Towards the Verlinde Formula for the Moduli of Flat $SU(2)$ Connections in a Real Polarization, via a Toric Degeneration

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Abstract

We review the geometric quantization of the moduli space $\mathcal{M}$ of flat $SU(2)$ connections on a compact Riemann surface in the real polarisation of Weitsman [Wei92][JW92]. We also use the methods of [RSW89] to construct a line bundle over the toric variety $\mathcal{P}$ associated to the moment polytope of Jeffrey and Weitsman's integrable system on the moduli space, which is compatible with the prequantum line bundle on $\mathcal{M}$. There is a degeneration of the moduli space to this toric variety due to Biswas and Hurtubise [BH21], and we discuss how this degeneration might be used to prove results about the real and Kaehler polarisations of the moduli space.

Résumé

Nous donnons un survol de la quantification géométrique de l'espace de modules $\mathcal{M}$ des connexions $SU(2)$ plate sur une surface de Riemann compacte, dans une polarisation réelle de Weitsman [Wei92][JW92]. Aussi, nous utilisons les méthodes de [RSW89] pour construire un fibré en droites sur la variété torique $\mathcal{P}$ associée à l'application moment de le système intégrable de Jeffrey et Weitsman sur l'espace de modules, qui est compatible avec le fibré en droites préquantique sur $\mathcal{M}$. Il y a un déformation de l'espace de modules à la variété torique, donnée par Biswas et Hurtubise [BH21], et nous discutons de la façon dont cette déformation peut être utilisée pour prouver des résultats concernant les polarisations réelle et complexe de l'espace de modules.

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Chapters 2 through 4 are entirely based on previous mathematical work. The author's contribution to these chapters is merely exposition and elaboration on previous work, as well as computations in some specific cases. Chapter 5 is also primarily based on the previous work of Biswas and Hurtubise, but with some new ideas contributed by the author, specifically in Sections 5.5 and 5.6. The conclusions in Chapter 6 are the authors own mathematical thought.

Contents

Chapter 1

Introduction

Moduli spaces of principal G-connections on Riemann surfaces have been a topic of much mathematical research, both due to their interesting complex geometry and their connections with gauge theories in physics. Of particular note are unitary connections ($G = U(n)$ or $SU(n)$), as unitary groups arise as the structure groups of the gauge theories of bosons in the standard model, and have particularly tractable moduli spaces. A pioneering work in this area is that of Atiyah and Bott [AB83], which we will reference frequently.

Understanding the *quantization* of these gauge theories is a ongoing area of research. One notable work in this area is a paper of Jeffrey and Weitsman [JW92] which discusses the geometric quantization of the space $\mathcal{M}$ of flat $SU(2)$ connections on a compact Riemann surface. In their paper, they describe a real polarization of $\mathcal{M}$ by decomposing the surface into *trinions*, or *pairs of pants*. At the boundary of two trinions in the decomposition, one has a closed curve in the surface, and a real polarization of the moduli space can be given by the holonomy of connections around these curves. These holonomies also give rise to Goldman flows, which almost give a toric Hamiltonian action on the space, but there are singularities for connections A which have a holonomy that is central in $SU(2)$ around one of the decomposition curves. Their paper counts the number of Bohr-Sommerfeld points in $\mathcal{M}$, showing that it is given by the Verlinde dimension. If their real polarizaton had been a smooth fibration, then a theorem of Sniatycki could be applied which says that the dimension of the quantization is given by the number of Bohr-Sommerfeld points in $\mathcal{M}$. This would match the known result for the Kaehler polarization of $\mathcal{M}$, that the dimension of the quantization is given by the Verlinde dimension [Fal94]. Unfortunately, due to the singularities at connections with central holonomies, this is not the case.

In order to complete the proof, one can try and build a smooth moduli space which is a toric variety, with a prequantum line bundle whose sections are the same as those which we wish to compute. Hurtubise and Jeffrey [HJS05][HJ00] construct a moduli space P using symplectic implosion, which is a toric variety, with a Hamiltonian system having the same moment polytope as that of the Hamiltonain system on $\mathcal{M}$. Furthermore, they also give a holomorphic description of the moduli space. Mehta and Seshadri [MS80] proved that unitary connections on a punctured Riemann surface with fixed holonomy at the fibres are in correspondence with the *parabolic vector bundles* on the unpunctured space. Considering a trinion as a thrice-punctured Riemann surface, we can study the moduli of unitary connections on a trinion in terms of parabolic vector bundles. Since we want to study unitary connections with any holonomies, we have to find a space $\mathcal{P}$ which includes all the parabolic structures with any holonomies, and this space will allow us to include the singular fibres of the real polarisation, at the cost of considering instead *framed parabolic sheaves*. Finally, Hurtubise and Jeffrey exhibit an isomorphism between P and $\mathcal{P}$.

Therefore, we have the moduli space $(\mathcal{M}, \omega)$ with prequantum line bundle $(\mathcal{L}, \nabla)$ for which we wish to compute the dimension of the polarisation, and the parabolic moduli space P, for which the dimension of the polarisation of a corresponding line bundle can be computed using the theory of toric varieties. It was expected that for toric varieties, an analogue of Sniatycki's theorem would hold, saying that the dimension of the polarisation is given by counting Bohr-Sommerfeld points. For toric varieties, the dimension of the Kaehler polarisation is given by the number of Bohr-Sommerfeld points, so such a result would prove that these polarisations have the same dimension for toric varieties. However, Hamilton proved that for toric symplectic manifolds, the dimension of the real polarisation is strictly less then that of the Kaehler polarisation, and that the difference is related to the singularities of the polarisation [Ham10]. To understand why this is the case, we aim to study the relationship between $\mathcal{M}$ and P.

The relationship between the spaces $\mathcal{M}$ and P is given in terms of a *degeneration* of the smooth Riemann surface to the punctured one, and the induced degeneration of the moduli spaces. Biswas and Hurtubise [BH21] provide a model for the degeneration of the Riemann surfaces and a corresponding degeneration of the moduli space of vector bundles. The degeneration of surfaces is a family over a neighbourhood of 0 in $\mathbb{C}$ of Riemann surfaces, which are smooth for $t \neq 0$ and which approach the punctured surface at $t = 0$. For the corresponding degeneration of moduli spaces, at $t = 0$, one obtains $\mathcal{P}$ and at $t \neq 0$ we have the moduli space $\mathcal{M}$ of holomorphic vector bundles which we hope to quantize.

Thus, the aim of this thesis is to construct the corresponding real polarisation and prequantum system on P, and investigate the relationship between the real and Kaehler quantisations on $\mathcal{M}$ and those on P. This document proceeds by introducing the moduli space $\mathcal{M}$ of unitary connections on a Riemann surface (Chapter 2), then describing the geometric quantization of this space in the real polarisation of Jeffrey and Weitsman (Chapter 3). Then we review the construction of the spaces P and $\mathcal{P}$, the toric variety of representations with weighted frames and of framed parabolic bundles introduced by Hurtubise and Jeffrey

(Chapter 4). Afterwards, we describe the degeneration of Biswas and Hurtubise, and how we can use Chern-Simons theory to build a bundle on P which naturally comes from the prequantum line bundle on $\mathcal{M}$ (Chapter 5). Finally, we conclude with a summary of the results and potential avenues for continued research (Chapter 6).

Chapter 2

Moduli Spaces of Flat Connections

Given a Riemann surface Σ and a unitary group $G = U(n)$ or $G = SU(n)$, we are interested in the moduli space $\mathcal{M}$ of connections on a principal G bundle over Σ, up to gauge equivalence. A detailed study of these spaces was made by Atiyah and Bott [AB83], from which we take much of the following discussion.

Thanks to the work of Narasimhan and Seshadri and Donaldson [Don83][NS65] there are multiple ways in which one can view $\mathcal{M}$. One equivalence is between flat unitary connections and irreducible representations of $\pi_1(\Sigma)$ into G. Gauge equivalence for the connections is accounted for by a quotient: $\mathrm{Hom}(\pi_1(\Sigma), G)/G$. Another equivalence is with holomorphic $SL(n, \mathbb{C})$ bundles over Σ, which we think of as Dolbeault operators $\overline{\partial}_E$ on a smooth complex vector bundle E. Different aspects of the geometry of $\mathcal{M}$ become clear in different pictures, so we will explain each of them here.

2.1 Flat Connections as Fundamental Group Representations

We begin with the correspondence between flat connections and representations of the fundamental group. For a Lie group G, given any G-connection A on a manifold Σ, the holonomy of A around a loop γ based at $p \in \Sigma$ gives us a map $\mathrm{Hol}_A(\gamma) : \mathrm{Loops}(p, \Sigma) \to G$. Generically, the holonomy is not invariant up to homotopy, so this map does not pass to a map on $\pi_1(\Sigma) \to G$. However, if one restricts to *flat* connections, which are those whose holonomy around any contractible loop is trivial, then one can pass to the quotient to get a map $\mathrm{Hol}_A(\gamma) : \pi_1(p, \Sigma) \to G$. Picking a different basepoint or trivialization conjugates the resulting morphism in G, so that we can associate to any flat connection A a map $\pi_1(\Sigma) \to G$ up to conjugation. This *holonomy representation* determines A up to gauge equivalence.

Let $\mathcal{A}$ denote the space of flat connections on the trivial principal bundle $P = G \times \Sigma$, let $\mathcal{G} = \mathbb{C}^\infty(P, G)^G$ be the gauge group, and let $\Phi : \mathcal{A} \to \mathrm{Hom}(\pi_1(\Sigma), G)/G$ denote the map taking A to Hol_A.

Lemma 2.1. *The map Φ is injective up to conjugation in G. That is, for any two connections $A, B \in \mathcal{A}$, if $\Phi(A) = \Phi(B)$, then $A \cong B \mod \mathcal{G}$.*

Proof. Suppose $A, B \in \mathcal{A}$ are flat connections with $\Phi(A) = \Phi(B) \mod G$. Explicitly, given any loop $\gamma \in \pi_1(\Sigma)$ based at $p \in \Sigma$, there exists an $h \in G$ such that

$$h^{-1}\mathrm{Hol}_A(\gamma)h = \mathrm{Hol}_B(\gamma). \tag{2.1}$$

To prove the lemma, we construct a gauge equivalence $f \in \mathcal{G}$ between A and B. For any $q \in \Sigma$ pick a curve $\sigma : p \to q$. Then for any $g \in G$ denote by $\Pi^A_\sigma g$ the parallel transport of g along σ. Let $f(q) = \Pi^B_\sigma \left(\Pi^A_\sigma\right)^{-1}$ for all $q \in \Sigma$, which we will show gives the required gauge equivalence. First we must show f is well defined; if τ is another curve from $p \to q$ then:

$$\begin{aligned} f(q)\Pi^A_\tau &= \Pi^B_\sigma \left(\Pi^A_\sigma\right)^{-1} \Pi^A_\tau \\ &= \Pi^B_\sigma \left(\Pi^A_\sigma\right)^{-1} \Pi^A_\sigma \mathrm{Hol}_A(\delta^{-1} \circ \tau) \\ &= \Pi^B_\sigma \mathrm{Hol}_B(\sigma^{-1} \circ \tau) \\ &= \Pi^B_\tau \\ f(q) &= \Pi^B_\tau (\Pi^A_\tau)^{-1} \end{aligned}$$

Therefore f is well defined, and moreover this calculation shows it takes A-horizontal vectors to B-horizontal vectors. It is easy to see f is smooth, and since it maps horizontal vectors to horizontal vectors, it must be an isomorphism of connections between A and B. □

This lemma tells us connections are determined up to gauge equivalence by their holonomy. Note that the proof did not use flatness of A or B, so it is true for all connections. To complete the correspondance between $\mathcal{M} = \mathcal{A}/\mathcal{G}$ and $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, it remains to show that given any map $\phi \in \mathrm{Hom}(\pi_1(\Sigma), G)$ one can find a connection whose holonomy matches ϕ.

Lemma 2.2. *The map $\Phi : \mathcal{A} \to \mathit{Hom}(\Pi_1(\Sigma), G)$ is surjective.*

Proof. Let $\phi : \pi_1(\Sigma) \to G$ be a group homomorphism. The universal cover $\tilde{\Sigma}$ of Σ is a $\pi_1(\Sigma)$ bundle $\pi : \pi_1(\Sigma) \times \Sigma \to \Sigma$. Then at a point $x \in \tilde{\Sigma}$, $\pi_1(\Sigma, \pi(x))$ acts on $\tilde{\Sigma}$ by monodromy, and $\pi_1(\Sigma)$ acts on G by

$$\gamma \cdot g = \phi(\gamma)g. \tag{2.2}$$

Thus, define a principal G-bundle over Σ by quotienting out this action:

$$P = \tilde{\Sigma} \times G/\pi_1(\Sigma). \tag{2.3}$$

The monodromy action is proper and free on the universal cover, and left multiplication in G is proper and free, so this quotient is a well-defined smooth manifold. Finally, one can put a connection on P with the correct holonomy. To do so, define a connection on $\tilde{\Sigma} \times G$ by picking the horizontal bundle in $T(\tilde{\Sigma} \times G)$ to be all vectors of the form $(v, 0)$; those with no G component. Then $\pi_1(\Sigma)$ preserves this space and the image in the quotient is a horizontal bundle defining a connection A on P.

Let γ be a loop in Σ starting at x. Then let $\gamma' : [0,1] \to P$ be defined by

$$\gamma'(t) = (\psi(\gamma), \gamma(t))). \tag{2.4}$$

If this is horizontal, then $\mathrm{Hol}_A(\gamma) = \gamma'(t) = \psi(\gamma)$ which completes the proof. If we lift under the quotient of $\pi_1(\Sigma)$ we get $\tilde{\gamma} \in \tilde{\Sigma} \times G$ with

$$\tilde{\gamma}(t) = ((v(t), \gamma(x)), \psi(\gamma)) \tag{2.5}$$

and $d/dt(\tilde{\gamma}) = (v'(t), 0)$, meaning γ' is horizontal. Note finally that since ψ is a group homomorphism, $\mathrm{Hol}_A(\gamma) = \mathrm{Hol}_A(e) = \mathbb{1}$ for any contractible loop, so A is flat. $\square$

Combining the previous two lemmas we have:

Theorem 2.3. *The map $\Phi : \mathcal{M} \to Hom(\pi_1(\Sigma), G)/G$ taking A to $Hol_A(-) \mod G$ is a bijection*

Thus, one may identify the set of flat connections with the set $\mathrm{Hom}(\pi_1(\Sigma), G)/G$. To build a moduli space, we want to endow $\mathcal{M}$ with a topology and some geometric structure. If $\pi_1(\Sigma)$ is finitely presented as

$$\pi_1(\Sigma) = \langle a_1, ..., a_N \mid R_1, ...R_N \rangle, \tag{2.6}$$

then consider $\mathrm{Hom}(\pi_1(\Sigma), G)$ as a subset of G^N by taking the generators to their images under any homomorphism. This lets $\mathrm{Hom}(\pi_1(\Sigma), G)$ inherit a topology from the Lie group topology on G, and $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ can be given the quotient topology.

Geometrically, $\mathrm{Hom}(\pi_1(\Sigma), G)$ corresponds to $G[a_1, ..., a_n]/\langle R_1, ..., R_N \rangle$, so when G is an algebraic group, $\mathrm{Hom}(\pi_1(\Sigma), G)$ is a variety. Then $\mathcal{M} = \mathrm{Hom}(\pi_1(\Sigma), G)/G$ is a quotient variety.

2.2 Unitary Representations on a Riemann Surface

Now we specialize to a compact connected Riemann surface Σ of genus g, and $G = U(n)$. Then the fundamental group is

$$\pi_1(\Sigma) = \{a_1, b_1, ..., a_g, b_g \mid \Pi_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = e\}. \tag{2.7}$$

Define $\xi : U(n)^{2g} \to U(n)$ by $\xi(A_1, B_1, ..., A_i, B_i) = \Pi_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1}$. Then $\mathcal{M} = \mathrm{Hom}(\pi_1(\Sigma), U(n))/U(n)$ is $\xi^{-1}(e)/U(n) \subset U^{2g}/U(n)$ and inherits a quotient topology from the topology of U^{2g}. In general, $\mathcal{M}$ is not smooth, but the subset of $\mathcal{M}$ consisting of irreducible representations will be a smooth manifold.

Lemma 2.4. *Let $\rho(a_i) = A_i, \rho(b_i) = B_i$, for the generators $(a_1, b_1, ..., a_g, b_g)$ of $\pi_1(\Sigma)$. Then a representation $\rho : \pi_1(\Sigma) \to U(n)$ is reducible if and only if all elements in the set $\{A_1, B_1, ..., A_2, B_2\}$ pairwise commute.*

Proof. Suppose the A_i, B_i all pairwise commute. Then by the spectral theorem for unitary matrices, they are all simultaneously diagonalizable. Thus they share at least one eigenspace W, which is invariant under all the A_i and B_i, and so ρ is reducible.

On the other hand, suppose ρ is reducible. Since unitary representations are semisimple, we can write the representation as $\bigoplus_{j=0}^{k} W_j$, with each W_j an irreducible subspace which is invariant under ρ. Then each W_j must be an eigenspace of each matrix A_i and B_i, and thus the matrices have the same eigenspaces and are simultaneously diagonalizable. Since simultaneously diagonalizable matrices commute, this means the A_i and B_i pairwise commute. $\square$

Let $\mathcal{R}$ denote the subset of $\mathcal{M}$ consisting of reducible representations, and $\mathcal{M}_s$ denote the subset of irreducible points. The condition that $[A, B] = 0$ is a closed condition, so $\mathcal{M}_s$ is open in $\mathcal{M}$ and $\mathcal{R}$ is closed.

Lemma 2.5. *$\mathcal{R}$ is compact.*

Proof. Let $p : \mathrm{Hom}(\pi_1(\Sigma), U(n)) \to \mathcal{M}$ denote the quotient by $U(n)$. Let $\tilde{\mathcal{R}} = p^{-1}(\mathcal{R})$. Then $\tilde{\mathcal{R}} \subset U(n)^{2g}$ is closed and thus since $U(n)$ is compact, $\tilde{\mathcal{R}}$ is compact. Then $\mathcal{R} = \tilde{\mathcal{R}}/U(n)$ is also compact. $\square$

Using this one can characterize the topology of $\mathcal{R}$.

Theorem 2.6. *The reducible part $\mathcal{R}$ of the moduli space $\mathcal{M}$ is homeomorphic to*

$$T^{2g}/W(T), \tag{2.8}$$

where $T \subset G$ is a maximal torus and $W(T)$ is its Weyl group, acting by the $2g$-diagonal action.

Proof. Let $\{a_i, b_i\}_{i=1}^{g}$ generate $\pi_1(\Sigma)$. For $[\rho] \in \mathcal{R}$, let $A_i = \rho(a_i)$ and $B_i = \rho(b_i)$. By Lemma 2.4, $\rho \in \mathcal{R}$ implies the A_i and B_i pairwise commute, and are hence contained in some maximal torus T, thus $(A_1, B_1, ..., A_g, B_g) \in T^{2g}$. To pass to the quotient $[\rho]$ under conjugation by $U(n)$, we need to quotient the Weyl group $W(T)$. Thus $[A_1, B_1, ..., A_g, B_g] \in T^{2g}/W(T)$. Diagrammatically, we have:

$$\begin{array}{ccc} \mathrm{Hom}(\pi_1(\Sigma), U(n)) & \longrightarrow & T^{2g} \\ \downarrow p & & q \downarrow \\ \mathcal{M} & \dashrightarrow & T^{2g}/W(T) \end{array}$$

Since the topology of $\mathrm{Hom}(\pi_1(\Sigma), U(n))$ and T^{2g} are their subspace topologies in U^{2g}, the upper arrow is continuous. Its composition with the quotient q gives a continuous map $\mathrm{Hom}(\pi_1(\Sigma), U(n)) \to T^{2g}/W(T)$, and by the universal property of the quotient topology, this means the map $\mathcal{M} \to T^{2g}/W(T)$ is continuous.

Next we show the map is bijective. For surjectivity, given any $(t_1, ..., t_{2g})$ define $\rho(a_i) = t_{2i-1}$ and $\rho(b_i) = t_{2i}$. The torus' commutativity $[t_i, t_j] = 0$ guarantees $\rho(a_i)$ will be a well-defined reducible representation of $\pi_1(\Sigma)$. For injectivity, if ρ and ρ' map to $[A_1, ...B_g]$ and $[A'_1, ..., B'_g]$ which are equal in in $T^{2g}/W(T)$ then it means there is an element $t \in W(T)$ for which $A'_i = tA_it^{-1}$ and $B'_i = tB_it^{-1}$. Therefore $\rho' = t\rho t$ and so $[\rho] = [\rho']$.

Finally, since $W(T)$ is finite, $T^{2g}/W(T)$ is Hausdorff; since $\mathcal{R}$ is compact, our mapping is a continuous bijection from a compact space to a Hausdorff space, hence a homeomorphism. □

When G or $\pi_1(\Sigma)$ is Abelian, $\mathcal{M} = \mathcal{R}$ and therefore Theorem 2.6 determines the entire moduli space. When $G = U(1)$ which is Abelian, $T = U(1) = \mathbb{C}^*$ and $W(T) = e$ so:

$$\mathcal{M} = \mathcal{R} \cong (\mathbb{C}^*)^{2g}. \tag{2.9}$$

In this case, $\mathcal{M}$ is the Jacobian variety of Σ, and equation 2.9 is the well-known result that the Jacobian of a compact connected Riemann surface is a torus.

When Σ has genus 1, $\pi_1(\Sigma) = \mathbb{Z}^2$ which is Abelian. Then

$$\mathcal{M} = \mathcal{R} \cong \frac{T^2}{W(T)}. \tag{2.10}$$

Now we would like to address the irreducible points. In general, $\mathcal{M}_0$ will be a smooth manifold [AB83, §7] but here we only prove it for $G = SU(2)$.

Theorem 2.7. *When $G = SU(2)$, $\mathcal{M}_0$ is a smooth manifold of (real) dimension $6g - 6$.*

Proof. This proof follows that of Michiels [**michiels˙moduli˙nodate**]. The strategy is to first show the map ξ is submersive on $\mathcal{M}_0$, so that $\xi^{-1}(e)$ is a smooth manifold, and then prove that $\mathcal{M}_0$ is a quotient of $\xi^{-1}(e)$ under a free action of a compact group with dimension $\dim SU(2)$. This will give a dimension count of

$$\dim \mathcal{M}_0 = \dim(\xi^{-1}(e)) - \dim(SU(2)) = (2g-1)\dim(SU(2)) - \dim(SU(2)) = 6g - 6. \tag{2.11}$$

Proving $\xi : SU(2)^{2g} \to SU(2)$ is submersive requires showing the rank of ξ is 3 at all irreducible points. Let $(A_1(t), ..., B_g(t)) = (A_1 + ta_1, ..., B_g + tb_g)$ for some $(A_1, ...B_g) \in SU^{2g}$ and some $(a_1, ..., b_g) \in \mathfrak{su}(2)^{2g}$. Then composing with ξ gives the curve

$$t \to \gamma(t) := \prod_{i=1}^{g} A_i(t)B_i(t)A_i(t)^{-1}B_i(t)^{-1}. \tag{2.12}$$

We compute the differential, first considering just one factor:

$$\frac{d}{dt}|_{t=0}A_i(t)B_i(t)A_i(t)^{-1}B_i(t)^{-1} = \mathrm{Ad}_{B_iA_iB_i^{-1}}(a_i) + \mathrm{Ad}_{B_iA_i}(b_i) - \mathrm{Ad}_{B_iA_i}(a_i) - \mathrm{Ad}_{B_i}(b_i)$$
$$= \mathrm{Ad}_{B_iA_i}\left((\mathrm{Ad}_{B_i^{-1}} - 1)a_i + (1 - \mathrm{Ad}_{A_i^{-1}})b_i\right)$$

Then the derivative of the entire product is given by the product rule:

$$\frac{d}{dt}|_{t=0}\gamma(t) = \sum_{i=1}^{g}\left[\mathrm{Ad}_{(\prod_{j>i}A_jB_jA_j^{-1}B_j^{-1})^{-1}B_iA_i}\left((\mathrm{Ad}_{B_i^{-1}} - 1)a_i + (1 - \mathrm{Ad}_{A_i^{-1}})b_i\right)\right]. \tag{2.13}$$

Fixing one value of $i \in 1, ..., g$, we can take $a_j = b_j = 0$ for $i \neq j$, to obtain

$$d\xi(a_1, ..., b_g) = \mathrm{Ad}_{(\prod_{j>i}A_jB_jA_j^{-1}B_j^{-1})^{-1}B_iA_i}\left((\mathrm{Ad}_{B_i^{-1}} - 1)a_i + (1 - \mathrm{Ad}_{A_i^{-1}})b_i\right). \tag{2.14}$$

If for any i the map $\mathfrak{g}^2 \to \mathfrak{g}$:

$$(a, b) \to (\mathrm{Ad}_{B_i^{-1}} - 1)a + (1 - \mathrm{Ad}_{A_i^{-1}})b \tag{2.15}$$

is surjective, then by varying a_i and b_i one obtains all of $\mathfrak{g}$, implying ξ would be surjective. Therefore, if instead ξ does not have full rank at an irreducible point $(A_1, ..., B_g)$, then for all i the above map $\mathfrak{g}^2 \to \mathfrak{g}$ is not surjective.

For $G = SU(2)$, the non-surjectivity of this map implies that A_i and B_i commute. If either is $\pm\mathbb{1}$ then they commute. Otherwise, $(\mathrm{Ad}_{B_i}^{-1} - 1)$ and $(1 - \mathrm{Ad}_{A_i}^{-1})$ have images given by the two planes perpendicular to the rotation axes of Ad_{B_i} and Ad_{A_i}. Since their sum is not surjective and $\dim\mathfrak{g} = 3$, their sum is dimension 2, meaning these planes coincide. Hence Ad_{A_i} and Ad_{B_i} share the same axis of rotation, implying A_i and B_i commute.

Since we can repeat this argument for each i, we conclude that if ξ is not full rank at $(A_1, ..., B_g)$ then $[A_i, B_i] = 0$ for all i and hence we can simplify the differential to

$$d\xi(a_1, ..., b_g) = \sum_{i=1}^{g}\left[\mathrm{Ad}_{B_iA_i}\left((\mathrm{Ad}_{B_i^{-1}} - 1)a_i + (1 - \mathrm{Ad}_{A_i^{-1}})b_i\right)\right]. \tag{2.16}$$

Since $[A_j, B_j] = 0$, the jth term in this sum has image given by the plane perpendicular to A_i (which is the same as that of B_i). Furthermore, because $d\xi$ is not full rank, we must have that for each j, the image is the same plane, as otherwise by the same dimensional count as above we'd have a contradiction. Thus, the $\{A_i, B_i\}_{i=1}^{g}$ all pairwise commute and so $(A_1, ..., B_g)$ is reducible. By the contrapositive, $\xi : G^{2g} \to G$ is a submersion on the irreducible points.

The action of $SU(2)$ on $\xi^{-1}(e)$ by conjugation is not free since $-\mathbb{1}$ acts trivially. Thus we define an action of $SU(2)/\pm\mathbb{1}$ by conjugation, which does act freely. Suppose $[C] \in SU(2)/\pm\mathbb{1}$ acts trivially on $(A_1, ..., B_g) \in \xi^{-1}(e)$. Then C commutes with all A_i and B_i, and since the point is irreducible, there is some pair in $(A_1, ..., B_g)$ that does not commute; call that pair (X, Y). Then C commutes with X and Y, which do not commute with eachother, so Ad_X and Ad_Y have different rotation axes, and Ad_C cannot have both; C must be ± 1. Thus the action of $SU(2)/\pm\mathbb{1}$ on $\xi^{-1}(e)$ is free.

Finally, the quotient $SU(2)/\pm\mathbb{1} \cong SO(3)$ is compact, and $\mathcal{M}_0 = \xi^{-1}(e)/SO(3)$. Since ξ is a submersion and $SO(3)$ is a compact group acting freely on it, the quotient $\mathcal{M}_0$ is a smooth manifold, with dimension $6g-6$ as computed at the beginning of the proof. □

Now that we have some understanding of the moduli space $\mathcal{M}$, we will pass to a holomorphic description of $\mathcal{M}$ in terms of *semi-stable* holomorphic vector bundles over Σ.

2.3 Semi-stable Holomorphic Bundles

When Σ is a Riemann surface, one can use its complex structure to augment the study of $\mathcal{M}$. Differential forms on a Riemann surface have a splitting, $\Omega^1(\Sigma) = \Omega^{1,0}(\Sigma) \oplus \Omega^{0,1}(\Sigma)$ which induces a splitting on the space $\mathcal{A}$ of connections on complex vector bundles E over Σ. As we will discuss, the $(0,1)$ part of a connection $A \in \Omega^1(\Sigma) \otimes \mathfrak{gl}(n,\mathbb{C})$ defines a *holomorphic structure* on E, and we can describe the moduli spaces of connections in terms of holomorphic structures.

Definition 2.8. *A holomorphic structure on a complex vector bundle E is a choice of trivializations $\{U_\alpha, \phi_\alpha\}$ for E, such that the transition functions*

$$T_{\alpha,\beta} = \phi_\alpha \circ \phi_\beta^{-1} : E|_{U_\alpha \cap U_\beta} \to E|_{U_\alpha \cap U_\beta},$$

are biholomorphic.

An equivalent and convenient characterization is as follows. Given a holomorphic structure, in every chart $\{U_\alpha\}$, with local frame $\{e_1, ..., e_n\}$ for E, one can define a local operator taking a section $s = s^i e_i$ to

$$\overline{\partial}_E(s) = \overline{\partial}(s^i) \otimes e_i,$$

where $\overline{\partial}$ is the usual Cauchy-Riemann operator on $\mathbb{C}$. Let us check this operator is well defined globally on E. On the intersection $U_\alpha \cap U_\beta$, with local frames $\{e_i\}$ and $\{f_i\}$, we have $s = s^i e_i = \tilde{s}^i f_i$, with $s^i = T_{\alpha\beta}{}^i_j \tilde{s}^j$. Since $T_{\alpha\beta}$ is biholomorphic, we have:

$$\begin{aligned}\overline{\partial}_E(s) = \overline{\partial}(s^i) \otimes e_i &= \overline{\partial}(T_{\alpha\beta}{}^i_j \tilde{s}^j) \otimes f_i \\ &= T_{\alpha\beta}{}^i_j \overline{\partial}(\tilde{s}^j) \otimes f_i.\end{aligned}$$

Hence $\overline{\partial}_E$ transforms with $T_{\alpha\beta}$ and it is globally well defined. We call $\overline{\partial}_E$ the *Dolbeault Operator* corresponding to the holomorphic structure on E. Conversely, if we have a differential operator $\overline{\partial}_E : \Gamma(E) \to$

$\Omega^{0,1}(\Sigma) \otimes \Gamma(E)$, we can define a holomorphic structure on E as operator defines local holomorphic structure by defining s to be holomorphic if $\overline{\partial}_E(s) = 0$, and these local structures can always be glued to give a global structure when Σ is a Riemann surface [AB83, §5].

Therefore, in order to study the space of holomorphic structures on E, we can equivalently study the space of Dolbeault operators on E. In a smooth local trivialization of E, we can write

$$\overline{\partial}_E = \overline{\partial} + B,$$

where $\overline{\partial}$ is the usual Cauchy-Riemann operator and $B \in \Omega^{0,1}(E, \text{End } E)$.

On an arbitrary complex manifold, there may be an obstruction to B's integrability, which lives in $\Omega^{0,2}(\Sigma)$. However $\dim \Sigma = 1$, so $\Omega^{0,2}(\Sigma) = 0$ and there is no constraints on B. Therefore the set of structures is an affine complex space with translations $\Omega^{0,1}(M, \text{End } E)$. We want to consider only equivalence classes of hermitian vector bundles, so we want to quotient out the action of $\text{Aut}(E) = \mathbb{C}^\infty(\Sigma, GL_n\mathbb{C})$ by change of basis. It is the space of such isomorphism classes, $N(n, k)$, that we wish to describe.

In order to put geometric structure on this space, we need to add an additional constraint.

Definition 2.9. *Let the* slope *of a bundle E be*

$$\mu := \deg(E)/rank(E),$$

where $\deg(E)$ *denotes the first Chern class of the line bundle* $\det E$*. Then E is said to be stable if, for every proper subbundle F of E, $\mu(F) < \mu(E)$. If the inequality is not strict, E is semi-stable.*

The Narasimhan-Seshadri correspondance tells us that to study the moduli space of flat $U(n)$ connections, one should restrict their focus to the subspace of semi-stable bundles.

Theorem 2.10 (Narasimhan-Seshadri)**.** *Let Σ be a compact connected Riemann surface with $g \geq 2$ and $G = U(n)$. Then*

1. *There is a correspondence between representations ρ up to conjugation and semi-stable holomorphic bundles E of degree zero up to gauge equivalence.*

2. *E is stable if and only if ρ is irreducible.*

Proof. The original proof of Narasimhan and Seshadri [NS65] is algebraic, and there is more recent proof of Donaldson [Don83] using the Yang-Mills functional on connections. □

This theorem in combination with Theorem 2.3 tells us that there are three equivalent sets we can use to describe the moduli space of flat connections. We can look at flat connections, representations of the fundamental group, or semi-stable holomorphic bundles.

For this reason, we will restrict our attention to only the subset of $N(n,k)$ consisting of semi-stable bundles; $N_{ss}(n,k)$. In particular, motivated by Theorem 2.10, we will denote the space of degree 0 semi-stable $SL(n,\mathbb{C})$ bundles as $\mathcal{M}$. The next result tells us that $N_{ss}(n,k)$ has a well-defined geometric structure.

Theorem 2.11. *For a compact connected Riemann surface Σ of genus g, there exists a connected complex projective variety $N_{ss}(n,k)$ of semi-stable holomorphic bundles. When n and k are co-prime, $N_{ss}(n,k)$ is a smooth manifold.*

Proof. Originally proven by Mumford [Mum04], see also an outline given by Thaddeus [Tha21, p. 4]. □

Remark: For $g=0$ there are no stable holomorphic bundles. It is a theorem of Grothendieck [Gro57, Theorem 2.1] that any holomorphic bundle E over $\mathbb{P}^1$ can be written as $E \cong \oplus_{i=1}^{\operatorname{rank} E} \mathcal{O}(n_i)$, which lets us verify that E is at best semi-stable, which occurs when all the n_i are equal.

Now let us focus only on $\mathcal{M}$. Just as in the representation picture, we will write $\mathcal{M}_0$ to denote the stable bundles. $\mathcal{M}_0$ is a smooth manifold [AB83, §7], and we can talk about its geometry. Being degree 0 means that the line bundle $\det E$ is topologically trivial, and a choice of global trivialization gives us an $SL(n,\mathbb{C})$ structure on E. To preserve this trivialization, we will restrict $\operatorname{Aut}(E)$ and $\operatorname{End} E$ to their intersections in $SL(n,\mathbb{C})$ and $\mathfrak{sl}(n,\mathbb{C})$ respectively.

An important property of stable bundles which we will make use of is the *stable implies simple lemma*:

Lemma 2.12 (Stable implies simple)**.** *If E is stable, then $H^0(\Sigma, End\ E) = \mathbb{C}$, and $H^0(\Sigma, \mathfrak{sl}(E)) = 0$.*

Proof. Suppose $f \in H^0(\Sigma, \operatorname{End}(E))$, $\lambda \in \mathbb{C}$. Then $\ker(f)$ and $\operatorname{im}(f)$ are subsheaves of E and we have the exact sequence

$$0 \to \ker(f) \to E \to \operatorname{im}(f) \to 0, \tag{2.17}$$

therefore $c_1(\ker(f))c_1(\operatorname{im})(f) = c_1(E)$ and $\operatorname{rank}(\ker(f))+\operatorname{rank} \operatorname{im}(f) = n$. Then either $\mu(\ker(f))$ or $\mu(\operatorname{im}(f))$ must be greater than or equal to $\mu(E)$, and hence either $\ker(f) = E$ or $\operatorname{im}(f) = E$ since E is stable.

Now for $\mathbb{1} \in H^0(\Sigma, \operatorname{End}(E))$, the argument above applied to $f - \lambda\mathbb{1}$, $\lambda \in \mathbb{C}$ shows that $f = \lambda\mathbb{1}$ and therefore $H^0(\Sigma, \operatorname{End}(E)) = \mathbb{C}$. Since $\mathbb{1}$ is not traceless, it is not in $\mathfrak{sl}(E)$, and hence $H^0(\Sigma, \mathfrak{sl}(E)) = 0$. □

Since stability is an open condition, one may consider deformations to compute the tangent space. At a bundle $(E, \overline{\partial}_E)$ with holomorphic structure given by transition functions $T_{\alpha,\beta}$, we can consider deforming the holomorphic structure to

$$T_{\alpha,\beta}(\epsilon) = T_{\alpha,\beta} + \epsilon t_{\alpha,\beta}, \tag{2.18}$$

where $t_{\alpha,\beta}$ is a Čech 1-cochain in End (E) and $\epsilon^2 = 0$. For this to remain a well-defined holomorphic structure, we require that $T_{\alpha,\beta}(\epsilon)$ satisfies the cocycle condition for all ϵ. That is, on $U_\alpha \cap U_\beta \cap U_\gamma$,

$$\begin{aligned} T_{\alpha,\beta}(\epsilon)T_{\beta,\gamma}(\epsilon) &= T_{\alpha,\gamma}(\epsilon) \\ (T_{\alpha,\beta} + \epsilon t_{\alpha,\beta})(T_{\beta,\gamma} + \epsilon t_{\beta,\gamma}) &= (T_{\alpha,\gamma} + \epsilon t_{\alpha,\gamma}) \\ T_{\alpha,\beta}T_{\beta,\gamma} + \epsilon(t_{\alpha,\beta}T_{\beta,\gamma} + T_{\alpha,\beta}t_{\beta,\gamma}) + \epsilon^2 t_{\alpha,\beta}t_{\beta,\gamma} &= T_{\alpha,\gamma} + \epsilon t_{\alpha,\gamma} \end{aligned}$$

using that $\epsilon^2 = 0$ and $T_{\alpha,\beta}$ satisfy the cocycle condition, we have

$$\begin{aligned} T_{\alpha,\gamma} + \epsilon(t_{\alpha,\beta}T_{\beta,\gamma} + T_{\alpha,\beta}t_{\beta,\gamma}) &= T_{\alpha,\gamma} + \epsilon t_{\alpha,\gamma} \\ t_{\alpha,\beta}T_{\beta,\gamma} + T_{\alpha,\beta}t_{\beta,\gamma} &= t_{\alpha,\gamma}. \end{aligned}$$

This condition tells us that $t_{\alpha,\beta}$ is a 1-cocycle in the sheaf End (E). When we quotient the action of of $\mathrm{Aut}(E)$, we find that the tangent space to $N(n,k)$ is $H^1(\Sigma, \mathrm{End}\ (E))$. Similarly, if we include an $SL(n,\mathbb{C})$ structure, we get the tangent space of $\mathcal{M}$, $T_E\mathcal{M} = H^1(\Sigma, \mathfrak{sl}(E))$.

Theorem 2.13. *If Σ has genus $g \geq 2$ and E is stable, then* $\dim H^1(\Sigma, \mathit{End}\ E) = n^2(g-1)+1$, *and* $\dim H^1(\Sigma, \mathfrak{sl}(E)) = (n^2-1)(g-1)$.

Proof. We can compute the dimension of $H^1(\Sigma, \mathrm{End}\ (E))$ via Hirzebruch-Riemann-Roch.

$$\dim H^0(\mathrm{End}\ (E)) - \dim H^1(\mathrm{End}\ (E)) = \int_\Sigma ch(L)\mathrm{Td}(\Sigma), \tag{2.19}$$

where $ch(V)$ is the Chern character and $\mathrm{Td}(\Sigma)$ is the Todd class of $T\Sigma$. We know from the stable implies simple lemma (2.12) that $H^0(\mathrm{End}\ E) = \mathbb{C}$. For a compact Riemann surface, the Todd class is $1 + c_1(T\Sigma)/2 = 1 + (1-g) = 2-g$, and for a vector bundle V the Chern character is $\mathrm{rank}(V) + c_1(V)$.

Since End $E = E \otimes E^*$, its rank is n^2 and its Chern class is

$$\begin{aligned} c_1(\mathrm{End}\ E) &= (\mathrm{rank}E)c_1(E) + (\mathrm{rank}E^*) \\ &= nc_1(E) - nc_1(E) \\ &= 0 \end{aligned}$$

Using these computations, equation 2.19 becomes

$$\begin{aligned} c_1(\mathrm{End}\ (E)) + \mathrm{rank}(\mathrm{End}\ (E))(1-g) &= \dim H^0(\mathrm{End}\ (E)) - \dim H^1(\mathrm{End}\ (E)) \\ \dim H^1(\mathrm{End}\ (E)) &= 1 - n^2(1-g) = n^2(g-1)+1. \end{aligned}$$

For $H^1(\Sigma, \mathfrak{sl}(E))$, we instead have from Lemma 2.12 that the dimension of $H^0 = 0$ and rank $\mathfrak{sl}(E) = (n^2 - 1)$ so we obtain:

$$\dim H^1(\mathfrak{sl}(n, \mathbb{C})) = 0 - (n-1)^2(1-g) = (n-1)^2(g-1).$$

□

When E has a hermitian metric $h : E \otimes E \to \mathbb{C}$, the conjugate Hodge star $\bar{\star} : \Omega^{0,1}(\Sigma) \to \Omega^{1,0}(\Sigma)$ combined with h allows us to define a hermitian inner product on $H^1(\text{End }(E))$. First h defines a metric on End E; if $A, B \in \text{End } E$, let

$$g(A, B) = \text{Tr }(A^\dagger B), \tag{2.20}$$

where $\dagger$ is defined in terms of h, by $h(Ae, e) = h(e, A^\dagger e)$ for all $e \in E$. Then for any $\alpha = A \otimes a$, $A \in \text{End } E$ and $a \in \Omega^{0,1}(\Sigma)$, we define:

$$\bar{*}_E \alpha = g(A, -) \otimes \bar{*} a, \tag{2.21}$$

and

$$\langle \alpha, \beta \rangle = \int_\Sigma \alpha \wedge_g \bar{*}_E \beta = \int_\Sigma g(A, B)\ a \wedge \bar{*} b. \tag{2.22}$$

In a local co-ordinate chart where $\alpha = Adz$ and $\beta = Bdz$, this takes the form

$$\langle \alpha, \beta \rangle = \int_\Sigma \text{Tr }(A^\dagger B)\ dz \wedge d\bar{z}. \tag{2.23}$$

The relationship between this space of connections and the space of holomorphic vector bundles is described by the Narasimhan-Seshadri theorem (2.10). In one direction, given a flat connection A on P, inducing a connection on the associated bundle E, we can decompose $A = A^{0,1} + A^{1,0}$. This allows us to take $\overline{\partial}_E = A^{0,1}$ as a complex structure on E corresponding to the flat connection A. The Narasimhan-Seshadri theorem guarantees that this structure will define a stable bundle, and also gives the converse direction; that flat stable structures $\overline{\partial}_E$ define flat unitary connections A.

2.4 Symplectic Picture

Another description of the moduli space $\mathcal{M}$ is in terms of a symplectic reduction, which makes the symplectic structure more clear. We will be considering an infinite dimensional symplectic manifold and taking a symplectic quotient, which requires more careful consideration then we provide here. Rigorous details of this picture can be read in Atiyah-Bott [AB83].

Again let Σ be the compact connected Riemann surface of genus g. Consider the trivial principal $G = SU(2)$ bundle P, over Σ and let $\mathcal{A}$ denote the space of smooth principal connections on P. In a fixed trivialization $P \cong G \times \Sigma$, a connection is determined by a form $A \in \Omega^1(\Sigma) \otimes \mathfrak{g}$. A connection is flat if and only if it has zero curvature, $0 = F_A := dA + A \wedge A$. The gauge group $\mathcal{G} = \mathrm{Hom}(\Sigma, G)$ acts on $\mathcal{A}$ as follows: for $g \in \mathcal{G}$,

$$g \circlearrowright A := g^{-1}Ag + g^{-1}dg. \tag{2.24}$$

Therefore to find the moduli space of gauge equivalence classes of connections, we want to consider a quotient $\mathcal{A}/\mathcal{G}$. This quotient will not be finite dimensional in general, so we want to impose the further constraint that $F_A = 0$.

The vector space $\mathcal{A}$ has a natural symplectic structure, which comes from the inner product (the Killing form) on the Lie Algebra $\mathfrak{g}$, $K : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$. If $A = \alpha \otimes X$ and $B = \beta \otimes Y$ then we can define

$$\omega(A, B) = \int_{\Sigma} K(X, Y)\alpha \wedge \beta. \tag{2.25}$$

If $\mathcal{A}$ were a finite dimensional symplectic manifold, to obtain the quotient of the flat connections $\mathcal{A}_0$ with $F_A = 0$ by $\mathcal{G}$, we could check that F_A is a moment map for the action of $\mathcal{G}$ and that ω is preserved by the action, to then obtain the symplectic quotient $\mathcal{M} = F_A^{-1}(0)/\mathcal{G} = \mathcal{A}_0/\mathcal{G}$. Although $\mathcal{A}$ is infinite-dimensional, this process still works, and yields a finite dimensional moduli space.

Theorem 2.14. *The symplectic structure ω defined above is invariant under the action of $\mathcal{G}$ on $\mathcal{A}$. Furthermore, the curvature F_A is a moment map for this action.*

Proof. Atiyah and Bott [AB83, §9], at the end of section 9. □

Then we consider the moduli space of flat connections by symplectic reduction:

$$\mathcal{M} = \mathcal{A} /\!/ \mathcal{G} = F^{-1}(0)/\mathcal{G} = \mathcal{A}_0/\mathcal{G}. \tag{2.26}$$

Symplectic reduction also gives us a symplectic structure on the quotient space $\mathcal{M}$, such that under the pullback by the quotient map, $q : \mathcal{A}_0 \to \mathcal{M}$, we recover the symplectic form ω for $\mathcal{A}$. This symplectic structure on $\mathcal{M}$ will be called the *Atiyah-Bott symplectic form* when we need to distinguish it from other forms on $\mathcal{M}$.

Chapter 3

Geometric Quantization when $G = SU(2)$

In Chapter 2, we built the space $\mathcal{M}$ of flat $SU(2)$ connections on a compact connected Riemann surface Σ of genus $g \geq 2$, which is a complex projective variety with dimension $3g - 3$. Furthermore, $\mathcal{M}$ is equipped with the Atiyah-Bott symplectic form ω, and in Section 3.1 we will equip $\mathcal{M}$ with a prequantum line bundle $(\mathcal{L}, \nabla)$ over $\mathcal{M}$ with curvature $2\pi i\omega$.

This system has two notable polarisations one can use to perform a geometric quantization. There is a Kaehler polarisation, whose quantization is known to have dimension computed by the *Verlinde formula*, and there is a real polarisation introduced by Weitsman [Wei92]. Jeffrey and Weitsman discuss the Bohr-Sommerfeld geometric quantization of $\mathcal{M}$ with this real polarisation. Bohr-Sommerfeld quantization typically requires a compact symplectic manifold (M, ω) and line bundle $\mathcal{L}$, with a real polarization of M. A real polarization of M is a map $\pi : M \to B$ onto a manifold of half dimension, such that $\omega|_{\pi^{-1}(b)} = 0$ for all $b \in B$. Supposing $\pi : M \to B$ is also a fibration, there will be a finite set of *Bohr-Sommerfeld points* b_i for which $\mathcal{L}$ restricted to the fibers L_{b_i} of π possesses global covariant constant sections. Let J_π denote the sheaf of sections of $\mathcal{L}$ which are covariant constant along the fibres of π. Then the Bohr-Sommerfeld quantization of a prequantum system $(M, \omega, \mathcal{L})$ (for π) is the vector space

$$\mathcal{H} = \bigoplus_{i=0}^{\dim M} H^i(M, J_\pi). \tag{3.1}$$

Here we aim to compute the dimension of $\mathcal{H}$. If B_s is the set of all Bohr-Sommerfeld points, and for each $b \in B_s$ S_b is the space of global covariant constant sections of $\mathcal{L}|_{\pi^{-1}(b)}$, then Sniatycki [Śni77] proves that there is a natural isomorphism:

$$\mathcal{H} \cong \bigoplus_{b \in B_s} S_b. \tag{3.2}$$

Since each S_b is one dimensional, counting $\dim \mathcal{H}$ boils down to counting the Bohr-Sommerfeld points.

For the prequantum system on $\mathcal{M}$ we're considering, the above theorem does not apply because $\mathcal{M}$ is not a smooth manifold and the polarisation we will describe is not a fibration. Sniatycki's theorem simply provides inspiration for investigating the Bohr-Sommerfeld set in $\mathcal{M}$, and Jeffrey and Weitsman show that Bohr-Sommerfeld fibres are associated to marked trivalent graphs satisfying the quantum Clebsch-Gordan conditions, and the number of such graphs is called the *Verlinde dimension*, counted by the Verlinde formula [JW92, Thm. 8.1].

3.1 Prequantum Line Bundle on the Moduli Space

A key part of a prequantum system is the prequantum line bundle $\mathcal{L}$ with curvature $2\pi i\omega$. Let us build such a bundle over our moduli space $\mathcal{M}$, following the paper of Ramadas, Singer and Weitsman [RSW89]. Given a connection 1-form A on a 3-manifold M, we can define the *Chern-Simons action*:

$$\mathrm{CS}(A) = \frac{k}{2\pi} \int_N A \wedge dA + \frac{2}{3} A \wedge A \wedge A, \; k \in \mathbb{Z}. \tag{3.3}$$

Under a change of gauge $g \cdot A = g^{-1}Ag + g^{-1}dg$, the Chern-Simons action transforms as

$$\mathrm{CS}(g \cdot A) = \mathrm{CS}(A) - k \int_N d\mathrm{Tr}\; \big((dg)g^{-1} \wedge A\big) - \frac{k}{3} \int_N \mathrm{Tr}\; ((g^{-1}dg) \wedge (g^{-1}dg) \wedge (g^{-1}dg)). \tag{3.4}$$

The third term on the right side will always be in $2\pi\mathbb{Z}$, and the second term is a total derivative which we can integrate using Stokes theorem over the boundary.

Using this action, we will define a function $\Theta : \mathcal{A} \times \mathcal{G} \to \mathbb{C}$. Pick any 3-manifold N for which $\partial N \cong \Sigma$. Given a pair $(A, g) \in \mathcal{A} \times \mathcal{G}$, we can always lift it to a choice of $(\tilde{A}, \tilde{g})$ on N, since $\pi_1(SU(2)) = \pi_2(SU(2)) = 0$. We define Θ as

$$\Theta(A, g) = \exp\left[i\mathrm{CS}(\tilde{A}) - i\mathrm{CS}(\tilde{g} \cdot A)\right]. \tag{3.5}$$

Equation (3.4) lets us simplify this;

$$\begin{aligned}\Theta(A,g) &= \exp(i\mathrm{CS}(\tilde{g}\cdot\tilde{A}) - \mathrm{CS}(\tilde{A})) \\ &= \exp\left[-ik\int\limits_N d\mathrm{Tr}\ \left((d\tilde{g})\tilde{g}^{-1}\wedge\tilde{A}\right) - 2\pi\kappa\right], \kappa\in\mathbb{Z} \\ &= \exp\left[-ik\int\limits_\Sigma \mathrm{Tr}\ (dg\ g^{-1}\wedge A)\right].\end{aligned}$$

Note that in the last equality, we use that the lifted pair $(\tilde{A},\tilde{g})$ restricts to (A,g) on $\partial N=\Sigma$ by definition. This computation shows us in particular that $\Theta(A,g)$ is well defined, as it does not depend on our choice of N or the lifting. One can further check that Θ is a cocycle:

$$\Theta(A,g)\Theta(g\cdot A,h) = \Theta(A,gh). \tag{3.6}$$

Thus we are ready to define $\mathcal{L}$.

Definition 3.1 (Chern-Simons Prequantum Line Bundle)**.** *For $(A,z)\in\mathcal{A}\times\mathbb{C}$, let $(A,z)\sim(g\cdot A,\Theta(A,g)z)$ under any $g\in\mathcal{G}$. Then define*

$$\mathcal{L} = \mathcal{A}\times\mathbb{C}/\sim, \tag{3.7}$$

which is a complex line bundle on $\mathcal{M}=\mathcal{A}/\mathcal{G}$. Since Θ is $U(1)$ valued, it is a hermitian line bundle.

Recall the Atiyah-Bott form, before quotienting out gauge transformations, is the form

$$\omega(\alpha,\beta) = \frac{i}{2\pi}\int\limits_\Sigma \mathrm{Tr}\ (\alpha\wedge\beta) \tag{3.8}$$

on $\mathcal{A}$. We can write it as $\omega = d\sigma$, where σ is the one-form

$$\sigma(\alpha) = \frac{i}{4\pi}\int\limits_\Sigma \mathrm{Tr}\ (A\wedge\alpha). \tag{3.9}$$

Identifying $\mathfrak{u}(1)=i\mathbb{R}$ allows the form $\sigma\in\Omega^1(\mathcal{A}_0,\mathbb{C})$ to define a connection one-form on $\mathcal{A}_0\times U(1)$. We want to check that σ agrees with the pull-back of a unitary connection on $\mathcal{L}$ with curvature ω, under the map

$$\mathcal{A}_0\to\mathcal{A}_0\times 1\hookrightarrow\mathcal{A}_0\times U(1)\to\mathcal{L}. \tag{3.10}$$

The last map comes from quotienting out the action of $\mathcal{G}$. For σ to pass to $\mathcal{L}$, we want that

$$\sigma_A(\alpha) = \sigma_{A^g}(\alpha) \mod \mathcal{G}, \tag{3.11}$$

where $\mod \mathcal{G}$ means the infinitesimal of $\mathcal{G}$ by adding $d\Theta(\alpha,g)$. We compute

$$\begin{aligned}\sigma_{A^g}(\alpha) &= \int_\Sigma \mathrm{Tr}\ \left[(g^{-1}Ag+g^{-1}dg)\wedge\alpha\right] \\ &= \int_\Sigma \mathrm{Tr}\ \left[g^{-1}Ag\wedge\alpha\right] + \int_\Sigma \mathrm{Tr}\ (g^{-1}dg\wedge\alpha) \\ &= \sigma_{A^g}(\alpha) + d\Theta(\alpha,g).\end{aligned}$$

Therefore, the connection σ passes to a connection on $\mathcal{L}$, with curvature ω.

There is another construction of a prequantum line bundle for $\mathcal{M}$, the *determinant line bundle* $\mathcal{L}_D$ of Quillen [Qui85]. Ramadas, Singer and Weitsman prove that

Theorem 3.2. *The line bundle $\mathcal{L}$ defined above is isomorphic to the determinant bundle $\mathcal{L}_D$ as a hermitian line bundle with connection on $\mathcal{M}$.*

Proof. Ramadas, Singer, Weitsman [RSW89, Theorem 2] □

3.2 Polarisation of the Moduli Space

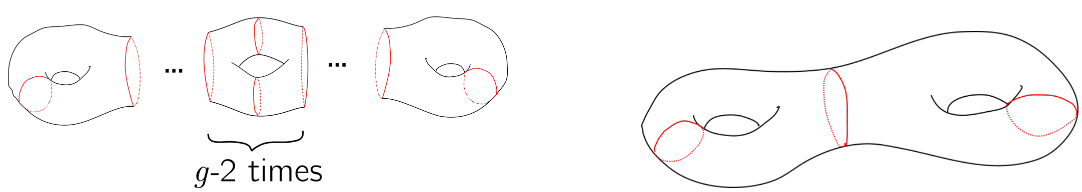
As before, let Σ be a compact Riemann surface and $\mathcal{M}$ the moduli space of flat $G = SU(2)$ connections on Σ. Following Jeffrey and Weitsman [JW92], we describe an action of T^{3g-3} on $\mathcal{M}$. Let C be a closed oriented curve in Σ and pick a basepoint $y \in C$. We can define a function $\tilde{f}_C : \mathcal{A} \to \mathbb{R}$ by

$$\tilde{f}_C(A) = \frac{1}{2}\mathrm{hol}_C(A), \tag{3.12}$$

where $\mathrm{hol}_C(A)$ means the holonomy of A around C from y to y. Since the holonomy is $\mathcal{G}$ invariant, this passes to $f_C : \mathcal{M} \to \mathbb{R}$. Σ admits a decomposition into *trinions* or *pairs of pants*, which are copies of a disc with two holes:

$$D = \{z \in \mathbb{C} \mid |z| \leq 2\} - \{z \mid |z-1| < 1/2\} \cup \{z \mid |z+1| < 1/2\}, \tag{3.13}$$

with marked points on the boundary of D.



(a) One decomposition for surfaces of genus $g \geq 2$.

(b) A different decomposition for $g = 2$.

Figure 3.1: Decomposition of a surface Σ into $2g-2$ trinions.

Suppose we are given such a decomposition of Σ into $2g-2$ trinions D_γ, $\gamma \in \{1, 2, ..., 2g-2\}$, joined along their boundaries and with the marked points on the boundaries coinciding for any trinions with non-trivial intersection. Then the boundary circles of D_γ give a collection C_i, $i \in \{1, 2, ..., 3g-3\}$ of closed oriented curves in Σ for which we get corresponding functions $f_i = f_{C_i} : \mathcal{M} \to \mathbb{R}$ using the above definition. Since these functions are the trace of $SU(2)$ matrices, they can be described by cosine of angles θ_i,

$$\theta_i(A) = \cos^{-1}(f_i(A)), \tag{3.14}$$

where θ_i is taken to lie in $[0, \pi]$. This defines a map $\theta = (\theta_1, ..., \theta_{3g-3}) : \mathcal{M} \to \mathbb{R}^{3g-3}$. These θ_i are smooth on $U_i := \theta_i^{-1}(0, \pi) \subset \mathcal{M}$, which is open and dense. Thus, the Hamiltonian flows of each θ_i are defined on $\mathcal{M}^s = \bigcap_{i=1}^{3g-3} U_i \subset \mathcal{M}$. These Hamiltonian flows are periodic with constant period, which means they induce a torus action on $\mathcal{M}^s$. Explicitly, if we let X_i denote the Hamiltonian vector field of θ_i, defined by

$$\iota_{X_i}\omega = d\theta_i, \tag{3.15}$$

and let e^{tX_i} be the corresponding vector field flow, then the action is given by $g = (\alpha_1, ..., \alpha_{3g-3}) \in T^{3g-3}$ acts by

$$A \to e^{\alpha_1 X_1 + ... + \alpha_{3g-3} X_{3g-3}} A. \tag{3.16}$$

The Lie algebra of T^{3g-3} is $\mathbb{R}^{3g-3}$ and we interpret $\theta(A)$ as being dual by $\langle \theta, X \rangle = \sum \theta_i X_i$. Then

$$d\left(\langle \theta(A), X \rangle\right) = d \sum \theta_i X_i = \sum X_i d\theta_i = \iota_X \omega, \tag{3.17}$$

which means θ is the moment map for the torus action. These functions f_i also give us a real polarization of $\mathcal{M}$. Let $B \subset \mathbb{R}^{3g-3}$ be the image of the f_i,

$$B = \{(f_i(E), ..., f_{3g-3}(E)) \mid E \in \mathcal{M}\}, \tag{3.18}$$

then the fibers of the map $\pi = (f_1, ..., f_{3g-3})$ foliate the smooth locus of $\mathcal{M}$, and the generic fibre is a Lagrangian subvariety.

Alternatively, one can describe the polarization using the picture of connections as representations of the fundamental group $\pi_1(\Sigma)$. First, a preliminary result. Let $T \subset SU(2)$ be a maximal torus.

Definition 3.3. *A connection A on Σ^g is said to be* adapted to a trinion decomposition *(a.t.d.) if there is a tubular neighbourhood $V_i \cong (-1, 1) \times S^1$ of each boundary circle C_i in the decomposition, such that in co-ordinates (s, θ) for V_i,*

$$A|_{V_i} = X_i d\theta, \tag{3.19}$$

where X_i is a constant element in $\mathfrak{t} = Lie(T)$.

Theorem 3.4. *For all $y \in \pi^{-1}(b)$, there exists an adapted to trinion decomposition connection A in the gauge equivalence class $y \mod \mathcal{G}$.*

Proof. First, any boundary circle C in a trinion decomposition has tubular neighbourhood $V \cong C \times [-1, 1]$ and co-ordinates (s, θ) in which the connection y takes the form:

$$y = R(s, \theta)d\theta + S(s, \theta)ds, \tag{3.20}$$

for some $R, S \in C^\infty(\Sigma, \mathfrak{su}(2))$. Suppose we have a gauge transformation $h(s, \theta)$ with $\partial_s h(s, \theta) = 0$, $h(0, 0) = \mathbb{1}$ and

$$\frac{\partial h}{\partial \theta} - Rh = 0. \tag{3.21}$$

Then the θ component of $h \cdot y$ will be

$$h^{-1}\left(\frac{\partial h}{\partial \theta} - Rh\right) = 0. \tag{3.22}$$

Such an h must exist as equation (3.21) is four linear first order ODEs for the components of the matrix h. One must only check that this solution will be an $SU(2)$ matrix for all s as required. At $\theta = 0$, $h = \mathbb{1}$, and the derivative is $-R(0,0) \in \mathfrak{su}(2)$; therefore the solution starts in $SU(2)$ and its derivative for all time is an $\mathfrak{su}(2)$ matrix, so the solution remains in $SU(2)$. Notice that $h(0) \neq h(2\pi)$ so this gauge transformation exists on $[0, 2\pi] \times [-1, 1]$ but it does not pass to V.

Now let $H = h(0, 2\pi)$. By the maximal torus theorem there exists some constant gauge transformation on V bringing H into T, so we reduce to $H \in T$. Let $X \in \mathfrak{t}$ be the element such that $\exp(2\pi X) = H^{-1}$ and define $f(s, \theta) = \exp(\theta X)$. Then $\partial_s f = 0$, $\partial_\theta f = Xf$ and

$$f \cdot h \cdot y = f^{-1}Xfd\theta + f^{-1}h^{-1}\left(\frac{\partial h}{\partial \theta} - Rh\right)f + S(s,\theta)ds = Xd\theta + S(s,\theta)ds. \tag{3.23}$$

Furthermore $fh(0,0) = \mathbb{1}$ and $fh(0, 2\pi) = H^{-1}H = \mathbb{1}$ so the gauge transformation fh satisfies the periodic boundary condition and is well defined on V. Thus it remains to find a gauge transformation sending the ds component to zero. Such a transformation must satisfy

$$\frac{\partial k}{\partial s} - Sk = 0, \tag{3.24}$$

with $k(-1, \theta) = \mathbb{1}$ and $\partial_\theta k = 0$. As before equation (3.24) is four first order ODEs with a unique solution, and the same argument as before shows k will be in $SU(2)$. The s co-ordinate has no periodic boundary condition, so k immediately passes to V, and the composition $g = kfh$ gives us our gauge transformation on V putting y in the desired form.

Repeating this process for each boundary circle C_i in our decomposition gives a set of local gauge transformations g_i on each V_i. Complete V_i to a cover $\{U_i\}$ of Σ and let $g_i = \mathbb{1}$ on the additional sets added. Within each V_i, pick a smaller tubular neighbourhood W_i, and let $\{\phi_i\}$ be a partition of unity for $\{U_i\}$ with $\phi_i = 1$ on W_i. Define the global gauge transformation $g = \sum g_i \phi_i$. Then for all i, $(g \cdot y)|_{W_i} = X_i d\theta_i$ and so $g \cdot y$ is an a.t.d. representative for $y \mod \mathcal{G}$. □

This lets us define subgroups of $G = SU(2)$, which correspond to stabilizers of flat connections. Suppose A is an a.t.d connection. Then the stabilizer of $A|_{C_i}$ in $\mathcal{G}(C_i) = \mathrm{Hom}(C_i, G)$ consists of constant maps, and can thus be identified with a subgroup H_i in G. If $\theta_i(A) \in \{0, \pi\}$, then $\mathrm{hol}_{C_i}(A) = \pm\mathrm{Id}$ and so $H_i = G$. Otherwise, $H_i = T$.

We can describe the fibre $\pi^{-1}(b)$ using these subgroups. Suppose A is a.t.d. and $[A] \in \pi^{-1}(b)$. Let $\tau_i \in H_i$ for each circle C_i, $i \in (1, 2, ..., 3g-3)$. Then define the map

$$\psi_A : \prod_{i=1}^{3g-3} H_i \to \pi^{-1}(b) \tag{3.25}$$

as follows. Denote the trinions composing Σ as D_γ, $\gamma \in 1, 2, ..., 2g-2$. For any circle C_i, let $D_{\gamma(i)}$, $D_{\gamma'(i)}$ be the trinions on either side. For $\tau = (\tau_1, \tau_2, ..., \tau_{3g-3})$, choose a collection of maps $\zeta_\gamma : D_\gamma \to g$ such that for every C_i, $\zeta_{\gamma(i)}$ and $\zeta_{\gamma'(i)}$ are constant on a tubular neighbourhood of C_i, and such that

$$\zeta_{\gamma(i)}|_{C_i} = \tau_i \zeta_{\gamma'(i)}|_{C_i}. \tag{3.26}$$

Here, adopt the convention that the orientation of the tubular neighbourhood is $v \wedge w$, where w is tangent to the oriented circle C_i and v is transverse to c_i and pointing *into* $D_{\gamma(i)}$, thus away from $D_{\gamma'(i)}$.

Now we define a connection A_τ on Σ by defining A_τ on each trinion: $A_\tau|_{D_\gamma} := \zeta_\gamma \circlearrowleft A|_{D_\gamma}$. Finally define $\psi_A(\tau) = [A_\tau]$. Next we ask, for $\tau, \tau' \in \prod_{i=1}^{3g-3} H_i$, when are A_τ and $A_{\tau'}$ gauge equivalent?

Let J_γ be the stabilizer of $A|_{D_\gamma}$ under $\mathcal{G}|_{D_\gamma} = \mathrm{Hom}(D_\gamma, G)$. Since A is a.t.d., this also consists of constant maps. $J_\gamma = Z(G) = \{\pm \mathrm{Id}\}$ if the holonomy is an irreducible representation of $SU(2)$, and otherwise $J_\gamma = T$ (resp G) if the holonomy reduces to T (resp $Z(G)$).

Jeffrey and Weitsman prove the following lemma and theorem:

Lemma 3.5. *If τ, τ' are in $\prod_{i=1}^{3g-3} H_i$, then $[A_\tau] = [A_{\tau'}]$ if and only if there is a set of gauge transformations $\Phi_\gamma : D_\gamma \to G$ such that:*

1. *$\Phi_\gamma \in J_\gamma$ for all γ.*
2. *For each boundary circle C_i, we have*

$$\Phi_{\gamma'(i)}|_{C_i} \tau_i = \tau_i' \Phi_{\gamma(i)}|_{C_i}.$$

Theorem 3.6. *The map $\psi_A : \prod_i H_i \to \pi^{-1}(b)$ is surjective and the group $\prod_\gamma J_\gamma$ has a natural action on $\prod_i H_i$ so that*

$$\pi^{-1}(b) = \left(\prod_i H_i\right) / \left(\prod_\gamma J_\gamma\right). \tag{3.27}$$

Proof. Jeffrey and Weitsman [JW92], lemma 2.4 and theorem 2.5 respectively. □

3.3 Moduli of Connections on a Trinion

In order to build the moduli space $\mathcal{M}$ by gluing together connections defined along a trinion decomposition, one must first understand the possible connections on one trinion D, denoted $\mathcal{M}(D)$. As in Section 2.2, $\mathcal{M}(D)$ can be described by the set $\mathrm{Hom}(\pi_1(D), G)/G$. For a trinion,

$$\pi_1(D) = \{[C_1], [C_2], [C_3] \mid [C_1][C_2][C_3] = 1\}, \tag{3.28}$$

where C_i are the three boundary curves of the trinion. We can again define the holonomy angle functions, first letting $\tilde{\theta}_i : \mathrm{Hom}(\pi_1(D), G) \to [0, \pi]$ be

$$\tilde{\theta}_i(\rho) = \cos^{-1}\left(\frac{1}{2}\mathrm{Tr}\ (\rho[C_i])\right), \tag{3.29}$$

and these maps will descend under the quotient by G to maps $\theta_i : \mathcal{M}(D) \to [0, \pi]$. Then Jeffrey and Weitsman prove [JW92, Proposition 3.1]:

Theorem 3.7. *The map $\theta = (\theta_1, \theta_2, \theta_3) : \mathcal{M}(D) \to [0, \pi]^3$ sends $\mathcal{M}(D)$ bijectively to the set satisfying the inequalities*

$$|\theta_i - \theta_j| \leq \theta_k \leq \min(\theta_i + \theta_j, 2\pi - (\theta_i + \theta_j)), \tag{3.30}$$

for every i, j, k a cyclic permutation of 1,2,3. These inequalities define a convex polytope in $\mathbb{R}^3$.

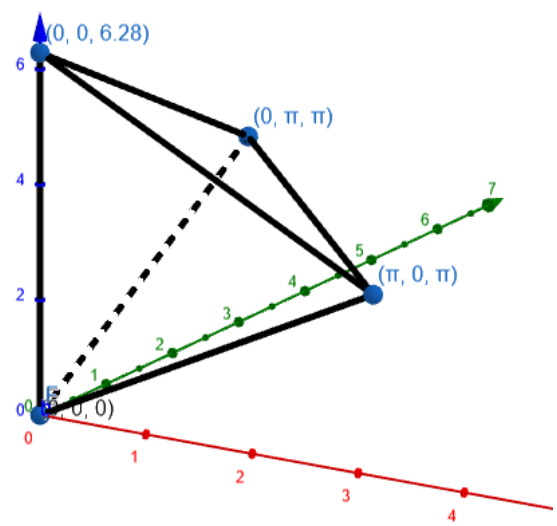


Figure 3.2: The polytope defined by the inequalities (3.30), generated using *Geogebra*.

Using this result, and the gluing process described in the last section, the image of $\mathcal{M}$ under the holonomy angles $\theta_1, ..., \theta_{3g-3}$ are the values satisfying the inequalities (3.30) on every trinion. Applying a theorem of Guillemin and Steinberg [GS83] to this case, one obtains:

Theorem 3.8. *Suppose $x \in \pi(\mathcal{M}) \subset B$ (equation 3.18) Then*

- *The Hamiltonian vector fields corresponding to the functions θ_i are linearly independent on the fibre $\pi^{-1}(x)$, if and only if x is a point where all the inequalities (3.30) are strict.*
- *In general, the number of linearly independent Hamiltonian vector fields on the fibre $\pi^{-1}(x)$ is equal to*

$3g-3-s$, where s is the number of independent linear equations out of the following satisfied by $\theta(x)$:

$$\theta_{i_{\sigma(1)}(\gamma)}(x) + \theta_{i_{\sigma(2)}(\gamma)}(x) - \theta_{i_{\sigma(3)}(\gamma)}(x) = 0, \tag{3.31}$$

$$\theta_{i_1(\gamma)}(x) + \theta_{i_2(\gamma)}(x) + \theta_{i_3(\gamma)}(x) = 2\pi. \tag{3.32}$$

where $\sigma : 1,2,3 \to 1,2,3$ is any cyclic permutation. These inequalities correspond to (3.30).

Furthermore

Lemma 3.9. *Let $x \in \mathcal{M}(D)$ and let $\theta(x)$ be the holonomy angles of x around the three boundary curves of D. Then x corresponds to a conjugacy class of reducible representations of $\pi_1(D)$ if and only if at least one of the equations (3.31),(3.32) above is satisfied.*

Motivated by this lemma, define *interior triples* in $[0,\pi]^3$ to be those for which none of the equations is satisfied, i.e those on the interior of the convex polytope. These triples correspond to points in $\mathcal{M}(D)$ which are conjugacy classes of irreducible representations of $\pi_1(D)$. Theorem 3.8 tells us that the fibre $\pi^{-1}(x)$ is a torus of dimension $3g-3$ if and only if $\theta(x)$ is an interior triple.

Theorem 3.10. *Let $x \in B$ and let A be a flat a.t.d. connection whose gauge equivalence class is in $\pi^{-1}(x)$. Further, assume that on every trinion the holonomy angles of x is an interior triple. Then the fibre $\pi^{-1}(x)$ is identified with $T^{3g-3}/(\mathbb{Z}_2)^{2g-2}$ under the map ψ_A defined in equation (3.25).*

Thus, the fibres corresponding to interior triples, which are the generic fibres, are tori of dimension $3g-3$.

3.4 Counting Bohr-Sommerfeld Points

With this real polarization of the moduli space, Jeffrey and Weitsman proceed to count the number of Bohr-Sommerfeld points. In this section we will summarize the results that we want to use going forward.

Definition 3.11. *Let $(\mathcal{M}, \omega)$ be a symplectic manifold with prequantum line bundle $\mathcal{L}$ and polarisation $\pi : \mathcal{M} \to B$. A point $b \in B$ is called a* Bohr-Sommerfeld *point if $\mathcal{L}|_{\pi^{-1}(b)}$ possess a one-dimensional family of global covariant constant sections.*

The quantization of a prequantum system is $\mathcal{H} = \bigoplus_{i=1}^{\dim M} H^i(M, \mathcal{J}_\pi)$, where $\mathcal{J}_\pi$ is the sheaf of sections of $\mathcal{L}$ covariant constant along the fibres of π. Sniatycki's theorem [Śni77] proves that when the polarisation of a symplectic manifold is a fibration, the dimension of $\mathcal{H}$ is given by the number of Bohr-Sommerfeld points. Although this result does not apply here, it still provides motivation for counting the Bohr-Sommerfeld points.

One characterization of the Bohr-Sommerfeld points is as the set of points b whose fibres $L_b = \pi^{-1}(b)$ satisfy the property that if $\pi_1(L_b)$ is generated by a set of loops, then the holonomies of the prequantum connection around those loops are all equal to $\mathbb{1}$. A basis of loops can be obtained in terms of a basis of a lattice of functions Λ on B, called the *period lattice*.

Definition 3.12. *For $\alpha \in T^*_x B$, let v_α denote the vertical vector field along the fibre L_x defined by $\pi^*(\alpha)$. Let f_α denote the diffeomorphism of the fibre L_x induced by flowing along v_α for time 1.*

Then the period lattice *in $T^*_x B$ is the set of α whose corresponding f_α is trivial.*

This is a lattice of dimension m, and one may show that there is a neighbourhood U of any point $x \in B$ on which there exist functions $\{H_i\}$ forming a lattice Λ under addition, such that the period lattice for all $x \in U$ is given by $\{(dH_i)_{x'}\}$ [Dui80]. This lattice will also be referred to as the *period lattice*.

Let $\{\tilde{\mu}_i\} \in C^\infty(B, \mathbb{R})$ be a basis of the period lattice, and define $\mu_i = \tilde{\mu}_i \circ \pi \in C^\infty(\mathcal{M}, \mathbb{R})$. Then the Hamiltonian flows of the μ_i have period 1, and the fundamental group $\pi_1(L_x)$ is generated by the loops γ_i which are the period 1 trajectories of the Hamiltonian flows of μ_i. The functions $(\mu_1, ..., \mu_m)$ define a moment map for a torus $(S^1)^m$ action on on $\mathcal{M}$, preserving the Lagrangian fibration [JW92, §4]. These functions correspond to a set of *action variables* to pair with our θ_i angle variables on $\mathcal{M}$. The period lattice is important, as Jeffrey and Weitsman show that the Bohr-Sommerfeld points correspond to integer values of a set of functions generating the period lattice [JW92, §5].

For Σ a connected compact Riemann surface of genus g, fix a trinion decomposition $\{D_\gamma\}$, and label the boundary loops of D_γ as $C_{i_1(\gamma)}$, $C_{i_2(\gamma)}$ and $C_{i_3(\gamma)}$. We call a boundary loop C_i *separating* if removing it disconnects Σ. Recall the co-ordinates θ_i for the points $x \in \mathcal{M}$ defined in equation 3.29.

Theorem 3.13 (Jeffrey and Weitsman, Theorem 8.1)**.** *The set P^{bs} of Bohr-Sommerfeld points in B for the line bundle $\mathcal{L}^k$ is given by the points $x \in B$ satisfying the conditions:*

1. *For each boundary circle C_i, $\theta_i(x) = \frac{\pi l_i}{k}$ for some $l_i \in \{0, 1, ..., k\}$, with l_i even if C_i is separating.*
2. *For each trinion D_γ, $l_{i_1(\gamma)} + l_{i_2(\gamma)} + l_{i_3(\gamma)} \in 2\mathbb{Z}$.*

From theorem 3.8, we know that $(\theta_1, ..., \theta_{3g-3})$ is the image of a point $x \in B$ if and only if the conditions (3.30) are satisfied for the triple $(\theta_{i_1(\gamma)}, \theta_{i_2(\gamma)}, \theta_{i_3(\gamma)})$ corresponding to each trinion D_γ. We can represent a trinion decomposition as a trivalent graph, with a vertex for each trinion and an edge for each boundary circle. Therefore, a Bohr-Sommerfeld point gives a labelled trivalent graphs, where the integer l_i is assigned the edge corresponding to boundary circle C_i. The set of Bohr-Sommerfeld points then corresponds with labelled trivalent graphs whose labelling satisfy certain conditions. Expanding out the conditions of Theorem 3.13 at each vertex with edges labeled l_1, l_2 and l_3, one has:

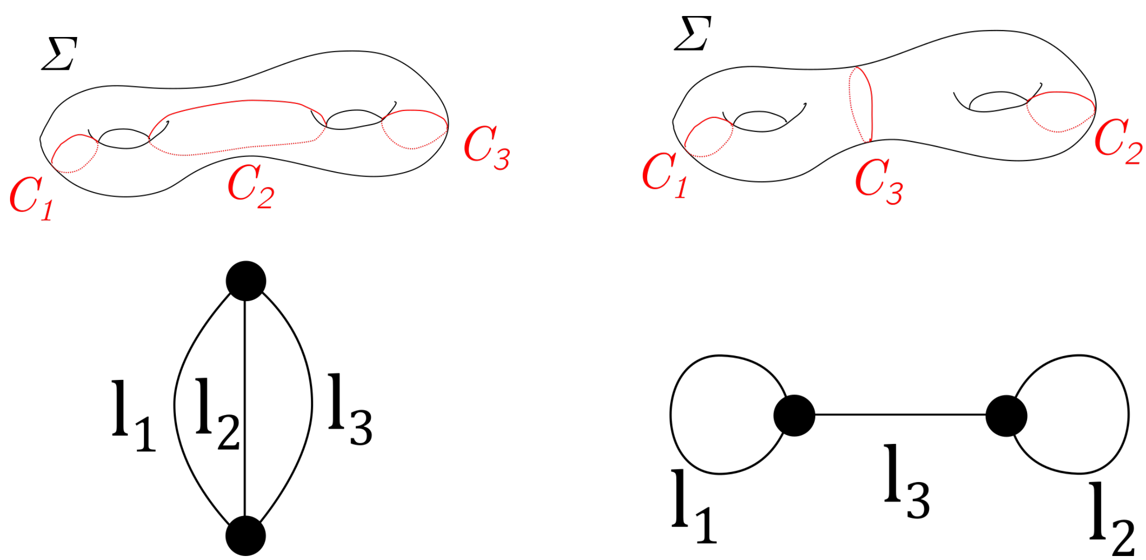
1. $|l_1 - l_2| \le l_3 \le l_1 + l_2$,

2. $l_1 + l_2 + l_3 \le 2k$,

3. $l_1 + l_2 + l_3 \in 2\mathbb{Z}$.

This gives the final result of Jeffrey and Weitsman:

Theorem 3.14 (Jeffrey and Weitsman, Theorem 8.3)**.** *Consider a fixed trinion decomposition of a compact connected Riemann surface Σ of genus g. It gives rise to a trivalent graph, and a real polarisation of the moduli space $\mathcal{M}$ of flat $SU(2)$ connections on Σ. There is one-to-one correspondence between the Bohr-Sommerfeld points of the polarisation and the set of integer labellings of the edges of the graph satisfying the conditions 1,2 in 3.13 and equation 3.30.*



(a) The symmetric decomposition and graph.

(b) The asymmetric decomposition and graph.

Figure 3.3: Decompositions of a genus 2 surface and the corresponding trivalent graphs.

Example: Let Σ be the compact Riemann surface of genus 2, which has two trinion decompositions into two trinions. Replacing the trinions with vertices and the boundary circles with edges, we obtain the graphs in Figure 3.3. Let us use Theorem 3.13 to find the number of Bohr-Sommerfeld points for the line bundle $\mathcal{L}$ over Σ with respect to the polarisations defined by these decompositions. For the symmetric decomposition, conditions 1 and 2 are the same at each vertex, so it suffices to consider them at one vertex. Since $k = 1$, each label must be either 0 or 1. Their sum must be even, which leaves us with four possibilities:

$$(l_1, l_2, l_3) = (0,0,0), (1,1,0), (0,1,1), \text{ or } (1,0,1). \tag{3.33}$$

Finally, one can check that each of these labellings satisfies $|l_1 - l_2| \le l_3 \le l_1 + l_2$, so these are all valid labellings. Therefore Theorem 3.13 tells us that there are four Bohr-Sommerfeld points of $\mathcal{L}$ in the polarisation defined by the symmetric decomposition of Σ.

For the asymmetric decomposition, the conditions at each vertex must be checked separately, as they are different. Denoting the edges at the two vertices by l_i^1 and l_i^2, we have that

$$(l_1^1, l_2^1, l_3^2) = (l_1, l_1, l_3), \qquad (l_1^2, l_2^2, l_3^2) = (l_2, l_2, l_3). \tag{3.34}$$

Once again since $k = 1$ each label must be either 0 or 1. Their sum at each vertex must be even, which means $2l_1 + l_3$ and $2l_2 + l_3$ must be even. This forces l_3 to be zero, and we're left with the following possibilities:

$$(l_1, l_2, l_3) = (0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0). \tag{3.35}$$

Once again, one can check that each of these labellings satisfies $|l_1^i - l_2^i| \leq l_3^i \leq l_1^i + l_2^i$, $(i = 1, 2)$ and is therefore a valid labelling of the graph. Thus Theorem 3.13 tells us that there are also four Bohr-Sommerfeld points of $\mathcal{L}$ in the polarisation defined by the asymmetric decomposition of Σ.

For higher values of k, a computer-assisted search finds that the number of labellings of these two graphs agree, and is equal to $10, 20, 35$ for $k = 2, 3, 4$ and so on.

One important consequence of this theorem is that the number of Bohr-Sommerfeld points can be computed by counting points in the intersection of the period lattice Λ with the moment polytope defined by equation 3.30. In the next chapter, we will discuss the construction of another moduli space P which is the toric variety corresponding to this moment polytope. From the theory of toric varieties, we know there is a line bundle over P corresponding to the polytope, with a basis of sections given by the Bohr-Sommerfeld points. It will then remain to show that this line bundle over P corresponds to the prequantum line bundle over $\mathcal{M}$ in a way which preserves the space of sections.

Chapter 4

Moduli Spaces of Parabolic Bundles

4.1 Extended Moduli Spaces of Connections

Let G be a unitary Lie group and let Σ be a compact connected genus g Riemann surface. Recall (Section 2.2) that the moduli space of flat connections $\mathcal{M}$ can be constructed as representations $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, which has its subset topology as $\xi^{-1}(e)/G \subset G^{2g}/G$, where

$$\xi(A_1, B_1, ..., A_g, B_g) = \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1}. \tag{4.1}$$

Now let us generalize to the case where Σ is permitted to have n punctures, $\{p_i\}_{i=1}^{n}$. Fix a point p in a neighbourhood of p_1, and let c_1 be a loop around p_1 based at p. For each $i \neq 1$, let c_i be a loop around p_i in Σ, based at a point q_i in a neighbourhood of p_i, and let $\{d_i\}_{i=2}^{n}$ be curves $d_i : p \to q_i$ in Σ. The path $k_j = d_j c_j d_j^{-1}$ is a loop based at p, and thus the fundamental group of Σ can be written [HJ00, Eqn. 2.2]:

$$\pi_1(\Sigma) = \left\langle a_i, b_i, c_1, k_j \;\middle|\; \prod_{i=1}^{g}(a_i b_i a_i^{-1} b_i^{-1}) c_1 \prod_{j=2}^{n} k_j = e \right\rangle. \tag{4.2}$$

Thus, we will redefine $\xi : G^{2g+2n-1} \to G$ to be

$$\xi(a_1, b_1, ..., a_g, b_g, c_1, ..., c_n, d_2, ..., d_n) = \prod_{i=1}^{g}(a_i b_i a_i^{-1} b_i^{-1}) c_1 \prod_{j=2}^{n} d_j c_j d_j^{-1}, \tag{4.3}$$

and consider the set $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, equipped with the subset topology $\xi^{-1}(e)/G \subset G^{2(g+n)-1}/G$.

Suppose $\{C_i\}_{i=1}^{2g-g}$ is a trinion decomposition for an unpunctured surface $\tilde{\Sigma}$. Then, if we collapse a curve C_j down to a single point p_j, we obtain a new surface Σ with one puncture, which we desingularize into two punctures. From Proposition 3.4, any element A in $\mathcal{M}$ can be represented by an adapted to trinion decomposition (a.t.d.) connection (Def. 3.3), meaning that A's holonomy around the curve c_i corresponding to the point p_i is in the maximal torus $\mathfrak{t}$. As a representation, this means we can choose a conjugacy class of A with holonomy around c_i in the fundamental alcove $\Delta \subset \mathfrak{t}$. If we repeat this for some n curves in the trinion decomposition, the moduli space $\mathcal{M}$ becomes the T-*extended* moduli space

$$\mathcal{M}^T := \{\left[(A_i, B_i)_{i=1}^g, C_1, (D_j, C_j)_{j=2}^n\right] \in G^{2g+n-1} \times \exp(\Delta)^n \mid \xi(A_i, B_i, D_j, C_j) = \mathbb{1}\}. \tag{4.4}$$

We can also define the G-*extended* moduli space:

$$\mathcal{M}^G := \{[(A_i, B_i)_{i=1}^g, C_1, (D_j, C_j)_{j=2^n}] \in G^{2g+n-1} \times G^n \mid \xi(A_i, B_i, D_j, C_j) = \mathbb{1}\}, \tag{4.5}$$

and the map $\exp : \Delta \to \exp(\Delta)$ induces an inclusion $\mathcal{M}^T \subset \mathcal{M}^G$.

Theorem 4.1. *The space $\mathcal{M}^G$ is a smooth manifold, isomorphic to $G^{2(g+n-1)}$.*

Proof. The relation $\xi = 1$ allows us to write C_1 as a function of the other variables, which exhibits $\mathcal{M}^G$ as the graph of a smooth function $G^{2(g+n-1)} \to G$. □

The conjugation action of G gives an action on $\mathcal{M}^G$, where an element $\sigma \in G$ acts by

$$A_j \to \sigma A_j \sigma^{-1}, \qquad B_j \to \sigma B_j \sigma^{-1}$$

$$D_j \to \sigma D_j, \qquad C_1 \to \sigma C_1 \sigma^{-1}$$

For the punctures $l = 2, ..., n$, there is also an action of G corresponding to changing the trivialization of the underlying bundle near the punctures. An element σ in the lth copy $(l = 2, ..., n)$ of G acts by

$$D_l \to D_l \sigma^{-1}, \qquad C_l \to \sigma C_l \sigma^{-1}. \tag{4.6}$$

This action of G^n on $\mathcal{M}^G$ restricts to give an action on $\mathcal{M}^T$. Next we would hope to give $\mathcal{M}^G$ some symplectic structure. It turns out, the appropriate structure is that of a *quasi-Hamiltonian G space*. It is known that $\mathcal{M}^G$ has a 2-form ω invariant under the G^k action, and an equivariant *moment map* $\Phi : \mathcal{M}^G \to G$, given by $(A_i, B_i, D_j, C_j) \to C_k^{-1}$ [AMM98].

4.2 Imploded Cross-Sections

To proceed, we will state some definitions and results about *imploded cross sections*, from e.g.[AMM98] and [JZ20]. The imploded cross-section is meant to take a quasi-Hamiltonian $G \times G$ space and reduce it to a quasi-Hamiltonian $G \times T$ space in two steps, similarly to a symplectic reduction. First, one restricts via the moment map to the fundamental alcove in $\mathfrak{t}$, and then one reduces the right G-action to a T-action by quotienting out the stabilizers of the maximal alcove in $\mathfrak{t}$.

Definition 4.2. *For a quasi-Hamiltonian $G \times G$-space M with moment map Φ, and fundamental alcove $\Delta \subset \mathfrak{g}$, we define the imploded cross section*

$$M_{impl} := \coprod_{\sigma} \frac{\Phi^{-1}(\sigma)}{[G_\sigma, G_\sigma]}, \tag{4.7}$$

where σ ranges over the faces of Δ, and G_σ is the stabilizer of σ under the action of G by conjugation at a point in the interior of σ.

Example: Consider the double $D(G) := G \times G$, which is a quasi-Hamiltonian $G \times G$ space with the action

$$(g_1, g_2) \cdot (a, b) = (g_1 a g_2^{-1}, g_2 b g_1^{-1}), \tag{4.8}$$

and moment-map $\Phi : D(G) \to G \times G$ given by $\Phi(a, b) \to (ab, a^{-1}b^{-1})$. Let us introduce new co-ordinates $u = a$ and $v = ba$ for $D(G)$, in which the action becomes

$$(g_1, g_2) \cdot (u, v) = (g_1 u g_2^{-1}, \mathrm{Ad}_{g_2} v), \tag{4.9}$$

and the moment map becomes

$$\Phi(u, v) = (\mathrm{Ad}_u v, v^{-1}). \tag{4.10}$$

For $G = SU(2)$, $T = S^1$, the implosion $D(G)_{impl}$ is smooth, isomorphic to S^4, and is a quasi-Hamiltonian $G \times T$-space. We only give a sketch here, the details can be found in [HJ00, Prop 2.29]. For $SU(2)$, the fundamental alcove is $\Delta = [0, 1]$, which has three faces, $\sigma_0 = (0, 1)$, and $\sigma_\pm = \pm 1$. For σ_0, we have $G_{\sigma_0} = T$ and hence $[G_{\sigma_0}, G_{\sigma_0}]$ is trivial. For $\sigma_\pm$, $G_\sigma = G = [G_\sigma, G_\sigma]$. For each face σ of Δ, the imploded cross-section has a stratum that is a quasi-Hamiltonian $G \times T$-space [AMM98, Theorem 5.1].

For the interior σ_0, we have $\Phi^{-1}(\sigma_0) = \{(u, v) \in D(G) \mid v^{-1} \in \sigma^0\} = G \times \sigma^0$, hence $\Phi^{-1}(\sigma_0)/[G_{\sigma_0}, G_{\sigma_0}] = G \times \sigma^0$. For the other faces $\sigma_\pm$, $\Phi^{-1}(\sigma_\pm) = \{(u, v) \in D(G) \mid v^{-1} = \pm 1\} = G$, hence $\Phi^{-1}(\sigma_\pm)/[G_{\sigma_\pm}, G_{\sigma_\pm}] = G/G = 1$. Therefore the imploded cross section is

$$D(G)_{impl} = (G \times \sigma^0) \coprod 1 \coprod 1. \tag{4.11}$$

One can identify this with S^4 by projecting the cylinder $(G \times \sigma^0) \cong S^3 \times (-1, 1)$ to S^4 without the north and south poles, and then identifying the two points with the poles.

The interior stratum $(G \times \sigma^0)$'s quasi-Hamiltonian structure is inherited from that of $D(G)$, and one must check explicitly that it extends to the other two points. The residual $G \times T$ action on $G \times \sigma^0$ is given by

$$(g, t) \cdot (u, v) = (gut^{-1}, v), \tag{4.12}$$

with moment maps $\Phi_G(u, v) = (uvu^{-1})$ and $\Phi_T(u, v) = v^{-1}$.

To compute the imploded cross-section of our extended moduli spaces, we will use the following result, which tells us that $D(G)_{impl}$ is a *universal* implosion [HJ00, Prop 2.32]:

Theorem 4.3. *Let M be a quasi-Hamiltonian $H \times G$-space, where $G = SU(2)$, and let $D(G)_{impl}$ be the imploded cross-section of $D(G)$. Then*

$$M_{impl} = M \times D(G)_{impl} \,//\, G = (m, \xi) \in M \times D(G)_{impl} \mid \Phi(m) = \Phi_G(\xi)/G. \tag{4.13}$$

The space M_{impl} is a quasi-Hamiltonian $H \times T$ space, and it is smooth over the locus of points $(m, \xi) \in M \times D(G)_{impl}$ where the stabiliser of the G-action is trivial.

Next we will construct the symplectic implosion of the space $\mathcal{M}^G$ which is a quasi-Hamiltonian G^n-space. There is an action of G associated to each curve in our trinion decomposition for Σ, with moment map Φ_j given by just C_j^{-1}. Performing the implosion once for each puncture, the resulting space $\mathcal{M}^G_{impl}$ will be denoted P, and it is a Hamiltonian T^n space.

After imploding the first $n-1$ punctures ($j = 2, ..., n$), we obtain the set

$$\left\{ (A_i, B_i, C_1, W_2, ..., W_n) \mid A_k, B_k, C_1 \in G, W_j \in D(G)_{impl}, \ \prod_{j=1}^{g} [A_j, B_j] C_1 \prod_{j=2}^{n} \Phi_G(W_j) = 1 \right\}. \tag{4.14}$$

In terms of the (u, v) co-ordinate for $D(G)$, $W_j = (u, v)$ with $u = D_j$ and $v = C_j$. To finish the imploded cross-section, it remains to implode the cross-section for the action of the first copy of G appropriately. Let $W_1 = (\mathbb{1}, C_1)$ in $G \times G$ and define an action of G by

$$(A_k, B_k, W_j = (u_j, v_j)) \to (gA_k g^{-1}, gB_k g^{-1}, (gu_j, v_j)). \tag{4.15}$$

Then the final imploded cross-section P becomes [HJ00]

$$P = \left\{ (A_k, B_k, W_1, ..., W_n) \mid \prod_{j=1}^{g} [A_j, B_j] \prod_{j=1}^{n} \Phi_G(W_j) = 1 \right\}. \tag{4.16}$$

There is a natural map from $\mathcal{M}$ to P:

Lemma 4.4. *There is a surjective map $\phi : \mathcal{M} \to P$ which is a bijection over the interior $(\Delta^0)^n$ of the moment polytope.*

Proof. For each puncture p_k, the corresponding imploded cross-section is given by the strata

$$\Phi_k^{-1}(\sigma)/[G_\sigma, G_\sigma] \tag{4.17}$$

for each face $\sigma \in \Delta$ in the fundamental chamber of the k-th copy of G. The inverse image $\Phi_k^{-1}(\Delta)$ are those elements with $C_k^{-1} \in T$, and therefore the elements in $\bigcap_k \Phi_k^{-1}(\Delta)$ is exactly $\mathcal{M}$. Then the quotient map on each strata defines a surjection $\phi : \mathcal{M} \to P$. Over the interior Δ^0 of each implosion, the quotient is trivial so ϕ is the identity map, which is bijective. □

Theorem 4.5. *Over* $(\Delta^0)^n$*,* $\phi : \mathcal{M} \to P$ *is a symplectomorphism.*

Proof. Hurtubise and Jeffrey [HJ00, Proposition 2.37]. □

Now associated to the compact Riemann surface Σ we have a symplectic variety P of representations with weighted frames, which is toric for $G = SU(2)$ and has a prequantum line bundle $\mathcal{L}_P$ corresponding to $\mathcal{L}$ on $\mathcal{M}$. Now we will go in detail on the other half of Jeffrey and Hurtubise's construction, building the complex variety $\mathcal{P}$ of *framed parabolic bundles* over a singular curve $\tilde{\Sigma}$ corresponding to contracting the loops in a trinion decomposition of Σ. We will see that P and $\mathcal{P}$ are diffeomorphic, meaning that we can compute sections of prequantum line bundles over $\mathcal{P}$ using the structure of P.

First we see how to embed $\mathcal{M}$ into projective space using determinants, as this construction will have us transform the data of $\mathcal{M}$ from holomorphic bundles over Σ to sheaves, which is the perspective we will use to construct $\mathcal{P}$.

4.3 Projective Embedding of $\mathcal{M}$

The moduli $\mathcal{M}$ of flat $SL(n, \mathbb{C})$ bundles can be embedded into projective space using the sections of the determinant bundle over $\mathcal{M}$. There is a construction of this embedding due to Bhosle [Bho89], which we describe following Thaddeus and Gieseker [Tha96, §7][Gie77].

First, fix a line bundle L of sufficiently high degree so that for all $E \in N(k, d)$, $E \otimes L$ is globally generated, and redefine E as $E \otimes L$. Then, for some large N we can write E as a quotient:

$$\phi : \mathcal{O}^N \to E. \tag{4.18}$$

This quotient induces a map from $\wedge^k(\mathcal{O}^N) \to \wedge^k(E)$ and the $SL(k, \mathbb{C})$ structure induces an isomorphism $\mu : \wedge^k(E) \cong L^k$. Hence, a quotient of the trivial bundle induces an element $\hat{\beta}$ in

$$V_1 := \mathrm{Hom}(H^0(\wedge^k(\mathcal{O}^N)), H^0(L^2)). \tag{4.19}$$

Now to pass to $\mathcal{M}$ we quotient by $GL(k, \mathbb{C})$. Suppose we have $\phi_2 = \Lambda^{-1}\phi_1\Lambda$. Then

$$\hat{\beta}_2 = \mu(\wedge^k \phi_2) = \mu(\wedge^k \Lambda^{-1}\phi_1\Lambda) = \det \Lambda \mu(\wedge^k \phi_1) = \det \Lambda \ \hat{\beta}_1. \tag{4.20}$$

Therefore the orbits of $GL(k, \mathbb{C})$ correspond to equivalence classes in $\mathbb{P}(V_1)$. It is this mapping, which we will denote $\iota : \mathcal{M} \to \mathbb{P}(V_1)$, which we claim is an embedding.

Lemma 4.6. *The map $\iota : \mathcal{M} \to \mathbb{P}(V_1)$ is injective.*

Proof. Suppose $(E_1, \overline{\partial}_{E_1})$ and $(E_2, \overline{\partial}_{E_2})$ are holomorphic vector bundles for which $\hat{\beta}_1 = \iota(E_1) = \iota(E_2) = \hat{\beta}_2$. Since the $SL(k, \mathbb{C})$ structure is unique up to $\mathbb{C}^*$, this means that

$$\wedge^k \phi_1 = \lambda \wedge^k \phi_2 \tag{4.21}$$

for some $\lambda \in \mathbb{C}^*$. Fixing a local trivialization of E_1, $E_1|_U \cong U \times \mathbb{C}^k$, and picking a local frame $e_1, .., e_k$, choose any sections $s_1, ..., s_k \in \mathcal{O}^N|_U$ so that $\phi_1(e_i) = s_i$. Let $s = s_1 \wedge s_2 \wedge ... \wedge s_k$. Then

$$e_1 \wedge ... \wedge s_k = \phi_1(s_1) \wedge ... \wedge \phi_1(s_k) = \wedge^k \phi_1(s) = \lambda \wedge^k \phi_2(s) = \lambda \phi_2(s_1) \wedge ... \wedge \phi_2(s_k) \tag{4.22}$$

Since $\{e_i\}$ was a local frame, the left-hand-side is non-zero, and the right-hand-side is also non-zero. Therefore $\{\phi_2(s_i)\}$ is a local frame trivializing E_2 on U. This map $e_i \to \phi_2(s_2)$ gives an isomorphism of E_1 with E_2. Note that this isomorphism is not unique as we could have picked other sections $\tilde{s}_i$ with $\phi_1(\tilde{s}_i) = e_i$. □

Theorem 4.7. *The map $\iota : \mathcal{M} \to \mathbb{P}(V_1)$ is an embedding.*

Proof. From the lemma, we know ι is injective, and thus it remains to show that $d\iota$ is injective. We give a proof following Thaddeus [Tha96, Prop 7.1].

Recall that the tangent space $T_\phi \mathcal{M} = H^1(\text{End } E)$ (Section 2.3), for $\phi : \mathcal{O}^N \to E$. Before quotienting $GL(k, \mathbb{C})$, the map ι was given by the sending $\phi \to \hat{\beta} \in V_1$ which is essentially $\phi \to \wedge^k \phi$. Thus, we expect the derivative $T_\phi \mathcal{M} \to V_1$ to be essentially $\psi \to \wedge^{k-1} \phi \wedge \psi$, for $\psi \in T_\phi \mathcal{M}$. Precisely, if we deform the quotient map ϕ to $\phi + \epsilon\psi$ with $\epsilon^2 = 0$, then we have

$$\wedge^k(\phi + \epsilon\psi) = \wedge^k(\phi) + \epsilon \wedge^{k-1} \phi \wedge \psi. \tag{4.23}$$

Then taking the $GL(k, \mathbb{C})$ quotient gives the map $d\iota$. Hence to show $d\iota$ is injective, we want to show that $\wedge^{k-1}\phi \wedge \psi = 0$ only if $\phi + \epsilon\psi = \phi \mod GL(k, \mathbb{C})$. To show this, we use the following linear algebraic lemma [Tha96, Lemma 7.2]

Lemma 4.8. *If $\phi : \mathbb{C}^n \to \mathbb{C}^k$ is a linear surjection and $\psi : \mathbb{C}^n \to \mathbb{C}^k$ is a linear map, then $\wedge^{k-1}\phi \wedge \psi = 0$ if and only if $\psi = f\phi$ for some $f \in$ End $\mathbb{C}^k$ with trace zero.*

This lemma tells us that if $\wedge^{k-1}\phi \wedge \psi = 0$ then $\psi = f\phi$ and hence $\phi + \epsilon\psi = \phi(1 + \epsilon f)$. Then since f is traceless, $1 + \epsilon f$ is in $GL(k, \mathbb{C})$, so $d\iota$ is injective. □

4.4 Parabolic Vector Bundles

Given a trinion decomposition of the Riemann surface Σ, let us fix the holonomies around each boundary loop. If we think of each trinion as a thrice-punctured sphere and consider the space of connections with fixed holonomies on the trinion, then due to Mehta and Seshadri (cite) there is a correspondence between the moduli of $\pi_1(\Sigma)$ representations into $SU(n)$ and that of rank-n holomorphic bundles with an $SL(n, \mathbb{C})$ structure and a parabolic structure at the punctures of Σ, which we call a parabolic bundle. Under this correspondence, the eigenvalues of the holonomy get translated into a set of weights for the parabolic structure.

We want to consider the moduli space of connections with all possible holonomies, and therefore we will want to fit all these moduli of parabolic bundles together, and in such a way that we can even include the $\theta_i = 0, \pi$ cases, which will correspond to weights 0 and 1. This is the construction of Hurtubise and Jeffrey which we will describe in the next section. First we lay out the basic definitions and results about parabolic bundles.

Definition 4.9. *A* parabolic bundle *over a complex manifold* Σ *is a holomorphic vector bundle* E *over* Σ *with a* parabolic structure*, which is a point of marked points* $\{p_1, ..., p_n\}$ *and for each point, a flag of subspaces in the fibre* E_{p_k}*.*

In particular if E has rank 2, then a parabolic structure on E is a choice of points $\{p_k\}$ and a sheaf homomorphism $\alpha : E \to S$ where $S := \bigoplus_k \mathbb{C}_{p_k}$.

There is an adapted notion of stability for parabolic vector bundles.

Definition 4.10. *Let* $\gamma_1, ..., \gamma_n \in [0, 1]$ *be a set of weights. For a subbundle (not necessarily proper)* F *of* E *we set* $\mu_i(F) = 1$ *if* $F_{p_i} \subset \ker \alpha_i$*, and* $\mu_i = 0$ *otherwise. Define* $\sigma(F) = \frac{1}{rk(E)}$ *if* $F = E$ *and* 0 *otherwise. Then we say a pair* (E, α) *is* stable *with respect to* γ *if*

$$rk(E) \deg(F) < rk(F) \left(\deg(E) - \sum_{i=1}^{n} \gamma_i \right) + rk(E) \sum_{i=1}^{n} (1 - \mu_i(F) + \sigma_i(F)) \gamma_i. \tag{4.24}$$

If the inequality is not strict, (E, α) *is* semi-stable.

Let us summarize some important results about weighted parabolic bundles.

Lemma 4.11. *If* (E, α) *is a parabolic bundle semi-stable with respect to weights* γ*, then:*

1. *The kernel of* α *is torsion free, and the torsion subsheaf of* E *is non-zero only at the* p_i*, equalling* 0 *or* $\mathbb{C}$ *at each* p_i*.*
2. *If* $\gamma_i > 0$*, then* $\alpha_i \neq 0$*.*
3. *If* $\gamma_i < 1$*, then* E *is torsion free at* p_i*.*
4. *If* $\gamma_i \in (0, 1)$*, one has a parabolic structure at* p_i*, and if all the weights are in* $(0, 1)$*, then* (E, α) *is stable with respect to* γ*, if and only if it is stable with respect to weights* $(1 - \gamma_i)/2$ *and* $(1 + \gamma_i)/2$*.*
5. *If* (E, α) *is locally free at* p_i *and* $\alpha \neq 0$*, then for* $\gamma_i = 0$*, there is a family* (E_t, α_t)*,* $t \in \mathbb{C}$*, of semi-stable pairs such that* $(E_t, \alpha_t) \cong (E, \alpha)$ *for* $t \neq 0$*, and* $\alpha_0 = 0$*.*
6. *If* (E, α) *is locally free at* p_i *and* $\alpha \neq 0$*, then for* $\gamma_i = 1$*, there is a family* (E_t, α_t)*,* $t \in \mathbb{C}$*, such that* $(E_t, \alpha_t) \cong (E, \alpha)$*,* $t \neq 0$ *and* E_0 *has torsion at* p_i*.*

Proof. Hurtubise and Jeffrey [HJ00, Lemma 4.3] □

One consequence of this lemma is that when $\gamma_i \in (0,1)$, the set of semi-stable parabolic bundles is torsion-free. The weights γ_i will correspond to holonomy angles of flat $SU(2)$ connections around curves in a trinion decomposition, and so weights in $(0,1)$ correspond to non-central holonomies. There are two edge cases to consider: when $\gamma_i = 0$, the parabolic structure vanishes, and when $\gamma_i = 1$ we acquire torsion. To fit all connections with all possible holonomies into a moduli space, each of these cases will need to be dealt with.

4.5 Moduli of Framed Parabolic Bundles

Here we construct a moduli space of parabolic vector bundles. This section closely follows Hurtubise and Jeffrey [HJ00, §4]. From now on, we restrict our attention to $G = SU(2)$. Let Σ be a compact connected Riemann surface with n punctures $\{p_i\}_{i=1}^n$. Fix a line bundle L of sufficiently high degree so that for all $E \in N(k,d), E \otimes L$ is a globally generated sheaf (and redefine $E = E \otimes L$). Then for some large N we can write E as a quotient of the trivial sheaf on Σ:

$$\phi : \mathcal{O}^N \to E. \tag{4.25}$$

Just as in Section 4.3 we have a mapping $\hat{\beta}$ taking E to the vector space $V_1 := \mathrm{Hom}(H^0(\wedge^k(\mathcal{O}^N)), H^0(L^2))$. Now we add the parabolic data. At a point p_i, the map $\alpha_i : E \to \mathbb{C}_{p_i}$ pulls back to $\hat{\alpha}_i = \alpha_i \circ \phi$ in $V_2 := H^0(\mathcal{O}^N)^*$. Since we are only interested in α_i up to (independent) scaling, the parabolic bundle (E, α) represents an equivalence class in

$$Z := \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times ... \times \mathbb{P}(V_2), \tag{4.26}$$

where there are n copies of $\mathbb{P}(V_2)$, one for each puncture. Then letting $\tilde{\mathcal{M}}$ denote the set of parabolic vector bundles on Σ, and $\tilde{\iota} : \tilde{\mathcal{M}} \to Z$ denote the map taking (E, α) to $(\hat{\beta}, \hat{\alpha})$, we get a closed subvariety $X = \tilde{\iota}\left(\tilde{\mathcal{M}}\right)$ in Z.

Now we also have weights $\gamma = (\gamma_1, ..., \gamma_n)$ for the parabolic structure. These vary the choice of polarization of X, namely the choice of line bundles on which the action of $SL(N, \mathbb{C})$ *linearises*. Let us recall what this means:

Definition 4.12. *Given a linear algebraic group G and a G-variety X, a line bundle $p : L \to X$* linearises *if there is an action of G on L such that for all $l \in L$, $g \in G$,*

$$p(g \cdot l) = g \cdot p(l), \tag{4.27}$$

and which restricts to a linear isomorphism $L_x \cong L_{g \cdot x}$ on fibres.

In this case $G \cong SL(N, \mathbb{C})$ which acts on V_1 and each copy of V_2.

Let $\pi_1 : Z \to \mathbb{P}(V_1)$ and $\pi_{2,i} : Z \to \mathbb{P}(V_2)$ denote the projections to the first factor and to the i-th factor of Z respectively. Let

$$L_0 = \pi_1^*(\mathcal{O}(N)), \quad \text{and} \quad L_{1,i} = \pi_1^*(\mathcal{O}(N-1)) \otimes \pi_{2,i}^*(\mathcal{O}(2)). \tag{4.28}$$

Then the linearisation corresponding to weights $\gamma = (\gamma_1, ..., \gamma_n)$ is

$$L_\gamma = (L_0)^{s_0} \otimes (\otimes_{i=1}^n (L_{1,i})^{s_{1,i}}), \tag{4.29}$$

where $s_0(\gamma_i) = s_{1,i}(1 - \gamma_i)$.

In summary, for each set of weights γ, we have a corresponding moduli space of parabolic bundles that are semi-stable with respect to those weights, with $\gamma_i \neq 0, 1$. Next we will fit these spaces together, and in such a way that we can include $\gamma_i = 0, 1$. To do this, we put a $(\mathbb{P}^1)^n$-bundle over X,

$$Y = \mathbb{P}(L_0 \oplus L_{1,1}) \oplus \mathbb{P}(L_0 \oplus L_{1,2}) \oplus ... \oplus \mathbb{P}(L_0 \oplus L_{1,n}). \tag{4.30}$$

We endow Y with the natural polarisation $\mathcal{O}(1, 1, ..., 1)$. Now Y contains all the stable points for the various choice of weights, which correspond to the various possible holonomies of the unitary connections. We still need to account for gauge equivalence, which suggests we take the $SL(N, \mathbb{C})$ quotient. This is not quite correct, as one must make an adjustment to account for the possibility that $\gamma_i = 0, 1$.

Consider a weighted parabolic bundles as a quadruple $(E, \alpha_i, A_i, \gamma)$, where $\alpha_i : E \to \mathbb{C}_{p_i}$ is the parabolic structure, A_i is a subspace of E_{p_i}, with $A_i = \ker \alpha_i$ whenever $\alpha_i \neq 0$ (equiv. $\gamma \neq 0$), and γ are the weights as usual. For $\gamma_i \neq 0$, we have not added any new information and when $\gamma_i = 0$ we are adding a projective class A_i for the parabolic structure even as the map vanishes. On the other hand when $\gamma_1 = 1$, the sheaves can acquire torsion. To handle this, we first need

Lemma 4.13 (H& J Lemma 4.11)**.** *Let E_t, $t \in \mathbb{C}$ be a family of coherent rank 2 sheaves over Σ, with E_t locally free at p for $t \neq 0$, and with E_0 having torsion subsheaf $\mathbb{C}_p$ near p. Let $\phi_t \in H^0(\Sigma, \wedge^2(E)^*)$ be a family of $SL(2, \mathbb{C})$ structures on E_t. Then ϕ_0 vanishes at p.*

Proof. If z is a local co-ordinate on Σ on an open set containing p, such that $z(p) = 0$, one can obtain E_t locally (up to reparameterization) from the exact sequence

$$\mathcal{O} \xrightarrow{(0,t^k,z)} \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \longrightarrow E_t \tag{4.31}$$

for some integer k. Then the $SL(2, \mathbb{C})$ structures ϕ_t are multiples of $e_1^* \wedge (-ze_2^* + t^k e_3^*)$, which vanishes when $z = t = 0$. □

Since Y is a projective bundle, one can freely tensor with line bundles and write Y as the bundle

$$Y = \bigoplus_{i=1}^{n} \mathbb{P}\left(\pi_1^*\left(\mathcal{O}(-1)\right) \oplus \pi_{2,i}^*\left(\mathcal{O}(-2)\right)\right). \tag{4.32}$$

In this form, when $\gamma \neq 1$ there is a natural lift of a parabolic bundle $(E, \alpha) \in X$ to Y, given by

$$\hat{E} = \left((\hat{\beta}, \hat{\alpha}_1^2), (\hat{\beta}, \hat{\alpha}_2^2), ..., (\hat{\beta}, \hat{\alpha}^2)\right), \tag{4.33}$$

which we want to extend to the torsion case where $\gamma = 1$. When there is torsion at say p_i ($\gamma_i = 1$), one can rescale the torsion subsheaf of E, modifying $\hat{\alpha}_i^2$ to $c\hat{\alpha}_i^2$ for some c. This rescaling should remain in the same equivalence class, and if $\hat{\beta} \neq 0$ then since $\hat{\beta}$ is not rescaled this would not be the case. Therefore we want that the i-th component of the lift of E in Y should be $(0, \hat{\alpha}_i^2)$. This is achieved as follows; recall that $\hat{\beta}$ is defined by composing

$$\wedge^2(\mathcal{O}^N) \xrightarrow{\phi} H^0(\Sigma, \wedge^2 E) \xrightarrow{\xi} H^0(\Sigma, L^2) \tag{4.34}$$

where $\phi : \mathcal{O}^N \to E$ is a quotient defining E. Recall that E was defined as $E_0 \times \mathcal{L}$ for some bundle E_0 with $SL(2, \mathbb{C})$ structure defining a map $\xi : H^0(\Sigma, \wedge^2(E_0)) \to H^0(\Sigma, \mathcal{O})$. Therefore, we have a commutative diagram:

$$\begin{array}{ccc} H^0(\Sigma, \wedge^2(E_0)) & \xrightarrow{\quad\xi\quad} & H^0(\Sigma, \mathcal{O}) \\ \Big\downarrow {\scriptstyle ev_{\wedge^2(E_0)}} & & \Big\downarrow {\scriptstyle ev_{\mathcal{O}}} \\ \wedge^2(E_0)|_{p_i} & \xleftarrow[\quad\xi^{p_i}\quad]{} & \mathbb{C} \end{array} \tag{4.35}$$

Thus one has $\xi = \mathrm{ev}_{\mathcal{O}}^{-1} \circ \xi^{p_i} \circ \mathrm{ev}_{\wedge^2(E)}$, and so for $\gamma_i = 1$ we use this definition to define ξ^{p_i} at the torsion points of E. Then, from Lemma 4.13, one has that $\hat{\beta}_i$ vanishes when there is torsion at p_i. Using this definition we can define a lift for all $(E, \alpha) \in X$ to Y by $\hat{E} = \left((\hat{\beta}, \hat{\alpha}_1^2), (\hat{\beta}, \hat{\alpha}_2^2), ..., (\hat{\beta}, \hat{\alpha}^2)\right)$.

Next Hurtubise and Jeffrey analyse which elements of Y are stable or semistable as weighted parabolic bundles. In Lemma 4.13 we saw that torsion in the kernel of α_i destabilised $(E, \alpha) \in X$. The same is true in Y:

Lemma 4.14 (Hurtubise and Jeffrey, Lemma 4.13)**.** *A semi-stable element $y \in Y$ corresponds to a parabolic bundle (E, α) with no torsion in the kernel of α.*

To allow the α_i to go to zero but still preserve the information of which projective class we have at p_i, we define a new map. Let $(E, \alpha) \in X$ with $SL(2, \mathbb{C})$ structure ϕ. Then we can map $(E, \alpha, \phi) \to Y$ by

$$(E, \alpha, \phi) \to \bigoplus_{i=1}^{n} \left((\phi(p_i)^N, \phi(p_i)^{N-1}\alpha_i^2\right), \tag{4.36}$$

and we define $\hat{Y}$ to be the closure in Y of the image of this map. In this closure, one obtains the points with $\alpha_i = 0$. Then since $\phi(p_i) \neq 0$, Lemma 4.13 guarantees E is torsion free at p_i. The next proposition addresses stability:

Theorem 4.15 (Hurtubise and Jeffrey, Prop. 4.14)**.** *Let*

$$y = ((b_1, a_1), (b_2, a_2), ..., (b_n, a_n)) \tag{4.37}$$

be a point in $\hat{Y}$. Let $\Gamma(y)$ be the set of $\gamma_i \in [0,1]$ such that $\gamma_i = 0$ if $a_i = 0$ and $\gamma_i = 1$ if $b_i = 0$. Then y is semi-stable if and only if for one element $\gamma \in \Gamma(y)$, $\pi(y) \in X$ is γ-semi-stable.

Therefore semi-stable elements in $\hat{Y}$ all project down to semi-stable elements of X for some choice of weights, and the GIT quotient $\hat{Y} /\!/ SL(N,\mathbb{C})$ corresponds to equivalence classes of quadruples $(E, \alpha_i, \hat{\alpha}_i, \phi)$ where (E, α) is a parabolic bundle, $\hat{\alpha}_i$ is a subspace of $E|_{p_i}$ (which is the kernal of α_i when $\alpha_i \neq 0$) and ϕ is an $SL(2,\mathbb{C})$ structure. However this is not quite the final moduli space we want to construct, as we have added the extra information of $\hat{\alpha}_i$ when $\alpha_i = 0$. At these points, $\hat{\alpha}_i \in \mathbb{P}_1 = \mathbb{P}(E|_{p_i})$. We want to collapse these extra $\mathbb{P}_1$s. To do this, embed V_1 into $W_1 = V_1^{\otimes N}$ so that a non-zero element l in $L_0 = \pi_1^*(\mathcal{O}(N))$ can be thought of as an element in W_1^* by taking $v_1 \otimes ... \otimes v_n$ to $l(v_1)l(v_2)...l(v_n)$. Similarly, embedding V_2 into $W_2 = V_1^{\otimes N-1} \otimes V_2 \otimes V_2$ allows us to think of non-zero elements in $L_{1,i}$ as elements of W_2^*. Then this maps $\hat{Y}$ to a subvariety $\tilde{Y}$ in $\prod_{i=1}^n \mathbb{P}(W_1 \otimes W_2)$, and the map collapses the unwanted $\mathbb{P}_1$s while being an embedding otherwise.

Finally, we let $\mathcal{P} = \tilde{Y} /\!/ SL(N,\mathbb{C})$ be the geometric quotient, and we call it the *moduli space of framed parabolic bundles.*

Remark: Here the quotient is meant to be taken in the sense of Geometric Invariant Theory; the resulting space consists of the stable and semi-stable orbits of the $SL(N,\mathbb{C})$ action on $\tilde{Y}$, where stable means the orbit is closed and the stabiliser is finite, semi-stable means 0 is not in the closure of the orbit, and unstable means 0 is in the closure. These definitions agree with Definitions 2.9 and 4.10 for vector bundles and parabolic sheaves.

4.6 Framed Parabolic Bundles on a Trinion

To understand the moduli space of framed parabolic bundles on a (punctured) Riemann surface, one can mirror the construction of section 3 and decompose the punctured Riemann surface into trinions. Towards this end, let us compute $\mathcal{P}$ for one trinion D, a copy of $\mathbb{P}^1$ with three marked points. Up to a birational map, let the three marked points be $z = 0, 1, \infty$.

Lemma 4.16. *A degree-0 framed parabolic sheaf (E, α) on D which is semi-stable must fall into one of four cases:*

1. *E is trivial; $E \cong \mathcal{O} \oplus \mathcal{O}$.*

2. *E is torsion-free but not trivial, and* $E \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$.

3. *E has torsion at one point p, and* $E \cong \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathbb{C}_p$.

4. *E has torsion at two points* p_1, p_2*, and* $E \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2}$.

Proof. Suppose that E has no torsion. Then $E \cong \mathcal{O}(j) \oplus \mathcal{O}(-j)$, $j \in \mathbb{Z}$. If E is semi-stable with respect to some weights γ, the semi-stability condition is that for all subbundles F of E,

$$2\deg(F) \leq rk(F)\left(0 - \sum_{i=1}^{n} \gamma_i\right) + 2\sum_{i=1}^{n}(1 - \mu_i(F) + \sigma(F))\gamma_i, \tag{4.38}$$

where we recall that $\sigma(F) = \frac{1}{rk(E)}$ if $F = E$ and 0 otherwise, and $\mu_i(F) = 1$ is $F_{p_i} \subset \ker \alpha_i$ and 0 otherwise. Consider $F = \mathcal{O}(j)$. The equation becomes

$$2j \leq \sum_{i=1}^{n} \gamma_i - 2\mu_i(F). \tag{4.39}$$

Since $\gamma_i \leq 1$, This can never be satisfied for $j \geq 2$, and therefore $E = \mathcal{O} \oplus \mathcal{O}$ or $E = \mathcal{O}(1) \oplus \mathcal{O}(-1)$ are the only choices which can yield a semi-stable parabolic sheaf.

When E has torsion at one of the marked points, say p, then let $E' = E/\mathrm{Tor}(E)$. which is a bundle with $E \cong \mathcal{O}(j-1) \oplus \mathcal{O}(-j)$. Let $F = \mathcal{O}(j-1) \oplus \mathbb{C}_p$. Then the stability condition for E and F is

$$2j \leq \sum_{i=1}^{3} \gamma_i - 2\mu_i(F) \tag{4.40}$$

The right-hand side is at most 3, and therefore if $j \geq 2$ then the sheaf $\mathcal{O}(j-1) \oplus \mathbb{C}_p$ is destabilising. Therefore we must have $E' \cong \mathcal{O}(-1) \oplus \mathcal{O}$, and $E \cong \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathbb{C}_p$.

If E has two torsion points, then $E' \cong \mathcal{O}(-j) \oplus \mathcal{O}(j-2)$ If $j \geq 2$, then $\mathcal{O}(j-2) \oplus \mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2}$ will be destabilizing. If $j = 0$, then $F = \mathcal{O} \oplus \mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2}$ has stability condition

$$4 \leq \sum_{i=1}^{3} \gamma_i - 2\mu_i(F). \tag{4.41}$$

The left hand side is at most 3, so F is destabilizing. So $E' \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is the only semistable choice, and $E \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2}$.

Finally, if E has torsion at all three marked points, $E' = \mathcal{O}(-j) \oplus \mathcal{O}(j-3)$. Then from Lemma 4.11, $\gamma_i = 1$ for all i, $\mu_i = 0$ for all i. Therefore the stability condition for the subsheaf $F = \mathcal{O}(j-3) \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_\infty$ becomes

$$2j \leq -3 + 2(3) = 3 \tag{4.42}$$

This rules out $j \geq 2$. For $j = 0, 1$ the subsheaf $\mathcal{O}(-j) \oplus \mathrm{Tor}(E)$ will be destabilizing; the stability condition is

$$2(3 - j) \leq 3. \tag{4.43}$$

Hence there are no semi-stable sheaves E with torsion at three points. □

Theorem 4.17. *For all degree-0 semi-stable parabolic sheaves (E, α) over D, $E \otimes \mathcal{O}(1)$ is generated by four global sections. In particular, there is an exact sequence:*

$$\mathcal{O}(-2) \oplus \mathcal{O}(-2) \xrightarrow{Az+Bz^{-1}} \mathcal{O}(-1)^{\oplus 4} \to E \to 0. \tag{4.44}$$

With A, B represented by 4×2 matrices. The map $\mathcal{O}(-2) \oplus \mathcal{O}(-2) \xrightarrow{Az+Bz^{-1}} \mathcal{O}(-1)^{\oplus 4}$ is injective as a map of bundles away from the torsion points of E.

Proof. From the lemma, $E \otimes \mathcal{O}(1)$ falls into one of four cases, each of which is a quotient of $\mathcal{O}^{\oplus 4}$. In each case, the sequence defining $E \otimes \mathcal{O}(1)$ can be written as a direct sum of the following exact sequences:

1. $\mathcal{O}(-1) \hookrightarrow \mathcal{O}^{\oplus 2} \twoheadrightarrow \mathcal{O}(1)$
2. $\mathcal{O}(-1)^{\oplus 2} \hookrightarrow \mathcal{O}^{\oplus 3} \twoheadrightarrow \mathcal{O}(1)$
3. $0 \hookrightarrow \mathcal{O} \twoheadrightarrow \mathcal{O}$
4. $\mathcal{O}(-1) \hookrightarrow \mathcal{O} \twoheadrightarrow \mathbb{C}_p$.

In any case, $E \otimes \mathcal{O}(1)$ is generated by four global sections, and the kernel is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Thus tensoring with $\mathcal{O}(-1)$ to recover E, we obtain the claimed exact sequence.

Since E is rank 2 away from torsion, and $E = \mathcal{O}(-1)^{\oplus 4}/\mathrm{im}(Az + Bz^{-1})$ by exactness, at every point p without torsion $\mathrm{im}(Az + Bz^{-1})|_p = \mathrm{im}(Ap + Bp^{-1})$ has dimension 2. The domain $\mathcal{O}(-2) \oplus \mathcal{O}(-2)|_p$ has dimension 2, and so by rank-nullity theorem the kernal of $Ap + Bp^{-1}$ has dimension $2 - 2 = 0$ and is hence injective. □

The 4×2 matrices A, B depend on E and in fact determine E. The parabolic structure $\alpha = (\alpha_0, \alpha_1, \alpha_\infty)$ is realized as maps $\mathcal{O}(-1)^{\oplus 4} \to \mathbb{C}_{p_i}$ for which $\mathrm{im}\,(\mathcal{O}(-2) \oplus \mathcal{O}(-2)) \subset \ker \alpha_{p_i}$. These maps are represented by row vectors V_0, V_1, V_∞ in $\mathbb{C}^4$ satisfying the conditions:

$$V_0 A = 0, \qquad V_1(A + B) = 0, \qquad V_\infty B = 0. \tag{4.45}$$

Finally, the framing ($SL(2, \mathbb{C})$ structure) is determined by the isomorphism

$$\wedge^2(E) \cong \left(\wedge^2 \mathcal{O}(-2) \oplus \mathcal{O}(-2)\right)^* \otimes \wedge^4 \mathcal{O}(-1)^{\oplus 4}, \tag{4.46}$$

so the data of A, B, V_0, V_1 and V_∞ determines a framed parabolic bundle.

There is an action of $GL(2,\mathbb{C}) \times GL(4,\mathbb{C})$ on this data, by

$$(g,G)\cdot(A,B,V_0,V_1,V_\infty) = (GAg^{-1}, GBg^{-1}, V_0G^{-1}, V_1G^{-1}, V_\infty G^{-1}), \tag{4.47}$$

and the orbits of this action are the isomorphism classes of framed parabolic bundles we are interested in. We define four functions which are projectively invariant under this action:

$$x = \det(A,B), \qquad y = \det\begin{pmatrix} V_0B \\ V_1A \end{pmatrix}, \qquad z = \det\begin{pmatrix} V_0B \\ V_\infty A \end{pmatrix}, \qquad w = \det\begin{pmatrix} V_1B \\ V_\infty A \end{pmatrix}. \tag{4.48}$$

The action of (g,G) on these functions pulls out the determinants of g and G, so we can instead consider just the orbits of equivalence classes of $(x,y,z,w) \mod \mathbb{C}^*$ under the action of $SL(2,\mathbb{C}) \times SL(4,\mathbb{C})$.

Theorem 4.18. *The map $\Phi : \mathcal{P}|_D$ taking a framed parabolic sheaf (E,α) to the class $[x:y:z:w]$ as defined in equations 4.48 is an isomorphism of $\mathcal{P}|_D$ with $\mathbb{P}^3$.*

Lemma 4.19. *A point in $\mathcal{P}|_D$ defined by (A,B,V_0,V_1,V_∞) is semi-stable if and only if one of the co-ordinates x,y,z or w is non-zero. This tells us Φ is well defined.*

Proof. If (A,B,V_0,V_1,V_∞) defines an unstable bundle, we want to show that all the co-ordinates vanish. Suppose the closure of $(A,B,V_0,V_1,V_\infty)'s$ orbit under $SL(2,\mathbb{C}) \times SL(4,\mathbb{C})$ contains $(0,0,0,0,0)$. First suppose the orbit itself contains 0. Then there is $G \in SL(4,\mathbb{C})$ such that $V_iG^{-1} = 0$, and since G is invertible this means $V_i = 0$ for all $V_0, V_1, V\infty$; this makes y,z,w all 0. Furthermore, if $g \in SL(2,\mathbb{C})$ such that $GAg^{-1} = GBg^{-1} = 0$ then $A = B = 0$ by invertibility of G and g. Hence $x = 0$; so if the orbit contains 0 then $x = y = z = w = 0$.

Now suppose that 0 is in the closure of the orbit. This means there is a sequence of matrices $(G_i, g_i)_{i=1}^\infty$ that approaches matrices that send (A,B,V_0,V_1,V_∞) to 0. By the previous argument, either G or g is not invertible, or all the co-ordinates x,y,z,w equal 0. Since the determinant is a continuous function on $SL(n,\mathbb{C})$, the limit of the determinant is the determinant of the limit, and since G_i and g_i all have determinant 1, so does their limit. They must be invertible, and hence $x = y = z = w = 0$.

Conversely, we can check semi-stability via the Hilbert-Mumford criterion. Consider a point whose co-ordinates all vanish. This tells us that there is a basis of $\mathbb{C}^2$ for which $V_iA = (a_i, 0)$ and $V_iB = (b_i, 0)$ for $i = 0, 1, \infty$. Pick a basis for $\mathbb{C}^4$ in which each V_i has the form $(*,*,*,0)$. Then let $n > 0$ and $G(z) = \mathrm{diag}(z^{-n}, z^{-n}, z^{-n}, z^{3n})$. $G(z)$ is a one parameter family in $SL(4,\mathbb{C})$ for which $G^{-1}(z)V_i \to 0$ as $z \to 0$. Let $g(z) = \mathrm{diag}(z^{-m}, z^m)$ for some $n < m < 3m$. Then

$$GAg^{-1}(z) = \begin{pmatrix} z^{m-n}A_{11} & z^{-m-n}A_{12} \\ z^{m-n}A_{21} & z^{-m-n}A_{22} \\ z^{m-n}A_{31} & z^{-m-n}A_{32} \\ z^{m+3n}A_{41} & z^{3n-m}A_{42} \end{pmatrix} \tag{4.49}$$

If the span of the V_i is 3-dimensional, then in our basis we must have $A_{12} = A_{22} = A_{32} = 0$ from equation 4.45. In this case, as $z \to 0$ we have $GAg^{-1}(z) \to 0$, so this family $(G, g)(z)$ is destabilizing. If the span is 2-dimensional, A_{32} may be non-zero. In this case, we modify $G(z) = \mathrm{diag}(z^{-n}, z^{-n}, z^n, z^n)$ and get a destablizing family. Similarly, if the span of the V_i is 1-dimensional, then $A_{12} = 0$ and we let $G(z) = \mathrm{diag}(z^{-3n}, z^n, z^n, z^n)$ to get a destabilizing family. All this holds for B as well.

In the last case with all $V_i = 0$, we need only consider A and B. Since $\det(A, B) = 0$, there is a basis in which

$$(A, B) = \begin{pmatrix} A_{11} & A_{12} & B_{11} & B_{12} \\ A_{21} & A_{22} & B_{21} & B_{22} \\ A_{31} & A_{32} & B_{31} & B_{32} \\ A_{41} & A_{42} & B_{41} & B_{42} \end{pmatrix} \tag{4.50}$$

has a row of zeros. Suppose without loss of generality it is the last row. Then let $0 < m < n$ and let $G(z) = \mathrm{diag}(z^n, z^n, z^n, z^{-3n})$ and $g(z) = \mathrm{diag}(z^{-m}, z^m)$. The family $(G, g)(z)$ will destabilize $(A, B, 0, 0, 0)$.

□

With this lemma we are ready to prove Theorem 4.18.

Proof. We will prove that Φ is bijective. To do this, we will perform a case-by-case analysis on the four possibilities for E given by Lemma 4.16.

Case 1: $E = \mathcal{O} \oplus \mathcal{O}$. Then $x = \det(A, B) \neq 0$, so we can choose bases such that

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then the stabiliser of (A, B) under the $SL(2, \mathbb{C})$ action (equation 4.47) is all of $SL(2, \mathbb{C})$. The conditions of (4.45) tell us that

$$V_0 = (0, 0, a, b), \qquad V_1 = (c, d, -c, d), \qquad V_\infty = (e, f, 0, 0). \tag{4.51}$$

Next we take the geometric quotient by the $SL(4, \mathbb{C})$ action (equation 4.47), which reduces to an action of $SL(2, \mathbb{C})$ on the three vectors $(a, b), (c, d), (e, f)$ in $\mathbb{C}^2$. When these vectors span $\mathbb{C}^2$, their stabiliser will be finite, so we obtain a stable point in the moduli space. The functions y, z, w map the set of stable points bijectively to $(\mathbb{C}^3 - \{0\})$. The semi-stable orbits are those for which the three vectors are linearly dependent. In this case, $y = z = w = 0$, so the semi-stable points are all sent to $(1, 0, 0, 0)$ in $\mathbb{P}^3$. Therefore, after the quotient, the set of bundles with $x \neq 0$ is in bijection with the affine set of points $(1, y, z, w)$ in $\mathbb{P}^3$.

Case 2: $E = \mathcal{O}(1) \oplus \mathcal{O}(-1)$. In this case, $\det(A,B) = 0$, and $\text{Im}(A) + \text{Im}(B)$ is three-dimensional. By Theorem 4.17, $(Az_0 + Bz_1)$ is injective for all $(z_0, z_1) \neq (0,0)$. Therefore we can find bases in which

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then from equations (4.45) we have

$$V_0 = (0,0,a,b), \qquad V_1 = (c,-c,c,d), \qquad V_\infty = (e,0,0,f). \tag{4.52}$$

This allows us to compute $(x,y,z,w) = (0,-ac,-ae,ec)$. If a,c,e are all non-zero then the stabiliser of the $SL(4,\mathbb{C})$ action is finite, so the points are stable and the projective co-ordinates are a bijection on this locus. If one of a,c or e is zero, then we obtain a semi-stable orbit with co-ordinates $[0:1:0:0]$, $[0:0:1:0]$ or $[0:0:0:1]$. If two are zero, we obtain an unstable point with co-ordinates $[0:0:0:0]$ which is not included in the moduli space.

Case 3: E has one torsion point p. Then $x = \det(A,B) = 0$, $\text{Im}(A) + \text{Im}(B)$ is three-dimensional, but $(Az_0 + Bz_1)$ will not be injective at some non-zero (z_0, z_1). From Theorem 4.17, $(Az_0 + Bz_1)$ is injective away from the torsion points, and from Lemma 4.11 know that the torsion subsheaf of E can only be non-zero at $0, 1$ or ∞. We will compute the case with $z_0/z_1 = 0$, the other two cases are similar. Then we can find bases such that

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

in which $V_0 = (0,a,b,g)$, $V_1 = (c,0,-c,d)$ and $V_\infty = (e,0,0,f)$ and $(x,y,z,w) = (0,ac,ae,0)$. Semi-stability ensures that $a \neq 0$ and either c or e is also non-zero. If they're both non-zero we obtain a stable point, otherwise we obtain a semi-stable point, with co-ordinates $[0:1:0:0]$ and $[0:0:1:0]$.

Case 4: E has two torsion points. Then $x = \det(A,B) = 0$ and $\text{Im}(A) + \text{Im}(B)$ is not three dimensional. Then since A, B are 4×2 and injective away from the torsion points, their image must be at least two dimensional. Once again, we only do one case, with torsion at $0, \infty$, as the other cases are similar. Then we can find bases such that

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and $V_0 = (0, a, b, c)$, $V_1 = (0, 0, d, e)$ and $V_\infty = (f, 0, g, h)$. The projective co-ordinates are $(0, 0, -ag, 0)$ and the unique closed orbit is that given by $V_0 = (0, a, 0, 0)$, $V_2 = (0, 0, 0, 0)$ and $V_3 = (f, 0, 0, 0)$. Permuting the choice of torsion point, this case contains three orbits corresponding to three points in the plane at infinity; $[0 : 1 : 0 : 0], [0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 1]$.

Summarizing, the sheaves in case 1 correspond to the set $\{[x : y : z : w] \in \mathbb{P}^3 \mid x \neq 0\} \cong \mathbb{C}^3$. Those in case 2 correspond to the set $\{[0 : y : z : w] \mid y, z, w \neq 0\} \cup \{[0 : 0 : 0 : 1], [0 : 1 : 0 : 0], [0 : 0 : 1 : 0]\}$. Those in case 3 correspond to the sets with two projective co-ordinates equal to 0, and those in case 4 correspond to the sets with three projective co-ordinate equal to 0. Cases 2, 3 and 4 all have semi-stable points, meaning non-closed orbits, whose projective co-ordinates are $[0 : 1 : 0 : 0], [0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 1]$. For Φ to be bijective, we must check that these non-closed orbits all share the same closure, so that they are all given by the same point in the geometric quotient P. This is true, because we can always obtain case 4 as a limit point of cases 2 and 3. For example, consider the family in $SL(2, \mathbb{C}) \times SL(4, \mathbb{C})$ given by

$$g_t = \mathbb{1}_{2\times 2}, \qquad G_t = \mathrm{diag}(1, 1, t^{-1}, t).$$

Given $(A, B, V_0, V_1, V_\infty)$ semi-stable of case 3, in the limit as $t \to \infty$ $(G_t, g_t) \cdot (A, B, V_0, V_1, V_\infty)$ will approach a semi-stable bundle of case 4, and so the closure of an orbit in case 4 contains that of case 3. □

Notice in particular that the four points $[1 : 0 : 0 : 0], [0 : 1 : 0 : 0], [0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 1]$ each correspond to a unique closed semi-stable orbit in P. The rest of $\mathbb{P}^3$ correspond to stable points. Each of the cases in the proof correspond to the four possible cases for E given by Lemma 4.16. In Case 1, $\det(A, B) \neq 0$ tells us $E = \mathcal{O} \oplus \mathcal{O}$, and there is a $\mathbb{C}^3$ of framed parabolic structures for the three punctures. In Case 2, there is no torsion and so the bundle is $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. In Case 3, there is one torsion point, so $E = \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathbb{C}_p$, and in Case 4, there are two torsion points so $E = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2}$.

Our moduli space is therefore $\mathcal{P}|_D \cong \mathbb{P}^3$ over the trinion D. We also have an action of $(\mathbb{C}^*)^3$ on $\mathcal{P}|_D$ where each factor scales the parabolic structure at one point; for $\lambda_i \in \mathbb{C}^*$,

$$(\lambda_1, \lambda_2, \lambda_3) \cdot (A, B, V_0, V_1, V_\infty) = (A, B, \lambda_1 V_0, \lambda_2 V_1, \lambda_3 V_\infty). \tag{4.53}$$

In terms of our co-ordinates for $\mathbb{P}^3$, the action is given by

$$(\lambda_1, \lambda_2, \lambda_3) \cdot [x : y : z : w] = [x : \lambda_1\lambda_2 y : \lambda_1\lambda_3 z : \lambda_2\lambda_3 w]. \tag{4.54}$$

Consider the line bundle $\mathcal{O}(1)$ of homogeneous linear functions over $\mathbb{P}^3$. The action on $\mathbb{P}^3$ linearizes on $\mathcal{O}(1)$; if $ax + by + cz + dw \in \mathcal{O}(1)(\mathbb{P}^3)$ then define

$$(\lambda_1, \lambda_2, \lambda_3) \cdot (ax + by + cz + dw) = ax + b\lambda_1\lambda_2 y + c\lambda_1\lambda_3 z + d\lambda_2\lambda_3 w. \tag{4.55}$$

4.7 Gluing $\mathcal{P}$ over Trinions

Given two Riemann surfaces Σ_1, Σ_2 with marked points p_1 and p_2, we can glue them together by identifying p_1 and p_2 to form $\Sigma = \Sigma_1 \coprod \Sigma_2/(p_1 \sim p_2)$. This will be a nodal complex curve. If $\mathcal{P}_1$ and $\mathcal{P}_2$ denote the moduli space of parabolic sheaves corresponding to Σ_1 and Σ_2, we can also consider gluing them together. For framed parabolic sheaves (E_i, α_i), with E_i over Σ_i and $\alpha_i : (E_i)|_{p_i} \to \mathbb{C}$, we can combine the two into a diagram:

$$(E_1)|_{p_1} \to \mathbb{C} \leftarrow (E_2)|_{p_2}. \tag{4.56}$$

We consider two such diagrams to be equivalent if the framed parabolic bundles on each surface are isomorphic, and if the parabolic structures agree, by which we mean that the induced determinant maps

$$E_1/\ker\alpha_1 \oplus E_2/\ker\alpha_2 \to E_1/\ker\alpha_1 \wedge E_2/\ker\alpha_2 \cong \mathbb{C} \tag{4.57}$$

are the same. This determinant map is essentially multiplying α_1 and α_2, so taking equivalence classes under this equivalence amounts to quotienting out the anti-diagonal action of $\mathbb{C}^*$ on the framed parabolic structure at p_0 and p_1, and we have $\mathcal{P}_\Sigma = \mathcal{P}_0 \times \mathcal{P}_1 /\!/ \mathbb{C}^*$.

In particular, given a compact Riemann surface Σ with trinion decomposition $\{D_\gamma\}_{\gamma=1}^{2g-2}$, we can pinch the boundary circles $C_{i\,i=1}^{3g-3}$ down to points to obtain a singular surface $\tilde{\Sigma}$, consisting of $2g-2$ copies of $\mathbb{P}^1$ with three marked points glued along those $3g-3$ total marked points. Then we can obtain the moduli space $\mathcal{P}$ for $\tilde{\Sigma}$ by gluing the moduli spaces $\mathcal{P}_\gamma$ for each trinion D_γ. From the previous section we know each $\mathcal{P}_\gamma$ is isomorphic to $\mathbb{P}^3$. Heuristically, this lets us estimate the dimension of $\mathcal{P}$. There is a total of $3(2g-2)$ dimensions for the $\mathbb{P}^3$ over each trinion, and we have to quotient an action of $\mathbb{C}^*$ along $3g-3$ curves. Therefore, we expect that $\mathcal{P}$ has dimension $3(2g-2) - 3g - 3 = 3g - 3$, which agrees with our dimension calculation for the moduli space $\mathcal{M}$ from section 2.

Given two trinions D_1, D_2 with moduli spaces $\mathcal{P}_1 \cong \mathcal{P}_2 \cong \mathbb{P}^3$, the Segre embedding lets us embed $\mathbb{P}^3 \times \mathbb{P}^3$ into $\mathbb{P}^{15}$. Let X denote the image of $\mathbb{P}^3 \times \mathbb{P}^3$ in $\mathbb{P}^{15}$. To recall, if $[x : y : z : w]$ and $[x' : y' : z' : w']$ are co-ordinates for each copy of $\mathbb{P}^3$, then the Segre embedding is the map

$$([x : y : z : w], [x' : y' : z' : w']) \to [xx' : xy' : xz' : xw' : yx' : ... : ww'], \tag{4.58}$$

where the right-hand side is ordered lexicographically. Suppose the punctured points are p_1, p_2, p_3 for D_1 and q_1, q_2, q_3 for D_2, and that we wanted to glue p_1 to q_1. The anti-diagonal action of $\mathbb{C}^*$ on $\mathbb{P}^3 \times \mathbb{P}^3$ is given on each copy of $\mathbb{P}^3$ from equation (4.54):

$$\lambda \cdot [x : y : z : w] \to [x : \lambda y : \lambda z : \lambda w], \qquad \lambda \cdot [x' : y' : z' : w'] \to [x' : \lambda^{-1} y' : \lambda^{-1} z' : \lambda^{-1} w'].$$

The GIT quotient is then defined as

$$X /\!/ \mathbb{C}^* := \mathrm{Proj}\left(\left(\bigoplus_{n\geq 0} \Gamma(X, (\mathcal{O}(1)_{\mathbb{P}^{15}}|_X)^n)\right)^{\mathbb{C}^*}\right). \tag{4.59}$$

Therefore to understand the quotient, we must find the ring of invariant functions under the action of $\mathbb{C}^*$. For the gluing we're considering now, the invariant sections are the quadratic functions with equal powers of λ and λ^{-1}:

$$\Gamma(X, (\mathcal{O}(1)_{\mathbb{P}^{15}}|_X))^{\mathbb{C}^*} = \text{span}(xx', xw', wx', ww', yy', zy', yz', zz'). \tag{4.60}$$

We can also glue two points on one trinion to each other. For example, if D has punctures p_1, p_2 and p_3 and we glue p_2 to p_3, then the ($\mathbb{C}^*$) action from equation (4.54) becomes

$$(1, \mu, mu^{-1}) \cdot [x : y : z : w] = [x : \mu y : \mu^{-1} w : \mu\mu^{-1} z] = [x : \mu y : \mu^{-1} w : z]. \tag{4.61}$$

In this case, the invariant functions are x, yw and z with degrees 1,2 and 1. Taking the homogeneous spectrum we get the *weighted projective space* $\mathbb{P}(1, 2, 1)$. This can be embedded into $\mathbb{P}^3$ as the cone over the twisted cubic [Rei02, Ex. 1.1]

Example: Let Σ be a genus 2 compact Riemann surface. Using the same notations as above, Σ can be decomposed into two trinions in two ways. First, one can glue p_i to q_i, which we will call the *symmetric* decomposition, and second one can glue p_1 to p_2, q_1 to q_2, and p_3 to q_3, which we call the *asymmetric* decomposition. Let us perform the quotients described above to find what $\mathcal{P}$ is for each of these decompositions of Σ.

First consider the symmetric decomposition. Then the anti-diagonal action of $(\mathbb{C}^*)^3$ on $\mathbb{P}^3 \times \mathbb{P}^3$ is given by

$$(\lambda, \mu, \nu) \cdot [x : y : z : w] = [x : \lambda\mu y : \lambda\nu z : \mu\nu w]$$
$$(\lambda, \mu, \nu) \cdot [x' : y' : z' : w'] = [x : \lambda^{-1}\mu^{-1} y : \lambda^{-1}\nu^{-1} z : \mu^{-1}\nu^{-1} w].$$

Embedding $\mathbb{P}^3 \times \mathbb{P}^3$ into $\mathbb{P}^{15}$ (and defining the image as X), to compute the quotient we must find the graded ring of homogeneous invariant functions of Segre co-ordinates under the $(\mathbb{C}^*)^3$ action. Consider the polynomial $(x^\alpha y^\beta z^\gamma w^\delta x'^{\alpha'} y'^{\beta'} z'^{\gamma'} w'^{\delta'})$; the power of λ, μ and ν one gains from the action of $(\mathbb{C}^*)^3$ can be computed by the matrix-vector multiplication

$$\begin{bmatrix} p_\lambda \\ p_\mu \\ p_\nu \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ \delta \\ \beta' \\ \gamma' \\ \delta' \end{bmatrix}. \tag{4.62}$$

Therefore, to find invariant polynomials, we can set $p_\lambda = p_\mu = p_\nu = 0$ and row-reduce the matrix to find the powers of each co-ordinate which yield an invariant polynomial. Putting the matrix into row-reduced echelon form, we obtain $[\mathbb{1}_3, -\mathbb{1}_3]$ which means the only possible invariant polynomials are those with $\beta = \beta'$, $\gamma = \gamma'$

and $\delta = \delta'$. For the polynomials to be homogeneous, we need $\alpha+\beta+\gamma+\delta = \alpha'+\beta'+\gamma'+\delta'$, and therefore $\alpha = \alpha'$ as well. Hence, the degree-1 polynomials in the Segre co-ordinates are

$$xx', yy', zz' \text{ and } ww'. \tag{4.63}$$

The ring of invariant functions is $\mathbb{C}[xx', yy', zz', ww']$, graded in the standard way, and the moduli space is given by the GIT quotient

$$\mathcal{P} = (\mathbb{P}_3 \times \mathbb{P}_3) \mathbin{/\!/} (\mathbb{C}^*)^3 = \operatorname{Proj}\left(\mathbb{C}[xx', yy', zz', ww']\right) = \mathbb{P}^3. \tag{4.64}$$

Now, let us consider the asymmetric decomposition. In this case, the $(\mathbb{C}^*)^3$ action is

$$\begin{aligned}(\lambda,\mu,\nu)\cdot[x:y:z:w] &= [x:y:\lambda\mu z:\lambda\mu^{-1}w]\\ (\lambda,\mu,\nu)\cdot[x':y':z':w'] &= [x:y:\lambda^{-1}\nu z:\lambda^{-1}\nu^{-1}w].\end{aligned}$$

The corresponding linear system for invariant polynomials is therefore

$$\begin{bmatrix}p_\lambda\\ p_\mu\\ p_\nu\end{bmatrix} = \begin{bmatrix}0 & 1 & 1 & 0 & -1 & -1\\ 0 & 1 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -1\end{bmatrix}\begin{bmatrix}\beta\\ \gamma\\ \delta\\ \beta'\\ \gamma'\\ \delta'\end{bmatrix}. \tag{4.65}$$

Therefore, the αs and βs are free variables, but $\gamma = \delta$ and $\gamma' = \delta'$. We also must have that $\gamma+\delta = \gamma'+\delta'$. The condition for homogeneity of degree gives one more constraint, that $d = \alpha+\beta+\gamma+\delta = \alpha+\beta+2\gamma$. Therefore, the set of degree-1 invariant polynomials is

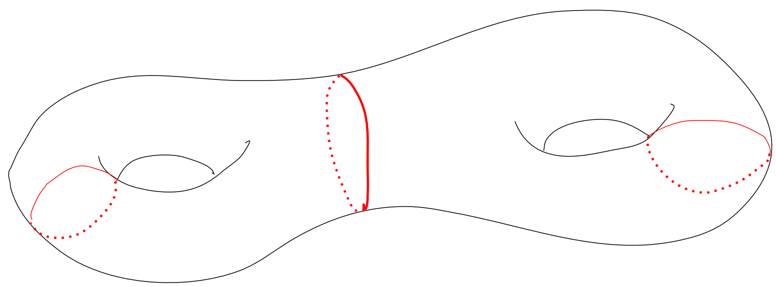
$$xx', yy', xy', yx'. \tag{4.66}$$

However in this case, the degree-2 invariant polynomials are not all generated by products of the degree-1 invariants. We have the additional invariants

$$(ww')(zz'), (wz')(zw'). \tag{4.67}$$

The degree-2 invariants are the same polynomial, and the degree-1 invariants have one relation. The graded ring of invariant polynomials is $R = \mathbb{C}[a,b,c,d,e]/\langle ab-cd\rangle$, where e has weight two. Taking the homogenous spectrum we obtain a variety inside the weighted projective space $\mathbb{P}(1,1,1,1,2)$ defined by the ideal $\langle ab-cd\rangle$.

For the symmetric gluing, we have $\mathcal{O}(1)(\mathcal{P}) = \operatorname{span}\{xx', yy', zz', ww'\}$ which is dimension 4. Similarly, $\dim\mathcal{O}(2)(\mathcal{P}) = 4 + \binom{4}{2} = 10$, from 4 squared functions and 6 cross-terms. For the asymmetric gluing, $\mathcal{O}(1)(\mathcal{P}) = \operatorname{span}\{xx', yy', xy', yx'\}$ which is also dimension 4. Then these give 4 squared functions and 5 cross-terms, as $xx'yy' = xy'yx'$. The missing section is $ww'zz' = wz'zw'$, giving again $\dim\mathcal{O}(2)(\mathcal{P}) = 10$. We see that in each of these gluings, the number of sections of $\mathcal{O}(n)$ agrees with the Verlinde formula of integer graph labellings that we computed in Section 3.4.



Chapter 5

Degenerating the Moduli Space

In Chapters 2, 3 and 4, we have described the moduli space $\mathcal{M}$ of flat $SU(2)$ connections on a compact Riemann surface Σ, and the corresponding spaces P and $\mathcal{P}$ of representations with weighted frames on Σ and framed parabolic sheaves on $\tilde{\Sigma}$, a singular curve corresponding to Σ via degeneration. Now in this section we will describe how this degeneration of Σ to $\tilde{\Sigma}$ induces a degeneration of $\mathcal{M}$ to $\mathcal{P}$. This degeneration of moduli spaces is due to Biswas and Hurtubise [BH21].

5.1 Degeneration of the Curves

The relationship between the moduli spaces $\mathcal{M}$ and $\mathcal{P}$ is given by degenerating the Riemann surface Σ to a nodal curve by smoothly shrinking the boundary curves of the trinion decomposition. First we describe a local model for this shrinking process.

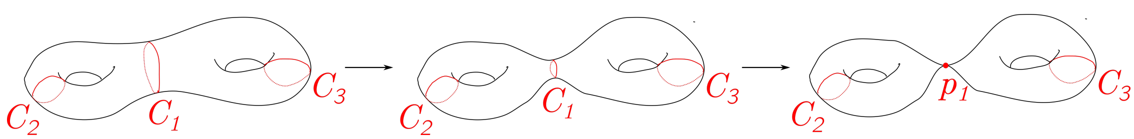


Figure 5.1: Smoothly shrinking one boundary curve to a point.

Let Σ be a compact connected Riemann surface of genus g as before. Let Σ_0 denote the nodal curve obtained by trinion decomposition from Σ, replacing each boundary circle of the trinions with a single point and then gluing at those points. Let $x_0 \in \Sigma_0$ be a nodal point corresponding to a boundary circle C_0. Let $\tilde{\Sigma}_0$ denote the desingularization of Σ_0. Let x_1 and x_2 denote the two points of $\tilde{\Sigma}_0$ which map to $x_0 \in \Sigma$.

Let B be the polydisk in $\mathbb{C}^2$ given by the product of two disks of radius 2 centred at the origin. Then define a family $\mathfrak{Q}$ of quadrics for $t \in U$ by

$$Q_t = \{(x, y) \in B \ \mid xy = t, t \in U\}. \tag{5.1}$$

For $t = 0$ we get the axes in $\mathbb{C}^2$ which is a local model for Σ_0 around x_0, and for $t \neq 0$ we get a cylinder which is a local model for Σ on a tubular neighbourhood of the boundary circle C_0. For $t \neq 0$, the boundary of Q_t is two circles, given by the equations

$$(x(t,\theta), y(t,\theta)) = \left(2e^{i\theta}, \frac{t}{2}e^{-i\theta}\right)$$
$$(x(t,\theta), y(t,\theta)) = \left(\frac{t}{2}e^{i\theta}, 2e^{-i\theta}\right).$$

There is a closed curve c_t in X_t given by

$$(x(t,\theta), y(t,\theta)) = \sqrt{t}(e^{i\theta}, e^{-i\theta}), \tag{5.2}$$

which approaches C_0 at $t = 1$ and x_0 at $t = 0$. By gluing this local model to the boundaries of the disjoint union $(U \times S^1) \coprod (U \times S^1)$ one obtains a family over U, with fibre Σ_0 at $t = 0$, and $\Sigma \cong \Sigma_t$ (topologically) at $t \neq 0$.

5.2 Local Model for the Connections

On our local model for the degeneration of the curve Σ, one also has a local model for the degeneration of a unitary connection ∇ on a vector bundle E over Σ. Theorem 3.4 guarantees that there exists some $\alpha \in \mathfrak{t}$ such that locally,

$$\nabla = d + i\frac{\alpha}{2}(d\theta_x - d\theta_y) = \partial + \frac{\alpha}{4}\left(\frac{dx}{x} - \frac{dy}{y}\right) + \overline{\partial} + \frac{\alpha}{4}\left(\frac{d\bar{x}}{\bar{x}} - \frac{d\bar{y}}{\bar{y}}\right), \tag{5.3}$$

where θ_x and θ_y are the arguments of the complex co-ordinates x and y. The holonomy $C := \mathrm{Hol}_{c_1}(\nabla)$ will be given in this gauge by $C = e^{-2\pi i\alpha}$. On our family Q_t, this connection is well defined for all $t \neq 0$, and it is isomonodromic, meaning that for all $t \neq 0$,

$$\mathrm{Hol}_{c_t}(\nabla) = \mathrm{Hol}_{c_1}(\nabla) = C. \tag{5.4}$$

It remains to describe the connection in the limit $t = 0$. Change co-ordinates on Q_t from (x, y) to (x, t), and the connection becomes

$$\nabla = d + i\frac{\alpha}{2}(2d\theta_x - d\theta_t). \tag{5.5}$$

Similarly in co-ordinates (y, t) we have

$$\nabla = d + i\frac{\alpha}{2}(-2d\theta_y + d\theta_t). \tag{5.6}$$

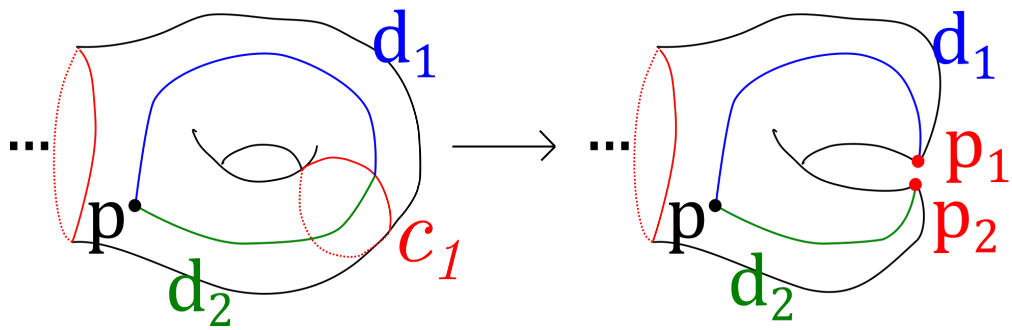


Figure 5.2: Example of d_1 and d_2, and the degeneration to $\tilde{\Sigma}_0$.

Since Q_t is defined by the equation $xy = t$, $d\theta_t$ is a normal component and one may project it out to obtain a partial connection ∇^p on Q_t. On the patch $x \neq 0$, using x co-ordinate:

$$\nabla^p = d + i\alpha d\theta_x. \tag{5.7}$$

This extends well to the limit $y \to 0$, despite the original connection ∇ having a singularity. On the patch $y \neq 0$ with y co-ordinate:

$$\nabla^p = d - i\alpha d\theta_y, \tag{5.8}$$

which also passes well to the limit $x \to 0$. Therefore we obtain a partial connection ∇^p defined everywhere on Q_0 except the nodal point x_0.

This local model can also be described in terms of the holonomy of the connection. Suppose that $\Sigma - c_1$ is not disconnected[1], and denote by $\tilde{\Sigma}_0$ the desingularized degenerated curve. In this case, $\tilde{\Sigma}_0$ has genus $g-1$. Let p_1, p_2 denote the punctures in $\tilde{\Sigma}$ corresponding to $c_0 = x_0$. Then $\tilde{\Sigma}$ is a twice punctured curve for which we have the extended moduli space $\mathcal{M}^G$ from Chapter 4:

$$\mathcal{M}^G = \left\{ \Big((A_i, B_i)_{i=1}^{g-1}, (C_1, C_2, D_1, D_2) \Big),\ \Big|\ \prod_{i=1}^{g-1} [A_i, B_i] D_1 C_1 D_1^{-1} D_2 C_2 D_2^{-1} = 1 \right\} / SU(n) \tag{5.9}$$

On the original curve Σ, a connection corresponds to a representation of $\pi_1(\Sigma)$; picking a point $p \in \Sigma$, let d_1 and d_2 be curves from p to c_1 such that $d_1 d_2^{-1}$ is a non-contractible loop based at p. Then let $(A_i, B_i)_{i=1}^{g-1}$ denote the standard generators for the rest of $\pi_1(\Sigma)$. In this notation, a connection on Σ is represented by holonomies $\Big((A_i, B_i)_{i=1}^{g-1}, C, D_1, D_2 \Big)$ around these curves.

On $\tilde{\Sigma}_0$ we have two circles around the punctures p_1 and p_2, denoted c_1 and c_2. Gluing open neighbourhoods of these points back together to obtain Σ identifies c_1 and c_2^{-1}, meaning a connection on $\tilde{\Sigma}_0$ which can be glued to form a connection on Σ must have the relation

$$D_2^{-1} D_1 C_1 D_1^{-1} D_2 = C_2^{-1} = C_1. \tag{5.10}$$

[1] A similar analysis can be performed for disconnecting curves, see [BH21, Page 10].

Therefore, $D_2^{-1}D_1 \in \mathrm{Stab}(C_1)$, or $D_1\mu^{-1} = D_2$ for some $\mu \in \mathrm{Stab}(C_1)$. Our local degeneration thus sends the connection $\nabla = (A_i, B_i, C, D_1, D_2)$ on Σ to the orbit under $\mu \in \mathrm{Stab}(C)$

$$(A_i, B_i, C, D_1, D_2) \to \left((A_i, B_i), (\mu C\mu^{-1}, D_1\mu^{-1})\right), = \mathrm{Orb}_{\mathrm{Stab}(C)}(\nabla), \tag{5.11}$$

in $\mathcal{M}^G$. If C is in $\exp\Delta^0$, then $\mathrm{Stab}(C) = \{e\}$ and the degeneration is bijective. Otherwise, when $C = \pm 1$, then $\mathrm{Stab}(C) = SU(2)$. Notice that the action of equation 5.11 matches the action of the k-th copy of G on $\mathcal{M}^G$ of equation 4.6. Therefore, these orbits are points in the strata of the imploded cross-section of $\mathcal{M}^G$. When we perform this local degeneration for each curve in the trinion decomposition, which corresponds to imploding $\mathcal{M}^G$ for each puncture, we will obtain an element in the imploded cross-section P.

5.3 Degeneration of the Moduli Space

Using this local model, one can describe the degeneration of the entire moduli space $\mathcal{M}$ as all of the boundary curves in a trinion decomposition of Σ are degenerated to a singular curve Σ_0. Let $\{c_i\}_{i=1}^{3g-3}$ be boundary curves in a trinion decomposition for Σ, which degenerate to points $\{p_i\}_{i=1}^{6g-6}$ in the desingularized curve; p_{2k-1} and p_{2k} each come from the degeneration of the curve c_k. On Σ_t for $t \neq 0$, we simply have the moduli space $\mathcal{M}_t$ of flat $SU(2)$ connections which corresponds to the moduli space of stable holomorphic vector bundles with $SL(2,\mathbb{C})$ structure over Σ_t. Therefore it remains to understand the moduli space over Σ_0 and the desingularisation $\tilde{\Sigma}_0$.

Let p be a nodal point in Σ_0 and x_1, x_2 the two points in $\tilde{\Sigma}_0$ which correspond to p. Any connection on Σ_0 has holonomy $A_i \in SU(2)$ around each nodal point which lives in the fundamental alcove of $SU(2)$. We can write the alcove as

$$\mathcal{A} = \{\exp\left(-2\pi i \;\mathrm{diag}(\gamma, -\gamma)\right) \mid \gamma \in [1/2, 1]\}, \tag{5.12}$$

and the logarithms of the holonomies are the values of γ. Under the Mehta-Seshadri correspondence these logarithms will define the weights for a parabolic structure of the $SL(2,\mathbb{C})$ bundle corresponding to the connection. The flags assigned to the points x_1 and x_2 will be determined by the eigenspaces of A_i.

Let $\Delta = [1/2, 1]$, which divides into three faces $\{1/2\}, \{1\}$ and $(1/2, 1)$. When $\gamma \in (1/2, 1)$, the corresponding holonomy matrix has two distinct eigenvalues γ and $-\gamma$. If $\gamma = 1/2$ then $A_i = -\mathbb{1}$, and if $\gamma = 1$ then $A_i = \mathbb{1}$.

Fix $\gamma \in [1/2, 1]$ and consider the space of connections with holonomy whose logarithm is γ about p_i. Quotienting out the gauge choice of $SL(2,\mathbb{C})$ framing around p_i, the Mehta-Seshadri theorem [MS80] tells us that this space of connections corresponds to the holomorphic moduli space of parabolic $SL(n,\mathbb{C})$ vector bundles E with parabolic structure at p_i of weight γ. The parabolic structure is the flag given by the largest eigenspace of the holonomy, i.e. that with eigenvalue γ. When $\gamma = 1/2$ or 1, then we acquire instead a framed parabolic sheaf with torsion at p_1 and the $SL(2,\mathbb{C})$ structure vanishes, as discussed in Chapter 4.

To obtain the entire moduli space of connections on Σ_0 and $\tilde{\Sigma}_0$, we need to allow the weights to vary and fit the space for each weight together. This was the construction of $\mathcal{P}$ from Chapter 4, the space of framed parabolic sheaves glued along a trinion decomposition, constructed by Hurtubise and Jeffrey. In summary, we have [BH21, Theorems 3.17, 4.1]

Theorem 5.1 (Biswas and Hurtubise)**.** *There is a family $\pi : \mathfrak{X} \to \mathbb{C}$, for which*

- *$\pi : \mathfrak{X} \to \mathbb{C}$ is a flat family of irreducible reduced schemes.*
- *For any $t \in \mathbb{C}^*$, the fibre $X_t = \pi^{-1}(t)$ is isomorphic to $\mathcal{M}$.*
- *The special fibre $\pi^{-1}(0)$ is $\mathcal{P}$, which is a toric variety.*

We refer to this bundle as the degeneration of $\mathcal{M}$ to $\mathcal{P}$*, or simply* the degeneration*.*

In the symplectic picture, one can rephrase this degeneration as a surjective map $\phi : \mathcal{M} \to P$, given by taking an element $A \in X_1 \cong \mathcal{M}$ to its corresponding degenerated element $A_0 \in X_0 \cong P$, given locally by the model in Section 5.2. In terms of the extended moduli spaces of Chapter 4, ϕ is the composition of the inclusion and symplectic implosion quotient maps:

$$\mathcal{M} = \mathcal{M}^T \hookrightarrow \mathcal{M}^G \twoheadrightarrow \coprod_{\sigma \in \Delta} \Phi^{-1}(\sigma) \twoheadrightarrow P, \tag{5.13}$$

described by Hurtubise and Jeffrey [HJ00, prop 2.37]. Note that $\coprod_{\sigma\in\Delta} \Phi^{-1}(\sigma) = \mathcal{M}^T$, just described in a way that respects the strata defined by the moment polytope Δ.

Lemma 5.2. *Given an element $A \in \mathcal{M}$, let $D(A)$ denote the element in P obtained by performing the local degeneration of Section 5.2 at each curve in a trinion decomposition for Σ. Then $D(A) = \phi(A)$, where $\phi(A)$ is the composition of maps*

$$\mathcal{M} = \mathcal{M}^T \hookrightarrow \mathcal{M}^G \twoheadrightarrow \coprod_{\sigma \in \Delta} \Phi^{-1}(\sigma) \twoheadrightarrow P, \tag{5.14}$$

where the surjections are the restriction and quotient maps of the imploded cross-section.

Proof. Fix one closed curve γ_k in the trinion decomposition. Then an element $A \in \mathcal{M}$ defines a holonomy $C_k \in T \subset SU(2)$ around γ_k, and we also have the local model for the degeneration $D(A)$ on a tubular neighbourhood around γ_k. First let us compute $\phi(A)$. Suppose $A = (A_i, B_i, C_j, D_j)$. Then to compute the quotient for the imploded cross-section, we have three faces $\{\sigma_0, \sigma_+, \sigma_-\}$ of $\Delta = [0,1]$ to consider. If

$C_j^{-1} \in \exp \sigma_0$ then it is quotiented by the stabiliser $[G_\sigma, G_\sigma] = \{e\}$ as discussed in Chapter 4. If $C_j^{-1} = \pm 1$, then it is quotiented by the stabiliser $[G_\sigma, G_\sigma] = G$, under the action

$$D_k \to D_k g^{-1},\ C_l \to g C_l g^{-1}. \tag{5.15}$$

In summary:

$$\phi(\nabla) = \begin{cases} (A_i, B_i, C_j, D_j), & C_k \in \sigma_0 \\ \mathrm{Orb}_{SU(2)}(\nabla), & C_k \in \sigma_\pm \end{cases} \tag{5.16}$$

On the other hand, the degeneration $D(A)$ takes (A_i, B_i, C_j, D_j) to the set

$$D(A) = \{((A_i, B_i, C_j, D_j)_{j \neq k}(C_k, D_k, D_k\mu) \mid \mu \in \mathrm{Stab}(C_k))\} \tag{5.17}$$

For C_k in σ_0, the stabilizer is $\{e\}$ and we have

$$D(A_i, B_i, C_j, D_k) = (A_i, B_i, C_j, D_k) = \phi(A). \tag{5.18}$$

If C_k is in $\sigma_\pm$, then the stabilizer is $SU(2)$ and we have

$$D(A_i, B_i, C_j, D_k) = \mathrm{Orb}_{SU(2)}(\nabla), \tag{5.19}$$

under the action of $SU(2)$ of equation 4.6 which is the same as that defined for the imploded cross-section. Hence on each stratum, $\phi(\nabla) = D(\nabla)$.

For each curve in the trinion decomposition, we can repeat this process, until it remains to perform the final quotient by the first copy of G. After these implosions, we arrive at the set

$$\left\{ (C_1, W_2, ..., W_n) \mid C_1 \in G, W_j \in D(G)_{impl}, C_1 \prod_{j=2}^{n} \Phi_G(W_j) = 1 \right\}. \tag{5.20}$$

Where $W_j = (D_j, C_j D_j) \in D(G)_{impl}$, $j \neq 1$ and $W_1 = (\mathbb{1}, C_1)$. The final copy of G acts by (Equation 4.15)

$$g \cdot (W_1, ..., W_n) = (C_1, gD_2, C_2, ..., gD_n, C_n). \tag{5.21}$$

As for the degeneration D, the final copy of G acts by conjugation on the representations $\mathrm{Hom}(\pi_1(G), G)$. However, as we already fixed the connection A to be in A.T.D. gauge, we know $C_j \in T$ and hence the conjugation action fixes C_j, and is given by

$$g \cdot (C_1, (C_j, D_j, D_j\mu)_{j=2}^n) = (C_1, (C_j, gD_j, gD_j\mu)_{j=2}^n), \tag{5.22}$$

which agrees with the action for the implosion. □

5.4 Toric Degenerations and Bohr-Sommerfeld Points

Theorem 5.1 gives us a relationship between the moduli space $\mathcal{M}$ of flat $SU(2)$ connections over the Riemann surface Σ, and a toric variety $\mathcal{P}$ via degeneration. Such degenerations to toric varieties have been studied in other contexts, where one can often prove a result about some space X by degenerating it to a toric variety for which the result holds. In this case, one may hope to prove that the number of Bohr-Sommerfeld points of $\mathcal{M}$ is equal to the number of fiberwise flat sections by proving Sniatycki's theorem for toric varieties and then using the degeneration of $\mathcal{M}$ to $\mathcal{P}$. If the degeneration preserved the space of sections, then by relating the fibrewise flat sections of $\mathcal{P}$ to the Bohr-Sommerfeld points you could obtain a similar relation for $\mathcal{M}$.

However, it turns out that Sniatycki's theorem does not hold for toric varieties. Hamilton proved in 2010 that if M is a compact toric symplectic manifold with a singular Lagrangian fibration and a prequantum line bundle $\mathcal{L}$, then the number of fibre-wise flat sections ($\dim H^0(\mathcal{J}_\pi, M)$) is equal to the number of *non-singular* Bohr-Sommerfeld points [Ham10, Theorem 8.3.2]. For example, when $g = 2$ we computed in Section 4.7 that the toric variety $\mathcal{P}$ corresponding to the symmetric decomposition is $\mathbb{P}^3$. This is also a compact symplectic manifold, so Hamilton's result applies. In this case, $\mathcal{P}$ had four Bohr-Sommerfeld points, corresponding to the vertices of its moment polytope, which are all singular. Therefore this result tells us that $\mathcal{P}$ has *no* fibre-wise flat sections. It is conjectured that the number of fibre-wise flat sections of $\mathcal{M}$ should agree with the Verlinde formula, so if the degeneration from $\mathcal{M}$ to $\mathcal{P}$ is section-preserving, this would contradict our expectations.

To investigate the behaviour of the holomorphic and fibre-wise flat sections of $\mathcal{L}$, we take inspiration from a result of Harada and Kaveh [HK15], which constructs an integrable system from a *toric degeneration* satisfying some additional hypotheses which do not apply to the degeneration of Biswas and Hurtubise.

Definition 5.3. *Let X be an n-dimensional quasi-projective irreducible reduced scheme. We call $\pi : \mathfrak{X} \to \mathbb{C}$ a* toric degeneration *of X if:*

1. *$\pi : \mathfrak{X} \to \mathbb{C}$ is a flat family of irreducible reduced schemes.*
2. *The family $\mathfrak{X}$ is trivial over $\mathbb{C}^*$, namely there exists a fibre-preserving isomorphism $\rho : X \times \mathbb{C}^* \to \mathfrak{X} - X_0$, such that for each $t \in \mathbb{C}^*$ $\rho_t : X \times \{t\} \to X_t$ is an isomorphism.*
3. *The fibre X_0 is a toric variety with respect to an action of $(\mathbb{C}^*)^n := \mathbb{T}$.*

Theorem 5.1 tells us that in this case, the moduli space $\mathcal{M}$ of flat $SU(2)$ connections serve as our quasi-projective scheme, and we have a toric degeneration to the moduli space of parabolic sheaves $\mathcal{P}$. Now let us describe the additional hypothesis that are required for Harada and Kaveh's result.

Suppose $\pi : \mathfrak{X} \to \mathbb{C}$ is a toric degeneration of X, and X has a Kaehler form ω. Let Ω denote a constant multiple of a Fubini-Study Kaehler form on $\mathbb{P}^N$, and equip $\mathbb{P}^N \times \mathbb{C}$ with the Kaehler structure $\Omega \times \omega_{std}$. Assume that:

1. The family $\mathfrak{X}$ is smooth away from the zero fibre X_0.

2. The family $\mathfrak{X}$ is embedded in $\mathbb{P}^N \times \mathbb{C}$ as an algebraic subvariety, for some projective space $\mathbb{P}^N$ such that:

   - the map $\pi : \mathfrak{X} \to \mathbb{C}$ is the restriction of $\mathfrak{X}$ to the projection of $\mathbb{P}^N \times \mathbb{C}$ to $\mathbb{C}$;
   - the action of $\mathbb{T}$ on X_0 extends to a linear action on $\mathbb{P}^N$.

   Let ω_t denote the restriction of $\Omega \times \omega_{std}$ to the fibre X_t embedded in $\mathbb{P}^N \times t$. Then

3. The map $\rho_1 : X \to X_1$ is an isomorphism of Kaehler manifolds; $\rho_1^*(\omega_1) = \omega$;

4. Let $T = (S^1)^n$ denote the compact subtorus of $\mathbb{T}$. The Kaehler form Ω on $\mathbb{P}^N$ is T-invariant and in particular the restriction ω_0 to the toric variety X_0 is a T-invariant Kaehler form.

Given a toric variety satisfying these conditions, Harada and Kaveh provide the following theorem:

Theorem 5.4 (Harada and Kaveh). *Let $\pi : \mathfrak{X} \to \mathbb{C}$ be a toric degeneration of X, and let ω be a Kaehler structure on X. If $\pi : \mathfrak{X} \to \mathbb{C}$ satisfies conditions 1-4 above, then:*

1. *There exists a surjective continuous map $\phi : X \to X_0$ which is a symplectomorphism restricted to a dense open subset $U \subset X$.*

2. *There exists a completely integrable system $\mu = (F_1, ..., F_n)$ on (X, ω) whose moment polytope Δ coincides with the moment polytope of (X_0, ω_0).*

3. *Let $U \subset X$ be the open dense subset of X from item 1. Then the integrable system of item 2 generates a Hamiltonian torus action on U, and the inverse image $\mu^{-1}(\Delta^\circ)$ of the interior of Δ under the moment map lies in the open subset U.*

For the degeneration $\mathcal{M} \to \mathcal{P}$, the spaces $\mathcal{M}_t$ are not smooth, so the theorem does not apply. Despite that, in this case we still have a surjective continuous map $\phi : \mathcal{M} \to \mathcal{P}$ which is a symplectomorphism on a dense open subset of $\mathcal{M}$, and we still have the moment polytopes of $\mathcal{M}$ and $\mathcal{P}$ coinciding.

5.5 Polarization and Prequantum Line Bundle on $\mathcal{P}$

The toric variety $\mathcal{P}$ is projective and has a very ample line bundle $\mathcal{L}_0$ whose sections are computed by counting lattice points in $\mathcal{P}$'s polytope. In this section, we want to investigate the space of holomorphic sections $H^0(\mathcal{P}, \mathcal{L}_0)$ and we want to define a polarization of $\mathcal{P}$ which will allow us to discuss the cohomology of fibre-wise flat sections, $\oplus_{k\geq 0} H^k(\mathcal{P}, \mathcal{J}_{\pi,0})$. In particular, we ask if the degeneration from $\mathcal{M}$ to $\mathcal{P}$ preserves these spaces or not.

From Section 4, $\mathcal{P}$ has a symplectic form ω, and the degeneration $\phi : \mathcal{M} \to \mathcal{P}$ is symplectomorphic on an open dense subset U of $\mathcal{M}$. Thus, it remains to equip $\mathcal{P}$ with a real polarisation $\pi_{\mathcal{P}}$ and a line bundle $\mathcal{L}_0$ with curvature $2\pi i\omega$. In both cases, we mirror the construction of the corresponding object for $\mathcal{M}$.

For $\mathcal{M}$ we defined functions $f_i : \mathcal{M} \to \mathbb{R}$ by taking the cosines of the holonomies of a connection around each loop in a given trinion decomposition of the surface Σ. Now for $\mathcal{P}$, we have Σ_0 with $n = 3g-3$-punctures and the extended moduli space $\mathcal{M}^G$; given an element $x = (C_1, (D_i, C_i)_{i=2}^n) \in \mathcal{M}^G$, it is the matrices $\{C_j\}$ which correspond to these holonomies. Taking the implosion, we obtained elements $W_j \in D(G)_{impl}$ with $W_j = (D_k, C_k)$, and so we want to define functions $f_k : \mathcal{M}^G \to \mathbb{R}$ by

$$\tilde{f}_k(C_1, (D_i, C_i)_{i=2}^n) = \frac{1}{\pi} \cos^{-1}(\text{Tr } C_k) \tag{5.23}$$

To get functions on P, one must check that these $\tilde{f}_k$ pass to the symplectic implosion. For the action of G_k on $\mathcal{M}^G$, with moment map $\Phi_k(C_1, (D_i, C_i)_{i=2}^n) = (C_k)^{-1}$, and a given face σ of Δ we have

$$\Phi_k^{-1}(\sigma_0) = \{(C_1, (D_i, C_i)_{i=2}^n) \in \mathcal{M}^G \mid (C_k)^{-1} \in \sigma\} \tag{5.24}$$

Hence $\tilde{f}_k$ is constant on on $\Phi_k^{-1}(\sigma)$. To perform the quotient of $[G_\sigma, G_\sigma]$, we have two cases: if $\sigma = \Delta^0$ then $[G_\sigma, G_\sigma] = \{e\}$ and so Φ_k passes to the quotient. If $\sigma \in \{0, 1\}$, then $[G_\sigma, G_\sigma] = SU(2)$ and we have

$$\tilde{f}_k(g \cdot (C_1, (D_i, C_i)_{i=2}^n)) = \frac{1}{\pi} \cos^{-1} \text{Tr } (\text{Ad}_g(C_k)) = \tilde{f}_k(C_1, (D_i, C_i)_{i=2}^n). \tag{5.25}$$

Hence $\tilde{f}_k$ is invariant and passes to the quotient by $[G_\sigma, G_\sigma]$. For $l \neq k$, the value of $\tilde{f}_l$ depends only on C_l, which is not acted on by G_k and therefore $\tilde{f}_l$ also passes to the quotient. It remains to check that these functions pass under the quotient by the first copy of G. Recall that after quotienting, we obtain the elements $(W_1, W_2, ..., W_n)$, where $W_i = (D_i, C_i) \in D(G)_{impl}$ $(D_1 = \mathbb{1})$. The action of the first G on W_k is given by (4.15):

$$(D_k, C_k) \to (gD_k, C_k)). \tag{5.26}$$

The functions $\tilde{f}_k$ do not depend on D_k and are therefore constant on the equivalence classes, and pass to the quotient to obtain functions $f_k : \mathcal{P} \to \mathbb{R}$.

From Equation 5.11 we know that $\phi : \mathcal{M} \to \mathcal{P}$ sends a connection with holonomy C_k around a trinion decomposition curve c_k to the orbit with $W_k = (D_k\mu^{-1}, \mu C_k \mu^{-1})$, $\mu^{-1} \in \text{Stab}(C_k)$. Therefore if we define $\theta_{k,P} = \cos^{-1}(f_k)$, $\theta_{k,P} : \mathcal{P} \to \mathbb{R}$, we have

$$\theta_{k,P} \circ \phi = \theta_k \tag{5.27}$$

where $\theta_k : \mathcal{M} \to \mathbb{R}$ were the functions defined in Section 3. Letting $\pi_P = (\theta_{1,P}, ..., \theta_{3g-3,P})$ we thus obtain a polarisation of $\mathcal{P}$ with $\pi_P \circ \phi = \pi$.

Next we build a prequantum line bundle on P. On the locus where $\phi : \mathcal{M} \to P$ is a symplectomorphism, we can simply define $\mathcal{L}_P = \mathcal{L}$, but we must discuss the extension to the rest of P. Recall that we defined a function $\Theta : \mathcal{A} \times \mathcal{G} \to U(1)$ for connections over Σ, which we computed to be (Equation (3.4))

$$\Theta(A, g) = \exp\left[-ik \int_{\Sigma} \text{Tr } (dgg^{-1} \wedge A)\right]. \tag{5.28}$$

For the singular curve Σ_0, we can define Θ exactly the same way, which allows us to define an equivalence relation on $\mathcal{A}_0 \times \mathbb{C}$ by $(A, z) \sim (g \cdot A, \Theta(A, g)z)$. Then we can define a line bundle $\mathcal{L}_0 = \mathcal{A}_0 \times \mathbb{C}/\sim$ over $\mathcal{A}_0/\mathcal{G}_0 = \mathcal{P}$.

We want this line bundle to be compatible with the degeneration $\phi : \mathcal{M} \to \mathcal{P}$, in the sense that $\phi^*\mathcal{L}_0 = \mathcal{L}$.

Lemma 5.5. *Let V_i be a tubular neighbourhood of C_i in Σ. Let V_0 denote the (disconnected) image of V_i under the surface degeneration. Then for every pair $(A, g) \in \mathcal{A} \times \mathcal{G}$ on Σ which degenerates to a pair (A_0, g_0) on Σ_0, there exists liftings $(\mathbf{A}, \mathbf{g})$ and $(\mathbf{A}_0, \mathbf{g}_0)$ such that*

$$\int_{V_i} Tr\ (\mathbf{g}^{-1} d\mathbf{g} \wedge \mathbf{A}) = \int_{V_0} Tr\ (\mathbf{g}_0^{-1} d\mathbf{g}_0 \wedge \mathbf{A}_0). \tag{5.29}$$

Proof. Recall that $V_i = Q_1 = \{(x, y) \in \mathbb{C}^2 \mid xy = 1\}$, and it degenerates to $V_0 = Q_0 = \{(x, y) \in \mathbb{C}^2 \mid xy = 0\} = Q_{0,x} \cup Q_{0,y}$, where $Q_{0,x}, Q_{0,y}$ are the locus with $x \neq 0$ and $y \neq 0$ respectively. The curve Q_1 has a loop γ_1, and cutting Q_1 along γ_1 disconnects it into the components with $x > y$ and $y > x$. Let Q_x denote where $x > y$, and Q_y where $y > x$.

Let $N = \{Q_t\}_{t \neq 0} \cong V_i \times (0, 1]$. Then one possible lifting of A from V_i to N is:

$$\mathbf{A} = \frac{\alpha}{2}\left(\frac{dx}{x} - \frac{dy}{y} + \frac{dt}{t}\right) \tag{5.30}$$

Now, since $t = xy$ we have $dt = ydx + xdy$ and therefore

$$\begin{aligned}\mathbf{A}_x &= \frac{\alpha}{2}\left(\frac{dx}{x} - \frac{dy}{y} + \frac{ydx + xdy}{xy}\right) \\ &= \alpha\, \frac{dx}{x}.\end{aligned}$$

Another possible lifting is

$$\mathbf{A}_y = \frac{\alpha}{2}\left(\frac{dx}{x} - \frac{dy}{y} - \frac{dt}{t}\right) = -\alpha\, \frac{dy}{y}. \tag{5.31}$$

These liftings are equal on the curve γ_1. We define the lifting we will use, $\mathbf{A}$ to be $\mathbf{A}_x$ on Q_x and $\mathbf{A}_y$ on Q_y.

For the gauge transformation $g(x, y)$ we can do a similar lifting process;

$$\mathbf{g}(x, y, t) = \begin{cases} g\left(x, t/x\right), & x > y \\ g\left(t/y, y\right), & y > x \end{cases}. \tag{5.32}$$

Then the integral over V_i becomes:

$$\int_{V_i} \text{Tr } (\mathbf{g}^{-1} d\mathbf{g} \wedge d\mathbf{A}) = \int_{Q_x} \text{Tr } (g^{-1}(x, 1/x) dg(x, 1/x) \wedge \alpha \ \frac{dx}{x}) - \int_{Q_y} \text{Tr } (g^{-1}(1/y, y) dg(1/y, y) \wedge \alpha \ \frac{dy}{y})$$

On the other hand, recall that A_0 is given by:

$$A_0 = \begin{cases} \alpha \frac{dx}{x}, & x \neq 0 \\ -\alpha \frac{dy}{y}, & y \neq 0 \end{cases} \tag{5.33}$$

which can be lifted to any three manifold N' in a horizontal manner, i.e. by not adding any tangential part. Similarly, our gauge transformation g_0 coming from g was defined by

$$g_0(x, y) = \begin{cases} g\,(x, 0)\,, & x \neq 0 \\ g\,(0, y)\,, & y \neq 0 \end{cases}. \tag{5.34}$$

Therefore the integral over V_0 becomes

$$\int_{V_0} \text{Tr } (\mathbf{g}_0^{-1} d\mathbf{g}_0 \wedge d\mathbf{A}_0) = \int_{Q_{x,0}} \text{Tr } (g^{-1}(x, 0) dg(x, 0) \wedge \alpha \ \frac{dx}{x}) - \int_{Q_{y,0}} \text{Tr } (g^{-1}(0, y) dg(0, y) \wedge \alpha \ \frac{dy}{y}).$$

Finally, all four of $Q_x, Q_y, Q_{x,0}$ and $Q_{y,0}$ are diffeomorphic to S^2 with two points removed, and a diffeomorphism between Q_x and $Q_{x,0}$ is given by our local model for the curves. This diffeomorphism pulls back $g(x, 0)$ to $g(x, x^{-1})$. The analogous equality holds for Q_y and $Q_{y,0}$. Thus these two integrals are equal. □

Theorem 5.6. *Given a pair $(A, g) \in \mathcal{A} \times \mathcal{G}$ over Σ, which degenerates to the pair (A_0, g_0) over Σ_0, we have*

$$\Theta(A, g) = \Theta(A_0, g_0). \tag{5.35}$$

In particular, this implies that if (A, g) and (B, h) both degenerate to (A_0, g_0), then $\Theta(A, g) = \Theta(B, g)$.

Proof. From Equation (3.4) we know

$$\Theta(A, g) = \exp\left[-ik \int_{\Sigma} \text{Tr } (dgg^{-1} \wedge A)\right]. \tag{5.36}$$

Let $\{C_i\}_{i=1}^{3g-3}$ be any trinion decomposition of Σ. Let V_i be a tubular neighbourhood in Σ of each C_i. Then on $U := \Sigma - \bigcup_{i=1}^{3g-3} V_i$, the degeneration does nothing, and we have $(A, g)|_U = (A_0, g_0)|_U$. Therefore, we have

$$\begin{aligned}
\Theta(A,g) &= \exp\left[-ik\int\limits_{\Sigma}\text{Tr }(dg\ g^{-1}\wedge A)\right] \\
&= \exp\left[-ik\int\limits_{U}\text{Tr }(dg\ g^{-1}\wedge A) - ik\sum_{i=1}^{3g-3}\int\limits_{V_i}\text{Tr }(dg\ g^{-1}\wedge A)\right] \\
&= \exp\left[-ik\int\limits_{U}\text{Tr }(dg_0\ g_0^{-1}\wedge A_0) - ik\sum_{i=1}^{3g-3}\int\limits_{V_{0,i}}\text{Tr }(dg_0\ g_0^{-1}\wedge A_0)\right],\ \text{(Lemma 5.5)} \\
&= \exp\left[-ik\int\limits_{\Sigma_0}\text{Tr }(dg_0\ g_0^{-1}\wedge A_0)\right] \\
&= \Theta(A_0,g_0).
\end{aligned}$$

□

Using Theorem 5.6 we can show $\mathcal{L}=\phi^*\mathcal{L}_0$. By definition

$$\phi^*\mathcal{L}_0 = \{(x,(y,z))\in\mathcal{M}\times\mathcal{L}_0 \mid \phi(x)=y\}. \tag{5.37}$$

Given a point $(x,z)\in\mathcal{L}$ we can map it to $(x,(\phi(x),z))$ in $\phi^*\mathcal{L}_0$. Since ϕ is surjective, this is surjective. Suppose then that for some (x,z) and $(x',z')\in\mathcal{L}$ we have

$$(x,(\phi(x),z)) = (x',(\phi(x'),z')). \tag{5.38}$$

Then $x=x'$ and $z=z' \mod \Theta$, namely $z'=\Theta(\phi(x),g_0)z$ for some $g_0\in\mathcal{G}_0$. Then to show $z=z'$ in $\mathcal{L}$, we need that there is some $g\in\mathcal{G}$ such that $\Theta(A,g)z'=z$, where $A\in\mathcal{A}$ represents $x\in\mathcal{M}$. The surjectivity of the degeneration gives the existence of some g that degenerates to g_0, and Theorem 5.6 tells us that $\Theta(A_0,g_0)z=\Theta(A,g)z=z'$ and therefore $z=z'$ in $\mathcal{L}$.

5.6 Holomorphic Sections of $\mathcal{L}$

Lemma 5.7. *The line bundle $\mathcal{L}_0$ on the toric moduli space $P(D)$ over a trinion D, has curvature ω equal to the Fubini-Study metric on $P(D)\cong\mathbb{P}^3$.*

Proof. Let $\mathcal{M}(D)$ denote the moduli space of flat $SU(2)$ connections over a single trinion. Then $\pi(\mathcal{M}(D)) = \Delta$ and over the interior Δ^0 we have the symplectomorphism ϕ. Thus, on on the dense torus $(\mathbb{C}^*)^3$ inside $P(D)$, we have $\phi^*\omega_P=\omega_{\mathcal{M}}$. Recall that the form on $\mathcal{M}(D)$ is the Atiyah-Bott form (Equation (2.25)).

The prequantum bundle $\mathcal{L}$ on $\mathcal{M}(D)$ is isomorphic to the determinant line bundle (Theorem 3.2) and the determinant line bundle's sections give an embedding ι of $\mathcal{M}(D)$ into projective space (Proposition 4.7). Therefore $\mathcal{L}$ is very ample over $\mathcal{M}(D)$ and $\mathcal{L} = \iota^*\mathcal{O}(1)$. The curvature ω_{fs} of $\mathcal{O}(1)$ is the Fubini-Study metric, so $\iota^*\omega_{fs} = \omega_{\mathcal{M}} = \phi^*\omega_P$ over $(\mathbb{C}^*)^3 \subset P(D)$.

Therefore, on $P(D) = \mathbb{P}^3$ we have two forms, ω_{fs} and ω_P, which agree on an open dense subset. Therefore they are equal everywhere on $P(D)$. □

Corollary 5.8. *The line bundle $\mathcal{L}_0$ over $P(D)$ is ample.*

Proof. From the lemma, the curvature of $\mathcal{L}_0$ is a positive (1,1) form. Then since $P(D)$ is a compact Kahler manifold, the Kodaira embedding theorem says $\mathcal{L}_0$ is ample. □

Theorem 5.9. *The line bundle $\mathcal{L}_0$ over $\mathcal{P}$ is ample.*

Proof. Let D be a trinion and $P(D)$ the toric moduli space for D. By Corollary 5.8, there exists large enough n for which $\mathcal{L}_{P(D)}^{\otimes^n}$ is very ample over $P(D)$. That is, since $P(D) = \mathbb{P}^3$,

$$\mathcal{O}(1) = \mathcal{L}_{P(D)}^{\otimes^n}. \tag{5.39}$$

From the discussion in section 4.7 we know that $\mathcal{P}$ can be constructed by gluing $\mathbb{P}^3$s along a trinion decomposition of the Riemann surface Σ. The decomposition consists of $2g-2$ copies of $\mathbb{P}^3$, and we must quotient by $3g-3$ actions of $\mathbb{C}^*$. Letting ι denote the Segre embedding (Equation 4.58) $\mathbb{P}^3 \coprod \mathbb{P}^3 \xrightarrow{\iota} X \subset \mathbb{P}^{15}$ and $p_i : X \to \mathbb{P}^3$ the projection to each factor, we have

$$\mathcal{O}(1)_{\mathbb{P}^{15}}|_X = p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1) = p_1^*\mathcal{L}_{P(D)}^{\otimes n} \otimes p_2^*\mathcal{L}_{P(D)}^{\otimes n} = \left(p_1^*\mathcal{L}_{P(D)} \otimes p_2^*\mathcal{L}_{P(D)}\right)^{\otimes n}. \tag{5.40}$$

Repeating this for all the copies, we can embed the product of the $\mathbb{P}^3$s into $\mathbb{P}^N$ for some large N, and we obtain an ample bundle

$$\tilde{L} = \bigotimes_{i=1}^{3g-3} p_i^*\mathcal{L}_{P(D)} \tag{5.41}$$

Taking the GIT quotient under $\mathbb{T} := (\mathbb{C}^*)^{3g-3}$, the result is given by the homogeneous spectrum of the invariant sections of $\tilde{\mathcal{L}}$, that is

$$\mathcal{P} = \mathrm{Proj}\left(\bigoplus_{k\geq 0} H^0(X, \tilde{L}^k)^{\mathbb{T}}\right). \tag{5.42}$$

Therefore, if we denote $q : X \to \mathcal{P}$ as the quotient, then $q^*\mathcal{O}(1)_{\mathcal{P}} = \tilde{\mathcal{L}}^{\otimes n}$.

On the other hand, the sections of $\tilde{\mathcal{L}}$ are generated by tensor products $\otimes_{i=1}^{3g-3}\sigma_i$ of sections in $\mathcal{L}_{P(D)}$ for each trinion D, and we can define generators of $\mathcal{L}_0$ over $\mathcal{P}$ by defining $\sigma(x) = \otimes_{i=1}^{3g-3}\sigma_i(\tilde{x_i})$, where $q(\tilde{x_i}) = x$. For most sections, this will not be well defined as the value of σ_i may depend on the choice of $\tilde{x_i}$, but a section will be invariant under the $\mathbb{T}$ action exactly if it does not depend on this choice: if $\tilde{x_i}$ and $\tilde{y_i}$ both map to x then there is some $t \in \mathbb{T}$ with $t \cdot \tilde{x_i} = \tilde{y_i}$ and therefore $\sigma_i(x_i) = \sigma_i(y_i)$. Thus $\mathcal{L}_0 \cong q^*\tilde{\mathcal{L}}$ and $\mathcal{L}^{\{} \otimes n\} \cong q^*\mathcal{O}(1)_{\mathcal{P}}$ meaning $\mathcal{L}_0$ is ample.

Finally, $\mathcal{L}_0$ is very ample as by the same argument as Lemma 5.6, its curvature is given on $\mathcal{P}$ by the Fubini-Study metric, and thus the curvature of $\mathcal{L}_0^n = n\omega_{fs}$. The curvature of $\mathcal{O}(1)$ is the Fubini-Study metric, so for these to be equal we must have $n = 1$. □

Corollary 5.10. *For $k \geq 0$, $H^k(\mathcal{P}, \mathcal{L}_0) = 0$.*

Proof. $\mathcal{L}_0$ is a very ample bundle on a complex projective variety, so $H^k(\mathcal{M}, \mathcal{L}) = 0$ for $k > 0$ [GD60, III, Prop 2.6.1]. □

The prequantum line bundle bundle $\mathcal{L}$ is also very ample over a complex projective variety, so $H^k(\mathcal{M}, \mathcal{L}) = 0$ for $k > 0$ as well. Therefore, if the sections $H^0(\mathcal{M}, \mathcal{L})$ and $H^0(\mathcal{P}, \mathcal{L})$ are the same, then the entire cohomologies will be the same. In fact, we know from Jeffrey and Weitsman that the dimension of $H^0(\mathcal{P}, \mathcal{L})$ is computed by the Verlinde formula, and it has also been shown that the dimension of $H^0(\mathcal{M}, \mathcal{L})$ is computed by the Verlinde formula [Fal94][Sch08]. This argument proves $H^0(\mathcal{M}, \mathcal{L}) \cong H^0(\mathcal{P}, \mathcal{L}_0)$, but in a non-canonical way. One may hope to find a direct isomorphism via the degeneration $\phi : \mathcal{M} \to \mathcal{P}$, and such an isomorphism would then provide another way to prove the Verlinde formula for $H^0(\mathcal{M}, \mathcal{L})$. If the degeneration were holomorphic, then this would be doable, however it is not. The degeneration of surfaces is not holomorphic, and since the holomorphic structure on the moduli space is determined by that of the surface, ϕ is not holomorphic.

Chapter 6

Conclusions and Further Discussion

The moduli space $\mathcal{M}$ of flat $SU(2)$ connections over a compact genus $g \geq 2$ Riemann surface Σ, has the Atiyah-Bott symplectic structure (Section 2.4) and a complex structure. The space can be polarised using the Kaehler polarisation associated to this complex structure, in which case the (level k) quantization of the space is defined to be the vector space $\mathcal{H}(k,g) = H^0(\mathcal{M}, \mathcal{L}^k)$ of sections of the prequantum line bundle over $\mathcal{M}$ (Section 3.1). In this case, it is known that the dimension of $\mathcal{H}$ is given by the Verlinde formula [Ver88][Sch08][Fal94][1], which can be given by counting integer labelling of graphs, or by the closed form

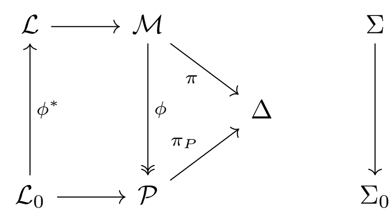
$$\dim \mathcal{H}(k,g) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin^2 \frac{j\pi}{k+2}\right)^{1-g}. \tag{6.1}$$

On the other hand, we have the real polarisation of the space defined by a given trinion decomposition of Σ, due to Weitsman [Wei92]. In this case, instead of holomorphic sections, one looks at the fibre-wise flat sections $\mathcal{J}_\pi$, namely those sections of $\mathcal{L}$ whose restrictions to the fibres of π are covariant constant. No such sections exist, but instead there will be a higher cohomology $H^n(\mathcal{M}, \mathcal{J}_\pi)$ with non-zero elements, and so in this case we define $\mathcal{H}_\pi = \bigoplus_{n \geq 0} H^n(\mathcal{M}, \mathcal{J}_\pi)$ to be the quantization associated to the polarisation π. This definition is inspired from the case of a smooth polarisation of a Kaehler manifold M, for which Sniyaticki's theorem provides an isomorphism between this space and the set of Bohr-Sommerfeld fibres of the polarisation π [Śni77].

Jeffrey and Weitsman provided strong evidence that these two polarisations yield the same quantization when they proved that the number of Bohr-Sommerfeld fibres of the real polarisation of $\mathcal{M}$ is given by the Verlinde formula [JW92]. However, Sniyaticki's theorem does not apply in that case, so we do not know that the number of Bohr-Sommerfeld fibres gives the dimension of $\mathcal{H}_\pi$.

[1] The author cannot find a copy of Falting's article, only other articles citing it as the original mathematical proof of the Verlinde formula.

The moduli space $\mathcal{M}$ with its real polarisation $\pi : \mathcal{M} \to \mathbb{R}^{3g-3}$ is also equipped with Hamiltonian functions for which π is the moment map. The image of $\mathcal{M}$ under π is a polytope $\Delta \in \mathbb{R}^{3g-3}$. Furthermore, the integer values of the Hamiltonians defines a lattice on the polytope, so it is natural to ask what the toric variety corresponding to this polytope is, and how it relates to $\mathcal{M}$. Hurtubise and Jeffrey constructed this toric variety $\mathcal{P}$ in two different ways, as a space of representations with weighted frames, and as a space of framed parabolic bundles (Section 4) [HJ00], in both cases over a *degenerated* Riemann surface Σ_0 with some punctures. The relationship between $\mathcal{M}$ and $\mathcal{P}$ was made explicit by Biswas and Hurtubise who showed that $\mathcal{P}$ arises as a degeneration of $\mathcal{M}$ as you degenerate Σ to Σ_0 by collapsing the boundary curves of the trinion decomposition defining the polarisation π (Section 5) [BH21]. Finally in Section 5.5 we construct a bundle $\mathcal{L}_0$ over $\mathcal{P}$ that corresponds to $\mathcal{L}$ over $\mathcal{M}$. To summarize the situation, we have the following diagram:



Moving forward there are some interesting questions one can ask about this degeneration of $\mathcal{M}$ to $\mathcal{P}$.

1. From the theory of toric varieties, we can compute the number of holomorphic sections of $\mathcal{L}_0$ by counting points in the moment polytope Δ corresponding to $\mathcal{P}$. This polytope is preserved by the degeneration, and so the result of Jeffrey and Weitsman that the count point is given by the Verlinde formula remains true for $\mathcal{P}$. Therefore, the space $H^0(\mathcal{P}, \mathcal{L}_0^k)$ has dimension given by the Verlinde formula. We also know that $H^0(\mathcal{M}, \mathcal{L}^k)$ has this dimension, and thus the dimension is preserved under the degeneration. However, we hope that there is a more direct isomorphism of $H^0(\mathcal{P}, \mathcal{L}_0^k)$ and $H^0(\mathcal{M}, \mathcal{L}^k)$ that can be constructed using the degeneration, and such an isomorphism would provide a new proof of the Verlinde formula for $\dim \mathcal{H}(k, g)$.

2. It has been proven that for toric symplectic manifolds with a singular polarisation π, that the dimension of $\mathcal{H}_\pi$ is given by counting the *non-singular* Bohr-Sommerfeld fibres of π [Ham10]. For $\mathcal{P}$, this count is strictly less than the Verlinde formula. Therefore, there are two possibilities:

(a) The degeneration preserves the fibre-wise flat cohomology: $\bigoplus_{n\geq 0} H^n(\mathcal{M}, \mathcal{J}_\pi) = \bigoplus_{n\geq 0} H^n(\mathcal{P}, \mathcal{J}_{\pi_P})$. In this case, we would conclude $\dim \mathcal{H}_\pi < \dim \mathcal{H}$, which suggests that there may be a more careful definition of the quantization required for real polarisations.

(b) The dimensions $\dim \mathcal{H} = \dim \mathcal{H}_\pi$ are equal, and therefore the degeneration does not preserve the fibre-wise flat cohomology. In this case, one would ask what cohomology elements are being lost, and if this data is preserved in another form.

Currently, it is expected that $\dim \mathcal{H} = \dim \mathcal{H}_\pi$ due to justifications from theoretical physics. In this case, the process of symplectically imploding the fibres with central holonomy must be collapsing some cohomology elements It may be possible that the number of cohomology elements lost during the degeneration process can be counted, and if this count is equal to the number of singular Bohr-Sommerfeld fibres then it would provide a proof that $\dim \mathcal{H} = \dim \mathcal{H}_\pi$.

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