum_{\lambda\in\mathfrak{b}} q^{\mathcal{N}\lambda/\mathcal{N}\mathfrak{b}} = \sum_{m=0}^{\infty} r_{\mathcal{B}}(m)q^m, \quad (q = e^{2\pi i z}) \tag{1.1}$$

where $\mathfrak{b}$ is any ideal in the class $\mathcal{B}$ and $\mathcal{N}$ is the norm function. The following result was proved by Hecke [11].

Theorem 1.1 *$E_{\mathcal{B}}$ is a modular form of weight 1 for $\Gamma_0(D)$, with character ϵ, where ϵ is the character of $(\mathbb{Z}/D\mathbb{Z})^\times$ defined by*

$$\epsilon(p) = \left(\frac{-D}{p}\right).$$

The following facts will be used later:

Proposition 1.2 -

(i) $r_{\mathcal{B}}(0) = \frac{1}{2u}$.

(ii) For any $m \geqslant 1$, $r_{\mathcal{B}}(m)$ is the number of ideals of $\mathcal{O}$ of norm m in the class $\mathcal{B}$.

Proof: (i) is clear.

(ii) : From the definition of $r_{\mathcal{B}}(m)$ we see that $2ur_{\mathcal{B}}(m)$ is the number of $\lambda \in \mathfrak{b}$ with

$$\mathcal{N}(\lambda) = m\mathcal{N}\mathfrak{b}.$$

For each $\lambda \in \mathfrak{b}$ with $\mathcal{N}(\lambda) = m\mathcal{N}\mathfrak{b}$, the ideal $(\lambda)\mathfrak{b}^{-1}$ is an ideal of $\mathcal{O}_K$ in the class $\mathcal{B}^{-1}$ and

$$\mathcal{N}((\lambda)\mathfrak{b}^{-1}) = m.$$

Conversely, if $\mathfrak{c} \subseteq \mathcal{O}_K$ is an ideal in the class $\mathcal{B}^{-1}$ with $\mathcal{N}(\mathfrak{c}) = m$, then $\mathfrak{c}\mathfrak{b}$ is a principal ideal. i.e, $\mathfrak{c}\mathfrak{b} = (\lambda)$ where $\lambda \in \mathfrak{b}$. Moreover,

$$\mathcal{N}(\lambda) = \mathcal{N}(\mathfrak{c})\mathcal{N}(\mathfrak{b}) = m\mathcal{N}(\mathfrak{b})$$

This gives a bijection

$$\{\lambda \in \mathfrak{b} : \mathcal{N}(\lambda) = m\mathcal{N}\mathfrak{b}\}/\mathcal{O}^\times \to \{\mathfrak{c} : \mathfrak{c} \in \mathcal{B}^{-1}, \mathcal{N}\mathfrak{c} = m\}$$

Now (ii) is a consequence of the bijection

$$\{\mathfrak{b} : \mathfrak{b} \in \mathcal{B}, \mathcal{N}\mathfrak{b} = m\} \to \{\mathfrak{b} : \mathfrak{b} \in \mathcal{B}^{-1}, \mathcal{N}\mathfrak{b} = m\}$$

in which $\mathfrak{b} \mapsto \bar{\mathfrak{b}}$, where $\bar{\mathfrak{b}}$ is the complex conjugate of $\mathfrak{b}$.

We also define

$$E(z) = \sum_{\mathcal{B}} E_{\mathcal{B}}(z) = \sum_{m=0}^{\infty} R(m)q^m \tag{1.2}$$

where the sum is over all ideal classes $\mathcal{B}$ of $\mathcal{O}$. Then from the above proposition, $R(0) = \frac{h}{2u}$, where h is the class number of K and for $m \geqslant 1$, $R(m)$ is the number of ideals of $\mathcal{O}$ of norm m.

1.2 Poisson summation formula

In our calculation we will use the Poisson summation formula several times. Recall that the Fourier transform on $\mathbb{R}$ is the operator on Lebesgue integrable functions given by

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(t)e^{-2\pi itu}dt$$

For example if $f(t) = e^{-\pi t^2}$ then $\hat{f}(u) = e^{-\pi u^2}$.

If f is also continuous and $\hat{f}$ is integrable , then the Fourier inversion formula says that

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(u)e^{2\pi itu}du.$$

If we define the space of Schwartz functions on $\mathbb{R}$ as:

$$S(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) | P(t)\frac{d^k f}{dt^k} \text{ is bounded for all } k \geq 0 \text{ and all polynomials } P \},$$

then the Fourier transform is a bijection on $S(\mathbb{R})$.

The Poisson summation formula in the one-dimensional case is:

Theorem 1.3 *If f is in $S(\mathbb{R})$ then*

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx}$$

Proof: See for example[12, page 211].

We wish to have a similar formula in higher dimensions.

The standard n dimensional torus is defined as:

$$T^n = \{x = (x_1, x_2, \dots, x_n)^t \in \mathbb{R}^n : 0 \leq x_i < 1\,,\, 1 \leq i \leq n\}.$$

A function f on T^n can be viewed as an $n-$periodic function on $\mathbb{R}^n$:

$$f(x+k) = f(x) \text{ for } x \in \mathbb{R}^n,\, k \in \mathbb{Z}^n\,.$$

Then standard theory of Fourier series on $L^2(T^n)$ says that for any $f \in L^2(T^n)$,

$$f(x) = \sum_{k \in \mathbf{Z}^n} \hat{f}(k)e^{2\pi ik\cdot x} \tag{1.3}$$

where $k \cdot x = k^t x$, the inner product of k and x :

$$k\cdot x = k_1x_1 + k_2x_2 + \cdots + k_nx_n \quad \text{if} \quad x = (x_1, x_2, \dots, x_n)^t, \quad k = (k_1, k_2, \dots, k_n)^t,$$

and

$$\hat{f}(k) = \int_{T^n} f(x)e^{-2\pi ik\cdot x}dx$$

We need to develop the Fourier series of $n-$periodic functions which are not necessarily on standard torus, but on $\mathbb{R}^n/\Lambda$ where Λ is an arbitrary lattice in $\mathbb{R}^n$. These are the functions f on $\mathbb{R}^n$ such that

$$f(x+\omega) = f(x), \quad x \in \mathbb{R}^n\ , \omega \in \Lambda.$$

Let

$$\Lambda = \mathbb{Z}\omega^1 + \mathbb{Z}\omega^2 + \cdots + \mathbb{Z}\omega^n,$$

where $\omega^1, \omega^2 \ldots, \omega^n$ is a basis for $\mathbb{R}^n$ and

$$P_\Lambda = (\omega^1, \omega^2 \ldots, \omega^n) \in Gl_n(\mathbb{Z}).$$

We define

$$g(x) = f(P_\Lambda x).$$

Then for $k \in \mathbb{Z}^n$

$$g(x+k) = f(P_\Lambda x + P_\Lambda k) = f(P_\Lambda x + k_1\omega_1 + k_2\omega_2 + \cdots + k_n\omega_n) = f(P_\Lambda x) = g(x).$$

Therefore g is a function on T^n, the standard torus. Hence by (1.3)

$$g(x) = \sum_{k\in\mathbb{Z}^n} \hat{g}(k) e^{2\pi i k\cdot x},$$

where

$$\hat{g}(k) = \int_{T^n} f(P_\Lambda x) e^{-2\pi i k\cdot x} dx = \frac{1}{S}\int_{T_\Lambda} f(x) e^{-2\pi i k\cdot P_\Lambda^{-1} x} dx,$$

where $T_\Lambda = P_\Lambda(T^n)$ is a fundamental region for T_Λ, and $S = \det P_\Lambda$ is its volume.

But

$$k \cdot P_\Lambda^{-1} x = k^t P_\Lambda^{-1} x = x^t \left(P_\Lambda^t\right)^{-1} k = x \cdot \left(P_\Lambda^t\right)^{-1} k.$$

Hence

$$\hat{g}(k) = \frac{1}{S}\int_{T_\Lambda} f(x)e^{-2\pi i((P^t_\Lambda)^{-1}k)\cdot x}dx\,.$$

We define $\hat{\Lambda} = (P^t_\Lambda)^{-1}\mathbb{Z}^n$, and for $\omega' = P^{-t}_\Lambda k$ in $\hat{\Lambda}$ we set

$$\hat{f}(\omega') = \hat{g}(k) = \frac{1}{S}\int_{T_\Lambda} f(x)e^{-2\pi i\omega'\cdot x}dx.$$

Then

$$f(P_\Lambda x) = g(x) = \sum_{k\in\mathbb{Z}^n} \hat{g}(k)e^{2\pi i k\cdot x}$$

implies that

$$\begin{aligned} f(x) &= \sum_{k\in\mathbb{Z}^n} \hat{f}((P^t_\Lambda)^{-1}k)e^{2\pi i k\cdot P^{-t}x} \\ &= \sum_{k\in\mathbb{Z}^n} \hat{f}((P^t_\Lambda)^{-1}k)e^{2\pi i(P^{-t}k)\cdot x} \\ &= \sum_{\omega'\in\Lambda} \hat{f}(\omega')e^{2\pi i\omega'\cdot x}. \end{aligned}$$

So :

Theorem 1.4 *Let Λ be a lattice in $\mathbb{R}^n$. Set $T_\Lambda = \mathbb{R}^n/\Lambda$ and let S be the volume of T_Λ. Also set*

$$\hat{\Lambda} = \{\omega' \in \mathbb{R}^n : \omega'\cdot\omega \in \mathbb{Z},\ \textit{for all}\ \omega\ \textit{in}\ \Lambda\ \}.$$

Let f be a function in $L^2(T_\Lambda)$. Then f can be expanded into a Fourier series

$$f(x) = \sum_{\omega'\in\Lambda'} \hat{f}(\omega')e^{2\pi i\omega'\cdot x}$$

where for $\omega' \in \hat{\Lambda}$ we define

$$\hat{f}(\omega') = \frac{1}{S}\int_{T_\Lambda} f(x)e^{-2\pi i\omega'\cdot x}dx.$$

Proof: Everything was proved except the fact that $\hat{\Lambda}$ defined in this theorem is indeed the lattice $(P_\Lambda^t)^{-1}\mathbb{Z}^n$. The proof of this fact is a direct calculation.

Theorem 1.5 *(Poisson summation formula) Let f be a function in $S(\mathbb{R}^n)$, the space of Schwartz functions on $\mathbb{R}^n$, and define*

$$\tilde{f}(\gamma) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\gamma\cdot x}dx$$

for $\gamma \in \mathbb{R}^n$. Then for any lattice Λ in $\mathbb{R}^n$ we have

$$\sum_{\omega\in\Lambda} f(x+\omega) = \frac{1}{S}\sum_{\omega'\in\hat{\Lambda}} \tilde{f}(\omega')e^{2\pi i\omega'\cdot x}$$

where S and $\hat{\Lambda}$ are as in the previous theorem.

Proof: Define

$$g(x) = \sum_{\omega\in\Lambda} f(x+\omega).$$

Then g is a function on $T_\Lambda = \mathbb{R}^n/\Lambda$ and by the previous theorem,

$$g(x) = \sum_{\omega'\in\hat{\Lambda}} \hat{g}(\omega')e^{2\pi i\omega'\cdot x},$$

where

$$\begin{aligned}\hat{g}(\omega') &= \frac{1}{S}\int_{T_\Lambda} g(x)e^{-2\pi i\omega'\cdot x}dx \\ &= \frac{1}{S}\sum_{\omega\in\Lambda}\int_{T_\Lambda} f(x+\omega)e^{-2\pi i\omega'\cdot x}dx \\ &= \frac{1}{S}\sum_{\omega\in\Lambda}\int_{T_\Lambda+\omega} f(x)e^{-2\pi i\omega'\cdot(x-\omega)}dx.\end{aligned}$$

Since $\omega'.\omega$ is an integer we have

$$\begin{aligned}\hat{g}(\omega') &= \frac{1}{S}\sum_{\omega\in\Lambda}\int_{T_\Lambda+\omega} f(x)e^{-2\pi i\omega'\cdot x}dx \\ &= \int_{\mathbb{R}^n} f(x)e^{-2\pi i\omega'\cdot x}dx = \frac{1}{S}\bar{f}(\omega').\end{aligned}$$

as required.

Chapter 2

Quaternion algebras

2.1 Introduction

In this chapter we will review some facts about quaternion algebras. The main reference for this chapter is [15]. Let F be a field with $\mathrm{char} F \neq 2$.

Definition 2.1 *A quaternion algebra H over F is a 4−dimensional algebra over F of the form*

$$H = F + Fi + Fj + Fij$$

where $i^2 = a$, $j^2 = b$, $ij = -ji$ and $a, b \in F^{\times}$.

We will write $H = \{a, b\}_F$. For $h = x + yi + zj + wij$ in H we define

$$\bar{h} = x - yi - zj - wij.$$

We then define the reduced trace $tr(h)$ and the reduced norm $n(h)$ of h by

$$tr(h) = h + \bar{h} = 2x$$

$$n(h) = h\bar{h} = x^2 - ay^2 - bz^2 + abw^2.$$

For any $h, k \in H$ and $\alpha, \beta \in F$ we have

(i) h is invertible iff $n(h) \neq 0$

(ii) $n(hk) = n(h)n(k)$

(iii) $tr(\alpha h + \beta k) = \alpha tr(h) + \beta tr(k)$

(iv) h satisfies the quadratic polynomial

$$(x - h)(x - \bar{h}) = x^2 - tr(h)x + n(h).$$

Examples

(i) The algebra of Hamilton quaternions is the quaternion algebra over $\mathbb{Q}$ defined by

$$\mathbb{H} = \{-1, -1\}_{\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}ij,$$

which is a division algebra.

(ii) The algebra $M(2, F)$ of all 2×2 matrices with entries in F is a quaternion algebra. Indeed $M(2, F) = \{1, 1\}_F$ by setting

$$i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ ij = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then for $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M(2, F)$ we have

$\bar{h} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$

$tr(h) = a + d$

$n(h) = ad - bc.$

A quadratic algebra over F is an F-algebra which is two-dimensional as an F-vector space.

Proposition 2.2 *[15, corollary I.2.2] If L is a quadratic algebra over F contained in H, then there exists $u \in H$ such that $H = L + Lu$, where*

$$u^2 = \theta \in K^\times, \; um = \bar{m}u, \; \textit{for all} \; m \in L, \; \textit{and} \; \bar{u} = -u.$$

Notation: With the notation above, we write

$$H = \{L, \theta\}.$$

Theorem 2.3 *[15, corollary I.2.4]*

(i) A quaternion algebra over F is either a division algebra or isomorphic to $M(2, F)$.

(ii) The quaternion algebra $\{L, \theta\}$ is isomorphic to $M(2, F)$ if and only if $L \simeq F \oplus F$, or θ is the norm of an element of L.

2.2 Orders and ideals

In this and the following sections F will denote either a p-adic field or a number field, and $\mathcal{O}_F$ will be the ring of integers of F.

Definition 2.4 *An* ideal *of H is a finitely generated $\mathcal{O}_F$-submodule I of H such that $I \otimes_{\mathcal{O}_F} F \cong H$.*

Definition 2.5 *An element $h \in H$ is called an* integer *if it satisfies the following equivalent conditions:*

(i) The ring $\mathcal{O}_F[h]$ is a finitely generated $\mathcal{O}_F$-module.

(ii) The norm $n(h)$ and the trace $tr(h)$ of h are in $\mathcal{O}_F$.

Definition 2.6 *A subset R of H is called an* order *if it satisfies the following equivalent conditions:*

(i) R is an ideal of H which is also a subring of H.

(ii) R is a subring of H containing $\mathcal{O}_F$, $FR = H$, and every element of R is an integer of H.

See [15, Proposition I.4.2] for the equivalence of (i) and (ii).

An order R of H is called a maximal order if it is not contained in any other order of H. The intersection of two maximal orders is called an Eichler order. Given an ideal I of H the subsets:

$$R_l(I) = \{h \in H : hI \subset I\}$$

$$R_r(I) = \{h \in H : Ih \subset I\}$$

are orders of H, and are called the left order and the right order of I respectively. We also define the inverse of I as:

$$I^{-1} = \{h \in H : IhI \subseteq I\}.$$

Then

$$I^{-1} = \{h \in H : Ih \subseteq R_l(I)\} = \{h \in H : hI \subseteq R_r(I)\}.$$

Definition 2.7 *Given an order R we define,*

(i) $R^\vee = \{x \in H : t(xR) \subset \mathcal{O}_F\}$.

(ii) $(R^\vee)^{-1}$ *is called the different ideal of R.*

(iii) The reduced norm $n((R^\vee)^{-1})$ *is called the reduced discriminant of R and is denoted by* $\mathrm{disc}(R)$.

Proposition 2.8 *[15, lemma I.4.7] If $\{e_1, e_2, e_3, e_4\}$ is an $\mathcal{O}_F$-basis for an order*

$$R = \mathcal{O}_F e_1 + \mathcal{O}_F e_2 + \mathcal{O}_F e_3 + \mathcal{O}_F e_4$$

of H, then

$$\mathrm{disc}(R) = |(\det(\mathrm{tr}(e_i \bar{e}_j)))|^{1/2}$$

2.3 Quaternion algebras over local fields

Theorem 2.9 *(Classification) [15, Theorem II.1.1] Over any local field $F \neq \mathbb{C}$ there exists a unique quaternion division algebra H (up to isomorphism).*

If F is not archimedean, then

$$H = \{L_{ur}, \pi\},$$

where L_{ur} is the (unique up to isomorphism) unramified quadratic extension of F (in a separable closure F_s of F), and π is a uniformizer in F. The valuation v on F can be extended to a valuation w on H by setting $w(h) = \frac{v(n(h))}{2}$ for $h \in H$.

The notations are as in the previous theorem:

Theorem 2.10 .

i) [15, lemma II.1.5] The valuation ring R_w *is the unique maximal order of* $\{L_{ur}, \pi\}$.

ii) [15, Theorem II.2.3] : The maximal orders of $M(2, F)$ *are the conjugates of* $M(2, \mathcal{O}_F)$.

iii) [15, lemma II.2.4] Any Eichler order of $M(2, F)$ *of level* π^n *is conjugate to*

$$R_n = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \pi^n \mathcal{O}_F & \mathcal{O}_F \end{pmatrix} .$$

2.4 Quaternion algebras over global fields

Let F be a number field. We let P_F be the set of places of F. For p in P_F we denote the completion of F at p by F_p. For any F-algebra L we denote

$$L_p = L \otimes F_p$$

where the tensor product is over F. If in particular L is a quadratic field extension of F then L_p is a field if and only if p does not split in L. If H is a quaternion algebra over F then by theorem 2.3, (for each $p \in P_F$), H_p is either a division algebra or is isomorphic to $M(2, F_p)$.

Definition 2.11 *The quaternion algebra H over F is said to be* ramified *at the place p of F (alternatively p is said to be ramified in H) if H_p is a division algebra. If H_p is isomorphic to $M(2, F_p)$, then H is said to be* split *at p.*

We denote the set of places of F which are ramified in H by $\mathrm{Ram}(H)$.

Theorem 2.12 *[15, Theorem III.3.1]*

i) The set $\mathrm{Ram}(H)$ is finite with even cardinality.

ii) $H = M(2, F)$ if and only if $H_p = M(2, F_p)$ for all places p of F.

iii) If S is a finite set of places of F with even cardinality, then there exists a unique (up to isomorphism) quaternion algebra H over F with $S = \mathrm{Ram}(H)$.

We define the (reduced) discriminant of H by

$$\mathrm{disc}(H) = \prod_{\substack{p \in \mathrm{Ram}(H) \\ p \text{ finite}}} p$$

Theorem 2.13 *[15, Theorem III.3.8] A quadratic extension L of F can be embedded in a quaternion algebra H over F if and only if L_p is a field for all $p \in \mathrm{Ram}(H)$. (I.e., all $p \in \mathrm{Ram}(H)$ are inert or ramified in L.)*

2.5 Quaternion algebras over $\mathbb{Q}$

A quaternion algebra H over $\mathbb{Q}$ has the form

$$H = \{a, b\}_{\mathbb{Q}} := \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}ij,$$

where

$$i^2 = a, \quad j^2 = b, \quad ij = -ji,$$

with $a, b \in \mathbb{Q}$. The algebra H ramifies at ∞ if and only if a and b are both negative. If H ramifies at ∞, then H is said to be a definite quaternion algebra. If H splits at ∞, then H is said to be indefinite. We define the Hilbert symbol $(a, b)_p$ by

$$(a, b)_p = \begin{cases} 1 & \text{if } \{a, b\} \text{ splits at } p \\ -1 & \text{if } \{a, b\} \text{ ramifies at } p. \end{cases}$$

Then we have,

Theorem 2.14 *[15, page37] Let p be an odd rational prime and $a, b \in \mathbb{Q}$. Then*

$$(a, b)_p = \begin{cases} 1 & \text{if } p \nmid ab \\ \left(\frac{a}{p}\right) & \text{if } p \nmid a \text{ and } p \parallel b, \end{cases}$$

where $\left(\frac{a}{p}\right)$ is the Legendre symbol, and $p \parallel b$ means that $p|b$ but $p^2 \nmid b$.

Definition 2.15 *For any lattice L of H and any prime p we define*

$$L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

L_p is called the localization of L at p.

The following proposition gives us a dictionary between global and local lattices.

Proposition 2.16 *[15, proposition III.5.1] Let X be a lattice of H. There is a bijection between the set of lattices L of H and the set*

$$\{(L_p) : L_p \text{ is a lattice of } H_p, \ L_p = X_p \text{ for almost all finite primes } p\}$$

of sequences of lattices, in which

$$\begin{array}{ccc} L & \longmapsto & (L_p) \\ (L_p) & \longmapsto & H\bigcap(\bigcap_{p \text{ finite}} L_p) \end{array}$$

A property $(\star)$ is called a local property for lattices if for every lattice L of H, the lattice L has the property $(\star)$, if and only if L_p has the property $(\star)$, for all primes p. We have:

Proposition 2.17 *[15, page 82] The following properties for a lattice L are all local properties:*

i) L is an ideal.

ii) L is an order.

iii) L is a maximal order.

iv) L is an Eichler order.

Definition 2.18 *The* level *of an Eichler order L is defined as*

$$l = \prod_{p finite} l_p,$$

where $l_p = p^{\alpha_p}$ is the level of L_p.

The following criterion is a very useful one:

Proposition 2.19 *[15, corollaryIII.5.3] An order R of H is a maximal order if and only if*

$$\mathrm{disc}(R) = \mathrm{disc}(H).$$

Chapter 3

Special values of L-functions

3.1 Introduction

In this chapter we will study the special values of a certain L−series of Rankin type. Our main reference in this chapter is [8], and we will follow the methods used in [8] in our proofs.

First we recall some notations from section 1.1: K is a quadratic imaginary field of discriminant $-D$, and $\mathcal{O}$ is its ring of integers. We let $\mathcal{A}$ be a fixed ideal class of $\mathcal{O}$ and set $u = u(-D)$ and $h = h(-D)$. The Dirichlet character associated to K is defined as:

$$\epsilon(p) = \left(\frac{-D}{p}\right),$$

which is an odd primitive character of conductor D [13, page 201]. The modular form associated to $\mathcal{A}$ is defined as

$$E_{\mathcal{A}}(z) = \frac{1}{2u}\sum_{\lambda\in\mathfrak{a}} q^{\mathcal{N}\lambda/\mathcal{N}\mathfrak{a}} = \frac{1}{2u} + \sum_{m=1}^{\infty} r_{\mathcal{A}}(m)q^m \quad (q = e^{2\pi i z})$$

where $\mathfrak{a}$ is any integral ideal in the class $\mathcal{A}$. By theorem 1.1, $E_{\mathcal{A}}(z)$ is a modular form of weight 1 and level D, with character ϵ. We also define

$$E(z) = \sum_{\mathcal{B}} E_{\mathcal{B}}(z) = \sum_{\mathcal{B}\in Pic(\mathcal{O})} E_{\mathcal{B}}(z) = \frac{h}{2u} + \sum_{m=1}^{\infty} R(m)q^m \ ,$$

where $Pic(\mathcal{O})$ is the class group of $\mathcal{O}$.

Now let $f \in S_2^{\text{new}}(\Gamma_0(N))$, where N is a positive integer with $(N, D) = 1$. Here $S_2^{\text{new}}(\Gamma_0(N))$ is the space of cusp forms of weight 2 of level N which are orthogonal (with respect to the Petersson product) to all oldforms. We recall that a modular form of level N is called an old form if it it is in the span of the forms $g(dZ)$ with g of level $N_1 < N$ and $dN_1 \mid N$. We also recall that the Petersson inner product of f with any modular form g of level N is defined as:

$$(f, g) = (f, g)_{\Gamma_0(N)} = \iint_{\Gamma_0(N)\backslash\mathfrak{H}} f(z)\overline{g(z)}dxdy \qquad (z = x + iy).$$

The space $S_2^{\text{new}}(\Gamma_0(N))$ is spanned by the newforms (Hecke eigenforms), but we do not assume that f is a newform. We let

$$f(z) = \sum a_n q^n$$

be the Fourier expansion of f, and

$$L(f, s) = \sum \frac{a_n}{n^s}$$

be the Hecke $L-$series of f. Given these data, the Dirichlet series $L_{\mathcal{A}}(f, s)$ is defined as the product of the Dirichlet $L-$function

$$L^{(N)}(2s-1, \epsilon) = \sum_{\substack{m=1 \\ (m,N)=1}} \frac{\epsilon(m)}{m^{2s-1}}$$

and the convolution of $L(f, s)$ with the zeta function $\sum_{m>0} r_{\mathcal{A}}(m) m^{-s}$. i.e.

$$L_{\mathcal{A}}(f, s) = \sum_{\substack{m \geq 1 \\ (m,N)=1}} \frac{\epsilon(m)}{m^{2s-1}} \sum_{m=1}^{\infty} \frac{a_m r_m}{m^s}. \tag{3.1}$$

Here we have set $r_m = r_{\mathcal{A}}(m)$.

Theorem 3.1 *[8, page 267] With notations as above, the Dirichlet series $L_{\mathcal{A}}(f, s)$ extends analytically to an entire function of s, and satisfies the functional equation*

$$L^*_{\mathcal{A}}(f, s) := \left(\frac{DN}{4\pi^2}\right)^2 \Gamma(s)^2 L_{\mathcal{A}}(f, s) = -\epsilon(N) L^*_{\mathcal{A}}(f, 2-s)$$

The above theorem shows that if $\epsilon(N) = +1$, then $L_{\mathcal{A}}(f, s)$ vanishes at $s = 1$. In this case [8] gives a formula for the derivative $L'_{\mathcal{A}}(f, 1)$. In the case when $\epsilon(N) = -1$, [8] gives a formula for $L_{\mathcal{A}}(f, 1)$. We will follow the methods used in [8] to give the formulas for $L_{\mathcal{A}}(f, 1)$ in the case when $\epsilon(N) = -1$.

In section 3.2, following [8], we use Rankin's method to obtain a formula for $L_{\mathcal{A}}(f, s)$ as Petersson scalar product (on $\Gamma_0(ND)$) of f with the product of a theta series and a non-holomorphic Eisenstein series. Then we will trace down the result to get $L_{\mathcal{A}}(f, s)$ as a Petersson product (on $\Gamma_0(N)$) of f with a modular form Φ_s. In section 3.3 we will calculate the coefficients of Φ_s

in the case where D is prime. For the more general case where D is not necessarily prime, but $D \equiv 3 \pmod 4$, we will state the final result without proof, referring to [8] for details.

3.2 Rankin's method

In this section we give an integral representation for $L_{\mathcal{A}}(f, s)$ using Rankin's method.

Let $\Gamma_\infty = \{\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$. Then for Re$(s)$ large enough we have

$$\begin{aligned}
\Gamma(s) \sum_{m=1}^{\infty} \frac{a_m r_m}{m^s} &= \left(\int_0^\infty e^{-y} y^s \frac{dy}{y} \right) \left(\sum_{m=1}^{\infty} \frac{a_m r_m}{m^s} \right) \\
&= \sum_{m=1}^{\infty} \int_0^\infty e^{-y} (\frac{y}{m})^s a_m r_m \frac{dy}{y} \\
&= \sum_{m=1}^{\infty} \int_0^\infty e^{-my} y^s a_m r_m \frac{dy}{y} \\
&= \sum_{m=1}^{\infty} \int_0^\infty a_m r_m e^{-4\pi m t} (4\pi)^s t^s \frac{dt}{t}
\end{aligned}$$

Therefore

$$(4\pi)^{-s}\Gamma(s) \sum_{m=1}^{\infty} \frac{a_m r_m}{m^s} = \int_0^\infty \left(\sum_{m=1}^{\infty} a_m r_m e^{-4\pi m y} \right) y^s \frac{dy}{y}$$

A direct calculation shows that the last expression is equal to

$$\begin{aligned}
&\int_0^\infty \left(\int_0^1 f(x+iy)\overline{E_{\mathcal{A}}(x+iy)} dx \right) y^s \frac{dy}{y} \\
&= \iint_{\Gamma_\infty \backslash \mathfrak{h}} f(z)\overline{E_{\mathcal{A}}(z)} y^{s+1} \frac{dxdy}{y^2} \\
&= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(ND)} \iint_{\gamma F_{ND}} f(z)\overline{E_{\mathcal{A}}(z)} y^{s+1} \frac{dxdy}{y^2}
\end{aligned}$$

where F_{ND} is a fundamental domain for the action of $\Gamma_0(ND)$ on $\mathfrak{h}$. For $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_0(ND)$ we have

$$\begin{aligned} f(\gamma z) &= (cz+d)^2 f(z) \\ \overline{E_{\mathcal{A}}(\gamma z)} &= \epsilon(d)(c\bar{z}+d)\overline{E_{\mathcal{A}}(z)} \\ \operatorname{Im}(\gamma z) &= \frac{y}{|cz+d|^2} \end{aligned}$$

Now using these equalities and the invariance of the measure $\frac{dxdy}{y^2}$ under $\mathrm{SL}_2(\mathbb{R})$, we get

$$\begin{aligned} &(4\pi)^{-s}\Gamma(s)\sum_{m=1}^{\infty}\frac{a_m r_m}{m^s} \\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(ND)} \iint_{F_{ND}} f(\gamma z)\overline{E_{\mathcal{A}}(\gamma z)}\operatorname{Im}(\gamma z)^{s+1}\frac{dxdy}{y^2} \\ &= \sum_{\gamma=\pm\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_0(ND)} \iint_{F_{ND}} f(z)(cz+d)^2\overline{E_{\mathcal{A}}(z)}(c\bar{z}+d)\epsilon(d)\frac{y^{s+1}}{|cz+d|^{2s+2}}\frac{dxdy}{y^2}. \end{aligned}$$

Therefore

$$(4\pi)^{-s}\Gamma(s)\sum_{m=1}^{\infty}\frac{a_m r_m}{m^s} = \sum_{\gamma=\pm\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_0(ND)} \iint_{F_{ND}} f(z)\overline{E_{\mathcal{A}}(z)}\frac{\epsilon(d)}{c\bar{z}+d}\frac{y^{s-1}}{|cz+d|^{2s-2}}dxdy. \tag{3.2}$$

Definition 3.2 *For given $M \geqslant 1$ the Eisenstein series $E_{MD}(s,z)$ of weight 1, level MD, and character ϵ is defined by*

$$E_{MD}(s,z) = \sum_{\substack{m\geqslant 1 \\ (m,M)=1}} \frac{\epsilon(m)}{m^{2s+1}} \sum_{\pm\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_0(MD)} \frac{\epsilon(d)}{cz+d}\frac{y^s}{|cz+d|^{2s}}$$

Now from (3.2), we get

$$(4\pi)^{-s}\Gamma(s) \sum_{\substack{m=1 \\ (m,N)=1}}^{\infty} \frac{\epsilon(m)}{m^{2s-1}} \sum_{m=1}^{\infty} \frac{a_m r_m}{m^s} =$$

$$\iint_{F_{ND}} f(z)\overline{E_{\mathcal{A}}(z)} \sum_{\substack{m=1 \\ (m,N)=1}}^{\infty} \frac{\epsilon(m)}{m^{2s-1}} \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_0(ND)} \frac{\epsilon(d)}{(c\bar{z}+d)} \frac{y^{s-1}}{|cz+d|^{2s-2}} dxdy \tag{3.3}$$

Hence we have proved :

Proposition 3.3

$$\begin{aligned}(4\pi)^{-s}\Gamma(s)L_{\mathcal{A}}(f,s) &= (f, E_{\mathcal{A}}(z)E_{ND}(\bar{s}-1,z))_{\Gamma_0(ND)} \\ &= \iint_{F_{ND}} f(z)\overline{E_{\mathcal{A}}(z)E_{ND}(\bar{s}-1,z)}dxdy.\end{aligned}$$

The method we just used to express the convolution of the L-series of two modular forms as a scalar product involving an Eisenstein series was first used by Rankin and Selberg in 1939 and is commonly referred to as "Rankin's method".

We now trace down the result given in the previous proposition to write $L_{\mathcal{A}}(f,s)$ as a Petersson scalar product over $\Gamma_0(N)$.

Definition 3.4 *For any modular form g of weight 2 and level ND we define,*

$$\mathrm{Tr}_N^{ND}\{g\} = \sum_{\gamma\in\Gamma_0(ND)\backslash\Gamma_0(N)} g|_2\gamma, \tag{3.4}$$

where for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$,

$$(g|_2\gamma)(z) = (\det\gamma)(cz+d)^{-2} g\left(\frac{az+b}{cz+d}\right).$$

It is easy to see that $\mathrm{Tr}_N^{ND}\{g\}$ is a modular form of level N.

Lemma 3.5 *With notation as above, we have*

$$(f,g)_{\Gamma_0(ND)} = \left(f, \mathrm{Tr}_N^{ND}\{g\}\right)_{\Gamma_0(N)}.$$

Proof:

$$\begin{aligned}
(f,g)_{\Gamma_0(ND)} &= \iint_{F_{ND}} f(z)\overline{g(z)}dxdy \\
&= \sum_{\gamma\in\Gamma_0(ND)\backslash\Gamma_0(N)} \iint_{\gamma F_N} f(z)\overline{g(z)}y^2\frac{dxdy}{y^2} \\
&= \sum_{\substack{\gamma\in\Gamma_0(ND)\backslash\Gamma_0(N)\\ \gamma=\left(\begin{smallmatrix}\cdot & \cdot\\ c & d\end{smallmatrix}\right)}} \iint_{F_N} f(\gamma z)\overline{g(\gamma z)}\frac{y^2}{|cz+d|^4}\frac{dxdy}{y^2} \\
&= \sum_{\substack{\gamma\in\Gamma_0(ND)\backslash\Gamma_0(N)\\ \gamma=\left(\begin{smallmatrix}\cdot & \cdot\\ c & d\end{smallmatrix}\right)}} \iint_{F_N} f(\gamma z)(cz+d)^{-2}\overline{g(\gamma z)(cz+d)^{-2}}dxdy \\
&= \sum_{\gamma\in\Gamma_0(ND)\backslash\Gamma_0(N)} \iint_{F_N} f(z)\overline{(g|_2\gamma)(z)}dxdy \\
&= \iint_{F_N} f(z)\overline{\left(\mathrm{Tr}_N^{ND}\{g\}\right)(z)}dxdy = (f, \mathrm{Tr}_N^{ND}\{g\}).
\end{aligned}$$

Now we have:

Lemma 3.6

$$(4\pi)^{-s}\Gamma(s)L_{\mathcal{A}}(f,s) = \left(f, \mathrm{Tr}_N^{ND}\{E_{\mathcal{A}}(z)E_{ND}(\bar{s}-1,z)\}\right).$$

Proof: This follows from proposition 3.3 and lemma 3.5.

Lemma 3.7 *For $M \geq 1$ we have:*

$$E_{MD}(s,z) = \frac{1}{2}\sum_{\substack{c,d\in\mathbf{Z}\\ c\equiv 0\ (MD)\\ (d,MD)=1}} \frac{\epsilon(d)}{cz+d}\frac{y^s}{|cz+d|^{2s}} = \sum_{r|M}\frac{\mu(r)\epsilon(r)}{r^{2s+1}}\left(\frac{r}{M}\right)^s E_D(s,\frac{M}{r}z),$$

where μ is the Mobius function.

Proof: The first equality is a direct result of the definition. We prove the second one. First suppose M is square-free. We prove the lemma in this case using induction on the number of prime divisors of M. If M is prime, then

$$\begin{aligned} E_{MD}(s,z) &= \frac{1}{2}\sum_{\substack{c,d\in\mathbf{Z}\\ c\equiv 0\ (MD)\\ (d,MD)=1}} \frac{\epsilon(d)}{cz+d}\frac{y^s}{|cz+d|^{2s}} \\ &= \frac{1}{2}\sum_{\substack{c,d\in\mathbf{Z}\\ c\equiv 0\ (MD)}} \frac{\epsilon(d)}{cz+d}\frac{y^s}{|cz+d|^{2s}} - \frac{1}{2}\sum_{\substack{c,d\in\mathbf{Z}\\ c\equiv 0\ (MD)}} \frac{\epsilon(Md)}{cz+Md}\frac{y^s}{|cz+Md|^{2s}} \\ &= \frac{1}{2}\sum_{\substack{c,d\in\mathbf{Z}\\ c\equiv 0\ (D)}} \frac{\epsilon(d)}{cMz+d}\frac{y^s}{|cMz+d|^{2s}} - \frac{1}{2}\sum_{\substack{c,d\in\mathbf{Z}\\ c\equiv 0\ (D)}} \frac{\epsilon(Md)}{cMz+Md}\frac{y^s}{|cMz+Md|^{2s}} \\ &= M^{-s}E_D(s,Mz) - \frac{\epsilon(M)}{M^{2s+1}}E_D(s,z). \end{aligned}$$

Now suppose that $M = Kp$ is square-free, and suppose we have

$$E_{KD}(s,z) = \sum_{r|K}\frac{\mu(r)\epsilon(r)}{r^{2s+1}}\left(\frac{r}{K}\right)^s E_D(s,\frac{K}{r}z).$$

Then

$$\begin{aligned}
E_{MD}(s,z) &= \frac{1}{2}\sum_{\substack{c,d\in\mathbf{Z}\\ c\equiv 0\ (MD)\\ (d,MD)=1}} \frac{\epsilon(d)}{cz+d}\frac{y^s}{|cz+d|^{2s}}\\
&= \frac{1}{2}\sum_{\substack{c,d\in\mathbf{Z}\\ c\equiv 0\ (MD)\\ (d,KD)=1}} \frac{\epsilon(d)}{cz+d}\frac{y^s}{|cz+d|^{2s}} - \frac{1}{2}\sum_{\substack{c,d\in\mathbf{Z}\\ c\equiv 0\ (MD)\\ (d,KD)=1}} \frac{\epsilon(pd)}{cz+pd}\frac{y^s}{|cz+pd|^{2s}}\\
&= \frac{1}{2}\sum_{\substack{c,d\in\mathbf{Z}\\ c\equiv 0\ (KD)\\ (d,KD)=1}} \frac{\epsilon(d)}{cpz+d}\frac{y^s}{|cpz+d|^{2s}} - \frac{1}{2}\sum_{\substack{c,d\in\mathbf{Z}\\ c\equiv 0\ (KD)\\ (d,KD)=1}} \frac{\epsilon(pd)}{cpz+pd}\frac{y^s}{|cpz+pd|^{2s}}\\
&= p^{-s}E_{KD}(s,pz) - \frac{\epsilon(p)}{p^{2s+1}}E_{KD}(s,z)\\
&= p^{-s}\sum_{r|K}\frac{\mu(r)\epsilon(r)}{r^{2s+1}}\left(\frac{r}{K}\right)^s E_D(s,\frac{Kp}{r}z) - \frac{\epsilon(p)}{p^{2s+1}}\sum_{r|K}\frac{\mu(r)\epsilon(r)}{r^{2s+1}}\left(\frac{r}{K}\right)^s E_D(s,\frac{K}{r}z)\\
&= \sum_{r|K}\frac{\mu(r)\epsilon(r)}{r^{2s+1}}\left(\frac{r}{M}\right)^s E_D(s,\frac{M}{r}z) + \sum_{r|K}\frac{\mu(rp)\epsilon(rp)}{(rp)^{2s+1}}\left(\frac{rp}{M}\right)^s E_D(s,\frac{M}{rp}z))\\
&= \sum_{r|M}\frac{\mu(r)\epsilon(r)}{r^{2s+1}}\left(\frac{r}{M}\right)^s E_D(s,\frac{M}{r}z).
\end{aligned}$$

In the case where M is not necessarily square-free, we have

$$E_{MD}(s,z) = \left(\frac{M_1}{M}\right)^s E_{M_1D}(s,\frac{M}{M_1}z)$$

where M_1 is the product of distinct prime divisors of M. Hence

$$\begin{aligned}
E_{MD}(s,z) &= \left(\frac{M_1}{M}\right)^s E_{M_1D}(s,\frac{M}{M_1}z)\\
&= \left(\frac{M_1}{M}\right)^s \sum_{r|M_1}\frac{\mu(r)\epsilon(r)}{r^{2s+1}}\left(\frac{r}{M_1}\right)^s E_D(s,\frac{M_1}{r}\frac{M}{M_1}z)\\
&= \sum_{\substack{r|M\\ r\ \text{square-free}}}\frac{\mu(r)\epsilon(r)}{r^{2s+1}}\left(\frac{r}{M}\right)^s E_D(s,\frac{M}{r}z)
\end{aligned}$$

Since $\mu(r) = 0$ for r not square-free , we have

$$E_{MD}(s,z) = \sum_{r|M} \frac{\mu(r)\epsilon(r)}{r^{2s+1}} \left(\frac{r}{M}\right)^s E_D(s, \frac{M}{r}z).$$

Now we can prove :

Theorem 3.8 *Define the Eisestein series $E_D(s,z)$ of level D and weight 1 and character ϵ as,*

$$E_D(s,z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ D|c}} \frac{\epsilon(d)}{cz+d} \frac{y^s}{|cz+d|^{2s}},$$

and let

$$\Phi_s(z) = \mathrm{Tr}_N^{ND} \left(E_{\mathcal{A}}(z) E_D(s, Nz)\right).$$

Then, with notations as in section 1, we have

$$(4\pi)^{-s}\Gamma(s)N^{s-1}L_{\mathcal{A}}(f,s) = (f, \Phi_{\bar{s}-1}).$$

Proof: Using the previous lemma we have

$$\mathrm{Tr}_N^{ND}\{E_{\mathcal{A}}(z)E_{ND}(\bar{s}-1,z)\} = \sum_{r|N} \frac{\mu(r)\epsilon(r)}{r^{2\bar{s}-1}} \left(\frac{r}{N}\right)^{\bar{s}-1} \mathrm{Tr}_N^{ND}\{E_{\mathcal{A}}(z)E_D(\bar{s}-1, \frac{N}{r}z)\}.$$

If $r|N$ and $r > 1$, $E_{\mathcal{A}}(z)E_D(\bar{s}-1, \frac{N}{r}z)$ is of level $\frac{ND}{r}$. Since $(N,D) = 1$, a complete set of coset representatives in $\Gamma_0(ND)\backslash\Gamma_0(N)$ is a complete set of coset representatives in $\Gamma_0(\frac{ND}{r})\backslash\Gamma_0(\frac{N}{r})$ as well. Hence

$$\mathrm{Tr}_N^{ND}\{E_{\mathcal{A}}(z)E_D(\bar{s}-1, \frac{N}{r}z)\} = \mathrm{Tr}_{\frac{N}{r}}^{\frac{ND}{r}}\{E_{\mathcal{A}}(z)E_D(\bar{s}-1, \frac{N}{r}z)\}$$

which is of level $\frac{N}{r}$. Therefore, since by our assumption $f \in S_2^{\text{new}}(\Gamma_0(N))$, for $r|N$ if $r > 1$ we have

$$\left(f, \text{Tr}_N^{ND}\{E_{\mathcal{A}}(z)E_D(\bar{s}-1, \frac{N}{r}z)\}\right) = 0.$$

Now using lemma 3.6 we have

$$(4\pi)^{-s}\Gamma(s)N^{s-1}L_{\mathcal{A}}(f,s) = \left(f, \text{Tr}_N^{ND}\{E_{\mathcal{A}}(z)E_D(\bar{s}-1, Nz)\}\right).$$

This proves the theorem.

3.3 Special values

In this section we calculate $L_{\mathcal{A}}(f,1)$ by computing the Fourier coefficients of Φ_0 defined in theorem 3.8. This calculation has been done in [8] for the case when $D \equiv 3 \pmod 4$. Using the same methods as in [7] we will do the calculation for the special case when D is a prime number. For the more general case we will state the result from [8] without proof. Therefore except in the last theorem, the number D will be assumed to be prime.

By theorem 3.8 we have

$$L_{\mathcal{A}}(f,1) = 4\pi \left(f, \text{Tr}_N^{ND}\{E_{\mathcal{A}}(z)E_D(0, Nz)\}\right)$$

On the other hand by [7, page 154] we have

$$E_D(0,z) = \frac{2\pi}{\sqrt{D}}E(z)$$

where as before

$$E(z) = \sum_{\mathcal{B}\in \mathrm{Pic}(\mathcal{O})} E_{\mathcal{B}}(z).$$

Therefore we have:

Proposition 3.9 *If D is prime, then*

$$L_{\mathcal{A}}(f,1) = \frac{8\pi^2}{\sqrt{D}}(f, G_{\mathcal{A}}),$$

where

$$G_{\mathcal{A}} = \mathrm{Tr}_N^{ND}\{E_{\mathcal{A}}(z)E(Nz)\}.$$

To calculate the Fourier coefficients of $G_{\mathcal{A}}$, first we need some lemmas :

Lemma 3.10 .

1) If $\gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ is in $SL_2(\mathbb{Z})$ and $c \not\equiv 0 (\mathrm{mod}\, D)$, then

$$(E_{\mathcal{A}}|_1\gamma)(z) = \frac{\epsilon(c)}{i\sqrt{D}} E_{\mathcal{A}}(\frac{z + c^*d}{D}),$$

where c^* is an inverse for $c (mod 4)$.

2) If $\gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ is in $\Gamma_0(N)$ and $c \not\equiv 0 (\mathrm{mod}\, D)$, then

$$E(Nz)|_1\gamma = \frac{\epsilon(N)\epsilon(c)}{i\sqrt{D}} E(N(\frac{z + c^*d}{D})).$$

Proof: 1) The D matrices

$$\beta_j = \begin{pmatrix} 0 & -1 \\ 1 & j \end{pmatrix} \qquad j = 0, \ldots, D-1$$

represent all non-trivial right cosets in $\Gamma_0(D)\backslash\Gamma_0(1)$ [12, page 259]. For each $0 \le j < D$ we have

$$\beta_j = \begin{pmatrix} 0 & -1 \\ 1 & j \end{pmatrix} = \frac{1}{D}\begin{pmatrix} 0 & -1 \\ D & 0 \end{pmatrix}\begin{pmatrix} 1 & j \\ 0 & D \end{pmatrix},$$

and hence

$$E_{\mathcal{A}}|_1\beta_j = E_{\mathcal{A}}|_1 \begin{pmatrix} 0 & -1 \\ D & 0 \end{pmatrix}\Bigg|_1 \begin{pmatrix} 1 & j \\ 0 & D \end{pmatrix}.$$

Using the Poisson summation formula(theorem 1.5), it can be shown that

$$E_{\mathcal{A}}|_1 \begin{pmatrix} 0 & -1 \\ D & 0 \end{pmatrix} = \frac{1}{i}E_{\mathcal{A}}.$$

Hence

$$(E_{\mathcal{A}}|_1\beta_j)(z) = \left(\frac{1}{i}E_{\mathcal{A}}|_1 \begin{pmatrix} 1 & j \\ 0 & D \end{pmatrix}\right)(z) = \frac{1}{i\sqrt{D}}E_{\mathcal{A}}(\frac{z+j}{D})$$

which means that 1) holds for β_j, $j = 0, \ldots, D-1$. Now

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & j \end{pmatrix} \tag{3.5}$$

for some $0 \le j < D$ and some $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0(D)$. Hence

$$(E_{\mathcal{A}}|_1\gamma)(z) = \left(E_{\mathcal{A}}|_1 \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)\Bigg|_1 \begin{pmatrix} 0 & -1 \\ 1 & j \end{pmatrix}(z)$$

$$= \epsilon(\delta)E_{\mathcal{A}}|_1 \begin{pmatrix} 0 & -1 \\ 1 & j \end{pmatrix}(z) = \frac{\epsilon(\delta)}{i\sqrt{D}}E_{\mathcal{A}}(\frac{z+j}{D})$$

But from 3.5 we see $c = \delta$ and $-\gamma + cj = -\gamma + \delta j = d$, which means that $cj = d(\text{mod } D)$, since $D|\gamma$. Hence $j = c^* d$. This completes the proof for 1).

2) we have

$$E(Nz)|_1\gamma = \left(E|_1 \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}\right)\Bigg|_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z)$$

$$= E|_1 \begin{pmatrix} Na & Nb \\ c & d \end{pmatrix}(z) = \left(E|_1 \begin{pmatrix} a & Nb \\ c/N & d \end{pmatrix}\right)\Bigg|_1 \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}(z)$$

Now since $E = \sum_B E_B$, from 1) and the above equality we have

$$E(Nz)|_1\gamma = \frac{\epsilon(c/N)}{i\sqrt{D}} E(\frac{Nz + Nc^* d}{D}) = \frac{\epsilon(c/N)\epsilon(c)}{i\sqrt{D}} E(N(\frac{z + c^* d}{D}))$$

as required.

Lemma 3.11 *The $D+1$ cosets of $\Gamma_0(ND)\backslash\Gamma_0(N)$ are represented by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ with $c \not\equiv 0(\text{mod } D)$ and $j = c^* d$ running through the D residue classes in $\mathbb{Z}/D\mathbb{Z}$.*

Proof: $\Gamma_0(N)$ acts transitively on $\mathbb{P}_1(\mathbb{Z}/D\mathbb{Z})$ by

$$(u, v)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (au + cv, bu + dv)$$

for $(u, v) \in \mathbb{P}_1(\mathbb{Z}/D\mathbb{Z})$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. The group $\Gamma_0(ND)$ is the isotropy group of $(0, 1) \in \mathbb{P}_1(\mathbb{Z}/D\mathbb{Z})$. Therefore we have a bijection

$$\Gamma_0(ND)\backslash\Gamma_0(N) \to \mathbb{P}_1(\mathbb{Z}/D\mathbb{Z})$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (0, 1)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (c, d).$$

But

$$(c,d) = \begin{cases} (0,1) & \text{if } c \equiv 0 (\text{mod } D) \\ (1, c^* d) & \text{if } c \not\equiv 0 (\text{mod } D). \end{cases}$$

This proves the lemma.

Now we can prove:

Proposition 3.12

$$G_{\mathcal{A}}(z) = g_{\mathcal{A}}(z) - \frac{\epsilon(N)}{D} \sum_{j=0}^{D-1} g_{\mathcal{A}}(\frac{z+j}{D})$$

where $g_{\mathcal{A}}(z) = E_{\mathcal{A}}(z) E(NZ)$.

Proof: By lemma 3.10 for each coset representative

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

given in Lemma 3.11 we have

$$\begin{aligned} g_{\mathcal{A}}|_2\gamma(z) &= (E_{\mathcal{A}}|_1\gamma)(z)(E(Nz)|_1\gamma) \\ &= \frac{\epsilon(c)}{i\sqrt{D}} E_{\mathcal{A}}(\frac{z+j}{D}) \frac{\epsilon(N)\epsilon(c)}{i\sqrt{D}} E(N(\frac{z+j}{D})) \\ &= -\frac{\epsilon(N)}{D} g_{\mathcal{A}}(\frac{z+j}{D}). \end{aligned}$$

Therefore

$$G_{\mathcal{A}}(z) = \sum_{\gamma \in \Gamma_0(ND) \backslash \Gamma_0(N)} g_{\mathcal{A}}|_2\gamma = g_{\mathcal{A}}(z) - \frac{\epsilon(N)}{D} \sum_{j=0}^{D-1} g_{\mathcal{A}}(\frac{z+j}{D}).$$

Using this proposition, now we prove:

Proposition 3.13 *The Fourier coefficients of $G_{\mathcal{A}} = \sum_{m=0}^{\infty} c_m q^m$ are given by:*

$$c_m = \frac{r_{\mathcal{A}}(m)h}{u} + \sum_{n=1}^{Dm/N} r_{\mathcal{A}}(Dm - nN)\delta'(n)R(n)$$

where $\delta'(n) = \begin{cases} -\epsilon(N) & \text{if } (n,D) = 1 \\ 1 - \epsilon(N) & \text{if } (n,D) \neq 1 \end{cases}$

Proof: Let $g_{\mathcal{A}}(z) = E_{\mathcal{A}}(z)E(Nz) = \sum_{m=0}^{\infty} b_m q^m$. A direct calculation shows that

$$\frac{1}{D}\sum_{j=0}^{D-1} g_{\mathcal{A}}(\frac{z+j}{D}) = \sum_{m=0}^{\infty} b_{mD} q^m.$$

Hence by previous proposition we have

$$c_m = b_m - \epsilon(N) b_{mD}.$$

By the definition of $g_{\mathcal{A}}$ we have

$$b_m = \sum_{l \geq 0} r_{\mathcal{A}}(m - lN)R(l).$$

For $m \geq 1$, we have $r_{\mathcal{A}}(m) = r_{\mathcal{A}}(Dm)$ and hence $R(m) = r(Dm)$. Hence

$$b_m = \sum_{l \geq 0} r_{\mathcal{A}}(mD - lDN)R(lD) = \sum_{\substack{l \geq 0 \\ D|l}} r_{\mathcal{A}}(mD - lN)R(l).$$

Therefore

$$\begin{aligned} c_m &= \sum_{\substack{l \geq 0 \\ D|l}} r_{\mathcal{A}}(mD - lN)R(l) - \epsilon(N)\sum_{l \geq 0} r_{\mathcal{A}}(mD - lN)R(l) \\ &= \sum_{n=0}^{mD/N} r_{\mathcal{A}}(mD - nN)\delta'(n)R(n) \\ &= \frac{r_{\mathcal{A}}(m)h}{u} + \sum_{n=1}^{mD/N} r_{\mathcal{A}}(mD - nN)\delta'(n)R(n)\,, \end{aligned}$$

since $R(0) = \frac{h}{2u}$ by definition.

Theorem 3.14 *With our notations as in section 1, if $D \equiv 3 (\mathrm{mod}\ 4)$ and $\epsilon(N) = -1$, then*

$$L_{\mathcal{A}}(f, 1) = \frac{8\pi^2}{\sqrt{D}} (f, \Phi_{\mathcal{A}}),$$

where

$$\Phi_{\mathcal{A}} = \sum_{m \geq 0} b_{m,\mathcal{A}} q^m,$$

is a modular form of weight 2 and level N, with

$$b_{m,\mathcal{A}} = \frac{r_{\mathcal{A}}(m)h}{u} + \sum_{0 < n \leq mD/N} r_{\mathcal{A}}(mD - nN)\delta(n)R_{[\mathcal{A}\mathfrak{n}]}(n).$$

Here $\mathfrak{n}$ is any integral ideal of $\mathcal{O}$ satisfying

$$\mathcal{N}(\mathfrak{n}) \equiv -N \pmod{D},$$

$[\mathcal{A}\mathfrak{n}]$ is the genus class of the ideal class $\mathcal{A}\{\mathfrak{n}\}$, and $R_{[\mathcal{A}\mathfrak{n}]}(n)$ is the number of integral ideals of $\mathcal{O}$ of norm n in the genus class $[\mathcal{A}\mathfrak{n}]$. Also $\delta(n) = 2^{\lambda_n}$, where λ_n is the number of primes dividing both D and n.

Proof: First let D be prime. Then for $n \geq 0$, $\delta(n) = \delta'(n)$, where $\delta'(n)$ is as defined in proposition 3.13. On the other hand, since in this case there are no elements of order 2 in $\mathrm{Pic}(\mathcal{O})$, there is only one genus class for $\mathcal{O}$. This means that

$$R_{[\mathcal{A}\mathfrak{n}]}(n) = R(n)$$

for all $n \geq 0$. Therefore for D prime the theorem follows from propositions 3.9and 3.13. For the proof in the general case, see [8, proposition IV.5.6].

Chapter 4

Theta series in quaternion algebras

4.1 Notations and basic assumptions

We recall that $N = N^-N^+$ is a positive integer, where $N^- = p_1p_2\dots p_s$ is the product of an odd number of distinct primes and $(N^-, N^+) = 1$. Let K be a quadratic imaginary field of discriminant $-D$ and $\mathcal{O} = \mathcal{O}_K$ the ring of integers of K. We set

$$D_0 = \begin{cases} D/4 & \text{if } D \equiv 0 \pmod{4} \\ D & \text{if } D \equiv 3 \pmod{4}, \end{cases}$$

and

$$\omega = \begin{cases} \sqrt{-D_0} & \text{if } D \equiv 0 \pmod{4} \\ \frac{1+\sqrt{-D}}{2} & \text{if } D \equiv 3 \pmod{4}. \end{cases}$$

Then we have

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}\omega.$$

Let H denote the (unique up to isomorphism) definite quaternion algebra of discriminant $-N^-$. We assume that,

- $p_1, p_2, \ldots, p_s$ are all inert in K. i.e, for $i = 1, 2, \ldots s$,

$$\begin{cases} \left(\frac{-D}{p_i}\right) = -1 & \text{if } p_i \neq 2 \\ -D \equiv 5 (\bmod 8) & \text{if } p_i = 2 \ . \end{cases}$$

- All prime divisors of N^+ are split in K.

4.2 Description of H

Our first goal is to give a concrete description of H which will be useful in doing calculations relative to K. For this we choose a prime $q \neq 2$ such that

$$q \equiv -N^- \, (\bmod D).$$

We need the following

Lemma 4.1 *With notations as above, we have*

$$\left(\frac{-D}{q}\right) = 1.$$

Proof: First suppose that D and N^- are both odd. Then $D \equiv 3 \pmod 4$ and hence $\left(\frac{-1}{D}\right) = -1$ and also $\left(\frac{-D}{N^-}\right) = -1$. Therefore,

$$\begin{aligned}\left(\frac{-D}{q}\right) &= \left(\frac{-1}{q}\right)\left(\frac{D}{q}\right) = \left(\frac{q}{D}\right) = \left(\frac{-N^-}{D}\right)\\ &=(-1)\left(\frac{N^-}{D}\right) = (-1)\left(\frac{-1}{N^-}\right)\left(\frac{D}{N^-}\right)\\ &=(-1)\left(\frac{-D}{N^-}\right) = (-1)(-1) = 1.\end{aligned}$$

Now suppose that $p_1 = 2$. Then by our assumptions $D \equiv 3 \pmod 8$, $\left(\frac{-1}{D}\right) = -1$, and $\left(\frac{-D}{N^-/2}\right) = 1$. Hence

$$\begin{aligned}\left(\frac{-D}{q}\right) &= \left(\frac{-1}{q}\right)\left(\frac{D}{q}\right) = \left(\frac{q}{D}\right) = \left(\frac{-N^-}{D}\right) = \left(\frac{-1}{D}\right)\left(\frac{2}{D}\right)\left(\frac{N^-/2}{D}\right)\\ &=(-1)(-1)\left(\frac{N^-/2}{D}\right) = \left(\frac{-1}{N^-/2}\right)\left(\frac{D}{N^-/2}\right) = \left(\frac{-D}{N^-/2}\right) = 1\end{aligned}$$

Finally, suppose that D is even. Then $D \equiv 0 \pmod 4$. If $D_0 = D/4$ is odd, then $D_0 \equiv 1 \pmod 4$, and we have

$$\begin{aligned}\left(\frac{-D}{q}\right) &= \left(\frac{-D_0}{q}\right) = \left(\frac{-1}{q}\right)\left(\frac{D_0}{q}\right) = \left(\frac{-1}{q}\right)\left(\frac{q}{D_0}\right) = \left(\frac{-1}{q}\right)\left(\frac{-N^-}{D_0}\right)\\ &=\left(\frac{-1}{q}\right)\left(\frac{-1}{D_0}\right)\left(\frac{N^-}{D_0}\right) = \left(\frac{-1}{q}\right)\left(\frac{D_0}{N^-}\right) = -\left(\frac{-1}{N^-}\right)\left(\frac{D_0}{N^-}\right)\\ &= -\left(\frac{-D_0}{N^-}\right) = 1.\end{aligned}$$

Here we have used the facts that $q \equiv -N^- \pmod 4$ which implies $\left(\frac{-1}{q}\right) = -\left(\frac{-1}{N^-}\right)$, and that $\left(\frac{-D_0}{N^-}\right) = \left(\frac{-D}{N^-}\right) = -1$. If $D \equiv 0 \pmod 8$ we set $D'' = D/8$.

Then using $\left(\frac{-D}{N^-}\right) = -1$, $\left(\frac{2}{q}\right) = \left(\frac{2}{N^-}\right)$, and $\left(\frac{-1}{q}\right) = -\left(\frac{-1}{N^-}\right)$, we have

$$\begin{aligned}
\left(\frac{-D}{q}\right) &= \left(\frac{-1}{q}\right)\left(\frac{2}{q}\right)\left(\frac{D''}{q}\right) = \left(\frac{-1}{q}\right)\left(\frac{2}{q}\right)(-1)^{\left(\frac{D''-1}{2}\right)\left(\frac{q-1}{2}\right)}\left(\frac{q}{D''}\right) \\
&=(-1)^{\left(\frac{D''-1}{2}\right)\left(\frac{q-1}{2}\right)}\left(\frac{-1}{q}\right)\left(\frac{2}{q}\right)\left(\frac{-N^-}{D''}\right) \\
&=(-1)^{\left(\frac{D''-1}{2}\right)\left(\frac{q-1}{2}\right)}\left(\frac{-1}{q}\right)\left(\frac{2}{q}\right)\left(\frac{-1}{D''}\right)\left(\frac{N^-}{D''}\right) \\
&=(-1)^{\left(\frac{D''-1}{2}\right)\left(\frac{q-1}{2}+\frac{N^--1}{2}\right)}\left(\frac{-1}{q}\right)\left(\frac{2}{q}\right)\left(\frac{-1}{D''}\right)\left(\frac{D''}{N^-}\right) \\
&=-(-1)^{\left(\frac{D''-1}{2}\right)\left(\frac{q-1}{2}+\frac{N^--1}{2}\right)}\left(\frac{-1}{D''}\right)\left(\frac{-2D''}{N^-}\right) \\
&=-(-1)^{\left(\frac{D''-1}{2}\right)\left(\frac{q-1}{2}+\frac{N^--1}{2}+1\right)}\left(\frac{-D}{N^-}\right) = (-1)^{\left(\frac{D''-1}{2}\right)\left(\frac{q+N^-}{2}\right)} = 1,
\end{aligned}$$

since $q \equiv -N^- \pmod 4$.

Remark: The above lemma is indeed a result of the fact that ϵ is a character with modulus D [3, page 237].

Proposition 4.2 *The definite quaternion algebra H of discriminant $-N^-$ can be written as*

$$H = K + Kj,$$

where $j^2 = -N^-q$, and $\alpha j = j\bar{\alpha}$ for all $\alpha \in K$. Here $(^-)$ denotes the complex conjugation.

Proof: We have

$$K + Kj = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}ij$$

where $i^2 = -D$, $j^2 = -N^-q$, $ij = -ji$. First we note that by section 2.5 the algebra $K + Kj$ is a definite quaternion algebra. i.e., it is ramified at ∞. By theorem 2.14, for any odd prime p, the algebra $K + Kj$ is ramified at p if and only if $(-D, -N^-q)_p = -1$, where for rational numbers a and b,

$$(a,b)_p = \begin{cases} 1 & p \nmid ab \\ \left(\frac{a}{b}\right) & p \nmid a,\ p \parallel b. \end{cases}$$

First we have by our assumptions,

$$(-D, -N^-q)_{p_i} = \left(\frac{-D}{p_i}\right) = -1 \quad \text{for } i = 1, 2, \ldots, s \text{ with } p_i \neq 2 \ .$$

For any prime divisor $p \neq 2$ of D we have $q \equiv -N^- \pmod p$, and hence

$$(-D, -N^-q)_p = \left(\frac{-N^-q}{p}\right) = \left(\frac{(N^-)^2}{p}\right) = 1.$$

Finally by the previous lemma,

$$(-D, -N^-q)_q = \left(\frac{-D}{q}\right) = 1.$$

Therefore

$$\{p : p \mid N^-,\ p \text{ is odd}\} \cup \{\infty\} \subseteq \mathrm{Ram}(K+Kj) \subseteq \{p : p \mid N^-,\ p \text{ is odd}\} \cup \{2, \infty\}.$$

By theorem 2.12 $\mathrm{Ram}(K + Kj)$ has even cardinality. Hence $K + Kj$ is ramified at 2 if and only if $2 \mid N^-$. Therefore $K + Kj$ is ramified exactly at $p_1, \ldots, p_s$ and ∞. This completes the proof of the proposition.

4.3 Maximal orders in H

We wish now to give a concrete description of a fixed maximal order in H.

First we let

$$R_1 = \{\alpha + \beta j : \quad \alpha,\, \beta \in \mathcal{O}_K\} = \mathbb{Z} + \mathbb{Z}\omega + \mathbb{Z}j + \mathbb{Z}\omega j,$$

where

$$\omega = \begin{cases} \sqrt{-D}/2 = \sqrt{-D_0} & \text{if } D \equiv 0 \pmod 4 \\ \frac{1+\sqrt{-D}}{2} & \text{if } D \equiv 3 \pmod 4. \end{cases}$$

Lemma 4.3 *R_1 is a non-maximal order in H with*

$$\operatorname{disc}(R_1) = N^- qD.$$

Proof: It is easily seen that R_1 is closed under multiplication and hence it is an order by definition 2.6. By proposition 2.8 the discriminant of R_1 is

$$\operatorname{disc}(R_1) = |\det\,(Tr e_i \bar{e_j})|^{1/2},$$

where

$$(e_1, e_2, e_3, e_4) = (1, \omega, j, \omega j)$$

is a $\mathbb{Z}$-basis for R_1.

If $D \equiv 0 \pmod 4$,

$$|Tr\,(e_i\bar{e_j})| = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 2D' & 0 & 0 \\ 0 & 0 & 2N^-q & 0 \\ 0 & 0 & 0 & 2N^-qD' \end{vmatrix} = 16D'^2N^{-2}q^2 = D^2N^{-2}q^2.$$

If $D \equiv 3 \pmod 4$,

$$|Tr\,(e_i\bar{e_j})| = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & \frac{D+1}{2} & 0 & 0 \\ 0 & 0 & 2N^-q & N^-q \\ 0 & 0 & N^-q & N^-q\,\frac{D+1}{2} \end{vmatrix} = N^{-2}q^2D^2.$$

Therefore,

$$\mathrm{disc}(R_1) = N^-Dq.$$

By proposition 2.19 every maximal order in H has $\mathrm{disc}(H) = N^-$ as its discriminant. Hence R_1 is not maximal.

Now we define

$$R_{N^-} = \left\{\alpha + \beta j : \quad \alpha \in (\sqrt{-D})^{-1},\ \beta \in (\sqrt{-D})^{-1}\mathfrak{q}^{-1},\ \alpha \equiv q\beta\,(\mathrm{mod}\,\mathcal{O}_K)\right\}$$

where $\mathfrak{q}$ is a prime ideal of $\mathcal{O}_K$ containing q.

To prove that R_{N^-} is a maximal order we need:

Lemma 4.4 *Let (e_1, e_2, e_3, e_4) and (e'_1, e'_2, e'_3, e'_4) be bases for two lattices in H, and let*

$$e'_i = \sum_{i=1}^{4} \alpha_{si} e_s \qquad \text{for} \quad i = 1, 2, 3, 4,$$

where $\alpha_{si} \in \mathbb{Q}$ for $\ i, s = 1, 2, 3, 4$. Then

$$\mathrm{disc}(e'_1, e'_2, e'_3, e'_4) = (\det(\alpha))\,\mathrm{disc}(e_1, e_2, e_3, e_4)$$

where $\alpha = (\alpha_{st})_{\substack{1\leqslant s\leqslant 4\\ 1\leqslant t\leqslant 4}}$.

Proof: We recall that by definition

$$\operatorname{disc}(e_1, e_2, e_3, e_4) = \left|\det\left((Tr(e_i\bar{e_j}))_{\substack{1\leqslant s\leqslant 4\\ 1\leqslant t\leqslant 4}}\right)\right|^{1/2}.$$

We have $e_i'\bar{e_j'} = \sum_{\substack{1\leqslant s\leqslant 4\\ 1\leqslant t\leqslant 4}} \alpha_{si}\alpha_{tj}e_s\bar{e}_t$. Hence

$$Tr\left(e_i'\bar{e_j'}\right) = \sum_{s,t} \alpha_{si}\alpha_{tj}Tr(e_s\bar{e}_t).$$

Thus

$$\left(Tr\left(e_i'\bar{e_j'}\right)\right)_{i,j} = \alpha^t \left(Tr(e_i\bar{e_j'})\right)_{i,j} \alpha.$$

This proves the lemma.

Now we can prove:

Lemma 4.5 *Let*

$$R_{N^-} := \left\{\alpha + \beta j : \quad \alpha \in (\sqrt{-D})^{-1}, \beta \in (\sqrt{-D})^{-1}\mathfrak{q}^{-1}, \alpha \equiv q\beta \,(\operatorname{mod} \mathcal{O}_K)\right\},$$

where

$$\mathfrak{q} = \begin{cases} (q, a+\omega) & \text{if } D \equiv 0 \,(\operatorname{mod} 4) \\ (q, \frac{a-1}{2}+\omega) & \text{if } D \equiv 3 \,(\operatorname{mod} 4), \end{cases}$$

with a odd and $a^2 \equiv -D \,(\operatorname{mod} q)$, is one the prime ideals of $\mathcal{O}_K$ containing q. Then R_{N^-} is a maximal order of H.

Proof: First we assume $D \equiv 0 \,(\operatorname{mod} 4)$. Then we have

$$\left(\sqrt{-D}\right)^{-1} = \frac{1}{2\sqrt{-D_0}}\left(\mathbb{Z} + \mathbb{Z}\sqrt{-D_0}\right) = \frac{1}{2D_0}\left(\mathbb{Z}\sqrt{-D_0} + \mathbb{Z}D_0\right),$$

and

$$\left(\sqrt{-D}\right)^{-1}\mathfrak{q}^{-1} = \frac{1}{2\,q\sqrt{-D_0}}\left(\mathbb{Z}q + \mathbb{Z}(a - \sqrt{-D_0})\right)$$
$$= \frac{1}{2\,q\,D_0}\left(\mathbb{Z}q\sqrt{-D_0} + \mathbb{Z}(a\sqrt{-D_0} + D_0)\right).$$

For $\alpha + \beta j \in R_{N^-}$ we have:

$$\beta = \frac{1}{2\,q\,D_0}(mq\sqrt{-D_0} + n(a\sqrt{-D_0} + D_0))$$
$$\alpha = q\beta + \lambda = q\beta + m' + n'\sqrt{-D_0}$$

where $\lambda \in \mathcal{O}_K$, $m, n, m', n' \in \mathbb{Z}$.

Hence

$$\begin{aligned}\alpha + \beta j =& \lambda + q\beta + \beta j\\ =& m' + n'\sqrt{-D_0} + \frac{1}{2D_0}\left(mq\sqrt{-D_0} + n(a\sqrt{-D_0} + D_0)\right)\\ &+ \frac{1}{2\,q\,D_0}\left(mq\sqrt{-D_0} + n(a\sqrt{-D_0} + D_0)\right)j\\ =& m' + n'\sqrt{-D_0} + \frac{1}{2D_0}m\left(q\sqrt{-D_0} + \sqrt{-D_0}j\right)\\ &+ \frac{1}{2D_0q}n\left(D_0q + a\sqrt{-D_0}q + a\sqrt{-D_0}j + D_0j\right).\end{aligned}$$

Hence R_{N^-} is a lattice in H with the $\mathbb{Z}$-basis $\{e_1, e_2, e_3, e_4\}$, where

$$\begin{aligned}e_1 &= 1\\ e_2 &= \sqrt{-D_0}\\ e_3 &= \frac{1}{2D_0}\left(q\sqrt{-D_0} + \sqrt{-D_0}j\right)\\ e_4 &= \frac{1}{2D_0q}\left(D_0q + aq\sqrt{-D_0} + D_0j + a\sqrt{-D_0}j\right).\end{aligned}$$

Thus using the previous two lemmas we have,

$$\operatorname{disc}(R_{N^-}) = |S| N^- qD,$$

where

$$S = \frac{1}{4D_0^2 q}\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q & 0 & 1 \\ D_0 q & aq & D_0 & a \end{vmatrix} = -\frac{1}{4D_0 q} = -\frac{1}{Dq}.$$

Therefore,

$$\operatorname{disc}(R_{N^-}) = N^-.$$

To prove that R_{N^-} is a maximal order it remains to show that R_{N^-} is closed under multiplication. Let $x = \lambda + q\beta + \beta j$, $y = \lambda' + q\beta' + \beta' j$ be two elements in R_{N^-} where λ, $\lambda' \in \mathcal{O}_K$, $\beta, \beta' \in \left(\sqrt{-D}\right)^{-1} \mathfrak{q}^{-1}$, $\lambda' = m' + n'\sqrt{-D_0}$. Then

$$\begin{aligned} xy \;&= \lambda\lambda' + (q\lambda\beta' + \lambda\beta' j) + q(\beta\beta' j + q\beta\beta') \\ &+ (q\lambda'\beta + \bar{\lambda}'\beta j) + (-N^- q\beta\bar{\beta}' + q\beta\bar{\beta}' j). \end{aligned}$$

We have $\lambda\lambda' \in R_{N^-}$, $\quad q\lambda\beta' + \lambda\beta' j \in R_{N^-}$, and

$$\begin{aligned} q\lambda'\beta + \bar{\lambda}'\beta j &= q\left(\bar{\lambda}' + 2n'\sqrt{-D_0}\right)\beta + \bar{\lambda}'\beta j \\ &= (q\bar{\lambda}'\beta + \bar{\lambda}'\beta j) + 2n'\sqrt{-D_0}q\beta \in R_{N^-}. \end{aligned}$$

Using the fact that $-N = q + cD$ for some $c \in \mathbb{Z}$, we have

$$\begin{aligned} & q(\beta\beta' j + q\beta\beta') + (-N^- q\beta\bar{\beta}' + q\beta\bar{\beta}' j) \\ &= q\beta\beta' j + q^2\beta\beta' + q^2\beta\bar{\beta}' + q\beta\bar{\beta}' j + cDq\beta\bar{\beta}' \\ &= q^2\beta(\beta' + \bar{\beta}') + q\beta(\beta' + \bar{\beta}')j + cDq\beta\bar{\beta}' \end{aligned}$$

since $\beta' + \bar{\beta}' = \frac{n}{q}$, we have $q\beta(\beta' + \bar{\beta}') \in \left(\sqrt{-D}\right)^{-1}\mathfrak{q}^{-1}$. Hence

$$q^2\beta(\beta' + \bar{\beta}') + q\beta(\beta' + \bar{\beta}')j \in R_{N^-}.$$

We also have

$$cDq\beta\bar{\beta}' \in Dq\left(\left(\sqrt{-D}\right)^{-1}\mathfrak{q}^{-1}\right)\left(\left(\sqrt{-D}\right)^{-1}\mathfrak{q}\right) \subseteq \mathcal{O}_K.$$

Therefore,

$$xy \in R_{N^-}$$

Thus we have proved the lemma for the case where $D \equiv 0 \pmod 4$.

The proof in the case where $D \equiv 3 \pmod 4$ is similar. For the disc(R_{N^-}) in this case we have

$$\text{disc}(R_{N^-}) = |S|\text{disc}(R_1) = |S|N^-qD,$$

where,

$$S = \frac{1}{qD^2}\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -q & 2q & -1 & 2 \\ \frac{-a+D}{2}q & aq & \frac{-a+D}{2} & a \end{vmatrix} = \frac{1}{qD}.$$

This completes the proof of the lemma.

4.4 Matrix representations for H

Here we give some matrix representations of H and its localizations which will be useful in our calculations. For this, we consider H as a 2-dimensional vector space on K with$\{1, j\}$ as a basis. The algebra H acts on itself by multiplication on the right. This action gives us an algebra monomorphism

$$\varphi : H \longrightarrow M_2(K),$$

where

$$\varphi(\alpha + \beta j) = \begin{pmatrix} \alpha & \beta \\ -N^{-}q\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

for $\alpha, \beta \in K$.

Let p be a rational prime which splits in K. Then

$$K_p := K \otimes \mathbb{Q}_p := \mathbb{Q}_p + \mathbb{Q}_p\sqrt{-D}.$$

If we fix λ as one of the two roots of $-D$ in $\mathbb{Q}_p$, then we get an algebra isomorphism,

$$K_p \xrightarrow{\simeq} \mathbb{Q}_p \oplus \mathbb{Q}_p$$

$$a + b\sqrt{-D} \longmapsto (a + b\lambda, a - b\lambda)$$

for $a, b \in \mathbb{Q}_p$ [4, theorem 9.1.1].

The conjugation on K_p simply switches the components in the direct sum:

$$\overline{(a, b)} = (b, a),$$

for $a, b \in \mathbb{Q}_p$. Using this we have the $\mathbb{Q}_p$-algebra isomorphism

$$H_p := H \otimes \mathbb{Q}_p = (\mathbb{Q}_p \oplus \mathbb{Q}_p) + (\mathbb{Q}_p \oplus \mathbb{Q}_p)j$$

In the same fashion as for definition of φ above, we can define the $\mathbb{Q}_p$-algebra monomorphism,

$$H_p \longrightarrow M_2(K_p).$$

Composing this with the homomorphism

$$\begin{pmatrix} (a,b) & (c,d) \\ (c',d') & (a',b') \end{pmatrix} \longrightarrow \begin{pmatrix} a & c \\ c' & a' \end{pmatrix}$$

we get an isomorphism:

$$\varphi_p : H_p \longrightarrow M_2(\mathbb{Q}_p).$$

We have proved:

Theorem 4.6 *Let p be a rational prime which splits in K and λ be a fixed root of $-D$ in $\mathbb{Q}_p$. Then the map*

$$\varphi_p : H_p \longrightarrow M_2(\mathbb{Q}_p),$$

given by:

$$\varphi_p(\alpha+\beta j) = \begin{pmatrix} a_1 + a_2\lambda & b_1 + b_2\lambda \\ -N^- q(b_1 - b_2\lambda) & a_1 - a_2\lambda \end{pmatrix}$$

for $\alpha + \beta j \in H_p, (\alpha, \beta \in K_p)$, with

$$\alpha = a_1 + a_2\sqrt{-D},\ \beta = b_1 + b_2\sqrt{-D} \quad (a_1, a_2, b_1, b_2 \in \mathbb{Q}_p),$$

is a $\mathbb{Q}_p$-algebra isomorphism.

4.5 Eichler orders and their ideals

Now we can give a concrete description of an Eichler order in H of level N^+. For this we let $\mathfrak{N}^+$ be an ideal of $\mathcal{O}_K$ of norm N^+. We set

$$R = R_{N^+,N^-}$$
$$:= \left\{\alpha + \beta j : \quad \alpha \in (\sqrt{-D})^{-1},\ \beta \in (\sqrt{-D})^{-1}\mathfrak{q}^{-1}\mathfrak{N}^+,\ \alpha \equiv q\beta \,(\operatorname{mod} \mathcal{O}_K)\right\}.$$

Then we have,

Lemma 4.7 *R is an Eichler order of H contained in R_{N^-} and of level N^+.*

Proof: By proposition 2.17, it is enough to prove that for every rational prime p, if $p \nmid N^+$, then $R_p := R \otimes \mathbb{Z}_p$ is a maximal order in H_p and if $p^e \parallel N^+$, then R_p is an Eichler order of level p^e in H_p. If $p \nmid N^+$ then it is clear that $R_p = (R_{N^-})_p$ is a maximal order of H_p. If $p^e \parallel N^+$, then since p splits in K, by theorem 4.6 we have the isomorphism,

$$R_p \simeq \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ -N^- q q^{-1} N^+ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^e \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix},$$

which means that R_p is an Eichler order of level p^e, by theorem 2.10. Therefore, using proposition 2.17 we get the result.

A left ideal of R is a $\mathbb{Z}$-lattice I in H such that

$$R_l(I) = \{h \in H : hI \subset I\} = R.$$

Given two left ideals I, J of R we define

$$\mathrm{Hom}(I, J) := \{h \in H : Ih \subseteq J\}.$$

We are interested in those left ideals of R which are generated by ideals of $\mathcal{O}$. Indeed for any ideal $\mathfrak{b}$ of $\mathcal{O}$, the product $R\mathfrak{b}$ is a left ideal of R. More precisely,

Lemma 4.8 *For any ideal $\mathfrak{b}$ of $\mathcal{O}$ with $(\mathfrak{b}, \sqrt{-D}) = 1$, we have,*

$$R\mathfrak{b} = \left\{\alpha + \beta j : \quad \alpha \in (\sqrt{-D})^{-1}\mathfrak{b},\ \beta \in (\sqrt{-D})^{-1}\mathfrak{q}^{-1}\mathfrak{N}^{+}\bar{\mathfrak{b}},\ \alpha \equiv q\beta \,(\mathrm{mod}\, \mathcal{O}_K)\right\}.$$

Proof: Let $x = \alpha + \beta j \in R$ where $\alpha \in (\sqrt{-D})^{-1}$, $\beta \in (\sqrt{-D})^{-1}\mathfrak{q}^{-1}\mathfrak{N}^{+}$ and $\alpha = q\beta \,(\mathrm{mod}\, \mathcal{O}_K)$. Then for $b = m + n\sqrt{-D} \in \mathfrak{b}$ we have

$$xb = \alpha b + \beta \bar{b} j,$$

with $\alpha b \in (\sqrt{-D})^{-1}\mathfrak{b}$, $\beta \bar{b} \in (\sqrt{-D})^{-1}\mathfrak{q}^{-1}\mathfrak{N}^{+}\bar{\mathfrak{b}}$ and

$$\begin{aligned}\alpha b - q\beta \bar{b} &= \alpha b - q\beta(b - 2n\sqrt{-D}) \\ &= (\alpha - q\beta)b + 2nq\beta\sqrt{-D} \in \mathcal{O}_K.\end{aligned}$$

Conversely, let $x = \alpha + \beta j$, $\alpha \in (\sqrt{-D})^{-1}\mathfrak{b}$, $\beta = \frac{\mu \bar{b}}{\sqrt{-D}} \in (\sqrt{-D})^{-1}\mathfrak{q}^{-1}\mathfrak{N}^{+}\bar{\mathfrak{b}}$ with $\mu \in \mathfrak{q}^{-1}$, $\bar{b} = m - n\sqrt{-D} \in \bar{\mathfrak{b}}$ and $\alpha = q\beta \,(\mathrm{mod}\, \mathcal{O}_K)$. Then assuming $\alpha = q\beta + \lambda$ with $\lambda \in \mathcal{O}_K$ we have,

$$\begin{aligned}x = \lambda + q\beta + \beta j &= \lambda + q\frac{\mu(b - 2n\sqrt{-D})}{\sqrt{-D}} + \frac{\mu \bar{b}}{\sqrt{-D}} j \\ &= \lambda - 2n\mu' + \left(\frac{\mu'}{\sqrt{-D}} + \frac{\mu}{\sqrt{-D}}\right) b,\end{aligned}$$

with $\mu' = q\mu \in \bar{\mathfrak{q}} \subseteq \mathcal{O}$. Now

$$\frac{\mu'}{\sqrt{-D}} + \frac{\mu}{\sqrt{-D}} j \in R,$$

and

$$\lambda - 2n\mu' = \alpha - \frac{\mu' b}{\sqrt{-D}} \in \mathcal{O} \cap \frac{\mathfrak{b}}{\sqrt{-D}} = \mathfrak{b}.$$

Hence $x \in R\mathfrak{b}$.

This proves the lemma.

Our next step is to give a description for $\mathrm{Hom}(R\mathfrak{b}, R\mathfrak{b}\mathfrak{a})$ for ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathcal{O}_K$. We have

Lemma 4.9 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be split prime ideals of $\mathcal{O}_K$ which are prime to $(\sqrt{-D})$. Then*

$$\mathrm{Hom}(R\mathfrak{b}, R\mathfrak{b}\mathfrak{a}) = \Big\{\alpha + \beta j : \alpha \in (\sqrt{-D})^{-1}\mathfrak{a},\ \beta \in (\sqrt{-D})^{-1}\mathfrak{q}^{-1}\mathfrak{N}^{+}\mathfrak{b}^{-1}\bar{\mathfrak{b}}\bar{\mathfrak{a}},$$
$$\alpha \equiv q\beta \,(\mathrm{mod}\, \mathcal{O}_{\mathfrak{p}}),\ \text{for } p \mid D\Big\}.$$

Proof: Let I be the set in the right hand side of the above equality. By lemma 4.8 we have

$$R\mathfrak{b} = \Big\{\alpha + \beta j : \alpha \in (\sqrt{-D})^{-1}\mathfrak{b},\ \beta \in (\sqrt{-D})^{-1}\mathfrak{q}^{-1}\mathfrak{N}^{+}\bar{\mathfrak{b}},\ \alpha \equiv q\beta \,(\mathrm{mod}\, \mathfrak{b}^{-1})\Big\},$$

and

$$R\mathfrak{b}\mathfrak{a} = \Big\{\alpha + \beta j : \alpha \in (\sqrt{-D})^{-1}\mathfrak{b}\mathfrak{a},\ \beta \in (\sqrt{-D})^{-1}\mathfrak{q}^{-1}\mathfrak{N}^{+}\bar{\mathfrak{b}}\bar{\mathfrak{a}},\ \alpha \equiv q\beta \,(\mathrm{mod}\, \mathfrak{b}^{-1})\Big\}.$$

Let a and b be rational primes such that $(a) = \mathfrak{a}\bar{\mathfrak{a}}$ and $(b) = \mathfrak{b}\bar{\mathfrak{b}}$ in $\mathcal{O}_K$. At any prime $p \neq a, b$ we have

$$(R\mathfrak{b})_p = (R\mathfrak{b}\mathfrak{a})_p = I_p = R_p$$

and hence

$$(\mathrm{Hom}\,(R\mathfrak{b}, R\mathfrak{b}\mathfrak{a}))_p = I_p = R_p.$$

If $a \neq b$, then using theorem 4.6, at a we have,

$$(R\mathfrak{b})_a \cong \begin{pmatrix} \mathbb{Z}_a & \mathbb{Z}_a \\ \mathbb{Z}_a & \mathbb{Z}_a \end{pmatrix}$$

and

$$(R\mathfrak{b}\mathfrak{a})_a \cong \begin{pmatrix} a\mathbb{Z}_a & \mathbb{Z}_a \\ a\mathbb{Z}_a & \mathbb{Z}_a \end{pmatrix}$$

and hence

$$(\mathrm{Hom}\,(R\mathfrak{b}, R\mathfrak{b}\mathfrak{a}))_a \cong \begin{pmatrix} a\mathbb{Z}_a & \mathbb{Z}_a \\ a\mathbb{Z}_a & \mathbb{Z}_a \end{pmatrix} \cong I_a$$

For b we will have,

$$(R\mathfrak{b})_b \cong \begin{pmatrix} b\mathbb{Z}_b & \mathbb{Z}_b \\ b\mathbb{Z}_b & \mathbb{Z}_b \end{pmatrix}$$

and

$$(R\mathfrak{b}\mathfrak{a})_b \cong \begin{pmatrix} b\mathbb{Z}_b & \mathbb{Z}_b \\ b\mathbb{Z}_b & \mathbb{Z}_b \end{pmatrix}.$$

Hence,

$$(\mathrm{Hom}\,(R\mathfrak{b}, R\mathfrak{b}\mathfrak{a}))_b \cong \begin{pmatrix} \mathbb{Z}_b & \frac{1}{b}\mathbb{Z}_b \\ b\mathbb{Z}_b & \mathbb{Z}_b \end{pmatrix} \cong I_b.$$

If $a = b$, then

$$(R\mathfrak{b})_b \cong \begin{pmatrix} b\mathbb{Z}_b & \mathbb{Z}_b \\ b\mathbb{Z}_b & \mathbb{Z}_b \end{pmatrix}$$

and

$$(R\mathfrak{b}^2)_b \cong \begin{pmatrix} b^2\mathbb{Z}_b & \mathbb{Z}_b \\ b^2\mathbb{Z}_b & \mathbb{Z}_b \end{pmatrix}.$$

Thus,

$$(\mathrm{Hom}\,(R\mathfrak{b}, R\mathfrak{b}^2))_b \cong \begin{pmatrix} b\mathbb{Z}_b & \frac{1}{b}\mathbb{Z}_b \\ b^2\mathbb{Z}_b & \mathbb{Z}_b \end{pmatrix} \cong I_b.$$

Now using proposition 2.16 we get the result.

Lemma 4.10 *Let $D' > 0$ be a divisor of D, and $\mathfrak{d}' = (D', \sqrt{-D})$ be the integral ideal of $\mathcal{O}_K$ of norm D'. Then*

$$R\mathfrak{d}' = \Big\{\alpha + \beta j : \ \alpha \in (\sqrt{-D})^{-1}\mathfrak{d}',\ \beta \in (\sqrt{-D})^{-1}\mathfrak{d}'\mathfrak{q}^{-1}\mathfrak{N}^{+},$$

$$\alpha \equiv q\beta (\text{mod } \mathcal{O}_{\mathfrak{p}}) \text{ for } p \mid D/D'$$

$$\text{and } \alpha \equiv -q\beta (\text{mod } \mathfrak{p}\mathcal{O}_{\mathfrak{p}}) \text{ for } p \mid D' \Big\},$$

where for any rational prime $p \mid D$ we set $\mathfrak{p} = (p, \sqrt{-D})$, and $\mathcal{O}_{\mathfrak{p}}$ is the localization of the ring $\mathcal{O}$ at the prime ideal $\mathfrak{p}$.

Proof: Let I be the set in the right hand side of the above equality. By proposition 2.16 it is enough to show that

$$(R\mathfrak{d}')_p = I_p$$

for all rational primes p, where for a lattice L in H we define

$$L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

If $p \nmid D'$, then $(R\mathfrak{d}')_p = I_p = R_p$. Let $p \mid D'$. Then $\mathfrak{d}'_p = p\mathcal{O}_p + \sqrt{-D}\mathcal{O}_p = (\sqrt{-D})\mathcal{O}_p = \mathfrak{p}_p$, and

$$R_p = \Big\{\alpha + \beta j : \quad \alpha,\ \beta \in (\sqrt{-D})^{-1}\mathcal{O}_p; \quad \alpha \equiv q\beta (\text{mod } \mathcal{O}_p)\Big\}.$$

Hence

$$\begin{aligned}(R\mathfrak{d}')_p &= R_p\mathfrak{p}_p = \left\{(\alpha+\beta j)(\sqrt{-D}): \quad \alpha,\ \beta \in \mathfrak{p}_p^{-1}, \quad \alpha \equiv q\beta (\mathrm{mod}\ \mathcal{O}_p)\right\} \\ &= \left\{(\alpha\sqrt{-D}) - (\beta\sqrt{-D})j: \quad \alpha,\ \beta \in \mathfrak{p}_p^{-1}, \quad \alpha \equiv q\beta (\mathrm{mod}\ \mathcal{O}_p)\right\} \\ &= \left\{\alpha+\beta j: \quad \alpha,\ \beta \in \mathcal{O}_p, \quad \alpha \equiv -q\beta (\mathrm{mod}\ \mathfrak{p}_p)\right\} \\ &= I_p\end{aligned}$$

This completes the proof of the lemma.

Definition 4.11 *For any positive divisor D' of D we define*

$$R^{D'} = \left\{\alpha+\beta j:\ \alpha \in (\sqrt{-D})^{-1},\ \beta \in (\sqrt{-D})^{-1}\mathfrak{q}^{-1}\mathfrak{N}^+, \right.$$
$$\left. \alpha \equiv \varepsilon_{D'}(p) q\beta \ (\mathrm{mod}\ \mathcal{O}_{\mathfrak{p}}), \text{for all } p \mid D\right\},$$

where for any prime divisor p of D, we set $\mathfrak{p} = (p, \sqrt{-D})$,

$$\epsilon_{D'}(p) = \begin{cases} 1 & p \mid D/D' \\ -1 & p \mid D', \end{cases}$$

and $\mathcal{O}_{\mathfrak{p}}$ is the localization of $\mathcal{O}$ at the the prime ideal $\mathfrak{p}$.

It is readily seen that $R^1 = R = R_{N^+,N^-}$ is the Eichler order defined before. In fact, we have,

Lemma 4.12 *Let D' be a positive divisor of D. With the above notations,*

i) The lattice

$$R^{D'}_{N^-} = \left\{\alpha+\beta j:\ \alpha \in (\sqrt{-D})^{-1},\ \beta \in (\sqrt{-D})^{-1}\mathfrak{q}^{-1}, \right.$$
$$\left. \alpha \equiv \varepsilon_{D'}(p) q\beta \ (\mathrm{mod}\ \mathcal{O}_{\mathfrak{p}}) \text{ for all } p \mid D\right\}$$

is a maximal order in H.

ii) The lattice $R^{D'}$ is an Eichler order of level N^+ in $R^{D'}_{N^-}$.

Proof: The proof is similar to the proof of lemma 4.5 using a local argument.

Lemma 4.13 *With notations as in the two previous lemmas we have*

$$\mathrm{Hom}(R\mathfrak{d}', R\mathfrak{d}') = R^{D'}.$$

Proof: We prove the lemma using a local argument. Let p be a rational prime. If $p \nmid D'$ then,

$$(R\mathfrak{d}') = R\mathfrak{d}' \otimes \mathbb{Z}_p = R_p$$

and hence

$$(\mathrm{Hom}(R\mathfrak{d}', R\mathfrak{d}'))_p = R_p = (R^{D'})_p.$$

Let $p \mid D'$. Then

$$(R\mathfrak{d}')_p = \{\alpha + \beta j : \ \alpha, \beta \in \mathcal{O}_p, \ \alpha \equiv -q\beta (\mathrm{mod}\ \mathfrak{p}_p)\}$$

and

$$(R^{D'})_p = \{\alpha + \beta j : \ \alpha \in (\sqrt{-D})^{-1}, \ \beta \in (\sqrt{-D})^{-1}, \ \alpha \equiv -q\beta \ (\mathrm{mod}\ \mathcal{O}_p)\}.$$

Let $x = \alpha' + \beta' j \in (R\mathfrak{d}')_p$, $y = \alpha + \beta j \in (R^{D'})_p$. Then

$$xy = (\alpha' + \beta' j)(\alpha + \beta j) = (\alpha\alpha' - Nq\bar{\beta}\beta') + (\alpha'\beta + \bar{\alpha}\beta')j.$$

from $\alpha' \equiv -q\beta'(\mathrm{mod}\ \mathfrak{p}_p)$ and $\alpha \equiv -q\beta(\mathrm{mod}\ \mathcal{O}_p)$, we have $\alpha\alpha' \equiv q^2\beta\beta'(\mathrm{mod}\ \mathcal{O}_p)$ and $q\alpha'\beta \equiv q\alpha\beta'(\mathrm{mod}\ \mathcal{O}_p)$. Hence

$$\alpha\alpha' - Nq\bar{\beta}\beta' \equiv \alpha\alpha' + q^2\bar{\beta}\beta' \equiv \alpha\alpha' - q^2\beta\beta' \equiv 0 \ (\mathrm{mod}\ \mathcal{O}_p),$$

and

$$\alpha'\beta + \bar{\alpha}\beta' \equiv \alpha'\beta - \alpha\beta' \equiv 0 \pmod{\mathcal{O}_p}.$$

We also have

$$\begin{aligned} \alpha\alpha' - Nq\bar{\beta}\beta' + q(\alpha'\beta + \bar{\alpha}\beta') &\equiv \alpha\alpha' + q^2\bar{\beta}\beta' + q(\alpha'\beta + \bar{\alpha}\beta') \equiv \\ \alpha'(\alpha + q\beta) + q\beta'(\overline{\alpha + q\beta}) &\equiv \alpha'(\alpha + q\beta) + q\beta'(\alpha + q\beta) \equiv \\ (\alpha + q\beta)(\alpha' + q\beta') &\equiv 0 \pmod{\mathfrak{p}_p}. \end{aligned}$$

Hence

$$xy \in (R\mathfrak{d}')_p$$

This means that $(R^{D'})_p \subseteq (\mathrm{Hom}(R\mathfrak{d}', R\mathfrak{d}'))_p$. Since $(R^{D'})_p$ is a maximal order, we have

$$(\mathrm{Hom}(R\mathfrak{d}', R\mathfrak{d}'))_p = (R^{D'})_p.$$

Now the lemma follows from proposition 2.16.

4.6 The main identity

We are now ready to state and prove our main theorem. As before, $\mathcal{A}$ is a fixed ideal class of $\mathcal{O}$ and $\mathfrak{a}$ is an integral ideal in $\mathcal{A}$. For any ideal class $\mathcal{B}$ of $\mathcal{O}$ we let $\mathfrak{b}$ be an integral ideal in $\mathcal{B}$. We define the theta series

$$\theta_{\mathcal{A},\mathcal{B}} = \sum_{x \in \mathrm{Hom}(R\mathfrak{b}, R\mathfrak{b}\mathfrak{a})} e^{2\pi i \rho(x)\tau},$$

where for $x \in \mathrm{Hom}(R\mathfrak{b}, R\mathfrak{b}\mathfrak{a})$ the integer $\rho(x)$ which we call the degree of x is defined as

$$\rho(x) = \frac{\mathcal{N}(x)}{\mathcal{N}(\mathrm{Hom}(R\mathfrak{b}, R\mathfrak{b}\mathfrak{a}))}.$$

By [6] $\theta_{\mathcal{A},\mathcal{B}}$ is a modular form of weight 2 and level $N = N^-N^+$, and is independent of the choices of $\mathfrak{a}$ and $\mathfrak{b}$ in the classes $\mathcal{A}$ and $\mathcal{B}$. To make the calculation easier we will choose $\mathfrak{a}$ and $\mathfrak{b}$ to be prime ideals which are split in K. Now we define

$$\theta_{\mathcal{A}} = \sum_{\mathcal{B}} \theta_{\mathcal{A},\mathcal{B}} = \sum_{m \geq 0} r(m) e^{2\pi i m \tau}. \tag{4.1}$$

Our main result (theorem 4.19) will be proved by showing that $\theta_{\mathcal{A}} = 2u^2\Phi_{\mathcal{A}}$, where $\Phi_{\mathcal{A}}$ is the theta series in theorem 3.14. This will be done by computing the Fourier coefficients of $\theta_{\mathcal{A}}$ and comparing with those of $\Phi_{\mathcal{A}}$.

Remark:

i) Theorem 4.19 was proved by Gross in the case where N and D are both prime [7]. A proof for the more general case where D is not necessarily prime has been suggested in [7]. The suggested proof needs to be modified, since the formula on the top of page 162 is incorrect.

ii) Hatcher has given similar formulas for the special values of $L_{\mathcal{A}}(f, s)$ in the case where N and D are prime, and f is of arbitrary weight [9]. Then in [10] she has extended her result to arbitrary D, using Gross' method. Her argument in the last paragraph on page 341 appears to be incomplete. The proof can be completed using an argument similar to the one we use in the proof of proposition 4.17.

For any divisor $D' > 0$ of D we let $\mathfrak{d}' = (D', \sqrt{-D})$ and $\{\mathfrak{d}'\}$ be the ideal class of $\mathfrak{d}'$. We note that

$$\theta_{\mathcal{A}} = \sum_{\mathcal{B} \in Pic(\mathcal{O})} \theta_{\mathcal{A},\ \{\mathfrak{d}'\}\mathcal{B}}.$$

Hence

$$\theta_{\mathcal{A}} = \frac{1}{2^g} \sum_{D'|D} \sum_{\mathcal{B} \in Pic(\mathcal{O})} \theta_{\mathcal{A},\{\mathfrak{d}'\}\mathcal{B}} = \frac{1}{2^g} \sum_{\mathcal{B} \in Pic(\mathcal{O})} \sum_{D'|D} \theta_{\mathcal{A},\{\mathfrak{d}'\}\mathcal{B}},$$

where g is the number of prime divisors of D. This simple observation will prove crucial in our calculations. So we record this as,

Lemma 4.14 *For any divisor $D' > 0$ of D we let $\mathfrak{d}' = (\ D', \sqrt{-D})$ and $\{\mathfrak{d}'\}$ be the ideal class of $\mathfrak{d}'$ in $\mathcal{O}$. For ideal classes $\mathcal{A}$ and $\mathcal{B}$ define*

$$\tilde{\theta}_{\mathcal{A},\mathcal{B}} = \sum_{D'|D} \theta_{\mathcal{A},\{\mathfrak{d}'\}\mathcal{B}} = \sum_{m \geq 0} r_{\mathcal{A},\mathcal{B}}(m) e^{2\pi i m \tau}.$$

Then we have

$$\theta_{\mathcal{A}} = \frac{1}{2^g} \sum_{\mathcal{B} \in Pic(\mathcal{O})} \tilde{\theta}_{\mathcal{A},\mathcal{B}}$$

Now we calculate $r_{\mathcal{A},\mathcal{B}}(m)$ for a given $m \geq 0$ and an ideal class $\mathcal{B}$.

Lemma 4.15 *Let $\mathcal{B}$ be an ideal class of $\mathcal{O}$ and $m \geq 0$ an integer. We have*

$$r_{\mathcal{A},\mathcal{B}}(m) = \sum_{0 \leq n \leq \frac{mAD}{N^-}} t_{\mathfrak{a}}(mAD - N^- n) \delta(n) t_{\mathfrak{q}^{-1}\mathfrak{N}+\mathfrak{b}^{-1}\bar{\mathfrak{b}}\bar{\mathfrak{a}}}\left(\frac{n}{q}\right),$$

where for any ideal I of $\mathcal{O}$ and any rational number s, $t_I(s)$ is the number of elements of norm s in the ideal I, and for any integer $n \geq 0$

$$\delta(n) = \prod_{\substack{p \text{ prime} \\ p|(n,D)}} 2$$

Proof: We let $\mathfrak{a}$ and $\mathfrak{b}$ be integral prime ideals in the classes $\mathcal{A}$ and $\mathcal{B}$ respectively, such that $\mathcal{N}(\mathfrak{a}) = A$ and $\mathcal{N}(\mathfrak{b}) = B$ both split in K. Then

$$\theta_{\mathcal{A},\{\mathfrak{d}'\}\mathcal{B}} = \sum_{x \in \mathrm{Hom}(R\mathfrak{d}'\mathfrak{b}, R\mathfrak{d}'\mathfrak{b}\mathfrak{a})} e^{2\pi i \rho(x) \tau},$$

and

$$\tilde{\theta}_{\mathcal{A},\mathcal{B}} = \sum_{D'|D} \sum_{x \in \mathrm{Hom}(R\mathfrak{d}'\mathfrak{b}, R\mathfrak{d}'\mathfrak{b}\mathfrak{a})} e^{2\pi i \rho(x)\tau}.$$

A local argument using lemma 4.13 shows that

$$\mathrm{Hom}(R\mathfrak{d}'\mathfrak{b}, R\mathfrak{d}'\mathfrak{b}\mathfrak{a}) = \mathrm{Hom}(R^{D'}\mathfrak{b}, R^{D'}\mathfrak{b}\mathfrak{a}) = \Big\{\alpha + \beta j : \alpha \in (\sqrt{-D})^{-1}\mathfrak{a},$$

$$\beta \in (\sqrt{-D})^{-1}\mathfrak{q}^{-1}\mathfrak{N}^{+}\mathfrak{b}^{-1}\bar{\mathfrak{b}}\bar{\mathfrak{a}},\ \alpha \equiv \epsilon_{D'}(p)q\beta \pmod{\mathcal{O}_{\mathfrak{p}}},\ \text{for all } p \mid D\Big\},$$

where for each prime divisor p of D we set $\mathfrak{p} = (p, \sqrt{-D})$. If $x = \alpha + \beta j$ with $\rho(x) = m$ be an element in $\mathrm{Hom}(R^{D'}\mathfrak{b}, R^{D'}\mathfrak{b}\mathfrak{a})$, for some $D' \mid D$, we set $\alpha' = \sqrt{-D}\alpha,\ \beta' \in \sqrt{-D}\beta$. This gives us a solution to the system

$$\begin{cases} \alpha' \in \mathfrak{a} \\ \beta' \in \mathfrak{q}^{-1}\mathcal{N}^{+}\mathfrak{b}^{-1}\bar{\mathfrak{b}}\bar{\mathfrak{a}} \\ \mathcal{N}(\alpha') + N^{-}q\mathcal{N}(\beta') = mAD. \end{cases} \tag{4.2}$$

This is because $\mathcal{N}(\mathrm{Hom}(R^{D'}\mathfrak{b}, R^{D'}\mathfrak{b}\mathfrak{a})) = A$ and hence

$$\rho(x) = \frac{\mathcal{N}(x)}{A} = \frac{\mathcal{N}(\alpha) + qN^{-}\mathcal{N}(\beta)}{A} = m.$$

We need to see how a solution to the system (4.2) contributes to elements in $\mathrm{Hom}(R^{D'}\mathfrak{b}, R^{D'}\mathfrak{b}\mathfrak{a})$, for various divisors D'. We start with a solution (α', β') to the system (4.2), with $q\mathcal{N}(\beta') = n$ and $\mathcal{N}(\alpha') = mAD - N^{-}n$. We set

$$D_1 = (D, n).$$

From $\mathcal{N}(\alpha') + N^{-}q\mathcal{N}(\beta') = mAD$ and the fact that $q = -N^{-} (\mathrm{mod}\ D)$ we have

$$\mathcal{N}(\alpha') \equiv q^2\mathcal{N}(\beta') \pmod{D}.$$

This implies that for any prime divisor p of D,

$$\alpha' = \eta_p q \beta' \, (\mathrm{mod}\ \mathfrak{p}\mathcal{O}_\mathfrak{p}), \tag{4.3}$$

where $\eta_p \in \{1, -1\}$ and $\mathfrak{p} = (p, \sqrt{-D})$. If $p \mid D_1$ then (4.3) is valid for both $\eta_p = 1$ and $\eta_p = -1$. But if $p \mid D/D_1$, then (4.3) is valid only for one choice of η_p. We let

$$D' = \prod_{\substack{p|D/D_1 \\ \eta_p = -1}} p,$$

and $x = \frac{\alpha'}{\sqrt{-D}} + \frac{\beta'}{\sqrt{-D}} j$. Since for any prime divisor p of D we have

$$\frac{\alpha'}{\sqrt{-D}} \equiv \eta_p q \frac{\beta'}{\sqrt{-D}} (\ \mathrm{mod}\ \mathcal{O}_\mathfrak{p}),$$

we have

$$x \in \mathrm{Hom}(R^{D'D''}\mathfrak{b}, R^{D'D''}\mathfrak{b}\mathfrak{a}),$$

for any divisor D'' of D_1. The number of divisors $D'' > 0$ of D_1 is $\delta(n)$. This means that every solution (α', β') to the system (4.2) with $q\mathcal{N}(\beta) = n$, contributes $\delta(n)$ to $r_{\mathcal{A},\mathcal{B}}(m)$. The number of such solutions (α', β') is

$$\begin{cases} t_\mathfrak{a}(mAD) & \text{if} \quad n = 0 \\ t_\mathfrak{a}(mAD - N^- n) t_{\mathfrak{q}^{-1}\mathfrak{N}^+ \mathfrak{b}^{-1}\bar{\mathfrak{b}}\bar{\mathfrak{a}}}(\frac{n}{q}) & \text{if} \quad 0 < n < \frac{mAD}{N^-} \\ t_{\mathfrak{q}^{-1}\mathfrak{N}^+ \mathfrak{b}^{-1}\bar{\mathfrak{b}}\bar{\mathfrak{a}}}(\frac{mAD}{N^- q}) & \text{if} \quad n = \frac{mAD}{N^-} \end{cases}$$

Adding up the contributions to $r_{\mathcal{A},\mathcal{B}}(m)$ from all n, gives the result.

We use the above lemma to prove:

Lemma 4.16 *For $m \geq 0$*

$$r_{\mathcal{A},\mathcal{B}}(m) = 4u^2 \sum_{0 \leq n \leq \frac{mD}{N}} r_\mathcal{A}(mD - Nn) \delta(n) r_{\mathcal{A}\{\mathfrak{q}\mathfrak{N}^{+-1}\}\mathcal{B}^2}(n),$$

where $r_{\mathcal{A}}$ and $r_{\mathcal{A}\{\mathfrak{q}\mathfrak{N}^{+-1}\}\mathcal{B}^2}$ are defined in (1.1), and $\delta(n)$ is as in lemma 4.15.

Proof: With notations as in the previous lemma, from the definition of $t_{\mathfrak{a}}(-)$ and $r_{\mathcal{A}}(-)$ and the proof of proposition 1.2 we have

$$t_{\mathfrak{a}}(mAD - N^{-}n) = 2ur_{\mathcal{A}^{-1}}(\frac{mAD - N^{-}n}{A}) = 2ur_{\mathcal{A}}(\frac{mAD - N^{-}n}{A})$$

where $2u = w$ is the number of units in $\mathcal{O}$. Similarly

$$t_{\mathfrak{q}^{-1}\mathfrak{N}^{+}\mathfrak{b}^{-1}\bar{\mathfrak{b}}\bar{\mathfrak{a}}}(\frac{n}{q}) = 2ur_{\mathcal{A}\{\mathfrak{q}\mathfrak{N}^{+-1}\}\mathcal{B}^2}(\frac{n}{AN^{+}}),$$

since

$$\{\mathfrak{q}^{-1}\mathfrak{N}^{+}\mathfrak{b}^{-1}\bar{\mathfrak{b}}\bar{\mathfrak{a}}\}^{-1} = \mathcal{A}\{\mathfrak{q}^{-1}\mathfrak{N}^{+^{-1}}\}\mathcal{B}^2.$$

Hence, from the previous lemma we have,

$$r_{\mathcal{A},\mathcal{B}}(m) = 4u^2 \sum_{0 \le n \le \frac{mAD}{N^{-}}} r_{\mathcal{A}}(\frac{mAD - N^{-}n}{A})\delta(n) r_{\mathcal{A}\{\mathfrak{q}\mathfrak{N}^{+-1}\}\mathcal{B}^2}(\frac{n}{AN^{+}})$$

If we set $l = \frac{n}{AN^{+}}$, we get the result.

Now we can calculate the coefficients of

$$\theta_{\mathcal{A}} = \sum_{m \ge 0} r(m) e^{2\pi i m \tau}.$$

Proposition 4.17 *For $m \ge 0$,*

$$r(m) = 2ur_{\mathcal{A}}(m)h + 2u^2 \sum_{n \ge 1} r_{\mathcal{A}}(mD - nN)\delta(n) r_{[\mathcal{A}\mathfrak{n}]}(n)$$

where h is the class number of $\mathcal{O}$, $\delta(n)$ is as defined in lemma 4.15, the ideal $\mathfrak{n}$ is any integral ideal of $\mathcal{O}$ with

$$\mathcal{N}(\mathfrak{n}) \equiv -N \pmod{D},$$

$[\mathcal{A}\mathfrak{n}]$ *is the genus class of the ideal class* $\mathcal{A}\{\mathfrak{n}\}$ *and* $r_{[\mathcal{A}\mathfrak{n}]}(n)$ *is the number of integral ideals of* $\mathcal{O}$ *in this genus class.*

Remark: Since any two ideals with the same norm (mod D) are in the same genus, the genus class $[\mathcal{A}\mathfrak{n}]$ in this proposition is independent of the choice of $\mathfrak{n}$, and $r_{[\mathcal{A}\mathfrak{n}]}(n)$ is equal to $R(n)$ or 0 depending on whether or not there is an ideal of norm n in $[\mathcal{A}\mathfrak{n}]$.

We now prove the proposition.

Proof: By lemma4.14,

$$r(m) = \frac{1}{2^g} \sum_{\mathcal{B}\in Pic(\mathcal{O})} r_{\mathcal{A},\mathcal{B}}(m).$$

By lemma 4.16

$$r_{\mathcal{A},\mathcal{B}}(m) = 2ur_{\mathcal{A}}(mD)2^g + 4u^2 \sum_{n\geq 1} r_{\mathcal{A}}(mD - nN)\delta(n) r_{\mathcal{A}\{\mathfrak{q}\mathfrak{N}^{+-1}\}\mathcal{B}^2}(n).$$

Hence

$$r(m) = 2ur_{\mathcal{A}}(mD)h + \frac{4u^2}{2^g} \sum_{n\geq 1} r_{\mathcal{A}}(mD - nN)\delta(n) \sum_{\mathcal{B}\in Pic(\mathcal{O})} r_{\mathcal{A}\{\mathfrak{q}\mathfrak{N}^{+-1}\}\mathcal{B}^2}(n).$$

But

$$\sum_{\mathcal{B}\in Pic(\mathcal{O})} r_{\mathcal{A}\{\mathfrak{q}\mathfrak{N}^{+-1}\}\mathcal{B}^2}(n) = 2^{g-1} r_{[\mathfrak{a}\mathfrak{q}\mathfrak{N}^{+-1}]}(n).$$

Therefore, using the fact that $r_{\mathcal{A}}(mD) = r_{\mathcal{A}}(m)$, we have

$$r(m) = 2ur_{\mathcal{A}}(m)h + 2u^2 \sum_{n\geq 1} r_{\mathcal{A}}(mD - nN)\delta(n) r_{[\mathfrak{a}\mathfrak{q}\mathfrak{N}^{+-1}]}(n).$$

Now we note that

$$[\mathfrak{a}\mathfrak{q}\mathfrak{N}^{+-1}] = [\mathfrak{a}\mathfrak{q}\mathfrak{N}^{+}] = [\mathcal{A}\mathfrak{n}].$$

This is because, clearly $[\mathfrak{N}^{+^{-1}}] = [\mathfrak{N}^+]$, and $[\mathfrak{q}\mathfrak{N}^+] = [\mathfrak{n}]$, since

$$\mathcal{N}(\mathfrak{q}\mathfrak{N}^+) \equiv \mathcal{N}(\mathfrak{n}) \equiv -N \pmod{D}.$$

This complete the proof of the proposition.

Comparing proposition 4.17 and theorem 3.14 we have

Proposition 4.18 *With notations as in (4.1) and theorem 3.14 we have,*

$$\theta_{\mathcal{A}} = 2u^2\Phi_{\mathcal{A}}.$$

Finally we have our main result:

Theorem 4.19 *Let K be a quadratic imaginary field of discriminant $-D \equiv 1 \pmod{4}$, and let $\mathcal{A}$ be an ideal class of K. Let f be a modular form of weight 2 and level $N = N^+N^-$, where N satisfies the conditions stated in section 4.1, and let $L_{\mathcal{A}}(f,s)$ be as defined in (3.1). Then we have*

$$L_{\mathcal{A}}(f,1) = \frac{4\pi^2}{u^2\sqrt{D}}(f,\theta_{\mathcal{A}}),$$

where u is half the number of units in the ring of integers of K, and $\theta_{\mathcal{A}}$ is the theta series associated to K and an Eichler order of level N^+ in the (unique) quaternion algebra of discriminant N^- as defined in (4.1).

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