A GLOBAL COMPACTNESS THEOREM FOR CRITICAL p-LAPLACE EQUATIONS WITH WEIGHTS

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Abstract

Let $n \geq 2$, fix $1 < p < n$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain, with possibly non-smooth boundary, containing the origin. We investigate the compactness of Palais-Smale sequences for a class of critical p-Laplace equations with weights. More precisely, we establish a Struwe-type decomposition result for Palais-Smale sequences extending the recent result of Mercuri-Willem [17] to weighted equations. In sharp contrast to the model case of the critical p-Laplace equation, all bubbling must occur at the origin. Furthermore, we do not impose any smoothness assumptions on the boundary of Ω and our Palais-Smale sequence may be sign-changing.

Abrégé

Soit $n \geq 2$, $1 < p < 2$ et $\Omega \subset \mathbb{R}^n$ un ouvert borné, avec possiblement un bord non lisse, contenant l'origine. Nous étudions la compacité des suites de Palais-Smale pour une classe d'équations critiques avec p-laplacien et poids. Plus précisément, nous établissons un résultat de décomposition de type Struwe pour les suites de Palais-Smale qui généralise un résultat récent de Mercuri-Willem [17] aux équations à poids. De manière très diffénte du cas modèle de l'équation avec p-laplacien, tout phénomène de bulle doit se produire à l'origine. De plus, nous n'imposons aucune condition de régularité sur le bord de Ω et nos suites de Palais-Smale peuvent changer de signe.

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Chapter 1

Introduction and Background

In this thesis we address questions relating to the analysis of partial differential equations. More specifically, we are interested in compactness properties for critical elliptic problems with weights. Informally, given a sequence of functions approximating a critical point of an energy functional, it is natural to hope for the existence of a subsequence that converges strongly to a solution of the equation. However, even when such a property fails, one can still sometimes glean information about the PDE by asking *why* and *how* the loss of compactness can occur. In our case, although compactness may fail, we are indeed capable of answering the latter question. It is the purpose of this work to describe this possible loss of compactness. Moreover, we seek to provide an asymptotic and energy expansion of this approximating sequence in terms of a solution to the original problem that our sequence indeed encodes.

In the section that follows, we offer a brief overview of the current literature on compactness theorems for critical elliptic problems in $\mathbb{R}^n$. Especially, we discuss the model case studied by M. Struwe in 1984 (Struwe [20]) which was the first decomposition result of its type. This has the added benefit of introducing the topic of this thesis in the more familiar setting of a classical Sobolev space. Afterwards, we consider the generalized problem with the p-Laplace operator, carefully noting the additional assumptions required to obtain an analogous compactness result. We also take the time to touch upon applications of compactness theorems in the context of PDE. Having provided sufficient context, we then introduce our weighted problem and its setting. Additionally, we provide a summary of the essential results and tools we develop in this document.

1.1 Struwe-type Decompositions: The Unweighted Setting

Let $n \geq 3$ be an integer. Given $1 < p < n$, we let

$$p^* := \frac{np}{n-p}$$

denote the Sobolev conjugate exponent of p. Consider the critical Laplace problem

$$\begin{cases} -\Delta u \equiv \lambda u + |u|^{2^*-2} u & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and $\lambda \in \mathbb{R}$. This problem has the associated energy functional

$$\phi : H_0^1(\Omega) \to \mathbb{R}, \quad u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{1}{2}\int_\Omega \lambda |u|^2 \,\mathrm{d}x - \frac{1}{2^*}\int_\Omega |u|^{2^*} \,\mathrm{d}x$$

with derivative

$$\langle \phi'(u), h \rangle = \int_\Omega \nabla u \cdot \nabla h \,\mathrm{d}x - \int_\Omega \lambda u h \,\mathrm{d}x - \int_\Omega |u|^{2^*-2} uh \,\mathrm{d}x, \quad \forall u, h \in H_0^1(\Omega).$$

Define $\mathcal{D}^{1,2}(\mathbb{R}^n)$ to be the space of all functions $u \in L^{2^*}(\mathbb{R}^n)$ such that ∇u exists in the weak sense on $\mathbb{R}^n$ and $\nabla u \in L^2(\mathbb{R}^n)$. We topologize $\mathcal{D}^{1,2}(\mathbb{R}^n)$ by giving it the inner product

$$\langle f, g \rangle_{\mathcal{D}^{1,2}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \nabla f \cdot \nabla g \,\mathrm{d}x.$$

It is well known that smooth functions of compact support are dense in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and that the Gagliardo-Nirenberg-Sobolev inequality holds on $\mathcal{D}^{1,2}(\mathbb{R}^n)$ (see Willem

[24]). Namely, there exists a constant $C > 0$ such that

$$\|u\|_{L^{2^*}(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^n)}, \quad \forall u \in \mathcal{D}^{1,2}(\mathbb{R}^n).$$

In 1984, M. Struwe established a global compactness theorem for the problem in (1.1). More precisely, Struwe showed (see Struwe [19]-[20]) that a Palais-Smale sequence for (1.1) must, up to a subsequence, converge to a solution of (1.1) perturbed by finitely many "bubbles" solving the limiting problem

$$\begin{cases} -\Delta u \equiv u\,|u|^{2^*-2} & \text{in } \mathbb{R}^n, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^n). \end{cases} \tag{1.2}$$

Formally, M. Struwe proved the following result:

Theorem A (Struwe [20], 1984)**.** *Let $n \geq 3$, $c \in \mathbb{R}$, and $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain. Let (u_α) be a sequence in $H_0^1(\Omega)$ such that*

$$\phi(u_\alpha) \to c \quad \textit{and} \quad \phi'(u_\alpha) \to 0 \textit{ in } H^{-1}(\Omega)$$

as $\alpha \to \infty$. Then, after passing to a subsequence, there exists a solution v^0 of (1.1), finitely many non-trivial functions $v^1, \ldots, v^k \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ solving (1.2), and associated sequences $(y_\alpha^{(j)}), (\lambda_\alpha^{(j)})$ in Ω and $(0, \infty)$, respectively, such that $\lambda_\alpha^{(j)} \to 0$ for each $j = 1, \ldots, k$ and

$$u_\alpha - v_0 - \sum_{j=1}^{k} \left(\lambda_\alpha^{(j)}\right)^{\frac{2-n}{n}} v^j \left(\frac{x - y_\alpha^{(j)}}{\lambda_\alpha^{(j)}}\right) \to 0 \quad \textit{in } \mathcal{D}^{1,2}(\mathbb{R}^n)$$

as $\alpha \to \infty$.[1]

Informally, the aforementioned result asserts that, up to a subsequence, a Palais-Smale sequence for (1.1) decomposes at the energy level into a solution of (1.1) and finitely many bubbles solving (1.2). Put otherwise, Theorem A offers an asymptotic

[1] The terms $\left(\lambda_\alpha^{(j)}\right)^{\frac{2-n}{n}} v^j \left(\frac{x-y_\alpha^{(j)}}{\lambda_\alpha^{(j)}}\right)$ are often called *bubbles*.

expansion of the Palais-Smale sequence in the energy space $\mathcal{D}^{1,2}(\mathbb{R}^n)$. Results of this type are often called "Struwe-type decompositions", and have proven to be useful when addressing questions of existence and multiplicity (see, for instance, Barletta-Candito-Marano-Perera [3], Clapp-Rios [9], Clapp-Weth [10], Devillanova-Solimini [13], and Vétois [23]).

In 2010, Mercuri-Willem [17] extended Theorem A to p-Laplace equations with critical nonlinearities. More precisely, Mercuri-Willem showed that a Struwe-type decomposition continues to hold for Palais-Smale sequences of the problem

$$\begin{cases} -\Delta_p u + a\,|u|^{p-2}\,u \equiv |u|^{p^*-2}\,u \quad \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega) \end{cases} \tag{1.3}$$

where $a \in L^{n/p}(\mathbb{R}^n)$ is arbitrary, but fixed, provided $(u_\alpha)_- \to 0$ strongly in $L^{p^*}(\mathbb{R}^n)$, as $\alpha \to \infty$. Here, Δ_p denotes the p-Laplace operator

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\,\nabla u).$$

In this case, the bubbles are explicit and known to be of the form

$$\left[\frac{\lambda^{\frac{1}{p-1}} N^{\frac{p-1}{p^2}} \left(\frac{N-p}{p-1}\right)^{1/p'}}{\lambda^{\frac{p}{p-1}} + |x-x_0|^{\frac{p}{p-1}}}\right]^{\frac{N-p}{p}} \qquad \text{for some } x_0 \in \mathbb{R}^N \text{ and } \lambda > 0. \tag{1.4}$$

This classification was established by

- Caffarelli-Gidas-Spruck [7] in 1989 for $p = 2$;
- Damascelli-Merchán-Montoro-Sciunzi [11] in 2014 for $\frac{2N}{N+2} \leq p < 2$;
- Vétois [22] in 2016 when $1 < p < 2$ (using a symmetry result of Damascelli-Ramaswamy [12] from 2001);
- Sciunzi [18] in 2016 for $2 \leq p < N$ (using a priori estimates from Vétois [22] in 2016).

For more on this classification result, we urge the reader to consult the introduction of the paper [18] of Sciunzi.

The assumption that $(u_\alpha)_- \to 0$ in $L^{p^*}(\mathbb{R}^n)$ is required to rule out to possibility of bubbling on the boundary of Ω. Furthermore, we should note that the proof put forth by Mercuri-Willem in [17] differs significantly from that found in Struwe [19], and relies largely upon a duality argument. This approach has its roots in a paper of Brézis-Coron [5] and was later refined in the book *Minimax Methods* of Willem (where the case $2 < p < 2^*$ was treated – see Willem [25] for more).

1.2 The Weighted Case

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and fix a dimension $n \geq 2$. For any $1 < p < n$, we let q be the critical Caffarelli-Kohn-Nirenberg exponent given by the rule

$$q := \frac{np}{n - p(1 + a - b)}, \tag{1.5}$$

whilst subject to the constraints

$$a < \frac{n-p}{p} \quad \text{and} \quad a \leq b < a + 1. \tag{1.6}$$

In particular, condition (1.6) implies that $\max(ap, qb) < n$ and $p < q \leq p^*$. Therefore, the weight functions $x \mapsto |x|^{-ap}$ and $x \mapsto |x|^{-bq}$ both belong to $L^1_{\text{loc}}(\mathbb{R}^n)$. In the special case where Ω contains the origin, we will be interested in the compactness of Palais-Smale sequences for the following weighted critical p-Laplace problem:

$$\begin{cases} -\operatorname{div}\left(|x|^{-ap} |\nabla u|^{p-2} \nabla u\right) = |x|^{-bq} |u|^{q-2} u & \text{in } \Omega, \\ u \in \mathcal{D}^{1,p}(\Omega, |x|^{-ap}). \end{cases} \tag{1.7}$$

In the above, $\mathcal{D}^{1,p}(\Omega, |x|^{-ap})$ denotes the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,p}(\Omega, |x|^{-ap})} := \left(\int_\Omega |\nabla u|^p |x|^{-ap} \, \mathrm{d}x \right)^{1/p}.$$

Namely, we ask whether a Struwe-type decomposition holds for Palais-Smale sequences associated to the problem (1.7). We note that the norm described above arises naturally from the Caffarelli-Kohn-Nirenberg inequality, which states that there exists a constant $C = C(a, b, n, p) > 0$ having the property that

$$\left(\int_{\mathbb{R}^n} |u(x)|^q \, |x|^{-bq} \, \mathrm{d}x \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p \, |x|^{-ap} \right)^{1/p} \tag{CKN}$$

for all functions $u \in C_c^\infty(\mathbb{R}^n)$.

1.3 Thesis Structure

In the next chapter, we develop a theory for the space $\mathcal{D}^{1,p}(\Omega, |x|^{-ap})$ in full generality. First, we identify $\mathcal{D}^{1,p}(\Omega, |x|^{-ap})$, up to isometric isomorphism, with an explicit function space. We then give conditions under which $\mathcal{D}^{1,p}(\Omega, |x|^{-ap})$ can be embedded into Lebesgue spaces, and provide criteria under which this embedding is compact. In other words, we prove a Rellich-Kondrachov Embedding Theorem for the Sobolev space $\mathcal{D}^{1,p}(\Omega, |x|^{-ap})$. We also give a classification of all bounded linear functionals on $\mathcal{D}^{1,p}(\Omega, |x|^{-ap})$ by way of a Riesz Representation Theorem. Finally, given a bounded sequence (u_α) in $\mathcal{D}^{1,p}(\Omega, |x|^{-ap})$, we establish conditions under which one can extract a subsequence whose gradients converge pointwise and touch upon the homogeneity/rescaling properties of $\mathcal{D}^{1,p}(\Omega, |x|^{-ap})$.

In Chapter 3, we extend Theorem A to the problem (1.7), i.e. we provide a Struwe-type decoposition result for the weighted critical p-Laplace equation in (1.7) in the case $a \neq b$. Our approach is based on the proof found in Mercuri-Willem [17], however, we no longer require that Ω have smooth boundary and we make no sign assumption on the Palais-Smale sequence. Even so, the presence of weights, particularly in the operator, introduces new difficulties. Especially, we are forced to work with a measure that is no longer translation invariant, thereby breaking the traditional rescaling law used in Mercuri-Willem [17] and Struwe [19]-[20]. On the other hand, these weights allow us to give a more precise description of the bubbling.

Indeed, Palais-Smale sequences for (1.7) can only produce bubbles at the origin. Moreover, the weights allow us to rule out boundary bubbling without appealing to a non-existence result (which is the case in both Mercuri-Willem [17] and Struwe [19]-[20]). Additionally, the weights appear in the limiting problem our bubbles solve.

In the final chapter, we discuss open problems relating to the compactness of Palais-Smale sequences for (1.7). In particular, we pay special attention to the limit case $a = b$ and the difficulties that arise when extending the arguments used in the main proof to this special case. We also comment on what phenomena we expect to arise when $a = b$.

Chapter 2

Homogeneous Sobolev Spaces with Critical Weights

In this chapter, we study the functional analytic properties of $\mathcal{D}^{1,p}(\Omega, |x|^{-ap})$ and analyze the behaviour of bounded sequences within this same space. In doing so, we extend well known classical results, including the Riesz Representation Theorem and the Rellich-Kondrachov Compactness Theorem, to this class of weighted homogeneous Sobolev spaces. Throughout this entire chapter, we fix an exponent $1 < p < n$, a point $x_0 \in \mathbb{R}^n$, and assume that (1.6) is satisfied. Unless otherwise stated, we will denote by $U \subseteq \mathbb{R}^n$ a non-empty open set.

If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is smooth and compactly supported, then so must be the mapping $y \mapsto \varphi(y + x_0)$. Consequently, the Caffarelli-Kohn-Nirenberg inequality (CKN) asserts that

$$\left(\int_{\mathbb{R}^n} |\varphi(x)|^q \, |x - x_0|^{-bq} \, \mathrm{d}x\right)^{1/q} = \left(\int_{\mathbb{R}^n} |\varphi(y + x_0)|^q \, |y|^{-bq} \, \mathrm{d}y\right)^{1/q} \tag{2.1}$$

$$\leq C \left(\int_{\mathbb{R}^n} |\nabla\varphi(y + x_0)|^p \, |y|^{-ap} \, \mathrm{d}y\right)^{1/p} \tag{2.2}$$

$$= C \left(\int_{\mathbb{R}^n} |\nabla\varphi(x)|^p \, |x - x_0|^{-ap} \, \mathrm{d}x\right)^{1/p} \tag{2.3}$$

with $C > 0$ a constant independent of φ *and* the point x_0. Therefore, the Caffarelli-Kohn-Nirenberg inequality remains valid after a translation of the origin. We also note that the map

$$\varphi \mapsto \left(\int_{\mathbb{R}^n} |\nabla \varphi(x)|^p \, |x - x_0|^{-ap} \, \mathrm{d}x \right)^{1/p}$$

is a valid norm on $C_c^\infty(\mathbb{R}^n)$. In light of this, we define the following:

Definition 2.0.1. Let $U \subseteq \mathbb{R}^n$ be a non-empty open set and fix $x_0 \in \mathbb{R}$. We define $\mathcal{D}_a^{1,p}(U, x_0)$ to be the completion of $C_c^\infty(U)$ with respect to the norm

$$\|\cdot\|_{\mathcal{D}_a^{1,p}(U,x_0)} := \left(\int_{\mathbb{R}^n} |\nabla(\cdot)| \, |x - x_0|^{-ap} \, \mathrm{d}x \right)^{1/p}.$$

The continuous dual of $\mathcal{D}_a^{1,p}(U, x_0)$ is denoted $\mathcal{D}_a^{-1,p'}(U, x_0)$, with p' given by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Note that $\mathcal{D}_a^{1,p}(U, 0)$ is precisely the space $\mathcal{D}^{1,p}(U, |x|^{-ap})$ previously defined.

Remark 2.0.1. We should point out that when $a = 0$, the space $\mathcal{D}_a^{1,p}(\mathbb{R}^n, x_0)$ is isomorphic to (see Willem [24])

$$\mathcal{D}^{1,p}(\mathbb{R}^n) := \left\{ u \in L^{p^*}(\mathbb{R}^n) : \nabla u \in L^p(\mathbb{R}^n) \right\},$$

endowed with the norm

$$\|\cdot\| := \left(\int_{\mathbb{R}^n} |\nabla \cdot|^p \, \mathrm{d}x \right)^{1/p}.$$

Furthermore, the Caffarelli-Kohn-Nirenberg inequality in (CKN) reduces to the classical Gagliardo-Nirenberg-Sobolev inequality on $\mathcal{D}^{1,p}(\mathbb{R}^n)$.

In the section that follows, we give an explicit description of $\mathcal{D}_a^{1,p}(U, x_0)$. More precisely, we will isometrically identify $\mathcal{D}_a^{1,p}(U, x_0)$ with a subspace of those functions u belonging to a weighted L^q-space possessing weak derivatives of the first order on $U \setminus \{x_0\}$ such that ∇u lives in a suitable weighted L^p-space. Afterwards, we establish a Rellich-Kondrachov type compactness theorem for the space $\mathcal{D}_a^{1,p}(U, x_0)$. We also provide a complete characterization of the continuous linear functionals on $\mathcal{D}_a^{1,p}(U, x_0)$ by way of a Riesz-type representation theorem. Finally, we develop a condition that will (up to a subsequence) yield the pointwise convergence of the gradients almost everywhere on U. The homogeneity and rescaling properties of $\mathcal{D}_a^{1,p}(U, x_0)$ are also touched upon in the last section.

2.1 L^p-Asymptotics and Concentration Functions

We begin by recalling the Brézis-Lieb lemma, which improves the conclusions of Fatou's lemma when the sequence (f_α) is uniformly bounded in L^p.

Theorem 2.1.1 (Brézis-Lieb Lemma)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let (f_α) be a sequence of measurable functions on X converging pointwise almost everywhere to a measurable function f. If the sequence (f_α) is bounded in $L^p(X, \mathrm{d}\mu)$, then*

$$\lim_{\alpha\to\infty}\left(\int_X |f_\alpha|^p \,\mathrm{d}\mu - \int_X |f_\alpha - f|^p \,\mathrm{d}\mu\right) = \int_X |f|^p \,\mathrm{d}\mu.$$

Moreover, when the measure space $(X, \mathfrak{M}, \mu)$ is complete, one can drop the measurability assumption on f.

For the proof we refer the reader to the original paper Brézis-Lieb [6]. We also take note of the following elementary statement:

Lemma 2.1.1. *Let $(X, \mathfrak{M}, \mu)$ be a measure space, fix $1 \le p < \infty$, and let (u_α) be a bounded sequence in $L^p(X, \mathrm{d}\mu)$. Assume that (v_α) is a sequence in $L^p(X, \mathrm{d}\mu)$ such that $\|u_\alpha - v_\alpha\|_{L^p(X,\mathrm{d}\mu)} \to 0$ as $\alpha \to \infty$. Then,*

$$\left|\int_X |u_\alpha|^p \,\mathrm{d}\mu - \int_X |v_\alpha|^p \,\mathrm{d}\mu\right| \to 0$$

as $\alpha \to \infty$.

Proof. Clearly, (v_α) is bounded in $L^p(X, \mathrm{d}\mu)$. Thus, there exists some $M > 0$ such that $0 \leq \|u_\alpha\|_{L^p(X,\mathrm{d}\mu)}, \|v_\alpha\|_{L^p(X,\mathrm{d}\mu)} \leq M$ for all $\alpha \in \mathbb{N}$. By the reverse triangle inequality, we have

$$\left| \|u_\alpha\|_{L^p(X,\mathrm{d}\mu)} - \|v_\alpha\|_{L^p(X,\mathrm{d}\mu)} \right| \leq \|u_\alpha - v_\alpha\|_{L^p(X,\mathrm{d}\mu)} = o(1).$$

Then, using that the map $x \mapsto x^p$ is uniformly continuous on $[0, M]$,

$$\left| \|u_\alpha\|^p_{L^p(X,\mathrm{d}\mu)} - \|v_\alpha\|^p_{L^p(X,\mathrm{d}\mu)} \right| = \left| \int_X |u_\alpha|^p \, \mathrm{d}\mu - \int_X |v_\alpha|^p \, \mathrm{d}\mu \right| = o(1)$$

as $\alpha \to \infty$. □

We now recall a standard duality result (see, for instance, Hewitt-Stromberg [15] and Jakszto [16]) linking weak convergence and the pointwise almost everywhere convergence of a bounded sequence in $L^p(X, \mathrm{d}\mu)$.

Theorem 2.1.2. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and fix $1 < p < \infty$. Denote by p' the Hölder conjugate exponent of p. That is, let $p' \in (1, \infty)$ be such that*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let (f_α) be a bounded sequence in $L^p(X, \mathrm{d}\mu)$ converging μ-almost everywhere to a measurable function f on X. Then, for every fixed $g \in L^{p'}(X, \mathrm{d}\mu)$, one has

$$\int_X |(f_\alpha - f)g| \, \mathrm{d}\mu \to 0, \quad \text{as } \alpha \to \infty.$$

In particular, when the (f_α) are supported on a set of finite measure, $f_\alpha \to f$ strongly in $L^1(X, \mathrm{d}\mu)$. If $(X, \mathfrak{M}, \mu)$ is complete, then the measurability assumption on f may be dropped.

Proof. By Fatou's lemma we must have $f \in L^p(X, \mathrm{d}\mu)$. Especially, $(f_\alpha - f)$ forms a bounded sequence in $L^p(X, \mathrm{d}\mu)$. Fix a function $g \in L^{p'}(X, \mathrm{d}\mu)$ and let $\varepsilon > 0$ be given. For each $\alpha \in \mathbb{N}$ consider the measurable set

$$E_\alpha := \left\{ x \in X : |(f_\alpha(x) - f(x))\, g(x)| \le \varepsilon\, |g(x)|^{p'} \right\}.$$

Clearly, since $f_\alpha(x) \to f(x)$ for μ-a.e. $x \in X$ and $\varepsilon\, |g|^{p'} \in L^1(X, \mathrm{d}\mu)$, the Dominated Convergence Theorem asserts that

$$\lim_{\alpha\to\infty} \int_{E_\alpha} |(f_\alpha - f)\, g|\, \mathrm{d}\mu = 0. \tag{2.4}$$

On the other hand, given $\alpha \in \mathbb{N}$, an application of Hölder's inequality yields

$$\begin{aligned} \int_{X\setminus E_\alpha} |(f_\alpha - f)\, g|\, \mathrm{d}\mu &\le \left(\int_{X\setminus E_\alpha} |f_\alpha - f|^p\, \mathrm{d}\mu \right)^{1/p} \left(\int_{X\setminus E_\alpha} |g|^{p'}\, \mathrm{d}\mu \right)^{\frac{p-1}{p}} \\ &\le M^{1/p} \left(\int_{X\setminus E_\alpha} \varepsilon^{-1}\, |(f_\alpha - f)\, g|\, \mathrm{d}\mu \right)^{1-\frac{1}{p}}, \end{aligned}$$

with $M \ge 0$ given by

$$M := \sup_{\alpha\in\mathbb{N}} \int_X |f_\alpha - f|^p\, \mathrm{d}\mu.$$

In particular, for every $\alpha \in \mathbb{N}$, there holds

$$\int_{X\setminus E_\alpha} |(f_\alpha - f)\, g|\, \mathrm{d}\mu \le M\varepsilon^{1-p}.$$

Using this with (2.4), we infer that

$$\begin{aligned} \limsup_{\alpha\to\infty} \int_X |(f_\alpha - f)\, g|\, \mathrm{d}\mu &= \limsup_{\alpha\to\infty} \left(\int_{E_\alpha} |(f_\alpha - f)\, g|\, \mathrm{d}\mu + \int_{X\setminus E_\alpha} |(f_\alpha - f)\, g|\, \mathrm{d}\mu \right) \\ &= \limsup_{\alpha\to\infty} \int_{X\setminus E_\alpha} |(f_\alpha - f)\, g|\, \mathrm{d}\mu \\ &\le M\varepsilon^{1-p}. \end{aligned}$$

Letting $\varepsilon \nearrow \infty$ then yields the desired conclusion. □

Lemma 2.1.2. *Let (X, d) be a metric space and $\mathcal{F}$ be an equibounded and equicontinuous subfamily of $C(X, \mathbb{R})$. The function $s(x) := \sup_{f\in\mathcal{F}} f(x)$ is a continuous map $X \to \mathbb{R}$.*

Proof. Since the family $\mathcal{F}$ is equibounded, our function s is well defined. Now, given any $x, y \in X$ note that

$$s(x) = \sup_{f\in\mathcal{F}} f(x) \leq \sup_{f\in\mathcal{F}}(f(x) - f(y)) + \sup_{f\in\mathcal{F}} f(y).$$

Thus,

$$s(x) - s(y) \leq \sup_{f\in\mathcal{F}} |f(x) - f(y)| .$$

Similarly, $s(y) - s(x) \leq \sup_{f\in\mathcal{F}} |f(x) - f(y)|$ whence

$$|s(x) - s(y)| \leq \sup_{f\in\mathcal{F}} |f(x) - f(y)| .$$

The continuity of s then follows from the equicontinuity of $\mathcal{F}$. □

Our only application of this will be the following:

Proposition 2.1.3. *Let $U \subseteq \mathbb{R}^n$ be a non-empty set and fix $f \in L^1(\mathbb{R}^n)$. Define the Lévy concentration function of f, denoted Q_f, by the following:*

$$Q_f : [0, \infty) \to [0, \infty), \quad Q_f(r) := \sup_{y\in U} \int_{B(y,r)} |f| \,\mathrm{d}x.$$

Then, Q_f is continuous on $[0, \infty)$.

Proof. Since $f \in L^1(\mathbb{R}^n)$, we have $\int_{B(y,r)} |f| \,\mathrm{d}x \leq \|f\|_{L^1(\mathbb{R})} < \infty$ for all $y \in U$ and $r \geq 0$. Thus, Q_f is a well defined function. In light of the previous lemma, we will be done provided the family $\{Q_{f,y}\}_{y\in U}$ defined by

$$Q_{f,y} : [0, \infty) \to [0, \infty), \quad r \mapsto \int_{B(y,r)} |f| \,\mathrm{d}x$$

is equicontinuous and equibounded on an arbitrary compact interval $[a, b] \subset [0, \infty)$. The equiboundedness of our family $\{Q_{f,y}\}_{y \in U}$ follows at once from the assumption that $f \in L^1(\mathbb{R}^n)$. As for equicontinuity, start by fixing $r, s \geq 0$. Without loss of generality, we can assume that $r \geq s$. Then, for any $y \in U$,

$$|Q_{f,y}(r) - Q_{f,y}(s)| = \int_{B(y,r) \setminus B(y,s)} |f| \, \mathrm{d}x.$$

Given $\varepsilon > 0$, we can find $\delta > 0$ such that $\int_E |f| \, \mathrm{d}x < \varepsilon$ whenever $E \subset [0, \infty)$ is Lebesgue measurable with $m(E) < \delta$. Now, it is easy to check that

$$\begin{aligned} m(B(y,r) \setminus B(y,s)) &= \omega_n(r^n - s^n) \\ &= \omega_n(r - s)(r^{n-1} + sr^{n-2} + \cdots + s^{n-2}r + s^{n-2}) \\ &\leq C\, |r - s| \, . \end{aligned}$$

Here, ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $C > 0$ is a suitable constant depending only on n, a, and b. Thus, $\{Q_{f,y}\}_{y \in U}$ is equibounded and equicontinuous on $[a, b]$. By the previous lemma, Q_f is continuous there. □

Finally, we will require (in the proof of Lemma 3.2.3) the following basic convergence result:

Proposition 2.1.4. *Let $(X, \mathfrak{M}, \mu)$ be a finite measure space and let (f_α) be a sequence of measurable functions on X converging almost everywhere to a measurable function f. Then, $f_\alpha \to f$ in measure as $\alpha \to \infty$. That is,*

$$\lim_{\alpha \to \infty} \mu\left(\{x \in X : |f_\alpha(x) - f(x)| \geq \varepsilon\}\right) = 0$$

for each fixed $\varepsilon > 0$.

Proof. Let $\delta, \varepsilon > 0$ be given. By Egoroff's theorem, there exists a measurable set $E \in \mathfrak{M}$ such that $f_\alpha \to f$ uniformly on E and $\mu(X \setminus E) < \delta$. Let $N \in \mathbb{N}$ be so large that $|f_\alpha(x) - f(x)| < \varepsilon$ for all $\alpha \geq N$ and all $x \in E$. Then, for each $\alpha \geq N$,

$$\{x \in X : |f_\alpha(x) - f(x)| \geq \varepsilon\} \subseteq X \setminus E$$

whence $\mu(\{x \in X : |f_\alpha(x) - f(x)| \geq \varepsilon\}) \leq \mu(X \setminus E) < \delta$. $\square$

2.2 Weighted Lebesgue Spaces

Let $(X, \mathfrak{M}, \mu)$ be a measure space and fix $1 \leq p < \infty$. Let $\omega : X \to [0, \infty]$ be a measurable function that is both positive and finite almost everywhere on X. We define $L^p(X, \omega)$ to be the weighted L^p-space consisting of all measurable maps $f : X \to \mathbb{R}$ with the property that

$$\int_X |f(x)|^p \, \omega(x) \mathrm{d}\mu(x) < \infty.$$

We endow this space with the norm

$$\|\cdot\|_{L^p(X,\omega)} := \left(\int_X |\cdot(x)|^p \, \omega(x) \mathrm{d}\mu(x) \right)^{1/p}.$$

Simply considered as L^p-spaces, weighted Lebesgue spaces have no special properties. Indeed, the weighted space $L^p(X, \omega)$ is precisely the space $L^p(X, \mathrm{d}\nu)$ with the measure ν given by

$$\nu(E) := \int_E \omega \, \mathrm{d}\mu, \quad E \in \mathfrak{M}.$$

However, when working with weighted Lebesgue measures in $\mathbb{R}^n$, it is natural to ask whether smooth functions of compact support remain dense in the resulting L^p-spaces; a question which appears to have not been directly answered in many references. Below we show that, under relatively weak assumptions, the answer is affirmative.

Theorem 2.2.1. *Let $1 \leq p < \infty$ and assume that $\omega \in L^1_{loc}(\mathbb{R}^n)$ is positive almost everywhere on $\mathbb{R}^n$. Then, $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, \omega)$.*

Proof. Let $\varepsilon > 0$ be given and fix a function $f \in L^p(\mathbb{R}^n, \omega)$. By the Monotone Convergence Theorem, we can find $R > 0$ such that

$$\int_{\{|x|>R\}} |f(x)|^p \, \omega(x) \mathrm{d}x < \frac{\varepsilon^p}{2^{p+1}},$$
$$\int_{\{|f|>R\}} |f(x)|^p \, \omega(x) \mathrm{d}x < \frac{\varepsilon^p}{2^{p+1}}.$$

Then, the function $g := f \mathbf{1}_{\{|x| \le R, |f| \le R\}}$ is bounded, compactly supported, and satisfies

$$\begin{aligned} \|f - g\|_{L^p(\mathbb{R}^n,\omega)} &= \left(\int_{\mathbb{R}^n} |f(x)|^p \, \mathbf{1}_{\{|x|>R\} \cup \{|f|>R\}} \omega(x) \mathrm{d}x \right)^{1/p} \\ &\le \left(\int_{\mathbb{R}^n} |f(x)|^p \left(\mathbf{1}_{\{|x|>R\}} + \mathbf{1}_{\{|f|>R\}} \right) \omega(x) \mathrm{d}x \right)^{1/p} \\ &< \left(\frac{\varepsilon^p}{2^{p+1}} + \frac{\varepsilon^p}{2^{p+1}} \right)^{1/p} \\ &= \frac{\varepsilon}{2}. \end{aligned}$$

Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a standard mollifier and define, for $\delta > 0$, the mollification $g_\delta := g * \eta_\delta$. Here, η_δ is given by

$$\eta_\delta(x) = \frac{1}{\delta^n} \eta \left(\frac{x}{\delta} \right).$$

Then, each g_δ is smooth and compactly supported. In fact, since g is compactly supported, there exists a compact set $K \subset \mathbb{R}^n$ such that $\operatorname{supp}(g_\delta), \operatorname{supp}(g) \subseteq K$ for all $\delta > 0$ sufficiently small. Furthermore, Hölder's inequality shows that

$$|g_\delta(x)| \le \|\eta_\delta\|_{L^1(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R}^n)} \le \|\eta\|_{L^1(\mathbb{R}^n)} R < \infty$$

for all $x \in \mathbb{R}^n$ and every $\delta > 0$. Since $g_\delta(x) \to g(x)$ for almost every $x \in \mathbb{R}^n$, the Dominated Convergence Theorem asserts that

$$\lim_{\delta \searrow 0} \int_{\mathbb{R}^n} |g_\delta(x) - g(x)|^p \, \omega(x) \mathrm{d}x = 0.$$

Hence, for $\delta > 0$ suitably small, we have $\|g_\delta - g\|_{L^p(\mathbb{R}^n,\omega)} < \frac{\varepsilon}{2}$. To summarize, we have that

$$\|f - g_\delta\|_{L^p(\mathbb{R}^n,\omega)} \leq \|f - g\|_{L^p(\mathbb{R}^n,\omega)} + \|g - g_\delta\|_{L^p(\mathbb{R}^n,\omega)} < \varepsilon.$$

Since $g_\delta \in C_c^\infty(\mathbb{R}^n)$, this is precisely what had to be shown. □

Remark 2.2.1. As a consequence of this theorem, if ν is any locally finite measure on the Lebesgue σ-algebra that is absolutely continuous with respect to the Lebesgue measure, then $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, \mathrm{d}\nu)$. Indeed, by the Radon-Nikodym theorem, there exists a non-negative $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $L^p(\mathbb{R}^n, \mathrm{d}\nu) = L^p(\mathbb{R}^n, \omega)$.

Before proceeding further, we establish some shorthand that will make the proofs that follow more elegant.

Notation. Let $E \subseteq \mathbb{R}^n$ be a measurable set. For the sake of simplicity, we will write $L^p_a(E, x_0)$ to denote the weighted Lebesgue space $L^p(E, |x - x_0|^{-ap})$. Similarly, we shall use $L^q_b(E, x_0)$ instead of $L^q(E, |x - x_0|^{-bq})$.

2.3 An Explicit Description of $\mathcal{D}^{1,p}_a(U, x_0)$

We continue to assume that $1 < p < n$ and that (1.6) holds true for a fixed pair (a, b). Given an open set $U \subseteq \mathbb{R}^n$ and a point $x_0 \in \mathbb{R}^n$, we define $\mathcal{E}^{1,p}_a(U, x_0)$ to be the real vector space consisting of all functions $u \in L^q_b(U, x_0)$ possessing weak derivatives of the first order on $U \setminus \{x_0\}$ such that $\nabla u \in L^p_a(U, x_0)$. In symbolic terms,

$$\mathcal{E}^{1,p}_a(U, x_0) := \left\{u \in L^q_b(U, x_0) : \nabla u \in L^p_a(U \setminus \{x_0\}, x_0)\right\}.$$

We endow this space with the *strong norm*

$$\|u\|_{\mathcal{E}^{1,p}_a(U,x_0)} := \|u\|_{L^q_b(U,x_0)} + \|\nabla u\|_{L^p_a(U,x_0)}.$$

Finally, let $\mathcal{E}^{1,p}_{a,0}(U, x_0)$ denote the closure of $C_c^\infty(U)$ in $\mathcal{E}^{1,p}_a(U, x_0)$.

Our first assertion is that the Caffarelli-Kohn-Nirenberg inequality in (2.1)-(2.3) continues to hold for all functions $u \in \mathcal{E}_{a,0}^{1,p}(U, x_0)$. Thankfully, this is immediate from the definition of $\mathcal{E}_{a,0}^{1,p}(U, x_0)$.

Lemma 2.3.1. *The Caffarelli-Kohn-Nirenberg inequality is valid in $\mathcal{E}_{a,0}^{1,p}(U, x_0)$. More precisely, there exists a constant $C > 0$ such that, for all functions $u \in \mathcal{E}_{a,0}^{1,p}(U, x_0)$,*

$$\left(\int_U |u(x)|^q \, |x - x_0|^{-bq} \, \mathrm{d}x \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p \, |x - x_0|^{-ap} \, \mathrm{d}x \right)^{1/p}. \tag{2.5}$$

In particular, $\|\cdot\|_{\mathcal{E}_{a,0}^{1,p}(U,x_0)} := \|\nabla \cdot\|_{L_a^p(U,x_0)}$ is an equivalent norm on $\mathcal{E}_{a,0}^{1,p}(U, x_0)$. Furthermore, the constant C does not depend on the singular point x_0.

Proof. Given a function $u \in \mathcal{E}_{a,0}^{1,p}(U, x_0)$, there exists by definition a sequence (φ_α) in $C_c^\infty(U)$ converging to u in $\mathcal{E}_a^{1,p}(U, x_0)$ as $\alpha \to \infty$. Consequently, $\varphi_\alpha \to u$ and $\nabla \varphi_\alpha \to \nabla u$ in $L_b^q(U, x_0)$ and $L_a^p(U, x_0)$ (respectively) as $\alpha \to \infty$. Since (2.5) is valid for each φ_α by our calculations in (2.1)-(2.3), the claim follows. □

In light of this lemma, we will henceforth give $\mathcal{E}_{a,0}^{1,p}(U, x_0)$ the simpler norm $\|\cdot\|_{\mathcal{E}_{a,0}^{1,p}(U,x_0)}$ defined above. Before giving an explicit description of $\mathcal{D}_a^{1,p}(U, x_0)$, we require another elementary result regarding Banach spaces. Despite being very easy, the proof is provided for the sake of completeness.

Lemma 2.3.2. *Let $\mathcal{Y}$ be a dense subspace of a Banach space $(\mathcal{X}, \|\cdot\|)$ and denote by $\mathcal{Y}^*$ the completion of $\mathcal{Y}$ with respect to $\|\cdot\|$. There exists an isometric isomorphism $T : \mathcal{Y}^* \to \mathcal{X}$.*

Proof. A general element of $\mathcal{Y}^*$ is an equivalence class $[(y_\alpha)]$, where (y_α) is a Cauchy sequence in $\mathcal{Y}$. In particular, (y_α) converges, as $\alpha \to \infty$, to some $y \in \mathcal{X}$. If (y'_α) is another representative of $[(y_\alpha)]$, then $y_\alpha - y'_\alpha \to 0$ in $\mathcal{Y}$ as $\alpha \to \infty$. Consequently, $y'_\alpha \to y$ in $\mathcal{X}$ as $\alpha \to \infty$. It follows that the map

$$T : \mathcal{Y}^* \to \mathcal{X}, \quad [(y_\alpha)] \mapsto y$$

is well defined. Clearly, T is linear and, by the density of $\mathcal{Y}$ in $\mathcal{X}$, is surjective.

Next we prove that T is an isometry. Indeed, fix a point $[(y_\alpha)]$ in $\mathcal{Y}^*$ and set $y := \lim y_\alpha$, with the limit being understood in $\mathcal{X}$. Then, by the continuity of norms,

$$\|T([(y_\alpha)])\| = \|y\| = \lim_{\alpha\to\infty} \|y_\alpha\| = \|[(y_\alpha)]\| \, .$$

This shows that T is an isometry and so the proof is complete. $\square$

This lemma readily gives way to the following identification result:

Theorem 2.3.1. *Give $\mathcal{E}^{1,p}_{a,0}(U, x_0)$ the norm $\|\cdot\|_{\mathcal{E}^{1,p}_{a,0}(U,x_0)}$. There exists an isometric isomorphism $\mathcal{D}^{1,p}_a(U, x_0) \to \mathcal{E}^{1,p}_{a,0}(U, x_0)$.*

Proof. Assume for the moment that $\mathcal{E}^{1,p}_a(U, x_0)$ is complete. In particular, $\mathcal{E}^{1,p}_{a,0}(U, x_0)$ is a Banach space. Since $C^\infty_c(U)$ is dense in $\mathcal{E}^{1,p}_{a,0}(U, x_0)$ with respect to the norm

$$\|\cdot\|_{\mathcal{D}^{1,p}_a(U,x_0)} := \left(\int_U |\nabla(\cdot)|^p \, |x - x_0|^{-ap} \, \mathrm{d}x \right)^{1/p},$$

we see from Lemma 2.3.2 that there exists an isometric isomorphism

$$\mathcal{D}^{1,p}_a(U, x_0) \to \mathcal{E}^{1,p}_{a,0}(U, x_0).$$

Hence, we are reduced to proving that $\mathcal{E}^{1,p}_a(U, x_0)$ is Banach. To this end, let (u_α) be a Cauchy sequence in $\mathcal{E}^{1,p}_a(U, x_0)$ with respect to the strong norm $\|\cdot\|_{\mathcal{E}^{1,p}_a(U,x_0)}$. Then, the sequences (u_α) and (∇u_α) are Cauchy in $L^q_b(U, x_0)$ and $L^p_a(U, x_0)$, respectively. Since these are complete, we can find functions $u \in L^q_b(U, x_0)$ and $v^1, \dots, v^n \in L^p_a(U, x_0)$ such that

$$\lim_{\alpha\to\infty} u_\alpha = u \quad \text{in } L^q_b(U, x_0)$$

and

$$\lim_{\alpha\to\infty} \partial_i u_\alpha = v^i \quad \text{in } L^p_a(U, x_0)$$

as $\alpha \to \infty$ for all $i = 1, \dots, n$. We now claim that $\partial_i u = v^i$ for each $i = 1, \dots, n$ on $U \setminus \{x_0\}$. Fix a test function $\varphi \in C^\infty_c(U)$ such that $\operatorname{supp}(\varphi) \subset U \setminus \{x_0\}$. Then, there exists $c > 0$ such that $|x - x_0|^{-ap} \geq c$ and $|x - x_0|^{-bq} \geq c$ for all $x \in \operatorname{supp}(\varphi)$. Since

$u_\alpha \to u$ in $L_b^q(U, x_0)$,

$$\begin{aligned}\int_{\mathrm{supp}(\varphi)} |u_\alpha - u|^q \,\mathrm{d}x &\leq \frac{1}{c}\int_{\mathrm{supp}(\varphi)} |u_\alpha - u|^q \,|x - x_0|^{-bq}\,\mathrm{d}x \\ &\leq \frac{1}{c}\int_U |u_\alpha - u|^q \,|x - x_0|^{-bq}\,\mathrm{d}x \to 0, \quad \text{as } \alpha \to \infty.\end{aligned}$$

Similarly, we see that $\partial_i u_\alpha \to v^i$ in $L^p(\mathrm{supp}(\varphi))$ for each $i = 1, \ldots, n$. Thus, two successive applications of Hölder's inequality show that

$$\begin{aligned}\int_U u\partial_i\varphi \mathrm{d}x = \lim_{\alpha\to\infty}\int_U u_\alpha \partial_i \varphi \mathrm{d}x &= -\lim_{\alpha\to\infty}\int_U \varphi\partial_i u_\alpha \mathrm{d}x \\ &= -\int_U \varphi v^i \mathrm{d}x\end{aligned}$$

for all $i = 1, \ldots, n$. It follows from here that $\nabla u = (v^1, \ldots, v^n)$ in the weak sense on $U \setminus \{x_0\}$. In particular, $u \in \mathcal{E}_a^{1,p}(U, x_0)$. Since $u_\alpha \to u$ and $\nabla u_\alpha \to (v^1, \ldots, v^n)$ in $L_b^q(U, x_0)$ and $L_a^p(U, x_0)$ respectively, we infer that $u_\alpha \to u$ in $\mathcal{E}_a^{1,p}(U, x_0)$. This completes the proof. □

Henceforth, we will identify the weighted Sobolev space $\mathcal{D}_a^{1,p}(U, x_0)$ with the explicit space $\mathcal{E}_{a,0}^{1,p}(U, x_0)$. Especially, we see that $\mathcal{D}_a^{1,p}(U, x_0)$ consists of functions in $L_b^q(U, x_0)$ having first order weak derivatives away from the singular point x_0. Furthermore, these functions must obey the Caffarelli-Kohn-Nirenberg inequality (2.5). As a matter of fact, when $a, b \geq 0$, the functions $u \in \mathcal{D}_a^{1,p}(U, x_0)$ have first order weak derivatives on the entire set U.

Corollary 2.3.3. *When $a \geq 0$, the elements of $\mathcal{D}_a^{1,p}(U, x_0)$ are differentiable in the weak sense on the entire set U.*

Proof. To see this, first note that $a \geq 0$ forces $b \geq 0$. Therefore, the weight functions $x \mapsto |x - x_0|^{-ap}$ and $x \mapsto |x - x_0|^{-bq}$ are bounded below by positive constants on every compact subset of $\mathbb{R}^n$. Then, if (φ_α) is a sequence in $C_c^\infty(U)$ converging to u in $\mathcal{D}_a^{1,p}(U, x_0)$, we have

$$\varphi_\alpha \to u \text{ in } L_b^q(U, x_0)$$

and

$$\nabla \varphi_\alpha \to \nabla u \text{ in } L^p_a(\Omega, x_0).$$

Now, given $\varphi \in C^\infty_c(U)$ we let Λ be its support in U. Following the argument in Theorem 2.3.1, we see that $\varphi_\alpha \to u$ in $L^q(\Lambda)$ and that $\nabla\varphi_\alpha \to \nabla u$ in $L^p(\Lambda)$. Then, using Hölder's inequality as in the proof of Theorem 2.3.1, we obtain

$$\int_U u \partial_i \varphi \, \mathrm{d}x = - \int_U \varphi \partial_i u \, \mathrm{d}x$$

for all $i = 1, \ldots, n$. □

2.4 A Rellich-Kondrachov Embedding Theorem

In this section we will show that $\mathcal{D}^{1,p}_a(U, x_0)$ can be compactly embedded into weighted Lebesgue spaces when the open set U is bounded. Of course, given a bounded sequence in $\mathcal{D}^{1,p}_a(U, x_0)$, this will allow us to extract a subsequence converging almost everywhere on the set U. Formally, the main result of this section is as follows:

Theorem 2.4.1. *Let $U \subseteq \mathbb{R}^n$ be bounded and fix $\theta \in (-\infty, bq]$.*

(1) Given any $1 \le r \le q$, there exists a constant $C > 0$ such that

$$\|u\|_{L^r(U, |x-x_0|^{-\theta})} \le C \|u\|_{\mathcal{D}^{1,p}_a(U,x_0)}$$

for all $u \in \mathcal{D}^{1,p}_a(U, x_0)$.

(2) Assume that $1 \le r < q$. If (u_α) is a bounded sequence in $\mathcal{D}^{1,p}_a(U, x_0)$, there exists a subsequence (u_β) converging strongly to a function $u \in L^r(U, |x - x_0|^{-\theta})$ and pointwise almost everywhere on U.

As a first step in proving this theorem, we will require a simple extension result for $\mathcal{D}^{1,p}_a(U, x_0)$.

Proposition 2.4.1. *Let V be an open set which contains the closure of U. Given a function $u \in \mathcal{D}^{1,p}_a(U, x_0)$, let $\overline{u}$ and $\overline{\nabla u}$ be the elements of $L^q_b(V, x_0)$ and $L^p_a(V, x_0)$ (respectively) given by*

$$\begin{cases} \overline{u} = u \text{ a.e. in } U \text{ and } \overline{u} = 0 \text{ a.e. on } V \setminus U; \\ \overline{\nabla u} = \nabla u \text{ a.e. in } U \text{ and } \overline{\nabla u} = 0 \text{ a.e. on } V \setminus U. \end{cases}$$

Then, $\overline{u} \in \mathcal{D}^{1,p}_a(V, x_0)$ and $\nabla \overline{u} = \overline{\nabla u}$ in U. Moreover, the linear map

$$T : \mathcal{D}^{1,p}_a(U, x_0) \to \mathcal{D}^{1,p}_a(V, x_0), \quad u \mapsto \overline{u}$$

is an isometry.

Proof. Given $u \in \mathcal{D}^{1,p}_a(U, x_0)$, let (u_α) be a sequence in $C^\infty_c(\Omega)$ converging to u in $\mathcal{D}^{1,p}_a(U, x_0)$. By (2.5), we have both $\nabla u_\alpha \to \nabla u$ in $L^p_a(U, x_0)$ and $u_\alpha \to u$ in $L^q_b(U, x_0)$, as $\alpha \to \infty$. Passing to a subsequence, we may also assume that this convergence takes place almost everywhere on U. Now, (u_α) and (∇u_α) all have compact support in $U \subseteq V$. Thus, (u_α) also forms a Cauchy sequence in $\mathcal{D}^{1,p}_a(V, x_0)$. Passing to yet another subsequence, we may assume that

(i) $u_\alpha \to v$ in $\mathcal{D}^{1,p}_a(V, x_0)$ as $\alpha \to \infty$;

(ii) $u_\alpha \to v$ pointwise a.e. on V;

(iii) and $\nabla u_\alpha \to \nabla v$ pointwise a.e. on V.

Now, this means that $v = u$ and $\nabla v = \nabla u$ a.e. on U. However, as each u_α is supported on U, we also have $v = 0$ and $\nabla v = 0$ a.e. on $V \setminus U$. These facts combined show that $v = \overline{u}$ and $\nabla v = \overline{\nabla u}$ almost everywhere in V. In particular, $\overline{u} \in \mathcal{D}^{1,p}_a(V, x_0)$. Since T is clearly linear, the proof is complete. □

We are now properly equipped to give the proof of Theorem 2.4.1.

Proof of Theorem 2.4.1. Let $1 \leq r \leq q$. First, note that on every compact subset Λ of $\mathbb{R}^n$, there exists a constant $c > 0$ such that $c\,|x - x_0|^{-\theta} \leq |x - x_0|^{-bq}$ everywhere on $\Lambda \setminus \{x_0\}$. Now, for any $u \in \mathcal{D}^{1,p}_a(U, x_0)$, Hölder's inequality together with the fact that the measure $|x - x_0|^{-\theta}\, \mathrm{d}x$ is locally finite on $\mathbb{R}^n$ shows that

$$\begin{aligned}\left(\int_U |u(x)|^r \, |x - x_0|^{-\theta} \, \mathrm{d}x\right)^{1/r} &\le C \left(\int_U |u(x)|^q \, |x - x_0|^{-\theta} \, \mathrm{d}x\right)^{1/q} \\ &\le C c^{-1/q} \, \|u\|_{L^q_b(U,x_0)} \, .\end{aligned}$$

for a suitable constant $C > 0$ independent of u. Invoking Lemma 2.3.1, part (1) of Theorem 2.4.1 is verified.

We must now verify the compactness of this embedding when $1 \le r < q$. Let (u_α) be a bounded sequence in $\mathcal{D}^{1,p}_a(U, x_0)$ so that, by Lemma 2.3.1, (u_α) is bounded in $L^q_b(U, x_0)$ and (∇u_α) is bounded in $L^p_a(U, x_0)$. Citing Proposition 2.4.1, after an extension by zero, we can assume that (u_α) is a bounded sequence in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, x_0)$ that is supported on U. Now, for $k \ge 1$ we denote by A_k the open annulus

$$B(x_0, k+1) \setminus \overline{B(x_0, 1/k)}$$

in $\mathbb{R}^n$. Since $q > p$, (u_α) forms a bounded sequence in $W^{1,p}(A_k)$ for each $k \ge 1$. Then, using the classical Rellich-Kondrachov compactness theorem, we can construct a family $\{(u_{\alpha,k})\}_{k=1}^{\infty}$ of subsequences of (u_α) such that

(i) $(u_{\alpha,k+1})$ is a subsequence of $(u_{\alpha,k})$ for each $k \ge 1$,

(ii) $(u_{\alpha,k})$ converges strongly to v_k in $L^p(A_k)$,

(iii) $(u_{\alpha,k})$ to v_k pointwise almost everywhere on A_k.

Clearly, $v_{k+1} = v_k$ almost everywhere on each A_k. Thus, the diagonal sequence $(u_{\alpha,\alpha})$ converges almost everywhere to a well defined measurable function u on $\mathbb{R}^n$. By (1), it is clear that $(u_{\alpha,\alpha})$ is bounded in $L^q(U, |x - x_0|^{-\theta})$. Thus, using Theorem 2.1.2 with exponent $q/r > 1$, we infer that

$$\lim_{\alpha \to \infty} \int_U |u_{\alpha,\alpha}(x) - u(x)|^r \, |x - x_0|^{-\theta} \, \mathrm{d}x = 0.$$

Finally, we have from (1) that $(u_{\alpha,\alpha})$ is bounded in $L^r(U, |x-x_0|^{-\theta})$. It now follows from Fatou's lemma that $u \in L^r(U, |x-x_0|^{-\theta})$. □

Next, we check that elements of $\mathcal{D}^{1,p}_a(U, x_0)$ behave as expected when multiplied by a cutoff function. Namely, given an open set V whose closure is contained in U and a cutoff function $\eta \in C^\infty_c(V)$, can we guarantee that $u\eta \in D^{1,p}(V, x_0)$? The answer is affirmative, but the argument relies on the following natural embedding:

Proposition 2.4.2. *Let $\Lambda \subset \mathbb{R}^n$ be compact. There exists a constant $C > 0$ such that*

$$\left(\int_\Lambda |u(x)|^p \, |x-x_0|^{-ap} \, \mathrm{d}x \right)^{1/p} \le C \, \|u\|_{\mathcal{D}^{1,p}_a(\mathbb{R}^n, x_0)}$$

for all $u \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, x_0)$. That is, $\mathcal{D}^{1,p}_a(\mathbb{R}^n, x_0) \hookrightarrow L^p_{loc}(\mathbb{R}^n, |x-x_0|^{-ap})$.

Proof. If $0 \le a \le b$ or $a < 0 \le b$ then, since $p < q$, we have $ap \le bq$ whence the claim follows from Theorem 2.4.1-(1). If instead $a \le b < 0$, we write

$$\int_\Lambda |u(x)|^p \, |x-x_0|^{-ap} \, \mathrm{d}x = \int_\Lambda |u(x)|^p \, |x-x_0|^{-ap\theta} \, |x-x_0|^{-ap(1-\theta)} \, \mathrm{d}x$$

with $\theta = b/a$. Then, $\theta \in (0,1]$ and Hölder's inequality with exponent q/p yields

$$\begin{aligned} \int_\Lambda |u(x)|^p \, |x-x_0|^{-ap} \, \mathrm{d}x &\le C \left(\int_\Lambda |u(x)|^q \, |x-x_0|^{-ap\theta \cdot \frac{q}{p}} \, \mathrm{d}x \right)^{p/q} \\ &= C \left(\int_\Lambda |u(x)|^q \, |x-x_0|^{-bq} \, \mathrm{d}x \right)^{p/q} \end{aligned}$$

with C given by

$$C = \left(\int_\Lambda |x-x_0|^{-ap(1-\theta)\frac{q}{q-p}} \right)^{1-p/q}.$$

Now, since $a < 0$ and $\theta \in (0,1]$, it is clear that $C < \infty$. Thus, in any case, we obtain

$$\|u\|_{L^p(\Lambda, |x-x_0|^{-ap})} \le C \, \|u\|_{\mathcal{D}^{1,p}_a(\mathbb{R}^n, x_0)}$$

for a suitable constant $C > 0$ independent of u. □

Proposition 2.4.3. *Let $V \subseteq \mathbb{R}^n$ be an open set such that $\bar{V} \subseteq U$. Let $\eta \in C_c^\infty(V)$ and fix a function $u \in \mathcal{D}_a^{1,p}(U, x_0)$. Then, $u\eta \in \mathcal{D}_a^{1,p}(V, x_0)$.*

Proof. After an extension by zero, we may again assume without loss of generality that $U = \mathbb{R}^n$. First, we note that $u\eta \in \mathcal{E}_a^{1,p}(V, x_0)$. Indeed, fix a test function $\varphi \in C_c^\infty(V)$ with

$$\Lambda := \mathrm{supp}(\varphi) \subseteq V \setminus \{x_0\}.$$

Then, using $\eta\varphi \in C_c^\infty(U)$ as a test function, we infer that

$$\int_U \varphi\eta\partial_i u \mathrm{d}x = -\int_U u\partial_i(\eta\varphi)\mathrm{d}x = -\int_U (u\varphi)\partial_i\eta \mathrm{d}x - \int_U (u\eta)\partial_i\varphi \mathrm{d}x$$

whence

$$\int_V (u\eta)\partial_i\varphi \mathrm{d}x = -\int_V (u\partial_i\eta + \eta\partial_i u)\,\varphi \mathrm{d}x$$

for all $i = 1, \dots, n$. In particular,

$$\nabla(u\eta) = u\nabla\eta + \eta\nabla u \quad \text{in } V \setminus \{x_0\}.$$

Consequently, $u\eta$ is weakly differentiable on $V \setminus \{x_0\}$. Since η is bounded, we clearly have $\eta u \in L_b^q(V, x_0)$. By Proposition 2.4.2, we see that $\nabla(u\eta)$ belongs to $L_a^p(V, x_0)$. All that remains is to check that $u\eta \in \mathcal{E}_{a,0}^{1,p}(V, x_0)$. To this end, let (u_α) be a sequence in $C_c^\infty(U)$ converging to u in $\mathcal{E}_a^{1,p}(U, x_0)$. Clearly, $u_\alpha\eta \in C_c^\infty(V)$ for each index $\alpha \in \mathbb{N}$. Moreover, for

$$M := \sup_{x\in\mathbb{R}^n} |\eta(x)|^q\,,$$

one has

$$\int_V |u\eta - u_\alpha\eta|^q\, |x - x_0|^{-bq}\, \mathrm{d}x \leq M \int_U |u - u_\alpha|^q\, |x - x_0|^{-bq}\, \mathrm{d}x \to 0,$$

as $\alpha \to \infty$. Similarly, we have $\nabla(u\eta - u_\alpha\eta) = \eta\nabla u + u\nabla\eta - \eta\nabla u_\alpha - u_\alpha\nabla\eta$ so that

$$\begin{aligned}\|\nabla(u\eta - u_\alpha\eta)\|_{L^p_a(V,x_0)} \leq \|\eta(\nabla u_\alpha - \nabla u)\|_{L^p_a(V,x_0)} \\ + \|\nabla\eta(u_\alpha - u)\|_{L^p_a(V,x_0)}.\end{aligned}$$

Since $\eta, \nabla\eta$ are bounded and have compact support in $V \subseteq U$, another application of Proposition 2.4.2 shows that both of these terms converge to zero as $\alpha \to \infty$. That is, $u_\alpha\eta \to u\eta$ strongly in $\mathcal{E}^{1,p}_a(V, x_0)$. This completes the proof. □

In fact, by the same argument, we obtain a more general but weaker result:

Lemma 2.4.4. *Let $u \in W^{1,1}_{loc}(\mathbb{R}^n \setminus \{x_0\})$ and let $\eta \in C^\infty_c(\mathbb{R}^n)$ vanish in a neighbourhood of x_0. Then $\eta u \in W^{1,1}(\mathbb{R}^n)$.*

Proof. Fix a test function $\varphi \in C^\infty_c(\mathbb{R}^n)$ and note that the product $\eta\varphi$ is a smooth compactly supported function vanishing near x_0. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n}(u\eta)\nabla\varphi \mathrm{d}x &= -\int_{\mathbb{R}^n}(u\varphi)\nabla\eta\,\mathrm{d}x + \int_{\mathbb{R}^n}(u\varphi)\nabla\eta\,\mathrm{d}x + \int_{\mathbb{R}^n}(u\eta)\nabla\varphi\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^n}(u\varphi)\nabla\eta\,\mathrm{d}x + \int_{\mathbb{R}^n} u\nabla(\eta\varphi)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^n}(u\varphi)\nabla\eta\,\mathrm{d}x - \int_{\mathbb{R}^n}(\eta\varphi)\nabla u\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^n}(u\nabla\eta + \eta\nabla u)\,\varphi\,\mathrm{d}x\end{aligned}$$

and we see that $u\eta$ is weakly differentiable with $\nabla(u\eta) = u\nabla\eta + \eta\nabla u$. Since $u, \nabla u$ are locally integrable away from x_0 and $\eta \in C^\infty_c(\mathbb{R}^n)$ vanishes in a neighbourhood of x_0, the claim follows. □

This gives way to the following simple lemma that will play a key role within the proof of our main result.

Lemma 2.4.5. *Let $\Lambda \subset \mathbb{R}^n$ be compact and let (u_α) be a bounded sequence in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, x_0)$ converging pointwise almost everywhere to 0 on $\mathbb{R}^n$. Then, there exists a subsequence (u_β) converging strongly to 0 in $L^p(\Lambda, |x - x_0|^{-ap})$ as $\beta \to \infty$.*

Proof. Choose a bounded open set $U \supset \Lambda$ and let $\eta \in C_c^\infty(U)$ be a cutoff function such that $\eta \equiv 1$ in a neighbourhood of Λ. We first assert that $\mathcal{D}_a^{1,p}(U, x_0)$ is compactly embedded into $L_a^p(U, x_0)$. Since $1 < p < q$, this follows immediately from Theorem 2.4.1 if either $0 \le a \le b$ or $a < 0 \le b$.[1] Thus, we are reduced to checking that the embedding is still compact when $a \le b < 0$. For $\varepsilon > 0$ small, we define

$$\theta_\varepsilon := \frac{bq}{a(q-\varepsilon)}.$$

Note that θ_ε is continuous in $\varepsilon > 0$ sufficiently small and that, as $\varepsilon \searrow 0$,

$$ap(1-\theta_\varepsilon) \cdot \frac{q-\varepsilon}{q-p-\varepsilon} \to \frac{pq(a-b)}{q-p} \le 0 < n.$$

Thus, for all $\varepsilon > 0$ small, we have $ap(1-\theta_\varepsilon) \cdot \frac{q-\varepsilon}{q-p-\varepsilon} < n$. Using Hölder's inequality as in the proof of Proposition 2.4.2, we obtain

$$\int_U |w(x)|^p \, |x - x_0|^{-ap} \, \mathrm{d}x \le C \left(\int_U |w(x)|^{q-\varepsilon} \, |x - x_0|^{-ap\theta_\varepsilon \cdot \frac{q-\varepsilon}{p}} \, \mathrm{d}x \right)^{p/(q-\varepsilon)}$$

for all measurable w, with $0 < C < \infty$ given by

$$C = \left(\int_U |x - x_0|^{-ap(1-\theta_\varepsilon) \cdot \frac{q-\varepsilon}{q-p-\varepsilon}} \right)^{1-\frac{p}{q-\varepsilon}}$$

which is finite for $\varepsilon > 0$ sufficiently small. Put otherwise, there exists a constant $C > 0$ such that

$$\left(\int_U |w(x)|^p \, |x - x_0|^{-ap} \, \mathrm{d}x \right)^{1/p} \le C \left(\int_U |w(x)|^{q-\varepsilon} \, |x - x_0|^{-bq} \right)^{1/(q-\varepsilon)}$$

for all measurable w. Again, by Theorem 2.4.1, $\mathcal{D}_a^{1,p}(U, x_0)$ is compactly embedded into $L^{q-\varepsilon}(U, |x - x_0|^{-bq})$ which proves our first assertion.

It remains to prove the lemma. By Proposition 2.4.3, $v_\alpha := u_\alpha \eta$ is an element of

[1] In either case, take $\theta = ap \le bq$ in the statement of Theorem 2.4.1.

$\mathcal{D}_a^{1,p}(U,x_0)$. In fact, we know from the proof of this result that

$$\nabla v_\alpha = \eta \nabla u_\alpha + u_\alpha \nabla \eta \quad \text{in } U \setminus \{x_0\}.$$

It is then clear from Proposition 2.4.2 that the sequence (v_α) is *bounded* in $\mathcal{D}_a^{1,p}(U,x_0)$. Furthermore, $v_\alpha \to 0$ almost everywhere on U as $\alpha \to \infty$. Since (v_α) is bounded in $\mathcal{D}_a^{1,p}(U,x_0)$, the argument above allows us to extract a subsequence (v_β) converging strongly to 0 in $L_a^p(U,x_0)$ as $\beta \to \infty$. As $\eta \equiv 1$ on Λ, it readily follows that $u_\alpha \to 0$ strongly in $L^p(\Lambda, |x-x_0|^{-ap})$. □

We remark that after studying the weak compactness of $\mathcal{D}_a^{1,p}(U,x_0)$, it will be possible to give a more elegant and precise reformulation of this lemma.

2.5 The Dual Space $\mathcal{D}_a^{-1,p'}(U,x_0)$

Recall that, by definition, $\mathcal{D}_a^{-1,p'}(U,x_0)$ is the vector space of all continuous linear functionals $\mathcal{D}_a^{1,p}(U,x_0) \to \mathbb{R}$ endowed with the operator norm. Here, p' denotes the Hölder conjugate exponent of p, i.e.

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

It is an elementary fact that the topological dual of any normed space is, in fact, a Banach space. In this section we study the behaviour of continuous linear functionals on $\mathcal{D}_a^{1,p}(U,x_0)$ and discuss their link to those on $L_a^p(U,x_0)$. More precisely, we show that $\mathcal{D}_a^{1,p}(U,x_0)$ is a reflexive Banach space and establish a Riesz-type representation theorem for the bounded linear functionals on $\mathcal{D}_a^{1,p}(U,x_0)$. Thus, we give a complete description of those $\phi \in \mathcal{D}_a^{-1,p'}(U,x_0)$.

Proposition 2.5.1. *Let (u_α) be a bounded sequence in $\mathcal{D}_a^{1,p}(U,x_0)$ and fix a point $u \in \mathcal{D}_a^{1,p}(U,x_0)$. Then, $u_\alpha \rightharpoonup u$ in $\mathcal{D}_a^{1,p}(U,x_0)$ if and only if $\nabla u_\alpha \rightharpoonup \nabla u$ in $L_a^p(U,x_0)$.*

Proof. First, we note that the linear operator

$$T : \mathcal{D}_a^{1,p}(U, x_0) \to L_a^p(U, x_0), \quad u \mapsto \nabla u,$$

is an isometry. Let now E denote the image of $\mathcal{D}_a^{1,p}(U, x_0)$ under T; we claim that $u_\alpha \rightharpoonup u$ in $\mathcal{D}_a^{1,p}(U, x_0)$ if and only if $\nabla u_\alpha \rightharpoonup \nabla u$ in E, where E is interpreted as a subspace of $L_a^p(U, x_0)$. First, assume that u_α converges weakly to u in $\mathcal{D}_a^{1,p}(U, x_0)$. Then, given a continuous linear functional $\psi : E \to \mathbb{R}$, the composite $\phi := \psi \circ T$ is a continuous linear functional on $\mathcal{D}_a^{1,p}(U, x_0)$. Since $u_\alpha \rightharpoonup u$ in $\mathcal{D}_a^{1,p}(U, x_0)$,

$$\phi(u_\alpha) \to \phi(u), \quad \text{as } \alpha \to \infty.$$

Put otherwise, this means that $\psi(\nabla u_\alpha) \to \psi(\nabla u)$ as $\alpha \to \infty$. Conversely, assume that $\nabla u_\alpha \rightharpoonup \nabla u$ in E. Given a continuous linear functional ϕ on $\mathcal{D}_a^{1,p}(U, x_0)$, define $\psi := \phi \circ T^{-1}$, which is a continuous linear functional on E. Clearly,

$$\psi(\nabla u_\alpha) \to \psi(\nabla u), \quad \text{as } \alpha \to \infty.$$

However, this is equivalent to writing $\phi(u_\alpha) \to \phi(u)$. We infer that $u_\alpha \rightharpoonup u$ in $\mathcal{D}_a^{1,p}(U, x_0)$.

In short, we have shown that weak convergece in $\mathcal{D}_a^{1,p}(U, x_0)$ is equivalent to the weak convergence of the gradients in the subspace E of $L_a^p(U, x_0)$. Now, since every continuous linear functional φ on E admits a continuous linear extension to the whole space $L_a^p(U, x_0)$ by the Hahn-Banach theorem, the assertion readily follows. □

Lemma 2.5.2. *Let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|$ and let E be a closed subspace of $\mathcal{X}$. If (x_α) is a sequence in E that converges weakly to $x \in \mathcal{X}$, then $x \in E$. Especially, $x_\alpha \rightharpoonup x$ in E.*

Proof. If $x \notin E$, the Hahn-Banach theorem guarantees the existence of a continuous linear functional $\phi : X \to \mathbb{R}$ vanishing on E such that $\phi(x) \neq 0$. However, this contradicts the fact that $\phi(x_\alpha) \to \phi(x)$ as $\alpha \to \infty$. Hence, we have $x \in E$. □

Given a bounded sequence in $\mathcal{D}_a^{1,p}(U, x_0)$, it is natural to ask whether there exists a subsequence that converges weakly in $\mathcal{D}_a^{1,p}(U,x_0)$.

Theorem 2.5.1. *Let (u_α) be a bounded sequence in $\mathcal{D}_a^{1,p}(U,x_0)$. There exists a function $u \in \mathcal{D}_a^{1,p}(U,x_0)$ and a subsequence (u_β) such that*

(1) $u_\beta \rightharpoonup u$ in $\mathcal{D}_a^{1,p}(U,x_0)$;

(2) $u_\beta \to u$ pointwise almost everywhere on U.

For simplicity and elegance, we divide the proof into two main steps.

Step 1. *Every bounded sequence (u_α) in $\mathcal{D}_a^{1,p}(U,x_0)$ has a subsequence converging weakly to a function in this same space.*

Proof of Step 1. First, note that (∇u_α) is bounded in $L_a^p(U,x_0)$. Now, as $L_a^p(U,x_0)$ is reflexive, there exists a vector $w \in L_a^p(U,x_0)$ and a subsequence (u_β) such that $\nabla u_\beta \rightharpoonup w$ in $L_a^p(U,x_0)$. Next, let

$$T : \mathcal{D}_a^{1,p}(U,x_0) \to L_a^p(U,x_0), \quad u \mapsto \nabla u$$

be the isometry used in the proof of Proposition 2.5.1. Since $\mathcal{D}_a^{1,p}(U,x_0)$ is complete, the image $\mathrm{Im}(T)$ is a closed subspace of $L_a^p(U,x_0)$. Citing Lemma 2.5.2, $\mathrm{Im}(T)$ is weakly closed whence $w \in \mathrm{Im}(T)$. Then, $u := T^{-1}(w)$ is the weak limit of (u_β) in $\mathcal{D}_a^{1,p}(U,x_0)$ by Proposition 2.5.1.

□

Next, we show that bounded weakly convergent sequences have subsequences that converge almost everywhere.

Step 2. *Let (u_α) be a sequence converging weakly to u in $\mathcal{D}_a^{1,p}(U,x_0)$. There exists a subsequence (u_β) that converges pointwise almost everywhere to u on U.*

Proof of Step 2. After an extension by zero outside of U, it follows from Proposition 2.5.1 that $u_\alpha \rightharpoonup u$ in $\mathcal{D}_a^{1,p}(\mathbb{R}^n,x_0)$. Thus, we can assume without loss of generality that $U = \mathbb{R}^n$. Now, for $k \geq 1$, we define A_k to be the open annulus

$$A_k := B(x_0,k+1) \setminus \overline{B(x_0,1/k)}$$

and observe that (u_α) forms a bounded sequence in $W^{1,p}(A_k)$. Following the proof used in Theorem 2.4.1, we can construct a family $\{(u_{\alpha,k})\}_{k\geq 1}$ of subsequences of (u_α) such that

(i) $(u_{\alpha,k+1})$ is a subsequence of $(u_{\alpha,k})$ for all $k \geq 1$;

(ii) $u_{\alpha,k}$ converges pointwise almost everywhere on A_k and *strongly* to a function $w_k \in L^p(A_k)$, for all $k \geq 1$.

In particular, $u_{\alpha,k}$ converges weakly to w_k in $L^p(A_k)$. Now, let ℓ be a continuous linear functional on $L^p(A_k)$. By (2.5) and L^p-inclusion theory, there exists a constant $C > 0$ such that, after a possible relabeling,

$$|\ell(u)| \leq C \|u\|_{L^p(A_k)} \leq C \|u\|_{L^q(A_k)} \leq C \|u\|_{L^q(A_k,|x-x_0|^{-bq})} \leq C \|u\|_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,x_0)}.$$

Put otherwise, the continuous linear functionals on $L^p(A_k)$ restrict to continuous linear functionals on $\mathcal{D}^{1,p}_a(\mathbb{R}^n, x_0)$. This implies that $u_{\alpha,k} \rightharpoonup u$ on $L^p(A_k)$ whence $w_k = u$ almost everywhere on A_k, for each $k \geq 1$. Clearly, the diagonal subsequence $\{u_{\alpha,\alpha}\}$ is the subsequence we seek. □

Finally, combining Step 1 and Step 2 implies Theorem 2.5.1 at once.

Corollary 2.5.3. *The space $\mathcal{D}^{1,p}_a(U, x_0)$ is a reflexive Banach space.*

As pointed out in the previous section, Theorem 2.5.1 yields the following reformulation of Lemma 2.4.5:

Theorem 2.5.2. *Let (u_α) be a bounded sequence in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, x_0)$. There exists a function $u \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, x_0)$ and a subsequence (u_β) converging strongly to u in $L^p_{loc}(\mathbb{R}^n, |x - x_0|^{-ap})$ and pointwise almost everywhere on $\mathbb{R}^n$, as $\beta \to \infty$. That is, the natural embedding $\mathcal{D}^{1,p}_a(\mathbb{R}^n, x_0) \hookrightarrow L^p_{loc}(\mathbb{R}^n, |x - x_0|^{-ap})$ is compact.*

Proof. By Theorem 2.5.1, we can assume that $u_\alpha \rightharpoonup u$ in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, x_0)$ and that $u_\alpha(x) \to u(x)$ for almost every $x \in \mathbb{R}^n$. Applying Lemma 2.4.5 to $u_\alpha - u$, given any ball $B = B(0, k)$, we can extract a subsequence of (u_α) converging strongly to u in $L^p(B, |x - x_0|^{-ap})$. The theorem then follows from a standard diagonal argument. □

2.5.1 A Riesz-Representation Theorem for $\mathcal{D}_a^{1,p}(U,x_0)$

We now provide a complete characterization of the bounded linear functionals on $\mathcal{D}_a^{1,p}(U,x_0)$. Let

$$T : \mathcal{D}_a^{1,p}(U,x_0) \to L_a^p(U,x_0), \quad u \mapsto \nabla u$$

be the canonical isometry and denote by $\mathrm{Im}(T)$ the image of T in $L_a^p(U,x_0)$. If ϕ is a continuous linear functional on $\mathcal{D}_a^{1,p}(U,x_0)$, then the composite

$$\phi \circ T^{-1} : \mathrm{Im}(T) \to \mathbb{R}$$

is a continuous linear functional on $\mathrm{Im}(T)$. Hence, by the Hahn-Banach theorem, $\phi \circ T^{-1}$ admits an extension to a continuous linear functional φ on $L_a^p(U,x_0)$. In fact, we can assume that $\|\varphi\|_{\mathrm{op}} = \|\phi \circ T^{-1}\|_{\mathrm{op}}$. Citing the classical Riesz-representation theorem, there exists $g = (g^1,\dots,g^n) \in L^{p'}(U,|x-x_0|^{-ap})$ such that

$$\varphi(f) = \int_U f \cdot g\,|x-x_0|^{-ap}\,\mathrm{d}x, \quad \forall f \in L_a^p(U,x_0)$$

and $\|g\|_{L^{p'}(U,|x-x_0|^{-ap})} = \|\varphi\|_{\mathrm{op}}$. In particular, for all $\nabla u \in \mathrm{Im}(T)$, we have

$$(\phi \circ T^{-1})(\nabla u) = \varphi(\nabla u) = \int_U \nabla u \cdot g\,|x-x_0|^{-ap}\,\mathrm{d}x.$$

Put otherwise, we have

$$\phi(u) = \int_U \nabla u \cdot g\,|x-x_0|^{-ap}\,\mathrm{d}x$$

for all $u \in \mathcal{D}_a^{1,p}(U,x_0)$. Finally, we observe that

$$\begin{aligned}\|g\|_{L^{p'}(U,|x-x_0|^{-ap})} = \|\varphi\|_{\mathrm{op}} = \left\|\phi \circ T^{-1}\right\|_{\mathrm{op}} &= \sup_{\substack{v \in \mathrm{Im}(T)\\ v \neq 0}} \frac{|(\phi \circ T^{-1})(v)|}{\|v\|_{L_a^p(U,x_0)}}\\ &= \sup_{\substack{u \in \mathcal{D}_a^{1,p}(U,x_0)\\ u \neq 0}} \frac{|\phi(u)|}{\|u\|_{\mathcal{D}_a^{1,p}(U,x_0)}}\end{aligned}$$

so that $\|g\|_{L^{p'}(U,|x-x_0|^{-ap})} = \|\phi\|_{\mathrm{op}}$. To summarize, we have established the following generalization of the Riesz-Representation Theorem for $W_0^{1,p}(U)$ (see Adams-Fournier [2]).

Theorem 2.5.3. *If ϕ is a continuous linear functional on $\mathcal{D}_a^{1,p}(U, x_0)$, there exist functions $g_1, \ldots, g_n \in L^{p'}(U, |x - x_0|^{-ap})$ such that*

$$\phi(u) = \sum_{j=1}^{n} \int_U g_i(x)\partial_i u(x) \, |x - x_0|^{-ap} \, \mathrm{d}x \tag{2.6}$$

for all $u \in \mathcal{D}_a^{1,p}(U, x_0)$. Furthermore, $\|\phi\|_{op} = \|(g_1, \ldots, g_n)\|_{L^{p'}(U,|x-x_0|^{-ap})}$.

2.6 Pointwise Convergence of the Gradients

In the previous section, we showed that bounded sequences in $\mathcal{D}_a^{1,p}(U, x_0)$ have subsequences that converge weakly and almost everywhere on U. In this section, we instead give conditions under which we can find a subsequence whose *gradients* converge pointwise. Following the ideas put forth in Mercuri-Willem [17], we require the following lemma of Alves [1].

Lemma 2.6.1. *Fix $1 < p < \infty$ and define $A : \mathbb{R}^n \to \mathbb{R}^n$ by the rule $A(y) := |y|^{p-2} y$. Let $\omega : U \to [0, \infty)$ be positive almost everywhere on U and Lebesgue measurable. Let μ be the measure given by $\mathrm{d}\mu := \omega \mathrm{d}x$. If (u_α) is bounded in $L^p(U, \mathrm{d}\mu)$ and u_α converges pointwise almost everywhere to a function u, then*

$$\lim_{\alpha \to \infty} \int_U |A(u_\alpha) - A(u) - A(u_\alpha - u)|^{p/(p-1)} \, \mathrm{d}\mu = 0.$$

Proof. The case $p \geq 2$ can be treated using a straightforward adaptation of the argument used in Lemma 3 of Alves [1]. When $1 \leq p < 2$, the proof from Lemma 3.2 in Mercuri-Willem [17] can be used without modification. □

We have the following notable immediate consequence:

Corollary 2.6.2. *Fix $1 < p < \infty$ and let A, ω be as in Lemma 2.6.1. Let μ be the measure given by $\mathrm{d}\mu := \omega \mathrm{d}x$. If $u_\alpha \to u$ strongly in $L^p(U, \mathrm{d}\mu)$ and pointwise almost everywhere as $\alpha \to \infty$, then*

$$\lim_{\alpha \to \infty} \int_U |A(u_\alpha) - A(u)|^{p/(p-1)} \, \mathrm{d}\mu = 0.$$

That is, $A(u_\alpha) \to A(u)$ in $L^{\frac{p}{p-1}}(U, \mathrm{d}\mu)$ as $\alpha \to \infty$.

Proof. By Lemma 2.6.1 it is clear that $\|A(u_\alpha) - A(u) - A(u_\alpha - u)\|_{L^{\frac{p}{p-1}}(\Omega, \mathrm{d}\mu)} = o(1)$ as $\alpha \to \infty$. Since $u_\alpha \to u$ strongly in $L^p(\Omega, \mathrm{d}\mu)$, we also have

$$\begin{aligned} \|A(u_\alpha - u)\|_{L^{\frac{p}{p-1}}(\Omega, \mathrm{d}\mu)} &= \left(\int_\Omega |u_\alpha - u|^{(p-1) \cdot \frac{p}{p-1}} \, \mathrm{d}\mu \right)^{(p-1)/p} \\ &= \|u_\alpha - u\|_{L^p(\Omega, \mathrm{d}\mu)}^{p-1} = o(1). \end{aligned}$$

The claim therefore follows from the Minkowski inequality for L^p-spaces. □

In order to establish our next result, we first make a crucial observation. For any $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, x_0)$ and $c \in \mathbb{R}$, we claim that

$$\nabla u \equiv 0 \quad \text{a.e. on } \{x \in \mathbb{R}^n : u(x) = c\}.$$

To see this, consider for each $k \in \mathbb{N}$ the open annulus

$$A_k = \left\{ x \in \mathbb{R}^n : \frac{1}{k} < |x - x_0| < k + 1 \right\}$$

and notice that $u \in W^{1,p}(A_k)$. It is therefore known that

$$m\left(A_k \cap \{x \in \mathbb{R}^n : u(x) = c, \nabla u(x) \neq 0\}\right) = 0,$$

with m being the Lebesgue measure on $\mathbb{R}^n$. Taking the union over all $k \in \mathbb{N}$, it readily follows that $m(\{x \in \mathbb{R}^n : u(x) = c, \nabla u(x) \neq 0\}) = 0$. This means that ∇u

vanishes almost everywhere on any level set of u.

Lemma 2.6.3. *Let $T : \mathbb{R} \to \mathbb{R}$ be the Lipschitz continuous function defined by the rule*

$$T(x) := \begin{cases} x & \text{if } |x| \leq 1, \\ \frac{x}{|x|} & \text{if } |x| > 1, \end{cases} \tag{2.7}$$

and fix a function $u \in \mathcal{D}^{1,p}_a(U, x_0)$. Then, $T(u) \in \mathcal{D}^{1,p}_a(U, x_0)$.

Remark 2.6.1. We mollify the function T using a sequence of standard mollifiers $\eta_{1/k}$ to obtain a sequence T_k of smooth functions that approximate T. It is well known (see Evans [14]) that since T is continuous, $T_k \to T$ locally uniformly. In fact, since T is constant for $x \leq -1$ and $x \geq 1$, it follows that $T_k \to T$ uniformly on all of $\mathbb{R}$.

Proof. Since T is piecewise smooth and $u \in W^{1,1}_{\text{loc}}(U \setminus \{x_0\})$, the composite $T(u)$ is weakly differentiable away from the singularity x_0 (see, for instance, Brézis [4] or Ziemer [26]). Moreover, the chain rule $\nabla(T(u)) = T'(u)\nabla u$ must hold away from x_0. Using these facts, it is clear that $T(u) \in \mathcal{E}^{1,p}_a(Ux_0)$. Indeed, since $|T(u)| \leq |u|$ and $|\nabla(T(u))| \leq |\nabla u|$,

$$\|T(u)\|_{L^q_b(U,x_0)} < \infty, \quad \|\nabla(T(u))\|_{L^p_a(U,x_0)} < \infty.$$

Mollifying T, we obtain a sequence of smooth functions (T_k) such that

(1) $T_k \to T$ uniformly on $\mathbb{R}$ as $k \to \infty$;

(2) $|T'_k(x)| \leq 1$ for all $x \in \mathbb{R}$ and all $k \in \mathbb{N}$;

(3) $T'_k(x) \to T'(x)$ as $k \to \infty$ for all $x \neq \pm 1$.[2]

Note that this last two properties hold because, since $T \in W^{1,\infty}(\mathbb{R})$, the derivative of T_k is precisely the mollification of T'. Furthermore, we can ensure that $T_k(0) = 0$

[2] For each $x \neq \pm 1$, the sequence $T'_k(x)$ is eventually equal to $T'(x)$.

for all $k \geq 1$. Next, let (u_α) be a sequence in $C_c^\infty(U)$ converging to u in $\mathcal{D}_a^{1,p}(U,x_0)$ and pointwise almost everywhere. Given any $k, \alpha \in \mathbb{N}$, we have

$$\begin{aligned}\|T_k(u_\alpha) - T(u)\|_{\mathcal{D}_a^{1,p}(U,x_0)} &\leq \|T_k(u_\alpha) - T(u_\alpha)\|_{\mathcal{D}_a^{1,p}(U,x_0)} + \|T(u_\alpha) - T(u)\|_{\mathcal{D}_a^{1,p}(U,x_0)} \\ &\leq \|T_k(u_\alpha) - T(u_\alpha)\|_{\mathcal{D}_a^{1,p}(U,x_0)} \\ &\quad + \|T'(u_\alpha)(\nabla u_\alpha - \nabla u)\|_{L_a^p(U,x_0)} \\ &\quad\quad + \|(T'(u_\alpha) - T'(u))\,\nabla u\|_{L_a^p(U,x_0)}\end{aligned}$$

We now fix $\varepsilon > 0$ and consider the set

$$E = \{x \in U : u(x) = \pm 1\}.$$

Since ∇u vanishes almost everywhere on E,

$$\|(T'(u_\alpha) - T'(u))\,\nabla u\|_{L_a^p(U,x_0)}^p = \int_{E^c} |T'(u_\alpha) - T'(u)|^p\,|\nabla u|^p\,|x - x_0|^{-ap}\,\mathrm{d}x.$$

Because $|T'(u_\alpha) - T'(u)|^p\,|\nabla u|^p \leq 2^p\,|\nabla u|^p$ and T' is continuous away from ± 1,

$$\lim_{\alpha \to \infty} \|(T'(u_\alpha) - T'(u))\,\nabla u\|_{L_a^p(U,x_0)}^p$$

by the Dominated Convergence Theorem. Therefore, for all $\alpha \in \mathbb{N}$ large,

$$\|(T'(u_\alpha) - T'(u))\,\nabla u\|_{L_a^p(U,x_0)} < \frac{\varepsilon}{3}. \tag{2.8}$$

Since $u_\alpha \to u$ in $\mathcal{D}_a^{1,p}(U,x_0)$, we also have

$$\|u_\alpha - u\|_{\mathcal{D}_a^{1,p}(U,x_0)} < \frac{\varepsilon}{3}$$

for all α large. Therefore, for all but finitely many α,

$$\|T'(u_\alpha)(\nabla u_\alpha - \nabla u)\|_{L_a^p(U,x_0)} \leq \|u_\alpha - u\|_{\mathcal{D}_a^{1,p}(U,x_0)} < \frac{\varepsilon}{3} \tag{2.9}$$

Fix any $\alpha \in \mathbb{N}$ such that (2.8)-(2.9) hold true. As ∇u_α vanishes on the level set $\{u_\alpha = \pm 1\}$, we have

$$\begin{aligned}
&\lim_{k\to\infty} \int_U |T_k'(u_\alpha) - T'(u_\alpha)|^p \, |\nabla u_\alpha(x)|^p \, |x - x_0|^{-ap} \, \mathrm{d}x \\
&\qquad = \lim_{k\to\infty} \int_U |T_k'(u_\alpha) - T'(u_\alpha)|^p \, |\nabla u_\alpha(x)|^p \, \mathbf{1}_{\{u_\alpha(x) \neq \pm 1\}} \, |x - x_0|^{-ap} \, \mathrm{d}x \\
&\qquad = 0
\end{aligned}$$

by the Dominated Convergence Theorem. It follows that

$$\|T_k(u_\alpha) - T(u_\alpha)\|_{\mathcal{D}_a^{1,p}(U,x_0)} < \frac{\varepsilon}{3}.$$

for all $k \in \mathbb{N}$ sufficiently large. Combining the above with equations (2.8) and (2.9), we have

$$\|T_k(u_\alpha) - T(u)\|_{\mathcal{D}_a^{1,p}(U,x_0)} < \varepsilon$$

As $\varepsilon > 0$ was arbitrary, this procedure outlines the construction of a subsequence (u_β) of (u_α) and a subsequence (T_β) of (T_k) such that

$$\lim_{\beta\to\infty} \|\nabla (T_\beta(u_\beta) - T(u))\|_{L_a^p(U,x_0)} = 0.$$

Since u_α has compact support in U and $T_k(0) = 0$ for each k, $T_k(u_\alpha) \in C_c^\infty(U)$. Especially, $T_k(u_\alpha) \in \mathcal{E}_{a,0}^{1,p}(U, x_0)$. Now, it follows from the above that $(T_\beta(u_\beta))$ is Cauchy in $\mathcal{E}_{a,0}^{1,p}(U, x_0)$. Therefore, there exists a function $v \in \mathcal{E}_{a,0}^{1,p}(U, x_0)$ such that

$$\lim_{\beta\to\infty} \|T_\beta(u_\beta) - v\|_{\mathcal{E}_a^{1,p}(U,x_0)} = 0.$$

Since $T_\beta \to T$ uniformly on $\mathbb{R}$ and $u_\beta \to u$ pointwise a.e. on U,

$$T_\beta(u_\beta) \to T(u)$$

almost everywhere on U, as $\beta \to \infty$. Ergo, $v = T(u)$ and $T(u) \in \mathcal{E}_{a,0}^{1,p}(U, x_0)$. $\square$

Thanks to this, the following is within reach:

Lemma 2.6.4. *Let (u_α) be a bounded sequence in $\mathcal{E}_a^{1,p}(U,x_0)$ converging pointwise almost everywhere to $u \in \mathcal{E}_a^{1,p}(U,x_0)$, as $\alpha \to \infty$. Let $T : \mathbb{R} \to \mathbb{R}$ be the Lipschitz continuous function defined in (2.7). If*

$$\lim_{\alpha\to\infty} \int_U \left(|\nabla u_\alpha|^{p-2} \nabla u_\alpha - |\nabla u|^{p-2} \nabla u\right) \cdot \nabla T(u_\alpha - u)\, |x - x_0|^{-ap}\, \mathrm{d}x = 0, \tag{2.10}$$

then there exists a subsequence (u_β) such that $\nabla u_\beta \to \nabla u$ pointwise almost everywhere on U.

Proof. We adapt the proof from Szulkin-Willem [21]. For each $\alpha \in \mathbb{N}$ we consider the set

$$E_\alpha := \{x \in U : |u_\alpha(x) - u(x)| \leq 1\}.$$

Then, (2.10) can be written as

$$\lim_{\alpha\to\infty} \int_{E_\alpha} \left(|\nabla u_\alpha|^{p-2} \nabla u_\alpha - |\nabla u|^{p-2} \nabla u\right) \cdot (\nabla u_\alpha - \nabla u)\, |x - x_0|^{-ap}\, \mathrm{d}x = 0. \tag{2.11}$$

Since the integrand is non-negative and $\mathbf{1}_{E_\alpha}(x) \to 1$ for almost every $x \in U$, passing to a subsequence if necessary, we see that

$$\left(|\nabla u_\alpha|^{p-2} \nabla u_\alpha - |\nabla u|^{p-2} \nabla u\right)(\nabla u_\alpha - \nabla u) \to 0$$

pointwise almost everywhere on U. Appealing to Lemma 2.1 from Szulkin-Willem [21], we infer that $\nabla u_\alpha \to \nabla u$ pointwise a.e. on U. □

Thus, we obtain a weighted version of Theorem 3.3 in Mercuri-Willem [17].

Theorem 2.6.1. *Let (u_α) be a bounded sequence in $\mathcal{D}_a^{1,p}(U,x_0)$ converging pointwise almost everywhere to $u \in \mathcal{D}_a^{1,p}(U,x_0)$ as $\alpha \to \infty$. Let T be defined as in Lemma 2.6.4 and assume that (U_k) is an increasing sequence of open subsets of U such that*

$\bigcup_{k\geq 1} U_k = U$. *Assume further that*

$$\lim_{\alpha\to\infty}\int_{U_k}\left(|\nabla u_\alpha|^{p-2}\nabla u_\alpha - |\nabla u|^{p-2}\nabla u\right)\cdot\nabla T(u_\alpha - u)\,\mathrm{d}\mu = 0$$

for each $k \geq 1$, *where* $\mathrm{d}\mu = |x-x_0|^{-ap}\,\mathrm{d}x$. *Then, there exists a subsequence* (u_β) *with the following properties:*

(1) $\nabla u_\beta \to \nabla u$ *pointwise almost everywhere on* U;

(2) $\lim_{\beta\to\infty}\left(\|u_\beta\|^p_{\mathcal{D}^{1,p}_a(U,x_0)} - \|u_\beta - u\|^p_{\mathcal{D}^{1,p}_a(U,x_0)}\right) = \|u\|^p_{\mathcal{D}^{1,p}_a(U,x_0)}$;

(3) and

$$|\nabla u_\beta|^{p-2}\nabla u_\beta - |\nabla u_\beta - \nabla u|^{p-2}(\nabla u_\beta - \nabla u) \to |\nabla u|^{p-2}\nabla u$$

strongly in $L^{\frac{p}{p-1}}(U,\mathrm{d}\mu)$.

Proof. By using a diagonal argument together with successive applications of Lemma 2.6.4, it is easy to see that (1) holds true. Moreover, we may assume by Theorem 2.5.1 that $u_\alpha \rightharpoonup u$ in $\mathcal{D}^{1,p}_a(U,x_0)$. Next, we note that (2) is a direct consequence of the Brézis-Lieb Lemma (Theorem 2.1.1). Finally, (3) follows from Lemma 2.6.1. □

2.7 Homogeneity

For simplicity take a function $u \in \mathcal{E}^{1,p}_a(\mathbb{R}^n, 0)$, fix $x_0 \in \mathbb{R}^n$, and let $\lambda > 0$. Consider the following rescaling of u:

$$v(x) := \lambda^\gamma u(\lambda(x + x_0)), \tag{2.12}$$

with $\gamma > 0$ being the *homogeneity exponent* defined by

$$\gamma := \frac{n - bq}{q} = \frac{n}{q} - b = \frac{n - p(1 + a - b)}{p} - b = \frac{n - p(1 + a)}{p}. \tag{2.13}$$

Since ∇u exists in the weak sense away from the origin, it is not hard to verify that ∇v exists weakly away from the singular point $-x_0$. Now, after a simple change of variables, we find that

$$\int_{\mathbb{R}^n} |v(x)|^q |x+x_0|^{-bq} \,\mathrm{d}x = \lambda^{\gamma q} \int_{\mathbb{R}^n} |u(\lambda(x+x_0)|^q |x+x_0|^{-bq} \,\mathrm{d}x \tag{2.14}$$

$$= \lambda^{n-bq} \int_{\mathbb{R}^n} |u(\lambda(x+x_0)|^q |x+x_0|^{-bq} \,\mathrm{d}x \tag{2.15}$$

$$= \lambda^{-bq} \int_{\mathbb{R}^n} |u(z)|^q \left|\frac{z}{\lambda}\right|^{-bq} \mathrm{d}z \tag{2.16}$$

$$= \int_{\mathbb{R}^n} |u(z)|^q |z|^{-bq} \,\mathrm{d}z. \tag{2.17}$$

Similarly, a straightforward calculation yields

$$\int_{\mathbb{R}^n} |\nabla v(x)|^p |x+x_0|^{-ap} \,\mathrm{d}x = \lambda^{(\gamma+1)p} \int_{\mathbb{R}^n} |\nabla u(\lambda(x+x_0))|^p |x+x_0|^{-ap} \,\mathrm{d}x \tag{2.18}$$

$$= \lambda^{n-ap} \int_{\mathbb{R}^n} |\nabla u(\lambda(x+x_0))|^p |x+x_0|^{-ap} \,\mathrm{d}x \tag{2.19}$$

$$= \int_{\mathbb{R}^n} |\nabla u(z)|^p |z|^{-ap} \,\mathrm{d}z. \tag{2.20}$$

Therefore, we see that the rescaling in (2.12) satisfies

$$\|v\|_{L_b^q(\mathbb{R}^n,-x_0)} = \|u\|_{L_b^q(\mathbb{R}^n,0)} \quad \text{and} \quad \|\nabla v\|_{L_a^p(\mathbb{R}^n,-x_0)} = \|\nabla u\|_{L_a^p(\mathbb{R}^n,0)}.$$

In particular, $v \in \mathcal{E}_a^{1,p}(\mathbb{R}^n, -x_0)$. This invariance property (known as homogeneity) will live at the heart of our main result. Equally important, however, is the following simple observation:

Lemma 2.7.1. *Let $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$, fix $x_0 \in \mathbb{R}^n$ and let $\lambda > 0$. If we define $v \in \mathcal{E}_a^{1,p}(\mathbb{R}^n, -x_0)$ by (2.12), then $v \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$.*

Proof. By definition of $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$, there exists a sequence (φ_α) of smooth functions with compact support in $\mathbb{R}^n$ converging to u in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$ as $\alpha \to \infty$. Clearly, for each $\alpha \in \mathbb{N}$, the function $\psi_\alpha(x) := \lambda^\gamma \varphi_\alpha(\lambda(x+x_0))$ also belongs to $C_c^\infty(\mathbb{R}^n)$. By the calculations carried out above, it follows that $\psi_\alpha \to v$ in $\mathcal{E}_a^{1,p}(\mathbb{R}^n, -x_0)$. Therefore, $v \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$ as was asserted. □

Chapter 3

A Compactness Result for Critical Weighted p-Laplace Equations

We now turn towards the original problem discussed within the first chapter. Namely, we consider the problem (1.7) when $\Omega \subset \mathbb{R}^n$ is a non-empty bounded domain containing the origin. Note that we do not impose any smoothness or regularity assumptions on the boundary of Ω. In this chapter, we first discuss the basic technical properties of Palais-Smale sequences for this problem and identify the possible limiting problems that our "bubbles" will solve. We then apply these technical results and the theory developed in the previous chapter to conduct a concentration/compactness analysis for Palais-Smale sequences associated to (1.7). The results presented within this chapter are based on a work in progress and part of a paper in preparation (Chernysh [8]).

3.1 The Weak Formulation of The Problem in Ω

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain containing the origin. For the sake of clarity with regards to the current literature, recall that $\mathcal{D}_a^{1,p}(\Omega, 0)$ is precisely the space $\mathcal{D}^{1,p}(\Omega, |x|^{-ap})$ from (1.7).

Definition 3.1.1. A function $u \in \mathcal{D}_a^{1,p}(\Omega, 0)$ is said to be a *weak solution* to the weighted critical p-Laplace problem (1.7) whenever one has

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla h \, |x|^{-ap} \, \mathrm{d}x = \int_{\Omega} |u|^{q-2} uh \, |x|^{-bq} \, \mathrm{d}x \tag{3.1}$$

for all test functions $h \in C_c^{\infty}(\Omega)$.

Consider now the *nonlinear* functional $\phi : \mathcal{D}_a^{1,p}(\Omega, 0) \to \mathbb{R}$ given by the rule

$$\phi(u) := \frac{1}{p} \int_{\Omega} \frac{|\nabla u|^p}{|x|^{ap}} \mathrm{d}x - \frac{1}{q} \int_{\Omega} \frac{|u|^q}{|x|^{bq}} \mathrm{d}x.$$

By (2.5), it is easy to see that ϕ is continuous on $\mathcal{D}_a^{1,p}(\Omega, 0)$. In fact, it is readily seen that ϕ is Fréchet differentiable on the space $\mathcal{D}_a^{1,p}(\Omega, 0)$ with derivative

$$\langle \phi'(u), h \rangle = \int_{\Omega} \Big(|\nabla u|^{p-2} \nabla u \cdot \nabla h \, |x|^{-ap} - |u|^{q-2} uh \, |x|^{-bq} \Big) \, \mathrm{d}x \tag{3.2}$$

for all $u, h \in \mathcal{D}_a^{1,p}(\Omega, 0)$. Note that by Hölder's inequality, for each fixed function $u \in \mathcal{D}_a^{1,p}(\Omega, 0)$, the map $\langle \phi'(u), \cdot \rangle$ is a bounded linear functional on $\mathcal{D}_a^{1,p}(\Omega, 0)$. Thus, since $C_c^{\infty}(\Omega)$ is dense in $\mathcal{D}_a^{1,p}(\Omega, 0)$ by definition, we see that u is a weak solution to (1.7) if and only if

$$\langle \phi'(u), h \rangle = 0, \quad \forall h \in \mathcal{D}_a^{1,p}(\Omega, 0).$$

The functional ϕ is called the *energy functional* associated to the problem (1.7).

In light of this, whilst attempting to find a solution to the problem (1.7), it is natural to search for critical points of the energy functional ϕ. With this in mind, the following definition is easily motivated.

Definition 3.1.2 (Palais-Smale Sequences)**.** Let (u_α) be a sequence in $\mathcal{D}_a^{1,p}(\Omega, 0)$. We say that (u_α) is a Palais-Smale sequence for (1.7) provided each of the following hold true:

(I) $\phi(u_\alpha)$ is bounded uniformly in α;

(II) $\phi'(u_\alpha) \to 0$ strongly in $\mathcal{D}_a^{-1,p'}(\Omega, 0)$ as $\alpha \to \infty$.

In particular, subsequences of Palais-Smale sequences are again Palais-Smale.

It is not a priori obvious that a Palais-Smale sequence (for (1.7)) is bounded in $\mathcal{D}_a^{1,p}(\Omega, 0)$. In the next result, we show that this is indeed the case.

Proposition 3.1.1. *A Palais-Smale sequence* (u_α) *for* (1.7) *is bounded.*

Proof. We follow the argument used in Struwe [19, Lemma 2.3]. Let (u_α) be a Palais-Smale sequence for (1.7). A simple calculation shows that

$$\begin{aligned} \|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} &= p\phi(u_\alpha) + \frac{p}{q}\int_\Omega \frac{|u_\alpha|^q}{|x|^{bq}}\mathrm{d}x \\ &\leq C + \frac{p}{q}\int_\Omega \frac{|u_\alpha|^q}{|x|^{bq}}\mathrm{d}x \end{aligned}$$

where we have used that $\phi(u_\alpha)$ is bounded in α. Next, observe that

$$p\phi(u_\alpha) - \langle \phi'(u_\alpha), u_\alpha\rangle = \left(1 - \frac{p}{q}\right)\int_\Omega \frac{|u_\alpha|^q}{|x|^{bq}}\mathrm{d}x$$

where the left hand side is bounded in absolute value by

$$C + o(1)\,\|u_\alpha\|_{\mathcal{D}_a^{1,p}(\Omega,0)}$$

by virtue of (I) and (II). Put otherwise, we have

$$\int_\Omega \frac{|u_\alpha|^q}{|x|^{bq}}\mathrm{d}x \leq \widetilde{C} + o(1)\,\|u_\alpha\|_{\mathcal{D}_a^{1,p}(\Omega,0)}$$

for a suitable constant $\widetilde{C} > 0$. It then follows that

$$\begin{aligned} \|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} &\leq C + \frac{p}{q}\int_\Omega \frac{|u_\alpha|^q}{|x|^{bq}}\mathrm{d}x \\ &\leq C + \frac{p}{q}\widetilde{C} + o(1)\,\|u_\alpha\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \end{aligned}$$

whence (u_α) is bounded in $\mathcal{D}_a^{1,p}(\Omega, 0)$. □

Using this, we show that in the limit Palais-Smale sequences for ϕ are uniformly bounded from below by zero in energy. More precisely, we have the following:

Lemma 3.1.2. *Let (u_α) be a Palais-Smale sequence for (1.7). Then, (u_α) has non-negative limiting energy. More precisely,*

$$\liminf_{\alpha\to\infty} \phi(u_\alpha) \geq 0.$$

Proof. Since (u_α) is bounded and a Palais-Smale sequence, it is readily seen that

$$\langle \phi'(u_\alpha), u_\alpha \rangle = \int_\Omega |\nabla u_\alpha|^p \, |x|^{-ap} \, \mathrm{d}x - \int_\Omega |u_\alpha|^q \, |x|^{-bq} \, \mathrm{d}x \to 0$$

as $\alpha \to \infty$. Therefore, given $\varepsilon > 0$, we have

$$\|\nabla u_\alpha\|^p_{L^p_a(\Omega,0)} - \|u_\alpha\|^q_{L^q_b(\Omega,0)} > -\varepsilon$$

for all $\alpha \in \mathbb{N}$ large. Consequently, for all such α,

$$\begin{aligned} \phi(u_\alpha) &= \frac{\|\nabla u_\alpha\|^p_{L^p_a(\Omega,0)}}{p} - \frac{\|u_\alpha\|^q_{L^q_b(\Omega,0)}}{q} \\ &> \left(\frac{1}{p} - \frac{1}{q}\right) \|u_\alpha\|^q_{L^q_b(\Omega,0)} - \frac{\varepsilon}{p} \\ &\geq -\frac{\varepsilon}{p}. \end{aligned}$$

Hence, we obtain

$$\liminf_{\alpha\to\infty} \phi(u_\alpha) \geq -\frac{\varepsilon}{p}.$$

Since $\varepsilon > 0$ was arbitrary, the claim follows. □

3.2 Technical Lemmas

Here we establish two main ingredients for the proof of our main result. The first lemma allows us to extract a solution to the original problem (1.7) in Ω, while the second result extracts the bubbles our Palais-Smale sequence may encode.

Lemma 3.2.1. *Let* (u_α) *in* $\mathcal{D}_a^{1,p}(\Omega, 0)$ *be a Palais-Smale sequence for* ϕ *such that, as* $\alpha \to \infty$,

(1) $u_\alpha \rightharpoonup u$ *in* $\mathcal{D}_a^{1,p}(\Omega, 0)$;

(2) $u_\alpha \to u$ *a.e. on* Ω;

(3) $\phi(u_\alpha) \to c$.

Then, passing to a subsequence if necessary, there holds

(i) $\nabla u_\alpha \to \nabla u$ *a.e. on* Ω *and* $\phi'(u) = 0$;

(ii) $\|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} - \|u_\alpha - u\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} = \|u\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} + o(1)$;

(iii) $\phi(u_\alpha - u) \to c - \phi(u)$ *as* $\alpha \to \infty$.

Furthermore, one has

$$\phi'(u_\alpha - u) \to 0$$

strongly in $\mathcal{D}_a^{-1,p'}(\Omega, 0)$.

Proof. We borrow ideas from Mercuri-Willem [17]. Let $T : \mathbb{R} \to \mathbb{R}$ be the Lipschitz continuous function defined in (2.7) and note that $|T'| \leq 1$ almost everywhere. Now, since the functions $|x|^{-ap}$ and $|x|^{-bq}$ are bounded above and below locally by positive constants away from the origin, $\mathcal{D}_a^{1,p}(\Omega, 0) \subseteq W^{1,p}_{\text{loc}}(\Omega \setminus \{0\})$. Furthermore, it is a consequence of Lemma 2.6.3 that T maps $\mathcal{D}_a^{1,p}(\Omega, 0)$ back to itself, i.e. there holds $T(v) \in \mathcal{D}_a^{1,p}(\Omega, 0)$ for all $v \in \mathcal{D}_a^{1,p}(\Omega, 0)$.

Let us now define $v_\alpha := u_\alpha - u$. Since T is continuous and bounded, the Dominated Convergence Theorem implies that

$$\int_\Omega |T(v_\alpha)|^r \, |x|^{-bq} \, \mathrm{d}x \to 0, \quad \text{as } \alpha \to \infty \tag{3.3}$$

for every $0 < r < \infty$. As (v_α) is bounded in $\mathcal{D}_a^{1,p}(\Omega, 0)$, it is easy to check that $(T(v_\alpha))$ is bounded in $\mathcal{D}_a^{1,p}(\Omega, 0)$ as well. Citing Theorem 2.5.1 and passing to a subsequence, we may assume that $T(v_\alpha) \rightharpoonup \eta$ in $\mathcal{D}_a^{1,p}(\Omega,0)$ and pointwise almost everywhere on Ω. Since $v_\alpha \to 0$ almost everywhere on Ω, we must have $\eta = 0$. Using this, we will show that, up to a subsequence, $\nabla u_\alpha \to \nabla u$ pointwise almost everywhere on Ω.

With the hope of applying Theorem 2.6.1, let us consider for each $\alpha \in \mathbb{N}$ the expression

$$\Gamma_\alpha := \int_\Omega \left(|\nabla u_\alpha|^{p-2} \nabla u_\alpha - |\nabla u|^{p-2} \nabla u \right) \cdot \nabla T(u_\alpha - u) \, |x|^{-ap} \, \mathrm{d}x.$$

Clearly, Γ_α can be written as follows:

$$\begin{aligned} \langle \phi'(u_\alpha), T(u_\alpha - u) \rangle - \underbrace{\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla T(u_\alpha - u) \, |x|^{-ap} \, \mathrm{d}x}_{=:I_1} \\ + \underbrace{\int_\Omega |u_\alpha|^{q-2} u_\alpha T(u_\alpha - u) \, |x|^{-bq} \, \mathrm{d}x}_{:=I_2}. \end{aligned}$$

Now, since $\phi'(u_\alpha)$ converges strongly to zero in $\mathcal{D}_a^{-1,p'}(\Omega,0)$ and

$$T(u_\alpha - u) = T(v_\alpha)$$

is bounded in $\mathcal{D}_a^{1,p}(\Omega,0)$, the first term $\langle \phi'(u_\alpha), T(u_\alpha - u) \rangle$ converges to 0. Next, the estimate

$$\begin{aligned} |I_1| &\le \left(\int_\Omega |\nabla T(u_\alpha - u)|^p \, |x|^{-ap} \, \mathrm{d}x \right)^{1/p} \left(\int_\Omega |\nabla u|^p \, |x|^{-ap} \, \mathrm{d}x \right)^{(p-1)/p} \\ &= \|T(v_\alpha)\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \left(\int_\Omega |\nabla u|^p \, |x|^{-ap} \, \mathrm{d}x \right)^{(p-1)/p} \end{aligned}$$

shows that the map

$$f \mapsto \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla f \, |x|^{-ap} \, \mathrm{d}x$$

is a continuous linear functional on $\mathcal{D}_a^{1,p}(\Omega,0)$. Using that $T(v_\alpha) \rightharpoonup 0$ in $\mathcal{D}_a^{1,p}(\Omega,0)$, it follows that $I_1 \to 0$ as well. To handle the final term, we observe that, by Hölder's inequality,

$$
\begin{aligned}
|I_2| &\leq \int_\Omega |u_\alpha|^{q-1} |T(v_\alpha)| \, |x|^{-bq} \, \mathrm{d}x \\
&\leq \left(\int_\Omega |T(v_\alpha)|^q \, |x|^{-bq} \, \mathrm{d}x \right)^{1/q} \left(\int_\Omega |u_\alpha|^q \, |x|^{-bq} \, \mathrm{d}x \right)^{(q-1)/q}
\end{aligned}
$$

where (u_α) is bounded in $L^q_b(\Omega, 0)$ by (CKN) and

$$
\int_\Omega |T(v_\alpha)|^q \, |x|^{-bq} \, \mathrm{d}x \to 0
$$

as $\alpha \to \infty$ by (3.3). Consequently, we see that $\Gamma_\alpha \to 0$ as $\alpha \to \infty$. Applying Theorem 2.6.1, we can assume that $\nabla u_\alpha \to \nabla u$ almost everywhere on Ω as $\alpha \to \infty$. Next, we assert that $\phi'(u)$ vanishes on $\mathcal{D}^{1,p}_a(\Omega, 0)$. That is, we claim that

$$
\langle \phi'(u), h \rangle = 0
$$

for all $h \in \mathcal{D}^{1,p}_a(\Omega, 0)$. Since $\phi'(u_\alpha) \to 0$ strongly in $\mathcal{D}^{-1,p'}_a(\Omega, 0)$, it would be enough to check that

$$
\langle \phi'(u_\alpha), h \rangle \to \langle \phi'(u), h \rangle
$$

as $\alpha \to \infty$ for each *fixed* $h \in \mathcal{D}^{1,p}_a(\Omega, 0)$. Fixing $h \in \mathcal{D}^{1,p}_a(\Omega, 0)$, we calculate

$$
\begin{aligned}
&\langle \phi'(u_\alpha), h \rangle - \langle \phi'(u), h \rangle \\
&\qquad = \underbrace{\int_\Omega \left(|\nabla u_\alpha|^{p-2} \nabla u_\alpha - |\nabla u|^{p-2} \nabla u \right) \cdot \nabla h \, |x|^{-ap} \, \mathrm{d}x}_{=: J_1} \\
&\qquad - \underbrace{\int_\Omega \left(|u_\alpha|^{q-2} u_\alpha - |u|^{q-2} u \right) h \, |x|^{-bq} \, \mathrm{d}x}_{=: J_2}
\end{aligned}
$$

where both J_1 and J_2 converge to 0 as $\alpha \to \infty$ by Theorem 2.1.2. It follows that $\phi'(u) = 0$ and (i) is therefore proven. The second conclusion follows at once from

Theorem 2.6.1-(ii). Combining this with the Brézis-Lieb lemma gives

$$\begin{aligned}\phi(v_\alpha) &= \int_\Omega \left[\frac{|x|^{-ap} |\nabla v_\alpha|^p}{p} - \frac{|x|^{-bq} |v_\alpha|^q}{q} \right] \mathrm{d}x \\ &= \frac{1}{p} \|u_\alpha - u\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} - \frac{1}{q} \|u_\alpha - u\|^q_{L^q_b(\Omega,0)} \\ &= \frac{1}{p} \left(\|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} - \|u\|^p_{\mathcal{D}^{1,p}_a(\Omega,0)} \right) - \frac{1}{q} \left(\|u_\alpha\|^q_{L^q_b(\Omega,0)} - \|u\|^q_{L^q_b(\Omega,0)} \right) \\ &\qquad + o(1) \\ &= \phi(u_\alpha) - \phi(u) + o(1) \\ &\to c - \phi(u).\end{aligned}$$

Finally, fix $h \in \mathcal{D}^{1,p}_a(\Omega, 0)$ with $\|h\| = 1$. The last claim follows by applying Hölder's inequality and Lemma 2.6.1. □

3.2.1 An Iterative Bubbling Lemma

We now identify the limiting problems that our bubbles can solve. The results that we present in this section will allow us to iterate within the proof of our main theorem.

Given a point $x_0 \in \mathbb{R}^n$ we define the functional

$$\phi_{x_0,\infty}(u) := \int_{\mathbb{R}^n} \left(\frac{|\nabla u(x)|^p}{p} |x + x_0|^{-ap} - \frac{|u(x)|^q}{q} |x + x_0|^{-bq} \right) \mathrm{d}x$$

for $u \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, -x_0)$. As before, $\phi_{x_0,\infty}$ is Fréchet differentiable on $\mathcal{D}^{1,p}_a(\mathbb{R}^n, -x_0)$ with derivative given by

$$\begin{aligned}\left\langle \phi'_{x_0,\infty}(u), h \right\rangle = &\int_{\mathbb{R}^n} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla h(x) \, |x + x_0|^{-ap} \, \mathrm{d}x \\ &- \int_{\mathbb{R}^n} |u(x)|^{q-2} u(x) h(x) \, |x + x_0|^{-bq} \, \mathrm{d}x\end{aligned}$$

for $u, h \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, -x_0)$. The map $\phi_{x_0,\infty}$ is the energy functional corresponding to the limiting problem

$$\begin{cases} -\operatorname{div}\left(|x+x_0|^{-ap}\,|\nabla u|^{p-2}\,\nabla u\right) = |x+x_0|^{-bq}\,|u|^{q-2}\,u \quad \text{in } \mathbb{R}^n, \\ u \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0). \end{cases} \tag{3.4}$$

Before proceeding further we check that every non-trivial critical point of $\phi_{x_0,\infty}$ has strictly positive energy. More precisely, we establish the following result:

Proposition 3.2.2. *Let $C > 0$ be such that (2.5) holds true for all $x_0 \in \mathbb{R}^n$. Fix a point $x_0 \in \mathbb{R}^n$ and let $u \neq 0$ be a critical point of $\phi_{x_0,\infty}$. Then,*

$$C^{-\frac{n}{1+a-b}}\left(\frac{1+a-b}{n}\right) \leq \phi_{x_0,\infty}(u).$$

Proof. Since $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$ is a critical point of $\phi_{x_0,\infty}$, we have

$$0 = \left\langle \phi'_{x_0,\infty}(u), u\right\rangle = \int_{\mathbb{R}^n} |\nabla u|^p\,|x+x_0|^{-ap}\,\mathrm{d}x - \int_{\mathbb{R}^n} |u|^q\,|x+x_0|^{-bq}\,\mathrm{d}x.$$

Therefore, by (2.5),

$$C^{-p}\,\|u\|^p_{L^q_b(\mathbb{R}^n,-x_0)} \leq \|\nabla u\|^p_{L^p_a(\mathbb{R}^n,-x_0)} = \|u\|^q_{L^q_b(\mathbb{R}^n,-x_0)}$$

whence

$$0 < C^{-p} \leq \|u\|^{q-p}_{L^q_b(\mathbb{R}^n,-x_0)}.$$

Or, rather,

$$0 < C^{-\frac{pq}{(q-p)}} \leq \|u\|^q_{L^q_b(\mathbb{R}^n,-x_0)}.$$

It follows that

$$\begin{aligned} \phi_{x_0,\infty}(u) &= \frac{\|\nabla u\|^p_{L^p_a(\mathbb{R}^n,-x_0)}}{p} - \frac{\|u\|^q_{L^q_b(\mathbb{R}^n,-x_0)}}{q} \\ &= \frac{\|u\|^q_{L^q_b(\mathbb{R}^n,-x_0)}}{p} - \frac{\|u\|^q_{L^q_b(\mathbb{R}^n,-x_0)}}{q} \end{aligned}$$

whence

$$
\begin{aligned}
\phi_{x_0,\infty}(u) = \|u\|^q_{L^q_b(\mathbb{R}^n,-x_0)}\left(\frac{1}{p}-\frac{1}{q}\right) &\geq C^{\frac{pq}{-(q-p)}}\left(\frac{1}{p}-\frac{1}{q}\right) \\
&= C^{-\frac{n}{1+a-b}}\frac{1+a-b}{n} \\
&> 0.
\end{aligned}
$$

This completes the proof. □

We now give an iterative result for $\phi_{x_0,\infty}$ that is similar to Lemma 3.2.1. First, however, we introduce the notion of a *bubble* for problem (1.7).

Definition 3.2.1. Let (λ_α) be a sequence of positive real numbers converging to 0 as $\alpha \to \infty$. A 0-*bubble*[1] (B_α) associated to (λ_α) is a sequence of functions in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$ of the form

$$B_\alpha(x) := \lambda_\alpha^{-\gamma} v\left(\frac{x}{\lambda_\alpha}\right)$$

with $v \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$ solving

$$
\begin{cases}
-\operatorname{div}\left(|x|^{-ap}\,|\nabla u|^{p-2}\,\nabla u\right) = |x|^{-bq}\,|u|^{q-2}\,u \quad \text{in } \mathbb{R}^n, \\
u \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)
\end{cases}
$$

and $\gamma > 0$ given by (2.13).

Remark 3.2.1. We note that, up to a subsequence, every 0-bubble (B_α) must satisfy

(i) $B_\alpha \rightharpoonup 0$ in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$ as $\alpha \to \infty$;

(ii) $B_\alpha(x) \to 0$ for almost every $x \in \mathbb{R}^n$;

(iii) $\nabla B_\alpha(x) \to 0$ pointwise almost everywhere on $\mathbb{R}^n$.

[1] The term 0-bubble is used to emphasize that our bubbles always concentrate near the origin.

Indeed, fix $\varepsilon > 0$ and compute

$$\begin{aligned}\int_{B(0,\varepsilon)^{\mathrm{c}}} |\nabla B_\alpha(x)|^p\,|x|^{-ap}\,\mathrm{d}x &= \lambda_\alpha^{-(\gamma+1)p} \int_{B(0,\varepsilon)^{\mathrm{c}}} \left|\nabla v\left(\frac{x}{\lambda_\alpha}\right)\right|^p |x|^{-ap}\,\mathrm{d}x \\ &= \lambda_\alpha^{-(\gamma+1)p+n-ap} \int_{B\left(0,\frac{\varepsilon}{\lambda_\alpha}\right)^{\mathrm{c}}} |\nabla v\,(z)|^p\,|z|^{-ap}\,\mathrm{d}z \\ &= \int_{B\left(0,\frac{\varepsilon}{\lambda_\alpha}\right)^{\mathrm{c}}} |\nabla v\,(z)|^p\,|z|^{-ap}\,\mathrm{d}z\end{aligned}$$

where the last term converges to 0 by the monotone convergence theorem. Passing to a subsequence, we infer that $\nabla B_\alpha(x) \to 0$ pointwise almost everywhere on $\mathbb{R}^n \setminus B(0,\varepsilon)$. As $\varepsilon > 0$ was arbitrary, a diagonal argument gives the existence of a subsequence such that $\nabla B_\alpha \to 0$ almost everywhere on $\mathbb{R}^n$. Since (∇B_α) is bounded in $L^p_a(\mathbb{R}^n, 0)$ by a simple change of variables, applying Theorem 2.1.2 and Proposition 2.5.1 implies that $B_\alpha \rightharpoonup 0$ in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$. By Theorem 2.5.1, there is a further subsequence converging to zero pointwise almost everywhere.

Lemma 3.2.3. *Fix a bounded sequence (u_α) in $\mathcal{D}^{1,p}_a(\Omega, 0)$ and extend it by zero outside of Ω. Fix $x_0 \in \mathbb{R}^n$ and let $(\lambda_\alpha) \subset (0,\infty)$ be such that $\lambda_\alpha \to 0$ as $\alpha \to \infty$. Given $\alpha \in \mathbb{N}$, let us define*

$$v_\alpha(x) := \lambda_\alpha^\gamma u_\alpha(\lambda_\alpha(x + x_0))$$

with $\gamma > 0$ as in (2.13) and assume that

(1) $v_\alpha \rightharpoonup v$ in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, -x_0)$ and pointwise almost everywhere as $\alpha \to \infty$;

(2) $\phi(u_\alpha) \to c$ and $\phi'(u_\alpha) \to 0$ strongly in the dual of $\mathcal{D}^{1,p}_a(\Omega, 0)$.

Then, passing to a subsequence if necessary, we have $\nabla v_\alpha \to \nabla v$ pointwise almost everywhere on $\mathbb{R}^n$. Furthermore, let $\phi_{x_0,\infty}$ be the energy functional associated to the limiting problem (3.4). Then $\phi'_{x_0,\infty}(v) = 0$. Finally, the sequence in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$

$$w_\alpha(z) := u_\alpha(z) - (\lambda_\alpha)^{-\gamma} v\left(\frac{z}{\lambda_\alpha} - x_0\right)$$

satisfies

(i) $\|u_\alpha\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)} - \|w_\alpha\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)} = \|v\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,-x_0)} + o(1)$;

(ii) $\phi_{0,\infty}(w_\alpha) \to c - \phi_{x_0,\infty}(v)$;

(iii) $\phi'_{0,\infty}(w_\alpha) \to 0$ *strongly in* $\mathcal{D}^{-1,p'}_a(\Omega, 0)$.

Proof. We adapt the argument from Mercuri-Willem [17]. Given $k \in \mathbb{N}$, let B_k denote the open ball $B(0,k) \subset \mathbb{R}^n$. As a first step, we claim that $\phi'_{x_0,\infty}(v_\alpha) \to 0$ strongly in $\mathcal{D}^{-1,p'}_a(B_k, -x_0)$ as $\alpha \to \infty$. Indeed, fix $h \in C^\infty_c(B_k)$ and define

$$h_\alpha(z) := \lambda_\alpha^{\frac{p(1+a)-n}{p}} h\left(\frac{z}{\lambda_\alpha} - x_0\right) = \lambda_\alpha^{-\gamma} h\left(\frac{z}{\lambda_\alpha} - x_0\right)$$

Clearly,

$$\nabla h_\alpha(z) = \lambda_\alpha^{a-\frac{n}{p}} \nabla h\left(\frac{z}{\lambda_\alpha} - x_0\right).$$

Then, each h_α is supported in $\lambda_\alpha B(x_0,k)$ which approaches zero as $\alpha \to \infty$. Thus, for all α large, we have $h_\alpha \in C^\infty_c(\Omega)$ (using that Ω is an open set containing the origin). Now, we compute

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla v_\alpha|^{p-2} \nabla v_\alpha \cdot \nabla h \, |x+x_0|^{-ap} \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \lambda_\alpha^{\left(\frac{n}{p}-a\right)(p-1)} |\nabla u_\alpha(\lambda_\alpha(x+x_0))|^{p-2} \nabla u_\alpha(\lambda_\alpha(x+x_0)) \cdot \nabla h(x) \, |x+x_0|^{-ap} \, \mathrm{d}x \\
&= \lambda_\alpha^{n-ap-n/p+a} \int_{\mathbb{R}^n} |\nabla u_\alpha(z)|^{p-2} \nabla u_\alpha(z) \cdot \lambda_\alpha^{\frac{n}{p}-a} \nabla h_\alpha(z) \, |z|^{-ap} \, \lambda_\alpha^{ap-n} \mathrm{d}z \\
&= \int_{\mathbb{R}^n} |\nabla u_\alpha(z)|^{p-2} \nabla u_\alpha(z) \cdot \nabla h_\alpha(z) \, |z|^{-ap} \, \mathrm{d}z.
\end{aligned}$$

Similarly, a simple change of variables yields

$$\begin{aligned}
&\int_{\mathbb{R}^n} |v_\alpha(x)|^{q-2} v_\alpha(x) h(x) \, |x+x_0|^{-bq} \, \mathrm{d}x \\
&= \lambda_\alpha^{\gamma(q-1)} \int_{\mathbb{R}^n} |u_\alpha(\lambda_\alpha(x+x_0)|^{q-2} u_\alpha(\lambda_\alpha(x+x_0)) h(x) \, |x+x_0|^{-bq} \, \mathrm{d}x
\end{aligned}$$

$$= \lambda_\alpha^{\gamma(q-1)} \int_{\mathbb{R}^n} |u_\alpha(z)|^{q-2} u_\alpha(z) \lambda_\alpha^{\gamma} h_\alpha(z) \lambda_\alpha^{bq-n} |z|^{-bq} \, \mathrm{d}z$$
$$= \lambda_\alpha^{\gamma q - n + bq} \int_{\mathbb{R}^n} |u_\alpha(z)|^{q-2} u_\alpha(z) h_\alpha(z) |z|^{-bq} \, \mathrm{d}z$$
$$= \int_{\mathbb{R}^n} |u_\alpha(z)|^{q-2} u_\alpha(z) h_\alpha(z) |z|^{-bq} \, \mathrm{d}z.$$

In this last step, we have used that

$$\gamma q = n - bq.$$

Combining these two identities shows that, for all $\alpha \in \mathbb{N}$ large,

$$\begin{aligned} \left| \left\langle \phi'_{x_0,\infty}(v_\alpha), h \right\rangle \right| = |\langle \phi'(u_\alpha), h_\alpha \rangle| &\leq \|\phi'(u_\alpha)\|_{\mathcal{D}_a^{-1,p'}(\Omega,0)} \|h_\alpha\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \\ &= \|\phi'(u_\alpha)\|_{\mathcal{D}_a^{-1,p'}(\Omega,0)} \|h\|_{\mathcal{D}^{1,p}(B_k,-x_0)} . \end{aligned}$$

It follows that $\phi'_{x_0,\infty}(v_\alpha) \to 0$ strongly in $\mathcal{D}_a^{-1,p'}(B_k, -x_0)$. Using this, we will extract a subsequence such that $\nabla v_\alpha \to \nabla v$ pointwise on $\mathbb{R}^n$, almost everywhere. Let now $\rho \in C_c^\infty(\mathbb{R}^n)$ be a bump function such that

$$\begin{cases} 0 \leq \rho \leq 1 & \text{on } \mathbb{R}^n, \\ \rho \equiv 1 & \text{in } B_k, \\ \rho \equiv 0 & \text{outside } B_{k+1}. \end{cases}$$

Consider the vector-valued map

$$f_\alpha := |\nabla v_\alpha|^{p-2} \nabla v_\alpha - |\nabla v|^{p-2} \nabla v,$$

which satisfies $f_\alpha \cdot (\nabla v_\alpha - \nabla v) \geq 0$ almost everywhere (see Szulkin-Willem [21]). Let

T be as in (2.7). For each fixed $k \in \mathbb{N}$, an easy computation shows that

$$\begin{aligned}\int_{B_k} f_\alpha \cdot \nabla T(v_\alpha - v)\,|x+x_0|^{-ap}\,\mathrm{d}x &\le \int_{\mathbb{R}^n} f_\alpha \cdot \rho \nabla T(v_\alpha - v)\,|x+x_0|^{-ap}\,\mathrm{d}x \\ &= \underbrace{\int_{\mathbb{R}^n} f_\alpha \cdot \nabla\left[\rho T(v_\alpha - v)\right]|x+x_0|^{-ap}\,\mathrm{d}x}_{=:I_1} \\ &\quad - \underbrace{\int_{\mathbb{R}^n} f_\alpha \cdot T(v_\alpha - v)\nabla\rho\,|x+x_0|^{-ap}\,\mathrm{d}x}_{=:I_2}.\end{aligned}$$

By Hölder's inequality and the fact that $\operatorname{supp}(\rho) \subseteq B_{k+1}$, it is easily seen that

$$|I_2| \le C\,\|f_\alpha\|_{L^{\frac{p}{p-1}}(\mathbb{R}^n,|x+x_0|^{-ap})}\left(\int_{B_{k+1}} |T(v_\alpha - v)|^p\,|x+x_0|^{-ap}\,\mathrm{d}x\right)^{1/p}.$$

Now, as (∇v_α) is uniformly bounded in $L^p_a(\mathbb{R}^n, -x_0)$, we see that

$$\|f_\alpha\|_{L^{\frac{p}{p-1}}(\mathbb{R}^n,|x+x_0|^{-ap})}$$

can be bounded independently of $\alpha \in \mathbb{N}$. Since T is bounded, continuous, and $v_\alpha \to v$ pointwise almost everywhere on $\mathbb{R}^n$, the Dominated Convergence Theorem ensures that

$$\int_{B_{k+1}} |T(v_\alpha - v)|^p\,|x+x_0|^{-ap}\,\mathrm{d}x \to 0, \quad \text{as } \alpha \to \infty.$$

We infer that $I_2 \to 0$ as $\alpha \to \infty$. It remains to check that I_1 also vanishes in the limit. To achieve this, note that we can write

$$\begin{aligned}I_1 = \left\langle \phi'_{x_0,\infty}(v_\alpha), \rho T(v_\alpha - v)\right\rangle &+ \underbrace{\int_{\mathbb{R}^n} |v_\alpha|^{q-2}\,v_\alpha \rho T(v_\alpha - v)\,|x+x_0|^{-bq}\,\mathrm{d}x}_{=:J_1} \\ &- \underbrace{\int_{\mathbb{R}^n} |\nabla v|^{p-2}\,\nabla v \cdot \nabla(\rho T(v_\alpha - v))\,|x+x_0|^{-ap}\,\mathrm{d}x}_{=:J_2}.\end{aligned}$$

Clearly, $\left\langle \phi'_{x_0,\infty}(v_\alpha), \rho T(v_\alpha - v)\right\rangle \to 0$ as $\alpha \to \infty$ since $\phi'_{x_0,\infty}(v_\alpha) \to 0$ strongly in $\mathcal{D}_a^{-1,p'}(B_m, -x_0)$ for each $m \geq 1$.[2] Next, using (2.5) together with Hölder's inequality shows that

$$|J_1| \leq \|v_\alpha\|^{q-1}_{L^q_b(\mathbb{R}^n,-x_0)} \left(\int_{\mathbb{R}^n} \rho(x)^q \left|T(v_\alpha - v)\right|^q |x + x_0|^{-bq} \,\mathrm{d}x \right)^{1/q} \to 0$$

by virtue of the Dominated Convergence Theorem. Finally, treating J_2, we have

$$\begin{aligned} J_2 = &\int_{\mathbb{R}^n} |\nabla v|^{p-2} \nabla v \cdot \nabla \rho T(v_\alpha - v)\, |x + x_0|^{-ap} \,\mathrm{d}x \\ &+ \int_{\mathbb{R}^n} \rho\, |\nabla v|^{p-2} \nabla v \cdot T'(v_\alpha - v)\, (\nabla v_\alpha - \nabla v)\, |x + x_0|^{-ap} \,\mathrm{d}x \end{aligned}$$

with this first term also converging to zero by Hölder's inequality and the Dominated Convergence Theorem. The second integral can be written as

$$\begin{aligned} &\int_{\mathbb{R}^n} \rho\, |\nabla v|^{p-2} \nabla v \cdot \nabla(v_\alpha - v)\, |x + x_0|^{-ap} \,\mathrm{d}x \\ &- \int_{\{|v_\alpha - v| > 1\}} \rho\, |\nabla v|^{p-2} \nabla v \cdot \nabla(v_\alpha - v)\, |x + x_0|^{-ap} \,\mathrm{d}x. \end{aligned}$$

Since $v_\alpha \rightharpoonup v$ in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$ as $\alpha \to \infty$, it is clear from Hölder's inequality that

$$\lim_{\alpha\to\infty} \int_{\mathbb{R}^n} \rho\, |\nabla v|^{p-2} \nabla v \cdot \nabla(v_\alpha - v)\, |x + x_0|^{-ap} \,\mathrm{d}x = 0.$$

Now, another application of Hölder's inequality yields for a suitable $C > 0$

$$\int_{\{|v_\alpha - v| > 1\}} \left|\rho\, |\nabla v|^{p-2} \nabla v \cdot \nabla(v_\alpha - v)\right| |x + x_0|^{-ap} \,\mathrm{d}x \tag{3.5}$$

$$\leq C\, \|v_\alpha - v\|_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,-x_0)} \left(\int_{B_{k+1} \cap \{|v_\alpha - v| > 1\}} |\nabla v|^p\, |x + x_0|^{-ap} \,\mathrm{d}x \right)^{(p-1)/p}. \tag{3.6}$$

[2]Since ρ is a test function, T is bounded and (v_α) is uniformly bounded in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$, the sequence $(\rho T(v_\alpha - v))_{\alpha\in\mathbb{N}}$ is bounded in $\mathcal{D}^{1,p}(B_m, -x_0)$ for sufficiently large m.

Using that $v_\alpha \to v$ pointwise almost everywhere on $\mathbb{R}^n$, it follows that this last integral converges to zero as $\alpha \to \infty$. Indeed, the ball B_{k+1} has finite measure with respect to $\mathrm{d}\mu = |x + x_0|^{-ap}\,\mathrm{d}x$ whence

$$\mu\left(\{x \in B_{k+1} : |v_\alpha(x) - v(x)| > 1\}\right) \to 0$$

as $\alpha \to \infty$. Especially, $J_2 \to 0$. To summarize, we have shown that

$$\lim_{\alpha\to\infty} \int_{B_k} f_\alpha \cdot \nabla T(v_\alpha - v)\, |x + x_0|^{-ap}\,\mathrm{d}x = 0$$

for all $k \in \mathbb{N}$. Citing Theorem 2.6.1, we may assume after passing to a subsequence that $\nabla v_\alpha \to \nabla v$ pointwise almost everywhere on $\mathbb{R}^n$. Furthermore, this result says that

$$\|v_\alpha\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,-x_0)} - \|v_\alpha - v\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,-x_0)} = \|v\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,-x_0)} + o(1) \tag{3.7}$$

and

$$|\nabla v_\alpha|^{p-2}\,\nabla v - |\nabla v_\alpha - \nabla v|^{p-2}\,(\nabla v_\alpha - \nabla v) \to |\nabla v|^{p-2}\,\nabla v \tag{3.8}$$

in $L^{p'}(\mathbb{R}^n, |x + x_0|^{-ap})$. By a simple change of variables, it is clear that (3.7) implies (i) directly. Next, write

$$\begin{aligned}
\frac{1}{p}\int_{\mathbb{R}^n} |\nabla w_\alpha(x)|^p\,|x|^{-ap}\,\mathrm{d}x &= \frac{1}{p}\int_{\mathbb{R}^n} \left|\nabla u_\alpha(x) - \lambda_\alpha^{-(\gamma+1)}\nabla v\left(\frac{x}{\lambda_\alpha} - x_0\right)\right|^p |x|^{-ap}\,\mathrm{d}x \\
&= \frac{\lambda_\alpha^{-(\gamma+1)p}}{p}\int_{\mathbb{R}^n} \left|\nabla v_\alpha\left(\frac{x}{\lambda_\alpha} - x_0\right) - \nabla v\left(\frac{x}{\lambda_\alpha} - x_0\right)\right|^p |x|^{-ap}\,\mathrm{d}x \\
&= \frac{\lambda_\alpha^{-(\gamma+1)p}}{p}\int_{\mathbb{R}^n} |\nabla v_\alpha(z) - \nabla v(z)|^p\,\lambda_\alpha^{-ap+n}\,|z + x_0|^{-ap}\,\mathrm{d}z \\
&= \frac{1}{p}\int_{\mathbb{R}^n} |\nabla v_\alpha(z) - \nabla v(z)|^p\,|z + x_0|^{-ap}\,\mathrm{d}z.
\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{1}{q}\int_{\mathbb{R}^n}|w_\alpha(x)|^q\,|x|^{-bq}\,\mathrm{d}x &= \frac{1}{q}\int_{\mathbb{R}^n}\left|u_\alpha(x)-\lambda_\alpha^{-\gamma}v_\alpha\left(\frac{x}{\lambda_\alpha}-x_0\right)\right|^q|x|^{-bq}\,\mathrm{d}x\\ &= \frac{\lambda_\alpha^{-\gamma q}}{q}\int_{\mathbb{R}^n}\left|v_\alpha\left(\frac{x}{\lambda_\alpha}-x_0\right)-v\left(\frac{x}{\lambda_\alpha}-x_0\right)\right|^q|x|^{-bq}\,\mathrm{d}x\\ &= \frac{\lambda_\alpha^{-\gamma q+n-bq}}{q}\int_{\mathbb{R}^n}|v_\alpha(z)-v(z)|^q\,|z+x_0|^{-bq}\,\mathrm{d}z\\ &= \frac{1}{q}\int_{\mathbb{R}^n}|v_\alpha(z)-v(z)|^q\,|z+x_0|^{-bq}\,\mathrm{d}z.\end{aligned}$$

Combining these, the Brézis-Lieb lemma ensures that

$$\begin{aligned}\phi_{0,\infty}(w_\alpha)=\phi_{x_0,\infty}(v_\alpha-v) &= \phi_{x_0,\infty}(v_\alpha)-\phi_{x_0,\infty}(v)+o(1)\\ &= \phi(u_\alpha)-\phi_{x_0,\infty}(v)+o(1)\\ &= c-\phi_{x_0,\infty}(v)+o(1).\end{aligned}$$

Next, we assert that $\phi'_{x_0,\infty}(v)=0$. That is, we claim that v solves (3.4). Equivalently, we show that $\left\langle\phi'_{x_0,\infty}(v),h\right\rangle=0$ for all $h\in C_c^\infty(\mathbb{R}^n)$. Now, since $\phi'_{x_0,\infty}(v_\alpha)\to 0$ strongly in $\mathcal{D}_a^{-1,p'}(B_k)$ for each $k\geq 1$, it suffices to check that

$$\lim_{\alpha\to\infty}\left\langle\phi'_{x_0,\infty}(v_\alpha),h\right\rangle=\left\langle\phi'_{x_0,\infty}(v),h\right\rangle.$$

Certainly, by Theorem 2.1.2 with $\mathrm{d}\mu=|x+x_0|^{-ap}\,\mathrm{d}x$, we see that

$$\int_{\mathbb{R}^n}\frac{|\nabla v_\alpha(x)|^{p-2}\,\nabla v_\alpha(x)\cdot\nabla h(x)}{|x+x_0|^{ap}}\mathrm{d}x\to\int_{\mathbb{R}^n}\frac{|\nabla v(x)|^{p-2}\,\nabla v(x)\cdot\nabla h(x)}{|x+x_0|^{ap}}\mathrm{d}x.$$

Similarly, as $\alpha\to\infty$,

$$\int_{\mathbb{R}^n}\frac{|v_\alpha(x)|^{q-2}\,v_\alpha(x)h(x)}{|x+x_0|^{bq}}\mathrm{d}x\to\int_{\mathbb{R}^n}\frac{|v(x)|^{q-2}\,v(x)h(x)}{|x+x_0|^{bq}}\mathrm{d}x$$

whence $\left\langle\phi'_{x_0,\infty}(v_\alpha),h\right\rangle\to\left\langle\phi'_{x_0,\infty}(v),h\right\rangle$. Finally, in order to prove claim (iii) we fix an arbitrary $g \in C_c^\infty(\Omega)$ with $\|g\|_{\mathcal{D}_a^{1,p}(\Omega,0)} = 1$. After rescaling, we see that

$$\left\langle \phi'_{0,\infty}(w_\alpha), g \right\rangle = \left\langle \phi'_{x_0,\infty}(v_\alpha - v), g_\alpha \right\rangle \tag{3.9}$$

where $g_\alpha(x) = \lambda_\alpha^{\frac{n-p(1+a)}{p}} g(\lambda_\alpha(x + x_0))$. Then, applying Hölder's inequality together with Theorem 2.6.1-(3) and the usual rescaling, it follows that

$$\begin{aligned}\left\langle \phi'_{0,\infty}(w_\alpha), g \right\rangle &= \left\langle \phi'_{x_0,\infty}(v_\alpha - v), g_\alpha \right\rangle \\ &= \left\langle \phi'_{x_0,\infty}(v_\alpha), g_\alpha \right\rangle - \left\langle \phi'_{x_0,\infty}(v), g_\alpha \right\rangle + o(1) \\ &= \langle \phi'(u_\alpha), g \rangle + o(1)\end{aligned}$$

uniformly for $g \in C_c^\infty(\Omega)$ with $\|g\|_{\mathcal{D}_a^{1,p}(\Omega,0)} = 1$. We infer that $\phi'_{0,\infty}(w_\alpha) \to 0$ strongly in $\mathcal{D}_a^{-1,p'}(\Omega, 0)$ and the proof is complete. □

3.3 Convergence of Domains

In this section we formalize some of the notation that will be helpful in the proof of our main result. We continue to denote by Ω an arbitrary bounded domain (i.e. a non-empty open connected bounded set) containing the origin. In particular, there exists $\delta > 0$ such that $B(0, \delta) \subseteq \Omega$. Let now (λ_α) be a sequence in $(0, \infty)$ converging to some $\lambda \geq 0$ as $\alpha \to \infty$ and fix a point $x_0 \in \mathbb{R}^n$. For each $\alpha \in \mathbb{N}$, let us consider

$$\Omega_\alpha := \frac{\Omega}{\lambda_\alpha} - x_0 = \left\{ \frac{x}{\lambda_\alpha} - x_0 : x \in \Omega \right\}.$$

Since the map $x \mapsto \frac{x}{\lambda_\alpha} - x_0$ is a homeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ for each $\alpha \in \mathbb{N}$, it is clear that Ω_α is a bounded domain for every such index and that

$$\overline{\Omega_\alpha} = \frac{\overline{\Omega}}{\lambda_\alpha} - x_0, \quad \forall \alpha \in \mathbb{N}.$$

Definition 3.3.1. Let $\Omega_\infty \subseteq \mathbb{R}^n$ be a domain. We say that $\Omega_\alpha \to \Omega_\infty$ as $\alpha \to \infty$ provided the following two conditions are met:

(1) for each compact set $K \subseteq \Omega_\infty$, there exists $N \in \mathbb{N}$ such that $K \subseteq \Omega_\alpha$ for all $\alpha \geq N$;

(2) if $K \subseteq \left(\overline{\Omega_\infty}\right)^{\complement}$ is compact, then $K \cap \Omega_\alpha = \varnothing$ for all $\alpha \in \mathbb{N}$ sufficiently large.

In our case there are only two possible limiting domains Ω_∞ that are of interest.

Proposition 3.3.1. *Let (λ_α) and Ω_α be as above. Define*

$$\Omega_\infty := \begin{cases} \mathbb{R}^n & \text{if } \lambda = 0, \\ \dfrac{\Omega}{\lambda} - x_0 & \text{if } \lambda > 0. \end{cases}$$

Then, $\Omega_\alpha \to \Omega_\infty$ as $\alpha \to \infty$.

Proof. We first treat the case where $\lambda = \lim_{\alpha\to\infty} \lambda_\alpha = 0$. Clearly,

$$\Omega_\alpha \supseteq \frac{B(0,\delta)}{\lambda_\alpha} - x_0 = B\left(0, \frac{\delta}{\lambda_\alpha}\right) - x_0 = B\left(-x_0, \frac{\delta}{\lambda_\alpha}\right)$$

which eventually contains every fixed compact set K. Hence, $\Omega_\alpha \to \mathbb{R}^n$.

Let us now assume that we are in the case $\lambda > 0$ and fix a compact set $K \subseteq \Omega_\infty$. Then, $\lambda(K + x_0)$ is a compact subset of Ω. Since Ω is open, the Lebesgue Number Lemma guarantees the existence of $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq \Omega$ for all $x \in \lambda(K+x_0)$. Let $M > 0$ be such that

$$\sup_{z\in K} |z + x_0| \leq M.$$

Since $\lambda_\alpha \to \lambda$ as $\alpha \to \infty$, we can find $N \in \mathbb{N}$ such that

$$|\lambda_\alpha - \lambda| < \frac{\varepsilon}{M}, \quad \forall \alpha \geq N.$$

Fix now a point $z \in K$; for all $\alpha \geq N$ we obtain

$$\begin{aligned} |\lambda(z + x_0) - \lambda_\alpha(z + x_0)| &= |(\lambda - \lambda_\alpha)(z + x_0)| \\ &\leq |\lambda - \lambda_\alpha|\, M \\ &< \varepsilon \end{aligned}$$

whence $\lambda_\alpha(z+x_0) \in B(\lambda(z+x_0),\varepsilon) \subseteq \Omega$. Thus, whenever $\alpha \geq N$,

$$z \in \frac{\Omega}{\lambda_\alpha} - x_0 = \Omega_\alpha.$$

It follows that $K \subseteq \Omega_\alpha$ for every $\alpha \geq N$.

Finally, assume that $K \subseteq \left(\overline{\Omega_\infty}\right)^{\complement}$ is compact; we claim that $K \cap \Omega_\alpha = \varnothing$ for all $\alpha \in \mathbb{N}$ sufficiently large. Otherwise, passing to a subsequence if necessary, there exists a sequence (z_α) in K such that $z_\alpha \in \Omega_\alpha$ for each $\alpha \in \mathbb{N}$. Since K is compact, we may assume that $z_\alpha \to z \in K$ as $\alpha \to \infty$. Now, for every index α there exists $x_\alpha \in \Omega$ such that

$$z_\alpha = \frac{x_\alpha}{\lambda_\alpha} - x_0.$$

Passing to yet another subsequence, we can assume that $x_\alpha \to x$ in $\overline{\Omega}$. Therefore,

$$z = \lim_{\alpha\to\infty} z_\alpha = \lim_{\alpha\to\infty}\left[\frac{x_\alpha}{\lambda_\alpha} - x_0\right] = \frac{x}{\lambda} - x_0.$$

Hence, $z \in \overline{\Omega_\infty}$ which is a contradiction. □

Finally, we verify the following elementary property:

Lemma 3.3.2. *Fix $r > 0$. Within the setting of the previous proposition, there exist countably many balls $\{B(x_j,r)\}_j$, with $x_j \in \Omega_\infty$, and countably many balls $\{B(y,\varepsilon_y)\}_y$ such that*

(i) $\overline{B(y,\varepsilon_y)} \subseteq \left(\overline{\Omega_\infty}\right)^{\complement}$ for each y;

(ii) $\bigcup_j B(x_j,r) \cup \bigcup_y B(y,\varepsilon_y) = \mathbb{R}^n$.

Proof. If $\lambda_\alpha \to 0$ then $\Omega_\infty = \mathbb{R}^n$ for which the assertion is trivial. We may therefore assume that $\lambda_\alpha \to \lambda > 0$. Clearly, the family

$$\{B(y,r) : y \in \Omega_\infty\}$$

forms an open cover of the compact set $\overline{\Omega_\infty}$. Therefore, it admits a finite subcover Σ of $\overline{\Omega_\infty}$. Now, given a point in $\left(\overline{\Omega_\infty}\right)^{\complement}$, there exists $\varepsilon_y > 0$ such that $\overline{B(y,\varepsilon_y)} \subset \left(\overline{\Omega_\infty}\right)^{\complement}$.

Then, the family

$$\Sigma \cup \left\{ B(y, \varepsilon_y) : y \notin \overline{\Omega_\infty} \right\}$$

is an open cover of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is second countable and therefore Lindelöf, we may pass to a countable subcover. □

3.4 The Global Compactness Theorem

We now show that the global compactness theorem established by Mercuri-Willem in [17] for unweighted critical p-Laplace equations can be extended to the weighted problem (1.7). Again, we do not impose any regularity assumptions on the boundary of Ω and we continue to assume that the conditions in (1.6) hold.

Theorem 3.4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain containing the origin and let (u_α) be a Palais-Smale sequence for (1.7). Assume $a \neq b$ and let $\gamma > 0$ be the homogeneity exponent from (2.13). Then, there exists a subsequence (u_β) of (u_α) along with*

(1) a solution v_0 of (1.7);

(2) finitely many non-trivial functions $w_1, \ldots, w_k$;

(3) sequences $(\lambda_\beta^{(j)})$ in $(0, \infty)$ for $j = 1, \ldots, k$,

such that

$$\lambda_\beta^{(j)} \to 0, \quad \text{as } \beta \to \infty,$$

and each w_j solves

$$\begin{cases} -\operatorname{div}\left(|x|^{-ap} |\nabla u|^{p-2} \nabla u\right) = |x|^{-bq} |u|^{q-2} u & \text{in } \mathbb{R}^n, \\ u \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0). \end{cases}$$

Furthermore, as $\beta \to \infty$,

$$\left\| u_\beta - v_0 - \sum_{j=1}^{k} \left(\lambda_\beta^{(j)}\right)^{-\gamma} w_j\left(\frac{\cdot}{\lambda_\beta^{(j)}}\right)\right\|_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,0)} \to 0,$$

$$\|u_\beta\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} \to \|v_0\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} + \sum_{j=1}^{k} \|w_j\|^p_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,0)},$$

$$\phi(u_\beta) \to \phi(v_0) + \sum_{j=1}^{k} \phi_{0,\infty}(w_j).$$

The Proof of Theorem 3.4.1

We take inspiration from the proof of Theorem 2 in Mercuri-Willem [17].

Step 1. Referring to Proposition 3.1.1, the sequence (u_α) is bounded in $\mathcal{D}_a^{1,p}(\Omega, |x|^{-ap})$. Therefore, by applying Theorem 2.5.1, we may assume that u_α converges weakly to some $v_0 \in \mathcal{D}_a^{1,p}(\Omega, 0)$, with pointwise convergence almost everywhere on Ω, as $\alpha \to \infty$. Next, invoking Lemma 3.2.1 shows that the sequence (u_α^1) defined by $u_\alpha^1 := u_\alpha - v_0$ satisfies

(i) $\|u_\alpha^1\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} = \|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} - \|v_0\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} + o(1)$;

(ii) $\phi(u_\alpha^1) \to c - \phi(v_0)$;

(iii) $\phi'(u_\alpha^1) \to 0$ in $\mathcal{D}_a^{-1,p'}(\Omega, 0)$.

Moreover, this same lemma asserts that $\phi'(v_0) = 0$ and $\nabla u_\alpha \to \nabla v_0$ almost everywhere on $\mathbb{R}^n$ as $\alpha \to \infty$.

Step 2. We claim that the proof is complete in the case where $u_\alpha^1 \to 0$ strongly in $L_b^q(\Omega, 0)$. First note that the boundedness of (u_α^1) in $\mathcal{D}_a^{1,p}(\Omega, 0)$ gives us that

$$\left|\left\langle \phi'(u_\alpha^1), u_\alpha^1\right\rangle\right| \leq \left\|\phi'(u_\alpha^1)\right\|_{\mathcal{D}_a^{-1,p'}(\Omega,0)} \sup_{\alpha\in\mathbb{N}} \left\|u_\alpha^1\right\|_{\mathcal{D}_a^{1,p}(\Omega,0)} \to 0$$

as $\alpha \to \infty$. Put otherwise,

$$\lim_{\alpha\to\infty} \int_\Omega \left(|x|^{-ap} \left|\nabla u^1_\alpha(x)\right|^p - |x|^{-bq} \left|u^1_\alpha(x)\right|^q \right) \mathrm{d}x = 0$$

whence we see that $\nabla u^1_\alpha \to 0$ strongly in $L^p_a(\Omega, 0)$. By definition, this implies that $u_\alpha \to v_0$ strongly in $\mathcal{D}^{1,p}_a(\Omega, 0)$ at which point the criteria of our theorem are satisfied for $k = 0$ since

$$\phi(v_0) = \lim_{\alpha\to\infty} \phi(u_\alpha).$$

Step 3. In light of this previous step, we may assume without harm that u^1_α does *not* converge strongly to 0 in $L^q_b(\Omega, 0)$ as $\alpha \to \infty$. Passing to a subsequence if necessary, let us assume that

$$\inf_{\alpha\geqslant 1} \int_\Omega \left|u^1_\alpha\right|^q |x|^{-bq} \,\mathrm{d}x > \delta$$

with $\delta > 0$ such that

$$0 < \delta < \left(\frac{S_p}{2^p}\right)^{\frac{p}{q-p}}. \tag{3.10}$$

Here, $S_p > 0$ is any positive constant such that, for all $\xi \in \mathbb{R}^n$,

$$S_p \|w\|^p_{L^q_b(\mathbb{R}^n,\xi)} \le \|\nabla w\|^p_{L^p_a(\mathbb{R}^n,\xi)}, \quad \forall w \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, \xi).$$

Note that the existence of such a S_p is guaranteed by (2.5). By Proposition 2.4.1, after an extension by zero outside Ω, we can interpret (u^1_α) as a sequence in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, 0)$.

Let us now consider the family $\{Q_\alpha\}_{\alpha\geqslant 1}$ of Lévy concentration functions each defined by

$$Q_\alpha(r) := \sup_{y\in\bar\Omega} \int_{B(y,r)} \left|u^1_\alpha\right|^q |x|^{-bq} \,\mathrm{d}x.$$

We observe that every Q_α is continuous on $[0, \infty)$ as a direct consequence of Proposition 2.1.3. Clearly,

$$Q_\alpha(0) = 0 \quad \text{and} \quad \lim_{r\nearrow\infty} Q_\alpha(r) > \delta$$

for all $\alpha \ge 1$. It then follows from the Intermediate Value Theorem that, for each

$\alpha \in \mathbb{N}$, there exists a smallest $\lambda_\alpha^1 > 0$ with the property that

$$Q_\alpha(\lambda_\alpha^1) = \sup_{y \in \bar{\Omega}} \int_{B(y,\lambda_\alpha^1)} \left|u_\alpha^1\right|^q |x|^{-bq} \, \mathrm{d}x = \delta.$$

Clearly, since Ω is bounded and we are choosing the smallest possible $\lambda_\alpha^1 > 0$ at each stage, the sequence (λ_α^1) is bounded by $2 \operatorname{diam}(\Omega)$. In particular, passing to a subsequence, it can be assumed that $\lambda_\alpha^1 \to \lambda^1 \geq 0$ as $\alpha \to \infty$. Finally, for each index $\alpha \in \mathbb{N}$, there exists (by the Dominated Convergence Theorem) $y_\alpha^1 \in \overline{\Omega}$ such that

$$Q_\alpha(\lambda_\alpha^1) = \sup_{y \in \bar{\Omega}} \int_{B(y,\lambda_\alpha^1)} \left|u_\alpha^1\right|^q |x|^{-bq} \, \mathrm{d}x = \int_{B(y_\alpha^1,\lambda_\alpha^1)} \left|u_\alpha^1\right|^q |x|^{-bq} \, \mathrm{d}x = \delta.$$

As above, we can assume that $y_\alpha^1 \to y^1 \in \overline{\Omega}$.

Step 4. We claim that the sequence of points $y_\alpha^1/\lambda_\alpha^1$ is bounded in $\mathbb{R}^n$. Arguing by contradiction and passing to a subsequence if necessary, we may assume that

$$\frac{y_\alpha^1}{\lambda_\alpha^1} \to \infty, \quad \text{as } \alpha \to \infty,$$

and that $|y_\alpha^1| > 4\lambda_\alpha^1 > 0$ for each $\alpha \in \mathbb{N}$. Fix now a cutoff function $\eta \in C_c^\infty(\mathbb{R}^n)$ having the property that

$$\begin{cases} 0 \leq \eta \leq 1 & \text{in } \mathbb{R}^n, \\ \eta \equiv 1 & \text{on } \overline{B\left(0, \frac{1}{2}\right)}, \\ \eta \equiv 0 & \text{outside } B(0,1). \end{cases}$$

It is readily verified that

$$\begin{cases} \eta\left(\dfrac{2}{|y_\alpha^1|}\left(z - y_\alpha^1\right)\right) \equiv 0 & \text{if } \left|z - y_\alpha^1\right| \geq \dfrac{|y_\alpha^1|}{2}, \\ \eta\left(\dfrac{2}{|y_\alpha^1|}\left(z - y_\alpha^1\right)\right) \equiv 1 & \text{if } \left|z - y_\alpha^1\right| \leq \dfrac{|y_\alpha^1|}{4}. \end{cases} \tag{3.11}$$

Let us also make the elementary observation that

$$B(y_\alpha^1, \lambda_\alpha^1) \subset B\left(y_\alpha^1, \frac{|y_\alpha^1|}{4}\right) \subset B\left(y_\alpha^1, \frac{|y_\alpha^1|}{2}\right) \tag{3.12}$$

for all $\alpha \in \mathbb{N}$. Clearly, if $z \in B(y_\alpha^1, |y_\alpha^1|/2)$ then

$$\frac{|y_\alpha^1|}{2} \leq |z| \leq \frac{3\,|y_\alpha^1|}{2}. \tag{3.13}$$

We now consider the function

$$v_\alpha(x) := \left(\frac{\lambda_\alpha^1}{|y_\alpha^1|}\right)^b \left(\lambda_\alpha^1\right)^\gamma u_\alpha^1(\lambda_\alpha^1 x + y_\alpha^1)\eta\left(\frac{2\lambda_\alpha^1 x}{|y_\alpha^1|}\right)$$

with $\gamma > 0$ defined as in (2.13). Note that v_α is weakly differentiable on $\mathbb{R}^n$ by Lemma 2.4.4.[3] In particular, ∇v_α exists almost everywhere on $\mathbb{R}^n$ and v_α obeys the product rule. Now, a straightforward calculation shows that

$$\nabla v_\alpha(x) = \left(\frac{\lambda_\alpha^1}{|y_\alpha^1|}\right)^b \left(\lambda_\alpha^1\right)^{\gamma+1} \nabla u_\alpha^1(\lambda_\alpha^1 x + y_\alpha^1)\eta\left(\frac{2\lambda_\alpha^1 x}{|y_\alpha^1|}\right) \tag{3.14}$$

$$+ 2\left(\frac{\lambda_\alpha^1}{|y_\alpha^1|}\right)^{b+1} \left(\lambda_\alpha^1\right)^\gamma u_\alpha^1(\lambda_\alpha^1 x + y_\alpha^1)\nabla\eta\left(\frac{2\lambda_\alpha^1 x}{|y_\alpha^1|}\right). \tag{3.15}$$

Denote by T_1 and T_2 be the terms in (3.14) and (3.15), respectively. Then, using a change of variables together with (3.13), we obtain

$$\begin{aligned}
\|T_1\|_{L^p(\mathbb{R}^n)}^p &= \int_{\mathbb{R}^n} \left(\frac{\lambda_\alpha^1}{|y_\alpha^1|}\right)^{bp} \left(\lambda_\alpha^1\right)^{(\gamma+1)p} \left|\nabla u_\alpha^1(\lambda_\alpha^1 x + y_\alpha^1)\right|^p \eta\left(\frac{2\lambda_\alpha^1 x}{|y_\alpha^1|}\right)^p \mathrm{d}x \\
&= \left(\frac{\lambda_\alpha^1}{|y_\alpha^1|}\right)^{bp} \left(\lambda_\alpha^1\right)^{p(\gamma+1)} \int_{\mathbb{R}^n} \left|\nabla u_\alpha^1(z)\right|^p \eta\left(\frac{2\lambda_\alpha^1}{|y_\alpha^1|}\cdot\frac{z - y_\alpha^1}{\lambda_\alpha^1}\right)^p \left(\lambda_\alpha^1\right)^{-n} \mathrm{d}z \\
&= \left(\frac{\lambda_\alpha^1}{|y_\alpha^1|}\right)^{bp} \left(\lambda_\alpha^1\right)^{p(\gamma+1)-n} \int_{\mathbb{R}^n} \left|\nabla u_\alpha^1(z)\right|^p \eta\left(\frac{2\,(z - y_\alpha^1)}{|y_\alpha^1|}\right)^p \mathrm{d}z
\end{aligned}$$

[3] Indeed, since ∇u_α^1 exists away from 0, the map $x \mapsto u_\alpha^1(\lambda_\alpha^1 x + y_\alpha^1)$ has weak derivatives of the first order away from $-y_\alpha^1/\lambda_\alpha^1$. Since $\eta\left(\frac{2\lambda_\alpha^1 x}{|y_\alpha^1|}\right)$ is supported away from this point, the claim follows.

$$
\begin{aligned}
&= \left(\frac{\lambda_\alpha^1}{|y_\alpha^1|}\right)^{bp} \left(\lambda_\alpha^1\right)^{p(\gamma+1)-n} \int_{B\left(y_\alpha^1, \frac{|y_\alpha^1|}{2}\right)} \left|\nabla u_\alpha^1(z)\right|^p \eta \left(\frac{2\,(z-y_\alpha^1)}{|y_\alpha^1|}\right)^p \mathrm{d}z \\
&\leq \frac{(\lambda_\alpha^1)^{bp+(\gamma+1)p-n}}{|y_\alpha^1|^{bp}} \int_{B\left(y_\alpha^1, \frac{|y_\alpha^1|}{2}\right)} \left|\nabla u_\alpha^1(z)\right|^p |z|^{-ap} |z|^{ap} \,\mathrm{d}z \\
&\leq C \frac{(\lambda_\alpha^1)^{bp+(\gamma+1)p-n}}{|y_\alpha^1|^{bp-ap}} \int_{B\left(y_\alpha^1, \frac{|y_\alpha^1|}{2}\right)} \left|\nabla u_\alpha^1(z)\right|^p |z|^{-ap} \,\mathrm{d}z
\end{aligned}
$$

for a constant $C > 0$ independent of $\alpha \in \mathbb{N}$. This implies that

$$
\begin{aligned}
\|T_1\|_{L^p(\mathbb{R}^n)}^p &\leq C \frac{(\lambda_\alpha^1)^{bp+(\gamma+1)p-n}}{|y_\alpha^1|^{(b-a)p}} \|u_\alpha\|_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,0)}^p \\
&= C \frac{(\lambda_\alpha^1)^{(b-a)p}}{|y_\alpha^1|^{(b-a)p}} \|u_\alpha\|_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,0)}^p .
\end{aligned}
$$

where in this last step we have used that

$$
\begin{aligned}
bp + (\gamma+1)p - n &= bp + \gamma p + p - n \\
&= bp + [n - p(1+a)] + p - n \\
&= bp - ap.
\end{aligned}
$$

To summarize, since (u_α) is bounded in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$,

$$
\|T_1\|_{L^p(\mathbb{R}^n)}^p \leq C \left(\frac{|y_\alpha^1|}{\lambda_\alpha^1}\right)^{(a-b)p} \|u_\alpha\|_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,0)}^p \to 0
$$

as $\alpha \to \infty$. Next we treat T_2. In a similar vein,

$$
\begin{aligned}
\|T_2\|_{L^p(\mathbb{R}^n)}^p &= 2^p \left(\frac{\lambda_\alpha^1}{|y_\alpha^1|}\right)^{p(b+1)} \left(\lambda_\alpha^1\right)^{\gamma p} \int_{\mathbb{R}^n} \left|u_\alpha^1(\lambda_\alpha^1 x + y_\alpha^1)\right|^p \left|\nabla\eta\left(\frac{2\lambda_\alpha^1 x}{|y_\alpha^1|}\right)\right|^p \mathrm{d}x \\
&= 2^p \frac{(\lambda_\alpha^1)^{p(b+1)+\gamma p-n}}{|y_\alpha^1|^{p(b+1)}} \int_{\mathbb{R}^n} \left|u_\alpha^1(z)\right|^p \left|\nabla\eta\left(\frac{2\lambda_\alpha^1 \cdot \frac{z-y_\alpha^1}{\lambda_\alpha^1}}{|y_\alpha^1|}\right)\right|^p \mathrm{d}z
\end{aligned}
$$

$$= 2^p \frac{(\lambda_\alpha^1)^{p(b+1)+\gamma p-n}}{|y_\alpha^1|^{p(b+1)}} \int_{\mathbb{R}^n} \left|u_\alpha^1(z)\right|^p \left|\nabla\eta\left(\frac{2(z-y_\alpha^1)}{|y_\alpha^1|}\right)\right|^p \mathrm{d}z$$
$$\leq C \frac{(\lambda_\alpha^1)^{p(b+1)+\gamma p-n}}{|y_\alpha^1|^{p(b+1)}} \int_{B\left(y_\alpha^1, \frac{|y_\alpha^1|}{2}\right)} \left|u_\alpha^1(z)\right|^p \mathrm{d}z$$

with $C := 2^p \sup_{\mathbb{R}^n} |\nabla\eta|$. Then, using Hölder's inequality,

$$\|T_2\|_{L^p(\mathbb{R}^n)}^p \leq C \frac{(\lambda_\alpha^1)^{p(b+1)+\gamma p-n}}{|y_\alpha^1|^{p(b+1)}} \left(\int_{B\left(y_\alpha^1, \frac{|y_\alpha^1|}{2}\right)} \left|u_\alpha^1(z)\right|^q \mathrm{d}z\right)^{p/q} \left(\int_{B\left(y_\alpha^1, \frac{|y_\alpha^1|}{2}\right)} \mathrm{d}z\right)^{1-p/q}$$
$$= \widetilde{C} \frac{(\lambda_\alpha^1)^{p(b+1)+\gamma p-n}}{|y_\alpha^1|^{p(b+1)}} \left(\frac{|y_\alpha^1|}{2}\right)^{n-\frac{np}{q}} \left(\int_{B\left(y_\alpha^1, \frac{|y_\alpha^1|}{2}\right)} \left|u_\alpha^1(z)\right|^q |z|^{-bq} |z|^{bq} \mathrm{d}z\right)^{p/q}$$

with $\widetilde{C} > 0$ a constant independent of $\alpha \in \mathbb{N}$. Invoking the Caffarelli-Kohn-Nirenberg inequality then implies that

$$\|T_2\|_{L^p(\mathbb{R}^n)}^p \leq \widetilde{C} \frac{(\lambda_\alpha^1)^{p(b+1)+\gamma p-n}}{|y_\alpha^1|^{p(b+1)}} |y_\alpha^1|^{n-\frac{np}{q}+bp} \left(\int_{B\left(y_\alpha^1, \frac{|y_\alpha^1|}{2}\right)} \left|u_\alpha^1(z)\right|^q |z|^{-bq} \mathrm{d}z\right)^{p/q}$$
$$\leq C \frac{(\lambda_\alpha^1)^{p(b-a)}}{|y_\alpha^1|^{p(b+1)-n+\frac{np}{q}-bp}} \left\|u_\alpha^1\right\|_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,0)}^p$$
$$= C \frac{(\lambda_\alpha^1)^{p(b-a)}}{|y_\alpha^1|^{p(b-a)}}$$
$$= C \left(\frac{\lambda_\alpha^1}{|y_\alpha^1|}\right)^{p(b-a)}$$

after a possible relabeling of C. We infer that $T_2 \to 0$ in $L^p(\mathbb{R}^n)$ as $\alpha \to \infty$. Combining all our work, it follows that $\nabla v_\alpha \to 0$ strongly in $L^p(\mathbb{R}^n)$. Now, a change of variables now familiar to us shows that, after a possible relabeling of some constant

$C_\alpha > 0$, there holds

$$\begin{aligned}\int_{\mathbb{R}^n} |v_\alpha(x)|^{p^*}\,\mathrm{d}x &= C_\alpha \int_{\mathbb{R}^n} \left| u_\alpha^1(\lambda_\alpha^1 x + y_\alpha^1)\eta\left(\frac{2\lambda_\alpha^1 x}{|y_\alpha^1|}\right)\right|^{p^*} \mathrm{d}x \\ &= C_\alpha \int_{\mathbb{R}^n} \left|u_\alpha^1(z)\right|^{p^*} \left|\eta\left(\frac{2(z-y_\alpha^1)}{|y_\alpha^1|}\right)\right|^{p^*} \mathrm{d}z \\ &\le C_\alpha \int_{B\left(y_\alpha^1,\frac{|y_\alpha^1|}{2}\right)} \left|u_\alpha^1(z)\right|^{p^*} \mathrm{d}z.\end{aligned}$$

Using that $u_\alpha^1 \in \mathcal{D}_a^{1,p}(\mathbb{R}^n, 0)$, it is clear that $u_\alpha^1 \in L^q\left(B\left(y_\alpha^1, |y_\alpha^1|/2\right)\right)$. Recalling that $q > p$, we obtain $u_\alpha^1 \in L^p\left(B\left(y_\alpha^1, |y_\alpha^1|/2\right)\right)$. Similarly, $\nabla u_\alpha^1 \in L^p\left(B\left(y_\alpha^1, |y_\alpha^1|/2\right)\right)$ and so $u_\alpha^1 \in W^{1,p}\left(B\left(y_\alpha^1, |y_\alpha^1|/2\right)\right)$. By the Sobolev Embedding Theorem, we infer that $u_\alpha \in L^{p^*}\left(B\left(y_\alpha^1, |y_\alpha^1|/2\right)\right)$ so that $v_\alpha \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. Since $\nabla v_\alpha \to 0$ strongly in $L^p(\mathbb{R}^n)$, it follows that $v_\alpha \to 0$ in $L^{p^*}(\mathbb{R}^n)$. Especially, because $q \le p^*$, we have that $v_\alpha \to 0$ strongly in $L^q_{\mathrm{loc}}(\mathbb{R}^n)$.

On the other hand, for each index $\alpha \in \mathbb{N}$ a straightforward calculation yields

$$\begin{aligned}\int_{B(0,1)} |v_\alpha(x)|^q\,\mathrm{d}x &= \left(\frac{\lambda_\alpha^1}{|y_\alpha^1|}\right)^{bq} \left(\lambda_\alpha^1\right)^{\gamma q} \int_{B(0,1)} \left|u_\alpha^1(\lambda_\alpha^1 x + y_\alpha^1)\right|^q \eta\left(\frac{2\lambda_\alpha^1 x}{|y_\alpha^1|}\right)^q \mathrm{d}x \\ &= \left(\frac{\lambda_\alpha^1}{|y_\alpha^1|}\right)^{bq} \left(\lambda_\alpha^1\right)^{\gamma q - n} \int_{B(y_\alpha^1,\lambda_\alpha^1)} \left|u_\alpha^1(z)\right|^q \eta\left(\frac{2\left(z-y_\alpha^1\right)}{|y_\alpha^1|}\right)^q \mathrm{d}z \\ &= \left(\frac{\lambda_\alpha^1}{|y_\alpha^1|}\right)^{bq} \left(\lambda_\alpha^1\right)^{\gamma q - n} \int_{B(y_\alpha^1,\lambda_\alpha^1)} \left|u_\alpha^1(z)\right|^q \mathrm{d}z\end{aligned}$$

where we have used (3.11)-(3.12) in this last step. Therefore, for a constant $C > 0$ independent of α, we obtain

$$\begin{aligned}\int_{B(0,1)} |v_\alpha(x)|^q\,\mathrm{d}x &= \left(\frac{\lambda_\alpha^1}{|y_\alpha^1|}\right)^{bq} \left(\lambda_\alpha^1\right)^{\gamma q - n} \int_{B(y_\alpha^1,\lambda_\alpha^1)} \left|u_\alpha^1(z)\right|^q |z|^{-bq} |z|^{bq}\,\mathrm{d}z \\ &\ge C\left(\lambda_\alpha^1\right)^{bq} \left(\lambda_\alpha^1\right)^{\gamma q - n} \int_{B(y_\alpha^1,\lambda_\alpha^1)} \left|u_\alpha^1(z)\right|^q |z|^{-bq}\,\mathrm{d}z \\ &= C\delta > 0.\end{aligned}$$

But this contradicts the fact that $v_\alpha \to 0$ strongly in $L^q_{\mathrm{loc}}(\mathbb{R}^n)$.

In light of Step 4, we may assume after passing to a subsequence that

$$\frac{y^1_\alpha}{\lambda^1_\alpha} \to x_0 \in \mathbb{R}^n, \quad \text{as } \alpha \to \infty.$$

We then consider the family of functions defined by

$$v^1_\alpha(x) = \left(\lambda^1_\alpha\right)^\gamma u^1_\alpha(\lambda^1_\alpha(x + x_0)).$$

Since every u^1_α is supported on Ω, it is clear that v^1_α has support in

$$\Omega_\alpha := \frac{\Omega}{\lambda^1_\alpha} - x_0.$$

As a direct consequence of Lemma 2.7.1, $v^1_\alpha \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, -x_0)$ for each $\alpha \in \mathbb{N}$. Furthermore, by the rescaling property in (2.18)-(2.20), the sequence (v^1_α) is bounded in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, -x_0)$. Once again citing Theorem 2.5.1, we may assume without loss of generality that $v^1_\alpha \rightharpoonup v^1$ in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, -x_0)$ with $v^1_\alpha(x) \to v^1(x)$ almost everywhere as $\alpha \to \infty$. Next, for each fixed $\alpha \in \mathbb{N}$, $\phi'(u^1_\alpha)$ is a continuous linear functional on $\mathcal{D}^{1,p}_a(\Omega, 0)$. By Theorem 2.5.3 we can find functions $f^{(1)}_\alpha, \ldots, f^{(n)}_\alpha \in L^{p'}(\Omega, |x|^{-ap})$ such that

$$\left\langle \phi'(u^1_\alpha), h \right\rangle = \sum_{i=1}^n \int_\Omega f^{(i)}_\alpha(x) \partial_i h(x) \, |x|^{-ap} \, \mathrm{d}x, \quad \forall h \in \mathcal{D}^{1,p}_a(\Omega, 0).$$

Moreover, since $\phi'(u^1_\alpha) \to 0$ strongly, we have

$$\sum_{i=1}^n \int_\Omega \left| f^{(i)}_\alpha(x) \right|^{p'} |x|^{-ap} \, \mathrm{d}x \to 0, \quad \text{as } \alpha \to \infty. \tag{3.16}$$

Step 5. For each $h \in \mathcal{D}^{1,p}_a(\Omega_\alpha, -x_0)$ define

$$h_\alpha(x) := h\left(\frac{x}{\lambda^1_\alpha} - x_0\right).$$

As in Lemma 2.7.1, it is not hard to check that $h_\alpha \in \mathcal{D}^{1,p}_a(\Omega, 0)$. Next, consider the functions

$$g^{(i)}_\alpha(x) := \left(\lambda^1_\alpha\right)^{\frac{n-ap}{p'}} f^{(i)}_\alpha\left(\lambda^1_\alpha(x+x_0)\right)$$

with p' being the Hölder conjugate exponent of p. Note that by (3.16),

$$\begin{aligned}
&\sum_{i=1}^n \int_{\Omega_\alpha} \left|g^{(i)}_\alpha(x)\right|^{p'} |x+x_0|^{-ap}\,\mathrm{d}x \\
&\qquad = \sum_{i=1}^n \int_{\Omega_\alpha} \left(\lambda^1_\alpha\right)^{n-ap} \left|f^{(i)}_\alpha(\lambda^1_\alpha(x+x_0))\right|^{p'} |x+x_0|^{-ap}\,\mathrm{d}x \\
&\qquad = \sum_{i=1}^n \int_{\Omega} \left(\lambda^1_\alpha\right)^{-ap} \left|f^{(i)}_\alpha(z)\right|^{p'} |z|^{-ap} \left(\lambda^1_\alpha\right)^{ap}\,\mathrm{d}z \\
&\qquad = \sum_{i=1}^n \int_{\Omega} \left|f^{(i)}_\alpha(z)\right|^{p'} |z|^{-ap}\,\mathrm{d}z \\
&\qquad = o(1)
\end{aligned}$$

as $\alpha \to \infty$. Furthermore, given any $h \in \mathcal{D}^{1,p}_a(\Omega_\alpha, -x_0)$, we claim that

$$\left\langle \phi'_{x_0,\infty}(v^1_\alpha), h\right\rangle = \sum_{i=1}^n \int_{\Omega_\alpha} g^{(i)}_\alpha(x)\partial_i h\,(x)\,|x+x_0|^{-ap}\,\mathrm{d}x. \tag{3.17}$$

Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|\nabla v^1_\alpha\right|^{p-2} \nabla v^1_\alpha \cdot \nabla h\,|x+x_0|^{-ap}\,\mathrm{d}x \\
&\quad = \left(\lambda^1_\alpha\right)^{(\gamma+1)(p-1)} \int_{\mathbb{R}^n} \left|\nabla u^1_\alpha(\lambda^1_\alpha(x+x_0))\right|^{p-2} \nabla u^1_\alpha(\lambda^1_\alpha(x+x_0)) \cdot \nabla h(x)\,|x+x_0|^{-ap}\,\mathrm{d}x \\
&\quad = \left(\lambda^1_\alpha\right)^{(\gamma+1)(p-1)-n+ap} \int_{\mathbb{R}^n} \left|\nabla u^1_\alpha(z)\right|^{p-2} \nabla u^1_\alpha(z) \cdot \nabla h\left(\frac{z}{\lambda^1_\alpha} - x_0\right) |z|^{-ap}\,\mathrm{d}x \\
&\quad = \left(\lambda^1_\alpha\right)^{(\gamma+1)(p-1)-n+ap+1} \int_{\mathbb{R}^n} \left|\nabla u^1_\alpha(z)\right|^{p-2} \nabla u^1_\alpha(z) \cdot \nabla h_\alpha\,(z)\,|z|^{-ap}\,\mathrm{d}x \\
&\quad = \left(\lambda^1_\alpha\right)^{-\gamma} \int_{\Omega} \left|\nabla u^1_\alpha(z)\right|^{p-2} \nabla u^1_\alpha(z) \cdot \nabla h_\alpha\,(z)\,|z|^{-ap}\,\mathrm{d}x
\end{aligned}$$

where we have used that u_α^1 is supported on Ω and (2.13). In a similar vein one has

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|v_\alpha^1(x)\right|^{q-2} v_\alpha(x)h(x)\left|x+x_0\right|^{-bq} \mathrm{d}x \\
&= \left(\lambda_\alpha^1\right)^{\gamma(q-1)} \int_{\mathbb{R}^n} \left|u_\alpha^1(\lambda_\alpha^1(x+x_0))\right|^{q-2} u_\alpha^1(\lambda_\alpha^1(x+x_0))h(x)\left|x+x_0\right|^{-bq} \mathrm{d}x \\
&= \left(\lambda_\alpha^1\right)^{\gamma(q-1)-n+bq} \int_{\mathbb{R}^n} \left|u_\alpha^1(z)\right|^{q-2} u_\alpha^1(z)h\left(\frac{z}{\lambda_\alpha^1}-x_0\right)|z|^{-bq} \mathrm{d}z \\
&= \left(\lambda_\alpha^1\right)^{\gamma(q-1)-n+bq} \int_{\Omega} \left|u_\alpha^1(z)\right|^{q-2} u_\alpha^1(z)h\left(\frac{z}{\lambda_\alpha^1}-x_0\right)|z|^{-bq} \mathrm{d}z \\
&= \left(\lambda_\alpha^1\right)^{-\gamma} \int_{\Omega} \left|u_\alpha^1(z)\right|^{q-2} u_\alpha^1(z)h_\alpha(z)\,|z|^{-bq} \mathrm{d}z.
\end{aligned}$$

Combining these last two identities yields

$$\begin{aligned}
\left\langle \phi'_{x_0,\infty}(v_\alpha^1), h\right\rangle &= \left(\lambda_\alpha^1\right)^{-\gamma} \left\langle \phi'(u_\alpha^1), h_\alpha\right\rangle \\
&= \left(\lambda_\alpha^1\right)^{-\gamma} \sum_{i=1}^n \int_\Omega f_\alpha^{(i)}(z)\partial_i h_\alpha(z)\,|z|^{-ap} \mathrm{d}z \\
&= \left(\lambda_\alpha^1\right)^{-\gamma-1} \sum_{i=1}^n \int_\Omega f_\alpha^{(i)}(z)\partial_i h\left(\frac{z}{\lambda_\alpha^1}-x_0\right)|z|^{-ap} \mathrm{d}z \\
&= \left(\lambda_\alpha^1\right)^{-\gamma-1-ap+n} \sum_{i=1}^n \int_{\Omega_\alpha} f_\alpha^{(i)}(\lambda_\alpha^1(x+x_0))\partial_i h\,(x)\left|x+x_0\right|^{-ap} \mathrm{d}x \\
&= \left(\lambda_\alpha^1\right)^{(n-ap)-\left(\frac{n-ap}{p}\right)} \sum_{i=1}^n \int_{\Omega_\alpha} f_\alpha^{(i)}(\lambda_\alpha^1(x+x_0))\partial_i h\,(x)\left|x+x_0\right|^{-ap} \mathrm{d}x \\
&= \sum_{i=1}^n \int_{\Omega_\alpha} g_\alpha^{(i)}(x)\partial_i h\,(x)\left|x+x_0\right|^{-ap} \mathrm{d}x.
\end{aligned}$$

This verifies (3.17) and ends the proof of Step 5.

Step 6. We claim that $v^1 \neq 0$. By way of contradiction, let us assume that $v^1 = 0$ so that $v_\alpha^1 \rightharpoonup 0$ weakly in $\mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)$ and pointwise almost everywhere on $\mathbb{R}^n$ as $\alpha \to \infty$. Appealing to Theorem 2.5.2, we can assume that $v_\alpha^1 \to 0$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^n, |x+x_0|^{-ap})$. In particular, (v_α^1) is bounded in $L^p_{\mathrm{loc}}(\mathbb{R}^n, |x+x_0|^{-ap})$. Recall that $\lambda^1 := \lim_{\alpha\to\infty} \lambda^1_\alpha \geq 0$. Define

$$\Omega_\infty := \begin{cases} \mathbb{R}^n & \text{if } \lambda^1 = 0, \\ \dfrac{\Omega}{\lambda^1} - x_0 & \text{if } \lambda^1 > 0, \end{cases}$$

and fix a point $y \in \Omega_\infty$. Clearly (see Proposition 3.3.1), $y \in \Omega_\alpha$ for all α sufficiently large. Let now $h \in C^\infty_c(\mathbb{R}^n)$ be such that $\operatorname{supp}(h) \subset B(y,1)$. Combining Hölder's inequality and (2.5) gives

$$\begin{aligned} &\int_{\mathbb{R}^n} |h|^p \left|v^1_\alpha\right|^q |x+x_0|^{-bq}\,\mathrm{d}x \\ &\quad= \int_{\operatorname{supp}(h)} \left|v^1_\alpha h\right|^p \left|v^1_\alpha\right|^{q-p} |x+x_0|^{-bq}\,\mathrm{d}x \\ &\quad\leq \left(\int_{\operatorname{supp}(h)} \left|v^1_\alpha\right|^q |x+x_0|^{-bq}\,\mathrm{d}x\right)^{\frac{q-p}{q}} \left(\int_{\operatorname{supp}(h)} \left|hv^1_\alpha\right|^q |x+x_0|^{-bq}\,\mathrm{d}x\right)^{p/q} \\ &\quad\leq S_p^{-1} \left(\int_{\operatorname{supp}(h)} \left|v^1_\alpha\right|^q |x+x_0|^{-bq}\,\mathrm{d}x\right)^{\frac{q-p}{p}} \left(\int_{\operatorname{supp}(h)} \left|\nabla(hv^1_\alpha)\right|^p |x+x_0|^{-ap}\,\mathrm{d}x\right) \\ &\quad\leq S_p^{-1}\delta^{\frac{q-p}{p}} \left(\int_{\operatorname{supp}(h)} \left|\nabla(hv^1_\alpha)\right|^p |x+x_0|^{-ap}\,\mathrm{d}x\right), \end{aligned} \tag{3.18}$$

where we have used that

$$\sup_{y\in\overline{\Omega_\alpha}} \int_{B(y,1)} \left|v^1_\alpha(x)\right|^q |x+x_0|^{-bq}\,\mathrm{d}x = \sup_{w\in\overline{\Omega}} \int_{B(w,\lambda^1_\alpha)} \left|u^1_\alpha(z)\right|^q |z|^{-bq}\,\mathrm{d}z = \delta.$$

Next, using that $v^1_\alpha \to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^n, |x+x_0|^{-ap})$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \left|\nabla(hv^1_\alpha)\right|^p |x+x_0|^{-ap}\,\mathrm{d}x &\leq 2^{p-1} \int_{\mathbb{R}^n} |h|^p \left|\nabla v^1_\alpha\right|^p |x+x_0|^{-ap}\,\mathrm{d}x + o(1) \\ &= 2^{p-1} \int_{\mathbb{R}^n} \left|\nabla v^1_\alpha\right|^{p-2} \nabla v^1_\alpha \cdot \nabla\left(|h|^p v^1_\alpha\right) |x+x_0|^{-ap}\,\mathrm{d}x + o(1). \end{aligned}$$

This implies that

$$\int_{\mathbb{R}^n} \left|\nabla(hv_\alpha^1)\right|^p |x+x_0|^{-ap}\,\mathrm{d}x \le 2^{p-1}\int_{\mathbb{R}^n} \left|\nabla v_\alpha^1\right|^{p-2}\nabla v_\alpha^1\cdot\nabla\left(|h|^p\, v_\alpha^1\right)|x+x_0|^{-ap}\,\mathrm{d}x + o(1)$$
$$= 2^{p-1}\left[\left\langle \phi'_{x_0,\infty}(v_\alpha^1), |h|^p\, v_\alpha^1\right\rangle + \int_{\mathbb{R}^n}\left|v_\alpha^1\right|^q |h|^p\, |x+x_0|^{-bq}\right] + o(1).$$

As $(|h|^p\, v_\alpha^1)$ is bounded in $\mathcal{D}_a^{1,p}(\Omega_\alpha, -x_0)$, one has by (3.17) that

$$\left\langle \phi'_{x_0,\infty}(v_\alpha^1), |h|^p\, v_\alpha^1\right\rangle = \sum_{i=1}^n \int_{\Omega_\alpha} g_\alpha^{(i)}\partial_i\left(|h|^p\, v_\alpha^1\right)|x+x_0|^{-ap}\,\mathrm{d}x = o(1)$$

as $\alpha\to\infty$. Therefore, combining this with (3.18), we find

$$\int_{\mathbb{R}^n} \left|\nabla(hv_\alpha^1)\right|^p |x+x_0|^{-ap}\,\mathrm{d}x \le 2^{p-1}\int_{\mathbb{R}^n}\left|v_\alpha^1\right|^q |h|^p\, |x+x_0|^{-bq} + o(1)$$
$$\le 2^{p-1} S_p^{-1}\delta^{\frac{q-p}{p}}\left(\int_{\mathrm{supp}(h)}\left|\nabla(hv_\alpha^1)\right|^p |x+x_0|^{-ap}\,\mathrm{d}x\right) + o(1)$$
$$\le \frac{1}{2}\int_{\mathbb{R}^n}\left|\nabla(hv_\alpha^1)\right|^p |x+x_0|^{-ap}\,\mathrm{d}x + o(1).$$

It follows that $\int_{\mathbb{R}^n}|\nabla(hv_\alpha^1)|^p\,|x+x_0|^{-ap}\,\mathrm{d}x\to 0$ as $\alpha\to\infty$. Taking h to be such that $h\equiv 1$ on $B(y,1/2)$, we see that $\nabla v_\alpha^1\to 0$ in $L_a^p\left(B\left(y,1/2\right),-x_0\right)$ for all $y\in\Omega_\infty$. Furthermore, if K is a compact set contained in the complement of $\overline{\Omega_\infty}$, then v_α^1 vanishes on K for all α large. By Lemma 3.3.2, this implies that $\nabla v_\alpha^1\to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^n, |x+x_0|^{-ap})$ as $\alpha\to\infty$.

Given a compact set $\Lambda\subset\mathbb{R}^n$, we can choose a cutoff function $\eta\in C_c^\infty(\mathbb{R}^n)$ that is equal to 1 in a neighbourhood of Λ. As before, by Proposition 2.4.3, one has $\eta v_\alpha^1\in\mathcal{D}_a^{1,p}(\mathbb{R}^n,-x_0)$. Since $v_\alpha^1,\nabla v_\alpha^1\to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^n,|x+x_0|^{-ap})$, it readily follows that $\eta v_\alpha^1\to 0$ strongly in $\mathcal{D}_a^{1,p}(\mathbb{R}^n,-x_0)$. Applying the Caffarelli-Kohn-Nirenberg inequality then implies that $v_\alpha^1\to 0$ strongly in $L_b^q(\Lambda,-x_0)$. Since Λ was arbitrary,

$$v_\alpha^1\to 0 \text{ strongly in } L^q_{\mathrm{loc}}(\mathbb{R}^n,|x+x_0|^{-bq})$$

as $\alpha \to \infty$. On the other hand, for all $\alpha \in \mathbb{N}$ sufficiently large, one has

$$B\left(\frac{y_\alpha^1}{\lambda_\alpha^1} - x_0, 1\right) \subseteq B(0,2)$$

whence a familiar change of variables gives

$$\begin{aligned}
\int_{B(0,2)} \left|v_\alpha^1\right|^q |x+x_0|^{-bq} \,\mathrm{d}x &\geq \int_{B\left(\frac{y_\alpha^1}{\lambda_\alpha^1} - x_0, 1\right)} \left|v_\alpha^1\right|^q |x+x_0|^{-bq} \,\mathrm{d}x \\
&= (\lambda_\alpha^1)^{\gamma q} \int_{B\left(\frac{y_\alpha^1}{\lambda_\alpha^1} - x_0, 1\right)} \left|u_\alpha^1(\lambda_\alpha^1(x+x_0))\right|^q |x+x_0|^{-bq} \,\mathrm{d}x \\
&= (\lambda_\alpha^1)^{\gamma q - n + bq} \int_{B(y_\alpha^1, \lambda_\alpha^1)} \left|u_\alpha^1(z)\right|^q |z|^{-bq} \,\mathrm{d}x \\
&= \delta > 0.
\end{aligned}$$

This is a contradiction.

Step 7. We claim that $\lambda^1 = \lim_{\alpha\to\infty} \lambda_\alpha^1 = 0$. Once again, we argue by contradiction and assume that $\lambda^1 > 0$. Fix a test function $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and note that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \nabla v_\alpha^1(x) \cdot \varphi(x) \, |x+x_0|^{-ap} \,\mathrm{d}x \\
&\quad = (\lambda_\alpha^1)^{(\gamma+1)} \int_{\mathbb{R}^n} \nabla u_\alpha^1(\lambda_\alpha^1(x+x_0)) \cdot \varphi(x) \, |x+x_0|^{-ap} \,\mathrm{d}x \\
&\quad = (\lambda_\alpha^1)^{(\gamma+1)-n+ap} \int_{\mathbb{R}^n} \nabla u_\alpha^1(z) \cdot \varphi\left(\frac{z}{\lambda_\alpha^1} - x_0\right) |z|^{-ap} \,\mathrm{d}z.
\end{aligned}$$

As the sequence (λ_α^1) is bounded, there exists a compact set $\Lambda \subset \mathbb{R}^n$ such that

$$\operatorname{supp}\left(\varphi\left(\frac{\cdot}{\lambda_\alpha^1} - x_0\right)\right) \subseteq \Lambda, \quad \forall \alpha \in \mathbb{N}.$$

Hence,

$$\left|\int_{\mathbb{R}^n} \nabla v_\alpha^1(x) \cdot \varphi(x) \, |x+x_0|^{-ap} \,\mathrm{d}x\right| \leq M(\lambda_\alpha^1)^{(\gamma+1)-n+ap} \int_\Lambda \left|\nabla u_\alpha^1(z)\right| |z|^{-ap} \,\mathrm{d}z.$$

where $M := \|\varphi\|_{L^\infty(\mathbb{R}^n)}$. Since $\nabla u_\alpha^1 \to 0$ pointwise almost everywhere on $\mathbb{R}^n$ and is bounded in $L_a^p(\mathbb{R}^n, 0)$, an application of Theorem 2.1.2 shows that

$$\int_\Lambda \left|\nabla u_\alpha^1(z)\right| |z|^{-ap} \, \mathrm{d}z \to 0, \quad \text{as } \alpha \to \infty,$$

Finally, since $\lambda_\alpha^1 \to \lambda^1 > 0$, we see that $(\lambda_\alpha^1)^{(\gamma+1)-n+ap}$ is bounded. It follows that

$$\int_{\mathbb{R}^n} \nabla v_\alpha^1(x) \cdot \varphi(x) \, |x + x_0|^{-ap} \, \mathrm{d}x \to 0, \quad \text{as } \alpha \to \infty.$$

Let $\varepsilon > 0$ be given and fix $g \in L^{p'}(\mathbb{R}^n, |x + x_0|^{-ap})$. By Theorem 2.2.1, we can find $\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\|g - \varphi\|_{L^{p'}(\mathbb{R}^n, |x+x_0|^{-ap})} < \varepsilon$. Then, by our calculations above, we find that

$$\begin{aligned}
\int_{\mathbb{R}^n} \nabla v_\alpha^1(x) \cdot g(x) \, |x + x_0|^{-ap} \, \mathrm{d}x &= \int_{\mathbb{R}^n} \nabla v_\alpha^1(x) \cdot (g - \varphi)(x) \, |x + x_0|^{-ap} \, \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^n} \nabla v_\alpha^1(x) \cdot \varphi(x) \, |x + x_0|^{-ap} \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \nabla v_\alpha^1(x) \cdot (g - \varphi)(x) \, |x + x_0|^{-ap} \, \mathrm{d}x \\
&\quad + o(1)
\end{aligned}$$

as $\alpha \to \infty$. By Hölder's inequality, we obtain

$$\left| \int_{\mathbb{R}^n} \nabla v_\alpha^1(x) \cdot g(x) \, |x + x_0|^{-ap} \, \mathrm{d}x \right| \le C\varepsilon + o(1)$$

with $C := \sup_{\alpha \in \mathbb{N}} \|v_\alpha^1\|_{\mathcal{D}_a^{1,p}(\mathbb{R}^n, -x_0)} < \infty$. It follows that

$$\limsup_{\alpha \to \infty} \left| \int_{\mathbb{R}^n} \nabla v_\alpha^1(x) \cdot g(x) \, |x + x_0|^{-ap} \, \mathrm{d}x \right| \le C\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we infer that

$$\lim_{\alpha \to \infty} \int_{\mathbb{R}^n} \nabla v_\alpha^1(x) \cdot g(x) \, |x + x_0|^{-ap} \, \mathrm{d}x = 0$$

for all $g \in L^{p'}(\mathbb{R}^n, |x+x_0|^{-ap})$. That is, $\nabla v_\alpha^1 \rightharpoonup 0$ in $L^p_a(\mathbb{R}^n, -x_0)$. By Proposition 2.5.1, this means that $v_\alpha^1 \rightharpoonup 0$ in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, -x_0)$, which contradicts Step 6.

Step 8. Since $v^1 \in \mathcal{D}^{1,p}_a(\mathbb{R}^n, -x_0)$ and $\Omega_\alpha \to \mathbb{R}^n$ in the sense of Definition 3.3.1 (see Proposition 3.3.1), we can assume that there exist functions $\psi_\alpha \in C_c^\infty(\Omega_\alpha)$ such that $\psi_\alpha \to v^1$ strongly in $\mathcal{D}^{1,p}_a(\mathbb{R}^n, -x_0)$ as $\alpha \to \infty$. Passing to a subsequence if necessary, we claim that the sequence in $C_c^\infty(\Omega)$ given by

$$\widetilde{\psi}_\alpha(x) := \left(\lambda_\alpha^1\right)^{-\gamma} \psi_\alpha\left(\frac{x}{\lambda_\alpha^1} - x_0\right) \tag{3.19}$$

satisfies the following:

(i) $\left(\widetilde{\psi}_\alpha\right)$ is bounded in $\mathcal{D}^{1,p}_a(\Omega, 0)$ and converges weakly to 0 as $\alpha \to \infty$;

(ii) $\widetilde{\psi}_\alpha \to 0$ and $\nabla\widetilde{\psi}_\alpha \to 0$ pointwise almost everywhere on $\mathbb{R}^n$.

Certainly, we first note that

$$\left\|\widetilde{\psi}_\alpha\right\|_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)} = \|\psi_\alpha\|_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,-x_0)}$$

for each $\alpha \in \mathbb{N}$. Therefore, by Theorem 2.5.1, we may pass to a subsequence converging weakly and pointwise almost everywhere to a function $\psi \in \mathcal{D}^{1,p}_a(\Omega, 0)$. On the other hand, an easy change of variables shows that

$$\widetilde{\psi}_\alpha(\cdot) - \left(\lambda_\alpha^1\right)^{-\gamma} v^1\left(\frac{\cdot}{\lambda_\alpha^1} - x_0\right) \to 0 \quad \text{in } \mathcal{D}^{1,p}_a(\mathbb{R}^n, 0).$$

Our assertions then readily follow from the observations made in Remark 3.2.1.

Step 9. We now begin our iteration process. Applying Lemma 3.2.3, we get that the sequence

$$u_\alpha^2(x) := u_\alpha^1(x) - \left(\lambda_\alpha^{(1)}\right)^{-\gamma} v^1\left(\frac{x}{\lambda_\alpha^{(1)}} - x_0^{(1)}\right),$$

for $x_0^{(1)} := x_0$ and $\lambda_\alpha^{(1)} := \lambda_\alpha^1$, satisfies

$$\begin{aligned}\left\|u_n^2\right\|^p_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,0)} = \|u_\alpha\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} \\ - \|v_0\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)} - \left\|v^1\right\|^p_{\mathcal{D}^{1,p}(\mathbb{R}^n,-x_0^{(1)})} + o(1),\end{aligned}$$

and

$$\begin{cases}\phi_{0,\infty}(u_\alpha^2) \to c - \phi(v_0) - \phi_{x_0^{(1)},\infty}(v^1), \\ \phi'_{0,\infty}(u_\alpha^2) \to 0 \text{ in } \mathcal{D}_a^{-1,p'}(\Omega,0).\end{cases}$$

Next, consider the auxiliary sequence

$$\widetilde{u}_\alpha^2(x) := u_\alpha^1(x) - \widetilde{\psi}_\alpha(x)$$

which is bounded in $\mathcal{D}_a^{1,p}(\Omega,0)$. Because $\psi_\alpha \to v^1$ strongly in $\mathcal{D}_a^{1,p}(\mathbb{R}^n,-x_0^{(1)})$,

$$\begin{cases}\phi\left(\widetilde{u}_\alpha^2\right) \to c - \phi(v_0) - \phi_{x_0^{(1)},\infty}(v^1), \\ \phi'(\widetilde{u}_\alpha^2) \to 0 \text{ in } \mathcal{D}_a^{-1,p'}(\Omega,0).\end{cases}$$

We may therefore apply Steps 2-9 to this new sequence $(\widetilde{u}_\alpha^2)$, at each stage removing another solution v^j. By Lemma 3.1.2 and Proposition 3.2.2, this procedure can only happen finitely many times before $\widetilde{u}_\alpha^k \to 0$ strongly in $L_b^q(\Omega,0)$, at which point we will find ourselves in Step 2. To summarize, there exists

(1) a subsequence (u_β) of (u_α);

(2) a solution v_0 of (1.7);

(3) finitely many non-trivial functions $v^1,\dots,v^k$;

(4) sequences $(\lambda_\beta^{(j)}) \subset (0,\infty)$ for $j = 1,\dots,k$;

such that each v^j solves

$$\begin{cases} -\operatorname{div}\left(\left|x+x_0^{(j)}\right|^{-ap}|\nabla u|^{p-2}\nabla u\right)=\left|x+x_0^{(j)}\right|^{-bq}|u|^{q-2}u \quad \text{in } \mathbb{R}^n, \\ u\in\mathcal{D}_a^{1,p}(\mathbb{R}^n,-x_0^{(j)}) \end{cases}$$

and $\lim\lambda_\beta^{(j)}=0$. Furthermore, as $\beta\to\infty$,

$$\left\|u_\beta-v_0-\sum_{j=1}^k\left(\lambda_\alpha^{(j)}\right)^{-\gamma}v^j\left(\frac{\cdot}{\lambda_\beta^{(j)}}-x_0^{(j)}\right)\right\|_{\mathcal{D}_a^{1,p}(\mathbb{R}^n,0)}\to 0,$$

$$\|u_\beta\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)}\to\|v_0\|^p_{\mathcal{D}_a^{1,p}(\Omega,0)}+\sum_{j=1}^k\left\|v^j\right\|^p_{\mathcal{D}^{1,p}(\mathbb{R}^n,-x_0^{(j)})},$$

$$\phi(u_\beta)\to\phi(v_0)+\sum_{j=1}^k\phi_{x_0^{(j)},\infty}(v^j).$$

Step 10. For each index $j=1,\ldots,k$ we define the function

$$w_j(x):=v^j\left(x-x_0^{(j)}\right).$$

Since v^j is a solution to the problem

$$\begin{cases} -\operatorname{div}\left(\left|x+x_0^{(j)}\right|^{-ap}|\nabla u|^{p-2}\nabla u\right)=\left|x+x_0^{(j)}\right|^{-bq}|u|^{q-2}u \quad \text{in } \mathbb{R}^n, \\ u\in\mathcal{D}_a^{1,p}(\mathbb{R}^n,-x_0^{(j)}), \end{cases}$$

an easy change of variables shows that w_j solves

$$\begin{cases} -\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right)=|x|^{-bq}|u|^{q-2}u \quad \text{in } \mathbb{R}^n, \\ u\in\mathcal{D}_a^{1,p}(\mathbb{R}^n,0). \end{cases}$$

Furthermore,

$$\left\|v^j\right\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,-x_0^{(j)})} = \|w_j\|^p_{\mathcal{D}^{1,p}_a(\mathbb{R}^n,0)} \quad \text{and} \quad \phi_{x_0^{(j)},\infty}(v^j) = \phi_{0,\infty}(w_j).$$

Since

$$v^j\left(\frac{\cdot}{\lambda_\beta^{(j)}} - x_0^{(j)}\right) = w_j\left(\frac{\cdot}{\lambda_\beta^{(j)}}\right),$$

the theorem readily follows from the conclusions of Step 9.

Chapter 4

Conclusion

A natural follow up to Theorem 3.4.1 is to ask what occurs in the limit case where $a = b$. Given that this would include the model unweighted p-Laplace problem

$$\begin{cases} -\Delta_p u + a\,|u|^{p-2}\,u \equiv |u|^{p^*-2}\,u & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega) \end{cases}$$

as a special case, we expect many of the same difficulties present in Merucri-Willem [17] to arise, in particular involving the boundary of the domain Ω.

By inspecting the proof of Theorem 3.4.1, we see that the conclusions drawn in Step 4 fail when $a = b$. Instead, Step 4 only implies that the sequence (v_α) is *bounded* in $\mathcal{D}^{1,p}(\mathbb{R}^n)$. Therefore, we cannot obtain the desired contradiction if we allow for the possibility that $a = b$. As a consequence, we anticipate bubbles different from those obtained when $a \neq b$. Additionally, it is a priori possible that the sequences $(y_\alpha^{(j)})$ converge to a point in $\overline{\Omega}$ different from 0. In particular, without any sign assumptions on the Palais-Smale sequence, it is reasonable to expect potential bubbling on the boundary of Ω. Here, we suspect the bubbles to solve the same type of limiting problem, but in a half space. Therefore, in order to obtain an extension of Theorem 3.4.1 including the extremal case $a = b$, we strongly suspect that one would need to impose regularity conditions upon the boundary of Ω.

We also point out that, when $a = b$, the sequence of points $y_\alpha^{(j)}/\lambda_\alpha^{(j)}$ obtained in the proof of Theorem 3.4.1 need not be bounded, as far as the author can tell. In this case, the rescaling law exploited in the main proof is no longer well defined and we must consequently find a different rescaling of u. However, as Steps 5 - 8 rely on precisely this rescaling law, the remainder of the proof will also require modification. Namely, we must extract a bubble directly from the sequence (v_α) used in the proof of Step 4. Interestingly, such a bubble would likely live in $\mathcal{D}^{1,p}(\mathbb{R}^n)$ and solve a problem not involving weights (either in $\mathbb{R}^n$ or in a half-space). The treatment of this limit case is a work in progress and the subject of a paper in preparation.

Finally, in future endeavours, we hope to use the compactness result in Theorem 3.4.1 to address questions relating to the existence of non-trivial solutions to (1.7). To achieve this, we hope to provide conditions under which one can construct suitable Palais-Smale sequences for (1.7) that converge strongly to a non-trivial solution. In addition, we also hope to treat questions of multiplicity for the problem (1.7) in future works.

Bibliography

[1] Alves, C. Existence of Positive Solutions for a Problem With Lack of Compactness Involving the p-Laplacian. Nonlinear Analysis 51, 2002, 1187-1206.

[2] Adams, R. A. and Fournier, J. Sobolev Spaces, 2nd ed. Pure and Applied Mathematics, V. 140. Amsterdam: Academic Press, 2003.

[3] Barletta G.; Candito P.; Marano S.A.; Perera K. Multiplicity Results for Critical p-Laplacian Problems. Annali Di Matematica Pura Ed Applicata, 2017, 196 (4), 1431-1440.

[4] Brézis H. Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext. New York: Springer, 2011.

[5] Brezis, H. and Coron, J. Convergence of Solutions of H-Systems or How to Blow Bubbles. Archive for Rational Mechanics and Analysis, vol. 89, no. 1, 1985, pp. 21-56.

[6] Brézis, H. and Lieb, E. A Relation Between Pointwise Convergence of Functions and Convergence of Functionals. Proceedings of the American Mathematical Society, 1983.

[7] Caffarelli, L.; Gidas , B. and Spruck, J. Asymptotic Symmetry and Local Behavior of Semilinear Elliptic Qquations With Critical Sobolev Growth. Communications on Pure and Applied Mathematics, 42(3), 1989, 271-297.

[8] Chernysh, E. A Global Compactness Theorem for Critical p-Laplace Equations with Weights. In preparation.

[9] Clapp, M. and Lopez Rios, L. Entire Nodal Solutions to the Pure Critical Exponent Problem for the P-Laplacian. Journal of Differential Equations, 2018, 265 (3), 891-905.

[10] Clapp, M. and Weth, T. Multiple Solutions for the Brézis-Nirenberg Problem. Adv. Differential Equations 10, 2005, no. 4, 463–480.

[11] Damascelli, L.; Merchán S.; Montoro, L. and Sciunzi B. Radial Symmetry and Applications for a Problem Involving the $-\Delta_p(\cdot)$ Operator and Critical Nonlinearity in $\mathbb{R}^N$. Advances in Mathematics, 256, 2014, 313-335.

[12] Damascelli, L. and Ramaswamy, M. Symmetry of C^1 Solutions of p-Laplace Equations in $\mathbb{R}^N$. Advanced Nonlinear Studies, 1, 2001, no. 1, 40-64.

[13] Devillanova, G. and Solimini, S. Concentration Estimates and Multiple Solutions to Elliptic Problems at Critical Growth. Adv. Differential Equations 7 (2002), no. 10, 1257–1280.

[14] Evans, L. Partial Differential Equations. AMS Graduate Texts in Mathematics.

[15] Hewitt, E. and Stromberg, K. R. Real and Abstract Analysis: A Modern Treatment of the Theory of Functions of a Real Variable. Graduate Texts in Mathematics, 25. Springer-Verlag: New York, 1965.

[16] Jakszto, M. Another Proof That L^p-bounded Pointwise Convergence Implies Weak Convergence. Real Analysis Exchange, Volume 36, Number 2 (2010), 479-482

[17] Mercuri, C. and Willem, M. A Global Compactness Result for the p-Laplacian Involving Critical Nonlinearities. Discrete and Continuous Dynamical Systems. Volume 28, Number 2, October 2010.

[18] Sciunzi, B. Classification of Positive $\mathcal{D}^{1,p}(\mathbb{R}^N)$-Solutions to the Critical p-Laplace Equation in $\mathbb{R}^N$. Advances in Mathematics, Volume 291, 2016.

[19] Struwe, M. Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. Fourth edition. Springer, A Series of Modern Surveys in Mathematics. 1990.

[20] Struwe, M. A Global Compactness Result for Elliptic Boundary Value Problems Involving Limiting Nonlinearities. Mathematische Zeitschrift, 187, 1984, 511-517.

[21] Szulkin, A. and Willem, M. On Some Weakly Coercive Quasilinear Problems With Forcing. Journal d'Analyse Mathmatique, 2020.

[22] Vétois, J. A Priori Estimates and Application to the Symmetry of Solutions for Critical p-Laplace Equations. Journal of Differential Equations 260, 2016, no. 1, 149-161.

[23] Vétois, J. Muliple Solutions for Nonlinear Elliptic Equations on Compact Riemannian Manifolds. International Journal of Mathematics 18, 2007, no. 9, 1071-1111.

[24] Willem, M. Functional Analysis: Fundamentals and Applications. Cornerstones. New York: Birkhaüser, 2013.

[25] Willem, M. Minimax Theorems. Birkhaüser, 1996.

[26] Ziemer, W. Weakly Differentiable Functions: Sobolev Spaces and Functions of Bounded Variation. Springer-Verlag, 1989.