# Musical Rhythms in the Euclidean Plane

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2008-12-01

A thesis submitted to McGill University
in partial fulfilment of the requirements of
the degree Doctorate of Philosophy

*Your file Votre référence*
*ISBN:* 978-0-494-66652-4
*Our file Notre référence*
*ISBN:* 978-0-494-66652-4

# Dedication

In memory of my father.

# Abstract

This thesis contains a collection of results in computational geometry that are inspired from music theory literature. The solutions to the problems discussed are based on a representation of musical rhythms where pulses are viewed as points equally spaced around the circumference of a circle and onsets are a subset of the pulses. All our results for rhythms apply equally well to scales, and many of the problems we explore are interesting in their own right as distance geometry problems on the circle.

In this thesis, we characterize two families of rhythms called *deep* and *Euclidean*. We describe three algorithms that generate the unique Euclidean rhythm for a given number of onsets and pulses, and show that Euclidean rhythms are formed of repeating patterns of a Euclidean rhythm with fewer onsets, followed possibly by a different rhythmic pattern. We then study the conditions under which we can transform one Euclidean rhythm to another through five different *operations*. In the context of measuring rhythmic similarity, we discuss the *necklace alignment problem* where the goal is to find rotations of two rhythms and a perfect matching between the onsets that minimizes some norm of the circular distance between the matched points. We provide $o(n^2)$-time algorithms to this problem using each of the $\ell_1$, $\ell_2$, and $\ell_\infty$ norms as distance measures. Finally, we give a polynomial-time solution to the *labeled beltway problem* where we are given the ordering of a set of points around the circumference of a circle and a labeling of all distances defined by pairs of points, and we want to construct a rhythm such that two distances with a common onset as endpoint have the same length if and only if they have the same label.

# Abrégé

Cette thèse contient un ensemble de résultats en géométrie algorithmique qui ont été inspirés par la littérature en théorie de la musique. Nos résultats se basent sur une façon de reprsenter un rythme musical où les pulsations sont symbolisées par des points répartis uniformément sur un cercle, et les onsets sont des sous-ensembles des pulsations. Tous nos résultats sur les rythmes s'appliquent également aux notes, et de nombreux problèmes que nous avons étudis sont intéressants en tant que tels, c'est-a-dire formuls comme des problèmes de distance géométrique sur le cercle.

Dans cette thèse nous caracétrisons deux familles de rythmes, les rythmes profonds (deep) et les rythmes Euclidiens. Nous décrivons trois algorithmes qui génèrent l'unique rythme Euclidien correspondant un nombre donné d'onsets et de pulsations. Nous montrons que les rythmes Euclidiens sont constitués de motifs répétitifs formés par d'autres (sous-)rythmes Euclidiens définis par un nombre moindre d'onsets, parfois suivi par un motif rythmique différent. Ensuite, nous étudions les conditions nécessaires pour pouvoir transformer un rythme Euclidien en un autre, en utilisant cinq types d'oprations bien définies. Dans le cadre de la mesure de la similitude entre rythmes, nous étudions le problème de l'alignement de collier (necklace alignment), où l'objectif est de trouver la rotation d'un rythme et un alignement parfait entre les onsets qui minimise la norme d'une distance circulaire particulière. Nous présentons trois algorithmes dont le temps d'exécution est en $o(n^2)$, respectivement pour les normes $\ell_1$, $\ell_2$ et $\ell_\infty$. Finalement, nous proposons un algorithme polynomial pour le problème du *labeled beltway*: étant donné un ensemble ordonné de points autours d'un cercle, et un labeling de toutes les distances définies par les paires de points,

nous voulons construire un rythme tel que deux distances qui partent d'un même onset ont la même longueur si et seulement si ces distances ont le même label.

# Statement of Originality

All results in this thesis are original contributions to knowledge, except when due acknowledgements are made.

The results in Chapter 8 appear only in this thesis, while the remaining results have appeared previously in conference proceedings and journal publications.

The results in Chapters 3 and 4 have been presented at the 18*th Canadian Conference on Computational Geometry* [60], and the full version of the paper will appear in *Computational Geometry: Theory and Application* [61]. Results in Chapters 5 and 6 have been presented at the 16*th* and 18*th Fall Workshop on Computational Geometry* [85, 165], the *Kyoto International Conference on Computational Geometry and Graph Theory 2007*, and the *Canadian Conference on Computer Science and Software Engineering* [86]. Full versions of the papers containing results from these two chapters will appear in the *Journal of Mathematics and Music* [87, 88]. Results in Chapter 7 were presented at the 14*th Annual Symposium on Algorithms* and published in the proceedings of the same conference [27].

The results in Chapter 8 were achieved in collaboration with David Bremner, Erik D. Demaine, and Godfried Toussaint. The discussion on applications of the theory of rhythmic operations to interlocking Euclidean rhythms in Section 6.6 is due to Paco Gómez-Martín. The list of rhythms in Appendices A, B, and C are compiled by Godfried Toussaint. The note on measuring *ugliness* in linear time in Appendix D is an observation made by Luc Devroye. Finally, the proof of Case 3 of Theorem 4.2.5 in Section 4.2.3 is based on a suggestion by Dmitri Tymoczko.

# Acknowledgements

This work would not have been possible without my supervisor Godfried Toussaint. I thank him for his constant encouragement, his helpful advice, and for his contagious excitement about music and geometry; I also thank him for giving me the opportunity to attend many conferences and workshops where I met most of my co-authors.

I thank Luc Devroye for his friendship. I thank him for taking me as his student with no questions asked.

I thank all my co-authors. With no exceptions, working with them has been a thrill.

I thank Vašek Chvátal, first and foremost for his friendship. He has inspired me in many ways with his lively personality and his unique views on different topics. He has an exceptional talent for transferring knowledge in a clear and absorbing way; often without realizing, he has taught me something new every single time I met him, in areas ranging from literature and arts to mathematics. I also thank him for introducing me to many fascinating people, and for his course *Discrete Mathematics of Paul Erdős* which I enjoyed tremendously.

I thank Victor Campos for bringing an eccentric twist to our daily office routine and for being the zany personality that he is. I thank him for the fun we had during his short stay in Montréal, and for the perverse pleasure we got out of sweating over the assignments for Vašek's course. He truly was my daily "shot in the arm".

I thank Conor Meagher for being the perfect officemate; I thank him particularly for his useful streetwise advice on the strangest everyday problems, from providing tips on how to fix my bike to dealing with drug-snorting homeless hobos camping under my window.

I thank members of the *McGill lunch group* for the sometimes twisted and obscene, but always amusing lunchtime conversations: Luc Devroye, Godfried Toussaint, Vida Dujmovic, Victor Campos, and all others who passed through McGill during my tenure there and sometimes joined us for our daily lunch ritual. I also thank Luc Devroye for documenting the topics of these conversations and our campus-life days on his excellent controversial web page.

I thank Stefan Langerman for hosting me in Brussels; Bernard Chazelle for our funny and exciting political correspondence; Yiannis Rekleitis for his friendship, care, and support; Bruce Reed for his patience, loyalty, and advice; Fausto Errico for being.

I thank the SOCS secretaries; in particular I thank Lucy for for always wearing a smile, and for her useful tips on how to go about getting various paperwork done.

I thank the donors of the espresso machine in the SOCS lounge; going through grad school would have been distressing without their precious donation.

I thank all my friends who supported me throughout the past four years; I thank Lalai in particular.

I thank Manuel Kueseyan for his friendship, and for having such a profound and positive impact on my views on life, people, and the world around.

I thank all writers and poets who gave me pleasant distraction from research during my years at McGill. In particular, I thank Zoulal Kazanjian, Haroutyun Gosdantian, Rawi Hage, and others for their wonderful work.

I thank Narbeh for his perseverance, love, and dedication.

Thank you Beirut for the continual rush you bring to my life. Thank you Yerevan for finally being something like a home. Thank you Montréal for being so warm, so lively, and so colorful despite the thick blankets of snow.

No matter where I go in Life, I know I will always be in Montréal.

Finally, as I owe everything to my family, I thank my brother for always being there, and I thank my parents for their love, their patience, and for the support and unbounded freedom they have given me from first the day I landed in this world.

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# Chapter 1

# Introduction

How would you place four points in the plane so that there is a distance determined by pairs of points that appears once, a distance that appears twice, and a distance that appears three times? With a few trials you may be able to find an embedding that satisfies these constraints; one solution is shown in Figure 1–1.

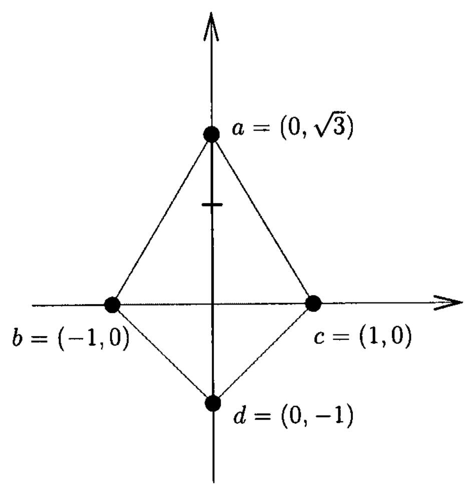



Figure 1–1: Each distance appears a unique number of times: $|ad| = 1 + \sqrt{3}$ appears once, $|bd| = |dc| = \sqrt{2}$ appears twice, and $|ab| = |bc| = |ca| = 2$ appears three times.

Now, can you do the same with any number of points? That is, is it possible to have $n$ points in general position (no three on a line or four on a circle) such that for every $i = 1, \ldots, n-1$, there exists a distance that appears exactly $i$ times? Paul

Erdős asked this question in 1989. Today, almost 20 years later, we can answer his question only for $n$ up to 8. This problem in distance geometry led to the study of geometric properties of musical rhythms, which is the topic of this thesis. The connection is not very obvious of course, but the path from geometry to music theory is nevertheless not too long. Since the solution to Erdős's question does not seem to be trivial, to say the least, we can try to solve a variation of the problem: restrict the points to a circle (you may think of it as dimension 1.5). It turns out that points on a circle with similar restrictions on distance multiplicities have been studied in music theory and are called *deep scales*. This was our first result in computational music theory and is rightfully the first result described in this thesis.

The problems in music theory that we restrict ourselves to in this work are related to time intervals within rhythms and pitch intervals within scales. We ignore all considerations to tempo (speed of a musical piece), acoustics (the physics of sound), harmonics, and all problems related to human musical perception. The problems discussed fall into two main categories: the study of properties of musical rhythms and scales, and their transformations given certain restrictions on their interval durations.

**Motivation.** Historically, mathematics and music have interacted for millenia. In some cases, musicians have composed pieces with no consideration to mathematics but where mathematical ideas may be detected, while in other cases, musicians have explicitly employed mathematical ideas such as Fibonacci numbers and the golden ratio in their musical compositions. An interesting example of the latter is Schoenberg's twelve-tone technique; Schoenberg developed a method for music composition which ensures that all the 12 notes of the diatonic scale appear exactly the same number of times throughout the piece, without emphasizing any [92]. The result is a disturbing musical feel that has made such musical pieces popular in soundtracks of horror movies.

More indirect applications of mathematics to music composition are the modeling and classification tools of musical elements. Such tools help composers visualize the space of musical possibilities. For example, Tymoczko [178] has developed a visualization where every possible musical chord is represented as a point in space, and segments connecting chords describe how to transform one chord to another. According to Tymoczko, composers sometimes like to combine harmonic consistency with efficient voice-leading; one can do this only if the chords divide the octave nearly evenly, or if they are clustered close together in the geometric visualization. This is clear when we look at the geometry, but not so obvious when we are thinking musically [1] . Geometric representations of musical elements may also be useful for teaching: teaching music via geometric visualization, or teaching geometry via musical rhythms and scales [166].

The study of properies of music transformations help us understand and formalize improvisation techniques used by musicians, and perhaps devise new techniques based on mathematical rules. Transformations are also useful in phylogenetic analysis of musical rhythms [63, 171], that is, the study of evolutionary relationship between various rhythms.

[1] Personal communication with Tymoczko, 2008.

More recently, and with the prevalence of computers, other applications of mathematical ideas to musical ends have surfaced. Notable applications include music retrieval from large databases having as query an arbitrary piece of melody, music comparison and recognition, computer-aided music composition, etc. We will detail these applications further in forthcoming chapters.

It should be noted however that finding direct applications of the problems discussed here to music theory is not the purpose of this thesis. We do not claim that our results will provide new tools for musicians, nor try to show connections between music and mathematics that are not necessarily there. The core topic of this work is the study of algorithmic and geometric properties of sequences within the framework of music analysis. Some of the ideas in this work may find their way in musical applications, but they are also of independent interest theoretically because of the new tools and techniques developed for solving these problems.

**How to read this thesis.** While reading this thesis, the reader must keep in mind that it is written by a person with a background in computer science, and with almost no formal knowledge in music theory except the very little seen in various dance classes and learned during the evolution of this thesis. Music however remains the inspiration to this work, and the context in which the mathematical results are presented.

In the next chapter, we define terminology and notation that are common in all the subsequent chapters. Additional notation specific to a chapter as well as review of literature for each topic discussed will be defined in the relevant chapter.

We believe that history, in general, is an integral element for understanding the present and its problems. In the context of this work, we view the history of our topic as an important part of the topic itself. Thus, we start in Chapter 2 with a brief historical account of the interaction between mathematics and music, and introduce the objects of our study: rhythms and scales. We then present our first result on deep rhythms in Chapter 3, followed by a longer chapter on maximally even rhythms (Chapter 4). In Chapter 5 we further investigate the properties of maximally even rhythms and follow it with a chapter where we describe some operations on rhythms and the conditions under which these operations preserve the maximal evenness property (Chapter 6). Chapter 7 describes algorithms for aligning rhythms while minimizing certain distance norms and shows connections to problems on convolutions and sorting $X + Y$ matrices in computer science. Our final result is about reconstructing rhythms given some restrictions on time intervals (Chapter 8).

**Main contributions of this thesis.** The main contributions of this thesis are:

1. Characterize the rhythms that have the *deep* property: each distinct distance between onsets occurs with a unique multiplicity, and these multiplicities form an interval $1, 2, \ldots, k-1$. Our characterization shows that deep rhythms form a subclass of generated rhythms, which in turn proves a useful property called shelling.
2. Demonstrate relationships between the classical Euclidean algorithm for finding the greatest common divisor of two numbers and many other fields of study, particularly in the context of music and distance geometry. Specifically, we

show how the structure of the Euclidean algorithm defines a family of rhythms that encompass over forty timelines from traditional world music.

3. Prove that *Euclidean rhythms* have the mathematical property that their onset patterns are distributed as evenly as possible: they maximize the sum of the Euclidean distances between all pairs of onsets. We describe three algorithms that generate rotations of the unique Euclidean rhythm for a given number of onsets and pulses, and show the connection between deep and Euclidean rhythms.
4. Show that Euclidean rhythms are composed of a repeating pattern $P$, followed possibly by a pattern $T$. Both $P$ and $T$ are shown to have the Euclidean property; furthermore, $T$ is a subsequence of $P$. We also show that pattern $P$ is *minimal*: it cannot be rewritten as a repeating pattern with fewer pulses.
5. Show how to transform one rhythm to another using an *operation*, with the additional restriction that both the original and resulting rhythms are Euclidean. We define five operations: shadow, complementation, concatenation, alternation, and decomposition, and study the conditions under which they preserve the Euclidean property.
6. Design a $o(n^2)$-time algorithm to find rotation of a rhythm with respect to another and a perfect matching between onsets that minimizes the circular distance between matched onsets. The distance measures we consider are the $\ell_1$, $\ell_2$, and $\ell_\infty$ norms. We also show connections of this problem to convolutions and sorting $X + Y$ matrices.

7. Given the ordering of a set of points around the circumference of a circle and constraints involving pairs of distances with a common endpoint, show a polynomial-time algorithm that reconstructs a rhythm satisfying the ordering constraint and such that two distances sharing an onset have the same length if and only if these distances have the same label.

# Chapter 2

# Music and Mathematics

In 1787 Wolfgang Mozart designed a game where the idea is to compose a musical piece by choosing and pasting together pre-written musical measures. Each pre-written musical bit is chosen by rolling a pair of dice and looking up the rolled sum in a table to determine which measure to pick [110] (see Figure 2–1). Mozart, however, was not the first to design such a dice game [1] .

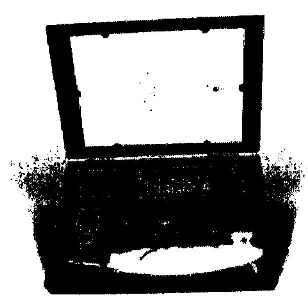


Figure 2–1: Today, you can buy *Mozart's Musical Dice Game - Luxury Edition* from Vienna's Leopold Museum.

Musical dice games were popular during the 18th century, and the first person known to have published such a game is Kirnberger. Due to the popularity of the game, other designers quickly followed; a few notable names are Hoegi, who claimed that it is possible to "compose ten thousand, all different" minuets using his game, and Haydn, who

[1] It should be noted that Mozart's authorship of this game is disputed. For more information and for a complete list of musical dice games in the 18th century, see [95].

claimed that one may compose "un infinito numero" of minuet trios [92]. Both figures for the number of composable pieces are incorrect of course, as this number is above ten thousand and definitely finite. Nevertheless, these games were perhaps the first random music generators, conceived more than a century before the first computer was built. They also constitute a close connection between music and mathematics — a field towards which the 18th century displayed great public enthusiasm.

However, the relationship between music and mathematics dates back to much earlier than the 18th century. In fact, the two fields have been intimately intertwined for over 2,500 years. Perhaps the first person to have mentioned such a connection is Pythagoras of Samos (6th century B.C), who noticed that the ratio between a string and the frequency of the tone it makes when plucked remains constant as the length of the string is varied [20]. He divided the length of a string into halves, thirds, quarters, and fifths, thus creating the first four overtones: an octave, a perfect fifth, a perfect fourth, and a major third. In music theory, these overtones have become the building blocks of musical harmony, which deals with how pitches relate to one another.

The interaction between musical arts and mathematics has continued to present times, when musical pieces are often composed with the help of computer algorithms (see for example [51]). A thorough historical snapshots of this interaction between the two fields can be found in the work of H. S. M. Coxeter [54]. Until recently however, most of this interaction has been in the domain of pitch and scales; in music theory, much attention has been devoted to the study of intervals used in pitch scales [75],

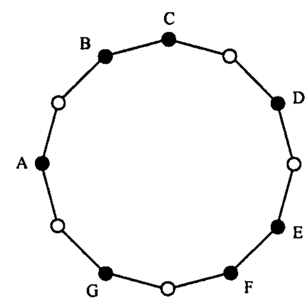



Figure 2–2: The Diatonic scale.

but relatively little work has been devoted to the analysis of time duration intervals of rhythm. Some notable recent exceptions are the books by Simha Arom [7], Justin London [113], and Christopher Hasty [94].

So what are rhythms and scales?

## 2.1 Rhythms and Scales

A musical *scale* is a collection of musical notes sorted by pitch. The ordering of these notes gives a measure of musical *distance*, whereby the intervals defined by two consecutive pitches in this ordering are not necessarily equal. In Western music, perhaps the most famous musical scale is the *Diatonic* scale, composed of seven notes (the white keys of the piano). See Figure 2–2.

Similar to a scale, a *rhythm* is a sorted collection of beats that occur at intervals which are not always regular. The interval between two onsets represents the amount of elapsed time between the first and the second. Thus, unlike pitch intervals, interonset intervals define a measure of time. The time dimension of rhythms and the pitch dimension of scales have an intrinsically cyclic nature, cycling every measure and every octave, respectively. As such, musical rhythms and scales can both be seen as two-way infinite binary sequences [169]. It is also generally assumed that the two-way infinite bit sequence is periodic with some period $n$, so that the information can be compacted down to an $n$-bit string. In a rhythm, each bit represents one unit of time called a *pulse* (for example, the length of a sixteenth note), a one bit represents a played note or *onset* (for example, a sixteenth note), and a zero bit represents a silence (for example, a sixteenth rest). In a scale, each bit represents a pitch, and zero or one represents whether the pitch is absent or present in the scale. Here we assume that all time intervals between onsets in a rhythm are multiples of a fixed time unit, and that all tone intervals between pitches in a scale are multiples of a fixed tonal unit (in logarithm of frequency).

In this thesis, we consider rhythms and scales that match this cyclic nature of the underlying space. In the case of rhythms, such cyclic rhythms are also called *timelines* or *claves*, rhythmic phrases or patterns that are repeated throughout a piece and serve as a rhythmic reference point [130, 179]. In what follows, we use the term "rhythm" to mean "timeline". The infinite bit sequence representation of a cyclic rhythm or scale is just a cyclic repetition of some $n$-bit string, corresponding to the timespan of a single measure or the log-frequency span of a single octave. To properly represent the cyclic nature of this string, we imagine assigning the bits to $n$ points equally spaced around a circle of circumference $n$ [120]. A rhythm or scale can therefore be represented as a subset of these $n$ points. We use $k$ to denote the size of this subset; that is, $k$ is the number of onsets in a rhythm or pitches in a scale. For uniformity of terminology, the remainder of this thesis speaks primarily about rhythms, but the notions and results apply equally well to scales.

## 2.2 Representation

We use four representations of rhythms of timespan $n$. The first representation is the commonly used *box-like* representation, also known as the Time Unit Box System (TUBS), which is a sequence of $n$ '×'s and '·'s where '×' represents an onset and '·' denotes a silence [169]. This notation was used and taught by Philip Harland at the University of California, Los Angeles, in 1962, and it was made popular in the field of ethnomusicology by James Koetting [108]. However, such box notation has been used in Korea for hundreds of years [98]. The second representation of rhythms and scales we use is the *clockwise distance sequence*, which is a sequence of integers that sum up to $n$ and represent the lengths of the intervals between consecutive pairs of onsets, measuring clockwise arc-lengths or distances around the circle of circumference $n$. Note that the clockwise distance sequence notation requires that the rhythm starts with an onset, so it cannot be used to represent all rhythms; however, it is useful for rhythmic analysis that disregards the starting position of the sequence. The third representation of rhythms and scales writes the onsets as a *subset* of the set of all pulses, numbered $0, 1, \ldots, n-1$, with a subscript of $n$ on the right-hand side of the subset to denote the timespan. Clough and Douthett [43] use this notation to represent scales. For example, the Cuban clave *Son* rhythm can be represented as $[\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \cdot \times \cdot \times \cdot \cdot \cdot]$ in box-like notation, $(3, 3, 4, 2, 4)$ in clockwise distance sequence notation, and $\{0, 3, 6, 10, 12\}_{16}$ in subset notation. Finally, the fourth representation is a graphical *clock diagram* [169], such as Figure 4–1, in which the zero label denotes the start of the rhythm and time flows in a clockwise direction. In such clock diagrams we usually connect adjacent onsets by line segments, forming a polygon. We consider two rhythms distinct if their sequence of zeros and ones differ at every bit position, starting from the leftmost bit. That is, two rhythms that do not have the same sequence are different. Moreover, rhythms that have the same sequence but differ in the starting beat are also considered to be different. In this case we say that one rhythm is a *rotation* of the other and that the two rhythms are instances of the same *necklace*: a cyclic sequence of onsets and pulses with no regards to a starting beat. From a mathematical perspective, cyclic binary sequences that are instances of the same necklace are considered to be the same mathematical object; from a music perspective however, rhythms that have the same sequence of pulses and onsets but different starting beats sound very different, and thus are considered to be different musical objects. The rhythmic properties we discuss below are in truth properties of necklaces; however, we will talk mainly about rhythms to stress this musical distinction.

We now define some precise mathematical notation for describing rhythms. These notations are common in all the chapters; further definitions will come up as needed.

## 2.3 Basic Definitions and Notations

Let $\mathbb{Z}^+$ denote the set of positive integers. For $k, n \in \mathbb{Z}^+$, let $\gcd(k, n)$ denote the greatest common divisor of $k$ and $n$. If $\gcd(k, n) = 1$, we call $k$ and $n$ *relatively prime*. For integers $a < b$, let $[a, b] = \{a, a+1, a+2, \ldots, b\}$. We let $\log n$ denote the base 2 logarithm of $n$.

Let $C$ be a circle in the plane, and consider any two points $x, y$ on $C$. The *chordal distance* between $x$ and $y$, denoted by $\overline{d}(x, y)$, is the length of the line segment $\overline{xy}$;

that is, $\overline{d}(x, y)$ is the Euclidean distance between $x$ and $y$. The *clockwise distance* from $x$ to $y$, or of the ordered pair $(x, y)$, is the length of the clockwise arc of $C$ from $x$ to $y$, and is denoted by $\overset{\circlearrowright}{d}(x, y)$. Finally, the *geodesic distance* between $x$ and $y$, denoted by $\overset{\smile}{d}(x, y)$, is the length of the shortest arc of $C$ between $x$ and $y$; that is, $\overset{\smile}{d}(x, y) = \min\{\overset{\circlearrowright}{d}(x, y), \overset{\circlearrowright}{d}(y, x)\}$.

A *rhythm of timespan n* is a subset of $\{0, 1, \ldots, n-1\}$, representing the set of pulses that are onsets in each repetition. For clarity, we write the timespan $n$ as a subscript after the subset: $\{\ldots\}_n$ (this is the subset notation described earlier). Geometrically, if we locate $n$ equally spaced points clockwise around a circle $C_n$ of circumference $n$, then we can view a rhythm of timespan $n$ as a subset of these $n$ points. We consider an element of $C_n$ to simultaneously be a point on the circle and an integer in $\{0, 1, \ldots, n-1\}$. When $n$ is an arbitrary real number, and onsets can be at arbitrary (not necessarily integer) points along the circle, then we call such rhythms *continuous rhythms.* Similarly, we say that rotations of a continuous rhythm are instances of the same *continuous necklace.*

The *rotation* of a rhythm $R$ of timespan $n$ by an integer $\Delta \geq 0$ is the rhythm $\{(i + \Delta) \bmod n : i \in R\}_n$ of the same timespan $n$. The *scaling* of a rhythm $R$ of timespan $n$ by an integer $\alpha \geq 1$ is the rhythm $\{\alpha i : i \in R\}_{\alpha n}$ of timespan $\alpha n$.

Let $R = \{r_0, r_1, \ldots, r_{k-1}\}_n$ be a rhythm of timespan $n$ with $k$ onsets sorted in clockwise order. Throughout this thesis, an onset $r_i$ will mean $(r_{i \bmod k}) \bmod n$. Observe that the clockwise distance $\overset{\circlearrowright}{d}(r_i, r_j) = (r_j - r_i) \bmod n$. This is the number of points on $C_n$ that are contained in the clockwise arc $(r_i, r_j]$ and is also known as the *chromatic length* [43].

The *clockwise distance sequence* of $R$ is the circular sequence $(d_0, d_1, \ldots, d_{k-1})$ where $d_i = \overset{\circlearrowright}{d}(r_i, r_{i+1})$ for all $i \in [0, k-1]$. Observe that each $d_i \in \mathbb{Z}^+$ and $\sum_i d_i = n$.

**Observation 1.** There is a one-to-one relationship between rhythms with $k$ onsets and timespan $n$ and circular sequences $(d_0, d_1, \ldots, d_{k-1})$ where each $d_i \in \mathbb{Z}^+$ and $\sum_i d_i = n$.

The *geodesic distance multiset* of a rhythm $R$ is the multiset of all nonzero pairwise geodesic distances; that is, it is the multiset $\{\overset{\smile}{d}(r_i, r_j) : r_i, r_j \in R, r_i \neq r_j\}$. The geodesic distance multiset has cardinality $\binom{k}{2}$. Finally, the *multiplicity* of a distance $d$ is the number of occurrences of $d$ in the geodesic distance multiset.

We now start with a short and technical chapter about deep rhythms.

# Chapter 3

# Deep Rhythms

*Deepness* is a property of rhythms and scales that pertains to the number of occurrences of the distances defined by pairs of onsets. Consider a rhythm with $k$ onsets and timespan $n$, represented as a set of $k$ points on a circle of circumference $n$. Now measure the arc-lengths (geodesic distances along the circle) between all pairs of onsets. A musical scale or rhythm is *Winograd-deep* if every distance $1, 2, \ldots, \lfloor n/2 \rfloor$ has a unique multiplicity (number of occurrences). For example, the rhythm [× × × · × · ] is Winograd-deep because distance 1 appears twice, distance 2 appears thrice, and distance 3 appears once.

The notion of deepness in scales was introduced by Winograd in an oft-cited but unpublished class project report from 1966 [187], disseminated and further developed by the class instructor Gamer in 1967 [79, 80], and considered further in numerous papers and books, such as [44, 101]. Equivalently, a scale is *Winograd-deep* if the number of pitches it has in common with each of its cyclic shifts (rotations) is unique. This equivalence is the Common Tone Theorem [101, page 42], originally described by Winograd [187] (who in fact uses this definition as his primary definition of "deep"). Deepness is one property of the ubiquitous Western *diatonic* 12-tone major scale

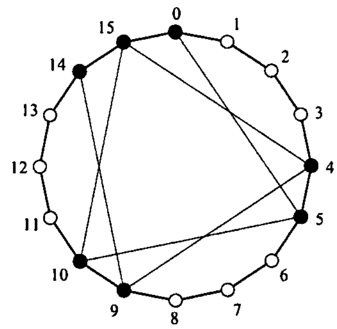



Figure 3–1: A rhythm with $k = 7$ onsets and timespan $n = 16$ that is Winograd-deep and thus Erdős-deep. Distances ordered by multiplicity from 1 to 6 are 2, 7, 4, 1, 6, and 5. The dotted line shows how the rhythm is generated by multiples of $m = 5$.

[× · × · ×× · × · × · ×] [101], and it captures some of the rich structure that perhaps makes this scale so attractive.

Winograd-deepness translates directly from scales to rhythms. For example, the diatonic major scale is equivalent to the famous Cuban rhythm *Bembé* [134, 171]. Figure 3–1 shows a graphical example of a Winograd-deep rhythm. However, the notion of Winograd-deepness is rather restrictive for rhythms, because it requires half of the pulses in a timespan (rounded to a nearest integer) to be onsets. In contrast, for example, the popular Bossa-Nova rhythm [× · · × · · × · · · × · · × · · ] $= \{0, 3, 6, 10, 13\}_{16}$ illustrated in Figure 4–1 has only five onsets in a timespan of sixteen. Nonetheless, if we focus on just the distances that appear at least once between two onsets, then the multiplicities of occurrence are all unique and form an interval starting at 1: distance 4 occurs once, distance 7 occurs twice, distance 6 occurs thrice, and distance 3 occurs four times.

We therefore define a rhythm (or scale) to be *Erdős-deep* if it has $k$ onsets and, for every multiplicity $1, 2, \ldots, k-1$, there is a nonzero arc-length/geodesic distance determined by the points on the circle with exactly that multiplicity. The same definition is made by Toussaint [173]. Every Winograd-deep rhythm is also Erdős-deep, so this definition is strictly more general.

To further clarify the difference between Winograd-deep and Erdős-deep rhythms, it is useful to consider which distances can appear. For a rhythm to be Winograd-deep, all the distances between 1 and $k-1$ must appear a unique number of times. In contrast, to be an Erdős-deep rhythm, it is only required that each distance that appears must have a unique multiplicity. Thus, the Bossa-Nova rhythm is not Winograd-deep because distances $1, 2$ and 5 do not appear.

The property of Erdős deepness involves only the distances between points in a set, and is thus a feature of *distance geometry*—in this case, in the discrete space of $n$ points equally spaced around a circle. In 1989, Paul Erdős [67] considered the analogous question in the plane, asking whether there exist $n$ points in the plane (no three on a line and no four on a circle) such that, for every $i = 1, 2, \ldots, n-1$, there is a distance determined by these points that occurs exactly $i$ times. Solutions have been found for $n$ between 2 and 8, but in general the problem remains open. Palásti [132] considered a variant of this problem with further restrictions—no three points form a regular triangle, and no one is equidistant from three others—and solved it for $n = 6$.

In this chapter we characterize all rhythms that are Erdős-deep. In particular, we prove that all deep rhythms, besides one exception, are *generated*, meaning that the rhythm can be represented as $\{0, m, 2m, \ldots, (k-1)m\}_n$ for some integer $m$, where all arithmetic is modulo $n$. In the context of scales, the concept of "generated" was defined by Wooldridge [188] and used by Clough et al. [44]. For example, the rhythm in Figure 3–1 is generated with $m = 5$. Our characterization generalizes a similar characterization for Winograd-deep scales proved by Winograd [187], and independently by Clough et al. [44].

In the pitch domain, generated scales are very common. The Pythagorean tuning is a good example: all its pitches are generated from the fifth of ratio $3:2$ modulo the octave. Another example is the equal-tempered scale, which is generated with a half-tone of ratio $\sqrt[12]{2}$ [13]. Generated scales are also of interest in the theory of the well-formed scales [35].

Generated rhythms have an interesting property called *shellability.* If we remove the "last" generated onset 14 from the rhythm in Figure 3–1, the resulting rhythm is still generated, and this process can be repeated until we run out of onsets. In general, every generated rhythm has a *shelling* in the sense that it is always possible to remove a particular onset and obtain another generated rhythm.

Most African drumming music consists of rhythms operating on three different strata: the unvarying timeline usually provided by one or more bells, one or more rhythmic motifs played on drums, and an improvised solo (played by the lead drummer) riding on the other rhythmic structures. Shellings of rhythms are relevant to the improvisation of solo drumming in the context of such a rhythmic background. The solo improvisation must respect the style and feeling of the piece which is usually determined by the timeline. One common technique to achieve this effect is to "borrow" notes from the timeline, and to alternate between playing subsets of notes from the timeline and from other rhythms that interlock with the timeline [1, 4]. In the words of Kofi Agawu [1], "It takes a fair amount of expertise to create an effective improvisation that is at the same time stylistically coherent". The borrowing of notes from the timeline may be regarded as a fulfillment of the requirements of style coherence. Another common method is to make frugal transformations to the timeline or improvise on a rhythm that is functionally related to the timeline [112]. Although such an approach does not give the performer wide scope for free improvisation, it is efficient in certain drumming contexts. In the words of Christophe Waterman [33], "individuals improvise, but only within fairly strict limits, since varying the constituent parts too much could unravel the overall texture".

Of course, some subsets of notes of a rhythm may be better choices than others. One might often want to select sets of rhythms that share a common property. For example, if a rhythm is deep, one might want to select subsets of the rhythm that are also deep. Furthermore, a shelling seems a natural way to decrease or increase the density of the notes in an improvisation that respects these constraints. For example, in the *Bembé* bell timeline [× · × · ×× · × · × · ×], which is deep, one possible shelling is [× · × · ×× · × · × · ·], [× · × · × · · × · × · ·], [× · × · · · · × · × · ·], [× · × · · · · × · · · ·]. All five rhythms sound good and are stylistically coherent. In fact the shelled rhythms are used in African drum music [40]. To our knowledge, shellings have not been studied from the musicological point of view. However, they may be useful both for theoretical analysis as well as providing formal rules for improvisation techniques. Rhythmic transformations are further discussed in Chapter 6.

One of the consequences of our characterization is that every Erdős-deep rhythm has a shelling. More precisely, it is always possible to remove a particular onset that preserves the Erdős-deepness property.

Winograd [187], and independently Clough et al. [44], characterize all Winograd-deep scales: up to rotation, they are the scales that can be generated by the first $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor + 1$ multiples (modulo $n$) of a value that is relatively prime to $n$, plus one exceptional scale $\{0, 1, 2, 4\}_6$. In this chapter, we prove a similar (but more general) characterization of Erdős-deep rhythms: up to rotation and scaling, they are the rhythms generable as the first $k$ multiples (modulo $n$) of a value that is relatively prime to $n$, plus the same exceptional rhythm $\{0, 1, 2, 4\}_6$. The key difference is that the number of onsets $k$ is now a free parameter, instead of being forced to be either $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor + 1$. Our proof follows Winograd's, but differs in one case (the second case of Theorem 3.2.3).

## 3.1 Definitions

A rhythm is *Erdős-deep* if it has (exactly) one distance of multiplicity $i$, for each $i \in [1, k-1]$. Note that these multiplicities sum to $\sum_{i=1}^{k-1} i = \binom{k}{2}$, which is the cardinality of the geodesic distance multiset, and hence these distances are all the distances in the rhythm. Every geodesic distance is between 0 and $\lfloor n/2 \rfloor$. A rhythm is *Winograd-deep* if every two distances from $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ have different multiplicity.

A *shelling* of an Erdős-deep rhythm $R$ is an ordering $s_1, s_2, \ldots, s_k$ of the onsets in $R$ such that $R - \{s_1, s_2, \ldots, s_i\}$ is an Erdős-deep rhythm for $i = 0, 1, \ldots, k$. (Every rhythm with at most two onsets is Erdős-deep.)

## 3.2 Characterization of Deep Rhythms

Our characterization of Erdős-deep rhythms is in terms of two families of rhythms. The main rhythm family consists of the generated rhythms $D_{k,n,m} = \{im \bmod n : i = 0, 1, \ldots, k-1\}_n$ of timespan $n$, for certain values of $k$, $n$, and $m$. The one exceptional rhythm is $F = \{0, 1, 2, 4\}_6$ of timespan 6.

**Fact 3.2.1.** *$F$ is Erdős-deep.*

**Lemma 3.2.2.** *If $k \leq \lfloor n/2 \rfloor + 1$ and $m$ and $n$ are relatively prime, then $D_{k,n,m}$ is Erdős-deep.*

*Proof.* The multiset of clockwise distances in $D_{k,n,m}$ is $\{(jm - im) \bmod n : i < j\} = \{(j-i)m \bmod n : i < j\}$. There are $k - p$ choices of $i$ and $j$ such that $j - i = p$, so there are exactly $p$ occurrences of the clockwise distance $(pm) \bmod n$ in the multiset. Each of these clockwise distances corresponds to a geodesic distance—either $(pm) \bmod n$ or $(-pm) \bmod n$, whichever is smaller (at most $n/2$). We claim that these geodesic distances are all distinct. Then the multiplicity of each geodesic distance $(\pm pm) \bmod n$ is exactly $p$, establishing that the rhythm is Erdős-deep.

For two geodesic distances to be equal, we must have $\pm pm \equiv \pm qm \pmod{n}$ for some (possibly different) choices for the $\pm$ symbols, and for some $p \neq q$. By (possibly) multiplying both sides by $-1$, we obtain two cases: (1) $pm \equiv qm \pmod{n}$ and (2) $pm \equiv -qm \pmod{n}$. Because $m$ is relatively prime to $n$, $m$ has a multiplicative inverse modulo $n$ (see Lemma 4.2.2 on page 60). Multiplying both sides of the congruence by this inverse, we obtain (1) $p \equiv q \pmod{n}$ and (2) $p \equiv -q \pmod{n}$. Since $0 \leq i < j < k \leq \lfloor n/2 \rfloor + 1$, we have $0 \leq p = j - i < \lfloor n/2 \rfloor + 1$, and similarly for $q$: $0 \leq p, q \leq \lfloor n/2 \rfloor$. Thus, the first case of $p \equiv q \pmod{n}$ can happen only when $p = q$, and the second case of $p + q \equiv 0 \pmod{n}$ can happen only when $p = q = 0$ or when $p = q = n/2$. Either case contradicts that $p \neq q$. Therefore the geodesic distances arising from different values of $p$ are indeed distinct, proving the lemma. □

We now state and prove our characterization of Erdős-deep rhythms, which is up to rotation and scaling. Rotation preserves the geodesic distance multiset and therefore Erdős-deepness (and Winograd-deepness). Scaling maps each geodesic distance $d$ to $\alpha d$, and thus preserves multiplicities and therefore Erdős-deepness (but not Winograd-deepness). Note that the rhythm $D_{k,n,m}$ is a rotation by $-m(k-1) \bmod n$ of the rhythm $D_{k,n,n-m}$; to avoid this duplication we restrict $m$ to be equal to at most $\lfloor n/2 \rfloor$.

**Theorem 3.2.3.** *A rhythm is Erdős-deep if and only if it is a rotation of a scaling of either the rhythm $F$ or the rhythm $D_{k,n,m}$ for some $k, n, m$ with $k \leq \lfloor n/2 \rfloor + 1$, $1 \leq m \leq \lfloor n/2 \rfloor$, and $m$ and $n$ are relatively prime.*

*Proof.* Since a rotation of a scaling of an Erdős-deep rhythm is Erdős-deep, the "if" direction of the theorem follows from Fact 3.2.1 and Lemma 3.2.2.

Consider an Erdős-deep rhythm $R$ with $k$ onsets. By the definition of Erdős-deepness, $R$ has one nonzero geodesic distance with multiplicity $i$ for each $i = 1, 2, \ldots, k-1$. Let $m$ be the geodesic distance with multiplicity $k-1$. Since $m$ is a geodesic distance, $1 \leq m \leq \lfloor n/2 \rfloor$. Also, $k \leq \lfloor n/2 \rfloor + 1$ (for any Erdős-deep rhythm $R$), because all nonzero geodesic distances are between 1 and $\lfloor n/2 \rfloor$ and therefore at most $\lfloor n/2 \rfloor$ nonzero geodesic distances occur. Thus $k$ and $m$ are suitable parameter choices for $D_{k,n,m}$.

Consider the graph $G_m = (R, E_m)$ with vertices corresponding to onsets in $R$ and with an edge between two onsets of geodesic distance $m$. By the definition of geodesic distance, every vertex $i$ in $G_m$ has degree at most 2: the only onsets at geodesic distance exactly $m$ from $i$ are $(i - m) \bmod n$ and $(i + m) \bmod n$. Thus, the graph $G_m$ is a disjoint union of paths and cycles. The number of edges in $G_m$ is the multiplicity of $m$, which we supposed was $k - 1$, which is 1 less than the number of vertices in $G_m$. Thus, the graph $G_m$ consists of exactly one path and any number of cycles.

The cycles of $G_m$ have a special structure because they correspond to subgroups generated by single elements in the cyclic group $(\mathbb{Z}/(n), +)$. Namely, the onsets corresponding to vertices of a cycle in $G_m$ form a regular $(n/a)$-gon, with a geodesic distance of $a = \gcd(m, n)$ between consecutive onsets. ($a$ is called the index of the subgroup generated by $m$.) In particular, every cycle in $G_m$ has the same length $r = n/a$. Since $G_m$ is a simple graph, every cycle must have at least 3 vertices, so $r \geq 3$.

The proof consists of four cases depending on the length of the path and on how many cycles the graph $G_m$ has. The first two cases will turn out to be impossible; the third case will lead to a rotation of a scaling of rhythm $F$; and the fourth case will lead to a rotation of a scaling of $D_{k,n,m}$.

First suppose that the graph $G_m$ consists of a path of length at least 1 and at least one cycle. We show that this case is impossible because the rhythm $R$ can have no geodesic distance with multiplicity 1. Suppose that there is a geodesic distance with multiplicity 1, say between onsets $i_1$ and $i_2$. If $i$ is a vertex of a cycle, then both $(i+m) \bmod n$ and $(i-m) \bmod n$ are onsets in $R$. If $i$ is a vertex of the path, then one or two of these are onsets in $R$, with the case of one occurring only at the endpoints of the path. If $(i_1+m) \bmod n$ and $(i_2+m) \bmod n$ were both onsets in $R$, or $(i_1-m) \bmod n$ and $(i_2-m) \bmod n$ were both onsets in $R$, then we would have another occurrence of the geodesic distance between $i_1$ and $i_2$, contradicting that this geodesic distance has multiplicity 1. Thus, $i_1$ and $i_2$ must be opposite endpoints of the path. If the path has length $\ell$, then the clockwise distance between $i_1$ and $i_2$ is $(\ell m) \bmod n$. This clockwise distance (and hence the corresponding geodesic distance) appears in every cycle, of which there is at least one, so the geodesic distance has multiplicity more than 1, a contradiction. Therefore this case is impossible.

Second suppose that the graph $G_m$ consists of a path of length 0 and at least two cycles. We show that this case is impossible because the rhythm $R$ has two geodesic distances with the same multiplicity. Pick any two cycles $C$ and $C'$, and let $d$ be the smallest positive clockwise distance from a vertex of $C$ to a vertex of $C'$. Thus $i$ is a vertex of $C$ if and only if $(i+d) \bmod n$ is a vertex of $C'$. Since the cycles are disjoint, $d < a$. Since $r \geq 3$, $d < n/3$, so clockwise distances of $d$ are also geodesic distances of $d$. The number of occurrences of geodesic distance $d$ between a vertex of $C$ and a vertex of $C'$ is either $r$ or $2r$, the case of $2r$ arising when $d = a/2$ (that is, $C'$ is a "half-rotation" of $C$). The number of occurrences of geodesic distance $d' = \min\{d+m, n-(d+m)\}$ is the same—either $r$ or $2r$, in the same cases. (Note that $d < a \leq n-m$, so $d+m < n$, so the definition of $d'$ correctly captures a geodesic distance modulo $n$.) The same is true of geodesic distance $d'' = \min\{d-m, n-(d-m)\}$. If other pairs of cycles have the same smallest positive clockwise distance $d$, then the number of occurrences of $d$, $d'$, and $d''$ between those cycles are also equal. Since the cycles are disjoint, geodesic distance $d$ and thus $d+m$ and $d-m$ cannot be $(pm) \bmod n$ for any $p$, so these geodesic distances cannot occur between two vertices of the same cycle. Finally, the sole vertex $x$ of the path has geodesic distance $d$ to onset $i$ (which must be a vertex of some cycle) if and only if $x$ has geodesic distance $d'$ to onset $(i+m) \bmod n$ (which must be a vertex of the same cycle) if and only if $x$ has geodesic distance $d''$ to onset $(i-m) \bmod n$ (which also must be a vertex of the same cycle). Therefore the multiplicities of geodesic distances $d$, $d'$, and $d''$ must be equal. Since $R$ is Erdős-deep, we must have $d = d' = d''$. To have $d = d'$, either $d = d+m$ or $d = n-(d+m)$, but the first case is impossible because $d > 0$ by nonoverlap of cycles, so $2d + m = n$. Similarly, to have $d = d''$, we must have $2d - m = n$. Subtracting these two equations, we obtain that $2m = 0$, contradicting that $m > 0$. Therefore this case is also impossible.

Third suppose that the graph $G_m$ consists of a path of length 0 and exactly one cycle. We show that this case forces $R$ to be a rotation of a scaling of rhythm $F$ because otherwise two geodesic distances $m$ and $m'$ have the same multiplicity. The number of occurrences of geodesic distance $m$ in the cycle is precisely the length $r$ of the cycle. Similarly, the number of occurrences of geodesic distance $m' = \min\{2m, n-2m\}$ in the cycle is $r$. The sole vertex $x$ on the path cannot have geodesic distance $m$ or $m'$ to any other onset (a vertex of the cycle) because $x$ would then be on the cycle. Therefore the multiplicities of geodesic distances $m$ and $m'$ must be equal. Since $R$ is Erdős-deep, $m$ must equal $m'$, which implies that either $m = 2m$ or $m = n - 2m$. The first case is impossible because $m > 0$. In the second case, $3m = n$, that is, $m = \frac{1}{3}n$. Therefore, the cycle has $r = 3$ vertices, say at $\Delta, \Delta + \frac{1}{3}n, \Delta + \frac{2}{3}n$. The fourth and final onset $x$ must be midway between two of these three onsets, because otherwise its geodesic distance to the three vertices are all distinct and therefore unique. No matter where $x$ is so placed, the rhythm $R$ is a rotation by $\Delta + c\frac{1}{3}n$ (for some $c \in \{0, 1, 2\}$) of a scaling by $n/6$ of the rhythm $F$.

Finally suppose that $G_m$ has no cycles, and consists solely of a path. We show that this case forces $R$ to be a rotation of a scaling of a rhythm $D_{k,n',m'}$ with $1 \le m' \le \lfloor n'/2 \rfloor$ and with $m'$ and $n'$ relatively prime. Let $i$ be the onset such that $(i-m) \bmod n$ is not an onset (the "beginning" vertex of the path). Consider rotating $R$ by $-i$ so that 0 is an onset in the resulting rhythm $R-i$. The vertices of the path in $R-i$ form a subset of the subgroup of the cyclic group $(\mathbb{Z}/(n), +)$ generated by the element $m$. Therefore the rhythm $R - i = D_{k,n,m} = \{(im) \bmod n : i = 0, 1, \ldots, k-1\}_n$ is a scaling by $a$ of the rhythm $D_{k,n/a,m/a} = \{(im/a) \bmod (n/a) : i = 0, 1, \ldots, k-1\}_n$. The rhythm $D_{k,n/a,m/a}$ has an appropriate value for the third argument: $m/a$ and $n/a$ are relatively prime ($a = \gcd(m, n)$) and $1 \le m/a \le \lfloor n/2 \rfloor / a \le \lfloor (n/a)/2 \rfloor$. Also, $k \le \lfloor (n/a)/2 \rfloor + 1$ because the only occurring geodesic distances are multiples of $a$ and therefore the number $k - 1$ of distinct geodesic distances is at most $\lfloor (n/a)/2 \rfloor$. Therefore $R$ is a rotation by $i$ of a scaling by $a$ of $D_{k,n/a,m/a}$ with appropriate values of the arguments. □

**Corollary 3.2.4.** *A rhythm is Erdős-deep if and only if it is a rotation of a scaling of the rhythm $F$ or it is a rotation of a rhythm $D_{k,n,m}$ for some $k, n, m$ satisfying $k \le \lfloor n/2g \rfloor + 1$ where $g = \gcd(m, n)$.*

*Proof.* First we show that any Erdős-deep rhythm has one of the two forms in the corollary. By Theorem 3.2.3, there are two flavors of Erdős-deep rhythms, and the corollary directly handles rotations of scalings of $F$. Thus it suffices to consider a rhythm $R$ that is a rotation by $\Delta$ of a scaling by $\alpha$ of $D_{k,n,m}$ where $k \leq \lfloor n/2 \rfloor + 1$, $1 \leq m \leq \lfloor n/2 \rfloor$, and $m$ and $n$ are relatively prime. Equivalently, $R$ is a rotation by $\Delta$ of $D_{k,n',m'}$ where $n' = \alpha n$ and $m' = \alpha m$. Now $g = \gcd(n', m') = \alpha$, so $n'/g = n$. Hence, $k \leq \lfloor n'/2g \rfloor + 1$ as desired. Thus we have rewritten $R$ in the desired form.

It remains to show that every rhythm in one of the two forms in the corollary is Erdős-deep. Again, rotations of scalings of $F$ are handled directly by Theorem 3.2.3. So consider a rotation of $D_{k,n,m}$ where $k \leq \lfloor n/2g \rfloor + 1$. The value of $m$ matters only modulo $n$, so we assume that $0 \leq m \leq n - 1$.

First we show that, if $\lfloor n/2 \rfloor + 1 \leq m \leq n - 1$, then $D_{k,n,m}$ can be rewritten as a rotation of the rhythm $D_{k,n,m'}$ where $m' = n - m \leq \lfloor n/2 \rfloor$. By reversing the order in which we list the onsets in $D_{k,n,m} = \{im \bmod n : i = 0, 1, \ldots, k-1\}_n$, we can write $D_{k,n,m} = \{(k - 1 - i)\, m \bmod n : i = 0, 1, \ldots, k - 1\}_n$. Now consider rotating the rhythm $D_{k,n,n-m} = \{i\,(n - m) \bmod n : i = 0, 1, \ldots, k - 1\}_n$ by $(k - 1)m$. We obtain the rhythm $\{[i\,(n - m) + (k - 1)\, m] \bmod n : i = 0, 1, \ldots, k - 1\}_n = \{[(k - 1 - i)\, m + in] \bmod n : i = 0, 1, \ldots, k-1\}_n = \{(k-1-i)\, m \bmod n : i = 0, 1, \ldots, k-1\}_n = D_{k,n,m}$ as desired.

Thus it suffices to consider rotations of $D_{k,n,m}$ where $1 \leq m \leq \lfloor n/2 \rfloor$ and $k \leq \lfloor n/2g \rfloor + 1$. The rhythm $D_{k,n',m'}$, where $n' = n/g$ and $m' = m/g$, is Erdős-deep by Theorem 3.2.3 because $n'$ and $m'$ are relatively prime, $k \leq \lfloor n'/2 \rfloor + 1$, and

$1 \leq m' \leq \lfloor n'/2 \rfloor$. But $D_{k,n,m}$ is the scaling of $D_{k,n',m'}$ by the integer $g$, so $D_{k,n,m}$ is also Erdős-deep. □

An interesting consequence of this characterization is the following:

**Corollary 3.2.5.** *Every Erdős-deep rhythm has a shelling.*

*Proof.* If the Erdős-deep rhythm is $D_{k,n,m}$, we can remove the last onset from the path, resulting in $D_{k-1,n,m}$, and repeat until we obtain the empty rhythm $D_{0,n,m}$. At all times, $k$ remains at most $\lfloor n/2 \rfloor + 1$ (assuming it was originally) and $m$ remains between 1 and $\lfloor n/2 \rfloor$ and relatively prime to $n$. On the other hand, $F = \{0, 1, 2, 4\}_6$ has the shelling $4, 2, 1, 0$ because $\{0, 1, 2\}_6$ is Erdős-deep. □

We can generalize this characterization of Erdős-deep rhythms to continuous rhythms, where where $n$ and the position of the onsets along the circle are arbitrary real numbers. In this case we have two kinds of rhythms. First, if $m$ and $n$ are rational multiples of each other, we can scale the rhythm by some rational $p$ such that $pm$ and $pn$ are integers, and apply Theorem 3.2.3 using $pn$ and $pm$ to characterize all deep rhythms where $m$ is a rational multiple of $n$. Second, if $m$ and $n$ are irrational multiples of each other, we can show that every $D_{k,n,m}$ is Erdős-deep. The complete characterization of continuous Erdős-deep rhythms is as follows:

**Theorem 3.2.6.** *A continuous rhythm is Erdős-deep if and only if it is a rotation of a scaling of $D_{k,n,m}$ with $k \leq \lfloor n/2 \rfloor + 1$, $0 < m \leq n/2$, and where $m$ and $n$ are either (1) irrational multiples of each other, or (2) rational multiples such that for some rational $p$, integers $pm$ and $pn$ are relatively prime.*

*Proof.* To prove the "if" direction, we show that all geodesic distances defined by $D_{k,n,m}$ are distinct; hence we need to prove that the multiplicity of each geodesic distance $(\pm pm) \bmod n$ is exactly $p$. First assume that $m$ and $n$ are irrational multiples of each other, i.e., there is no rational number that divides both $m$ and $n$. Suppose two geodesic distances $\pm pm \equiv \pm qm \pmod n$ for some (possibly different) choices for the $\pm$ symbols, and for some $p \neq q$. Then we can write $\pm pm = \pm qm + rn$ for some integer $r$. This in turn implies that $m = \frac{r}{\pm p \mp q} n$, which contradicts the fact that $m$ and $n$ are irrational multiples of each other. Therefore, when $m$ and $n$ are irrational multiples of each other, the geodesic distances arising from different values of $p$ are distinct, proving that $D_{k,n,m}$ is Erdős-deep.

If $m$ and $n$ are rational multiples of each other, then so are each of the geodesic distances $2m, 3m, \ldots (k-1)m \pmod n$ with $n$. In this case, there exists a rational $p$ such that $pn$ and $pm$ are both integers. We can now apply Theorem 3.2.3 using $pn$ and $pm$, and generate all deep rhythms where $m$ is a rational multiple of $n$.

For the "only if" direction, consider a continuous Erdős-deep rhythm $R$ with $k$ onsets and period $n$, and with some geodesic distance $m$ having multiplicity $k-1$. Consider the graph $G_m = (R, E_m)$ as defined in the proof of Theorem 3.2.3 (with vertices corresponding to onsets in $R$ and with an edge between two onsets of geodesic distance $m$). If $m$ and $n$ are rational multiples of each other, then we can scale $R$ by some rational $p$ and apply Theorem 3.2.3 to show that $R$ is a scaling by $1/p$ of $D_{k,pn,pm}$ where $pm$ and $pn$ are relatively prime integers and $1 \leq pm \leq \lfloor pn/2 \rfloor$, so $0 < m \leq n/2$.

If $m$ and $n$ are irrational multiples of each other, then there is no rational number $r$ such that $n = rm$. This means that $G_m$ cannot contain a cycle, so consists of a single path of length $k-1$. As in the proof of Theorem 3.2.3, we can rotate $R$ by $-i$ so that 0 is an onset in the resulting rhythm $R-i$. The vertices of the path in $R-i$ form a subset of the subgroup of the cyclic group $(\mathbb{Z}/(n), +)$ generated by the element $m$. Therefore the rhythm $R - i = D_{k,n,m} = \{(im) \bmod n : i = 0, 1, \ldots, k-1\}_n$ where $m$ and $n$ are irrational multiples of each other and $0 < m \leq n/2$. □

This concludes our characterization of deep rhythms. In the next chapter, we study a different property of rhythms called *maximal evenness*, and show the connections between this family and deep rhythms.

# Chapter 4

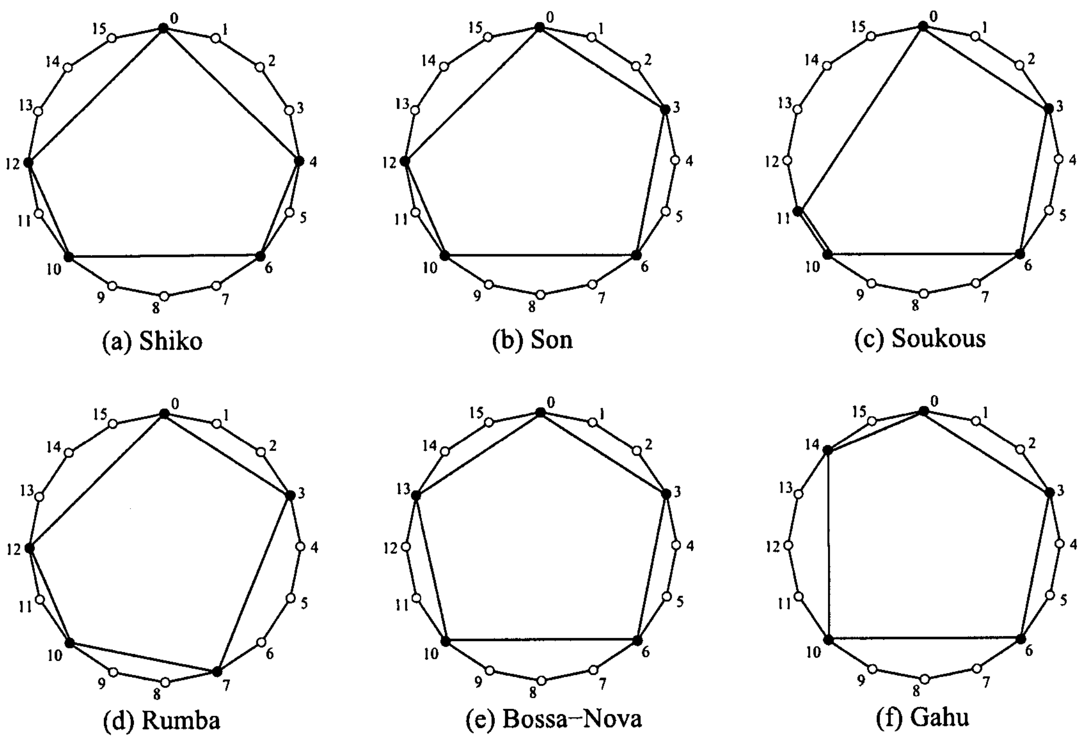
# Maximally Even Rhythms

Roughly speaking, maximally even rhythms are a family of rhythms where the onsets are distributed among the pulses *as evenly as possible*. This property of the onsets being "spread out as much as possible" is know in music theory as *maximal evenness* and was first introduced by Clough and Douthett in 1991 [43]. In this chapter, we present three algorithms that generate maximally even rhythms (for some fixed definition of evenness); we then show that for a given number of pulses and onsets, these algorithms produce the unique rhythm (up to rotation) with maximum evennes. Finally, we show that that maximally even rhythms with $n$ and $k$ relatively prime are Erdős-deep. But first, we investigate the notion of maximal evenness a bit further.

Consider the following three 12/8-time rhythmic patterns expressed in box-like notation: [× · × · × · × · × · × ·], [× · × · ×× · × · × · ×], and [× · · · ×× · · ×× × · ]. The first rhythm contains beats that are distributed perfectly. Such rhythms are found throughout the world, and are most easily identified and incorporated in music and dance. However, in many cultures, rhythms that are not perfectly even are preferred. It is intuitively clear that the first rhythm is more *even* (well spaced) than the second rhythm, and that the second rhythm is more even than the third rhythm. In fact, the second rhythm is the internationally most well known of all African timelines. It is traditionally played on an iron bell, and is known mainly by its Cuban name *Bembé* [171]. Traditional rhythms tend to exhibit such properties of evenness to some degree. The reason for such evenness in traditional rhythms is that many of these timelines often have a *call-and-response structure*, meaning that the rhythmic pattern is divided into two parts: the first poses a rhythmic question, usually by creating rhythmic tension, and the second part answers this question by releasing that tension. One way of creating rhythmic tension is through *syncopation*. Seyer et al. [150] define syncopation as "the shifting of an expected accent, moving it from the usual strong beat to a beat that is usually weak". This definition implies that the rhythm (or meter) must create an *expectation* in order to later break this expectation and introduce an element of surprise by moving the accent to an unpredicted location. From a rhythmic perspective, moving an accent may be accomplished by moving an onset from a strong beat to a weak one. Let us for example consider the clave Son [× · · × · · × · · · × · × · · ·]. The strong beats of the underlying 4/4 meter occur at positions 0, 4, 8, and 12. This clave creates rhythmic tension through syncopation, which appears twice in the clave; the first is between the second and third onsets, and the second is between the third and fourth onsets. The former syncopation is strong because the strong beat at position 4 is closer to the second onset than to the third onset, while the latter is weak syncopation because the strong beat at position 8 lies halfway between the third and fourth onsets [84]. Claves played with instruments that produce unsustained notes often use syncopation and accentuation to bring about rhythmic tension. Many clave rhythms create syncopation by evenly distributing onsets in contradiction with the pulses of the underlying meter. For example, in the clave Son, the first three onsets are equally spaced at the distance of three sixteenth pulses, which forms a contradiction because 3 does not divide 16. Then the response of the clave answers with an offbeat onset, followed by an onset on the fourth strong beat of a 4/4 meter, releasing that rhythmic tension.

On the other hand, a rhythm that is too even (most onsets are equally spaced), such as the rhythm [× · × · × · × · × · × ·], is less interesting from a syncopation point of view. Indeed, in the most interesting rhythms with $k$ onsets and timespan $n$, $k$ and $n$ are relatively prime. This property is natural because the rhythmic contradiction is easier to obtain if the onsets do not coincide with the strong beats of the meter. Also, we find that many timelines have an onset on the last strong beat of the meter, as does the clave Son. This is a natural way to respond in the call-and-response structure. A different case is that of the Bossa-Nova clave [× · · × · · × · · · × · · × · ·]. This clave tries to break the feeling of the pulse and, although it is very even, it produces a cycle that perceptually does not coincide with the beginning of the meter.

The prevalence of evenness in world rhythms has led to the study of mathematical measures of evenness in the new field of mathematical ethnomusicology [37, 175, 176], where they may help to identify, if not explain, cultural preferences of rhythms in traditional music. Furthermore, evenness in musical chords plays a significant role in the efficacy of voice leading (creating simultaneous melodies by combining chords with a series of notes) as discussed in the work of Tymoczko [90, 178].



(a) Shiko (b) Son (c) Soukous

(d) Rumba (e) Bossa–Nova (f) Gahu

Figure 4–1: The six fundamental African and Latin American rhythms which all have equal sum of pairwise geodesic distances; yet intuitively, the Bossa-Nova rhythm is more "even" than the rest.

The notion of *maximally even sets* with respect to scales represented on a circle was introduced by Clough and Douthett [43]. According to Block and Douthett [23], Douthett and Entringer went further by constructing several mathematical measures of the amount of *evenness* contained in a scale; see [23, page 40]. One of their evenness measures simply sums the interval arc-lengths (geodesics along the circle) between all pairs of onsets (or more precisely, onset points). This measure differentiates between rhythms that differ widely from each other. For example, the two 4-onset rhythms [× · · · × · · · × · · · × · · ·] and [× · × · × · · × · · · · · · · ·] yield evenness values of 32 and 23, respectively, reflecting clearly that the first rhythm is more evenly spaced than the second. However, the measure is too coarse to be useful for comparing rhythm timelines such as those studied in [169, 171]. For example, all six fundamental 4/4-time clave/bell patterns discussed in [169] and shown in Figure 4–1 have an equal pairwise sum of geodesic distances, namely 48, yet the *Bossa-Nova* clave is intuitively more even than, say, the *Soukous* and *Rumba* claves. Jiang [? ] characterizes the configurations of points on a circle that maximize the sum of arc distances and gives a linear-time algorithm to compute this sum for a given pointset. This characterization explains why the sum of arc-lengths is not a very sensitive measure of evenness.

Another distance measure that has been considered is the sum of pairwise chordal distances between adjacent onsets, measured by Euclidean distance between points on the circle. It can be shown that the rhythms maximizing this measure of evenness are precisely the rhythms with maximum possible area. Rappaport [140] shows that many of the most common chords and scales in Western harmony correspond to these maximum-area sets. This evenness measure is finer than the sum of pairwise arc-lengths, but it still does not distinguish half the rhythms in Figure 4–1. Specifically, the Son, Rumba, and Gahu claves have the same occurrences of arc-lengths between consecutive onsets, so they also have the same occurrences (and hence total) of distances between consecutive onsets.

The measure of evenness we consider here is the sum of all pairwise Euclidean distances between points on the circle, as described by Block and Douthett [23]. It is worth pointing out that Fejes Tóth [167] showed in 1956 that a configuration of points on a circle maximizes this sum when the points are the vertices of a regular polygon. This measure is also more discriminating than the others, and is therefore the preferred measure of evenness. For example, this measure distinguishes all of the six rhythms in Figure 4–1, ranking the Bossa-Nova rhythm as the most even, followed by the Son, Rumba, Shiko, Gahu, and Soukous. Intuitively, the rhythms with a larger sum of pairwise chordal distances have more "well spaced" onsets.

In what follows, we first investigate the notion of maximum evennes in various disciplines (Section 4.1). Then in Section 4.2, we study the mathematical and computational aspects of rhythms that maximize evenness and characterize rhythms with maximum evenness. Section 4.3 shows connections between maximally even and Erdős-deep rhythms. We close the chapter with final remarks and open problems.

## 4.1 Euclid and Evenness in Various Disciplines

In this section, we first describe Euclid's classic algorithm for computing the greatest common divisor of two integers. Then, through an unexpected connection to timing systems in neutron accelerators, we see how the same type of algorithm can be used as an approach to maximizing "evenness" in a binary string with a specified number of zeroes and ones. This algorithm defines an important family of rhythms, called *Euclidean rhythms*, which appear throughout world music. Euclidean rhythms are known by different names in several areas of mathematics. In the algebraic combinatorics of words they are called *Sturmian* words [114]. Lunnon and Pleasants call them *two-distance* sequences [116], and de Bruijn calls them *Beatty* sequences [57, 58]. See also the geometry of Markoff numbers [149]. In the remainder of the section, we see how ideas similar to these rhythms have been used in algorithms for drawing digital straight lines, designing calendars, and in combinatorial strings called Euclidean strings.

### 4.1.1 The Euclidean Algorithm for Greatest Common Divisors

The Euclidean algorithm for computing the greatest common divisor of two integers is one of the oldest known algorithms (circa 300 B.C.). It was first described by Euclid in Proposition 2 of Book VII of *Elements* [69, 76]. Donald Knuth [107] calls this algorithm the "granddaddy of all algorithms, because it is the oldest nontrivial algorithm that has survived to the present day".

The idea of the algorithm is simple: repeatedly replace the larger of the two numbers by their difference until both are equal. This final number is then the greatest common divisor. For example, consider the numbers 5 and 13. First, $13-5 = 8$; then $8 - 5 = 3$; next $5 - 3 = 2$; then $3 - 2 = 1$; and finally $2 - 1 = 1$. Therefore, the greatest common divisor of 5 and 13 is 1; in other words, 5 and 13 are relatively prime.

The algorithm can also be described succinctly in a recursive manner as follows [52]. Let $k$ and $n$ be the input integers with $k < n$.

**Algorithm** EUCLID($k, n$)

1. **if** $k = 0$ **then return** $n$
2. **else return** EUCLID($n$ mod $k, k$)

Running this algorithm with $k = 5$ and $n = 13$, we obtain EUCLID(5, 13) = EUCLID(3, 5) = EUCLID(2, 3) = EUCLID(1, 2) = EUCLID(0, 1) = 1. Note that this division version of Euclid's algorithm skips one of the steps (5, 8) made by the original subtraction version.

### 4.1.2 Evenness in Timing Systems in Neutron Accelerator

One of our main musical motivations is to find rhythms with a specified timespan and number of onsets that maximize evenness. Bjorklund [21, 22] was faced with a similar problem of maximizing evenness, but in a different context: the operation of components such as high-voltage power supplies of spallation neutron source (SNS) accelerators used in nuclear physics. In this setting, a timing system controls a collection of gates over a time window divided into $n$ equal-length intervals (in the case of SNS, each interval is 10 seconds). The timing system can send signals to enable a gate during any desired subset of the $n$ intervals. For a given number $n$ of time intervals, and another given number $k < n$ of signals, the problem is to distribute the pulses as evenly as possible among these $n$ intervals. Bjorklund [22] represents this problem as a binary sequence of $k$ ones and $n - k$ zeroes, where each bit represents a time interval and the ones represent the times at which the timing system sends a signal. The problem then reduces to the following: construct a binary sequence of $n$ bits with $k$ ones such that the $k$ ones are distributed as evenly as possible among the $n - k$ zeroes.

One simple case is when $k$ evenly divides $n$ (without remainder), in which case we should place ones every $n/k$ bits. For example, if $n = 16$ and $k = 4$, then the solution is [1000100010001000]. This case corresponds to $n$ and $k$ having a common divisor of $k$. More generally, if the greatest common divisor between $n$ and $k$ is $g$, then we would expect the solution to decompose into $g$ repetitions of a sequence of $n/g$ bits. Intuitively, a string of maximum evenness should have this kind of symmetry, in which it decomposes into more than one repetition, whenever such symmetry is possible. This connection to greatest common divisors suggests that a rhythm of maximum evenness might be computed using an algorithm like Euclid's. Indeed, Bjorklund's algorithm closely mimics the structure of Euclid's algorithm.

We describe Bjorklund's algorithm by using one of his examples. Consider a sequence with $n = 13$ and $k = 5$. Since $13 - 5 = 8$, we start by considering a sequence consisting of 5 ones followed by 8 zeroes which should be thought of as 13 sequences of one bit each:

[1][1][1][1][1][0][0][0][0][0][0][0][0]

If there is more than one zero the algorithm moves zeroes in stages. We begin by taking zeroes one at a time (from right to left), placing a zero after each one (from left to right), to produce five sequences of two bits each, with three zeroes remaining:

[10] [10] [10] [10] [10] [0] [0] [0]

Next we distribute the three remaining zeros in a similar manner, by placing a [0] sequence after each [10] sequence:

[100] [100] [100] [10] [10]

Now we have three sequences of three bits each, and a remainder of two sequences of two bits each. Therefore we continue in the same manner, by placing a [10] sequence after each [100] sequence:

[10010] [10010] [100]

The process stops when the remainder consists of only one sequence (in this case the sequence [100]), or we run out of zeroes (there is no remainder). The final sequence is thus the concatenation of [10010], [10010], and [100]:

[1001010010100]

We could proceed further in this process by inserting [100] into [10010] [10010]. However, Bjorklund argues that, because the sequence is cyclic, it does not matter (hence his stopping rule). For the same reason, if the initial sequence has a group of ones followed by only one zero, the zero is considered as a remainder consisting of one sequence of one bit, and hence nothing is done. Bjorklund [22] shows that the final sequence may be computed from the initial sequence using $O(n)$ arithmetic operations in the worst case.

A more convenient and visually appealing way to implement this algorithm by hand is to perform the sequence of insertions in a vertical manner as follows. First take five zeroes from the right and place them under the five ones on the left:

1 1 1 1 1 0 0 0

0 0 0 0 0

Then move the three remaining zeroes in a similar manner:

1 1 1 1 1

0 0 0 0 0

0 0 0

Next place the two remainder columns on the right under the two leftmost columns:

1 1 1

0 0 0

0 0 0

1 1

0 0

Here the process stops because the remainder consists of only one column. The final sequence is obtained by concatenating the three columns from left to right:

1 0 0 1 0 1 0 0 1 0 1 0 0

Bjorklund's algorithm applied to a string of $n$ bits consisting of $k$ ones and $n-k$ zeros has the same structure as running $\textsc{Euclid}(k,n)$. Indeed, Bjorklund's algorithm uses the repeated subtraction form of division, just as Euclid did in his *Elements* [69]. It is also well known that applying the algorithm $\textsc{Euclid}(k,n)$ to two $O(n)$ bit numbers (binary sequences of length $n$) causes it to perform $O(n)$ arithmetic operations in the worst case [52]. The connection between the algorithms of Bjorklund and Euclid is studied in further detail in Section 5.1.

### 4.1.3 Euclidean Rhythms

The binary sequences generated by Bjorklund's algorithm, as described in the preceding, may be considered as one family of rhythms. Furthermore, because Bjorklund's algorithm is a way of visualizing the repeated-subtraction version of the Euclidean algorithm, we call these rhythms *Euclidean rhythms.* The Euclidean algorithm has been connected to music theory previously by Viggo Brun [31]. Brun used the Euclidean algorithm to calculate the lengths of strings in musical instruments between two lengths $l$ and $2l$, so that all pairs of adjacent strings have the same length ratios. In contrast, we relate the Euclidean algorithm to rhythms and scales in world music.

Throughout the remainder of this thesis we denote the Euclidean rhythm by $E(k,n)$, where $k$ is the number of ones (onsets) and $n$ is the length of the sequence (number of pulses). For example, $E(5,13)$ = [1001010010100]. The zero-one notation is not ideal for representing binary rhythms because it is difficult to visualize the locations of the onsets as well as the duration of the interonset intervals. In the more iconic box notation, the preceding rhythm is written as $E(5,13) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot]$. It should be emphasized that Euclidean rhythms are merely the result of applying Bjorklund's algorithm and do not privilege a priori the resulting rhythm over any of its other rotations.

The rhythm $E(5,13)$ is in fact used in Macedonian music [8], but having a timespan of 13 (and defining a measure of length 13), it is rarely found in world music. For contrast, let us consider two widely used values of $k$ and $n$; in particular, what is $E(3,8)$? Applying Bjorklund's algorithm to the corresponding sequence [11100000],

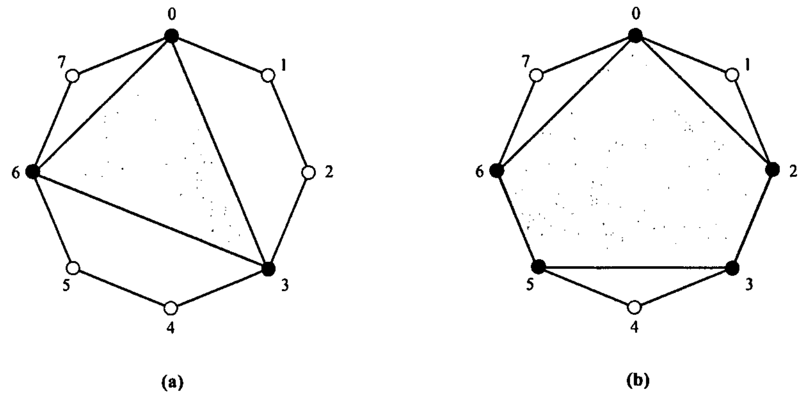



Figure 4–2: (a) The Euclidean rhythm $E(3,8)$ is the Cuban *tresillo*. (b) The Euclidean rhythm $E(5,8)$ is the Cuban *cinquillo*.

the resulting Euclidean rhythm is $E(3,8) = [\times \cdot \cdot \times \cdot \cdot \times \cdot]$. Figure 4–2(a) shows a clock diagram of this rhythm. The Euclidean rhythm $E(3,8)$ is one of the most famous on the planet. In Cuba, it goes by the name of the *tresillo*, and in the USA, it is often called the *Habanera* rhythm. It was used in hundreds of *rockabilly* songs during the 1950's. It can often be heard in early rock-and-roll hits in the left-hand patterns of the piano, or played on the string bass or saxophone [29, 74, 127]. A good example is the bass rhythm in Elvis Presley's *Hound Dog* [29]. The tresillo pattern is also found widely in West African traditional music. For example, it is played on the *atoke* bell in the *Sohu*, an *Ewe* dance from Ghana [103]. The tresillo can also be recognized as the first bar (first eight pulses) of the ubiquitous two-bar clave Son shown in Figure 4–1(b).

In the two examples $E(5,13)$ and $E(3,8)$, there are fewer ones than zeros. If instead there are more ones than zeros, Bjorklund's algorithm yields the following steps with, for example, $k = 5$ and $n = 8$:

[1 1 1 1 1 0 0 0]

[10] [10] [10] [1] [1]

[101] [101] [10]

[1 0 1 1 0 1 1 0]

The resulting Euclidean rhythm is $E(5,8) = [\times \cdot \times\times \cdot \times\times \cdot]$. Figure 4–2(b) shows a clock diagram for this rhythm. It is another famous rhythm. In Cuba, it goes by the name of the *cinquillo* and it is intimately related to the tresillo [74]. It has been used in jazz throughout the 20th century [139], and in rockabilly music. For example, it is the hand-clapping pattern in Elvis Presley's *Hound Dog* [29]. The cinquillo pattern is also widely used in West African traditional music [137, 169], as well as Egyptian [89] and Korean [98] music.

We show in this chapter that Euclidean rhythms maximize evenness, which should come as no surprise given how we designed the family of rhythms.

### 4.1.4 Euclidean Rhythms in Traditional World Music

In this section, we list all the Euclidean rhythms found in world music that we have collected so far, restricting attention to those in which $k$ and $n$ are relatively prime. In some cases, the Euclidean rhythm is a rotated version of a commonly used rhythm; this makes the two rhythms instances of the same necklace. Figure 4–3 illustrates an example of two rhythms that are instances of the same necklace. We provide this list because it is interesting ethnomusicological data on rhythms. We make no effort to establish that Euclidean rhythms are more common than their rotations, and leave the problem of defining which rhythms are preferred over others as an open problem to ethnomusicologists.

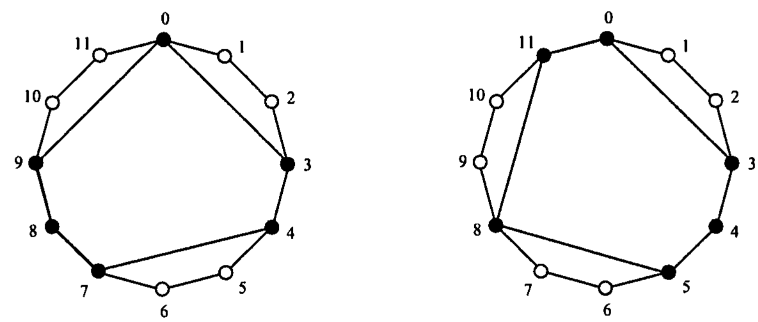



Figure 4–3: These two rhythms are instances of the same rhythm necklace.

Rhythms in which $k$ and $n$ have a common divisor larger than 1 are common all over the planet in traditional, classical, and popular genres of music. For example, $E(4,12) = [\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot]$ is the 12/8-time *Fandango* clapping pattern in the Flamenco music of southern Spain, where '$\times$' denotes a loud clap and '$\cdot$' denotes a soft clap [63]. However, the string itself is periodic. A sequence $\{a_0, a_1, \ldots, a_{n-1}\}$ is said to be *periodic with period* $p$ if it satisfies $a_i = a_{(i+p) \bmod n}$ for the smallest possible value of $p < n$ and for all $i = 0, \ldots, n-1$. In our example, $E(4,12)$ is periodic with period 3, even though it appears in a timespan of 12. For this reason, we restrict ourselves to the more interesting Euclidean rhythms that do not decompose into repetitions of shorter Euclidean rhythms. We are also not concerned with rhythms that have only one onset ($[\times \cdot]$, $[\times \cdot \cdot]$, etc.), and similarly with any repetitions of these rhythms (for example, $[\times \cdot \times \cdot]$).

There are surprisingly many Euclidean rhythms with $k$ and $n$ relatively prime that are found in world music. Appendix A includes more than 40 such rhythms uncovered so far.

### 4.1.5 Aksak Rhythms

Euclidean rhythms are closely related to a family of rhythms known as *aksak* rhythms, which have been studied from the combinatorial point of view for some time [8, 26, 42]. Béla Bartók [14] and Constantin Brăiloiu [26], respectively, have used the terms *Bulgarian rhythm* and *aksak* to refer to those meters that use units of durations 2 and 3, and no other durations. Furthermore, the rhythm or meter must contain at least one duration of length 2 and at least one duration of length 3. Arom [8] refers to these durations as *binary cells* and *ternary cells*, respectively.

Arom [8] generated an inventory of all the theoretically possible *aksak* rhythms for values of $n$ ranging from 5 to 29, as well as a list of those that are actually used in traditional world music. He also proposed a classification of these rhythms into several classes, based on structural and numeric properties. Three of his classes are considered here:

1. An *aksak* rhythm is *authentic* if $n$ is *prime.*
2. An *aksak* rhythm is *quasi-aksak* if $n$ is *odd* but not prime.
3. An *aksak* rhythm is *pseudo-aksak* if $n$ is *even.*

A quick perusal of the Euclidean rhythms listed in Appendix A reveals that *aksak* rhythms are well represented. Indeed, all three of Arom's classes (authentic, quasi-aksak, and pseudo-aksak) make their appearance. There is a simple characterization of those Euclidean rhythms that are *aksak.* From the iterative subtraction algorithm of Bjorklund it follows that if $n = 2k$ all cells are binary (duration 2). Similarly, if $n = 3k$ all cells are ternary (duration 3). Therefore, to ensure that the Euclidean rhythm contains both binary and ternary cells, and no other durations, it follows that $n$ must be between $2k$ and $3k$.

Of course, not all *aksak* rhythms are Euclidean. Consider the Bulgarian rhythm with interval sequence (3322) [8], which is also the metric pattern of *Indian Lady* by Don Ellis [104]. Here $k = 4$ and $n = 10$, and $E(4, 10) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot]$ or (3232), a periodic rhythm.

Appendix B lists some examples of Aksak rhythms.

### 4.1.6 Drawing Digital Straight Lines

Euclidean rhythms and necklace patterns also appear in the computer graphics literature on drawing digital straight lines [105]. The problem here consists of efficiently converting a mathematical straight line segment defined by the $x$ and $y$ integer coordinates of its endpoints, to an ordered sequence of pixels that most faithfully represents the given straight line segment. Figure 4–4 illustrates an example of a digital straight line (shaded pixels) determined by the two given endpoints $p$ and $q$. All the pixels intersected by the segment $(p, q)$ are shaded. If we follow either the lower or upper boundary of the shaded pixels from left to right we obtain the interval sequences (43333) or (33334), respectively. Note that the upper pattern corresponds to $E(5, 16)$, a *Bossa-Nova* variant. Indeed, Harris and Reingold [93] show that the well-known Bresenham algorithm [28] is described by the Euclidean algorithm. Reinhard Klette and Azriel Rosenfeld [105] have written an excellent survey of the properties of digital straight lines and their many connections to geometry and number theory.

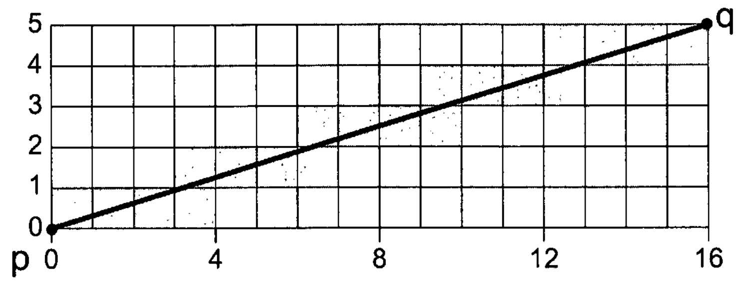



Figure 4–4: The shaded pixels form a digital straight line determined by the points $p$ and $q$.

### 4.1.7 Calculating Leap Year in Calendar Design

For thousands of years human beings have observed and measured the time it takes between two consecutive sunrises, and between two consecutive spring seasons. These measurements inspired different cultures to design calendars [10, 142]. Let $T_y$ denote the duration of one revolution of the earth around the sun, more commonly known as a year. Let $T_d$ denote the duration of one complete rotation of the earth, more commonly known as a day. The values of $T_y$ and $T_d$ are of course continually changing, because the universe is continually reconfiguring itself. However the ratio $T_y/T_d$ is approximately 365.242199..... It is very convenient therefore to make a year last 365 days. The problem that arises both for history and for predictions of the future, is that after a while the 0.242199..... starts to contribute to a large error. One simple solution is to add one extra day every 4 years: the so-called Julian calendar. A day with one extra day is called a leap year. But this assumes that a year is 365.25 days long, which is still slightly greater than 365.242199...... So now we have an error in the opposite direction albeit smaller. One solution to this problem is the Gregorian calendar [151]. The Gregorian calendar defines a leap year as one divisible by 4, except those divisible by 100, except those divisible by 400. With this rule a year becomes 365 + 1/4 - 1/100 + 1/400 = 365.2425 days long, a better approximation.

Another solution is provided by the Jewish calendar which uses the idea of cycles [10]. Here a regular year has 12 months and a leap year has 13 months. The cycle has 19 years including 7 leap years. The 7 leap years must be distributed as evenly as possible in the cycle of 19. The cycle is assumed to start with Creation as year 1. If the year modulo 19 is one of 3, 6, 8, 11, 14, 17, or 19, then it is a leap year. For example, the year $5765 = 303 \cdot 19 + 8$ and so is a leap year. The year 5766, which begins at sundown on the Gregorian date of October 3, 2005, is 5766 = 303x19 + 9, and is therefore not a leap year. Applying Bjorklund's algorithm to the integers 7 and 19 yields $E(7,19) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \cdot]$. If we start this rhythm at the 7th pulse we obtain the pattern $[\cdot \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \times]$, which describes precisely the leap year pattern 3, 6, 8, 11, 14, 17, and 19 of the Jewish calendar. In this sense the Jewish calendar is an instance of a Euclidean necklace.

### 4.1.8 Euclidean Strings

In this section we explore the relationship between Euclidean rhythms and Euclidean strings, which were introduced by Ellis et al. [66] as part of the study of the combinatorics of words and sequences. We use the same terminology and notation introduced in [66]. Euclidean strings result from a mathematical algorithm and represent a different arbitrary convention as to how to choose a canonical rhythm that

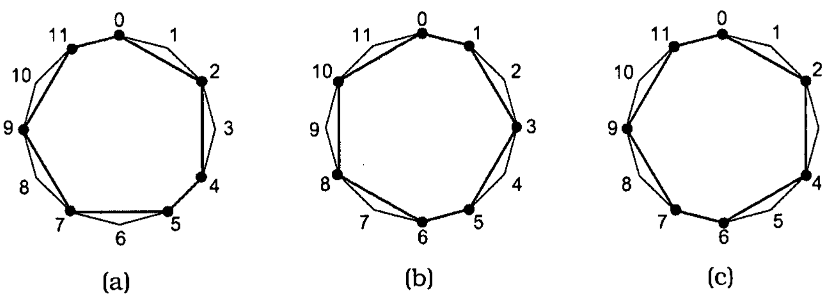



Figure 4–5: Two right-rotations of the *Bembé* string: (a) the *Bembé*, (b) rotation by one unit, (c) rotation by seven units.

represents the necklace. Whether there is anything musically meaningful about these conventions is left to ethnomusicologists to decide.

Let $P = (p_0, p_1, ..., p_{n-1})$ denote a string of non-negative integers. Let $\rho(P)$ denote the right rotation of $P$ by one position; that is, $\rho(P) = (p_{n-1}, p_0, p_1, ..., p_{n-2})$. Let $\rho^d(P)$ denote the right rotation of $P$ by $d$ positions. If $P$ is considered as a cyclic string, a right rotation corresponds to a clockwise rotation. Figure 4–5 illustrates the $\rho(P)$ operator with $P$ equal to the *Bembé* bell-pattern of West Africa [171]. Figure 4–5(a) shows the *Bembé* bell-pattern, Figure 4–5(b) shows $\rho(P)$, which is a hand-clapping pattern from West Africa [134], and Figure 4–5(c) shows $\rho^7(P)$, which is the *Tambú* rhythm of Curaçao [147].

Ellis et al. [66] define a string $P = (p_0, p_1, ..., p_{n-1})$ to be *Euclidean* if incrementing $p_0$ by 1 and decrementing $p_{n-1}$ by 1 yields a new string $\tau(P)$ that is the rotation of $P$. In other words, $P$ and $\tau(P)$ are instances of the same necklace. Therefore, if we represent rhythms as binary sequences, Euclidean rhythms cannot be Euclidean strings because all Euclidean rhythms begin with a 'one'. Increasing $p_0$ by one makes it a 'two', which is not a binary string. Therefore, to explore the relationship between Euclidean strings and Euclidean rhythms, we will represent rhythms by their clockwise distance sequences, which are also strings of non-negative integers. As an example, consider $E(4,9) = [\times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (2223)$. Now $\tau(2223) = (3222)$, which is a rotation of $E(4,9)$, and thus (2223) is a Euclidean string. Indeed, for $P = E(4,9)$, $\tau(P) = \rho^3(P)$. As a second example, consider the West African clapping-pattern shown in Figure 4–5(b) given by $P = (1221222)$. We have $\tau(P) = (2221221) = \rho^6(P)$, the pattern shown in Figure 4–5(c), which also happens to be the mirror image of $P$ about the $(0,6)$ axis. Therefore $P$ is a Euclidean string; however, note that $P$ is not a Euclidean rhythm. Nevertheless, $P$ is a rotation of the Euclidean rhythm $E(7,12) = (2122122)$.

Ellis et al. [66] have many results about Euclidean strings. They show that Euclidean strings exist if, and only if, $n$ and $p_0 + p_1 + ... + p_{n-1}$ are relatively prime, and that when they exist they are unique. They also show how to construct Euclidean strings using an algorithm that has the same structure as the Euclidean algorithm. In addition they relate Euclidean strings to many other families of sequences studied in the combinatorics of words [2, 114].

Let $R(P)$ denote the reversal (or mirror image) of $P$; that is, $R(P) = (p_{n-1}, p_{n-2}, \ldots, p_1, p_0)$. Now we may determine which of the Euclidean rhythms used in world music listed in Appendix A, are Euclidean strings or *reverse* Euclidean strings. The length of a Euclidean string is defined as the number of integers it contains. In the rhythm domain, this translates to the number of onsets in a rhythm.

Furthermore, strings of length one are Euclidean strings, trivially. Therefore all the trivial Euclidean rhythms with only one onset, such as $E(1,2) = [\times\ \cdot\,] = (2)$, $E(1,3) = [\times\ \cdot\ \cdot\,] = (3)$, and $E(1,4) = [\times\ \cdot\ \cdot\ \cdot\,] = (4)$, etc., are both Euclidean strings as well as reverse Euclidean strings. Appendix C gives a list of Euclidean rhythms that are also Euclidean strings.

The Euclidean rhythms that often appear in classical music and jazz are also Euclidean strings (the first group of Appendix C). Furthermore, this group is not popular in African music. The Euclidean rhythms that are neither Euclidean strings nor reverse Euclidean strings (group three of the appendix) fall into two categories: those consisting of clockwise distances 1 and 2, and those consisting of clockwise distances 2 and 3. The latter group is used only in Bulgaria, and the former is used in Africa. Finally, the Euclidean rhythms that are reverse Euclidean strings (the second group of the appendix) appear to have a much wider use. Finding musicological explanations for these mathematical properties raises interesting ethnomusicological questions.

The Euclidean strings defined in [66] determine another family of rhythms, many of which are also used in world music but are not necessarily Euclidean rhythms. For example, (1221222) is an Afro-Cuban bell pattern. Therefore it would be interesting to explore empirically the relation between Euclidean strings and world music rhythms, and to determine formally the exact mathematical relation between Euclidean rhythms and Euclidean strings.

## 4.2 Algorithms and Characterization of Maximally Even Rhythms

In this section we first describe three algorithms that generate even rhythms. We then characterize rhythms with maximum evenness and show that, for given numbers of pulses and onsets, the three described algorithms generate the unique rhythm with maximum evenness. As mentioned earlier, the measure of evenness considered here is the pairwise sum of chordal distances. More formally, it is

$$\sum_{0\leq i<j\leq k-1} \overline{d}(r_i, r_j).$$

The even rhythms characterized in this section were studied by Clough and Meyerson [45, 46] for the case where the numbers of pulses and onsets are relatively prime. This was subsequently expanded upon by Clough and Douthett [43]. We revisit these results and provide an additional connection to rhythms (and scales) that are obtained from the Euclidean algorithm. Most of these results are stated in [43]. However our proofs are new, and in many cases are more streamlined.

### 4.2.1 Characterization

We first present three algorithms for computing a rhythm with $k$ onsets, timespan $n$, for any $k \leq n$, that possess large evenness.

The first algorithm is by Clough and Douthett [43]:

**Algorithm** CLOUGH-DOUTHETT$(k, n)$

1. **return** $\{\lfloor \frac{in}{k} \rfloor : i \in [0, k-1]\}$

Since $k \leq n$, the rhythm output by CLOUGH-DOUTHETT$(k, n)$ has $k$ onsets as desired.

The second algorithm is a geometric heuristic implicit in the work of Clough and Douthett [43]:

**Algorithm** SNAP$(k, n)$

1. Let $D$ be a set of $k$ evenly spaced points on $C_n$ such that $D \cap C_n = \emptyset$.
2. For each point $x \in D$, let $x'$ be the first point in $C_n$ clockwise from $x$.
3. **return** $\{x' : x \in D\}$

Since $k \leq n$, the clockwise distance between consecutive points in $D$ in the execution of SNAP$(k, n)$ is at least that of consecutive points in $C_n$. Thus, $x' \neq y'$ for distinct $x, y \in D$, so SNAP returns a rhythm with $k$ onsets as desired.

The third algorithm is a recursive algorithm in the same mold as Euclid's algorithm for greatest common divisors. The algorithm uses the clockwise distance sequence notation described in the introduction. The resulting rhythm always defines the same necklace as the Euclidean rhythms from Section 4.1.3; that is, the only difference is a possible rotation.

**Algorithm** EUCLIDEAN$(k, n)$

1. **if** $k$ evenly divides $n$ **then return** $(\underbrace{\tfrac{n}{k}, \tfrac{n}{k}, \ldots, \tfrac{n}{k}}_{k})$
2. $a \leftarrow n \bmod k$
3. $(x_1, x_2, \ldots, x_a) \leftarrow$ EUCLIDEAN$(a, k)$
4. **return** $(\underbrace{\lfloor\tfrac{n}{k}\rfloor, \ldots, \lfloor\tfrac{n}{k}\rfloor}_{x_1-1}, \lceil\tfrac{n}{k}\rceil; \underbrace{\lfloor\tfrac{n}{k}\rfloor, \ldots, \lfloor\tfrac{n}{k}\rfloor}_{x_2-1}, \lceil\tfrac{n}{k}\rceil; \ldots; \underbrace{\lfloor\tfrac{n}{k}\rfloor, \ldots, \lfloor\tfrac{n}{k}\rfloor}_{x_a-1}, \lceil\tfrac{n}{k}\rceil)$

As a simple example, consider $k = 5$ and $n = 13$. The sequence of calls to EUCLIDEAN$(k, n)$ follows the same pattern as the EUCLID algorithm for greatest common divisors from Section 4.1.1, except that it now stops one step earlier: $(5, 13)$, $(3, 5)$, $(2, 3)$, $(1, 2)$. At the base of the recursion, we have EUCLIDEAN$(1, 2) = (2) = [\times \cdot]$. At the next level up, we obtain EUCLIDEAN$(2, 3) = (1, 2) = [\times\times \cdot]$. Next we obtain EUCLIDEAN$(3, 5) = (2; 1, 2) = [\times \cdot \times\times \cdot]$. Finally, we obtain EUCLIDEAN$(5, 13) = (2, 3; 3; 2, 3) = [\times \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot]$. (For comparison, the Euclidean rhythm from Section 4.1.2 is $E(5, 13) = (2, 3, 2, 3, 3)$, a rotation by 5).

We now show that algorithm EUCLIDEAN$(k, n)$ produces a circular sequence of $k$ integers that sum to $n$ (which is thus the clockwise distance sequence of a rhythm with $k$ onsets and timespan $n$). We proceed by induction on $k$. If $k$ evenly divides $n$, then the claim clearly holds. Otherwise $a$ $(= n \bmod k) > 0$, and by induction $\sum_{i=1}^{a} x_i = k$. Thus the sequence that is produced has $k$ terms and sums to

$$\begin{aligned} a\left\lceil \frac{n}{k} \right\rceil + \left\lfloor \frac{n}{k} \right\rfloor \sum_{i=1}^{a} (x_i - 1) &= a\left\lceil \frac{n}{k} \right\rceil + (k - a)\left\lfloor \frac{n}{k} \right\rfloor \\ &= a\left(1 + \left\lfloor \frac{n}{k} \right\rfloor\right) + (k - a)\left\lfloor \frac{n}{k} \right\rfloor \\ &= a + k\left\lfloor \frac{n}{k} \right\rfloor \\ &= n \end{aligned}$$

The following theorem is the main contribution of this chapter.

**Theorem 4.2.1.** *Let $n \geq k \geq 2$ be integers. The following are equivalent for a rhythm $R = \{r_0, r_1, \ldots, r_{k-1}\}_n$ with $k$ onsets and timespan $n$:*

(A) *$R$ has maximum evenness (sum of pairwise interonset chordal distances);*

(B) *$R$ is a rotation of the* CLOUGH-DOUTHETT$(k,n)$ *rhythm;*

(C) *$R$ is a rotation of the* SNAP$(k,n)$ *rhythm;*

(D) *$R$ is a rotation of the* EUCLIDEAN$(k,n)$ *rhythm; and*

($\star$) *for all $\ell \in [1,k]$ and $i \in [0,k-1]$, the ordered pair $(r_i, r_{i+\ell})$ has clockwise distance $\overset{\circ}{d}(r_i, r_{i+\ell}) \in \{\lfloor \frac{\ell n}{k} \rfloor, \lceil \frac{\ell n}{k} \rceil\}$.*

*Moreover, up to a rotation, there is a unique rhythm that satisfies these conditions.*

Note that the evenness of a rhythm equals the evenness of the same rhythm played backwards. Thus, if $R$ is the unique rhythm with maximum evenness, then $R$ is the same rhythm as $R$ played backwards (up to rotation).

The proof of Theorem 4.2.1 proceeds as follows. In Section 4.2.2 we prove that each of the three algorithms produces a rhythm that satisfies property ($\star$). Then in Section 4.2.3 we prove that there is a unique rhythm that satisfies property ($\star$). Thus the three algorithms produce the same rhythm, up to rotation. Finally in Section 4.2.4 we prove that the unique rhythm that satisfies property ($\star$) maximizes evenness.

### 4.2.2 Properties of the Algorithms

We now prove that each of the algorithms has property ($\star$). Clough and Douthett [43] proved the following.

*Proof* (B) $\Rightarrow$ ($\star$). Say $R = \{r_0, r_1, \ldots, r_{k-1}\}_n$ is the CLOUGH-DOUTHETT$(k,n)$ rhythm. Consider an ordered pair $(r_i, r_{i+\ell})$ of onsets in $R$. Let $p_i = in \bmod k$ and let $p_\ell = \ell n \bmod k$. By symmetry we can assume that $r_i \leq r_{(i+\ell) \bmod k}$. Then the clockwise distance $\overset{\circlearrowright}{d}(r_i, r_{i+\ell})$ is

$$\left\lfloor \frac{(i+\ell)n}{k} \right\rfloor - \left\lfloor \frac{in}{k} \right\rfloor = \left\lfloor \frac{in}{k} \right\rfloor + \left\lfloor \frac{\ell n}{k} \right\rfloor + \left\lfloor \frac{p_i + p_\ell}{k} \right\rfloor - \left\lfloor \frac{in}{k} \right\rfloor = \left\lfloor \frac{\ell n}{k} \right\rfloor + \left\lfloor \frac{p_i + p_\ell}{k} \right\rfloor ,$$

which is $\lfloor \frac{\ell n}{k} \rfloor$ or $\lceil \frac{\ell n}{k} \rceil$, because $\lfloor \frac{p_i + p_\ell}{k} \rfloor \in \{0, 1\}$. □

A similar proof shows that the rhythm $\{\lceil \frac{in}{k} \rceil : i \in [0, k-1]\}$ satisfies property ($\star$). Observe that ($\star$) is equivalent to the following property.

($\star\star$) If $(d_0, d_1, \ldots, d_{k-1})$ is the clockwise distance sequence of $R$, then for all $\ell \in [1, k]$, the sum of any $\ell$ consecutive elements in $(d_0, d_1, \ldots, d_{k-1})$ equals $\lceil \frac{\ell n}{k} \rceil$ or $\lfloor \frac{\ell n}{k} \rfloor$.

*Proof* (C) $\Rightarrow$ ($\star\star$). Let $(d_0, d_1, \ldots, d_{k-1})$ be the clockwise distance sequence of the rhythm determined by SNAP$(k,n)$. For the sake of contradiction, suppose that for some $\ell \in [1, k]$, the sum of $\ell$ consecutive elements in $(d_0, d_1, \ldots, d_{k-1})$ is greater than $\lceil \frac{\ell n}{k} \rceil$. The case in which the sum is less than $\lfloor \frac{\ell n}{k} \rfloor$ is analogous. We can assume that these $\ell$ consecutive elements are $(d_0, d_1, \ldots, d_{\ell-1})$. Using the notation defined in the statement of the algorithm, let $x_0, x_1, \ldots, x_\ell$ be the points in $D$ such that $\overset{\circlearrowright}{d}(x'_i, x'_{i+1}) = d_i$ for all $i \in [0, \ell-1]$. Thus $\overset{\circlearrowright}{d}(x'_1, x'_{\ell+1}) \geq \lceil \frac{\ell n}{k} \rceil + 1$. Now $\overset{\circlearrowright}{d}(x_{\ell+1}, x'_{\ell+1}) < 1$. Thus $\overset{\circlearrowright}{d}(x'_1, x_{\ell+1}) > \lceil \frac{\ell n}{k} \rceil \geq \frac{\ell n}{k}$, which implies that $\overset{\circlearrowright}{d}(x_1, x_{\ell+1}) > \frac{\ell n}{k}$. This contradicts the fact that the points in $D$ were evenly spaced around $C_n$ in the first step of the algorithm. □

*Proof* (D) $\Rightarrow$ ($\star\star$). We proceed by induction on $k$. Let $R = \text{EUCLIDEAN}(k, n)$. If $k$ evenly divides $n$, then $R = (\frac{n}{k}, \frac{n}{k}, \ldots, \frac{n}{k})$, which satisfies (D). Otherwise, let $a = n \bmod k$ and let $(x_1, x_2, \ldots, x_a) = \text{EUCLIDEAN}(a, k)$. By induction, for all $\ell \in [1, a]$, the sum of any $\ell$ consecutive elements in $(x_1, x_2, \ldots, x_a)$ equals $\lfloor \frac{\ell k}{a} \rfloor$ or $\lceil \frac{\ell k}{a} \rceil$. Let $S$ be a sequence of $m$ consecutive elements in $R$. By construction, for some $1 \le i \le j \le a$, and for some $0 \le s \le x_i - 1$ and $0 \le t \le x_j - 1$, we have

$$S = (\underbrace{\lfloor \tfrac{n}{k} \rfloor, \ldots, \lfloor \tfrac{n}{k} \rfloor}_{s}, \lceil \tfrac{n}{k} \rceil, \underbrace{\lfloor \tfrac{n}{k} \rfloor, \ldots, \lfloor \tfrac{n}{k} \rfloor}_{x_{i+1}-1}, \lceil \tfrac{n}{k} \rceil, \ldots, \underbrace{\lfloor \tfrac{n}{k} \rfloor, \ldots, \lfloor \tfrac{n}{k} \rfloor}_{x_{j-1}-1}, \lceil \tfrac{n}{k} \rceil, \underbrace{\lfloor \tfrac{n}{k} \rfloor, \ldots, \lfloor \tfrac{n}{k} \rfloor}_{t}) \ .$$

It remains to prove that $\lfloor \frac{mn}{k} \rfloor \le \sum S \le \lceil \frac{mn}{k} \rceil$.

We first prove that $\sum S \ge \lfloor \frac{mn}{k} \rfloor$. We can assume the worst case for $\sum S$ to be minimal, which is when $s = x_i - 1$ and $t = x_j - 1$. Thus by induction,

$$m + 1 = \sum_{\alpha=i}^{j} x_\alpha \le \left\lceil \frac{(j-i+1)k}{a} \right\rceil \ .$$

Hence,

$$\frac{am}{k} \le \frac{a}{k} \left\lceil \frac{(j-i+1)k}{a} \right\rceil - \frac{a}{k} \le \frac{a}{k} \left( \frac{(j-i+1)k + a - 1}{a} \right) - \frac{a}{k} = j - i + 1 - \frac{1}{k} \ .$$

Thus $\lfloor \frac{am}{k} \rfloor \le j - i$ and

$$\begin{aligned} \sum S &= m \left\lfloor \frac{n}{k} \right\rfloor + j - i \\ &\ge m \left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{am}{k} \right\rfloor = \left\lfloor m \left\lfloor \frac{n}{k} \right\rfloor + \frac{am}{k} \right\rfloor = \left\lfloor \frac{m}{k} \left( k \left\lfloor \frac{n}{k} \right\rfloor + a \right) \right\rfloor = \left\lfloor \frac{mn}{k} \right\rfloor \ . \end{aligned}$$

Now we prove that $\sum S \leq \lfloor \frac{mn}{k} \rfloor$. We can assume the worst case for $\sum S$ to be maximal, which is when $s = 0$ and $t = 0$. Thus by induction,

$$m - 1 = \sum_{\alpha=i+1}^{j-1} x_\alpha \geq \left\lfloor \frac{(j-i-1)k}{a} \right\rfloor .$$

Hence

$$\frac{am}{k} \geq \frac{a}{k} \left\lfloor \frac{(j-i-1)k}{a} \right\rfloor + \frac{a}{k} \geq \frac{a}{k} \left( \frac{(j-i-1)k - a + 1}{a} \right) + \frac{a}{k} = j - i - 1 + \frac{1}{k} .$$

Thus $\lceil \frac{am}{k} \rceil \geq j - i$ and

$$\begin{aligned} \sum S &= m \left\lfloor \frac{n}{k} \right\rfloor + j - i \\ &\leq m \left\lfloor \frac{n}{k} \right\rfloor + \left\lceil \frac{am}{k} \right\rceil = \left\lceil m \left\lfloor \frac{n}{k} \right\rfloor + \frac{am}{k} \right\rceil = \left\lceil \frac{m}{k} \left( k \left\lfloor \frac{n}{k} \right\rfloor + a \right) \right\rceil = \left\lceil \frac{mn}{k} \right\rceil . \end{aligned}$$

□

### 4.2.3 Uniqueness

In this section we prove that there is a unique rhythm satisfying the conditions in Theorem 4.2.1. The following well-known number-theoretic lemmas will be useful. Two integers $x$ and $y$ are *inverses* modulo $m$ if $xy \equiv 1 \pmod{m}$.

**Lemma 4.2.2** ([162, page 55]). *An integer $x$ has an inverse modulo $m$ if and only if $x$ and $m$ are relatively prime. Moreover, if $x$ has an inverse modulo $m$, then it has an inverse $y \in [1, m-1]$.*

**Lemma 4.2.3.** *If $x$ and $m$ are relatively prime, then $ix \not\equiv jx \pmod{m}$ for all distinct $i, j \in [0, m-1]$.*

*Proof.* Suppose $ix \equiv jx \pmod{m}$ for some $i, j \in [0, m-1]$. By Lemma 4.2.2, $x$ has an inverse modulo $m$. Thus $i \equiv j \pmod{m}$, and $i = j$ because $i, j \in [0, m-1]$. □

**Lemma 4.2.4.** *For all relatively prime integers $n$ and $k$ with $2 \le k \le n$, there is an integer $\ell \in [1, k-1]$ such that:*

(a) $\ell n \equiv 1 \pmod{k}$,

(b) $i\ell \not\equiv j\ell \pmod{k}$ *for all distinct* $i, j \in [0, k-1]$, *and*

(c) $i\lfloor \frac{\ell n}{k} \rfloor \not\equiv j\lfloor \frac{\ell n}{k} \rfloor \pmod{n}$ *for all distinct* $i, j \in [0, k-1]$.

*Proof.* By Lemma 4.2.2 with $x = n$ and $m = k$, $n$ has an inverse $\ell$ modulo $k$. This proves (a). Thus $k$ and $\ell$ are relative prime by Lemma 4.2.2 with $x = \ell$ and $m = k$. Hence (b) follows from Lemma 4.2.3. Let $t = \lfloor \frac{\ell n}{k} \rfloor$. Then $\ell n = kt + 1$. By Lemma 4.2.3 with $m = n$ and $x = t$ (and because $k \le n$), to prove (c) it suffices to show that $t$ and $n$ are relatively prime. Let $g = \gcd(t, n)$. Thus $\ell \frac{n}{g} = k \frac{t}{g} + \frac{1}{g}$. Since $\frac{n}{g}$ and $\frac{t}{g}$ are integers, $\frac{1}{g}$ is an integer and $g = 1$. This proves (c). □

The following theorem is the main result of this section.

**Theorem 4.2.5.** *For all integers $n$ and $k$ with $2 \le k \le n$, there is a unique rhythm with $k$ onsets and timespan $n$ that satisfies property $(\star)$, up to a rotation.*

*Proof.* Let $R = \{r_0, r_1, \ldots, r_{k-1}\}_n$ be a $k$-onset rhythm that satisfies $(\star)$. Recall that the index of an onset is taken modulo $k$, and that the value of an onset is taken modulo $n$. That is, $r_i = x$ means that $r_{i \bmod k} = x \bmod n$.

Let $g = \gcd(k, n)$. We consider three cases for the value of $g$.

**Case 1.** $g = k$: Since $R$ satisfies property ($\star$) for $\ell = 1$, every ordered pair $(r_i, r_{i+1})$ has clockwise distance $\frac{n}{k}$. By a rotation of $R$ we may assume that $r_0 = 0$. Thus $r_i = \frac{in}{k}$ for all $i \in [0, k-1]$. Hence $R$ is uniquely determined in this case.

**Case 2.** $g = 1$ (see Figure 4–6): By Lemma 4.2.4(a), there is an integer $\ell \in [1, k-1]$ such that $\ell n \equiv 1 \pmod{k}$. Thus $\ell n = (k-1)\lfloor\frac{\ell n}{k}\rfloor + \lceil\frac{\ell n}{k}\rceil$. Hence, of the $k$ ordered pairs $(r_i, r_{i+\ell})$ of onsets, $k-1$ have clockwise distance $\lfloor\frac{\ell n}{k}\rfloor$ and one has clockwise distance $\lceil\frac{\ell n}{k}\rceil$. By a rotation of $R$ we may assume that $r_0 = 0$ and $r_{k-\ell} = n - \lceil\frac{\ell n}{k}\rceil$. Thus $r_{i\ell} = i\lfloor\frac{\ell n}{k}\rfloor$ for all $i \in [0, k-1]$; that is, $r_{(i\ell) \bmod k} = (i\lfloor\frac{\ell n}{k}\rfloor) \bmod n$. By Lemma 4.2.4(b) and (c), this defines the $k$ distinct onsets of $R$. Hence $R$ is uniquely determined in this case.

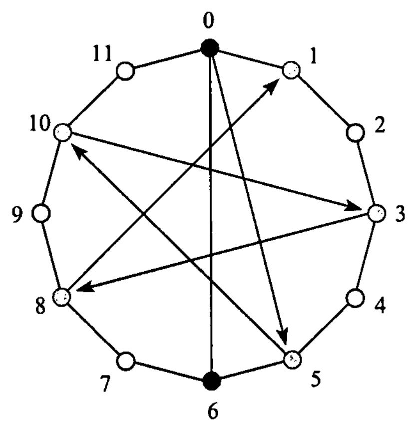



Figure 4–6: Here we illustrate Case 2 with $n = 12$ and $k = 7$. Thus $\ell = 3$ because $3 \times 12 \equiv 1 \pmod{7}$. We have $\lceil\frac{\ell n}{k}\rceil = 6$ and $\lfloor\frac{\ell n}{k}\rfloor = 5$. By a rotation we may assume that $r_0 = 0$ and $r_{k-\ell} = r_4 = 6$ (the blue or darker dots). Then as shown by the arrows, the positions of the other onsets are implied.

**Case 3.** $g \in [2, k-1]$ (see Figure 4–7): Let $k' = \frac{k}{g}$ and let $n' = \frac{n}{g}$. Observe that both $k'$ and $n'$ are integers. Since $R$ satisfies ($\star$) and $\lceil\frac{k'n}{k}\rceil = \lfloor\frac{k'n}{k}\rfloor = n'$, we have

$\overset{\circlearrowright}{d}(r_i, r_{i+k'}) = n'$ for all $i \in [0, k-1]$. Thus

$$r_{ik'+j} = in' + r_j \tag{4.1}$$

for all $i \in [0, g-1]$ and $j \in [0, n'-1]$.

Now $\gcd(n', k') = 1$ by the maximality of $g$. By Lemma 4.2.4(a), there is an integer $\ell' \in [1, k'-1]$ such that $\ell' n' \equiv 1 \pmod{k'}$. Thus $\ell' n' = (k'-1)\lfloor \frac{\ell' n'}{k'} \rfloor + \lceil \frac{\ell' n'}{k'} \rceil$, implying $\ell' n = (k-g)\lfloor \frac{\ell' n'}{k'} \rfloor + g\lceil \frac{\ell' n'}{k'} \rceil$. Hence, of the $k$ ordered pairs $(r_i, r_{i+\ell'})$ of onsets, $k-g$ have clockwise distance $\lfloor \frac{\ell' n'}{k'} \rfloor$ and $g$ have clockwise distance $\lceil \frac{\ell' n'}{k'} \rceil$. By a rotation of $R$ we may assume that $r_0 = 0$ and $r_{\ell'} = \lceil \frac{\ell' n'}{k'} \rceil$. By Equation (4.1) with $j = 0$ and $j = \ell'$, we have

$$r_{ik'} = in' \text{ and } r_{ik'+\ell'} = in' + \left\lceil \frac{\ell' n'}{k'} \right\rceil \tag{4.2}$$

for all $i \in [0, g-1]$. This accounts for the $g$ ordered pairs $(r_i, r_{i+\ell'})$ with clockwise distance $\lceil \frac{\ell' n'}{k'} \rceil$. The other $k-g$ ordered pairs $(r_i, r_{i+\ell'})$ have clockwise distance $\lfloor \frac{\ell' n'}{k'} \rfloor$. Define

$$L_0 = 0 \text{ and } L_j = \left\lceil \frac{\ell' n'}{k'} \right\rceil + (j-1)\left\lfloor \frac{\ell' n'}{k'} \right\rfloor \text{ for all } j \in [1, k'-1] .$$

Thus by Equation (4.2),

$$r_{ik'+j\ell'} = in' + L_j$$

for all $i \in [0, g-1]$ and $j \in [0, k'-1]$; that is, $r_{(ik'+j\ell') \bmod k} = (in' + L_j) \bmod n$.

To conclude that $R$ is uniquely determined, we must show that over the range $i \in [0, g-1]$ and $j \in [0, k'-1]$, the numbers $ik' + j\ell'$ are distinct modulo $k$, and the numbers $in' + L_j$ are distinct modulo $n$.

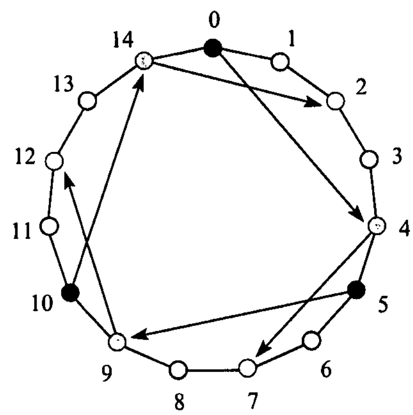



Figure 4–7: Here we illustrate Case 3 with $n = 15$ and $k = 9$. Thus $g = 3$, $n' = 5$ and $k' = 3$. We have $\ell' = 2$ because $2 \times 5 \equiv 1 \pmod 3$. Thus $\lceil \frac{\ell' n'}{k'} \rceil = 4$ and $\lfloor \frac{\ell' n'}{k'} \rfloor = 3$. We have $L_0 = 0$, $L_1 = 4$ and $L_2 = 7$. A rotation fixes the first $g = 3$ onsets (the blue or darker). As shown by the arrows, these onsets imply the positions of the next three onsets (medium or green dots), which in turn imply the positions of the final three onsets (the light or yellow dots).

First we show that the numbers $ik' + j\ell'$ are distinct modulo $k$. Suppose that

$$ik' + j\ell' \equiv pk' + j\ell' \pmod k \tag{4.3}$$

for some $i, p \in [0, g-1]$ and $j, q \in [0, k'-1]$. Since $k = k'g$, we can write $(ik' + j\ell') \bmod k$ as a multiple of $k'$ plus a residue modulo $k'$. In particular,

$$(ik' + j\ell') \bmod k = k'\left((i + \lfloor \tfrac{j\ell'}{k'} \rfloor) \bmod g\right) + (j\ell' \bmod k') \ .$$

Thus Equation (4.3) implies that

$$k'\left((i + \lfloor \tfrac{j\ell'}{k'} \rfloor) \bmod g\right) + (j\ell' \bmod k') = k'\left((p + \lfloor \tfrac{q\ell'}{k'} \rfloor) \bmod g\right) + (q\ell' \bmod k') \ . \tag{4.4}$$

Hence $j\ell' \equiv q\ell' \pmod{k'}$. Thus $j = q$ by Lemma 4.2.4(c). By substituting $j = q$ into Equation (4.4), it follows that $i \equiv p \pmod{g}$. Thus $i = p$ because $i, p \in [0, g-1]$. This proves that the numbers $ik' + j\ell'$ are distinct modulo $k$.

Now we show that the numbers $in' + L_j$ are distinct modulo $n$. The proof is similar to the above proof that the numbers $ik' + j\ell'$ are distinct modulo $k$.

Suppose that

$$in' + L_j \equiv pn' + L_q \pmod{n} \tag{4.5}$$

for some $i, p \in [0, g-1]$ and $j, q \in [0, k'-1]$. Since $n = n' \cdot g$, we can write $(in' + L_j) \bmod n$ as a multiple of $n'$ plus a residue modulo $n'$. In particular,

$$(in' + L_j) \bmod n = n' \left( (i + \left\lfloor \tfrac{L_j}{n'} \right\rfloor) \bmod g \right) + (L_j \bmod n') \ .$$

Thus Equation (4.5) implies that

$$n' \left( (i + \left\lfloor \tfrac{L_j}{n'} \right\rfloor) \bmod g \right) + (L_j \bmod n') = n' \left( (p + \left\lfloor \tfrac{L_q}{n'} \right\rfloor) \bmod g \right) + (L_q \bmod n') \ . \tag{4.6}$$

Hence $L_j \equiv L_q \pmod{n'}$. We claim that $j = q$. If $j = 0$ then $L_j = 0$, implying $L_q = 0$ and $q = 0$. Now assume that $j, q \geq 1$. In this case, $L_j = j\lfloor \frac{\ell' n'}{k'} \rfloor + 1$ and $L_q = q\lfloor \frac{\ell' n'}{k'} \rfloor + 1$. Thus

$$j \left\lfloor \tfrac{\ell' n'}{k'} \right\rfloor \equiv q \left\lfloor \tfrac{\ell' n'}{k'} \right\rfloor \pmod{k'} \ .$$

Hence $j = q$ by Lemma 4.2.4(c). By substituting $j = q$ into Equation (4.6), it follows that $i \equiv p \pmod{g}$. Thus $i = p$ because $i, p \in [0, g-1]$. This proves that the numbers $in' + L_j$ are distinct modulo $n$.

Therefore $R$ is uniquely determined. □

We have shown that each of the three algorithms generates a rhythm with property ($\star$), and that there is a unique rhythm with property ($\star$). Thus all of the algorithms produce the same rhythm, up to rotation. It remains to prove that this rhythm has maximum evenness.

### 4.2.4 Rhythms with Maximum Evenness

We start with a technical lemma. Let $v, w$ be points at geodesic distance $d$ on a circle $C$. Clearly, $\overline{d}(v, w)$ is a function of $d$, independent of $v$ and $w$. Let $f(C, d) = \overline{d}(v, w)$.

**Lemma 4.2.6.** *For all geodesic lengths $x \leq d$ on a circle $C$, we have $f(C, x) + f(C, d - x) \leq 2 \cdot f(C, \frac{d}{2})$, with equality only if $d = 2x$.*

*Proof.* We may assume that $C$ is a unit circle. Consider the isosceles triangle formed by the center of $C$ and a geodesic of length $d$ ($\leq \pi$). We have $\frac{1}{2}f(C, d) = \sin \frac{d}{2}$. Thus $f(C, d) = 2 \sin \frac{d}{2}$. Thus our claim is equivalent to $\sin x + \sin(d - x) \leq 2 \sin \frac{d}{2}$ for all $x \leq d$ ($\leq \pi/2$). In the range $0 \leq x \leq d$, $\sin x$ is increasing, and $\sin(d - x)$ is decreasing at the opposite rate. Thus $\sin x + \sin(d - x)$ is maximized when $x = d - x$. That is, when $d = 2x$. The result follows. □

For a rhythm $R = \{r_0, r_1, \ldots, r_{k-1}\}_n$, for each $\ell \in [1, k]$, let $S(R, \ell)$ be the sum of chordal distances taken over all ordered pairs $(r_i, r_{i+\ell})$ in $R$. That is, let $S(R, \ell) = \sum_{i=0}^{k-1} \overline{d}(r_i, r_{i+\ell})$. Property (A) says that $R$ maximizes $\sum_{\ell=1}^{k} S(R, \ell)$. Before we characterize rhythms that maximize the sum of $S(R, \ell)$, we first concentrate on rhythms that maximize $S(R, \ell)$ for each particular value of $\ell$. Let $D(R, \ell)$ be the multiset of clockwise distances $\{\overset{\circ}{d}(r_i, r_{i+\ell}) : i \in [0, k - 1]\}$. Then $S(R, \ell)$ is determined by $D(R,\ell)$. In particular, $S(R,\ell) = \sum\{f(C_n,d) : d \in D(R,\ell)\}$ (where $\{f(C_n,d) : d \in D(R,\ell)\}$ is a multiset).

**Lemma 4.2.7.** *Let $1 \leq \ell \leq k \leq n$ be integers. A $k$-onset rhythm $R = \{r_0, r_1, \ldots, r_{k-1}\}_n$ maximizes $S(R,\ell)$ if and only if $|\mathring{d}(r_i, r_{i+\ell}) - \mathring{d}(r_j, r_{j+\ell})| \leq 1$ for all $i,j \in [0,k-1]$.*

*Proof.* Suppose that $R = \{r_0, r_1, \ldots, r_{k-1}\}_n$ maximizes $S(R,\ell)$. Let $d_i = \mathring{d}(r_i, r_{i+\ell})$ for all $i \in [0,k-1]$. Suppose on the contrary that $d_p \geq d_q + 2$ for some $p, q \in [0,k-1]$. We may assume that $q < p$, $d_p = d_q + 2$, and $d_i = d_q + 1$ for all $i \in [q+1, p-1]$. Define $r'_i = r_i + 1$ for all $i \in [q+1,p]$, and define $r'_i = r_i$ for all other $i$. Let $R'$ be the rhythm $\{r'_0, r'_1, \ldots, r'_{k-1}\}_n$. Thus $D(R,\ell) \setminus D(R',\ell) = \{d_p, d_q\}$ and $D(R',\ell) \setminus D(R,\ell) = \{d_p - 1, d_q + 1\}$. Now $d_p - 1 = d_q + 1 = \frac{1}{2}(d_p + d_q)$. By Lemma 4.2.6, $f(C_n, d_p) + f(C_n, d_q) < 2 \cdot f(C_n, \frac{1}{2}(d_p + d_q))$. Thus $S(R,\ell) < S(R',\ell)$, which contradicts the maximality of $S(R,\ell)$.

For the converse, let $R$ be a rhythm such that $|\mathring{d}(r_i, r_{i+\ell}) - \mathring{d}(r_j, r_{j+\ell})| \leq 1$ for all $i,j \in [0,k-1]$. Suppose on the contrary that $R$ does not maximize $S(R,\ell)$. Thus some rhythm $T = (t_0, t_1, \ldots, t_{k-1})$ maximizes $S(T,\ell)$ and $T \neq R$. Hence $D(T,\ell) \neq D(R,\ell)$. Since $\sum D(R,\ell) = \sum D(T,\ell)$ $(= \ell n)$, we have $\mathring{d}(t_i, t_{i+\ell}) - \mathring{d}(t_j, t_{j+\ell}) \geq 2$ for some $i,j \in [0,k-1]$. As we have already proved, this implies that $T$ does not maximize $S(T,\ell)$. This contradiction proves that $R$ maximizes $S(R,\ell)$. □

Since $\sum_{i=0}^{k-1} \mathring{d}(r_i, r_{i+\ell}) = \ell n$ for any rhythm with $k$ onsets and timespan $n$, Lemma 4.2.7 can be restated as follows.

**Corollary 4.2.8.** *Let $1 \leq \ell \leq k \leq n$ be integers. A $k$-onset rhythm $R = \{r_0, r_1, \ldots, r_{k-1}\}_n$ maximizes $S(R,\ell)$ if and only if $\mathring{d}(r_i, r_{i+\ell}) \in \{\lceil \frac{\ell n}{k} \rceil, \lfloor \frac{\ell n}{k} \rfloor\}$ for all $i \in [0,k-1]$.* □

*Proof* ($\star$) $\Rightarrow$ (A). If ($\star$) holds for some rhythm $R$, then by Corollary 4.2.8, $R$ maximizes $S(R,\ell)$ for *every* $\ell$. Thus $R$ maximizes $\sum_\ell S(R,\ell)$. □

*Proof* (A) $\Rightarrow$ ($\star$). By Theorem 4.2.5, there is a unique rhythm $R$ that satisfies property ($\star$). Let $R$ denote the unique rhythm that satisfies property ($\star$). Suppose on the contrary that there is a rhythm $T = (t_0, t_1, \ldots, t_{k-1})$ with property (A) but $R \neq T$. Thus there exists an ordered pair $(t_i, t_{i+\ell})$ in $T$ with clockwise distance $\overset{\circlearrowright}{d}(t_i, t_{i+\ell}) \notin \{\lfloor \frac{\ell n}{k} \rfloor, \lceil \frac{\ell n}{k} \rceil\}$. By Corollary 4.2.8, $S(T,\ell) < S(R,\ell)$. Since $T$ has property (A), $\sum_{\ell=1}^{k} S(T,\ell) \geq \sum_{\ell=1}^{k} S(R,\ell)$. Thus for some $\ell'$ we have $S(T,\ell') > S(R,\ell')$. But this is a contradiction, because $S(R,\ell') \geq S(T,\ell')$ by Corollary 4.2.8. □

This completes the proof of Theorem 4.2.1. We now show that Theorem 4.2.1 can be generalized for other metrics that satisfy Lemma 4.6. To formalize this idea we introduce the following definition. A function $g : [0,\pi] \rightarrow \mathbb{R}^+ \cup \{0\}$ is *halving* if for all geodesic lengths $x \leq d \leq \pi$ on the unit circle, $g(x) + g(d-x) \leq 2 \cdot g(\frac{d}{2})$, with equality only if $d = 2x$. For example, chord length is halving, but geodesic distance is not (because we have equality for all $x$). Observe that the proof of Lemma 4.2.7 and Corollary 4.2.8 depend on this property alone. Thus we have the following generalization of Theorem 4.2.1.

**Theorem 4.2.9.** *Let $n \geq k \geq 2$ be integers. Let $g$ be a halving function. The following are equivalent for a rhythm $R = (r_0, r_1, \ldots, r_{k-1})$ with $n$ pulses and $k$ onsets:*

(A) *$R$ maximises $\sum_{i=0}^{k-1} \sum_{j=i+1}^{k-1} g(\overset{\circlearrowright}{d}(r_i, r_j))$,*

(B) *$R$ is determined by the* Clough-Douthett$(k,n)$ *algorithm,*

(C) *$R$ is determined by the* Snap$(k,n)$ *algorithm,*

(D) *$R$ is determined by the* EUCLIDEAN$(k, n)$ *algorithm,*

($\star$) *for all $\ell \in [1, k]$ and $i \in [0, k-1]$, the ordered pair $(r_i, r_{i+\ell})$ has clockwise distance $d(r_i, r_{i+\ell}) \in \{\lfloor \frac{\ell n}{k} \rfloor, \lceil \frac{\ell n}{k} \rceil\}$.*

*Moreover, up to a rotation, there is a unique rhythm that satisfies these conditions.*

## 4.3 Connection Between Deep and Even Rhythms

A connection between maximally even scales and Winograd-deep scales is shown by Clough et al. [44]. They define a *diatonic scale* to be a maximally even scale with $k = (n+2)/2$ and $n$ a multiple of 4. They show that diatonic scales are Winograd-deep. We now prove a similar result for Erdős-deep rhythms.

**Lemma 4.3.1.** *A rhythm $R$ of maximum evenness satisfying $k \leq \lfloor n/2 \rfloor + 1$ is Erdős-deep if and only if $k$ and $n$ are relatively prime.*

*Proof.* Recall that by property ($\star$) one of the unique characterizations of an even rhythm of maximum evenness can be stated as follows. For all $1 \leq \ell \leq k$, and for every ordered pair $(r_i, r_{i+\ell})$ of onsets in $R$, the clockwise distance $d(r_i, r_{i+\ell}) \in \{\lfloor \frac{\ell n}{k} \rfloor, \lceil \frac{\ell n}{k} \rceil\}$.

For the case where $k$ and $n$ are relatively prime, by Lemma 4.2.2, there exists a value $\ell < k$ such that $\ell n \equiv 1 \pmod{k}$. Thus we can write $\ell n = k\lfloor \ell n/k \rfloor + 1$. Let $m = \lfloor \ell n/k \rfloor$. Now consider the set $\{im \bmod n : i = 0, 1, \ldots, k-1\}_n$. By Lemma 4.2.4(c), we get $k$ distinct values, so $R$ can be realized as $D_{k,n,m} = \{im \bmod n : i = 0, 1, \ldots, k-1\}_n$. Thus, by Lemma 3.2.2, $R$ is Erdős-deep.

Observe that $F = \{0, 1, 2, 4\}_6$ does not maximize evenness because $\overset{\circlearrowright}{d}(0, 2) = 2$ and $\overset{\circlearrowright}{d}(2, 0) = 4$ yet $\ell = 2$. Hence, any rhythm that maximizes evenness and that is deep must also be generated.

Now consider the case where $n$ and $k$ are not relatively prime. We show that the assumption that $R$ is deep leads to a contradiction. Thus, assuming that $R$ is deep implies that there is a value $m$ such that $R$ can be realized as $D_{k,n,m} = \{im \bmod n : i = 0, 1, \ldots, k-1\}_n$. This in turn implies that there exists an integer $\ell$ such that $km = \ell n + 1$, that is, $\ell n \equiv 1 \pmod{k}$. However, for this to happen, $n$ and $k$ must be relatively prime, a contradiction.

Thus we have shown that $R$ is Erdős-deep if and only if $k$ and $n$ are relatively prime. $\square$

## 4.4 Further Remarks and Open Problems

As discussed in the beginning of this chapter, evenness is an important and desirable property of rhythms and scales; it is thus worth investigating this property further. In this section, we propose two measures of evenness that are different than the ones described in this chapter and that might lead to interesting theoretical and/or musically meaningful results.

Mathematically, there are many ways to quantify the "evenness" of a rhythm. However when it comes to defining useful measures of evenness, there are two properties that are highly desirable. The first property is sensitivity: the measure has to be sensitive enough to discriminate between rhythms that are very different. The second property is computational efficiency: how fast can we compute it? Therefore, we would like to define measures based on how well they discriminate between rhythms, how fast they can be computed, and also how useful they are in practice. We now propose two measures that are ripe for further exploration, both musically and mathematically.

The first measure is called the *degree of irregularity* of a rhythm. The notion was first introduced by Hitt and Zhang [97] as a way of quantifying how far away a given cyclic polygon is from the regular polygon with the same number of edges. Let $P$ be a cyclic polygon having $\theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ as the ordered set of its central angles (angles between the center of the circle and the endpoints of each of the $n$ edges), with $0 < \theta_i < \pi$ for all $i = 1, 2, \ldots, n$. Hitt and Zhang [97] define the degree of irregularity of $P$, denoted by $I(P)$, to be:

$$I(P) = n \sin(\pi/n) - \sum_{i=1}^{n} \sin(\theta_i/2).$$

A rhythm $R$ represented as a clock diagram is essentially a cyclic polygon, and a rhythm that divides the timeline evenly is the regular cyclic polygon. The degree of irregularity of $R$ tells us how "uneven" $R$ is with respect to the perfectly even rhythm, which in this case is a regular cyclic polygon with the same number of onsets and pulses. Note that $I(R)$ is a positive value invariant under scaling and is equal to 0 only when $R$ is perfectly even. The major drawback of this measure is that $I(R)$ is independent of the permutation of the central angles (or edges) of the cyclic polygon it defines. Thus, for a rhythm with $k$ onsets and $n$ pulses, there may be up to $k!$ rhythms with the same degree of irregularity.

One way of quantifying the evenness of a rhythm is by measuring how far it is from the maximally even rhythm (maximizing the sum of pairwise interonset chordal distances) with the same number of onsets and pulses. Toussaint [170, 174] proposes a few distance measures for comparing and classifying rhythms; some examples of the proposed distance measures are hamming distance, edit distance, swap distance, earth mover's distance, ... etc. Using these definitions of distance, it is thus possible to compute the evenness of a rhythm based on how far it is from the perfectly even rhythm. It is not clear whether these different distance measures produce distinct total orderings of rhythms with the same number of onsets and pulses. We now define a measure of evenness that is different than the ones discussed above. This measure was first described by Bjorklund in [21] as the measure of *ugliness* of a binary string. Because we can represent musical rhythms as binary strings, the ugliness property can be transposed to rhythms. Let $\delta_j(i)$ be the forward distance between the $i$th and $(i+j)$th 1-bits of a binary string (onsets of a rhythm). The ugliness of a binary string is equal to

$$\frac{1}{k}\sum_{j=1}^{k-1}\sum_{i=0}^{k-1}\left(\delta_j(i)-\frac{jn}{k}\right)^2, \tag{4.7}$$

The ugliness of a rhythm $R$ measures how far away $R$ is from the perfectly even rhythm with the same number of pulses and onsets. In this case, a perfectly even rhythm is the one that maximizes the sum of pairwise Euclidean distances between onsets. This measure of evenness is invariant under rotation, but not under the permutation of the bits; the ugliness of a perfectly even pattern (such as 10101010) is zero. It should be noted here that the total ordering of all possible rhythms with $n$ pulses and $k$ onsets under the ugliness measure is different than the ordering under the swap distance measure [170, 174]. For example, the ugliness of rhythm [$\times\times\cdot\times\cdot\times\times\times\cdot\cdot$] is 0.51 while the swap distance from the perfectly even rhythm [$\times\times\cdot\times\cdot\times\times\cdot\times\cdot$] is 1; whereas the ugliness of [$\times\times\cdot\times\times\cdot\times\times\cdot\cdot$] is 0.44 while the swap distance from the perfectly even rhythm is 2.

Bjorklund naively computes the ugliness of a binary pattern in quadratic-time, which is not very efficient in terms of computability of the measure. However, it is possible to reduce the running time for computing the ugliness of a rhythm to linear-time by simplifying equation (4.7). See Appendix D for details.

# Chapter 5

# Structural Properties of Euclidean Rhythms

Maximally even rhythms are an important class of rhythms that seem to dominate popular Western music, as well as rhythms throughout the world. In this chapter, we study some structural properties of this family of rhythms. We restrict our attention to Euclidean rhythms defined in Section 4.1.3 in the previous chapter: these are the maximally even rhythms generated by Bjorklund's algorithm (Section 4.1.2). In particular, we show that Euclidean rhythms are composed of a pattern $P$ repeated a certain number of times, plus a (shorter) pattern $T$ repeated at most once. We also show that each of $P$ and $T$ are themselves Euclidean rhythms, and that pattern $P$ is *minimal*; that is, for a given Euclidean rhythm $R$ there is no way of splitting $R$ into repeated patterns $P'$ (possibly followed by a shorter pattern $T'$) such that the number of bits in $P'$ is less than the number of bits in $P$.

In this chapter we focus on the internal structure of Euclidean rhythms. We first take a closer look at Bjorklund's and Euclid's algorithms; we then prove two technical lemmas in Section 5.3 and show how the first relates to Bezout's theorem [53]. In the same section, we show the main contribution of this chapter: that patterns $P$ and $T$ are Euclidean and that $P$ is minimal.

## 5.1 Bjorklund and Euclid Revisited

Consider Bjorklund's algorithm described in Section 4.1.2. Observe that this algorithm consists of two steps: an initialization step, performed once at the beginning; and, a subtraction step, performed repeatedly until the stopping condition is satisfied. At all times Bjorklund's algorithm maintains two lists $A$ and $B$ of strings of bits, with $a$ and $b$ representing the number of strings in each list respectively.

1. **Initialization step.** In this step the algorithm builds the string $\{1, \overset{k}{\ldots}, 1, 0, \overset{n-k}{\ldots}, 0\}$, and sets $A$ as the first $a = \min\{k, n-k\}$ bits of that string, and $B$ as the remaining $b = \max\{k, n-k\}$ bits. Next the algorithm removes $\lfloor b/a \rfloor$ strings of $a$ consecutive bits from $B$, starting with the rightmost bit, and places them under the $a$-bit strings in $A$ one below the other (see Figure 5–1, steps (1) and (2)). Lists $A$ and $B$ are then redefined: $A$ is now composed of $a$ strings (the $a$ columns in $A$), each having $\lfloor b/a \rfloor + 1$ bits, and $B$ is composed of $b \mod a$ strings of 0-bits. Finally, the algorithm sets $b = b \mod a$.
2. **Subtraction step.** At a subtraction step, the algorithm removes $\lfloor a/b \rfloor$ strings of $b$ consecutive bits (or columns) from $B$ and $A$, starting with the rightmost bit of $B$ and continuing with the rightmost bit of $A$, and places them at the bottom-left of the strings in $A$ one below the other. Lists $A$ and $B$ are then redefined as follows: $A$ is composed of the first $b$ strings (starting with the leftmost bit), while $B$ is composed of the remaining $a \mod b$ strings. Finally, the algorithm sets $b = a \mod b$ and $a = b$ (before $b$ was redefined). see Figure 5–1, step (3).

The algorithm stops when, after the end of a subtraction step, list $B$ is empty or consists of one string. The output is produced by returning the strings of $A$ from left to right followed by the strings of $B$, if any; see Figure 5–1, step (4).

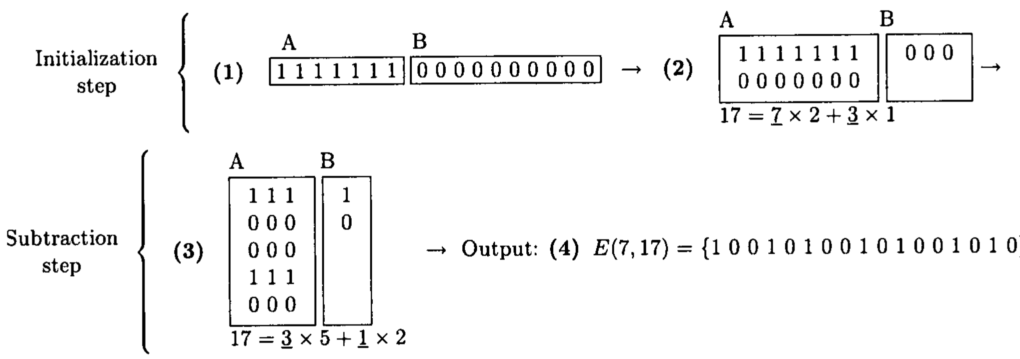



Figure 5–1: Bjorklund's algorithm for generating the Euclidean rhythm $E(7,17)$. In the initialization step, $a$ is set to 7, and $b$ is set to 10. In Step (2), the algorithm removes $\lfloor 10/7 \rfloor = 1$ string of 7 bits from $B$ and places it under the string in $A$. List $A$ is now composed of seven 2-bit strings [10], while list $B$ is composed of three 1-bit strings [0] (think of each string as one column in box $A$ or $B$). The new values of $a$ and $b$ are 7 and 3 respectively (the underlined digits in Step (2)). In the subtraction step (Step (3)), the algorithm takes $\lfloor 7/3 \rfloor = 2$ strings of 3 bits (columns) each from $B$ and $A$ (starting with the rightmost column in $B$) and places them at the bottom-left of $A$, one below the other. The algorithm now stops because $B$ is formed of the single string [10].

Let us establish now the relationship between the algorithms of Euclid and Bjorklund. Both perform the same operations, the former on numbers, while the latter on strings of bits. At some subtraction step, Bjorklund's algorithm first performs the division $\lfloor a/b \rfloor$ by moving bits from $B$ to $A$, after which it sets $b$ as the number of strings in $A$ and $a \bmod b$ as the number of strings in $B$. This is exactly what Euclid's algorithm does at a subtraction step: it sets $b$, $a \bmod b$ as the new pair (for $a > b$). The initialization step of Bjorklund's algorithm is equivalent to the first step of Euclid's algorithm when $k \leq n-k$, and to the first two steps of Euclid's algorithm when $k > n-k$. This is because in the latter case, the initialization step sets $a = n-k$ and $b = k$; after moving the bits needed to build lists $A$ and $B$, the number of strings turns out to be $n-k$ and $k \bmod (n-k)$, exactly what Euclid's algoirhtm produces after the second step.

For example, consider computing $\gcd(27,10)$ using Euclid's algorithm. The sequence generated during its execution is $\{\gcd(10,7), \gcd(7,3), \gcd(3,1)\}$. When applying Bjorklund's algorithm the sequence formed by the number of strings at each step is also $\{\gcd(10,7), \gcd(7,3), \gcd(3,1)\}$. On the other hand, if we compute $\gcd(27,17)$, the sequence associated with Euclid's algorithm is $\{\gcd(17,10), \gcd(10,7), \gcd(7,3), \gcd(3,1)\}$, whereas the sequence associated with Bjorklund's algorithm is $\{\gcd(10,7), \gcd(7,3), \gcd(3,1)\}$. This example also illustrates the fact that the sequence provided by Bjorklund's algorithm is the same for both $E(k,n)$ and $E(n-k,n)$.

We can keep track of the dimensions of the lists $A$ and $B$ at each iteration of Bjorklund's algorithm using Euclid's algorithm for computing the greatest common divisor of $n$ and $k$. This can be accomplished by rewriting the quotient in terms of the remainder (the underlined numbers in the example below) at every step of Euclid's algorithm and grouping the terms apropriately. For the example in Figure 5–1 we have:

$$\begin{aligned}17 &= \underline{7}\times 1+\underline{10}\times 1\\ &= \underline{7}\times 1+(\underline{7}\times 1+\underline{3}\times 1)\times 1\\ &= \underline{7}\times 2+\underline{3}\times 1\\ &= (\underline{3}\times 2+\underline{1}\times 1)\times 2+\underline{3}\times 1\\ &= \underline{3}\times 5+\underline{1}\times 2\end{aligned}$$

The algorithm stops at this point because we have a remainder of $\underline{1}$. The last expression in the above example tells us that $E(7,17)$ has $\underline{3}$ strings of length 5 each, and $\underline{1}$ string of length 2.

Once Bjorklund's algorithm is completed, we obtain two lists $A$ and $B$ that form the Euclidean rhythm $E(k,n)$. It follows that $E(k,n)$ is composed of multiple copies of a pattern $P$ given by the strings of $A$, followed (possibly) by a single pattern $T$ given by the only string in list $B$. We call $P$ the *main pattern* of $E(k,n)$ and $T$ the *tail pattern*. We now introduce some notation that will be used throughout the remainder of the chapter.

## 5.2 Definitions

Let $E_{CD}(k,n)$ denote the Euclidean rhythm with $k$ onsets and $n$ pulses generated by the Clough-Douthett algorithm (Section 4.2.1):

$$E_{CD}(k,n)=\left\{\left\lfloor\frac{in}{k}\right\rfloor : i=0,1,\ldots,k-1]\right\}. \tag{5.1}$$

For any two rhythms $R_1$ and $R_2$ let $R_1\oplus R_2$ denote the rhythm composed of the pulses of $R_1$ followed by the pulses $R_2$; this is known as the *concatenation* of $R_1$ and $R_2$ and is discussed further in Section 6.3 of the following chapter. The length of the main pattern $P$ will be denoted by $\ell_p$, the length of the tail pattern $T$ by $\ell_t$, and the number of times $P$ is repeated in $E(k,n)$ by $p$. For any $k$ and $n$, the following equality holds:

$$n = \ell_p \times p + \ell_t \tag{5.2}$$

Analogously, call $k_p$ the number of onsets of the main pattern $P$ and $k_t$ the number of onsets of the tail pattern $T$. When $\gcd(k,n) = 1$, $k$ can be written as:

$$k = k_p \times p + k_t \tag{5.3}$$

Additionally, if $\gcd(k,n) = d > 1$, it follows that $p = d$ and $\ell_t = 0$. Hence, $n = \ell_p \times d$, and $k = k_p \times p$.

## 5.3 The Subpatterns of $E(k,n)$

One question to ask about Euclidean rhythms is whether patterns $P$ and $T$ are themselves Euclidean. In this section we show that this indeed is the case. We first prove two technical lemmas.

**Lemma 5.3.1.** *Let $E(k,n)$ be a Euclidean rhythm where $1 \leq k < n$ and let $g = \gcd(k,n)$. The following equalities hold:*

*(a) If $g > 1$, then $nk_p - k\ell_p = 0$.*

*(b) If $g = 1$, then $nk_p - k\ell_p = \pm 1$ and $\ell_t k_p - k_t \ell_p = \pm 1$.*

*Proof.* If $g > 1$, then $n = g\ell_p$ and $k = gk_p$, and a simple computation proves this case:

$$g = \frac{n}{\ell_p} = \frac{k}{k_p} \Longrightarrow nk_p - k\ell_p = 0.$$

For the case $g = 1$, we prove the result by induction on $k$. If $k = 1$, then $E(1, n)$ is the rhythm $\{10 \overset{n-1}{\ldots} 0\}$ with $P = \{10 \overset{n-2}{\ldots} 0\}$ and $T = \{0\}$. For this rhythm $l_p = n - 1, k_p = 1$ and the required equality holds since $n \cdot 1 - 1 \cdot (n-1) = 1$.

For the inductive step, assume the statement is true for all values less than $k$. Consider first the case $n - k < k$. When the initialization step of Bjorklund's algorithm is executed, it produces a list $A$ of $n - k$ strings of $\lfloor \frac{n}{n-k} \rfloor$ bits each, and a list $B$ of $r = n \bmod (n - k)$ strings of one bit. In other words, it performs the division:

$$n = (n-k)\left\lfloor \frac{n}{n-k} \right\rfloor + r \tag{5.4}$$

At this point we replace each string in list $A$ by the symbol $\Theta$. This replacement transforms lists $A$ and $B$ into the string $\{\Theta \overset{n-k}{\ldots} \Theta 1 \overset{r}{\ldots} 1\}$. If we apply Bjorklund's algorithm to this string, we obtain a Euclidean rhythm $E^*(n-k, n-k+r)$ (here the symbols $\Theta$ play the role of onsets and the 1's those of rests). When we perform the inverse replacement on $E^*(n-k, n-k+r)$, that is, when $\Theta$ is replaced by the original string in $A$, we get the Euclidean rhythm $E(k, n)$. Figure 5–2 shows the entire process.

By equation 5.4, it follows that $r < n - k$; this implies that $n - k + r - (n-k) < n - k$. Therefore, we can apply the induction hypothesis to $E^*(n-k, n-k+r)$. Thus, we have:

$$(n-k+r)k_p^{'} - (n-k)l_p^{'} = \pm 1,$$

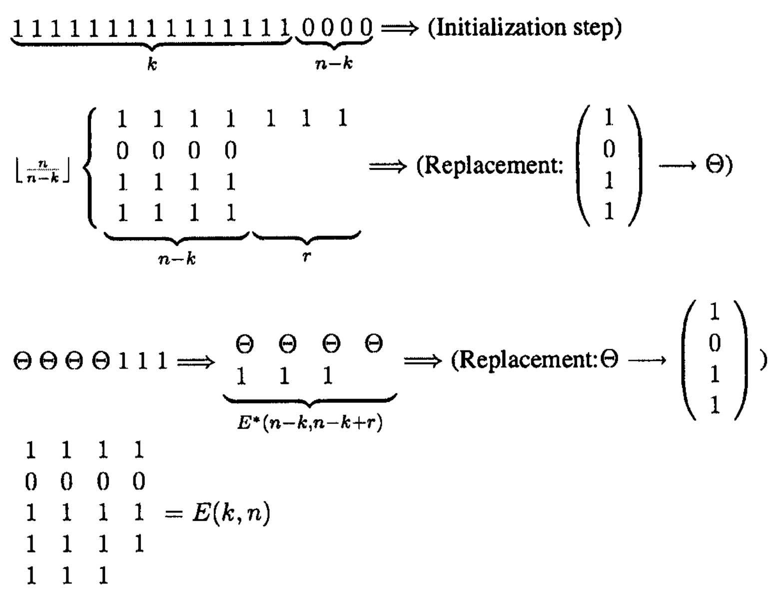



Figure 5–2: Proof of Lemma 5.3.1, case $k > n - k$.

where $k_p'$ is the number of onsets (that is, Θ's) in the main pattern of $E^*(n-k, n-k+r)$, and $l_p'$ is the number of pulses. The following equations show the relationship between $k_p', l_p'$ and $k_p, l_p$.

- $k_p' = l_p - k_p$, since the number of Θ's in the main pattern of $E^*(n-k, n-k+r)$ is equal to the number of zeroes in $E(k,n)$.
- $l_p = k_p' \lfloor \frac{n}{n-k} \rfloor + l_p' - k_p'$, where the first term of the right-hand side accounts for the expansion of Θ, while the last two terms account for the number of ones in $E^*(n-k, n-k+r)$.

Applying the induction hypothesis to $E^*(n-k, n-k+r)$, together with equation (5.4) we obtain:

$$\pm 1 \quad = \quad (n-k+r)k_p' - (n-k)l_p'$$

$$
\begin{aligned}
&= rk_p' - (n-k)(l_p' - k_p') \\
&= rk_p' - (n-k)\left(l_p - k_p'\left\lfloor\frac{n}{n-k}\right\rfloor\right) \\
&= rk_p' - (n-k)l_p + k_p'(n-r) \\
&= nk_p' - (n-k)l_p \\
&= kl_p - n(l_p - k_p') = kl_p - nk_p.
\end{aligned}
$$

We now turn to the case when $n - k > k$. The proof is similar to the previous case. As above, the initialization step of Bjorklund's algorithm is first applied to the input string $\{1 \overset{k}{\ldots} 10 \overset{n-k}{\ldots} 0\}$. Each string in list $A$ is now replaced by the symbol $\mathcal{I}$. This yields the string $\{\mathcal{I} \overset{k}{\ldots} \mathcal{I}0 \overset{r}{\ldots} 0\}$, where $r = n \bmod k$. Next we execute Bjorklund's algorithm on string $\{\mathcal{I} \overset{k}{\ldots} \mathcal{I}0 \overset{r}{\ldots} 0\}$, which produces a Euclidean rhythm $E^*(k, k+r)$ composed of $\mathcal{I}$'s and 0's. Figure 5–3 illustrates these transformations. Note that we cannot apply induction here as the number of onsets in $E^*(k, k+r)$ is at least $k$. However, since $k + r - k = r < k$ we can apply the result of the first case to $E^*(k, k+r)$ and write the following:

$$(k+r)k_p' - kl_p' = \pm 1,$$

where $k_p'$ is the number of onsets in the main pattern of $E^*(k, k+r)$, and $l_p'$ the number of pulses.

Again, we need to relate the main patterns of $E^*(k, k+r)$ and $E(k, n)$ in order to derive the final formula.

- $k_p' = k_p$, since the number of $\mathcal{I}$'s in $E^*(k, k+r)$ is equal to the number of ones in $E(k, n)$.

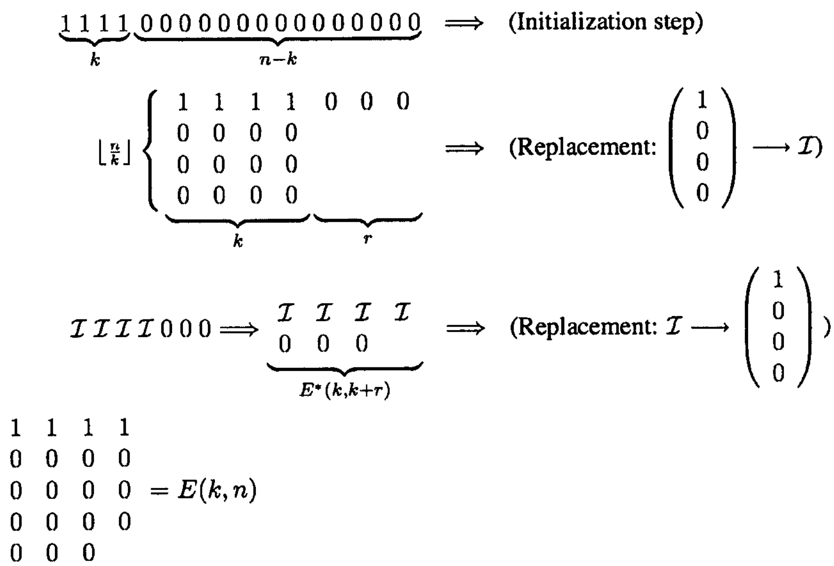



Figure 5–3: Proof of Lemma 5.3.1, case $k < n - k$.

- $l_p = k_p^{'} \lfloor \frac{n}{k} \rfloor + l_p^{'} - k_p^{'}$, where the first term on the right-hand side accounts for the expansion of $\mathcal{I}$, while the last two terms account for the number of zeroes in $E^*(k, k+r)$.

We can now carry out a similar manipulation as above to prove the result:

$$\begin{aligned}
\pm 1 &= (k+r)k_p^{'} - kl_p^{'} \\
&= rk_p^{'} - k(l_p^{'} - k_p^{'}) \\
&= rk_p^{'} - k\left(l_p - k_p^{'} \left\lfloor \frac{n}{k} \right\rfloor\right) \\
&= rk_p^{'} - kl_p + k_p^{'}(n - r) \\
&= nk_p^{'} - kl_p \\
&= nk_p - kl_p.
\end{aligned}$$

Finally, we prove that the equality $\ell_t k_p - k_t \ell_p = \pm 1$ holds. Using equations (5.2) and (5.3) together with the above result we obtain:

$$\begin{aligned} \pm 1 &= nk_p - k\ell_p \\ &= (p\ell_p + \ell_t)k_p - (pk_p + k_t)\ell_p \\ &= p\ell_p k_p + \ell_t k_p - pk_p\ell_p - k_t\ell_p \\ &= \ell_t k_p - k_t \ell_p \end{aligned}$$

This completes the proof of the lemma. □

We now show a connection between Lemma 5.3.1 above and what is called Bezout's theorem [53].

**Theorem 5.3.2** (Bezout's Theorem). *Given two integers $a$ and $b$, there exists two integers $x$ and $y$ such that:*

$$ax + by = \gcd(a, b).$$

When $k$ and $n$ are relatively prime, Lemma 5.3.1 states that $k_p$ and $\ell_p$ are the absolute value of their Bezout coefficients. It also states that the absolute value of the Bezout coefficients of $k_p$ and $\ell_p$ are $\ell_t$ and $k_t$ respectively.

Note that when $n = 1 \bmod k$ the tail pattern of $E(k, n)$ is just $\{0\}$ (by convention, we will consider that $\{0\}$ is the Euclidean rhythm $E(0, 1)$). This is due to the fact that in this particular case Bjorklund's algorithm performs only the initialization step. Otherwise, it follows from Bjorlklund's algorithm that the tail pattern consists of the first $\ell_t$ pulses of $P$.

**Observation 2.** Given a fixed integer $j \geq 0$, the rhythm $\left\{ \left\lfloor \frac{(i+j)n}{k} \right\rfloor \bmod n : i = 0, \ldots, k-1 \right\}$ is $E_{CD}(k,n)$ starting from the onset at position $j$. The rhythm $\{j + \lfloor \frac{in}{k} \rfloor \bmod n : i = 0, \ldots, k-1\}$ is a rotation of $E_{CD}(k,n)$ to the right by $j$ positions.

For example, take rhythm $E_{CD}(7,24) = \{\lfloor \frac{24i}{7} \rfloor \bmod 24 : i = 0, \ldots, 6\} = \{0,3,6,10,13,17,20\}$. The rhythm $\left\{ \left\lfloor \frac{(i+3)24}{7} \right\rfloor \bmod 24 : i = 0, \ldots, 6 \right\}$ is $\{10,13,17,20,0,3,6\}$, which is just a reordering of $E_{CD}(7,24)$ with exactly the same onsets. On the other hand, the formula $\{3 + \lfloor \frac{24i}{7} \rfloor \bmod 24 : i = 0, \ldots, 6\} = \{3,6,9,13,16,20,23\}$ produces a rhythm with onsets at different positions from those of $E_{CD}(7,24)$. This rhythm is a rotation of $E_{CD}(7,24)$ by 3 positions to the right.

A rhythm of the form $\left\{ -\lfloor \frac{jn}{k} \rfloor + \left\lfloor \frac{(i+j)n}{k} \right\rfloor \bmod n : i = 0, \ldots, k-1 \right\}$ is a rotation of $E_{CD}(k,n)$ – it is $E_{CD}(k,n)$ starting from position $j$. For example, the rhythm $R = \left\{ -\lfloor \frac{3 \cdot 24}{7} \rfloor + \left\lfloor \frac{(i+3)24}{7} \right\rfloor \bmod 24 : i = 0, \ldots, 6 \right\}$ is $\{0,3,7,10,14,17,20\}$. By comparing the clockwise distance sequences of $E_{CD}(7,24) = (3\ 3\ 4\ 3\ 4\ 3\ 4)$ and $R = (3\ 4\ 3\ 4\ 3\ 3\ 4)$, it is proved that $R$ is $E_{CD}(7,24)$ when listed from the onset at position 3.

Finally, consider the clockwise distance sequence of $E_{CD}(k,n)$, say, $\{d_0, d_1, \ldots, d_{k-1}\}$. Distances $d_i$ are equal to $\left\lfloor \frac{n(i+1)}{k} \right\rfloor - \lfloor \frac{ni}{k} \rfloor$ and this expression can only take the values of $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$. As shown in Section 4.2.1, $E(k,n)$ and $E_{CD}(k,n)$ are the same rhythm up to rotation. Therefore, there exists an index $s$ such that $\{d_s, d_{s+1}, \ldots, d_{k-1}, d_0, d_1, \ldots, d_{s-1}\}$ is the clockwise distance sequence of $E(k,n)$. Set $m = \sum_{i=0}^{s} d_i$.

Thus, the position of the $i$-th onset of $E(k,n)$ is given by the following formula:

$$-m + \left\lfloor \frac{n(i+s)}{k} \right\rfloor. \tag{5.5}$$

For example, the clockwise distance sequence of $E(7,17)$ is $C_1 = (3,2,3,2,3,2,2)$, while the clockwise distance sequence of $E_{CD}(7,17)$ is $C_2 = (2,2,3,2,3,2,3)$. A rotation of $C_2$ to the left by 4 transforms $C_2$ into $C_1$. Then, $s = 1$ and $m = 2+2$. Therefore, the formula $-4 + \left\lfloor \frac{17\cdot(i+2)}{7} \right\rfloor$, for $i = 0,\ldots,6$ generates the onsets of $E(7,17) = \{0,3,5,8,10,13,15\}$.

We now proceed to prove our second technical lemma.

**Lemma 5.3.3.** *Let $E(k,n) = P \oplus \overset{p}{\ldots} \oplus P \oplus T$ be a Euclidean rhythm. Let $E^*(k,n)$ be a rotation of $E(k,n)$ such that $E^*(k,n) = P' \oplus \overset{p}{\ldots} \oplus P' \oplus T'$, where $|P'| = \ell_p$ and $|T'| = \ell_t$. Then, $P$ is a rotation of $P'$.*

*Proof.* If there is no $P'$ that is a subrhythm of $P \oplus P$, then $P'$ must be a subrhythm of $P \oplus T$. In this case the number of subrhythms $P$ in $E(k,n)$ is at most two. If $E(k,n)$ has only one pattern $P$, then $n \equiv 1 \bmod k$. In this case $T = \{0\}$ and $P$ is the rhythm $\{1\ 0 \overset{\lfloor n/k \rfloor}{\ldots} 0\}$. For $P'$ to have $\ell_p$ pulses, $P'$ must start on the second pulse of $P$. This forces $P$ to be equal to an only-rest rhythm, which leads to a contradiction since $E^*(k,n)$ would also come to an only-rest rhythm. If $E(k,n)$ has two patterns $P$, then at least one $P'$ has its first $j$ pulses in $P$ and its last $n-j$ pulses in $T$. From Bjorklund's algorithm we know that when there is more than one pattern $P$, the tail $T$ of $E(k,n)$ is composed of the first $\ell_t$ pulses of $P$ (when $n \not\equiv 1 \bmod k$). In this case, $P'$ consists of the last $\ell_p - j$ pulses of $P$, for some $j$ with $j \leq \ell_t$, followed by the first $j$ pulses of $P$. By Observation 2, $P'$ is a rotation of $P$. □

**Theorem 5.3.4.** *The main pattern of rhythm $E(k,n)$ is Euclidean up to rotation.*

*Proof.* Consider first the case $\gcd(k,n) > 1$. Using (5.1) and the expressions (5.2) and (5.3) we get:

$$\begin{aligned}
E_{CD}(k,n) &= \left\{0, \left\lfloor \frac{n}{k} \right\rfloor, \ldots, \left\lfloor \frac{(k-1)n}{k} \right\rfloor \right\} \\
&= \left\{0, \left\lfloor \frac{\ell_p}{k_p} \right\rfloor, \ldots, \left\lfloor \frac{(k-1)\ell_p}{k_p} \right\rfloor \right\} \\
&= \left\{0, \left\lfloor \frac{\ell_p}{k_p} \right\rfloor, \ldots, \left\lfloor \frac{(k_p-1)\ell_p}{k_p} \right\rfloor, \left\lfloor \frac{\ell_p k_p}{k_p} \right\rfloor, \left\lfloor \frac{\ell_p(k_p+1)}{k_p} \right\rfloor, \ldots, \right. \\
&\quad \left\lfloor \frac{\ell_p k_p + \ell_p(k_p-1)}{k_p} \right\rfloor, \ldots\ldots, \left\lfloor \frac{\ell_p k_p(p-1)}{k_p} \right\rfloor, \left\lfloor \frac{\ell_p(k_p(p-1)+1)}{k_p} \right\rfloor, \ldots, \\
&\quad \left. \left\lfloor \frac{\ell_p k_p(p-1) + \ell_p(k_p-1)}{k_p} \right\rfloor \right\} \\
&= \left\{0, \left\lfloor \frac{\ell_p}{k_p} \right\rfloor, \ldots, \left\lfloor \frac{(k_p-1)\ell_p}{k_p} \right\rfloor, \ell_p, \ell_p + \left\lfloor \frac{\ell_p}{k_p} \right\rfloor, \ldots, \ell_p + \left\lfloor \frac{\ell_p(k_p-1)}{k_p} \right\rfloor, \right. \\
&\quad \left. , \ldots\ldots, \ell_p(p-1), \ldots, \ell_p(p-1) + \left\lfloor \frac{\ell_p(k_p-1)}{k_p} \right\rfloor \right\} \\
&= E_{CD}(k_p,\ell_p) \oplus E_{CD}(k_p,\ell_p) \oplus \cdots \oplus E_{CD}(k_p,\ell_p). \qquad (5.6)
\end{aligned}$$

It is clear from the above expansions that $E_{CD}(k,n)$ is the concatenation of $p$ copies of $E_{CD}(k_p,\ell_p)$. Rhythms $E_{CD}(k,n)$ and $E(k,n)$ are both Euclidean up to a fixed rotation, and by Lemma 5.3.3 this implies that $P$ and $E_{CD}(k_p,\ell_p)$ are rotations of each other. Consequently, $P$ is a Euclidean rhythm.

When $\gcd(k,n) = 1$, we split the proof into two subcases depending on the value of the expression $nk_p - \ell_p k$, which by Lemma 5.3.1 can be equal to either $+1$ or $-1$.

Consider first the case when $nk_p - \ell_p k = 1 \Rightarrow n = \frac{\ell_p k+1}{k_p}$. We first show that $\lfloor \frac{in}{k} \rfloor = \left\lfloor \frac{i\ell_p}{k_p} \right\rfloor$ for all $i = 0, 1, \ldots, k-1$. This means that we need to show that:

$$\left\lfloor \frac{in}{k} \right\rfloor - \left\lfloor \frac{i\ell_p}{k_p} \right\rfloor = \left\lfloor \frac{i(\ell_p k+1)}{kk_p} \right\rfloor - \left\lfloor \frac{i\ell_p}{k_p} \right\rfloor = \left\lfloor \frac{i\ell_p}{k_p} + \frac{i}{kk_p} \right\rfloor - \left\lfloor \frac{i\ell_p}{k_p} \right\rfloor = 0.$$

The above expression is true if the following inequality holds:

$$\frac{i\ell_p}{k_p} + \frac{i}{kk_p} < \left\lfloor \frac{i\ell_p}{k_p} \right\rfloor + 1,$$

Let $r_i \equiv i\ell_p \bmod k_p$. Then,

$$\begin{aligned} \frac{i\ell_p}{k_p} + \frac{i}{kk_p} < \left\lfloor \frac{i\ell_p}{k_p} \right\rfloor + 1 \;\; &\Rightarrow \;\; i\ell_p + \frac{i}{k} < k_p \left\lfloor \frac{i\ell_p}{k_p} \right\rfloor + k_p \\ &\Rightarrow \;\; \frac{i}{k} < k_p \left\lfloor \frac{i\ell_p}{k_p} \right\rfloor - i\ell_p + k_p \\ &\Rightarrow \;\; \frac{i}{k} < k_p - r_i \end{aligned}$$

The greatest value that $\frac{i}{k}$ can take is $\frac{k-1}{k}$, which is always strictly less than 1; on the other hand, the smallest value that $k_p - r_i$ can take is 1. Therefore, the above inequality always holds and $\lfloor \frac{in}{k} \rfloor = \left\lfloor \frac{i\ell_p}{k_p} \right\rfloor$ for all $i = 0, 1, \ldots, k-1$. Together with (5.6), this implies that $E_{CD}(k, n)$ is formed by the concatenation of $p$ copies of $E_{CD}(k_p, \ell_p)$ followed by the concatenation of the sequence $\left\{ p\ell_p + \left\lfloor \frac{i\ell_p}{k_p} \right\rfloor : i = 0, 1, \ldots, k_t - 1 \right\}$. Since $E(k, n)$ and $E_{CD}(k, n)$ differ by a rotation, by Lemma 5.3.3 it follows that $P$ is a rotation of $E_{CD}(k_p, \ell_p)$.

Now suppose that $nk_p - \ell_p k = -1$. For this case we will first show that $\left\lfloor \frac{i\ell_p}{k_p} \right\rfloor + \left\lfloor \frac{k_t \ell_p}{k_p} \right\rfloor = \left\lfloor \frac{(i+k_t)n}{k} \right\rfloor$ for $i = 0, \ldots, k-1$. We start by observing that the multiplicative inverse of $\ell_p \bmod k_p$ is exactly $k_t$. This result is deduced from the equality $\ell_t k_p - \ell_p k_t = -1$, which was proved in Lemma 5.3.1.

If we write $n = \frac{\ell_p k - 1}{k_p}$ and perform some algebraic manipulations, we arrive at the following equality:

$$\left\lfloor \frac{(i+k_t)n}{k} \right\rfloor = \left\lfloor (i+k_t)\frac{\ell_p k - 1}{k k_p} \right\rfloor = \left\lfloor \frac{i\ell_p}{k_p} - \frac{i}{kk_p} + \frac{k_t\ell_p}{k_p} - \frac{k_t}{kk_p} \right\rfloor = \left\lfloor \frac{i\ell_p}{k_p} + \frac{k_t\ell_p}{k_p} - \frac{i+k_t}{kk_p} \right\rfloor.$$

Let $r_i$ be the remainder of the integer division of $i\ell_p$ by $k_p$. Since $k_t\ell_p = 1 \bmod k_p$, we can write:

$$\left\lfloor \frac{i\ell_p}{k_p} + \frac{k_t\ell_p}{k_p} - \frac{i+k_t}{kk_p} \right\rfloor = \left\lfloor \frac{i\ell_p}{k_p} \right\rfloor + \left\lfloor \frac{k_t\ell_p}{k_p} \right\rfloor + \left\lfloor \frac{r_i+1}{k_p} - \frac{i+k_t}{kk_p} \right\rfloor.$$

Finally, the expression we seek to prove is reduced to the following equation:

$$\left\lfloor \frac{(i+k_t)n}{k} \right\rfloor - \left\lfloor \frac{i\ell_p}{k_p} \right\rfloor - \left\lfloor \frac{k_t\ell_p}{k_p} \right\rfloor = \left\lfloor \frac{r_i+1}{k_p} - \frac{i+k_t}{kk_p} \right\rfloor.$$

Therefore, we must prove that the inequality $0 \le \frac{r_i+1}{k_p} - \frac{i+k_t}{kk_p} < 1$ is true. By multiplying the inequality by $kk_p$, we obtain:

$$0 \le \frac{r_i+1}{k_p} - \frac{i+k_t}{kk_p} < 1 \Rightarrow 0 \le k(r_i+1) - (i+k_t) < kk_p$$

The greatest value the expression $k(r_i+1)-(i+k_t)$ can attain is $k(k_p-1+1)-(0+1) = kk_p - 1 < kk_p$. To show that $k(r_i+1)-(i+k_t)$ is always non-negative, we distinguish two cases. If $k_p$ divides $i$, then $r_i = 0$ and the smallest value of our expression is $k(0+1)-(pk_p+k_t) = 0$. If $k_p$ does not divide $i$, then $r_i \ge 1$ and the smallest value of our expression is $k(1+1)-(k-1+k_t) = k - k_t + 1 > 0$. Therefore, the above inequality is always true, and hence $\left\lfloor \frac{i\ell_p}{k_p} \right\rfloor + \left\lfloor \frac{k_t\ell_p}{k_p} \right\rfloor = \left\lfloor \frac{(i+k_t)n}{k} \right\rfloor$ for $i = 0, \ldots, k-1$. From

Observation 2 we know that $\left\{\left\lfloor \frac{(i+k_t)n}{k} \right\rfloor \bmod n : i = 0, \dots, k-1\right\}$ is $E_{CD}(k,n)$ starting from the onset at position $k_t$, and $\left\{\left\lfloor \frac{i\ell_p}{k_p} \right\rfloor + \left\lfloor \frac{k_t\ell_p}{k_p} \right\rfloor \bmod n,\ i = 0, \dots, k-1\right\}$ is a rotation of $E_{CD}(k,n)$ by $\left\lfloor \frac{k_t\ell_p}{k_p} \right\rfloor$, which moreover is in the hypothesis of Lemma 5.3.3. Hence, $P$ is the Euclidean rhythm $E_{CD}(k_p, \ell_p)$ up to rotation. This completes the proof of the theorem. □

It remains to prove that rhythm $\left\{p\ell_p + \left\lfloor \frac{i\ell_p}{k_p} \right\rfloor : i = 0, 1, \dots, k_t - 1\right\}$, is in fact $E(k_t, \ell_t)$ up to rotation. The next theorem settles this question.

**Theorem 5.3.5.** *The tail pattern of rhythm $E(k,n)$ is Euclidean up to rotation.*

*Proof.* The tail pattern $T$ has nonzero length when $\gcd(k,n) = 1$. If $n = 1 \bmod k$, from Bjorklund's algorithm we know that the tail is the Euclidean rhythm $\{0\}$. Assume now that $n \neq 1 \bmod k$. This implies that $k_t \neq 0$. The proof of this case is very similar to the proof of the previous theorem. We again split the proof into two subcases based on the value of $\ell_p k_t - \ell_t k_p$, which by Lemma 5.3.1 can be 1 or $-1$.

Assume first that $\ell_p k_t - \ell_t k_p = 1$. We will prove that $\left\lfloor \frac{i\ell_p}{k_p} \right\rfloor = \left\lfloor \frac{i\ell_t}{k_t} \right\rfloor$, for $i = 0, \dots, k_p - 1$.

Thus, we need to show that:

$$\left\lfloor \frac{i\ell_p}{k_p} \right\rfloor - \left\lfloor \frac{i\ell_t}{k_t} \right\rfloor = 0 \Rightarrow \left\lfloor \frac{i(\ell_t k_p + 1)}{k_t k_p} \right\rfloor - \left\lfloor \frac{i\ell_t}{k_t} \right\rfloor = 0 \Rightarrow \left\lfloor \frac{i\ell_t}{k_t} + \frac{i}{k_t k_p} \right\rfloor - \left\lfloor \frac{i\ell_t}{k_t} \right\rfloor = 0.$$

The above expression is true if the following inequality holds:

$$\frac{i\ell_t}{k_t} + \frac{i}{k_t k_p} < \left\lfloor \frac{i\ell_t}{k_t} \right\rfloor + 1.$$

Substituting $r_i = i\ell_t \bmod k_t$ in the above inequality, with an argument similar to the one in the proof of Theorem 5.3.4, we can show that this inequality is always true and hence $T$ is Euclidean. For conciseness we omit the details.

For the case $\ell_p k_t - \ell_t k_p = -1$, we will show that $\left\lfloor \frac{i\ell_t}{k_t} \right\rfloor + \left\lfloor \frac{\alpha\ell_t}{k_t} \right\rfloor = \left\lfloor \frac{(i+\alpha)\ell_p}{k_p} \right\rfloor$, for $i = 0, \ldots, k_p - 1$ and for some value $\alpha$. Let $r_i \equiv i\ell_t \bmod k_t$ and $\alpha\ell_t \equiv 1 \bmod k_t$. On the other hand, since $\ell_p k_t - \ell_t k_p = -1 \Rightarrow k_p \ell_t \equiv 1 \bmod k_t$, then $\alpha \equiv k_p \bmod k_t$. This fact falls out from equality $\ell_p k_t - \ell_t k_p = -1$ (Lemma 5.3.1) when we take $\bmod k_t$ of both sides of the equation. From the two expressions for $\alpha$ we get that $1 \leq \alpha \leq k_t - 1$. Now,

$$\begin{aligned}
\left\lfloor \frac{(i+\alpha)\ell_p}{k_p} \right\rfloor &= \left\lfloor (i+\alpha)\frac{\ell_t k_p - 1}{k_t k_p} \right\rfloor \\
&= \left\lfloor \frac{i\ell_t}{k_t} - \frac{i}{k_t k_p} + \frac{\alpha\ell_t}{k_t} - \frac{\alpha}{k_t k_p} \right\rfloor \\
&= \left\lfloor \frac{i\ell_t}{k_t} + \frac{\alpha\ell_t}{k_t} - \frac{i+\alpha}{k_t k_p} \right\rfloor \\
&= \left\lfloor \frac{i\ell_t}{k_t} \right\rfloor + \left\lfloor \frac{\alpha\ell_t}{k_t} \right\rfloor + \left\lfloor \frac{r_i+1}{k_t} - \frac{i+\alpha}{k_t k_p} \right\rfloor.
\end{aligned}$$

If the following inequality holds,

$$0 \leq \frac{r_i+1}{k_t} - \frac{i+\alpha}{k_t k_p} < 1 \Longrightarrow 0 \leq k_p(r_i+1) - (i+\alpha) < k_t k_p,$$

then $\left\lfloor \frac{i\ell_t}{k_t} \right\rfloor + \left\lfloor \frac{\alpha\ell_t}{k_t} \right\rfloor = \left\lfloor \frac{(i+\alpha)\ell_p}{k_p} \right\rfloor$. The upper bound of $k_p(r_i+1) - (i+\alpha)$ is $k_p(k_t - 1 + 1) - (0+1) = k_p k_t - 1 < k_p k_t$. To show that $k_p(r_i+1) - (i+\alpha)$ is always nonnegative, we analyze two subcases. If $k_t$ does not divide $i$, then since $\ell_t$ and $k_t$ are relatively prime, $r_i$ must be at least 1. Thus, the lower bound for our expression in this case is $k_p(1+1) - (k_p - 1 + k_t - 1) = k_p - k_t + 2 > 0$. Now suppose $k_t$ divides $i$; then, $r_i = 0$

and the greatest value of $i$ that is divisible by $k_t$ is $k_t \left\lfloor \frac{k_p}{k_t} \right\rfloor$. Thus, the lower bound in this case is $k_p(0+1) - \left(k_t \left\lfloor \frac{k_p}{k_t} \right\rfloor + \alpha\right) = k_p - (k_p - \alpha + \alpha) = 0$. This completes the proof of the theorem. □

If $P$ admits a decomposition $P = Q \oplus \overset{q}{\dots} \oplus Q$, for certain $q > 1$, then $E(k,n)$ can be written as the concatenation of a pattern $Q$ having fewer pulses. We will now show that, in fact, $P$ does not admit such a decomposition, and therefore is *minimal.* First we show that the rhythm obtained by removing the tail of a Euclidean rhythm remains Euclidean. Note that this fact does not follow immediately from the preceding two theorems, as the main pattern in a Euclidean rhythm depends on its number of pulses and onsets; removing the tail changes these numbers, and it is thus not clear what the main pattern in a Euclidean rhythm with fewer pulses looks like.

**Theorem 5.3.6.** *Let $k$ and $n$ be two integers with* $\gcd(k,n) = 1$. *If* $E(k,n) = P \oplus \overset{p}{\dots} \oplus P \oplus T$ *is the decomposition given by Bjorklund's algorithm, then rhythm* $P \oplus \overset{p}{\dots} \oplus P$ *is a rotation of* $E(k_p p, \ell_p p)$.

*Proof.* It is sufficient to prove the result for the Clough-Douthett representation $E_{CD}(k_p, \ell_p)$ of $P$. By concatenating $p$ copies of $E_{CD}(k_p, \ell_p)$ we obtain:

$$\begin{aligned}
& E_{CD}(k_p,\ell_p) \oplus \overset{p}{\dots} \oplus E_{CD}(k_p,\ell_p) \\
= \ & \left\{0, \left\lfloor \frac{\ell_p}{k_p} \right\rfloor, \dots, \left\lfloor \frac{(k_p-1)\ell_p}{k_p} \right\rfloor, \ell_p, \ell_p + \left\lfloor \frac{\ell_p}{k_p} \right\rfloor, \dots, \ell_p + \left\lfloor \frac{(k_p-1)\ell_p}{k_p} \right\rfloor, \dots, \right. \\
& \left. \ell_p(p-1), \ell_p(p-1) + \left\lfloor \frac{\ell_p}{k_p} \right\rfloor, \dots, \ell_p(p-1) + \left\lfloor \frac{(k_p-1)\ell_p}{k_p} \right\rfloor \right\} \\
= \ & \left\{0, \left\lfloor \frac{p\ell_p}{pk_p} \right\rfloor, \dots, \left\lfloor \frac{(k_p-1)p\ell_p}{pk_p} \right\rfloor, \ell_p, \ell_p + \left\lfloor \frac{p\ell_p}{pk_p} \right\rfloor, \dots, \ell_p + \left\lfloor \frac{(k_p-1)p\ell_p}{pk_p} \right\rfloor, \dots, \right. \\
& \left. \ell_p(p-1), \ell_p(p-1) + \left\lfloor \frac{p\ell_p}{pk_p} \right\rfloor, \dots, \ell_p(p-1) + \left\lfloor \frac{(k_p-1)p\ell_p}{pk_p} \right\rfloor \right\}
\end{aligned}$$

$$\begin{aligned}
&= \left\{0, \left\lfloor\frac{p\ell_p}{pk_p}\right\rfloor, \ldots, \left\lfloor\frac{(k_p-1)p\ell_p}{pk_p}\right\rfloor, \left\lfloor\frac{\ell_p pk_p}{pk_p}\right\rfloor, \left\lfloor\frac{\ell_p pk_p}{pk_p}+\frac{p\ell_p}{pk_p}\right\rfloor, \ldots,\right. \\
&\quad \left\lfloor\frac{\ell_p pk_p}{pk_p}+\frac{(k_p-1)p\ell_p}{pk_p}\right\rfloor, \ldots, \left\lfloor\frac{pk_p\ell_p(p-1)}{pk_p}\right\rfloor, \left\lfloor\frac{pk_p\ell_p(p-1)}{pk_p}+\frac{p\ell_p}{pk_p}\right\rfloor, \ldots, \\
&\quad \left.\left\lfloor\frac{pk_p\ell_p(p-1)}{pk_p}+\frac{(k_p-1)p\ell_p}{pk_p}\right\rfloor\right\} \\
&= \left\{0, \left\lfloor\frac{p\ell_p}{pk_p}\right\rfloor, \ldots, \left\lfloor\frac{(k_p-1)p\ell_p}{pk_p}\right\rfloor, \left\lfloor\frac{\ell_p pk_p}{pk_p}\right\rfloor, \ldots, \left\lfloor\frac{(pk_p-1)p\ell_p}{pk_p}\right\rfloor\right\} \\
&= E_{CD}(pk_p, p\ell_p).
\end{aligned}$$

Therefore, the concatenation of $p$ copies of $P$ is a rotation of $E(k_p p, \ell_p p)$. □

**Theorem 5.3.7.** *The main pattern $P$ of $E(k,n)$ is minimal.*

*Proof.* Assume first that $\gcd(k,n) > 1$. If $Q$ is a pattern such that $P = Q \oplus \overset{q}{\ldots} \oplus Q$, for some $q \geq 1$, then the number $qp$ must divide both $n$ and $k$. Since in this case $p = \gcd(k,n)$, it follows that $q = 1$, and therefore, $P = Q$.

Assume now that $\gcd(k,n) = 1$. By Theorem 5.3.6, removing the tail pattern of $E(k,n)$ will produce the rhythm $E(k_p p, \ell_p p)$ up to rotation. By the previous case, the main pattern $P$ of $E(k_p p, \ell_p p)$ cannot be written as the concatenation of copies of a shorter pattern $Q$. Thus, the main pattern of $E(k,n)$ is minimal. □

This concludes our study of structural properties of Euclidean rhythms. Besides satisfying our theoretical curiosity about the mathematics behind this family of rhythms, understanding these properties helps us understand various transformations of these rhythms throughout the progression of musical pieces. In the following chapter, we study transformations of maximally even rhythms that preserve this evenness property.

# Chapter 6

# Evenness Preserving Operations on Rhythms

An *operation* transforms one musical rhythm into another based on a set of rules. Rhythmic transformations are common in musical pieces, where they are often used to vary the flavor of the piece while at the same time staying within the same musical theme during its progression. For example, a jazz soloist must respect the style and feeling of the piece, and thus play an improvised variation based on the foundation of the main theme [92]. One way of realizing such an improvisation is by taking the base rhythm and transforming it to another through one or more operations. It is generally desirable that such rhythmic transformations preserve some properties of the original rhythm. One example of a rhythmic operation is the *shelling* operation described in Chapter 3; when applied (carefully) to a deep rhythm, the shelling operation produces another deep rhythm with fewer onsets. Operations are thus important for improvisation as well as music composition. They are also useful for music analysis, for example to understand rhythmic transformations and generate new ones, as well as to provide formal rules for improvisation techniques.

In this chapter, we are interested in analyzing transformations of maximally even rhythms through the study of *interlocking rhythms.* In many musical traditions, the concept of interlocking rhythms is intuitive more than well defined. Although some rhythms are classified as interlocking in certain contexts, the label is applied almost intuitively; nonetheless, we see interlocking rhythms in many musical traditions. Roughly speaking, interlocking rhythms are a set of rhythms that, when played together, cover the rhythmic timespan (possibly with overlap). For example, in Afro-Cuban music interlocking rhythms are found in styles such as the *son* or the *guaracha* where the timbal plays an interlocking rhythm called *cáscara* [146, 179].

In percussion, especially in funk and jazz, playing the complement of a rhythm very quietly, more felt than heard, is common. This is referred to as playing the ghost notes. Interlocking rhythms are also related to complementary rhythmic canons. Rhythmic canons were first introduced by the French composer Olivier Messian as part of his compositional ideas. A rhythmic canon is a rhythmic pattern played several times, each at a different entry time; it is called complementary when, on each pulse, no more than one pattern has an onset. Rhythmic canons were first studied by Vuza [181–184] and later by Andreatta [3]. Hall [91] studied the relationship between asymmetric rhythms and *tiling canons*, complementary canons whose union tiles (covers with no overlap) the entire timespan.

In Western classical music interlocking rhythms became more frequent in the early common practice period. For example, in Tchaikovsky's *Romeo and Juliet* Fantasy-Overture, interlocking rhythms are found in the development section – between measures 333-342 the trombone plays a solo melody, while the rest of the orchestra plays on the complement of its onsets.

Another outstanding example of interlocking rhythms in Western classical music is from the *allegretto* of Beethoven's string quartet Op. 59, no. 2. Here, for more than half of the movement the first violin plays a melody whose rhythmic pattern is [× · · ×], while the second violin and the viola play the complementary rhythmic pattern [ · ×× · ].

Interlocking rhythms also occur in music from Cuba, Spain, Brazil, the Caribbean, and Germany. For some examples of interlocking rhythms from these regions, see [4, 6, 111, 122, 179]. A few authors have also studied the psychological aspects of interlocking rhythms considered within a melodic context [64, 121, 156].

In this chapter we define five operations on musical rhythms and study the conditions under which a given operation preserves the Euclidean property. The operations we describe are shadow, complementation, concatenation, alternation, and decomposition. Later, in Section 6.5 we study the problem of decomposing Euclidean rhythms into Euclidean rhythms with a smaller number of onsets. Finally, we relate these mathematical properties to interlocking Euclidean rhythms in Section 6.6.

## 6.1 Shadow

Several ethnomusicologists have argued that African drumming, handclapping, and mallet performance are best understood as a motor activity. For example, a hand (arm) is raised and then dropped to strike the instrument.

According to Jay Rahn [139], one possible mechanism for the tacit motor mediation of attack points of onsets is the peaking of the gesture at the temporal midpoint between two sounds. He calls the sequence of midpoints of the onsets of a rhythm the *shadow* of the rhythm. For example, the Cuban *tresillo* rhythm given by [× · · · · · × · · · · · × · · · ] has the shadow [ · · · × · · · · · × · · · · × · ], which yields the shadow rhythm [× · · · · · × · · · · × · · · · ] (Figure 6–1).

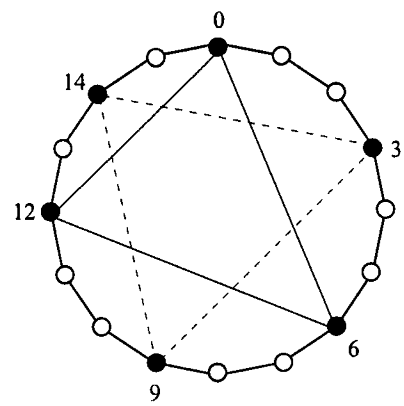



Figure 6–1: The Cuban tresillo and its shadow.

Performed on a rhythm, the shadow operation increases the evenness of the new rhythm. One question that comes to mind is: what happens to the sequence obtained when we continue to perform this operation on every rhythm resulting from the shadow of another? It turns out that this is a geometric problem that has been investigated in the mathematics literature [148].

The study of the properties of sequences of polygons generated by performing iterative processes on an initial polygon has received much attention, and the shadow operation is just one of many operations that has been investigated. Perhaps the most studied sequence is the one sometimes referred to as Kasner polygons [148]. Given a polygon $P^0$, the Kasner descendant $P^1$ of $P^0$ is obtained by placing the vertices of $P^1$ at the midpoints of the edges of $P^0$. Fejes Tóth [168] was interested in the more general problem of sequences of Kasner polygons where each polygon $P^t$ in the sequence is obtained by dividing every edge of $P^{t-1}$ with a ratio $\alpha : (1-\alpha)$ in the clockwise (or counterclockwise) direction and making the division points the vertices of $P^t$ for $t = 1, 2, \ldots$. He proved that if $\alpha = 1/2$ (Kasner polygon), then the sequence converges to a regular polygon when $P^0$ is a convex pentagon or a convex hexagon. He conjectured that for any $\alpha$ and any initial convex polygon, the sequence converges to a regular polygon. Reichardt [141] showed that if $\alpha = 1/2$, every convex polygon converges to the regular polygon. Later, Lükő [115] proved that for any $\alpha \in [0, 1]$ and for any convex polygon $P^0$, the sequence $P^0, P^1, P^2, \ldots$ converges to a regular polygon, thus settling the more general conjecture of Fejes Tóth. More results on Kasner polygons can be found in [11, 56].

The shadow sequence of a rhythm we study here is similar to the Kasner sequence. Hitt and Zhang [97] show that given any convex cyclic polygon $P^0$, its shadow sequence converges to a regular polygon. From their proof, it follows that the area of each $P^i$ is greater than or equal to the area of $P^{i-1}$ for any $i > 0$, with equality resulting only when $P^i$ is regular.

In their proof of the convergence of the shadow sequence, Hitt and Zhang make use of doubly stochastic matrices and Schur-convex functions. Below we provide a simpler proof that uses a different approach and is more intuitive. This proof also gives a bound on the rate of convergence. We then show how our results extend to the general shadow sequence of a rhythm, where at each step every arc is split into a fixed ratio $\alpha$ of the arc lengths in the clockwise or counterclockwise direction.

**Theorem 6.1.1.** *The shadow sequence of a cyclic polygon converges to a regular polygon in such a way that the variance of the interval lengths decreases at each step by at least one half.*

*Proof.* Let $P$ be a polygon inscribed in a unit circle, and let $\langle a_0, a_1, \dots, a_{n-1}\rangle$ be the sequence of intervals or edge lengths of $P$, where by *length* we mean the geodesic distance along the circle between two consecutive vertices of $P$. Thus, $\sum_{i=0}^{n-1} a_i = 1$. Let $P^t$ denote the polygon $P$ after $t$ shadow operations, and $a_i^t$ denote its corresponding edge lengths, for $i = 0, 1, \dots, n-1$. At any step $t$, we can write the edge lengths of $P^{t+1}$ as the sequence:

$$\left\{ \frac{a_i^{(t)} + a_{i+1}^{(t+1)}}{2} : i = 0, 1, \dots, n-1 \right\}.$$

Also, the average edge length of $P^t$ is $1/n$ for any $t$. Since the edge lengths sum to 1 at any step, we can treat the sequence of edges as a random variable and compute its variance. We will show that the variance $V^{t+1}$ of the sequence of edge lengths at time $t+1$ decreases by a constant fraction of the variance at time $t$. For simplicity, we will assume $a_i^t = a_i$; thus,

$$\begin{aligned}
V^{t+1} &= \frac{1}{n}\sum_{i=0}^{n-1}\left(a_i^{t+1}\right)^2 - \frac{1}{n^2} \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{a_i + a_{i+1}}{2}\right)^2 - \frac{1}{n^2} \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\frac{a_i^2 + a_{i+1}^2 + 2a_i a_{i+1}}{4} - \frac{1}{n^2} \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\frac{a_i^2 + a_{i+1}^2}{4} + \frac{1}{n}\sum_{i=0}^{n-1}\frac{a_i a_{i+1}}{2} - \frac{1}{n^2} \\
&= \frac{1}{2n}\sum_{i=0}^{n-1} a_i^2 + \frac{1}{2n}\sum_{i=1}^{n} a_i a_{i+1} - \frac{1}{n^2} \\
&= \frac{1}{2n}\sum_{i=0}^{n-1} a_i^2 - \frac{1}{2n^2} + \frac{1}{2n}\sum_{i=0}^{n-1} a_i a_{i+1} - \frac{1}{2n^2}
\end{aligned}$$

$$= \frac{1}{2}\left(\frac{1}{n}\sum_{i=0}^{n-1} a_i^2 - \frac{1}{n^2}\right) + \frac{1}{2n}\sum_{i=0}^{n-1} a_i a_{i+1} - \frac{1}{2n^2}$$

$$= \frac{1}{2}V^{(t)} + \frac{1}{2n}\sum_{i=0}^{n-1} a_i a_{i+1} - \frac{1}{2n^2}$$

To show that $V^{t+1}$ is at most a fraction of $V^t$ at any step $t$, we find the maximum value of $\sum_{i=0}^{n-1} a_i a_{i+1}$ subject to the constraint $\sum_{i=0}^{n-1} a_i = 1$. Using Lagrange multipliers it can easily be determined that the maximum value of the above sum is attained when all the $a_i$'s have the same value, that is, when they are all equal to $1/n$. To find this maximal point, we solve:

$$\frac{\partial}{\partial a_j}\left(\sum_{i=0}^{n-1} a_i a_{i+1}\right) + \lambda\left(\sum_{i=0}^{n-1} a_i - 1\right) = 0$$

Differentiating these $n$ equations (for $j = 0, 1, \ldots, n-1$) we obtain, $a_{i-1}+a_{i+1}+\lambda = 0$.

Using the constraint $\sum_{i=0}^{n-1} a_i = 1$ we find that $a_i = \frac{1}{n}$.

Thus, we have:

$$V^{t+1} = \frac{1}{2}V^t + \frac{1}{2n}\sum_{i=0}^{n-1} a_i a_{i+1} - \frac{1}{2n^2}$$

$$< \frac{1}{2}V^t + \frac{1}{2n}\sum_{i=0}^{n-1} \frac{1}{n}\cdot\frac{1}{n} - \frac{1}{2n^2} = \frac{1}{2}V^t$$

Therefore, after every shadow step the variance is at least halved; and since the variance is always bounded below by zero, then it converges to zero as $t$ goes to infinity. This in turn implies that every edge length converges to the mean value, which is $1/n$. Thus, the shadow sequence of any cyclic polygon converges to the regular polygon in such a way that the variance of the edge lengths decreases by at least one half at every step. □

Let us now consider the general case, where each polygon $P^{t+1}$ in the shadow sequence is obtained by dividing every arc of $P^t$ with a ratio $\alpha : (1-\alpha)$ $(0 < \alpha < 1)$ in the clockwise (or counterclockwise) direction, and making the division points the vertices of $P^{t+1}$ for all $t \geq 0$. In this case, at any step $t$ we can write the edge lengths of $P^{t+1}$ as the sequence: $\left\{(1-\alpha)a_i^t + \alpha a_{i+1}^t : i = 0, 1, \ldots, n-1\right\}$. We can extend the proof of Theorem 6.1.1 to show that the variance of the edge lengths of the generalized shadow sequence decreases at every step by at least $2\alpha^2 - 2\alpha + 1$. By doing calculations similar to those made in the proof of Theorem 6.1.1, we can show that:

$$\begin{aligned} V^{t+1} &= \frac{1}{n}\sum_{i=0}^{n-1}(a_i^{t+1})^2 - \frac{1}{n^2} \\ &= \frac{1}{n}\sum_{i=0}^{n-1}((1-\alpha)a_i + \alpha a_{i+1})^2 - \frac{1}{n^2} \\ &< (2\alpha^2 - 2\alpha + 1)V^t \end{aligned}$$

Note that $0 < 2\alpha^2 - 2\alpha + 1 < 1$ for any $0 < \alpha < 1$. Therefore, after every shadow step the variance decreases by a constant fraction that depends on $\alpha$. Thus, again the variance converges to zero as $t$ goes to infinity, and every edge length converges to $1/n$.

In general, however, we want to represent a rhythm as a cyclic polygon with vertices that lie on integer coordinates (pulses). Let $R$ be a rhythm represented as a cyclic polygon with integer vertices (pulses). If an onset $v$ of the shadow does not lie on an integer point, then we move $v$ to the nearest integer coordinate in the clockwise direction. The result is the *discrete shadow* of $R$. Consider the clockwise distance sequence notation $(a_0, a_1, \ldots, a_{k-1})$ of a rhythm $R$ where $a_i$ is the geodesic distance between two consecutive onsets in the clockwise direction for all $i = 0, 1, \ldots, k-1$. Then the discrete shadow of $R$ is $(a'_0, a'_1, \ldots, a'_{k-1})$, where each $a'_i = \lfloor a_i/2 \rfloor + \lceil a_{i+1}/2 \rceil$.

**Theorem 6.1.2.** *The discrete shadow of a Euclidean rhythm $R$ is a rotation of $R$.*

*Proof.* It was proved in Section 4.2.1 that a Euclidean rhythm generated by Bjorklund's algorithm has the form (in clockwise distance sequence notation):

$$(\underbrace{\lfloor \tfrac{n}{k} \rfloor, \ldots, \lfloor \tfrac{n}{k} \rfloor}_{x_1}, \lceil \tfrac{n}{k} \rceil; \underbrace{\lfloor \tfrac{n}{k} \rfloor, \ldots, \lfloor \tfrac{n}{k} \rfloor}_{x_2}, \lceil \tfrac{n}{k} \rceil; \ldots; \underbrace{\lfloor \tfrac{n}{k} \rfloor, \ldots, \lfloor \tfrac{n}{k} \rfloor}_{x_p}, \lceil \tfrac{n}{k} \rceil)$$

for $x_1, x_2, \ldots, x_p > 0$. Let $a = \lfloor \frac{n}{k} \rfloor$ and $b = \lceil \frac{n}{k} \rceil$. Then a Euclidean rhythm $R$ has the form

$$(\underbrace{a, \ldots, a}_{x_1}, b; \underbrace{a, \ldots, a}_{x_2}, b; \ldots; \underbrace{a, \ldots, a}_{x_p}, b).$$

Let $R'$ be the discrete shadow of $R$. Then $R'$ is the rhythm:

$$\begin{aligned}
R' &= (\underbrace{\lfloor \tfrac{a}{2} \rfloor + \lceil \tfrac{a}{2} \rceil, \ldots; \lfloor \tfrac{a}{2} \rfloor + \lceil \tfrac{a}{2} \rceil}_{x_1-1}, \lfloor \tfrac{a}{2} \rfloor + \lceil \tfrac{b}{2} \rceil, \lfloor \tfrac{b}{2} \rfloor + \lceil \tfrac{a}{2} \rceil; \\
&\quad \underbrace{\lfloor \tfrac{a}{2} \rfloor + \lceil \tfrac{a}{2} \rceil, \ldots; \lfloor \tfrac{a}{2} \rfloor + \lceil \tfrac{a}{2} \rceil}_{x_2-1}, \lfloor \tfrac{a}{2} \rfloor + \lceil \tfrac{b}{2} \rceil, \lfloor \tfrac{b}{2} \rfloor + \lceil \tfrac{a}{2} \rceil; \ldots\ldots; \\
&\quad \underbrace{\lfloor \tfrac{a}{2} \rfloor + \lceil \tfrac{a}{2} \rceil, \ldots; \lfloor \tfrac{a}{2} \rfloor + \lceil \tfrac{a}{2} \rceil}_{x_p-1}, \lfloor \tfrac{a}{2} \rfloor + \lceil \tfrac{b}{2} \rceil, \lfloor \tfrac{b}{2} \rfloor + \lceil \tfrac{a}{2} \rceil) \\
&= (\underbrace{a, \ldots, a}_{x_1-1}, \lfloor \tfrac{a}{2} \rfloor + \lceil \tfrac{b}{2} \rceil, \lfloor \tfrac{b}{2} \rfloor + \lceil \tfrac{a}{2} \rceil; \underbrace{a, \ldots, a}_{x_2-1}, \lfloor \tfrac{a}{2} \rfloor + \lceil \tfrac{b}{2} \rceil, \lfloor \tfrac{b}{2} \rfloor + \lceil \tfrac{a}{2} \rceil; \\
&\quad \ldots\ldots \underbrace{a, \ldots, a}_{x_p-1}, \lfloor \tfrac{a}{2} \rfloor + \lceil \tfrac{b}{2} \rceil, \lfloor \tfrac{b}{2} \rfloor + \lceil \tfrac{a}{2} \rceil)
\end{aligned}$$

If $k$ divides $n$, then $a = b$ and $\lfloor \frac{a}{2} \rfloor + \lceil \frac{b}{2} \rceil = \lfloor \frac{b}{2} \rfloor + \lceil \frac{a}{2} \rceil = a$; in this case, $R$ and $R'$ are the same Euclidean rhythm $(a, a, \ldots, a)$. Now suppose $k$ does not divide $n$; then $b = a + 1$. Two cases arise:

1. If $a$ is even, we have:
   - $\lfloor \frac{a}{2} \rfloor + \lceil \frac{b}{2} \rceil = \lfloor \frac{a}{2} \rfloor + \lceil \frac{a+1}{2} \rceil = \frac{a}{2} + \frac{a}{2} + 1 = a + 1 = b.$
   - $\lfloor \frac{b}{2} \rfloor + \lceil \frac{a}{2} \rceil = \lfloor \frac{a+1}{2} \rfloor + \lceil \frac{a}{2} \rceil = \frac{a}{2} + \frac{a}{2} = a.$

   Therefore, $R'$ is the rhythm $(\underbrace{a, \ldots, a}_{x_1-1}, b, a; \underbrace{a, \ldots, a}_{x_2-1}, b, a; \ldots; \underbrace{a, \ldots, a}_{x_p-1}, b, a)$. Rotating $R'$ to the right by one position we obtain $(a, \underbrace{a, \ldots, a}_{x_1-1}, b; a, \underbrace{a, \ldots, a}_{x_2-1}, b; \ldots; a, \underbrace{a, \ldots, a}_{x_p-1}, b)$, which is rhythm $R$.

2. If $a$ is odd, we have:
   - $\lfloor \frac{a}{2} \rfloor + \lceil \frac{b}{2} \rceil = \lfloor \frac{a}{2} \rfloor + \lceil \frac{a+1}{2} \rceil = \lfloor \frac{a}{2} \rfloor + \lfloor \frac{a}{2} \rfloor + 1 = a.$
   - $\lfloor \frac{b}{2} \rfloor + \lceil \frac{a}{2} \rceil = \lfloor \frac{a+1}{2} \rfloor + \lceil \frac{a}{2} \rceil = \lceil \frac{a}{2} \rceil + \lceil \frac{a}{2} \rceil = a + 1 = b.$

   Therefore, $R'$ is the following rhythm:

$$\begin{aligned} & (\underbrace{a, \ldots, a}_{x_1-1}, a, b; \underbrace{a, \ldots, a}_{x_2-1}, a, b; \ldots; \underbrace{a, \ldots, a}_{x_p-1}, a, b;) \\ = \ & (\underbrace{a, \ldots, a}_{x_1}, b; \underbrace{a, \ldots, a}_{x_2}, b; \ldots; \underbrace{a, \ldots, a}_{x_p}, b;) \end{aligned}$$

   which, again, is rhythm $R$.

Therefore, the discrete shadow of every Euclidean rhythm is a rotation of itself.

□

**Corollary 6.1.3.** *The discrete shadow of a maximally even rhythm $R$ is a rotation of $R$.*

## 6.2 Complement

The study of the complementary sets of sets of intervals in the context of pitch (scales and chords) has received a lot of attention in music theory [125]. The complement of rhythm, on the other hand, has scarcely been explored. Consider the Cuban *cinquillo* rhythm given by [× · ×× · ×× · ]. Its complementary rhythm is [ · × · · × · · ×], which is a rotation of the famous Cuban *tresillo* rhythm given by [× · · × · · × · ] (see Figure 6–2).

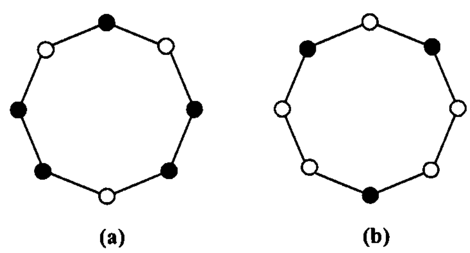


Figure 6–2: (a) The Cuban cinquillo (b) The complement of the Cuban cinquillo is a rotation of the Cuban tresillo.

Complementary sets have many applications, such as the composition of rhythmic complementary canons [181]. Clough and Douthett [43] show that the complement of a maximally even rhythm is maximally even. This theorem was later proved independently by other authors such as Bruckstein [30], who presents several self-similarity properties of digital straight lines which indirectly show that the complement of a Euclidean rhythm is also Euclidean. Here we present a simple proof of the complementation theorem that is different from that of Clough and Douthett but that uses digital straight lines.

**Theorem 6.2.1** ([30, 43, 93]). *The complement of a Euclidean rhythm is Euclidean up to rotation.*

*Proof.* Harris and Reingold [93] showed that Euclid's algorithm for computing the greatest common divisor of two integers generates digital straight lines described by the Bresenham algorithm [28]. Euclid's algorithm also generates Euclidean rhythms (Section 4.2.1); thus, a sequence of 0-1 bits of a Euclidean rhythm corresponds to the sequence of 0-1 bits of a digital straight line (up to rotation). Without loss of generality, assume that a 0-bit of a digital straight line corresponds to a vertical segment and a 1-bit corresponds to a horizontal segment.

Let $R$ be a Euclidean rhythm corresponding to the digital line $L$ defined by the equation $y = ax$ (we assume the line passes through the center of our coordinate system). If we rotate $L$ by 90°, then the equation of the rotated line $L'$ becomes $x = ay$. To draw the digital line $L'$, we can merely interchange the $x$ and $y$ axis and plot line $L : y = ax$. This means that $L$ and $L'$ are the same digital line and hence are both described by Euclid's algorithm. However, when we rotate $L$, the 0-bits of $R$ become 1-bits and the 1-bits become 0-bits. Hence we get the complement of $R$. Since both $R$ and its complement correspond to the "same" digital line drawn by the same sequence of vertical and horizontal segments, they are both Euclidean. □

**Corollary 6.2.2.** *The complement of a maximally even rhythm is maximally even.*

## 6.3 Concatenation

The *concatenation* of two rhythms $R_1$ and $R_2$, denoted by $R_1 \oplus R_2$, is a new rhythm $R$ formed by the pulses of $R_1$ followed by those of $R_2$. For example, $E(3,7) \oplus E(4,6) = [\times \cdot \times \cdot \times \cdot \cdot] \oplus [\times \cdot \times\times \cdot \times] = [\times \cdot \times \cdot \times \cdot \cdot \times \cdot \times\times \cdot \times]$. Concatenation of Euclidean rhythms were extensively used in proofs of theorems in Chapter 5.

The following theorem examines the conditions under which the concatenation of Euclidean rhythms is itself Euclidean.

**Theorem 6.3.1.** *For any Euclidean rhythm $E(k,n)$ with $1 \leq k \leq n$, and any natural number $c > 1$.*

1. *$E(ck, cn)$ is the concatenation of $c$ copies of a rotation of $E(k,n)$. If $\gcd(k,n) = 1$, the main pattern of $E(ck, cn)$ is a rotation of $E(k,n)$.*
2. *$E(k,n) \oplus E(ck, cn)$ is a rotation of $E((c+1)k, (c+1)n)$.*

*Proof.* We will use the Clough-Douthett form of $E(k,n)$ to prove (1).

$$
\begin{aligned}
E_{CD}(ck, cn) &= \Big\{0, \left\lfloor \frac{cn}{ck} \right\rfloor, \ldots, \left\lfloor \frac{(k-1)cn}{ck} \right\rfloor, \cdots, \left\lfloor \frac{(ck-k)cn}{ck} \right\rfloor, \left\lfloor \frac{(ck-k+1)cn}{ck} \right\rfloor, \\
&\quad \cdots, \left\lfloor \frac{(ck-1)cn}{ck} \right\rfloor \Big\} \\
&= \Big\{0, \left\lfloor \frac{n}{k} \right\rfloor, \ldots, \left\lfloor \frac{(k-1)n}{k} \right\rfloor, \cdots, \left\lfloor \frac{(ck-k)n}{k} \right\rfloor, \left\lfloor \frac{(ck-k+1)n}{k} \right\rfloor, \\
&\quad \cdots, \left\lfloor \frac{(ck-1)n}{k} \right\rfloor \Big\} \\
&= \Big\{0, \left\lfloor \frac{n}{k} \right\rfloor, \ldots, \left\lfloor \frac{(k-1)n}{k} \right\rfloor, \cdots, n(c-1), n(c-1) + \left\lfloor \frac{n}{k} \right\rfloor, \cdots \\
&\quad \cdots, n(c-1) + \left\lfloor \frac{(k-1)n}{k} \right\rfloor \Big\} \\
&= E_{CD}(k,n) \oplus \overset{c}{\ldots} \oplus E_{CD}(k,n).
\end{aligned}
$$

If $\gcd(k,n) = 1$, the greatest common divisor of $ck$ and $cn$ is $c$. Therefore, the main pattern of $E(ck, cn)$ has $n$ pulses and $k$ onsets. Since by Theorem 5.3.4 the main pattern is Euclidean, then by Lemma 5.3.3, it has to be a rotation of $E(k,n)$.

As for (2), we have the following computation:

$$
\begin{aligned}
& E_{CD}(k,n) \oplus E_{CD}(ck,cn) \\
= & \left\{0, \left\lfloor \frac{n}{k} \right\rfloor, \left\lfloor \frac{2n}{k} \right\rfloor, \ldots, \left\lfloor \frac{(k-1)n}{k} \right\rfloor, n, n + \left\lfloor \frac{cn}{ck} \right\rfloor, n + \left\lfloor \frac{2cn}{ck} \right\rfloor, \ldots, \right. \\
& \left. n + \left\lfloor \frac{(ck-1)cn}{ck} \right\rfloor \right\} \\
= & \left\{0, \left\lfloor \frac{n}{k} \right\rfloor, \left\lfloor \frac{2n}{k} \right\rfloor, \ldots, \left\lfloor \frac{(k-1)n}{k} \right\rfloor, \left\lfloor \frac{kn}{k} \right\rfloor, \left\lfloor \frac{(k+1)n}{k} \right\rfloor, \ldots, \left\lfloor \frac{((c+1)k-1)n}{k} \right\rfloor \right\} \\
= & \left\{0, \left\lfloor \frac{(c+1)n}{(c+1)k} \right\rfloor, \left\lfloor \frac{2(c+1)n}{(c+1)k} \right\rfloor, \ldots, \left\lfloor \frac{(k-1)(c+1)n}{(c+1)k} \right\rfloor, \left\lfloor \frac{k(c+1)n}{(c+1)k} \right\rfloor, \right. \\
& \left. \left\lfloor \frac{(k+1)(c+1)n}{(c+1)k} \right\rfloor, \ldots, \left\lfloor \frac{((c+1)k-1)(c+1)n}{(c+1)k} \right\rfloor \right\} \\
= & E_{CD}((c+1)k, (c+1)n).
\end{aligned}
$$

□

In general it is not true that the concatenation of two arbitrary Euclidean rhythms is Euclidean. The concatenation of rhythms $E(3,7) \oplus E(4,6)$ is $[\times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \times \cdot \times]$, which is not equal to $E(7,13)$ or any of its rotations.

## 6.4 Alternation

Starting from the $j$-th onset, the *alternation* operation keeps every $c$-th onset of a rhythm and transforms the remaining onsets into rests. More formally, let $\{u_0, u_1, \ldots, u_{k-1}\}$ be the sorted sequence of the positions of the $k$ onsets of a rhythm $R$. The *$j$-alternation of order $c$* of $R$, denoted by $A_{j,c}(R)$, is the rhythm whose pulses are all rests except the onsets $u_j, u_{j+c}, u_{j+2c}, \ldots, u_{j+vc}$, for $0 \leq j < c \leq k-1$ and $v = \lfloor \frac{k-1-j}{c} \rfloor$.

In general alternations of a rhythm may not be rotation invariant. The alternation $A_{j,c}$ of a rhythm $R$ and a rotation of $R$ might not produce the same rhythm (or any of its rotations). To see this, consider the alternation $A_{0,2}$ of rhythms $E(7,17)$ and $E_{CD}(7,17)$.

$$E(7,17) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot]$$
$$A_{0,2}(E(7,17)) = [\times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot]$$

$$E_{CD}(7,17) = [\times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot]$$
$$A_{0,2}(E_{CD}(7,17)) = [\times \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot \cdot]$$

Clearly, the two alternations are not rotations of each other.

Euclidean rhythms fulfill very precise constraints on the durations between onsets; the lengths of these durations are changed after an alternation operation. In general, an arbitrary alternation might destroy the Euclidean property of a rhythm. In this section we will determine the conditions under which the alternation of a Euclidean rhythm remains Euclidean.

Note that when $c$ does not divide $k$, we can obtain two alternations having different numbers of onsets by varying the value of $j$. Set $r = k \bmod c$; there are $r$ alternations with $\lceil k/c \rceil$ onsets (these are $A_{0,c} \ldots, A_{r-1,c}$), and $c-r$ alternations with $\lfloor k/c \rfloor$ onsets (the remaining $A_{r,c} \ldots, A_{c-1,c}$). Indeed, we can write:

$$\begin{aligned} 1 + \left\lfloor \frac{k-1-j}{c} \right\rfloor = 1 + \left\lfloor \frac{k-1-j+r-r}{c} \right\rfloor &= 1 + \left\lfloor \frac{k-r}{c} + \frac{r-1-j}{c} \right\rfloor \\ &= 1 + \left\lfloor \frac{k}{c} \right\rfloor + \left\lfloor \frac{r-1-j}{c} \right\rfloor . \end{aligned}$$

If $j \leq r-1$, then $\lfloor \frac{r-1-j}{c} \rfloor$ is 0, and thus $1 + \lfloor \frac{k-1-j}{c} \rfloor = \lceil \frac{k}{c} \rceil$. When $r - 1 < j \leq c - 1$, the quotient $\frac{r-1-j}{c}$ is negative, and $\lfloor \frac{r-1-j}{c} \rfloor = -1$, which implies that $1 + \lfloor \frac{k-1-j}{c} \rfloor = \lfloor \frac{k}{c} \rfloor$.

Consider the rhythm $E(8,17)$ and its 3-alternations:

$$
\begin{aligned}
E(8,17) &= [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] \\
A_{0,3}(E(8,17)) &= [\times \cdot \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot] \\
A_{1,3}(E(8,17)) &= [\cdot \cdot \times \cdot \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \cdot \times \cdot \cdot] \\
A_{2,3}(E(8,17)) &= [\cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \cdot \cdot]
\end{aligned}
$$

Alternations $A_{0,3}(E(8,17))$ and $A_{1,3}(E(8,17))$ have $\lceil 8/3 \rceil = 3$ onsets, whereas alternation $A_{2,3}(E(8,17))$ has $\lfloor 8/3 \rfloor = 2$ onsets. Here $r = 2 = 8 \bmod 3$, and therefore there are 2 alternations with 3 onsets and one alternation with 2 onsets. Note that $A_{2,3}(E(8,17))$ is not Euclidean.

In what follows we make use of another characterization of Euclidean rhythms that relates to the clockwise distances between onsets. Corollary 4.2.8 from Chapter 4 tells us that a rhythm is Euclidean (up to rotation) if and only if the clockwise distance of the ordered pair of onsets $(r_i, r_{i+\ell})$ is equal to either $\lfloor \frac{\ell n}{k} \rfloor$ or $\lceil \frac{\ell n}{k} \rceil$ for all $\ell = 1, 2, \ldots, k$ and $i = 0, 1, \ldots, k-1$. In particular, when $\ell = 1$ this characterization states that the clockwise distances between consecutive onsets is either $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$. We can now show the following.

**Theorem 6.4.1.** *The $j$-alternation of order $c$ of any rotation of $E(ck,n)$ is Euclidean up to rotation for $c > 1$.*

*Proof.* Let $E^*(ck,n)$ be a rotation of $E(ck,n)$. Denote by $r_i^{'}$ the onsets of $E^*(ck,n)$. Since $E^*(ck,n)$ is a rotation of $E(ck,n)$, there exist two integers $j \geq 0$ and $m$ such that the formula $m + \left\lfloor \frac{n(i+j)}{ck} \right\rfloor$ generates the onsets of $E^*(ck,n)$. In addition, denote by $r_i$ the onsets of $E(ck,n)$. Finally, let us compute the clockwise distances for the alternation. Let $\ell$ be a positive integer.

$$\begin{aligned} r_{i+\ell}^{'} - r_i^{'} &= m + \left\lfloor \frac{nj}{ck} + \frac{nc(i+\ell)}{ck} \right\rfloor - \left( m + \left\lfloor \frac{nj}{ck} + \frac{nci}{ck} \right\rfloor \right) \\ &= r_{j+c(i+\ell)} - r_{j+ci} \in \left\{ \left\lfloor \frac{nc\ell}{ck} \right\rfloor, \left\lceil \frac{nc\ell}{ck} \right\rceil \right\} = \left\{ \left\lfloor \frac{n\ell}{k} \right\rfloor, \left\lceil \frac{n\ell}{k} \right\rceil \right\} \end{aligned}$$

By Corollary 4.2.8, the last equality implies that the alternations of $E^*(ck,n)$ are rotations of $E(k,n)$. □

We now prove a result on the alternations of $E_{CD}(k,n)$. We first show the following lemma.

**Lemma 6.4.2.** *The alternations of order 2 of $E_{CD}(k,n)$ are Euclidean up to rotation if and only if $k$ is even.*

*Proof.* First suppose $k$ is even, and consider the two alternations $A_{0,2}(E_{CD}(k,n))$ and $A_{1,2}(E_{CD}(k,n))$. By Theorem 6.4.1, both of these alternations are Euclidean.

Now assume for the sake of contradiction that $k$ is odd. By Corollary 4.2.8 the clockwise distance sequence of $E(k,n)$ is formed by only two distances $a$ and $b$, where $b = a + 1$ and $a \geq 1$. Moreover, we have the constraint that the consecutive clockwise sequence of a Euclidean rhythm cannot have distances that differ by more than 1. The proof is hence divided into four cases:

1. Rhythm $E(k,n)$ has the form $(\underbrace{b,b,\ldots,b}_{k-1},c)$, where $c$ is either $a$ or $b$. When we keep the odd onsets, we obtain $(\underbrace{2b,2b,\ldots,2b}_{(k-1)/2},c)$, which contains distances that differ by more than 1 (there is a distance equal to $2b-c>1$). Thus, this alternation cannot be a Euclidean rhythm.
2. Rhythm $E(k,n)$ has the form $(b,a,a,\ldots,a)$ (starts with distance $b$ followed by $a$'s). When we keep the odd onsets, we obtain $(\underbrace{b+a,2a,\ldots,2a}_{(k-1)/2},a)$, which is not Euclidean (there is a distance equal to $a+b-a>1$).
3. Rhythm $E(k,n)$ has the form $(b,\ldots,a,\ldots,b)$ (starts and ends with $b$, having at least one $a$ in between). If we take the odd onsets, the last distance wraps around and results in a distance of $2b+a$ or $3b$ in the alternation. On the other hand, the distance $a$ in $E(k,n)$ is transformed into either a distance $2a$ or $a+b$ in the alternation. Hence, there is a pair of distances in the alternation that differs by more than 1, and hence the alternation cannot be Euclidean.
4. Rhythm $E(k,n)$ is of the form $(\underbrace{a,a,\ldots,a}_{k-1},b\}$. If we take the even onsets, we obtain a rotation of rhythm $(\underbrace{2a,\ldots,2a}_{(k-1)/2},2a+b)$, which is not Euclidean (there is a distance equal to $2a+b-2a>1$).

□

Note that the proof of Theorem 6.4.2 is an argument on the clockwise distance sequence of $E_{CD}(k,n)$ and it can be applied to $E(k,n)$ or any of its rotations in a straightforward manner. Consequently, we state the following corollary.

**Corollary 6.4.3.** *The alternations of order 2 of any rotation of $E(k,n)$ are Euclidean up to rotation if and only if $k$ is even.*

**Theorem 6.4.4.** *The alternations of order $c$ of $E_{CD}(k,n)$ are Euclidean up to rotation if and only if $c$ divides $k$ (where $1 \leq k < n$).*

*Proof.* When $c$ divides $k$, the fact that the alternation $A_{j,c}(E_{CD}(k,n))$ is Euclidean for all $j < c$ follows directly from Theorem 6.4.1.

Assume now for the sake of contradiction that $c$ does not divide $k$ and $A_{j,c}(k,n)$ is Euclidean for all $j \leq c$. Let $r = k \bmod c$. Because $c$ does not divide $k$, $r \neq 0$. In this case there are $r$ alternations having $\lceil k/c \rceil$ onsets and $c-r$ alternations having $\lfloor k/c \rfloor$ onsets. Consider two of these alternations, $A_{0,c}$ and $A_{r,c}$. Since $A_{0,c}$ is Euclidean by hypothesis, by Corollary 4.2.8 its consecutive clockwise distances can only belong to the set $\{\lfloor \frac{cn}{k} \rfloor, \lceil \frac{cn}{k} \rceil, \lfloor \frac{rn}{k} \rfloor, \lceil \frac{rn}{k} \rceil\}$. This is because $A_{0,c}$ is formed by changing $c-1$ onsets into rests from each consecutive block of $c$ onsets, followed by a final block where only $r-1$ onsets are changed into rests. The blocks of $c$ onsets generate distances $\{\lfloor \frac{cn}{k} \rfloor, \lceil \frac{cn}{k} \rceil\}$, while the block of $r$ onsets generates distances $\{\lfloor \frac{rn}{k} \rfloor, \lceil \frac{rn}{k} \rceil\}$. Consecutive clockwise distances of a Euclidean rhythm can only take two distinct values (Corollary 4.2.8), and therefore at least one of the following equalities must hold:

$$\left\lfloor \frac{cn}{k} \right\rfloor = \left\lfloor \frac{rn}{k} \right\rfloor, \left\lfloor \frac{cn}{k} \right\rfloor = \left\lceil \frac{rn}{k} \right\rceil, \left\lceil \frac{cn}{k} \right\rceil = \left\lfloor \frac{rn}{k} \right\rfloor, \text{or} \left\lceil \frac{cn}{k} \right\rceil = \left\lceil \frac{rn}{k} \right\rceil.$$

Since $r < c$, the first three equalities $\lfloor \frac{cn}{k} \rfloor = \lfloor \frac{rn}{k} \rfloor$, $\lceil \frac{cn}{k} \rceil = \lceil \frac{rn}{k} \rceil$ and $\lceil \frac{cn}{k} \rceil = \lfloor \frac{rn}{k} \rfloor$ lead to contradictions. Consider the equality $\lfloor \frac{cn}{k} \rfloor = \lceil \frac{rn}{k} \rceil$:

$$\left\lfloor \frac{cn}{k} \right\rfloor = \left\lceil \frac{rn}{k} \right\rceil \Rightarrow \left\lfloor (c-r+r)\frac{n}{k} \right\rfloor = \left\lfloor \frac{rn}{k} \right\rfloor + 1$$

$$\begin{aligned}
&\Rightarrow \left\lfloor (c-r)\frac{n}{k} + r\frac{n}{k} \right\rfloor = \left\lfloor \frac{rn}{k} \right\rfloor + 1 \\
&\Rightarrow \left\lfloor (c-r)\frac{n}{k} \right\rfloor + \left\lfloor \frac{rn}{k} \right\rfloor + \alpha = \left\lfloor \frac{rn}{k} \right\rfloor + 1 \\
&\Rightarrow \left\lfloor (c-r)\frac{n}{k} \right\rfloor + \alpha = 1,
\end{aligned}$$

where $\alpha = \lfloor \frac{(c-r) \bmod k + r \bmod k}{k} \rfloor$. If $\alpha = 1$, we obtain $\left\lfloor \frac{(c-r)n}{k} \right\rfloor = 0$, which lead to a contradiction. So, suppose $\alpha = 0$. Then,

$$\lfloor (c-r)\tfrac{n}{k} \rfloor = 1 \Longrightarrow 1 \leq (c-r)\tfrac{n}{k} < 2.$$

For this inequality to be true, $n < 2k$ and $r = c - 1$ must hold. This is the only case that, for the moment, does not cause a contradiction.

When considering $A_{r,c}$, the possible clockwise distances are $\{\lfloor \frac{cn}{k} \rfloor, \lceil \frac{cn}{k} \rceil, \lfloor \frac{(c+r)n}{k} \rfloor, \lceil \frac{(c+r)n}{k} \rceil\}$. The distances $\{\lfloor \frac{cn}{k} \rfloor, \lceil \frac{cn}{k} \rceil\}$ come from blocks of $c$ onsets of $A_{r,c}$. The last block of $A_{r,c}$ wraps around and produces two blocks: one with $c$ onsets located at the end of the rhythm, and another with $r$ onsets located at the beginning. The distances for these blocks are $\{\lfloor \frac{(c+r)n}{k} \rfloor, \lceil \frac{(c+r)n}{k} \rceil\}$. Again, at least one of the following equalities must hold:

$$\left\lfloor \frac{cn}{k} \right\rfloor = \left\lfloor \frac{(c+r)n}{k} \right\rfloor, \left\lfloor \frac{cn}{k} \right\rfloor = \left\lceil \frac{(c+r)n}{k} \right\rceil, \left\lceil \frac{cn}{k} \right\rceil = \left\lceil \frac{(c+r)n}{k} \right\rceil, \text{or} \left\lceil \frac{cn}{k} \right\rceil = \left\lfloor \frac{(c+r)n}{k} \right\rfloor.$$

Since $c < c + r$, equalities $\lfloor \frac{cn}{k} \rfloor = \left\lfloor \frac{(c+r)n}{k} \right\rfloor$, $\lceil \frac{cn}{k} \rceil = \left\lceil \frac{(c+r)n}{k} \right\rceil$ and $\lfloor \frac{cn}{k} \rfloor = \left\lceil \frac{(c+r)n}{k} \right\rceil$ lead to a contradiction.

Finally, consider equality $\lceil \frac{cn}{k} \rceil = \left\lfloor \frac{(c+r)n}{k} \right\rfloor$.

$$\begin{aligned}
\left\lceil \frac{cn}{k} \right\rceil = \left\lfloor \frac{(c+r)n}{k} \right\rfloor \quad & \Rightarrow \quad \left\lfloor \frac{cn}{k} \right\rfloor + 1 = \left\lfloor c\frac{n}{k} + r\frac{n}{k} \right\rfloor \\
& \Rightarrow \quad \left\lfloor \frac{cn}{k} \right\rfloor + 1 = \left\lfloor c\frac{n}{k} \right\rfloor + \left\lfloor r\frac{n}{k} \right\rfloor + \alpha \\
& \Rightarrow \quad \left\lfloor r\frac{n}{k} \right\rfloor + \alpha = 1,
\end{aligned}$$

where $\alpha = \lfloor \frac{cn \bmod k + cr \bmod k}{k} \rfloor$; hence $\alpha$ can only take the values 1 or 0. If $\alpha = 1$, then $\lfloor r\frac{n}{k} \rfloor = 0$, which leads to a contradiction. Suppose $\alpha = 0$. In this case $\lfloor r\frac{n}{k} \rfloor = 1$. This equality is true when $r = 1$ and $n < 2k$. Equality $r = 1$ contradicts equality $r = c - 1$ derived in the previous case, except for the value of $c = 2$. However, in Lemma 6.4.2 it was already proved that this case cannot occur either. This concludes the proof. □

**Theorem 6.4.5.** *The alternations of order $c$ of any rotation of $E(k,n)$ are Euclidean if and only if $c$ divides $k$.*

*Proof.* Let $E^*(k,n)$ be a rotation of $E(k,n)$. There exist two integers $j_0 \geq 0$ and $m$ such that the sequence $\{m + \left\lfloor \frac{n(j_0+i)}{k} \right\rfloor : i = 0, \ldots, k-1\}$ produces the onsets of $E^*(k,n)$. Furthermore, the onsets of the $j$-alternation of order $c$ of $E^*(k,n)$ are given by $\{m + \left\lfloor \frac{n(j_0+j+ic)}{k} \right\rfloor : i = 0, \ldots, k-1\}$. Denote by $r_i'$ the onsets of $A_{j,c}(E^*(k,n))$ and by $r_i$ those of $E(k,n)$. First, we will compute the distance between two consecutive onsets in $A_{0,c}(E^*(k,n))$ for any $i = 0, \ldots, \lfloor \frac{k}{c} \rfloor - 1$.

$$\begin{aligned}
r_{i+1}' - r_i' &= m + \left\lfloor \frac{n(j_0 + (i+1)c)}{k} \right\rfloor - \left( m + \left\lfloor \frac{n(j_0 + ic)}{k} \right\rfloor \right) \\
&= r_{j_0+ci+c} - r_{j_0+ci} \in \left\{ \left\lfloor \frac{nc}{k} \right\rfloor, \left\lceil \frac{nc}{k} \right\rceil \right\}
\end{aligned}$$

For the case $i = \lfloor \frac{k}{c} \rfloor$, that is, for the distance between the last onset and the first one, we have:

$$\begin{aligned} r'_{i+1} - r'_i &= m + \left\lfloor \frac{n(j_0 + k)}{k} \right\rfloor - \left( m + \left\lfloor \frac{n(j_0 + \lfloor \frac{k}{c} \rfloor c)}{k} \right\rfloor \right) \\ &= m + \left\lfloor \frac{n(j_0 + k}{k} \right\rfloor - \left( m + \left\lfloor \frac{n(j_0 + k - r)}{k} \right\rfloor \right) \\ &= r_{j_0+k} - r_{j_0+k-r} \in \left\{ \left\lfloor \frac{nr}{k} \right\rfloor, \left\lceil \frac{nr}{k} \right\rceil \right\} \end{aligned}$$

We now look at the alternation $A_{r,c}(E^*(k,n))$. For $i = 0, \ldots, \lfloor \frac{k}{c} \rfloor - 1$, the consecutive distances are:

$$\begin{aligned} r'_{i+1} - r'_i &= m + \left\lfloor \frac{n(j_0 + r + (i+1)c)}{k} \right\rfloor - \left( m + \left\lfloor \frac{n(j_0 + r + ic)}{k} \right\rfloor \right) \\ &= r_{j_0+r+ci+c} - r_{j_0+r+ci} \in \left\{ \left\lfloor \frac{nc}{k} \right\rfloor, \left\lceil \frac{nc}{k} \right\rceil \right\} \end{aligned}$$

For $i = \lfloor \frac{k}{c} \rfloor - 1$, we have:

$$\begin{aligned} r'_{i+1} - r'_i &= m + \left\lfloor \frac{n(j_0 + r + k)}{k} \right\rfloor - \left( m + \left\lfloor \frac{n(j_0 + r + (\lfloor \frac{k}{c} \rfloor - 1)c)}{k} \right\rfloor \right) \\ &= \left\lfloor \frac{n(j_0 + r + k)}{k} \right\rfloor - \left\lfloor \frac{n(j_0 + r + k - r - c)}{k} \right\rfloor \\ &= r_{j_0+r+k} - r_{j_0+k-c} \in \left\{ \left\lfloor \frac{n(c+r)}{k} \right\rfloor, \left\lceil \frac{n(c+r)}{k} \right\rceil \right\} \end{aligned}$$

For both alternations $A_{0,c}(E^*(k,n))$ and $A_{r,c}(E^*(k,n))$ we obtained the same set of consecutive distances: $\{\lfloor \frac{cn}{k} \rfloor, \lceil \frac{cn}{k} \rceil, \lfloor \frac{rn}{k} \rfloor, \lceil \frac{rn}{k} \rceil\}$ for $A_{0,c}(E^*(k,n))$ and $\{\lfloor \frac{cn}{k} \rfloor, \lceil \frac{cn}{k} \rceil, \lfloor \frac{(c+r)n}{k} \rfloor, \lceil \frac{(c+r)n}{k} \rceil\}$ for $A_{r,c}(E^*(k,n))$. Using similar arguments as in the proof of Theorem 6.4.4, we can complete the proof of our theorem. □

## 6.5 Decomposition

Let $R_1$ and $R_2$ be two rhythms having the same number of pulses and represented as binary strings. The *union* of $R_1$ and $R_2$ is the rhythm $R$ obtained by performing the logical *OR* operation between the $i$th bit of $R_1$ and the $i$th bit of $R_2$ for all $i = 0, 2, \ldots, n-1$. For example, the union of the two rhythms [× · × · ×× · × · × · ×] and [× · × · × · · × · × · ·] is the rhythm [× · × · ×× · × · × · ×]. We say that $R$ can be *decomposed* into the two rhythms $R_1$ and $R_2$. A union of two rhythms $R_1$ and $R_2$ is *disjoint* if the bits in position $i$ in $R_1$ and $R_2$ are not both 1 for any $0 \leq i \leq n-1$. We can think of the alternation operation as a form of rhythm decomposition. The following result follows from Theorem 6.4.1 from the previous section:

**Theorem 6.5.1.** *$E(k, n)$ can be decomposed into the disjoint union of $d$ copies of $E(\frac{k}{d}, n)$ up to rotation, where $d$ is a divisor of $k$.*

The following two results concern the decomposition of a rhythm in terms of its complement.

**Theorem 6.5.2.** *A Euclidean rhythm $E(k, n)$ with $\lfloor n/2 \rfloor < k \leq n$ can be decomposed into the union of $\lfloor \frac{n}{n-k} \rfloor$ copies of the rotations of the rhythm $E(n-k, n)$. Such a union is disjoint if and only if $n-k$ divides $n$.*

*Proof.* Consider the complement $E^*(n-k, n)$ of the rhythm $E(k, n)$. We want to find a set of rhythms whose union has a 1-bit at every 0-bit position of $E^*(n-k, n)$. Let $E_i^*(n-k, n)$ denote the rhythm $E^*(n-k, n)$ rotated $i$ steps in the clockwise direction, and consider the union $R$ of the rhythms $E_1^*(n-k, n)$, $E_2^*(n-k, n)$, ...,

$E^*_x(n-k,n)$ where $x = \lfloor \frac{n}{n-k} \rfloor - 1$. Rhythm $R$ has at most $k$ onsets:

$$(n-k)\left(\left\lfloor \frac{n}{n-k} \right\rfloor - 1\right) = n - r - (n-k) = k - r \leq k,$$

where $r = n \bmod (n-k)$. The number of consecutive onsets between pairs of 0-bits is exactly $\lfloor \frac{n}{n-k} \rfloor - 1$. By Corollary 4.2.8, the distance between two consecutive onsets in $E(n-k,n)$ is either $\lfloor \frac{n}{n-k} \rfloor$ or $\lceil \frac{n}{n-k} \rceil$. Hence, $R$ has every 1-bit coinciding with a 0-bit position in $E^*(n-k,n)$. If $n-k$ divides $n$, then the distance between any two consecutive onsets in $E^*(n-k,n)$ is exactly $\frac{n}{n-k}$, and $R$ has exactly $k$ onsets; it follows that $R = E(k,n)$ and is the union of $\lfloor \frac{n}{n-k} \rfloor$ rhythms. If $n-k$ does not divide $n$, then the distance between any two consecutive onsets in $E^*(n-k,n)$ is either $\lfloor \frac{n}{n-k} \rfloor$ or $\lceil \frac{n}{n-k} \rceil$, and $R$ has fewer than $k$ onsets (and hence is not equal to any rotation of $E(k,n)$). The onsets of $E(k,n)$ that are missing in $R$ are those between two consecutive onsets of $E^*(n-k,n)$ that are at distance $\lceil \frac{n}{n-k} \rceil$. Let $E_{-1}(n-k,n)$ be the rotation of rhythm $E^*(n-k,n)$ by one bit in the counter-clockwise direction. If we take the union of $R$ and $E_{-1}(n-k,n)$, this will add the missing onsets, and the rhythm resulting from this union will be the rhythm $E(k,n)$. Note that $R \cup E_{-1}(n-k,n)$ is not necessarily disjoint, and the total number of rhythms in this union is equal to $\lfloor \frac{n}{n-k} \rfloor - 1 + 1 = \lfloor \frac{n}{n-k} \rfloor$. □

Our final result on decomposition of rhythms is the following:

**Theorem 6.5.3.** *A Euclidean rhythm $E(k,n)$ with $\lfloor n/2 \rfloor < k < n$ is the union of rotations of two disjoint Euclidean rhythms $E(n-k,n)$ and $E(2k-n,n)$.*

*Proof.* Consider the rhythm $E(2(n-k),n)$. Applying Theorem 6.4.4 to $E(2(n-k),n)$, we can conclude that $E(2(n-k),n)$ is the disjoint union of rotations of

$A_{0,2}(n-k,n)$ and $A_{1,2}(n-k,n)$. Moreover, by Theorem 6.4.1, $E(2(n-k),n)$ is the union of two copies of rotations of $E(n-k,n)$. On the other hand, $E(2(n-k),n)$ is the complement of $E(2k-n,n)$ up to rotation, which implies that the two rhythms are disjoint. Therefore, by taking the complement of one of the alternations we obtain a decomposition of $E(k,n)$ into the disjoint union of two rotations of $E(n-k,n)$ and $E(2k-n,n)$. □

## 6.6 Musical Connections

In this section we interpret the mathematical results of this chapter in musical terms. As mentioned in the introduction, we are interested in studying interlocking and Euclidean rhythms; here we combine both ideas and study interlocking rhythms that are Euclidean.

First, we will define three types of interlocking rhythms with the same timespan. *Complementary interlocking rhythms* are a set of rhythms that have no common onset and exactly one onset on any pulse of the timespan; *Disjoint interlocking rhythms* are interlocking rhythms that have no onset at a common position; and *non-disjoint interlocking rhythms* are a family of rhythms such that every rhythm in the family has no more than half of its onsets in common with other rhythms in the family.

We will first consider complementary interlocking rhythms formed by Euclidean rhythms. We already know that the complement of a Euclidean rhythm is Euclidean up to rotation. Therefore, it is always possible to build pairs of complementary interlocking Euclidean rhythms. Next, as its most straightforward generalization, we may ask when a Euclidean rhythm can generate a tiling canon. According to Hall and Klingsberg [91], a tiling canon is a canon of periodic rhythms that has exactly one-note onset (in some voice) per pulse. If $E(k,n)$ is a tiling canon, $k$ necessarily divides $n$, and $c = \frac{n}{k}$ is the number of voices (or the number of times $E(k,n)$ is played). By applying our results on alternations (Theorems 6.4.1 and 6.5.1) to rhythm $E(n,n)$, we conclude that alternations $A_{1,c}(k,n), \ldots, A_{c,c}(k,n)$ are all clockwise rotations of $E(k,n)$ and that they tile a timespan of $n$ pulses. It is worth noting that the resulting tiling canon is rather uninteresting musically. Because $k$ divides $n$, rhythm $E(k,n)$ consists solely of an onset played regularly every $\frac{n}{k}$ pulses. The tiling canon is then composed of consecutive clockwise rotations of $E(k,n)$. For instance, consider a timespan of 12 pulses and a tiling canon of 4 onsets. The number of voices is $3 = 12/4$ and the tiling canon is:

$$A_{0,3} = [\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot]$$
$$A_{1,3} = [\cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot]$$
$$A_{2,3} = [\cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times].$$

When the number of voices $c$ does not divide $n$, tiling canons cannot be built. Let us generalize the concept of tiling canons so that this case can be subsumed. We define a *tiling quasi-canon* as a set of $c$ periodic rhythms that tiles a timespan of $n$ pulses and whose number of onsets for each pair of rhythms differs at most by one. Note that this definition includes that of usual tiling canons. Euclidean rhythms, even when $c$ does not divide $n$, admit tiling quasi-canons. Indeed, the equality $n = c\lfloor n/c \rfloor + r$, where $r = n \bmod c$, can be rearranged as $n = (c-r)\lfloor n/c \rfloor + r\lceil n/c \rceil$. This equality implies that a timespan of $n$ pulses can be tiled with $c - r$ rotations of $E(\lfloor n/c \rfloor, n)$ and $r$ rotations of $E(\lceil n/c \rceil, n)$. It is enough to apply Theorem 6.4.1 to rhythm $E((c-r)\lfloor n/c \rfloor, n)$ and its complement $E(r\lceil n/c \rceil, n)$. For example, consider the problem of finding a tiling quasi-canon with 4 voices on a timespan of 19 pulses. Given that $r = 19 \bmod 4 = 3$, the tiling quasi-canon is formed by $c - r = 1$ rhythm of $\lfloor 19/4 \rfloor = 4$ onsets and $r = 3$ rhythms of $\lceil 19/4 \rceil = 5$ onsets, as shown below.

$$
\begin{aligned}
E(4,19) &= [\times\cdot\cdot\cdot\cdot\times\cdot\cdot\cdot\cdot\times\cdot\cdot\cdot\cdot\times\cdot\cdot\cdot] \\
E(5,19) &= [\cdot\times\cdot\cdot\times\cdot\cdot\cdot\times\cdot\cdot\cdot\times\cdot\cdot\cdot\times\cdot\cdot] \\
E(5,19) &= [\cdot\cdot\times\cdot\cdot\cdot\times\cdot\cdot\times\cdot\cdot\cdot\times\cdot\cdot\cdot\times\cdot] \\
E(5,19) &= [\cdot\cdot\cdot\times\cdot\cdot\cdot\times\cdot\cdot\cdot\times\cdot\cdot\times\cdot\cdot\cdot\times]
\end{aligned}
$$

Unlike tiling canons, rotations of the tiling quasi-canon are not consecutive.

If we relax the complementarity constraint, we may consider disjoint interlocking rhythms. When $k < \lfloor n/2 \rfloor$ the decomposition of $E(k,n)$ is not interesting. Often such a decomposition does not exist or is trivial (it is the union of rotations of $E(1,n)$). For example, $E(4,11) = [\times\cdot\cdot\times\cdot\cdot\times\cdot\cdot\times\cdot]$ cannot be decomposed into Euclidean rhythms with a smaller number of onsets, except for the trivial union of four copies of $E(1,11)$. Moreover, the interesting case is the decomposition of a dense rhythm in terms of sparser rhythms. Theorem 6.5.2 shows that, when $k > \lfloor n/2 \rfloor$, $E(k,n)$ can be decomposed into the union of two disjoint interlocking rhythms, namely, rotations of $E(n-k,n)$ and $E(2k-n,n)$. Below is an example of such a decomposition for rhythm $E(9,11)$.

$$
\begin{aligned}
E(9,11) &= [\times\cdot\times\times\times\times\cdot\times\times\times\times] \\
E(7,11) &= [\times\cdot\times\times\cdot\times\cdot\times\times\cdot\times] \\
E(2,11) &= [\cdot\cdot\cdot\cdot\times\cdot\cdot\cdot\cdot\times\cdot] \\
\overline{E(9,11)} &= [\cdot\times\cdot\cdot\cdot\cdot\times\cdot\cdot\cdot\cdot]
\end{aligned}
$$

Interestingly enough, rhythm $E(9,11)$ can also be decomposed into the union of rotations of rhythms $E(3,11)$ because 3 divides 9.

Note that this result also provides a new way for finding complementary interlocking rhythms. Since the complement of a Euclidean rhythm is Euclidean, for $k > \lfloor n/2 \rfloor$ there exists a decomposition of the timespan as the union of $E(2k-n,n)$ and two copies of $E(n-k,n)$, up to rotation.

Finally, we analyze the union of Euclidean rhythms when this union is not required to be disjoint. Recall from the introduction that interlocking rhythms are defined as having no more than half of their onsets in common. Theorem 6.5.2 ensures that, when $k < \lfloor n/2 \rfloor$, there is a decomposition of $E(k,n)$ into not necessarily disjoint rhythms. If $n-k$ divides $n$, such a decomposition is disjoint; otherwise, the number of onsets in common is $n \bmod (n-k)$, a number that is strictly less than $n-k < n/2$. For example, consider $E(9,11)$.

$$
\begin{aligned}
E(9,11) &= [\times \cdot \times\times\times\times \cdot \times\times\times\times] \\
E(2,11) &= [\cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot] \\
E(2,11) &= [\cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot \cdot] \\
E(2,11) &= [\cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot] \\
E(2,11) &= [\cdot \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times] \\
E(2,11) &= [\times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \cdot]
\end{aligned}
$$

As a tangible musical example, consider the bembé, a style from the Afro-Cuban musical tradition. Its instrumentation includes at least one bell that plays a timeline also called bembé $[\times \cdot \times \cdot \times\times \cdot \times \cdot \times \cdot \times]$, which is a rotation of the Euclidean rhythm $E(7,12)$. Sometimes in the bembé ensemble there are two bell players, one playing the bembé itself, also known as the standard pattern, and the other playing one of its rotations [40, 171, 179].

This concludes our analysis of interlocking rhythms. There are however some interesting outstanding questions. For example, there is no characterization for the decomposition of a Euclidean rhythm into other Euclidean rhythms (up to rotation). It would also be interesting to find further conditions for the concatenation of Euclidean rhythms to be Euclidean. Equally interesting would be finding other ways of decomposing Euclidean rhythms apart from those given by alternations.

# Chapter 7

# Necklace Swaps, Convolutions, and $X + Y$

In the previous chapter, we were interested in transforming one rhythm into another while retaining some rhythmic property – in our case maximal evenness – without worrying about how different or similar the initial and final rhythms sound. We now turn our attention to the question of how similar two rhythms with the same number of pulses and onsets are. There is no single answer to this question. A variety of methods for measuring similarity of rhythms exist, many of which have been thoroughly studied in the music literature. A measure that quantifies the distance between different rhythms is essential to any algorithm that compares, queries, or recognizes rhythms. Similarity measures find applications in methods for retrieving music from large databases using techniques such as *query by humming* [83], or for finding music copyright infringements [55]. Thus, good measures of similarity are useful and important tools for music analysis.

Toussaint describes a variety of distance measures for rhythm comparison [170, 174]. The *hamming distance* is one measure that compares two strings of characters and is widely used in coding theory. It essentially measures the number of places where two strings do not match. One drawback of the hamming distance is that it does not measure how far the mismatching characters are from each other. In music, two rhythms where the positions of mismatching onsets are far apart sound much different than rhythms where mismatching onsets are close. Thus, for rhythm similarity analysis it is preferable to use a measure that expresses the extent of the mismatch.

A measure that reflects the distance between mismatches better is the *edit distance*, which in a way is a generalization of the hamming distance. The edit distance allows insertions and deletions of characters. Applications of the edit distance to measuring music similarity can be found in [123, 129]. One drawback of the edit distance is its costly running time. While the hamming distance can trivially be computed in linear time, the edit distance has a quadratic-time algorithm using dynamic programming. A different generalization of the hamming distance is the *fuzzy hamming distance* that allows shifting of characters in addition to insertions and deletion [24, 25], and can be computed in linear time [100]. A restricted version of the fuzzy hamming distance is the *swap distance* [170, 171] introduced by Toussaint, that can similarly be computed in linear time [100, 174]. The swap distance between two strings is the minimum number of swaps (interchange of two adjacent characters) required to convert one string to another. For example, one can convert the Tambù rhythm $[\times \cdot \times \cdot \times \cdot \times\times \cdot \times \cdot \times]$ to the Yoruba rhythm $[\times \cdot \times \cdot \times\times \cdot \times \cdot \times\times \cdot]$ by swapping the fourth and seventh onsets of the first with their left neighbor; this makes the swap distance between these two rhythms equal to 2. In comparing some rhythmic similarity measures, Toussaint [172] shows that the swap distance performs better than other measures based on how well rhythms may be recognized with them, how well they model human perception and cognition, and how efficiently they can be computed. The distance measures that Toussaint considers are the hamming distance, the Euclidean interval vector distance, the interval-difference vector distance, the chronotonic distance, and the swap distance.

The similarity measures discussed so far are useful for comparing linear strings. In a musical context however, rhythms are cyclic binary strings, and in some applications (such as music retrieval and rhythm phylogeny [63, 172]) we are interested in finding the best alignment of two cyclic binary strings over all possible rotations. In other words, we want to minimize the distance between two rhythms over all possible rotations of one with respect to the other, for some definition of *distance*. Toussaint calls this measure the *necklace swap distance* [174] since it is the swap distance between two rhythmic necklaces. The necklace swap distance between two cyclic strings can be computed trivially in $O(n^2)$ time by using a linear-time algorithm for computing the swap distance of two strings for every possible alignment of the necklaces. Ardila et al. [5] show that this distance may be computed in $O(k^2)$ time, where $k$ is the number of onsets in each rhythm. Toussaint highlighted as an interesting open question whether the necklace swap distance can be computed in $o(n^2)$ time. In this chapter, we answer this question by giving $o(n^2)$-time algorithms for finding the best alignment of two continuous necklaces using each of the $\ell_1$, $\ell_2$, and $\ell_\infty$ norms as distance measures. The $\ell_1$ norm is equivalent to solving the necklace swap distance problem, and thus solves the open question posed by Toussaint. We also show a surprising connection to convolutions, for which we also obtain improved running times. Thus, in the remainder of this chapter we will temporarily step out of the musical setting and talk about the more general *necklace alignment problem*,

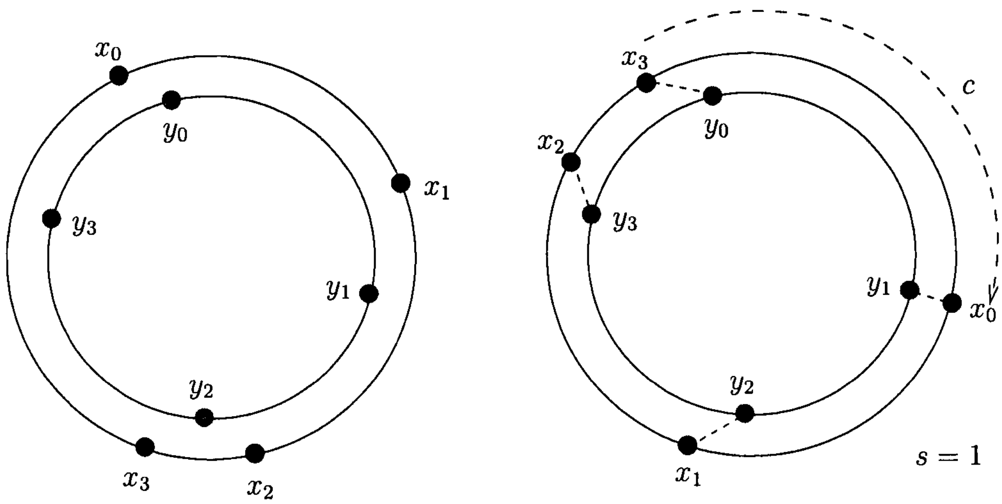



Figure 7–1: An example of necklace alignment: the input (left) and one possible output (right).

one variation of which (the $\ell_1$ norm) computes the necklace-swap distance in $o(n^2)$ time.

## 7.1 The Necklace Alignment Problem

In the necklace alignment problem we are given two continuous necklaces, each represented by a set of $n$ points on the unit-circumference circle, and our goal is to find rotations of the necklaces, and a perfect matching between the beads of the two necklaces, that minimizes some norm of the circular distances between matched beads. In particular, the $\ell_1$ norm minimizes the average absolute circular distance between matched beads, the $\ell_2$ norm minimizes the average squared circular distance between matched beads, and the $\ell_\infty$ norm minimizes the maximum circular distance between matched beads.

Our approach is based on reducing the necklace alignment problem to another important problem, convolution, for which we also obtain improved algorithms. The $(+, \cdot)$ convolution of two vectors $\mathbf{x} = \langle x_0, x_1, \ldots, x_{n-1} \rangle$ and $\mathbf{y} = \langle y_0, y_1, \ldots, y_{n-1} \rangle$, is the vector $\mathbf{x} * \mathbf{y} = \langle z_0, z_1, \ldots, z_{n-1} \rangle$ where $z_k = \sum_{i=0}^{k} x_i \cdot y_{k-i}$. While any $(\oplus, \odot)$ convolution with specified addition and multiplication operators (here denoted $\mathbf{x} \overset{\odot}{\underset{\oplus}{*}} \mathbf{y}$) can be computed in $O(n^2)$ time, the $(+, \cdot)$ convolution can be computed in $O(n \log n)$ time using the Fast Fourier Transform [50, 81, 96], because the Fourier transform converts convolution into elementwise multiplication. Indeed, fast $(+, \cdot)$ convolution was one of the early breakthroughs in algorithms, with applications to polynomial and integer multiplication [19], batch polynomial evaluation [53, Problem 30-5], 3SUM [12, 68], string matching [49, 73, 99, 102], matrix multiplication [47], and even juggling [34].

As we show in Theorems 7.2.1, 7.3.2, and 7.4.2, respectively, $\ell_2$ necklace alignment reduces to standard $(+, \cdot)$ convolution, $\ell_\infty$ necklace alignment reduces to $(\min, +)$ [and $(\max, +)$] convolution, and $\ell_1$ necklace alignment reduces to $(\text{median}, +)$ convolution (whose $k$th entry is $\text{median}_{i=0}^{k} (x_i + y_{k-i})$). The $(\min, +)$ convolution problem has appeared frequently in the literature, already appearing in Bellman's early work on dynamic programming in the early 1960s [18, 72, 118, 124, 144, 164]. Its name varies among "minimum convolution", "min-sum convolution", "inf-convolution", "infimal convolution", and "epigraphical sum". To date, however, no worst-case $o(n^2)$-time algorithms for this convolution, or the more complex $(\text{median}, +)$ convolution, has been obtained. In this chapter, we develop worst-case $o(n^2)$-time algorithms for $(\min, +)$ and $(\text{median}, +)$ convolution in the real RAM model of computation. It should be noted that $o(n^2)$-time algorithms for $(\min, +)$ and $(\text{median}, +)$ convolution in the nonuniform linear decision tree model appear in [27].

More formally, in the *necklace alignment problem*, the input is a number $p$ representing the $\ell_p$ norm, and two sorted vectors of $n$ real numbers, $\mathbf{x} = \langle x_0, x_1, \ldots, x_{n-1} \rangle$ and $\mathbf{y} = \langle y_0, y_1, \ldots, y_{n-1} \rangle$, representing the two necklaces. See Figure 7–1. Canonically, we assume that each number $x_i$ and $y_i$ is in the range $[0, 1)$, representing a point on the unit-circumference circle (parameterized clockwise from some fixed point).

The optimization problem involves two parameters. The first parameter, the *offset* $c \in [0, 1)$, is the clockwise rotation angle of the first necklace relative to the second necklace. The second parameter, the *shift* $s \in \{0, 1, \ldots, n\}$, defines the perfect matching between beads: bead $i$ of the first necklace matches with bead $(i+s) \bmod n$ of the second necklace. (Here we use the property that an optimal perfect matching between the beads does not cross itself.)

The goal of the $\ell_p$ necklace alignment problem is to find the offset $c \in [0, 1)$ and the shift $s \in \{0, 1, \ldots, n\}$ that minimize

$$\sum_{i=0}^{n-1} \left| x_i - y_{(i+s) \bmod n} + c \right|^p$$

or, in the case $p = \infty$, that minimize

$$\max_{i=0}^{n-1} \left| x_i - y_{(i+s) \bmod n} + c \right| .$$

Although not obvious from the definition, the $\ell_1$, $\ell_2$, and $\ell_\infty$ necklace alignment problems all have trivial $O(n^2)$ solutions. In each case, as we show, the optimal offset $c$ can be computed in linear time for a given shift value $s$ (sometimes even independent of $s$). The optimization problem is thus effectively over just $s \in \{0, 1, \ldots, n\}$, and the objective costs $O(n)$ time to compute for each $s$, giving an $O(n^2)$-time algorithm.

**Related work.** Although necklaces are studied throughout mathematics, mainly in combinatorial settings, we are not aware of any work on the necklace alignment problem before Toussaint [174]. He introduced $\ell_1$ necklace alignment, calling it the *cyclic swap-distance* or *necklace swap-distance* problem, with a restriction that the beads lie at integer coordinates. Colannino et al. [48] consider some different distance measures between two sets of points on the real line in which the matching does not have to match every point. They do not, however, consider alignment under such distance measures.

The only subquadratic results for $(\min, +)$ convolution concern two special cases. First, the $(\min, +)$ convolution of two convex sequences or functions can be trivially computed in $O(n)$ time by a simple merge, which is the same as computing the Minkowski sum of two convex polygons [144]. This special case is already used in image processing and computer vision [72, 118]. Second, Bussieck et al. [32] proved that the $(\min, +)$ convolution of two *randomly permuted* sequences can be computed in $O(n \log n)$ expected time. Our results are the first to improve the worst-case running time for $(\min, +)$ convolution.

**Connections to $X + Y$.** Necklace alignment problems, and their corresponding convolution problems, are also intrinsically connected to problems on $X+Y$ matrices. Given two lists of $n$ numbers, $X = \langle x_0, x_1, \ldots, x_{n-1} \rangle$ and $Y = \langle y_0, y_1, \ldots, y_{n-1} \rangle$, $X + Y$ is the matrix of all pairwise sums, whose $(i, j)$th entry is $x_i + y_j$. A classic unsolved problem [62] is whether the entries of $X + Y$ can be sorted in $o(n^2 \log n)$

time. Fredman [78] showed that $O(n^2)$ comparisons suffice in the nonuniform linear decision tree model, but it remains open whether this can be converted into an $O(n^2)$-time algorithm in the real RAM model. Steiger and Streinu [161] gave a simple algorithm that takes $O(n^2 \log n)$ time while using only $O(n^2)$ comparisons.

The $(\min, +)$ convolution is equivalent to finding the minimum element in each antidiagonal of the $X + Y$ matrix, and similarly the $(\max, +)$ convolution finds the maximum element in each antidiagonal. We show that $\ell_\infty$ necklace alignment is equivalent to finding the antidiagonal of $X+Y$ with the smallest *range* (the maximum element minus the minimum element). The $(\text{median}, +)$ convolution is equivalent to finding the median element in each antidiagonal of the $X + Y$ matrix. We show that $\ell_1$ necklace alignment is equivalent to finding the antidiagonal of $X + Y$ with the smallest *median cost* (the total distance between each element and the median of the elements). Given the apparent difficulty in sorting $X + Y$, it seems natural to believe that the minimum, maximum, and median elements of every antidiagonal cannot be found, and that the corresponding objectives cannot be minimized, any faster than $O(n^2)$ total time. Figure 7–2 shows a sample $X + Y$ matrix with the maximum element in each antidiagonal marked, with no apparent structure. Nonetheless, we show that $o(n^2)$ algorithms are possible.

In this chapter, we give subquadratic algorithms in the standard real RAM model for the $\ell_1$, $\ell_2$, and $\ell_\infty$ necklace alignment problems, and for the $(\min, +)$ and $(\text{median}, +)$ convolution problems, using techniques of Chan [36]. Despite the roughly logarithmic factor improvements for $\ell_1$ and $\ell_\infty$, these results do not use word-level bit tricks.

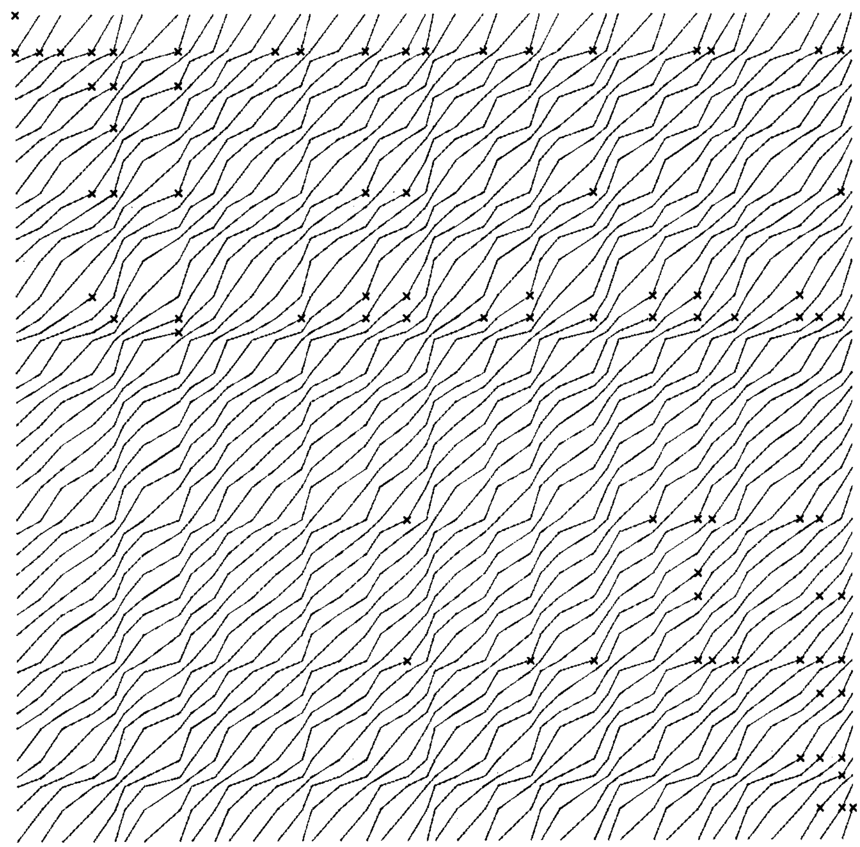


Figure 7–2: An $X + Y$ matrix. Each polygonal line denotes an antidiagonal of the matrix, with a point at coordinates $(x, y)$ denoting the value $x + y$ for $x \in X$ and $y \in Y$. An $\times$ denotes the maximum element in each antidiagonal.

1. $O(n \log n)$-time algorithm on the real RAM for $\ell_2$ necklace alignment (Section 7.2).
2. $O(n^2/\log n)$-time algorithm on the real RAM for $\ell_\infty$ necklace alignment and $(\min, +)$ convolution (Section 7.3).
3. $O(n^2(\log\log n)^2/\log n)$-time algorithm on the real RAM for $\ell_1$ necklace alignment and $(\text{median}, +)$ convolution (Section 7.4).

## 7.2 $\ell_2$ Necklace Alignment and $(+, \cdot)$ Convolution

In this section, we show how $\ell_2$ necklace alignment reduces to standard convolution, leading to an $O(n \log n)$-time algorithm.

**Theorem 7.2.1.** *The $\ell_2$ necklace alignment problem can be solved in $O(n \log n)$ time.*

*Proof.* The objective expands algebraically to

$$\begin{aligned}
&\sum_{i=0}^{n-1} \left(x_i - y_{(i+s) \bmod n} + c\right)^2 \\
&= \sum_{i=0}^{n-1} \left(x_i^2 + y_{(i+s) \bmod n}^2 + 2cx_i - 2cy_{(i+s) \bmod n} + c^2\right) - 2\sum_{i=0}^{n-1} x_i y_{(i+s) \bmod n} \\
&= \sum_{i=0}^{n-1} \left(x_i^2 + y_i^2 + 2cx_i - 2cy_i + c^2\right) - 2\sum_{i=0}^{n-1} x_i y_{(i+s) \bmod n} \\
&= \left[\sum_{i=0}^{n-1} \left(x_i^2 + y_i^2\right) + 2c\sum_{i=0}^{n-1} (x_i - y_i) + nc^2\right] - 2\sum_{i=0}^{n-1} x_i y_{(i+s) \bmod n}.
\end{aligned}$$

The first term depends solely on the inputs and the variable $c$, while the second term depends solely on the inputs and the variable $s$. Thus the two terms can be optimized separately. The first term can be optimized in $O(n)$ time by solving for when the derivative, which is linear in $c$, is zero. The second term can be computed, for each $s \in \{0, 1, \ldots, n-1\}$, in $O(n \log n)$ time using $(+, \cdot)$ convolution (and therefore optimized in the same time). Specifically, define the vectors

$$\begin{aligned}
\mathbf{x}' &= \langle x_0, x_1, \ldots, x_{n-1}; \underbrace{0, 0, \ldots, 0}_{n} \rangle, \\
\mathbf{y}' &= \langle y_{n-1}, y_{n-2}, \ldots, y_0; y_{n-1}, y_{n-2}, \ldots, y_0 \rangle.
\end{aligned}$$

Then, for $s' \in \{0, 1, \ldots, n-1\}$, the $(n+s')$th entry of the convolution $\mathbf{x}' * \mathbf{y}'$ is

$$\sum_{i=0}^{n+s'} x'_i y'_{n+s'-i} = \sum_{i=0}^{n-1} x_i y_{(i-s'-1) \bmod n},$$

which is the desired entry if we let $s' = n - 1 - s$. We can compute the entire convolution in $O(n \log n)$ time using the Fast Fourier Transform. □

## 7.3 $\ell_\infty$ Necklace Alignment and $(\min, +)$ Convolution

First we show the relation between $\ell_\infty$ necklace alignment and $(\min, +)$ convolution. We need the following basic fact:

**Fact 7.3.1.** *For any vector $\mathbf{z} = \langle z_0, z_1, \ldots, z_{n-1} \rangle$, the minimum value of $\max_{i=0}^{n-1} |z_i + c|$ is*

$$\frac{1}{2} \left( \max_{i=0}^{n-1} z_i - \min_{i=0}^{n-1} z_i \right),$$

*which is achieved when $c = -\frac{1}{2} \left( \min_{i=0}^{n-1} z_i + \max_{i=0}^{n-1} z_i \right)$.*

Instead of using $(\min, +)$ convolution directly, we use two equivalent forms, $(\min, -)$ and $(\max, -)$ convolution:

**Theorem 7.3.2.** *The $\ell_\infty$ necklace alignment problem can be reduced in $O(n)$ time to one $(\min, -)$ convolution and one $(\max, -)$ convolution.*

*Proof.* For two necklaces $\mathbf{x}$ and $\mathbf{y}$, we apply the $(\min, -)$ convolution to the following vectors:

$$\begin{aligned}
\mathbf{x}' &= \langle x_0, x_1, \ldots, x_{n-1}; \underbrace{\infty, \infty, \ldots, \infty}_{n} \rangle, \\
\mathbf{y}' &= \langle y_{n-1}, y_{n-2}, \ldots, y_0; y_{n-1}, y_{n-2}, \ldots, y_0 \rangle.
\end{aligned}$$

Then, for $s' \in \{0, 1, \ldots, n-1\}$, the $(n+s')$th entry of $\mathbf{x}' \mathbin{\bar{*}_{\min}} \mathbf{y}'$ is

$$\min_{i=0}^{n+s'}(x'_i - y'_{n+s'-i}) = \min_{i=0}^{n-1}(x_i - y_{(i-s'-1) \bmod n}),$$

which is $\min_{i=0}^{n-1}(x_i - y_{(i+s) \bmod n})$ if we let $s' = n - 1 - s$. By symmetry, we can compute the $(\max, -)$ convolution $\mathbf{x}'' \mathbin{\bar{*}_{\max}} \mathbf{y}'$, where $\mathbf{x}''$ has $-\infty$'s in place of $\infty$'s, and use it to compute $\max_{i=0}^{n-1}(x_i - y_{(i+s) \bmod n})$ for each $s \in \{0, 1, \ldots, n-1\}$. Applying Fact 7.3.1, we can therefore minimize $\max_{i=0}^{n-1} |x_i - y_{(i+s) \bmod n} + c|$ over $c$, for each $s \in \{0, 1, \ldots, n-1\}$. By brute force, we can minimize over $s$ as well using $O(n)$ additional comparisons and time. □

Our results use the following geometric lemma from Chan's work on all-pairs shortest paths:

**Lemma 7.3.3.** [36, Lemma 2.1] *Given $n$ points $p_1, p_2, \ldots, p_n$ in $d$ dimensions, each colored either red or blue, we can find the $P$ pairs $(p_i, p_j)$ for which $p_i$ is red, $p_j$ is blue, and $p_i$ dominates $p_j$ (i.e., for all $k$, the $k$th coordinate of $p_i$ is at least the $k$th coordinate of $p_j$), in $2^{O(d)}n^{1+\varepsilon} + O(P)$ time for arbitrarily small $\varepsilon > 0$.*

**Theorem 7.3.4.** *The $(\min, -)$ convolution of two vectors of length $n$ can be computed in $O(n^2/\log n)$ time.*

*Proof.* Let $\mathbf{x}$ and $\mathbf{y}$ denote the two vectors of length $n$, and let $\mathbf{x} \mathbin{\bar{*}_{\max}} \mathbf{y}$ denote their $(\max, -)$ convolution. (Symmetrically, we can compute the $(\min, -)$ convolution.) For each $\delta \in \{0, 1, \ldots, d-1\}$, for each $i \in \{0, d, 2d, \ldots, \lfloor n/d \rfloor d\}$, and for each

$j \in \{0, 1, \ldots, n-1\}$, we define the $d$-dimensional points

$$p_{\delta,i} = (x_{i+\delta} - x_i,\ x_{i+\delta} - x_{i+1},\ \ldots,\ x_{i+\delta} - x_{i+d-1}),$$
$$q_{\delta,j} = (y_{j-\delta} - y_i,\ y_{j-\delta} - y_{i-1},\ \ldots,\ y_{j-\delta} - y_{j-d-1}).$$

(To handle boundary cases, define $x_i = \infty$ and $y_j = -\infty$ for indices $i, j$ outside $[0, n-1]$.) For each $\delta \in \{0, 1, \ldots, d-1\}$, we apply Lemma 7.3.3 to the set of red points $\{p_{\delta,i} : i = 0, d, 2d, \ldots, \lfloor n/d \rfloor d\}$ and the set of blue points $\{q_{\delta,j} : j = 0, 1, \ldots, n-1\}$, to obtain all dominating pairs $(p_{\delta,i}, q_{\delta,j})$.

Point $p_{\delta,i}$ dominates $q_{\delta,j}$ precisely if $x_{i+\delta} - x_{i+\delta'} \geq y_{j-\delta} - y_{j-\delta'}$ for all $\delta' \in \{0, 1, \ldots, d-1\}$ (ignoring the indices outside $[0, n-1]$). By re-arranging terms, this condition is equivalent to $x_{i+\delta} - y_{j-\delta} \geq x_{i+\delta'} - y_{j-\delta'}$ for all $\delta' \in \{0, 1, \ldots, d-1\}$. If we substitute $j = k - i$, we obtain that $(p_{\delta,i}, q_{\delta,k-i})$ is a dominating pair precisely if $x_{i+\delta} - y_{k-i-\delta} = \max_{\delta'=1}^{d-1}(x_{i+\delta'} - y_{k-i-\delta'})$. Thus, the set of dominating pairs gives us the maximum $M_k(i) = \max\{x_i - y_{k-i}, x_{i+1} - y_{k-i+1}, \ldots, x_{\min\{i+d,n\}-1} - y_{\min\{k-i+d,n\}-1}\}$ for each $i$ divisible by $d$ and for each $k$. Also, there can be at most $O(n^2/d)$ such pairs for all $i, j, \delta$, because there are $O(n/d)$ choices for $i$ and $O(n)$ choices for $j$, and if $(p_{\delta,i}, q_{\delta,j})$ is a dominating pair, then $(p_{\delta',i}, q_{\delta',j})$ cannot be a dominating pair for any $\delta' \neq \delta$. (Here we assume that the max is achieved uniquely, which can be arranged by standard perturbation techniques or by breaking ties consistently [36].) Hence, the running time of the $d$ executions of Lemma 7.3.3 is $d2^{O(d)}n^{1+\varepsilon} + O(n^2/d)$ time, which is $O(n^2/\log n)$ if we choose $d = \alpha \log n$ for a sufficiently small constant $\alpha > 0$. We can rewrite the $k$th entry $\max_{i=0}^{k}(x_i - y_{k-i})$ of $\mathbf{x} \underset{\max}{\bar{*}} \mathbf{y}$ as $\max\{M_k(0), M_k(d), M_k(2d), \ldots, M_k(\lceil k/d \rceil d)\}$, and thus we can compute it in $O(k/d) = O(n/d)$ time. Therefore all $n$ entries can be computed in $O(n^2/d) = O(n^2/\log n)$ time. □

Combining Theorems 7.3.2 and 7.3.4, we obtain the following result:

**Corollary 7.3.5.** *The $\ell_\infty$ necklace alignment problem can be solved in $O(n^2/\log n)$ time.*

It should be noted that any improvement to this approach beyond $O(n^2/\log n)$ would probably require an improvement to Lemma 7.3.3, which would in turn improve the fastest known algorithm for all-pairs shortest paths in dense graphs, the $O(n^3/\log n)$-time algorithm of [36].

## 7.4 $\ell_1$ Necklace Alignment and (median, +) Convolution

First we show the relation between $\ell_1$ necklace alignment and (median, +) convolution. We need the following basic fact:

**Fact 7.4.1.** *For any vector $\mathbf{z} = \langle z_0, z_1, \ldots, z_{n-1}\rangle$, $\sum_{i=0}^{n-1} |z_i + c|$ is minimized when $c = -\operatorname{median}_{i=0}^{n-1} z_i$.*

Instead of using (median, +) convolution directly, we use the equivalent form, (median, −) convolution:

**Theorem 7.4.2.** *The $\ell_1$ necklace alignment problem can be reduced in $O(n)$ time to one (median, −) convolution.*

*Proof.* For two necklaces $\mathbf{x}$ and $\mathbf{y}$, we apply the (median, −) convolution to the following vectors, as in the proof of Theorem 7.3.2:

$$\begin{aligned}\mathbf{x}' &= \langle x_0, x_0, x_1, x_1, \ldots, x_{n-1}, x_{n-1}; \underbrace{\infty, -\infty, \infty, -\infty, \ldots, \infty, -\infty}_{2n}\rangle,\\ \mathbf{y}' &= \langle y_{n-1}, y_{n-1}, y_{n-2}, y_{n-2}, \ldots, y_0, y_0; y_{n-1}, y_{n-1}, y_{n-2}, y_{n-2}, \ldots, y_0, y_0\rangle.\end{aligned}$$

Then, for $s' \in \{0, 1, \ldots, n-1\}$, the $2(n+s')+1$st entry of $\mathbf{x}' \mathop{\bar{*}}_{\text{med}} \mathbf{y}'$ is

$$\operatorname*{median}_{i=0}^{2(n+s')+1} (x'_i - y'_{2(n+s')+1-i}) = \operatorname*{median}_{i=0}^{n-1} (x_i - y_{(i-s'-1) \bmod n}),$$

which is $\text{median}_{i=0}^{n-1}(x_i - y_{(i+s) \bmod n})$ if we let $s' = n-1-s$. Applying Fact 7.4.1, we can therefore minimize $\text{median}_{i=0}^{n-1} |x_i - y_{(i+s) \bmod n} + c|$ over $c$, for each $s \in \{0, 1, \ldots, n-1\}$. By brute force, we can minimize over $s$ as well using $O(n)$ additional comparisons and time. □

Our results for (median, −) convolution use the following result of Frederickson and Johnson:

**Theorem 7.4.3.** [77] *The median element of the union of $k$ sorted lists, each of length $n$, can be computed in $O(k \log n)$ time and comparisons.*

Our result for (median, −) convolution is the following:

**Theorem 7.4.4.** *The* (median, −) *convolution of two vectors of length $n$ can be computed in $O(n^2 (\log \log n)^2 / \log n)$ time.*

*Proof.* Let $\mathbf{x}$ and $\mathbf{y}$ denote the two vectors of length $n$, and let $\mathbf{x} \mathop{\bar{*}}_{\text{med}} \mathbf{y}$ denote their (median, −) convolution. For each permutation $\pi$ on the set $\{0, 1, \ldots, d-1\}$, for each $i \in \{0, d, 2d, \ldots, \lfloor n/d \rfloor d\}$, and for each $j \in \{0, 1, \ldots, n-1\}$, we define the $(d-1)$-dimensional points

$$\begin{aligned} p_{\pi,i} &= (x_{i+\pi(0)} - x_{i+\pi(1)},\ x_{i+\pi(1)} - x_{i+\pi(2)},\ \ldots,\ x_{i+\pi(d-2)} - x_{i+\pi(d-1)}), \\ q_{\pi,j} &= (y_{j-\pi(0)} - y_{j-\pi(1)},\ y_{j-\pi(1)} - y_{j-\pi(2)},\ \ldots,\ y_{j-\pi(d-2)} - y_{j-\pi(d-1)}), \end{aligned}$$

(To handle boundary cases, define $x_i = \infty$ and $y_j = -\infty$ for indices $i, j$ outside $[0, n-1]$.) For each permutation $\pi$, we apply Lemma 7.3.3 to the set of red points

$\{p_{\pi,i} : i = 0, d, 2d, \ldots, \lfloor n/d \rfloor d\}$ and the set of blue points $\{q_{\pi,j} : j = 0, 1, \ldots, n-1\}$, to obtain all dominating pairs $(p_{\pi,i}, q_{\pi,j})$.

Point $p_{\pi,i}$ dominates $q_{\pi,j}$ precisely if $x_{i+\pi(\delta)} - x_{i+\pi(\delta+1)} \geq y_{j-\pi(\delta)} - y_{j-\pi(\delta+1)}$ for all $\delta \in \{0, 1, \ldots, d-2\}$ (ignoring the indices outside $[0, n-1]$). By re-arranging terms, this condition is equivalent to $x_{i+\pi(\delta)} - y_{j-\pi(\delta)} \geq x_{i+\pi(\delta+1)} - y_{j-\pi(\delta+1)}$ for all $\delta \in \{0, 1, \ldots, d-2\}$, i.e., $\pi$ is a sorting permutation of $\langle x_i - y_j, x_{i+1} - y_{j-1}, \ldots, x_{i+d-1} - y_{j-d+1} \rangle$. If we substitute $j = k - i$, we obtain that $(p_{\pi,i}, q_{\pi,k-i})$ is a dominating pair precisely if $\pi$ is a sorting permutation of the list $L_k(i) = \langle x_i - y_{k-i}, x_{i+1} - y_{k-i+1}, \ldots, x_{\min\{i+d,n\}-1} - y_{\min\{k-i+d,n\}-1} \rangle$. Thus, the set of dominating pairs gives us the sorted order of $L_k(i)$ for each $i$ divisible by $d$ and for each $k$. Also, there can be at most $O(n^2/d)$ total dominating pairs $(p_{\pi,i}, q_{\pi,j})$ over all $i, j, \pi$, because there are $O(n/d)$ choices for $i$ and $O(n)$ choices for $j$, and if $(p_{\pi,i}, q_{\pi,j})$ is a dominating pair, then $(p_{\pi',i}, q_{\pi',j})$ cannot be a dominating pair for any permutation $\pi' \neq \pi$. (Here we assume that the sorted order is unique, which can be arranged by standard perturbation techniques or by breaking ties consistently [36].) Hence, the running time of the $d!$ executions of Lemma 7.3.3 is $d!\, 2^{O(d)} n^{1+\varepsilon} + O(n^2/d)$ time, which is $O(n^2 \log\log n / \log n)$ if we choose $d = \alpha \log n / \log\log n$ for a sufficiently small constant $\alpha > 0$. By Theorem 7.4.3, we can compute the median of $L_k(0) \cup L_k(d) \cup L_k(2d) \cup \cdots \cup L_k(\lceil k/d \rceil d)$, i.e., $\text{median}_{i=0}^{k}(x_i - y_{k-i})$, in $O((k/d)\log d) = O((n/d)\log d)$ comparisons. Also, in the same asymptotic number of comparisons, we can binary search to find where the median fits in each of the $L_k(\lambda)$ lists, and therefore which differences are smaller and which differences are larger than the median. This median is the $k$th entry of $\mathbf{x} \underset{\text{med}}{\bar{*}} \mathbf{y}$.

Therefore all $n$ entries can be computed in $O(n^2(\log d)/d) = O(n^2(\log\log n)^2/\log n)$ time. □

Combining Theorems 7.4.2 and 7.4.4, we obtain the following result:

**Corollary 7.4.5.** *The $\ell_1$ necklace alignment problem can be solved in $O(n^2(\log\log n)^2/\log n)$ time.*

As before, this approach likely cannot be improved beyond $O(n^2/\log n)$, because such an improvement would require an improvement to Lemma 7.3.3, which would in turn improve the fastest known algorithm for all-pairs shortest paths in dense graphs [36].

In contrast to (median, +) convolution, (mean, +) convolution is trivial to compute in linear time by inverting the two summations.

## 7.5 Conclusion

Although motivated by measuring music similarity, the necklace alignment problem is interesting in its own right from the theoretical point of view. The convolution problems we consider here have connections to many classic problems, and it would be interesting to explore whether the structural information extracted by our algorithms could be used to devise faster algorithms for these classic problems. For example, does the antidiagonal information of the $X + Y$ matrix lead to a $o(n^2 \log n)$-time algorithm for sorting $X + Y$?

# Chapter 8

# Rhythm Reconstruction from Interval Content

In the problems discussed so far, we either characterized rhythms having some desired properties, or we were given a rhythm that we wanted to transform into or compare to another rhythm. Solutions to these problems involved studying the intervals between pairwise onsets of the rhythms and how they relate to one another. In the final chapter of this thesis, we introduce a class of problems that, in a way, go in the reverse direction. In this reverse setting, we are given constraints on interonset intervals and are required to reconstruct rhythms satisfying these constraints.

Consider a rhythm represented as a clock diagram. Every pair of onsets determines an interonset geodesic distance (or interval). We will call the multiset of these geodesic distances between every pair of onsets the *interval content* of the rhythm. It is clear that every rhythm uniquely determines an interval content; however, the converse is not necessarily true. The patterns [× · × · · · × · · · · ×] and [×× · · × · × · · · · ·], which are two well known tetrachords, are neither rotations nor mirror images of each other, and yet they have the same interval content (Figure 8–1). Therefore, there is an inherent problem in uniquely reconstructing a rhythmic pattern from its multiset of distances.

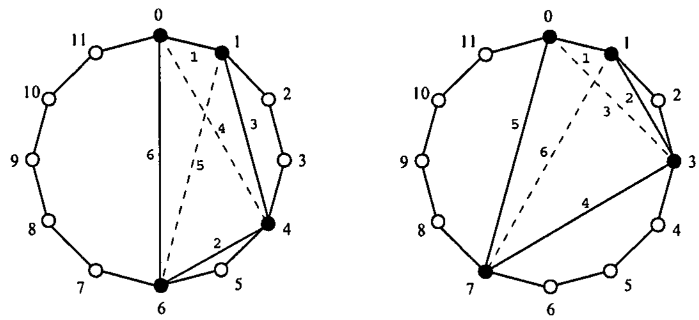


Figure 8–1: Two rhythms with the same interval content that are neither rotations nor mirror images of each other.

In music theory, music perception, and music information retrieval, questions concerning the existence and constructibility of melodies and rhythms from partial information is a well studied problem. One popular method of encoding rhythms and melodies with only partial information is via rhythmic and melodic *contours.* Rhythmic contours have been used for the analysis of rhythms, the description of general stylistic features of music, the design of algorithms for automatic classification of musical genres, and also for the study of perceptual discrimination of rhythms [119, 152, 158]. The rhythmic contour is defined as the pattern of successive relative changes of durations in a rhythm. Some authors represent the rhythmic contour as a sequence of integers reflecting these changes; others simply describe the changes in a qualitative manner, observing whether a duration becomes longer, shorter, or remains the same. As an example, consider the rhythmic contour of the clave son rhythm expressed in box notation as [× · · × · · × · · · × · × · · ·]. First, we determine its cyclic ordered set of duration intervals (3,3,4,2,4). The difference-encoding of this set is given by $\{0, 1, -2, 2, -1\}$. The rhythmic contour is concerned with only the pattern of direction changes of the durations, which may be encoded as $\{0, 1, -1, 1, -1\}$. Alternately we can write $\{+--+0-\}$ [126, 136]. A pitch (melodic contour) is defined in the same way [153]. Two types of contours have received a great deal of attention in the music literature: the *adjacent* contours (edge lengths of adjacent onsets) described in the preceding, and the *full* contours that use the entire matrix of distances between all pairs of points on the circle. An interesting composition problem in music theory is whether a given set of intervals (adjacent or full) admits a realization as a melody or rhythm [138]. Demaine et al. [59] give efficient algorithms for determining whether a rhythm may be reconstructed.

It must be noted here that reconstruction problems also have applications in areas such as crystallography and DNA sequencing [109, 157]. One variation, called the *turnpike problem*, dates back to the 1930's in the area of X-ray crystallography, where the objective was to reconstruct the coordinates of the atoms in a crystal. More recently, reconstruction problems have become important in the field of DNA *sequencing* — determining the pattern of the amino acids that constitute a strand of DNA. Given a DNA molecule, exposing it to a special kind of enzyme (called *restriction enzyme*) divides it into pieces of different lengths. The lengths of these fragments can be measured with standard techniques, and the challenge is to reconstruct the original ordering of these fragments in the DNA molecule. In molecular biology, this reconstruction problem is called the *partial digest* problem and is equivalent to the turnpike problem in crystallography.

From a theoretical perspective, problems related to reconstructing sets from interpoint distances are, in general, computationally challenging. Even when the points are restricted to a line (turnpike or partial digest problem), the complexity of the problem remains unknown. One of the main difficulties of these problems lies in the fact that while a given pointset uniquely defines a multiset of distances, the inverse is not always true: there may be many pointsets defining a given multiset of distances; such pointsets are known as *homometric* sets. Lemke et al. [109] study the computational and combinatorial complexity of pointset reconstruction problems. For a given set of $\binom{n}{2}$ distances, they give upper and lower bounds on the number of mutually noncongruent and homometric $n$-point sets that realize these distances in $\mathbb{R}^d$. They also show that the decision problem of whether a multiset of $\binom{n}{2}$ distances is realized by $n$ points in $\mathbb{R}^d$ (for arbitrary dimensions $d$) is NP-complete.

When a rhythm is represented as a clock diagram, the edges defined by onsets consecutive along the circumference of a circle form a cyclic polygon. One question is how one may construct such a polygon given the lengths of its sides. In [117] Macnab describes a method to construct such a cyclic polygon: given an ordered list of edge lengths $(a_0, a_1, \ldots, a_{n-1})$, start with a large enough circle and place $n$ vertices $v_0, v_1, \ldots v_{n-1}$ on the circumference such that the distance of the chord between $v_i$ and $v_{i+1}$ is equal to $a_i$ for all $i \in [0, n-2]$. Then, decrease the radius of the circle continuously until the first and last vertices $v_0$ and $v_{n-1}$ coincide. Macnab claims that this heuristic produces the desired cyclic polygon. However, Pinelis [133] gives a simple counterexample to Macnab's heuristic with only three edge lengths, and provides a fix. Pinelis's example shows that making the circle smaller sometimes causes the diameter to be equal to the longest edge of the polygon. If the first and last vertices do not coincide at this point, the circle cannot be made smaller because the longest edge cannot fit in a smaller circle. As a fix, Pinelis suggests *increasing* the radius of the circle after this point until the first and last vertices coincide. It is worthwhile to mention here that a heuristic identical to that of Macnab's also appears in a paper by Pak [131]. Pinelis moves on to show that given an ordered sequence of edge lengths $(a_0, a_2, \ldots, a_{n-1})$ such that each $a_i$ is strictly less than the sum of the rest, there is a unique convex cyclic polygon with the given edge lengths. Changing the order of the edges around the circle will result in a different cyclic polygon, but with the same area: the area of a cyclic polygon can be seen as the sum of the areas of the triangles determined by the two endpoints of an edge of the polygon and the center of the circle, and rearranging the triangles will keep the area unchanged. Since any triangle is cyclic, its edge lengths determine a unique triangle. For polygons with a higher number of edges however, it is necessary to have the cyclic condition in order to guarantee uniqueness of the area of the polygon described by the given lengths.

The question of whether constructing a cyclic polygon given its edge lengths is possible with only a ruler and compass is settled by Schreiber [154] who proves that such a construction with a ruler and compass is indeed not possible in general, even if the edge lengths are equal. In fact, Carl Gauss [82] proved around the year 1800 that a regular polygon can be constructed with a ruler and compass only if the number of its vertices is the product of a power of two and any number of distinct Fermat primes (a prime number of the form $2^{2^i} + 1$). Later in 1836, Pierre Wantzel proved that Gauss's condition is also necessary [186].

In an attempt to construct approximate regular polygons with a ruler and compass, Treatman and Wickham [177] describe a construction that uses cyclic polygons. To construct a regular polygon with $n$ sides, first place $n$ points $v_0, v_1, \ldots, v_{n-1}$ at arbitrary positions around the circumference of the circle such that the arc lengths between two consecutive points are sorted counterclockwise in a nondecreasing order. Inserting edges between consecutive points $v_i$ and $v_{i+1}$ produces a cyclic polygon $P^0$. Then for every $i = 0, 1, \ldots, n-1$, move $v_i$ to the center of the arc $\widehat{v_{i-1}v_i}$ and let the resulting polygon be $P^1$. This can be done using a ruler and compass. By repeating this operation, one can construct a sequence of polygons $P^0, P^1, P^2, \ldots$. Treatman and Wickham show that the limit of the such a sequence $P^i$ as $i$ goes to infinity is a regular polygon, and that the area of $P^i$ is at most the area of $P^{i+1}$. Hitt and Zhang [97] describe a similar operation, that we call the shadow operation in Section 6.1, and show that the shadow sequence of any convex cyclic polygon converges to a regular polygon. Note that every shadow polygon in the sequence can be constructed by a ruler and compass.

Reconstruction problems have connections with other areas such as crystallography and DNA sequencing [109, 157]; we omit further discussion of connections with such areas and move to the discussion of the main problem in this chapter. In what follows, we describe a reconstruction problem we call the *labeled beltway problem*, and provide a polynomial-time solution.

## 8.1 The Labeled Beltway Problem

Lemke et al. [109] call the version of the reconstruction problem where the points lie on a circle the *beltway problem.* Here we consider a variation of the beltway problem, which we call *labeled beltway,* where the edges defined by the points along the circle are assigned labels. More formally, given a set of points and their clockwise order around a circle, we want to find an embedding of the points on the circumference of a unit circle subject to some constraints. The constraints involve the geodesic distance between pairs of points. The distance constraints are given by a labeling of the edges defined by pairs of points such that two edges having the same label have the same length. Note that if together with the ordering we were given all the distances between pairs of points, the problem is easy in the plane, as well as on the circle. If the points are embedded on a circle, then we have two cases: if the lengths of the convex hull edges sum to 1, then the given edge lengths measure the clockwise distance between pairs of points consecutive around the circumference, and we can embed. Otherwise, the longest edge of the convex hull equals the sum of the lengths of the remaining edges; in this case, invert the length of the longest edge around 1; all the edges now have clockwise distances and we can again embed.

We specify the constraints on the geodesic distances between pairs of points by associating these distances with edges of the complete graph on the $n$ points. In the labeled beltway problem, each vertex assigns a label to each incident edge, and two edges incident to the same vertex are constrained to have equal geodesic lengths if and only if they have the same label assigned by that vertex. Thus, incident pairs of edges have an isometry or anisometry constraint, while nonincident edges have no direct constraints. In this chapter we give a solution to the labeled beltway problem by reducing it to solving a set of linear equations and nonequations.

## 8.2 Reduction to a Set of Linear Equations

Let $\{p_0, p_1, \ldots, p_{n-1}\}$ be a set of points embedded on a circle of unit circumference such that $p_{(i+1) \bmod n}$ is the closest point to $p_i$ in the clockwise traversal along the circumference starting from $p_i$ for $i = 0, 1, \ldots, n-1$. We say that an edge $p_i p_j$ is *oriented clockwise* if its geodesic distance is the length of the traversal from $p_i$ to $p_j$ along the circumference of the circle; it is *oriented counterclockwise* if its geodesic distance is the length of the traversal from $p_j$ to $p_i$ along the circumference. If the length of an edge is $\frac{1}{2}$, then the edge can be oriented either way; but otherwise, its orientation is forced.

We now state the constraints on the edges of the labeled complete graph more formally as follows: for every three distinct points $p_i, p_j, p_k$ on the circle, if the edge $p_i p_j$ has the same label as $p_j p_k$, then the geodesic distance between $p_i$ and $p_j$ is equal to the distance between $p_j$ and $p_k$, that is, $\overset{\frown}{d}(p_i, p_j) = \overset{\frown}{d}(p_j, p_k)$ (*isometry constraints*); moreover, two edges with different labels must have different geodesic lengths (*anisometry constraints*).

We show how to reconstruct the points given isometry and anisometry constraints between pairs of edges that do not cross (that is, edges whose endpoints appear together in the cyclic order). In particular, such constraints capture all constraints, solving the problem. Our solution is based on representing the constraints by linear equations ($\sum_i a_i x_i = b_1$), linear inequalities ($\sum_i a_i x_i \leq b_i$ and $\sum_i a_i x_i < b_i$), and linear nonequations ($\sum_i a_i x_i \neq b_i$) on $n$ variables. We then solve this system by reducing to a sequence of linear programs with only linear equations and inequalities.

We parameterize the desired embedding of the $n$ points on the circle by variables $x_0, x_1, \ldots, x_{n-1}$, where $x_i = \overset{\circ}{d}(p_i, p_{i+1})$, the clockwise distance from $p_i$ to $p_{i+1}$. These $n$ variables determine an embedding up to rotation provided that they satisfy the following constraints (defining an open $(n-1)$-simplex):

$$\begin{aligned} \sum_{i=0}^{n} x_i &= 1 \\ x_i &> 0 \qquad \text{for } i = 0, 1, \ldots, n-1 \end{aligned}$$

The positivity constraints force the points to embed to distinct locations in the correct cyclic order.

The challenge in representing an isometry or anisometry constraint among geodesic distances is that a geodesic distance between two points $p_i$ and $p_j$ may be realized by either the clockwise distance $x_i + x_{i+1} + \cdots + x_{j-1}$ or the counterclockwise distance $x_j + x_{j+1} + \cdots + x_{i-1}$. If we knew the orientation of every edge, then we could write the isometry or anisometry constraint as a linear equation or nonequation. There are $n$ choices for the orientations of the $n-1$ edges incident to a vertex $p_i$, depending on which wedge between consecutive edges contains the center of the circle. Considering all vertices, there are at most $n^n$ choices for the orientations, each leading to a system of linear equations and nonequations. On the other hand, consider the $n$ pairwise crossing edges $p_i p_{i+n/2}$; the orientation of each can be chosen independently. Hence, there are at least $2^n$ possible orientations.

For the edges that do not cross, the isometry and anisometry constraints can be reconstructed easily. Two edges $p_i p_j$ and $p_k p_l$ do not cross if their endpoints appear in the order $p_i, p_j, p_k, p_l$ along the circumference. Fortunately, for constraints between such noncrossing edges, we can effectively determine the orientations of the edges to obtain a single system of linear equations and nonequations. To describe these constraints we first need the following lemma:

**Lemma 8.2.1.** *If the points $p_i, p_j, p_k, p_l$ appear in clockwise order around the circle (with the possibility that $j = k$ or $l = i$ but not both) then we cannot have both edges $p_i p_j$ and $p_k p_l$ oriented counterclockwise.*

*Proof.* Suppose both edges are oriented counterclockwise. This implies that $\overset{\circlearrowleft}{d}(p_i, p_j) \geq \frac{1}{2}$ and $\overset{\circlearrowleft}{d}(p_k, p_l) \geq \frac{1}{2}$. Because of the ordering of the points around the circle, both $p_k$ and $p_l$ lie on the clockwise arc from $p_j$ to $p_i$. Thus,

$$\overset{\circlearrowleft}{d}(p_j, p_i) = \overset{\circlearrowleft}{d}(p_j, p_k) + \overset{\circlearrowleft}{d}(p_k, p_l) + \overset{\circlearrowleft}{d}(p_l, p_i).$$

This equality cannot hold because $\overset{\circlearrowleft}{d}(p_j, p_i) \leq \frac{1}{2}$ while the right-hand-side is strictly greater than half. Thus, at least one of the edges must be oriented clockwise. □

We start with the isometry constraints, for which we need the following lemma:

**Lemma 8.2.2.** *If the points $p_i, p_j, p_k, p_l$ appear in clockwise order around the circle (with the possibility that $j = k$ or $l = i$ but not both) such that $\overset{\circlearrowright}{d}(p_k, p_l) = \overset{\circlearrowright}{d}(p_i, p_j)$, then both edges $p_i p_j$ and $p_k p_l$ must be oriented clockwise.*

*Proof.* By Lemma 8.2.1 we know that we cannot orient both edges counterclockwise. Suppose only $p_k p_l$ is oriented counterclockwise; this means that $\overset{\circlearrowright}{d}(p_k, p_l) = \overset{\circlearrowleft}{d}(p_l, p_k) \leq$

$\frac{1}{2}$ and $\overset{\text{Ù}}{d}(p_i, p_j) = \overset{\text{Ò}}{d}(p_i, p_j)$. Because of the ordering of the points $p_i, p_j, p_k, p_l$ around the circle, we have

$$\overset{\text{Ò}}{d}(p_l, p_k) = \overset{\text{Ò}}{d}(p_l, p_i) + \overset{\text{Ò}}{d}(p_i, p_j) + \overset{\text{Ò}}{d}(p_j, p_k) \leq \frac{1}{2}.$$

Because at least one of the distances $\overset{\text{Ò}}{d}(p_l, p_i)$ and $\overset{\text{Ò}}{d}(p_j, p_k)$ is greater than zero, $\overset{\text{Ò}}{d}(p_l, p_k) > \overset{\text{Ò}}{d}(p_i, p_j)$ which implies that $\overset{\text{Ù}}{d}(p_k, p_l) > \overset{\text{Ù}}{d}(p_i, p_j)$; contradiction. Similarly, if only $p_i p_j$ is oriented counterclockwise, then this would imply that $\overset{\text{Ù}}{d}(p_i, p_j) > \overset{\text{Ù}}{d}(p_k, p_l)$. Therefore, if $\overset{\text{Ù}}{d}(p_i, p_j) = \overset{\text{Ù}}{d}(p_k, p_l)$ then both edges must be oriented clockwise. □

Thus, if two edges $p_i p_j$ and $p_k p_l$ are noncrossing and their edge lengths must be equal, then by Lemma 8.2.2 we can assume that the edges are oriented clockwise and we force this by adding the linear inequalities

$$\sum_{i \leq r < j} x_r \leq \frac{1}{2},$$
$$\sum_{k \leq s < l} x_s \leq \frac{1}{2}$$

Now that we have forced the edges to be oriented clockwise, we can force the equality of their lengths with a linear equation:

$$\sum_{i \leq r < j} x_r \quad = \quad \sum_{k \leq s < l} x_s$$

For the anisometry constraints, we first show the following lemma:

**Lemma 8.2.3.** *If the points $p_i, p_j, p_k, p_l$ appear in clockwise order around the circle (with the possibility that $j = k$ or $l = i$ but not both) such that at least one of the edges $p_i p_j$ and $p_k p_l$ must be oriented counterclockwise, then $\overset{\circlearrowright}{d}(p_i, p_j) \neq \overset{\circlearrowright}{d}(p_k, p_l)$.*

*Proof.* Assume that $p_k p_l$ is oriented counterclockwise; then, $\overset{\circlearrowright}{d}(p_l, p_k) \leq \frac{1}{2}$. By Lemma 8.2.1, $p_i p_j$ must be oriented clockwise. Then, we have $\overset{\circlearrowright}{d}(p_l, p_k) = \overset{\circlearrowright}{d}(p_l, p_i) + \overset{\circlearrowright}{d}(p_i, p_j) + \overset{\circlearrowright}{d}(p_j, p_k) \leq \frac{1}{2}$. Because at least one of $\overset{\circlearrowright}{d}(p_l, p_i)$ or $\overset{\circlearrowright}{d}(p_j, p_k)$ is strictly positive, then $\overset{\circlearrowright}{d}(p_k, p_l) > \overset{\circlearrowright}{d}(p_i, p_j)$. Thus, the two edges have distinct clockwise lengths. □

Now consider an anisometry constraint between two noncrossing edges $p_i p_j$ and $p_k p_l$. We represent this constraint by a linear nonequation:

$$\sum_{i \leq r < j} x_r \neq \sum_{k \leq s < l} x_s$$

To show that the above nonequation holds independent of the orientation of each of the edges, we need to show that it holds if and only if the desired geodesic distances are distinct. First, if both edges can be oriented clockwise, then this constraint on the clockwise distances is equivalent to the distinctness of the geodesic lengths. Second, if one of the edges must be oriented counterclockwise, then by Lemma 8.2.2, the geodesic lengths must be different; and by Lemma 8.2.3, the linear nonequation must be satisfied.

## 8.3 Solving Systems of Linear Equations and Nonequations

The previous section yields a linear system like the following.

$$Ax \leq \frac{1}{2} \tag{8.1}$$

$$Mx = f \tag{8.2}$$

$$Nx \neq g \tag{8.3}$$

$$x_i > 0 \tag{8.4}$$

Geometrically, (8.1), (8.2) and (8.4) define a (partly open) convex polyhedron $P$, while each row of (8.3) defines a forbidden hyperplane. To solve this system, we will use the following idea: initially we ignore the forbidden hyperplanes and find a relative interior point in the feasible region $P$. Then we check if this point lies on a forbidden hyperplane; if it does not lie on any forbidden hyperplane, we have a solution. Otherwise, if the point lies on $h_i$, then we recurse on one side of $h_i \cap P$. Recursing on one side of $h_i$ ensures that we never pick a point on the same hyperplane more than once. Thus, in the worst case we will eventually run out of forbidden hyperplanes and one side of the last eliminated hyperplane will be a region with no forbidden points. We can now find a feasible point in this region.

After a suitable (and polynomial time computable) change of coordinates, we may assume that $P$ has interior points. Let $h(a, \mu)$ denote the hyperplane $\{x \mid \langle a, x \rangle = \mu\}$.

**Proposition 8.3.1.** *If $h(a, \mu) \cap P$ has relative interior points, then for any sufficiently small $\varepsilon > 0$, $h(a, \mu + \varepsilon)$ and $h(a, \mu - \varepsilon)$ both have relative interior points.*

The idea is that given a point $q$ in the relative interior of $P \cap h(a, \mu)$ (if such a $q$ does not exist, then $h(a, \mu)$ is redundant and can be discarded), choose $\varepsilon$ sufficiently small so that $h(a, \mu + \varepsilon) \cap P$ has relative interior, and does not collide with any possible parallel forbidden hyperplanes. If a solution exists, then there is one on the hyperplane $h(a, \mu + \varepsilon)$. To avoid calculation of $\varepsilon$, we add the inequality $a^T x > \mu$ to our system, replacing the nonequation $a^T x \neq \mu$. If this new system is infeasible, it means that $P \subset h(a, \mu)$, and the original problem is infeasible.

Our basic computational step is thus to find a relative interior point of a polyhedron defined by strict and nonstrict inequalities. This is equivalent to feasibility testing for systems of strict linear inequalities, which can be solved by linear programming.

The inequalities and (non)equations of our system describe the constraints for pairs of noncrossing edges. We have added a constant number of constraints per pair of noncrossing edges having the same or different labels. Because we have a total of $\binom{n}{2}$ edges, pairing them gives us $\Theta(n^4)$ constraints. The linear programs required can be solved in time polynomial in $n$ and the number of bits required of the output. In our case, $\Theta(n)$ bits suffice to disambiguate all the distinct distances, at which point the solution will be correct; so we can solve the linear programs in time polynomial in $n$. The number of times we solve such a system is at most equal to the number of forbidden hyperplanes, which is a polynomial function of $n$. Thus we have a polynomial-time solution to the labeled beltway problem.

# Chapter 9

# Conclusion

In this thesis we have shown a variety of mathematical and geometric problems that have connections to computational music theory. Although most of these problems were not initially derived from musical considerations and are primarily solved for mathematical objectives, they may find applications in constructing practical tools for music classification, retrieval, comparison, etc., as well as for music-theoretic analysis of rhythms and scales. One example in the latter case is the study of deep rhythms described in Chapter 3, which initially arose as a problem in distance geometry, but which also appears independently in one form as Gamer's Common Tone Theorem in the music literature. On the other hand, sometimes problems that arise in music analysis contexts may find unexpected connections with theoretical problems in mathematics and computer science, and hence may improve on results in this latter area. An example is the necklace alignment problem in Chapter 7, which initially was a problem in rhythmic comparison and whose solutions improved results on convolutions. Therefore, the problems studied in this thesis contribute in a way to both areas of theoretical computer science and computational music theory.

Irrespective of the connections between these two areas however, if we move away from the musical context it is clear that the objects we study here are in essence cyclic binary sequences. Although such sequences have been extensively studied in the computer science literature, what sets our work apart is our geometric representation of these objects. This representation allows us to visualize the related problems differently, and to make use of geometric tools and properties (such as area, distance, polygons, etc.) to solve them.

The two areas of music and mathematics feed each other with endless open problems. This thesis leaves us with a few that we list below.

**Open Problem 9.0.2** (Measures of evenness). *Define new measures of evenness based on how well they discriminate between rhythms, how fast they can be computed, and how useful they are in practice. Two possibly useful measures were discussed in Section 4.4.*

**Open Problem 9.0.3** (Reconstructing even rhythms). *Suppose we are given a set of intervals that represent the geodesic distances between* consecutive *onsets of a rhythm. Is it possible to find an ordering of these intervals that maximizes the evenness of the rhythm described by this ordering? Geometrically, the problem is equivalent to rearranging the edges of a cyclic polygon such that evenness is maximized, for some definition of evenness.*

**Open Problem 9.0.4** (Constructing approximate cyclic polygons). *In the introduction to Chapter 8, it was discussed that, in general, the problem of constructing a cyclic polygon given the lengths of its edges is not possible with a ruler and compass. Is the problem of reconstructing such polygons possible in the real RAM model* *of computation? If not, is it possible to design an algorithm that constructs a good approximation of the desired polygon?*

**Open Problem 9.0.5** (Rhythm reconstruction). *Given an integer $r$, design and algorithm that places points around a unit circle such that the number of distinct geodesic distances realized by these points is equal to $r$. Is such a construction possible for any $r$?*

**Open Problem 9.0.6** (Global beltway). *The global beltway problem is a variation on the beltway problem discussed in Chapter 8. In this variation, constraints on the edges are defined globally such that any two edges are constrained to have the same geodesic length if and only if they have the same label. To restate the problem more formally, we are given the ordering of a set of points around the circumference of a unit circle and some global constraints involving the distances defined by pairs of points. We want to embed these points such that the distance constraints are satisfied. These distance constraints are as follows: for every four distinct points $p_i, p_j, p_k, p_l$ on the circle, if the length $p_ip_j$ has the same label as $p_kp_l$, then the geodesic distance between $p_i$ and $p_j$ is equal to the distance between $p_k$ and $p_l$; moreover, two distances with different labels must have different lengths. Is this problem NP-complete?*

## APPENDIX A

# Euclidean Rhythms in Traditional World Music

Below is a list of Euclidean rhythms that can be found in traditional world music. We restrict out attention to rhythms where $k$ and $n$ are relatively prime.

- $E(2,3) = [\times\times\cdot\,] = (12)$ is a common Afro-Cuban drum pattern when started on the second onset as in $[\times\cdot\times]$. For example, it is the conga rhythm of the (6/8)-time *Swing Tumbao* [106]. It is common in Latin American music, as for example in the *Cueca* [180], and the *coros de clave* [145]. It is common in Arabic music, as for example in the *Al Táer* rhythm of Nubia [89]. It is also a rhythmic pattern of the Drum Dance of the Slavey Indians of Northern Canada [9].
- $E(2,5) = [\times\cdot\times\cdot\cdot\,] = (23)$ is a rhythm found in Greece, Namibia, Rwanda and Central Africa [8]. It is also a 13th century Persian rhythm called *Khafif-e-ramal* [189], as well as the rhythm of the Macedonian dance *Makedonka* [155]. Tchaikovsky used it as the metric pattern in the second movement of his *Symphony No. 6* [104]. Started on the second onset as in $[\times\cdot\cdot\times\cdot\,]$ it is a rhythm found in Central Africa, Bulgaria, Turkey, Turkestan and Norway [8]. It is also the metric pattern of Dave Brubeck's *Take Five*, as well as *Mars* from *The Planets* by Gustav Holst [104]. B as in $[\times\cdot\cdot\times\cdot\times\cdot\cdot\times\cdot\cdot]$, it is a Serbian rhythmic

pattern [8]. When it is started on the fourth (last) onset it is the *Daasa al kbiri* rhythmic pattern of Yemen [89].

- $E(4,15) = [\times\cdot\cdot\cdot\times\cdot\cdot\cdot\times\cdot\cdot\cdot\times\cdot\cdot]$ = (4443) is the metric pattern of the *pañcam savārī tāl* of North Indian music [41].
- $E(5,6) = [\times\times\times\times\times\cdot]$ = (11112) yields the *York-Samai* pattern, a popular Arabic rhythm [160]. It is also a handclapping rhythm used in the *Al Medēmi* songs of Oman [65].
- $E(5,7) = [\times\cdot\times\times\cdot\times\times]$ = (21211) is the *Nawakhat* pattern, another popular Arabic rhythm [160]. In Nubia it is called the *Al Noht* rhythm [89].
- $E(5,8) = [\times\cdot\times\times\cdot\times\times\cdot]$ = (21212) is the Cuban *cinquillo* pattern discussed in the preceding [74], the *Malfuf* rhythmic pattern of Egypt [89], as well as the Korean *Nong P'yŏn* drum pattern [98]. Started on the second onset, it is a popular Middle Eastern rhythm [185], as well as the *Timini* rhythm of Senegal, the *Adzogbo* dance rhythm of Benin [39], the *Spanish Tango* [70], the *Maksum* of Egypt [89], and a 13th century Persian rhythm, the *Al-saghil-al-sani* [189]. When it is started on the third onset it is the *Müsemmen* rhythm of Turkey [17]. When it is started on the fourth onset it is the *Kromanti* rhythm of Surinam.
- $E(5,9) = [\times\cdot\times\cdot\times\cdot\times\cdot\times]$ = (22221) is a popular Arabic rhythm called *Agsag-Samai* [160]. Started on the second onset, it is a drum pattern used by the *Venda* in South Africa [137], as well as a Rumanian folk-dance rhythm [135]. It is also the rhythmic pattern of the *Sigaktistos* rhythm of Greece [89], and the *Samai aktsak* rhythm of Turkey [89]. Started on the third onset, it is the rhythmic pattern of the *Nawahiid* rhythm of Turkey [89].

- $E(5,11) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (22223)$ is the metric pattern of the Savārī tāla used in the Hindustani music of India [113]. It is also a rhythmic pattern used in Bulgaria and Serbia [8]. In Bulgaria is is used in the *Kopanitsa* [143]. This metric pattern has been used by Moussorgsky in *Pictures at an Exhibition* [104]. Started on the third onset, it is the rhythm of the Macedonian dance *Kalajdzijsko Oro* [155], and it appears in Bulgarian music as well [8].
- $E(5,12) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot] = (32322)$ is a common rhythm played in the Central African Republic by the *Aka* Pygmies [7, 37, 38]. It is also the *Venda* clapping pattern of a South African children's song [134], and a rhythm pattern used in Macedonia [8]. Started on the second onset, it is the *Columbia* bell pattern popular in Cuba and West Africa [106], as well as a drumming pattern used in the *Chakacha* dance of Kenya [15], and also used in Macedonia [8]. Started on the third onset, it is the *Bemba* bell pattern used in Northern Zimbabwe [134], and the rhythm of the Macedonian dance *Ibraim Odža Oro* [155]. Started on the fourth onset, it is the *Fume Fume* bell pattern popular in West Africa [106], and is a rhythm used in the former Yugoslavia [8]. Finally, when started on the fifth onset it is the *Salve* bell pattern used in the Dominican Republic in a rhythm called *Canto de Vela* in honor of the Virgin Mary [71], as well as the drum rhythmic pattern of the Moroccan *Al Kudám* [89].
- $E(5,13) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot] = (32323)$ is a Macedonian rhythm which is also played by starting it on the fourth onset as follows: $[\times \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot]$ [8].

- $E(5,16) = [\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \cdot] = (33334)$ is the *Bossa-Nova* rhythm necklace of Brazil. The actual Bossa-Nova rhythm usually starts on the third onset as follows: $[\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot \times \cdot \cdot]$ [169]. However, other starting places are also documented in world music practices, such as $[\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot]$ [16].
- $E(6,7) = [\times \times \times \times \times \times \cdot] = (111112)$ is the *Póntakos* rhythm of Greece when started on the sixth (last) onset [89].
- $E(6,13) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (222223)$ is the rhythm of the Macedonian dance *Mama Cone pita* [155]. Started on the third onset, it is the rhythm of the Macedonian dance *Postupano Oro* [155], as well as the *Krivo Plovdivsko Horo* of Bulgaria [143].
- $E(7,8) = [\times \times \times \times \times \times \times \cdot] = (1111112)$, when started on the seventh (last) onset, is a typical rhythm played on the *Bendir* (frame drum), and used in the accompaniment of songs of the *Tuareg* people of Libya [160].
- $E(7,9) = [\times \cdot \times \times \times \cdot \times \times \times] = (2112111)$ is the *Bazaragana* rhythmic pattern of Greece [89].
- $E(7,10) = [\times \cdot \times \times \cdot \times \times \cdot \times \times] = (2121211)$ is the *Lenk fahhte* rhythmic pattern of Turkey [89].
- $E(7,12) = [\times \cdot \times \times \cdot \times \cdot \times \times \cdot \times \cdot] = (2122122)$ is a common West African bell pattern. For example, it is used in the *Mpre* rhythm of the *Ashanti* people of Ghana [171]. Started on the seventh (last) onset, it is a *Yoruba* bell pattern of Nigeria, a *Babenzele* pattern of Central Africa, and a *Mende* pattern of Sierra Leone [163].

- $E(7,15) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (2222223)$ is a Bulgarian rhythm when started on the third onset [8].
- $E(7,16) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot] = (3223222)$ is a *Samba* rhythm necklace from Brazil. The actual Samba rhythm is $[\times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot]$ obtained by starting $E(7,16)$ on the last onset, and it coincides with a Macedonian rhythm [8]. When $E(7,16)$ is started on the fifth onset it is a clapping pattern from Ghana [134]. When it is started on the second onset it is a rhythmic pattern found in the former Yugoslavia [8].
- $E(7,17) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot] = (3232322)$ is a Macedonian rhythm when started on the second onset [155].
- $E(7,18) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot] = (3232323)$ is a Bulgarian rhythmic pattern [8].
- $E(8,17) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (22222223)$ is a Bulgarian rhythmic pattern which is also started on the fifth onset [8].
- $E(8,19) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot] = (32232232)$ is a Bulgarian rhythmic pattern when started on the second onset [8].
- $E(9,14) = [\times \cdot \times \times \cdot \times \times \cdot \times \times \cdot \times \times \cdot] = (212121212)$, when started on the second onset, is the rhythmic pattern of the *Tsofyan* rhythm of Algeria [89].
- $E(9,16) = [\times \cdot \times \times \cdot \times \cdot \times \cdot \times \times \cdot \times \cdot \times \cdot] = (212221222)$ is a rhythm necklace used in the Central African Republic [7]. When it is started on the second onset it is a bell pattern of the *Luba* people of Congo [128]. When it is started on the fourth onset it is a rhythm played in West and Central Africa [74], as well as a cow-bell pattern in the Brazilian *samba* [159]. When it is started on the penultimate

onset it is the bell pattern of the *Ngbaka-Maibo* rhythms of the Central African Republic [7].

- $E(9,22)$ =[× · · × · × · · × · × · · × · × · · × · × ·] = (323232322) is a Bulgarian rhythmic pattern when started on the second onset [8].
- $E(9,23)$ =[× · · × · × · · × · × · · × · × · · × · × · ·] = (323232323) is a Bulgarian rhythm [8].
- $E(11,12)$ =[× × × × × × × × × × × ·] = (11111111112), when started on the second onset, is the drum pattern of the *Rahmāni* (a cylindrical double-headed drum) used in the *Sōt silām* dance from *Mirbāt* in the South of Oman [65].
- $E(11,24)$ =[× · · × · × · × · × · × · · × · × · × · × · × ·] = (32222322222) is a rhythm necklace of the *Aka* Pygmies of Central Africa [7]. It is usually started on the seventh onset. Started on the second onset, it is a Bulgarian rhythm [8].
- $E(13,24)$ =[× · × × · × · × · × · × · × × · × · × · × · × ·] = (2122222122222) is another rhythm necklace of the *Aka* Pygmies of the upper *Sangha* [7]. Started on the penultimate onset, it is the *Bobangi* metal-blade pattern used by the *Aka* Pygmies.
- $E(15,34)$ = [× · · × · × · × · × · · × · × · × · × · · × · × · × · × · · × · × ·] = (322232223222322) is a Bulgarian rhythmic pattern when started on the penultimate onset [8].

APPENDIX B

# Aksak Rhythms

The following Euclidean rhythms are *authentic aksak*:

- $E(2,5) = [\times \cdot \times \cdot \cdot] = (23)$ (classical music, jazz, Greece, Macedonia, Namibia, Persia, Rwanda).
- $E(3,7) = [\times \cdot \times \cdot \times \cdot \cdot] = (223)$ (Bulgaria, Greece, Sudan, Turkestan).
- $E(4,11) = [\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot] = (3332)$ (Southern India rhythm), (Serbian necklace).
- $E(5,11) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (22223)$ (classical music, Bulgaria, Northern India, Serbia).
- $E(5,13) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot] = (32323)$ (Macedonia).
- $E(6,13) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (222223)$ (Macedonia).
- $E(7,17) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot] = (3232322)$ (Macedonian necklace).
- $E(8,17) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (22222223)$ (Bulgaria).
- $E(8,19) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot] = (32232232)$ (Bulgaria).
- $E(9,23) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot] = (323232323)$ (Bulgaria).

The following Euclidean rhythms are *quasi-aksak*:

- $E(4,9) = [\times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (2223)$ (Greece, Macedonia, Turkey, Zaïre).
- $E(7,15) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (2222223)$ (Bulgarian necklace).

The following Euclidean rhythms are *pseudo-aksak*:

- $E(3,8) = [\times \cdot \cdot \times \cdot \cdot \times \cdot] = (332)$ (Central Africa, Greece, India, Latin America, West Africa, Sudan).
- $E(5,12) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot] = (32322)$ (Macedonia, South Africa).
- $E(7,16) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot] = (3223222)$ (Brazilian, Macedonian, West African necklaces).
- $E(7,18) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot] = (3232323)$ (Bulgaria).
- $E(9,22) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot] = (323232322)$ (Bulgarian necklace).
- $E(11,24) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot] = (32222322222)$ (Central African and Bulgarian necklaces).
- $E(15,34) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot] = (322232223222322)$ (Bulgarian necklace).

## APPENDIX C

# Euclidean Strings

In the lists that follow the Euclidean rhythms are shown in their box-notation format as well as in the clockwise distance sequence representation. The styles of music that use these rhythms is also included. Finally, if only a rotated version of the Euclidean rhythm is played, then it is still included in the list but referred to as a necklace.

The following Euclidean rhythms are Euclidean strings:

- $E(2,3) = [\times\times\cdot] = (12)$ (West Africa, Latin America, Nubia, Northern Canada).
- $E(2,5) = [\times\cdot\times\cdot\cdot] = (23)$ (classical music, jazz, Greece, Macedonia, Namibia, Persia, Rwanda), (*authentic aksak*).
- $E(3,4) = [\times\times\times\cdot] = (112)$ (Brazil, Bali rhythms), (Colombia, Greece, Spain, Persia, Trinidad necklaces).
- $E(3,7) = [\times\cdot\times\cdot\times\cdot\cdot] = (223)$ (Bulgaria, Greece, Sudan, Turkestan), (*authentic aksak*).
- $E(4,5) = [\times\times\times\times\cdot] = (1112)$ (Greece).
- $E(4,9) = [\times\cdot\times\cdot\times\cdot\times\cdot\cdot] = (2223)$ (Greece, Macedonia, Turkey, Zaïre), (*quasi-aksak*).
- $E(5,6) = [\times\times\times\times\times\cdot] = (11112)$ (Arab).

- $E(5,11) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (22223)$ (classical music, Bulgaria, Northern India, Serbia), (*authentic aksak*).
- $E(5,16) = [\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \cdot] = (33334)$ (Brazilian, West African necklaces).
- $E(6,7) = [\times\times\times\times\times\times \cdot] = (111112)$ (Greek necklace)
- $E(6,13) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (222223)$ (Macedonia), (*authentic aksak*).
- $E(7,8) = [\times\times\times\times\times\times\times \cdot] = (1111112)$ (Libyan necklace).
- $E(7,15) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (2222223)$ (Bulgarian necklace), (*quasi-aksak*).
- $E(8,17) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (22222223)$ (Bulgaria), (*authentic aksak*).

The following Euclidean rhythms are reverse Euclidean strings:

- $E(3,5) = [\times \cdot \times \cdot \times] = (221)$ (Korean, Rumanian, Persian necklaces).
- $E(3,8) = [\times \cdot \cdot \times \cdot \cdot \times \cdot] = (332)$ (Central Africa, Greece, India, Latin America, West Africa, Sudan), (*pseudo-aksak*).
- $E(3,11) = [\times \cdot \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot] = (443)$ (North India).
- $E(3,14) = [\times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot] = (554)$ (North India).
- $E(4,7) = [\times \cdot \times \cdot \times \cdot \times] = (2221)$ (Bulgaria).
- $E(4,11) = [\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot] = (3332)$ (Southern India rhythm), (Serbian necklace), (*authentic aksak*).
- $E(4,15) = [\times \cdot \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot] = (4443)$ (North India).
- $E(5,7) = [\times \cdot \times\times \cdot \times\times] = (21211)$ (Arab).
- $E(5,9) = [\times \cdot \times \cdot \times \cdot \times \cdot \times] = (22221)$ (Arab).

- $E(5,12) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot\,] = (32322)$ (Macedonia, South Africa), (*pseudo-aksak*).
- $E(7,9) = [\times \cdot \times \times \times \cdot \times \times \times] = (2112111)$ (Greece).
- $E(7,10) = [\times \cdot \times \times \cdot \times \times \cdot \times \times] = (2121211)$ (Turkey).
- $E(7,16) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot\,] = (3223222)$ (Brazilian, Macedonian, West African necklaces), (*pseudo-aksak*).
- $E(7,17) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot\,] = (3232322)$ (Macedonian necklace), (*authentic aksak*).
- $E(9,22) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot\,] = (323232322)$ (Bulgarian necklace), (*pseudo-aksak*).
- $E(11,12) = [\times \cdot \times \times \times \times \times \times \times \times \times \times] = (11111111112)$ (Oman necklace).
- $E(11,24) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot\,] = (32222322222)$ (Central African and Bulgarian necklaces), (*pseudo-aksak*).

The following Euclidean rhythms are neither Euclidean nor reverse Euclidean strings:

- $E(5,8) = [\times \cdot \times \times \cdot \times \times \cdot\,] = (21212)$ (Egypt, Korea, Latin America, West Africa).
- $E(5,13) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot\,] = (32323)$ (Macedonia), (*authentic aksak*).
- $E(7,12) = [\times \cdot \times \times \cdot \times \cdot \times \times \cdot \times \cdot\,] = (2122122)$ (West Africa), (Central African, Nigerian, Sierra Leone necklaces).
- $E(7,18) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot\,] = (3232323)$ (Bulgaria), (*pseudo-aksak*).
- $E(8,19) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot\,] = (32232232)$ (Bulgaria), (*authentic aksak*).
- $E(9,14) = [\times \cdot \times \times \cdot \times \times \cdot \times \times \cdot \times \times \cdot\,] = (212121212)$ (Algerian necklace).

- $E(9,16) = [\times \cdot \times\times \cdot \times \cdot \times \cdot \times\times \cdot \times \cdot \times \cdot] = (212221222)$ (West and Central African, and Brazilian necklaces).
- $E(9,23) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot] = (323232323)$ (Bulgaria), (*authentic aksak*).
- $E(13,24) = [\times \cdot \times\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot] = (2122222122222)$ (Central African necklace).
- $E(15,34) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot] = (322232223222322)$ (Bulgarian necklace), (*pseudo-aksak*).

APPENDIX D

# Measure of Ugliness in Linear Time

Let the forward distance vector of a rhythm be denoted by $(d_0, d_1, \ldots, d_{k-1})$. We know that $\sum_{i=0}^{k-1} d_i = n$. Define

$$D_i = \begin{cases} \sum_{j=0}^{k-1} d_j, & i < k \\ \\ \sum_{j=0}^{i-k} d_j + n, & k \leq i < 2k \end{cases}$$

so that the $D_i$'s are the cumulative values with a circular definition. Bjorklund's $\delta_j(i) = D_{i+j-1} - D_{i-1}$, and his measure of ugliness is proportional to:

$$\sum_{j=1}^{k-1} \sum_{i=1}^{k} (D_{i+j-1} - D_{i-1} - \alpha_j)^2 \tag{D.1}$$

where $\alpha_j = \frac{jn}{k}$. We will rewrite equation (D.1) in a form that makes it easy to compute it in $O(n)$ time.

Manipulating equation (D.1) we get:

$$\sum_{j=1}^{k} \sum_{i=1}^{k} (D_{i+j-1} - D_{i-1} - \alpha_j)^2 - \sum_{i=1}^{k} (D_{i+k-1} - D_{i-1} - \alpha_j)^2 \tag{D.2}$$

The second term of equation (D.2) can be computed in linear time. We will thus concentrate on the first term of the equation and first expand it to:

$$\sum_{j=1}^{k}\sum_{i=1}^{k}(D_{i+j-1} - D_{i-1})^2 + \sum_{j=1}^{k}\sum_{i=1}^{k}\alpha_j^2 - 2\sum_{j=1}^{k}\alpha_j\sum_{i=1}^{k}(D_{i+j-1} - D_{i-1}) \tag{D.3}$$

$$= (A) + (B) + (C)$$

Clearly, the second term

$$(B) = \sum_{j=1}^{k}\sum_{i=1}^{k}\alpha_j^2 = k\sum_{j=1}^{k}\alpha_j^2$$

Next, because $D_{i+k} = D_i + n$ for all $0 \leq j < k$, we have:

$$\sum_{i=1}^{k}(D_{i+j-1} - D_{i-1}) = nj \tag{D.4}$$

Thus, the third term of equation (D.3)

$$(C) = -2\sum_{j=1}^{k}\alpha_j\sum_{i=1}^{k}(D_{i+j-1} - D_{i-1}) = -2\sum_{j=1}^{k}\alpha_j \cdot nj$$

Therefore, the only variable element in equation (D.3) is the first term $(A)$, as the other two can be precomputed. Expanding $(A)$ we get:

$$(A) = \sum_{j=1}^{k}\sum_{i=1}^{k}(D_{i+j-1} - D_{i-1})^2 = \sum_{j=1}^{k}\sum_{i=1}^{k}(D_{i+j-1}^2 + D_{i-1}^2) - 2\sum_{j=1}^{k}\sum_{i=1}^{k}D_{i+j-1}D_{i-1} \tag{D.5}$$

We can simplify equation (D.5) by using the following three relations:

$$\sum_{j=1}^{k}D_{i+j-1} = \sum_{j=1}^{k}D_{j-1} + ni,$$

already used in equation (D.4),

$$\sum_{i=1}^{k} D_{i+j-1}^2 = \sum_{i=1}^{j}(D_{i-1}+n)^2 + \sum_{i=j+1}^{k} D_{i-1}^2 = \sum_{i=1}^{k} D_{i-1}^2 + 2n\sum_{i=1}^{j} D_{i-1} + n^2 j,$$

and

$$\sum_{j=1}^{k}\sum_{i=1}^{k} D_{i+j-1}D_{i-1} = \left(\sum_{i=1}^{k} D_i\right)^2 + n\sum_{i=1}^{k} iD_i$$

We can now rewrite equation (D.5) as:

$$k\sum_{i=1}^{k} D_{i-1}^2 + 2n\sum_{j=1}^{k}\sum_{i=1}^{j} D_{i-1} + n^2\frac{k(k-1)}{2} - 2\left(\sum_{i=1}^{k} D_i\right)^2 - 2n\sum_{i=1}^{k} iD_i$$

Note that $\sum_{j=1}^{k}\sum_{i=1}^{k} D_{i-1} = \sum_{i=1}^{k-1} D_{i-1}\sum_{j=i}^{k-1} 1 = \sum_{i=1}^{k-1}(k-i)D_{i-1}$
Therefore, the computation of (4) takes linear time.

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