A CHARACTERIZATION OF THE CATEGORY OF TOPOLOGICAL SPACES

by

DANA I. SCHLOMIUK

A Thesis submitted to the Faculty of Graduate Studies and Research of McGill University in partial fulfilment for the degree of Doctor of Philosophy.

Department of Mathematics,
McGill University.

March 1967

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PREFACE

In his paper "An Elementary Theory of the Category of Sets" [6], F. W. Lawvere has shown that by starting from the primitive notions of category theory, the notions of set theory can be recaptured. He obtained a characterization of the category of sets as a complete category satisfying a finite set of elementary axioms.

In this thesis we consider an analogous problem for the category of topological spaces and continuous mappings.

The first chapter is devoted to the construction of an elementary theory of this category. We introduce the notions of discrete space, subspace, open subspace, and we state elementary axioms involving these notions. An important role in defining "open subspace" is played by an axiom which implies the existence of an object with three endomorphisms.

In Chapter II we prove a metatheorem which gives a characterization of the category of topological spaces. This metatheorem says that any complete category, satisfying our elementary system is equivalent to the topological category. The proof shows that the traditional construction of this category, its translation in Lawvere's system and our axiomatization lead essentially to the same thing. An alternative and direct proof is indicated. The role played by the final elementary axiom of our system in obtaining an equivalence of categories appears clear from this proof. This axiom is more complex in character than the others. When it is deleted, a full embedding into the category of topological spaces is obtained.

I wish to express my gratitude to Professor B. Rattray for his interest in my work, fruitful remarks and many interesting conversations.

I also wish to thank S. Baron for discussions related to the topic.

Above all, I should like to thank my director of research, Professor J. Lambek, for introducing me to the subject of category theory, for his constant encouragement, patience and the time he generously gave me while writing the manuscript.

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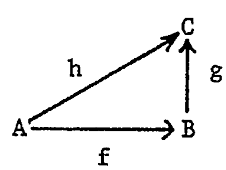
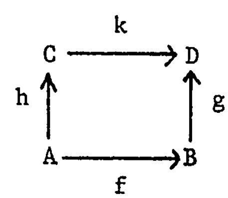
PRELIMINARIES

The notion of abstract category in the sense of Eilenberg-MacLane [1] is assumed to be known. We note that the definition of this notion can be formalized [7], using a language with one sort of variable symbols (the mappings), two unary function symbols (domain and co-domain) and one ternary relation symbol (composition). Moreover, all the axioms which form this definition are elementary, i.e. all quantifiers range over individual variables.

We shall use the notation of Lawvere. An object is defined as being a mapping which is also a domain or co-domain. In order to distinguish objects from general mappings, we shall denote them by capital letters. The symbol $A \xrightarrow{f} B$ stands for: "f is a mapping whose domain is A and whose co-domain is B". Whenever we have a pair of mappings f, g such that $A \xrightarrow{f} B \xrightarrow{g} C$, the mapping resulting from the composition of f with g will be denoted by fg.

Diagrams such as





are said to be commutative if h = fg, respectively fg = hk.

Familiarity is assumed with the basic notions of category theory as encountered in the standard texts [2], [8], [9]. Another prerequisite is [6].

We shall use the following notations, where the equality symbol is actually an abuse of language. The expressions containing it are to be interpreted as follows:

$k = Eq(f,g)$ means: k is an equalizer* of f with g

$q = Coeq\,(f,g)$ means: q is a coequalizer of f with g

$A = \sum_{j \in J} A_j$ means: there exists a family of mappings

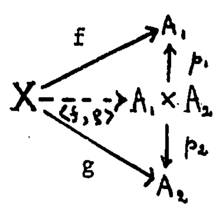
$\left\{A_j \xrightarrow{i_j} A\right\}_{j \in J}$ such that $A, \left\{i_j\right\}_{j \in J}$ is a sum.

$A = \prod_{j \in J} A_j$ means: there exists a family of mappings

$\left\{A \xrightarrow{p_j} A_j\right\}_{j \in J}$ such that $A, \left\{p_j\right\}_{j \in J}$ forms a product.

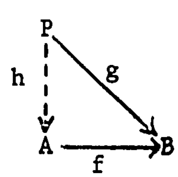
(In the case of two objects A_1, A_2 we write $A = A_1 + A_2$ respectively $A = A_1 \times A_2$).

For every pair of mappings $A_1 \xrightarrow{f} X$, $A_2 \xrightarrow{g} X$ the unique mapping for which the diagram below commutes will be denoted by $\langle f,g \rangle$.



Every equalizer of a pair of mappings will be called a regular monomorphism and every coequalizer will be called a regular epimorphism [5].

An object P is projective if for every epimorphism $A \xrightarrow{f} B$ and mapping $P \xrightarrow{g} B$ there exists a mapping h such that the diagram below is commutative.



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* Sometimes the word kernel is used in place of equalizer.

An object G is a generator if for every pair of mappings $A \underset{g}{\overset{f}{\rightrightarrows}} B$ such that $f \neq g$ there exists a mapping $G \xrightarrow{h} A$ such that $hf \neq hg$.

We shall denote by $\mathcal{S}$ the category of sets and by $\mathcal{T}$ the category of topological spaces and continuous maps.

CHAPTER I

AN ELEMENTARY THEORY OF THE CATEGORY OF TOPOLOGICAL SPACES

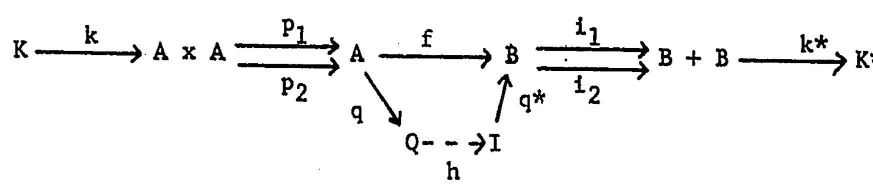
In what follows we list a number of elementary axioms, all of which hold for $\mathcal{T}$ and we prove from them a number of theorems.

The first group of axioms will be formed by those which define an abstract category.

The next group of axioms is a part of Lawvere's elementary system for the category of sets [6] .

Axiom 1. There exists an initial and a terminal object, every pair of objects has a product and a sum, every pair of mappings has an equalizer and a coequalizer.

The above axiom implies the existence of pullbacks and pushouts. Also, for every mapping $A \xrightarrow{f} B$ we can carry out the following construction



where $k = Eq(p_1 f,\ p_2 f)$; $k^* = Coeq\ (fi_1,\ fi_2)$

$q = Coeq\ (kp_1,\ kp_2)$; $q^* = Eq\ (i_1 k^*,\ i_2 k^*)$

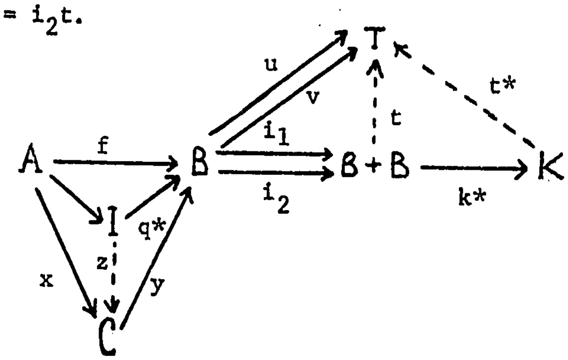
and h is the unique mapping for which the above diagram is commutative.

The mappings q, q* are called respectively, the regular coimage of f and the regular image of f. We write q = Coim f and q* = Im f.

We shall use the following property of q*:

Proposition 1. The regular image of a mapping f is the smallest regular monomorphism through which f factors, i.e. if f factors through a regular monomorphism y, then there exists a mapping z such that $q^* = zy$.

Proof: Suppose that f is a mapping $(A \xrightarrow{f} B)$ such that $f = xy$ and $y = Eq(u, v)$. Let t be the unique mapping in the diagram below such that $u = i_1 t$, $v = i_2 t$.



Since $fi_1 t = fu = xyu = xyv = fi_2 t$, there exists a mapping t* such that $k^* t^* = t$. Therefore, $q^* u = q^* i_1 t = q^* i_1 k^* t^* = q^* i_2 k^* t^* = q^* i_2 t = q^* v$. But $y = Eq(u, v)$, hence there exists a mapping z such that

$zy = q^*$ q.e.d.

<u>Corollary</u>. f is a regular monomorphism if and only if f = Im f.

To simplify our notations, following Lawvere, we shall make the following assumption.

<u>Assumption.</u> There exists a unique initial and a unique terminal object. The initial object will be denoted by 0 and the terminal object by 1.

<u>Definition 1.</u> x is an element of A, written $x \in A$, if and only if $1 \xrightarrow{x} A$.

<u>Axiom 2.</u> The object 1 is a projective generator.

We note that an immediate consequence of this axiom is that if A has exactly one element, A = 1.

The fact that 1 is projective means that the epimorphisms are onto mappings. In the system of axioms given by Lawvere for the category of sets this assumption was redundant since it followed from the axiom of choice. Lawvere's axiom of choice does not hold for $\mathcal{T}$ but in $\mathcal{T}$ epimorphisms are onto mappings, so we included this in our axiom.

The following proposition is an immediate consequence of Axiom 2.

Proposition 2. Let f be a mapping with domain A and co-domain B. Then:

i) f is an epimorphism if and only if for every $x \in B$ there exists a $y \in A$ such that $yf = x$.

ii) f is a monomorphism if and only if for every pair of elements x, y of A such that $x \neq y$, we have $xf \neq yf$.

Definition 2. A mapping f is a bijection if and only if it is both an epimorphism and a monomorphism.

Definition 3. A mapping $A \xrightarrow{f} B$ is called a constant mapping if there exists a $u \in B$ such that $A \xrightarrow{f} B = A \longrightarrow 1 \xrightarrow{u} B$.

Axiom 3. Every non-zero object has elements.

We remark that for every oject $A \neq 0$ the mapping $A \xrightarrow{t} 1$ is an epimorphism since by the above axiom there exists an $x \in A$ such that $xt = 1$. This implies that if $A \neq 0$; $x, y \in B$ and $x \neq y$, the constant mappings $A \longrightarrow 1 \xrightarrow{x} B$, $A \longrightarrow 1 \xrightarrow{y} B$ are different.

<u>Axiom 4.</u> Every element of a sum A+B can be factored through one of the two injections i_A, i_B, i.e. if $x \in A+B$ there exists a t such that $x = ti_A$ or $x = ti_B$.

<u>Axiom 5.</u> There exists an object with more than one element.

<u>Definition 4.</u> $2 = 1 + 1$.

The following propositions are immediate consequences of the axioms [6].

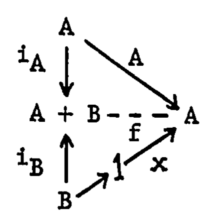
<u>Proposition 3.</u> 0 has no element.

<u>Proposition 4.</u> The two injections i_o and i_1, $1 \overset{i_o}{\underset{i_1}{\rightrightarrows}} 2$ are different and they are the only elements of 2.

<u>Proposition 5.</u> If $x \in A + B$, then x cannot be factored through both injections i_A and i_B, i.e. at most one of the equations $x = ti_A$ and $x = ti_B$ has a solution t.

<u>Proposition 6.</u> The injections i_A and i_B are monomorphisms.

Proof: If $A = 0$, this is clear. If $A \neq 0$, by Axiom 3 there exists $x \in A$. Let f be the unique mapping for which the diagram below commutes.



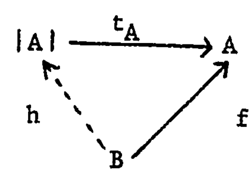
Then $i_A f = A$ and hence i_A is a monomorphism. Similarly i_B is a monomorphism q.e.d.

The next group of axioms involve the notion of discrete space which we define as follows:

Definition 5. An object A is called a discrete space if for every $x \in A$ there exists a mapping $A \xrightarrow{f_x} 2$ such that for every $y \in A$, $y \neq x$, we have $xf_x \neq yf_x$.

Clearly the objects 0, 1 are discrete spaces. Also 2 is discrete since 2 has only two elements and the identy on 2 satisfies the property of f in the above definition for each one of them.

Axiom 6. For every object A, there exists a discrete space $|A|$ together with a mapping $|A| \xrightarrow{t_A} A$ such that for every discrete space B and mapping $B \xrightarrow{f} A$, there exists a unique mapping h for which the diagram below is commutative.



Since $|A|$, t_A is a solution of a universal mapping problem, the object $|A|$ is defined up to isomorphism and the operation $|\ \ |$ is functorial. (The mapping $|A| \xrightarrow{t_A} A$ is a reflection [9] of A into the discrete spaces, and the functor $|\ \ |$ is an adjoint of the inclusion functor.)

If A is a discrete space, the mapping A satisfies the property of t_A in the above axiom and hence we write $|A| = A = t_A$.

Proposition 7. The mapping $|A| \xrightarrow{t_A} A$ is a bijection.

Proof: By Proposition 2, in order to show that t_A is a monomorphism, it is sufficient to prove that for every pair x, y of elements of A such that $xt_A = yt_A$, we have x = y. This follows from Axiom 6 since 1 is discrete and hence the mapping xt_A (= yt_A) must factor uniquely through t_A. t_A is also an epimorphism. This follows from Proposition 2, the fact that 1 is discrete and Axiom 6.

<u>Corollary.</u> The functor $| \ |$ is faithful, i.e. for every pair of mappings $A \xrightarrow{f} B$, $A \xrightarrow{g} B$ such that $|f| = |g|$, we have f = g.

Proof: If $|f| = |g|$ then $t_A f = |f| t_B = |g| t_B = t_A g$ and since t_A is an epimorphism, f = g.

In the following theorem we study some properties of the discrete spaces.

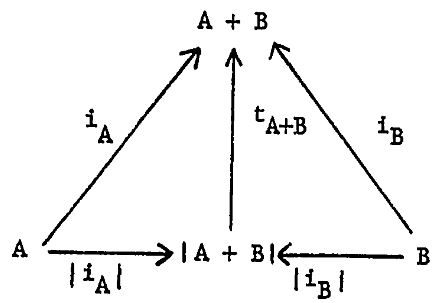
<u>Theorem 1.</u> a) If $A \xrightarrow{m} B$ is a monomorphism and B is discrete, then A is discrete.

b) If A and B are discrete, then A + B is discrete.

c) If $B \xrightarrow{h} Q$ is a regular epimorphism and B is discrete, then Q is discrete.

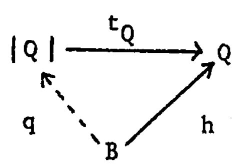
Proof: a) Suppose $x \in A$ and consider u = xm. Then $u \in B$ and since B is discrete, there exists a mapping $B \xrightarrow{f_u} 2$ such that for every $v \in B$, $v \neq u$ we have $vf_u \neq uf_u$. Let $f_x = mf_u$. If $y \in A$ and $y \neq x$, since m is a monomorphism, we have $ym \neq xm$ (= u) and hence $yf_x = ymf_u \neq xmf_u = xf_x$. q.e.d.

b) Since A and B are discrete, we have $A = |A| = t_A$, $B = |B| = t_B$. Consider the pair $|A + B|$, t_{A+B} defined by Axiom 6.



Since A + B is a sum, there exists a unique mapping $A + B \xrightarrow{u} |A + B|$ such that $i_A u = |i_A|$, $i_B u = |i_B|$ and hence $i_A u t_{A+B} = i_A$, $i_B u t_{A+B} = i_B$. Therefore, $ut_{A+B} = A + B$ which implies that u is a monomorphism. By a), since |A + B| is discrete, we have that A + B is discrete.

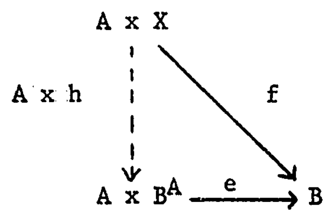
c) Suppose h = Coeq (f, g) where $A \overset{f}{\underset{g}{\rightrightarrows}} B$. Since B is discrete, by Axiom 6 there exists a unique mapping q for which the diagram below is commutative.



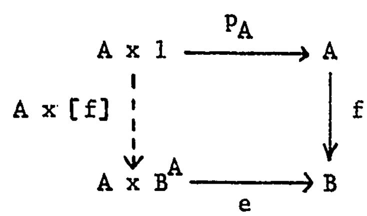
Then $fqt_Q = fh = gh = gqt_Q$ and since t_Q is a monomorphism it follows that fq = gq. This implies the existence of a unique mapping $Q \xrightarrow{u} |Q|$ such that hu = q. Therefore, $hut_Q = qt_Q = h$ and since h is an epimorphism $ut_Q = Q$. It follows that u is a monomorphism and since Q is discrete, by a), Q is discrete. q.e.d.

The category of discrete topological spaces, being isomorphic to the category of sets, satisfies Lawvere's axioms [6]. It is therefore natural to inquire which among them are already satisfied by the "discrete spaces" (Definition 5) and to formulate the remaining ones for such objects. A quick look at the axioms for the category of sets tells us that we need to add the axioms of exponentiation, infinity and choice.

<u>Axiom 7.</u> For every discrete spaces A and B, there exists a discrete space B^A and a mapping $A \times B^A \xrightarrow{e} B$ (called the evaluation mapping) such that for every discrete space X and mapping $A \times X \xrightarrow{f} B$, there exists a unique mapping $X \xrightarrow{h} B^A$ for which the diagram below is commutative.

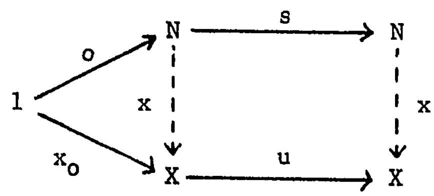


The elements of B^A are in one-to-one correspondence with the mappings from A to B. This consequence of the above axiom is obtained by taking X = 1 and remarking that $A \times 1 \xrightarrow{p_A} A$ is an isomorphism. Using Lawvere's notation, we shall denote by $[f] \in B^A$ the unique mapping for which the diagram below commutes.



The operation B^A can be extended to a functor which is contravariant in the exponent. For every pair of mappings $A' \xrightarrow{f} A$ and $B \xrightarrow{g} B'$, the action of the induced mapping $B^A \xrightarrow{g^f} B'^{A'}$ on $[u] \in B^A$ is $[u](g^f) = [fug]$.

Axiom 8. There exists a discrete space N together with mappings $1 \xrightarrow{o} N$ and $N \xrightarrow{s} N$ such that for every discrete space X, $x_o \in X$ and mapping $X \xrightarrow{u} X$, there exists a unique mapping $N \xrightarrow{x} X$ for which the diagram below commutes.



Axiom 9. If A and B are discrete spaces and A has at least one element, then for every mapping $A \xrightarrow{f} B$, there exists a mapping g such that fgf = f.

Theorem Schema. If Ø is a theorem of the elementary theory of the category of sets and Ø is obtained from Ø by replacing "object" with "discrete space", then Ø is a theorem in our system.

Proof: It suffices to prove the theorem for the case when Ø is an axiom. In the case of Axiom 1 this follows from Theorem 1 and the fact that the functor | |, being an adjoint, preserves products and equalizers. In the case of Axiom 7 [6], this follows from Theorem 1 and Axiom 4. The fact is clear in all the remaining cases. q.e.d.

For the development of our theory we shall need the following propositions obtained from the above schema.

Proposition 8. For every pair of discrete spaces A, B a mapping $A \xrightarrow{f} B$ is an isomorphism if and only if it is a bijection.

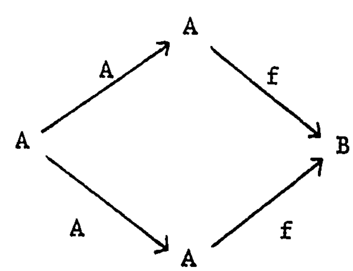
Proposition 9. If f is a mapping with discrete domain and co-domain, and f = qhq* is the standard factorization of f, then h is an isomorphism.

The above proposition follows from the Theorem schema (∅ is taken to be the factorization theorem in Lawvere's system) and the observation that q is actually the regular discrete coimage of f and q* is the regular discrete image of f.

<u>Proposition 10.</u> Every monomorphism $A \xrightarrow{m} D$ into a discrete space D has a complement, i.e. there exists a monomorphism $A' \xrightarrow{m'} D$ such that D, m, m' form a sum. Moreover m has a characteristic function $D \xrightarrow{\varphi_m} 2$, i.e. $m = Eq\,(\varphi_m,\ D \longrightarrow 1 \xrightarrow{i_1} 2)$.*

<u>Proposition 11.</u> A mapping f is a monomorphism (epimorphism) if and only if f is a monomorphism (epimorphism).

Proof: Since | | is faithful, it reflects monomorphisms and epimorphisms [9] , i.e. if f is mono (epi) then so is f. Since | | is an adjoint functor, it must preserve monomorphisms. This follows from the fact that an adjoint functor preserves pull-back diagrams [9] and f is a monomorphism if and only if the diagram below is a pull-back.



It remains to show that | | preserves epimorphisms. Suppose f is an epimorphism. By Proposition 2, in order to show that | f | is epi it will suffice to prove that | f | is surjective. Both t_A and f are

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* We assume that a fixed labelling of the injections into 2 has been chosen, i.e. $1 \underset{i_1}{\overset{i_0}{\rightrightarrows}} 2$.

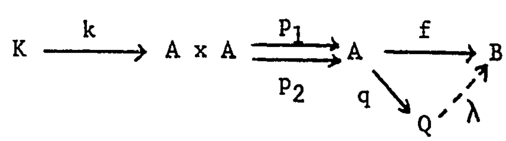
surjective and hence for every $y \in |B|$, there exists $x \in |A|$ such that $xt_A f = yt_B \in B$. Therefore $x\,|f|\,t_B = xt_A f = yt_B$ and since t_B is a monomorphism $x\,|f| = y$. q.e.d.

<u>Corollary.</u> For every discrete space D and bijection $D \xrightarrow{d} A$ we have $d \cong t_A$.

Proof: Since D is discrete, $D = |D| = t_D$ and hence $D \xrightarrow{d} |A|$. The above proposition together with the fact that d is a bijection implies that $|d|$ is a bijection. But D and $|A|$ are discrete and hence by Proposition 8, $|d|$ is an isomorphism. q.e.d.

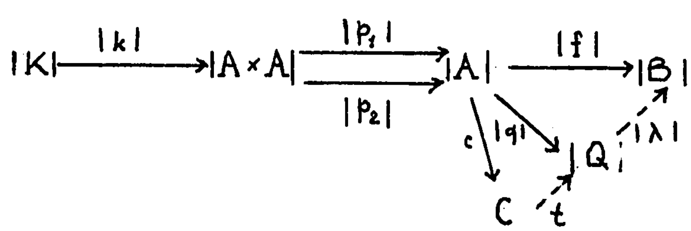
<u>Theorem 2.</u> The functor $|\ |$ preserves regular coimages.

Proof: Consider the construction of the regular coimage q of a mapping $A \xrightarrow{f} B$.



(A x A, p_1, p_2 form a product, $k = \mathrm{Eq}\,(p_1 f,\ p_2 f)$, $q = \mathrm{Coeq}\,(kp_1,\ kp_2)$ and λ denotes the unique mapping for which $q\lambda = f$).

Let us apply to the above construction the functor $|\ |$. Since $|\ |$ preserves products and kernels, the regular (discrete) coimage of $|f|$ is a mapping $c = \mathrm{Coeq}\,(|kp_1|,\ |kp_2|)$. Since $|kp_1 q| = |kp_2 q|$, there exists a unique mapping t such that $ct = |q|$. We obtain the following diagram.



We want to show that t is an isomorphism. For this, since C, $|Q|$ are discrete, by Proposition 8 it will suffice to show that t is a bijection. Clearly q is an epimorphism. By Proposition 11, $|q|$ is an epimorphism and hence so is t. It remains to show that t is a monomorphism. Suppose $x, y \in C$ and $xt = yt$. We shall prove that $x = y$.

Since c is an epimorphism by Proposition 2, there exist $u \in |A|$, $v \in |A|$ such that $uc = x$, $vc = y$. Consider $z = \langle u, v\rangle \in |A \times A|$. Then $z\,|p_1 f| = u\,|f| = uct\,|\lambda| = xt\,|\lambda| = yt\,|\lambda| = vct\,|\lambda| = v\,|f| = z\,|p_2 f|$. Since $|k| = \mathrm{Eq}\,(|p_1 f|, |p_2 f|)$, there exists a $w \in |K|$ such that $w|k| = z$. But $c = \mathrm{Coeq}\,(|kp_1|, |kp_2|)$ and hence we have $x = w|kp_1|c = w|kp_2|c = y$. q.e.d.

Corollary. If q is the regular coimage of f, then the mapping λ such that $f = q\lambda$ is a monomorphism.

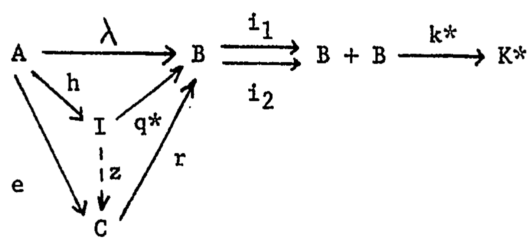
Proof: By the above theorem, q is a regular (discrete) coimage of $|f|$. This together with Proposition 9 implies that $|\lambda|$ is a monomorphism. q.e.d.

It may easily be seen that the dual of the above corollary also holds in $\mathcal{T}$. This follows from the fact that the forgetful functor from $\mathcal{T}$ into $\mathcal{S}$ preserves regular images and reflects epimorphisms. In order to carry out this informal argument in our system we should first show that $|\;\;|$ preserves regular images. It is the author's feeling that this cannot be proved from the axioms given so far. We therefore add the following axiom, which is a weaker form of the statement dual to the above corollary.

Axiom 10. Every monomorphism can be factored into an epimorphism followed by a regular monomorphism.

Proposition 12. If q is the regular coimage of f, q^* is the regular image of f, and h is the unique mapping such that $f = qhq^*$, then h is a bijection.

Proof: By the above corollary the mapping $\lambda = hq^*$ is a monomorphism and hence h is a monomorphism. Since q is a epimorphism, clearly q^* is a regular image of λ. By the above axiom $\lambda = er$ where e is an epimorphism and r is a regular monomorphism. According to Proposition 1 there exists a z such that $q^* = zr$.

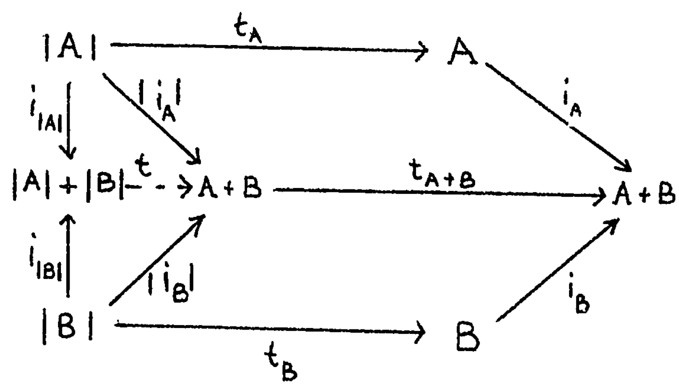


On the other hand $eri_1k^* = \lambda i_1k^* = \lambda i_2k^* = eri_2k^*$ and since e is an epimorphism, we have $ri_1k^* = ri_2k^*$. But $q^* = Eq(i_1k^*, i_2k^*)$. Therefore there exists a mapping w such that $wq^* = r$. From this and $q^* = zr$ it follows that $wzr = r$ and $zwq^* = q^*$. Since r and q^* are monomorphisms we have $wz = C$ and $zw = I$. Consequently h is an epimorphism and this completes the proof.

Corollary. If q^* is the regular image of f and λ is the unique mapping for which $f = \lambda q^*$, then λ is an epimorphism.

Theorem 3. The functor $|\ |$ preserves sums.

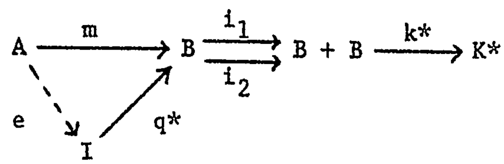
Proof: Suppose $A + B$, i_A, i_B form a sum of A, B and let $|A| + |B|$, $i_{|A|}$, $i_{|B|}$ be a sum of $|A|$, $|B|$. By Theorem 1 b) $|A| + |B|$ is discrete. Let t be the unique mapping such that $i_{|A|}t = |i_A|$ and $i_{|B|}t = |i_B|$.



We shall prove that t is an isomorphism and hence $|A + B|$, $|i_A|$, $|i_B|$ form a sum. Since both $|A| + |B|$ and $|A + B|$ are discrete, by Proposition 8 it will suffice to show that t is a bijection. In order to prove that t is surjective consider $x \in |A + B|$. Then $xt_{A+B} \in A + B$ and hence by Axiom 4, xt_{A+B} factors through one of the injections i_A, i_B. Suppose $xt_{A+B} = ui_A$, $u \in A$. Then $x = |xt_{A+B}| = |u||i_A| = |u|i_{|A|}t$ and $|u|i_{|A|} \in |A| + |B|$ which proves that t is surjective. t is also injective for suppose $x, y \in |A| + |B|$ and $xt = yt$. By Axiom 4 both x and y must factor through one of the injections into $|A| + |B|$. Because of symmetry we need only consider two of the four possible cases. If $x = ui_{|A|}$ and $y = vi_{|B|}$ then $ut_Ai_A = ui_{|A|}tt_{A+B} = xtt_{A+B} = ytt_{A+B} = vi_{|B|}tt_{A+B} = vt_Bi_B$. Let $w = ut_Ai_A = vt_Bi_B$. Then $w \in A + B$ and w factors through both i_A and i_B which contradicts Proposition 5. This leaves us with the case $x = ui_{|A|}$ and $y = vi_{|A|}$. Then $u|i_A| = ui_{|A|}t = xt = yt = vi_{|A|}t = v|i_A|$. But i_A is a monomorphism because it is an injection into a sum and hence $|i_A|$ is a monomorphism. This implies that u = v and hence x = y. q.e.d.

<u>Theorem 4.</u> The functor | | preserves regular images.

Proof: We only need to prove the theorem for mappings which are monomorphisms. The general case will follow from Theorem 2 and its corollary. So suppose $A \xrightarrow{m} B$ is a monomorphism. Consider the construction of the regular image of m:



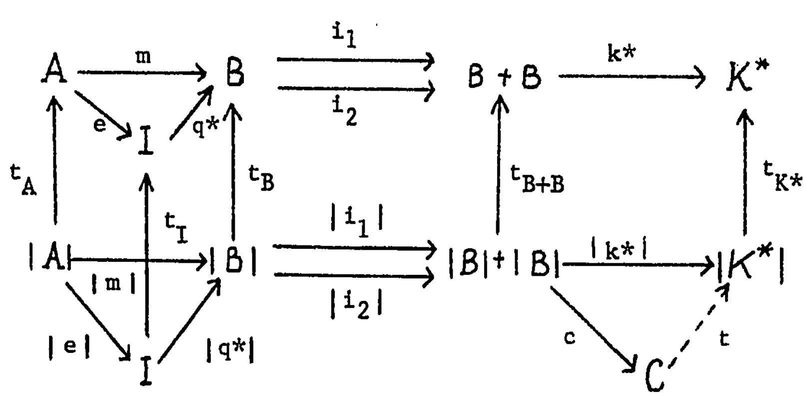
$k^* = \text{Coeq}(mi_1, mi_2)$

$q^* = \text{Eq}(i_1k^*, i_2k^*)$

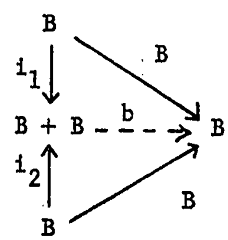
$m = eq^*$

By the corollary of Proposition 12 the mapping e is an epimorphism. Let us apply to the above construction the functor | | . Since | | preserves kernels and by the preceding theorem $|B + B| = |B| + |B|$, in order to show that $|q^*|$ is the regular image of $|m|$ it will suffice to prove that $k^* = \text{Coeq}(|mi_1|, |mi_2|)$.

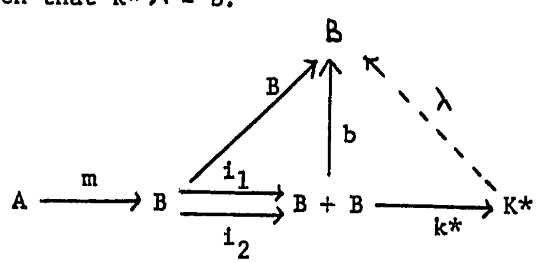
Let $|B| + |B| \xrightarrow{c} C$ be the coequalizer of $|mi_1|$, $|mi_2|$. By Theorem 1 c) C is discrete. Let t be unique mapping for which $ct = |k^*|$.



We shall prove that t is an isomorphism by showing that t is a bijection. Since $ct = |k^*|$ and $|\ \ |$ preserves epimorphisms, t is an epimorphism. t is also a monomorphism because t is injective. To see this consider $x, y \in C$ such that $xt = yt$. Since c is an epimorphism there exists $x_1, y_1 \in |B|+|B|$ such that $x_1c = x$, $y_1c = y$. Let $z = x_1t_{B+B}$, $w = y_1t_{B+B}$; $z, w \in B + B$. Then $zk^* = x_1t_{B+B}k^* = x_1|k^*|t_{K^*} = x_1ctt_{K^*} = xtt_{K^*}$ and $wk^* = y_1t_{B+B}k^* = y_1ctt_{K^*} = ytt_{K^*}$. Since $xt = yt$ we have that $zk^* = wk^*$. By Axiom 4, z and w must factor through one of the injections, i_1, i_2. From the four possible situations we need only consider two since the other two can be treated similarly. Suppose $z = ui_1$ and $w = vi_2$ where $u, v \in B$. Let b be the unique mapping for which the diagram below commutes.



$mi_1b = m = mi_2b$ and since $k^* = \text{Coeq}\ (mi_1, mi_2)$, there exists a unique mapping λ such that $k^*\lambda = b$.



This implies that $u = ui_1b = zb = zk^*\lambda = wk^*\lambda = wb = vi_2b = v$, and hence $ui_1k^* = zk^* = wk^* = vi_2k^* = ui_2k^*$. Since $q^* = \text{Eq}\ (i_1k^*, i_2k^*)$, it follows that there exists a $u_1 \in I$ such that $u = u_1q^*$. Since e is surjective, there exists $a \in A$ such that $ae = u_1$ and hence $am = aeq^* = u$. Therefore $x = x_1c = |z|c = |ui_1|c = |a||m||i_1|c$ and $y = y_1c = |w|c = |ui_2|c = |a||m||i_2|c$. But $c = \text{Coeq}\,(|mi_1|, |mi_2|)$, so $|mi_1|c = |mi_2|c$ $\therefore$ $x = y$.

Consider now the case $z = ui_1$, $w = vi_1$. We have $zk^* = wk^*$ and hence $u = ui_1b = ui_1k^*\lambda = zk^*\lambda = wk^*\lambda = vi_1b = v$. Therefore $z = w$ which implies $x = y$. q.e.d.

In $\mathcal{T}$ a subspace of a topological space (X, τ) is a topological space (A, τ_A) where A is a subset of X and τ_A is the topology induced by τ on A. The inclusion mapping i is continuous and clearly the map $(A, \tau_A) \xrightarrow{i} (X, \tau)$ is a monomorphism in $\mathcal{T}$. Hence, to every subspace of (X, τ) we can associate a monomorphism defined on it and with codomain (X, τ). Not every monomorphism into (X, τ) has as domain a subspace (or a space isomorphic to a subspace) of (X, τ). For example consider the space (A, δ) where δ is the discrete topology on A. If we denote by j the inclusion map $A \subseteq X$, then j is continuous, the map $(A, \delta) \xrightarrow{j} (X, \tau)$ is a monomorphism in $\mathcal{T}$ but (A, δ) is not necessarily a subspace of (X, τ). This shows that the notion of monomorphism is not restrictive enough to define "subspace" in our system. The notion which appears to be suitable for this purpose is that of extremal monomorphism and was introduced by Isbell [3] .

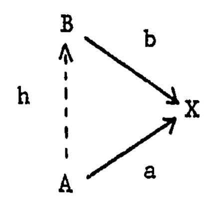
Definition 6 (Isbell). A monomorphism f is called extremal if for every factorization f = em where e is an epimorphism and m a monomorphism, e must be an isomorphism.

It can easily be seen that in $\mathcal{T}$ the monomorphisms associated with subspaces are extremal and every extremal monomorphism has its domain isomorphic to a subspace. This justifies the following definition.

<u>Definition 7.</u> A mapping is called a subspace of an object X, written $a \subseteq X$, if a is an extremal monomorphism whose co-domain is X.

Clearly X, $0 \longrightarrow X$ and every $x \in X$ are subspaces of X. Also for every discrete space D and monomorphism $A \xrightarrow{m} D$, m is a subspace of D.

<u>Definition 8.</u> A mapping a is a subspace of a mapping b, written $a \subseteq b$, if a and b have the same co-domain X, $a \subseteq X$, $b \subseteq X$ and there exists a mapping h such that the diagram below commutes.



Clearly the mapping h above is a subspace of B.

The use of $\subseteq$ for a more general situation creates no ambiguity. If $a \subseteq b$ and a is an element we write $a \in b$.

The following theorem will provide an alternative definition for subspaces.

<u>Theorem 5.</u> A monomorphism is regular if and only if it is extremal.*

Proof: Suppose a is a regular monomorphism, i.e. there exists f and g such that a = Eq (f, g) and assume that a = em where e is an epimorphism and m is a monomorphism. Then emf = af = ag = emg and since e is epi, mf = mg. Since a = Eq (f, g) there exists a v such that m = va.

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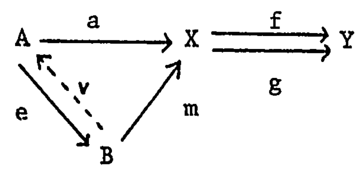
* It follows from the proof of the theorem that regular always implies extremal. The proof of this implication was given by Isbell [3] .

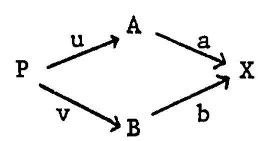


Therefore m = vem and since m is mono ve = B. Similarly ev = A and hence e is an isomorphism which proves that e is extremal.

Suppose now that a is an extremal monomorphism. By Proposition 12, a = qhq* where q is the regular coimage of a, q* is the regular image of a and h is a bijection. Let e = qh. Then a = eq* where e is an epimorphism and q* is a monomorphism. Since a is extremal, e must be an isomorphism which implies that a is regular. q.e.d.

We shall use the above theorem to prove some properties for the subspaces.

<u>Definition 9.</u> If $a \subseteq X$, $b \subseteq X$ and the diagram below is a pull-back diagram,

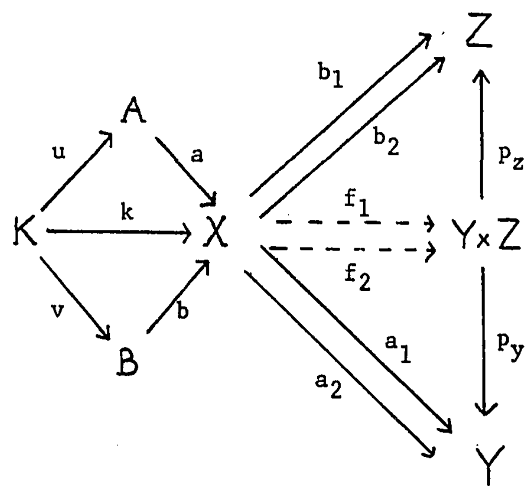


then the mapping ua (= vb) is called the intersection of a with b and is denoted by $a \cap b$.

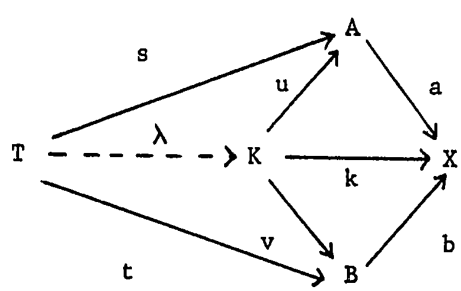
<u>Theorem 6.</u> The intersection of two subspaces of an object X is a subspace of X. In fact, if a = Eq (a_1, a_2) and b = Eq (b_1, b_2), then $a \cap b$ = Eq $(\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle)$.

Proof: Suppose a = Eq (a_1, a_2) and b = Eq (b_1, b_2) and consider k = Eq $(\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle)$. We shall prove that $k = a \cap b$.

For this we need two mappings u and v such that u, v, a, b form a pull-back diagram and $k = ua$. Let $f_1 = \langle a_1, b_1 \rangle$ and $f_2 = \langle a_2, b_2 \rangle$.

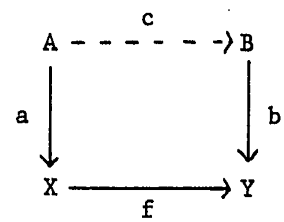


Since $ka_1 = kf_1p_y = kf_2p_y = ka_2$ there exists a mapping u such that $k = ua$. Similarly $kb_1 = kb_2$ and hence $k = vb$ for some v. If for two mappings $T \xrightarrow{s} A$ and $T \xrightarrow{t} B$ we have that $sa = tb$, then $saf_1p_y = saa_1 = saa_2 = saf_2p_y$ and $saf_1p_z = sab_1 = tbb_1 = tbb_2 = sab_2 = saf_2p_z$. Since Y x Z, p_y, p_z form a product it follows that $saf_1 = saf_2$. But $k = Eq(f_1, f_2)$. Therefore there exists a unique mapping λ such that $sa = \lambda k$.



Hence $sa = \lambda ua$ and since a is a monomorphism $s = \lambda u$. Also $tb = sa = \lambda k = \lambda vb$ and since b is a monomorphism $t = \lambda v$ which proves that $k = a \cap b$.

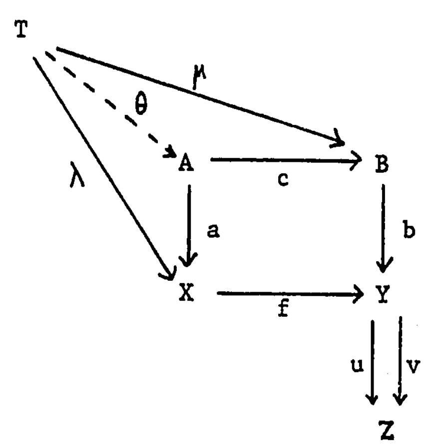
Definition 10. If $b \subseteq Y$ and $X \xrightarrow{f} Y$ then an inverse image of b through f is a mapping $A \xrightarrow{a} X$ with the property that there exists a mapping c such that the diagram below is a pull-back diagram.



Whenever a is an inverse image of b, we write $a = f^{-1}(b)$.

Theorem 7. If $b \subseteq Y$ and $X \xrightarrow{f} Y$, then the inverse image of b through f is a subspace of X. In fact, if $b = Eq\ (u, v)$ then $f^{-1}(b) = Eq\ (fu, fv)$.

Proof: Let $a = f^{-1}(b)$. Since b is a monomorphism, a is a monomorphism [9] . We shall prove that $a = Eq\ (fu, fv)$. Clearly $afu = cbu = cbv = afv$. Suppose λ is a mapping such that $\lambda fu = \lambda fv$. Since $b = Eq\ (u, v)$, there exists a mapping μ for which $\lambda f = \mu b$. This implies the existence of a unique θ with the properties $\lambda = \theta a$ and $\mu = \theta c$.



It follows that a = Eq (fu, fv). q.e.d.

<u>Proposition 13.</u> Suppose $a \subseteq X$, b is a monomorphism with co-domain X and $|b| \subseteq |a|$. Then there exists a mapping u such that b = ua.

Proof: Since $a \subseteq X$ there exists f, g such that a = Eq (f, g). $|b| \subseteq |a|$ means that there exists a mapping d with the property $|b| = d|a|$. Therefore $|bf| = |b||f| = d|a||f| = d|af|$ and $|bg| = |b||g| = d|a||g| = d|ag|$. But af = ag and hence $|bf| = |bg|$. By the faithfulness of $|\ |$ we must have bf = bg. Therefore there exists a u such that b = ua. q.e.d.

<u>Proposition 14.</u> If $b \subseteq X$ and $a \subseteq X$, then $b \subseteq a$ if and only if, for every $x \in X$, $x \in b$ implies $x \in a$.

Proof: We remark that this proposition is a theorem in Lawvere's system and hence, by the Theorem scheme, for discrete spaces it is also provable from our axioms. Suppose that for every $x \in X$, if $x \in b$, then $x \in a$. Then for every $x \in |X|$, $x \in |b|$ implies $x \in |a|$. By the above remark we must have $|b| \subseteq |a|$, which together with Proposition 13 implies the existence of a u such that b = ua. Since a and b are subspaces of X, we have $b \subseteq a$. The opposite implication is obvious.

We shall now investigate the relationship between the regular subobjects* of an object X and the subobjects of $|X|$.

Consider the restriction of the functor $|\ |$ to the subspaces of an object X. Since $|\ |$ preserves monomorphisms, $|\ |$ carries the

* By a (regular) subobject of X we mean an equivalence class of (regular) monomorphisms into X.

subspaces of X into monomorphisms with co-domain $|X|$. If $u \subseteq X$, $v \subseteq X$ and $u \cong v$ (i.e. $u \subseteq v$ and $v \subseteq u$) then $|u| \subseteq |v|$. Hence $|\ \ |$ carries a regular subobject of X into a subobject of $|X|$.

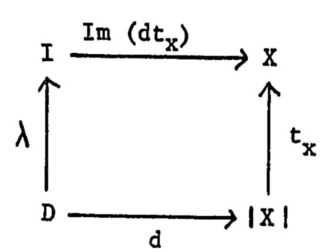
Consider the mapping Imf, i.e. the regular image of a mapping f. Clearly $\mathrm{Imf} \subseteq X$. In particular if $D \xrightarrow{d} |X|$ is a monomorphism and $|X| \xrightarrow{t_X} X$ is the mapping defined by Axiom 6, then $\mathrm{Im}\ (dt_X) \subseteq X$. Consider the application $\mathrm{Im}\ (-t_X)$ from the monomorphisms $|X|$ to the subspaces of X. If d and d' are monomorphisms with co-domain $|X|$ such that $d \cong d'$ then by Proposition 13 $\mathrm{Im}\ (dt_X) \cong \mathrm{Im}\ (d't_X)$. Therefore $\mathrm{Im}\ (-t_X)$ carries a subobject of $|X|$ into a regular subobject of X.

Let us make the convention to write $\lambda = t_X$ whenever λ is a mapping satisfying the property of t_X in Axiom 6.

The following proposition shows that the applications $|\ \ |$ and $\mathrm{Im}\ (-t_X)$ are inverse to one another.

<u>Proposition 15.</u> For every object X, discrete space D and monomorphism $D \xrightarrow{d} |X|$, we have $|\mathrm{Im}\ (dt_X)| = d$. If $a \subseteq X$, then $\mathrm{Im}\ (|a|t_X) = a$.

Proof: By the corollary of Proposition 12, $dt_X = \lambda q^*$ where $q^* = \mathrm{Im}\ (dt_X)$ and λ is an epimorphism. Since both d and t_X are monomorphisms, λ is also a monomorphism. Therefore λ is a bijection.



Since D is discrete and λ is a bijection, by the corollary of Proposition 11, λ satisfies the property of t_I (Axiom 6) and hence we may write

$\lambda = t_I$. It follows that $|I| = D$ and hence $\mathrm{Im}|(dt_x)| = d$. Suppose now that a is a subspace of X. By the above result $\mathrm{Im}\,(|a|t_x) = |a|$ which immediately implies that $\mathrm{Im}\,(|a|t_x) = a$.

We shall next define "open subspaces" in our system. The notion of open set is primitive in topology. In order to recapture this notion in our system, we shall enrich our language by adding to it a new undefined term namely an individual constant which we shall denote by o.

__Definition 11.__ E is the co-domain of the mapping o. Our next axiom will make precise the meaning of o.

__Axiom 11.__ The object E has exactly three endomorphisms and o is an element of E.

$\mathcal{T}$ is a model of the above axiom because we can interpret E as a space with two points o, 1 and three open sets (the empty set and the sets $\{o\}$ and $\{o, 1\}$). Clearly this space has exactly three endomorphisms, i.e. the identity and the two constant mappings. We shall interpret o as the open point of this space.

__Proposition 16.__ The object E has two elements.

Proof: Clearly $E \neq 0$, $E \neq 1$ and hence E has at least two elements. It follows from Axiom 3 that two different elements of E generate different constant endomorphisms of E. Also if t is the mapping from E into 1, then $tx \neq E$ for all $x \in E$ (otherwise $E = 1$). This together with the fact that E has only three endomorphisms, implies that E has exactly two elements. q.e.d.

We shall denote by c the element of E different from o. In $\mathcal{T}$ there is a one-to-one correspondence between the open subsets of a topological space X and the mappings $X \longrightarrow E$ namely, for every open subset A of X we consider the mapping

$$\Psi_A(x) = \begin{cases} o & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases}$$

Clearly the correspondence $A \longrightarrow \Psi_A$ is one-to-one. It is also onto because if $X \xrightarrow{\Psi} E$ is a continuous mapping then $A = \Psi^{-1}(o)$ is open and $\Psi_A = \Psi$. This shows that the following definition of "open subspaces" is adequate.

<u>Definition 12.</u> A subspace a of X is called open if there exists a mapping $X \xrightarrow{\Psi_a} E$ such that $a = \mathrm{Eq}\,(\Psi_a, X \longrightarrow 1 \xrightarrow{o} E)$.

Clearly the mappings $0 \longrightarrow X$ and X are open subspaces of X and o is an open subspace of E.

The following proposition says that every mapping is "continuous".

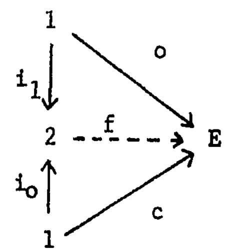
<u>Proposition 17.</u> For every mapping $X \xrightarrow{f} Y$, if b is an open subspace of Y, then the inverse image of b through f is an open subspace of X.

Proof: Since b is open in X there exists a mapping Ψ_b such that $b = \mathrm{Eq}\,(\Psi_b, Y \longrightarrow 1 \xrightarrow{o} E)$. Let $a = f^{-1}(b)$ and $\Psi_a = f\Psi_b$. Then by Theorem 6, $a = \mathrm{Eq}\,(f\Psi_b, X \xrightarrow{f} Y \longrightarrow 1 \xrightarrow{o} E) = \mathrm{Eq}\,(\Psi_a, X \longrightarrow 1 \xrightarrow{o} E)$ and hence a is open. q.e.d.

<u>Proposition 18.</u> Every monomorphism with discrete co-domain D is an open subspace of D.

Proof: Let $a \subseteq D$. By Proposition 10, a has a characteristic function

φ_a, i.e. a = Eq (φ_a, $D \longrightarrow 1 \xrightarrow{i_1} 2$). It follows that $a = \varphi^{-1}(i_1)$. We shall prove that i_1 is open and hence by Proposition 17, a is open. Let f be the unique mapping for which the diagram below commutes.

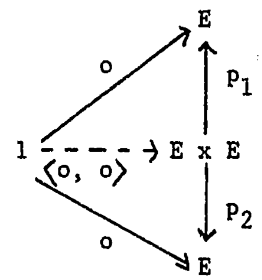


Clearly $f^{-1}(o) = i_1$ and since o is open, by Proposition 17 i_1 is open. q.e.d.

Having defined the notion of open subspace, we now want to make each object X into a "topological space". Since the open subspaces of X are in a one-to-one correspondence with the mappings $X \longrightarrow E$, we must make sure that we have enough such mappings to guarantee the properties satisfied by the open subsets of a topological space. For instance, the intersection of two arbitrary open subsets must be open. We shall see later that by postulating the existence of some mapping of E x E into E, the above property can be proved in the resulting system of axioms.

The following observation motivates our next axiom. Consider the product E x E, p_1, p_2. In $\tilde{\mathcal{T}}$ the interpretation of E x E is the product space. The open subsets of this space are: the empty set, the total set, $p_1^{-1}(o) = \{(o, o), (o, 1)\}$, $p_2^{-1}(o) = \{(o, o), (1, o)\}$ and the set $p_1^{-1}(o) \cap p_2^{-1}(o) = \{(o, o)\}$. Since E x E must have five open subspaces, we should have five mappings from E x E into E. The axioms stated so far guarantee the existence of only four such mappings, i.e. $E \times E \underset{p_2}{\overset{p_1}{\rightrightarrows}} E$ and $E \times E \longrightarrow 1 \underset{c}{\overset{o}{\rightrightarrows}} E$. We need an additional mapping corresponding to the open set $p_1^{-1}(o) \cap p_2^{-1}(o) = \{(o,o)\}$. For this reason we add to the system the following axiom.

<u>Axiom 12.</u> The unique mapping $\langle o, o\rangle$ for which the diagram below commutes is an open subspace of E x E.

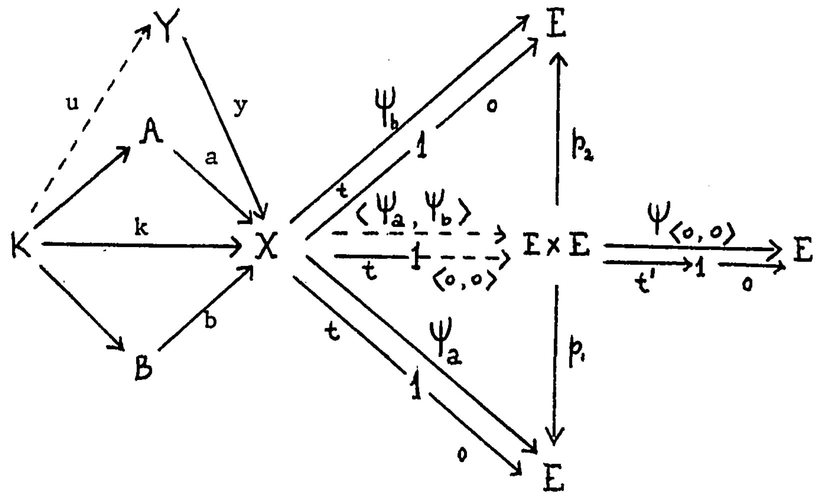


<u>Theorem 8.</u> The intersection of two open subspaces of an object X is an open subspace of X.

Proof: Suppose that a and b are open subspaces of X. This means that there exists mappings $X \overset{\Psi_a}{\underset{\Psi_b}{\rightrightarrows}} E$ such that the diagrams below are equalizer diagrams.

$$A \xrightarrow{a} X \overset{\Psi_a}{\underset{t}{\rightrightarrows}} 1 \xrightarrow{o} E \qquad\qquad B \xrightarrow{b} X \overset{\Psi_b}{\underset{t}{\rightrightarrows}} 1 \xrightarrow{o} E$$

Clearly $\langle to, to\rangle = t\langle o, o\rangle$. This together with Theorem 6 implies $a \cap b = Eq(\langle\Psi_a, \Psi_b\rangle, t\langle o, o\rangle)$. Let $k = a \cap b$. We must show that k is open and hence we seek a mapping $X \xrightarrow{\Psi_k} E$ such that $k = Eq(\Psi_k, to)$. Since by Axiom 12 $\langle o, o\rangle$ is open, there exists a mapping $\Psi_{\langle o, o\rangle}$ such that $\langle o, o\rangle = Eq(\Psi_{\langle o, o\rangle}, E \times E \xrightarrow{t'} 1 \xrightarrow{o} E)$.
Let $\Psi_k = \langle\Psi_a, \Psi_b\rangle\Psi_{\langle o, o\rangle}$.



$k\Psi_k = k\langle \Psi_a, \Psi_b \rangle \Psi_{\langle o,\ o \rangle} = kt\langle o,\ o \rangle \Psi_{\langle o,\ o\rangle} = kto.$

Suppose $y\Psi_k = yto$ for some mapping $Y \xrightarrow{y} X$. Therefore $y\langle \Psi_a, \Psi_b \rangle \Psi_{\langle o,o \rangle}$

$= yto = y\langle \Psi_a, \Psi_b \rangle t'o$ and since $\langle o,\ o \rangle = Eq\ (\Psi_{\langle o,\ o\rangle}, t'o)$,

$y\langle \Psi_a, \Psi_b \rangle = Y \longrightarrow 1 \xrightarrow{\langle o,o \rangle} E \times E.$ But $Y \longrightarrow 1 = yt$, so $y\langle \Psi_a, \Psi_b \rangle =$

$yt\langle o,o \rangle$ and, since $k = Eq\ (\langle \Psi_a, \Psi_b \rangle, t\langle o,o \rangle)$, there exists a unique u such that $uk = y$ which proves that $k = Eq\ (\Psi_k, to)$. q.e.d.

In order to define a "topology" on each object X by open subspaces, an arbitrary union of such subspaces must be open in X. Since this property is not elementary, we shall define the "topology" of X by an "interior operation" on the subspaces of X.

In $\mathcal{T}$ the interior of a subset A of X is the largest open subset of X contained in A. This definition has an elementary counterpart in our system and it makes sense once we know that for every $a \subseteq X$ we do have a largest open subspace of X contained in a. The next axiom guarantees exactly this.

Axiom 13. For every $a \subseteq X$, there exists an open subspace $a°$ of X such that:

i) $a^\circ \subseteq a$

ii) for every open subspace b of X such that $b \subseteq a$, we have $b \subseteq a^\circ$.

Clearly a° is determined up to an isomorphism. If a is open we may write $a^\circ = a$. Consequently $(a^\circ)^\circ = a^\circ$ and $X^\circ = X$. By the definition of a° we also have $a^\circ \subseteq a$. Hence, to show that "$\circ$" is an "interior operation" we need only verify that it is distributive with respect to intersection.

<u>Proposition 19.</u> If $a \subseteq X$, $b \subseteq X$, then $(a \cap b)^\circ = a^\circ \cap b^\circ$.

Proof: a° and b° are open subspaces of X and hence by Theorem 8 $a^\circ \cap b^\circ$ is open. According to the definition of intersection $a^\circ \cap b^\circ \subseteq a^\circ$ and $a^\circ \cap b^\circ \subseteq b^\circ$. Also $a^\circ \subseteq a$ and $b^\circ \subseteq b$. Hence $a^\circ \cap b^\circ \subseteq a$ and $a^\circ \cap b^\circ \subseteq b$. This implies that $a^\circ \cap b^\circ \subseteq a \cap b$. We shall show that $a^\circ \cap b^\circ$ is the largest open subspace of $a \cap b$ and hence $a^\circ \cap b^\circ = (a \cap b)^\circ$. Suppose that c is an open subspace of X such that $c \subseteq a \cap b$. Clearly $c \subseteq a$, $c \subseteq b$ and hence (since c is open), $c \subseteq a^\circ$ and $c \subseteq b^\circ$. This implies that $c \subseteq a^\circ \cap b^\circ$. q.e.d.

In the system of axioms stated so far every object has a "topology" and every mapping $X \xrightarrow{f} Y$ is "continuous" with respect to the topologies on X and Y. We now need an axiom to the effect that we have enough "continuous" mappings in the system. Since we may think of $| \ |$ as being a forgetful functor built into the system, any mapping $|X| \xrightarrow{d} |Y|$ may be interpreted as a set mapping. Supposing d is continuous with respect to the topologies of X and Y, i.e. d^{-1} carries open subsets of Y into open subsets of X, d must be the set-mapping of some mapping from X into Y. This forms the content of our next axiom.

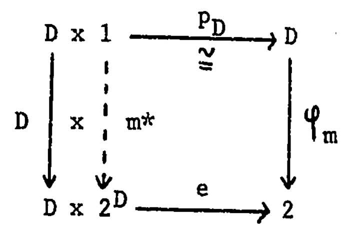
<u>Axiom 14.</u> Let X and Y be objects and $|X| \xrightarrow{d} |Y|$ a mapping with the following property: whenever v is an open subspace of Y and u is a subspace of X such that $d^{-1}(|v|) = |u|$, then u is also open. Then there exists a mapping $X \xrightarrow{f} Y$ such that $|f| = d$.

We note that since $|\ \ |$ preserves pull-back diagrams we have $|f^{-1}(v)| \cong |u|$ and hence by Proposition 13, $f^{-1}(v) \cong u$.

We also remark that by the faithfulness of $|\ \ |$, the mapping f, whose existence is postulated in the above axiom, is unique.

Our final elementary axiom will grant the existence of enough "topological spaces" in our system. In order to state this as an elementary axiom, we shall introduce some definitions and notations.

We recall that every monomorphism m with discrete co-domain D has a characteristic function $D \xrightarrow{\varphi_m} 2$, i. e. $m = Eq\,(\varphi_m,\ D \longrightarrow 1 \xrightarrow{i_1} 2)$. By the exponentiation axiom there exists a unique mapping m* such that the diagram below commutes.



Clearly, if m and n are isomorphic monomorphisms into a discrete space D then $\varphi_m = \varphi_n$ and hence $m^* = n^*$. Moreover, for every $x \in 2^D$, there exists $m \subseteq D$ such that $x = m^*$. The correspondence $m \longrightarrow m^*$ induces a boolean structure on the elements of 2^D with the operations $\cap$, $\cup$, ' defined as follows:

$$m^* \cap n^* = (m \cap n)^*, \quad m^* \cup n^* = (m \cup n)^*, \quad (m^*)' = (m')^*$$

(m' denotes the complement of m) and $0^* = (0 \longrightarrow D)^*$.

If X is an object and $u \subseteq X$ we shall write $u^* = |u|^*$.

We shall later make use of the following property of the (contravariant) exponential functor 2^D, defined on the discrete spaces:

If f is a mapping with discrete domain D and discrete co-domain D' and m is a monomorphism into D, then $m^* 2^f = (f^{-1}(m))^*$.

<u>Definition 13.</u> If D is a discrete space, a mapping $2^D \xrightarrow{I} 2^D$ is called an interior operation on D if

i) $I^2 = I \qquad (I^2 = I\,I)$

ii) $D^*I = D^*$

iii) for every $x \in 2^D$, $xI \cap x = xI$

iv) for every pair $x, y \in 2^D$, $(x \cap y)I = xI \cap yI$.

Every interior operation defined on a set S determines a topology and every topology on the set S determines uniquely an interior operation. For this reason, in our system, we must have as many "topological spaces" X on a "set" D (i.e. $X = D$), as we have interior operations on D. This condition may be formulated as follows:

<u>Axiom 15.</u> For every discrete space D and every interior operation I on D, there exists an object X such that

i) $|X| = D$

ii) for every subspace u of X, $u^*I = (u^\circ)^*$.

We remark that the object X whose existence is stated in the above axiom is determined up to isomorphism by the interior operation

I on D. For, if X and Y are two objects corresponding to the same interior operation I on D, the identity on $D = |X| = |Y|$ may be lifted, according to Axiom 14, to a mapping $X \xrightarrow{f} Y$ which has an inverse.

Suppose that X is an object determined according to Axiom 15 by an interior operation I on D. Then u is an open subspace of X if and only if $u^* I = u^*$. This follows from the fact that u is open in X if and only if $u^\circ = u$.

The above axiom is the final elementary axiom of our system. We may develop this elementary theory and recapture in the system various topological notions and theorems.

CHAPTER II

THE CHARACTERIZATION METATHEOREM

It is the purpose of this chapter to show that the elementary system of axioms which we have constructed, together with one non-elementary axiom, form a characterization of $\mathcal{T}$.

The proof of this result is informal but it could easily be formalized within a sufficiently strong set theory or presumably in the category of categories [7].

Our discussion will be restricted to locally small categories. By this we mean categories with the property that for every pair of objects A and B, the class of all mappings from A to B is a set.

The Characterization Metatheorem. If $\mathcal{C}$ is a locally small category such that:

i) $\mathcal{C}$ is a model of the elementary axioms 1 - 15.

ii) for every family $\{A_j\}_{j \in J}$ of objects of $\mathcal{C}$ there exists a sum and a product in $\mathcal{C}$.

Then, $\mathcal{C}$ is equivalent to $\mathcal{T}$.

Proof: Let $\mathcal{D}$ be the full subcategory of $\mathcal{C}$ whose objects are all the discrete spaces. By the Theorem Schema, $\mathcal{D}$ is a model for the elementary axioms of Lawvere's system.

Since $\mathcal{C}$ has arbitrary products and the functor $|\ | : \mathcal{C} \longrightarrow \mathcal{D}$ is product preserving, $\mathcal{D}$ must have products.

By Theorem 1 b), the sum of two discrete spaces is discrete. The proof of Theorem 1 b) can also be used to show that if $\{A_j\}_{j \in J}$

is a family of discrete spaces and $\sum_{j\in J} A_j$, $\{i_j\}_{j \in J}$ form a sum, then $\sum_{j\in J} A_j$ is discrete. This, together with the fact that arbitrary sums exist in $\mathcal{C}$, implies that $\mathcal{D}$ has arbitrary sums. Hence, all axioms of Lawvere for the category of sets hold for $\mathcal{D}$ and consequently the functor $H^1:\mathcal{D}$ $\mathcal{S}$ $(H^1(d) = \{x \mid x \in D\} \in \mathcal{S})$ is an equivalence of categories [6] .

Consider the category $\mathcal{T}_{\mathcal{D}}$ defined as follows:

The objects of $\mathcal{T}_{\mathcal{D}}$ are ordered pairs (D, I) consisting of an object $D \in \mathcal{D}$ and an interior operation I on D.

The morphisms of $\mathcal{T}_{\mathcal{D}}$ with domain (D, I) and co-domain (D', I') are mappings $D \xrightarrow{f} D'$ in $\mathcal{D}$ satisfying the continuity condition with respect to the interior operations I, I'. By this we mean that for every $x \in 2^{D'}$ such that $xI' = x$ we have $x2^f I = x2^f$. This condition may be written in the compact form $I'2^f I = I'2^f$.

Clearly $\mathcal{T}_{\mathcal{D}}$ is a category. The identities in $\mathcal{T}_{\mathcal{D}}$ are ordered pairs of the form (D, 2^D). (Clearly, the mapping 2^D is an interior operation on D.)

We shall prove that $\mathcal{C}$ is equivalent to $\mathcal{T}$ by showing that $\mathcal{C}$ is equivalent to $\mathcal{T}_{\mathcal{D}}$ and that $\mathcal{T}_{\mathcal{D}}$ is equivalent to $\mathcal{T}$.

Consider the functor $H^1:\mathcal{D} \longrightarrow \mathcal{S}$. H^1 is both faithful and full. (This follows from the proof of Lawvere's metatheorem [6] .)

Suppose $X \in \mathcal{C}$, $D = |X|$ and let s be the function defined on $H^1(2^D)$ such that $s(x^*) = (x^\circ)^*$. Then s is a morphism in $\mathcal{S}$ and since H^1 is full there exists a mapping $2^D \xrightarrow{I} 2^D$ such that $s = H^1(I)$. Furthermore, if x is a subspace of X we have $x^*I = (x^\circ)^*$ for, $x^*I = H^1(I)(x^*) = s(x^*) = (x^\circ)^*$. This also implies that x is open in X if and only if $x^*I = x^*$. I is an interior operation on D, because:

i) Clearly $s^2 = s$ and hence $H^1(I^2) = \left(H^1(I)\right)^2 = s^2 = s = H^1(I)$ and so $I^2 = I$ since H^1 is faithful.

ii) $D^*I = X^*I = (X^\circ)^* = X^* = D^*$

iii) For every $x^* \in 2^D$ $(x \subseteq X)$, we have $x^*I \cap x^* = (x^\circ)^* \cap x^* = (x^\circ \cap x)^* = (x^\circ)^* = x^*I$.

iv) For every pair of elements x^*, y^* of 2^D $(x, y \subseteq X)$ we have $(x^* \cap y^*)I = (x \cap y)^*I = (x \cap y)^{\circ *} = (x^\circ \cap y^\circ)^* = (x^\circ)^* \cap (y^\circ)^* = x^*I \cap y^*I$.

We define a functor $F: \mathcal{C} \longrightarrow \mathcal{T}_{\mathfrak{D}}$ as follows: $F(X) = (D, I)$, where $D = |X|$ and I is the interior operation on D, obtained above.

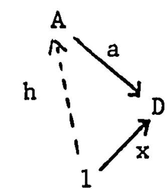
If $X \xrightarrow{f} Y$ and $F(X) = (D, I)$, $F(Y) = (D', I')$ then the mapping $D \xrightarrow{|f|} D'$ is in $\mathcal{T}_{\mathfrak{D}}$. To see this, consider $y \subseteq Y$ and let $x = f^{-1}(y^\circ)$. Since y° is open in Y, by Proposition 17, x is open in X and hence $x^*I = x^*$. The functor $|\ |$ preserves inverse images, so we have $|f|^{-1}(|y^\circ|) = |x|$ and hence $y^*I'2^{|f|} = (y^\circ)^*2^{|f|} = x^*$. Therefore $y^*I'2^{|f|}I = x^*I = x^* = y^*I'2^{|f|}$. This equality holds for all $y^* \in 2^{D'}$ and hence $I'2^{|f|}I = I'2^{|f|}$, i.e. $|f| \in \mathcal{T}_{\mathfrak{D}}$. Let $F(f) = |f|$. Clearly $F: \mathcal{C} \longrightarrow \mathcal{T}_{\mathfrak{D}}$ is a functor. We shall now define a functor $G: \mathcal{T}_{\mathfrak{D}} \longrightarrow \mathcal{C}$. For this consider $(D,I) \in \mathcal{T}_{\mathfrak{D}}$ and let X be the object assigned by Axiom 15 to the pair (D, I). (X is uniquely determined up to isomorphism.) We define $G(D,I) = X$. If $D \xrightarrow{d} D'$ is a morphism in $\mathcal{T}_{\mathfrak{D}}$ from (D,I) into (D',I'), and $X = G(D,I)$, $Y = G(D',I')$, then d satisfies the condition in Axiom 14 with respect to X and Y. In order to prove this we first note that $|X| = D$, $|Y| = D'$. Furthermore, if v is an open subspace of Y, then clearly $v^*I' = v^*$. Suppose u is a subspace of X such that $d^{-1}(|v|) = |u|$. Then, since $d \in \mathcal{S}_{\mathcal{D}}$, we have $v{*}2^f I = v{*}2^f = u{*}$ and hence $u{*}I = v{*}2^f I^2 = v{*}2^f = u{*}$. This implies that u is open in X. According to Axiom 14 there exists a (unique) mapping $X \xrightarrow{f} Y$ such that $|f| = d$. Let $G(d) = f$. Clearly, G is a functor. We shall show that $F \circ G$ and $G \circ F$ are naturally equivalent to the identities on $\mathcal{S}_{\mathcal{D}}$ and $\mathcal{C}$ respectively. Suppose $X = G(D,I)$. Then for every $x \subseteq X$ we have $x{*}I = (x^\circ)^*$. Let $F(X) = (D,I')$. By the definition of I' for every $x \subseteq X$, $x{*}I' = (x^\circ)^*$ and hence $x{*}I = (x^\circ)^* = x{*}I'$ for all $x^* \in 2^D$ which implies that $I = I'$. Therefore $F \circ G(D,I) = (D,I)$ for all (D,I), i.e. $F \circ G = \mathcal{S}_{\mathcal{D}}$. Suppose $(D,I) = F(X)$. Clearly for all $x \subseteq X$, we have $x{*}I = (x^\circ)^*$. Let $G(D,I) = Y$. Then $|Y| = D$ and $y{*}I = (y^\circ)^*$. Since both X and $Y = G \circ F(X)$ satisfy Axiom 15 with respect to the pair (D,I), they are solutions of a universal mapping problem and so we have $G \circ F(X) \cong X$ and this isomorphism is natural. Therefore, $F\colon \mathcal{C} \longrightarrow \mathcal{S}_{\mathcal{D}}$ is an equivalence of categories. The proof of the metatheorem will be complete once we show that $\mathcal{S}_{\mathcal{D}}$ is equivalent to $\mathcal{S}$.

<u>Metatheorem.</u> Let $\mathcal{D}$ be any locally small category such that $\mathcal{D}$ is a model of Lawvere's system of axioms for the category of sets. Then $\mathcal{S}_{\mathcal{D}}$ * is equivalent to $\mathcal{S}$.

Proof: Since $\mathcal{D}$ is a model of Lawvere's system, for every object $D \in \mathcal{D}$ the lattice of subobjects of D is complete, i.e. for every family of subobjects of D there exists a union and an intersection. Let $(D,I) \in \mathcal{S}_{\mathcal{D}}$ and let $\mathcal{O}_D = \{a \subseteq D \mid a{*}I = a^*\}$. Clearly $D \in \mathcal{O}_D$ and $0 \longrightarrow D \in \mathcal{O}_D$. Since I is an interior operation on D, if $a, b \in \mathcal{O}_D$ then $a \cap b \in \mathcal{O}_D$.

* $\mathcal{S}_{\mathcal{D}}$ is constructed from $\mathcal{D}$ as in the previous metatheorem.

$\mathcal{O}_D$ is also closed under arbitrary unions for let $\{a_j\}_{j\in J}$ be a family of elements of $\mathcal{O}_D$. Then $\bigcup_{j\in J} a_j \subseteq D$ and clearly $(\bigcup_{j\in J} a_j)*I \subseteq (\bigcup_{i\in J} a_j)*$. Also $a_i* = a_i*I \subseteq (\bigcup_{j\in J} a_j)*I$ for all $i \in J$ and hence $(\bigcup_{j\in J} a_j)* \subseteq (\bigcup_{j\in J} a_j)*I$. But this implies that $(\bigcup_{j\in J} a_j)*I = (\bigcup_{j\in J} a_j)*$, i.e. $\bigcup_{j\in J} a_j \in \mathcal{O}_D$. For every $a \subseteq D$ let $V_a = \{x \in D \mid x \in a\}$, i.e. $x \in V_a$ if there exists a mapping h for which the diagram below is commutative.



Let $\mathcal{T} = \{V_a \mid a \in \mathcal{O}_D\}$. Since $H'(D) = V_D \in \mathcal{T}$, $V_a \cap V_b = V_{a\cap b}$ and $\bigcup_{j\in J} V_{a_j} = V_{\bigcup_{j\in J} a_j}$, $\mathcal{T}$ is closed under finite intersections and arbitrary unions. Hence $\mathcal{T}$ defines a topology on H'(D). Let $F(D,I) = (H'(D), \mathcal{T})$. If $D \xrightarrow{f} D'$ is a morphism in $\mathcal{T}_{\mathcal{D}}$, H'(f) is continuous because if $V_{a_1} \in \mathcal{T}'$ and $a = f^{-1}(a_1)$, clearly $[H'(f)]^{-1}(V_{a_1}) = \{x \mid xf \in V_{a_1}\} = V_a$ and since $a_1*I' = a_1*$, we have $a*I = a_1*2^f I = a_1*2^f = a*$ which means that $V_a \in \mathcal{T}$. Let $F(f) = H'(f)$. Clearly F thus defined is a functor from $\mathcal{T}_{\mathcal{D}}$ into $\mathcal{T}$. Since H' is faithful, F is faithful. Since H' is full, for every continuous function $\alpha : (H'(D), \mathcal{T}) \longrightarrow (H'(D'), \mathcal{T}')$ there exists a mapping $D \xrightarrow{f} D'$ such that $\alpha = H'(f)$. The continuity of α implies that $f \in \mathcal{T}_{\mathcal{D}}$ because if $a_1 \subseteq D'$, $a_1*I = a_1*$ and $a = f^{-1}(a_1)$ then $V_a = \alpha^{-1}(V_{a_1})$. But $V_{a_1} \in \mathcal{T}'$ and since α is continuous $V_a \in \mathcal{T}$ and hence $a*I = a*$. Therefore $a_1*2^f I = a*I = a* = a_1*2^f$ and hence $f \in \mathcal{T}_{\mathcal{D}}$. This means that F is full.

To show that F is an equivalence it will suffice to prove that for every topological space $(X, \mathcal{T})$ there exists an object (D,I) in $\mathcal{T}_{\mathcal{D}}$

such that $(X, \mathcal{T}) \cong F(D,I)$. Let $(X, \mathcal{T}) \in \mathfrak{T}$ and let $D = \sum_X 1 \in \mathfrak{D}$. For every open subset U of X let $\sum_U 1 \xrightarrow{a_U} D$ be the mapping induced by the inclusion $U \subseteq X$. The family $\mathcal{O} = \{a_U | U$ is open in $X\}$ is closed under finite intersection and arbitrary unions. $\mathcal{O}$ defines an operation on the elements of 2^D $(a^* \longrightarrow (\bigcup_{a_U \subseteq a} a_U)^*)$ which can be lifted to an interior operation I on D such that $a^*I = a^*$ if and only if $a \in \mathcal{O}$. Then $F(D,I) = (H^1(\sum_X 1), \mathcal{T}')$ is isomorphic to X. Indeed the map $i: X \longrightarrow H^1(\sum_X 1)$ which associates to every $x \in X$ the injection $1 \xrightarrow{i_x} \sum_X 1$ is one-to-one, onto and since for every open subset U of X, $i(U) = V_{a_U}$, i is a homoemorphism. q.e.d.

The proof of the characterization metatheorem shows that the three constructions of the topological category, i.e. the traditional construction, the translation of this construction in Lawvere's system and our axiomatization lead essentially to the same thing.

We give below an outline of an alternive direct proof of the characterization metatheorem.

The functor $H^1: \mathcal{C} \longrightarrow \mathcal{S}$ can be lifted to a functor $F: \mathcal{C} \longrightarrow \mathfrak{T}$ in the following way:

Let C be an object of $\mathcal{C}$ and for every subspace a of C let $V_a = \{x \in C | x \in a\}$. Then $\mathcal{T} = \{V_a | a \subseteq C, a^\circ = a\}$ defines a topology on $H^1(C)$. It can easily be seen that if $C \xrightarrow{f} C'$ is a mapping in $\mathcal{C}$, then $H^1(f)$ is continuous. Let $F(C) = (H^1(C), \mathcal{T})$ and $F(f) = H^1(f)$. Clearly F is a functor and because H^1 is faithful so is F. F is also full. To prove this we consider a mapping $F(C) \xrightarrow{\lambda} F(C')$ in $\mathfrak{T}$ and we apply to it the forgetful functor U, i.e. $H^1(C) \xrightarrow{U(\lambda)} H'(C')$ is a mapping in $\mathcal{S}$. Since for every object $C \in \mathcal{C}$, the mapping

$|C| \xrightarrow{t_C} C$ is a bijection, $H^1(t_C)$ is an isomorphism. Let μ be the unique mapping for which the diagram below commutes.

$$\begin{array}{ccc} H^1(|C|) & \xrightarrow[\cong]{H^1(t_C)} & H^1(C) \\ \mu \Big\downarrow & & \Big\downarrow U(\lambda) \\ H^1(|C'|) & \xrightarrow[H^1(t_{C'})]{\cong} & H^1(C') \end{array}$$

Since $H^1 : \mathcal{D} \longrightarrow \mathcal{S}$ is full there exists a mapping d such that $H'(d) = \mu$. It can be shown that d satisfies the condition in Axiom 14 with respect to C and C' and hence there exists a mapping $C \xrightarrow{f} C'$ such that $|f| = d$. It is easy to see that $F(f) = \lambda$ which proves that F is full. Hence F is an embedding.

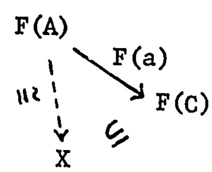
We note that Axiom 15 was not used when proving that F is an embedding.

To complete the proof of the metatheorem it suffices to show that for every topological space X there exists an object $A \in \mathcal{C}$ with the property $F(A) \cong X$. In order to show this, we make use of the following proposition obtained by S. Baron as an application of the embedding lemma [4]:

"Every topological space can be embedded in a product of copies of the indiscrete space with two points and the topological space with two points and three open subsets."

We first remark that Axiom 15 guarantees the existence of an "indiscrete space" with two points, $\overline{E}$, in our system. It is easy to see that F(E) is a space with two points and three open subsets, F($\overline{E}$) is an indiscrete space with two points. Furthermore, using Axiom 15 we can show that F preserves products and equalizers.

The proof will be completed once we show that for every subspace X of a topological space F(C), there exists a subspace a of C such that X is the image of F(a) in F(C).



This follows from the fact that every subset of $H^1(C)$ determines a subset of $H^1(|C|)$ which in turn is determined by a monomorphism into $|C|$. Clearly to every such monomorphism there corresponds a subspace a of C which has the required property.

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