Russell Grant Woods

Certain properties of $\beta X-X$ for σ-compact X

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ABSTRACT

A study is made of the topological space $\beta X-X$, where βX denotes the Stone-Cech compactification of the σ-compact Hausdorff space X. A homomorphism is defined from the Boolean algebra $R(X)$ of all regular closed subsets of X into $R(\beta X-X)$. Under certain conditions, the image under this mapping of a certain subalgebra of $R(X)$ is isomorphic to the Boolean algebra of all open-and-closed subsets of $\beta\underline{N}-\underline{N}$, where $\underline{N}$ is the countable discrete space. This result is used to obtain new properties of the projective cover of $\beta X-X$ and of the set of remote points of βX. The Lebesgue dimension of $\beta X-X$ is studied, and a new proof is given of the (known) theorem that $\beta\underline{R}^n-\underline{R}^n$ has dimension n (where $\underline{R}^n$ denotes Euclidean n-space). Let $\underline{R}^+$ denote the space of non-negative real numbers. Then $\beta\underline{R}^+-\underline{R}^+$ is an indecomposable continuum containing decomposable subcontinua. The space $\beta X-X$ is not connected im kleinen at any point.

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A THESIS

PRESENTED TO THE

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IN PARTIAL FULFILLMENT

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INTRODUCTION

This thesis is devoted to a study of certain properties of the topological space $\beta X-X$, where X is a σ-compact Hausdorff topological space and βX denotes the Stone-Cech compactification of X.

In chapter I a summary is given of known results that will be used throughout the thesis.

In chapter II a homomorphism is defined from the Boolean algebra $R(X)$ of all regular closed subsets of X into $R(\beta X-X)$. It is then shown, assuming the continuum hypothesis, that if the ring of real-valued continuous functions defined on the σ-compact space X has cardinality $2^{\aleph_0}$, then the image under the above homomorphism of a certain subalgebra of $R(X)$ is isomorphic to the Boolean algebra of all open-and-closed subsets of $\beta\underline{\underline{N}}-\underline{\underline{N}}$, where $\underline{\underline{N}}$ denotes the countable discrete space. The proof relies on Parovicenko's characterization of this latter Boolean algebra. Still assuming the continuum hypothesis, the projective cover of $\beta X-X$, i.e. the Stone space of $R(\beta X-X)$, is shown to be homeomorphic to the projective cover of $\beta\underline{\underline{N}}-\underline{\underline{N}}$, and an alternative proof of this is given. The chapter closes with a necessary condition that a compact space have a projective cover homeomorphic to that of $\beta\underline{\underline{N}}-\underline{\underline{N}}$.

Chapter III is concerned primarily with the properties of the set of remote points of βX, i.e. those points that are not in the βX-closure of any discrete subspace of X. The concept of a remote point of $\beta\underline{\underline{R}}$ (where $\underline{\underline{R}}$ denotes the space of real numbers) was first defined by Fine and Gillman, who proved, assuming the continuum hypothesis, that $\beta\underline{\underline{R}}$ has a set of $2^{2^{\aleph_0}}$ remote points which form a dense subset of $\beta\underline{\underline{R}}-\underline{\underline{R}}$. Plank extended these results to show if X is a σ-compact metric space without isolated points, then βX has $2^{2^{\aleph_0}}$ remote points which form a dense subset of $\beta X-X$. Plank gives an explicit formula for this subset.

Chapter III begins with a generalization of some results of of Gleason, who demonstrated the existence of an irreducible map from the projective cover of a compact space Y onto Y . It is shown that if $\mathcal{A}$ is a suitable subalgebra of $R(Y)$, then there is an irreducible mapping from the Stone space of $\mathcal{A}$ onto Y . Under somewhat stronger conditions on Y and $\mathcal{A}$, it is shown that there is associated with $\mathcal{A}$ a dense subset of Y that can be embedded densely in the Stone space of $\mathcal{A}$. This and the results of chapter II are combined with Plank's results to show that if X is a σ-compact metric space without isolated points, then, assuming the continuum hypothesis, the remote points of $\beta X-X$ can be embedded densely in $\beta\underline{\underline{N}}-\underline{\underline{N}}$. Call a space strongly countably compact if the closure in X of every countable subset of X is compact. Under the above assumptions it is shown that both the remote points of βX and their complement in $\beta X-X$ are strongly countably compact. Thus $\beta X-X$ can be decomposed into two disjoint dense strongly countably compact subspaces. A similar decomposition is obtained for $\beta\underline{\underline{N}}-\underline{\underline{N}}$.

In chapter IV a necessary and sufficient condition is obtained that the (Lebesgue) dimension of $\beta X - X$ be equal to the non-negative integer n. This criterion is used to obtain a new proof of the known result, first proved by Jerison, that the dimension of $\beta \underline{\underline{R}}^n - \underline{\underline{R}}^n$ is n.

In chapter V it is shown that $\beta \underline{\underline{R}}^+ - \underline{\underline{R}}^+$ (where $\underline{\underline{R}}^+$ denotes the space of non-negative real numbers) is an indecomposable continuum, but contains proper subcontinua that are decomposable. If $n > 1$ then $\beta \underline{\underline{R}}^n - \underline{\underline{R}}^n$ is a decomposable continuum. It is also shown that if X is σ-compact then $\beta X - X$ is not connected im kleinen at any point, and hence is not locally connected at any point.

Originality may be claimed for all results in chapters II to V with the exception of the following: 2.1, 2.10, 2.14, 2.16, 3.16, 3.17, and 4.11. Theorem 4.11 is due to Jerison, but our proof of it is new. Lemma 3.20 was discovered independently by Mandelker and myself. The concept of a basic subalgebra (1.86) also appears to be new.

CONTENTS

I PRELIMINARIES

In this chapter we shall give a summary of the definitions and results that will be needed in later chapters. No original material appears in this chapter, with the exception of the concept of a basic subalgebra of $R(X)$ (1.86), which to our knowledge is new.

A. Set-theoretic Notation and Conventions

1.1 Notation

(i) The set of non-negative integers will be denoted by N.

(ii) The countable discrete space will be denoted by $\underline{\underline{N}}$.

(iii) The space of real numbers will be denoted by $\underline{\underline{R}}$, the non-negative real numbers will be denoted by $\underline{\underline{R}}^+$, and $\underline{\underline{R}}^n$ will denote Euclidean n-space.

1.2 Notation

The cardinality of a set A will be denoted by $|A|$.

1.3 Notation

The cardinal number of N will be denoted by $\aleph_0$, and $\aleph_1$ will denote the first uncountable cardinal.

1.4 Remark

By the continuum hypothesis we shall mean the assumption that $\aleph_1 = 2^{\aleph_0}$. The use of the continuum hypothesis in a proof will be indicated by the appearance of the symbol "[CH]" immediately preceding the statement of the theorem.

1.5 Notation

If $\mathfrak{F}$ is a family of sets, the set $\bigcap_{F \in \mathfrak{F}} F$ will be denoted by $\bigcap \mathfrak{F}$.

B. Topological Prerequisites

Results for which a reference is not supplied can be found in the text by Gillman and Jerison [GJ] . References will be given for the other results that are quoted.

1.6 Definition

A Hausdorff space X is said to be completely regular if, given any closed subset A of X and any point $p \in X$ such that $p \notin A$, there exists a continuous real-valued function f defined on X such that $f(p) = 0$ and $f[A] = \{1\}$.

Throughout this thesis all topological spaces will be assumed to be completely regular Hausdorff spaces.

1.7 Notation

Let S be a subset of a space X. The closure, interior, and boundary of S with respect to X will be denoted respectively by $cl_X S$, $int_X S$, and $bd_X S$ $(bd_X S = cl_X S - int_X S)$.

1.8 Definition

(i) A closed subset S of a space X is said to be regular closed if $S = cl_X(int_X S)$.

(ii) An open subset V of a space X is said to be regular open if $\mathrm{int}_X(\mathrm{cl}_X V) = V$.

1.9 <u>Proposition</u>

If $\mathcal{B}$ is a base for the closed subsets of X , then so is $\{\mathrm{cl}_X(\mathrm{int}_X B) : B \in \mathcal{B}\}$.

<u>Proof</u>: Let A be a closed subset of X . If $p \notin A$, then as X is completely regular there exists a continuous real-valued function f on X such that $f(p) = 0$ and $f[A] = \{1\}$. Thus $f^{-1}([1/2 , 3/2])$ is a closed subset of X not containing p , so as $\mathcal{B}$ is a base for the closed subsets of X , there exists $B(p) \in \mathcal{B}$ such that $p \notin B(p)$ and $f^{-1}([1/2 , 3/2]) \subseteq B(p)$. Thus $A \subseteq f^{-1}((1/2 , 3/2)) \subseteq \mathrm{int}_X B(p)$ and so $A = \bigcap_{p \notin A} \mathrm{cl}_X(\mathrm{int}_X B(p))$. The proposition follows.

1.10 <u>Definition</u>

If X is a locally compact, non-compact space that can be written as the union of countably many compact subspaces, then X is said to be a σ-compact space.

Note that by our convention σ-compact spaces are not compact.

The next two results can be found in Dugundji [D] .

1.11 <u>Theorem</u>

A non-compact space X is σ-compact if and only if X can be expressed in the form $\bigcup_{n=0}^{\infty} V_n$ where for each $n \in N$, V_n is open in in X and $cl_X V_n$ is compact and contained in V_{n+1} . Without loss of generality we may also assume that each V_n is regular open and that $V_{n+1} - cl_X V_n$ is non-empty.

1.12 Theorem

A σ-compact space is normal.

1.13 Proposition

A closed subset S of a σ-compact space X is compact if and only if there exists $n \in N$ such that $S \subseteq V_n$.

Proof: If $S \subseteq V_n$ then S is a closed subset of the compact space $cl_X V_n$; conversely, if $S \not\subseteq V_n$ for each $n \in N$, then $(S \cap V_n)_{n \in N}$ is an open cover of S that has no finite subcover.

1.14 Definition

The ring of all continuous real-valued functions defined on a space X will be denoted by $C(X)$. The ring of all continuous bounded real-valued functions defined on X will be denoted by $C^*(X)$.

1.15 Definition

A subset S of a space X is said to be C-embedded (C^*-embedded) in X if, given any $f \in C(S)$ ($f \in C^*(S)$), there exists $g \in C(X)$ such that the restriction of g to S is f .

1.16 Proposition

In a normal space every closed subset is C-embedded.

1.17 <u>Definition</u>

Any subset of a space X that is of the form $f^{-1}(0)$ for some $f \in C(X)$ is called a zero-set of X .

1.18 <u>Notation</u>

The family of all zero-sets of X will be denoted by $Z(X)$.

1.19 <u>Proposition</u>

A Hausdorff space X is completely regular if and only if $Z(X)$ forms a base for the closed sets of X .

1.20 <u>Proposition</u>

The family $Z(X)$ is a lattice under set-theoretic union and intersection, and is closed under countable intersection.

1.21 <u>Proposition</u>

If X is a metric space then every closed subset of X is a zero-set.

1.22 <u>Definition</u>

A cozero-set of X is the complement of some zero-set of X .

1.23 <u>Definition</u>

A z-filter on a space X is a non-empty subfamily $\mathcal{F}$ of $Z(X)$ satisfying the following conditions:

(i) $\emptyset \notin \mathcal{F}$

(ii) If Z_1 and $Z_2 \in \mathcal{F}$, then $Z_1 \cap Z_2 \in \mathcal{F}$.

(iii) If $Z_1 \in \mathcal{F}$ and $Z_1 \subseteq Z_2$ then $Z_2 \in \mathcal{F}$.

A z-filter that is not properly contained in any other z-filter is called a z-ultrafilter. By Zorn's lemma every z-filter is contained in some z-ultrafilter.

1.24 Theorem

Corresponding to every completely regular space X there is a compact space βX which contains X as a dense subspace and which is characterized up to homeomorphism by each of the following equivalent conditions:

(i) X is C*-embedded in βX .

(ii) For any Z_1 , $Z_2 \varepsilon Z(X)$, $cl_{\beta X}(Z_1 \cap Z_2) = cl_{\beta X}Z_1 \cap cl_{\beta X}Z_2$.

The space βX is called the Stone-Cech compactification of X .

1.25 Theorem

The family $\{cl_{\beta X}Z : Z \varepsilon Z(X)\}$ is a base for the closed subsets of βX .

1.26 Notation

The space $\beta X - X$ will be denoted by X^* .

1.27 Theorem

If X is a normal space and if A and B are any two closed subsets of X then $cl_{\beta X}(A \cap B) = cl_{\beta X}A \cap cl_{\beta X}B$.

Proof: Obviously $cl_{\beta X}(A \cap B) \subseteq cl_{\beta X}A \cap cl_{\beta X}B$. Conversely, suppose that $p \notin cl_{\beta X}(A \cap B)$. By 1.25 there exists $(Z_\alpha)_\alpha \subseteq Z(X)$ such that $p \varepsilon \bigcap_\alpha cl_{\beta X}Z_\alpha$ and $(\bigcap_\alpha cl_{\beta X}Z_\alpha) \cap cl_{\beta X}(A \cap B) = \emptyset$. As βX is compact, there exist $(Z_i)_{1 \le i \le n}$

in $Z(X)$ such that $p \in \bigcap_{i=1}^{n} cl_{\beta X}Z_i$ and $[\bigcap_{i=1}^{n} cl_{\beta X}Z_i] \cap cl_{\beta X}(A \cap B) = \emptyset$. Put $Z_0 = \bigcap_{i=1}^{n} Z_i$. By 1.20 $Z_0 \in Z(X)$ and by 1.24 (ii) $p \in cl_{\beta X}Z_0$; obviously $cl_{\beta X}Z_0 \cap cl_{\beta X}(A \cap B) = \emptyset$. Thus $Z_0 \cap A \cap B = \emptyset$. As X is normal, the disjoint closed sets $Z_0 \cap A$ and B are contained in disjoint zero-sets; it follows from 1.24 (ii) that $cl_{\beta X}(Z_0 \cap A) \cap cl_{\beta X}B = \emptyset$. If $p \notin cl_{\beta X}B$ we are finished; if not, $p \notin cl_{\beta X}(Z_0 \cap A)$. A repetition of the above argument shows that there exists $Z' \in Z(X)$ such that $Z' \cap Z_0 \cap A = \emptyset$ and $p \in cl_{\beta X}Z'$. Thus $p \in cl_{\beta X}Z_0 \cap cl_{\beta X}Z' = cl_{\beta X}(Z_0 \cap Z')$ (by 1.24 (ii)), and so $p \notin cl_{\beta X}A$ (as above). Thus $p \notin cl_{\beta X}A \cap cl_{\beta X}B$ and the theorem follows.

1.28 <u>Theorem</u>

The space X is open in βX if and only if X is locally compact. Thus $\beta X-X$ is compact if and only if X is locally compact.

1.29 <u>Theorem</u>

The space $\beta \underline{N}$ has cardinality $2^{2^{\aleph_0}}$.

1.30 <u>Theorem</u>

A subset S of a space X is C*-embedded in X if and only if $cl_{\beta X}S = \beta S$.

There is a natural one-to-one mapping from βX onto the family of all z-ultrafilters on X . This relationship is summarized in the following theorem.

1.31 Theorem

If $p \in \beta X$, then the family $\mathcal{A}^p = \{Z \in Z(X) : p \in \mathrm{cl}_{\beta X} Z\}$ is a z-ultrafilter (1.24) on X. Conversely, if $\mathcal{A}$ is a z-ultrafilter on X, then $\bigcap_{Z \in \mathcal{A}} \mathrm{cl}_{\beta X} Z$ is a unique point of βX, and $\bigcap_{Z \in \mathcal{A}^p} \mathrm{cl}_{\beta X} Z = \{p\}$ for each $p \in \beta X$.

1.32 Proposition

If f is a continuous mapping of a space X onto a space Y whose restriction to a dense subset T is a homeomorphism, then f carries $X - T$ onto $Y - f[T]$.

1.33 Definition

A point p of a space X is called a P-point of X if every G_δ-set of X that contains p is a neighborhood of p. If every point of X is a P-point, then X is called a P-space.

1.34 Notation

The set of all P-points of a space X will be denoted by $P(X)$.

1.35 Proposition

(i) A space X is a P-space if and only if every zero-set of X is open.

(ii) A point p is a P-point of X if and only if every zero-set containing p is a neighborhood of p.

(iii) A P-point of X is a P-point in any subspace of X that contains it.

(iv) A compact P-space is finite.

(v) For any space X , $P(X) = \bigcap_{Z \in Z(X)} [X - \mathrm{bd}_X Z]$.

1.36 Definition

A space X is called an F-space if every cozero-set of X is C*-embedded in X .

1.37 Theorem

If X is σ-compact then $\beta X-X$ is a compact F-space.

1.38 Theorem

If D is a countable subset of an F-space X then D is C*-embedded in X .

1.39 Definition

A space X is said to be realcompact if, for every $p \in \beta X-X$, there exists $Z \in Z(\beta X)$ such that $p \in Z \subseteq \beta X-X$.

1.40 Proposition

Every σ-compact space is realcompact.

The following two results are due to Fine and Gillman [FG_1, lemma 3.1] , although 1.42 is not explicitly proved.

1.41 Theorem

If X is locally compact and realcompact then every zero-set of $\beta X-X$ is regular closed.

1.42 Proposition

Every zero-set of a space X is regular closed if and only if every non-empty G_δ-set of X has a non-empty interior.

Proof: First note that if $f \in C(X)$ then $f^{-1}(0) = \bigcap_{n=1}^{\infty} f^{-1}(-1/n\ ,\ 1/n)$ and so $f^{-1}(0)$ is a G_δ-set.

Suppose $cl_X(int_X Z) \neq Z$ for some $Z \in Z(X)$. Choose $p \in Z - cl_X(int_X Z)$. Then as X is completely regular there exists $Z' \in Z(X)$ such that $p \in Z' \subseteq X - cl_X(int_X Z)$ (choose $f \in C(X)$ such that $f(p) = 0$ and $f[cl_X(int_X Z)] = \{1\}$ and put $Z' = f^{-1}(0)$). Thus $Z \cap Z' \in Z(X)$ but $int_X(Z \cap Z') = \emptyset$, so $Z \cap Z'$ is a non-empty G_δ-set with an empty interior.

Conversely, suppose that every zero-set of X is regular closed. Let $(V_n)_{n \in N}$ be a countable family of open subsets of X , and let $p \in \bigcap_{n=0}^{\infty} V_n$. As above there exists $Z_n \in Z(X)$ such that $p \in Z_n \subseteq V_n$. Thus $p \in \bigcap_{n=0}^{\infty} Z_n \subseteq \bigcap_{n=0}^{\infty} V_n$ and by 1.20 , $\emptyset \neq \bigcap_{n=0}^{\infty} Z_n \in Z(X)$. Thus $\emptyset \neq int_X(\bigcap_{n=0}^{\infty} Z_n)$ and so $int_X(\bigcap_{n=0}^{\infty} V_n) \neq 0$.

The following result is due to Rudin [R] and Parovičenko [Pa] for $X = \underline{N}$; the more general case follows similarly by use of 1.41 .

1.43 Theorem

Let X be locally compact and realcompact and let $\mathfrak{F}$ be a family of dense open subsets of X . If $|\mathfrak{F}| \leq \aleph_1$, then $\bigcap \mathfrak{F}$ is dense in $\beta X - X$.

An immediate consequence of 1.43 is the following result:

1.44 Theorem [CH]

If X is locally compact and realcompact and if $|C(X)| = 2^{\aleph_0}$, then $\beta X - X$ has a dense set of $2^{2^{\aleph_0}}$ P-points.

1.45 Notation

Let S be a partially ordered set and let A and B be subsets of S. Then "$A < B$" will mean that $a<b$ for each $a \in A$ and $b \in B$. If $s \in S$ then $A < s$ and $s < B$ will mean, respectively, that $a<s$ for each $a \in A$ and $s<b$ for each $b \in B$.

1.46 Definition

A totally ordered set S is called an η_1-set if, given any empty, finite, or countably infinite subsets A and B of S such that $A < B$, there exists $c \in S$ such that $A < c < B$.

1.47 Proposition

The cardinality of any η_1-set is at least $2^{\aleph_0}$.

1.48 Definition

If S is a partially ordered set, then a maximal chain in S is a totally ordered subset of S that is not properly contained in any other totally ordered subset of S.

By Zorn's lemma each totally ordered subset of S is contained in some maximal chain of S.

The following theorem is due to Rudin [R] and Parovicenko [Pa] .

1.49 Theorem [CH]

The Boolean algebra of all open-and-closed subsets of $\beta\underline{\underline{N}}-\underline{\underline{N}}$ has cardinality $2^{\aleph_0}$, and maximal chains in this Boolean algebra (excluding $\emptyset$ and $\beta\underline{\underline{N}}-\underline{\underline{N}}$) are η_1-sets. Further, if B is any other Boolean algebra with these two properties, then B is isomorphic to the Boolean algebra of all open-and-closed subsets of $\beta\underline{\underline{N}}-\underline{\underline{N}}$.

The following notion is due to Fine and Gillman [FG_2] , who proved, assuming the continuum hypothesis, the existence of a set of remote points in $\beta\underline{\underline{R}}$ that is dense in $\beta\underline{\underline{R}}-\underline{\underline{R}}$.

1.50 Definition

A point $p \in \beta X$ is called a remote point of βX if p is not in the βX-closure of any discrete subspace of X .

The idea of demonstrating the existence of remote points in $\beta\underline{\underline{R}}$ by use of 1.43 is due to Negrepontis. Plank [Pl , theorems 5.4 , 5.5] has proved the following more general result.

1.51 Theorem

If X is a separable, locally compact non-compact metric space without isolated points, then the set of remote points of βX is precisely the set $\bigcap_{Z \in Z(X)} [(\beta X-X) - (cl_{\beta X}(bd_X Z) - X)]$, which is identical with the set $\bigcap_{Z \in Z(X)} [(\beta X-X) - bd_{X^*}(cl_{\beta X} Z - X)]$. Assuming the continuum hypothesis, this is a dense subset of $\beta X - X$ of cardinality $2^{2^{\aleph_0}}$.

1.52 Notation

The set of remote points of βX will be denoted by $T(X^*)$.

1.53 Remark

Since a locally compact, non-compact metric space is separable if and only if it is σ-compact, the phrase "separable, locally compact non-compact" in 1.51 may be replaced by "σ-compact".

The following result was proved by Plank [Pl , theorem 6.2] for $X = \underline{\underline{R}}$. His proof can be adapted with virtually no changes to yield the following result.

1.54 Theorem [CH]

Let X be a σ-compact metric space without isolated points. Then the sets $P(X^*) \cap T(X^*)$, $[X^* - P(X^*)] \cap T(X^*)$, $P(X^*) \cap [X^* - T(X^*)]$, and $X^* - [P(X^*) \cup T(X^*)]$ are all dense in $\beta X - X$ and have cardinality $2^{2^{\aleph_0}}$.

1.55 Definition

(i) A space X is said to be extremally disconnected if every open subset of X has an open closure.

(ii) A space X is said to be basically disconnected if every cozero-set of X has an open closure.

1.56 Proposition

A space X is basically disconnected if and only if, given a cozero-set U and an open set V disjoint from U, the set $cl_X U \cap cl_X V$ is empty.

1.57 Proposition

If X is an arbitrary space, its subspace $P(X)$ is basically disconnected.

1.58 Proposition

The space X is extremally disconnected if and only if βX is extremally disconnected.

1.59 Proposition

If X is compact and $p \varepsilon X$, then the connected component of X that contains p is the intersection of all the open-and-closed subsets of X that contain p.

1.60 Definition

A compact connected space is called a continuum.

1.61 Definition

A continuum is said to be indecomposable if it cannot be written as the union of two proper subcontinua.

The following result appears in Hocking and Young [HY , theorem 3.41]

1.62 Theorem

A continuum K is indecomposable if and only if every proper subcontinuum of K has an empty interior.

1.63 Definition

A space X is said to be locally connected at a point p if, for every open set U containing p , there is a connected open set V containing p and contained in U .

1.64 Definition

A space X is said to be connected im kleinen at a point p if for each open set U containing p , there is an open set V containing p and lying in U such that if y is any point of V , then there is a connected subset of U containing p and y .

The following result appears in Hocking and Young [HY] .

1.65 Proposition

If X is locally connected at p , then it is connected im kleinen at p , but the converse is untrue in general.

1.66 Definition

(i) A cover of a space X is a finite collection of open subsets of X whose union is X .

(ii) A collection $\mathcal{V}$ of open subsets of a space X is said to be a refinement of a cover $\mathcal{U}$ of X if $\mathcal{V}$ is a cover of X and if every member of $\mathcal{V}$ is a subset of some member of $\mathcal{U}$.

1.67 Definition

The order of a cover $\mathcal{U}$ of a space X - abbreviated ord$\mathcal{U}$ - is defined as follows:

$$\mathrm{ord}\,\mathcal{U} = \sup\{n \in \mathbb{N} : \text{there exists } (U_i)_{1\le i\le n+1} \subseteq \mathcal{U} \text{ such that } \bigcap_{i=1}^{n+1} U_i \neq \emptyset\}$$

1.68 <u>Definition</u>

The Lebesgue dimension of a space X - abbreviated $\dim X$ - is defined as follows:

$$\dim X = \min\{n \in N : \text{every cover of } X \text{ has a refinement of order} \leq n\}$$

If the above set is empty then $\dim X = \aleph_0$.

1.69 <u>Theorem</u>

If X and Y are normal spaces and if X is C^*-embedded in Y then $\dim X \leq \dim Y$.

1.70 <u>Theorem</u>

If X is normal then $\dim X = \dim(\beta X)$.

The following result can be found in Hurewicz and Wallman [HW , theorem IV 3] .

1.71 <u>Theorem</u>

If $A \subseteq \underline{R}^n$ then $\dim A = n$ if and only if $\operatorname{int}_{\underline{R}^n} A \neq \emptyset$.

C. <u>Boolean Algebras</u>

The basic reference for the following material is Sikorski [S] . All Boolean algebras are assumed to contain 1 .

1.72 <u>Definition</u>

Consider the following conditions on a subset F of a Boolean algebra B :

(i) $0 \notin F$

(ii) If $x \in F$ and $y \in F$ then $x \wedge y \in F$.

(iii) If $x \in F$ and $x \leq y$ then $y \in F$.

If F satisfies these three conditions then F is called a filter. A filter that is not properly contained in any other filter is called an ultrafilter.

By Zorn's lemma every filter is contained in some ultrafilter.

1.73 <u>Definition</u>

Let B be a Boolean algebra. If every subset of B has a supremum in B , then B is said to be complete. If every countable subset of B has a supremum in B , then B is said to be σ-complete.

1.74 <u>Definition</u>

Let B be a Boolean algebra and let S be a subset of B . If, for every $x \in B$ such that $x \neq 0$, there exists $y \in S$ such that $0 \neq y \leq x$, then S is said to be a dense subset of B . If S is also a subalgebra of B , then S is said to be a dense subalgebra of B .

1.75 <u>Theorem</u>

Let B_1 and B_2 be complete Boolean algebras, let U_1 be a dense subalgebra of B_1 , and let U_2 be a dense subalgebra of B_2 . If $f:U_1 \to U_2$ is a Boolean algebra isomorphism of U_1 onto U_2 , then there exists a Boolean algebra isomorphism $g:B_1 \to B_2$ such that the restriction of g to U_1 is f .

1.76 <u>Definition</u>

Let B be a Boolean algebra and U a subalgebra of B . Then

U is said to be a σ-subalgebra of B if, for every countable subset S of U, the supremum in B of S belongs to U whenever it exists.

Obviously a σ-subalgebra of a complete Boolean algebra is σ-complete.

1.77 Definition

Let B be a Boolean algebra and S a subset of B. Then S is said to generate a subalgebra A of B if A is the smallest subalgebra of B that contains S. The subalgebra of B generated by S will be denoted by $\langle S \rangle$.

1.78 Theorem

Let S be a subset of a Boolean algebra B. Then $\langle S \rangle$ consists of all elements of the form $\bigvee_{j=1}^{m} \bigwedge_{i=1}^{n} x_{ij}$, where $m, n \in N$ and for each pair i, j, either x_{ij} or its complement belongs to S.

1.79 Definition

Let B be a Boolean algebra and S a subset of B. Then S is said to σ-generate a subalgebra A of B if A is the smallest σ-subalgebra of B that contains S. The subalgebra of B σ-generated by S will be denoted by σS.

Obviously if B is complete and $S \subseteq B$ then σS is σ-complete.

1.80 Theorem

If S is a subset of the Boolean algebra B, then $|\sigma S| \leq |S|^{\aleph_0}$.

D. Boolean Algebras of Regular Closed Sets

Results for which no reference is given can be found in Sikorski [S] .

1.81 Notation

The family of all regular closed subsets of a space X will be denoted by $R(X)$.

1.82 Theorem

The family $R(X)$ is a complete Boolean algebra under the following definitions of $\leq$, $\bigvee$, $\bigwedge$, and complement $'$:

(i) $A \leq B$ if and only if $A \subseteq B$.

(ii) $\bigvee_{i=1}^{n} A_i = \bigcup_{i=1}^{n} A_i$

(iii) $\bigwedge_{i=1}^{n} A_i = cl_X(\bigcap_{i=1}^{n} int_X A_i)$

(iv) $A' = cl_X(X - A)$.

Henceforth the symbols $\bigvee$, $\bigwedge$, and $'$, when applied to regular closed sets, are to be interpreted according to the above definitions.

1.83 Notation

The family of all open-and-closed subsets of a space X will be denoted by $B(X)$.

1.84 Proposition

The family $B(X)$ is a subalgebra of $R(X)$ and if A , $B \in B(X)$, then $A \wedge B = A \cap B$ and $A' = X - A$.

1.85 Proposition

If X is a compact totally disconnected space then $B(X)$ is dense (see 1.74) in $R(X)$.

1.86 Definition

A subalgebra $\mathcal{A}$ of $R(X)$ is called a basic subalgebra of $R(X)$ if $\mathcal{A}$ is a base for the closed subsets of X .

1.87 Theorem

If $\mathcal{A}$ is a basic subalgebra of $R(X)$, if V is open in X , and if $p \in V$, then there exists $A \in \mathcal{A}$ such that $p \in \mathrm{int}_X A \subseteq A \subseteq V$.

Proof: As $\mathcal{A}$ is a base for the closed subsets of X , there exists $\mathcal{F} \subseteq \mathcal{A}$ such that $X - V = \bigcap \mathcal{F}$. Thus

$$V = X - \bigcap \mathcal{F} = \bigcup_{F \in \mathcal{F}} (X - F) = \bigcup_{F \in \mathcal{F}} \mathrm{int}_X(F') .$$

Thus V has been expressed as a union of members of $\{\mathrm{int}_X A : A \in \mathcal{A}\}$. This family is thus a base for the open subsets of X , and hence the theorem follows from the complete regularity of X .

1.88 Proposition

Every basic subalgebra $\mathcal{A}$ of $R(X)$ is dense in $R(X)$.

Proof: Let $B \in R(X)$ and assume that $B \neq \emptyset$. Then $\mathrm{int}_X B \neq \emptyset$ so by 1.87 there exists $A \in \mathcal{A}$ such that $\emptyset \neq A \subseteq \mathrm{int}_X B$. The proposition follows.

The converse of 1.88 is not true in general.

1.89 Theorem

A space X is extremally disconnected if and only if $B(X)$ is complete and is a base for the open subsets of X.

1.90 Theorem

Let U be a Boolean algebra and let $S(U)$ be the family of all ultrafilters (1.72) on U. For each $x \varepsilon U$, let $\lambda(x) = \{\alpha \varepsilon S(U) : x \varepsilon \alpha\}$. If a topology τ is assigned to $S(U)$ by letting $\{\lambda(x) : x \varepsilon U\}$ be an open base for τ, then $(S(U), \tau)$ is a compact totally disconnected space and the map $x \to \lambda(x)$ is a Boolean algebra isomorphism from U onto $B(S(U))$. The space $S(U)$ is called the Stone space of U.

1.91 Definition

Let $f:X \to Y$ be a continuous mapping of the space X onto the space Y. If $f[A] \neq Y$ for each proper closed subset A of X, then f is said to be irreducible.

1.92 Proposition

If $f:X \to Y$ is an irreducible closed mapping of X onto Y, and if D is dense in Y, then $f^{-1}[D]$ is dense in X.

Proof: As f is closed, we have $f[cl_X(f^{-1}[D])] \supseteq cl_Y D = Y$. It follows from the irreducibility of f that $cl_X(f^{-1}[D]) = X$.

The following result is due to Gleason [G, theorem 3.2], who first investigated projective covers of compact spaces.

1.93 Theorem

Let X be a compact space. Then there exists an irreducible continuous map f sending $S(R(X))$ onto X ; the map f is defined by $f(y) = \bigcap_{\substack{A \varepsilon R(X) \\ y \varepsilon \lambda(A)}} A$ for each $y \varepsilon S(R(X))$, where $\lambda : R(X) \to B(S(R(X)))$ is the canonical isomorphism defined in 1.90 . Furthermore, if K is any other compact extremally disconnected space and if $g:K \to X$ is an irreducible mapping of K onto X , then there exists a homeomorphism $h:K \to S(R(X))$ such that $g = f \circ h$.

The space $S(R(X))$ is called the projective cover of X .

II THE PROJECTIVE COVER OF $\beta X-X$

Let X be any σ-compact space. In this chapter we will define a Boolean algebra homomorphism $*: R(X) \to R(\beta X-X)$ and examine the properties of this homomorphism. In particular we will show that if $\mathcal{S}$ is a basic subalgebra (1.86) of $R(X)$ such that $|\mathcal{S}| = 2^{\aleph_0}$ and with the property that if $(S_n)_{n\in N} \subseteq \mathcal{S}$ and if $\bigcup_{n=0}^{\infty} S_n \in R(X)$ then $\bigcup_{n=0}^{\infty} S_n \in \mathcal{S}$, then the image of $\mathcal{S}$ under this homomorphism is isomorphic to $B(\beta\underline{\underline{N}}-\underline{\underline{N}})$. It will then follow that if X is a σ-compact space with $|C(X)| = 2^{\aleph_0}$, then the projective cover of $\beta X-X$ is homeomorphic to that of $\beta\underline{\underline{N}}-\underline{\underline{N}}$.

Throughout this chapter we will assume that X is a σ-compact space (see 1.10) and observe the notational conventions stated in 1.11.

2.1 Notation

Let A be a closed subset of X. Then A^* will denote the set $(\mathrm{cl}_{\beta X}A) - X$. Note that this is consistent with the notation defined in 1.26. If A and B are closed subsets of X, the following results are immediate:

(i) $(A\cup B)^* = A^* \cup B^*$.

(ii) $(A\cap B)^* = A^*\cap B^*$ (see 1.12 and 1.27)

(iii) $A^* = \emptyset$ if and only if A is compact.

2.2 Proposition

Let $(A_n)_{n\in N}$ be a countable family of closed subsets of X, and define the positive integer k_n by:

$$k_n = \min\{j \in N : A_n \cap V_j \neq \emptyset\}$$

for each $n \in N$. Then:

(i) If $\lim\limits_{n\to\infty} k_n = \infty$, then $\bigcup\limits_{n=0}^{\infty} A_n$ is closed.

(ii) If $\lim\limits_{n\to\infty} k_n = \infty$ and $A_n \in R(X)$ for each $n \in N$, then $\bigcup\limits_{n=0}^{\infty} A_n \in R(X)$ and $\bigcup\limits_{n=0}^{\infty} A_n = \bigvee\limits_{n=0}^{\infty} A_n$.

<u>Proof</u>: (i) Let $p \in cl_X(\bigcup\limits_{n=0}^{\infty} A_n)$ and let U be an open subset of X containing p . There exists $i \in N$ such that $p \in V_i$; thus as $U \cap V_i$ is open, it follows that $(U \cap V_i) \cap (\bigcup\limits_{n=0}^{\infty} A_n) \neq \emptyset$. As $\lim\limits_{n\to\infty} k_n = \infty$, there exists $m \in N$ such that $n \geq m$ implies $A_n \cap V_i = \emptyset$. Thus $(U \cap V_i) \cap (\bigcup\limits_{n=0}^{m-1} A_n) \neq \emptyset$ and so $U \cap (\bigcup\limits_{n=0}^{m-1} A_n) \neq \emptyset$. As U was an arbitrary open set containing p , it follows that p belongs to the closed set $\bigcup\limits_{n=0}^{m-1} A_n$, and so $p \in \bigcup\limits_{n=0}^{\infty} A_n$. Thus $\bigcup\limits_{n=0}^{\infty} A_n$ is closed.

(ii) If $p \in \bigcup\limits_{n=0}^{\infty} A_n$, there exists $k \in N$ such that $p \in A_k$. As $A_k \in R(X)$, it follows that $p \in cl_X(int_X A_k) \subseteq cl_X(int_X[\bigcup\limits_{n=0}^{\infty} A_n])$. As $\bigcup\limits_{n=0}^{\infty} A_n$ is closed by (i) , it follows that $\bigcup\limits_{n=0}^{\infty} A_n \in R(X)$. Obviously $\bigcup\limits_{n=0}^{\infty} A_n = \bigvee\limits_{n=0}^{\infty} A_n$.

2.3 <u>Lemma</u>

Let A and B be closed subsets of X .

(i) $A^* \subseteq B^*$ if and only if there exists $n \in N$ such that $A - B \subseteq V_n$.

(ii) If $A^* \subsetneq B^*$ and if $B \in R(X)$, then $(\mathrm{int}_X B) - (A \cup \mathrm{cl}_X V_n) \neq \emptyset$ for each $n \in N$.

Proof: (i) Suppose that $A - B \subseteq V_n$ for some $n \in N$. Then as $\mathrm{cl}_X V_n$ is compact, $\mathrm{cl}_X(A - B)$ is also compact. As A is closed, it follows that $A = (A \cap B) \cup \mathrm{cl}_X(A - B)$. By 2.1,

$$\begin{aligned} A^* &= (A \cap B)^* \cup [\mathrm{cl}_X(A - B)]^* \\ &= (A \cap B)^* \quad \text{(by 2.1 (iii))} \\ &= A^* \cap B^* \quad \text{(by 2.1 (ii))}. \end{aligned}$$

Thus $A^* \subseteq B^*$.

Conversely, suppose that $A - B$ is not contained in V_n for any $n \in N$. Then there exists a sequence $(n_i)_{i \in N}$ of positive integers, with $\lim_{i \to \infty} n_i = \infty$, such that $(A - B) \cap (V_{n_{i+1}} - \mathrm{cl}_X V_{n_i}) \neq \emptyset$ for each $i \in N$. Let $p_i \in (A - B) \cap (V_{n_{i+1}} - \mathrm{cl}_X V_{n_i})$ for each $i \in N$, and put $S = (p_i)_{i \in N}$. By 2.2 S is closed, and obviously S is not contained in any V_n; hence by 1.13 S is not compact and so by 2.1 (iii), $S^* \neq \emptyset$. Obviously $S \subseteq A - B$, so $S^* \subseteq A^*$ and by 2.1 (ii), $S^* \cap B^* = \emptyset$. Consequently $A^* - B^* \neq \emptyset$ and (i) follows.

(ii) By (i) it follows that for each $n \in N$, $B - (A \cup V_{n+1}) \neq \emptyset$ and so $B - (A \cup \mathrm{cl}_X V_n) \neq \emptyset$. Thus $[\mathrm{cl}_X(\mathrm{int}_X B)] \cap [X - (A \cup \mathrm{cl}_X V_n)] \neq \emptyset$ and so $(\mathrm{int}_X B) \cap [X - (A \cup \mathrm{cl}_X V_n)] \neq \emptyset$.

2.4 Proposition

If A is a closed subset of X, then $cl_{X^*}(X^* - A^*) = [cl_X(X - A)]^*$.

Proof: Since $A \cup cl_X(X - A) = X$, by 2.1 (i) it follows that $A^* \cup [cl_X(X - A)]^* = X^*$. Thus $X^* - A^* \subseteq [cl_X(X - A)]^*$, and as $[cl_X(X - A)]^*$ is closed in X^*, it follows that

$$cl_{X^*}(X^* - A^*) \subseteq [cl_X(X - A)]^* \qquad \ldots \quad (1)$$

Conversely, suppose that $x \notin cl_{X^*}(X^* - A^*)$. Since by 1.25 the family $\{cl_{\beta X}Z : Z \in Z(X)\}$ is a base for the closed subsets of βX, it follows that $\{X^* - S^* : S$ closed in $X\}$ is a base for the open sets of X^*. Thus since X^* is completely regular, there exists a closed subset B of X such that $x \in X^* - B^*$ and also $(X^* - B^*) \cap cl_{X^*}(X^* - A^*) = \emptyset$; thus $(X^* - B^*) \cap (X^* - A^*) = \emptyset$ and thus by 2.1 (i), $X^* = (A \cup B)^*$. By 2.3 (i) there exists $i \in N$ such that $X - (A \cup B) \subseteq V_i$. Thus

$$[cl_X(X - A)] \cap (X - B) \subseteq cl_X V_i \qquad \ldots \quad (2);$$

for if not, $[cl_X(X - A)] \cap (X - B) \cap (X - cl_X V_i) \neq \emptyset$ and as $(X - B) \cap (X - cl_X V_i)$ is open, it would follow that $(X - A) \cap (X - B) \cap (X - cl_X V_i) \neq \emptyset$, which contradicts $X - (A \cup B) \subseteq V_i$. It follows from (2) that $[cl_X(X - A)] - B \subseteq V_{i+1}$, and so by 2.3 (i) we have $[cl_X(X - A)]^* \subseteq B^*$. As $x \in X^* - B^*$, it follows that $x \notin [cl_X(X - A)]^*$. Thus $[cl_X(X - A)]^* \subseteq cl_{X^*}(X^* - A^*)$, and combining this with (1) yields the proposition.

2.5 Proposition

If A is a closed subset of X, then $cl_{X*}(int_{X*}A^*) = [cl_X(int_X A)]^*$.

Proof: Since $int_{X*}A^* = X^* - cl_{X*}(X^* - A^*)$, by 2.4 it follows that $int_{X*}A^* = X^* - [cl_X(X - A)]^*$ and so

$$\begin{aligned} cl_{X*}(int_{X*}A^*) &= cl_{X*}(X^* - [cl_X(X - A)]^*) \\ &= [cl_X(X - [cl_X(X - A)])]^* \qquad \text{(by 2.4)} \\ &= [cl_X(int_X A)]^* . \end{aligned}$$

The following result is an immediate consequence of 2.5 .

2.6 Corollary

If $A \in R(X)$ then $A^* \in R(X^*)$.

2.7 Theorem

The map $A \to A^*$ is a Boolean algebra homomorphism from $R(X)$ into $R(X^*)$.

Proof: By 2.6 the map $A \to A^*$ is well-defined. Suppose that A and B belong to $R(X)$. Then by 2.1 (i) ,

$$A^* \vee B^* = A^* \cup B^* = (A \cup B)^* = (A \vee B)^* \qquad \text{(see 1.82)} .$$

Using 2.4 it can be seen that

$$(A')^* = [cl_X(X - A)]^* = cl_{X*}(X^* - A^*) = (A^*)' .$$

Thus the map preserves suprema and complements and hence is a Boolean algebra homomorphism.

Note that the kernel of this map is the family of compact regular closed subsets of X .

2.8 Notation

(i) If $\mathcal{F}$ is a subfamily of $R(X)$, then $[\mathcal{F}]^*$ will denote the family $\{F^* : F \in \mathcal{F}\}$.

(ii) The family $\{cl_X(int_X Z) : Z \in Z(X)\}$ will be denoted by $G(X)$. Recall (1.21) that if X is a metric space then every closed subset of X is a zero-set, and so $G(X) = R(X)$.

The following result will be needed later.

2.9 Proposition

The family $[G(X)]^*$ is a base for the closed subsets of X^* .

Proof: Since $\{Z^* : Z \in Z(X)\}$ is a base for the closed subsets of X^* (see 1.25) , it follows from 1.9 that $\{cl_{X^*}(int_{X^*} Z^*) : Z \in Z(X)\}$ is a base for the closed subsets of X^* . The proposition then follows from 2.5 .

The proof of the following result mimics that of [GJ , lemma 13.5] .

2.10 Lemma

Let $\mathcal{A}$ be any subalgebra of $R(X)$ and let $\mathcal{E}$ be any countable subset of $[\mathcal{A}]^*$. Then $\mathcal{E}$ has a family $(E_n)_{n \in N}$ of preimages in $\mathcal{A}$ - that is, $\mathcal{E} = (E_n^*)_{n \in N}$ - such that if E_i , $E_j \in (E_n)_{n \in N}$, then $E_i^* \subseteq E_j^*$ implies $E_i \subseteq E_j$.

Proof: Let $(F_n)_{n \in N}$ be any indexed family of preimages of $\mathcal{E}$. The family

$(E_n)_{n\varepsilon N}$ will be defined inductively. Put $E_0 = F_0$. For a fixed $n \varepsilon N$, assume that for each $k < n$, E_k has been defined so that:

(i) $E_j^* \subseteq E_k^*$ implies $E_j \subseteq E_k$ for any $j , k < n$.

(ii) $E_k^* = F_k^*$ for each $k < n$.

Let $H = \sup \{E_j : E_j^* \subseteq F_n^*\}$ and let $K = \inf \{E_j : E_j^* \supseteq F_n^*\}$. From the induction hypotheses, $H \subseteq K$. Define $E_n = (H \vee F_n) \wedge K$, with the convention that H or K is simply omitted in case the set defining it is empty. Then $(E_k)_{0 \leq k \leq n}$ satisfies (i) and (ii) , and so $(E_n)_{n\varepsilon N}$ has the desired properties.

2.11 Theorem

Let $\mathcal{S}$ be a basic subalgebra of $R(X)$ with the property that if $(S_n)_{n\varepsilon N} \subseteq \mathcal{S}$ and if $\bigcup_{n=0}^{\infty} S_n \varepsilon R(X)$, then $\bigcup_{n=0}^{\infty} S_n \varepsilon \mathcal{S}$. Then maximal chains in $[\mathcal{S}]^* - \{\emptyset , X^*\}$ are η_1-sets (see 1.46 and 1.48) .

Proof: Let $\mathcal{A}$ and $\mathcal{B}$ be chains (with respect to set-theoretic inclusion) in $[\mathcal{S}]^* - \{\emptyset , X^*\}$, and assume that both $\mathcal{A}$ and $\mathcal{B}$ have cardinality no greater than $\aleph_0$. Then in order to prove the theorem, it suffices to show that if $\mathcal{A} < \mathcal{B}$, then there exists $C \varepsilon \mathcal{S}$ such that $\mathcal{A} < C^* < \mathcal{B}$ (see 1.45 for notation). As $\mathcal{S}$ is a basic subalgebra of $R(X)$, it can be assumed without loss of generality that $X - V_n \varepsilon \mathcal{S}$; for as $(\mathrm{int}_X S)_{S\varepsilon\mathcal{S}}$ is a base for the open subsets of X (see 1.87) , there exists, for each $n \varepsilon N$, a family $(S_\alpha)_{\alpha\varepsilon\Sigma} \subseteq \mathcal{S}$ such that $V_{n+1} = \bigcup_{\alpha\varepsilon\Sigma} \mathrm{int}_X S_\alpha$. Thus $\mathrm{cl}_X V_n \subseteq \bigcup_{\alpha\varepsilon\Sigma} \mathrm{int}_X S_\alpha$, and so by the compactness of $cl_X V_n$ there exist $\alpha_1, \cdots, \alpha_k \in \Sigma$ such that $cl_X V_n \subseteq \bigcup_{i=1}^{k} int_X S_{\alpha_i}$. Thus $cl_X V_n \subseteq \bigvee_{i=1}^{k} S_{\alpha_i} \subseteq cl_X V_{n+1}$, and we may replace $X - V_n$ by $(\bigvee_{i=1}^{k} S_{\alpha_i})'$.

Let $\mathcal{A} = (A^*_n)_{n \in N}$ and $\mathcal{B} = (B^*_n)_{n \in N}$. There are several cases to consider.

<u>Case 1</u> Assume that $\mathcal{A}$ either is empty or has a largest member, and that $\mathcal{B}$ has no smallest member. Let A^* be the largest member of $\mathcal{A}$ (for some $A \in \mathcal{S}$), and put $A^* = \emptyset$ if $\mathcal{A}$ is empty. By replacing B^*_n by $\bigwedge_{i=0}^{n} B^*_i$ if necessary, and noting that $\mathcal{B}$ has no smallest member, we can assume that $A^* \subsetneq B^*_{n+1} \subsetneq B^*_n$ for each $n \in N$. Thus by 2.10 we can assume that $A \subsetneq B_{n+1} \subsetneq B_n$ for each $n \in N$. By 2.3 (ii) it is evident that $(int_X B_n) - (A \cup cl_X V_n) \neq \emptyset$ for each $n \in N$. Thus for each $n \in N$, there exists $k_n \in N$ such that the open set

$$(int_X B_n) \cap (X - A) \cap (X - cl_X V_n) \cap V_{k_n}$$

is non-empty. As $\mathcal{S}$ is a basic subalgebra of $R(X)$, by 1.87 there exists $S_n \in \mathcal{S}$ such that

$$\emptyset \neq S_n \subseteq (int_X B_n) \cap (X - A) \cap (X - cl_X V_n) \cap V_{k_n} \quad \cdots \quad (1)$$

Put $E = \bigcup_{n=0}^{\infty} S_n$. As $S_n \subseteq X - cl_X V_n$ for each $n \in N$, by 2.2 (ii) $E \in R(X)$. Thus by hypothesis $E \in \mathcal{S}$. By (1), $E - V_n \supseteq S_n - V_n \neq \emptyset$ for each $n \in N$, so by 2.3 (i) $E^* \neq \emptyset$. As $S_n \cap A = \emptyset$ for each $n \in N$ (see (1)), it follows that $E \cap A = \emptyset$, and so $E^* \cap A^* = \emptyset$.

Put $C = A \cup E$; then it follows from the above remarks that $A^* \subsetneqq C^*$ and $C \in \mathcal{S}$. As $A \subset B_j$ for each $j \in N$, it follows that

$$C - B_j = E - B_j = \bigcup_{n=0}^{\infty} (S_n - B_j) \qquad \cdots \qquad (2) .$$

But by (1) , $S_n - B_j \subseteq (\mathrm{int}_X B_n) - B_j$. If $n \geq j$ then $B_n \subset B_j$ and so $S_n - B_j$ is empty. Thus by (2) ,

$$C - B_j = \bigcup_{n=0}^{j-1} (S_n - B_j) \subseteq \bigcup_{n=0}^{j-1} V_{k_n} \quad ;$$

the second inclusion follows from (1) . Thus $C - B_j \subseteq V_m$ where $m = \max_{0 \leq n \leq j-1} \{k_n\}$. Thus by 2.3 (i) , $C^* \subseteq B_j^* \subsetneqq B_{j-1}^*$ for each $j \in N$. Thus $\mathcal{A} < C^* < \mathcal{B}$.

<u>Case 2</u> Assume that $\mathcal{B}$ either is empty or has a smallest member, and that $\mathcal{A}$ has no largest member. Let B^* be the smallest member of $\mathcal{B}$ (and put $B^* = X^*$ if $\mathcal{B}$ is empty). As in case 1 , since $\mathcal{A}$ has no largest member we can assume that $A_n^* \subsetneqq A_{n+1}^* \subsetneqq B^*$ for each $n \in N$, and thus by 2.10 that $A_n \subsetneqq A_{n+1} \subsetneqq B$ for each $n \in N$.

As $A_0^* \subsetneqq B^*$, it follows by 2.3 (i) that $(B - A_0) \cap (X - \mathrm{cl}_X V_0) \neq \emptyset$ and so we can choose $p_0 \in (B - A_0) \cap (X - \mathrm{cl}_X V_0)$. Put $m_0 = 0$ and choose $m_1 \in N$ so that $p_0 \in V_{m_1}$. Thus $m_1 > m_0$. Inductively, suppose that we have chosen $p_i \in X$, $0 \leq i \leq n-1$, and $m_i \in N$, $0 \leq i \leq n$, such that:

(i) $m_{i+1} > m_i$, $0 \leq i \leq n-1$

(ii) $p_i \in (B - A_i) \cap (X - cl_X V_{m_i}) \cap V_{m_{i+1}}$, $0 \le i \le n-1$ ... (3)

As $A_n^* \subsetneqq B^*$, by 2.3 (i) there exists a point $p_n \in (B - A_n) \cap (X - cl_X V_{m_n})$ and an integer $m_{n+1} \in N$ such that $p_n \in V_{m_{n+1}}$. Thus $(p_n)_{n \in N}$ and $(m_n)_{n \in N}$ satisfy (i) and (ii) . Put

$$C = \bigcup_{n=0}^{\infty} [A_n \wedge (X - V_{m_{n+1}})] \qquad \cdots \qquad (4)$$

By (i) , $\lim_{n \to \infty} m_n = \infty$; hence by 2.2 (ii) it follows that $C \in R(X)$. As $A_n \wedge (X - V_{m_{n+1}}) \in \mathcal{S}$ for each $n \in N$, it follows from the hypotheses that $C \in \mathcal{S}$. It is obvious from (4) that for each $n \in N$, $C^* \supseteq A_n^* \wedge (X - V_{m_{n+1}})^*$. By 2.3 (i) , $(X - V_{m_{n+1}})^* = X^*$ and so $C^* \supseteq A_n^* \supsetneqq A_{n-1}^*$ for each $n \in N$. Thus $\mathcal{A} < C^*$.

On the other hand, it is obvious from (4) that $C^* \subseteq (\bigvee_{n=0}^{\infty} A_n)^* \subseteq B^*$. Furthermore, for fixed $i \in N$, the set

$$(B - A_i) \cap (X - cl_X V_{m_i}) \cap V_{m_{i+1}} \cap [A_n \wedge (X - V_{m_{n+1}})]$$

is empty for each $n \in N$; for if $n \ge i$, then $V_{m_{i+1}} \cap (X - V_{m_{n+1}}) = \emptyset$, and if $n < i$, then $(B - A_i) \cap A_n = \emptyset$. Thus by (3) and (4) , $p_i \notin C$ for each $i \in N$. The set $S = (p_i)_{i \in N}$ is closed by 2.2 (i) since $p_i \in X - cl_X V_{m_i}$; since S is disjoint from C , it follows from 2.1 that $S^* \cap C^* = \emptyset$. But $S^* \neq \emptyset$ by 1.13 , and so $C^* \subsetneqq C^* \cup S^*$. By (3) $S^* \subseteq B^*$ and so $C^* \subsetneqq B^*$. Thus $\mathcal{A} < C^* < \mathcal{B}$.

<u>Case 3</u> Assume that $\mathcal{A}$ either is empty or has a largest element, and that $\mathcal{B}$ either is empty or has a smallest element. Let A^* be the largest element of $\mathcal{A}$ and let B^* be the smallest element of $\mathcal{B}$ (put $A^* = \emptyset$ if $\mathcal{A}$ is empty and $B^* = X^*$ if $\mathcal{B}$ is empty). Then $A^* \subsetneqq B^*$. By 2.3 (ii), $(\mathrm{int}_X B) - A$ is not contained in any V_n. By a simple induction, for each $n \in N$ we can find $k_n \in N$ such that the open set $(\mathrm{int}_X B) \cap (X - A) \cap (X - \mathrm{cl}_X V_{k_n}) \cap V_{k_{n+1}}$ is non-empty. As $\mathcal{S}$ is a basic subalgebra of $R(X)$, for each $n \in N$ we can find $S_n \in \mathcal{S}$ such that $\emptyset \neq S_n \subseteq (\mathrm{int}_X B) \cap (X - A) \cap (X - \mathrm{cl}_X V_{k_{2n-1}}) \cap V_{k_{2n}}$. Also, for each $n \in N$ there exists $p_n \in (\mathrm{int}_X B) \cap (X - A) \cap (X - \mathrm{cl}_X V_{k_{2n}}) \cap V_{k_{2n+1}}$. If we set $C = A \cup (\bigcup_{n=0}^{\infty} S_n)$, then by arguments of a type previously seen, $C \in \mathcal{S}$ and $A^* \subsetneqq C^*$. If $(p_n)_{n\in N} = S$, then $C^* \cap S^* = \emptyset$ and $\emptyset \neq S^* \subseteq B^*$. Thus $\mathcal{A} < C^* < \mathcal{B}$.

<u>Case 4</u> Assume that $\mathcal{A}$ has no largest member and $\mathcal{B}$ has no smallest member. As in cases 1 and 2, it can be assumed that $A^*_n \subsetneqq A^*_{n+1} \subsetneqq B^*_{m+1} \subsetneqq B^*_m$ for each $n, m \in N$. By 2.10 it can also be assumed that $A_n \subsetneqq A_{n+1} \subsetneqq B_{m+1} \subsetneqq B_m$ for each $n, m \in N$. Put $C = \bigvee_{n=0}^{\infty} [A_n \wedge (X - V_n)]$. As in earlier cases, it is evident that $C \in \mathcal{S}$. Obviously $C^* \supseteq A^*_n \wedge (X - V_n)^* = A^*_n \supsetneqq A^*_{n-1}$ for each $n \in N$, and $C^* \subseteq (\bigvee_{n=0}^{\infty} A_n)^* \subseteq B^*_m \subsetneqq B^*_{m-1}$ for each $m \in N$. Thus $\mathcal{A} < C^* < \mathcal{B}$. This completes the proof of the theorem.

2.12 Theorem [CH]

Let X be a σ-compact space and assume that $|C(X)| = 2^{\aleph_0}$. Then $[\sigma G(X)]^*$ is a basic subalgebra of $R(X^*)$ and is isomorphic to $B(\beta\underline{\underline{N}}-\underline{\underline{N}})$ (see 1.83 and 2.8 for notation). In particular the projective covers of $\beta X-X$ and $\beta\underline{\underline{N}}-\underline{\underline{N}}$ are homeomorphic.

Proof: Since $|C(X)| = 2^{\aleph_0}$, it follows that $|G(X)| = 2^{\aleph_0}$; thus by 1.80 $|\sigma G(X)| = 2^{\aleph_0}$ and so $[\sigma G(X)]^*$ has cardinality no greater than $2^{\aleph_0}$. By 1.9 and 1.19 $\sigma G(X)$ is a basic subalgebra of $R(X)$, and since $\sigma G(X)$ is σ-complete (see 1.73) it satisfies all the conditions on $\mathcal{S}$ required in 2.11. Thus by 2.11 maximal chains in $[\sigma G(X)]^* - \{\emptyset, X^*\}$ are η_1-sets. Since by 1.47 every η_1-set has cardinality at least $2^{\aleph_0}$, and since (as noted above) $[\sigma G(X)]^*$ has cardinality no greater than $2^{\aleph_0}$, it follows that $[\sigma G(X)]^*$ is a Boolean algebra of cardinality $2^{\aleph_0}$. It then follows from 1.49 that $[\sigma G(X)]^*$ is isomorphic to $B(\beta\underline{\underline{N}}-\underline{\underline{N}})$. As $G(X) \subseteq \sigma G(X)$, it follows from 2.9 that $[\sigma G(X)]^*$ is a basic subalgebra of $R(X^*)$. Consequently by 1.87 $[\sigma G(X)]^*$ is dense in $R(X^*)$. As $\beta\underline{\underline{N}}-\underline{\underline{N}}$ is totally disconnected, by 1.85 $B(\beta\underline{\underline{N}}-\underline{\underline{N}})$ is dense in $R(\beta\underline{\underline{N}}-\underline{\underline{N}})$; thus by 1.75 $R(\beta\underline{\underline{N}}-\underline{\underline{N}})$ and $R(\beta X-X)$ are isomorphic. It follows immediately from 1.93 that $\beta\underline{\underline{N}}-\underline{\underline{N}}$ and $\beta X-X$ have homeomorphic projective covers.

2.13 Remarks

(i) Note again that if X is a σ-compact space in which every closed set is a zero-set, and if $|C(X)| = 2^{\aleph_0}$, then $\sigma G(X) = G(X) = R(X)$ and so $[R(X)]^*$ is isomorphic to $B(\beta\underline{\underline{N}}-\underline{\underline{N}})$.

(ii) Not only do $\beta\underline{\underline{N}}-\underline{\underline{N}}$ and $\beta X-X$ have homeomorphic projective covers, but we can in fact make the stronger statement that there is a continuous irreducible map from $\beta\underline{\underline{N}} - \underline{\underline{N}}$ onto $\beta X - X$. This will become clear in remark 3.7

It was noticed, after the proofs in 2.11 and 2.12 had been found, that one of the conclusions of 2.12 - namely that if X is σ-compact and $|C(X)| = 2^{\aleph_0}$, then $\beta X - X$ and $\beta\underline{\underline{N}} - \underline{\underline{N}}$ have homeomorphic projective covers - can be deduced from the following three known results:

(i) [CH] If X is a σ-compact space and $|C(X)| = 2^{\aleph_0}$, then $P(\beta X - X)$ and $P(\beta\underline{\underline{N}} - \underline{\underline{N}})$ are homeomorphic [CN , theorem 3.6] (see 1.34 for notation) .

(ii) [CH] If X is locally compact and realcompact and if $|C(X)| = 2^{\aleph_0}$, then $P(\beta X - X)$ is dense in $\beta X - X$ (1.44) .

(iii) If S is a dense subspace of Y, then $R(S)$ and $R(Y)$ are isomorphic.

We include a proof of (iii) below as we cannot find a reference for it.

2.14 Theorem

If S is a dense subspace of Y, then $R(S)$ and $R(Y)$ are isomorphic.

Proof: Suppose $A \in R(Y)$. Then $A \cap S \in R(S)$; for suppose $x \in A \cap S$. If W is any open subset of Y containing x, then $W \cap cl_Y(int_Y A) \neq \emptyset$, and so $W \cap int_Y A \neq \emptyset$. As S is dense in Y, it follows that $(W \cap int_Y A) \cap S \neq \emptyset$, i.e. $(W \cap S) \cap (S \cap int_Y A) \neq \emptyset$. Thus $x \in cl_S(int_S(A \cap S))$

and so $A \cap S \in R(S)$. Thus the mapping $h:R(Y) \to R(S)$ defined by $h(A) = A \cap S$ is a well-defined mapping from $R(Y)$ into $R(S)$. If A and B are in $R(Y)$, then

$$h(A \vee B) = (A \cup B) \cap S = (A \cap S) \cup (B \cap S) = h(A) \vee h(B)$$

and so h preserves suprema.

We now claim that if $A \in R(Y)$, then $(\mathrm{int}_Y A) \cap S = \mathrm{int}_S(A \cap S)$. Obviously $(\mathrm{int}_Y A) \cap S \subseteq \mathrm{int}_S(A \cap S)$. Conversely, if $x \in \mathrm{int}_S(A \cap S)$ there exists W open in Y such that $x \in W$ and $W \cap S \subseteq A \cap S$. If $W - A \neq \emptyset$, then $(W - A) \cap S \neq \emptyset$ as S is dense in Y . This is a contradiction, so $W \subseteq A$ and hence $x \in (\mathrm{int}_Y A) \cap S$. It follows that

$$\begin{aligned}
h(A') &= S \cap \mathrm{cl}_Y(Y - A) \\
&= S \cap (Y - \mathrm{int}_Y A) \\
&= S - [(\mathrm{int}_Y A) \cap S] \\
&= S - \mathrm{int}_S(S \cap A) \\
&= \mathrm{cl}_S(S - (S \cap A)) \\
&= [h(A)]'
\end{aligned}$$

and so h preserves complements. Thus h is a Boolean algebra homomorphism.

Suppose that A and B are in $R(Y)$ and that $A \neq B$. Then either $(\mathrm{int}_Y A) - B \neq \emptyset$ or $(\mathrm{int}_Y B) - A \neq \emptyset$; assume that $(\mathrm{int}_Y A) - B \neq \emptyset$. As S is dense in Y , it follows that $(S \cap \mathrm{int}_Y A) - (S \cap B) \neq \emptyset$, and so $h(A) \neq h(B)$. Thus h is one-to-one.

Finally, suppose that $B \in R(S)$. Then there exists W open in Y such that $W \cap S = \mathrm{int}_S B$. As $B = \mathrm{cl}_S(\mathrm{int}_S B)$, it follows that $\mathrm{cl}_Y B = \mathrm{cl}_Y(\mathrm{int}_S B) = \mathrm{cl}_Y(W \cap S) = \mathrm{cl}_Y W$ and so $\mathrm{cl}_Y B \in R(Y)$. Obviously $B \subseteq S \cap \mathrm{cl}_Y B$; and as $S - B$ is open in S , there exists an open subset U of Y such that $U \cap S = S - B$. Thus $U \cap \mathrm{cl}_Y B = \emptyset$ and so $S \cap \mathrm{cl}_Y B \subseteq B$. It follows that $h(\mathrm{cl}_Y B) = B$ and so h maps $R(Y)$ onto $R(S)$. Thus $R(Y)$ and $R(S)$ are isomorphic Boolean algebras.

An attempt was made to characterize topologically those compact spaces whose projective covers are homeomorphic to that of $\beta\underline{\underline{N}} - \underline{\underline{N}}$. Although this attempt was unsuccessful, the following partial result was obtained.

2.15 <u>Theorem</u> [CH]

Let Y be a compact space whose projective cover is homeomorphic to that of $\beta\underline{\underline{N}} - \underline{\underline{N}}$. Then dense G_δ-sets of Y have non-empty interiors.

<u>Proof</u>: If Y and $\beta\underline{\underline{N}} - \underline{\underline{N}}$ have homeomorphic projective covers, then $R(Y)$ and $R(\beta\underline{\underline{N}} - \underline{\underline{N}})$ are isomorphic. Thus $R(Y)$ contains a dense copy $\mathfrak{F}$ of $B(\beta\underline{\underline{N}} - \underline{\underline{N}})$ (see 1.85) . Let $G = \bigcap_{n=0}^{\infty} U_n$ be a dense G_δ-set in Y , where $(U_n)_{n \in N}$ is a countable family of dense open subsets of Y . As $\mathfrak{F}$ is dense in $R(Y)$, there exists $F_0 \in \mathfrak{F}$ such that $\emptyset \neq F_0 \subseteq U_0$ (see 1.74) . As U_1 is dense in Y , it follows that $(\mathrm{int}_Y F_0) \cap U_1$ is non-empty. Thus there exists $F_1 \in \mathfrak{F}$ such that $\emptyset \neq F_1 \subseteq (\mathrm{int}_Y F_0) \cap U_1$.

Inductively, suppose we have found $(F_k)_{0 \le k \le n}$ in $\mathfrak{F}$ such that

$\emptyset \neq F_i \subseteq (\mathrm{int}_Y F_{i-1}) \cap U_i \quad (1 \leq i \leq n)$. Then as U_{n+1} is dense in Y , it follows that $(\mathrm{int}_Y F_n) \cap U_{n+1} \neq \emptyset$ and so there exists $F_{n+1} \in \mathfrak{F}$ such that $\emptyset \neq F_{n+1} \subseteq (\mathrm{int}_Y F_n) \cap U_{n+1}$.

Thus we have a sequence $(F_n)_{n \in N} \subseteq \mathfrak{F}$ such that $\emptyset \neq F_n \subseteq (\mathrm{int}_Y F_{n+1}) \cap U_n$ for each $n \in N$. Thus $(F_n)_{n \in N}$ is a countable chain of non-empty members of $\mathfrak{F}$, and since $\mathfrak{F}$ is isomorphic to $B(\beta\underline{\underline{N}} - \underline{\underline{N}})$, whose maximal chains are η_1-sets (see 1.46) , it follows by 1.49 that there exists $H \in \mathfrak{F}$ such that $\emptyset \neq H \subseteq \bigcap_{n=0}^{\infty} F_n \subseteq \bigcap_{n=0}^{\infty} U_n = G$. As $\mathrm{int}_Y H \neq \emptyset$, it follows that $\mathrm{int}_Y G \neq \emptyset$.

2.16 Remark

It is not necessarily true that if Y is compact and has a projective cover homeomorphic to that of $\beta\underline{\underline{N}} - \underline{\underline{N}}$, then every non-empty G_δ-set of Y has a non-empty interior. As an example, let Y be the projective cover of $\beta\underline{\underline{N}} - \underline{\underline{N}}$. Then Y is extremally disconnected (see 1.82 , 1.89 , and 1.93) . It follows from 1.55 (i) that every regular closed subset of Y is open-and-closed. If every non-empty G_δ-set of Y had a non-empty interior, then by 1.42 every zero-set of Y would be regular closed and hence open-and-closed. It follows from 1.35 (i) that Y would be a compact P-space; hence by 1.35 (iv) Y would be finite, which is impossible. Thus the projective cover of $\beta\underline{\underline{N}} - \underline{\underline{N}}$ contains non-empty G_δ-sets with empty interiors.

III THE REMOTE POINTS OF βX

3.1 Definition

Let X be a compact space and let α be an arbitrary cardinal number. Then X will be called a Baire α-space if, given a family $\mathfrak{J}$ of dense open subsets of X such that $|\mathfrak{J}| \leq \alpha$, the set $\bigcap \mathfrak{J}$ is dense in X.

The Baire category theorem states that every compact space is a Baire $\aleph_0$-space. Theorem 1.43 says that if X is locally compact and realcompact, then $\beta X-X$ is a Baire $\aleph_1$-space. We shall be concerned primarily with Baire $\aleph_1$-spaces of this sort.

The following theorem is a generalization of a result due to Gleason (1.93) . Our proof mimics the proof of [G , theorem 3.2] .

3.2 Theorem

Let Y be a compact space and let $\mathcal{R}$ be a basic subalgebra of $R(Y)$. Then there exists an irreducible surjection $f:S(\mathcal{R}) \to Y$ defined by $f(x) = \bigcap \{A \in \mathcal{R} : x \in \lambda(A)\}$ (see 1.90 and 1.91 for notation and terminology) .

Proof: We first show that f is well-defined. Since Y is compact, if $f(x)$ were empty there would exist $(A_i)_{1\leq i\leq n} \subseteq \mathcal{R}$ such that $\bigcap_{i=1}^{n} A_i = \emptyset$ and $x \in \bigcap_{i=1}^{n} \lambda(A_i)$. But then $\bigwedge_{i=1}^{n} A_i = \emptyset$, and as λ is an isomorphism, this implies that $\bigcap_{i=1}^{n} \lambda(A_i) = \emptyset$, which is a contradiction. Thus $f(x)$

is not empty. If y and z are distinct points of Y belonging to $\bigcap\{A \in \mathcal{A}: x \in \lambda(A)\}$, then since $\mathcal{A}$ is a basic subalgebra of $R(Y)$, by 1.87 there exists $E \in \mathcal{A}$ such that $y \in \mathrm{int}_Y E$ and $z \notin E$. Since $z \in \bigcap\{A \in \mathcal{A}: x \in \lambda(A)\}$, it follows that $x \notin \lambda(E)$. Thus as $S(\mathcal{A})$ is totally disconnected, it follows that $x \in \lambda(E)' = \lambda(E')$. Since $y \in \mathrm{int}_Y E$ it follows that $y \notin E'$, and since $y \in \bigcap\{A \in \mathcal{A}: x \in \lambda(A)\}$ it follows that $x \notin \lambda(E')$, a contradiction. Thus f is a well-defined map.

To show that f is continuous, let $x \in S(\mathcal{A})$ and let W be any open subset of Y that contains $f(x)$. As $\mathcal{A}$ is a basic subalgebra of $R(Y)$, there exists $A \in \mathcal{A}$ such that $f(x) \in \mathrm{int}_Y A \subseteq A \subseteq W$. Thus $x \in \lambda(A)$ and if $z \in \lambda(A)$, then $f(z) \in A \subseteq W$. Thus $x \in \lambda(A) \subseteq f^{-1}(W)$ and so f is continuous.

To show that f maps $S(\mathcal{A})$ onto Y, let $y \in Y$ and consider the family $\mathcal{F} = \{A \in \mathcal{A} : y \in \mathrm{int}_Y A\}$. Then $\bigwedge_{i=1}^{n} A_i \neq \emptyset$ for every finite subfamily $(A_i)_{1 \le i \le n} \subseteq \mathcal{F}$, and since $\mathcal{A}$ is a basic subalgebra of $R(Y)$, it follows from 1.87 that $\bigcap \mathcal{F} = \{y\}$. Evidently $\mathcal{F}$ is a filter on $\mathcal{A}$ and so it is contained in some ultrafilter $\mathcal{U}$ on $\mathcal{A}$ (see 1.72); thus by 1.90 the intersection $\bigcap_{U \in \mathcal{U}} \lambda(U)$ is a unique point x of $S(\mathcal{A})$. Obviously $f(x) = y$.

To show that f is irreducible, let K be any proper closed subset of $S(\mathcal{A})$. Then there exists a non-empty $A \in \mathcal{A}$ such that $\lambda(A) \subseteq S(\mathcal{A}) - K$. Thus $K \subseteq \lambda(A')$ and so $f[K] \subseteq A'$. It follows that

$f[K] \cap \mathrm{int}_Y A = \emptyset$, and so $f[K] \neq Y$. Thus f is irreducible and the theorem is proved.

3.3 Definition

Let Y be a space and let $\mathfrak{F}$ be a subfamily of $R(Y)$. Then $H(\mathfrak{F})$ is defined to be the set $\bigcap_{F \in \mathfrak{F}} (Y - \mathrm{bd}_Y F)$.

Note that $H(\mathfrak{F})$ may be empty.

3.4 Proposition

Let Y be a space and let $\mathfrak{F}$ be a subfamily of $R(Y)$. Then $H(\mathfrak{F}) = H(<\mathfrak{F}>)$ (see 1.77 for notation) .

Proof: Since $\mathfrak{F} \subseteq <\mathfrak{F}>$, it follows that

$$H(<\mathfrak{F}>) = \bigcap_{E \in <\mathfrak{F}>} (Y - \mathrm{bd}_Y E) \subseteq \bigcap_{F \in \mathfrak{F}} (Y - \mathrm{bd}_Y F) = H(\mathfrak{F}) .$$

Conversely, if $E \in <\mathfrak{F}>$, there exists by 1.78 a finite subfamily $(F_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ of $\mathfrak{F}$ such that $E = \bigvee_{j=1}^{n} (\bigwedge_{i=1}^{m} \varepsilon_{ij} F_{ij})$, where $\varepsilon_{ij} F_{ij}$ equals either F_{ij} or F_{ij}' . It is well-known that if $B , C \in R(Y)$, then $\mathrm{bd}_Y (B \cup C) \subseteq \mathrm{bd}_Y B \cup \mathrm{bd}_Y C$ and $\mathrm{bd}_Y B = \mathrm{bd}_Y (B')$; thus $\mathrm{bd}_Y (B \wedge C) \subseteq \mathrm{bd}_Y B \cup \mathrm{bd}_Y C$ also. Consequently $\mathrm{bd}_Y E \subseteq \bigcup_{i=1}^{m} (\bigcup_{j=1}^{n} \mathrm{bd}_Y F_{ij})$ and so $\bigcap_{i=1}^{m} (\bigcap_{j=1}^{n} [Y - \mathrm{bd}_Y F_{ij}]) \subseteq Y - \mathrm{bd}_Y E$. Thus $\bigcap_{F \in \mathfrak{F}} (Y - \mathrm{bd}_Y F) \subseteq H(<\mathfrak{F}>)$ and the result follows.

3.5 Remark

There seems to be some formal similarity between the notions of a basic subalgebra $\mathcal{A}$ of $R(Y)$ and the associated subset $H(\mathcal{A})$ of Y and the concepts, defined by Plank [Pl , definitions 2.2 and 3.1] , of a β-subalgebra A of $C(X)$ and the associated set of A-points of $\beta X - X$. However, the exact relationship between these concepts is unclear.

3.6 Theorem

Let Y be a compact space and $\mathcal{A}$ a basic subalgebra of $R(Y)$.

(i) There exists a topological embedding $g : H(\mathcal{A}) \to S(\mathcal{A})$ such that $f \circ g$ is the natural inclusion of $H(\mathcal{A})$ in Y (where f is the mapping defined in 3.2)

(ii) If Y is a Baire α-space and if $|\mathcal{A}| \leq \alpha$ then $H(\mathcal{A})$ is dense in Y , $g[H(\mathcal{A})]$ is dense in $S(\mathcal{A})$, and $f[S(\mathcal{A}) - g[H(\mathcal{A})]] = Y - H(\mathcal{A})$.

Proof: (i) Let $y \in H(\mathcal{A})$. If $A \in \mathcal{A}$, then from the definition of $H(\mathcal{A})$ it is apparent that $y \in A$ if and only if $y \in A - \mathrm{bd}_Y A = \mathrm{int}_Y A$. Define $\mathcal{U}(y) = \{\lambda(A) : A \in \mathcal{A}$ and $y \in A\}$ (see 1.90 for notation). This is an ultrafilter (1.72) on $B(S(\mathcal{A}))$; for if $\lambda(A_1)$ and $\lambda(A_2)$ belong to $\mathcal{U}(y)$ then $y \in \mathrm{int}_Y A_1 \cap \mathrm{int}_Y A_2 = \mathrm{int}_Y(A_1 \wedge A_2)$. Thus $\lambda(A_1 \wedge A_2) = \lambda(A_1) \cap \lambda(A_2)$ is a member of $\mathcal{U}(y)$. Obviously $\emptyset \notin \mathcal{U}(y)$ and if $\lambda(A_1) \in \mathcal{U}(y)$ and $\lambda(A_1) \subseteq \lambda(A_2)$ then $\lambda(A_2) \in \mathcal{U}(y)$. Thus $\mathcal{U}(y)$ is a filter on $B(S(\mathcal{A}))$. Finally, if $\lambda(A) \notin \mathcal{U}(y)$ for some $A \in \mathcal{A}$, then $y \notin A$ and since $\mathcal{A}$ is a basic subalgebra of $R(Y)$, there exists $B \in \mathcal{A}$ such that $y \in B$ and $A \cap B = \emptyset$. Thus $\lambda(B) \in \mathcal{U}(y)$ and $\lambda(A) \cap \lambda(B) = \emptyset$; hence $\mathcal{U}(y)$ is an ultrafilter on $B(S(\mathcal{A}))$. Thus $\bigcap \mathcal{U}(y)$ is a single point of $S(\mathcal{A})$ and so we can define $g(y)$ by $g(y) = \bigcap \mathcal{U}(y)$.

Suppose that x and y are distinct members of $H(\mathfrak{A})$. Since $\mathfrak{A}$ is a basic subalgebra of $R(Y)$, there exists $A \in \mathfrak{A}$ such that $y \in A$ and $x \in A'$. Thus $g(y) \in \lambda(A)$, $g(x) \in \lambda(A')$, and $\lambda(A) \cap \lambda(A') = \emptyset$. Thus g is one-to-one.

We next claim that if $y \in H(\mathfrak{A})$ and if $B \in \mathfrak{A}$, then $g(y) \in \lambda(B)$ if and only if $y \in B$. It is obvious from the definition of g that $y \in B$ implies $g(y) \in \lambda(B)$. Conversely, if $y \notin B$ then $y \in B'$ and so $g(y) \in \lambda(B') = \lambda(B)'$; thus $g(y) \notin \lambda(B)$ and our claim is valid.

We now show that g is continuous. It follows from the previous paragraph that

$$\begin{aligned} g^{-1}[\lambda(A)] &= \{y \in H(\mathfrak{A}) : g(y) \in \lambda(A)\} \\ &= H(\mathfrak{A}) \cap A \end{aligned}$$

for each $A \in \mathfrak{A}$. Since $\{\lambda(A) : A \in \mathfrak{A}\}$ is a base for the closed subsets of $S(\mathfrak{A})$ and since $H(\mathfrak{A}) \cap A$ is closed in $H(\mathfrak{A})$, it follows that g is continuous.

Finally, for each $A \in \mathfrak{A}$ we have

$$\begin{aligned} g[H(\mathfrak{A}) \cap A] &= \{g(y) : y \in H(\mathfrak{A}) \cap A\} \\ &= \lambda(A) \cap g[H(\mathfrak{A})] . \end{aligned}$$

Since $\mathfrak{A}$ is a basic subalgebra of $R(Y)$, the family $\{H(\mathfrak{A}) \cap A : A \in \mathfrak{A}\}$, which is identical with $\{H(\mathfrak{A}) \cap \mathrm{int}_Y A : A \in \mathfrak{A}\}$, is a base for the open sets of $H(\mathfrak{A})$, and so g is an open mapping onto its range. It follows that $H(\mathcal{A})$ and $g[H(\mathcal{A})]$ are homeomorphic, and so g is a topological embedding.

Suppose that $y \in H(\mathcal{A})$. Since $g(y) \in \lambda(A)$ if and only if $y \in A$ for each $A \in \mathcal{A}$, it follows that $f(g(y)) = \bigcap\{A \in \mathcal{A} : y \in A\} = y$.

(ii) Since $|\mathcal{A}| \leq \alpha$, the family $\mathfrak{F} = \{Y - bd_Y A : A \in \mathcal{A}\}$ is a family of not more than α dense open subsets of Y . Since Y is a Baire α-space, the set $H(\mathcal{A}) = \bigcap \mathfrak{F}$ is dense in Y . Thus if A is any member of $\mathcal{A}$, it follows that $(int_Y A) \cap H(\mathcal{A}) \neq \emptyset$. Choose $y \in (int_Y A) \cap H(\mathcal{A})$; then $g(y) \in \lambda(A) \cap g[H(\mathcal{A})]$, as seen above. As $\{\lambda(A) : A \in \mathcal{A}\}$ is a base for the open subsets of $S(\mathcal{A})$, it follows that $g[H(\mathcal{A})]$ is dense in $S(\mathcal{A})$. Finally, since by (i) the restriction of f to the dense subset $g[H(\mathcal{A})]$ of $S(\mathcal{A})$ is a homeomorphism onto $H(\mathcal{A})$, it follows from 1.32 that $f[S(\mathcal{A}) - g[H(\mathcal{A})]] = Y - H(\mathcal{A})$.

3.7 <u>Remarks</u>

(i) $H(\mathcal{A}) = S(\mathcal{A}) = Y$ if and only if $\mathcal{A} = B(Y)$ and Y is compact and totally disconnected.

(ii) [CH] Let X be σ-compact and suppose $|C(X)| = 2^{\aleph_0}$. If we put $Y = \beta X - X$ and $\mathcal{A} = [\sigma G(X)]^*$ (see 2.8 for notation) , then the conditions of 3.6 (i) and (ii) are satisfied for $\alpha = \aleph_1$ (see 1.40 and 1.43) . Since by 2.12 $[\sigma G(X)]^*$ is isomorphic to $B(\beta\underline{\underline{N}} - \underline{\underline{N}})$, it follows that $S(\mathcal{A})$ is homeomorphic to $\beta\underline{\underline{N}} - \underline{\underline{N}}$. Thus there is a continuous irreducible surjection from $\beta\underline{\underline{N}} - \underline{\underline{N}}$ onto $\beta X - X$.

Recall the definition of remote points given in 1.50 . Using the characterization of the set of remote points of βX given in 1.51 , we obtain the following statement.

3.8 Theorem [CH]

Let X be a σ-compact metric space without isolated points. Then $T(X^*)$ can be embedded densely in $\beta\underline{\underline{N}}-\underline{\underline{N}}$ (see 1.52 for notation) .

Proof: Since X is separable (1.53) , clearly $|C(X)| = 2^{\aleph_0}$. By 2.12 and 2.13 (i) , it follows that $[R(X)]^*$ is a basic subalgebra of $R(X^*)$ and is isomorphic to $B(\beta\underline{\underline{N}}-\underline{\underline{N}})$. Thus $S([R(X)]^*)$ is homeomorphic to $\beta\underline{\underline{N}}-\underline{\underline{N}}$, and by 3.6 (i) there is an embedding g of $H([R(X)]^*)$ into $\beta\underline{\underline{N}}-\underline{\underline{N}}$. Since $|[R(X)]^*| = 2^{\aleph_0}$ and X^* is a Baire $\aleph_1$-space (1.43) , it follows from 3.6 (ii) that $g[H([R(X)]^*)]$ is dense in $\beta\underline{\underline{N}}-\underline{\underline{N}}$. Using 1.51 and recalling (1.21) that in a metric space $R(X) \subseteq Z(X)$, we have

$$T(X^*) \subseteq \bigcap_{A\in R(X)} [X^* - \mathrm{bd}_{X^*}A^*] = H([R(X)]^*) .$$

As $T(X^*)$ is dense in X^* (1.51) and hence in $H([R(X)]^*)$, it follows that $g[T(X^*)]$ is dense in $g[H([R(X)]^*)]$ and thus in $\beta\underline{\underline{N}}-\underline{\underline{N}}$. As g is a topological embedding, $g[T(X^*)]$ is homeomorphic to $T(X^*)$ and dense in $\beta\underline{\underline{N}}-\underline{\underline{N}}$.

3.9 Definition

A space X will be called strongly countably compact if the closure in X of every countable subset of X is compact.

3.10 Proposition

A strongly countably compact space is countably compact.

Proof: Let D be a countably infinite subset of the strongly countably compact space X. Then $cl_X D$ is compact and thus contains limit points of D. Thus each countably infinite subset of X has limit points, and so X is countably compact.

3.11 Example

As an example of a countably compact space that is not strongly countably compact, consider the space $Y = \beta\underline{\underline{N}} - \{p\}$, where $p \notin \underline{\underline{N}}$. Since $\underline{\underline{N}}$ is a countable dense subset of Y and Y is not compact, it follows that Y is not strongly countably compact. By 1.58, $\beta\underline{\underline{N}}$ is extremally disconnected and thus is an F-space (1.36). Thus if D is any countably infinite subspace of Y, it follows from 1.38 and 1.30 that $cl_{\beta\underline{\underline{N}}} D$ is homeomorphic to $\beta\underline{\underline{N}}$ and thus by 1.29 has cardinality $2^{2^{\aleph_0}}$. Thus D has limit points in $\beta\underline{\underline{N}}$ other than p, and thus has Y-limit points. Thus Y is countably compact.

3.12 Theorem

Let X be a σ-compact metric space without isolated points. Then both $T(X^*)$ and $X^* - T(X^*)$ are strongly countably compact. Thus assuming the continuum hypothesis, X^* can be decomposed into two disjoint dense strongly countably compact subspaces.

Proof: We first prove that $T(X^*)$ is strongly countably compact. Let $D = (p_n)_{n \varepsilon N}$ be any countable subset of $T(X^*)$, and let $\mathcal{A}^{p_n}$ be the z-ultrafilter on X associated with p_n (see 1.31). Put $\mathcal{F} = \bigcap_{i=0}^{\infty} \mathcal{A}^{p_i}$ and set $K = \bigcap_{F \varepsilon \mathcal{F}} F^*$. Then $p_n \varepsilon F^*$ for each $n \varepsilon N$ and each $F \varepsilon \mathcal{F}$, and so D is a subset of K, which is closed in X^*. It thus suffices to show that $K \subseteq T(X^*)$. Suppose that $q \notin T(X^*)$. Then by 1.51 there exists a closed nowhere dense subset Z of X such that $q \in Z^*$. Since p_n is a remote point, it follows that $Z \notin \mathcal{A}^{p_n}$ for each $n \in N$, and so we can choose $A_n \in \mathcal{A}^{p_n}$ such that $Z \cap A_n = \emptyset$. Without loss of generality we can assume that $A_n \subseteq X - V_n$ for each $n \in N$ (see 1.11). It follows from 2.2 (i) that $\bigcup_{i=0}^{\infty} A_i$ is closed in X and as $A_n \subseteq \bigcup_{i=0}^{\infty} A_i$ for each $n \in N$, evidently $\bigcup_{i=0}^{\infty} A_i \in \mathcal{F}$. Thus $K \subseteq (\bigcup_{i=0}^{\infty} A_i)^*$. But $Z \cap (\bigcup_{i=0}^{\infty} A_i) = \emptyset$ so by 2.1 (ii) we have $Z^* \cap (\bigcup_{i=0}^{\infty} A_i)^* = \emptyset$. Thus $q \notin K$ and so $K \subseteq T(X^*)$.

We next prove that $X^* - T(X^*)$ is strongly countably compact. Let $D = (p_n)_{n \in N}$ be a countable subset of $X^* - T(X^*)$. By 1.51 we can, for each $n \in N$, find a closed nowhere dense subset Z_n of X such that $p_n \in Z_n^*$, and without loss of generality we can assume that $Z_n \subseteq X - V_n$. By 2.2 (i) it follows that $\bigcup_{i=0}^{\infty} Z_i$ is closed, and obviously $p_n \in (\bigcup_{i=0}^{\infty} Z_i)^*$ for each $n \in N$. Thus $\mathrm{cl}_{X^*} D \subseteq (\bigcup_{i=0}^{\infty} Z_i)^*$. Applying the Baire category theorem to the locally compact space X, we see that $\mathrm{int}_X(\bigcup_{i=0}^{\infty} Z_i) = \emptyset$; thus it follows from 1.51 that $(\bigcup_{i=0}^{\infty} Z_i)^* \subseteq X^* - T(X^*)$. Hence $\mathrm{cl}_{X^*} D$ is a subset of $X^* - T(X^*)$.

Since, assuming the continuum hypothesis, both $T(X^*)$ and $X^* - T(X^*)$ are dense subsets of X^* (1.53), the final assertion follows.

3.13 Theorem [CH]

Let X be a σ-compact metric space without isolated points, and let g be the embedding of 3.8 . Then $(\beta\underline{\underline{N}}-\underline{\underline{N}}) - g[T(X^*)]$ is a dense, strongly countably compact subspace of $\beta\underline{\underline{N}}-\underline{\underline{N}}$ of cardinality $2^{2^{\aleph_0}}$. Thus $\beta\underline{\underline{N}}-\underline{\underline{N}}$ can be decomposed into two disjoint, dense strongly countably compact subspaces.

Proof: Let f be the irreducible map from $\beta\underline{\underline{N}}-\underline{\underline{N}}$ onto $\beta X-X$ described in 3.7 (ii) . By 3.6 and 3.8 , the restriction of f to the dense subset $g[T(X^*)]$ is a homeomorphism sending $g[T(X^*)]$ onto $T(X^*)$. It follows from 1.32 that $f[(\beta\underline{\underline{N}}-\underline{\underline{N}}) - g[T(X^*)]] = X^* - T(X^*)$, which by 1.54 is dense in X^* . As f is irreducible, by 1.92 the set $f^{-1}[X^* - T(X^*)] = (\beta\underline{\underline{N}}-\underline{\underline{N}}) - g[T(X^*)]$ is dense in $\beta\underline{\underline{N}}-\underline{\underline{N}}$, and as $X^* - T(X^*)$ has cardinality $2^{2^{\aleph_0}}$, so does $(\beta\underline{\underline{N}}-\underline{\underline{N}}) - g[T(X^*)]$.

Let $D = (p_n)_{n\epsilon N}$ be a countable subset of $(\beta\underline{\underline{N}}-\underline{\underline{N}}) - g[T(X^*)]$. Then as seen above, the set $(f(p_n))_{n\epsilon N}$ is a countable (or finite) subset of $X^* - T(X^*)$ and so by 3.12 it follows that $cl_{X^*}[(f(p_n))_{n\epsilon N}]$ is a subset of $X^* - T(X^*)$. Thus

$$\begin{aligned} cl_{\underline{\underline{N}}^*}D &\subseteq f^{-1}[cl_{X^*}[(f(p_n))_{n\epsilon N}]] \\ &\subseteq f^{-1}[X^* - T(X^*)] \\ &= (\beta\underline{\underline{N}}-\underline{\underline{N}}) - g[T(X^*)] \ . \end{aligned}$$

Finally, since $g[T(X^*)]$ is homeomorphic to $T(X^*)$, it follows from 3.12 and the above that $g[T(X^*)]$ and $(\beta\underline{\underline{N}}-\underline{\underline{N}}) - g[T(X^*)]$ are the subspaces of $\beta\underline{\underline{N}}-\underline{\underline{N}}$ whose existence is claimed in the statement of the theorem.

The following result generalizes a portion of theorems 3.12 and 3.13 .

3.13a <u>Theorem</u> [CH]

If X is a σ-compact space and $|C(X)| = 2^{\aleph_0}$, then $\beta X - X$ can be partitioned into two disjoint, dense strongly countably compact subspaces.

<u>Proof</u>: Since $|C(X)| = 2^{\aleph_0}$, the family of $2^{\aleph_0}$ cozero-sets of X^* forms a base for the open subsets of X^* . Hence by 1.44 there exists a dense subset D of X^* consisting of $\aleph_1$ P-points of X^* . Put

$$S = \{x \in X^* : \text{there exists } E \subseteq D \text{ such that } |E| = \aleph_0 \text{ and } x \in cl_{X^*}E\}.$$

As D has $\aleph_1$ countable subsets - say $(E_\alpha)_{\alpha<\omega_1}$ - we can write S in the form $S = \bigcup_{\alpha<\aleph_1} cl_{X^*}E_\alpha$.

Now $int_{X^*}(cl_{X^*}E_\alpha) = \emptyset$ for each $\alpha < \omega_1$; for if not, let $E_\alpha = (x_i)_{i\in N}$. By 2.12 we can, for each $i \in N$, find $A_i^* \in [R(X)]^*$ such that

$$\emptyset \neq int_{X^*}A_i^* \subseteq int_{X^*}(cl_{X^*}E_\alpha) - \{p_i\} .$$

By case 1 of 2.11 there exists $B_i^* \in [R(X)]^*$ such that

$$\emptyset \neq int_{X^*}B_i^* \subseteq \bigcap_{i=0}^{\infty} int_{X^*}A_i^* \subseteq cl_{X^*}E_\alpha - E_\alpha ,$$

which is a contradiction. Thus $int_{X^*}(cl_{X^*}E_\alpha) = \emptyset$ and so S is a union of $\aleph_1$ closed nowhere dense subsets of X^* . By 1.43 it follows that $X^* - S$ is dense in X^* . Of course S is dense in X^* also.

Now S is strongly countably compact, for let $(x_n)_{n\in N} \subseteq S$. Then there exists, for each $n \in N$, a subset E_n of D such that $|E_n| = \aleph_0$ and $x_n \in cl_{X^*}E_n$. Thus

$$(x_n)_{n\in N} \subseteq \bigcup_{n=0}^{\infty} cl_{X^*}E_n \subseteq cl_{X^*}(\bigcup_{n=0}^{\infty} E_n) \subseteq S$$

(as $\bigcup_{n=0}^{\infty} E_n$ is a countable subset of D). Thus $cl_{X^*}(x_n)_{n\in N} \subseteq S$, and so S is strongly countably compact.

Finally, suppose that there exists a countable subset A of $X^* - S$ such that $S \cap cl_{X^*}A \neq \emptyset$. Then there exists a countable subset E of D such that $cl_{X^*}E \cap cl_{X^*}A \neq \emptyset$. Now $cl_{X^*}E \subseteq S$ so the set $(cl_{X^*}E) \cap A = \emptyset$. As each point of E is a P-point of X^*, it is evidently not a limit point of any countable subset of X^*, and so $E \cap cl_{X^*}A = \emptyset$. Thus E and A are disjoint open-and-closed subsets of the countable subspace $A \cup E$ of X^*. As X^* is an F-space (see 1.37), $A \cup E$ is C*-embedded in X^* (see 1.38). Thus

$$cl_{X^*}E \cap cl_{X^*}A = cl_{X^*}(E \cap A) = \emptyset$$

(see 1.30 and 1.27). This is a contradiction and so $S \cap cl_{X^*}A = \emptyset$ for each countable subset A of $X^* - S$. Thus $X^* - S$ is strongly countably compact and $\{S, X^* - S\}$ is the desired decomposition of X^*.

3.14 Proposition

Let X be any σ-compact space and let A be a closed subset of X. Then $(\mathrm{bd}_X A)^* = \mathrm{bd}_{X^*} A^*$.

Proof: Since A^* is closed in X^*, it follows that

$$\begin{aligned} \mathrm{bd}_{X^*} A^* &= A^* - \mathrm{int}_{X^*} A^* \\ &= A^* \cap \mathrm{cl}_{X^*}(X^* - A^*) \\ &= A^* \cap [\mathrm{cl}_X(X - A)]^* && \text{(see 2.4)} \\ &= [A \cap \mathrm{cl}_X(X - A)]^* && \text{(see 2.1 (ii))} \\ &= (\mathrm{bd}_X A)^* . \end{aligned}$$

3.15 Remarks

(i) [CH] We are now in a position to indicate somewhat simpler proofs of two of Plank's results. First, it is evident that the equivalence of the two characterizations of $T(X^*)$ that appear in 1.51 follows immediately from 3.14. Second, in his proof of the fact that $[X^* - P(X^*)] \cap T(X^*)$ is a dense subset of X^* of cardinality $2^{2^{\aleph_0}}$, Plank constructs a compact infinite set of remote points by appealing to a somewhat complex result of Fine and Gillman [FG_2, lemma 2.3]. This can be avoided by taking the closure in X^* of a countable set of remote points (see 3.12).

(ii) [CH] In [M, section 4] Mandelker calls a subset A of βX a round subset of βX if for any $Z \in Z(X)$, if $\mathrm{cl}_{\beta X} Z$ contains A then it is a neighborhood of A. If we say that p is a round point of βX if and only if $\{p\}$ is a round subset of βX, then it is evident that the set of round points of βX that are not in X is precisely the set $\bigcap_{Z \varepsilon Z(X)} [X^* - bd_{X^*}Z^*]$. Obviously the proof of 3.8 can be adapted to show that if X is a σ-compact space such that $|C(X)| = 2^{\aleph_0}$ and $R(X) \subseteq Z(X)$, then the set of round points of βX that are not in X can be embedded densely in $\beta \underline{\underline{N}} - \underline{\underline{N}}$.

Recall that P-points were defined in 1.33 .

3.16 <u>Lemma</u>

If X is a dense subspace of T , then $P(X) = X \cap P(T)$.

<u>Proof</u>: It is immediate from 1.35 (iii) that $X \cap P(T) \subseteq P(X)$. Conversely, let $p \varepsilon P(X)$, let $Z \varepsilon Z(T)$, and suppose that $p \varepsilon Z$. Then $Z \cap X \varepsilon Z(X)$ and so by 1.35 (ii) there exists an open subset W of T such that $p \varepsilon W \cap X \subseteq Z \cap X$. If $W - Z \neq \emptyset$, then since $W - Z$ is open in T and since X is dense in T , it follows that $(W \cap X) - (Z \cap X) \neq \emptyset$, which gives a contradiction. Thus $W \subseteq Z$ and so Z is a neighborhood (in T) of p . As Z was an arbitrary zero-set of T , it follows from 1.35 (ii) that $p \varepsilon P(T)$.

In [CN , theorem 3.6] , Comfort and Negrepontis have shown, assuming the continuum hypothesis, that if X and Y are two σ-compact spaces and if $|C(X)| = |C(Y)| = 2^{\aleph_0}$, then $P(\beta X-X)$ and $P(\beta Y-Y)$ are homeomorphic. We can, using the results developed in this chapter, prove the following weaker result.

3.17 Theorem [CH]

If X is a σ-compact metric space without isolated points, then there exists a dense subset of $P(\beta X-X)$ of cardinality $2^{2^{\aleph_0}}$ that is homeomorphic to a dense subset of $P(\beta\underline{\underline{N}}-\underline{\underline{N}})$.

Proof: By 1.54 , $P(X^*)\cap T(X^*)$ is a dense subset of X^* of cardinality $2^{2^{\aleph_0}}$. If g is the embedding of $T(X^*)$ in $\beta\underline{\underline{N}}-\underline{\underline{N}}$ defined in 3.8 , it follows from the fact that $g[T(X^*)]$ is dense in $\beta\underline{\underline{N}}-\underline{\underline{N}}$ that $P(g[T(X^*)]) = g[P(X^*)\cap T(X^*)]$ is a dense subset of $\beta\underline{\underline{N}}-\underline{\underline{N}}$ of cardinality $2^{2^{\aleph_0}}$. By 3.16 , $P(g[T(X^*)]) = g[T(X^*)]\cap P(\beta\underline{\underline{N}}-\underline{\underline{N}})$ and the theorem follows.

If X is a σ-compact metric space without isolated points, then $\{A^*\cap T(X^*) : A \in R(X)\}$ is evidently a family of open-and-closed subsets of $T(X^*)$ that forms a basis for the open sets of $T(X^*)$. This raises the question of whether $T(X^*)$ has the stronger property of being basically disconnected (1.55 (ii)) . The following proposition provides the answer.

3.18 Proposition

Let X be locally compact and realcompact, and let S be a dense subset of $\beta X-X$ with the property that $S\cap[X^* - P(X^*)] \neq \emptyset$. Then S is not basically disconnected. Thus, assuming the continuum hypothesis, if $|C(X)| = 2^{\aleph_0}$ then $P(X^*)$ is the largest dense basically disconnected subset of $\beta X-X$.

Proof: By hypothesis there exists a non-empty $Z \in Z(X^*)$ such that

$S \cap bd_{X^*}Z \neq \emptyset$ (see 1.35 (v)) . Let $x \in S \cap bd_{X^*}Z$. As S is dense in X^* , the set $S \cap (X^* - Z)$ is a non-empty cozero-set of S . By 1.41 $Z = cl_{X^*}(int_{X^*}Z)$ and so $S \cap (int_{X^*}Z) \neq \emptyset$. Since $S \cap (X^* - Z)$ and $S \cap (int_{X^*}Z)$ are disjoint, by 1.56 it suffices to show that $cl_S[S \cap (X^* - Z)] \cap cl_S[S \cap (int_{X^*}Z)]$ is non-empty. Let W be any open subset of X^* containing x . As $x \in cl_{X^*}(int_{X^*}Z)$, it follows that $W \cap int_{X^*}Z \neq \emptyset$. As S is dense in X^* , we see that $S \cap W \cap int_{X^*}Z \neq \emptyset$ and so $x \in cl_S[S \cap int_{X^*}Z]$. Since $x \in cl_{X^*}(X^* - Z)$, it follows in a similar manner that $x \in cl_S[S \cap (X^* - Z)]$, and so S is not basically disconnected. Assuming the continuum hypothesis, if $|C(X)| = 2^{\aleph_0}$ then by 1.44 and 1.57 $P(X^*)$ is a dense basically disconnected subspace of X^* , and the final statement of the theorem follows.

Combining 3.18 and 1.53 , we immediately obtain the following result.

3.19 Corollary

If X is a σ-compact metric space without isolated points, then $T(X^*)$ is not basically disconnected.

We conclude chapter III by showing that the set of remote points of $\beta\underline{\underline{R}}$ is precisely the set $H([R(\underline{\underline{R}})]^*)$. To do this we need the following lemma, which was proved independently and approximately simultaneously by ourselves and Mandelker [M , lemma 2.3] .

3.20 Lemma

Let K be a closed nowhere dense subset of $\underline{\underline{R}}$. Then there exists $A \in R(\underline{\underline{R}})$ such that $K \subseteq bd_{\underline{\underline{R}}}A$.

<u>Proof</u>: Without loss of generality we can assume that K is unbounded, as any bounded closed nowhere dense subset of $\underline{R}$ is a subset of some unbounded closed nowhere dense subset of $\underline{R}$. We can thus write $\underline{R} - K$ in the form $\bigcup_{i=1}^{\infty}(a_i,b_i)$ where $a_i < b_i$ and $i \neq j$ implies that $(a_i,b_i) \cap (a_j,b_j) = \emptyset$. Fix i and form a "Cantor set" from $[a_i,b_i]$ by deleting "open middle third" intervals in the standard way. This is accomplished in a sequence of steps; at the n th step 2^{n-1} intervals are deleted. Inductively, we call the "open middle third" intervals that we delete from $[a_i,b_i]$ either "red" or "green" according to the following rule:

(i) $(a_i + (b_i - a_i)/3 \, , \, a_i + 2(b_i - a_i)/3)$ is red.

(ii) Suppose that the n th stage of deleting "open middle third" intervals has been completed and that colours have been assigned to these. At the $(n+1)$ st stage we delete 2^n open intervals and assign each interval I a colour as follows: if there is a previously deleted interval to the left of I, assign I the colour that is the opposite of the colour of the previously deleted interval that lies closest to the left of I. If there is no previously deleted interval to the left of I, then the colour of I is taken to be the opposite of the colour of the previously deleted interval that lies closest to the right of I.

Let R_i be the union of the red subintervals of $[a_i,b_i]$, and let G_i be the union of the green subintervals of $[a_i,b_i]$. Put $R = \bigcup_{i=1}^{\infty} R_i$ and $G = \bigcup_{i=1}^{\infty} G_i$. Then as $R_i \cup G_i$ is dense in (a_i,b_i), $R \cup G$ is dense in $\bigcup_{i=1}^{\infty} (a_i, b_i) = \underline{\underline{R}} - K$ and thus is dense in $\underline{\underline{R}}$. Obviously $R \cap G = \emptyset$ and $K \subseteq \underline{\underline{R}} - (R \cup G)$. From the construction of R and G it is easily seen that $\underline{\underline{R}} - (R \cup G) = \mathrm{bd}_{\underline{\underline{R}}}(\mathrm{cl}_{\underline{\underline{R}}} G)$, and of course $\mathrm{cl}_{\underline{\underline{R}}} G \in R(\underline{\underline{R}})$. The lemma follows immediately.

3.21 <u>Theorem</u>

$$T(\underline{\underline{R}}^*) = H([R(\underline{\underline{R}})]^*) .$$

<u>Proof</u>: Recall (1.51) that $T(\underline{\underline{R}}^*) = \bigcap_{Z \in Z(\underline{\underline{R}})} [\underline{\underline{R}}^* - (\mathrm{bd}_{\underline{\underline{R}}} Z)^*]$ and that $H([R(\underline{\underline{R}})]^*) = \bigcap_{A \in R(\underline{\underline{R}})} [\underline{\underline{R}}^* - (\mathrm{bd}_{\underline{\underline{R}}} A)^*]$. As $R(\underline{\underline{R}}) \subseteq Z(\underline{\underline{R}})$, obviously $T(\underline{\underline{R}}^*) \subseteq H([R(\underline{\underline{R}})]^*)$. Conversely, if $Z \in Z(\underline{\underline{R}})$ then by 3.20 there exists $A \in R(\underline{\underline{R}})$ such that $\mathrm{bd}_{\underline{\underline{R}}} Z \subseteq \mathrm{bd}_{\underline{\underline{R}}} A$; thus $\underline{\underline{R}}^* - (\mathrm{bd}_{\underline{\underline{R}}} A)^* \subseteq \underline{\underline{R}}^* - (\mathrm{bd}_{\underline{\underline{R}}} Z)^*$. Thus $H([R(\underline{\underline{R}})]^*) \subseteq T(\underline{\underline{R}}^*)$ and the theorem follows.

3.22 <u>Questions</u>

(i) Is $H([R(X)]^*) = T(X^*)$ for any σ-compact metric space X without isolated points?

(ii) If X and Y are two σ-compact metric spaces without isolated points, are $T(X^*)$ and $T(Y^*)$ homeomorphic?

IV THE DIMENSION OF $\beta X-X$

In this chapter we develop, for a σ-compact space X, a relatively simple characterization of the Lebesgue covering dimension of $\beta X-X$. This characterization is then used to show that for each positive integer n, the Lebesgue covering dimension of $\beta\underline{\underline{R}}^n - \underline{\underline{R}}^n$ is n. Throughout this chapter we assume that X is σ-compact (and hence normal).

4.1 Lemma

If B is closed in $\beta X-X$ and W is open in $\beta X-X$, and if $B \subseteq W$, then there exists $A \in R(X)$ such that $B \subseteq \mathrm{int}_{X^*}A^* \subseteq A^* \subseteq W$.

Proof: Since X is σ-compact, it follows from 2.12 that $[R(X)]^*$ is a basic subalgebra of $R(X^*)$; thus by 1.87, the family $\{\mathrm{int}_{X^*}A^* : A \in R(X)\}$ is a base for the open subsets of X^*. As B and $X^* - W$ are disjoint closed subsets of the normal space X^*, there exists $(A_\alpha)_{\alpha\in\Sigma} \subseteq R(X)$ such that

$$B \subseteq \bigcup_{\alpha\in\Sigma} \mathrm{int}_{X^*}A^*_\alpha \subseteq \mathrm{cl}_{X^*}[\bigcup_{\alpha\in\Sigma} \mathrm{int}_{X^*}A^*_\alpha] = K \subseteq W .$$

As B is compact there exist $\alpha_1, \cdots, \alpha_n \in \Sigma$ such that $B \subseteq \bigcup_{i=1}^n \mathrm{int}_{X^*}A^*_{\alpha_i} \subseteq K \subseteq W$. It follows from 2.1 (i) and 2.5 that

$$\mathrm{cl}_{X^*}[\bigcup_{i=1}^n \mathrm{int}_{X^*}A^*_{\alpha_i}] = (\bigcup_{i=1}^n A_{\alpha_i})^*.$$

Put $A = \bigcup_{i=1}^n A_{\alpha_i}$; then $A \in R(X)$ and obviously $B \subseteq \mathrm{int}_{X^*}A^* \subsetneq A^* \subseteq W$.

4.2 Definition

A cover of $\beta X-X$ all of whose members are of the form $\mathrm{int}_{X^*}A^*$

(for some $A \in R(X)$) will be called an R-cover of $\beta X - X$. If $\mathcal{V}$ is both a refinement of a cover $\mathcal{U}$ of $\beta X - X$ and also an R-cover of $\beta X - X$, then $\mathcal{V}$ will be called an R-refinement of $\mathcal{U}$. (see 1.66 for terminology).

The proof of the following lemma mimics the proof of [GJ , theorem 16.6].

4.3 Lemma

Let $\mathcal{U} = (U_i)_{1 \le i \le k}$ be a cover of $\beta X - X$. Then there exists an R-refinement $\mathcal{A} = (\mathrm{int}_{X^*} A_i^*)_{1 \le i \le k}$ of $\mathcal{U}$ such that $A_i^* \subseteq U_i$ for each i from 1 to k .

Proof: The family $(A_i)_{1 \le i \le k}$ is defined inductively. Put $B_1 = X^* - \bigcup_{i=2}^{k} U_i$. Then as $\mathcal{U}$ is a cover of X^* , it follows that B_1 and $X - U_1$ are disjoint closed subsets of X^* . By 4.1 we can find $A_1 \in R(X)$ such that $B_1 \subseteq \mathrm{int}_{X^*} A_1^* \subseteq A_1^* \subseteq U_1$. Then $\{\mathrm{int}_{X^*} A_1^* , U_2 , \cdots , U_k\}$ is a cover of X^* .

Suppose that we have defined $A_1 , \cdots , A_{i-1} \in R(X)$ such that:

(i) $A_j^* \subseteq U_j$, $1 \le j \le i-1$.

(ii) The family $\{\mathrm{int}_{X^*} A_1^* , \cdots , \mathrm{int}_{X^*} A_{i-1}^* , U_i , \cdots , U_k\}$ is a cover of X^* .

Put $B_i = X^* - (\mathrm{int}_{X^*} A_1^* \cup \cdots \cup \mathrm{int}_{X^*} A_{i-1}^* \cup U_{i+1} \cup \cdots \cup U_k\}$ if $i \ne k$, and put $B_k = X^* - \bigcup_{j=1}^{k-1} \mathrm{int}_{X^*} A_j^*$. By (ii) B_i and $X^* - U_i$ are disjoint closed subsets of X^* ; thus by 4.1 there exists $A_i \in R(X)$ such that $B_i \subseteq \mathrm{int}_{X^*} A_i^* \subseteq A_i^* \subseteq U_i$. Then (i) and (ii) are both satisfied when $i-1$ is replaced by i . This induction yields the desired $\mathcal{A}$.

4.4 Proposition

The following two statements are equivalent:

(i) The space $\beta X - X$ has a cover $\mathcal{U}$ such that every refinement of $\mathcal{U}$ has order (see 1.67) not less than n .

(ii) The space $\beta X - X$ has an R-cover $\mathcal{W}$ such that every R-refinement of $\mathcal{W}$ has order not less than n .

Proof: Let $\mathcal{U}$ be a cover of X^* such that every refinement of $\mathcal{U}$ has order not less than n . By 4.3 there exists an R-refinement $\mathcal{W}$ of $\mathcal{U}$. As every refinement of $\mathcal{W}$ is a refinement of $\mathcal{U}$, every R-refinement of $\mathcal{W}$ has order not less than n .

Conversely, if $\mathcal{W}$ is an R-cover of X^* such that every R-refinement of $\mathcal{W}$ has order not less than n , let $\mathcal{V}$ be any refinement of $\mathcal{W}$. By 4.3 $\mathcal{V}$ has an R-refinement $\mathcal{A}$, and $\mathcal{A}$ is an R-refinement of $\mathcal{W}$. Thus there exist $A_1 , \cdots , A_{n+1} \varepsilon \mathcal{A}$ such that $\bigcap_{i=1}^{n+1} A_i \neq \emptyset$. As $\mathcal{A}$ is a refinement of $\mathcal{V}$, we can find $V_1 , \cdots , V_{n+1} \varepsilon \mathcal{V}$ such that $A_i \subseteq V_i$ for each i from 1 to $n+1$. Thus $\bigcap_{i=1}^{n+1} V_i \neq \emptyset$ and so $\operatorname{ord} \mathcal{V} \geq n$. Thus $\mathcal{W}$ is the desired $\mathcal{U}$.

4.5 Proposition

The dimension of $\beta X - X$ is not less than n if and only if there exists an R-cover $\mathcal{W}$ of $\beta X - X$ such that every R-refinement of $\mathcal{W}$ has order not less than n .

Proof: From the definition of dimension (1.68) it is evident that

$\dim X^* \geq n$ if and only if it is false that every cover of X^* has a refinement of order not greater than $n-1$. Thus $\dim X^* \geq n$ if and only if X^* has a cover $\mathcal{U}$ such that every refinement of $\mathcal{U}$ has order not less than n. By 4.4 this occurs if and only if there exists an R-cover $\mathcal{W}$ of X^* such that every R-refinement of $\mathcal{W}$ has order not less than n.

4.6 Proposition

If Y is a normal space, then every cover of Y has a refinement all of whose members are regular open sets.

Proof: Suppose that B is closed in Y, W is open in Y, and $B \subseteq W$. Then as Y is normal, there exist an open set U and a closed set K such that $B \subseteq U \subseteq K \subseteq W$. Thus $B \subseteq U \subseteq cl_Y U \subseteq W$. Thus there exists $A \in R(Y)$ - namely $A = cl_Y U$ - such that $B \subseteq int_Y A \subseteq A \subseteq W$. A repetition of the proof of 4.3 now yields the proposition, since $int_Y A$ is a regular open set.

4.7 Proposition

Let X be a σ-compact space and let $(A_i)_{1 \leq i \leq k} \subseteq R(X)$. Then $(int_{X^*} A_i^*)_{1 \leq i \leq k}$ is an R-cover of $\beta X - X$ if and only if $X - \bigcup_{i=1}^{k} int_X A_i$ is compact.

Proof: The family $(int_{X^*} A_i^*)_{1 \leq i \leq k}$ is a cover of X^* if and only if $X^* - \bigcup_{i=1}^{k} int_{X^*} A_i^* = \emptyset$, i.e. if and only if $\bigcap_{i=1}^{k} (X^* - int_{X^*} A_i^*) = \emptyset$. But

$$\bigcap_{i=1}^{k} (X^* - int_{X^*} A_i^*) = \bigcap_{i=1}^{k} cl_{X^*}(X^* - A_i^*)$$

$$= [\bigcap_{i=1}^{k} cl_X(X - A_i)]^* \qquad \text{(by 2.1 (ii) and 2.4)}$$

$$= [\bigcap_{i=1}^{k} (X - int_X A_i)]^*$$

$$= [X - \bigcup_{i=1}^{k} int_X A_i]^* \ .$$

The proposition now follows from 2.1 (iii) .

4.8 Proposition

Let $(A_i)_{1 \le i \le m}$ and $(E_i)_{1 \le i \le k}$ be subfamilies of $R(X)$, and let $\mathcal{E} = (int_{X^*} E_i^*)_{1 \le i \le k}$ be an R-cover of $\beta X - X$. Then the family $\mathcal{A} = (int_{X^*} A_i^*)_{1 \le i \le m}$ is an R-refinement of $\mathcal{E}$ if and only if there exists a compact subset K of X such that $X - \bigcup_{i=1}^{m} int_X A_i \subseteq K$ and for each i from 1 to m , there exists $j_i \in N$, $1 \le j_i \le k$, such that $A_i - E_{j_i} \subseteq K$.

Proof: By 4.6 $\mathcal{A}$ is a cover of X^* if and only if $X - \bigcup_{i=1}^{m} int_X A_i$ is compact. By definition $\mathcal{A}$ is a refinement of $\mathcal{E}$ if and only if $int_{X^*} A_i^* \subseteq int_{X^*} E_{j_i}^*$ for each i from 1 to m and some j_i $(1 \le j_i \le k)$. Since by 2.5 A_i^* and $E_{j_i}^*$ are regular closed, this occurs if and only if $A_i^* \subseteq E_{j_i}^*$. By 2.3 (i) $A_i^* \subseteq E_{j_i}^*$ if and only if $A_i - E_{j_i}$ is contained in some compact subset K_i of X . Put $K = (X - \bigcup_{i=1}^{m} int_X A_i) \cup (\bigcup_{i=1}^{m} K_i)$ and we are done.

4.9 Proposition

Let $\mathcal{A} = (int_{X^*} A_i^*)_{1 \le i \le k}$ be an R-cover of $\beta X - X$. Then $ord \mathcal{A} \ge n$ if and only if there exist $n+1$ members of $\mathcal{A}$ - say $(int_{X^*} A_{i_j}^*)_{1 \le j \le n+1}$ -

such that $\bigcap_{j=1}^{n+1} \mathrm{int}_X A_{i_j}$ is not contained in any compact subset of X .

Proof: Evidently $\mathrm{ord}\,\mathcal{R} \geq n$ if and only if there exists a subfamily $(\mathrm{int}_{X^*} A^*_{i_j})_{1 \leq j \leq n+1}$ of $\mathcal{R}$ such that $\bigcap_{j=1}^{n+1} \mathrm{int}_{X^*} A^*_{i_j} \neq \emptyset$. By 1.82 this occurs if and only if $\bigwedge_{j=1}^{n+1} A^*_{i_j} \neq \emptyset$, which by 2.7 is equivalent to $(\bigwedge_{j=1}^{n+1} A_{i_j})^* \neq \emptyset$. By 2.1 (iii) and 1.82 this occurs if and only if $\bigcap_{j=1}^{n+1} \mathrm{int}_X A_{i_j}$ is not contained in any compact subset of X .

4.10 Theorem

Let X be a σ-compact space. Then $\dim X^* \geq n$ if and only if there exists a cover $\mathcal{U} = (U_i)_{1 \leq i \leq k}$ of X such that every refinement of $\mathcal{U}$ contains $n+1$ sets whose intersection is not contained in any compact subset of X .

Proof: Suppose that the cover $\mathcal{U}$ exists as described. By 4.6 there exists a refinement $\mathcal{W} = (W_i)_{1 \leq i \leq k}$ of $\mathcal{U}$ all of whose members are regular open sets. Since $X - \bigcup_{i=1} W_i = \emptyset$, by 4.7 the family $\mathcal{E} = (\mathrm{int}_{X^*}(\mathrm{cl}_X W_i)^*)_{1 \leq i \leq k}$ is an R-cover of X^* . If $\mathcal{R} = (\mathrm{int}_{X^*} A^*_i)_{1 \leq i \leq m}$ is an R-refinement of $\mathcal{E}$, then by 4.8 there exists a compact subset K of X such that $X - \bigcup_{i=1}^{m} \mathrm{int}_X A_i \subseteq K$ and such that for each i from 1 to m , there exists an integer j_i $(1 \leq j_i \leq k)$ such that $A_i - \mathrm{cl}_X W_{j_i} \subseteq K$. If $(\mathrm{int}_X A_i) \cap (X - \mathrm{int}_X[\mathrm{cl}_X W_{j_i}]) \cap (X - K) \neq \emptyset$, i.e. if $(\mathrm{int}_X A_i) \cap \mathrm{cl}_X(X - \mathrm{cl}_X W_{j_i}) \cap (X - K) \neq \emptyset$, then it follows that

$(\mathrm{int}_X A_i - \mathrm{cl}_X W_{j_i}) \cap (X - K) \neq \emptyset$, which is a contradiction. Thus by 1.8 it follows that $\mathrm{int}_X A_i - W_{j_i} = \mathrm{int}_X A_i - \mathrm{int}_X(\mathrm{cl}_X W_{j_i}) \subseteq K \quad (1 \leq i \leq m)$. Consider the family $\mathfrak{F}$ of open subsets of X defined as follows:

$$\mathfrak{F} = ([\mathrm{int}_X A_i] \cap [X - K])_{1 \leq i \leq m} \cup (S \cap W_i)_{1 \leq i \leq k}$$

where S is an open subset of X such that $K \subseteq S$ and $\mathrm{cl}_X S$ is compact (such an S exists as X is σ-compact) . Since $X - \bigcup_{i=1}^{m} \mathrm{int}_X A_i$ is a subset of K and $(\mathrm{int}_X A_i) \cap (X - K) \subseteq W_{j_i}$, it follows that $\mathfrak{F}$ is a refinement of $\mathcal{W}$, and hence of $\mathcal{U}$. Thus by hypothesis $\mathfrak{F}$ contains $n+1$ sets whose intersection is not contained in any compact subset of X . As $\mathrm{cl}_X S$ is compact, it follows that there exist sets $A_{i_1}, \cdots, A_{i_{n+1}}$ such that $\bigcap_{j=1}^{n+1} \mathrm{int}_X A_{i_j}$ is not contained in any compact subset of X . By 4.9 $\mathrm{ord}\, \mathcal{R} \geq n$, and as $\mathcal{R}$ was an arbitrary R-refinement of $\mathcal{E}$, it follows from 4.5 that $\dim X^* \geq n$.

Conversely, suppose that $\dim X^* \geq n$. Then by 4.5 there exists an R-cover $\mathcal{E}$ of X^* such that every R-refinement of $\mathcal{E}$ contains $n+1$ sets with non-empty intersection. If $\mathcal{E} = (\mathrm{int}_{X^*} E_i^*)_{1 \leq i \leq k}$, then by 4.7 the set $X - \bigcup_{i=1}^{k} \mathrm{int}_X E_i$ is compact. Let W be any open subset of X such that $\mathrm{cl}_X W$ is compact and $X - \bigcup_{i=1}^{k} \mathrm{int}_X E_i \subseteq W$. Then $\{W , \mathrm{int}_X E_1 , \cdots , \mathrm{int}_X E_k\}$ is a cover $\mathfrak{F}$ of X . Let $(U_i)_{1 \leq i \leq m}$ be a refinement of $\mathfrak{F}$. Since $\bigcup_{i=1}^{m} \mathrm{int}_X(\mathrm{cl}_X U_i) = X$, by 4.7 the family

$\mathcal{W} = (\mathrm{int}_{X*}[\mathrm{cl}_X U_i]*)_{1\le i\le m}$ is an R-cover of $X*$. Since $\mathrm{cl}_X W$ is compact, for each i such that $(\mathrm{cl}_X U_i)* \neq \emptyset$, there exists an integer j_i $(1 \le j_i \le k)$ such that $U_i \subseteq \mathrm{int}_X E_{j_i}$. Thus $\mathrm{cl}_X U_i \subseteq \mathrm{cl}_X(\mathrm{int}_X E_{j_i}) = E_{j_i}$ and so $\mathrm{int}_{X*}(\mathrm{cl}_X U_i)* \subseteq \mathrm{int}_{X*} E^*_{j_i}$. Thus $\mathcal{W}$ is an R-refinement of $\mathcal{E}$, and so there exist $U_{k_1}, \cdots, U_{k_{n+1}}$ such that $\bigcap_{i=1}^{n+1} \mathrm{int}_{X*}(\mathrm{cl}_X U_{k_i})* \neq \emptyset$. A repetition of the proof of 4.9 shows that $T = \mathrm{int}_X[\bigcap_{i=1}^{n+1} \mathrm{cl}_X U_{k_i}]$ is not contained in any compact subset of X . Using the notation of 1.11 , suppose that $\bigcap_{i=1}^{n+1} U_{k_i} \subseteq V_s$ for some $s \in N$. Then as the open set $T \cap (X - \mathrm{cl}_X V_s)$ is non-empty and meets $\mathrm{cl}_X U_{k_1}$, it follows that the open set $T \cap (X - \mathrm{cl}_X V_s) \cap U_{k_1}$ is non-empty. Since this meets $\mathrm{cl}_X U_{k_2}$, it follows that $T \cap (X - \mathrm{cl}_X V_s) \cap U_{k_1} \cap U_{k_2} \neq \emptyset$. A repetition of this argument shows that $T \cap (X - \mathrm{cl}_X V_s) \cap (\bigcap_{i=1}^{n+1} U_{k_i}) \neq \emptyset$, which is a contradiction. Thus $\bigcap_{i=1}^{n+1} U_{k_i} - V_s \neq \emptyset$ for all $s \in N$, and so $\bigcap_{i=1}^{n+1} U_{k_i}$ is not contained in any compact subset of X . As $(U_i)_{1\le i\le m}$ was an arbitrary refinement of $\mathcal{F}$, it follows that $\mathcal{F}$ is the desired cover $\mathcal{U}$ of X .

4.11 <u>Theorem</u> **(Jerison)**

For each positive integer n , $\dim(\beta\underline{\underline{R}}^n - \underline{\underline{R}}^n) = n$.

<u>Proof</u>: For each $k \in N$ put $S_k = \{x \in \underline{\underline{R}}^n : 2k+1 \le \|x\| \le 2k+2\}$ (where $\|x\|$ denotes the norm of x) , and let $K = \bigcup_{k=1}^{\infty} S_k$. By 2.2 (i) K is

closed in the metric space $\underline{\underline{R}}^n$ and thus by 1.16 K is C*-embedded in $\underline{\underline{R}}^n$. It follows from 1.30 that $cl_{\beta\underline{\underline{R}}^n}K$ is homeomorphic to βK and thus K^* is homeomorphic to $\beta K-K$. Since K^* is closed in the normal space $(\underline{\underline{R}}^n)^*$, it follows from 1.16 that it is C*-embedded in $(\underline{\underline{R}}^n)^*$; thus by 1.69 it follows that $\dim K^* \leq \dim (\underline{\underline{R}}^n)^*$. By 1.70 and 1.71, $\dim (\beta\underline{\underline{R}}^n) = \dim \underline{\underline{R}}^n = n$. Since $(\underline{\underline{R}}^n)^*$ is closed in $\beta\underline{\underline{R}}^n$, by 1.16 it is C*-embedded in $\beta\underline{\underline{R}}^n$ and thus by 1.69 $\dim (\underline{\underline{R}}^n)^* \leq \dim \underline{\underline{R}}^n = n$. Hence to prove the theorem it suffices to show that $\dim K^* \geq n$. Since K^* is homeomorphic to $\beta K-K$, by 4.10 it suffices to exhibit a cover $\mathcal{U}$ of K such that every refinement of $\mathcal{U}$ contains $n+1$ sets whose intersection is not contained in any compact subset of K.

As $int_{\underline{\underline{R}}^n}S_k \neq \emptyset$, it follows from 1.71 that $\dim S_k = n$. Thus there exists a cover $\mathcal{U}_k = (U_i^k)_{1\leq i\leq s}$ of S_k such that every refinement of $\mathcal{U}_k$ contains $n+1$ members whose intersection is non-empty. If $j, k \in N$ then S_j and S_k are homeomorphic and so we can assume that each $\mathcal{U}_k$ contains the same finite number s of sets. Put $\mathcal{U} = (\bigcup_{k=1}^{\infty} U_i^k)_{1\leq i\leq s}$. As each S_k is open-and-closed in K, each member of $\mathcal{U}$ is an open subset of K, and thus $\mathcal{U}$ is a cover of K. We also note that if $m \in N$ and $1 \leq j \leq s$, then $S_m \cap (\bigcup_{k=1}^{\infty} U_j^k) = U_j^m$ (since the (S_k) are pairwise disjoint).

Suppose that $\mathcal{W}$ is a refinement of $\mathcal{U}$ such that any $n+1$ members of $\mathcal{W}$ have a bounded intersection. As $\mathcal{W}$ has only finitely many subfamilies containing $n+1$ members, there exists $m \in N$ such that

$(\bigcap_{i=1}^{n+1} W_{j_i}) \cap S_m = \emptyset$ for any $n+1$ members $W_{j_1}, \cdots, W_{j_{n+1}}$ of $\mathcal{W}$. Consider the family $\mathcal{F} = (W \cap S_m)_{W \in \mathcal{W}}$ of open subsets of S_m. Obviously $\mathcal{F}$ is a cover of S_m. As $\mathcal{W}$ is a refinement of $\mathcal{U}$, each $W \in \mathcal{W}$ is contained in a set of the form $\bigcup_{k=1}^{\infty} U_i^k$. Thus $W \cap S_m \subseteq (\bigcup_{k=1}^{\infty} U_i^k) \cap S_m = U_i^m$, and so $\mathcal{F}$ is a refinement of $\mathcal{U}_m$. By our choice of m, any $n+1$ members of $\mathcal{F}$ have an empty intersection; this contradicts our choice of $\mathcal{U}_m$, and so $\mathcal{W}$ cannot exist. Thus $\mathcal{U}$ is the type of cover of K required in the hypotheses of 4.10 and so $n \leq \dim(\beta K - K) = \dim K^*$. Hence by our previous remarks, it follows that $\dim(\underline{\underline{R}}^n)^* = n$.

4.12 <u>Corollary</u>

If C is a cozero-set of $\beta\underline{\underline{R}}^n - \underline{\underline{R}}^n$, then $\dim C = n$.

<u>Proof</u>: Since $(\underline{\underline{R}}^n)^*$ is an F-space (see 1.37), it follows from 1.36 that C is C^*-embedded in $(\underline{\underline{R}}^n)^*$; hence by 1.69 $\dim C \leq n$. Since by 2.12 $[R(\underline{\underline{R}}^n)]^*$ is a basic subalgebra of $R((\underline{\underline{R}}^n)^*)$, by 1.87 there exists $A \in R(\underline{\underline{R}}^n)$ such that $\emptyset \neq A^* \subseteq C$. Thus $\mathrm{int}_{\underline{\underline{R}}^n} A$ is unbounded, and hence there exists a sequence $(S_k)_{k \in N}$ of n-cubes, each contained in $\mathrm{int}_{\underline{\underline{R}}^n} A$, such that any compact subset of $\underline{\underline{R}}^n$ contains only finitely many of these cubes. Put $B = \bigcup_{k=0}^{\infty} S_k$. As any two n-cubes are homeomorphic, the argument used in 4.11 to prove that $\dim K^* \geq n$ can be used here to show that $\dim B^* \geq n$. Since by 1.16 B^* is C^*-embedded in $(\underline{\underline{R}}^n)^*$, it is also C^*-embedded in C and so by 1.69 $n \leq \dim B^* \leq \dim C \leq n$. Thus $\dim C = n$.

4.13 Remark

Theorem 4.11 was first proved by Jerison [J] by constructing an essential mapping of $\beta\mathbb{R}^n - \mathbb{R}^n$ onto $[-1, 1]^n$. Our method of proof seems to have a greater range of application; for example, it is used to prove theorem 4.12.

4.14 Question

If W is an arbitrary open subset of $\beta\mathbb{R}^n - \mathbb{R}^n$, what is the dimension of W? It is evident that if W is normal then the argument employed in 4.12 can be used to show that $\dim W \geq n$, but it is not obvious that $\dim W \leq n$.

V CONNECTED SUBSETS OF $\beta X-X$

5.1 Theorem

Let X be a σ-compact space. Then $\beta X-X$ is not connected im kleinen (see 1.64) at any point, and thus is not locally connected at any point.

Proof: Using the notation of 1.11, put $A = \bigcup_{n=1}^{\infty}(cl_X V_{4n-2} - V_{4n-3})$ and $B = \bigcup_{n=1}^{\infty}(cl_X V_{4n} - V_{4n-1})$. It follows from 2.2 (i) that A and B are closed, and evidently $A \cap B = \emptyset$. Thus by 2.1 (iii), $A^* \cap B^* = \emptyset$. Let p be any point of X^*. Either $p \varepsilon X^* - A^*$ or $p \varepsilon X^* - B^*$; assume without loss of generality that $p \varepsilon X^* - A^*$. For $n = 1, 2, \cdots$, put $E_n = cl_X V_{8n-3} - V_{8n-6}$ and $F_n = cl_X V_{8n+1} - V_{8n-2}$, and set $E = \bigcup_{n=1}^{\infty} E_n$ and $F = \bigcup_{n=1}^{\infty} F_n$. By 2.1 (i) E and F are closed in X; evidently $E \cap F = \emptyset$ and $A^* \cup E^* \cup F^* = X^*$. Hence $X^* - A^* \subseteq E^* \cup F^*$ and $E^* \cap F^* = \emptyset$. Thus either $p \varepsilon E^* - F^*$ or $p \varepsilon F^* - E^*$; suppose without loss of generality that $p \varepsilon E^* - F^*$. Then $p \varepsilon (X^* - A^*) \cap (X^* - F^*) \subseteq int_{X^*} E^*$. Let W be any open subset of X^* contained in $int_{X^*} E^*$ and containing p. Since by 2.12 $\langle\{(cl_X[int_X Z])^* : Z \varepsilon Z(X)\}\rangle$ is a basic subalgebra of $R(X^*)$, there exists $Z \varepsilon Z(X)$ such that $p \varepsilon (cl_X[int_X Z])^* \subseteq W \subseteq E^*$. Thus by 2.3 (i) there exists $k \varepsilon N$ such that $cl_X(int_X Z) - E \subseteq V_k$. As $\emptyset \neq (cl_X[int_X Z])^* \subseteq E^*$, it is possible to partition the positive integers greater than k into two disjoint families N_1 and N_2 such that $cl_X(int_X Z) \cap E_n \neq \emptyset$ for infinitely many $n \varepsilon N_1$, and also for infinitely many $n \in N_2$. Put $S = \bigcup_{n\in N_1} E_n$ and $T = \bigcup_{n\in N_2} E_n$. Then S and T are disjoint, and by 2.2 (i) they are closed in X . As $N - (N_1 \cup N_2)$ is finite, it follows from 2.3 (i) that $(S \cup T)^* = E^*$ and neither $S^* \cap [cl_X(int_X Z)]^*$ nor $T^* \cap [cl_X(int_X Z)]^*$ is empty. Without loss of generality assume that $p \in S^* \cap [cl_X(int_X Z)]^*$, and pick $y \in T^* \cap [cl_X(int_X Z)]^*$; then $y \in W$. As $\{S^* \cap E^* , T^* \cap E^*\}$ is a partition of E^* into disjoint open-and-closed subsets, it follows that no connected subset of $int_{X^*} E^*$ contains both p and y . As W was an arbitrary open subset of $int_{X^*} E^*$ containing p , it follows that X^* is not connected im kleinen at p . As p was an arbitrary point of X^* , it follows that X^* is not connected im kleinen at any point. Thus by 1.65 X^* is not locally connected at any point.

5.2 <u>Lemma</u>

Let X be a σ-compact space and let $Z \in Z(X)$. If A and B are subsets of Z^* such that $A \cap B = \emptyset$ and $A \cup B = Z^*$, then there exist closed subsets E and F of X such that $A = E^*$ and $B = F^*$.

<u>Proof</u>: As B is closed in X^* , there exists a family $\{W_\alpha\}_{\alpha\in\Sigma}$ of closed subsets of X such that $B = \bigcap_{\alpha\in\Sigma} W_\alpha^*$. Thus $A \cap [\bigcap_{\alpha\in\Sigma} W_\alpha^*] = \emptyset$ and since X^* is compact, there exist indices $\alpha_1 , \cdots , \alpha_n \in \Sigma$ such that $A \cap [\bigcap_{i=1}^{n} W_{\alpha_i}^*] = \emptyset$. Hence $A \cap [\bigcap_{i=1}^{n} Z^* \cap W_{\alpha_i}^*] = \emptyset$, and as $A \cup B = Z^*$, it follows that $\bigcap_{i=1}^{n} Z^* \cap W_{\alpha_i}^* \subseteq B$. But obviously $B \subseteq \bigcap_{i=1}^{n} Z^* \cap W_{\alpha_i}^*$ and so it follows from 2.1 (iii) that

$$B = \bigcap_{i=1}^{n} [Z^* \cap W^*_{\alpha_i}] = [\bigcap_{i=1}^{n} Z \cap W_{\alpha_i}]^* \quad .$$

Thus $\bigcap_{i=1}^{n} Z \cap W_{\alpha_i}$ is the desired F ; E is constructed in a similar manner.

5.3 Theorem

The space $\beta\underline{\underline{R}}^+ - \underline{\underline{R}}^+$ is an indecomposable continuum, but contains decomposable subcontinua. However, if $n > 1$ then $\beta\underline{\underline{R}}^n - \underline{\underline{R}}^n$ is decomposable (see 1.60 and 1.61 for terminology).

Proof: It is well-known (for instance [GJ , problem 6L.4]) that $(\underline{\underline{R}}^+)^*$ is a continuum; it remains to show that it is indecomposable. By 1.62 it suffices to show that every proper subcontinuum of $(\underline{\underline{R}}^+)^*$ has an empty interior. Let E be such a subcontinuum. Then there exists a family $\mathfrak{F} \subseteq Z(\underline{\underline{R}}^+)$ such that $E = \bigcap_{F \in \mathfrak{F}} F^*$. As $E \neq (\underline{\underline{R}}^+)^*$, there exists $Z \in Z(\underline{\underline{R}}^+)$ such that $E \subseteq Z^* \neq (\underline{\underline{R}}^+)^*$. Thus it follows from 2.3 (i) that both Z and $\underline{\underline{R}}^+ - Z$ are unbounded subsets of $\underline{\underline{R}}^+$.

Now E is contained in a connected component K of Z^* ; for suppose that K_1 and K_2 were distinct components of Z^* and that $p \in E \cap K_1$ and $q \in E \cap K_2$. As Z^* is compact, it follows from 1.59 that K_1 is the intersection of all the open-and-closed subsets of Z^* that contain p ; hence as $q \notin K_1$ there exists an open-and-closed subset A of Z^* such that $p \in A$ and $q \in Z^* - A$. Then $\{E \cap A \ , E \cap (Z^* - A)\}$ is a partition of E into non-empty open-and-closed subsets, which contradicts the assumption that E is connected.

Suppose that $\mathrm{int}_{(\underline{\underline{R}}^+)^*}E \neq \emptyset$. Since by 2.12 $[R(\underline{\underline{R}}^+)]^*$ is a basic subalgebra of $R((\underline{\underline{R}}^+)^*)$, there exists $B \in R(\underline{\underline{R}}^+)$ such that $\emptyset \neq B^* \subseteq E$. Since E is contained in a connected component of Z^* , so is B^* . As $B^* \subseteq Z^*$, it follows from 2.3 (i) that there exists a positive integer n_0 such that $B \cap [n_0 , \infty) \subseteq Z \cap [n_0 , \infty)$. As both $R^+ - Z$ and Z are unbounded, we can choose a sequence $\{\lambda_n\}_{n\varepsilon N}$ of real numbers as follows:

(i) $\lambda_0 = 0$ and $\lambda_{n+1} \geq \lambda_n + 1$ for each $n \varepsilon N$.

(ii) $\{\lambda_n\}_{n\varepsilon N} \cap Z = \emptyset$.

(iii) $Z \cap [\lambda_n , \lambda_{n+1}] \neq \emptyset$ for each $n \varepsilon N$.

Another sequence $\{\alpha_n\}_{n\varepsilon N}$ of real numbers is now chosen inductively as follows:

(i) $\alpha_0 = 0$

(ii) $\alpha_n = \min \{\lambda_i : \lambda_i > \alpha_{n-1}$ and $[\alpha_{n-1} , \lambda_i] \cap B \neq \emptyset\}$.

Let $Z_1 = Z \cap (\bigcup_{n=0}^{\infty} [\alpha_{2n} , \alpha_{2n+1}])$ and $Z_2 = Z \cap (\bigcup_{n=0}^{\infty} [\alpha_{2n+1} , \alpha_{2n+2}])$.

It follows from 2.2 (i) that Z_1 and Z_2 are both closed, and they are obviously disjoint since $\{\lambda_n\}_{n\varepsilon N} \cap Z = \emptyset$. It follows from 2.1 (iii) and the choice of the $\{\alpha_n\}$ that $B^* \cap Z_1^*$ and $B^* \cap Z_2^*$ are both non-empty. Since $Z^* = Z_1^* \cup Z_2^*$ and $Z_1^* \cap Z_2^* = \emptyset$, this contradicts our earlier result that E is contained in a connected subset of Z^* . Thus $\mathrm{int}_{(\underline{\underline{R}}^+)^*}E = \emptyset$ and so $(\underline{\underline{R}}^+)^*$ is indecomposable.

To complete the proof of the theorem we construct a proper decomposable subcontinuum of $(\underline{\underline{R}}^+)^*$. Put $Z_1 = \bigcup_{n=1}^{\infty} [4n+1 , 4n+2]$ and let

$Z_2 = \bigcup_{n=1}^{\infty} [4n+2\ ,\ 4n+3]$. Let $\{r_n\}$ and $\{s_n\}$ be sequences in $\underline{\underline{R}}^+$ with the following properties:

(i) $4n+1 < r_n < 4n+2 < s_n < 4n+3$ for each positive integer n .

(ii) $\lim_{n\to\infty} (4n+2-r_n) = \lim_{n\to\infty} (4n+2-s_n) = 0$.

Let $\mathcal{U}$ be an ultrafilter on N such that $\bigcap \mathcal{U} = \emptyset$. Then there exist unique points p , r , and s in $(\underline{\underline{R}}^+)^*$ satisfying the following relations:

$$r \in \bigcap\{cl_{\beta\underline{\underline{R}}^+}\{r_n : n \in U\} : U \in \mathcal{U}\}$$
$$p \in \bigcap\{cl_{\beta\underline{\underline{R}}^+}\{4n+2 : n \in U\} : U \in \mathcal{U}\}$$
$$s \in \bigcap\{cl_{\beta\underline{\underline{R}}^+}\{s_n : n \in U\} : U \in \mathcal{U}\}$$

Now r and p are both in Z_1^* ; we shall show that they are in the same connected component of Z_1^* . If they were not, since Z_1^* is compact it follows from 1.59 that there exist non-empty closed subsets A and B of Z_1^* such that $p \in A$, $r \in B$, $A \cup B = Z_1^*$, and $A \cap B = \emptyset$. By 5.2 there exist closed subsets S and T of $\underline{\underline{R}}^+$ such that $A = S^*$ and $B = T^*$. It follows from 2.3 (i) that $S \cap T$ and $Z_1 - (S \cup T)$ are both bounded. As $r \notin S^*$, it follows that $\bigcap\{\{r_n : n \in U\}^* : U \in \mathcal{U}\} \cap S^* = \emptyset$. Since $(\underline{\underline{R}}^+)^*$ is compact, there exist $U_1 , \cdots , U_k \in \mathcal{U}$ such that $\bigcap_{i=1}^{k} \{r_n : n \in U_i\}^* \cap S^* = \emptyset$; thus by 2.1 (iii) , $[\{r_n : n \in \bigcap_{i=1}^{k} U_i\} \cap S]^* = \emptyset$. Put $\bigcap_{i=1}^{k} U_i = U_\alpha$; then U_α is a member of $\mathcal{U}$ with the property that

$\{r_n : n \in U_\alpha\} \cap S$ is bounded. Let $F = \{n \in N : n \in U_\alpha$ and $r_n \in S\}$. Then as any bounded subset of $\{r_n\}$ is finite, F is finite and so $N - F \in \mathcal{U}$. Thus $U_\beta = U_\alpha \cap (N - F) \in \mathcal{U}$, and if $n \in U_\beta$ then $r_n \notin S$.

Replacing $\{r_n\}$ by $\{4n+2\}_{n \in N}$ and S by T, we see that a repetition of the above argument shows that there exists $U_\gamma \in \mathcal{U}$ such that if $n \in U_\gamma$ then $4n+2 \notin T$. Put $U_\delta = U_\beta \cap U_\gamma$; then $U_\delta \in \mathcal{U}$ and hence U_δ is infinite. As both $S \cap T$ and $Z_1 - (S \cup T)$ are bounded, there exists a positive integer m such that $S \cap T \cap [m, \infty) = \emptyset$ and $(S \cup T) \cap [m, \infty) = Z_1 \cap [m, \infty)$. As U_δ is infinite, we can choose $n_0 \in U_\delta$ such that $n_0 > m$. As $[4n_0+1, 4n_0+2]$ is connected, either $[4n_0+1, 4n_0+2] \subseteq S$ or $[4n_0+1, 4n_0+2] \subseteq T$. The former situation is impossible as $r_{n_0} \notin S$, while the latter situation is impossible since $4n_0+2 \notin T$. Thus we have a contradiction, and so r and p are in the same connected component E_1 of Z_1.

A repetition of the above argument shows that there exists a connected component E_2 of Z_2 containing both p and s. Since $p \in E_1 \cap E_2$ and since E_1 and E_2 are closed in $(\underline{R}^+)^*$, the set $E_1 \cup E_2$ is a subcontinuum of $(\underline{R}^+)^*$. As $r \in E_1 - E_2$ and $s \in E_2 - E_1$, it is evident that $E_1 \cup E_2$ is the union of two proper subcontinua E_1 and E_2, and hence is decomposable.

Finally, if $n > 1$ let $A = \{(x_1, \cdots, x_n) \in \underline{R}^n : x_1 \geq 0\}$. Then both A and $\underline{R}^n - A$ are unbounded, and so A^* is a proper compact subset of $(\underline{R}^n)^*$. If A^* were not connected, then by 5.2 there would exist closed subsets S and T of $\underline{\underline{R}}^n$ such that $S^* \cup T^* = A^*$ and $S^* \cap T^* = \emptyset$. By 2.3 (i) there would exist $r > 0$ such that the set $S \cap T \cap \{x \in A : \|x\| \geq r\}$ is empty and

$$(S \cup T) \cap \{x \in A : \|x\| \geq r\} = \{x \in \underline{\underline{R}}^n : \|x\| \geq r\} .$$

This is clearly impossible, and hence A^* is a proper subcontinuum of $(\underline{\underline{R}}^n)^*$. Similarly $[cl_{\underline{\underline{R}}^n}(\underline{\underline{R}}^n - A)]^*$ is also a proper subcontinuum of $(\underline{\underline{R}}^n)^*$ and evidently $(\underline{\underline{R}}^n)^* = A^* \cup [cl_{\underline{\underline{R}}^n}(\underline{\underline{R}}^n - A)]^*$. Thus $(\underline{\underline{R}}^n)^*$ is decomposable.

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