THE EFFECT OF JET ENTRAINMENT ON LOSS
OF THRUST FOR A TWO-DIMENSIONAL
SYMMETRICAL JET-FLAP AEROFOIL

by

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SUMMARY

Jet-drag, defined as the form drag associated with jet entrainment in otherwise inviscid fluid, has been analyzed by replacing a two-dimensional incompressible jet with a suitable continuous distribution of sinks. Exact solutions for the jet drag, with the flow incompressible and the jet turbulent, are presented for uncambered struts of various shapes (including an ellipse and a circular cylinder), various thickness ratios, and with blowing slots of various widths in both quiescent and uniform streaming flow. In the latter cases the jet is blowing in the direction of the external flow. The extension of the theory to cross-stream blowing (as in the jet-flap) is discussed.

The present theory compares favourably with the few available experimental results when the jet velocity to streaming-flow velocity is sufficiently large.
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a - radius of the transformation circle (Figure 2.2)

b - radius of the circle transformed into a strut (Figure 2.2)

c - aerofoil chord

$C_J$ - jet momentum coefficient $= \frac{J}{\frac{1}{2}\rho U_\infty^2 c}$

$C_M$ - jet mass flow coefficient $= \frac{M}{\frac{1}{2}\rho U_\infty c}$

D - jet-drag per unit span; the form drag associated with the low pressure induced by jet entrainment

E - the rate of increase of vortical fluid in the jet per unit span per unit length of jet axis

J - jet momentum per unit span $= \rho U_J^2 t$

M - mass flow emanating from the slot $= \rho U_J t$

$Q_L$ - the strength of the $L^{th}$ sink

s - distance measured along the axis of the jet or the mixing layer downstream of the slot

t - width of the blowing slot

T - the maximum thickness of the aerofoil

u - time mean axial velocity in the jet

$U_J$ - jet velocity at the slot (assumed uniform), and in the potential core

$U_m$ - maximum velocity in the jet

$U_\infty$ - free stream velocity

V - the component of the inflow velocity towards and normal to the jet axis

W - complex potential function

x - real axis in the physical complex plane

y - distance measured normal to the axis of the jet, or the imaginary axis in the physical complex plane
$z = x + iy$ - coordinate system in the complex physical plane

$\zeta$ - the displacement between the centre of the transformation circle and the centre of the transformed circle defining the shape of the strut (Figure 2.2)

$\zeta = \xi + i\eta$ - coordinate system in the complex circle plane

$\rho$ - density of the fluid

$\sigma$ - growth parameter determining the rate of growth of a jet

**Subscripts**

$\bar{c}$ - identifies the position at which the potential core ends and the jet is assumed to become fully developed

$J$ - refers to the fully developed jet (region II in Figure 2.1) in the absence of streaming flow

$J_\infty$ - refers to the fully developed jet (region II in Figure 2.1) in the presence of a uniform external stream which is parallel to the jet

$L, m$ - identify the location of the $L^{th}$ and $m^{th}$ sink respectively

M.R. - refers to the mixing region (region I in Figure 2.1) in the absence of an external stream

M.R.$\infty$ - refers to the mixing region (region I in Figure 2.1) in the presence of a uniform external stream which is parallel to the jet
INTRODUCTION

The design of an efficient lift augmenting device with low drag is a problem which has received considerable attention because of its practical importance in the take-off and landing of aircraft. For take-off one of the most promising devices is that of the jet flap; high lift coefficients are obtained while most of the jet momentum is recovered as thrust. Lift augmentation over and above that associated with the downward component of the jet momentum at the trailing edge is obtained by an increase of circulation around the wing; this increase is called supercirculation. Initially the jet flap arrangement consisted of a thin jet sheet ejected from the trailing edge of an aerofoil and inclined to its chord.\(^1\)

The arrangement was later modified to include a small trailing edge flap over which the jet was blown and which enabled the jet angle to be controlled by altering the flap deflection. The latter arrangement produced an increase of lift for a given jet angle and jet momentum\(^2\) due to the contribution of the deflected flap itself\(^3\) as well as jet entrainment.\(^4\)

Numerous theoretical analyses have been developed for predicting the increment of lift and moment created by supercirculation round aerofoils. At first these analyses were a mere analogy with the conventional solid flap which led to the descriptive name: the jet-flap. Stratford\(^5\) replaced the initial straight portion of the jet by a finite straight solid flap and this was followed by an analogy with a finite curved
solid flap. A more realistic approach was made by Maskell and Gates, Spence and others who replaced the jet by a vortex sheet springing from the trailing edge and curving backwards in the direction of the stream, asymptotically approaching the stream direction at infinity. In most cases the problem was linearized for small incidence and jet deflection, and quite successful predictions of lift were obtained even for large jet deflections. The initial N.G.T.E. jet flap experiments indicated that the thrust acting on a two-dimensional aerofoil due to the jet exceeded the horizontal component of the jet momentum thus leading to the thrust hypothesis which Stratford stated as follows:

"Provided that the main stream speed is sufficient to prevent separation near the leading edge, the total forward thrust on the aerofoil will be almost independent of the deflection of the jet."

This hypothesis was substantiated theoretically by Woods and Maskell and Gates for ideal non-mixing jets with large jet to streaming-flow velocity. A complete experimental justification of this hypothesis could not be obtained since the skin friction and form drag associated with the boundary-layer development alone could not be isolated. By assuming that the drag associated with the boundary layer development depends on the lift and is independent of how this is produced, Foley reinterpreted the jet thrust as the reduction of drag due to the jet at constant lift, over and above that which an unblown aerofoil would experience if it could attain that lift without
flow separation. To use this new definition Foley extrapolated the lift-drag curves of an unblown aerofoil with fully attached flow past the point where the flow would normally separate and thus estimated the increase of drag associated with lift. Foley's interpretation of Stratford's jet thrust hypothesis was largely but not completely verified by experiment for jet angles as large as 60°. Foley never succeeded in measuring a reduction in drag equal to 100% of the jet momentum even for very small jet angles. This result substantiates the belief that the mixing of the jet with the surrounding fluid contributes to the overall drag experienced by a blown aerofoil. This component of drag is called the "jet-drag".

Stratford(11,12) was the first to analyze the mixing effects of the jet and realized that a strong jet entrains external fluid by inducing an external flow perpendicular to its axis; in this sense the jet is similar to a continuous sink. Stratford determined the total sink strength for the entire jet by assuming that the mixing takes place at two slightly different pressures, and that the velocities of the jet and the surrounding fluid are parallel in the mixing region. The jet drag acting on the aerofoil was then calculated by replacing the jet with a single sink located where the velocity, in the absence of the sink, would have been the free stream velocity at the position of mixing.(12)

Payne(13) criticized Stratford's analysis and extended it theoretically. He used an empirical equation to determine an overall sink strength associated with jet entrainment and
positioned the sink at the trailing edge of the aerofoil. Payne chose his sink strength on the basis of experiments in axisymmetric jets. Since the entrainment in the axisymmetric jet is much smaller near the blowing slot than in the two-dimensional jet (14) Payne's choice of sink strength is unfortunate.

The effect of jet entrainment on the lift and moment of an aerofoil was analysed in reference (14), by using a continuous sink distribution obtained by using a simple extension of Görtler's analysis for a two-dimensional jet. The effect of finite slot size was accounted for by matching the mass flow at the slot to that in a fully developed jet. A more detailed examination of the mixing layers near the slot was given in reference (14), where the flow induced by both two-dimensional and axisymmetric jets issuing normally to an infinite plate in the absence of an external stream was analysed. Once again a suitable distribution of sinks was used and this inviscid flow model was well substantiated experimentally.

In the present analysis the 'jet-drag' (i.e. the drag associated with the region of low pressure in the vicinity of the trailing edge due to jet entrainment, in otherwise inviscid flow) is calculated for a symmetrical strut by replacing the jet with a continuous sink extending from the trailing edge to infinity.

The effects of altering the shape of the strut, its thickness-to-chord ratio, the slot width-to-chord ratio and the jet velocity to the streaming-flow velocity ratio are analysed for a jet emanating parallel to an external stream, as well as
for quiescent surroundings. The flow is assumed to be incompressible and the jet always turbulent.

A qualitative explanation for the loss of thrust when the jet is inclined at large angles to the chord of the aerofoil is also given, and in particular the contribution of jet entrainment to this loss is discussed.
2. ANALYSIS

2.1 The Distribution of Sinks which Represent the Entrainment Effect of a Two-Dimensional Incompressible Jet on an External Fluid.

A viscous jet is often analysed theoretically as if it consisted of two distinct regions. Region I, near the exit of the jet, consists of a core of uniform velocity and two mixing layers which spread linearly in the downstream direction. In region II, the mixing layers have joined and the mixing takes place across the entire jet. The flows are turbulent for all Reynolds numbers of practical aeronautical interest. Both regions have been studied separately in quiescent surroundings and each is individually self-preserving since the flow as a whole possesses no characteristic length. Hence each

![Diagram showing regions I and II of a jet]

FIGURE 2.1
region has a unique velocity profile when the velocity at any section is non-dimensionalized with respect to the mean velocity at the centre, and when the lateral coordinate in the jet or the mixing layer is non-dimensionalized with respect to distance downstream of a suitably chosen hypothetical origin. This origin is the slot itself for a mixing layer with small upstream boundary layers. The application of these separate analyses to an actual jet is not entirely satisfactory since they do not provide a smooth change from one region to the other. However, in analyzing the flow induced by a jet issuing normally to an infinite plate in still surroundings this discontinuity appeared to be unimportant. \(^{(14)}\)

The rate of increase of flux of vortical fluid in the jet per unit span per unit length of jet axis is given by:

\[
E = \frac{d}{ds} \int_{-h}^{h} u dy \quad \ldots (2.1)
\]

where \(u\) is the time mean axial velocity in the jet

\(h\) defines the mean distance of the laminar superlayer from the jet axis. \(^{(15)}\)

\(s\) is the distance measured along the jet axis downstream of the blowing slot.

Differentiating equation (2.1), making use of the continuity equation in two-dimensional incompressible flow and assuming that the jet is parallel to a uniform stream of velocity \(U_\infty\)

\[
E = 2V + 2U_\infty \frac{dh}{ds} \quad \ldots (2.2)
\]
where \( V \) is the average component of the inflow velocity perpendicular to the jet axis at \( h \).

Only the first term on the right hand side of equation (2.2) produces changes of pressure outside the boundaries of the jet. This term may therefore be simulated by a continuous sink of strength \( 2V \), extending from the blowing slot to infinity and placed on the axis of the jet, provided that the average jet thickness at any downstream station is small compared with the distance of that station from the blowing slot.

The second term on the right hand side of equation (2.2), which vanishes when the surrounding fluid is quiescent, represents the spreading effect of the jet.

The inflow velocity at the outer edge of region I in quiescent surroundings as determined by Tollmien \(^{(16)}\) is 0.032 \( U_J \) (where \( U_J \) is the axial velocity in the central core). This result is consistent with the measurements of Liepmann and Laufer \(^{(17)}\).

Hence the associated sink strength per unit distance of the jet axis in region I is:

\[
2V_{M,R} = 0.064 U_J \quad \cdots (2.3)
\]

Region I terminates about six slot widths downstream of the blowing slot. Values close to this number may be obtained from Tollmien's \(^{(16)}\) analysis supported by experiment, or by matching \(^{(18)}\) say, the fully developed jet velocity profile due to Görtler \(^{(19)}\) to the velocity profiles for the mixing region measured by Liepmann and Laufer \(^{(17)}\).
The axial velocity distribution in region II in the absence of an external stream is given by Görtler:

\[ u = \left( \frac{3J_0}{4\rho S} \right)^{1/2} \text{sech}^2 \frac{\sigma y}{s} \]  \hspace{1cm} \ldots(2.4)

where \( J \) is the jet momentum per unit span
\( \sigma \) is an experimental constant determined from measurements of growth of a jet (\( \sigma = 7.7 \) for a free jet in quiescent fluid, \( \sigma = 13.5 \) for the outer part of a wall jet in quiescent fluid)
\( s \) is the distance along the jet axis measured downstream of the slot.*
\( y \) is the distance normal to the jet axis.

From equation (2.4) the inflow into the jet per unit length of the jet axis is:

\[ 2V_J = \left( \frac{3J_0}{4\sigma \rho S} \right)^{1/2} \]  \hspace{1cm} \ldots(2.5)

At present there is little known about the characteristics of a jet issuing into an external stream, and this is particularly true when the jet is not parallel to the flow. The flow in general is not self preserving, and the entrainment velocity at the outer edge of the jet, \( V_{J\infty} \) (the additional subscript \( \infty \) denotes the presence of an external stream), decreases as the ratio of the local maximum jet velocity \( U_m \) to that of the external stream \( U_\infty \) decreases. Once the velocity variations across the flow become very small \( \left( \frac{U_m - U_\infty}{U_\infty} \to 0 \right) \) the flow,

* Note: the hypothetical origin for the jet has been taken at the slot which is consistent with measurements from slots with relatively thin internal boundary layers.(33)
if symmetrical, is very nearly self preserving (being a negative weak wake) and from that point onwards \( V_{J\infty} = 0 \).

In determining the circulation induced by a jet blowing over the surface of an aerofoil\(^{4}\) the mean jet velocity profile was approximated by effectively superimposing half a free jet on the streaming flow\(^{20}\). The appropriate value of the growth parameter \( \sigma \) was taken from Patel's\(^{21}\) results for self preserving wall jets in streaming flow. Downstream of the trailing edge however, the jet becomes free and Patel's results are no longer valid; indeed the correct choice of \( \sigma' \) can not be obtained entirely from experimental results due to the lack of reliable data. The theoretical analyses for a jet in streaming flow made by either Abramovich\(^{22}\) or Squire and Trouncer\(^{23}\) are most cumbersome to use and are known to be at variance with experiment\(^{24}\). Küchemann and Weber\(^{25}\) give a more convenient equation for an axisymmetric jet in streaming flow which is arrived at by consideration of the flow in a mixing layer. Based on their approach the following velocity profile for a two-dimensional jet in uniform streaming flow \( U_{\infty} \) is assumed to be:

\[
\frac{u - U_{\infty}}{U_J - U_{\infty}} = \left[ \frac{3\sigma}{4(\%)(1 - \frac{U_{\infty}}{U_J})} \right]^{1/2} \text{sech}^2 \frac{\sigma y}{s} \quad \ldots(2.6)
\]

where \( \sigma \) remains a constant = 7.7 for a free jet and \( \sigma = 13.5 \) for a wall jet.

Equation (2.6) does not satisfy the condition

\[ \int_{0}^{\infty} u(u - U_{\infty}) dy = \text{const.} \] which is arrived at from considerations of
momentum and continuity, and thus it is strictly incorrect far downstream of the slot. In addition local pressure gradients in the vicinity of the slot may have some effect on the growth of the jet. However equation (2.6) should be satisfactory where \( \frac{U_J}{U_\infty} \) is sufficiently large, and comparison at \( y = 0 \) with existing experimental data is fair (Figure 1) at least for \( \frac{U_J}{U_\infty} > 3 \) and distances up to 200 slot widths. It should be noted that the experimental results of Foley\(^{(10)}\) shown in Figure 1 are for an inclined jet with \( \frac{S}{c} \) measured along the axis of the jet which is defined as the locus of points of maximum velocity \( U_m \). These results are also in fair agreement with equation (2.6).

From equation (2.6) the inflow into the jet is:

\[
2 y_\infty = \left[ \frac{3 U_J (U_J - U_\infty)}{4 \sigma (s/c)} \right]^{\frac{1}{2}} = U_\infty \left[ \frac{3 (C_J - C_M)}{2 \sigma (s/c)} \right]^{\frac{1}{2}} \quad \ldots (2.7)
\]

where \( C_J \) and \( C_M \) are the momentum and the mass flow coefficients respectively, and \( c \) is the chord of the aerofoil.

Similarly the sink strength per unit length per unit span of the mixing region (region I) may be written as:

\[
2 V_{M,R,\infty} = 0.064 (U_J - U_\infty) \quad \ldots (2.8)
\]

and the distance from the slot at which the core ends;\(^{(25)}\)

\[
S_c = \frac{6 t}{1 - U_\infty/U_J} = \frac{6 t}{1 - \sqrt{\frac{6 t}{c U_J}}} \quad \ldots (2.9)
\]

where \( c \) is the aerofoil chord.
2.2 The Effect of Sinks on the Flow Around a Symmetrical Thick Joukowski Strut at Zero Incidence.

Consider an uncambered strut of chord $2b(1 + \frac{a^2}{b^2 - \varepsilon^2})$, thickness $2b(1 - \frac{a^2}{b^2 + \varepsilon^2})$ and a bluff trailing edge from which the jet is emerging. Such a strut resembles closely many symmetrical aerofoil sections which have been tested in jet flap experiments. The sinks representing the jet extend along the x-axis downstream of the trailing edge. The strut is transformed into a circle of radius $b$ by the transformation:

$$z = (\xi - \varepsilon) + \frac{a^2}{(\xi - \varepsilon)} \quad \ldots (2.10)$$

where $\varepsilon$ is a parameter defining the shape of the strut shown in Figure (2.2).

For the strut to have a bluff trailing edge $\varepsilon < (b - a)$. 
The transformation is made single-valued by considering the flow external to the circle only. In order to keep the circle as a streamline an image of each sink is added at the inverse point and a source of the same strength at the origin. The complex potential describing the flow in the $\zeta$ plane is:

$$W(\zeta) = U_0 (\zeta + \frac{\zeta^2}{\zeta} - \sum_{m} \frac{Q_m}{2\pi i} \left[ \ln \zeta - \ln(\zeta - \zeta_m) - \ln(\zeta - \frac{\zeta^2}{\zeta_m}) \right] \ldots (2.11)$$

where $Q_m$ is the strength of the $m^{th}$ sink located on the real axis at $\zeta_m$.

Due to the symmetry of the flow about the real axis the resultant force must be real (i.e. drag).

The force in the physical plane is calculated directly from Blasius' theorem:

$$D + i L = \frac{i\rho}{2} \oint_{\text{surf}} (\frac{dW}{dz})^2 dz = \frac{i\rho}{2} \oint_{\text{surf}} (\frac{dW}{d\zeta})^2 d\zeta d\zeta \ldots (2.12)$$

where \( \frac{d\zeta}{dz} = 1 + \frac{\sigma^2}{(\zeta_0 - \zeta)^2 - \alpha^2} \)

Thus differentiating equation (2.11), squaring the derivative and performing the contour integration around the body, i.e. excluding the sinks downstream of the trailing edge, the "jet-drag" is given by the following equation:

$$\frac{D}{J} = \frac{2}{\frac{\sigma}{U_0}} \sum_{m} Q_m \left\{ \frac{\sigma^2}{2\epsilon_m} \left[ \frac{\sigma}{(\epsilon_m - \epsilon)^2 - \alpha^2} + 1 \right] - \frac{\sigma^2}{(\epsilon_m - \epsilon)^2 - \alpha^2} \right\} -$$

$$- \frac{\rho}{2\pi J} \sum_{m} \sum_{l} Q_m Q_l \left\{ \frac{1}{\epsilon_l} + \frac{\epsilon_m}{\epsilon_m - \epsilon_l} + \frac{[0.5(\epsilon_m + \epsilon_l) - \epsilon] \sigma^2}{[(\epsilon_l - \epsilon)^2 - \alpha^2][(\epsilon_m - \epsilon)^2 - \alpha^2]} +$$

$$+ \frac{\sigma^2}{\epsilon_m [(\epsilon_m - \epsilon)^2 - \alpha^2]} + \frac{\sigma^2}{[(\epsilon_m - \epsilon)^2 - \alpha^2](\epsilon_m - \epsilon_l)} \right\} \ldots (2.13)$$
where the subscripts \( L \) and \( m \) refer to the \( L \text{th} \) and \( m \text{th} \) sink respectively.

By letting \( \varepsilon = 0 \) in equation (2.13) one obtains the "jet-drag" about an ellipse and by letting \( a = \varepsilon = 0 \) the "jet-drag" about a circular cylinder is obtained. In this way it is easy to establish the effect of the slope of the aerofoil surface near the trailing edge as well as the effect of its thickness. When the surrounding fluid is quiescent one gets the static jet-drag acting on a two-dimensional symmetric aerofoil, which in fact is the only kind of drag existing in stagnant surrounding fluid.

2.3 Jet-Drag About a Circular Cylinder

This is the simplest case and will be analyzed in considerable detail.

When \( a = \varepsilon = 0 \) in equation (2.10) the transformation becomes a trivial one (i.e. \( \zeta = z \)) hence equation (2.13) may be reduced to:

\[
\frac{D}{J} = \frac{b}{U_\infty} \sum_{m} \psi_m \left( \frac{1}{\xi_m} \right) - \frac{\rho}{2\pi J} \sum_{m} \int \int \psi_L \psi_m \left( \frac{1}{\xi_L} + \frac{\xi}{\xi - \xi_m \xi} \right) \ldots \tag{2.14}
\]

since \( \epsilon = 2b \).

Consider first the case when the surroundings are quiescent, i.e. \( U_\infty = 0 \). The first term in equation (2.14) then vanishes. The discrete sinks are replaced by the actual continuous sink distribution extending from the slot at the trailing edge;
\[
\frac{D}{J} = \frac{\rho b^2}{2\pi J} \int_{b}^{\infty} \int_{b}^{x_L} Q(x_m)Q(x)\frac{dx_m dx}{x_L (x_m x_L - L^2)} \] ...

where \( Q(x_L) = Q(x_m) = 2V_{M,R} = 0.064 \) \( U_J \) for \( b < x_L < b + 6t \);
and for \( b < x_m < b + 6t \)
\[
Q(x_L) = 2V_J = \left[ \frac{3J}{4 \sigma \rho (x_L - b)} \right]^{\frac{1}{2}} \text{ for } x_L > b + 6t
\]
\[
Q(x_m) = 2V_J = \left[ \frac{3J}{4 \sigma \rho (x_m - b)} \right]^{\frac{1}{2}} \text{ for } x_m > b + 6t
\]

\( t \) is the width of the blowing slot.

Equation (2.15) with the appropriate boundary conditions contains four surface integrals all of which may be integrated at least once analytically in a closed form. Although the integrand in equation (2.15) diverges when both \( x_m \) and \( x_L \) tend to \( b \) the integral exists. The result is presented in graphical form in Figure 2. The drag is plotted as a fraction of the total jet momentum for different values of \( \frac{t}{2b} \), where \( 2b \) represents the chord of the cylinder.

Since in wind tunnel tests \( \frac{t}{c} \) is usually smaller than 0.005 the error introduced by neglecting entrainment in the mixing region is less than 3\% of the total jet drag, consequently \( 2V_{M,R} \) may be neglected in many cases. Furthermore in the presence of an external stream and with a significant jet flap effect, \( C_J \) is of the order of unity. Thus from equation (2.9) the extent of the mixing region \( s_\infty \), remains effectively unaltered and the entrainment in the mixing region may also be neglected in the streaming flow case.

Hence the "jet-drag" on a circular cylinder in streaming flow is:
\[ \frac{D}{J} = \frac{b}{c_f U_\infty} \int_{\xi \in \xi_0}^{\infty} Q(\xi m) \frac{d\xi m}{\xi m^2} \]

\[ - \frac{\rho \beta^2}{2 \pi \xi J} \int_{\xi \in \xi_0}^{\infty} Q(\xi m) d\xi m \int_{\xi \in \xi}^{\infty} Q(\xi) \frac{d\xi}{\xi \left( b^2 - \xi m \xi \right)} \]

\[ ... (2.16) \]

where \( \xi_m = b + 6t \)

\[ Q(x) = 2V_{J\infty}(x) = U_\infty \sqrt{\frac{\Gamma (C_J - C_m)(2b)}{\delta \sigma (x-b)^2}} \]

In Figure 4, \( \frac{D}{J} \) is plotted against \( C_J \) for various values of \( \frac{t_b}{2b} \) using \( \sigma = 7.7 \). It may be observed from the figure that for large \( C_J \) the effect of an external stream becomes unimportant and \( \frac{D}{J} \) tends to a constant for a given \( \frac{t}{C} \). This effect is evident from equation (2.16), since for large \( C_J \) the second term which is proportional to \( C_J \) becomes dominant.

2.4 Jet-Drag About a Symmetric Strut and an Elliptic Cylinder.

Replacing the discrete sinks in equation (2.13) by a continuous sink distribution, and making use of the simplifications discussed in the former section the static jet drag on a strut is:

\[ \frac{D}{J} = \frac{\rho}{2 \pi \xi J} \int_{\xi \in \xi_0}^{\infty} \int_{\xi \in \xi_0}^{\infty} Q(\xi) Q(\xi m) \frac{[\xi m \xi - \xi_m + \xi_m \xi - \xi] + [\alpha \xi (\xi + \xi_m) - \xi] \xi_m \xi (b^2 \xi_m \xi)}{\xi m \xi [(\xi^2 - \xi_m^2)(\xi_m \xi - \xi)^2 - \alpha^2]} d\xi m d\xi \]

\[ ... (2.17) \]

where \( Q(\xi) d\xi \) is obtained by transforming

\[ Q(x) dx = 2V_J dx = \left\{ \frac{\Gamma J}{4 \sigma \rho (x-(b^2 - \xi m \xi))} \right\}^{\frac{1}{2}} dx \]
into the $\zeta$ plane using equation (2.10) thus:

$$Q(\zeta)d\zeta = \left(\frac{3}{4\sigma}\frac{1}{(\zeta-e)(\zeta-b)(\zeta-e-\alpha)(\zeta-b-\alpha)}\right)^{\frac{1}{2}} \left[\frac{(\zeta-e)^2 - \alpha^2}{(\zeta-e)^2}\right] d\zeta$$

$\zeta_0$ is the point at which the potential core terminates and the jet becomes fully developed in the $\zeta$ plane. $\epsilon$ determines the shape of the strut, e.g. an increase of $\epsilon$ moves the point of maximum thickness closer to the leading edge.

Similarly the jet drag in the presence of streaming flow is:

$$D = \left[\frac{3b}{\lambda c}\left(\frac{c}{c'} - \frac{c_0}{c_0'}\right)\right]^{\frac{1}{2}} \int_{\zeta_e}^{\infty} \left[\frac{\xi_m^2 (b^2 - a^2) - 2b\xi_m b^2 + b^2 e^2}{\xi_m^2 (\xi_m - e)(\xi_m - b)(\xi_m - e - \alpha)(\xi_m - b - \alpha)}\right] d\xi_m$$

$$+ \frac{3}{8\pi \sigma} \left(1 - \frac{c_0}{c'}\right) \int_{\zeta_e}^{\infty} \int_{\zeta_e}^{\infty} \frac{\left[(\xi_m - e)^2 - \alpha^2\right]\left[(\xi_m - b)^2 - \alpha^2\right]}{V(\xi_m - e)(\xi_m - b)(\xi_m - e - \alpha)(\xi_m - b - \alpha)} d\xi_m d\xi_0$$

$$\cdots (2.18)$$

where $c$ is the chord of the strut and the appropriate sink strength has already been substituted.

By letting $\epsilon = 0$ in equations (2.17) and (2.18) the strut becomes an ellipse. The effect of the local shape of the strut near the blowing slot may then be investigated.

The static jet drag is plotted non-dimensionally in Figure 2 against various $\frac{t}{c}$ for elliptic cylinders of different thickness ratios and may be compared with the static jet drag acting on a circular cylinder which is plotted in the same figure. The calculations are repeated for a symmetrical strut.
having a thickness ratio of 25% and various $\epsilon$ and $\frac{t}{c}$, in Figure 3. The result for an elliptic aerofoil of the same thickness ratio is repeated for comparison in Figure 3.

The jet-drag for all the above configurations was calculated again with an external stream and is shown in Figures 5 to 10.
3. DISCUSSION

It has been shown that the thickness of the aerofoil, the width of the blowing slot, the shape of the aerofoil particularly near the trailing edge, and the jet-to-external-stream velocity ratio, are all of importance in determining the "jet-drag".

The variation of jet-drag with thickness ratio for various slot widths in quiescent surroundings is shown in Figure 2. It is obvious from this figure that the jet drag increases with bluffness of the body. For a bluff body in streaming flow, however, the overall drag may be appreciably reduced when a jet is blowing at the trailing edge since separation of the boundary layer is delayed by the favourable pressure gradient induced by the jet (Figure 11). In fact by blowing hard enough the flow may not separate at all. Even when separation is prevented by a proper geometric design of an aerofoil, jet entrainment may still reduce the overall drag by delaying transition in the boundary layer. However these viscous effects are specifically excluded in the present treatment.

The analysis shows a dependence of the jet drag on the slope of the aerofoil near the blowing slot. Figure 3 shows the loss of thrust in quiescent surroundings for different struts having the same thickness ratio (25%) but various positions of maximum thickness: the maximum thickness moves forward and the strut becomes more streamlined as $\epsilon$ increases. The effect of the slope of the aerofoil upstream of the blowing slot may be observed
from experiments made by Quanbeck\(^{26}\) which indicate a reduced thrust recovery with increased slope. The present analysis is qualitatively in agreement with the theoretical work of Payne\(^{13}\) with respect to the effect that the slope of the aerofoil near the trailing edge has on the jet-drag. It seems, however, that Payne over-emphasized this effect in representing the jet entrainment by a single sink positioned at the trailing edge for in doing so he localized the region of reduced pressure to the immediate vicinity of the blowing slot.

The present analysis is in agreement with Stratford's\(^{11}\) suggestion that in order to reduce the "jet-drag" in streaming flow, a jet of large mass flow for a given jet momentum should be used (equation 2.18).

The analysis proves that the jet momentum, even in quiescent surroundings, cannot be determined from balance measurements since not all the momentum is recovered as thrust. Foley\(^{10}\), who determined the jet momentum by measuring the mass flow into the wing and the pressure difference between the plenum chamber and the surroundings, found that the total force acting on the aerofoil in quiescent surroundings was equivalent to only 94\% of the estimated jet momentum. A static jet-drag of 0.06J is higher than would have been estimated from the present analysis for the same \( \frac{t}{c} \) and thickness ratio, however, the jet momentum estimated by Foley is probably too large due to boundary layer development within the slot. Experiments\(^{14}\) indicate that such boundary layer development
accounts for 3% - 6% loss in jet momentum calculated on the basis of ideal isentropic expansion for blowing slots of similar contraction ratio but this error should be smaller in Foley's case since the mass flow was determined by a metering nozzle.

Based on experiments with a 12.5% thick symmetrical aerofoil and an undeflected jet having the same temperature as the surrounding fluid, Dimmock\(^{(1,27)}\) suggested the following empirical formulae for estimating the jet drag:

\[
\frac{D}{J} = 0.06 \quad \text{for} \quad C_J < 0.1
\]

\[
\frac{D}{J} = 0.017 \quad \text{for} \quad 0.1 < C_J < 0.5
\]

\[
\frac{D}{J} = 0.0104 \quad \text{for} \quad 0.5 < C_J
\]

This jet drag was calculated from the experimental results on the assumption that the drag associated with the boundary-layer development was independent of the jet momentum.

Dimmock's experimental results are replotted for comparison with the present theory in Figure 12. The theoretical curves drawn in Figure 12 are for a symmetrical strut of 12.5% thickness ratio, \(\frac{C}{D} = 0.15\), \(\frac{t}{C} = 0.002\) and \(\frac{t}{C} = 0.00225^*\); the agreement is very satisfactory.

When a jet emerges at an angle relative to the external stream it is gradually deflected in the direction of the stream. The penetration distance of the jet normal to the external stream

---

*The aerofoil used by Dimmock had a chord of 8" but its slot width, t, varied between 0.018" and 0.016". The width of the slot is not stated in the references and it was obtained through correspondence with the author.*
depends on $C_J$ as well as the angle $\theta$ between the jet and the stream at the trailing edge. For $\theta \leq 60^\circ$ and $C_J \leq 1$ the jet is only slightly offset from the chord of the aerofoil when at zero incidence, and Foley (10) has shown experimentally that the thrust recovery is practically independent of $\theta$. The thrust recovery decreases rapidly when $\theta$ is increased above $60^\circ$ and is practically zero (28) when $\theta \approx 110^\circ$.

Tsongas (28) who made total pressure traverses across the jet for various $\theta$'s attributed the deterioration of thrust recovery to a "wake" which he found to be situated above the upper edge of the jet. This wake appeared to be very significant at large $\theta$ and it increased in width with an increase of $C_J$ although no separation anywhere from the aerofoil or the flap could be detected. Since the traverse was made perpendicular to the jet axis it is very doubtful that the actual total pressure was measured towards the outside of the jet since the streamlines in that region are sharply inclined to the jet axis. (29) As the entrainment increases with $C_J$ the "streamlines outside the jet are inclined more steeply to the jet and the error in Tsongas' total pressure measurement increased. This error was interpreted by Tsongas as a real increase in the deficit of total pressure.
For this case a more convincing explanation is as follows:

(i) Momentum is not completely conserved in a curved jet and this is even true for inviscid jets\(^{(8)}\) unless \(\frac{U_J}{U_\infty}\) is very large.

(ii) The entrainment of the jet close to the aerofoil is considerably increased. On the lower side of the jet the entrainment is probably enhanced, in comparison with the undeflected jet for the same \(\frac{U_J}{U_\infty}\), due to convex curvature of the jet boundary which is generally associated with a certain centrifugal instability of fluid particles in the jet,\(^{(23)}\) as well as due to the stagnation effect of the jet sheet which slows up the upcoming external stream locally. Smoke tunnel pictures given by Attinello\(^{(30)}\) and Davidson\(^{(29)}\) reveal that entrainment is even stronger on the upper side of the jet than on the lower side. (The ratio of entrained streamlines on the upper side of the jet to those on the lower side per unit length of jet axis in reference 30, is roughly 2:1). At first this appears to be a strange result since the upper edge of the jet is concave and it is generally accepted\(^{(18)}\) that the entrainment on the concave side of a jet is reduced. For a jet-flapped aerofoil at large \(C_J\) and \(\theta\) the overall circulation around the aerofoil is large with particularly strong vorticity near the blowing slot.\(^{(8)}\) The presence of this vorticity enhances the entrainment of the upper side of the jet and
over-rides curvature effects by inducing a downwash velocity roughly perpendicular to the axis of the jet.

(iii) The force acting on a body due to a sink outside the body increases when the distance between the sink and the body decreases. A certain length of a line sink representing the jet in the curved case is on the average closer to the body than a corresponding straight line sink which is aligned with the chord. This effect is of particular importance since the sink strength decreases with distance from the slot measured along the jet axis.

The increase of sink strength and the proximity of the sinks to the aerofoil combine to increase the jet drag and reduce the thrust. This appears to be an important contribution to the loss of thrust when there is no upstream boundary-layer separation. In general, however, an increase of $C_J$ or $\theta$ leads eventually to the formation of a leading edge separation bubble and attendant losses of momentum in the separated shear layer which further increase the drag.

Finally it should be noted that the increase of entrainment on the upper side of the deflected jet produces an increase of circulation over and above that predicted by the analysis based on the deflection of an inviscid jet. This may well explain why Spence's first order theory for lift produced by a jet flap is in agreement with experiment for a large deflection angle $\theta$. 
4. CONCLUSIONS

The effect of jet entrainment for an undeflected incompressible turbulent jet blowing in the downstream direction at the trailing edge of a two-dimensional body has been analyzed by replacing the jet with a suitable distribution of sinks. Exact solutions have been obtained for Joukowski uncambered struts of various shapes, thicknesses and with blowing slots of various widths in both quiescent and streaming flow when \( \frac{U_J}{U_\infty} \) is sufficiently large. The theory is in substantial agreement with the few available experimental results and is probably sufficiently accurate for practical purposes when \( \frac{U_J}{U_\infty} \) exceeds about 5.

It is shown that the jet-drag increases with a decrease of slot-width-to-chord ratio, with increase of thickness ratio, and with increase in bluffness of the body (i.e. increase of the slope in the vicinity of the trailing edge). There is a small but significant loss of thrust associated with jet entrainment in quiescent surroundings which has also been determined theoretically.

At present insufficient information is available concerning:

(i) the behaviour of a wall jet in an arbitrary pressure gradient;

(ii) the behaviour of a free jet in an external stream;

(iii) the effects of curvature on the development of the jet;

(iv) the characteristics of a wall jet leaving the surface and becoming a free-jet.
When this information becomes available a mathematical model should be able to predict the effects of jet entrainment on lift, drag and moment round a two-dimensional aerofoil with a deflected blown flap.

It is possible to conclude, however, that in a real two-dimensional incompressible jet the total thrust is never equal to the jet momentum, and this is particularly evident for large jet deflections ($\theta > 60^\circ$), due in part to a considerable increase of entrainment into the jet when it is inclined at large angles.
<table>
<thead>
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<th>Reference</th>
<th>Author(s)</th>
<th>Title and Details</th>
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Kistler, A.L.


Laufer, J.


Eichelbrenner, E.A.


Trouncer, J.

Shapiro, A.H.


Experimental Check on the Assumed Maximum Velocity Decay in Streaming Flow

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<td>6.7</td>
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Wall Jet $\sigma = 13.5$

Free Jet $\sigma = 7.7$
The Loss of Thrust (i.e., the Jet-Drag) on Circular and Elliptic Cylinders of Various $\frac{T}{c}$ and $\frac{t}{c}$ with Trailing Edge Blowing in Quiescent Surroundings.
The Loss of Thrust (i.e. the Jet-Drag) on Symmetrical Struts having $a = 0.25$ but various shapes ($\varepsilon \neq 0$) and various $\Delta$ in Quiescent Surroundings.
The Jet-Drag on Circular Cylinders of Various $t/c$ in Streaming Flow.
The Jet-Drag on Elliptic Cylinders of $\frac{t}{c} = 0.25$ and Various $\frac{t}{c}$ in Streaming Flow.
The Jet-Drag on Elliptic Cylinders of $\delta_c = 0.15$ and Various $t/c$ in Streaming Flow.
The Jet-Drag on Elliptic Cylinders of

\( \frac{T}{C} = 0.10 \) and Various \( \frac{t}{c} \) in Streaming Flow.
The Jet-Drag on Symmetrical Struts of $\frac{T}{c} = 0.25$, $\frac{c}{b} = 0.05$ and Various $\frac{t}{c}$ in Streaming Flow.
The Jet-Drag on Symmetrical Struts of
\( \frac{T}{c} = 0.25 \), \( \frac{b}{b} = 0.10 \) and Various \( \frac{t}{c} \) in
Streaming Flow.
The Jet-Drag on Symmetrical Struts of
\( \frac{t}{c} = 0.25, \frac{\xi}{D} = 0.15 \) and Various \( \frac{t}{c} \) in
Streaming Flow.
The Delay of Separation and Reduction of Wake Size with the Increase of Jet Momentum Coefficient

$C_J = 0$

$C_J \approx 0.29$

$C_J \approx 0.37$
Variation of the Jet-Drag with Momentum Coefficient for a Two-Dimensional Symmetrical Aerofoil of $\frac{T}{C} = 0.125$ at Zero Incidence with an Undeflected Jet. (Ref. 39)
CALCULATION OF INTEGRAL PARAMETERS OF A COMPRESSIBLE TURBULENT BOUNDARY LAYER USING A CONCEPT OF MASS ENTRAINMENT

by

N.M. Standen

Report 64-14

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TABLE OF CONTENTS

List of Figures
List of Symbols
Summary
Introduction
Analysis
A. Momentum Equation
B. Temperature Distribution
C. Auxiliary Equation
D. Solution of the Equations
Comparison with Experiment
A. Mach 3 Flat Plate
B. Mach 6 Flat Plate
C. Mach 3 Circular Arc Surface
D. Mach 6 Isentropic Surface
E. Modification of Temperature-Velocity Relation
Conclusions
References
Appendices
Figures
LIST OF FIGURES

Fig. 1  Correlation of Functions F and $H_{\Delta - \Delta^*}$, from Ref. (8)

Fig. 2  Correlation of Shape Factors $H_{\Delta - \Delta^*}$ and $H_{\Delta}$, from Ref. (8)

Fig. 3  Velocity profiles at start of compression, Isentropic surface, Ref. (6)

Fig. 4  Velocity profiles in final stages of compression, Isentropic surface, Ref. (6)

Fig. 5  Temperature-velocity relations, Mach 3 flat plate and start of Compression on Circular Arc surface. (a) $Tw/To \approx 0.835$ (b) $Tw/To \approx 0.460$

Fig. 6  Variation of Momentum and Displacement Thicknesses on Mach 3 flat plate of Ref. (6). Uncooled $Tw/To \approx 0.835$; Cooled $Tw/To \approx 0.460$

Fig. 7  Temperature-velocity relations, Mach 6 flat plate of Ref. (6). (a) $Tw/To \approx 0.76$ (b) $Tw/To \approx 0.28$

Fig. 8  Variation of Momentum and Displacement Thicknesses on Mach 6 flat plate of Ref. (6). Uncooled $Tw/To \approx 0.76$; Cooled $Tw/To \approx 0.28$

Fig. 9  Temperature-velocity relations near end of compression, Mach 3 Circular Arc surface. (a) $Tw/To \approx 0.835$ (b) $Tw/To \approx 0.460$

Fig. 10  Variation of Momentum Thickness on Circular Arc surface of Ref. (6). Uncooled $Tw/To \approx 0.835$; Cooled $Tw/To \approx 0.460$

Fig. 11  Variation of Displacement Thickness on Circular Arc surface of Ref. (6). Uncooled $Tw/To \approx 0.835$; Cooled $Tw/To \approx 0.460$

Fig. 12  Temperature-velocity relations at start of compression, Isentropic surface of Ref. (6). (a) $Tw/To \approx 0.82$ (b) $Tw/To \approx 0.45$
Fig. 13 Temperature-velocity relations near end of compression, Isentropic surface of Ref. (6).  
(a) $\frac{T_w}{T_o} = 0.82$  (b) $\frac{T_w}{T_o} = 0.45$

Fig. 14 Variation of Momentum Thickness on Isentropic Compression surface of Ref. (6)

Fig. 15 Variation of Displacement Thickness on Isentropic Compression surface of Ref. (6)
### LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
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<tr>
<td>$a, b, c,$</td>
<td>Constant Coefficients</td>
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<td>$a_e, a_o$</td>
<td>Sound speed at temperatures $T_e$ and $T_o$ respectively</td>
</tr>
<tr>
<td>$C_f$</td>
<td>Local Skin Friction Coefficient</td>
</tr>
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<td>$F$</td>
<td>Non-dimensional Mass Entrainment Rate, (Ref. 8)</td>
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<td>$H$</td>
<td>Compressible Shape Factor</td>
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<td>$H_{tr}$</td>
<td>Transformed Shape Factor</td>
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<tr>
<td>$H_i$</td>
<td>Equivalent Incompressible Shape Factor</td>
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<td>$H_{\Delta - \Delta^*}$</td>
<td>Shape Factor associated with Entrainment Rate, Ref.(8)</td>
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<td>Mach Number at outer edge of Boundary Layer</td>
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<td>Total Temperature at Height $y$ in Boundary Layer</td>
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<td>$U$</td>
<td>Transformed Longitudinal Velocity</td>
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<td>$x$</td>
<td>Longitudinal Coordinate Parallel to Wall</td>
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X  Transformed Longitudinal Coordinate
y  Coordinate Normal to Wall
Y  Transformed Normal Coordinate
γ  Specific Heat Ratio
δ  Boundary Layer Thickness
δ*  Compressible Displacement Thickness
δ*_{tr}  Transformed Displacement Thickness
δ*_{i} = Δ*  Incompressible Displacement Thickness
θ  Compressible Momentum Thickness
θ_{i}  Transformed or Incompressible Momentum Thickness
μ  Absolute Viscosity
ρ  Density
Δ  Transformed Boundary Layer Thickness

Subscripts
e  Outer edge of Boundary Layer
e  Evaluated at Total Temperature
r  Evaluated at Reference Temperature
i  Incompressible
The problem of determining the growth of a turbulent boundary layer under conditions occurring on ramjet air-intakes is discussed. A modified Stewartson transformation is employed to transform the compressible integral momentum equation to an equivalent incompressible plane, adopting approximate temperature-velocity relations of the Crocco and van Driest form. A semi-empirical auxiliary equation, developed by M.R. Head for incompressible flow and using the concept of mass entrainment into the boundary layer, is rearranged for use in the transformed plane. The theoretical results obtained by this method are then compared to McLafferty's lag-length theory, and to experimental data obtained by C.E. Kepler and R.L. O'Brien. It is seen that the mass entrainment theory is in good qualitative agreement with the reported data, and in reasonable quantitative agreement. The latter may be improved by postulating a relationship between temperature and velocity in the boundary layer which is in closer agreement with experiment. It appears that the mass entrainment theory indicates the point of separation of the compressible, turbulent boundary layer, in accordance with conventional incompressible separation criteria.
INTRODUCTION

The study of turbulent boundary layers is generally recognized to be a task of more than slight complexity, for the random eddying motion of the turbulent fluid is not amenable to simple mathematical description. In addition, such phenomena as shearing stresses and heat transfer across a turbulent layer are no longer proportional to parameters which are properties of the fluid alone, as in the case of laminar flows.

The shear stress distribution is not known analytically for turbulent motion. Therefore, the averaged or integral method is widely used in analyzing turbulent boundary layers, since a knowledge of the shear stress variation is not required in this method. For most engineering applications, the solutions of the integral equations provide sufficient information. The momentum thickness yields a measure of the drag on the surface due to the viscosity of the fluid and the displacement thickness accounts for the modification of the inviscid flow field about the surface again due to the fluid viscosity. The behavior of these parameters in many cases also yields information concerning the separation of the flow from the surface.
Such prediction of separation is primarily based on the magnitude and behavior of the ratio of displacement thickness to momentum thickness (Ref. 1). This ratio, known as the shape factor, is a parameter of the integral momentum equation and must be determined in the course of calculation. This entails the use of an auxiliary equation describing the variation of the shape factor. Such an equation is usually empirical or semi-empirical, since the concepts of conservation of mass, momentum and energy in the boundary layer do not yield a relation involving the shape factor variation.

Several auxiliary equations, used in conjunction with the von Karman (integral) momentum equation, have yielded satisfactory results for incompressible flows (Ref. 1). For compressible flow, however, largely due to the lack of experimental data in the supersonic Mach Number range, few original auxiliary equations have been developed. A notable exception is the lag-length theory of G.H. McLafferty and R.E. Barber (Ref. 2). For the most part, auxiliary equations for compressible turbulent boundary layers have been obtained through the use of a mathematical transformation. Using such a transformation, the compressible integral momentum equation is reduced to the incompressible form and is then used in conjunction
with an auxiliary equation obtained from experiments with incompressible fluids. The transformation thus allows the application of the large amount of incompressible data to compressible flows. It has been demonstrated (Ref. 3 and Ref. 4) that a modified form of the Stewartson-Illingworth transformation may be applied to the compressible turbulent momentum equation, thus reducing it to the incompressible form. Perhaps the best known application of such a method is that of Reshotko and Tucker (Ref. 5).

Methods involving the use of transformations have not met with a great deal of success in the past, largely because of difficulties with the auxiliary equations. In addition, the variation of density across the boundary layer, which must be considered in compressible flows, is difficult to describe analytically. Such a description requires the solution of the energy equation, usually in a form yielding a relation between temperature and velocity in the boundary layer. Under certain conditions, such a solution may be achieved, (Ref. 1) but the simultaneous occurrence of heat transfer to the surface and pressure gradients in the flow direction violate the required conditions. A temperature-velocity relationship including the effects of heat transfer and pressure
gradient is unavailable at this time.

In addition to such phenomena as compressibility of the fluid, adverse pressure gradients in the flow direction and heat transfer between the fluid and its bounding surface, other complexities such as centrifugal forces acting on the fluid (which have been neglected in the order of magnitude analysis of the boundary layer equations) and surface roughness are often engineering realities which cannot be ignored. An example of knowing accurately the behavior of boundary layers under these conditions is found in the study of supersonic combustion ramjets. The inlet diffuser of such a ramjet is a non-adiabatic surface over which flows high Mach Number air under an adverse pressure gradient. Due to the high recovery temperatures of high speed flight, the inlet surface must necessarily be cooled, resulting in heat transfer from the fluid to the surface. These are the basic conditions experienced by a turbulent boundary layer on a ramjet diffuser, and consequently comprise the parameters of this analysis.

**ANALYSIS**

In the following analysis, an ideal gas has been assumed, with a specific heat ratio of 1.4. The
experiment with which the theoretical results were compared was performed at a total temperature of about 400°F and a maximum Mach Number of 6, (Ref. 6) and it was felt that the ideal gas assumptions were sufficiently accurate in this thermodynamic regime. Real gas effects may be included without too much alteration or difficulty.

(A) Momentum Equation

The integral momentum equation for compressible turbulent flow is obtained by integrating the Prandtl momentum equation across the boundary layer thickness (Ref. 1). If the turbulent flow properties are represented as the sum of time-mean values and fluctuating components, then certain terms involving the fluctuating components appear in the integral momentum equation (Ref. 7). These terms usually may be neglected except in the region of separation or in the presence of large centrifugal forces acting on the fluid (Ref. 6 and 7). In Ref. 6 it is observed that these fluctuation terms, which include the variation of static pressure in the direction normal to a curved compression surface, may be neglected for moderately curved surfaces. Thus, the compressible integral momentum equation becomes
\[
\frac{d\Theta}{dx} + \frac{\Theta}{M_e} \frac{dM_e}{dx} \left( \frac{2 + H - M_e^2}{T_0/T_e} \right) = \frac{C_f}{2}
\] (2)

Normal pressure gradients due to centrifugal forces may be neglected on compression surfaces where the radii of curvature are large in comparison with the boundary layer thickness.

The skin friction coefficient \( C_f \) must also be defined for turbulent flow. In the present analysis, the Ludwig-Tillman equation for incompressible flow is used, with fluid properties evaluated at Eckert's reference temperature, following the procedure of Ref. 5. The resulting expression employing Sutherland's law of viscosity is

\[
\frac{C_f}{2} = \left[ 0.123 \exp(-1.56H_1)(U_e \Theta_i \rho_0/\mu)^{0.268} \right] \times
\]

\[
\times \left[ \frac{T_e}{T_r} \left( \frac{T_0}{T_r} \right)^{0.402} \left( \frac{T_0}{T_r + 198} \right)^{0.268} \right]
\] (3)

The momentum equation is now transformed to a form similar to the integral momentum equation for incompressible flow. A modified Stewartson transformation (Ref. 3 and 4) is used. Defining the transformation for the normal coordinate by
\[ dY = \frac{\rho^a_e}{\rho_0^{a_0}} \, dy \]  \hspace{1cm} (4)

and equating the compressible stream function to the transformed stream function, there results

\[ U = \frac{a_0}{a_e} \, u \]  \hspace{1cm} (5)

Under the transformation, the velocity ratio \( u/u_e \) is equal to the ratio of transformed velocities \( U/U_e \).

Employing the Stewartson transformation to the integral boundary layer quantities, then, and recalling that the static pressure remains constant through the boundary layer, the transformed integral parameters are defined (Ref. 4) as

\[ \Theta_i = \left( \frac{T_e}{T_0} \right)^3 \Theta \]  \hspace{1cm} (6)

\[ \delta_{tr}^* = \int_0^\Delta \left( \frac{T_s}{T_0} - \frac{U}{U_e} \right) dY \]  \hspace{1cm} (See App. A)

\[ H_{tr} = \frac{\delta_{tr}^*}{\Theta_i} \quad ; \quad H = \frac{T_0}{T_e} H_{tr} + \frac{T_0}{T_e} - 1 \]

Substituting equations (6) into the
compressible momentum equation (2), the transformed momentum equation is obtained

\[
\frac{d\theta_i}{dx} + \frac{\theta_i}{M_e} \frac{dM_e}{dx} \left(2 + H_{tr}\right) = \frac{C_f}{2} \left(\frac{T_e}{T_o}\right)^3
\]  

(7)

This transformed equation is still not of the form of the incompressible momentum equation, however, since the transformation of the longitudinal coordinate "x" is undefined, and since the friction coefficient term is not equivalent to the incompressible skin friction. In addition, the transformed shape factor must be related to an equivalent incompressible shape factor.

(B) Temperature Distribution

A relationship between temperature and velocity at any point in the boundary layer is now required in order to relate the transformed shape factor to an equivalent incompressible shape factor (Appendix B).

Such a temperature-velocity relationship was obtained by Crocco as an exact solution of the momentum and energy equations under zero pressure gradient, assuming a Prandtl Number of unity. Van Driest also obtained a similar relationship for a non unit Prandtl Number, although he assumed that the thermal boundary
layer was the same thickness as the velocity layer. The first of these analyses resulted in a linear relationship between temperature and velocity, of the form

\[
\frac{T_s}{T_o} = a + b \frac{u}{u_e}
\]  

(8)

with "a" and "b" constant.

Van Driest's equation was a quadratic in velocity ratio,

\[
\frac{T_s}{T_o} = a + b \frac{u}{u_e} + c \left( \frac{u}{u_e} \right)^2
\]  

(9)

The second derivative of this relation is positive, after evaluation of the constants from the boundary conditions.

However, under adverse pressure gradients, temperature-velocity curves obtained from experiments (Ref. 6) exhibit a negative second derivative (Fig. 6). It appears that the temperature distribution is at least of second order in velocity ratio, and has coefficients which yield negative second derivatives when the pressure gradient is considered.

The difficulty in obtaining an equation for
the temperature distribution suggests the use of a
closer, although less accurate approximation. Conse-
quently, both the Crocco and Van Driest temperature
distributions were used. Since the temperature
distribution appears in integral or averaged relations
only, the resulting error is not too severe.

(C) Auxiliary Equation

In either the compressible or transformed
momentum equations, it is still necessary to evaluate
the shape parameter H. Since this shape factor varies
with the growth of the boundary layer, an equation
defining its variation must be obtained.

A concept of the rate of entrainment of
external flow into the incompressible turbulent
boundary layer, suggested by M.R. Head (Ref. 8), has
led to the formulation of an auxiliary equation
describing the shape parameter variation. Head's
auxiliary equation is a more promising approach to the
problem since it involves the investigation of a
physical phenomenon which is the basis of boundary layer
growth.

In his derivation, (Ref. 8), Head assumes
that the rate of entrainment into a turbulent boundary
layer depends upon a boundary layer thickness
parameter, the free stream velocity and the velocity distribution in the outer portion of the layer. Using non-dimensional terms, Head arrived at a form of the auxiliary equation

\[
\frac{d}{dX} (\Delta - \Delta^*) = F - (\Delta - \Delta^*)/M_e \frac{dM_e}{dX} \quad (10)
\]

Using the experimental results of several papers, Head obtained an empirical correlation between the function \(F(H_{\Delta-\Delta^*})\) and the shape factor \(H_{\Delta-\Delta^*} = (\Delta - \Delta^*)/\theta_1\), and a correlation between \(H_{\Delta-\Delta^*}\) and \(H_i\), Figs. (1) and (2). It should be emphasized that these results were obtained for incompressible flow.

(D) Solution of the Equations

The transformation of the momentum equation is completed by defining the \(x\)-coordinate transformation as

\[
\frac{dX}{dx} = \frac{T_e}{T_r} \left( \frac{T_e}{T_o} \right)^3 \left( \frac{\mu_r}{\mu_o} \right)^{268} \quad (\text{Ref. (3) and Appendix C})
\]

and by relating the transformed and equivalent incompressible shape factors (Appendix B).

Head's auxiliary equation, Eqn. (10), is put into workable form by fitting equations to the curves
in Figs. (1) and (2). The equations so obtained are

\[ H_{\Delta-\Delta^*} = 1.535 \ (H_i - 0.7)^{-2.715} + 3.3 \]  \hspace{1em} (11)

corresponding to Fig. (2), and

\[ F = 0.0306 \ (H_{\Delta-\Delta^*} - 3.0)^{-0.653} \]  \hspace{1em} (12)

corresponding to Fig. (1).

The resulting form of the auxiliary equation is

\[ \frac{dH_i}{dx} = - \frac{(H_i - 0.7)^{3.715}}{4.17} \left[ \frac{F}{\theta_i} \frac{dX}{dx} - \frac{H_{\Delta-\Delta^*}}{M_e} \frac{dMe}{dx} - \frac{H_{\Delta-\Delta^*}}{\theta_i} \frac{d\theta_i}{dx} \right] \]  \hspace{1em} (13)

where \( H_{\Delta-\Delta^*} \) and \( F \) are given by Eqn. (11) and (12). The derivation of this form is presented in Appendix C.

The momentum and auxiliary equations were solved using a simultaneous numerical solution of the Runge-Kutta type. The integral parameters so computed were then transformed to the compressible plane by means of Eqn. (6).
COMPARISON WITH EXPERIMENT

A comparison of the calculation results with experiment will be limited to four sets of data obtained by Kepler and O'Brien (Ref. 6). This data consists of boundary layer measurements on flat plates at Mach Numbers of 3 and 6, a circular arc compression surface at an initial Mach Number of 3 and an isentropic compression ramp at Mach 6. Two different rates of wall cooling have been applied to each surface. Although these measurements appear to be quite precise, any conclusions which are drawn on the basis of such a limited comparison must be regarded as preliminary observations.

A. Mach 3 Flat Plate

The cross plot of the velocity and temperature profiles reported by Kepler and O'Brien is shown in Fig.(5). The Crocco temperature-velocity relation appears to be a good approximation to both the cooled and uncooled wall conditions. The quadratic temperature-velocity relationship (Eqn.(9) and Appendix B(b)) would afford a better fit for the uncooled wall case, but at a Mach Number of 3 and at the high wall temperature the difference between the linear and quadratic relations is small. For convenience, therefore, the Crocco relation
was used.

The variation of momentum thickness and displacement thickness with distance along the flat plate is presented in Fig. (6). The theoretical variation is in good qualitative and quantitative agreement with experiment, any discrepancies being within the magnitude of the repeatability of the experiment. The McLafferty lag-length prediction of the uncooled momentum thickness is not shown as it is coincident with the corresponding curve of the mass entrainment theory.

B. Mach 6 Flat Plate

The calculation results for the flat plate at Mach 6, Fig. (8), do not show as good agreement with experiment as in the Mach 3 case. It is difficult to compare the theory with the experiment qualitatively since only three measurements of the boundary layer parameters were made. However, it is seen from experiment that the momentum thickness of the boundary layer over the cooled wall is greater than the momentum thickness of the flow over the uncooled wall. It appears that the magnitudes of these thicknesses become nearly equal in the downstream direction but such a trend cannot be established due both to the lack of further experimental points and to the experimental error involved,
represented by the repeatability of the measurement. The mass entrainment theory indicates that the cooled momentum thickness increases at a slower rate than the uncooled thickness, and that the magnitude of the cooled boundary layer momentum thickness eventually becomes less than the magnitude of the uncooled momentum thickness. On the other hand, McLafferty's lag-length theory predicts the same trends exhibited by the experiment. For the uncooled wall case, the mass entrainment theory and the lag-length theory are in close agreement.

Neither the mass entrainment theory nor the lag-length theory yield good quantitative agreement with the experimental variation of displacement thickness, although both exhibit the same qualitative behavior as the experimental data. The magnitude of the cooled displacement thickness as given by the mass entrainment theory could be increased and thus improved by using a temperature-velocity relationship in closer agreement with experiment. The experimental temperature relation is illustrated in Fig.(7).

The temperature-velocity curve obtained from experiment is well approximated, in the uncooled wall case, by the Crocco relation. In the absence of a pressure gradient, the quadratic relation should yield the best description of the actual temperature-velocity
curve, as in Fig.(5a) and (12a). However, a small adverse pressure gradient was reported to exist on the flat plate at Mach 6 and this factor caused the deviation of the temperature-velocity curve from that predicted by the quadratic relation to a form more closely described by the Crocco linear equation. The temperature distribution in the boundary layer over the cooled wall is at variance with even the Crocco relation, suggesting that another factor in addition to the adverse pressure gradient must be considered. This is discussed more fully in a later section.

The existence of the adverse pressure gradient was taken into account in both the momentum and auxiliary equations. Following the argument set forth in the ANALYSIS, the Crocco temperature-velocity relation was not corrected for adverse pressure gradient effects in the cooled wall case.

C. Mach 3 Circular Arc Surface

The Mach Number distribution on the circular arc surface, reported in Ref. 6, was described analytically by a series of linear and semi-logarithmic equations. Each of these equations was chosen to represent best the experimental Mach Number data within a particular interval on the compression surface.

The temperature-velocity relations obtained
from the data of Ref. 6 are illustrated in Fig. (5) and (9). Figure (5) shows the relationship at the beginning of compression, under a negligible pressure gradient. It is identical to the case of the flat plate at Mach 3. Figure (9) depicts the temperature-velocity relationship at the end of compression, where the adverse pressure gradient is present, for both the cooled and uncooled wall conditions. It is seen in Fig. (9) that the experimental points are reasonably well approximated by the Crocco linear relation in the uncooled boundary layer. The temperature relation in the cooled boundary layer, however, is in poor agreement with Crocco's linear relation, and vividly exhibits the negative second derivative characteristic of the temperature-velocity profiles from Ref. 6 under adverse pressure gradients with considerable heat transfer.

The curves of the variation of momentum thickness, Fig. (10), and displacement thickness, Fig. (11), with distance along the circular arc are in poor quantitative agreement with experiment. Qualitatively, however, the mass entrainment theory does bear some resemblance to the experimental variations. In this respect it is more accurate than McLafferty's lag-length theory, also shown in these figures. McLafferty's theory does predict values of the momentum and displacement thicknesses which are nearer in magnitude to the
experimental results than is the mass entrainment theory. This is more or less expected since the lag-length theory was formulated on the basis of experimental results obtained in the Mach 3 range on various compression surfaces, some of which were similar to the circular arc under consideration.

The experimental momentum thickness, Fig.(10), shows a larger value initially for the cooled wall than for the uncooled surface. The cooled momentum thickness remains greater than the uncooled thickness to a distance of approximately 7.6 inches along the surface (1.6 inches downstream of the start of compression). At this point the uncooled momentum thickness, increasing rapidly, exceeds the cooled values. This same trend is predicted by the mass entrainment theory, although the slopes of the theoretical curves for both wall cooling rates are negative at the start of compression, while the slopes of the experimental curves are nearly zero, or slightly positive in the uncooled case.

The same qualitative agreement is apparent concerning the displacement thickness variation, Fig.(11). According to the data of Ref. 6, the magnitude of the uncooled displacement thickness is greater than that of the cooled displacement thickness at all points on the
compression surface. This same behavior is predicted by the mass entrainment theory, which also indicates a more rapidly increasing magnitude of the uncooled displacement thickness in comparison to the cooled displacement thickness. This comparison is also evident in the experimental data. Again, however, the mass entrainment theory predicts decreasing values of both uncooled and cooled displacement thickness after the start of compression, whereas the experimental values in this region show only a small decrease in the cooled boundary layer, and nearly constant values for the uncooled flow.

Improvement in the values of cooled displacement thickness could be achieved if an analytic relation between temperature and velocity, more closely approximating the experimental relation at all points on the compression surface, were used. Mathematically, from Eqn. (A5), it is seen that values of the total temperature ratio T_s/T_o greater than those predicted by the Crocco relation would yield larger values of the transformed displacement thickness, and thus, in equation (A4), larger values of the compressible displacement thickness. The increase of displacement thickness values would be reflected to a lesser degree in the values of the momentum thickness, which would also tend to
increase. Since the values of both the momentum thickness and displacement thickness at the start of compression \((x = 6.0 \text{ in.})\) are fixed as initial or starting conditions in the calculation, the increase of the integral parameters at succeeding points on the surface would reduce the negative slopes of the theoretical curves for the cooled boundary layer, and bring the curves into closer agreement with experiment. This improvement is discussed more fully in a later section.

Little improvement in the curves of the uncooled thicknesses could be expected by means of this correction, however, since the Crocco temperature-velocity equation provides good agreement with the experimental relations throughout the compression. Indeed, it is not expected that an adjustment to the cooled temperature relation would induce sufficient improvement in the curves of the cooled thicknesses to provide satisfactory agreement with experiment. Some additional factor must be considered in the flow over this particular compression surface, and this factor should be common to both the uncooled and cooled conditions. It is suggested that the consideration of centrifugal forces in the boundary layer may provide the necessary correction to the curves. An analysis of the magnitude of terms in the integral momentum equation
at the start of compression for the uncooled flow case indicated that a relatively small positive addition to the right side of Eqn. (7) would change the slope of the compressible momentum thickness curve from a negative value to a small positive value. An order of magnitude analysis indicated that the inclusion of centrifugal force effects in the momentum equation would provide at least a partial correction in the desired direction. However, the difficulty of describing these effects analytically in the integral momentum equation precluded their immediate application in the circular arc case. Of the four experimental surfaces considered in this study, it is expected that centrifugal force effects would be of importance only on the circular arc, due to the relatively smaller ratio of the boundary layer thickness to radius of curvature of that surface.

D. Mach 6 Isentropic Surface

An examination of the velocity profiles reported by Kepler and O'Brien, Figs. (3) and (4) indicates a full profile for both the uncooled and cooled surfaces at the initiation of compression, and inflected profiles for both wall temperatures in the final stages of compression.

Using such profiles in conjunction with total
temperature profiles obtained by Kepler and O'Brien at five stations on the compression surface, temperature-velocity curves were plotted. Examples of these are given in Figs. (12) and (13). It was observed that the van Driest temperature-velocity relation provided good agreement with the experimental points over more than half of the compression surface for the uncooled wall condition. In the case of the cooled wall, neither the van Driest nor the Crocco relation lay among the experimental points, but the Crocco distribution was the nearer of the two. Consequently in accordance with the argument set forth in ANALYSIS, the van Driest relation was used for the calculation of the uncooled boundary layer, and the Crocco equation for the cooled layer.

The calculation results are shown in Figs. (14) and (15). These curves indicate a better qualitative than quantitative agreement with experiment, although the uncooled wall results are in reasonable proximity to the experimental points. This is believed to be due to the closer agreement between the van Driest temperature distribution and experiment in the uncooled wall case, than between the Crocco relation and experiment in the cooled boundary layer. As a check on this assumption, the van Driest relation, giving even poorer agreement with experiment than the Crocco equation (Fig.(13)) was
used in the cooled wall calculation. The values of both the momentum thickness and displacement thickness were found to be lower at all points on the compression surface than those obtained using the Crocco relation. Although the difference was not great (of the order of 5%) it was significant enough to indicate that a closer approximation to the true temperature-velocity relationship would yield better values of the integral parameters.

The calculation of the integral parameters using McLafferty's lag-length procedure was performed by Kepler and O'Brien, and is shown in Figs. (14) and (15). It is noted that the results of the lag-length theory indicate no increase in either the momentum thickness or displacement thickness in the final stages of compression, as is shown by the experimental data. The mass-entrainment method, however, does indicate such an increase, or at least a levelling-off, of the integral values in this region. The value of the equivalent incompressible shape factor, $H_j$, was between 2.4 and 5.0 at this point, and was increasing rapidly. Such behavior of the shape factor in incompressible flow is generally accepted as a criterion of incipient separation. Although Kepler and O'Brien reported no occurrence of separation in their experiment (Ref. (6)),
the existence of inflected profiles and the increase of integral values suggest that separation was approaching. As was pointed out in Ref. (6), the expansion of the flow at the end of the compression surface, occurring when the flow was returned to its original direction, could well have influenced the behavior of the boundary layer in this region. The influence would be exhibited mainly in the subsonic portion of the boundary layer near the wall, and would have the effect of relaxing the inflected velocity profile, thus discouraging separation.

The calculation of the boundary layer presented here is a first order consideration. The addition of the displacement thickness to the surface profile would modify the Mach Number distribution, especially near the end of compression where the displacement thickness increases rapidly. In this region, the modification would result in a locally increased adverse pressure gradient. One would expect that this result, when included in the calculation, would tend to increase the theoretical values of the integral parameters in this area.

E. Modification of Temperature-Velocity Relation

In the preceding discussion, it has been noted that the existence of an adverse pressure gradient causes the experimental temperature-velocity relationship
in the boundary layer to deviate from the flat plate van Driest quadratic relation. It is also noted that this deviation is always in a direction such that the total temperature ratio is greater than the van Driest value at a given value of the velocity ratio. This deviation is expected, since the quadratic temperature-velocity equation (Eqn.(9) and App.B(b)) was derived on the basis of zero pressure gradient. The effect of an adverse pressure gradient is to retard the fluid near the surface and so reduce the fluid velocity across most of the boundary layer (Fig.4), causing only minor changes in the temperature profile (Ref. 6). The modification of the velocity profile under the adverse pressure gradient results in a larger total temperature ratio at a given velocity ratio than for the zero pressure gradient, or flat plate case.

For the near-adiabatic (uncooled) surfaces described in Ref. 6, the linear Crocco equation (Eqn.(8)) provides reasonably satisfactory agreement with the experimental temperature-velocity curves in the presence of adverse pressure gradients (Fig.(9a) and (13a)). It is emphasized that such agreement is not due to the inclusion of pressure gradient effects in the Crocco equation, for this equation was derived also on the basis of zero pressure gradient. The agreement is
due to the approximate linearity of the experimental data points, and thus is largely fortuitous.

The experimental temperature-velocity relations in the boundary layers over the cooled compression surfaces of Ref. 6 are another matter, however. The linear Crocco relation yields either poor agreement or no agreement at all. Since the cooled flow over either of the compression surfaces is under the same external pressure gradient as the uncooled flow (neglecting the modification of the inviscid flow field due to the boundary layer), the temperature-velocity relation for both rates of heat transfer would be expected to show the same deviation from the quadratic equation. Indeed, the process of cooling the boundary layer increases the velocity of the fluid in the layer as compared to the uncooled flow. On the basis of the argument presented above, the increase of velocity should reduce the deviation of the experimental temperature-velocity relation. Such is not the case, however, and the explanation for the reported temperature-velocity relationships is most likely to be found in the method by which the wall was cooled. As described in Ref. 6, the boundary layer developed initially over an uncooled surface upstream of the test surface. At some distance upstream of the model cooling was applied to the
area on which the boundary layer was growing, and was continued to the end of the test surface. Kepler and O'Brien pointed out that under this system only the portion of the boundary layer near the wall would be cooled. The upper regions would have insufficient time to adjust to the heat transfer at the wall surface. Consequently, the major portion of the boundary layer away from the wall would exhibit higher total temperatures than would be found at corresponding levels in a fully cooled boundary layer. In fact, the temperature profiles reported in Ref. 6 indicate total temperatures in the upper two thirds of the cooled boundary layers equal or nearly equal to those in the corresponding uncooled boundary layers. This factor is believed to be principally responsible for the extreme deviations from theory of the cooled temperature-velocity relations.

It is suggested that the reported temperature-velocity curves be approximated by a quadratic relation of the form of Eqn.(9). Such an approximation should not be of a higher degree than a quadratic since the derivation of Appendix B would not then apply.

The coefficient of the second order term (the coefficient "c" in Eqn.(9)) must be negative, in order that the second derivative of the expression be negative and thus describe the correct curvature of the
relationship. The constant term (term "a" in Eqn. (9)) must necessarily equal the ratio of wall temperature to free stream total temperature in order to satisfy the boundary condition at the wall. The remaining coefficient (coefficient "b" in Eqn. (9)) is then defined by the boundary condition at the outer edge of the boundary layer, namely, that the temperature ratio be unity when the velocity ratio is unity. The determination of the value of the coefficient "c" should include consideration of the effects of pressure gradient and heat transfer.
CONCLUSIONS

The conclusions reached in the preceding discussion may be summarized as follows:

(1) The concept of mass entrainment by a turbulent boundary layer appears to provide the basis of a suitable auxiliary equation for calculation of the shape parameter \( H \) of the boundary layer. Empirical relations describing such an entrainment in incompressible flow are applicable to compressible flows as well, through a suitably defined mathematical transformation. The good qualitative agreement between experiment and theory obtained through this concept suggests that the entrainment relationship should be investigated further, and established on a better theoretical and mathematical basis.

(2) Better quantitative theoretical results may be expected if a temperature-velocity relationship, providing closer agreement with experiment than the Crocco or van Driest form, is used. This implies the inclusion of pressure gradient and heat transfer effects in such a relationship.
(3) The mass entrainment method indicates separation of the boundary layer in a region where inflected velocity profiles and increasing values of integral parameters were observed in experiment. Separation was indicated by the behavior of the incompressible shape factor $H_i$, in accordance with the usual criteria for incompressible flow. Further comparison with experiments must be undertaken before the capability of this method to predict incipient separation is established.
REFERENCES


APPENDIX A

Derivation of Transformed Integral Parameters using Stewartson's Transformation

(a) Momentum Thickness

By definition

\[ \Theta = \int_0^\delta \frac{\rho u}{\rho_0 u_e} (1 - \frac{u}{u_e}) \, dy \]

Substituting equation (4), and recalling that \( u/u_e = U/U_e \), we have

\[ \Theta = \frac{\rho_0 a_0}{\rho_0 a_e} \int_0^\Delta \frac{U}{U_e} (1 - \frac{U}{U_e}) \, dY \]

where \( \Delta \) is the transformed boundary layer thickness.

With \( \gamma = 1.4 \), this becomes

\[ \Theta = \left( \frac{T_0}{T_e} \right)^3 \Theta_i \quad \text{A1} \]

where

\[ \Theta_i = \int_0^\Delta \frac{U}{U_e} (1 - \frac{U}{U_e}) \, dY \quad \text{A2} \]

so

\[ \Theta_i = \Theta \left( \frac{T_e}{T_0} \right)^3 \quad \text{A3} \]
(b) Displacement Thickness

By definition
\[ \delta^* = \int_0^\delta (1 - \frac{u \rho}{\rho_e u_e}) \, dy \]

or
\[ \delta^* = \int_0^\delta \frac{\rho}{\rho_e} \left( \frac{\rho_e}{\rho} - \frac{u}{u_e} \right) \, dy \]

Again using equation (4), we obtain

\[ \delta^* = \frac{\rho_o a_o}{\rho_e a_e} \int_0^\Delta \left( \frac{\rho_e}{\rho} - \frac{U}{U_e} \right) \, dY \]

By assuming constant static pressure normal to the wall,

\[ \frac{\rho_e}{\rho} = \frac{T}{T_e} \]

and

\[ \delta^* = \left( \frac{T_o}{T_e} \right)^3 \int_0^\Delta \left( \frac{T}{T_e} - \frac{U}{U_e} \right) \, dY \]

now
\[ \frac{T}{T_e} = \frac{T_o}{T_e} \left[ \frac{T_s}{T_o} - \frac{u_e^2}{2 c_p T_o} \left( \frac{u}{u_e} \right)^2 \right] \]

so
\[ \frac{T}{T_e} - \frac{U}{U_e} = \frac{T_o}{T_e} \frac{T_s}{T_o} - \left( \frac{T_o}{T_e} - 1 \right) \left( \frac{U}{U_e} \right)^2 - \frac{U}{U_e} \]
\[
\frac{T_{e} - U}{U_{e}} = \frac{T_{o}}{T_{e}} \frac{T_{s}}{T_{o}} - \frac{T_{o}}{T_{e}} \frac{U}{U_{e}} + \frac{T_{o}}{T_{e}} \frac{U}{U_{e}} - \left(\frac{T_{o}}{T_{e}} - 1\right) \left(\frac{U}{U_{e}}\right)^{2} - \frac{U}{U_{e}}
\]

Gathering terms

\[
\frac{T_{e} - U}{U_{e}} = \frac{T_{o}}{T_{e}} \left(\frac{T_{s}}{T_{o}} - \frac{U}{U_{e}}\right) + \left(\frac{T_{o}}{T_{e}} - 1\right) \frac{U}{U_{e}} \left(1 - \frac{U}{U_{e}}\right)
\]

Then

\[
\delta^{*} = \left(\frac{T_{o}}{T_{e}}\right)^{3} \int_{0}^{\Delta} \left[ \frac{T_{o}}{T_{e}} \left(\frac{T_{s}}{T_{o}} - \frac{U}{U_{e}}\right) + \left(\frac{T_{o}}{T_{e}} - 1\right) \frac{U}{U_{e}} \left(1 - \frac{U}{U_{e}}\right)\right] dY
\]

or

\[
\delta^{*} = \left(\frac{T_{o}}{T_{e}}\right)^{3} \left[ \frac{T_{o}}{T_{e}} \delta_{tr}^{*} + \left(\frac{T_{o}}{T_{e}} - 1\right) \Theta_{i} \right]
\]

using Eqn. (A 2) and defining

\[
\delta_{tr}^{*} = \int_{0}^{\Delta} \left(\frac{T_{s}}{T_{o}} - \frac{U}{U_{e}}\right) dY
\]

Now, using Eqn. (A 4) and (A 1), we obtain

\[
H = \frac{\delta^{*}}{\Theta_{i}} = \frac{T_{o}}{T_{e}} H_{tr} + \frac{T_{o}}{T_{e}} - 1
\]

where

\[
H_{tr} = \frac{\delta_{tr}^{*}}{\Theta_{i}}
\]
APPENDIX B

Derivation of Relations between Shape Factors

(a) Crocco Temperature Distribution

From Eqn. (8) we have

\[ \frac{T_s}{T_o} = a + b \frac{u}{U_e} = a + b \frac{U}{U_e} \]

where, from boundary conditions,

\[ a = \frac{T_w}{T_o}, \quad b = 1 - \frac{T_w}{T_o} = 1 - a \]

Then, in Eqn. (A 5),

\[ \delta_{tr}^* = \int_0^\Delta (a + (1-a) \frac{U}{U_e} - \frac{U}{U_e}) \, dy \]

\[ = \int_0^\Delta a (1 - \frac{U}{U_e}) \, dy \]

or

\[ \delta_{tr}^* = a \delta_i^* = \frac{T_w}{T_o} \delta_i^* \]

where

\[ \delta_i^* = \int_0^\Delta (1 - \frac{U}{U_e}) \, dy \]

as usual.

Now, from Eqn. (6)

\[ H_{tr} = \frac{\delta_{tr}^*}{\delta_i^*} = \frac{T_w}{T_o} \frac{\delta_i^*}{\delta_i^*} = \frac{H_i}{T_o} \]

\[ B_{1} \]
Then, from Eqn. (A 6)

\[ H = \frac{T_w}{T_e} H_i + \frac{T_o}{T_e} - 1 \]

(b) van Driest Temperature Distribution

From Eqn. (9) we have

\[ \frac{T_s}{T_o} = a + b \frac{u}{u_e} + c \left( \frac{u}{u_e} \right)^2 = a + b \frac{U}{U_e} + c \left( \frac{U}{U_e} \right)^2 \]

where the coefficients are defined as

\[ a = \frac{T_w}{T_o}; \quad b = \frac{T_w}{T_o} - \frac{T_w}{T_o}; \quad c = 1 - \frac{T_w}{T_o} \]

and thus \[ b = \frac{T_w}{T_o} - a = 1 - c - a \]

Then, in Eqn. (A 5),

\[ \delta^*_{tr} = \left( \int_0^\Delta \left[ a + \frac{U}{U_e} (1-c-a) + c \left( \frac{U}{U_e} \right)^2 - \frac{U}{U_e} \right] dY \right) \]

\[ = \left( \int_0^\Delta \left[ a \left( 1 - \frac{U}{U_e} \right) - c \frac{U}{U_e} (1 - \frac{U}{U_e}) \right] dY \right) \]

so \[ \delta^*_{tr} = a \delta^*_i - c \theta_i \]

Then, from Eqn. (6)
\[ H_{tr} = a H_i - c = \frac{T_w}{T_0} H_i + \frac{T_{aw}}{T_0} - 1 \] 

And from Eqn. (A 6)

\[ H = \frac{T_w}{T_e} H_i + \frac{T_{aw}}{T_e} - 1 \]
APPENDIX C

Development of Momentum and Auxiliary Equations

(a) Momentum Equation

From Eqn. (7)

\[
\frac{d\Theta}{dx} + \frac{\Theta}{M_e} (2 + H_{tr}) \frac{dM_e}{dx} = \frac{C_f}{2} \left( \frac{T_e}{T_0} \right)^3
\]

or

\[
\frac{d\Theta}{dx} + \frac{\Theta}{M_e} (2 + H_{tr}) \frac{dM_e}{dx} = \frac{dx}{dx} \frac{C_f}{2} \left( \frac{T_e}{T_0} \right)^3 = \frac{C_f}{2} i
\]

by transforming the longitudinal coordinate "x".

Thus

\[
\frac{dx}{dx} = \frac{C_f}{2} \left( \frac{T_e}{T_0} \right)^3
\]

where

\[
\frac{C_f}{2} i = 0.123 e^{-1.561 H_1} \left( \frac{U_e \Theta_i \rho_0}{\mu_0} \right)^{-0.268}
\]

Then, using Eqn. (3),

\[
\frac{dx}{dx} = \frac{T_e}{T_r} \left( \frac{T_r}{T_0} \right)^{0.402} \left( \frac{T_0 + 198}{T_r + 198} \right)^{0.268} \left( \frac{T_e}{T_0} \right)^3
\]

or

\[
\frac{dx}{dx} = \frac{T_e}{T_r} \left( \frac{T_e}{T_0} \right)^3 \left( \frac{\mu_r}{\mu_0} \right)^{0.268}
\]
(b) Head's Auxiliary Equation

From Eqn. (10)

\[
\frac{d}{dx} (\Delta - \Delta^*) = F - (\Delta - \Delta^*) \frac{dM_e}{dx}
\]

But \[ H_{\Delta - \Delta^*} = \frac{\Delta - \Delta^*}{\theta_i} \]

So \[ \frac{d}{dx} (H_{\Delta - \Delta^*} \theta_i) = F - \frac{H_{\Delta - \Delta^*} \theta_i}{M_e} \frac{dM_e}{dx} \]

or

\[ \frac{d}{dx} (H_{\Delta - \Delta^*} \theta_i) = F \frac{dX}{dx} - \frac{\theta_i H_{\Delta - \Delta^*}}{M_e} \frac{dM_e}{dx} \]

Then

\[ \theta_i \frac{dH_{\Delta - \Delta^*}}{dx} = F \frac{dX}{dx} - \frac{\theta_i H_{\Delta - \Delta^*}}{M_e} \frac{dM_e}{dx} - H_{\Delta - \Delta^*} \frac{d\theta_i}{dx} \]

or

\[ \frac{dH_{\Delta - \Delta^*}}{dx} = \frac{F}{\theta_i} \frac{dX}{dx} - \frac{H_{\Delta - \Delta^*}}{M_e} \frac{dM_e}{dx} - \frac{H_{\Delta - \Delta^*}}{\theta_i} \frac{d\theta_i}{dx} \]

But

\[ \frac{dH_{\Delta - \Delta^*}}{dx} = \frac{\partial H_{\Delta - \Delta^*}}{\partial H_i} \frac{dH_i}{dx} \]

and, from Eqn. (11)

\[ \frac{\partial H_{\Delta - \Delta^*}}{\partial H_i} = -4.17 (H_i - 0.7) -3.715 \]
So \[
\frac{dH_i}{dx} = - \left( \frac{H_i - 0.7}{4.17} \right)^{3.715} \left[ \frac{F}{\theta_i} \frac{dX}{dx} - \frac{H_{\Delta - \Delta^*}}{M_e} \frac{dM_e}{dx} \right] - \frac{H_{\Delta - \Delta^*}}{\theta_i} \frac{d\theta_i}{dx} \]

where \( \frac{dX}{dx} \) is given by Eqn. (C 1)
Fig. 1. Correlation of Functions $F$ and $H_{\Delta - \Delta^*}$, from Ref. (8)

Fig. 2. Correlation of Shape Factors $H_{\Delta - \Delta^*}$ and $H_1$, from Ref. (8)
Fig. 3. Velocity profiles at start of Compression, Isentropic surface, Ref. (6).

Fig. 4. Velocity profiles in final stages of Compression, Isentropic surface, Ref. (6).
Fig. 5. Temperature-velocity relations, Mach 3 flat plate and start of compression on Circular Arc surface. (a) $T_w/T_o = 0.835$
(b) $T_w/T_o = 0.460$
Fig. 6. Variation of Momentum and Displacement Thicknesses on Mach 3 flat plate of Ref. (6). Uncooled $T_w/T_o = 0.835$; Cooled $T_w/T_o = 0.460$
Fig. 7. Temperature-velocity relations, Mach 6 flat plate of Ref. (6). Uncooled $T_w/T_0 = 0.835$; Cooled $T_w/T_0 = 0.460$.
Fig. 8. Variation of Momentum and Displacement Thicknesses on Mach 6 flat plate of Ref. (6). Uncooled Tw/To = 0.76; Cooled Tw/To = 0.28.
Fig. 9. Temperature-velocity relations near end of compression, Mach 3 Circular Arc surface.
(a) Tw/To = .835  (b) Tw/To = .460
Fig. 10. Variation of Momentum Thickness on Circular Arc surface of Ref. (6). Uncooled $T_w/T_o = 0.835$; Cooled $T_w/T_o = 0.460$
Fig. 11. Variation of Displacement Thickness on Circular Arc surface of Ref. (6). Uncooled $\frac{T_w}{T_o} = 0.835$; Cooled $\frac{T_w}{T_o} = 0.460$. 

UNCOOLED COOLED

EXPT. --- THEORY EXPT. --- THEORY

--- McLafferty LAG-LENGTH
Fig. 12. Temperature-velocity relations at start of compression, Isentropic surface of Ref. (6).
(a) $T_w/T_0 = 0.82$  
(b) $T_w/T_0 = 0.45$
Fig. 13. Temperature-velocity relations near end of compression, Isentropic surface of Ref. (6). (a) $T_w/T_o \approx 0.82$  (b) $T_w/T_o \approx 0.45$
**Fig. 14.** Variation of Momentum Thickness on Isentropic Compression Surface of Ref. (6)
Fig. 15. Variation of Displacement Thickness on Isentropic Compression Surface of Ref. (6)
INFLUENCE OF TUBE ORIENTATION
ON COMBINED FREE AND FORCED
LAMINAR CONVECTION HEAT TRANSFER

by

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ABSTRACT

Combined free and forced-convection inside inclined circular tubes is studied theoretically. The case considered is that of fully developed laminar flow with constant pressure gradient, and constant heat flux. Fluid properties are considered constant except for the variation of density in the buoyancy terms. Upward flow only is considered. Velocity and temperature fields are calculated by perturbation analysis in terms of power series of Rayleigh numbers. A detailed analysis of the final equations is made to determine the range of values of non-dimensional parameter such as Rayleigh and Reynolds numbers over which the mathematical results are valid. Nusselt numbers are calculated based on bulk temperature difference and in final form are also expressed in terms of power series of Rayleigh numbers. Rayleigh number appears to be the most dominant parameter in equations of velocity and temperature fields and Nusselt number. However, Rayleigh and Reynolds number product and Prandtl number also influence the equations independently. As the tube inclination varies from horizontal, the Nusselt number increases up to a maximum which may occur before the vertical position is reached. The angle at which this maximum occurs appears to be a function of Rayleigh, Reynolds and Prandtl number, and in most instances lies between 20° and 60° of tube inclination.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>1</td>
</tr>
<tr>
<td>Nomenclature</td>
<td>2</td>
</tr>
<tr>
<td>Introduction</td>
<td>5</td>
</tr>
<tr>
<td>Calculation of Nusselt Number</td>
<td>13</td>
</tr>
<tr>
<td>Discussion</td>
<td>15</td>
</tr>
<tr>
<td>Conclusions</td>
<td>18</td>
</tr>
<tr>
<td>Appendix A</td>
<td>19</td>
</tr>
<tr>
<td>Appendix B</td>
<td>22</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>24</td>
</tr>
<tr>
<td>References</td>
<td>25</td>
</tr>
<tr>
<td>Diagrams (Figs. 1 to 16)</td>
<td>End</td>
</tr>
</tbody>
</table>
NOMENCLATURE

Roman Letter Symbols

\[ A = \text{axial temperature gradient (assumed constant), } \frac{\partial t}{\partial x} \text{ } ^\circ F/ft. \]
\[ a = \text{tube radius in feet.} \]
\[ C = \left( \frac{\partial p}{\partial x} + \rho_w g_x \right), \text{axial pressure gradient in fluid, } \text{lb/ft}^3. \]
\[ c_p = \text{specific heat of fluid at constant pressure, BTU/lb. } ^\circ F. \]
\[ g = \text{acceleration due to gravity, } \text{ft/sec}^2. \]
\[ g_x, g_r, g \theta = \text{components of acceleration due to gravity in three coordinate directions, } \text{ft/sec}^2. \]
\[ h = \frac{q}{(T_w - T_b)} = \text{heat transfer coefficient for fully developed flow based on bulk temperature difference, } \text{BTU/sec-ft}^2 \text{ } ^\circ F. \]
\[ N_{Gr} = \frac{\beta g A a^4}{\sqrt{2}} \text{ Grashof number, dimensionless.} \]
\[ N_{Nu} = 2ah/k \text{ Nusselt number, dimensionless.} \]
\[ N_{Nu_b} = \text{Nusselt number based on bulk temperature difference, dimensionless.} \]
\[ N_{Pr} = \frac{c_p \mu}{k} \text{ Prandtl number, dimensionless.} \]
\[ N_{Ra} = N_{Gr} \times N_{Pr} = \beta g A a^4 \beta^2 \frac{c_p \mu}{k} \text{ Rayleigh number, dimensionless} \]
\[ N_{Re} = -\frac{Ca^3}{(4\pi \gamma^2)} \text{ Reynolds number based on pipe diameter, dimensionless.} \]
\[ p = p'(x) + P(r, \theta) \text{, static fluid pressure, } \text{lb/ft}^2. \]
\[ q = \text{wall heat flux density, average over the circumference } \text{BTU/sec-ft}^2. \]
\[ R = \frac{r}{a} \text{, radial distance from the centre line of the tube, dimensionless.} \]
\[ r = \text{radial distance in cylindrical coordinate system, measured from the centre line of the tube, ft.} \]
Roman Letter Symbols

A = axial temperature gradient (assumed constant), °F/ft.

a = tube radius in feet.

C = \( \frac{\partial p}{\partial x} + \rho_w g_x \), axial pressure gradient in fluid, lb/ft^3.

c\(_p\) = specific heat of fluid at constant pressure, BTU/lb °F.

g = acceleration due to gravity, ft/sec^2.

\( \varepsilon_x, \varepsilon_r, \varepsilon_\theta \) = components of acceleration due to gravity in three coordinate directions, ft/sec^2.

h = \( \frac{q}{(T_w - T_b)} \) = heat transfer coefficient for fully developed flow based on bulk temperature difference, BTU/sec.ft^2 °F.

\( N_{Gr} \) = \( \beta g \alpha_a^4 / \gamma^2 \), Grashof number, dimensionless.

\( N_{Nu} \) = \( 2ah/k \), Nusselt number, dimensionless.

\( N_{Nu_b} \) = Nusselt number based on bulk temperature difference, dimensionless.

\( N_{Pr} \) = \( c_p \mu / k \), Prandtl number, dimensionless.

\( N_{Ra} \) = \( N_{Gr} \times N_{Pr} \) = \( \beta g \alpha_a^4 / \rho^2 \), Rayleigh number, dimensionless

\( N_{Re} \) = \( \alpha a^3 / (4 \rho \gamma^2) \), Reynolds number based on pipe diameter, dimensionless.

p = \( p'x + P(r, \theta) \), static fluid pressure, lb/ft^2.

q = wall heat flux density, average over the circumference BTU/sec.ft^2.

R = \( r/a \), radial distance from the centre line of the tube, dimensionless.

r = radial distance in cylindrical coordinate system, measured from the centre line of the tube, ft.
\[ T = \text{temperature of the fluid at any point, } ^\circ F. \]

\[ T_b = \text{bulk temperature of the fluid at any section, } ^\circ F. \]

\[ T_0 = \text{temperature of the tube wall at beginning of the fully developed flow, } ^\circ F. \]

\[ T^* = (T_w - T)/(A a N_{Pr}), \text{ difference between the wall temperature and any point of the fluid at same section, dimensionless.} \]

\[ T^*_{b} = (T_w - T_b)/(A a N_{Pr}), \text{ difference between the wall temperature and bulk temperature of the fluid at the same section, dimensionless.} \]

\[ v = \text{velocity of a fluid particle, ft/sec.} \]

\[ v_r = \text{fluid velocity measured along the radial coordinate of tube, ft/sec.} \]

\[ v_x = \text{axial velocity of fluid measured along the x-axis of tube, ft/sec.} \]

\[ v_{\theta} = \text{angular velocity of fluid measured along the angular coordinate of tube, ft/sec.} \]

\[ v_{xav} = \text{average velocity along the x-axis of tube, ft/sec.} \]

\[ v_x = v_x/(\nu/a), \text{ axial velocity, dimensionless.} \]

\[ v_{xav} = v_{xav}/(\nu/a), \text{ average axial velocity, dimensionless.} \]

\[ x = \text{axial coordinate of the tube measured in upward direction (direction of flow), ft.} \]

**Greek Letter Symbols**

\[ \alpha = \text{tube inclination measured from the horizontal position, degrees.} \]

\[ \beta = \text{volumetric coefficient of thermal expansion of the fluid, } 1/^\circ F. \]

\[ \Delta = \text{difference between two points.} \]

\[ \nabla^2 = \text{Laplacian in cylindrical coordinates.} \]

\[ = \frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2} \]
\( \nabla^4 \) = Laplacian of the Laplacian in cylindrical coordinates

\[
\nabla^4 = \frac{\partial^4}{\partial R^4} + (2/R) \frac{\partial^3}{\partial R^3} - (1/R^2) \frac{\partial^2}{\partial R^2} + (1/R^3) \frac{\partial}{\partial R} + (2/R^2) \frac{\partial^4}{\partial \theta^2 \partial R^2} \\
- (2/R^3) \frac{\partial^3}{\partial \theta^2 \partial R} + (4/R^4) \frac{\partial^2}{\partial \theta^2} + (1/R^4) \frac{\partial^4}{\partial \theta^4}
\]

\( k \) = thermal conductivity of fluid, BTU/sec.sq.ft. \( ^{\circ} \)F/ft.

\( \theta \) = angular position in cylindrical coordinate system, measured from the top point of the tube circumference, degrees.

\( \mu \) = dynamic viscosity of fluid, lb/ft.sec.

\( \nu \) = kinematic viscosity of fluid, ft\(^2\)/sec.

\( \rho \) = mass density of the fluid, lb.sec\(^2\)/ft\(^4\).

\( \rho_w \) = mass density of the fluid at wall, lb.sec\(^2\)/ft\(^4\).

\( \psi \) = Stokes stream function, dimensionless.

**Subscripts**

0, 1, 2 - refer to zero, 1st and 2nd order perturbations.

0, 1, 2 - refer to points at various positions along the axis of the tube.
INTRODUCTION

In any convective heat transfer process, density differences arise due to differences in temperature, and under the influence of a gravitational force field natural-convection effects result. In a forced-convection case, associated with large Reynolds numbers connected with large flow velocities, where the forces and momentum transport rates are very large, the effects of natural-convection are negligible. If, on the other hand, buoyancy forces arising from density differences are relatively large, (as exemplified by large Grashof numbers) the forced convection effects may be ignored. However, in many cases of practical interest, both the effects of forced-convection and natural-convection may be of comparable order. An indication of the relative magnitude of the two effects can be obtained from the differential equations describing the flow. With the comparatively small velocities associated with laminar motion, the heat transfer is substantially affected by buoyancy forces and the resulting velocity fields. In this circumstance, in addition to Grashof, Reynolds, and Prandtl numbers, the parameters describing the geometry of heat transfer surface and flow orientation to the gravitational field are also important. The problem that has been most extensively studied is that of vertical round tubes, where gravitational force is parallel to the tube axis. Various aspects of combined free and forced-

convection inside vertical circular tubes, ducts and channels were studied in references [1-20]. References [21-27] deal with the influence

1 Numbers in brackets refer to similarly numbered references in bibliography at end of paper.
of free-convection on forced flow in horizontal circular tubes and channels. Numerous studies of the influence of free-convection on forced-flow for external flows of boundary layer type are also available in the literature cited [28-31]. The case of pure free-convection inside vertical and inclined tubes, with both ends closed or open, has been investigated by references [32-34].

The present analytical study of combined free and forced-convection inside inclined tubes springs from an interest in its applications to flat plate solar collectors, which are normally placed at an inclined position. It is desired to know the influence of tube inclination on heat transfer. The case considered is that of uniform heat flux, which results in uniform temperature gradient along the wall. This happens to be approximately so in solar collectors. The analysis, however, is a generalized one.

**FORMULATION OF THE PROBLEM**

Consider a tube of radius "a" inclined at an angle \( \alpha \) to the horizontal, as shown in Fig. 1. There is a uniform heat flux "q" around the circumference and per unit length of the tube. This heat flux could be due to solar energy absorbed by the flat plate collector in solid contact with the tube, or resistance heating of the tube, etc. On the slow laminar motion of the fluid, flowing under external pressure, buoyancy forces are superimposed due to differences in density arising out of differences in temperature. These buoyancy forces create a secondary flow, distorting the normal Poiseuille flow to a form of helical
motion as shown in Fig. 2. Due to the buoyancy effects and the circulation of the fluid inside the tube, the circumferential distribution of tube temperature at any section will be no longer constant. The fact, however, that the thermal conductivity of the tube material is usually much higher than that of the fluid, will tend to minimize the circumferential temperature variation, so that a constant tube wall temperature at any section may be assumed. In addition, the specific heat, thermal conductivity and viscosity can be considered constant throughout the fluid. Density is also considered constant throughout except for its variation in the buoyancy term. Pressure gradient is assumed constant. For the case of uniform heat flux and under the above conditions, the temperature gradient within the fluid and at the tube surface becomes constant beyond the entrance length, as shown in Fig. 3. Also the temperature difference \( (T_w - T) \), between tube wall and fluid at any section, is constant along the tube, barring the entrance length. Since this temperature difference creates buoyancy forces, resulting in secondary flow that gives rise to radial and angular velocities, therefore \( v_r, v_\theta \), and \( v_x \), (the radial, angular and axial velocities respectively) become independent of the axial distance \( x \).

**ANALYSIS**

Using the cylindrical polar coordinate system where \( \theta \) is measured from the top vertical position of the circumference, \( x \) is measured from the point of fully developed flow, and with the assumptions made in the previous section, the governing equations for laminar flow can be written as:
Continuity Equation

\[ \frac{\partial (r v_r)}{\partial r} + \frac{\partial v_{\theta}}{\partial \theta} = 0 \]  

Momentum Equations

Momentum equations in \( r, \theta \), and \( x \) directions are respectively:

\[ \rho \left( v_r \frac{\partial v_r}{\partial r} + \frac{v_{\theta}}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_{\theta}^2}{r} \right) = \]

\[- \frac{\partial P}{\partial r} + \mu \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_{\theta}}{\partial \theta} \right) - \rho g_r \]  

\[ \rho \left( v_r \frac{\partial v_{\theta}}{\partial r} + \frac{v_{\theta}}{r} \frac{\partial v_{\theta}}{\partial \theta} + \frac{v_r v_{\theta}}{r} \right) = \]

\[- \frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left( \frac{\partial^2 v_{\theta}}{\partial r^2} + \frac{1}{r} \frac{\partial v_{\theta}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v_{\theta}}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{v_{\theta}}{r^2} \right) + \rho g_{\theta} \]  

\[ \rho \left( v_r \frac{\partial v_x}{\partial r} + \frac{v_{\theta}}{r} \frac{\partial v_x}{\partial \theta} \right) = \]

\[- \frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r} \frac{\partial v_x}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v_x}{\partial \theta^2} \right) - \rho g_x \]  

In the above equations, density in the buoyancy terms is to be expanded as \( \rho = \rho_w \left[ 1 + \beta (T_w - T) \right] \), where \( \beta \) is the coefficient of thermal expansion of fluid. Buoyancy force is calculated using the difference between fluid temperature \( T \) and the temperature of the wall \( T_w \). Gravitational components \( g_r, g_{\theta} \) and \( g_x \) are \( g \cos \alpha \cos \theta \), \( g \cos \alpha \sin \theta \) and \( g \sin \alpha \) respectively. Pressure gradient along the tube is considered constant, while across a section it is a function of \( r \) and \( \theta \). Temperature of the wall at any section is calculated as \( T_w = T_o + (\partial t/\partial x)x \). Where \( T_o \) is the wall temperature at the point starting the fully developed flow, and \( \partial T/\partial x = A \) being a constant.
Energy Equation

The energy equation is:

\[ c_p \left( \frac{V_r}{r} \frac{\partial T}{\partial r} + \frac{V_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{V_x}{\partial x} \right) = k \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} \right) \]  \hspace{1cm} (5)

In this equation, conduction along tube axis, dissipation and pressure terms are ignored.

The pressure terms from equation (2) and (3) can be eliminated by differentiating equation (2) with respect to \( \theta \) and equation (3) with respect to \( r \) and then reducing them to one equation. The resulting equation, along with equations (1), (4) and (5) can now be reduced and non-dimensionalized with the help of the stream function \( \psi \) expressed as:

\[ r \frac{V_r}{V} = \frac{\partial \psi}{\partial \theta} \] \hspace{1cm} (6a)

\[ \frac{V_\theta}{V} = -\frac{\partial \psi}{\partial r} \] \hspace{1cm} (6b)

and the parameters,

- dimensionless radius \( R = \frac{r}{a} \) \hspace{1cm} (7a)
- dimensionless axial velocity \( V_x = \frac{V_x}{(V/a)} \) \hspace{1cm} (7b)
- dimensionless temperature \( T^* = \frac{(T_w - T)}{(A a N_{pr})} \) \hspace{1cm} (7c)

The momentum equations (2) and (3) become:

\[ \nabla^4 \psi + \frac{1}{R^2} \left( \frac{\partial \psi}{\partial R} \cdot \frac{\partial}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \cdot \frac{\partial}{\partial R} \right) \nabla^2 \psi = \]

\[ N_{Ra} \left( \frac{\partial T^*}{\partial R} \sin \theta + \frac{1}{R} \frac{\partial T^*}{\partial \theta} \cos \theta \right) \cos \alpha \] \hspace{1cm} (8)
Equation (4) is reduced to:

$$\nabla^2 V_x + \frac{1}{R} \left( \frac{\partial \psi}{\partial R} \cdot \frac{\partial}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \cdot \frac{\partial}{\partial R} \right) V_x + 4 N_{Re} \frac{N_{Ra}}{N_{Re}} \frac{T^*}{\sin \alpha} = 0 . \quad (9)$$

and the energy equation becomes:

$$\nabla^2 T^* + \frac{N_{Pr}}{R} \left( \frac{\partial \psi}{\partial R} \cdot \frac{\partial}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \cdot \frac{\partial}{\partial R} \right) T^* + V_x = 0 \quad \ldots \quad (10)$$

The boundary conditions for the above equations are:

$$T^* = \frac{V_x}{\partial \psi/\partial R = \partial \psi/\partial \theta} = 0 \text{ at } R = 1 \quad \ldots \quad (11)$$

and

$$T^*, \frac{V_x}{R^{-1}} \frac{\partial \psi/\partial \theta}, \frac{\partial \psi/\partial R}{\partial \theta} \text{ are finite, at } R = 0 \quad \ldots \quad (12)$$

Equations (8), (9) and (10) are non-linear, simultaneous partial differential equations. Their solution is extremely difficult, however a perturbation approach similar to Morton's [26] is made.

In the absence of any exact solution for the dependent variables \( \psi, V_x \) and \( T^* \) in equations (8), (9) and (10), these unknowns are expanded in a power series of \( \frac{N_{Ra}}{N_{Re}} \). In literature, Rayleigh number is generally chosen for such an expansion, since this is the significant parameter indicative of velocity and temperature fields with free-convection effects.

$$\psi = \psi_0 + \frac{N_{Ra}}{N_{Re}} \psi_1 + \frac{N_{Ra}^2}{N_{Re}} \psi_2 + \frac{N_{Ra}^3}{N_{Re}} \psi_3 + \ldots \quad \ldots \quad (13)$$

$$V_x = V_{x_0} + \frac{N_{Ra}}{N_{Re}} V_{x_1} + \frac{N_{Ra}^2}{N_{Re}} V_{x_2} + \frac{N_{Ra}^3}{N_{Re}} V_{x_3} + \ldots \quad \ldots \quad (14)$$

$$T^* = T_{0}^* + \frac{N_{Ra}}{N_{Re}} T_{1}^* + \frac{N_{Ra}^2}{N_{Re}} T_{2}^* + \frac{N_{Ra}^3}{N_{Re}} T_{3}^* + \ldots \quad \ldots \quad (15)$$
To ensure convergence of the series, the numerical value of \( N_{Ra} \) must be small. For a desired accuracy of the result, number of terms chosen in the series depends upon the numerical value of \( N_{Ra} \).

The equations (13) to (15) are now substituted in equations (8) to (10) and the terms of the like powers of \( N_{Ra} \) are grouped together. Taking the terms up to second order of Rayleigh number only, the three groups of equations are:

The terms with zero order of \( N_{Ra} \) are:

\[
\nabla^2 V_{x_0} + 4 N_{Re} = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (16)
\]

\[
\nabla^2 T_0 + V_{x_0} = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (17)
\]

The terms with first power of \( N_{Ra} \) are:

\[
\nabla^4 \Psi_1 = \left( \frac{\partial T_0}{\partial R} \sin \Theta + \frac{1}{R} \frac{\partial T_0}{\partial \Theta} \cos \Theta \right) \cos \lambda \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (18)
\]

\[
\nabla^2 V_{x_1} + \frac{1}{R} \left( \frac{\partial \Psi_1}{\partial R} \frac{\partial V_{x_0}}{\partial \Theta} - \frac{\partial \Psi_1}{\partial \Theta} \frac{\partial V_{x_0}}{\partial R} \right) - T_0 \sin \lambda = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (19)
\]

\[
\nabla^2 T_1 + \frac{N_{Pr}}{R} \left( \frac{\partial \Psi_1}{\partial R} \frac{\partial T_0}{\partial \Theta} - \frac{\partial \Psi_1}{\partial \Theta} \frac{\partial T_0}{\partial R} \right) + V_{x_1} = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (20)
\]

The terms with 2nd power of \( N_{Ra} \) are:

\[
\nabla^4 \Psi_2 + \frac{1}{R} \left\{ \frac{\partial \Psi_1}{\partial R} \frac{\partial \left( \nabla^2 \Psi_1 \right)}{\partial \Theta} - \frac{\partial \Psi_1}{\partial \Theta} \frac{\partial \left( \nabla^2 \Psi_1 \right)}{\partial R} \right\} =
\cos \lambda \left( \frac{\partial T_1}{\partial R} \sin \Theta + \frac{1}{R} \frac{\partial T_1}{\partial \Theta} \cos \Theta \right) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (21)
\]
\n
The boundary conditions, equations (11) and (12) apply to all equations (16) through (23).

Equations (16) and (17) are for normal Poiseuille flow, without free-convection effects. Their solution is available in standard texts. The remaining equations are solved by substituting the forced flow solution in the equations of first order in Rayleigh number and proceeding step by step.

The final equations up to second order in \( \text{N}_\text{Ra} \) are rather lengthy and are given in the Appendix A as equations (A1), (A2) and (A3).

For the case of \( \mathcal{L} = 0 \), these equations reduce to those of Morton (26) except for some slight numerical differences and more importantly, differences in some minus signs in the temperature equations which will probably affect his final expression for the mean Nusselt number.
CALCULATION OF NUSSELT NUMBER

Nusselt number is the dimensionless parameter indicative of the
rate of energy convection from a surface.

\[ N_{Nu_b} = \frac{2a}{k} \frac{q}{T_W - T_b} \]

(24)

Where \( h \), the convective heat transfer coefficient is based on bulk
temperature of the fluid.

Nusselt number can also be expressed as:

\[ N_{Nu_b} = \frac{2a}{k} \left( \frac{T(r, \theta)}{v_x(r, \theta)} \right) \frac{rdrd\theta}{2\pi} \]

(25)

Where \( T_b = \frac{\int \int T(r, \theta) v_x(r, \theta) rdrd\theta}{\int \int v_x(r, \theta) rdrd\theta} \)

(26)

And in dimensionless form, \( \frac{T^*}{T_b} = \frac{\int \int T^*(R, \theta) v_x(R, \theta) RdRd\theta}{\int \int v_x(R, \theta) RdRd\theta} \)

(27)

From the equation (A2) the dimensionless average axial velocity is
evaluated as:

\[ V_{x_{av}} = \frac{\int \int v_x(R, \theta) RdRd\theta}{\int \int RdRd\theta} \]

(28)

\[ = 2N_{Re} \left\{ \frac{1}{h} - (1.125/576) N_{Ra} \sin \alpha - 0.052486(N_{Ra} N_{Re} \cos \alpha/4608)^2 \right\} \]
Making an energy balance over two cross-sections of the tube, $\Delta x$ apart gives,

$$q = \frac{\kappa a \nabla c_p V_{x_{av}} (T_{b2} - T_{b1})}{2 \kappa a \Delta x}$$

$$= \frac{(1/2) \nabla \rho c_p A V_{x_{av}}}{2 \kappa a \Delta x}$$

From equation (25) and (31),

$$N_{Nu_b} = \frac{V_{x_{av}}}{T_b^*}$$

Equation (32) is given in detail in Appendix B.
Equations A1, A2 and A3 indicate that the three velocity components as well as the temperature at any location on a cross-section of the tube are a function of $N_{Ra}$, $N_{Re}$, $N_{Pr}$ and of the tube inclination. In order to obtain a clear physical picture of the limitations of these equations, numerical calculations were performed to study the convergence of these equations for various parameters. It should be stressed here that the criterion used was that of reasonably rapid convergence, considering that the solutions obtained are developed only up to the second order in $N_{Ra}$.

The analysis showed that convergence limits for the velocity, temperature and Nusselt number equations may be considerably different. In the present case, for instance, limits of convergence of the Nusselt number and temperature equations may, in some cases (higher $N_{Pr}$), be considerably lower than those of the velocity equation. This also appears to be the case in Morton's analysis (Ref. 26, which considers the problem of horizontal tubes). A detailed check of his results indicates, moreover, that his final Nusselt number equation is convergent in a range of values of $N_{Ra} \times N_{Re}$ in which the temperature equation is divergent, and therefore his results have doubtful physical meaning in this range. It was to avoid this mathematical pitfall that a study of validity limits of equations A1, A2, A3 and B1 was made.

This study revealed that, for inclined tubes, both the Nusselt number and the temperature equations were convergent within practically the same limits of the dimensionless parameters, while for the velocity equation the limits were considerably higher.
Figure 4 shows quantitatively the results obtained for temperature and Nusselt number equations at two values of Prandtl number (0.75 and 5.0). To interpret these figures, the reader should select a radial line corresponding to the tube inclination. The intersection of this line with the limit lines will give the upper limit of $N_{Ra}$ (read on the vertical axis) and $N_{Ra} \times N_{Re}$ (on the horizontal axis). At large inclinations (nearly vertical tubes), $N_{Ra}$ is the only controlling parameter and all the equations are valid within the same limits. At lower inclinations (nearly horizontal), the significant parameters are the Prandtl number and the product $N_{Ra} \times N_{Re}$. The upper limit of $N_{Ra} \times N_{Re}$ for temperature and Nusselt number equations decreases rapidly as $N_{Pr}$ increases, as clearly shown in Figure 4.

The dependence on Prandtl number is not so pronounced in the velocity expression. In Figure 5, the Rayleigh number limitations are about the same as in the case of $N_{Nu}$ and $T^*$ equation. The range of $N_{Ra} \times N_{Re}$ is, however, much higher.

Figures 6 - 9 show the variations of temperature and axial velocity at the vertical tube centre line for Prandtl numbers 5.0 and 0.75. Figure 10 is a contour map showing a typical temperature distribution across the tube. These plots show that Prandtl number has a strong influence on distortion of temperature and velocity profiles. Maximum temperature and velocity occur below the centre line for the horizontal position, but as tube inclination increases, the location of the maximum values shifts upward. Depending upon the value of Prandtl number, this maximum could be located above the centre line for some tube inclinations. As the inclination is increased, the distortion of the profiles is reduced until they become
symmetrical about the centre line in the vertical position. Exact solutions
available for this case (heating in vertical upward flow) indicate
that the effect of natural convection is to decrease the centre line
velocity and temperature. A similar result is also obtained by the present
perturbation analysis.

Equation B1 in Appendix B shows the influence of the various
parameters on the Nusselt number. For the horizontal case ($\alpha = 0^\circ$),
Nusselt number depends on the Rayleigh number only.

Figures 11 and 12 show the plots of Nusselt number against Rayleigh
number for various values of $N_{Ra} \times N_{Re}$. Figure 11 gives the results for
$N_{Pr} = 5.0$ and Figure 12 for $N_{Pr} = 0.75$. At low Prandtl numbers, the heat
transfer is affected only slightly by the product $N_{Ra} \times N_{Re}$, but at higher
values this dependence is very significant for low tube inclinations. The
influence of both $N_{Pr}$ and $N_{Ra} \times N_{Re}$ diminishes with tube inclination, until
it disappears completely at $\alpha = 90^\circ$ (vertical tubes).

Figures 13 and 14 show that for any combination of Rayleigh,
Reynolds and Prandtl numbers, there is an optimum value of tube inclination
that gives the maximum value of Nusselt number. For most instances this
maximum appears to lie between $20^\circ$ and $60^\circ$ of tube inclination. This is
somewhat similar to Larson and Hartnett's $[33]$ findings for free convection
inside inclined tube closed at both ends.

The optimum tube inclination mentioned in the preceding paragraph
is plotted against the product $N_{Ra} \times N_{Re}$ in Figures 15 and 16. The curves
are similar at both Prandtl numbers investigated, but an interesting effect
of Rayleigh number can be observed. At high values of $N_{Ra} \times N_{Re}$ (for a
given $N_{Pr}$), the optimum tube inclination increases slightly when $N_{Ra}$ is
increased, but the reverse is true in the low $N_{Ra} \times N_{Re}$ range.
CONCLUSIONS

The analysis shows that for inclined tubes, the velocity and temperature profiles as well as Nusselt numbers are functions of Rayleigh, Reynolds and Prandtl numbers. At low values of these parameters, for the horizontal and vertical case, the results seem to be in agreement with other theoretical work already published. Nusselt number increases with tube inclination, compared to the horizontal case. For most combinations of $N_{Ra}$, $N_{Re}$ and $N_{Pr}$, the maximum value of Nusselt number seems to lie between $20^\circ$ and $60^\circ$ of tube inclination.
APPENDIX A

The three equations in dimensionless form, up to second order in Rayleigh number are:

Equation of Stokes Stream Function

\[ \psi = \psi_0 + \frac{N_{\text{Ra}}}{N_{\text{Re}}} \psi_1 + \frac{2}{N_{\text{Re}}} \psi_2 + \ldots \quad \text{where } \psi_0 = 0 \]

\[ = \frac{N_{\text{Ra}} N_{\text{Re}} \cos \alpha}{4608} (-10R + 21R^3 - 12R^5 + R^7) \sin \theta \]

\[ + \frac{N_{\text{Ra}}^2 N_{\text{Re}}^2 \cos \alpha}{(4608)^2} (1.204108R^2 - 2.061786R^4 + 0.15006R^6 + 1.10R^8 - 0.425R^{10} \]

\[ + 0.034285R^{12} - 0.001607R^{14}) \sin 2\theta \]

\[ + \frac{N_{\text{Ra}}^2 N_{\text{Re}}^2 \cos \alpha}{(4608)^2} N_{\text{Pr}} (3.75643R^2 - 9.160715R^4 + 7.5R^6 - 2.6R^8 + 0.5625R^{10} \]

\[ - 0.06R^{12} + 0.001785R^{14}) \sin 2\theta \]

\[ + \frac{N_{\text{Ra}}^2 N_{\text{Re}} \sin 2\alpha}{36864} (1.244166R^2 - 2.652083R^3 + 1.583333R^5 - 0.1875R^7 \]

\[ + 0.0125R^9 - 0.000416R^{11}) \sin \theta \quad \ldots \quad \text{(A1)} \]

Velocity Equation in Axial Direction

\[ V_x = V_{x_0} + \frac{N_{\text{Ra}}}{N_{\text{Re}}} V_{x_1} + \frac{2}{N_{\text{Re}}} V_{x_2} + \ldots \]

\[ = N_{\text{Re}} (1 - R^2) + \frac{N_{\text{Ra}} N_{\text{Re}}^2 \cos \alpha}{184320} \cos \theta (-49R + 100R^3 - 70R^5 + 20R^7 - R^9) \]

\[ + \frac{N_{\text{Ra}} N_{\text{Re}} \sin \alpha}{576} (-19 + 27R^2 - 9R^4 + R^6) \]

\[ + \frac{N_{\text{Ra}}^2 N_{\text{Re}}^2 \cos \alpha}{(4608)^2} (\cos 2\theta)(0.115356R^2 - 0.340952R^4 + 0.432723R^6 \]

\[ - 0.31125R^8 + 0.13125R^{10} - 0.02964R^{12} + 0.00241R^{14} \]

\[ - 0.00073R^{16}) \]
\[
\begin{aligned}
&+ \frac{N_{Ra}^2 N_{Re}^3 \cos \angle \cos \theta}{(4608)^2} (\cos \theta) N_{Pr} (0.51357R^2 - 1.252143R^4 + 1.145089R^6 \\
&\quad - 0.5R^8 + 0.108333R^{10} - 0.016071R^{12} + 0.00125R^{14} \\
&\quad - 0.000028R^{16}) \\
&+ \frac{N_{Ra}^2 N_{Re}^2 \sin 2\angle}{(4608)^2} (\cos \theta) (215.361347R - 472.019904R^3 + 390.999983R^5 \\
&\quad - 164.99992R^7 + 34.35R^9 - 3.84R^{11} + 0.148566R^{13}) \\
&+ \frac{N_{Ra}^2 N_{Re}^2 \sin 2\angle}{(4608)^2} (\cos \theta) N_{Pr} (39.561429R - 79.5R^3 + 60R^5 - 26R^7 \\
&\quad + 6.75R^9 - 0.84R^{11} + 0.028571R^{13}) \\
&+ \frac{N_{Ra}^2 N_{Re}^3 \cos \angle}{(4608)^2} (-0.614718 + 3.0625R^2 - 6.340625R^4 + 7.058333R^6 \\
&\quad - 4.560937R^8 + 1.7125R^{10} - 0.344791R^{12} + 0.028571R^{14} \\
&\quad - 0.000833R^{16}) \\
&+ \frac{N_{Ra}^2 N_{Re} \sin \angle}{(4608)^2} (21029.76 - 30384R^2 + 10944R^4 - 1728R^6 + 144R^8 - 5.76R^{10}) \\
&+ --- \\
\end{aligned}

\text{Temperature Equation}

\[
T^* = T^*_0 + N_{Ra} T^*_1 + N_{Ra}^2 T^*_2 + ---
\]

\[
= \frac{N_{Re}}{16} (3 - 4R^2 + R^4) + \frac{N_{Ra} N_{Re}^2 \cos \angle}{22118400} (\cos \theta) (-381R + 735R^3 - 500R^5 \\
&\quad + 175R^7 - 30R^9 + R^{11}) \\
&+ \frac{N_{Ra} N_{Re}^2 \cos \angle}{22118400} (\cos \theta) N_{Pr} (-1325R + 3000R^3 - 2600R^5 + 1125R^7 - 210R^9 + 10R^{11}) \\
&+ \frac{N_{Ra} N_{Re} \sin \angle}{36864} (-211 + 304R^2 - 108R^4 + 16R^6 - R^8) \\
&+ N_{Ra}^2 N_{Re}^3 \cos \angle
\]
\[ \begin{align*}
&+ \frac{N_{Ra}^2 N_{Re}^3 \cos^2 \zeta}{(4608)^2} (\cos 2\theta) N_{Pr} \left(0.052219 R^2 - 0.131837 R^4 + 0.127947 R^6 \\
&\quad - 0.058686 R^8 + 0.009667 R^{10} + 0.001011 R^{12} - 0.000334 R^{14} \\
&\quad + 0.000014 R^{16} - 0.000001 R^{18}\right) \\
&+ \frac{N_{Ra}^2 N_{Re}^3 \cos^2 \zeta}{(4608)^2} (\cos 2\theta) N_{Pr} \left(0.112114 R^2 - 0.350796 R^4 + 0.476476 R^6 \\
&\quad - 0.369308 R^8 + 0.173893 R^{10} - 0.049954 R^{12} + 0.008157 R^{14} \\
&\quad - 0.000597 R^{16} + 0.000015 R^{18}\right) \\
&+ \frac{N_{Ra}^2 N_{Re}^2 \sin 2\zeta}{(4608)^2} (\cos \theta) \left(13.600061 R - 26.920158 R^3 + 19.667496 R^5 \\
&\quad - 8.145832 R^7 + 2.062499 R^9 - 0.28625 R^{11} + 0.022857 R^{13} \\
&\quad - 0.000663 R^{15}\right) \\
&+ \frac{N_{Ra}^2 N_{Re}^2 \sin 2\zeta}{(4608)^2} (\cos \theta) N_{Pr} \left(42.527436 R - 97.235154 R^3 + 87.102492 R^5 \\
&\quad - 41.268747 R^7 + 10.012494 R^9 - 1.22375 R^{11} + 0.087856 R^{13} \\
&\quad - 0.002627 R^{15}\right) \\
&+ \frac{N_{Ra}^2 N_{Re}^3 \cos^2 \zeta}{(4608)^2} N_{Pr} \left(-0.063482 + 0.153679 R^2 - 0.191406 R^4 + 0.176128 R^6 \\
&\quad - 0.110286 R^8 + 0.045609 R^{10} - 0.011892 R^{12} + 0.001759 R^{14} \\
&\quad - 0.000111 R^{16} + 0.000002 R^{18}\right) \\
&+ \frac{N_{Ra}^2 N_{Re}^3 \cos^2 \zeta}{(4608)^2} N_{Pr} \left(-0.040842 + 0.198437 R^2 - 0.399765 R^4 + 0.434149 R^6 \\
&\quad - 0.279147 R^8 + 0.111343 R^{10} - 0.028038 R^{12} + 0.004131 R^{14} \\
&\quad - 0.000273 R^{16} + 0.000005 R^{18}\right) \\
&+ \frac{N_{Ra}^2 N_{Re}^3 \cos^2 \zeta}{(4608)^2} N_{Pr} \left(-0.13391 + 0.6901 R^2 - 1.505856 R^4 + 1.821189 R^6 \\
&\quad - 1.343423 R^8 + 0.624218 R^{10} - 0.178906 R^{12} + 0.028683 R^{14} \\
&\quad - 0.002148 R^{16} + 0.000057 R^{18}\right) \\
&+ \frac{N_{Ra}^2 N_{Re}^2 \sin 2\zeta}{(4608)^2} \left(3636.84 - 5257.44 R^2 + 1899 R^4 - 304 R^6 + 27 R^8 - 1.44 R^{10} \\
&\quad + 0.04 R^{12}\right) + --- \quad \ldots (A3)
\end{align*} \]
Detailed expression for Nusselt number.

\[ N_{Nu_b} = \frac{V_{x_{av}}}{T_b} \cdot \frac{1}{\left( \frac{L}{2} \right)} \cdot \left( \frac{1 + 1RN + 2RN + \ldots}{1 + 34.909116 (1RD + 2RD) + \ldots} \right) \]  

\[ = \frac{48}{11} \left( \frac{1 + 1RN + 2RN + \ldots}{1 + 34.909116 (1RD + 2RD) + \ldots} \right) \]  

\[ \ldots \]  

\[ \text{(B1)} \]

Where

1RN = First order effect in Rayleigh number in the numerator

\[ = - (N_{Ra \ Sin\alpha} \cdot 16.5/576) \]  

\[ \ldots \]  

\[ \text{(B2)} \]

2RN = Second order effect in Rayleigh number in the numerator

\[ = \left\{ - 0.20994 (N_{Ra \ Re \ Cos\alpha})^2 + 18163.2 (N_{Ra \ Sin\alpha})^2 \right\}/4608^2 \]  

\[ \ldots \]  

\[ \text{(B3)} \]

1RD = First order effect in Rayleigh number in the denominator

\[ = 1RD_1 + 1RD_2 \]  

\[ \ldots \]  

\[ \text{(B4)} \]

2RD = Second order effect in Rayleigh number in the denominator

\[ = 2RD_1 + 2RD_2 + 2RD_3 + 2RD_4 + 2RD_5 \]  

\[ \ldots \]  

\[ \text{(B5)} \]

Where

1RD_1 = 63.06666 N_{Ra \ Sin\alpha}/36864 + 543.048214 (N_{Ra \ Sin\alpha}/4608)^2 \]  

\[ \ldots \]  

\[ \text{(B6)} \]

1RD_2 = (0.00105 + 0.003281 N_{Pr}) (N_{Ra \ Re \ Cos\alpha}/4608)^2 \]  

\[ \ldots \]  

\[ \text{(B7)} \]

2RD_1 = \left\{ \begin{align*} & 1086.096428 - 32.47384 \ N_{Ra \ Sin\alpha} \\ & + 1030844.2 (N_{Ra \ Sin\alpha}/4608)^2 \} (N_{Ra \ Re \ Sin\alpha}/4608)^2 \end{align*} \]  

\[ \ldots \]  

\[ \text{(B8)} \]

2RD_2 = (0.00034 - 0.000618 N_{Pr} + 0.000255 N_{Pr}^2) x \]  

\[ x \ N_{Ra \ Sin\alpha} (N_{Ra \ Re \ Cos\alpha}/4608)^2 \]  

\[ \ldots \]  

\[ \text{(B9)} \]
2RD3 = - (196.29141 - 276.640184 N_{Pr} + 128.267224 N_{Pr}^2) \times \left( N_{Re} \sin \alpha \cos \alpha \right)^2 \left( N_{Ra}/4608 \right)^4 \quad \ldots \quad (B10)

2RD4 = - (0.016054 + 0.003028 N_{Pr} + 0.009552 N_{Pr}^2) \times \left( N_{Ra} N_{Re} \cos \alpha /4608 \right)^2 \quad \ldots \quad (B11)

2RD5 = (0.005553 + 0.001188 N_{Pr} + 0.003826 N_{Pr}^2 + 0.000095 N_{Pr}^3) \times \left( N_{Ra} N_{Re} \cos \alpha /4608 \right)^4 \quad \ldots \quad (B12)
ACKNOWLEDGMENTS

The authors are indebted to Dr. Austin Whillier, Brace Research Institute of Solar Energy, McGill University, for suggesting this problem. Thanks are also due to Mr. Richard Peene of Sir George Williams University who programmed the calculations. Facilities provided by the Computing Centre of Sir George Williams University, where all the numerical computations were performed, are greatly appreciated. Financial assistance of the National Research Council of Canada is gratefully acknowledged.
REFERENCES


Fig. 1 COORDINATE SYSTEM
Fig. 2  FLOW VISUALIZATION
Fig. 3 SURFACE FLUID TEMPERATURE RISE
Fig. 4  SUGGESTED LIMITS OF $N_{RA}$ AND $N_{RA} \times N_{RE}$ FOR TEMPERATURE AND NUSSELT NUMBER EQUATIONS A3 AND B1 RESPECTIVELY.
Fig. 5  SUGGESTED LIMITS OF $N_{RA}$ AND $N_{RA} \times N_{RE}$ FOR AXIAL VELOCITY EQUATION A2. LIMITS VALID FOR $N_{PR} = 0.75$ TO $5.0$
Fig. 6 DIMENSIONLESS TEMPERATURE DIFFERENCE VARIATION AGAINST DIMENSIONLESS RADIUS AT VERTICAL \( \theta \) FOR VARIOUS TUBE INCLINATIONS.
Fig. 7 DIMENSIONLESS TEMPERATURE DIFFERENCE VARIATION AGAINST DIMENSIONLESS RADIUS AT VERTICAL $\phi$ FOR VARIOUS TUBE INCLINATIONS.
Fig. 8 DIMENSIONLESS AXIAL VELOCITY VARIATION AGAINST DIMENSIONLESS RADIUS AT VERTICAL $\phi$ FOR VARIOUS TUBE INCLINATIONS.
Fig. 9 DIMENSIONLESS AXIAL VELOCITY VARIATION AGAINST DIMENSIONLESS RADIUS AT VERTICAL Θ FOR VARIOUS TUBE INCLINATIONS.
Fig. 10 CONTOUR MAP OF DIMENSIONLESS TEMPERATURE T* IN A HORIZONTAL TUBE

N_{RA} = 20 \quad N_{RE} = 30 \quad N_{PR} = 5

\theta = 0^\circ
Fig. II NUSSELT NUMBER AGAINST RAYLEIGH NUMBER AT VARIOUS TUBE INCLINATIONS.
Fig. 12 Nusselt Number Against Rayleigh Number at Various Tube Inclinations.
Fig. 13 VARIATION OF NUSSELT NUMBER WITH TUBE INCLINATION AT A PRANDTL NUMBER OF 5.0
Fig. 14 VARIATION OF NUSSELT NUMBER WITH TUBE INCLINATION AT A PRANDTL NUMBER OF 0.75
Fig. 15. VARIATION OF OPTIMUM TUBE INCLINATION FOR MAXIMUM NUSSELT NUMBER AGAINST VARIOUS VALUES OF $N_{RA}$ AND $N_{RA} \times N_{RE}$.
Fig. 16 VARIATION OF OPTIMUM TUBE INCLINATION FOR MAXIMUM NUSSLELT NUMBER AGAINST VARIOUS VALUES OF $N_{RA}$ AND $N_{RA} \times N_{RE}$
Computer Program for Calculation of the Dancoff Correction by Monte Carlo Simulation by R. Wiedmer

Report 64-17

Faculty of Engineering

McGill University [Dept. of Mech. Eng'g. Research Labs.]

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ABSTRACT

A computer program was devised for calculation of the Dancoff Correction by the "Reverse" Method. The program is constructed so that upon changing the input data, different reactor geometries may be considered. The geometry discussed in this report serves as an illustrative example only.

The program was written in FORTRAN IV language for the IBM 7040 Computer.
ACKNOWLEDGEMENTS

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The present report is largely based on original work done by S. S. Hyder who devised a Monte Carlo Method for simulation of neutron streaming in a nuclear reactor.\(^1\)
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>GENERAL CONSIDERATIONS</td>
<td>2</td>
</tr>
<tr>
<td>THE MAIN PROGRAM</td>
<td>8</td>
</tr>
<tr>
<td>THE SUB-PROGRAMS</td>
<td>13</td>
</tr>
<tr>
<td>GENERAL FLOW-DIAGRAM</td>
<td>21</td>
</tr>
<tr>
<td>DETAILED FLOW-DIAGRAM</td>
<td>22</td>
</tr>
<tr>
<td>PROGRAM LISTING</td>
<td>23</td>
</tr>
<tr>
<td>APPENDIX I</td>
<td>31</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>32</td>
</tr>
</tbody>
</table>
INTRODUCTION

A reverse Monte Carlo Method was devised by Bach, Freibergs and Hyder (2), for calculation of the Dancoff Correction, $T$, in a heterogeneous reactor. In this method one may consider neutrons as being born on the surface of the fuel rod for which $T$ is required, and follow their path to either their first collision in moderator or their interaction with a neighbouring fuel rod.

The method was applied to an infinite hexagonal lattice of fuel rods, surrounded by air gaps, in an infinite moderator. $T$ was computed for various values of the parameters $\Sigma_1 a$, $c/a$ and $V_1/V_o$, which denote respectively the fuel rod radius in units of mean free paths, the ratio of the radii of fuel rod plus air gap to fuel rod, and the ratio of moderator volume to fuel volume.
The main features of the methodology used are outlined in this section in some detail for the sake of completeness, though they have been outlined briefly by Hyder (1).

Due to symmetry in the geometry of a hexagonal lattice, the whole of space can be represented by a unit cell, Figure 2, which is a portion of space containing every geometrical and physical property of the system at least once. The unit cell is bounded by reflecting surfaces, so that upon "rotation" about these surfaces any desired portion of space is represented.

Evidently the smallest unit cell is that which contains all the properties of space, but at least one of them only once. For the present case the smallest unit cell would have the form shown in Figure 1, and would suffice for calculation of the total Dancoff Correction.

We have, however, chosen the next larger unit cell, so that the contribution of the nearest and next nearest neighbours to the total Dancoff Correction could be determined if desired.

The unit cell is divided internally into regions called zones, so as to contain one homogeneous medium in each zone; fuel, moderator or void. The zones are defined by the
surfaces bounding them, i.e. by plane and cylindrical surfaces in the present case. The equations of the surfaces are established in accordance with the current values of $\Sigma_1 a$, $c/a$ and $V_1/V_0$. 
LEGEND
5 BOUNDARY NUMBER
V ZONE NUMBER

NOTES
1. REFLECTING PROPERTIES
   ASSIGNED TO BOUNDARIES
   1, 2, 3, 4 AND 5.

2. MEDIUM IN THE ZONES:
   I, MODERATOR
   II, III, IV, FUEL
   V, VI, VII, VOID

PLAN OF UNIT CELL
(FOR Σ₁a = 1.5, c/a = 1.4, Vᵢ/Vᵦ = 2.5)

FIG. 2
All surfaces must be defined so as to be without discontinuities throughout the space of the unit cell, and all zones thus obtained must be treated separately even though some adjacent zones might contain the same medium. For example, if region II in Figure 3a contains moderator, it must, by extension of surfaces 5 and 6 be treated as separate zones, as shown in Figure 3b.

Each zone is assigned a number by the input data. The most advantageous method of numbering the zones is discussed in the description of FUNCTION NZON.

The input data further supplies a one-dimensional matrix GAP (I), where I is the zone number, which designates the medium in every zone. Thus for GAP (I) = 0, 1, 2, the medium is void, moderator or fuel respectively.

The surfaces, or boundaries, are also assigned numbers. Plane surfaces are given by the general equation

\[ AX + BY + CZ = D \]  

...(1)

and spherical surfaces by the equation:

\[ A(x - x1)^2 + B(y - y1)^2 + C(z - z1)^2 = D \]  

...(2)

which for vertical cylinders reduces to

\[ A(x - x1)^2 + B(y - y1)^2 = D \]  

...(3)
where $x_1, y_1$ are the coordinates of the cylinder's axis, $\sqrt{D}$ is its radius, and $A, B$ are unity.

The boundary parameters are supplied by the input as subscripted variables, viz. $A(J), B(J), C(J)$ and $D(J)$ where $J$ is the boundary number. A blank is entered for those parameters which are known to change for different cell dimensions. The latter are computed by the main program according to the current values of $\Sigma_1a, c/a$ and $V_1/V_0$. Since there is uniformity in the vertical direction, the height of the unit cell is arbitrarily set at 50 units, i.e. $D(1) = 0.0$, $D(2) = 50.0$.

Also supplied by the input data is a one-dimensional matrix, $JE(J)$, which describes the reflection properites of boundary $J$. From it we obtain the following information:

\[
JE(J) = 1 \quad - \quad \text{reflection in the X direction only.}
\]
\[
JE(J) = 2 \quad - \quad \text{reflection in the Y direction only.}
\]
\[
JE(J) = 3 \quad - \quad \text{reflection in the Z direction only.}
\]
\[
JE(J) = 4 \quad - \quad \text{reflection oblique in the X-Y plane}
\]
\[
JE(J) = 5 \quad - \quad \text{no reflection.}
\]

Finally there are two matrices, $Q(I, J)$ and $G(I, J)$, entered by the input data which are of use in zone searching.

$Q(I, J)$ is constructed so as to give the number of the new zone which a particle enters upon leaving zone $I$ across boundary $J$. It is zero for boundaries not "visible" from Zone $I$.

$G(I, J)$ determines whether a particle in zone $I$ lies inside or outside boundary $J$. It gives a value of $+1$ and $-1$
for these cases respectively and is zero for boundaries not visible from zone I.

The specific use of the matrices just described will be seen in the description of the program. For the geometry in Figure 2 these matrices are as follows:

\[ \text{GAP}(I) = 1, 2, 2, 2, 0, 0, 0. \]
\[ \text{JE}(J) = 3, 3, 1, 2, 4, 5, 5, 5, 5, 5, 5, 5. \]

\[
\begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & 1 & 5 & 0 & 6 & 0 & 7 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\
3 & 3 & 0 & 3 & 3 & 0 & 0 & 0 & 6 & 0 & 0 \\
\end{array}
\]

\[ \text{Q}(I, J) =
\begin{array}{ccccccccccc}
4 & 4 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 7 \\
5 & 5 & 5 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\
6 & 6 & 0 & 6 & 6 & 0 & 0 & 1 & 3 & 0 & 0 \\
7 & 7 & 7 & 0 & 7 & 0 & 0 & 0 & 0 & 1 & 4 \\
\end{array}
\]

\[
\begin{array}{ccccccccccc}
-1 & 1 & -1 & 1 & 1 & -1 & 0 & -1 & 0 & -1 & 0 \\
-1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{array}
\]

\[ \text{G}(I, J) =
\begin{array}{ccccccccccc}
-1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\
-1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\
\end{array}
\]
THE MAIN PROGRAM

To begin with, the parameters and constants which define the dimensions and properties of the unit cell are read in and printed by SUBROUTINE REED. (Frequent reference to the flow-diagrams and program listings might prove helpful in following this text.) These include A, B, C, D, GAP, JE, Q and G as described above. In addition a range of values for $\Sigma_1 a(I_1)$, c/a(I_2) and $V_1/V_0(I_3)$ are read in, where the integer variables I_1, I_2 and I_3 control a particular set of parameters. Finally a constant, K_l, is supplied, which specifies the number of particles to be processed for a given Dancoff Correction.

The program proceeds to execute three successive DO-LOOPS controlled by I_1, I_2, I_3 which select values for $\Sigma_1 a$, c/a, and $V_1/V_0$ respectively. Each of these parameters defines a number of constants for boundary equations which are subsequently computed.

Thus $\Sigma_1 a$ defines D(7), D(9), D(11),
c/a defines D(6), D(8), D(10)
and $V_1/V_0$ defines D(4), X1(8), X1(9), Y1(6), Y1(7),
    Y1(8) and Y1(9).

The unit cell is now completely established and the program enters the phase where a particle is generated and pursued. This phase is repeated K_l times.
A particle is born by SUBROUTINE XYZ at a position \( \vec{r}(X, Y, Z) \) on the surface of the central rod. SUBROUTINE ANGLE then finds the direction of motion \( \vec{\Omega}(U, V, W) \), where U, V, W are the direction cosines with respect to the X, Y and Z axis.

The path-length, PL, is defined as the distance a particle would travel to its first collision in moderator, and is given by the equation

\[
PL = -P1 \times \text{ALOG}(R) \quad \ldots(4)
\]

where P1 is the mean free path in moderator and R is a random number between zero and one.

In a further operation PL is set equal to PL2 so as to preserve the original PL's for any other considerations one might chose. PL2 is used in all subsequent computations.

At birth a particle is located either in zone I when \( c/a = 1 \), or in zone VII when \( c/a > 1 \). A call to FUNCTION NZON confirms the number of the zone in which the particle is located, NX1, to be one or seven as the case may be.

The program goes on to compute PL3, the distance to the nearest boundary from the particles location at \( \vec{r} \) along its direction of motion \( \vec{\Omega} \), and to establish the number, NB, of that boundary. This is accomplished by SUBROUTINE SURFAC. In order to assure that the particle is moved over the new boundary, in case it gets that far, and that it will be in a new zone after translation along \( \vec{\Omega} \), a small increment, \( \Delta \), is added to PL3.
In working with various computers, viz. IBM 1410, IBM 7090 and IBM 7040, it was found that $\Delta$ should be not less than $10^{4-L}$ where $L$ is the number of decimal digits in the floating point fraction; i.e. for the IBM 7040 $\Delta$ is $10^{-4}$.

Before being able to follow the particle on its way to the new boundary, the medium in the zone has to be known. This is given by GAP(NX1).

In summary we have thus far established the parameters $\tilde{r}$, $\tilde{\Omega}$, NX1, PL2, PL3, NB and GAP. These are necessary and sufficient to follow the particle on any course within the unit cell, and are subject to re-evaluation each time the particle has been moved over a new boundary.

The establishment of the medium in the zone through GAP(NX1) leads the program to branch into three alternative courses. These are as follows:

**Case 1:** GAP(NX1) = 0, the zone represents a void. In this case the particle experiences a free ride from $\tilde{r}$ along $\tilde{\Omega}$ to and over the new boundary, NB, by the translation

\[
X = X + U \times PL3 \\
Y = Y + V \times PL3 \\
Z = Z + W \times PL3
\] ...

(5)

The remaining pathlength, PL2, is still the same.

**Case 2:** GAP(NX1) = 1, the medium is moderator. Here a decision is made as to whether the particle has sufficient
pathlength to reach the new boundary. If PL2 < PL3, the particle suffers a collision in the moderator, and a new particle is started by SUBROUTINE XYZ. If PL2 \geq PL3, the particle experiences a translation over the new boundary similar to that in Case 1, given by equations (5). This time, however, the particle uses up some of its pathlength. The remaining pathlength is established by the operation:

\[ PL2 = PL2 - PL3 \]  \hspace{1cm} \text{(6)}

**Case 3:** GAP(NX1) = 2, the medium is fuel. In this event the particle has hit a shielding rod, the tally is increased by one and a new particle is started by SUBROUTINE XYZ.

After a particle has been moved over a new boundary, cases 1 and 2, the nature of NB is examined. If JE(NB) < 5, NB is a reflecting boundary. A call to SUBROUTINE COORD effects the proper reflection of the particles present coordinates \( \tilde{P}(X, Y, Z) \) and \( \tilde{N}(U, V, W) \). If JE(NB) = 5, NB is not reflecting and the above step is omitted.

The number of the new zone in which the particle is presently located, NX1, is predicted by Q(NX1, NB). This information is checked with a call to FUNCTION NZON, for Q might occasionally predict a wrong zone number, (see appendix I). If FUNCTION NZON returns the predicted zone number the program continues; if it returns an invalid zone number, the program
enters a zone searchloop which calls on FUNCTION NZON to test systematically all zones until the particle has been "found".

(For clarification one might choose at this point to read the description of FUNCTION NZON).

Thus far, after having crossed a boundary, the parameters \( \hat{r}, \hat{n} \), NX1 and PL2 have been re-established. The program returns to call SUBROUTINE SURFAC for PL3 and NB, then re-enters cases 1, 2 or 3 according to GAP(NX1).

The process repeats itself until the particle either suffers a moderator collision in case 2, or interacts with a fuel rod, case 3.

After K1 particles have been processed, the Dancoff Correction is calculated and printed as the fraction TALLY/K1 and the program returns to consider a new set of values for \( \Sigma_1a \), c/a and \( v_1/v_0 \).

For diagnostic reasons, the main program contains many print statements inserted at points of interest and controlled by a constant P2, supplied by the input data. When P2 is zero, all the print statements are ignored except those which print the final result. When P2 is not zero, all print statements are executed and the path of particles can be followed in detail.
SUBROUTINE REED reads and prints the input data.

SUBROUTINE XYZ establishes the position vector \( \vec{r} \) \((X, Y, Z)\) of a particle at birth, where \(X, Y\) and \(Z\) are the co-ordinates in a 3-dimensional cartesian system with the \(Z\)-axis vertical.

A particle is born with uniform probability density anywhere on the surface of the central fuel rod. Its position in the \(X-Y\) plane is, due to the geometry, most easily established in cylindrical co-ordinates and then transformed to cartesian co-ordinates.

![Diagram of cylindrical coordinates](image)

**FIG. 4**

The procedure is as follows: \(\text{THETA}\) in Fig. 4 is computed by the equation

\[
\text{THETA} = R \times \pi / 6, \quad (7)
\]

where \(R\) is a random number between zero and one.

The radius, \(RRR\), is obtained by adding a small increment to the rod radius \(RR\), so as to assure that the particle lies in the zone just outside the central rod.
i.e. \[ RRR = RR + .001 \] (8)

The coordinates are then transformed to the cartesian system according to the equations

\[ X = RRR \times \sin(\theta) \]
\[ Y = RRR \times \cos(\theta). \] (9)

The Z-coordinate is obtained by simply multiplying D (2), the height of the unit cell, by a random number between zero and one.

**SUBROUTINE ANGLE** assigns the direction of motion, \( \hat{\mathbf{n}}(U,V,W) \), to a particle at birth, where U, V, W are the direction cosines with respect to the X, Y and Z axis. The probability distribution of \( \hat{\mathbf{n}} \) is forward peaked in the X-Y plane according to a cosine distribution normal to the tangent at \( \hat{r} \); Fig. 5. In the Z-direction \( \hat{\mathbf{n}} \) has a uniform probability distribution.

![Diagram](image)

**Fig. 5**

The direction cosines are first established with respect to the X1, Y1, Z1 coordinates defined by THETA from SUBROUTINE XYZ. Hence we have
\( V_1 = \text{SQRT} \left( R_{N1} \right) \)
\[ \phi = R_{N2} \times \pi, \]
\[ U_1 = \text{SQRT} \left( 1. - R_{N1} \right) \times \cos \left( \phi \right), \quad (10) \]
\[ W_1 = \text{SQRT} \left( 1. - R_{N1} \right) \times \sin \left( \phi \right), \]

Where \( R_{N1}, R_{N2} \) are random numbers between zero and one.

Transformation to the regular coordinate axis is carried out by the following equations:

\[ U = U_1 \times \cos \left( \theta \right) + V_1 \times \sin \left( \theta \right) \]
\[ V = V_1 \times \cos \left( \theta \right) - U_1 \times \sin \left( \theta \right) \]
\[ W = W_1 \]

**SUBROUTINE COORD** executes reflection of the coordinates \( \vec{r}(X,Y,Z) \) and \( \vec{v}(U,V,W) \) from the outer boundaries of the system, so as to keep the particle moving within the unit cell. It might be envisaged that the particle, upon reflection, does in fact enter the adjacent region of space, and that this region is again represented by the unit cell upon "rotation" about the reflecting boundary.

If we let \( \vec{v}(U,V,W) \) and \( \vec{v}_1(U_1,V_1,W_1) \) be the incident and reflected direction vectors of a particle, and \( \hat{n}(U_n,V_n,W_n) \) the normal unit vector to the reflecting boundary, then \( \vec{v} \) and \( \vec{v}_1 \) are related by equations

\[ \hat{n} \cdot \vec{v} = -\hat{n} \cdot \vec{v}_1, \]
\[ \hat{n} \cdot (\vec{v} \times \vec{v}_1) = 0 \quad (12) \]

Solving these equations yield the following results for various \( JE \) values:

\( JE = 1 \)
\[ U_1,V_1,W_1 = -U,V,W \]
\( \hat{n} \parallel \) to X-axis

\( JE = 2 \)
\[ U_1,V_1,W_1 = U,-V,W \]
\( \hat{n} \parallel \) to Y-axis

\( JE = 3 \)
\[ U_1,V_1,W_1 = U,V,-W \]
\( \hat{n} \parallel \) to Z-axis
\[ \text{JE} = 4 \quad x_1, y_1, z_1 = -(u_n x + v_n x), (v_n y - u_n x), z \]

for \( \hat{n} \) oblique in the X-Y plane.

Reflection of the position coordinates \( \hat{r} (x, y, z) \)
is governed by a mathematical relationship similar to that
for \( \hat{\Omega} \). When \( \hat{r} (x, y, z) \) and \( \hat{r}_1 (x_1, y_1, z_1) \) are the original and
reflected coordinates, we have for

\[ \text{JE} = 1 \quad x_1, y_1, z_1 = -x, y, z \quad \hat{n} \parallel \text{to X-axis} \]

\[ \text{JE} = 2 \quad x_1, y_1, z_1 = x, -y, z \quad \hat{n} \parallel \text{to Y-axis} \]

\[ \text{JE} = 3 \quad x_1, y_1, z_1 = x, y, -z \quad \hat{n} \parallel \text{to Z-axis} \]

\[ \text{JE} = 4 \quad x_1, y_1, z_1 = -(u_n y + v_n x), (v_n y - u_n x), z \]

for \( \hat{n} \) oblique in the X-Y plane.

The necessity for reflecting the position coordinates
arises from addition of the increment \( \Delta L \) to PL3 which, as may
be recalled, moves the particle over the target boundary.

This scheme presents no difficulties for boundaries inside the
unit cell, (for which JE = 5). However, upon striking an outer
boundary the particle is thus moved beyond the space of the unit
cell and would be "lost" unless reflected back inside.

Following reflection of \( \hat{r} \), the subroutine continues
to test the validity of the new coordinates \( x, y \) and \( z \). It reflects
\( \hat{r} \) again if necessary and keeps on repeating this process until the
coordinates are found to place the particle inside the unit cell.

The reason for this is seen when considering reflection of \( \hat{r} \) near
an acute corner of the unit cell; Fig. 6. A single reflection in
such a case might place the particle again outside the system, and
two or several successive reflections are needed to place it
inside the unit cell.
Repeated reflections are not necessary for the present geometry since the acute corners are occupied by fuel rods. The situation might arise, however, when working with different geometries.

FUNCTION NZON (II) carries out a test to see whether the particle with position $\mathbf{r}(X,Y,Z)$ lies inside or outside zone II. The tentative value of II is supplied by the mainprogram. The test lies in computing the parameter TEST for every boundary J surrounding zone II. Plane boundaries are tested by the equation

$$\text{TEST} = G(II,J) \left[ D(J) - A(J)X - B(J)Y - C(J)Z \right]$$

(14)

and circular boundaries by

$$\text{TEST} = G(II,J) \left[ D(J) - A(J)(X-X1(J))^2 - B(J)(Y-Y1(J))^2 - C(J)(Z-Z1(J))^2 \right].$$

(15)

$C(J)$ is zero for vertical cylinders, hence the last term in (15) vanishes for the present geometry.

It may be recalled that G has values of +1, -1 and 0. TEST is zero for G = 0 and is of no interest since in this case boundary J does not bound zone II. Hence TEST is calculated only for G = +1 or -1.

The particle is known to be on the inside of boundary.
J when TEST is positive, and on the outside of boundary J when TEST is negative. It is easily seen that a particle is located inside zone II if and only if TEST is positive for all J surrounding zone II. In this case the function returns the value NZON = II and the main program sets NX1 = NZON(II).

If TEST is not positive for all J surrounding zone II the function returns the value NZON = NI + 1, where NI is the highest zone number. The main program recognizes the invalid number NI + 1, and sets a new value for II to be tested.

It was seen that SUBROUTINE COORD exits only on condition that the particles lie inside the unit cell. Hence TEST is positive for the outer boundaries of the system (No's 1 to 5 incl.) for all II, and is not computed for those boundaries. Therefore TEST is computed for boundaries 6 to NJ inclusive, where NJ is the highest boundary number, with those boundaries omitted for which G(II,J) is zero.

It so happens for the present geometry that all reflecting surfaces are plane and all non-reflecting surfaces are cylindrical. Hence we can establish whether a surface is plane or cylindrical by merely considering the value of JE(J). If JE(J) < 5, equation (14) is used for calculation of TEST and if JE(J) = 5 equation (15) is used.

If in other geometries the boundaries cannot be classified according to their form by JE(J), a further one-dimensional matrix, say KE(J) has to be supplied by the input data, which decides for every J whether equation (14) or (15) is to be used.

In general FUNCTION NZON is called after a particle has been moved over a new boundary. In this case the main program will first set II = Q(NX1,NB) and when the function confirms
that NZON = II the mainprogram sets NX1 = NZON(II) and continuous.

On the other hand, when the function returns NZON = NI + 1 the particle is not in zone II and the mainprogram enters a zone-search loop, setting II successively equal to 1, 2, 3... NI. The loop terminates when the particle has been "found", i.e. when the function returns a zone number other than NI + 1.

It is now easily deduced how the zones should be numbered; namely in decreasing order of the probability of finding the particle in it, which is in general proportional to the volume in the zone. When this scheme is observed the zone search loop terminates on the average sooner, and a minimum of time is spent.

**SUBROUTINE SURFAC** calculates the distance, PL3, from the position of a particle at \( \vec{r}(X, Y, Z) \) along its direction of motion \( \vec{\Omega}(U, V, W) \) to the nearest boundary, NB. It further establishes the number of NB.

When J is a plane surface, the distance S(J) from \( \vec{r} \) along \( \vec{\Omega} \) to J is computed by the operation

\[
S(J) = \frac{D(J) - A(J)X - B(J)Y - C(J)Z}{A(J)U + B(J)V + C(J)W}.
\]  

(16)

Equation (16) follows from simple geometry and is true for all plane surfaces, whether inclined or perpendicular to the coordinate axis.

Calculation of S(J) for cylindrical surfaces is governed by the identity

\[
(X1 + U* S(J))^2 + (Y1 + V* S(J))^2 = D(J),
\]  

(17)

which upon solving yields

\[
S(J) = \frac{T \pm \sqrt{T^2 - T3}}{W*W} - 1.
\]  

(18)
where
\[ T = U \times XX1 + V \times YY1, \]
\[ XX1 = X - X1(J) \]
\[ YY1 = Y - Y1(J) \]
\[ T3 = T^2 - T1 \]
and
\[ T1 = \left[ A(J) \times (XX1 \times XX1) + B(J) \times (YY1 \times YY1) \right. \]
\[ \left. - D(J) \right] \times \left[ 1. - W^2 \right]. \]

As was the case in FUNCTION NZON the matrix JE(J) decides whether J is a plane or cylindrical boundary and chooses equation (16) or (18) accordingly for computation of S(J). In other geometries this might have to be done by a matrix KE(J) as suggested above.

The rather lengthy computations for S(J) are kept to a minimum by omitting them for surfaces for which Q(NX1, J) = 0. In this case S(J) is of no interest since surface J does not bound zone NX1. When Q(NX1, J) = 0, S(J) is arbitrarily set equal to a value larger than any possible line path in the system.

After S(J) has been computed for all J surrounding zone NX1, the smallest positive value among S(J) is selected, say S(5). In a final operation S(5) is set equal to PL3 and 5 is set equal to NB.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Zone number</td>
</tr>
<tr>
<td>NI</td>
<td>Highest zone number</td>
</tr>
<tr>
<td>NX1</td>
<td>Number of zone in which particle is located</td>
</tr>
<tr>
<td>J</td>
<td>Boundary number</td>
</tr>
<tr>
<td>NJ</td>
<td>Highest boundary number</td>
</tr>
<tr>
<td>NB</td>
<td>Number of nearest boundary; synonymous with new boundary or target boundary</td>
</tr>
<tr>
<td>GAP(I)</td>
<td>Medium in zone I</td>
</tr>
<tr>
<td>JE(J)</td>
<td>Reflection condition of boundary J</td>
</tr>
<tr>
<td>Pl</td>
<td>Mean free path in moderator</td>
</tr>
<tr>
<td>PL</td>
<td>Distance neutron will travel through moderator on first flight</td>
</tr>
<tr>
<td>PL2</td>
<td>Remaining distance of first flight through moderator</td>
</tr>
<tr>
<td>PL3</td>
<td>Distance to target boundary</td>
</tr>
<tr>
<td>X,Y,Z</td>
<td>Position coordinates of a particle</td>
</tr>
<tr>
<td>U,V,W</td>
<td>Direction cosines of a particle</td>
</tr>
<tr>
<td>A,B,C,D</td>
<td>Parameters of boundary equations</td>
</tr>
<tr>
<td>Q(I,J),G(I,J)</td>
<td>See under General Considerations</td>
</tr>
<tr>
<td>SIGA</td>
<td>Radius of a fuel rod in mean free paths (called $\Sigma$, a in text)</td>
</tr>
<tr>
<td>CDIVR</td>
<td>Gap radius/fuel radius (called c/a in text)</td>
</tr>
<tr>
<td>VEE</td>
<td>Moderator volume/fuel volume (called $V_1/V_o$ in text)</td>
</tr>
<tr>
<td>DANCOR</td>
<td>Dancoff Correction</td>
</tr>
</tbody>
</table>
GENERAL
FLOW DIAGRAM
FOR
REVERSE MONTE CARLO PROGRAM
DETAILED FLOW DIAGRAM
FOR
REVERSE MONTE CARLO PROGRAM
CALCULATION OF THE DANCOFF CORRECTION

PARTICLES BORN AT CENTRAL ROD

DIMENSION A(24), B(24), C(24), D(24), JE(24), G(24, 24), Q(24, 24),
1GAP(24), X1(24), Y1(24), CDIVR(12), VEE(12), DANCOR(12), SIGA(12),
2S(24)

COMMON A, B, C, D, X1, Y1, JE, G, Q, GAP, NB, NX1, PL, PL3, X, Y, Z, U, V, W, R1, R2,
1K1, P2, R, CEE, VEE, DEE, CDIVR, SIGA, NI, NJ, D2, D1, RR, THETA

INTEGER GAP

104 FORMAT (5H X = , F13.8, 3X, 5H Y = , F13.8, 3X, 5H Z = , F13.8)
105 FORMAT (5H U = , F13.8, 3X, 5H V = , F13.8, 3X, 5H W = , F13.8)
106 FORMAT (5H NX1 = , I4, 12X, 5H GAP = , I4)
107 FORMAT (5H PL3 = , F13.8, 3X, 5H NB = , I4)
108 FORMAT (5H JE = , I4)
118 FORMAT (1H1, 33X, 62H THE DANCOFF CORRECTION AS A FUNCTION OF SIGA,
1C/A, AND V1/V0.)
119 FORMAT (1HL, 3X, 4HSIGA, 11X, 3HC/A, 52X, 5HV1/V0)
120 FORMAT (1HK, 28X, 7(F9.1, 5X))
121 FORMAT (5X, F3.1, 12X, F3.1, 8X, 7(F8.6, 6X))
122 FORMAT (1HK)
123 FORMAT (6H SIG2 =, F10.6, 3HRR =, F10.6, 5HD(6)=, F10.6, 5HD(9)=, F10.6,
16HCIV2=, F10.6, 4HCEE=, F10.6, 5HD(7)=, F10.6, 5HD(8)=, F10.6)
124 FORMAT (5H VE2 =, F10.6, 4HDEE=, F10.6, 4HDE2=, F10.6, 5HD(4)=, F10.6,
16HX1(8)=, F10.6, 6HX1(9)=, F10.6, 6HY1(8)=, F10.6, 6HY1(9)=, F10.6)
1251 FORMAT (6H A(5)=, F10.6)
1261 FORMAT (3H X =, F10.6, 3H Y =, F10.6, 3H Z =, F10.6)
1264 FORMAT (3H K =, I6)
450 FORMAT (5H GAP =, I13)

NI = 7
NJ = 11

CALL REED
PRINT 118
PRINT 119
PRINT 120, (VEE(I), I=1, 7)
P1 = 2.65
R1 = 0.625731
R2 = SQRT(3.)
A(5) = R2
DO 46 II = 5, 5
SIG2 = SIGA(II)
RR = SIG2*P1
RSQ = RR**RR
D(7) = RSQ
D(9) = RSQ
D(11) = RSQ
PRINT 122
DU 46 I2 = 4, 4
CDIV2 = CDIVR(12)
CEE = CDIV2*RR
CSQ = CEE*CEE
D(6) = CSQ
D(8) = CSQ
D(10) = CSQ
402 DO 45 I3 = 1, 3, 2
VE2 = VEE(I3)


DEE=RR*SQRT((6.28318/R2)*(VE2+(CEE*CEE)/(RR*RR)))
COPHI=R2/2.
D3=2.*DEE*COPHI
D1=D3*COPHI
D2=D3/2.
D(4)=D1
X1(8)=D2
X1(9)=D2
Y1(6)=DEE
Y1(7)=DEE
Y1(8)=D1
Y1(9)=D1

512 K=0
TALLY=0.
11 CALL XYZ
12 CALL ANGLE
7 R=ABS(RANDR(R1))
IF(R) 14,7,14
14 PL=-P1*ALOG(R)
IF(P2) 126,126,125
125 PRINT 104, X,Y,Z
PRINT 105, U,V,W
PRINT 1051, PL
126 PL2=PL
II=7
NX1=NZON(II)
IF (NX1-7) 16,2020,16
16 DO 19 II=1,NI
NX1=NZON(II)
NI1=NI+1
IF(NX1-NI1) 2020,19,2020
19 CONTINUE
2020 CALL SURFAC
2021 IF(P2) 2023,2023,2022
2022 PRINT 107,PL3,NB
PRINT 106,NX1,GAP(NX1)
2023 PL3=PL3+.001
21 JGAP=GAP(NX1)+1
GO TO (30,40,23), JGAP
23 TALLY=TALLY+1.
GO TO 43
30 X=X+U*PL3
Y=Y+V*PL3
Z=Z+W*PL3
231 CALL COORD
IF(P2) 34,34,232
232 PRINT 104, X,Y,Z
PRINT 105, U,V,W
PRINT 1051, PL
PRINT 108, JE(NB)
34 NX1=Q(NX1,NB)
II=NX1
NX1=NZON(II)
IF (P2) 343,343,342
342 PRINT 106,NX1,GAP(NX1)
343 IF (NX1-NI1) 2020, 16, 16
40 IF (PL2-PL3) 43, 43, 41
41 PL2 = PL2 - PL3
410 X = X + U * PL3
     Y = Y + V * PL3
     Z = Z + W * PL3
     IF (P2) 231, 231, 411
411 PRINT 105, U, V, W
     PRINT 1051, PL
     PRINT 104, X, Y, Z
     GO TO 231
43 K = K + 1
     IF (K - K1) 11, 44, 44
44 AK = K
     DANCOR(I3) = TALLY / AK
45 CONTINUE
46 PRINT 121, SIG2, CDIV2, (DANCOR(I3), I3=1, 7)
     STOP
     END
SUBROUTINE REED
DIMENSION A(24), B(24), C(24), D(24), JE(24), G(24, 24), Q(24, 24),
       GAP(24), X1(24), Y1(24), CDIVR(12), VEE(12), DANCOR(12), SIGA(12),
       2S(24)
COMMON A, B, C, D, X1, Y1, JE, G, Q, GAP, NB, NX1, PL, PL3, X, Y, Z, U, V, W, R1, R2,
       K1, P2, R, CEE, VEE, DEE, CDIVR, SIGA, NI, NJ, D2, D1, RR, THETA
INTEGER GAP
1 FORMAT (11F4.0)
2 FORMAT (11I3)
3 FORMAT (24F3.0)
4 FORMAT (7I3)
5 FORMAT (5F4.1)
6 FORMAT (4F4.1)
7 FORMAT (7F4.1)
8 FORMAT (16,F4.0)
9 FORMAT (7H1, A=, 11(F8.5, 3X))
10 FORMAT (7H1, B=, 11(F8.5, 3X))
11 FORMAT (7H1, C=, 11(F8.5, 3X))
12 FORMAT (7H1, D=, 11(F8.5, 3X))
13 FORMAT (7H1, JE=, 11(I2, 9X))
14 FORMAT (7H1, GAP=, 7(I2, 9X))
15 FORMAT (7H1, SIGA=, 5(F8.5, 3X))
16 FORMAT (7H1, CDIVR=, 4(F8.5, 3X))
17 FORMAT (7H1, V1/V0=, 7(F8.5, 3X))
18 FORMAT (4H1, K1=, I6)
19 FORMAT (4H1, P2=, F6.0)
20 FORMAT (7HK, Q=, 3X, 11F3.0/10X, 11F3.0)
21 FORMAT (7HK, G=, 3X, 11F3.0/10X, 11F3.0)
22 FORMAT (4H1, NI=, 15, 5X, 3HNJ=, I3)
   READ 1, (A(J), J=1, NJ), (B(J), J=1, NJ), (C(J), J=1, NJ),
   (D(J), J=1, NJ), (X1(J), J=1, NJ), (Y1(J), J=1, NJ)
   READ 2, (JE(J), J=1, NJ)
   READ 3, (G(I, J), I=1, NI, J=1, NJ)
   READ 3, (Q(I, J), I=1, NI, J=1, NJ)
   READ 4, (GAP(I), I=1, NI)
   READ 5, (SIGA(I1), I1=1, 5)
   READ 6, (CDIVR(I2), I2=1, 4)
   READ 7, (VEE(I3), I3=1, 7)
   READ 8, K1, P2
PRINT 9, (A(J), J=1, NJ)
PRINT 10, (B(J), J=1, NJ)
PRINT 11, (C(J), J=1, NJ)
PRINT 12, (D(J), J=1, NJ)
PRINT 13, (JE(J), J=1, NJ)
PRINT 14, (GAP(I), I=1, NI)
PRINT 15, (SIGA(I1), I1=1, 5)
PRINT 16, (CDIVR(I2), I2=1, 4)
PRINT 17, (VEE(I3), I3=1, 7)
PRINT 18, K1
PRINT 19, P2
PRINT 22, NI, NJ
PRINT 20, (G(I, J), J=1, NJ), (I=1, NI)
PRINT 21, (Q(I, J), J=1, NJ), (I=1, NI)
RETURN
END
SUBROUTINE XYZ
  DIMENSION A(24), B(24), C(24), D(24), JE(24), G(24, 24), Q(24, 24),
  LGAP(24), X1(24), Y1(24), CDIVR(12), VEE(12), DANCOR(12), SIGA(12),
  2S(24)
  COMMON A, B, C, D, X1, Y1, JE, G, Q, GAP, NB, NX1, PL, PL3, X, Y, Z, U, V, W, R1, R2,
  1K1, P2, R, CEE, VEE, DEE, CDIVR, SIGA, NI, NJ, D2, D1, RR, THETA
  INTEGER GAP
  126 FORMAT (3H X=, F10.6, 3H Y=, F10.6, 3H Z=, F10.6)
  127 FORMAT (19H HAS ARRIVED AT XYZ)
  128 FORMAT (5H NX1=, I3)
  R=ABS(RANDR(R1))
  THETA=R*0.52359
  RRR=RR+0.001
  X=RRR*SIN(THETA)
  Y=RRR*COS(THETA)
  3 R=ABS(RANDR(R1))
  Z=R*D(2)
  RETURN
  END

SUBROUTINE ANGLE
  DIMENSION A(24), B(24), C(24), D(24), JE(24), G(24, 24), Q(24, 24),
  LGAP(24), X1(24), Y1(24), CDIVR(12), VEE(12), DANCOR(12), SIGA(12),
  2S(24)
  COMMON A, B, C, D, X1, Y1, JE, G, Q, GAP, NB, NX1, PL, PL3, X, Y, Z, U, V, W, R1, R2,
  1K1, P2, R, CEE, VEE, DEE, CDIVR, SIGA, NI, NJ, D2, D1, RR, THETA
  INTEGER GAP
  RN1=ABS(RANDR(R1))
  VONE=SQRT(RN1)
  RN2=ABS(RANDR(R1))
  PHI=RN2*2. *3.14159
  UONE=SQRT((1.-RN1)*COS(PHI))
  W=SQRT((1.-RN1)*SIN(PHI))
  V=VONE*COS(THETA)-UONE*SIN(THETA)
  U=UONE*COS(THETA)+VONE*SIN(THETA)
  RETURN
  END
SUBROUTINE COORD

DIMENSION A(24), B(24), C(24), D(24), JE(24), G(24,24), Q(24,24),
GAP(24), X1(24), Y1(24), CDIVR(12), VEE(12), DANCOR(12), SIGA(12),
2S(24)

COMMON A, B, C, D, X1, Y1, JE, G, Q, GAP, NB, NX1, PL, PL3, X, Y, Z, U, V, W, R1, R2,
IK1, P2, R, CEE, VEE, DEE, CDIVR, SIGA, NI, NJ, D2, DI, RR, THETA

INTEGER GAP

1 IF(Z) 2, 2, 3
2 Z=-Z
3 IF(Z-D(2)) 5, 5, 4
4 Z=2.*D(2)-Z
5 IF(Y) 6, 7, 7
6 Y=-Y
7 IF(Y-D(4)) 9, 9, 8
8 Y=2.*D(4)-Y
9 IF(X) 10, 11, 11
10 X=-X
11 XX=Y/R2
   IF(X-XX) 13, 13, 12
12 X2=(Y*R2-X)/2.
   Y=(X*R2+Y)/2.
   X=X2
   GO TO 5
13 JJ=JE(NB)
   GO TO (31, 32, 33, 34, 35), JJ
31 U=-U
   GO TO 35
32 V=-V
   GO TO 35
33 W=-W
   GO TO 35
34 UU=(R2*V-U)/2.
   V=(R2*U+V)/2.
   U=UU
35 RETURN

END
FUNCTION NZON(N)
DIMENSION A(24), B(24), C(24), D(24), JE(24), G(24, 24), Q(24, 24), 
GAP(24), X1(24), Y1(24), CDIVR(12), VEE(12), DANCOR(12), SIGA(12), 
R2(24)
COMMON A, B, C, D, X1, Y1, JE, G, Q, GAP, NB, NX1, PL, PL3, X, Y, Z, U, V, W, R1, R2, 
1K1, P2, R, CEE, VEE, DEE, CDIVR, SIGA, NI, NJ, D2, DI, RR, THETA
INTEGER GAP
1 DO 7 J=6,NJ
  IF(Q(N,J)) 7,7,2
2 IF(JE(J)-5) 3,4,3
3 TEST=G(N,J)*(D(J)-A(J)*X-B(J)*Y-C(J)*Z)
  GO TO 6
4 XX=X-X1(J)
  YY=Y-Y1(J)
  TEST=G(N,J)*(D(J)-A(J)*XX*XX-B(J)*YY*YY))
6 IF(TEST) 8,7,7
7 CONTINUE
NZON=N
GO TO 9
8 NZON=NI+1
9 RETURN
END
SUBROUTINE SURFAC
DIMENSION A(24), B(24), C(24), D(24), E(24), F(24), G(24, 24), H(24, 24),
I(24, 24), J(24), K(24), L(24), M(24), N(24), O(24), P(24), Q(24, 24),
R(24, 24), S(24), T(24), U(24), V(24), W(24), X(24), Y(24), Z(24),
INTEGER GAP
1 DO 10 J = 1, NJ
2 IF (Q(NJ, J)) 7, 7, 3
3 IF (J - 1) 4, 4, 4
   GO TO 10
5 XX1 = X - X1(J)
   YY1 = Y - Y1(J)
   T = U * XX1 + V * YY1
6 T1 = (A(J) * (XX1 * XX1) + B(J) * YY1 * YY1) - D(J) * (1. - W * W)
7 T3 = T * T - T1
   IF (T3) 7, 52, 52
8 T2 = SQRT(T3)
   S1 = (T - T2) / (W * W - 1.)
   S2 = (T + T2) / (W * W - 1.)
   IF (S1) 6, 6, 9
7 IF (S2) 7, 7, 8
9 S(J) = 105.
   GO TO 10
8 S(J) = S2
   GO TO 10
9 IF (S2) 11, 11, 10
10 IF (S1 - S2) 11, 11, 12
11 S(J) = S1
   GO TO 10
12 S(J) = S2
13 CONTINUE
14 DO 20 K = 1, NJ
   IF (S(K)) 20, 20, 15
15 DO 19 J = 1, NJ
   IF (S(J)) 18, 18, 16
16 IF (K - J) 17, 17, 17
17 IF (S(K) - S(J)) 18, 18, 20
18 IF (J - NJ) 19, 21, 21
19 CONTINUE
20 CONTINUE
21 IF (K - NJ) 23, 23, 22
22 K = K - 1
23 PL3 = S(K)
   NB = K
RETURN
END
APPENDIX I

It was mentioned that the Q-matrix might occasionally predict an incorrect zone number. An example of such a case is given.

We consider the particle striking a reflecting boundary near its intersection with another boundary, Figure 7.

Here $Q(1, 5)$ predicts the new zone number as 1, whereas the addition of $\Delta$ and subsequent reflection place the particle in zone 6.

Other incorrect Q-values arise for the case where $c/a = 1$, i.e. when zones V, VI and VII do not exist. This situation is not accounted for in the design of the Q-matrix.
References

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