Generalized Ornstein-Uhlenbeck Processes in Catalytic Media

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DEDICATION

To the memory of my father

Fidel Pérez Maldonado

To my mother

Trinidad Abarca Ruano

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ABSTRACT

A class of infinite dimensional Ornstein-Uhlenbeck processes that arise as solutions of stochastic partial differential equations with noises generated by measure-valued catalytic processes is investigated.

The first topic is the determination of the covariance structure of the processes and identification of the Hilbert space in which the solutions live and have continuous paths. Example of catalysts which are singular measures or measure-valued processes are given and results on the behavior of the corresponding catalytic Ornstein-Uhlenbeck processes are obtained.

The second main topic is the study of the special case in which the catalyst is given by a super-Brownian motion. Continuity theorems are established and results on the regularity properties on and off the catalyst are obtained.

The third main topic is to prove that the catalytic Ornstein-Uhlenbeck process with super-Brownian catalyst in one dimension arises as a high density fluctuation limit of a super-Brownian motion with a super-Brownian catalyst and with immigration using some of the techniques established before, particularly those based on properties of Sobolev spaces and Laplace functionals.

RÉSUMÉ

Nous étudions une classe de processus d'Ornstein-Uhlenbeck en dimension infinie qui apparaissent comme solutions d'équations aux dérivées partielles stochastiques ayant des bruits générés par des processus catalyseurs á valeurs de mesures.

Dans un premier temps, nous déterminons la structure de covariance des processus et identifions les space de Hilbert dans lesquels se trouvent les solutions et dans lesquels elles ont des réalisations continues. Nous donnos des examples de catalyseurs qui sont soit des mesures singulières, soit des processus à valeurs en mesure, et nous obtenons des résultats sur le comportement du processus d'Ornstein-Uhlenbeck catalytiques correspondant.

Ensuite, nous étudions le cas particulier, où le catalysateur est un super-mouvement Brownien. Nous établissons des théoremès de continuité, et nous obtenons des résultats sur les propriétés de regularité à l'interieur et à l'extérieur du domaine du catalyseur.

Finalement, nous prouvons que le processus d'Ornstein-Uhlenbeck catalytique ayant un catalyseur super-Brownien en une dimension apparait comme la limite de fluctuations à hâute densité d'un super-mouvement Brownien avec un catalyseur super-Brownien et un terme d'immimigration. Pour ce faire, nous utiliserons des téchniques précédemment établies, en particulier celles basées sur les propriétés des spaces de Sobolev et la transformation de Laplace.

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CHAPTER 1
Introduction

The classical one-dimensional Ornstein-Uhlenbeck (OU) process given by the solution of the stochastic differential equation (SDE):

$$dX_t = -\alpha X_t\, dt + dB_t$$

where B_t is a standard Brownian motion and $\alpha > 0$, is perhaps the simplest example of a stochastic (ordinary) differential equation, it is the equation for the Brownian motion of a particle with friction, its origin can be found in [OU 30], [Wa 45] and [Do 42]. It was established from the very beginning that under the assumption $\mathbb{E}X_0^2 < \infty$ then the expectation, variance and covariance of X_t have simple expression and it is easy to characterize it as a Gaussian process.

The next natural steps were generalizations to finite linear systems of SDE's and then to infinite dimensions. The latter inevitably leads to the treatment of the original equation as a *stochastic partial differential equation* (SPDE) involving an infinite dimensional Wiener process or space-time white noise. The whole development culminated with what is nowadays known as *generalized Ornstein-Uhlenbeck processes* (see [DZ 92], [Ktz 07] and [Wal 85]) and which are described by a SPDE of the form: $dX_t = AX_t\, dt + B\, dW_t$ for some linear operators A, B and space-time white noise W_t.

At the same time, it was necessary to study in more detail the class of *Gaussian processes*, which are the counterpart of Gaussian random variables, they can be characterized as random processes whose finite-dimensional distributions are Gaussian, equivalently their Laplace transform is of the form $\exp(i\langle u, a\rangle - \frac{1}{2}\langle Bu, u\rangle)$ where a is the *mean* and B is a linear self-adjoint non-negative definite operator called the *correlation operator.* These processes have simple properties and one tries whenever possible to determine first if a given processes is Gaussian or equivalent to one in a certain sense.

On the other hand, the study of SPDE's, the nature of their solutions and the need to understand their microscopic structure gave raise to a new class of processes. The natural way to proceed is to consider systems of particles evolving according to a system of stochastic differential equations and then to consider its limit as measures, this procedure originated the *measure-valued processes* (see [Da 93]). It was surprisingly shown in [Da 75] that the solution of a perturbed 2-dimensional heat equation with zero boundary conditions is not even a measure but a distribution. This stresses the fact that the question of the state space for solutions is an important problem.

In this context, two classes of measure-valued processes are particularly interesting: the (W, κ, Φ)-*super-processes* and the super-Brownian motion. The former can have *càdlàg* paths once specified the branching rate κ and branching mechanism Φ, and the latter describes the high density limit of independent particles undergoing a Brownian motion and dying or splitting in two.

At the microscopic level, the super-Brownian motion (SBM) has interesting properties that make it suitable to model the catalyst process. Catalytic processes are of interest because they shed some light on the intrinsic structure of some processes, for example in chemical and biological systems. Such processes involve reactions that take place with different degrees of intensity depending of the presence of another component which plays the role of catalyst. We will introduce the class of *catalytic Ornstein-Uhlenbeck processes* described by a SPDE of the form

$$dX_t = AX_t + \sigma(t, x)dW_t,$$

here X_t is a function or generalized function on $\mathbb{R}^d$ or a bounded subset of $\mathbb{R}^d$ and σ describes a spatially inhomogeneous catalyst having compact support or even being a signed measure.

In the case that $\sigma(t, \cdot)$ is given by SBM we will show that these are infinite dimensional generalizations of the class of affine processes which are defined as process whose Laplace functional has a vanishing non-homogeneous term and which recently aroused much interest in mathematical finance [Sh 02], [Du 03].

The objectives of this thesis are several. The first objective is to establish the relation among some class of generalized OU process in catalytic media, their covariance function and the space of continuity of the paths. Here, the media might be a single static or moving point or a measure also changing with the time either deterministic or random and particularly super-Brownian motion.

It is important to mention that the problem of finding a Hilbert space of functions or generalized functions on the underlying space ($\mathbb{R}^d$ or $[0, 1]^d$) for the continuity of the paths, is particularly challenging. It will be seen, that even without trying to find an optimal space (in the sense of the minimal space), establishing one such a space requires a lot of different techniques. The general strategy to solve this problem will be this: starting with L_2, adjust the underlying measure or enlarge the space L_2 in such a way that the Kolmogorov test is fulfilled. Recall that this test requires:

$$\mathbb{E}\,\|X(t) - X(s)\|^{\alpha} \leq C|t - s|^{1+\beta}$$

for some positive constants C, α, β, and all $t, s, \in [0, T]$, (see [Kat 99] pp 53). When the process X_t is a Gaussian process with marginal distributions in $(H, \|\cdot\|)$ with zero mean, then a sufficient condition is the existence of a $M > 0$ and $\gamma \in (0, 1]$ such that $\mathbb{E}(\|X_t - X_s\|^2) \leq M(t-s)^{\gamma}, \forall t, s \geq 0$, in which case, X has a $\alpha-$Hölder version for any $\alpha \in (0, \frac{1}{2\gamma})$, (see [DZ 92] pp. 83). This sufficient condition can be also verified when the covariance function $\rho(s,t) \doteq \mathbb{E}\langle X_s, X_t\rangle$ is locally Hölder continuous: for each $N \in \mathbb{N}$ there exists $\theta = \theta(N) > 0$ and $C = C(N)$ such that, for $|s|, |t| \leq N$, $|\rho(s,t) - \rho(t,t)| \leq C\,|s-t|^{\theta}$ (see [Wil 86] I p. 61).

There are some cases where the techniques require the analysis of higher moments and after somewhat lengthy calculations one can show that the inequalities are satisfied, but one has to keep in mind that inequalities lie at the heart of applied mathematics.

The second objective is to study the regularity of the paths of catalytic OU processes. Here, several examples of catalytic media are given, including single point and moving catalysts, it will also be seen that out of the support of the catalyst, the process behaves as a pure diffusion and is smooth but that at fixed times it is not differentiable on the support of the catalyst, a result which is intuitive and whose proof relies on the inequality $\mathbb{E}\left|X_t(x_1) - X_t(x_2)\right| \geq a\left|x_1 - x_2\right|^{\alpha}$, $a, \alpha > 0$, which is remarkably similar to Kolmogorov's inequality.

The third objective is to do the same analysis for the quenched and annealed cases of the catalytic OU process.

The 4-th objective, we will to show that the catalytic Ornstein-Uhlenbeck process with super-Brownian catalyst in one dimension arises as a high density fluctuation limit of a super-Brownian motion with a super-Brownian catalyst and with immigration. The main ingredients of the proof will be the Rellich's embedding theorem [Fo 99], Jakubowski's theorem, the Joffe-Metivier criterion [Da 93] and the convergence of finite-dimensional distributions using Laplace transforms. This is done in detail for the two-dimensional distributions, it required several pages of careful computations and it is a pleasure to verify its correctness.

1.1 Thesis Outline

Chapter 2 Some basic but interesting classes of generalized OU processes are reviewed in order to introduce some techniques and to indicate the technical difficulties involved.

Chapter 3 Here some perturbations with bounded support as well as single point moving perturbations are studied and the space for solutions is identified and continuity of the paths is established.

Chapter 4 The super-Brownian motion as catalyst is introduced and the corresponding catalytic OU process is characterized by its Laplace transform, with this tool, some annealed and quenched cases are studied as well as some of their functionals, at the end we show that the process is not differentiable on the support of the catalyst.

Chapter 5 One of the main results is proven, namely that the catalytic Ornstein-Uhlenbeck process with super-Brownian catalyst in one dimension arises as a high density fluctuation limit of a super-Brownian motion with a super-Brownian catalyst and with immigration.

CHAPTER 2
Fundamental models and techniques

In this chapter we review some basic results of stochastic analysis and fundamental classes of infinite dimensional stochastic processes.

2.1 General Concepts

Definition 2.1.1. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space. A *stochastic process* $\{X_t(\omega)\}_{t\in T}$ is a family of random variables with values on a Polish space E called the *state space.* T is assumed to be $[0,\infty)$ unless otherwise stated. Sometimes we write $X_t(\omega)$ as $X_t, X(t)$, or $X(t,\omega)$.

For a fixed sample point $\omega \in \Omega$, the function $t \mapsto X_t(\omega); t \geq 0$ is the *sample path* (realization, trajectory) of the process X associated with ω.

In the present discussion, two stochastic processes will be of central importance, namely the *Wiener Process* and the *Ornstein Uhlenbeck Process,* which will be defined next.

Definition 2.1.2. Given a Hilbert space U, and a symmetric nonnegative operator $Q \in L(U)$ with $\mathrm{Tr}\ Q < \infty$, there exits a complete orthonormal system $\{e_k\} \in U$, and a bounded sequence of nonnegative real numbers λ_k such that

$$Qe_k = \lambda_k e_k, \qquad k = 1, 2, \ldots$$

A U-valued stochastic process $\{W(t)\}_{t\geq 0}$, is called a *Q-Wiener process* if

(i) $W(0) = 0$.

(ii) W has continuous trajectories.

(iii) W has independent increments.

(iv) $\mathcal{L}(W(t) - W(s)) = \mathcal{N}(0, (t-s)Q), \ t \geq s \geq 0.$

If a process $W(t), t \in [0,T]$ satisfies (i)-(iv) for $t, s \in [0,T]$ then we say that W is a Q-Wiener process on $[0,T]$.

The following is a basic properties of a Q-Wiener process whose proof can be found in any book of infinite dimensional stochastic processes (see for example [DZ 92]):

Proposition 2.1.1 ([DZ 92] pp. 87). *Assume that W is a Q-Wiener process, with $Tr\, Q < \infty$. Then:*

(i) W is a Gaussian process on U and

$$\mathbb{E}(W(t)) = 0, \qquad Cov\ (W(t)) = tQ, \quad t \geq 0$$

(ii) For arbitrary t, W has the expansion

$$W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\beta_j(t)e_j \tag{2.1.1}$$

where

$$\beta_j = \frac{1}{\sqrt{\lambda_j}} \langle W(t), e_j \rangle , \quad j = 1, 2, \dots$$

are real valued Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$ and the series (2.1.1) converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

The next is an important result, for it assures the existence of certain Q-Wiener processes:

Proposition 2.1.2 ([DZ 92] pp. 88). *For arbitrary trace class symmetric nonnegative operator on Q on a separable Hilbert space U there exists a Q-Wiener process $\{W(t)\}_{t\geq 0}$.*

Remark 2.1.1. If Tr $Q = \infty$, the Q-Wiener process can be shown to exist in a larger space and is known as a *cylindrical Wiener process.*

Given a stochastic process $\{X_t(\omega)\}_{t\in T}$ its integral with respect to a Wiener process is the process defined by:

$$(X \bullet W)_t = \int_0^t X_s \, dW_s \, .$$

The theory of stochastic integration with respect to a Wiener process is now a well established technique, its construction and main properties can be found in the literature (see for instance: [DZ 92], [Wal 85]).

We collect below the most important facts and use them to introduce some definitions and unless otherwise stated, all Hilbert spaces are assumed to be real and separable.

Definition 2.1.3. Given a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. We consider two Hilbert spaces H and U, we assume that there exists a complete orthonormal system $\{e_k\}$ in U, a bounded sequence of nonnegative real numbers λ_k such that

$$Qe_k = \lambda_k \, e_k, \qquad k = 1, 2, \cdots$$

and a sequence β_k of real independent Brownian motions such that

$$\langle W(t), u \rangle = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \, \langle e_k, u \rangle \, \beta_k, \quad u \in U, t \geq 0.$$

We will consider the following linear equation

$$(SPDE) \qquad \begin{cases} dX(t) &= [AX(t) + f(t)]dt + BdW(t) \\ X(0) &= \xi \end{cases}$$

where $A : D(A) \subset H \to H$ and $B : D(B) \subset U \to H$ are linear operators, f is an H-valued stochastic process. We will assume that the deterministic Cauchy problem

$$\dot{u}(t) = Au(t), \qquad u(0) = x \in H$$

is such that

(i) A generates a strongly continuous semigroup $S(\cdot)$ in H,

(ii)$B \in L(U; H)$

Depending on the elements involved in (SPDE) and its difficulty, one can obtain some of the different solutions defined below:

Definition 2.1.4. *Strong, weak and mild solutions.*

An H-valued predictable process $X(t),\ t \in [0, T]$, is said to be a *strong* solution to (SPDE) if X takes values in $D(A), \mathbb{P}_T$ a.s. and:

(i)

$$\int_0^T |AX(s)|\ ds < \infty,\ \mathbb{P}_T,$$

(ii)

$$X(t) = x + \int_0^t [AX(s) + f(s)]ds + BW(t), \qquad \mathbb{P} - a.s.$$

An H-valued predictable process $X(t)$, $t \in [0, T]$, is said to be a *weak solution* if the trajectories of X are $\mathbb{P}$-a.s. Bochner integrable and if for all $\zeta \in D(A^*)$ and all $t \in [0,T]$ we have

$$\langle X(t), \zeta \rangle = \langle x, \zeta \rangle + \int_0^t [\langle X(s), A^*\zeta \rangle + \langle f(s), \zeta \rangle] \, ds$$
$$+ \langle BW(t), \zeta \rangle, \qquad \mathbb{P} - a.s.$$

An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a *mild solution* if X takes values in $D(B)$, $\mathbb{P}_T$-a.s, the following holds

(i)

$$\mathbb{P}\left(\int_0^T |X(s)| \, ds\right) = 1$$

(ii)

$$\mathbb{P}\left(\int_0^T \|B(X(s))\|_{L_2^0}^2 \, ds < \infty\right) = 1$$

where $\|\cdot\|_{L_2^0}^2$ denotes the Hilbert-Schmidt norm of the operator, and for arbitrary $t \in [0,T]$:

(iii).

$$X(t) = S(t) + \int_0^t S(t-s)f(s) \, ds + \int_0^t S(t-s)B(X(s)) \, dW(s)$$

where $S(\cdot)$ is the semigroup generated by the operator A.

The existence of each solution is given under the following conditions:

Proposition 2.1.3 ([DZ 92], pp. 121). *Assume :*

(i) A generates a strongly continuous semigroup $S(\cdot)$ in H.

(ii) $B \in L(U,H)$.

(iii)

$$\int_0^T \|S(r)B\|_{L_2^0}^2 \, dr = \int_0^t Tr[S(r)BQB^*S^*(r)] \, dr \leq \infty$$

then the problem (SPDE) has exactly one weak solution which is given by:

$$X_t = S(t)\xi + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)BdW(s), \qquad t \in [0,T]$$

Proposition 2.1.4 ([DZ 92], pp. 147,148). *Assume that either:*

(i) $Tr\, Q < +\infty, U = H, B = I$ *and* $A \in L_2(H)$.

(ii) $f \in C^1([0,T] : H) \cap C([0,T]) : D(A)),\ \mathbb{P} - a.s.$

(iii) $\xi \in D(A),\ \mathbb{P} - a.s.$

or

(i) $(-A)^\beta \in L_2(H)$ *for some* $\beta \in (1/2, 1)$.

(ii) $f \in C^\alpha([0,T] : H) \cap C([0,T]) : D(A))$ *for some* $\alpha \in (0,1),\ \mathbb{P} - a.s.$

(iii) $\xi \in D(A),\ \mathbb{P} - a.s.$

then, the problem (SDPE) has a unique strong solution.

Proposition 2.1.5 ([DZ 92], pp. 156). *Assume that* $A : D(A) \subset H \to H$ *is the infinitesimal generator of a* C_0 *semigroup* $S(\cdot)$ *in* H. *Then a strong solution is always a weak solution and a weak solution is a mild solution of (SPDE). Conversely if* X *is a mild solution of (SPDE) and*

$$\mathbb{E} \int_0^T \|B(X(S))\|^2_{L_2^0}\, ds < +\infty.$$

Then X *is also a weak solution of (SPDE).*

In our discussion, the *state space* will be either the d-dimensional Euclidean space or a suitable infinite-dimensional space of functions, as it is well known, this is a huge vast area, but for us, it will be enough to know any of the spaces listed below.

Notation We will denote by $\mathcal{H} = L_2(\Omega, \mathcal{F}, \mathbb{P})$, the space of square integrable functions defined on Ω with values in $\mathbb{R}^d$. Hilbert spaces will usually denoted by H, or $\mathbb{H}$.

If two different measures μ_1 and μ_2 are defined on the same measure space Ω, we will refer to the spaces of square integrable functions as L_{2,μ_1} and L_{2,μ_2} respectively and will drop the measure when the underlying measure is the Lebesgue measure.

Definition 2.1.5. Given a Hilbert space $\mathbb{H}_0$, with a basis $\{e_k\}$ then

$$\mathbb{H}_0 = \left\{ x = \sum_{k=1}^{\infty} a_k e_k : \sum_{k=1}^{\infty} a_k^2 < \infty \right\}$$

we define the *weighted Hilbert space* $\mathbb{H}_1$ with weights $\{w_k : w_k \in \mathbb{R}\}$ as:

$$\mathbb{H}_1 = \left\{ x = \sum_{k=1}^{\infty} a_k e_k : \|x\|_1 = \sum_{k=1}^{\infty} (a_k w_k)^2 < \infty \right\}$$

and

$$\mathbb{H}_{-1} = \left\{ x = \sum_{k=1}^{\infty} a_k e_k : \|x\|_{-1} = \sum_{k=1}^{\infty} \frac{a_k^2}{w_k^2} < \infty \right\}$$

If $w_k^2 \geq w_0^2 > 0$, then

$$\mathbb{H}_1 \subseteq \mathbb{H}_0 \subseteq \mathbb{H}_{-1}$$

and

$$\mathbb{H}_1' = \mathbb{H}_{-1}.$$

The use of weighted Hilbert spaces will be a useful technique that will be used frequently when there is a need to enlarge the state space of a given process. An important class of weighted spaces are the *Sobolev spaces* $\mathbb{H}^p$ which are constructed starting with $L_2[0,1]$ in such a way that $\mathbb{H}^p \subseteq L_2[0,1]$ and have some differentiability structure. Further details of these spaces are given in Sect. 2.4.

2.2 Gaussian Processes in Hilbert Spaces

Gaussian processes will appear frequently in this work, such as the quenched case of catalytic process. Gaussian random variables have finite second moments and consequently they can be studied by Hilbert space methods. Below we give their definition and summarize their most useful properties.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. An H-valued stochastic process on $[0,\infty)$ is said to be *Gaussian* if, for and $n \in \mathbb{N}$ and for arbitrary positive numbers $t_1, t_2, \ldots, t_n$, the H^n-valued random variable $(X(t_1), \ldots, X(t_n))$ is Gaussian.

Proposition 2.2.1. *(Continuity of the paths for Gaussian processes) Let X be a Gaussian process on H. Assume that there exists $M > 0$ and $\gamma \in (0,1]$ such that*

$$\mathbb{E}(\|X(t) - X(s)\|^2) \leq M(t-s)^{\gamma}, \qquad \forall t, s \geq 0. \tag{2.2.2}$$

Then X has an $\alpha-$Hôlder continuous version, for any $\alpha \in (0, (1/2)\gamma)$.

Proof. For any Gaussian random variable X with $\mathbb{E}X = 0$ and any natural number m, we have $\mathbb{E}X^{2m} = C_m(\mathbb{E}X^2)^m$ and so it follows from (2.2.2) that

$$\mathbb{E}(\|X(t) - X(s)\|^{2m}) \leq C_m(t-s)^{m\gamma}, \qquad \forall t, s \geq 0.$$

since m is arbitrary, one can always choose m large enough to apply the Kolmogorov test. $\square$

Definition 2.2.1. Let X be a Gaussian process in a Hilbert space H. Let $m(t) = \mathbb{E}(X(t))$, and

$$Q(t) = \mathbb{E}(X(t) - m(t)) \otimes (X(t) - m(t)), \qquad t \geq 0$$

$$B(t,s) = \mathbb{E}(X(t) - m(t)) \otimes (X(s) - m(s)), \qquad s,t \geq 0$$

the process is said to be *stationary* if and only if

(i) $m(t+r) = m(t), \qquad \forall\, t, r \geq 0$

(ii) $B(t+r, s+r) = B(t,s), \qquad \forall\, t, s, r \geq 0$

Continuity of the paths for Gaussian systems can also be established using other criteria than the expectation of higher moment. Let $X(t)$ be a Gaussian process with covariance $B(s,t)$ as defined above and fix $T \in \mathbb{R}^+$ and let ϕ be the function defined as follows:

$$\phi(h) = \sup_{(s,t)\in[0,T], |s-t|\leq h} \sqrt{B(s,s) - 2B(s,t) + B(t,t)}$$

Theorem 2.2.2. *([XFe 74]) Assume the integral $\int_0^\infty \phi(e^{-x^2})dx$ is finite, then the Gaussian process $X(t)$ has continuous paths almost surely and for every integer $p \geq 2$ and every real $x \geq \sqrt{1 + 4\ln p}$, we have:*

$$\mathbb{P}\left[\sup_{t\in[0,T]} |X(t)| \geq x \left(\sup_{[0,T]\times[0,T]} \sqrt{B(s,t)} + (2+\sqrt{2}) \int_1^\infty \phi(p^{-u^2})du\right)\right] \leq \frac{5}{2}p^{2n} \int_x^\infty e^{-\frac{u^2}{2}} du$$

Theorem 2.2.3. *(Fernique's inequality). Let $X(t)$ be a Gaussian process with values on a Polish space E, $S = [a,b]$; with the notation of the above theorem, and* *setting:*

$$\|f\| = \sup_{S\times S} |f|$$

we have the following inequality:

$$\mathbb{P}\left[\sup_{t\in S} |X(t)| \geq .x \left(\sqrt{\|B\|} + (2+\sqrt{2}) \int_1^\infty \phi(\frac{b-a}{2} p^{-u^2}) du\right)\right] \leq \frac{5}{2} p^{2n} \int_x^\infty e^{-\frac{u^2}{2}} du$$

For the proof of the above results see [XFe 74] pp. 48-51.

2.3 Wiener perturbation of the heat equation

Let us first study the heat equation under different perturbations and establish the relation between the covariance function and the space of solutions.

2.3.1 Perturbation of the heat equation with Q-Wiener process with trace class Q

Let $U = H = L^2(\mathcal{O})$, where $\mathcal{O}$ is a bounded open set of $\mathbb{R}^d$, in this section $\mathcal{O}$ will be the unit interval $(0,1)$, we are looking for solutions of $X(t) \in H$ of the following stochastic partial differential equation:

$$\text{(SPE)} \qquad \begin{cases} dX(t,\xi) = \Delta X(t,\xi) + dW_Q(t,\xi) & t \geq 0, \quad \xi \in \mathcal{O} \\ X(t,\xi) = 0 & t \geq 0, \quad \xi \in \partial\mathcal{O} \\ X(0,\xi) = 0 & \xi \in \mathcal{O} \end{cases}$$

where Δ is the Laplacian with Dirichlet boundary conditions and W_Q is a Q-Wiener process with trace class Q.

Remark 2.3.1. $X(t)$ takes values in H , which is an infinite dimensional space, note also that the corresponding deterministic equation only has the trivial solution, whereas (SPE) has a nontrivial solution in the space $L^2(\mathcal{O})$.

In (SPE), we write $X(t,\xi) = [X(t)](\xi)$ that is, the value of the function at the point ξ, and similarly for $W(t,\xi)$ (where $W(t,\cdot)$ can in fact be a generalized function).

The first step to solve the problem is to find the suitable space of solutions.

The original problem can be expressed as the following Cauchy problem:

Find X(t) in $L_2(\mathcal{O})$ such that:

$$\text{(CP)} \qquad \begin{cases} dX(t,\xi) = \Delta X(t,\xi)dt + dW(t,\xi) & t \geq 0, \quad \xi \in \mathcal{O} \\ X(0,\xi) = 0 & \xi \in \mathcal{O} \end{cases}$$

Where $D(\Delta) = H^2(O) \cap H_0^1(O)$ (see section 2.4 for definition and properties of these spaces).

Let us first verify a sufficient condition for the existence of a solution, namely:

$$\int_0^t \|S(r)B\|_{L_2^0}^2 \, dr = \int_0^t Tr[S(r)BQB^*S^*(r)] \, dr < +\infty$$

this follows from the fact that Q is positive, trace class and so

$$\|Q\|_1 = \mathrm{Tr[Q]}$$

Also SQS* is positive since $\langle SQS^*x, x\rangle = \langle QS^*x, S^*x\rangle$ and so

$$\begin{aligned}\mathrm{Tr}[S(r)QS^*(r)] &= \|S(r)QS^*(r)\|_1 \\ &\leq \|S(r)\|\,\|Q\|_1\,\|S^*(r)\| \\ &= \|S(r)\|\,\mathrm{Tr}[Q]\,\|S^*(r)\| \\ &< +\infty\end{aligned}$$

Assuming in the last inequality that Tr Q $< +\infty$ and the growth property of a C_0-semigroup: $\|S(r)\| \leq Me^{\omega r}$, from which it follows that $\int_0^t \|S(r)\|^2_{L_2^0}$ dr, and so the solution is given by:

$$\begin{aligned}X(t,\xi) &= S(t)X(0,\xi) + \int_o^t S(t\text{-}r)\ dW(r,\xi) \\ &= \int_0^t S(t\text{-}r)\ dW(r,\xi)\end{aligned}$$

2.3.2 The heat equation perturbed by space-time white noise

The case Q=I corresponds to space time white noise. This was studied by Dawson [Da 72], Walsh [Wal 81] and Kotelenez [Ktz 87], DaPrato and Zabczyk [DZ 92] and others.

First in order to compute $\|S(r)\|_{L_2^0}$, it is convenient to find the exact expression of the semigroup S(t).

For that, observe that the functions

$$\phi_k(x) = \sqrt{2}\ \mathrm{Sin}\ k\pi x$$

are the eigenfunctions of the Laplacian $\Delta = \mathrm{A}$ with the given boundary conditions and form a complete orthonormal system of $\mathrm{H}^2(\mathcal{O})$, therefore:

$$\begin{aligned}\Delta\phi_k &= -\mathrm{k}^2\pi^2\phi_k \\ &\doteq -\mu_k\,\phi_k \qquad\qquad \text{for k=1,2,}\ldots\end{aligned}$$

From the spectral mapping theorem

$$\sigma(\mathrm{S(t)}) \setminus \{0\} = e^{\mathrm{t}\sigma(\mathrm{A})}$$

or equivalently

$$\mathrm{S(t)}\,\phi_k = e^{-\mu_k t}\,\phi_k$$

So

$$\begin{aligned}||\mathrm{S(t)}||^2_{L_2^0} &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |\langle \mathrm{S(t)}\,\phi_i, \mathrm{S(t)}\,\phi_j\rangle|^2 \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} e^{-2\mu_i}\,\delta_{ij} \\ &= \sum_{i=0}^{\infty} e^{-2\mu_i t}\end{aligned}$$

Hence

$$\int_0^t ||\mathrm{S(t)}||^2_{L_2^0}\ \mathrm{dr} = \sum_{k=0}^{\infty}\frac{[1-e^{-2\mu_k t}]}{2\mu_k}$$

which according to Weyl's lemma is convergent only if $d = 1$ (see [Lib 04]).

With the above results, one can find an explicit expression of the semigroup S(t) and also of the solution X(t). Let us write the space-time Wiener process

W(s) in the form $W(s) = \sum_k b_k(s)\phi_k$ where $b_k(s) \doteq \langle W(s), \phi_k \rangle$ are independent Wiener processes, from which, the solution can be written as

$$
\begin{aligned}
X(t) &= \int_0^t S(t\text{-}s) \sum_k \phi_k db_k(s) \\
&= \int_0^t \sum_k S(t\text{-}s)\phi_k db_k(s) \\
&= \int_0^t \sum_k e^{-\mu_k(t-s)} \phi_k db_k(s) \\
&= \sum_k \left[\int_0^t e^{-\mu_k(t-s)} db_k(s) \right] \phi_k \\
&= \sum_{k=0}^{\infty} V_k(t)\phi_k
\end{aligned}
$$

where by definition:

$$
V_k(t) \doteq \int_0^t e^{-\mu_k(t-s)} db_k(s) \tag{2.3.3}
$$

2.3.3 Covariance structure

The next point of interest, is to determine the covariance matrix and to examine its properties in order to get information about the space of solutions, for that reason, we need the following:

Definition 2.3.1. Given two stochastic processes X(t) and Y(t) having finite 2nd moments, the <u>Covariance</u> of $X(t)$ is defined by:

$$
\text{Cov}(X(t)) = \mathbb{E}[[X(t) - \mathbb{E}X(t)] \otimes [X(t) - \mathbb{E}X(t)]]
$$

and Correlation of $X(t)$ and $Y(t)$ by:

$$\mathrm{Cor}(\mathrm{X(t),Y(s)}) = \mathbb{E}[[\mathrm{X(t)} - \mathbb{E}\mathrm{X(t)}] \otimes [\mathrm{Y(s)} - \mathbb{E}\mathrm{Y(s)}]]$$

Remark 2.3.2. In the above definition, the operator tensor product $a \otimes b$ of any two elements a, b in a given Hilbert space H is defined by $(a \otimes b)(c) = a \langle b, c \rangle$, if H is finite dimensional and $x \in H$ is considered as a column vector, then $a \otimes b$ coincides with $x^T x$. And it also follows that Cov and Cor are operators from H into H, further, the former operator is symmetric, positive definite and trace class.

We now use (2.3.3) to find an explicit representation of the operators Cov and Cor for the solution of the equation (2.3.1) with $Q = I$ with respect to the basis $\{\phi_k\}$:

$$\begin{aligned}\mathrm{Cov}(\mathrm{X(t)}) &= \mathbb{E}[\mathrm{X(t)} \otimes \mathrm{X(t)}] \\ &= [\mathbb{E}(\mathrm{V}_i\mathrm{V}_j)] = [\delta_{ij}\mathbb{E}(\mathrm{V}_i\mathrm{V}_j)]\end{aligned}$$

where, by the Itô isometry:

$$\begin{aligned}\mathbb{E}(\mathrm{V}_i\mathrm{V}_i) &= \mathbb{E}\left(\int_0^t e^{-\mu_i(t-s)}\mathrm{db}_i(\mathrm{s}) \int_0^t e^{-\mu_i(t-s)}\mathrm{db}_i(\mathrm{s})\right) \\ &= \int_0^t e^{-2\mu_i(t-s)}\,\mathrm{ds} \\ &= \frac{[1 - e^{-2\mu_i t}]}{2\mu_i}\end{aligned} \tag{2.3.4}$$

It is also interesting to find, for a given t, the covariance Cov(X(t,ξ),X(t,ζ)), this can also be done using the representation found above, that is:

$$X(t,\xi) = \sum_k V_k(t)\phi_k(\xi)$$
$$X(t,\zeta) = \sum_k V_k(t)\phi_k(\zeta)$$

where $\phi_k(\xi) = \sqrt{2}\sin \mathrm{k}\,\pi\,\xi$ and $\phi_k(\zeta) = \sqrt{2}\sin \mathrm{k}\,\pi\,\zeta$ are now scalars and $V_k(t)$ are the stochastic integrals given by (2.3.3). With this, we can compute:

$$\begin{aligned}
\mathrm{Cov}(X(t,\xi)) &= \mathbb{E}[X(t,\xi)X(t,\zeta)] \\
&= \mathbb{E}\left[\sum_k V_k(t)\phi_k(\xi)\ \sum_l V_l(t)\phi_l(\zeta)\right] \\
&= \mathbb{E}\left[\sum_k\sum_l V_k(t)\phi_k(\xi)V_l(t)\phi_l(\zeta)\right] \\
&= \sum_k\sum_l \mathbb{E}V_k(t)\phi_k(\xi)V_l(t)\phi_l(\zeta) \\
&= \sum_k \phi_k(\xi)\phi_k(\zeta)\mathbb{E}(V_k(t)V_k(t))
\end{aligned}$$

The third equality can be done because the sum $\sum_k V_k(t)\phi_k(\xi) = X(t)$ is a. s. bounded, and the next one using several times Lebesgue's MCT and BCT, using now (2.3.4) one gets:

$$\begin{aligned}
\mathrm{Cov}(X(t,\xi),(X(t,\zeta)) &= \sum_k \phi_k(\xi)\phi_k(\zeta)\frac{1-e^{-2\mu_k t}}{2\mu_k} \\
&= \sum_k \frac{\phi_k(\xi)\phi_k(\zeta)}{2\mu_k} - \sum_k e^{-2\mu_k}\frac{\phi_k(\xi)\phi_k(\zeta)}{2\mu_k}
\end{aligned} \tag{2.3.5}$$

From this expression one can establish properties of the long run behavior of the solution of (CP), recall that the Brownian bridge going from 0 to 0 at time 1 is specified by:

$$\mathbb{E}X(t,\xi) = 0$$
$$\mathrm{Cov}(X(t,\xi), X(t,\zeta)) = \xi \wedge \zeta - \xi\zeta$$

Note also, that the function $\xi \wedge \zeta - \xi\zeta$ is the Green function of the operator Δ on the square $[0,1] \times [0,1]$, that is, the solution of the problem

$$\frac{d^2}{dx^2}\phi(\xi) = \psi(\xi), \qquad \xi \in \mathbb{R}$$
$$\phi(0) = \phi(1) = 0$$

is given by:

$$\phi(\xi) = \int_0^1 (\xi \wedge \zeta - \xi\zeta)\psi(\zeta)\, \mathrm{d}\zeta$$

Proposition 2.3.1. *The solution X(t) of (CP), converges in the L_2 norm as $t \to \infty$ to a Brownian bridge going from 0 to 1.*

Proof. The second term of (2.3.5) decays very fast to zero because of the factor $e^{-2\mu_k t}$, whereas the first term can be written as:

$$\sum_k \frac{\phi_k(\xi)\phi(\zeta)}{2\mu_k} = \sum_k \frac{(\sin k\pi\xi)(\sin k\pi\zeta)}{k^2\pi^2}$$

which is the Fourier series of the function $\xi \wedge \zeta - \xi\zeta$, with respect to the orthonormal basis $\{\sqrt{2}\mathrm{Sin}\, k\pi\xi, \sqrt{2}\mathrm{Sin}\, k\pi\zeta\}$. □

2.4 The state space of solutions and continuity of the paths

As noted earlier, the space of solutions of (CP) is $\mathrm{H}^2(\mathcal{O}) \bigcap \mathrm{H}_0^1(\mathcal{O})$, for this reason, we summarize below the definition and basic properties of the Sobolev spaces on the interval [0,1].

Given a function $\phi \in \mathrm{L}^2[0,1]$, the series:

$$\sum_{-\infty}^{+\infty} a_m e^{i2\pi mt}$$

where:

$$a_m := \int_0^1 \phi(t) e^{-i2\pi mt}\, \mathrm{dt}$$

is called the Fourier series of ϕ, its coefficients a_m are called the Fourier coefficients of ϕ. On $\mathrm{L}^2[0,1]$, as usual, the mean square norm is introduced by the scalar product

$$(\phi, \psi) := \int_0^1 \phi(t)\overline{\psi}\, \mathrm{dt}$$

We denote by f_m the trigonometrical monomials:

$$f_m(t) := e^{i2\pi mt}$$

for t $\in \mathbb{R}$ and $m \in \mathbb{Z}$, the set $\{f_m : m \in \mathbb{Z}\}$ is an orthonormal system. Here, Parseval's equality assumes the following form:

$$\sum_{-\infty}^{+\infty} |a_m|^2 = \int_0^1 |\phi(t)|^2\, \mathrm{dt} = ||\phi||_2^2$$

Definition 2.4.1. Let $0 \leq p < \infty$. We denote by $\mathrm{H}^p[0,1]$ the space of all functions $\phi \in \mathrm{L}^2[0,1]$ with the property:

$$\sum_{m=-\infty}^{+\infty} (1+m^2)^p \left|a_m\right|^2 < \infty \tag{2.4.6}$$

for the Fourier coefficients a_m of ϕ. The space $\mathrm{H}^p[0,1]$, is called a Sobolev space. And we will abbreviate $\mathrm{H}^p[0,1]$ by H^p

The following properties of the Sobolev spaces will be needed later, for further details and proofs, see [Kr 99].

Theorem 2.4.1. *The Sobolev space $H^p[0,1]$ is a Hilbert space with the scalar product*

$$(\phi, \psi) = \sum_{m=-\infty}^{+\infty} (1+m^2)^p a_m \overline{b_m}$$

and the norm on $H^p[0,1]$ is defined by:

$$\left\|\phi\right\|_p := \left\{ \sum_{m=-\infty}^{+\infty} (1+m^2)^p \left|a_m\right|^2 \right\}^2$$

Theorem 2.4.2. *If $q > p$ then $H^q[0,1]$ is dense in $H^p[0,1]$, with compact imbedding from $H^q[0,1]$ into $H^p[0,1]$.*

Proof. From $(1+m^2)^p < (1+m^2)^q$ for $m \in \mathbb{Z}$ it follows that $\mathrm{H}^p \supset \mathrm{H}^q$ with bounded imbedding

$$\left\|\phi\right\|_p \leq \left\|\phi\right\|_q$$

and the denseness is a consequence of the denseness of the trigonometric polynomials is H^p □

Theorem 2.4.3. *Let $p > 1/2$, then the Fourier series for $\phi \in H^p[0,1]$ converges absolutely and uniformly. Its limit is continuous and has period 1 and coincides with ϕ almost everywhere, the imbedding $H^p[0,1] \hookrightarrow C[0,1]$ is compact.*

Proof. For the series of $\phi \in \mathrm{H}^p[0,1]$, by the Cauchy-Schwarz inequality, we conclude that:

$$\left\{ \sum_{m=-\infty}^{\infty} \left| a_m e^{i2\pi mt} \right|^2 \right\} \leq \sum_{m=-\infty}^{\infty} \frac{1}{(1+m^2)^p} \sum_{m=-\infty}^{\infty} (1+m^2)^p \left| a_m \right|^2$$

the key observation is that the first summation on the right hand side converges for $p > 1/2$ and one can apply Dini's theorem. □

Denote by $C^k_{[0,1]}$ the space of k-times continuously functions from $\mathbb{R}$ to $\mathbb{C}$ differentiable having period 1.

Theorem 2.4.4. *For $k \in \mathbb{N}$ we have $C^k_{[0,1]} \subset H^k$ and on $C^k_{[0,1]}$ the norm $\|\cdot\|_k$ is equivalent to:*

$$\|\phi\|_{k,k} := \left(\int_0^1 \left(|\phi(t)|^2 + \left|\phi^{(k)}(t)\right|^2 \right) dt \right)^{1/2}$$

Theorem 2.4.5. *For $0 \leq p < \infty$, we denote by $H^{-p}[0,1]$ the dual space of $H^p[0,1]$, i. e. the space of bounded linear functionals on $H^p[0,1]$ endowed with the norm:*

$$\|F\|_p := \left\{ \sum_{m=-\infty}^{\infty} (1+m^2)^{-p} \left| c_m \right|^2 \right\}^{1/2}$$

where $c_m = F(f_m)$. Conversely, to each sequence $(c_m) \subset \mathbb{C}$ satisfying

$$\sum_{m=-\infty}^{\infty} (1+m^2)^{-p} \left| c_m \right|^2 < \infty$$

there exists a bounded linear functional $F \in L^2[0,1]$ with $F(f_m) = c_m$.

Remark 2.4.1. According to this definition, the delta function belongs to the space H^{-1} since, the delta function has a Fourier series expansion with coefficients a_m, with $|a_m| \leq 1$, and so:

$$\sum_{m=-\infty}^{+\infty} (1+m^2)^{-1} |a_m| < \infty$$

Theorem 2.4.6. *For each function $g \in L^2[0,1]$, the dual pairing:*

$$G(\phi) = \int_0^1 \phi(t)\overline{g(t)}\, dt \qquad \phi \in H^p[0,1]$$

defines a linear functional $G \in H^{-p}[0,1]$.

In this sense $L^2[0,1]$ is a subspace of the dual space $H^{-p}[0,1]$ and the trigonometrical polynomials are dense in $H^{-p}[0,1]$. This fact can be represented the following way:

$$\cdots \supset H^{-2} \supset H^{-1} \supset L^2 \supset H^1 \supset H^2 \cdots.$$

Next we define the Sobolev space of functions which are suitable for dealing with boundary conditions.

Definition 2.4.2. Let $p \in \mathbb{R}$, the Sobolev space $H_0^p(\mathcal{O})$ is defined as the completion of the set $C_0^\infty(\mathcal{O})$ of infinitely differentiable functions with support contained in $\mathcal{O}$ under the norm (2.4.6).

With these tools, the question of continuity of the paths for the case Q=I can be solved, notice first, that the operator:

$$\Delta : \mathrm{H}^2(\mathcal{O}) \cap \mathrm{H}_0^1(\mathcal{O}) \rightarrow \mathrm{H}^2(\mathcal{O}) \cap \mathrm{H}_0^1(\mathcal{O})$$

is well defined in the sense of distributions, and using a well known theorem (see [DZ 92] pp. 193), we have the following:

Proposition 2.4.7. *The solution of (CP) has continuous paths on* $L_2([0,1])$.

Proof. For this case, one only has to verify the condition:

$$\int_0^s t^{-\alpha} \|\mathrm{S(t)}\|^2_{L_2(H)} \, \mathrm{dt} < \infty \qquad \text{for some } s > 0 \text{ and } \alpha \in (0,1)$$

where:

$$\|\mathrm{S(t)}\|^2_{L_2(H)} = \sum_{n=0}^{\infty} e^{-n^2\pi^2 t}$$

so that:

$$\begin{aligned} \int_0^s t^{-\alpha} \|\mathrm{S(t)}\|^2_{L_2(H)} \, \mathrm{dt} &= \int_0^s t^{-\alpha} \sum_{n=0}^{\infty} e^{-n^2\pi^2 t} \, \mathrm{dt} \\ &= \sum_{n=0}^{\infty} \int_0^s t^{-\alpha} e^{-n^2\pi^2 t} \, \mathrm{dt} \end{aligned} \tag{2.4.7}$$

The last equality by using the monotone convergence theorem. The terms of the last summation can be bounded above by applying Hölder's inequality:

$$\int_0^s t^{-\alpha} e^{-n^2\pi^2 t} \, \mathrm{dt} \leq \left(\int_0^s t^{-\alpha p} \mathrm{dt} \right)^{\frac{1}{p}} \left(\int_0^s e^{-n^2\pi^2 q t} \, \mathrm{dt} \right)^{\frac{1}{q}} \tag{2.4.8}$$

with $\frac{1}{p} + \frac{1}{q} = 1$, the first integral on the right side is finite for $1 > \alpha p$, with value:

$$\int_0^s t^{-\alpha p} \, \mathrm{dt} = \frac{s^{1-\alpha p}}{1 - \alpha p}$$

the rightmost integral of (2.4.8) is readily seen to be:

$$\int_0^s e^{-n^2\pi^2 qt}\,\mathrm{dt} = \frac{1-e^{-n^2\pi^2 qs}}{n^2\pi^2 q}$$
$$\leq \frac{1}{n^2\pi^2 q}$$

and so

$$\left(\int_0^s e^{-n^2\pi^2 qt}\,\mathrm{dt}\right)^{\frac{1}{q}} \leq \frac{1}{(n^2\pi^2 q)^{\frac{1}{q}}}$$

Hence, for the sum (2.4.7) to converge, it is enough to find α, p, q such that:

$$\alpha \in (0,1)$$
$$\frac{1}{p} + \frac{1}{q} = 1$$
$$q < 2$$
$$\frac{1}{p} > \alpha$$

one can verify that $q = \frac{3}{2}, p = 3, \alpha = \frac{1}{4}$ satisfy the above conditions and so the equation (2.4.8) is finite and therefore, the solution of (CP) has continuous paths in $L_2([0,1])$. □

The following result from Iscoe and McDonald (see [IsM 90]) provides a result of continuity in l^2 of Ornstein-Uhlenbeck processes. Given the stochastic differential equation:

$$dX_t = -AX_t dt + \sqrt{2a}dB_t$$

where A is a constant self-adjoint , positive definite operator on L^2 having a complete orthonormal family of eigenvectors ϕ_k corresponding to its set of positive eigenvalues λ_k, $k = 1, 2, \ldots$, and a is constant, positive operator such that

$(\phi_k, \sqrt{a}\phi_k) = \sqrt{a_k}$, $(\phi_k, \sqrt{a}\phi_j) = 0, i \neq j$, The diagonal system

$$dx_k(t) = -\lambda_k x_k(t)dt + \sqrt{2a}dB_k(t), \quad k = 1, 2, \ldots, \tag{2.4.9}$$

where $x_k(t) = (X_t, \phi_k)$, $B_k(t) = (B_t, \phi_k)$, $x_k(0) \sim N(0, a_k/\lambda_k)$, enables us to write $x_t = \{x_k(t)\}_{k=1}^{\infty}$ as a vector of independent Ornstein-Uhlenbeck i.e. mean zero Gaussian processes defined by

$$\mathbb{E}x_k(t)x_k(s) = \frac{a_k}{\lambda_k}\exp(-\lambda_k |t-s|)$$

If $\sum_{k=1}^{\infty} a_k/\lambda_k < \infty$, then for each fixed $t, x_t \in l^2$ a.s. (that is, $\sum_{k=1}^{\infty} |x_k(t)|^2 < \infty$ a.s.) and the continuity of the paths is given by the following:

Theorem 2.4.8. *Let $f(x)$ be a positive function on $[x_1, \infty)$ such that $f(x)/x$ is nondecreasing for $x \geq x_1 > 0$ and such that*

$$\int_{x_1}^{\infty} \frac{dx}{f(x)} < \infty$$

Suppose also that

$$\sum_k \frac{a_k}{\lambda_k} < \infty$$

and

$$\sup_k \frac{f(a_k \vee x_1)}{\lambda_k \vee 1} < \infty$$

Then x_t is continuous in l^2 a.s.

In Iscoe, Marcus, McDonald, Talagrand and Zinn [ISMTZ] a sufficient condition for continuity was found that is sharper than the above and necessary in certain special cases.

The above technique cannot be extended to the case $\mathbb{R}^d$ since Δ does not have a discrete spectrum on $\mathbb{R}^d$. However if $A := \Delta - |x|^2$ then $-A$ has a discrete spectrum in $\mathbb{R}_+$ with eigenfunctions $\phi_k = h_{k_1}, \ldots h_{k_d}$ which give a CONS for $\mathbb{H}_0 := L^2(\mathbb{R})$ where h_k are the normalized Hermite functions (see [RT 03]).

For $\gamma \geq 0$ set

$$\mathcal{H}_\gamma := \left\{ f \in \mathbb{H}_0 : \left| (-A)^{\frac{\gamma}{2}} f \right|_0 < \infty \right\}$$

where $(-A)^{\frac{\gamma}{2}}$ is the fractional power of $-A$. $\mathcal{H}_\gamma$ endowed with the scalar product

$$\langle f, g \rangle_\gamma := \left\langle (-A)^{\frac{\gamma}{2}} f, (-A)^{\frac{\gamma}{2}} g \right\rangle_0$$

becomes a separable Hilbert space. Let $\mathcal{H}_{-\gamma} = \mathcal{H}'_\gamma$.

The imbedding $\mathcal{H}_\gamma \subset \mathcal{H}_0$ is Hilbert-Schmidt if and only if $\gamma > d$, that is,

$$\sum_n |\phi_n^\gamma|_\alpha^2 < \infty$$

The semigroup $S(t)$ generated by A can be restricted to any $\mathcal{H}_\gamma$ and extended to a strongly continuous semigroup on the corresponding spaces $\mathcal{H}_{-\gamma}$ by duality preserving the norm (see [RT 03]). Then the imbedding $\mathcal{H}_0 \subset \mathcal{H}_{-\gamma}$ is Hilbert Schmidt. The Wiener process associated to space time white noise $W(\cdot)$ is regular on $\mathcal{H}_{-\gamma}$.

Now consider

$$\text{(OU-Rd)} \qquad \begin{cases} dX(t,x) = AX(t,x)dt + W(dx,dt) & t \geq 0, \quad x \in \mathbb{R}^d \\ X(0,x) = X_0 & x \in \mathbb{R}^d \end{cases}$$

so that we can extend the given SDE onto $\mathcal{H}_{-\gamma}$ and solve it there via the stochastic convolution:

$$X(t) = S(t)X_0 + \int_0^t S(t-s)dW_s \tag{2.4.10}$$

We have the following:

Theorem 2.4.9. *Given any positive $\gamma > d$, the SDE given by (OU-Rd) has continuous paths in the space $\mathcal{H}_{-\gamma}$.*

Proof. Let W be a Q-Wiener process, continuity of the paths follow easily from the condition:

$$\begin{aligned}\int_0^T \mathrm{Tr}\,[S(t)QS^*(t)]\,dt &\le \int_0^T \|S(t)Q\|_1 \|S(t)^*\|\,dt \\ &\le \int_0^T \|Q\|_1 \|S(t)\| \|S(t)^*\|\,dt \\ &\le \int_0^T \|Q\|_1\,dt = T\,\|Q\|_1 < +\infty\end{aligned}$$

since Q is trace class in $\mathcal{H}_{-\gamma}$. □

Remark 2.4.2. Using the same procedure based on a scale of Schwartz distributions, we can also prove that the real-valued space-time white noise in $\mathbb{R}$ has continuous paths in the Sobolev space $H_{-(1/2+\delta)}$, following the guidelines given in the appendix, with $Q = Id$ and $U = L_2$ it follows that $U_0 = Q^{1/2}(U) = Id(L_2) = L_2$ and the embedding $H_0 = L_2 \hookrightarrow H_{-(1/2+\delta)}$ being compact. We will use this result later in Ch. 5.

2.5 Hausdorff dimension and support of a measure

For the catalytic OU process, we will see latter that the Hausdorff dimension of the support will play an important role in the continuity of the paths, so, let us recall some concepts and definitions:

Definition 2.5.1. Suppose that (X, ρ) is a metric space , $p \geq 0$, and $\delta > 0$. For $A \in X$, let

$$H_{p,\delta}(A) = \inf\left\{\sum_{1}^{\infty}(\text{diam } B_j)^p : A \subset \bigcup_{1}^{\infty} B_j \text{ and diam } B_j \leq \delta\right\}$$

with the convention that $\inf \phi = \infty$. As δ decreases the infimum is being taken over a smaller family of coverings of A , so $H_{p,\delta}(A)$ increases. The limit

$$H_p(A) = \lim_{\delta \to 0} H_{p,\delta}(A)$$

is called the **p-dimensional Hausdorff (outer) measure** of A.

The following property follows immediately from the definition, details on the proof can be found in [FL 85], pp. 85,86.

Proposition 2.5.1. *If $H_p(A) < \infty$, then $H_q(A) = 0$ for all $q > p$. If $H_p(A) > 0$, then $H_p(A) = \infty$ for all $q < p$.*

According to the above Proposition, for any $A \subset X$ the numbers

$$\inf\{p \geq 0 : H_p(A) = 0\} \text{ and } \sup\{p \geq 0 : H_p(A) = \infty\}$$

are equal. Their common value is called the **Hausdorff dimension** of A.

Definition 2.5.2. A Borel measure μ on $\mathbb{R}^n$, of compact support and with $0 < \mu(\mathbb{R}^n) < \infty$, is called a *mass distribution.* The *t-potential* at a point x due to mass distribution μ is defined as

$$\phi_t(x) = \int \frac{d\mu(y)}{|x-y|^t}$$

The *t-energy* of μ is given by:

$$I_t(\mu) = \int \phi_t(x) d\mu(x) = \int\int \frac{d\mu(x) d\mu(y)}{|x-y|^t}$$

The following is an important result known as **Frostman's lemma**:

Proposition 2.5.2. *Let E be s Souslin subset of $\mathbb{R}^n$*

(a) If $I_t(\mu) < \infty$ for some mass distribution μ supported by E, then $t \leq$ dim E.

(b) If $t <$ dim E, then there exists a mass distribution μ with support in E such that $I_t(\mu) < \infty$.

Corollary 2.5.3. *Let E, t and μ as in (b) of the above theorem, then there exists $0 < K < \infty$ such that*

$$\int \frac{\mu(dx)}{|x-y|^t} < K \qquad \forall x \in \mathbb{R}$$

CHAPTER 3
Catalytic Wiener and Ornstein-Uhlenbeck processes

In this chapter, we introduce a more general class of processes than those studied before. The new class of processes differs from those described above in that the intensity of the perturbation or activity depends on the presence of another component which can be thought of as a catalyst and plays a role analogous to catalysts that increase the rate of chemical reactions.

3.1 Definitions and Properties

In mathematical terms, a catalyst can be described as a measure or mass μ which is static or can evolve on time μ_t, more precisely:

Definition 3.1.1. Let (E,$\mathcal{E}, \nu$) be a $\sigma-$finite measure space. A random set function W_ν on the sets $A \in \mathcal{E}$ of finite $\nu-$measure is called a white noise based on ν, if:

(i) W_ν(A) is a $N(0, \nu(A))$ random variable

(ii) if $A \cap B = \varnothing$, then W_ν(A) and W_ν(B) are independent and

$$W_\nu(A \cup B) = W_\nu(A) + W_\nu(B).$$

We will also denote W_ν by $W(d\nu, dt)$ or $\dot{W}(dx, dt)$. The existence of the white noise as well as a consistent theory of integration with respect to white noise based on a given measure can be found in the reference see [Wal 85].

The space-time white noise on $\mathbb{R}$ corresponds to the case $E = \mathbb{R} \times [0, \infty)$ and ν is Lebesgue measure on E.

Of importance for us is the isometry property; let f be a measurable function, then the stochastic integral:

$$X(t) = \int_0^t \int_{\mathbb{R}} f(s,x) W(dx, ds)$$

satisfies the property:

$$\mathbb{E}X^2(t) = \int_0^t \int_{\mathbb{R}} \mathbb{E}f^2(s,x) dx\, ds.$$

In the case $E = \mathbb{R}^d \otimes [0, \infty)$ and $\nu(dx, ds) = \mu(dx)ds$ we will also refer to such a processes as the Catalytic Wiener process $W_\mu(dt, dx)$ based on the catalyst $\mu \in M(\mathbb{R}^d)$. In this case the isometry is given by

$$X(t) = \int_0^t \int_{\mathbb{R}} f(s,x) W_\mu(dx, ds)$$

satisfies the property:

$$\mathbb{E}X^2(t) = \int_0^t \int_{\mathbb{R}} \mathbb{E}f^2(s,x) \mu(dx)\, ds.$$

Another class of processes closely related to the above processes are the measure-valued processes of which we give below its definition and basic properties:

Let (E, d) be a compact metric space, $C(E)$ the space of continuous functions, $\mathcal{E} = \mathcal{B}(E)$ the σ-algebra of Borel sets of E, and $M_1(E)$ the space of probability measures on E. We denote by $b\mathbb{E}$ (resp. $pb\mathcal{E}$) the bounded (resp. non-negative bounded) $\mathcal{E}$-measurable functions on E. If $\mu \in M_1(E)$ and $f \in b\mathcal{E}$, we define $\langle \mu, f \rangle := \int_E f d\mu$. Note that $M_1(E)$ endowed with the topology of the weak convergence is a compact metric space (recall that $\mu_n \overset{w}{\Rightarrow} \mu$ if and only if $\langle \mu_n, f \rangle \to \langle \mu, f \rangle \, \forall f \in bC(E)$).

Let Λ be the set of functions, strictly increasing Lipschitz continuous functions from $[0, \infty)$ onto $[0, \infty)$ such that

$$\gamma(\lambda) := \sup_{s>t\geq 0} \left| \log \frac{\lambda(s) - \lambda(t)}{s-t} \right| < \infty$$

For $\omega, \omega' \in D_E = D([0,\infty), E)$:

$$\rho := \inf_{\lambda \in \Lambda} \left(\gamma(\lambda) + \int_0^\infty e^{-u} \left(1 \wedge \sup_{t\geq 0} d(\omega(t\wedge u), \omega'(\lambda(t \wedge u))) \right) du \right)$$

it is easy to verify that ρ defines a metric on D_E. The resulting topology is called the *Skorohod topology* (J_1-topology) on D_E. Let $\mathcal{D} := \mathcal{B}(D[0,\infty); E))$ denote the σ-algebra of Borel subsets with respect to this topology. Let $X_t(\omega) := \omega(t)$ for $\omega \in \mathbb{D}$ and $\mathcal{D}_t^0 := \sigma(X_s :\leq s \leq t), \mathcal{D}_t := \bigcap_{\epsilon>0} \mathcal{D}_{t+\epsilon}^0 \subset \mathcal{D}$.

Then $(D_E, \mathcal{D}, (\mathcal{D}_t)_{t\geq 0}, (X_t)_{t\geq 0})$ denotes the *canonical stochastic process* on E.

Let $\mathbb{D} = D([0,\infty), M_1(E))$ be endowed with the usual Skorohod topology and $X_t : \mathbb{D} \to M_1(E), X_t(\omega) := \omega(t)$ for $\omega \in \mathbb{D}$.Let $\mathcal{D}_t^0 = \sigma\{X_s : 0 \leq t\}, \mathcal{D} = \vee \mathcal{D}_t^0 = \mathcal{B}(\mathbb{D}), \mathcal{D}_t^+ : \bigcap_{\epsilon>0} \mathcal{D}_{t+\epsilon}^0$. Then $(\mathbb{D}, (\mathcal{D}_t)_{t\geq 0}, \mathcal{D}, (X_t)_{t\geq 0})$ defines the *canonical probability-measure-valued process.*

By an $M_1(E)-$valued *stochastic process* we mean a family of probability measures $\{P_\mu : \mu \in M_1(E)\}$ on $(\mathbb{D}, \mathcal{D}, (\mathcal{D}_t)_{t\geq 0})$ such that:

(i) $P_\mu(X(0) = \mu) = 1$, that is $\Pi_0 P\mu = \delta_\mu$

(ii) the mapping $\mu \to P_\mu$ from $M_1(E)$ to $M_1(\mathbb{D})$ is measurable.

Let $\{\mu_t\}_{t\in[0,\infty)} \in C([0,\infty), M_1(\mathbb{R}^d))$ and consider the white noise on $[0,\infty)\times\mathbb{R}^d$ with

$$\nu(dx, ds) = \mu_s(dx)ds.$$

Now consider the equation:

$$\begin{aligned} dX_t &= \Delta X_t\, dt + dW_\mu \\ X_0 &= 0 \end{aligned} \tag{3.1.1}$$

here W_μ is a white noise based on the catalytic μ_t.

In the next chapter we consider the random case in which μ_t evolves in time according to a superbrownian motion, that is, we will take $\mu = \mu_t = \mathrm{Z}_t$, where Z_t is a (α, d, β)-superprocess. Here Z_t, W_μ and X_t are will be assumed to be defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}\}_{t\geq 0})$.

As before, the aim is to compute the covariance and Laplace functional of the process X_t. But before, we introduce some special processes and will try to study its relations and similarities.

3.2 Ornstein-Uhlenbeck with single point catalyst in $[0,1]$

We would like to carry out the same analysis we did in Chapter 2, but this time with a Brownian perturbation placed on a subset of $\mathcal{O} = (0,1)$, which could be expressed formally as the following Cauchy problem:

Find X(t) in $L_2([0,1])$ such that:

$$
\text{(CP1)} \qquad \begin{cases} dX(t,\xi) = \Delta X(t,\xi)dt + dW_{\delta_{1/2}}(t,\xi) & t \geq 0, \quad \xi \in [0,1] \\ X(0,\xi) = 0 & \xi \in [0,1] \end{cases}
$$

Note that $W_{\delta_{1/2}}(dx, ds) = \delta_{1/2}(dx)dB_s$ where B_t is a standard Brownian motion. Note also that $\delta_{1/2}$ is not a function, but this difficulty can be overcome using the sine series of the delta function, by computing first the Fourier series of the Heaviside function on (0,1).

$$
\mathrm{H}(x) = \begin{cases} -\frac{1}{2} & 0 \leq x \leq 1/2 \\ \frac{1}{2} & 1/2 < x \leq 1 \end{cases}
$$

The Fourier cosine expansion of such a Heaviside function is easily seen to be:

$$
2\left(-\mathrm{Cos}\ \pi x + \frac{\mathrm{Cos}\ 3\pi x}{3} - \frac{\mathrm{Cos}\ 5\pi x}{5} + \cdots\right)
$$

The above choice was done, because its derivative, which is the δ function at the midpoint has the expansion:

$$
2\left(\mathrm{Sin}\ \pi x - \mathrm{Sin}\ 3\pi x + \mathrm{Sin}\ 5\pi x + \cdots\right) = 2\sum_{k=1}^{\infty}(-1)^{k+1}\mathrm{Sin}\ (2\mathrm{k}\text{-}1)\pi x
$$

This is in terms of the eigenfunctions of the Laplacian with Dirichlet boundary conditions which give a basis for $L^2(0,1)$. Therefore, one can find its value under the semigroup and the solution of (CP1) is then given by:

$$
\begin{aligned}
X(t,x) &= \int_0^t \text{S(t-s)}\delta_{1/2}(x)\,\text{dB}_s \\
&= \frac{4}{\pi}\int_0^t \left(\text{S(t-s)}\sum_{k=1}^{\infty}(-1)^{k+1}\text{Sin (2k-1)}\pi x\right)\,\text{dB}_s \\
&= \frac{4}{\pi}\int_0^t \left(\sum_{k=1}^{\infty}(-1)^{k+1}\text{S(t-s)Sin (2k-1)}\pi x\right)\,\text{dB}_s \\
&= \int_0^t \left(\sum_{k=1}^{\infty} a_k(x)e^{-\psi_k(t-s)}\right)\,\text{dB}_s
\end{aligned}
\tag{3.2.2}
$$

where, by definition:

$$
a_k(x) = (-1)^{k+1}\text{Sin(2k-1)}\pi x
$$

$$
\psi_k = \mu_{2k-1} = \pi^2(2k-1)^2
$$

Hence:

$$
\begin{aligned}
\mathbb{E}(X(t,x)X(t,y)) &= \left(\frac{4}{\pi}\right)^2 \int_0^t \left(\sum_{k=1}^{\infty} a_k(x)e^{-\psi_k(t-s)}\right)\left(\sum_{l=1}^{\infty} a_l(y)e^{-\psi_l(t-s)}\right)\,\text{dB}_s \\
&= \left(\frac{4}{\pi}\right)^2 \int_0^t \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_k(x)a_l(y)e^{-(\psi_k+\psi_l)(t-s)}\,\text{ds} \\
&= \left(\frac{4}{\pi}\right)^2 \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{a_k(x)a_l(y)}{\psi_k+\psi_l}\left(1-e^{-(\psi_k+\psi_l)t}\right)
\end{aligned}
$$

As before the exponential term decays rapidly to zero and is therefore only a transient one, so in the long run, the stationary effect is given by:

$$\left(\frac{4}{\pi}\right)^2 \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{a_k(x)a_l(y)}{\psi_k+\psi_l}$$
$$= \frac{4^2}{\pi^4}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(-1)^{k+l}\frac{\text{Sin(2k-1)}\pi x \cdot \text{ Sin(2l-1)}\pi y}{(2k-1)^2+(2l-1)^2}$$

for the particular case $x = y = 1/2$, one gets:

$$\text{Cov(X(t,x)X(t,x))} = \frac{4^2}{\pi^4}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(-1)^{k+l}\frac{1}{(2k-1)^2+(2l-1)^2}$$

this covariance is seen to be infinite by comparison with the integral

$$\iint_{1\le x^2+y^2\le R^2} \frac{dxdy}{(2x-1)^2+(2y-1)^2}$$

after the transformation $(x, y) \mapsto \frac{1}{2}(x-1, y-1)$ and using polar coordinates, it becomes

$$\frac{1}{4}\int_0^{2\pi}\int_1^R \frac{rdrd\theta}{r^2} = \frac{\pi}{2}\ln R$$

which tends to infinity as $R \to \infty$.

However, when $(x, y) \neq (1/2, 1/2)$, the covariance is finite, to see this assume $y \neq 1/2$ and fix k, then the sum

$$\sum_{l=1}^{\infty}(-1)^{k+l}\frac{\text{Sin }(2k-1)\pi x \text{ Sin }(2l-1)\pi y}{(2k-1)^2+(2l-1)^2}$$
$$= (-1)^k\text{Sin }(2k-1)\pi x \sum_{l=1}^{\infty}(-1)^{k+1}\frac{\text{Sin }(2l-1)\pi y}{(2k-1)^2+(2l-1)^2}$$

is an alternating series, which changes sign after a certain period depending on y, and its summation is $\mathcal{O}\left(\frac{1}{(2k-1)^2}\right)$, this shows that the summation over $k \geq 1$ is finite.

Theorem 3.2.1. *The process given by (3.2.2) has a continuous version on* $L^2[0,1]$.

Proof. Let us write first (3.2.2) as:

$$X(t,x) = \int_0^t H(r,x)\,\mathrm{dB}_r$$

where:

$$H(r,x) = \sum_{k=1}^{\infty} a_k(x) e^{-\psi_k (t-r)}$$

and $a_k(x) = (-1)^{k+1}\mathrm{Sin}(2\mathrm{k}\text{-}1)\pi x$, $\psi_k = \mu_{2k-1}\pi^2(2k-1)^2$, so, we get for the increments:

$$\begin{aligned}
\mathbb{E}\,\|X(t) - X(s)\|_{L^2}^2 &= \mathbb{E}\int_0^1 \left(\int_s^t H(t-r,x)\,dB_r\right)^2 dx \\
&= \int_0^1 \int_s^t H^2(t-r,x)\,dr\,dx \\
&= \int_s^t \int_0^1 H^2(t-r,x)\,dx\,dr \\
&= \int_0^1 \|H(t-r,x)\|_{L^2}^2\,dr
\end{aligned}$$

from the definition, it follows:

$$
\begin{aligned}
\|H(t-r,x)\|_{L^2}^2 &= \sum_{k=1}^{\infty} \exp(-2\pi^2(2k-1)^2(t-r)) \\
&\leq C \int_{\mathbb{R}} \exp(-(t-r)x^2)\,dx \\
&= C(t-r)^{-1/2}
\end{aligned}
$$

so:

$$
\begin{aligned}
\mathbb{E}\,\|X(t)-X(s)\|_{L^2}^2 &= \int_0^1 \|H(t-r,x)\|_{L^2}^2\,dr \\
&\leq \int_s^t C(t-r)^{-1/2}\,dr = K(t-s)^{1/2}
\end{aligned}
$$

Therefore X has a $\alpha - \textit{Hölder}$ continuous version for any $\alpha \in (0, \frac{1}{4})$ by Proposition 2.2.1. □

3.2.1 Catalytic OU processes on $[0,1]$: uniform L_2 boundedness

We introduce the following:

Definition 3.2.1. A solution X_t of a OU process,is said to be *uniformly L^2-bounded*, if $\sup_{x\in\mathbb{R}} \mathbb{E}(X_t(x)^2) < \infty$.

We proved above that for the single point catalyst at $\frac{1}{2}$, $E(X(t,\frac{1}{2})^2) = \infty$ so that this process is not uniformly L_2-bounded.

Theorem 3.2.2.

(i) If μ is the Lebesgue measure, then X_t is uniformly L^2-bounded only if $d = 1$.

(ii) X_t is not uniformly L^2-bounded if $d = 1$ and $\mu = \delta_0$.

(iii) For any set of positive Hausdorff dimension $E \subset \mathbb{R}$ there exists a mass distribution μ supported by E such that X_t^μ (i.e. the OU process based on the measure μ) is uniformly L^2-bounded.

Proof. Denote by $p(t-s,x,y)$ the kernel corresponding to the Laplacian on [0,1] without boundary conditions Recall that for a OU processes with a Wiener perturbation based on the measure μ, we have:

$$\mathbb{E}(X(t,x)^2) = \int_0^t \int (p(t-s,x,y))^2 \mu(dy)\, ds$$

If μ is the Lebesgue measure:

$$\mathbb{E}(X(t,x)^2) = \int_0^t \frac{ds}{s^{d/2}}$$

which is finite only if $d = 1$, this proves (i).

Assume now, $d = 1$ and $\mu = \delta_0$, and comparing $\int p^2(t-s,x,y)ds$ with a gamma distribution, one obtains:

$$\mathbb{E}(X(t,x)^2) = C_1 \int \frac{1}{|x-y|}\, \mu(dy)$$

which clearly is not uniformly L^2-bounded thus proving (ii). Note that this corresponds to the fact that the covariance of a one point catalyst in $d = 1$ is infinite at the catalyst, as seen before.

For (iii), the inequality :

$$\ln \frac{1}{|x-y|} \leq \frac{const}{|x-y|^t} \qquad t > 0.$$

yields:

$$\mathbb{E}(X(t,x)^2) \le C \int \frac{\mu(dy)}{|x-y|^t}.$$

Now applying Corollary 2.5.3 gives the desired result. □

Corollary 3.2.3. *The catalytic OU process based on the Cantor set on* $[0,1]$ *is uniformly* L^2*-bounded.*

Proof. The Cantor set has Hausdorff dimension $\frac{\log 2}{\log 3}$. □

3.3 Wiener perturbation at the center of the domain in $\mathbb{R}^2$

A Fourier analysis can also be carried out for a Wiener perturbation at the center of the square $[0,1]^2$, but the situation here becomes extremely difficult to handle due to the complexity of the Fourier coefficients as well as to the many involved summations. We will present two different approaches to the problem and verify that they are both consistent.

To begin we note that given a bounded continuous function $f(\cdot)$ the solution to the equation

$$dX(t,\xi) = \frac{1}{2}\Delta X(t,\xi) + f(x)W(dt,dx), \quad X(0) =\equiv 0$$

is given by

$$X(t,x) = \int_0^t \int p(t-s,x,y)f(y)W(ds,dy).$$

In order to relate this to catalytic Ornstein-Uhlenbeck processes with singular catalysts in this section we obtain the case of a single point catalyst in $\mathbb{R}^2$ as a limit of processes with absolutely continuous catalysts.

We consider the catalytic Ornstein-Uhlenbeck equation:

$$\text{(NB)} \qquad \begin{cases} dX(t,x) &= \Delta X(t,x)dt + W_{\delta_{(0,0)}}(dt,dx), \qquad t \geq 0, \quad x \in \mathbb{R}^2 \\ X(0,x) &= 0 \quad x \in \mathbb{R}^2 \end{cases} .$$

The objective is to show that this process can be obtained as a limit (in sense of f.d.d.) of processes with absolutely continuous catalysts on a certain Hilbert space.

3.3.1 Two equivalent sequences of approximating processes

We first construct a sequence of **approximate process I** given by the equations:

$$\text{(NBn)} \qquad \begin{cases} dX_n(t,x) &= \Delta X_n(t,x) + D_n(x)W(dt,dx) \quad t \geq 0, \quad x \in \mathbb{R}^2 \\ X_n(0,x) &= 0 \qquad\qquad x \in \mathbb{R}^2 \end{cases}$$

As usual, W(dt,dx) is a space-time white noise and $D_n(x) \triangleq [p(1/n,x,0)]^{1/2}$ is the sequence, given by:

$$D_n \triangleq \frac{1}{(\frac{2\pi}{n})^{1/2}} \exp\left(-\frac{\|x\|^2}{\frac{2}{n}}\right).$$

This choice of an approximate delta sequence will simplify the analysis, since it can be operated with the semigroup of the Laplacian , namely the heat kernel.

For $x \neq (0,0)$, the solution of (NBn) is then given by the stochastic convolution:

$$X_n(t,x) = \int_0^t \int_{\mathbb{R}^2} p(t-s,x,y)D_n(y)W(ds,dy)$$

Lemma 3.3.1.

$$\lim_{n\to\infty} Cov(X_n(t,x), X_n(t,y)) = \frac{\exp(-(\|x\|^2 + \|y\|^2)/t)}{\|x\|^2 + \|y\|^2} \tag{3.3.3}$$

Proof. First note that $\mathrm{Cov}(X_n(t,x), X_n(t,y)) = \mathbb{E}(X_n(t,x)X_n(t,y))$ is given by

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \int_{\mathbb{R}^2} p(t-s_1,x,z_1)D_n(z_1)W(ds_1,dz_1)\right. \\
&\qquad \left.\cdot \int_0^t \int_{\mathbb{R}^2} p(t-s_2,y,z_2)D_n(z_2)W(ds_2,dz_2)\right] \\
&= \int_0^t \int_{\mathbb{R}^2} p(t-s,x,z)p(t-s,y,z)D_n^2(z)dz\, ds \\
&= \int_0^t \int_{\mathbb{R}^2} p(t-s,x,z)p(t-s,y,z)p(1/n,0,z)dz\, ds \\
&= \int_{\mathbb{R}^2} \left(\int_0^t p(t-s,x,z)p(t-s,y,z)ds\right) p(1/n,0,z)dz
\end{aligned}$$

The innermost integral is easily found to be:

$$t^2 \cdot \frac{p(t,x,z)p(t,y,z)}{\|x-z\|^2 + \|y-z\|^2}$$

With this, one can compute the covariance, the following way:

$$Cov(X_n(t,x), X_n(t,y)) = t^2 \int_{\mathbb{R}^2} \frac{p(t,x,z)p(t,y,z)p(1/n,0,z)}{\|x-z\|^2 + \|y-z\|^2} dz$$

Taking the limit as $n \to \infty$ and since $p(1/n,0,z) \to \delta_0(z)$ in the sense of distributions, we obtain the covariance of the limiting process to be:

$$
\begin{aligned}
\operatorname{Cov}(X(t,x),X(t,y)) &\doteq \lim_{n\to\infty} \mathbb{E}(X_n(t,x)X_n(t,y)) \\
&= \frac{t^2 p(t,x,0)p(t,y,0)}{\|x\|^2+\|y\|^2} \\
&= \frac{\exp(-(\|x\|^2+\|y\|^2)/t)}{\|x\|^2+\|y\|^2}
\end{aligned}
\tag{3.3.4}
$$

□

Remark 3.3.1. Similar to the one dimensional case, the covariance is infinite at the center where the Wiener perturbation is placed and it is positive and finite away from the origin.

We next consider a second **approximate process II** given by the equations:

$$
\text{(YBn)} \qquad \begin{cases} dY_n(t,x) &= \Delta Y_n(t,x) + p(1/n,x)dB_t \quad t \geq 0, \quad x \in \mathbb{R}^2 \\ Y_n(0,x) &= 0 \qquad\qquad x \in \mathbb{R}^2 \end{cases}
$$

where B_t is a standard Brownian motion.

Remark 3.3.2. Note that the perturbation is now a 1-dimensional Brownian motion, and that the approximate δ-sequence is now $p(1/n,x)$.

The covariance of this process is given by:,

$$
\begin{aligned}
&\mathrm{Cov}(Y_n(t,x),Y_n(t,y)) = \mathbb{E}(Y_n(t,x)Y_n(t,y)) = \\
&= \mathbb{E}\left[\int_o^t \int_{\mathbb{R}^2} p(t-s_1,x,z_1)p(1/n,z_1)dB_{s_1}dz_1 \right.\\
&\qquad \left. \cdot \int_o^t \int_{\mathbb{R}^2} p(t-s_2,y,z_2)p(1/n,z_2)dB_{s_2}dz_2\right] \\
&= \int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} p(t-s,x,z_1)p(t-s,y,z_2)p(1/n,z_1)p(1/n,z_2)dz_1dz_2\, ds \\
&= \int_0^t p(t-s+1/n,x,0)p(t-s+1/n,y,0)dz\, ds \\
&= \frac{\exp(-(\|x\|^2+\|y\|^2)/(t+2/n))}{\|x\|^2+\|y\|^2}
\end{aligned}
$$

therefore:

$$
\begin{aligned}
\lim_{n\to\infty} \mathrm{Cov}(Y_n(t,x),Y_n(t,y)) &\doteq \lim_{n\to\infty} \mathbb{E}(Y_n(t,x)Y_n(t,y)) \\
&= \frac{\exp(-(\|x\|^2+\|y\|^2)/t)}{\|x\|^2+\|y\|^2}
\end{aligned}
$$

which is the same as the obtained by the approximate process X_n.

The convergence of (YBn) to $X(t)$ is easier to prove, so that we will refer to the process (YBn) as the approximate process and will rename it X_n.

The next step is to prove the following convergence result. We first introduce the measure $\mu_\alpha(x) = (\|x\|^\alpha \wedge 1)dx$ on $\mathbb{R}^2$ and Hilbert space $H_\alpha = L_2(\mathbb{R}^2, \mu_\alpha)$.

Theorem 3.3.2. *For $\alpha > 0$, the processes $X_n(t,\cdot)$ converge in the sense of f.d.d. of H_α-valued processes to the solution $X(t,x)$ of (**NB**), which is given by the* *stochastic convolution*

$$\begin{aligned} X(t,x) &= \int_0^t S(t-s)\delta_{(0,0)} dB_t \\ &= \int_0^t \int_{\mathbb{R}^2} p(t-s,x,y)\delta_{(0,0)}(dy) dB_s \\ &= \int_0^t p(t-s,x,0) dB_s. \end{aligned}$$

Proof. We must show that for $t > 0$ that $X_n(t) \to X(x)$ in the mean square, i.e.

$$\lim_{n\to\infty} \mathbb{E}\|X_n(t) - X(t)\|_{H_\alpha}^2 = 0 \tag{3.3.5}$$

This follows from the Itô isometry:

$$\begin{aligned} \mathbb{E}|X_n(t,x) - X(t,x)|_2^2 &= \mathbb{E}\left| \int_0^t \left(p(1/n+t-s,x,0) - p(t-s,x,0)\right) dB_s \right|^2 \\ &= \int_0^t |p(1/n+t-s,x,0) - p(t-s,x,0)|^2 ds. \end{aligned}$$

The continuity of $p(t-s,x,0)$ implies the pointwise convergence:

$$\lim_{n\to\infty} p(1/n+t-s,x,0) = p(t-s,x,0),$$

In order to apply the dominated convergence theorem, note that

$$\begin{aligned} |p(1/n+t-s,x,0) - p(t-s,x,0)|^2 = |p(1/n&+t-s,x,0)|^2 \\ &+ |p(t-s,x,0)|^2 \\ &- 2(p(1/n+t-s,x,0), p(t-s,x,0)) \end{aligned}$$

together with the inequality $(t-s) < (1/n+t-s)$ implies:

$$p(1/n+t-s,x,0)p(t-s,x,0) \leqq p^2(t-s,x,0)$$

Noting that $\int_0^t p^2(t-s,x,0)ds \leq \frac{\text{const}\cdot\exp(-\|x\|^2/t)}{\|x\|^2}$ the Lebesgue dominated convergence theorem applies and

$$E[\|X_n(t) - X(t)\|^2_{H\alpha} \to 0.$$

Using the decomposition

$$X(t+s,x) = S_s X(t,x) + \int_t^{t+s} \int p(t+s-u,x,0)dB_s$$

it then follows that the finite dimensional distributions converge. □

Remark 3.3.3. Note that if $\mathcal{O}$ denotes the unit ball

$$\begin{aligned} &= \mathbb{E}\int_{\mathcal{O}} X_n^2(x,t)\, dx = \int_{\mathcal{O}} \mathbb{E}X_n^2(x,t)\, dx \\ &= C_1 \int_{\mathcal{O}} \frac{dx}{\|x\|^2} e^{-2\|x\|^2/(t+2/n)} \end{aligned}$$

and the last integral is infinite since $(0,0) \in \mathcal{O}$. However, taking the measure with density $\|x\|^\alpha$ for some $\alpha > 0$, we get:

$$\mathbb{E}\left\|X_n^2(t)\right\|_{H_\alpha} < M < \infty$$

Then using Chebyshev's inequality, the following is true for any $K > 0$:

$$\mathrm{P}[\|X_n(t)\|_{H_\alpha} \geq K] \leq \frac{M}{K^2} \qquad \forall n \in \mathbb{N}$$

and therefore yields:

$$\mathrm{P}[\|X(t)\|_{H_\alpha} \geq K] \leq \frac{M}{K^2}.$$

3.3.2 The question of state space and continuity of the paths

Theorem 3.3.3. *The solutions of* **(NB)***have a continuous version on the Hilbert space* H_1.

Proof. We first use polar coordinates to evaluate the following norm:

$$\begin{aligned}\|p(t-r,x,0)\|_{H_1}^2 &= C\int_0^{\infty}\frac{1}{(t-r)^2}\exp(-\frac{\rho^2}{t-r})\rho(\rho\wedge 1)d\rho \\ &= C\int_0^{1}\frac{\rho^2}{(t-r)^2}\exp(-\frac{\rho^2}{t-r})d\rho \\ &+ C\int_1^{\infty}\frac{1}{(t-r)^2}\exp(-\frac{\rho^2}{t-r})d\rho\end{aligned} \tag{3.3.6}$$

next, we estimate each of the above integrals, take $\beta \in (1, 3/2)$, inequality 3.5.14 yields:

$$\begin{aligned}\int_0^1 \frac{\rho^2}{(t-r)^2}\exp(-\frac{\rho^2}{t-r})d\rho &\leq C_1(t-r)^{\beta-2}\int_0^1 \rho^{2-2\beta}\,d\rho \\ &\leq C_2(t-r)^{\beta-2}\end{aligned}$$

similarly:

$$\begin{aligned}\int_1^{\infty} \frac{\rho}{(t-r)^2}\exp(-\frac{\rho^2}{t-r})d\rho &\leq C_3(t-r)^{\beta-2}\int_1^{\infty} \rho^{1-2\beta}\,d\rho \\ &\leq C_4(t-r)^{\beta-2}\end{aligned}$$

So:

$$\begin{aligned}\mathbb{E}\,\|X(t)-X(s)\|_{H_1}^2 &= \int_s^t \|p(t-r,x,0)\|_{H_1}^2\,dr \\ &\leq (C_2+C_4)\int_s^t (t-r)^{\beta-2}\,dr \\ &\leq C_5(t-s)^{\beta-1}\end{aligned}$$

the assertion follows since $\beta > 1$. □

Remarks

1. From

$$\|x\| \leq 1 \Rightarrow \|x\|^{1+\alpha} \leq 1, \; \forall \, \alpha > 0$$

follows the continuity of the paths on L^2-spaces whose measure has density $\|x\|^{1+\alpha}$, as well as measures with densities of the form

$$\|x\|^{1+\alpha} \exp(-\beta\|x\|), \; \alpha, \beta \geq 0$$

2. The same conclusion is valid for processes obtained by replacing Δ by any given operator $\mathcal{L}$ of the form:

$$\mathcal{L}u(r,x) \sum_{i,j=1}^{d} a_{ij}\partial_{ij}u(r,x) + \sum_{i=1}^{d} b_i(r,x)\partial_i u(r,x)$$

where $r \in \mathbb{R}, x \in E = \mathbb{R}^d$, and $a_{ij} = a_{ji}$ and are *elliptic* i.e. there exists $\kappa > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(r,x)t_i t_j \geq \kappa \sum t_i^2 \quad \forall \, (r,x) \in \mathbb{R}^{1+d}, t_1, \ldots, t_d \in \mathbb{R}$$

This is due to the simple fact, that the corresponding semigroup

$$P_t f(x) = \mathbb{E}_x f(X_t)$$

can be represented as

$$P_t f(x) = \int f(y) p(t,x,y) dy$$

where $p(t,x,y)$ is a symmetric function satisfying:

$$p(t,x,y) \leq c_1 t^{-d/2} e^{-c_2|x-y|^2/t}$$

for some c_1 and c_2 depending only on κ, (see [Ba 98] Th. 5.5 for further details)

3. We can now extend the above techniques to the unit ball $\mathcal{O} = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ with a Wiener perturbation placed at the origin, that is, consider the SPD:

$$\text{(UC)} \qquad \begin{cases} dX(t,x) & = \Delta X(t,x)dt + \delta_{(0,0)}(x)dW_t, \qquad t \geq 0, \quad x \in \mathcal{O} \\ X(0,x) & = 0 \quad x \in \mathcal{O} \\ X(t,x) & = 0 \quad x \in \partial\mathcal{O} \end{cases}$$

as well as its sequence of **approximate process** given by:

$$\text{(UCn)} \qquad \begin{cases} dX_n(t,x) & = \Delta X_n(t,x) + D_n(x)dW(t,x) \quad t \geq 0, \quad x \in \mathcal{O} \\ X_n(0,x) & = 0 \quad x \in \mathcal{O} \\ X_n(t,x) & = 0 \quad x \in \partial\mathcal{O} \end{cases}$$

This time, the delta sequence will be chosen to be:

$$D_n(x) = C\sqrt{p(1/n,0,x)\ J_0(\|x\|)}$$

where J_0 is the Bessel function of order zero and C is a normalizing constant, recall that the Bessel functions are radial solutions of the heat equation and also its eigenvalues under homogeneous boundary conditions and in particular, J_0 is positive on the unit circle and zero at the boundary.

Therefore, the solution is given by:

$$X_t^n = C\int_0^t \int_{\mathbb{R}^2} p(t-s,x,y)D_n(x)\, W(dy,ds)$$

hence:

$$\begin{aligned}
\mathbb{E}\,\|X_t^n\|_{L_2}^2 &= C^2\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} p^2(t-s,x,y)p(1/n,0,y)J_0(\|y\|)\,dx\,dy\,ds \\
&= C_1\int_0^t \frac{1}{(t-s)}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} p((t-s)/2,x,y)p(1/n,0,y)J_0(\|y\|)dx\,dy\,ds \\
&= C_1\int_0^t \frac{1}{t-s}\int_{\mathbb{R}^2} p(1/n,0,y)J_0(\|y\|)dy\,ds \\
&= C_1\int_0^t \frac{\exp\left(-\alpha/n\right)}{t-s} J_0(0)\,ds \\
&= C_1\exp\left(-\alpha/n\right)J_0(0)\int_0^t \frac{ds}{t-s} \\
&= \infty.
\end{aligned}$$

As before, considering instead H_1 :

$$\begin{aligned}
\mathbb{E}\,\|X_t^n\|_{H_1}^2 &= C\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} p^2(t-s,x,y)p^2(2/n,0,y)nJ_0(\|y\|)\ \|x\|\,dx\,dy\,ds \\
&= Cn\int_0^t \frac{n}{(t-s)}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} p((t-s)/2,x,y)\ \|x\|\,dx\,p(1/n,0,y)J_0(\|y\|)\,dy\,ds \\
&= Cn^2\int_0^t (t-s)^{-1/2}\int_{\mathbb{R}^2} p(1/n,0,y)J_0(\|y\|)dy\,ds \\
&= Cn^2\int_0^t \exp\left(-\alpha/n\right)J_0(0)(t-s)^{-1/2}\,ds \\
&= Cn^2\exp\left(-\alpha/n\right)J_0(0)\,t^{1/2}
\end{aligned}$$

which shows continuity of the paths.

We can also investigate higher moments of $\|X_t\|$, for that , recall the following property of the stochastic integration with respect to Brownian motion:

$$\mathbb{E}\left[\int_0^t f(s)\,dW_s\right]^4 = \frac{1}{2}\binom{4}{2}\left[\int_0^t f^2(s)\,ds\right]^2$$
$$\mathbb{E}\left[\int_0^t f(s)\,dW_s\right]^6 = \frac{1}{2}\binom{6}{2}\binom{4}{2}\left[\int_0^t f^2(s)\,ds\right]^3$$

with these facts and using the simplified notation $p(s,x) \triangleq p(t-s,x,0)$, it will be shown below that $\mathbb{E}\|X_t\|_{L_2}^4 = \infty$. In spite of its discouraging appearance, we will carry out the calculations since it introduces some techniques which are to be used later.

Proposition 3.3.4. *Let L_2 be the space of square-integrable functions w.r.t Lebesgue measure in $\mathbb{R}^2$, then for the solution X_t of (NB), $\mathbb{E}\|X_t\|_{L_2}^4 = \infty$.*

Proof.

$$\begin{aligned}
&\mathbb{E}\|X_t\|_{L_2}^4 = \\
&= \mathbb{E}\left[\int_{\mathbb{R}^2}\left(\int_0^t p(s,x,0)\,dW_s\right)^2 dx\right]^2 \\
&= \mathbb{E}\left[\int_{\mathbb{R}^2}\left(\int_0^t p(s,x_1)\,dW_s\right)^2 dx_1 \cdot \int_{\mathbb{R}^2}\left(\int_0^t p(s,x_2)\,dW_s\right)^2 dx_2\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}^4\times[0,t]^2} p(s_1,x_1)p(s_2,x_1)p(s_3,x_2)p(s_4,x_2)dW_{s_1}dW_{s_2}dW_{s_3}dW_{s_4}dx_1dx_2\right] \\
&= \int_{\mathbb{R}^4\times[0,t]^2} p(s_1,x_1)p(s_2,x_1)p(s_3,x_2)p(s_4,x_2)\mathbb{E}[dW_{s_1}dW_{s_2}dW_{s_3}dW_{s_4}]dx_1dx_2
\end{aligned}$$

Now using a Riemann sum analysis and the fact that the Wiener process in the above stochastic integrals is the same, as well as the properties given above, the four innermost integrals are equal to the following simplified sum:

$$= \int_0^t p^2(s,x_1)ds \cdot \int_0^t p^2(s,x_2)ds \qquad (s_1 = s_2 \wedge s_3 = s_4)$$

$$+ \left(\int_0^t p(s,x_1)p(s,x_2)\,ds\right)\left(\int_0^t p(s,x_1)p(s,x_2)\,ds\right) \qquad (s_1 = s_3 \wedge s_2 = s_4)$$

$$+ \left(\int_0^t p(s,x_1)p(s,x_2)\,ds\right)\left(\int_0^t p(s,x_1)p(s,x_2)\,ds\right) \tag{3.3.7}$$

$$= \left(\int_0^t p^2(s,x_1)ds\right) \cdot \left(\int_0^t p^2(s,x_2)ds\right) \tag{3.3.8}$$

$$+ 2\left(\int_0^t p(s,x_1)p(s,x_2)\,ds\right)^2 \tag{3.3.9}$$

After integration w.r.t. dx_1 and dx_2 (3.3.8), one gets:

$$\left(\int_{\mathbb{R}^2}\left(\int_0^t p^2(s,x)ds\right)dx\right)^2 = \left[\int_{\mathbb{R}^2}\int_0^t \frac{1}{(t-s)^2}\exp\left(-\frac{2\|x\|^2}{t-s}\right)ds\,dx\right]^2$$

$$= C_1\left[\int_0^t \frac{ds}{t-s}\right]^2$$

$$= \infty$$

and for (3.3.9) one gets:

$$\begin{aligned}
\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\int_0^t p(s,x_1)p(s,x_2)ds\right)^2 dx_1dx_2 &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\int_0^t \frac{e^{-\frac{\|x_1\|^2+\|x_2\|^2}{t-s}}}{(t-s)^2}ds\right)^2 dx_1dx_2 \\
&= \int_{\mathbb{R}^4} \frac{e^{-2r^2/t}}{r^4}dx \\
&= C_2\int_0^\infty \frac{e^{-2r^2/t}}{r}dr \\
&= \infty
\end{aligned}$$

□

A second option is to change the underlying measure of $\mathbb{R}^2$, as it is shown next, $d\mu = \|x\|\ dx$ leads to the desired conclusion:

$$\begin{aligned}
\mathbb{E}\,\|X_t\|^4_{L_2(\mu)} &\left(\int_{\mathbb{R}^2}\left(\int_0^t p^2(s,x)ds\right)\,d\mu(x)\right)^2 \\
&+2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\int_0^t p(s,x_1)p(s,x_2)ds\right)^2 d\mu(x_1)d\mu(x_2)
\end{aligned}$$

now, we compute each summand separately:

$$\begin{aligned}
\left(\int_{\mathbb{R}^2}\left(\int_0^t p^2(s,x)ds\right)\,dx\right)^2 &= \left[\int_{\mathbb{R}^2}\int_0^t \frac{\|x\|}{(t-s)^2}\exp\left(-\frac{2\,\|x\|^2}{t-s}\right)ds\,dx\right]^2 \\
&= \left[\int_0^t\int_{\mathbb{R}^2} \frac{\|x\|}{(t-s)^2}\exp\left(-\frac{2\,\|x\|^2}{t-s}\right)dx\,ds\right]^2 \\
&= C_1\left[\int_0^t (t-s)^{1/2}\,ds\right]^2 \\
&= C_1(t)^3.
\end{aligned}$$

For the second term, we observe that for $x_1, x_2 \in \mathbb{R}^2$ then $(x_1,x_2)\in\mathbb{R}^4$, $\|x_1\|^2 + \|x_2\|^2 = r^2$ and that the measure in $\mathbb{R}^2$ given by $d\mu = \|x\|\,dx$, induces the measure in $\mathbb{R}^4$ given by $d\mu(x) = r^2 Cos\theta_1 Cos\theta_2\, dx_1 dx_2$, so:

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\int_0^t p(s,x_1)p(s,x_2)ds\right)^2 d\mu(x_1)d\mu(x_2) =$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\int_0^t \frac{e^{-\frac{\|x_1\|^2+\|x_2\|^2}{t-s}}}{(t-s)^2} ds\right)^2 d\mu(x_1)d\mu(x_2)$$

$$= \int_{\mathbb{R}^4} \frac{e^{-2r^2/t}}{r^4} r^2 Cos\theta_1 Cos\theta_2 dx_1 dx_2$$

$$= C_2 \int_0^\infty r e^{-2r^2/t}\, dr \qquad \text{(polar coordinates)}$$

$$= C_3 t.$$

Now, fixing $T > 0$ large enough, we have:

$$\mathbb{E}\,\|X_t\|^4_{L_2(\mu)} = c_1 t^3 + c_2 t \leq c_3(T)\, t$$

.

3.4 Moving perturbations

The objective of this section is to develop the necessary tools and techniques to deal with perturbations changing in time. This will be further developed in the next chapter in which super-processes are going to play the role of catalysts.

As a first example, consider the effect of a Brownian perturbation based on a measure changing in time, namely, $W(dt, dx) = \delta_{ct}(dx)dW_t$, where W_t is a standard Brownian, $c > 0$ if $d = 1$, or is a given vector if $d > 1$. This is a perturbation moving along the line given by ct, as before, the solution is given by the stochastic convolution:

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} p(t-s,x,y)W(\delta_{cs}(y), ds) \qquad (3.4.10)$$

and the variance is given by:

$$\mathbb{E}X^2(t,x) = \frac{1}{(2\pi)^d}\int_0^t \frac{1}{(t-s)^d}\exp\left(-\frac{\|x-cs\|^2}{t-s}\right)\, ds.$$

The value of the covariance will depend on the dimension d of the space , if the perturbation hits or not the point x and in the former case, if at time t, the perturbation lies before, at, or after the point x. These values will be computed in the next theorems.

Theorem 3.4.1. *Let d=1, let $X(t,x)$ be given by (3.4.10). Then for any given $c>0$ and a point $x=ct_0$, the variance of $X(t,x)=X(t,ct_0)$ is given by:*

$$Var\ X(t,x) = \begin{cases} \int_0^t \frac{1}{t-s}\exp\left(-\frac{c^2(t_0-t)^2}{t-s}\right) ds < K < \infty & t<t_0 \\ \infty & t=t_0 \\ \textit{finite, but} \to \infty \textit{ as } t\to\infty & t>t_0 \end{cases}$$

Remark 3.4.1. In one dimension, any given point $x \geq 0$ is hit sometime by the perturbation δ_{ct}.

Proof. • Assume first that the perturbation is at the point x, that is $x=ct_0$, and $t=t_0$, so:

$$\begin{aligned}\mathbb{E}X^2(t_0,x) &= \int_0^{t_0}\frac{1}{t_0-s}\exp\left(-\frac{c^2(t_0-s)^2}{t_0-s}\right) ds \\ &= \int_0^{t_0}\frac{1}{t_0-s}\exp\left(-c^2(t_0-s)\right) ds = \infty\end{aligned}$$

The last identity results by comparing $\frac{1}{x}$ and e^{-x} as $x\to 0$.

- Before the perturbation hits the point, that is when $t < t_0$ then $x - cs = c(t_0 - t) + c(t_0 - s)$ and the inequality:

$$\begin{aligned}\exp\left(-\frac{\|x - cs\|^2}{t - s}\right) &= \exp\left(-\frac{c^2[(t_0 - t) + (t - s)]^2}{t - s}\right)\\ &\leq \exp\left(-\frac{c^2(t_0 - t)^2}{t - s}\right) ds\end{aligned}$$

Setting $M \triangleq c^2(t_0 - t)^2$, yields:

$$\begin{aligned}\mathbb{E}X^2(t, x) &\leq \int_0^t \frac{1}{t - s} \exp\left(-\frac{M}{t - s}\right) ds\\ &\leq \int_0^{t_0} \frac{1}{t - s} \exp\left(-\frac{M}{t - s}\right) ds = K < \infty\end{aligned}$$

- When the perturbation has passed the point: $t > t_0$, as before, one gets:

$$\begin{aligned}\exp\left(-\frac{\|x - cs\|^2}{t - s}\right) &= \exp\left(-\frac{c^2[(t_0 - t) + (t - s)]^2}{t - s}\right)\\ &\leq \exp\left(-\frac{M}{t - s}\right) \exp\left(2c^2(t - t_0)\right)\end{aligned}$$

The inequality is obtained by setting $M \triangleq c^2(t_0 - t)^2$ and noticing that $2c^2(t - t_0) \geq 0$ and $-c^2(t - s) \leq 0$, so that for $t > t_0$, the covariance is bounded by:

$$\mathbb{E}X^2(t, x) \leq \exp(2c^2(t - t_0)) \int_0^t \frac{1}{t - s} \exp\left(-\frac{M}{t - s}\right) ds < \infty$$

However this time, because of the factor $\exp(2c^2(t-t_0))$,the covariance cannot be uniformly bounded, in fact it goes to infinite as t does, to see this; from $t-s \leq t$ and $-c^2(t-s) \geq -c^2 t$, follows $2c^2(t-t_0)-c^2(t-s) \geq c^2(t-2t_0)$ and so:

$$\exp\left(-\frac{\|x-cs\|^2}{t-s}\right) \geq \exp\left(-\frac{c^2(t_0-t)^2}{t-s}\right)\exp(c^2(t-2t_0))$$

Therefore, the covariance satisfies the following inequality:

$$\int_0^t \frac{1}{t-s}\exp\left(-\frac{\|x-cs\|^2}{t-s}\right) ds \geq \frac{1}{t}\exp(2c^2(t-2t_0)) \int_0^t \exp\left(-\frac{2c^2(t_0-t)^2}{t-s}\right) ds$$

where the last integral is $O(t)$, so that

$$\begin{aligned} EX^2(t,x) &= \int_0^t \frac{1}{t-s}\exp\left(-\frac{\|x-cs\|^2}{t-s}\right) ds \\ &\approx O(\exp(2c^2(t-2t_0))) \to \infty \ \text{ as } t \to \infty. \end{aligned}$$

□

For the continuity of the paths, we have the following:

Theorem 3.4.2. *If $d=1$, the process $X(t,x)$ given by (3.4.10) has continuous paths in the space L_2.*

Proof. The key observation is that the variance is infinite only at the point ct which has zero Lebesgue measure and its expression is the same for the other points of the line, so that:

$$
\begin{aligned}
\mathbb{E}\left\|X_t\right\|_{L_2}^2 &= \int_{\mathbb{R}} \mathbb{E}X^2(t,x)\,dx \\
&= \int_{\mathbb{R}\setminus\{ct\}} \mathbb{E}X^2(t,x)\,dx \\
&= \int_{\mathbb{R}}\int_0^t \frac{1}{t-s}\exp\left(-\frac{\left\|x-ct\right\|^2}{t-s}\right) ds\,dx \\
&= \int_0^t\int_{\mathbb{R}} \frac{1}{t-s}\exp\left(-\frac{\left\|x-ct\right\|^2}{t-s}\right) dx\,ds \\
&= \int_0^t \frac{C\,ds}{(t-s)^{1/2}} = Ct^{1/2}
\end{aligned}
$$

□

When $d \geq 2$, a distinction has to be made according to $\overline{x} = \overline{c}t_0$ for some t_0 or $\overline{x} \neq \overline{c}t_0$ for all t_0, that is if the perturbation hits or not the point $\overline{x}$.

Theorem 3.4.3. *Let $d \geq 2$ and let $X(t,x)$ be given by (3.4.10), then for any given vector $\overline{c} \in \mathbb{R}^d$ and a point $\overline{x} = \overline{c}t_0$, the variance of $X(t,x) = X(t,\overline{c}t_0)$ is given by:*

$$
Var\ X(t,x) = \begin{cases} \int_0^t \frac{1}{t-s}\exp\left(-\frac{c^2(t_0-t)^2}{t-s}\right) ds < K < \infty & t < t_0 \\ \infty & t = t_0 \\ \textit{finite, but } \to \infty \textit{ as } t \to \infty & t > t_0 \end{cases}
$$

If the point does not lie in the line of the perturbation, i. e. $\overline{x} \neq \overline{c}t$, $\forall t \geq 0$, the variance is bounded by a constant.

Proof. Since the analysis is similar to the case $d = 1$, it will be assumed first , that the point lies in the line of the perturbation and so there are three cases:

- $t < t_0$, when the perturbation still does not reach the point:

$$\begin{aligned}\mathbb{E}X^2(t,\overline{c}t_0) &= \int_0^t \frac{1}{(t-s)^d}\exp\left(-\frac{2\|\overline{c}\|^2(t_0-s)^2}{t-s}\right)ds \\ &\leq \int_0^t \frac{1}{(t-s)^d}\exp\left(-\frac{M}{t-s}\right)ds \\ &\leq \int_0^{t_0} \frac{1}{(t-s)^d}\exp\left(-\frac{M}{t-s}\right)ds < \infty\end{aligned}$$

  in the second line $M = 2\left\|\overline{c}\right\|^2(t-t_0)^2$, and the last integral is finite for any $d \geq 2$.

- $t = t_0$, when the perturbation is at the point $\overline{x}$, we get the following

$$\mathbb{E}X^2(t_0,\overline{c}t_0) = \int_0^{t_0} \frac{1}{(t_0-s)^d}\exp\left(-\left\|\overline{c}\right\|^2(t_0-s)\right)ds = \infty$$

- When, $t > t_0$, the perturbation has passed the point:

$$\mathbb{E}X^2(t,\overline{c}t_0) \leq \exp(2\left\|\overline{c}\right\|^2(t-t_0))\int_0^t \frac{1}{(t-s)^d}\exp\left(-\frac{K`}{t-s}\right)ds < \infty$$

  And the same reasoning as in the case $d = 1$ shows:

$$\mathbb{E}X^2(t,\overline{c}t_0) \to \infty \quad \text{as } t \to \infty$$

- So far, the situation is identical to the one-dimensional case, except when the point x does not lie in the line $\overline{c}t$, this makes a difference in higher dimension, here we can assume $0 < \text{dist}(\overline{x},\overline{c}t) = K < \infty$, so:

$$\mathbb{E}X^2(t,\overline{x}) \leq \int_0^t \frac{1}{(t-s)^d}\exp\left(-\frac{K^2}{t-s}\right)ds < \infty$$

  uniformly on t, so that the variance does not go to infinity.

□

For the continuity of the paths, we will write $\overline{x} = (x_1, x_2)$, and assume without loss of generality that $\overline{c} = (c_1, 0)$, i.e. the perturbation moves along the x_1 axis.

Theorem 3.4.4. *When $d = 2$, the process 3.4.10 has continuous paths in the space L_2 endowed with the measure $d\mu = x_2^2 \exp(-x_1^2)\, dx_1 dx_2$.*

Proof. Similarly to the proof of 3.4.2, the main observation is that the variance is infinite only the set of zero Lebesgue measure $\{\overline{c}t\}$, and so:

$$\begin{aligned}
\mathbb{E}\left\|X_t\right\|^2_{L_2(\mu)} &= \int_{\mathbb{R}^2} \mathbb{E}X^2(t,x)\, d\mu \\
&= \int_{\mathbb{R}^2 \setminus \{\overline{c}t\}} \mathbb{E}X^2(t,x)\, d\mu \\
&= \int_{\mathbb{R}} \int_0^t \frac{1}{(t-s)^2} \exp\left(-\frac{2\left\|x-\overline{c}t\right\|^2}{t-s}\right) ds\, d\mu
\end{aligned}$$

for the particular choice of $\overline{c}$, we have $\left\|x - \overline{c}t\right\|^2 \geq x_2^2$ and so:

$$\begin{aligned}
\mathbb{E}\left\|X_t\right\|^2_{L_2(\mu)} &\leq \int_{\mathbb{R}^2} \int_0^t \frac{1}{(t-s)^2} \exp\left(-\frac{x_2^2}{t-s}\right) ds d\mu \\
&= C \int_{\mathbb{R}} \frac{1}{x_2^2} \exp\left(-\frac{x_2^2}{t}\right) \exp(-x_1^2) x_2^2\, dx_1 dx_2 \\
&= C \int_{\mathbb{R}} \exp(-x_1^2)\, dx_1 \cdot \int_{\mathbb{R}} \exp\left(-\frac{x_2^2}{t}\right) dx_2 \\
&\leq C_1 (t)^{1/2}
\end{aligned}$$

which shows the Hölder continuity of the paths. □

3.4.1 A randomly moving catalyst

In the next section we will consider the case of a catalyst given by a measure-valued stochastic process. In this section we consider the simple case of a randomly moving atom $\mu_t = \delta_{B_t}$ where B_t is a Brownian motion in $\mathbb{R}^d$ starting at the origin. In the case of a random catalyst there are two processes to consider. The first is the solution of the perturbed heat equation conditioned on a given realization of the catalyst process - this is called the *quenched* case. The second is the process with probability law obtained by averaging the laws of the perturbed heat equation with respect to the law of the catalytic process - the is called the *annealed* case.

We will now determine the behavior of the annealed process and show that it also depends on the dimension as expressed in the following:

Theorem 3.4.5. *Let B_t be a Brownian motion and let $X_{\delta_B(\cdot)}(t,x)$ be given by (3.4.10), with $\delta_{B(t)}$ instead of δ_{ct}, the annealed variance of $X_{\delta_B(\cdot)}(t,x)$ is given by:*

$$E\left[Var\ X_{\delta_B(\cdot)}(t,x)\right] = \begin{cases} 1/4 & d=1, \quad x=0 \\ <\infty & d=1, \quad x\neq 0 \\ \infty & d\geq 2 \end{cases}$$

Proof. Let us assume $d=1$ and by simplicity that $x=0$, then the second moments are computed as follows:

$$\begin{aligned} \mathbb{E}X^2(t,0) &= \mathbb{E}\int_0^t \frac{1}{2\pi(t-s)}\exp\left(-\frac{\|B(s)\|^2}{(t-s)}\right) ds \\ &= \int_0^t \frac{1}{2\pi(t-s)}\mathbb{E}\exp\left(-\frac{B^2(s)}{t-s}\right) ds \end{aligned}$$

The expectation in the last integral can be computed using the Laplace transform M_X of the χ_1^2 distribution as follows:

$$\begin{aligned}
\mathbb{E}\exp\left(-\frac{B^2(s)}{t-s}\right) &= \frac{1}{2\pi}\mathbb{E}\exp\left(-\frac{s}{t-s}\frac{B^2(s)}{s}\right) \\
&= \frac{1}{2\pi}M_X\left(-\frac{s}{t-s}\right) \\
&= \frac{1}{2\pi}\left(\frac{t-s}{t+s}\right)^{1/2}
\end{aligned}$$

So, a trigonometric substitution shows:

$$\mathbb{E}X^2(t,0) = \frac{1}{2\pi}\int_0^t \frac{1}{(t^2-s^2)^{1/2}}ds = \frac{1}{4}.$$

For $x \neq 0$ and $d = 1$, using a spatial shift we have

$$\begin{aligned}
\mathbb{E}X^2(t,x) = \mathbb{E}\int_0^t \frac{1}{2\pi(t-s)}\exp\left(-\frac{(B(s)-x)^2}{(t-s)}\right)ds \\
&= \int_0^t \frac{1}{2\pi(t-s)(2\pi s)^{1/2}}\int e^{-\frac{(y-x)^2}{t-s}}e^{-\frac{y^2}{2s}}dyds \\
&\leq \int_0^t \frac{1}{2\pi(t-s)(2\pi s)^{1/2}}\int e^{-\frac{(y-x)^2}{t-s}}dyds \\
&\leq \text{const}\int_0^t \frac{1}{(s(t-s))^{1/2}}ds < \infty.
\end{aligned}$$

When $d \geq 2$ and the perturbation is $\delta_{B(t)}$ as before, then $\frac{\|B(s)^2\|}{s}$ is distributed as χ_d^2 , and its Laplace transform is:

$$M_x(t) = \left[\frac{1}{1-2t}\right]^{d/2}$$

So:

$$\mathbb{E}X^2(t,x) = \int_0^t \frac{ds}{(t^2-s^2)^{d/2}}$$

Which is infinite when $d = 2, 3$ at $x = 0$ and hence everywhere. $\square$

Remark 3.4.2. Note that the annealed process is not Gaussian. A similar calculation can be used to show that $E(X^4(t,x)) \neq 3(E(X^2(t,x)))^2$.

3.5 Perturbations with compact support in $\mathbb{R}^d$

The purpose of this section is to establish some properties of stochastic processes influenced by a catalyst of bounded support, which will be used in the next chapter.

We begin with the one-dimensional case.

Theorem 3.5.1. *Let $d = 1$ and μ be a mass distribution, then the solutions to (BS) always have continuous paths in $L_2(\mathbb{R})$ w.r.t Lebesgue measure dx.*

Proof.

$$\begin{aligned}
\mathbb{E}\,\|X(t) - X(s)\|_{L_2}^2 &= \int_{\mathbb{R}} \int_s^t \int_{\mathbb{R}} p^2(t-r,x,y)\mu(dy)drdx \\
&= \int_s^t \int_{\mathbb{R}} \int_{\mathbb{R}} p^2(t-r,x,y)dx\mu(dy)dr \\
&= \int_s^t \int_{\mathbb{R}} \frac{\mu(dy)dr}{(t-r)^{1/2}} \\
&= \mu(\mathbb{R}) \int_s^t \frac{dr}{(t-r)^{1/2}} \\
&= 2\,\mu(\mathbb{R})(t-s)^{1/2}
\end{aligned}$$

and the continuity follows from Proposition 2.2.1. $\square$

We now consider the behavior off the support of the catalyst.

Proposition 3.5.2. *Consider the SPDE of the form*

$$(BS) \qquad \begin{cases} dX(t,x) = \Delta X(t,x)dt + W_\mu(dx,ds) & t \geq 0, \quad x \in \mathcal{O} \\ X(0,x) = 0 & x \in \mathcal{O} \end{cases}$$

where μ is a measure with compact support. Then the solution is $C^{1,2}$ and solves the heat equation off the support of μ.

Proof. The following are the four main ingredients of the proof

Leibniz's rule: assume $f(x,t)$ and $\partial f/\partial t$ are continuous, $a(t)$ and $b(t)$ differentiable functions, then

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)dx + f(b(t),t)b'(t) - f(a(t),t)a'(t)$$

as well as the following important properties of the heat kernel:

Poisson integral: Let w be a continuous function on $\mathbb{R}^d$ with compact support, then:

$$u(x,t) = \int_{\mathbb{R}^d} p(t,x,y)w(y)dy \tag{3.5.11}$$

defines an infinitely differentiable solution of the heat equation on $\mathbb{R}^d \times [0,\infty)$.

Derivatives of the heat kernel: $\frac{\partial^{k+\ell}}{\partial t^k \partial x^\ell} p(t,x,y)$ can be written as a finite linear combination of $(x-y)^i t^{-j} e^{-\frac{|x-y|^2}{2t}}$. Hence for any $M > 0$

$$\sup_{t>0, |x-y|\geq M} \left|\frac{\partial^{k+\ell}}{\partial t^k \partial x^\ell} p(t,x,y)\right| < \infty. \tag{3.5.12}$$

The noise process: $W_\mu(ds,dy)$ restricted to $0 \leq s \leq T$ is with probability one a (Schwartz) distribution on $\mathbb{R} \times \mathbb{R}^d$ with compact support, namely, $[0,T] \times \text{supp}(\mu)$. Therefore by a basic result of Schwartz [Sch], Theorem

XXVI, $W_\mu(ds, dy)$ can a.s. be represented by the sum of a finite number of derivatives of continuous functions having their supports in an arbitrary neighborhood of $[0, T] \times \text{supp}(\mu)$.

With this, the proposition can be proven. Note first that the solution of (BS) is given by the stochastic convolution

$$\begin{aligned} X(t) &= \int_0^t S(t-s)dW_\mu(s) \\ &= \int_0^t \int_{\mathbb{R}^d} p(t-s, x, y)dW_\mu(ds, dy) \end{aligned} \tag{3.5.13}$$

If $x \notin \text{supp}(\mu)$, then a.s. we can choose the neighbourhood of $\text{supp}(\mu)$ in the Schwartz representation of W_μ so that its closure does not contain x, that is, $\text{dist}(x, \text{supp}(\mu)) \geq M > 0$.

Therefore, Leibniz's rule can be applied to (3.5.13) to yield a.s. for $x \notin\in$ $\text{supp}(\mu)$:

$$\begin{aligned} \frac{\partial X}{\partial t}(t, x) &= \int_0^t \frac{\partial}{\partial t} \int_{\mathbb{R}^d} p(t-s, x, y) W_\mu(ds, dy) + \int_{\mathbb{R}^d} p(0, x, y) X(0, dy) \\ &= \int_0^t \Delta \int_{\mathbb{R}^d} p(t-s, x, y) W_\mu(ds, dy) + \int_{\mathbb{R}^d} \delta(x-y) X(0, y) \\ &= \Delta \int_0^t \int_{\mathbb{R}^d} p(t-s, x, y) W_\mu(ds, dy) \\ &= \Delta X(t, x) \end{aligned}$$

noting that the last integral of the second line vanishes and that $p(t, x, y)$ satisfies the heat equation. All interchanges of differentiation under the integrals are justified since by (3.5.12) all integrands involved are bounded continuous function of (x, t, s). □

Remark 3.5.1. Continuing this argument and using induction it can be shown directly that $X(t,x)$ is C^∞ away from the catalyst.

We will next show that the the solution is also continuous in $L_2(\nu)$ for any with bounded density and support disjoint from the support of μ. We will use the following easy to verify inequalities valid $\forall a \in \mathbb{R},\ \alpha > 0\,, t \geq 0$:

$$\begin{aligned} \exp\left(-\frac{a}{t}\right) &\leq \left(\frac{\alpha}{ae}\right)^{\alpha} t^{\alpha} \\ \frac{1}{a}\exp\left(-\frac{a}{t}\right) &\leq \frac{e^{-1}}{a^2}\, t \qquad \text{if } \alpha = 1. \end{aligned} \tag{3.5.14}$$

Since the support of μ is assumed bounded, we can write Supp $\mu \subset \{x : m \leq \|x\| \leq M\}$ for some m, $M \geq 0$, we will see below, that any measure ν whose support is bounded away from that of μ, for example taking

$$\text{Supp } \nu \subset \{x : \|x\| \leq m/2 \text{ or } 2M \leq \|x\| \leq 3M\}\,,$$

the idea behind, is that then:

$$\forall m \geq 0: \qquad \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\mu(dy)\nu(dx)}{\|x-y\|^m} < \infty$$

Theorem 3.5.3. *Let $d = 2, 3$, then the problem (BS) has continuous paths in the space $L_2(\nu)$ where ν is a measure whose support is bounded away from that of μ according to the above definition.*

Proof. For any $\alpha > 0$:

- d=2

$$
\begin{aligned}
\mathbb{E}\,\|X(t)-X(s)\|_{L_2(\nu)}^2 &= \int_s^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} p^2(t-r,x,y)\mu(dy)\nu(dx)dr \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_s^t p^2(t-r,x,y)dr\,\mu(dy)\nu(dx) \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{\|x-y\|^2}\exp\left(-\frac{\|x-y\|^2}{t-s}\right)\nu(dx)\mu(dy) \\
&\leq C(\alpha)(t-s)^\alpha \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\nu(dx)\mu(dy)}{\|x-y\|^{2+2\alpha}} \\
&= C_1(\alpha)(t-s)^\alpha
\end{aligned}
$$

- d=3

$$
\begin{aligned}
\mathbb{E}\,\|X(t)-X(s)\|_{L_2(\nu)}^2 &= \int_s^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} p^2(t-r,x,y)\mu(dy)\nu(dx)dr \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_s^t p^2(t-r,x,y)dr\,\mu(dy)\nu(dx) \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{(t-s)\,\|x-y\|^2}\exp\left(-\frac{\|x-y\|^2}{t-s}\right)\nu(dx)\mu(dy) \\
&+ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{\|x-y\|^4}\exp\left(-\frac{\|x-y\|^2}{t-s}\right)\nu(dx)\mu(dy)
\end{aligned}
$$

the first integral of the last equality can be bounded above using (3.5.14), with $\alpha = 1+\beta, \;\; \beta > 0$ one obtains:

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{(t-s)\,\|x-y\|^2} \exp\left(-\frac{\|x-y\|^2}{t-s}\right) \nu(dx)\mu(dy) \\
&\leq \iint\limits_{\text{Supp } \nu\times\mu} \frac{(t-s)^{1+\beta}}{(t-s)\,\|x-y\|^2} \left(\frac{1+\beta}{e\,\|x-y\|^2}\right)^{1+\beta} \nu(dx)\mu(dy) \\
&\leq C(\beta)(t-s)^{\beta} \iint\limits_{\text{Supp } \nu\times\mu} \frac{\nu(dx)\mu(dy)}{\|x-y\|^{4+2\beta}} \\
&\leq C_1(\beta)(t-s)^{\beta}.
\end{aligned}$$

For the second integral, using again (3.5.14), with $\alpha = \beta, \;\; \beta > 0$ yields:

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{\|x-y\|^4} \exp\left(-\frac{\|x-y\|^2}{t-s}\right) \nu(dx)\mu(dy) \leq \\
&\leq \iint\limits_{\text{Supp } \nu\times\mu} \frac{(t-s)^{\beta}}{\|x-y\|^4} \left(\frac{\beta}{e\,\|x-y\|^2}\right)^{\beta} \nu(dx)\mu(dy) \\
&\leq C(\beta)(t-s)^{\beta} \iint\limits_{\text{Supp } (\nu\times\mu)} \frac{\nu(dx)\mu(dy)}{\|x-y\|^{4+2\beta}} \\
&\leq C_1(\beta)(t-s)^{\beta}
\end{aligned}$$

So, taking $\beta \in (0,1]$, one obtains:

$$\mathbb{E}\,\|X(t) - X(s)\|^2_{L_2(\nu)} \leq 2C_1(\beta)(t-s)^{\beta}$$

this shows the continuity of the paths.

□

Remark 3.5.2. The same procedure can be used to show the continuity of the paths in dimensions higher than three.

CHAPTER 4
The Catalytic Ornstein-Uhlenbeck Process with Super-Brownian Catalyst

4.1 Outline

The main objective of this section is to study a natural class of catalytic Ornstein-Uhlenbeck processes with a random catalyst, namely super-Brownian motion. We begin by introducing the notion of the class of affine processes that gives a unified setting in which to view Ornstein-Uhlenbeck processes, superprocesses, and the Ornstein-Uhlenbeck process with super-Brownian catalyst. We then review the framework and basic properties of super-Brownian motion which we need and introduce the Ornstein-Uhlenbeck process with super-Brownian catalyst. The main results of this section are the identification of state space and sample path continuity for both the quenched and annealed processes as well as regularity properties of the process at fixed times.

4.2 Affine Processes and Semigroups

In recent developments, Affine processes have raised a lot of interest, due to their rich mathematical structure, as well as to their wide range of applications in branching processes, Ornstein-Uhlenbeck processes and mathematical finance.

In a general form, *Affine Processes* are the class of stochastic processes for which, the logarithm of the characteristic function of its transition semigroup has the form $\langle x, \psi(t,u)\rangle + \phi(t,u)$. Important finite dimensional examples of such processes are following SDE's:

- **Ornstein-Uhlenbeck process:** $z(t)$ satisfies the Langevin type equation

$$dz(t) = (b - \beta z(t))dt + \sqrt{2}a dB(t)$$

This is also known as a Vasieck model for interest rates in mathematical finance.

- **Continuous state branching immigration process:** $y(t) \geq 0$ satisfies

$$dy(t) = (b - \beta z(t))dt + \sigma\sqrt{2y(t)}dB(t)$$

with binary branching rate σ^2, linear decay rate β and immigration rate b. This is also called the Cox-Ingersoll-Ross model in mathematical finance.

- **General affine diffusion process in $\mathbb{R}^2$:** $r(t) = a_1 y(t) + a_2 z(t) + b$ where

$$\begin{aligned} dy(t) &= (b_1 - \beta_{11} y(t))dt \\ &\quad + \sigma_{11}\sqrt{2y(t)}dB_1(t) + \sigma_{12}\sqrt{2y(t)}dB_2(t) \\ dz(t) &= (b_2 - \beta_{21} y(t) - \beta_{22} z(t))dt \\ &\quad + \sigma_{21}\sqrt{2y(t)}dB_1(t) + \sigma_{22}\sqrt{2y(t)}dB_2(t) + \sqrt{2}a dB_0(t) \end{aligned}$$

It can be seen that the general affine semigroup can be constructed as the convolution of a *homogeneous* semigroup (one in which $\phi = 0$) with a *skew convolution semigroup* which corresponds to the constant term $\phi(t, u)$.

Definition 4.2.1. A transition semigroup $(Q(t)_{t\geq 0})$ with state space D is called a *homogeneous affine semigroup (HA-semigroup)* if for each $t \geq 0$ there exists a continuous complex-valued function $\psi(t,\cdot) := (\psi_1(t,\cdot), \psi_2(t,\cdot))$ on U such that:

$$\int_D \exp\{\langle u, \xi\rangle\} Q(t,x,d\xi) = \exp\{\langle x, \psi(t,u)\rangle\}, \qquad x \in D, u \in U. \tag{4.2.1}$$

The HA-semigroup $(Q(t)_{t\geq 0})$ given above is called *regular* if it is stochastically continuous and the derivative $\psi_t'(0,u)$ exists for all $u \in U$ and is continuous at $u = 0$.

Definition 4.2.2. A transition semigroup $(P(t))_{t\geq 0}$ on D is called a (general) affine semigroup with the HA-semigroup $(Q(t))_{t\geq 0}$ if its characteristic function has the representation

$$\int_D \exp\{\langle u, \xi\rangle\} P(t,x,d\xi) = \exp\{\langle x, \psi(t,u)\rangle + \phi(t,u)\}, \quad x \in D, u \in U$$

where ψ is given by (4.2.1) and $\phi(t,\cdot)$ is a continuous function on U satisfying $\phi(t,0) = 0$.

Further properties of the affine processes can be found in ([Du 03] and [Da 05]). We will see that the class of catalytic Ornstein-Uhlenbeck processes we introduce in this chapter forms a new class of infinite dimensional affine processes.

4.3 A brief review on super-processes and their properties

Given a measure μ on $\mathbb{R}^d$ and $f \in \mathcal{B}(\mathbb{R}^d)$, denote by $\langle \mu, f\rangle := \int_{\mathbb{R}^d} f \, d\mu$, let $M_F(\mathbb{R}^d)$ be the set of finite measures on $\mathbb{R}^d$ and let $C_0(\mathbb{R}^d)_+$ denote the continuous and positive functions, we also define:

$$C_p(\mathbb{R}^d) = \left\{f \in C(\mathbb{R}^d) : \|f(x)\cdot |x|^p\|_\infty < \infty, p > 0\right\}$$
$$M_p(\mathbb{R}^d) = \left\{\mu \in M(\mathbb{R}^d) : (1+|x|^p)^{-1}\, d\mu(x) \text{ is a finite measure}\right\}$$

Definition 4.3.1. The (α, d, β)-superprocess Z_t is the measure-valued process, whose Laplace functional is given by:

$$\mathbb{E}_\mu[\exp(-\langle\psi, Z_t\rangle)] = \exp[-\langle \mathrm{U}_t\psi, \mu\rangle] \qquad \mu \in M_p(\mathbb{R}^d),\ \psi \in C_p(\mathbb{R}^d)_+$$

where $\mu = Z_0$, and U_t is the nonlinear continuous semigroup given by the the mild solution of the evolution equation:

$$\begin{aligned} \dot{u}(t) &= \Delta_\alpha u(t) - u(t)^{1+\beta}, \qquad 0 < \alpha \leq 2, \quad 0 < \beta \leq 1 \\ u(0) &= \psi, \qquad \psi \in D(\Delta_\alpha)_+. \end{aligned} \tag{4.3.2}$$

where Δ_α is the generator of the α-symmetric stable process. That is, $u(t) := \mathrm{U}_t\psi$ satisfies the non-linear integral equation:

$$u(t) = \mathrm{U}_t\psi - \int_0^t \mathrm{U}_{t-s}(u^{1+\beta}(s))ds.$$

It has been shown in [Wat 68] that there exists such a process which is a finite measure-valued Markov process with sample paths in $D(\mathbb{R}_+, M_p(\mathbb{R}^d))$. The special case $\alpha = 2,\ \beta = 1$ is called super-Brownian motion and this has sample paths in $C(\mathbb{R}_+, M_p(\mathbb{R}^d))$

Consider the following differential operator on $M_p(\mathbb{R}^d)$:

$$LF(\mu) = \frac{1}{2}\int_{\mathbb{R}^d} \mu(dx)\frac{\delta^2 F}{\delta\mu(x)^2} + \int_{\mathbb{R}^d} \mu(dx)\Delta\left(\frac{\delta F}{\delta\mu}\right)(x) \tag{4.3.3}$$

Here the differentiation of F is defined by

$$\frac{\delta F}{\delta\mu(x)} = \lim_{\epsilon\downarrow 0}(F(\mu + \epsilon\delta_x) - F(\mu))/\epsilon$$

where δ_x denotes the Dirac measure at x. The domain $\mathcal{D}(L)$ of L will be chosen a class containing such functions $F(\mu) = f(\langle \mu, \phi_1 \rangle, \dots, \langle \mu, \phi_1 \rangle)$ with smooth functions $\phi_1, \dots, \phi_n$ defined on $\mathbb{R}^d$ having compact support and a bounded smooth function f on $\mathbb{R}^d$. As usual $\langle \mu, \phi \rangle := \int \phi(x)\mu(dx)$.

The super-Brownian motion is also characterized as the unique solution to the martingale problem given by $(L, \mathcal{D}(L))$. The process is defined on the probability space $(\Omega, \mathcal{F}, P_\mu, \{Z_t\}_{t \geq 0})$ and

$$P_\mu(Z(0) = \mu) = 1, \quad P_\mu(Z \in C([0, \infty), M_p(\mathbb{R}^d))) = 1.$$

4.3.1 The super-Brownian motion in $\mathbb{R}$

We now recall results of Konno and Shiga [Ksh 88] as applied to super-Brownian motion in $\mathbb{R}$.

Theorem 4.3.1. *(Konno and Shiga 88) Let $(\Omega, \mathcal{F}, P_\mu, X_t)$ be the super-Brownian motion taking values in $\mathcal{M}_p(R)$, $p > 1$ governed by L. Then for every $\mu \in \mathcal{M}_p$:*

(i) $\mathbb{P}_\mu$ almost surely, $Z_t(dx)$ is absolutely continuous with respect to dx for all $t > 0$ and its density $Z(t, \cdot)$ is an E-valued continuous function in $t > 0$. Here $E = \{f \in C(\mathbb{R}) : \left\|(1 + |x|^2)^{-p/2} f(x)\right\| < \infty \, \forall \, p > 0\}$

(ii) for $\delta > 0$ there exists an $\mathbb{H}_{-(\frac{1}{2}+\delta)}(\mathbb{R})$-valued standard Wiener process W_t (see Rem. 2.4.2) defined on an extension of the probability space $(\Omega, \mathcal{F}, P_\mu)$ such that:

$$Z_t(x) - Z_{t_0}(x) = \int_{t_0}^{t} \sqrt{Z_s(x)} W(ds, dx) + \int_{t_0}^{t} \Delta Z_s(x) ds \tag{4.3.4}$$

holds in the distribution sense for every $0 < t_0 < t$, P_μ*-almost surely. More precisely, (4.3.4) means that for every* $\phi \in \mathcal{S}(R)$

$$\langle Z_t, \phi\rangle - \langle Z_{t_0}, \phi\rangle = \int_{t_0}^{t}\int_R \sqrt{Z_s(x)}\phi(x)W(ds, dx) + \int_{t_0}^{t} \langle X_s(x), \Delta\phi\rangle\, ds$$

4.3.2 Some properties of super-Brownian motion in $\mathbb{R}^d$

In this subsection we review some basic properties of super-Brownian motion (SBM) $Z(t)$. The compact support property, that is, the closed support $S(Z_t)$ is compact for $t > 0$ if $S(Z_0)$, is compact was proved by Iscoe [Is 88]. Further results on the support process were obtained in [Da 89]. Here we summarize the results needed for the analysis of the catalytic OU process in a super-Brownian catalyst.

Let

$$h(t) = (t(\log t^{-1}) \vee 1))^{1/2}$$

and for a given $\mu \in M_F(\mathbb{R}^d)$, let P_μ denote the law of Z on $C([0,\infty), M_F)$.

Theorem 4.3.2. *([Da 89]) If* $\mu \in M_F(\mathbb{R}^d)$*, then for* P_μ*-a.s. for each* $c > 0$ *there is a* $\delta(\omega, c) > 0$ *such that if* $s, t \geq 0$ *satisfy* $0 < t - s < \delta(\omega, c)$ *then:*

$$S(Z_t) \subset S(Z_s)^{ch(t-s)}$$

where, given the set $A \subset \mathbb{R}^d$*, we define* $A^r \doteq \{x : d(x, A) \leq r\}$.

If K_1 and K_2 are non-empty compact subsets of $\mathbb{R}^d$ let

$$\rho_1(K_1, K_2) := \min[\sup_{x \in K_1} d(x, K_2), 1] \text{ and}$$

$$\rho(K_1, K_2) := \max(\rho_1(K_1, K_2), \rho_1(K_2, K_1))$$

$$\rho(K_1, \phi) = 1$$

Let $\mathbb{K}(\mathbb{R}^d)$ denote the set of non-empty compact subsets of $\mathbb{R}^d$ with metric ρ.

Theorem 4.3.3. *([Da 89])* $\{S(Z_t) : t > 0\}$ *is a right continuous process taking values in* $(\mathbb{K}(\mathbb{R}^d), \rho)$.

Remark 4.3.1. The latter results show that $S(Z_t)$ propagates with finite speed.

It is also well known (see, for example [Da 93]) that Z_t, $t > 0$ is a.s. a singular measure if $d = 2$ and if $d \geq 3$, then

$$\text{Hausdorff dim } S(Z_t) = 2 \qquad \text{for all } t \geq 0, \text{ a.s.}$$

4.3.3 Second moment measures of SBM

In this section we evaluate the second moment measures of SBM, $\mathbb{E}\,(Z_1(dx)Z_2(dy))$. These will be needed below in order to compute $\mathbb{E}[\|Z_t\|_{L_2}^4]$ in the study of the annealed catalytic OU process.

To accomplish this, note first that given any two random measures Z_1 and Z_2, then it is easy to verify that:

$$\mathbb{E}(\langle Z_1, A\rangle), \mathbb{E}(\langle Z_2, A\rangle), \quad A \in \mathcal{B}(\mathbb{R}^d)$$

$$\mathbb{E}(\langle Z_1, A\rangle)\,\langle Z_2, B\rangle), \quad A, B \in \mathcal{B}(\mathbb{R}^d)$$

are well defined measures which will be called the first and second moment measures. Similarly one can define the n-th moment measure of n given random variables denoted by:

$$\mathbb{E}\left(Z_1(dx)Z_2(dx_2), \cdots, Z_n(dx_n)\right).$$

Recall also that if a given measure μ is related to the measure dx in such a way that for some function $f \in bC(\mathbb{R})$ and for every Borel set $B \in \mathcal{B}$ we have:

$$\mu(B) = \int_B f(x)\, dx$$

then, necessarily:

$$\mu(dx) = f(x)dx$$

We now consider SBM Z_t with $Z_0 = \mu$. Then for $\lambda \in \mathbb{R}_+$ and $\phi \in C_+(\mathbb{R}^d)$:

$$\mathbb{E}_\mu \left[\exp(-\lambda \langle \phi, Z_t \rangle)\right] = \exp\left[- \langle u(\lambda, t), \mu \rangle\right] =: \exp(-F(\lambda))$$

where, $u(\lambda, t)$ satisfies the equation:

$$\begin{aligned} \dot{u}(t) &= \Delta u(t) - u^2(t) \\ u(\lambda, 0) &= \lambda\phi \end{aligned} \tag{4.3.5}$$

Then the first moment of the r. v. is computed according to

$$\mathbb{E}(\int \phi(x) Z_t(dx)) = \left[\frac{\partial e^{F(\lambda)}}{\partial \lambda}\right]_{\lambda=0} = F'(0)$$

since $F(0) = 0$. $F(\lambda)$ will be developed as a Taylor series below. First rewrite (4.3.5) as a Volterra integral equation of the second kind, namely:

$$u(t) + \int_0^t T_{t-s}(u^2(s))\, ds = T_t(\lambda\phi)$$

whose solution is given by its Neumann series:

$$u(t) = T_t(\lambda\phi) + \sum_{k=1}^{n}(-1)^k \mathbb{T}^k(T_t(\lambda\phi)) \tag{4.3.6}$$

where the operators T_t and $\mathbb{T}$ are defined by:

$$\begin{aligned} T_t(\lambda\phi(x)) &= \lambda \int_{\mathbb{R}} p(t,x,y)\phi(y)\, dy \\ \mathbb{T}(T_t(\lambda\phi)) &= -\int_0^t T_{t-s}[(T_s(\lambda\phi))^2]\, ds \\ &= -\lambda^2 \int_0^t T_{t-s}[(T_s\phi)^2]\, ds \\ &= -\lambda^2 \int_0^t \int_{\mathbb{R}} p(t-s,x,y) \left(\int_{\mathbb{R}} p(s,y,z)\phi(z)\, dz\right)^2 dy\, ds. \end{aligned} \tag{4.3.7}$$

We obtain $F(\lambda) = \sum_{n=1}^{\infty}(-1)^{n+1} c_n \lambda^n$, and:

$$\begin{aligned} c_1 &= \int_{\mathbb{R}} T_t\phi\, \mu(dx) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} p(t,x,y)\phi(y)\, dy\, \mu(dx) \end{aligned} \tag{4.3.8}$$

$$\begin{aligned} c_2 &= \int_{\mathbb{R}}\int_0^t \int_{\mathbb{R}} p(t-s,x,y) \left(\int_{\mathbb{R}} p(s,y,z)\phi(z)\, dz\right)^2 dy\, ds\, \mu(dx) \\ &= \int_0^t \int_{\mathbb{R}^4} p(t-s,x,y)p(s,y,z_1)p(s,y,z_1)\phi(z_1)\phi(z_2)\, dz_1\, dz_2\, dy\, \mu(dx)\, ds \end{aligned} \tag{4.3.9}$$

$$c_4 = \int_{\mathbb{R}} \int_0^t p(t-s,x,w) \left[\int_0^s \int_{\mathbb{R}} p(s-s_1,w,y) \left(\int_{\mathbb{R}} p(s,y,z)\phi(z)\,dz \right)^2 dy\,ds_1 \right]^2 ds\,dw\mu(dx) \tag{4.3.10}$$

and so on.

Remark 4.3.2. The above procedure can also be used for any $\alpha \in (0,2], \beta = 1$, any dimension $d \geq 1$ and any C_0 semigroup S_t.

Taking now $d = 1$, $Z_0 = dx$, a given Borel set $A \in \mathcal{B}(\mathbb{R})$ and $\phi = \mathbb{I}_A$, we obtain from (4.3.8):

$$\begin{aligned} \mathbb{E}\langle A, Z_t \rangle &= \int_{\mathbb{R}} \int_{\mathbb{R}} p(t,x,y)\phi(y) dy\, dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} p(t,x,y)\, dx\, \mathbb{I}_A(y)\, dy \\ &= \int_A dy \end{aligned}$$

i.e. $\mathbb{E}\{Z_t(dx)\}$ is the Lebesgue measure.

The covariance measure $\mathbb{E}(Z_t(dx_1)Z_t(dx_2))$ can now be computed applying the same procedure to $\phi = \lambda_1 \mathbb{I}_{A_1} + \lambda_2 \mathbb{I}_{A_2}$ in which case we conclude that $-\langle u(t), \mu \rangle = F(\lambda_1, \lambda_2)$ is again a Fourier series in λ_1, and λ_2, with constant coefficient equal to zero, and only the coefficient of $\lambda_1\lambda_2$ has to be computed:

$$\begin{aligned}
\mathbb{T}(T_t\phi) &= -\int_0^t T_{t-s}[(T_s\phi)^2]\,ds \\
&= -\int_0^t T_{t-s}[(T_s(\lambda_1\mathbb{I}_{A_1} + \lambda_2\mathbb{I}_{A_2}))^2]\,ds \\
&= -\int_0^t T_{t-s}[\lambda_1^2(T_s\mathbb{I}_{A_1})^2 + 2\lambda_1\lambda_2 T_s\mathbb{I}_{A_1}T_s\mathbb{I}_{A_2} + \lambda_2^2(T_s\mathbb{I}_{A_2})^2]\,ds \qquad (4.3.11)\\
&= -\lambda_1^2\int_0^t T_{t-s}[(T_s\mathbb{I}_{A_1})^2]\,ds - 2\lambda_1\lambda_2\int_0^t T_{t-s}[T_s\mathbb{I}_{A_1}\cdot T_s\mathbb{I}_{A_2}]ds \\
&\quad - \lambda_2^2\int_0^t T_{t-s}[(T_s\mathbb{I}_{A_2})^2]\,ds
\end{aligned}$$

So:

$$\begin{aligned}
\mathbb{E}(Z_t(A_1)Z_t(A_2)) &= \frac{1}{2}\left[\frac{\partial^2 F(\lambda_1,\lambda_2)}{\partial\lambda_1\partial\lambda_2}\right]_{\lambda_1=\lambda_2=0} \\
&= \int_0^t\int_{\mathbb{R}} T_{t-s}[T_s\mathbb{I}_{A_1}\cdot T_s\mathbb{I}_{A_2}]\,\mu(dx)\,ds.
\end{aligned} \qquad (4.3.12)$$

In order to obtain the density for the measure $\mathbb{E}(Z_t(dx_1)Z_t(dx_2))$, we write in detail the last integral as follows:

$$\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} p(t-s,x,y)\left(\int_{A_1} p(s,y,z_1)dz_1\cdot\int_{A_2} p(s,y,dz_2)dz_2\right)\mu(dx)\,ds\,dy$$

applying Fubini yields:

$$\int_{A_1}\int_{A_2}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} p(t-s,x,y)p(s,y,z_1)p(s,y,z_2)\,dy\,\mu(dx)\,ds\,dz_1\,dz_2$$

from which, we conclude that $\mathbb{E}(Z_t(dx_1)Z_t(dx_2))$ has the following density w.r.t. Lebesgue measure:

$$\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} p(t-s,x,y)p(s,y,z_1)p(s,y,z_2)\,dy\,\mu(dx)\,ds \qquad (4.3.13)$$

assuming $\mu(dx)$ is the Lebesgue measure in $\mathbb{R}$ and $dx_1 \cdot dx_2$ is the Lebesgue measure in $\mathbb{R}^2$ one obtains the second moment measure $\mathcal{M}_t(dx_1 dx_2)$ defined as:

$$\mathcal{M}_t(dx_1 dx_2) \doteq \left(\int_0^t p(2s, x_1, x_2) ds \right) \cdot dx_1 \, dx_2 \tag{4.3.14}$$

For the case $t_1 \neq t_2$, assume $t_1 < t_2$ then:

$$\begin{aligned}
&\mathbb{E}\left[\int \phi_1 Z_{t_1} \cdot \int \phi_2 Z_{t_2} \right] = \\
&= \mathbb{E}\left[\mathbb{E}\left[\int \phi_1 Z_{t_1} \cdot \int \phi_2 Z_{t_2} \right] | \sigma(Z_r) : 0 \leq r \leq t_1 \right] \\
&= \mathbb{E}\left[\int \phi_1 Z_{t_1} \cdot \mathbb{E}\left[\int \phi_2 Z_{t_2} | \mathcal{F}_{Z_{t_1}} \right] \right] \\
&= \mathbb{E}\left[\int \phi_1 Z_{t_1} \cdot T_{t_2 - t_1} \left[\int \phi_2 Z_{t_1} \right] \right] \\
&= \mathbb{E}\left[\int \phi_1 Z_{t_1} \cdot \int T_{t_2 - t_1}(\phi_2) Z_{t_1} \right] \\
&= \mathbb{E}[\langle \phi_1, Z_{t_1} \rangle \langle T_{t_2 - t_1}(\phi_2), Z_{t_1} \rangle]
\end{aligned}$$

so, replacing in (4.3.12) $\mathbb{I}_{A_2}$ by $T_{t_2 - t_1}(\mathbb{I}_{A_2})$ and performing the same analysis, we can define following measure on $\mathbb{R}^2$

$$\mathcal{M}_{t_1 t_2}(dx_1 dx_2) = \left(\int_0^{t_1} p(2s + t_2 - t_1, x_1, x_2) ds \right) \cdot dx_1 dx_2$$

notice that if $Z_0 = \delta_0(x)$, then 4.3.13 yields:

$$\mathcal{M}_{t_1 t_2}(dx_1 dx_2) = \left(\int_0^{t_1} \int_{\mathbb{R}} p(t_1 - s, 0, y) p(s, y, x_1) p(s, y, x_2) \, dy \, ds \right) \cdot dx_1 dx_2$$

Summarizing, we have proven the following:

Lemma 4.3.4. *Denote by $dx, dx_1 dx_2$ the Lebesgue measures on $\mathbb{R}$ and $\mathbb{R}^2$, the one dimensional super-Browinian motion Z_t induces the following first and second moment measures:*

(i) if $Z_0 = dx$, $\mathcal{M}_t(dx) = dx$ the Lebesgue measure.

(ii) if $Z_0 = \delta_0(x)$, then:

$$\mathcal{M}_t(dx) = p(t, x) \cdot dx$$

(iii) if $Z_0 = dx$:

$$\mathcal{M}_t(dx_1 dx_2) = \left(\int_0^t p(2s, x_1, x_2) ds \right) \cdot \, dx_1 \, dx_2 \qquad (4.3.15)$$

(iv) if $Z_0 = \delta_0(x)$:

$$\mathcal{M}_t(dx_1 dx_2) = \left(\int_0^t \int_{\mathbb{R}} p(t - s, 0, y) p(2s, x_1, x_2) \, dy \, ds \right) \cdot \, dx_1 \, dx_2 \qquad (4.3.16)$$

(v) if $t_1 < t_2$ and $Z_0 = dx$:

$$\mathcal{M}_{t_1 t_2}(dx_1 dx_2) = \left(\int_0^{t_1} p(2s + t_2 - t_1, x_1, x_2) ds \right) \cdot \, dx_1 \, dx_2 \qquad (4.3.17)$$

(vi) if $t_1 < t_2$ and $Z_0 = \delta_0(x)$:

$$\mathcal{M}_{t_1 t_2}(dx_1 dx_2) = \left(\int_0^{t_1} \int_{\mathbb{R}} p(t_1 - s, 0, y) p(s, y, x_1) p(s, y, x_2) \, dy ds \right) \cdot \, dx_1 \, dx_2 \qquad (4.3.18)$$

4.4 Catalytic OU with the $(\alpha, \mathbf{d}, \beta)$-superprocess as catalyst

The main object of this section is the catalytic OU process given by the solution of

$$dX(t, x) = \Delta X(t, x) dt + W_{Z_t}(dt, dx), \quad X_0(t, x) \equiv 0 \qquad (4.4.19)$$

where Z_t is the (α, d, β)-superprocess, in the following discussion, we will assume that the catalyst and the white noise are independent processes.

Proposition 4.4.1. *The variance of the process X(t) given by (4.4.19), is:*

$$\mathbb{E}X^2(t,x) = \int_0^t \int_{\mathbb{R}^d} p^2(t-s,x,u) Z_s(du)ds$$

and its covariance by:

$$\mathbb{E}X(t,x)X(t,y) = \int_0^t \int_{\mathbb{R}^d} p(t-s,x,u)p(t-s,y,u)Z_s(du)ds$$

Proof. In order to compute the covariance of X(t); recall that the solution of (3.1.1) is given by the stochastic convolution:

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} p(t-s,x,u) W_{Z_s}(ds,du)$$

From which, the covariance is computed as:

$$\mathbb{E}X^2(t,x) = \int_0^t \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} p(t-r,x,u)p(t-s,x,w)\mathbb{E}[W_{Z_r}(dr,du)W_{Z_s}(ds,dw)]$$

The <u>covariance measure</u> is defined by:

$$\begin{aligned} \mathrm{Cov}_{W_Z}(dr,ds;du,dw) &\doteq \mathbb{E}[W_{Z_r}(dr,du)W_{Z_s}(ds,dw)] \\ &= \delta_s(r)\ dr\ \delta_u(w)\ Z_r(dw) \end{aligned}$$

The second equality is due to the fact that W_{Z_r} is a white noise perturbation and has the property of independent increments in time and space. □

We will compute now the characteristic function $\mathbb{E}[\exp(i\langle\phi, X_t\rangle)]$, where by definition:

$$\langle\phi, X_t\rangle \doteq \int_{\mathbb{R}^d} X(t,x)\phi(x)\,\mathrm{d}x, \qquad \phi\in C(\mathbb{R}^d)$$

which is well-defined since X_t is a random field. For this, we need the following definitions and properties, for further details see [Is 86].

Definition 4.4.1. Given the (α, d, β)-superprocess Z_t, we define the weighted occupation time process Y_t by

$$<\psi, Y_t> = \int_0^t <\psi, Z_s> ds \quad \psi\in C_p(\mathbb{R}^d).$$

Remark 4.4.1. The definition coincides with the intuitive interpretation of Y_t as the measure-valued process satisfying:

$$Y_t(B) = \int_0^t Z_s(B)ds, \quad \text{for } B\in\mathcal{B}(\mathbb{R}^d)$$

Theorem 4.4.2. *It can be shown ([Is 86]) that, given $\mu\in M_p(\mathbb{R}^d)$ and $\phi,\psi\in C_p(\mathbb{R}^d)_+$, and $p<d+\alpha$ then the joint process $[Z_t, Y_t]$ has the following Laplace functional:*

$$E_\mu[\exp(-<\psi, Z_t> - <\phi, Y_t>)] = \exp[-<U_t^\phi\psi, \mu>], \;\; t\geq 0,$$

where U_t^ϕ is the strongly continuous semigroup associated with the evolution equation:

$$\begin{aligned}\dot{u}(t) &= \Delta_\alpha u(t) - u(t)^{1+\beta} + \phi\\ u(0) &= \psi\end{aligned} \tag{4.4.20}$$

A similar expression will be derived when ϕ is a function of time, but before, we need the following:

Definition 4.4.2. A deterministic non-autonomous Cauchy problem, is given by:

$$\text{(NACP)} \qquad \begin{cases} \dot{u}(t) = A(t)u(t) + f(t) & 0 \leq s \leq t \\ u(s) = x \end{cases}$$

where $A(t)$ is a linear operator which depends on t.

Similar to the autonomous case, the solution is given in terms of a two parameter family of operators $U(t,s)$, which is called the propagator or the evolution system of the problem (NACP), with the following properties:

- $U(t,t) = I, \quad U(t,r)U(r,s) = U(t,s).$
- $(t,s) \to U(t,s)$ is strongly continuous for $0 \leq s \leq t \leq T$.
- The solution of (NACP) is given by:

$$u(t) = U(t,s)x + \int_s^t U(t,r)f(r)dr \tag{4.4.21}$$

- in the autonomous case the propagator is equivalent to the semigroup $U(t,s) = T_{t-s}$.

Theorem 4.4.3. *Let $\mu \in M_p(\mathbb{R}^d)$, and $\Phi : \mathbb{R}^1_+ \to C_p(\mathbb{R}^d)_+$ be right continuous and piecewise continuous such that for each $t > 0$ there is a $k > 0$ such that $\Phi(s) \leq k \cdot (1 + |x|^p)^{-1}$ for all $s \in [0,t]$. Then*

$$\mathbb{E}_\mu \left[\exp \left(-\langle \Psi, Z_t \rangle - \int_0^t \langle \Phi(s), Z_s \rangle \, ds \right) \right] = \exp\left(-\langle U^{\Phi}_{t,t_0} \Psi, \mu \rangle \right)$$

where U^{Φ}_{t,t_0} *is the non-linear propagator generated by the operator* $Au(t) = \Delta_\alpha u(t) - u^{1+\beta}(t) + \Phi(t)$, *that is,* $u(t) = U^{\Phi}_{t,t_0}\Psi$ *satisfies the evolution equation:*

$$\begin{aligned} \dot{u}(s) &= \Delta_\alpha u(s) - u(s)^{1+\beta} + \Phi(s), \qquad t_0 \le s \le t \\ u(t_0) &= \Psi \ge 0. \end{aligned} \tag{4.4.22}$$

Proof. The existence of U^{Φ}_{t,t_0} follows from the fact that Φ is a Lipschitz perturbation of the maximal monotone non-linear operator $\Delta_\alpha(\cdot) - u^{1+\beta}(\cdot)$ (refer to [Ze 89] pp. 27).

Note that the solution $u(t)$ depends continuously on Φ, to see this, denote by V^α_t the subgroup generated by Δ_α, then the solution $u(t)$ can be written as:

$$u(t) = V_t(\Psi) + \int_0^t V^\alpha_{t-s}(\Phi(s) - u^{1+\beta}(s))\, ds$$

from which the continuity follows.

So we will assume first that Φ is a step function defined on the partition of $[0,t]$ given by $0 = s_0 < s_1 < \cdots < s_{N-1} < s_N = t$ and $\Phi(t) = \phi_k$ on $[s_{k-1}, s_k]$, $k = 1, \cdots, N$, we will denote the step function by Φ_N.

Note also that the integral:

$$\langle \Phi(t), Y_t \rangle = \int_0^t \langle \Phi(s), Z_s \rangle \, ds, \qquad \Phi : \mathbb{R}^1_+ \to C_p(\mathbb{R}^d)_+$$

can be taken a.s. in the sense of Riemann since we have enough regularity on the paths of Z_t.

Taking a Riemann sum approximation:

$$\mathbb{E}_\mu\left(\exp\left[-\langle\Psi, Z_t\rangle - \int_0^t \langle\Phi(s), Z_s\rangle\, ds\right]\right)$$
$$= \lim_{N\to\infty} \mathbb{E}_\mu\left(\exp\left[-\sum_{k=1}^{N-1}\left\langle\phi_k, Z_{\frac{k}{N}t}\right\rangle\frac{t}{N} - \left\langle\frac{t}{N}\phi_N + \Psi, Z_t\right\rangle\right]\right).$$

Denote by U_t^Φ and $U_t^{\Phi_N}$ the non-linear semigroups generated by $\Delta_\alpha u(t) - u^{1+\beta}(t) + \Phi(t)$ and $\Delta_\alpha u(t) - u^{1+\beta}(t) + \Phi_N(t)$ respectively. Conditioning and using the Markov property of Z_t we can calculate:

$$\mathbb{E}_\mu\left(\exp\left[-\langle\Psi, Z_t\rangle - \int_0^t \langle\Phi(s), Z_s\rangle\, ds\right]\right)$$
$$= \lim_{N\to\infty} \mathbb{E}_\mu\left(\exp\left[-\langle\Psi, Z_t\rangle - \sum_{k=1}^{N}\int_{s_{k-1}}^{s_k} \langle\phi_k, Z_s\rangle\, ds\right]\right)$$
$$= \lim_{N\to\infty} \mathbb{E}_\mu\left(\mathbb{E}_\mu\left(\exp\left[-\langle\Psi, Z_t\rangle - \sum_{k=1}^{N-1}\int_{s_{k-1}}^{s_k}\langle\phi_k, Z_s\rangle\, ds - \int_{s_{N-1}}^{s_N}\langle\phi_{N-1}, Z_s\rangle\, ds\right]\right.\right.$$
$$\left.\left. \mid \sigma(Z_s), 0 \le s \le s_{N-1}\right)\right)$$

$$= \lim_{N\to\infty} \mathbb{E}_\mu\left(\exp\left(-\sum_{k=1}^{N-1}\int_{s_{k-1}}^{s_k}\langle\phi_k, Z_s\rangle\, ds\right)\mathbb{E}_\mu\left[\exp(-\langle\Psi, Z_t\rangle - \int_{s_{N-1}}^{t}\langle\phi_N, Z_s\rangle\, ds)\right]\right.$$
$$\left. \mid \sigma(Z_s), 0 \le s \le s_{N-1}\right)$$
$$= \lim_{N\to\infty} \mathbb{E}_\mu\left(\exp\left(-\sum_{k=1}^{N-1}\int_{s_{k-1}}^{s_k}\langle\phi_k, Z_s\rangle\, ds\right)\mathbb{E}_{Z_{s_{N-1}}}\left[\exp(-\langle\Psi, Z_t\rangle - \int_{s_{N-1}}^{t}\langle\phi_N, Z_s\rangle\, ds)\right]\right)$$
$$= \lim_{N\to\infty} \mathbb{E}_\mu\left(\exp\left(-\sum_{k=1}^{N-1}\int_{s_{k-1}}^{s_k}\langle\phi_k, Z_s\rangle\, ds\right)\exp(-\langle U^{\Phi_N}_{s_N - s_{N-1}}\Psi, Z_{s_{N-1}}\rangle)\right)$$
$$= \lim_{N\to\infty} \mathbb{E}_\mu\left(\exp\left[-\langle U^{\Phi_N}_{s_N - s_{N-1}}\Psi, Z_t\rangle - \sum_{k=1}^{N-1}\int_{s_{k-1}}^{s_k}\langle\phi_k, Z_s\rangle\, ds\right]\right)$$

where $U^{\Phi_N}_{s-s_{N-1}}\Psi$ is the mild solution of the equation:

$$\dot{u}(s) = \Delta_\alpha u(s) - u(s)^{1+\beta} + \phi_{N-1}, \qquad s_{N-1} \leq s \leq s_N$$
$$u(s_{N-1}) = \Psi.$$

Performing this step N times, one obtains:

$$\begin{aligned}
&\mathbb{E}_\mu\left(\exp\left[-\langle\Psi, Z_t\rangle - \int_0^t \langle\Phi(s), Z_s\rangle\, ds\right]\right)\\
&= \lim_{N\to\infty} \mathbb{E}_\mu\left(\exp\left[-\left\langle U^{\Phi_N}_{s_1-s_0}\cdots U^{\Phi_N}_{s_{N-1}-s_{N-2}} U^{\Phi_N}_{s_N-s_{N-1}}\Psi, Z_{s_0}\right\rangle\right]\right)\\
&= \lim_{N\to\infty} \mathbb{E}_\mu\left(\exp\left[-\left\langle U^{\Phi_N}_t\Psi, Z_0\right\rangle\right]\right)\\
&= \lim_{N\to\infty} \exp\left[-\left\langle U^{\Phi_N}_0 U^{\Phi_N}_t\Psi, \mu\right\rangle\right]\\
&= \exp\left[-\left\langle U^{\Phi}_t\Psi, \mu\right\rangle\right]
\end{aligned}$$

where in the fourth line $U^{\Phi_N}_0 = \mathbb{I}$. The last equality follows because the solution depends continuously on Φ as noted above and the result follows by taking the limit as $N \to \infty$.

□

Now let X_t be the solution of (4.4.19). The previous result will allow us to compute the characteristic-Laplace functional $\mathbb{E}[\exp(i\langle\phi, X_t\rangle - \langle\lambda, Z_t\rangle)]$ of the pair $[X_t, Z_t]$.

Theorem 4.4.4. *The characteristic-Laplace functional of the joint process* $[X_t, Z_t]$ *is given by :*

$$\mathbb{E}_\mu[\exp(i\langle\phi, X_t\rangle - \langle\lambda, Z_t\rangle)] = \exp(-\langle u(t), \mu\rangle)$$

where $u(t)$ is the solution of the equation:

$$\begin{aligned}\frac{\partial u(s,x)}{\partial s} &= \Delta_\alpha u(s,x) - u^{1+\beta}(s,x) + G_\phi^2(t-s,x), \qquad 0 \le s \le t \\ u(0,x) &= \lambda\end{aligned} \tag{4.4.23}$$

with G_ϕ defined as:

$$G_\phi(t,s,z) = \int_{\mathbb{R}^d} p(t-s,x,z)\phi(x)dx$$

Proof. First recall the following property of the Gaussian processes:

$$\mathbb{E}[\exp(i\langle\phi, X_t\rangle)] = \exp(-\mathrm{Var}\,\langle\phi, X_t\rangle)$$

with:

$$\langle\phi, X_t\rangle \doteq \int_{\mathbb{R}^d} X(t,x)\phi(x)\,dx, \qquad \phi \in C(\mathbb{R}^d)$$

So

$$[\langle\phi, X_t\rangle^2] = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} X(t,x)X(t,y)\phi(x)\phi(y)\,dx\,dy$$

Hence:

$$\begin{aligned}\mathrm{Var}[\langle\phi, X_t\rangle] &= \mathbb{E}[\langle\phi, X_t\rangle^2] \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x)\mathbb{E}[X(t,x)X(t,y)]\phi(y)\,dx\,dy \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x)\Gamma_t(x,y)\phi(y)\,dx\,dy\end{aligned}$$

where by definition:

$$\begin{aligned}\Gamma_t(x,y) &= \mathbb{E}[X(t,x)X(t,y)] \\ &= \int_0^t\int_{\mathbb{R}^d} p(t-s,x,z)p(t-s,y;z)\,Z_s(dz)\,ds\end{aligned}$$

So:

$$\begin{aligned}\mathrm{Var}[\langle \phi, X_t \rangle] &= \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x) p(t-s,x,z) p(t-s,y,z) \phi(y)\, Z_s(dz)\, dx\, dy\, ds \\ &= \int_0^t \int_{\mathbb{R}^d} g_\phi(t-s,z)\, Z_s(dz)\, ds\end{aligned}$$

In the last line:

$$\begin{aligned}g_\phi(t-s,z) &\doteq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p(t-s,x,z) p(t-s,y,z) \phi(x) \phi(y)\, dx\, dy \\ &= \left[\int_{\mathbb{R}^d} p(t-s,x,z) \phi(x)\, dx \right]^2\end{aligned}$$

Note that the function:

$$G_\phi(t-s,z) \doteq \int_{\mathbb{R}^d} p(t-s,x,z) \phi(x)\, dx$$

considered as a function of s, satisfies the backward heat equation:

$$\frac{\partial}{\partial s} G_\phi(t-s,z) + \Delta_z G_\phi(t-s,z) = 0, \qquad 0 \le s \le t$$

with final condition:

$$G_\phi(t) = \phi(x)$$

So:

$$\mathbb{E}[\exp(i \langle \phi, X_t \rangle)] = \exp\left[- \int_0^t < G_\phi^2(t-s,z), Z_s(dz) > ds \right]$$

where G^2_ϕ is given by the above expression. With this, assuming $Z_0 = \mu$ we have the following expression for the Laplacian of the joint process $[X_t, Z_t]$:

$$\begin{aligned}
&\mathbb{E}_\mu[\exp(i\langle\phi, X_t\rangle - \langle\lambda, Z_t\rangle)] \\
&= \mathbb{E}_\mu[\mathbb{E}_\mu[\exp(i\langle\phi, X_t\rangle - \langle\lambda, Z_t\rangle)|\sigma(Z_s), 0 \le s \le t]] \\
&= \mathbb{E}_\mu[\exp(-\langle\lambda, Z_t\rangle)\mathbb{E}_\mu[\exp(i\langle\phi, X_t\rangle)|\sigma(Z_s), 0 \le s \le t]] && \text{(measurability)} \\
&= \mathbb{E}_\mu\left[\exp\left(-\int_0^t\int_{\mathbb{R}^d} G^2_\phi(t-s,z)Z_s(dz)\,ds - \langle\lambda, Z_t\rangle\right)\right] \\
&= \mathbb{E}_\mu\left[\exp\left(-\int_0^t \left\langle G^2_\phi(t-s,z), Z_s\right\rangle\,ds - \lambda\langle 1, Z_t\rangle\right)\right] && \text{(by definition)} \\
&= \exp(-\langle u(t), \mu\rangle) && \text{(by Theorem 4.4.3)}
\end{aligned}$$

and the result follows after renaming the variables. □

4.5 The state space and continuity of the paths

As in chapter three, it is of interest to determine the state space of the process $\{X_t\}$ and the continuity of its paths.

It has been seen above that this problem requires a variety of different techniques depending on the nature of the model. For processes with a random catalyst a distinction has to be made between the *Quenched* process, which is Gaussian conditioned on $\{Z_s : 0 \le s \le t\}$ and the *Annealed* process which is a compounded stochastic process and is not necessarily Gaussian. These two cases will of course require different treatments. For the quenched case, properties of X_t are obtained for a.e. realization or for a set of realizations of Z_t of positive probability whereas for the annealed case we obtain, for example, results on the annealed law $\int_{C([0,t],M_F(\mathbb{R}))} P_{\mu(\cdot)}(X(t) \in A)P((Z \in d(\mu(\cdot)))$, where $P_{\mu(\cdot)}(X(t) \in A)$

denotes the probability law for the Ornstein-Uhlenbeck process X_t in the catalyst $\{\mu(s),\ 0 \leq s \leq t\}$.

In the case $d = 1$ we will also consider both the bounded case $[0,1]$ and the case of the whole real line $\mathbb{R}$.

4.5.1 The Quenched OU process with super-Brownian catalyst

Let us consider the problem:

We begin with the the case where Z_t is super-Brownian motion on $[0,1]$ (with absorbing boundary conditions). Find a Hilbert space H, where the solutions of the following SDE lie

$$\text{(CSE)} \qquad \begin{cases} dX(t,x) = \Delta X(t,x)dt + W_{Z_t}(dt,dx) & t \geq 0, \quad x \in [0,1] \\ X(t,x) = 0 & t \geq 0, \quad x = 0,1 \\ X(0,x) = 0 & x \in [0,1] \end{cases}$$

that is, a Hilbert space H, such that $\mathbb{E}\,\|X_t\|_H^2 < \infty$.

For that purpose, recall that the trigonometric functions $e_k(x)\sqrt{2}\,Sin\,k\pi x$ form a complete orthonormal system (c.o.s.) of $L_2[0,1]$ with zero boundary values, and they are also the eigenvalues of the Laplace operator with $\Delta e_k = \lambda_k\, e_k$, $\lambda_k = -(k\pi)^2$.

Denoting by $T(t)$ the semigroup generated by the Laplacian with Dirichlet b.c., it follows from the spectral mapping theorem that:

$$
\begin{aligned}
T(t)e_k &= \int_0^1 p(t,x,y)e_k(y)\,dy \\
&= e^{\lambda_k t}\, e_k
\end{aligned}
$$

Using now the results of the last section, we get:

$$
\mathbb{E}[\langle X_t, e_k\rangle^2] = \int_0^t \int_0^1 g_{e_k}(t-s,z)\, Z_s(dz)\, ds
$$

where, as before:

$$
\begin{aligned}
g_{e_k} &= \left[\int_0^1 p(t-s,z,y)\, e_k(y)\, dy\right]^2 \\
&= [T(t-s)e_k(z)]^2 \\
&= e^{2\lambda_k (t-s)}\, Sin^2\, k\pi z
\end{aligned}
$$

Hence conditioned on $Z_s(x) \leq C_1$ (recall that we can assume this without loss of generality by Theorem 4.3.1),

$$
\begin{aligned}
\mathbb{E}[\|X(t)\|^2] &= \mathbb{E}\left[\sum_{k=0}^{\infty} \langle X(t), e_k\rangle^2\right] && \text{(Parseval's identity)} \\
&= \sum_{k=1}^{\infty} \mathbb{E}\left[\langle X(t), e_k\rangle^2\right] && \text{(MCT)} \\
&= \sum_{k=1}^{\infty} 2 \int_0^t e^{-2k^2\pi^2\,(t-s)} \int_0^1 Sin^2\, k\pi z\, Z_s(dz)\, ds \\
&\leq 2C_1 \int_0^t \sum_{k=1}^{\infty} e^{-2k^2\pi^2(t-s)}\, ds.
\end{aligned}
$$

Using now, the inequality established in Ch. 2, namely: $\sum_{k=1}^{\infty} e^{-2k^2\pi^2(t-s)} \leq C_2(t-s)^{-1/2}$, we get:

$$\mathbb{E}[\|X(t)\|^2] \leq Ct^{1/2}$$

so that, with probability one , the solutions lie in $L_2[0,1]$; in fact, the same procedure shows that :

$$\mathbb{E}[\|X(t)-X(s)\|^2] \leq C(t-s)^{1/2}.$$

Since conditioned on $\{Z_s : 0 \leq s \leq t\}$, the process X_t is Gaussian, we can also conclude, that the process X_t has continuous paths on $L_2[0,1]$.

We can also prove the following:

Theorem 4.5.1. *Now consider the process on $\mathbb{R}$, with super-Brownian catalyst Z_t with $S(Z_0)$ compact. Then:*

$$\mathbb{E}\,\|X_t\|^2 \leq C(\omega)t^{1/2}.$$

Proof. Expressing the solution of (4.4.19) in integral form:

$$X(t,x) = \int_0^t \int_{\mathbb{R}} p(t-s,x,y)\, W_{Z_s}(ds,dy).$$

We have:

$$\|X_t\|^2 = \int_{\mathbb{R}} \left[\int_0^t \int_{\mathbb{R}} p(t-s,x,y)\, W_{Z_s}(ds,dy) \right]^2 dx$$

Applying Fubini and Itô isometry, yields:

$$\begin{aligned}\mathbb{E}\left\|X_t\right\|^2 &= \int_{\mathbb{R}}\int_0^t\int_{\mathbb{R}} p^2(t-s,x,y)\,Z_s(dy)\,ds\,dx \\ &= \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} p^2(t-s,x,y)\,dx\,Z_s(dy)\,ds\end{aligned}$$

Now by the compact support property and Theorem 4.3.1, $\sup_{0\leq s\leq t}\sup_{\mathbb{R}} Z(s,x) \leq C(\omega)$ and $C(\omega) < \infty$, P_{Z_0}-a.s.

Since

$$\int_{\mathbb{R}} p^2(t-s,x,y)\,dx = \frac{1}{[4\pi(t-s)]^{1/2}}$$

we then have

$$\begin{aligned}\mathbb{E}\left\|X_t\right\|^2 &= C(\omega)\int_0^t \frac{1}{[4\pi(t-s)]^{1/2}}\,ds \\ &\leq C(\omega)t^{1/2}\end{aligned}$$

from which, the desired result follows. □

Theorem 4.5.2. *In dimension $d \geq 1$ and with $Z_0 \in M_F(\mathbb{R}^d)$, the problem:*

$$(COU\text{-}Rd) \qquad \begin{cases} dX(t,x) = \Delta X(t,x)dt + W_{Z_t}(dx,dt) & t \geq 0, \quad x \in \mathbb{R}^d \\ X(0,x) = 0 & x \in \mathbb{R}^d \end{cases}$$

has for a.e. realization of $\{Z_t\}$ continuous paths in a space of distributions $\mathcal{S}_{-n}$ for any $n > d/2$.

Proof. The proof follows the lines of Th. 2.4.9, noting first that the $Z_t(\cdot)$ has continuous paths in $\mathcal{M}(\mathbb{R}^d)$ with $\sup_{0\leq t\leq T} Z_t(\mathbb{R}^d) < \infty$. Therefore W_{Z_t} lives on $\mathcal{S}_{-n}$ for $n > \frac{d}{2}$ (see definition in remark below) since $\mathcal{S}_n \hookrightarrow \mathcal{S}_m$ is Hilbert-Schmidt if $n > m + d/2$ and $\mathcal{S}_0 = L^2(\mathbb{R}^d)$. The result now follows since the the semigroup T_t can be extended to a bounded operator on $\mathcal{S}_{-n}$ and $\sup_{0\leq t\leq T} \|T_t\|$ where $\|T_t\|$ is the operator norm on $\mathcal{S}_{-n}$ (see [RT 03], Theorem 2.4). □

Remark 4.5.1. Let $\mathcal{S}(\mathbb{R}^d)$, be the Schwartz space of rapidly decreasing functions. That is, each $f \in \mathcal{S}(\mathbb{R}^d)$ is infinitely differentiable and for each non-negative integer k and each non-negative integer-valued vector $\alpha = (\alpha_1, \ldots, \alpha_d)$ we have

$$\lim_{|x|\to\infty} |x|^k |\partial^\alpha f(x)| = 0$$

where

$$\partial^\alpha f(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} f(x_1, \ldots, x_d)$$

and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Let $\mathcal{S}'(\mathbb{R}^d)$ denote the dual space of $\mathcal{S}(\mathbb{R}^d)$ equipped with the strong topology and let $\mathcal{S}_n$ be the closure of $\mathcal{S}(\mathbb{R}^d)$ under the norm:

$$\|\phi\|_n^2 = \sum_q (2\,|q| + d)^n \langle \phi, h_q \rangle^2, \qquad n \in \mathbb{R}$$

where h_q, $q = (q_1, \ldots, q_d)$ are the basis for $L^2(\mathbb{R}^d)$ given by the d-fold tensor product of Hermite functions (see Appendix for details).

4.5.2 The Catalytic OU Process - Higher Moments

We can now use the results of the last section together with the techniques of Ch. 2, to determine properties of the solutions of (4.4.19), from:

$$X(t,x) = \int_0^t \int_{\mathbb{R}} p(t-s,x,y) W_{Z_s}(ds,dy)$$

Using the short forms $p = p(t-s,x,y)$, $dW^s = W_{Z_s}(ds,dy)$; $p_i(x) = p(t-s_i,x,y_i)$, $p_{ij} = p(t-s_i,x_j,y_i)$, $dW^{s_i} = W_{Z_{s_i}}(ds_i,dy_i)$, for , $i = 1,\cdots 4, j = 1,2$ as well as $\mathcal{D}_1 = [0,t]\times\mathbb{R}$, $\mathcal{D}_2 = [0,t]^2\times\mathbb{R}^2$, $\mathcal{D}_4 = [0,t]^4\times\mathbb{R}^4$, and noting that $p_{ij} = p_i(x_j)$, we can compute:

$$\begin{aligned}
&\mathbb{E}[||X_t||^4_{L_2} | Z_s, 0 \le s \le t] = \\
&= \mathbb{E}\left[\int_{\mathbb{R}} \left(\int_0^t \int_{\mathbb{R}} p(t-s,x,y) W_{Z_s}(ds,dy)\right)^2 dx\right]^2 \\
&= \mathbb{E}\left[\int_{\mathbb{R}} \left(\int_{\mathcal{D}_1} p\, dW^s\right)^2 dx_1\right]\left[\int_{\mathbb{R}} \left(\int_{\mathcal{D}_1} p\, dW^s\right)^2 dx_2\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}} \left(\int_{\mathcal{D}_1} p_1(x_1)\, dW^{s_1} \cdot \int_{\mathcal{D}_1} p_2(x_1)\, dW^{s_2}\right) dx_1\right] \\
&\quad \cdot \left[\int_{\mathbb{R}} \left(\int_{\mathcal{D}_1} p_3(x_2)\, dW^{s_3} \cdot \int_{\mathcal{D}_1} p_4(x_2)\, dW^{s_4}\right) dx_2\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}} \int_{\mathcal{D}_2} p_{11}p_{21}\, dW^{s_1} dW^{s_2}\, dx_1\right]\left[\int_{\mathbb{R}} \int_{\mathcal{D}_2} p_{32}p_{42}\, dW^{s_3} dW^{s_4}\, dx_2\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}^2} \int_{\mathcal{D}_4} p_{11}p_{21}p_{32}p_{42}\, dW^{s_1} dW^{s_2} dW^{s_3} dW^{s_4} dx_1 dx_2\right] \\
&= \int_{\mathbb{R}^2} \int_{\mathcal{D}_4} p_{11}p_{21}p_{32}p_{42}\, \mathbb{E}(dW^{s_1} dW^{s_2} dW^{s_3} dW^{s_4}) dx_1 dx_2 \\
&= \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3.
\end{aligned}$$

Each one of the above integrals can be evaluated using the independence of the increments of the Wiener process, according to the following cases:

Case 1: $s_1 = s_2 \wedge s_3 = s_4$, then: $p_{11}p_{21} = p_{11}^2 = p_1^2(x_1)$, $p_{32}p_{42} = p_{32}^2 = p_3^2(x_2)$, and $\mathbb{E}(dW^{s_1}dW^{s_2}dW^{s_3}dW^{s_4}) = Z_{s_1}(dy_1)ds_1 Z_{s_3}(dy_3)ds_3$, hence:

$$\begin{aligned}
\mathbb{I}_1 &= \int_{\mathbb{R}^2}\int_{\mathcal{D}_2} p_1^2(x_1)p_3^2(x_2)\, Z_{s_1}(dy_1)ds_1 Z_{s_3}(dy_3)ds_3 dx_1 dx_2 \\
&= \left[\int_{\mathbb{R}}\int_{\mathcal{D}_1} p_1^2(x_1)\, Z_{s_1}(dy_1)ds_1 dx_1\right]\left[\int_{\mathbb{R}}\int_{\mathcal{D}_1} p_3^2(x_2)\, Z_{s_3}(dy_3)ds_3 dx_2\right] \\
&= \left[\int_{\mathbb{R}}\int_{\mathcal{D}_1} p^2\, Z_s(dy)dsdx\right]^2 \\
&= \left[\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} p^2\, dx\, Z_s(dy)ds\right]^2 \\
&= C\left[\int_0^t\int_{\mathbb{R}} \frac{Z_s(dy)}{\sqrt{t-s}}ds\right]^2
\end{aligned}$$

Case 2: $s_1 = s_3 \wedge s_2 = s_4$, then: $p_{11}p_{32} = p_{11}p_{12} = p_1(x_1)p_1(x_2)$, $p_{21}p_{42} = p_{21}p_{22} = p_2(x_1)p_2(x_2)$, and $\mathbb{E}(dW^{s_1}dW^{s_2}dW^{s_3}dW^{s_4}) = Z_{s_1}(dy_1)ds_1 Z_{s_2}(dy_2)ds_2$, hence:

$$\begin{aligned}
\mathbb{I}_2 &= \int_{\mathbb{R}^2}\int_{\mathcal{D}_2} p_1(x_1)p_1(x_2)p_2(x_1)p_2(x_2)\, Z_{s_1}(dy_1)ds_1 Z_{s_2}(dy_2)ds_2 dx_1 dx_2 \\
&= \int_{\mathcal{D}_2}\left(\int_{\mathbb{R}} p_1(x_1)p_2(x_1)\, dx_1 \cdot \int_{\mathbb{R}} p_1(x_2)p_2(x_2)\, dx_2\right) Z_{s_1}(dy_1)ds_1 Z_{s_2}(dy_2)ds_2 dx_1 \\
&= \int_{\mathcal{D}_2} p^2(2t - s_1 - s_2, y_1, y_2)\, Z_{s_1}(dy_1)ds_1 Z_{s_2}(dy_2)ds_2
\end{aligned}$$

Case 3: $s_1 = s_4 \wedge s_2 = s_3$, then: $p_{11}p_{42} = p_{11}p_{12} = p_1(x_1)p_1(x_2)$, $p_{21}p_{32} = p_{21}p_{22} = p_2(x_1)p_2(x_2)$, and $\mathbb{E}(dW^{s_1}dW^{s_2}dW^{s_3}dW^{s_4}) = Z_{s_1}(dy_1)ds_1 Z_{s_2}(dy_2)ds_2$, and we get the same result as before, namely:

$$\mathbb{I}_3 = \int_{\mathcal{D}_2} p^2(2t - s_1 - s_2, y_1, y_2)\, Z_{s_1}(dy_1)ds_1 Z_{s_2}(dy_2)ds_2$$

Putting everything together, yields:

$$
\begin{aligned}
&\mathbb{E}[\|X_t\|_{L_2}^4 |Z_s, 0 \leq s \leq t] \\
&= C \left[\int_0^t \int_{\mathbb{R}} \frac{Z_s(dy)}{\sqrt{t-s}} ds \right]^2 \\
&+ 2 \int_{\mathcal{D}_2} p^2(2t - s_1 - s_2, y_1, y_2)\, Z_{s_1}(dy_1) ds_1 Z_{s_2}(dy_2) ds_2
\end{aligned}
\tag{4.5.24}
$$

So:

$$
\begin{aligned}
\mathbb{E}[\|X_t\|_{L_2}^4] &= \mathbb{E}\left(\mathbb{E}[\|X_t\|_{L_2}^4 |Z_s, 0 \leq s \leq t]\right) \\
&= C\mathbb{E} \left[\int_0^t \int_{\mathbb{R}} \frac{Z_s(dy)}{\sqrt{t-s}} ds \right]^2 \\
&+ 2\mathbb{E} \int_{\mathcal{D}_2} p^2(t - s_1 - s_2, y_1, y_2)\, Z_{s_1}(dy_1) ds_1 Z_{s_2}(dy_2) ds_2 \\
&= C\mathbb{E} \left[\int_0^t \int_{\mathbb{R}} \frac{Z_s(dy)}{\sqrt{t-s}} ds \right]^2 \\
&+ 2 \int_{\mathcal{D}_2} p^2(2t - s_1 - s_2, y_1, y_2)\, \mathbb{E}\left(Z_{s_1}(dy_1) Z_{s_2}(dy_2)\right)\; ds_1 ds_2.
\end{aligned}
\tag{4.5.25}
$$

To illustrate the application of these results we prove:

Proposition 4.5.3. *The quenched process on $\mathbb{R}^d, d \geq 1$ has continuous paths on $L_2(\nu)$, where ν is a finite measure with:*

$$dist(supp(\nu), supp(Z_t)) = m > 0$$

Proof. The above formulas can be reexpressed for higher dimension $d \geq 1$, and weighted spaces with the measure $\nu(dx)$:

$$\begin{aligned}
&\mathbb{E}[\|X_t\|^4_{L_2(\nu)} |Z_s, 0 \leq s \leq t] \\
&= C\left[\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p^2 \nu(dx)\, Z_s(dy) ds\right]^2 \\
&+ 2 \int_{[0,1]^2\times\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^d} p_1(x_1)p_2(x_1)\,\nu(dx_1)\cdot\int_{\mathbb{R}^d} p_1(x_2)p_2(x_2)\,\nu(dx_2)\right) Z_{s_1}(dy_1)ds_1 Z_{s_2}(dy_2)ds_2
\end{aligned} \tag{4.5.26}$$

for any $\alpha > 0$, inequality (3.5.14) , yields:

$$p^2 = \frac{\exp -(\|x-y\|^2 /(t-s)}{(t-s)^d} \leq C\frac{(t-s)^\alpha}{\|x-y\|^{2\alpha}}$$

$$p_1(x_1)p_2(x_1) = p(t-s_1, x_1, y_1)p(t-s_2, x_1, y_2) \leq C\frac{(t-s_1)^\alpha}{\|x_1-y_1\|^{2\alpha}}\frac{(t-s_2)^\alpha}{\|x_2-y_2\|^{2\alpha}}$$

then:

$$\begin{aligned}
&\mathbb{E}[\|X_t\|^4_{L_2(\nu)} |Z_s, 0 \leq s \leq t] \\
&\leq C_1 t^{2+2\alpha}\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\nu(dx)Z_s(dy)}{\|x-y\|^{2\alpha}}\right)^2 \\
&+ 2C_2 t^{2+2\alpha}\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\nu(dx_1)Z_{s_1}(dy_1)}{\|x_1-y_1\|^{2\alpha}}\right)\cdot\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\nu(dx_2)Z_{s_2}(dy_2)}{\|x_2-y_2\|^{2\alpha}}\right)
\end{aligned}$$

from which the assertion follows, since by hypothesis, all three integrals are finite. □

4.5.3 The Annealed O-U process with super-Brownian catalyst: State space and sample path continuity

Now, we apply the techniques developed above to prove the following result:

Proposition 4.5.4. *Let $X(t)$ be the solution of the stochastic equation:*

$$(CE) \qquad \begin{cases} dX(t,x) = \Delta X(t,x)dt + W_{Z_t}(dt,dx) & t \geq 0, \quad x \in \mathbb{R} \\ X(0,x) = 0 & x \in \mathbb{R} \end{cases}$$

(i) if $Z_0 = dx$, then $\mathbb{E}\,\|X(t)\|_{L_2}^4 = \infty$.

(ii) if $Z_0 = \delta_0(x)$, then the annealed $X(t)$ has continuous paths in $L_2(\mathbb{R})$.

Proof. It has been seen before that, the solution X_t satisfies:

$$\begin{aligned} \mathbb{E}[\|X_t\|_{L_2}^4] = \mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \frac{Z_s(dx)}{\sqrt{t-s}}ds\right]^2 \\ + 2\int_{\mathcal{D}_2} p^2(2t-s_1-s_2,x_1,x_2)\,\mathbb{E}\left(Z_{s_1}(dx_1)Z_{s_2}(dx_2)\right)\,ds_1 ds_2 \end{aligned} \tag{4.5.27}$$

The first integral is bounded above as follows:

$$\begin{aligned} &\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \frac{Z_s(dx)}{\sqrt{t-s}}ds\right]^2 = \\ &= \mathbb{E}\left[\left(\int_0^t \frac{1}{\sqrt{t-s_1}}\int_{\mathbb{R}} Z_{s_1}(dx_1)ds_1\right)\cdot\left(\int_0^t \frac{1}{\sqrt{t-s_2}}\int_{\mathbb{R}} Z_{s_2}(dx_2)ds_2\right)\right] \\ &= \mathbb{E}\left[\int_0^t\int_0^t\left(\frac{1}{\sqrt{(t-s_1)(t-s_2)}}\int_{\mathbb{R}}\int_{\mathbb{R}} Z_{s_1}(dx_1)Z_{s_2}(dx_2)\right)ds_1\,ds_2\right] \\ &= \int_0^t\int_0^{s_2}\left(\frac{1}{[(t-s_1)(t-s_2)]^{1/2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{E}[Z_{s_1}(dx_1)Z_{s_2}(dx_2)]\right)ds_1\,ds_2 \\ &+ \int_0^t\int_{s_2}^t\left(\frac{1}{[(t-s_1)(t-s_2)]^{1/2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{E}[Z_{s_1}(dx_1)Z_{s_2}(dx_2)]\right)ds_1\,ds_2 \end{aligned} \tag{4.5.28}$$

For the last two integrals, we assume $Z_0 = dx$, and $s_1 \leq s_2$, using (4.3.17) we obtain:

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[Z_{s_1}(dx_1)Z_{s_2}(dx_2)] = \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{s_1} p(2s + s_2 - s_1, x_1, x_2)ds \cdot dx_1 dx_2 \\
&= \int_0^{s_1} \int_{\mathbb{R}}\int_{\mathbb{R}} p(2s + s_2 - s_1, x_1, x_2)dx_1 dx_2 \cdot ds \\
&= \int_0^{s_1} \int_{\mathbb{R}} dx_2 \cdot ds \\
&= \infty.
\end{aligned}
$$

This shows (i). For (ii) assume now $Z_0 = \delta_0(x)$, and $s_1 \leq s_2$, using (4.3.18), one obtains:

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[Z_{s_1}(dx_1)Z_{s_2}(dx_2)] = \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\int_0^{s_1} \int_{\mathbb{R}} p(s_1 - s, 0, y)p(s, y, x_1)p(s, y, x_2)dy\, ds \right) dx_1 dx_2 \\
&= \int_0^{s_1} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} p(s_1 - s, 0, y)p(s, y, x_1)p(s, y, x_2)\, dx_1 dx_2\, dy\, ds \\
&= \int_0^{s_1} \int_{\mathbb{R}} p(s_1 - s, 0, y)\, dy\, ds \\
&= \int_0^{s_1} ds = s_1
\end{aligned}
$$

and similarly, for $s_2 \leq s_1$:

$$
\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[Z_{s_1}(dx_1)Z_{s_2}(dx_2)] = s_2
$$

so that (4.5.28) can be written as:

$$\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \frac{Z_s(dx)}{\sqrt{t-s}} ds\right]^2 =$$
$$= \int_0^t \int_0^{s_2} \frac{s_1 ds_1 ds_2}{[(t-s_1)(t-s_2)]^{1/2}} + \int_0^t \int_{s_2}^t \frac{s_2 ds_1 ds_2}{[(t-s_1)(t-s_2)]^{1/2}}$$

each one of the last integrals will be estimated below. A direct integration shows:

$$\frac{1}{(t-s_2)^{1/2}} \int_0^{s_2} \frac{s_1 \, ds_1}{(t-s_1)^{1/2}} \frac{1}{(t-s_2)^{1/2}} \left(\frac{4}{3}t^{3/2} + \frac{2}{3}(t-s_2)^{3/2} - 2t(t-s_2)^{1/2}\right)$$
$$\leq \frac{1}{(t-s_2)^{1/2}} \left(\frac{4}{3}t^{3/2} + \frac{2}{3}(t-s_2)^{3/2}\right)$$
$$= \frac{4}{3} \frac{t^{3/2}}{(t-s_2)^{1/2}} + \frac{2}{3}.$$

Hence:

$$\int_0^t \int_0^{s_2} \frac{s_1 ds_1 ds_2}{[(t-s_1)(t-s_2)]^{1/2}} \leq \frac{4}{3}t^2 + \frac{2}{3}t^2 = 2t^2$$

similarly:

$$\frac{s_2}{(t-s_2)^{1/2}} \int_{s_2}^t \frac{ds_1}{(t-s_1)^{1/2}} = s_2$$

and:

$$\int_0^t \int_{s_2}^t \frac{s_2 ds_1 ds_2}{[(t-s_1)(t-s_2)]^{1/2}} \leq \frac{t^2}{2}.$$

Both inequalities together imply:

$$\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \frac{Z_s(dx)}{\sqrt{t-s}} ds\right]^2 \leq C_1 t^2 \tag{4.5.29}$$

Similarly, we estimate below the second integral of (4.5.27):

$$\int_{\mathcal{D}_2} p^2(2t-s_1-s_2,x_1,x_2)\,\mathbb{E}\left(Z_{s_1}(dx_1)Z_{s_2}(dx_2)\right)\;ds_1ds_2 =$$

$$= \int_0^t\int_0^{s_2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} p^2(2t-s_1-s_2,x_1,x_2)\mathbb{E}\left(Z_{s_1}(dx_1)Z_{s_2}(dx_2)\right)\right)ds_1ds_2$$

$$+\int_0^t\int_{s_2}^t\left(\int_{\mathbb{R}}\int_{\mathbb{R}} p^2(2t-s_1-s_2,x_1,x_2)\mathbb{E}\left(Z_{s_1}(dx_1)Z_{s_2}(dx_2)\right)\right)ds_1ds_2$$

when $0 \le s_1 \le s_2$. Using (4.3.18) with $0 \le s \le s_1 \le s_2 \le t$, we obtain the following estimate for:

$$\mathbb{I}_1 \doteq \int_{\mathbb{R}}\int_{\mathbb{R}} p^2(2t-s_1-s_2,x_1,x_2)\mathbb{E}\left(Z_{s_1}(dx_1)Z_{s_2}(dx_2)\right)$$

$$= \frac{1}{\sqrt{2t-s_1-s_2}}\int\limits_{[0,s_2]\times\mathbb{R}^3} p(\frac{(2t-s_1-s_2)}{2},x_1,x_2)p(s_1-s,y)p(s,y,x_1)p(s,y,x_2)dx_1dx_2dyds$$

$$= \frac{1}{\sqrt{2t-s_1-s_2}}\int\limits_{[0,s_2]\times\mathbb{R}^2} p(\frac{(2t-s_1-s_2)}{2}+s,y,x_2)p(s_1-s,0,y)p(s,y,x_2)\,dx_2dyds$$

$$= \frac{1}{\sqrt{2t-s_1-s_2}}\int\limits_{[0,s_2]\times\mathbb{R}} p(\frac{(2t-s_1-s_2)}{2}+2s,0,0)p(s_1-s,0,y)\,dyds$$

$$= \frac{1}{\sqrt{2t-s_1-s_2}}\int_0^{s_2} p(\frac{(2t-s_1-s_2)}{2}+2s,0,0)ds$$

$$= \frac{\sqrt{2}}{\sqrt{2t-s_1-s_2}}\int_0^{s_2}\frac{1}{\sqrt{(2t-s_1-s_2)+4s}}ds$$

$$\le \frac{\sqrt{2}}{\sqrt{2t-s_1-s_2}}\int_0^{s_2}\frac{1}{\sqrt{4s}}ds$$

$$\le C_1\sqrt{\frac{s_2}{2t-s_1-s_2}}$$

so, we obtain:

$$
\begin{aligned}
\int_0^t \int_0^{s_2} \mathbb{I}_1 \, ds_1 ds_2 &\le C_1 \int_0^t \int_0^{s_2} \sqrt{\frac{s_2}{2t - s_1 - s_2}} ds_1 \, ds_2 \\
&= C_1 \int_0^t \sqrt{s_2} \left[2\sqrt{2t - s_2} - 2\sqrt{2t - 2s_2} \right] ds_2 \\
&\le C_2 \int_0^t \sqrt{s_2}\sqrt{t} ds_2 \\
&\le C_3 t^2
\end{aligned}
\tag{4.5.30}
$$

when $0 \le s_2 \le s_1$, using 4.3.18 with $0 \le s \le s_2 \le s_1 \le t$, and following the same steps above for the integral:

$$
\begin{aligned}
\mathbb{I}_2 &\doteq \int_{\mathbb{R}} \int_{\mathbb{R}} p^2(2t - s_1 - s_2, x_1, x_2) \mathbb{E} \left(Z_{s_2}(dx_1) Z_{s_1}(dx_2) \right) \\
&\le C_1 \sqrt{\frac{t}{2t - s_1 - s_2}}
\end{aligned}
$$

therefore:

$$
\begin{aligned}
\int_0^t \int_0^{s_2} \mathbb{I}_1 \, ds_1 ds_2 &\le C_4 \int_0^t \int_{s_2}^t \sqrt{\frac{t}{2t - s_1 - s_2}} ds_1 \, ds_2 \\
&= C_5 \int_0^t \sqrt{t} \left[2\sqrt{2t - 2s_2} - 2\sqrt{t - s_2} \right] ds_2 \\
&\le C_6 \int_0^t \sqrt{t}\sqrt{t} ds_2 \\
&= C_6 t^2.
\end{aligned}
\tag{4.5.31}
$$

Gathering (4.5.29), (4.5.30) and (4.5.31) yields:

$$
\mathbb{E}[\|X_t\|_{L_2}^4] \le Ct^2.
$$

In the same way we can obtain the estimates

$$
\mathbb{E}[\|X_t - X_s\|_{L_2}^4] \le C(t - s)^2.
$$

for the increments which shows the continuity of the paths using Kolmogorov's criteria. □

4.5.4 Further results on sample path continuity

In this section we establish the space of continuity of the paths of the catalytic Ornstein-Uhlenbeck on $[0,1]$ with zero boundary and initial conditions. Let us recall first that the set $\{\sin k\pi x\}$, $k = 1, 2 \ldots$ are the nonzero eigenvalues of the space of functions with zero boundary values, and further:

$$S(t) \sin k\pi x = \exp(-k^2\pi^2 t) \sin k\pi x, \qquad k = 1, 2 \ldots$$

denote by $\lambda_k = k^2\pi^2$, the completion of the space $C_0[0,1]$ under the L^2 norm is $L^2[0,1]$, so let $\phi_k(x) = \sin k\pi x$, $k = 1, 2 \ldots$ be a basis of $L^2[0,1]$, any element $f \in L^2[0,1]$ has the expansion:

$$f(x) = \sum_{k=1}^{\infty} a_k \phi_k(x)$$

so:

$$S(t)f(x) = \sum_{k=1}^{\infty} a_k \exp(-\lambda_k t)\phi_k(x)$$

this is the desired expression for the semigroup. It can also be expressed as an integral operator, using $a_k = \int_0^1 \sin \ k\pi y \, f(y)dy$, and applying the bounded convergence theorem yields:

$$\begin{aligned} S(t)f(x) &= \int_0^1 \left(\sum_{k=1}^{\infty} \exp(-\lambda_k t)\phi_k(x)\phi_k(y) \right) f(y)dy \\ &= \int_0^1 G(t,x,y)f(y)dy \end{aligned}$$

where the kernel $G(t,x,y)$ (usually called the Green function) is given by :

$$G(t,x,y) := \sum_{k=1}^{\infty} e^{-\lambda_k t}\phi_k(x)\phi_k(y)$$

With this property we can now prove the following particular case:

Theorem 4.5.5. *The annealed solution of the problem*

$$(COU\text{-}[0,1]) \qquad \begin{cases} dX(t,x) = \Delta X(t,x)dt + W_{Z_t}(dx,dt) & t \geq 0, \quad x \in [0,1] \\ X(t,x) = 0 & t = 0 \text{ or } x = 0,1 \end{cases}$$

with $Z_0 = \delta_{1/2}$ *is given by:*

$$X(t,x) = \int_0^t \int_0^1 \sum_{k\geq 1} e^{-\lambda_k (t-s)}\phi_k(x)\phi_k(y) W_{Z_s}(ds,dy) \tag{4.5.32}$$

has continuous paths in $L_2[0,1]$.

Proof. It is a well known fact, that the semigroup $S(t)$ of this problem is a positive semigroup smaller than the heat operator $T(t)$ of the same equation with no boundary conditions, in the sense that if $f(x) \geq 0$ then:

$$S(t)f(x) \leq T(t)f(x)$$

from which it follows, that the covariance measure of the process satisfies:

$$\begin{aligned}
\mathbb{E}(Z_t(dx_1)Z_t(dx_2)) &= \int_0^t \int_{\mathbb{R}} S_{t-s}[S_s\mathbb{I}_{A_1} \cdot S_s\mathbb{I}_{A_2}]\,\mu(dx)\,ds \\
&\leq \int_0^t \int_{\mathbb{R}} T_{t-s}[T_s\mathbb{I}_{A_1} \cdot T_s\mathbb{I}_{A_2}]\,\mu(dx)\,ds \\
&= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} p(t-s,0,y)p(s,y,x_1)p(s,y,x_2)\,dy\,ds\,dx_1dx_2 \\
&= t\,dx_1dx_2
\end{aligned}$$

following the same steps of Proposition 4.5.4, we arrive the desired conclusion. □

In fact, using a completely different technique we can prove a more general result for the quenched case:

Theorem 4.5.6. *In dimension $d \geq 1$ and with $Z_0 \in M_F(\mathbb{R}^d)$, the problem:*

$$(COU\text{-}[0,1]d) \qquad \begin{cases} dX(t,x) = AX(t,x)dt + W_{Z_t}(dx,dt) & t \geq 0, \quad x \in [0,1]^d \\ X(0,x) = 0 & x \in \partial[0,1]^d \end{cases}$$

has for almost every realization of $\{Z_t\}$ continuous paths in H_{-n} for any $n > d/2$, where H_{-n} is a certain space of distributions (see Appendix).

Proof. As in the above Theorem, the solution is given by:

$$X(t,x) = \int_0^t \int_0^1 \sum_{k \geq 1} e^{-\lambda_k (t-s)} \phi_k(x)\phi_k(y) W_{Z_s}(dy,ds)$$

Let

$$A_k(t) = \int_0^t \int_0^1 e^{-\lambda_k (t-s)} \phi_k(y) W_{Z_s}(dy,ds)$$

Then

$$X(t,x) = \sum_{k \geq 1} \phi_k(x) A_k(t)$$

we will show that $X(t)$ has continuous paths on some space H_n mentioned in the Appendix. H_n is isomorphic to the set of formal eigenfunction series

$$f = \sum_{k=1}^{\infty} a_k \phi_k$$

for which

$$\|f\|_n = \sum a_k^2 (1+\lambda_k)^n < \infty$$

Fix some $T > 0$, we first find a bound for $\mathbb{E}\left\{\sup_{t\leq T} A_k^2(t)\right\}$.

Let $V_k(t) = \int_0^t \int_0^1 \phi_k(x) W(Z_s(dx), ds)$, integrating by parts in the stochastic integral, we obtain:

$$\begin{aligned} A_k(t) &= \int_0^t e^{-\lambda_k(t-s)} dV_k(s) \\ &= V_k - \int_0^t \lambda_k e^{-\lambda_k(t-s)} V_k(s)\, ds. \end{aligned}$$

Thus:

$$\begin{aligned} \sup_{t\leq T} |A_k(t)| &\leq \sup_{t\leq T} |V_k(t)| + (1 + \int_0^t \lambda_k e^{\lambda_k(t-s)} ds) \\ &\leq 2 \sup_{t\leq T} |V_k(t)|\,. \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E}\left\{\sup_{t\leq T} A_k^2(t)\right\} &\leq 4\mathbb{E}\left\{\sup_{t\leq T} V_k^2(t)\right\} \\ &\leq 16\mathbb{E}\left\{V_k^2(t)\right\} \qquad \text{(Doob's inequality)} \\ &= 16 \int_0^1 \int_0^1 \int_0^T \phi_k(x)\phi_k(y) Z_s(dx) Z_s(dy) ds \\ &\leq 16 T Z_T^2[0,1] = C. \end{aligned}$$

Thus

$$\mathbb{E}\left(\sum_{k\geq 1} \sup_{t\leq T} A_k^2(t)(1+\lambda_k)^{-n}\right) \leq C \sum_{k\geq 1} (1+\lambda_k)^{-n}. \tag{4.5.33}$$

Using now:

$$\begin{aligned} \sum_{k\geq 1} (1+\lambda_k)^{-p} &< \infty \qquad \text{if } p > d/2 \\ \sup_{k\geq 1} \|\phi_k\|_\infty (1+\lambda_k)^{-p} &< \infty \qquad \text{if } p > d/2 \end{aligned}$$

then (4.5.33) is finite if $n > d/2$ and clearly:

$$\|X(t)\|_{-n}^2 = \sum_{k\geq 1} A_k^2(t)(1+\lambda_k)^{-n}$$

is a.s finite, hence $X(t) \in H_{-n}$. Moreover, if $s > 0$,

$$\|X(t+s) - X(t)\|_{-n}^2 = \sum (A_k(t+s) - A_k(t))^2(1+\lambda_k)^{-n}$$

The summands are continuous since W is and the sum is bounded by

$$4 \sum \sup_{t\leq T} A_j^2(t)(1+\lambda_k)^{-n}$$

for a.e. ω. Now $A_k(s) \to A_k(t)$ as $s \downarrow t$, hence $\|X_s - X_t\|_{-n} \to 0$ as $s \downarrow t$. if W is continuous, so is A_k, and we can let $s \uparrow t$ to establish X is also left continuous. □

4.6 The Identification Problem for the Catalyst

The question to be considered in this section is whether by observing the catalytic Ornstein-Uhlenbeck process it is possible to determine the support of the catalyst. In Theorem 3.5.1 we proved that the solution of the catalytic Ornstein-Uhlenbeck process is smooth off the support. In one dimension since $X(t,x)$ is a continuous function of x the set $X(t,x) > 0$ is an open set. In this section we prove that in $d = 1$ the OU process is non-differentiable in the interior of the support.

Theorem 4.6.1. *Given a catalytic OU $\{X(t,\cdot)\}_{t\geq 0}$ with super-Brownian catalyst $\{Z(t)\}_{t\geq 0}$ in $\mathbb{R}^1$ with $Z(0) = \delta_0$, $X(0,\cdot) \equiv 0$, then a.s. $X(t,\cdot)$ is non-differentiable at a.e. x in the interior of $S(Z(t))$.*

Proof. By Theorem 4.3.2 $S(Z(t))$ is compact and with positive probability has non-empty interior by Theorem 4.3.1. Let $x \in Int(S(Z(t))$. Then $Z(t,x) > 0$ and by the joint continuity of $Z(s,x)$ (by Theorem 4.3.1(i)) there exists $\delta > 0$ such that $Z(s,x) > \delta$ for $t - \delta < s \leq t$, $|y - x| < 2\delta$. Now consider

$$\begin{aligned}
&\mathbb{E}[(X(t,x) - X(t,y))^2] \\
&= (\mathbb{E}[X(t,x) - X(t,y)])^2 + Var[X(t,x) - X(t,y)]
\end{aligned}$$

Note that $\mathbb{E}[X(t,x)] \equiv 0$ so we only need to compute the variance. Now recall that by the Konno-Shiga representation

$$\begin{aligned}
X(t,x) &= \int_0^t \int p(t-s, x-y) W_Z(dy, ds) \\
&= \int_0^t \int p(t-s, x-y) W_{Z - \delta 1_{(x-2\delta, x+2\delta)\times(t-\delta,t)}}(dy, ds) \\
&+ \int_0^t \int p(t-s, x-y) W_{\delta 1_{(x-2\delta, x+2\delta)\times(t-\delta,t)}}(dy, ds)
\end{aligned}$$

where we can take $W_{Z-\delta 1_{(x-2\delta,x+2\delta)\times(t-\delta,t)}}$, $W_{\delta 1_{(x-2\delta,x+2\delta)\times(t-\delta,t)}}$ independent so that for $y_1, y_2 \in (x - \delta, x + \delta)$,

$$\begin{aligned}
&\mathbb{E}[X(t,y_1) - X(t,y_2)]^2 \geq \mathbb{E}[X_\delta(t,y_1) - X_\delta(t,y_2)]^2 \\
&= \mathbb{E}[X_\delta(t,y_1)]^2 + \mathbb{E}[X_\delta(t,y_2)]^2 - 2\mathbb{E}[X_\delta(t,y_1) \cdot X_\delta(t,y_2)] \\
&= \mathbb{E}[X_\delta^*(t,y_1)]^2 + \mathbb{E}[X_\delta^*(t,y_2)]^2 - 2\mathbb{E}[X_\delta^*(t,y_1) \cdot X_\delta^*(t,y_2)] + o(|y_1 - y_2|)
\end{aligned}$$

where X^* is driven by space-time white noise on all of $\mathbb{R}$ multiplied by $\sqrt{\delta}$. Here the $o(|y_1 - y_2|)$ term comes from the missing contribution from outside

$(x-2\delta, x+2\delta)$, that is:

$$X_\delta(t,x) = \int_0^t \int p(t-s,x,y) W_{\delta 1(x-2\delta,x+2\delta)\times(t-\delta,t)}(dy,ds)$$
$$X_\delta^*(t,x) = \int_{t-2\delta}^t \int p(t-s,x,y)\sqrt{\delta} W(dy,ds)$$

Hence:

$$\mathbb{E}[X_\delta^*(t,y_1)\cdot X_\delta^*(t,y_2)] = \delta \int_{t-2\delta}^t \int p(t-s,y_1,u)p(t-s,y_2,u)dsdu$$

and

$$\begin{aligned}
&\mathbb{E}[(X^*(t,y_1)-X^*(t,y_2))^2] \\
&= 2\delta \int_{t-\delta}^t \int p(2(t-s),u)^2\, duds + o(|y_1-y_2|) \\
&- 2\delta \int_{t-\delta}^t \int p(2(t-s),y_1-y_2-u)^2\, duds \\
&= 2\int_0^\delta \Big(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s}} e^{-\frac{|y_1-y_2|}{2s}}\Big)ds + o(|y_1-y_2|) \\
&\geq 2\int_0^\infty e^{-\beta s}\Big(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s}} e^{-\frac{|y_1-y_2|}{2s}}\Big)ds + o(|y_1-y_2|) \\
&= \frac{2}{\sqrt{\beta}}(1-e^{-2\sqrt{\beta}|y_1-y_2|}) + o(|y_1-y_2|) \\
&\geq c\,|y_1-y_2| + o(|y_1-y_2|)
\end{aligned}$$

for some $c > 0$. Here we used the Laplace transform:

$$\int_0^\infty e^{-\beta s}\frac{1}{\sqrt{\phi s}} e^{-\frac{\alpha^2}{4s}} ds = \frac{1}{\sqrt{\beta}} e^{-\sqrt{\beta}\alpha} \qquad \alpha \geq 0, \beta \geq 0$$

This satisfies the condition (3) in Yeh's theorem [Y 67] (also see [Be 69]) and implies that a.s. $X(t,x)$ is non-differentiable at a.e. $y \in (x-\delta, x+\delta)$. □

Remark 4.6.1. Berman [Be 69] has extended Yeh's results and proved not only non-differentiability but other properties including unbounded γ variation for $\gamma < 2$.

CHAPTER 5
Relations with catalytic branching

5.1 Super-Brownian motion in super-Brownian catalyst

A basic question in the study of chemical and biological process is the relation between description at *microscopic* and *macroscopic* levels. In this chapter, we consider the catalytic branching system (described in [Da 99]). In this system at the microscopic level the molecular reaction occurs only in the presence of a *catalyst* and from the macroscopic point of view, a catalyst can be considered as a spatially inhomogeneous rate function.

At the microscopic level we begin with a system of *reactant* particles and a spatial density field $\rho = \{\rho_t(x); t \geq 0, x \in \mathbb{R}^d\}$ representing the *catalyst*. We assume that the reactant particles move independently in $\mathbb{R}^d$ according to standard Brownian motions and in addition each particle located at the point x at time t may die or split into a random number of offsprings at rate proportional to the amount of catalyst $\rho_t(x)$ present. The newly created particles start moving independently at the position of their parent.

Let $N(t)$ denote the random number of particles at time t and $X_i(t)$ the location of the i-th particle at time t, so that the state of the reactant at time t is given by the measure-valued process $\sum_{i=1}^{N(t)} \delta_{x_i(t)}$. If we start at time s with a single particle at a, this system of branching Brownian motions is described by its Laplace transition functional

$$v(s,t,a) := \mathbb{E}_{s,\delta_a} \exp(-\sum_{i+1}^{N(t)} \phi(x_i(t))$$

which satisfies the catalytic reaction diffusion equation

$$-\frac{\partial v(s,t,a)}{\partial s} = \frac{1}{2}\Delta v(s,t,a) + \rho_s(a)(G(v(s,t,a)) - v(s,t,a)), \quad v|_{s=t} = e^{-\phi}$$

where ϕ is a nonnegative measurable function on $\mathbb{R}^d$. If we assume that the offspring distribution of the reactant has a finite second moment we obtain in the limit the *catalytic super-Brownian motion* $X = X^\rho = \{X_t^\rho; t \geq 0\}$ in $\mathbb{R}^d$, described by the log-Laplace function

$$v(s,t,a) = \log \mathbb{E}_{s,\delta_a} \exp\left(-\int X_t^\rho(db)\phi(b)\right)$$

which solves a special case of the catalytic diffusion equation introduced above, namely:

$$-\frac{\partial v(s,t,a)}{\partial s} = \frac{1}{2}\Delta v(s,t,a) - \rho_s(a)v^2(s,t,a), \quad s \leq t, \quad v|_{s=t} = \phi$$

Here the equation is written in backward form and $\rho_s(a)$ is understood as the generalized density function of the measure $\rho_s(da)$.

In the case in which $\rho_t(db) \equiv \gamma db$, where γ is a (strictly) positive constant, X^ρ is the *continuous super-Brownian motion (SBM)* with constant branching rate γ. However in applications the catalytic mass ρ can be a singular measure $\rho(db)$(e.g.

concentrated on a hyper-surface), may vary in time, $\rho_t(db)$, (varying medium), or even be sampled from a random object (*random medium*).

In this Chapter we begin with the case of super-Brownian motion in a super-Brownian catalyst constructed in [Da 95] and then show that in the one-dimensional case the high density fluctuations of this process lead to the catalytic Ornstein-Uhlenbeck process with super-Brownian catalyst.

5.2 Statement of the Fluctuation Limit theorem

Following the notation in Ch. 4, we will denote by $Z(t)$ a super-Brownian motion in $[0,1]$, which by the result of Konno and Shiga is absolutely continuous with a density that can be described by the stochastic equation:

$$\begin{aligned} dZ(t,x) &= \frac{1}{2}\Delta Z(t,x)dt + \sqrt{Z(t,x)}dW(t,x) \\ Z(0,x) &= \mu \end{aligned}$$

We now introduce a sequence, $X_k(t)$, of super-Brownian motions on $[0,1]$ with catalyst given by $Z(t)$ and reflecting boundary conditions. These are special cases of the processes constructed in [Da 95] and in the one dimensional case the processes $X_k(t)$ satisfy the Konno-Shiga stochastic partial differential equations:

$$\begin{aligned} dX_k(t,x) &= \frac{1}{2}\Delta X_k(t,x)dt + \beta(\theta_k - X_k(t,x))dt + \sqrt{Z(t,x)\frac{X_k(t,x)}{\theta_k}}dW(t,x) \\ X(0,x) &= \theta_k \end{aligned} \tag{5.2.1}$$

where Δ is the Laplacian with Dirichlet or reflecting boundary conditions on $[0,1]$. In other words the $X_k(t,x)$ are super-Brownian motions with immigration in the catalytic super-Brownian medium $Z(t)$.

We define the fluctuations around θ_k by

$$V_k(t,x) = X_k(t,x) - \theta_k.$$

The main result of this chapter is the following:

Theorem 5.2.1 (SBM in SBM catalyst with immigration). *Assume $\beta > 0, \theta_k \to \infty$ and $|V_k(0,x)| \to 0$ uniformly. Let V denote the solution of*

$$\begin{aligned} dV(t,x) &= (\frac{1}{2}\Delta V(t,x) - \beta V(t,x))dt + \sqrt{Z(t,x)}W(dx,dt) \qquad && t \geq 0 \\ V(0,x) &= 0. && x \in [0,1] \end{aligned}$$

Here $\frac{1}{2}\Delta$ is the generator of reflecting Brownian motion, that is with domain $D(\frac{1}{2}\Delta) = \{f \in C^2[0,1],\ f'(0) = f'(1) = 0\}$.

Then for almost every realization of Z, V_k converges to V in the following ways:

1. *in the sense of finite dimensional distributions (f.d.d.) on $L^2([0,1])$,*
2. *in the sense of weak convergence of processes, that is, probability laws on $C([0,\infty), H_{-(\frac{1}{2}+\delta)})$ for $\delta > 0$.*

Remark 5.2.1. The reason for the difference in 1 and 2 is that the proof of tightness works only in the larger space $H_{-1/2+\delta}$.

In order to formulate the next results we introduce the notation to make explicit the quenched and annealed processes. For a fixed realization of $Z = f(t,x)$

we denote the corresponding processes by X_k^f and X^f. The annealed process will be denoted by X^*.

Corollary 5.2.2. *The annealed processes X_k^* also converge to X^* in the same ways.*

This is an infinite dimensional analogue of a recent result of Dawson and Li [Da 05]. We will also consider the process without immigration.

Theorem 5.2.3 (SMB in SBM catalyst without immigration). *Consider the sequence of super-Brownian motions on $[0,1]$ with catalyst given by $Z(t)$, $X_k(t)$, and which satisfy*

$$
\begin{aligned}
dX_k(t,x) &= \frac{1}{2}\Delta X_k(t,x)dt - \beta X_k(t,x)dt + \sqrt{Z(t,x)\frac{X_k(t,x)}{\theta_k}}dW(t,x) \\
X(0,x) &= \theta_k
\end{aligned}
\tag{5.2.2}
$$

that is $X_k(t,x)$ are super-Brownian motions with no immigration in the super-Brownian catalyst $Z(t)$. Here $\frac{1}{2}\Delta$ again denotes the generator of reflecting Brownian motion.

Note that $E(X_k(t,x)) = \theta_k e^{-\beta t}$ and consider the fluctuations around the mean defined by $V_k(t,x) = X_k(t,x) - \theta_k e^{-\beta t}$. Assume that $\beta > 0$, $\theta_k \to \infty$ and $E\,|V_k(0,x)| \to 0$. Then in the same ways as in Theorem 5.2.1 V_k converges to the process given by the OU-type equation:

$$
\begin{aligned}
dV(t,x) &= (\frac{1}{2}\Delta V(t,x) - \beta V(t,x))dt + \sqrt{Z(t,x)}W(e^{-\beta t}dx, dt) \qquad && t \geq 0 \\
V(0,x) &= 0. && x \in [0,1]
\end{aligned}
$$

Remark 5.2.2. These results are stated and proved on $[0,1]$ but it can be similarly extended to $\mathbb{R}$ assuming that μ has a bounded density with compact support.

Following the tradition of probabilistic limit theorems we first prove a weak law of large numbers and then the analogue of the central limit theorem which is the fluctuation limit theorem.

To illustrate the key idea we first consider the one-dimensional case.

Theorem 5.2.4. *(A Weak Law of Large Numbers for the real-valued case) Consider the sequence of $\mathbb{R}_+$-valued processes associated to the stochastic differential equations:*

$$dX_k(t) = \beta(\theta_k - X_k(t))dt + \sqrt{\frac{X_k(t)}{\theta_k}}W(dt)$$

$$X_k(0) = \theta_k.$$

Assume that $\beta > 0$ and $\theta_k \to \infty$. Then

$$\left[\frac{X_k(t)}{\theta_k}\right] \to 1 \text{ in probability.}$$

Proof. The solution is given by :

$$\begin{aligned} X_k(t) &= e^{-\beta t}\theta_k + \beta\theta_k \int_0^t e^{-\beta(t-s)}ds + \int_0^t e^{-\beta(t-s)}\sqrt{\frac{X_k(s)}{\theta_k}}W(ds) \\ &= \theta_k + \int_0^t e^{-\beta(t-s)}\sqrt{\frac{X_k(s)}{\theta_k}}W(ds) \end{aligned}$$

Taking expectations:

$$\mathbb{E}(X_k(t)) = \theta_k$$

and so:

$$\mathbb{E}\left[\frac{X_k(t)}{\theta_k}\right] = 1 \quad \forall k \geq 1 \tag{5.2.3}$$

For the second moments, write first:

$$\begin{aligned}
& X_k(t)^2 = \\
& = \theta_k^2 + 2\sqrt{\theta_k}\int_0^t e^{\beta(t-s)}\sqrt{X_k(s)}W(ds) \\
& + \frac{e^{-2\beta t}}{\theta_k}\int_0^t\int_0^t e^{\beta(s_1+s_2)}\sqrt{X_k(s_1)X_k(s_2)}W(ds_1)W(ds_2)
\end{aligned}$$

Hence:

$$\begin{aligned}
& \mathbb{E}[X_k(t)]^2 \\
& = \theta_k^2 \\
& + \frac{e^{-2\beta t}}{\theta_k}\int_0^t\int_0^t e^{\beta(s_1+s_2)}\mathbb{E}[X_k(s_1)X_k(s_2)]^{1/2}\delta_{s_1}(s_2)d(s_2) \\
& = \theta_k^2 + \int_0^t e^{-\beta(2t-2s)}\mathbb{E}\frac{X_k(s)}{\theta_k}ds \\
& = \theta_k^2 + \int_0^t e^{-\beta(2t-2s)}ds \\
& = \theta_k^2 + \frac{\delta_{0l}}{2\beta}(1-e^{-2\beta t})
\end{aligned}$$

This means:

$$\mathbb{E}\left[\frac{X_k(t)}{\theta_k}\right]^2 \to 1, \text{ as } k \to \infty \tag{5.2.4}$$

together with (5.2.3) yields:

$$\text{Var}\left[\frac{X_k(t)}{\theta_k}\right] \to 0, \text{ as } k \to \infty$$

and by Chebyshev's inequality:

$$\left[\frac{X_k(t)}{\theta_k}\right] \to 1, \text{ in probability.}$$

□

Theorem 5.2.5. *(A weak law of large numbers for the $C([0,1])$-valued processes X_k) The sequence of processes X_k given by (5.2.1) satisfies:*

$$\left[\frac{X_k(t,x)}{\theta_k}\right] \to 1 \textit{ weakly in probability.}$$

Proof. Since by the results of [Ksh 88] $Z(s,y)$ is a.s. bounded on $[0,T] \times [0,1]$, it suffices to prove this with $|Z(s,y)| \leq C$. We define $Y_k(t,x) = X_k(t,x)/\theta_k$ and let $A = \frac{1}{2}\Delta - \beta\mathbb{I}$ with reflecting boundary conditions, that is, with domain $D(A) = \{f \in C^2[0,1] : f'(0) = f'(1) = 0\}$. Then equation (5.2.1) can be formulated as:

$$dY_k(t,x) = AY_k(t,x)dt + \beta dt + \frac{1}{\sqrt{\theta_k}}\sqrt{Z(t,x)Y_k(t,x)}\, W(dx,dt)$$

$$Y_k(0,x) = 1.$$

Denote the semigroup on $C([0,1])$ generated by A as $S(t)$, that is,

$$\forall \phi \in C_b(\mathbb{R}) : \quad S(t)\phi(x) = e^{-\beta t}\int_0^1 p(t,x,y)\phi(y)\, dy$$

where $p(t,x,y)$ is the transition density of reflecting Brownian motion.

Then the solution is given by:

$$\begin{aligned}
Y_k(t,x) &= S(t)(1) + \beta \int_0^t S(t-s)(1)ds + \frac{1}{\sqrt{\theta_k}} \int_0^t S(t-s)\sqrt{Z(s,y)Y_k(s,y)}\, W(ds,dy) \\
&= e^{-\beta t} \int_0^t p(t,x,y)dy + \beta \int_0^t \int_0^1 e^{-\beta(t-s)} p(t-s,x,y)dyds \\
&\quad + \frac{e^{-\beta t}}{\sqrt{\theta_k}} \int_0^t \int_0^1 e^{-\beta s} p(t-s,x,y)\sqrt{Z(s,y)Y_k(s,y)}\, W(ds,dy) \\
&= 1 + \frac{e^{-\beta t}}{\sqrt{\theta_k}} \int_0^t \int_0^1 e^{-\beta s} p(t-s,x,y)\sqrt{Z(s,y)Y_k(s,y)}\, W(ds,dy).
\end{aligned}$$

Then one obtains:

$$\mathbb{E}\left(Y_k(t,x)\right) = 1 \ \forall x \in [0,1]. \tag{5.2.5}$$

Now consider the second moments:

$$\begin{aligned}
&\mathbb{E}[Y_k(t,x_1)Y_k(t,x_2)] = 1+ \\
&+ \frac{e^{-2\beta t}}{\theta_k} \int_0^t \int_0^1 \int_0^t \int_0^1 e^{\beta(s_1+s_2)} p(t-s_1,x_1,y_1)p(t-s_2,x_2,y_2) \\
&\cdot \mathbb{E}[Z(s_1,y_1)Y_k(s_1,y_1)Z(s_2,y_2)Y_k(s_2,y_2)]^{1/2} \cdot \delta_{y_1}(y_2)\delta_{s_1}(s_2)d(y_2)d(s_2)
\end{aligned}$$

when $s_1 = s_2 = s$ and $y_1 = y_2 = y$, $\mathbb{E}Y_k(s,y) = 1$ and assuming $x_1 = x_2 = x$, we obtain:

$$\mathbb{E}[Y_k(t,x)]^2 \leq 1 + \frac{Ce^{-2\beta t}}{\theta_k} \int_0^t \int_0^1 e^{2\beta s} p^2(t-s,x,y)\, dsdy \tag{5.2.6}$$

the numerator in the last term is bounded and this shows:

$$\text{Var}\ [Y_k(t,x)] \to 0, \ \text{as } k \to \infty$$

□

Remark 5.2.3. This suggests that the sequence of random processes defined by $V_k := X_k - \theta_k$ satisfies:

$$dV_k(t,x) = \frac{1}{2}\Delta V_k(t,x) - \beta V_k(t,x)dt + \sqrt{\frac{Z(t,x)X_k(t,x)}{\theta_k}}\, W(dx,dt)$$

$$V_k(0,x) = 0$$

and converges to a process V given by the SPDE:

$$dV(t,x) = \frac{1}{2}\Delta V(t,x) - \beta V(t,x)dt + \sqrt{Z(t,x)}\, W(dx,dt)$$

$$V(0,x) = 0.$$

However since we have only proved convergence of $\frac{X_k(t,x)}{\theta_k} \to 1$ pointwise in probability the proof in the next section will be based on Laplace functional calculations.

Remark 5.2.4. One can also consider the case of absorbing boundary conditions $A = \frac{1}{2}\Delta - \beta\mathbb{I}$, $D(A) = \{f \in C^2[0,1] : f(0) = f(1) = 0\}$ and the solution $Y_k(t,x)$ of the problem:

$$dY_k(t,x) = AY_k(t,x)dt + (\beta + \pi^2)\sin \pi x dt + \frac{1}{\sqrt{\theta_k}}\sqrt{Z(t,x)Y_k(t,x)}\, W(dx,dt)$$

$$Y_k(0,x) = \sin\, \pi x.$$

The analogous result is that $Y_k(t,x)$ converges in probability to $\sin \pi x$ for all $x \in (0,1)$ and $t > 0$.

5.3 Proof of the Fluctuation Limit Theorem

5.3.1 Key ingredients

For the convergence of the processes we will denote the above sequence of processes by $\langle X_t^n \rangle, \langle Y_t^n \rangle$ and $\langle Z_t^n \rangle$ respectively and summarize below some facts about convergence of processes which will be used later. For the following basic facts about weak convergence refer to [EK].

Definition 5.3.1. Consider a sequence of stochastic process X_n defined on probability spaces $\{(\Omega_n, \mathcal{F}_n, P_n^*)\}_{n=1}^{\infty}$ and a process X defined on a probability space $\{(\Omega, \mathcal{F}, P^*)\}$ all with continuous paths on a Polish space E. Denote by P_n and P the induced measures on $C([0,\infty), E)$. We say that $\{X_n\}_{n\geq 0}$ converges in distribution to X if: $P_n \Rightarrow P$ in the topology of weak convergence of probability measures.

In order to prove weak convergence of such a sequence it suffices to verify

1. The finite dimensional distributions converge.
2. The probability measure $\{P_n\}_{n\in\mathbb{N}}$ are relatively compact in the set of probability measures on $C([0,\infty), E)$.

To prove relative compactness, by Prohorov's theorem it suffices to prove *tightness.* The sequence $\{P_n\}$ is *tight* if for every $\epsilon > 0$, there is a compact set $K \subset C([0,\infty), E)$ such that $P_n(K) \geq 1 - \epsilon$, for every $n \in \mathbb{N}$.

Theorem 5.3.1 ([Kat 99], Th. 4.15). *Let $\{X^n\}_{n=1}^{\infty}$ be a tight sequence of continuous process such that the finite-dimensional distributions of $\{X^n\}_{n=1}^{\infty}$*

converge to those of X*, that is given* $0 \le t_1 < \cdots \le t_d < \infty$ *then:*

$$(X^n_{t_1}, \ldots, X^n_{t_d}) \overset{\mathcal{D}}{\Rightarrow} (X_{t_1}, \ldots, X_{t_d})$$

Let P_n *denote the measures induced on* $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ *by* $\{X^n\}_{n=1}^{\infty}$*. Then* $\{P^n\}_{n=1}^{\infty}$ *converges weakly to the measure* P *induced by* X*.*

5.3.2 The Laplace Functional

In this section we develop the Laplace functional methods needed in the proof of the convergence of the finite dimensional distributions. To explain the idea we begin with the one-dimensional continuous branching process.

Proposition 5.3.2. *The continuous state branching (CSB) given by the SODE:*

$$\begin{aligned} dX_t &= -\beta X_t dt + \sqrt{X_t} dW_t \\ X_0 &= x \end{aligned} \tag{5.3.7}$$

has Laplace functional given by:

$$\mathbb{E}_x \exp(-\lambda X_t) = \exp(-u(t) X_0)$$

where $u(t)$ *satisfies the equation:*

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\beta u - u^2 \\ u(0) &= \lambda \end{aligned} \tag{5.3.8}$$

Proof. Define

$$M(\lambda, t, x) = \mathbb{E}_x(e^{-\lambda X(t)})$$

Using Ito's lemma

$$\frac{\partial M}{\partial t} = \beta\lambda \frac{\partial M}{\partial \lambda} + \frac{1}{2}\lambda^2 \frac{\partial M}{\partial \lambda} \tag{5.3.9}$$

with solution

$$M(\lambda, t, x) = e^{-\lambda x}$$

Let u be the solution of

$$\frac{\partial u(\lambda, t)}{\partial t} = -\beta u(\lambda, t) - \frac{1}{2}u^2(\lambda, t), \qquad u(\lambda, 0) = \lambda$$

from the above definitions:

$$M(u(\lambda, t-s), s, x)e^{-u(\lambda, t-s)x}$$

is a constant, since:

$$\frac{\partial M(u(\lambda, t-s), s, x)}{\partial s} = 0$$

Therefore

$$\mathbb{E}_x(e^{-\lambda X(t)}) = M(\lambda, t, x) = M(u(\lambda, t), 0, x)e^{-u(t)x}$$

□

Adding an immigration term βt to X_t, one obtains the continuous state branching with immigration process (CBI), and can verify (see [Li]) the following propositions:

Proposition 5.3.3. *The continuous branching process with immigration (CBI), given by the SPDE:*

$$\begin{aligned} dY_t &= \beta(1 - Y_t)dt + \sqrt{Y_t} dW_t \\ Y_0 &= \mu \end{aligned} \tag{5.3.10}$$

has Laplacian functional given by :

$$\mathbb{E}_\mu \exp(-\lambda Y(t)) = \exp(-\int_0^1 \beta u(s)\, d\mu)$$

$$\frac{\partial u}{\partial t} = -\beta u - u^2$$

$$u(0) = \lambda$$

Proposition 5.3.4. *Let f be a continuous function on $[0, \infty) \times [0, 1]$. Then the superprocess on $[0, 1]$ given by the SPDE:*

$$\begin{aligned} dX(t,x) &= \frac{1}{2}\Delta X(t,x)\, dt + \beta(\theta - X(t,x))dt + \sqrt{f(t,x)X(t,x)}W(dt,dx) \\ X(0,x) &= g(x), \quad x \in [0,1], \end{aligned} \tag{5.3.11}$$

has Laplacian functional given by :

$$\mathbb{E}_\mu \exp(-\int \phi(x) X(t,x) dx) = \exp(-\int u(t,x)\, g(x) dx - \beta\theta \int_0^t \int v_r(t,x) dx dr)$$

$$\text{where } \mu(dx) = g(x)dx$$

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u - \beta u - f(t,x)u^2 \quad u(0) = \phi \\ \frac{\partial v_r}{\partial s} &= \frac{1}{2}\Delta v_r - \beta v_r - f(s,x)v_r^2, \quad r \le s \le t \\ v_r(r,x) &= \phi(x). \end{aligned} \tag{5.3.12}$$

5.3.3 Convergence of the finite dimensional distributions

According to the above discussion, convergence in distribution of the finite-dimensional distributions plus tightness of $\{Z^n\}_{n\geq 1}$ imply convergence of the processes.

We first prove the convergence of the one-dimensional distributions, as noted in the introduction , we do it first for the case where there is no immigration term:

Theorem 5.3.5 (Convergence of the one-dimensional distributions, processes without immigration). *Fix $t \geq 0$ and assume $\beta > 0$, and let $X_k(t,x)$ be the sequence of super-Brownian motions on $[0,1]$ with catalyst $Z(t)$ and no immigration, given by:*

$$dX_k(t,x) = \frac{1}{2}\Delta X_k(t,x)dt - \beta X_k(t,x)dt + \sqrt{Z(t,x)\frac{X_k(t,x)}{\theta_k}}dW(t,x)$$
$$X(0,x) = \theta_k$$

then, the fluctuations around θ_k defined by $V_k(t,x) = X_k(t,x) - \theta_k e^{-\beta t}$ converge in distribution to the distribution of the solution $V(t,x)$ of the OU-type equation at time t (that is, weak convergence of probability laws on $L^2([0,1])$):

$$dV(t,x) = (\frac{1}{2}\Delta V(t,x) - \beta V(t,x))dt + \sqrt{Z(t,x)}W(e^{-\beta t}dx, dt) \qquad t \geq 0$$
$$V(0,x) = 0. \qquad x \in [0,1]$$

i.e.

$$V_k(t) \Rightarrow V(t)$$

Proof. Let $X_0^n = \theta_n\, dx$, and $\phi \in C_+([0,1])$. Since $Z(t,x)$ a.s. has a density w.r.t. Lebesgue measure, it suffices to prove this for a fixed function $Z(t,x) =$

$f(t,x)$, where $f(t,x) \in C_0(\mathbb{R})$. Then:

$$\begin{aligned}\mathbb{E}_\mu \exp - \langle \lambda\phi, X_t^n \rangle &= \mathbb{E}_\mu \exp -\lambda \int \phi(x) X_t^n(dx) \\ &= \exp - \langle u(t), \mu \rangle \\ &= \exp - \int_0^1 u(\lambda, t, x)\theta_n dx\end{aligned}$$

where u satisfies the equation:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u - \beta u - f(t,x)\frac{u^2}{\theta_n} \\ u(\lambda, 0, x) &= \lambda\phi(x)\end{aligned} \tag{5.3.13}$$

using the following notation for the semigroup:

$$T_t^\beta u(x) = e^{-\beta t} \int_0^1 p(t,x,y)u(y)\,dy$$

we get:

$$u(\lambda, t, x) = \lambda T_t^\beta \phi - \int_0^t T_{t-s}^\beta \left(\frac{f(s,x)u^2(\lambda, s, x)}{\theta_n} \right) ds.$$

Substitution of the same expression for $u(\lambda, s, x)$ on the right hand side of (5.3.13) yields:

$$\begin{aligned}u(\lambda,t,x) &= \lambda T_t^\beta \phi - \frac{1}{\theta_n}\int_0^t T_{t-s}^\beta f(s,x) \left(\lambda T_s^\beta \phi - \frac{1}{\theta_n}\int_0^s T_{s-r}^\beta f(r,x) u^2(\lambda, r)\, dr \right)^2 ds \\ &= \lambda T_t^\beta \phi - \frac{\lambda^2}{\theta_n}\left(\int_0^t T_{t-s}^\beta (f(s,x)(T_s^\beta \phi)^2)\, ds \right) + o(\theta_n^2)\end{aligned} \tag{5.3.14}$$

So, the Laplace transform for the fluctuation $V_t^n : X(t,x) - \theta_n e^{-\beta t}$ can be written as:

$$
\begin{aligned}
&\mathbb{E}_\mu \exp -\lambda \int_0^1 \phi(x) V_t^n(dx) = \\
&= \mathbb{E}_\mu \exp - \int_0^1 \phi(x)(\lambda X_t^n(t,dx) - \lambda\theta_n e^{-\beta t})dx \\
&= \exp\left(- \int_0^1 u(\lambda,t,x)\theta_n dx + \lambda\theta_n e^{-\beta t} \int_0^1 \phi(x)dx \right) \\
&= \exp\left(-\lambda\theta_n \int_0^1 T_t^\beta \phi(x)dx + \lambda^2 \int_0^1 \int_0^t T_{t-s}^\beta (f(s,x)(T_s^\beta\phi)^2) ds dx + \right. \\
&\quad \left. + \lambda\theta_n e^{-\beta t} \int_0^1 \phi(x)dx + o(\theta_n) \right).
\end{aligned}
$$

Observing the following relation:

$$
\begin{aligned}
\int_0^1 T_t^\beta \phi(x)dx &= e^{-\beta t} \int_0^1 \int_0^1 p(t,x,y)\phi(y)dydx \\
&= e^{-\beta t} \int_0^1 \phi(y)dy
\end{aligned}
\tag{5.3.15}
$$

and also:

$$
\begin{aligned}
&\int_0^t\int_0^1 T_{t-s}^\beta (f \cdot (T_s^\beta \phi)^2(x))dxds \\
&= \int_0^t\int_0^1 \left[e^{-\beta(t-s)} \int_0^1 p(t-s,x,y)f(s,y)\left(e^{-\beta s}\int_0^1 p(s,y,z)\phi(z)dz \right)^2 dy \right] dxds \\
&= \int_0^t \int_0^1 e^{-\beta(t+s)} f(t-s,y) \left(\int_0^1 p(s,z,y)\phi(z)dz \right)^2 dyds \\
&= \int_0^t \int_0^1 e^{-\beta(2t-u)} f(t-s,y) \left(\int_0^1 p(t-u,z,y)\phi(z)dz \right)^2 dydu \\
&= \int_0^t \int_0^1 e^{-\beta s} f(s,y) (T_{t-s}^\beta \phi(y))^2 dyds
\end{aligned}
\tag{5.3.16}
$$

and putting these facts together we obtain:

$$\mathbb{E}_\mu \exp - \left(\lambda \int_0^1 \phi(x) V_t^n(dx)\right) \to \exp \left(\lambda^2 \int_0^t \int_0^1 e^{-\beta s} f(s,x) (T_{t-s}^\beta \phi(x))^2 \, dx ds\right)$$

as $n \to \infty$.

On the other hand, the OU-type process given by the SPDE:

$$dV_t = \left(\frac{1}{2}\Delta V_t - \beta V_t\right) dt + \sqrt{f(t,x)} W(e^{-\beta t} dx, dt)$$

$$V_t = 0$$

is Gaussian centered, with covariance:

$$\begin{aligned}\mathbb{E}\left[\int_0^1 \phi(x) V_t(dx)\right]^2 &= \mathbb{E}\left[\int_0^1 \int_0^t T_{t-s}^\beta \phi(x) \sqrt{f(s,x)} W(e^{-\beta s} ds, dx)\right]^2 \\ &= \int_0^1 \int_0^t \left(T_{t-s}^\beta \phi(x)\right)^2 e^{-\beta s} f(s,x) ds dx\end{aligned}$$

so that:

$$\mathbb{E} \exp - \left(\lambda \int_0^1 \phi(x) V_t(dx)\right) = \exp \left(\lambda^2 \int_0^t \int_0^1 e^{-\beta s} f(s,x) (T_{t-s}^\beta \phi(x))^2 \, dx ds\right).$$

This shows that $V_t^n \Rightarrow V_t$, as $n \to \infty$. □

Theorem 5.3.6 (Convergence of the one-dimensional distributions, processes with immigration). *Fix $t \geq 0$ and assume $\beta > 0$, and let $X_k(t,x)$ be the sequence of super-Brownian motions on $[0,1]$ with catalyst $Z(t)$ and immigration $\beta\theta_k$, given by:*

$$dX_k(t,x) = \frac{1}{2}\Delta X_k(t,x) dt + \beta(\theta_k - X_k(t,x)) dt + \sqrt{Z(t,x)\frac{X_k(t,x)}{\theta_k}} dW(t,x)$$

$$X(0,x) = \theta_k,$$

and $V_k(t,x)$ the fluctuations around θ_k defined by $V_k(t,x) = X_k(t,x) - \theta_k e^{-\beta t}$. Then $V_k(t,x)$ converges in distribution to $V(t,x)$ where the latter is the catalytic OU-process given by the equation:

$$dV(t,x) = (\frac{1}{2}\Delta V(t,x) - \beta V(t,x))dt + \sqrt{Z(t,x)}W(dx,dt) \qquad t \geq 0$$

$$V(0,x) = 0. \qquad x \in [0,1]$$

that is,

$$V_k(t) \Rightarrow V(t).$$

Proof. Let $X_0^k = \mu = \theta_k\, dx$, and $\phi \in C_+([0,1])$. As above we take a fixed realization $Z(t,x) = f(t,x)$ where $f(t,x) \in C_0(\mathbb{R})$. Then the Laplace transform of $X_n(t,x)$ is given by (5.3.12) with $f(t,x)$ replaced by $\frac{f(t,x)}{\theta_k}$, θ by θ_k, ϕ by $\lambda\phi$, $\lambda > 0$ and $g(x) \equiv \theta_k$.

The Laplace functional of the fluctuation $V_k(t,x) := X_k(t,x) - \theta_k$ is given by:

$$\begin{aligned} &\mathbb{E}\exp(-\int \lambda\phi(x)V_k(t,x)dx) \\ &= \exp\left(-\left(\theta_k\int u(t,x)dx + \beta\theta_k\int_0^t\int v_s(t,x)dsdx - \theta_k\right)\right). \end{aligned} \tag{5.3.17}$$

Using the same notation for the semigroup, T_t^β, as in the last theorem, as well as the substitution given by eq. (5.3.14), we obtain:

$$\begin{aligned} u(\lambda,t,x) = \lambda T_t^\beta\phi - \frac{\lambda^2}{\theta_n}\left(\int_0^t T_{t-s}^\beta(f(s,x)(T_s^\beta\phi)^2)\,ds\right) \\ + \text{ higher order terms in } \frac{1}{\theta_k}. \end{aligned} \tag{5.3.18}$$

Setting $f(t) := f(t,x)$ and $F(s) = \int (T_s^\beta \phi)^2 dx$, we evaluate below the term $\theta_k \int v(t)dx + \beta\theta_k \int_0^t \int u(s)dsdx - \theta_k$ of eq. (5.3.17):

$$\theta_k \int v(t)dx + \theta_k \int_0^t \int u(s)dsdx - \theta_k =$$

$$= \int_0^1 \int_0^t (T_{t-s}^\beta (f(s)\cdot (T_s^\beta\phi)^2))dsdx + \beta \int_0^1 \int_0^t \int_0^{t-s} (T_{t-s-u}^\beta (f(u)\cdot (T_u^\beta \phi)^2))dudsdx$$

$$= \int_0^t (e^{-\beta(t-s)} f(t-s) \int_0^1 (T_s^\beta \phi)^2 dx)ds + \beta \int_0^t \int_0^{t-s} (e^{-\beta(t-s-u)} f(t-s-u) \int_0^1 (T_u^\beta\phi)^2 dx)duds$$

$$= e^{-\beta t}\left[\int_0^t e^{-\beta(t-s)} f(t-s)F(s)ds + (e^{\beta s}\int_0^{t-s} e^{\beta u} f(t-u)F(u)du)_0^t \right.$$

$$\left. - \int_0^t e^{\beta s} e^{\beta(t-s)} f(u)F(t-u)du\right] \qquad \text{(Integration by parts)}$$

$$= - \int_0^t \int_0^1 f(s,x)(T_{t-s}^\beta \phi)^2 dxds$$

Hence:

$$\mathbb{E}_\mu \exp - \left(\lambda \int_0^1 \phi(x) V_n(t,x)(dx)\right) \to \exp \left(\lambda^2 \int_0^t \int_0^1 f(s,x)(T_{t-s}^\beta \phi(x))^2 \, dxds\right)$$

as $n \to \infty$.

Similarly, the catalytic OU process given by the SPDE:

$$dV_t = \left(\frac{1}{2}\Delta V_t - \beta V_t\right) dt + \sqrt{f(t,x)} W(dx,dt)$$

$$V_t = 0$$

is Gaussian centered, with covariance:

$$\mathbb{E}\left[\int_0^1 \phi(x)V_t(dx)\right]^2 = \mathbb{E}\left[\int_0^1 \int_0^t T_{t-s}^\beta \phi(x)\sqrt{f(s,x)} W(ds,dx)\right]^2$$

$$= \int_0^1 \int_0^t \left(T_{t-s}^\beta \phi(x)\right)^2 f(s,x)dsdx$$

so that:

$$\mathbb{E}\exp - \left(\lambda \int_0^1 \phi(x)V_t(dx)\right) = \exp\left(\lambda^2 \int_0^t \int_0^1 f(s,x)(T^{\beta}_{t-s}\phi(x))^2\, dxds\right).$$

This shows that $V_t^n \Rightarrow V_t$, as $n \to \infty$. □

Theorem 5.3.7 (Convergence of the finite-dimensional distributions). *With the notation of the last theorem, given* $d \geq 1$ *and real numbers* $0 \leq t_1 \leq \cdots \leq t_d < \infty$, *then*

$$(V_{t_1}^n, \ldots, V_{t_d}^n) \Rightarrow (V_{t_1} \ldots, V_{t_d}).$$

Proof. We will give the proof with $d = 2$ and for the case without immigration. The cases $d \geq 3$ and with immigration follow in essentially the same way. Let us take $t_1 < t_2$, $\phi_1, \phi_2 \in C^{\infty}[0,1]$, $\mu = X_0^n = \theta_n \cdot dx$, then we have for the fluctuations $V_t^n = X_t^n - \theta_n e^{-\beta t}$:

$$\begin{aligned}\mathbb{E}_{\mu}\exp(-\langle\lambda_1\phi_1, V_{t_1}^n\rangle - \langle\lambda_2\phi_2, V_{t_2}^n\rangle) &= \exp(\langle\lambda_1\phi_1, \theta_n e^{-\beta t_1}\rangle + \langle\lambda_2\phi_2, \theta_n e^{-\beta t_2}\rangle) \\ &\cdot \mathbb{E}_{\mu}\exp(-\langle\lambda_1\phi_1, X_{t_1}^n\rangle - \langle\lambda_2\phi_2, X_{t_2}^n\rangle)\end{aligned} \tag{5.3.19}$$

Using the Markov property, the last factor of the above equation is computed below:

$$
\begin{aligned}
&\mathbb{E}_\mu \exp(-\langle \lambda_1\phi_1, X^n_{t_1}\rangle - \langle \lambda_2\phi_2, X^n_{t_2}\rangle)\mathbb{E}_\mu[\mathbb{E}_\mu(\exp(-\lambda_1 \langle \phi_1, X^n_{t_1}\rangle - \lambda_2 \langle \phi_2, X^n_{t_2}\rangle)|\mathcal{F}_{t_1})] \\
&= \mathbb{E}_\mu[\exp(-\lambda_1 \langle \phi_1, X^n_{t_1}\rangle)\mathbb{E}_\mu[\mathbb{E}_\mu \exp(-\lambda_2 \langle \phi_2, X^n_{t_2}\rangle |\mathcal{F}_{t_1}]] \\
&= \mathbb{E}_\mu[\exp(-\lambda_1 \langle \phi_1, X^n_{t_1}\rangle)\mathbb{E}_\mu \exp(-\lambda_2 \langle \phi_2, X^n_{t_2-t_1}\rangle)] \\
&= \mathbb{E}_\mu[\exp(-\lambda_1 \langle \phi_1, X^n_{t_1}\rangle) \exp(-\langle u(\lambda_2\phi_2, t_2-t_1)), X^n_{t_1}\rangle] \\
&= \mathbb{E}_\mu \exp - \langle \lambda_1\phi_1 + u(\lambda_2\phi_2, t_2-t_1), X^n_{t_1}\rangle \\
&= \exp\, - \langle u(\lambda_1\phi_1 + u(\lambda_2\phi_2, t_2-t_1),\, t_1), X^n_0\rangle \\
&= \exp\, -\theta_n \int u(\lambda_1\phi_1 + u(\lambda_2\phi_2, t_2-t_1),\, t_1)dx
\end{aligned}
\tag{5.3.20}
$$

where $u(\lambda\phi, t)$ satisfies:

$$
\begin{aligned}
\frac{\partial u(\lambda\phi, t)}{\partial t} &= \frac{1}{2}\Delta u(\lambda\phi, t) - \beta u(\lambda\phi, t) - f(t,x)\frac{u(\lambda\phi, t)^2}{\theta_n} \\
u(\lambda, 0) &= \lambda\phi.
\end{aligned}
$$

In order to simplify the notation we now set $f(t,x) = 1$ - the case with $f \in C([0,\infty) \times [0,1])$ follows in exactly the same way as we have done for the one dimensional marginal. As before, the integral representation of u is given by :

$$
\begin{aligned}
u(\lambda\phi, t) &= T^\beta_t(\lambda\phi) - \frac{1}{\theta_n}\int_0^t T^\beta_{t-s}(u^2(\lambda\phi, s))\, ds \\
T^\beta_t u(x) &:= e^{-\beta t}\int_0^1 p(t,x,y)u(y)\, dy.
\end{aligned}
$$

With this notation, we can compute the integrand of the last expression in (5.3.20) :

$$u(\lambda_2\phi_2, t_2 - t_1) = T^{\beta}_{t_2-t_1}(\lambda_2\phi_2) - \frac{1}{\theta_n}\int_0^{t_2-t_1} T^{\beta}_{t_2-t_1-s}(u^2(\lambda_2\phi_2, s))\, ds \tag{5.3.21}$$

$$\begin{aligned} u(\lambda_1\phi_1 + u(\lambda_2\phi_2, t_2 - t_1), t_1) &= T^{\beta}_{t_1}(\lambda_1\phi_1 + u(\lambda_2\phi_2, t_2 - t_1)) \\ &\quad - \frac{1}{\theta_n}\int_0^{t_1} T^{\beta}_{t_1-s}(u^2(\lambda_1\phi_1 + u(\lambda_2\phi_2, t_2 - t_1), s))\, ds \end{aligned} \tag{5.3.22}$$

In order to keep track of the terms $o(\theta_n^k)$, we will write shortly $u_2(r) := u(\lambda_2\phi_2, r)$ and express (5.3.21) as a Volterra integral equation:

$$u_2(r) - \theta_n^{-1}\int_0^r T^{\beta}_{r-s}(u_2^2(s))ds = \lambda_2 T^{\beta}_r\phi_2 \tag{5.3.23}$$

whose solution is given by the Neumann series:

$$u_2(r) = \sum_{k=0}^{\infty} A^k(\lambda_2 T^{\beta}_r\phi_2)$$

where the operator A^k is defined recursively as:

$$\begin{aligned} A^0(\lambda_2 T^{\beta}_r\phi_2) &= \mathbb{I}(\lambda_2 T^{\beta}_r\phi_2)\lambda_2 T^{\beta}_r\phi_2 \\ A^1(\lambda_2 T^{\beta}_r\phi_2) &= \theta_n^{-1}\int_0^r T^{\beta}_{r-s}(\lambda_2 T^{\beta}_s\phi_2)^2\, ds \\ &= \theta_n^{-1}\lambda_2^2\int_0^r T^{\beta}_{r-s}(T^{\beta}_s\phi_2)^2\, ds. \end{aligned}$$

Setting:

$$\begin{aligned} a_0(r) &:= \lambda_2 T^{\beta}_r\phi_2 \\ a_1(r) &:= \lambda_2^2\int_0^r T^{\beta}_{r-s}(T^{\beta}_s\phi_2)^2\, ds \end{aligned}$$

this means:

$$u_2(r) = a_0(r) + a_1(r)\theta_n^{-1} + O(\theta_n^{-2})$$

$$u_2^2(r) = a_0(r)^2 + 2a_0(r)a_1(r)\theta_n^{-1} + O(\theta_n^{-2})$$

For (5.3.22), let $u_1(r) := u(\lambda_1\phi_1 + u_2(t_2 - t_1), r)$ and write it in the form of a Volterra equation:

$$\begin{aligned} u_1(r) - \theta_n^{-1}\int_0^r T_{r-s}^\beta(u_1^2(s))dx &= T_r^\beta(\lambda_1\phi_1 + u_2(t_2 - t_1)) \\ &= T_r^\beta(\lambda_1\phi_1) + T_r^\beta(u_2(t_2 - t_1)) \\ &= \lambda_1 T_r^\beta\phi_1 + T_r^\beta a_0(t_2 - t_1) + \theta_n^{-1}T_r^\beta a_1(t_2 - t_1) + O(\theta_n^{-2}) \end{aligned}$$

which is the same integral equation as for u_2 with a different non-homogeneous term. In this case we have:

$$u_1(r) = \sum_{k=0}^{\infty} A^k(T_r^\beta(\lambda_1\phi_1 + u_2(t_2 - t_1)))$$

but this time the operator A^k is given by:

$$\begin{aligned} A^0(T_r^\beta(\lambda_1\phi_1 + u_2(t_2 - t_1))) &= \mathbb{I}(T_r^\beta(\lambda_1\phi_1 + u_2(t_2 - t_1))) \\ &= T_r^\beta(\lambda_1\phi_1 + u_2(t_2 - t_1)) \\ &= \lambda_1 T_r^\beta\phi_1 + T_r^\beta a_0(t_2 - t_1) + \theta_n^{-1}T_r^\beta a_1(t_2 - t_1) + O(\theta_n^{-2}) \\ A^1(T_r^\beta(\lambda_1\phi_1 + u_2(t_2 - t_1)))\theta_n^{-1} &\int_0^r T_{r-s}^\beta(T_s^\beta(\lambda_1\phi_1 + u_2(t_2 - t_1)))^2\, ds \\ &= \theta_n^{-1}\int_0^r T_{r-s}^\beta(\lambda_1 T_s^\beta\phi_1 + T_s^\beta a_0(t_2 - t_1))^2 ds + O(\theta_n^{-3}) \\ &= \theta_n^{-1}\int_0^r T_{r-s}^\beta(\lambda_1 T_s^\beta\phi_1 + T_s^\beta a_0(t_2 - t_1))^2 ds + O(\theta_n^{-3}) \\ &= \theta_n^{-1}\int_0^r T_{r-s}^\beta(\lambda_1 T_s^\beta\phi_1 + T_s^\beta a_0(t_2 - t_1))^2 ds + O(\theta_n^{-3}) \end{aligned}$$

setting:

$$b_0(r) := \lambda_1 T_r^\beta \phi_1 + T_r^\beta a_0(t_2 - t_1)$$

$$b_1(r) := T_r^\beta a_1(t_2 - t_1) +$$
$$+ \int_0^r T_{r-s}^\beta ((\lambda_1 T_s^\beta \phi_1)^2 + 2\lambda_1 T_s^\beta \phi_1 T_s^\beta a_0(t_2 - t_1) + (T_s^\beta a_0(t_2 - t_1))^2) ds$$

We obtain:

$$u(\lambda_1 \phi_1 + u_2(t_2 - t_1), t_1) = b_0(t_1) + \frac{1}{\theta_n} b_1(t_1) + o(\theta^{-2})$$

where:

$$b_0(t_1) = \lambda_1 T_{t_1}^\beta \phi_1 + \lambda_2 T_{t_2}^\beta \phi_2$$

$$b_1(t_1) = \lambda_2^2 T_{t_1}^\beta \int_0^{t_2 - t_1} T_{t_1 - s}^\beta (T_s^\beta \phi_2)^2 \, ds + \lambda_1^2 \int_0^{t_1} T_{t_1 - s}^\beta (T_s^\beta \phi_1)^2 \, ds$$
$$+ 2\lambda_1 \lambda_2 \int_0^{t_1} T_{t_1 - s}^\beta \left(T_s^\beta \phi_1 \cdot T_{t_2 - t_1 + s}^\beta \phi_2 \right) \, ds + \lambda_2^2 \int_0^{t_1} T_{t_1 - s}^\beta (T_{t_2 - t_1 + s}^\beta \phi_2)^2 \, ds$$

As in the one dimensional case, the term $b_0(t_1)$ vanishes and we get:

$$\mathbb{E}_\mu \exp(-\langle \lambda_1 \phi_1, Z_{t_1}^n \rangle - \langle \lambda_2 \phi_2, Z_{t_2}^n \rangle) \to \exp(\int_0^1 b_1(t_1) dx).$$

Using eqs. (5.3.15) and (5.3.16), we obtain:

$$\int_0^1 b_1(t) dx = \int_0^1 \int_{t_1}^{t_2} e^{-\beta s} (T_{t_2 - s}^\beta \lambda_2 \phi_2)^2 \, dx ds$$
$$+ \int_0^1 \int_0^{t_1} e^{-\beta s} (T_{t_1 - s}^\beta \lambda_1 \phi_1 + T_{t_2 - s}^\beta \lambda_2 \phi_2)^2 \, ds dx.$$

Now consider the two-dimensional distributions of the limit process Z_t. Using the notation $\psi_i := \lambda_i\phi_i$, $i = 1, 2$ and $dW_s = W(e^{-\beta t}dy, ds)$:

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^1 \lambda_1\phi_1 Z_{t_1} + \lambda_2\phi_2 Z_{t_2}\right]^2 = \\
&= \mathbb{E}\left[\int_0^1 \left(\int_0^{t_1} T^{\beta}_{t_1-s}\psi_1\, dW_s + \int_0^{t_2} T^{\beta}_{t_2-s}\psi_2 dW_s\right)\right]^2 \\
&= \mathbb{E}\left[\int_0^1 \left(\int_0^{t_1} T^{\beta}_{t_1-s}\psi_1\, dW_s + \int_0^{t_1} T^{\beta}_{t_2-s}\psi_2 dW_s + \int_{t_1}^{t_2} T^{\beta}_{t_2-s}\psi_2 dW_s\right)\right]^2 \\
&= \mathbb{E}\left[\int_0^1 \left(\int_0^{t_1} (T^{\beta}_{t_1-s}\psi_1 + T^{\beta}_{t_2-s}\psi_2) dW_s + \int_{t_1}^{t_2} T^{\beta}_{t_2-s}\psi_2 dW_s\right)\right]^2 \\
&= \int_0^1 \int_0^{t_1} (T^{\beta}_{t_1-s}\psi_1 + T^{\beta}_{t_2-s}\psi_2)^2\, e^{-\beta t} ds dy + \int_0^1 \int_{t_1}^{t_2} (T^{\beta}_{t_2-s}\psi_2)^2\, e^{-\beta t} ds dy \\
&= \int_0^1 \int_0^{t_1} (T^{\beta}_{t_1-s}\lambda_1\phi_1 + T^{\beta}_{t_2-s}\lambda_2\phi_2)^2\, e^{-\beta t} ds dy + \int_0^1 \int_{t_1}^{t_2} (T^{\beta}_{t_2-s}\lambda_2\phi_2)^2\, e^{-\beta t} ds dy \\
&= \int_0^1 b_1(t)(x) dx
\end{aligned}
$$

where, in the fourth equality, we used :

$$\mathbb{E}\int_0^{t_1} F_s dW_s \cdot \int_{t_1}^{t_2} F_s dW_s = 0.$$

This completes the proof of the convergence of the two-dimensional distributions. Repeated use of the Markov property and essentially the same calculations gives the proof of the n-tuple distributions for any n. □

Remark 5.3.1. The same procedure shows the convergence of the finite-dimensional distributions for both cases, without immigration and with immigration.

5.3.4 Proof of tightness

In the previous subsection we proved that for any $d \in \mathbb{N}$ and $\phi_1, \ldots, \phi_d \in C_+([0,1])$, $(\langle V_n(t_1), \phi_1\rangle, \ldots, \langle V_n(t_d), \phi_d\rangle) \Rightarrow (\langle V(t), \phi_1\rangle, \ldots, \langle V(t), \phi_d\rangle)$. Our objective is now to establish weak convergence of processes in the sense of weak convergence of the probability laws on $C([0,\infty), H_*)$ where the separable Hilbert space H_* is specified below. Since all the processes involved have a.s. continuous sample paths it suffices to prove tightness in $D([0,\infty), H_*)$, the Skorohod space of càdlàg paths (see [EK], Ch. 3, Prob. 25(d)).

For the tightness in $D([0,\infty), H_*)$ we will apply the following criterion of Jakubowski with $E = H_*$.

Theorem 5.3.8. *(Jakubowski's criterion for tightness for $D([0,\infty), E)$)*

Let (E, d) be a Polish space. Let $\mathbb{F}$ be a family of real continuous functions on E that separates points in E and is closed under addition, i.e. $f, g \in \mathbb{F} \Rightarrow f + g \in \mathbb{F}$. Given $f \in \mathbb{F}$, $\tilde{f} : D_E \to D([0,\infty), \mathbb{R})$ is defined by $(\tilde{f}(x))(t) := f(x(t))$. A sequence, $\{P_n\}$ of probability measures on D_E is tight iff the following two conditions hold:

(i) for each $T > 0$ and $\epsilon > 0$ there is a compact $K_{T,\epsilon} \subset E$ such that:

$$P_n(D([0,T], K_{T,\epsilon})) > 1 - \epsilon$$

(ii) The family $\{P_n\}$ is $\mathbb{F}$-weakly tight, i.e. for each $f \in \mathbb{F}$ the sequence $\left\{P_n \circ (\tilde{f})^{-1}\right\}$ of probability measures in $D([0,\infty), \mathbb{R})$ is tight.

The proof will then follow in two main steps. In step 1 we verify condition (i) of Jakubowski's theorem and then in step 2 we verify condition (ii).

Step 1.

We first identify a scale of Hilbert spaces that will be used to identify the compact subset K needed for condition (i) of Jakubowski's theorem.

For a real number s, let H_s be the Sobolev space given by the image of $L^2(\mathbb{R})$ under the operator $\Lambda^{-s} = (I - \Delta)^{-s/2}$ - see Appendix for details.

The strategy is to show first that $\sup_n \mathbb{E} \sup_{0 \leq t \leq T} \|V_t^n\|_{H_{-s}}^2 < M < \infty$ for $s > \frac{1}{2}$, that is the processes form a bounded set in H_{-s} and then we use the following theorem (see [Fo 99], pp. 305):

Theorem 5.3.9. ***(Rellich's Theorem).*** *Suppose that $\{f_k\}$ is a sequence of distributions in H_s that are all supported in a fixed compact set K and satisfy $\sup_k \|f_k\|_s < \infty$. Then there is a convergent subsequence $\{f_{k_j}\}$ in H_t for all $t < s$, $s, t \in \mathbb{R}$. That is, H_s is compactly embedded in H_t and bounded subsets of H_s are embedded into compact subsets of H_t.*

Since as above we can assume that $Z(t,x) \leq C$ on $[0,T] \times [0,1]$, without loss of generality we can replace $Z(t,x)$ by a constant in the following calculations since all bounds obtained will then dominate those with $Z(t,x)$.

Lemma 5.3.10. *Given the process X_t^n defined by the SPDE (5.2.1) with $\{\theta_n\}_{n \geq 1}$ such that $\theta_n \geq 1$ $\theta_n \to \infty$, and $W(dt,dx)$ space-time white noise on $[0,1]$ with values in H_{-s} for $s > \frac{1}{2}$, then the processes defined by the stochastic integrals:*

$$M_n(t,x) = \int_0^t \sqrt{\frac{X^n(s,x)Z(s,x)}{\theta_n}}\, W(ds,dx)$$

are H_{-s}-valued martingales uniformly bounded in n, that is, there is an M such that:

$$\mathbb{E} \sup_{0 \le t \le T} \|M_n(t)\|_{-s} < M < \infty$$

Proof. Assume that $\sup_{t\in[0,T],x\in[0,1]} Z(t,x) \le C$. Let $\{\phi_k\}$ be a c.o.n. system of $H_0[0,1]$, then the real-valued martingales:

$$\langle m_n(t,x), \phi_k(x)\rangle = \int_0^t \int_0^1 \sqrt{\frac{Z(s,x)X^n(s,x)}{\theta_n}}\, \phi_k(x)\, W(ds,x)$$

satisfy

$$\begin{aligned} &\mathbb{E}\,\langle m_n(t,x), \phi(x)\rangle \,\langle m_n(t,x), \psi(x)\rangle \\ &= \int_0^t \int_0^1 \phi(x)\psi(x) E(\frac{X^n(t,x)}{\theta_n}) ds \\ &\le Ct \int_0^1 \phi(x)\psi(x) dx. \end{aligned}$$

Since the embedding $H_0 \subset H_{-s}$ is Hilbert Schmidt for $s > 1/2$

$$\mathbb{E}\,\|M_n(t)\|_{-s}^2 \le Ct \sum \|\phi_i\|_{-s}^2 \; < \infty$$

and the result follows by Doob's inequality (see [MP], [MP2]).

□

Recall that the quadratic variation of the H_{-s}-martingale, $[M]_t$, is a process such that $\|M_t\|_{-s}^2 - [M]_t$ is a martingale.

We can now apply the following result of Kotelenez (see [Ktz 87], [Ktz 07]):

Theorem 5.3.11. *(Maximal Inequality for Stochastic Convolution Integrals) Let $m(\cdot)$ be an $\mathbb{H}$-valued square integrable càdlàg martingale with quadratic variation*

$[M]_t$ and $U(t,s)$ a strongly continuous two-parameter semigroup of bounded linear operators on $\mathbb{H}$. Suppose there is an $\eta \geq 0$ such that:

$$\|U(t,s)\|_{\mathcal{L}(\mathbb{H})} \leq e^{\eta(t-s)} \quad \forall 0 \leq t < \infty.$$

Then the $\mathbb{H}$-valued convolution integral $\int_0^t U(t,s)M(ds)$ has a càdlàg modification and for any bounded stopping time $\tau \leq T < \infty$:

$$\mathbb{E} \sup_{0 \leq t \leq \tau} \left\| \int_0^t U(t,s)m(ds) \right\|_{\mathbb{H}}^2 \leq e^{4T\eta} E[M]_\tau$$

Using the above result, we now prove:

Lemma 5.3.12. *Fix $T > 0$, $s > \frac{1}{2}$ and consider the processes $V_t^n = X_t^n - \theta_n$ where:*

$$dX_t^n = \frac{1}{2}\Delta X_t^n dt + \beta(\theta_n - X_t^n)dt + \sqrt{\frac{Z(t,x)X_t^n}{\theta_n}} W(ds,dx), \quad X_0^n = \theta_n.$$

Then

$$\mathbb{E} \sup_{0 \leq t \leq T} \|V_t^n\|_{-s}^2 \leq M < \infty \textit{ for some } M > 0.$$

Proof. Then:

$$V_t^n = X_t^n - \theta_n = \int_0^t T_{t-s}^\beta \left(\frac{Z(t,x)X_t^n(x)}{\theta_n} \right)^{1/2} W(dy,dx)$$

is a H_{-s} valued process. As before:

$$M_t^n = \int_0^t \left[\frac{Z(t,x)X_t^n}{\theta_n} \right]^{1/2} W(ds,dx)$$

is a H_{-s}-valued martingale with quadratic variation $[M^n]_t$ satisfying:

$$\mathbb{E}[M^n]_t = \int_0^t \sum_i \|\phi_i\|_{-s}^2 \int_0^1 \mathbb{E}\left[\frac{Z(t,x)X_t^n(x)}{\theta_n} \right] dx\, ds < \infty$$

since by theorem (5.2.5) $\mathbb{E}(Z(t,x)X_t^n(x)/\theta_n) \leq C$. Together with Theorem (5.3.11), this yields:

$$\mathbb{E} \sup_{0\leq t\leq T} \|V_t^n\|_{-s} \leq M < \infty.$$

□

Remark 5.3.2. Since M is independent of n we also obtain the following bound uniformly on n:

$$\sup_n \mathbb{E} \sup_{0\leq t\leq T} \|V_t^n\|_{-s} \leq M < \infty$$

Theorem 5.3.13. *Given $s > \frac{1}{2}$ and $\delta > 0$, the measures induced by the process V_t^n on $C([0,\infty), H_{-(s+\delta)})$ satisfy condition (i) of Jakubowski's criterion.*

Proof. Let M be as above. Given $\epsilon > 0$ and an integer m let $\mathbb{K} \subset H_{-s}$ be the bounded set defined by:

$$\mathbb{K} = \left\{h \in H_{-s} : \|h\|_{-s} \leq mM\right\}$$

Then, by Chebyshev's inequality :

$$P[V_t^n \in \mathbb{K}^c \text{ for some } 0 \leq t \leq T] \leq \sup_n P[\sup_{0\leq t\leq T} \|V_t^n\|_{-s} \geq mM] \leq \frac{1}{m}$$

choosing $m > 1/\epsilon$, follows:

$$P[\sup_n \sup_{0\leq t\leq T} \|V_t^n\|_{-s} > mC] \leq \frac{1}{m} < \epsilon.$$

By Rellich's theorem, the set

$$\mathbb{K} = \left\{h \in H_{-(s+\delta)} : \|h\|_{-(s)} \leq mM\right\}$$

is a compact subset of $H_{-(s+\delta)}$. This shows the first condition of (5.3.8) $\square$

Step 2.

For the second part we will use the Joffe-Metivier Criterion for tightness of D-semimartingales , see [JM] or the Appendix. Let us write first the $V_t^n = X_t^n - \theta_n$ in the form :

$$X_t^n - \theta_n = \int_0^t \left(\frac{1}{2}\Delta X_t^n\, dt + \beta(\theta_n - X_t^n)\right) dt + \int_0^t \sqrt{\frac{X_s^n Z_s}{\theta_n}}\, dW_s$$

Then, for a given eigenfunction $\phi \in C([0,1])$, $\Delta\phi = -\lambda\phi$:

$$\begin{aligned}
\langle V_t^n, \phi\rangle &= \int_0^t \left\langle \frac{1}{2}\Delta X_s^n, \phi\right\rangle ds + \int_0^t \langle \beta(\theta_n - X_s^n), \phi\rangle\, ds \\
&+ \int_0^t \int_0^1 \sqrt{\frac{X_s^n}{\theta_n}}\phi\, dW_s \\
&= \int_0^t (-(\lambda+\beta)\langle X_s - \theta_n, \phi\rangle)\, ds + \int_0^t \int_0^1 \sqrt{\frac{X_s^n Z_s}{\theta_n}}\phi\, dW_s \\
&= \int_0^t (-(\lambda+\beta)\langle V_s^n, \phi\rangle)\, ds + \int_0^t \int_0^1 \sqrt{\frac{X_s^n Z_s}{\theta_n}}\phi\, dW_s.
\end{aligned}$$

Defining the real-valued process $\mathbf{V}_t^n := \langle V_t^n, \phi\rangle$ and the martingale $\mathbf{M}_t^n := \int_0^t \int_0^1 \sqrt{\frac{X_s^n Z_s}{\theta_n}}\phi\, dW_s$ we can write the above process, the following way:

$$\mathbf{V}_t^n = \int_0^t (-(\lambda+\beta)\mathbf{V}_s^n)\, ds + \int_0^t d\mathbf{M}_s^n.$$

Now set: $b_n(\mathbf{V}_t^n) := -(\lambda+\beta)\mathbf{V}_t^n$ and $\sigma_n(\mathbf{M}_t^n) := 1$. Let $D(L) \subset C(\mathbb{R})$ be the space generated by the polynomials, for $\psi \in D(L)$ the process $\psi(\mathbf{V}_t^n)$ satisfies:

$$d\psi(\mathbf{V}_t^n) = \psi'(\mathbf{V}_t^n) b_n(\mathbf{V}_t^n)\, dt + \frac{1}{2}\psi''(\mathbf{V}_t^n)\, d[\mathbf{M}_t^n] + \psi'(\mathbf{V}_t^n)\, d\mathbf{M}_t^n \qquad (5.3.24)$$

where according to (5.2.5) the quadratic variation $[\mathbf{M}_t^n]$ satisfies:

$$[\mathbf{M}_t^n] = \int_0^t \int_0^1 \phi^2 \mathbb{E} \frac{X_s^n Z_s}{\theta_n} \, dx \, ds \leq Ct$$

that is, $d[M]_t^n \leq Cdt$ for some constant C. Then equation (5.3.24) can be written as:

$$d\psi(\mathbf{V}_t^n) = (\psi'(\mathbf{V}_t^n) b_n(\mathbf{V}_t^n) + \frac{1}{2}\psi''(\mathbf{V}_t^n)) \, dt + \psi'(\mathbf{V}_t^n) \, d\mathbf{M}_t^n.$$

Following the notation of Joffe-Metivier , let:

$$L(\psi, \mathbf{V}^n, t) := \psi'(\mathbf{V}_t^n) b_n(\mathbf{V}_t^n) + \frac{1}{2}\psi''(\mathbf{V}_t^n)$$

with $\psi(x) = x$ and $\psi^2(x) = x^2$, the local coefficients of first and second order are defined as follows:

$$\beta_n(z, t, \omega) := L(\psi, z, t, \omega) = -(\lambda + \beta) z$$

$$\alpha_n(z, t, \omega) := L(\psi^2, z, t, \omega) - 2z\beta_n(z, t, \omega) = 1$$

Then we can verify the conditions of the Joffe-Métivier criterion (see Appendix) with $A_t^n = t$:

(i) $\sup_n \mathbb{E} |\mathbf{V}_0^n|^2 = 0 < \infty$

(ii)

$$\begin{aligned} |\beta_n(z, t)|^2 + \alpha_n &= |(\lambda + \beta)|^2 z^2 + 1 \\ &= (\lambda + \beta)^2 \left[\frac{1}{(\lambda + \beta)^2} + z^2 \right] \\ &= K(C_t^n(\omega) + z^2) \end{aligned}$$

where in the last line $K := (\lambda + \beta)^2$ and $C_t^n(\omega) := \frac{1}{(\lambda+\beta)^2}$

for every $T > 0$:

$$\sup_n \sup_{t\in[0,T]} \mathbb{E}\,|C_t^n| < \frac{2}{(\lambda+\beta)^2} < \infty$$

$$\lim_{k\to\infty} \sup_n \mathbb{P}(\sup_{t\in[0,T]} C_t^n > k) = 0.$$

(iii) From $A_t^n = t$ follows $A_t^n - A_s^n = (t-s)$ and the condition is satisfied.

(iv) The process

$$\mathbf{V}_t^n = \int_0^t (\lambda+\beta)\mathbf{V}_t^n\, dt + \mathbf{M}_t^n$$

is such that $\mathbf{V}_t^n = 0$ and also:

$$\begin{aligned}
|\mathbf{V}_t^n|^2 &= \left[(\lambda+\beta)\int_0^t \mathbf{V}_s^n\, ds + \mathbf{M}_t^n\right]^2 \\
&\le 2(\lambda+\beta)^2 \left(\int_0^t \mathbf{V}_s^n\, ds\right)^2 + 2\,|\mathbf{M}_t^n|^2 \\
&\le 2(\lambda+\beta)^2 t \int_0^t |\mathbf{V}_s^n|^2\, ds + 2\,|\mathbf{M}_t^n|^2
\end{aligned}$$

taking the sup on $t \in [0,T]$:

$$\begin{aligned}
\sup_{t\in[0,T]} |\mathbf{V}_t^n|^2 &\le 2(\lambda+\beta)^2 T \int_0^t \sup_{r\in[0,s]} |\mathbf{V}_r^n|^2\, ds + 2 \sup_{t\in[0,T]} |\mathbf{M}_t^n|^2 \\
&\le 2(\lambda+\beta)^2 2T \int_0^T \sup_{r\in[0,s]} |\mathbf{V}_r^n|^2\, ds + 2 \sup_{t\in[0,2T]} |\mathbf{M}_t^n|^2
\end{aligned} \tag{5.3.25}$$

setting:

$$C_1 := 4(\lambda+\beta)T$$

$$C_2 := 2 \sup_{t\in[0,2T]} |\mathbf{M}_t^n|^2$$

$$v(t) := \sup_{s\in[0,t]} |\mathbf{V}_s^n|^2$$

then eq. (5.3.25) can be written as:

$$v(T) \leq C_1 \int_0^T v(s)\, ds + C_2$$

Gronwall's lemma implies:

$$v(T) \leq C_2(1 + C_1 T e^{C_1 T})$$

that is:

$$\sup_{t\in[0,T]} |\mathbf{V}_t^n|^2 \leq 2 \sup_{t\in[0,2T]} |\mathbf{M}_t^n|^2 \left(1 + (\lambda+\beta)^2 4T\right) e^{(\lambda+\beta)^2 4T}. \tag{5.3.26}$$

Now using the maximal inequality for martingales:

$$\mathbb{E}\left(\sup_{t\in[0,T]} |\mathbf{M}_t^n|^2\right) \leq 4 \sup_{t\in[0,T]} \mathbb{E}\,|\mathbf{M}_t^n|^2$$

and taking expectations on both sides of (5.3.26), yields:

$$\begin{aligned}
\mathbb{E} \sup_{t\in[0,T]} |\mathbf{V}_t^n|^2 &\leq \mathbb{E}\left(2 \sup_{t\in[0,2T]} |\mathbf{M}_t^n|^2 \left(1 + (\lambda+\beta)^2 4T\right) e^{(\lambda+\beta)^2 4T}\right) \\
&\leq 2(1 + (\lambda+\beta)^2 4T)\,\mathbb{E}\left(\sup_{t\in[0,2T]} |\mathbf{M}_t^n|^2\right) \\
&\leq 2(1 + (\lambda+\beta)^2 4T) \cdot 4 \sup_{t\in[0,T]} \mathbb{E}\,|\mathbf{M}_t^n|^2 .
\end{aligned}$$

Condition (iv) now follows with $K_T := 2(1 + (\lambda+\beta)^2 4T)\left(4 \sup_{t\in[0,T]} \mathbb{E}\,|\mathbf{M}_t^n|^2\right)$ and recalling that $\mathbb{E}\,|\mathbf{V}_t^n| = 0$.

(v) Finally:

$$
\begin{aligned}
\mathbb{E}\left(\sup_{t\in[0,T]} |\mathbf{M}_t^n|^2\right) &\leq 4 \sup_{t\in[0,T]} \mathbb{E}\,|\mathbf{M}_t^n|^2 \qquad \text{(martingale inequality)}\\
&\leq 2(1+(\lambda+\beta)^2 4T)\left(4 \sup_{t\in[0,T]} \mathbb{E}\,|\mathbf{M}_t^n|^2\right)\\
&= K_T.
\end{aligned}
$$

This verifies that the family P_n is $\mathbb{F}$-weakly tight with $\mathbb{F}$ given by finite linear combinations of eigenfunctions and therefore condition (ii) of Jakubowski's criterion is satisfied. This completes the proof that that the sequence of process V^n is tight in $D([0,\infty), H_{-(s+\delta)})$. As pointed out above, since all processes are continuous this implies tightness in $C([0,\infty), H_{-(s+\delta)})$ (see [EK] pp. 153 problem 25.)

Step 3

Combining steps 1 and 2 we have proved that $V_t^n \Rightarrow V_t$ in the sense of weak convergence of probability laws on $C([0,\infty), H_{-(s+\delta)})$.

Proof of the Corollary To prove the weak convergence it suffices to show that for every bounded continuous function, F, on the appropriate spaces in convergence statements (1) and (2),

$$\lim_{k\to\infty} E(F(X_k^*)) \to E(F(X^*)). \tag{5.3.27}$$

But since $E(F(X^*)) = E[E(F(X^Z))]$ and we have proved in Theorem 5.2.1 that

$$P(\{Z : \lim_{k\to\infty} E(F(X_k^Z)) = E(F(X^Z))\}) = 1 \tag{5.3.28}$$

(5.3.27) follows by bounded convergence.

CHAPTER 6
Concluding remarks

In this concluding chapter we briefly discuss some possible extensions of the results obtained in this thesis as well as some open problems.

The great variety of techniques explored in this work, particularly those concerning continuity of the paths of the solutions and computation of higher moments can be easily adapted to higher dimensions and extended to larger classes of catalytic processes. However others still seem to be very challenging, to mention a few:

- the OU with catalyst $\delta_{B(t)}$, here very few results have been obtained;
- the more general SBM (α, d, β), $\beta \neq 1$ would require new methods in the annealed case since, for example, fourth moments are expected to be infinite in this case.

Only one functional of the catalytic OU process in a SBM media based on the Laplace functional was obtained, other functionals are of interest but they were not included due to the length of the thesis.

The fact that the variance is infinite on the support of a catalyst has been already observed for one-point catalysts and it seems to be true for more general singular catalysts in higher dimensions. This conjecture is supported by a zero-one law of Gaussian processes which asserts that a Gaussian process has with probability zero or one, either continuous or locally unbounded paths. Unfortunately, the available criteria are still extremely difficult to verify.

The proof of the fluctuation limit theorem in Chapter 5 is based on explicit calculation of finite dimensional distributions and simple convergence of processes, this procedure is safe but needs long computations. Other possibilities using the Feymann-Kac formula or the martingale problem method could be explored.

The proof of the fluctuation limit theorem relies on the fact that SBM has a density in $d = 1$ and the situation in higher dimensions is challenging since densities do not exist. It is known that non-trivial SBM with SBM catalyst exists only in $d = 1, 2$ and 3, and is a pure deterministic diffusion in higher dimensions (see [Da 95]), whereas the OU process with SBM catalyst is non-trivial in all dimensions. The nature of the fluctuations in the intermediate dimensions $d = 2, 3$ and whether or not they are described by a catalytic OU process is an open problem.

Catalytic Ornstein-Uhlenbeck where there is a correlation between the moving particles ($Q \neq Id$) normally known as colored noise are easier to deal with than the case $Q = Id$ (white-noise), this is due to the fact that the technical difficulty involving the cylindrical Wiener process does not appear, this might be an interesting topic for future research.

Appendix

Nuclear and Hilbert-Schmidt operators

Let E, G be Banach spaces and let $L(E, G)$ be the Banach space of all linear bounded operators form E into G endowed with the supremum norm. Let us denote by E' and G' the dual spaces of E and G respectively. An element $T \in L(E, G)$ is said to be a *nuclear* operator if there exists two sequences $\{a_j\} \subset G, \{\phi_j\} \subset E'$ such that

$$\sum_{j=1}^{\infty} ||a_j|| \cdot ||\phi_j|| < +\infty$$

and T has the representation

$$Tx = \sum_{j=1}^{\infty} a_j \phi_j(x), \quad x \in E$$

The space of all nuclear operators from E into G, endowed with the norm

$$||T||_1 = \inf \left\{ \sum_{j=1}^{\infty} ||a_j|| \cdot ||\phi_j|| : Tx = \sum_{j=1}^{\infty} a_j \phi_j(x) \right\}$$

is a Banach space and will be denoted by $L_1(E, G)$. We write $L_1(E)$ instead of $L_1(E, E)$.

Theorem Let K be another Banach space; such that $T \in L_1(E, G)$ and $S \in L(G, K)$ then $TS \in L_1(E, K)$ and $||TS||_1 \leq ||T|| \, ||S||_1$

Let H be a separable Hilbert space and let $\{e_k\}$ be a CONS in H. If $T \in L_1(H,H)$ then we define the trace of T:

$$\mathrm{Tr} T = \sum_{j=1}^{\infty} \langle Te_j, e_j \rangle .$$

Theorem If $T \in L_1(H)$ then Tr T is a well-defined number independent of the choice of the orthonormal basis $\{e_k\}$. Further:

(i) $|\mathrm{Tr}\ T| \leq \|T\|_1$.

(ii) If $S \in L(H)$, then $TS, ST \in L_1(H)$ and

$$\mathrm{Tr}\ TS = \mathrm{Tr}\ ST \leq \|T\|_1 \|S\|$$

Proposition A nonnegative operator $T \in L(H)$ is nuclear if and only if for any orthonormal basis $\{e_k\}$ on H:

$$\sum_{j=1}^{\infty} \langle Te_j, e_j \rangle < +\infty$$

Moreover, in this case Tr $T = \|T\|_1$.

Let E and F be two separable Hilbert spaces with complete orthonormal basis $\{e_k\} \subset H$, $\{f_j\} \subset F$. A linear bounded operator $T : H \to E$ is said to be *Hilbert-Schmidt* if

$$\sum_{k=1}^{\infty} |Te_k|^2 < \infty$$

It is easy to verify that the set $L_2(E,F)$ of all Hilbert-Schmidt operators from E into F, equipped with the norm :

$$\|T\|_2 = \left(\sum_{k=1}^{\infty} |Te_k|^2 \right)^{1/2}$$

is a Hilbert space with scalar product given by:

$$\langle S, T \rangle = \sum_{k=1}^{\infty} \langle Se_k, Te_k \rangle$$

We write $L_2(E)$ instead of $L_2(E,E)$.

Proposition Let E, F, G be separable Hilbert spaces. If $T \in L_2(E,F)$ and $S \in L_2(F,G)$, then $ST \in L_1(E,G)$ and

$$\|ST\|_1 \leq \|S\|_2 \|T\|_2$$

The Cylindrical Wiener process

Let Q be a general bounded, self-adjoint, nonnegative operators Q on U. We want to consider the Wiener process with covariance operator Q, when Q is not nuclear we proceed as follows: assume without loss of generality that Q is strictly positive: $Qx \neq 0$ for $x \neq 0$. Let $U_0 = Q^{1/2}(U)$ with the induced norm $\|u\|_0 = \left\|Q^{-1/2}(u)\right\|, u \in U_0$, and let U_1 be an arbitrary Hilbert space such that U is embedded continuously into U_1 and the embedding of U_0 into U_1 is Hilbert-Schmidt. Let $\{g_j\}$ be an orthonormal and complete basis in U_0 and $\{\beta_j\}$ be a family of independent real valued standard Wiener processes.

Proposition The formula

$$W(t) = \sum_{j=1}^{\infty} g_j \beta_j(t), \quad t \geq 0$$

defines a Q_1-Wiener process on U_1 with Tr $Q_1 < \infty$. For arbitrary $a \in U$, the process

$$\langle a, W(t) \rangle = \sum_{j=1}^{\infty} \langle a, g_j \rangle \beta_j(t)$$

is a real valued Wiener process and

$$\mathbb{E} \langle a, W(t) \rangle \langle b, W(s) \rangle = (t \wedge s) \langle Qa, b \rangle, a, b \in U$$

Moreover we have Im $Q_1^{1/2} = U_0$ and

$$\|u\|_0 = \left\| Q_1^{-1/2} u \right\|_1.$$

Details of the proof can be found in [DZ 92] Prop 4.11, the main ingredient of which is the fact that the series defining $W(t)$ is convergent in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U_1)$ since:

$$\mathbb{E}\left(\left\| \sum_{j=n}^{m} g_j \beta_j(t) \right\|_1^2 \right) = \sum_{j=n}^{m} \|g_j\|_1^2, \qquad m \geq n \geq 1.$$

and the embedding $J : U_0 \to U_1$ is Hilbert-Schmidt which means $\sum_{j=1}^{\infty} \|g_j\|_1^2 < \infty$.

It is important to remark that the space U_1 is not uniquely defined, in fact, it is often the case that one has to find such a space U_1 and we show below some basic techniques to find it.

Some Hilbert-Schmidt embeddings

- Let M be a smooth compact d-dimensional differentiable manifold (e.g. $[0,1]$ or a domain in $\mathbb{R}^d$) with a smooth boundary and let L be a self-adjoint uniformly strongly elliptic second order differential operator with smooth coefficients, and smooth homogeneous boundary conditions. then $-L$ has a CONS of smooth eigenfunctions $\{\phi_n\}$ with eigenvalues $\{\lambda_n\}$ which satisfy $\sum_j (1+\lambda_j)^{-p} < \infty$ if $p > d/2$.

  Let E_0 be the set of f of the form $f(x) = \sum_{j=1}^{N} c_j \phi_j(x)$ with N finite, where the c_j are constants. For each integer n, positive or negative, define the space $H_n = \{f \in E_0 : \|f\|_n < \infty\}$ to be the completion of $E_)$ with respect to the norm given by :

$$\|f\|_n = \sum_j (1+\lambda_j)^n c_j^2$$

  Note that the embedding $J_0 : H_n \to H_m$ is HS if $n > m + d/2$. Indeed, set $e_j = (1+\lambda_j)^{-n/2}\phi_j$. The e_j form a CONS relative to $\|\cdot\|_n$, and

$$\sum_j \|e_j\|_m^2 = \sum_j (1+\lambda_j)^{m-n} < \infty.$$

  If $f, g \in H_n$, we can represent f by the formal series

$$f = \sum_j c_j \phi_j, \qquad g = \sum_j b_j \phi_j$$

  where $\sum (1+\lambda_j)^n c_j^2 = \|f\|_n < \infty$. Then H_n and H_{-n} are dual under the product $\langle f, g \rangle = \sum_j c_j b_j$.

The ϕ_j are smooth, so that the elements of H_n will be differentiable for large n. Since $E = \bigcap_n H_n, E$ will consist of C^∞ functions. Note that if $f \in H_n$, $Lf \in H_{n-2}$ since $Lf = \sum_j \lambda_j c_j \phi_j$.

- Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $E_0 = \mathcal{D}(\Omega)$ be the set of C^∞ functions of compact support in Ω. Let $\|\cdot\|_0$ be the usual L_2-norm on Ω and set

$$\|\phi\|_n = \|\phi\|_0^2 + \sum_{1 \le |\alpha| \le n} \|D^\alpha \phi\|_0^2$$

where α is a multi-index of length $|\alpha|$, and D^α is the partial derivative operator. Let E_n be the completion of E_0 in the topology induced by the norms $\|\cdot\|_n$.

In this case $H_0 = L^2(\Omega)$ and H_n is the classical Sobolev space (normally denoted by $W_0^{n,2}(\Omega)$. By Maurin's theorem, the embedding mapping

$$W_0^{m+k,2}(\Omega) \to W_0^{m,2}(\Omega)$$

is a HS operator. if $k > d/2$.

The spaces H_{-n} (dual of H_n) consist of derivatives: $f \in H_{-n}$ iff there exists $f_\alpha \in L^2$ such that

$$f = \sum_{|\alpha| \le n} D^\alpha f_\alpha$$

- If we let $\Omega = \mathbb{R}^d$ in the above example, we can use the Fourier transform to define the H_n. Let $E = \mathcal{S}(\mathbb{R}^d)$. If $u \in E$, define the Fourier transform $\hat{u}$ of u by

$$\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi x \cdot \xi} u(x) dx$$

If u is a tempered distribution, i.e. if $u \in \mathcal{S}'(\mathbb{R}^d)$, we can define $\hat{u}$ (as a distribution) by $\hat{u}(\phi) := u(\hat{\phi})$, $\phi \in E$. Define a norm on E by

$$\|u\|_t = \int_{\mathbb{R}^d} (1 + |\xi|^2)^t \, |\hat{u}(\xi)|^2 \, d\xi$$

and let H_t be the completion of E in the norm $\|\cdot\|_t$.

If u is a distribution whose Fourier transform $\hat{u}$ is a function, then $u \in H_t$ iff $\|u\|_t$ is finite. The space $H_o = L^2$ by Plancherel's theorem. For $t > 0$ the elements of H_t are functions. For $t < 0$ they are in general distributions. It can be shown that if t is an integer, say $t = n$, the norms $\|\cdot\|_n$ defined this way and those of the last example are equivalent. Note that $\|\cdot\|_t$ makes sense for all real t , positive or negative, integer or not, and it can be shown that the imbedding form H_t into H_s is HS if $t > s + d/2$.

The Hermite functions Let $E := \mathbb{S}(\mathbb{R}^d)$, be the Schwartz space of rapidly decreasing functions. Let

$$g_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$$

and set

$$h_k(x) = (\pi^{1/2} 2^k k!)^{-1/2} g_k(x) e^{-x^2}$$

the $\{g_k(x)\}_{k\geq 0}$ are called the Hermite polynomials, and the $\{h_k(x)\}_{k\geq 0}$ are the Hermite functions. The latter are a CONS in $L^2(\mathbb{R}^d)$.

Let $q = (q_1, \ldots, q_d)$ where the q_i are non-negative integers, and for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, set

$$h_q(x) = h_{q_1}(x_1) \cdots h_{q_d}(x_d)$$

Then $h_q \in \mathcal{S}(\mathbb{R}^d)$, and they form a CONS in $L^2(\mathbb{R}^d)$. If $\phi \in \mathcal{S}(\mathbb{R}^d)$, let $\hat{\phi}_q = \langle \phi, h_q \rangle$ and write:

$$\phi = \sum_q \hat{\phi}_q h_q$$

Define

$$\|\phi\|_n^2 = \sum_q (2|q| + d)^n \hat{\phi}_q^2$$

where $|q|^2 = q_1^2 + \cdots + q_d^2$. One can show $\|\phi\|_n < \infty$ if $\phi \in \mathcal{S}(\mathbb{R}^d)$. Let $\mathcal{S}_n$ be the completion of E in $\|\cdot\|_n$. Note that this makes sense for negative n, in fact for all real n and:

$$\|\phi\|_{-n}^2 = \sum_q (2|q| + d)^{-n} \hat{\phi}_q^2$$

further $\mathcal{S}_{-n}$ is dual to $\mathcal{S}_n$ under the inner product

$$\langle \phi, \psi \rangle = \sum_q \hat{\phi}_q \psi_q$$

The Hilbert-Schmidt embedding of such spaces is easily verified here, since the functions $e_q = (2|q| + d)^{-n/2} h_q$ are a CONS under $\|\cdot\|_n$ and if $m < n$:

$$\sum_q \|e_q\|_m = \sum_q (2|q| + d)^{(n-m)}$$

which is finite if $n > m + d/2$. Thus the embedding:

$$\mathcal{S}_n \hookrightarrow \mathcal{S}_m$$

is Hilbert-Schmidt if $n > m + \frac{d}{2}$.

Remark that the spaces $\mathcal{S}_n$ can be also described in terms of the spectral properties of the operator $(|x|^2 - \Delta)$, it is a well-known fact that

$$(|x|^2 - \Delta)h_q = (2\,|q| + d)h_q$$

The Joffe-Metivier Criterion

A *càdlàg* adapted process X, defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with values in $\mathbb{R}$ is called a *D-semimartingale* if there exists an increasing $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ function $A(t)$ a linear sub-space $D(L) \subset C(\mathbb{R})$, and a mapping $L : (D(L) \times \mathbb{R} \times [0, \infty) \times \Omega) \to \mathbb{R}$ with the following properties:

(ai) For every $(x, t, \omega) \in \mathbb{R} \times [0, \infty) \times \Omega)$ the mapping $\phi \to L(\phi, x, t, \omega)$ is a linear functional on $D(L)$ and $L(\phi, ., t, \omega) \in D(L)$.

(aii) For every $\phi \in D(L)$, $(x, t, \omega) \to L(\phi, x, t, \omega)$ is $\mathcal{B}(\mathbb{R}) \times \mathcal{P}-$measurable, where $\mathcal{P}$ is the predictable $\sigma-$algebra on $[0, \infty) \times \Omega$, ($\mathcal{P}$ is generated by the sets of the form $(s, t] \times F$ where $F \in \mathcal{F}_s$ and s, t are arbitrary).

(bi) For every $\phi \in D(L)$ the process M^ϕ defined by

$$M^\phi(t, \omega) := \phi(X_t(\omega)) - \phi(X_0(\omega)) - \int_0^t L(\phi, X_{s-}, s, \omega)\, dA_s$$

is a locally square integrable martingale on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$,

(bii) The functions $\psi(x) := x$ and ψ^2 belong to $D(L)$.

The functions

$$\beta(x, t, \omega) := L(\psi, x, t, \omega)$$

$$\alpha(x, t, \omega) := L((\psi)^2, x, t, \omega) - 2x\beta(x, t, \omega)$$

are called the *local coefficients of first and second order.*

Theorem (*Joffe-Metivier Criterion for tightness of D-semimartingales*)
Let $X^m = (\Omega^m, \mathcal{F}^m, \mathcal{F}_t^m, \mathbb{P}^m)$ be a sequence of D-semimartingales with common $D(L)$ and associated operators L^m, functions A^m, β_m, and α_m. Then the sequence $\{X^m : m \in \mathbb{N}\}$ is tight in $D([0,\infty), \mathbb{R})$ provided the following conditions hold:

(i) $\sup_m \mathbb{E}\,|X_0^m|^2 < \infty$.

(ii) there is a $K > 0$ and a sequence of positive adapted processes $\{C_t^m : t \geq 0\}$ (on Ω^m for each m) such that for every $m \in \mathbb{N}, x \in \mathbb{R}, \omega \in \Omega^m$

(a) $|\beta_m(x,t,\omega)|^2 + \alpha_m(x,t,\omega) \leq K(C_t^m(\omega) + x^2)$.

(b) for every $T > 0$

$$\sup_m \sup_{t\in[0,T]} \mathbb{E}[C_t^m] < \infty \text{ and } \lim_{k\to\infty} \sup_m \mathbb{P}^m(\sup_{t\in[0,T]} C_t^m \geq k) = 0$$

(iii) there exists a positive function γ on $[0,\infty]$ and a decreasing sequence of numbers $\{\delta_m\}$ such that $\lim_{t\to 0} \gamma(t) = 0$, $\lim_{m\to\infty} \delta_m = 0$, and for all $0 < s < t$ and all m.

$$(A^m(t) - A^m(s)) \leq \gamma(t-s) + \delta_m$$

Further, if we set $M_t^m := X_t^m - X_0^m - \int_0^t \beta_m(X_{s-}^m, s, .)\, dA_s^m$, then for each $T > 0$ there is a constant K_T and m_0 such that for all $m \geq m_0$,

(iv) $\mathbb{E}(\sup_{t\in[0,T]} |X_t^m|^2) \leq K_T(1 + \mathbb{E}\,|X_0^m|^2)$ and

(v) $\mathbb{E}(\sup_{t\in[0,T]} |M_t^m|^2) \leq K_T(1 + \mathbb{E}\,|X_0^m|^2)$

List of symbols and abbreviations

NB: catalytic OU without boundaries, 46

NBn: approximate process of NB, 46

SMB: super Brownian motion. $\mathcal{S}$: rapidly decreasing functions, 101

$\mathcal{S}'$: space of distributions, 101

$\mathcal{S}_n$: sub-space of $\mathcal{S}'$, 101

OU: Ornstein-Uhlenbeck. OU-Rd: OU process in $\mathbb{R}^d$, 31

T_t: semigroup of the Laplacian on $\mathbb{R}^d$ or $[0,1]$

UC: OU in the unit circle, 54

UCn: approximate process of UC, 54

YBn: approximate process of NB, 48

Z: SBM, super-Brownian motion.

REFERENCES

[Ba 98] R.F. Bass : *Diffusions and Elliptic Operators*, Springer 98.

[Be 69] S.M. Berman : *Harmonic analysis, local times and sample functions of Gaussian processes*, Trans. Amer. Math. Soc. **143**, 269-281 (1969).

[Be 05] J. Bertoin : *Levy Processes* , Cambridge 2005.

[Bi 95] P. Billingsley : *Probability and Measure*, Wiley 1995.

[Br 73] H. Brezis : *Operateurs Maximaux Monotones*, Elsevier 1973.

[DZ 92] G. Da Prato G. and J. Zabczyk: *Stochastic Equations in Infinite Dimensions*, Cambridge 1992.

[Da 72] D.A. Dawson: *Stochastic evolution equations*, Mathematical Biosciences 15, 287-316, 1972.

[Da 75] D. A. Dawson: *Stochastic evolution equations and realated measure-valued processes*, Journal of Multivariate Analysis **1975** pp. 1-52.

[Da 92] D. A. Dawson: *Stochastic Analysis and Related Topics*, Birkhäuser 1992.

[Da 99] D. A. Dawson and Fleischmann: *Catalytic and Mutually Catalytic Branching*, Progress in Probability, Vol.52, 89-110, Birkhäuser, 2002.

[Da 95] D. A. Dawson and Fleishmann: *A continuous super Brownian motion in a super-Brownian medium*, Journ. of Theoretical Probability, 10, 213-276, 1997.

[Da 93] D. A. Dawson: *Measure-valued Markov Processes*, Lecture Notes in Mathematics, Springer No. **1541**, 1993.

[Da 99] D. A. Dawson and E.A. Perkins: *Measure-Valued Processes and Renormalization of Branching Particle Systems*, AMS Matematical Surveys and Monographs. Vol. **64**.

[Da 04] D. A. Dawson, Zenghu Li and Bryon Schmuland: *Generalized Mehler Semigroups and Catalytic Branching Processes with Immigration, Kluwer*, Potential Analysis No. **21**, 2004.

[Da 05] D. A. Dawson, Y. Li and C. Mueller: *The Support of Measure-valued Branching processes in a Random Environment*, The Annals of Probability No. **23**, 1995.

[Da 89] D. A. Dawson, I. Iscoe and E. A. Perkins: *Super-Brownian Motion: Path Properties and Hitting Probabilities*, Prob. Th. Rel. Fields **83**, 1989.

[Da 05] D. A. Dawson and Zhenghu Li: *Skew Convolution Semigroups and Affine Markov Processes* , Ann. Probab. 34, 1103-1142, 2006.

[Do 42] Doob, J.L.: *The Brownian movement and stochastic equations* , Ann. Math. **43**, 351-369.

[Du 03] D. Duffie, D. Filipovic and W. Schachermayer : *Affine processes and applications in finance* Ann. Appl. Probab. **13**, 984-1053.

[Dy 02] E.B. Dynkin : *Diffusions, Superdiffusions and Partial Differential Equations*, AMS 2002.

[EK] S. Ethier and T.G. Kurtz: *Markov processes, characterization and convergence*, Wiley, New York, 1986.

[EV 97] L. C. Evans: *Partial Differential Equations*, AMS 1997.

[FL 85] K.J. Falconer: *The geometry of fractal sets*, Cambridge 1985.

[Fe 65] W. Feller : *An Introduction to Probability and its Applications*, Wiley 65.

[XFe 74] X. M. Fernique: *Lecture Notes in Mathematics, Springer* No. **480**, 1974.

[Fo 99] G.B. Folland : *Real Analysis*, Wiley 1999.

[Is 86] I. Iscoe: *A Weighted Occupation Time for a Class of Measure-Valued Branching Processes*, Prob. Th. Rel. Fields No. **71**, 85-116.

[Is 88] I. Iscoe: *On the supports of a measure-valued critical branching Brownian motion*, Ann. Prob. **16**, 200-221, 1988.

[IsM 90] I. Iscoe and D. McDonald : *Continuity of l^2 valued Ornstein-Uhlenbeck processes* , The Annals of Probability No. 1 Vol. 18, 1990.

[ISMTZ] I. Iscoe, M.B. Marcus, M. Talagrand, J. Zinn: *Continuity of ℓ^2-valued Ornstein-Uhlenbeck processes*, Ann. Probab. 18, 68-84.

[JM] A. Joffe and M. Métivier (1986), *Weak convergence of sequences of semi-martingales with application to multitype branching processes.* Adv. Appl. Probab. **18**, 20-65.

[Kat 99] I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus* Springer Verlag 1999.

[Ktz 87] P. Kotelenez : *A maximal inequality for stochastic convolution integrals on Hilbert spaces and space-time regularity of linear stochastic partial differential equations* , Stochastics, **21**, 345-358 (1987).

[Ktz 07] P. Kotelenez : *Stochastic Ordinary and Stochastic Partial Differential Equations; Transition from Microscopic to Macroscopic Equations* , Springer Verlag, 2007.

[Kr 99] R. Kress: *Linear Integral Equations*, Springer 99.

[Ksh 88] N. Konno and T. Shiga: *sotchastic Differential Equations for some Measure-Valued Diffusions*, Probability Theory and related fields 79, 201-225.

[Lib 04] Elliot H. Lieb and Michael Loss: *Analysis*, AMS 04, Vol. **14**.

[Li] Z. Li: *Branching processes with immigration and related topics*, Frontiers of Mathematics in China 1(2006), 73-97.

[MP] M. Métivier and J. Pellaumail: *Stochastic Integration*, Academic Press, 1980.

[MP2] M. Métivier and J. Pellaumail: *Semimartingales*, Walter de Gruyter, 1982.

[Ok 00] B. Oksendal : *Stochastic Differential Equations*, Springer 2000.

[Pa 83] A. Pazy : *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer 1983.

[RT 03] B. Rajeev and S. Thangavelu : *Probalilistic representations of solutions of the heat equation*, Proc. Indian Acad. Sci. (Math. Sci.) **113**, 321-332, 2003.

[Sch] L. Schwartz: *Théorie des distributions*, Hermwnn, 1966.

[Sh 02] S. Shreve : *Methods of mathematical finance II, the continuous-time case* Springer Verlag 2002.

[Wil 86] L.C.G. Rogers and D. Williams: *Diffusions, Markov Processes and Martingales*, Cambridge 1986.

[Sat 99] K. I. Sato : *Levy Processes and Infinitely Divisible Distributions*, Cambridge 1999.

[OU 30] G.E. Uhlenbeck, G.E. & L.S. Ornstein : *On the theroy of Brownian motion* Physical Review **36**, 1930, 823-841.

[Wal 81] J. B. Walsh: *A stochastic model of neural response*, Adv. in Appl. Prob. **13**,231-281, 1981.

[Wal 83] J. B. Walsh: *Regularity properties of a stochastic partial differential equation*, Seminar on Stochastic Processes 1983,257-290, Progr. Probab. Statist **7**, Birkhäuser, 1983.

[Wal 85] J. B. Walsh: *An Introduction to Stochastic Partial Differential Equations, Lecture Notes in Mathematics, Springer* No. **1180**, 1984.

[Wa 45] M.C. Wang & G.E. Uhlenbeck : *On the theory of Brownian motion II* Modern Physics **17**, 323-342, 1945.

[Wat 68] S. Watanabe *A limit theorem of branching processes and continuous state branching processes*, J. Math. Kyoto Univ. **8-1**,141-167(1968).

[Y 67] J. Yeh. : *Differentiability of sample functions in Gaussian processes*, Proc. Amer. Math. Soc. **18**,105-108(1967), Correction Proc. Amer. Math. Soc. **19**,843,(1968).

[Ze 89] E. Zeidler : *Nonlinear Functional Analysis and its Applications* **Vol.II/A**, Springer 1989.