

Claw-free graphs and two conjectures
on ω , Δ , and χ

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Abstract

This thesis concerns the relationship between four graph invariants: ω , χ_f , χ , and Δ . These are the clique number, the fractional chromatic number, the chromatic number, and the maximum degree, respectively¹. Trivially $\omega \leq \chi_f \leq \chi \leq \Delta + 1$. We seek to improve the upper bound on χ . We are motivated by a conjecture of Reed, which essentially states that χ is at most the average of its trivial upper and lower bounds:

Conjecture. *For any graph, $\chi \leq \lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$.*

We call this the Main Conjecture, and propose a Local Strengthening based on the neighbourhood of a single vertex:

Conjecture. *For any graph G , $\chi \leq \max_{v \in V(G)} \lceil \frac{1}{2}(d(v) + 1 + \omega(G[\bar{N}(v)]) \rceil$.*

We begin by showing that much of the early evidence supporting the Main Conjecture also supports the Local Strengthening. In particular, the variant of the Local Strengthening obtained by replacing χ by χ_f holds, as does the Local Strengthening when the stability number is two.

Guided by the first of these results we look towards line graphs, for which χ_f and χ agree asymptotically. We prove the Main Conjecture for line graphs, then we seek to generalize this result.

To do this we use recent results of Chudnovsky and Seymour, who characterized the structure of all claw-free graphs. We refine their results by introducing a graph reduction on certain types of homogeneous pairs of cliques that preserves the chromatic number. Thus we need only consider the problem of colouring *skeletal* claw-free graphs, which cannot be reduced. The structure of skeletal claw-free graphs is simpler than that of general claw-free graphs.

¹We use standard graph theory notation, which can be found in the glossary.

We generalize two results from line graphs to the class of quasi-line graphs. Namely, that the Main Conjecture holds, and that χ_f and χ agree asymptotically. We then consider all claw-free graphs. We prove the Main Conjecture for all claw-free graphs and we prove the Local Strengthening for claw-free graphs with a three-colourable complement. Our proofs yield polynomial-time colouring algorithms that achieve the conjectured bounds.

Résumé

Cette thèse a pour sujet la relation entre quatre invariants de graphes : ω , χ_f , χ , et Δ . Il s'agit respectivement du nombre de clique, du nombre chromatique fractionnaire, du nombre chromatique, et du degré maximum. Ces paramètres vérifient trivialement l'encadrement suivant : $\omega \leq \chi_f \leq \chi \leq \Delta + 1$, dans lequel on cherche à améliorer la borne supérieure sur χ . Une des principales motivations pour ce travail est une conjecture de Reed, qui dit essentiellement que χ est au plus la moyenne de ses bornes inférieures et supérieures triviales.

Conjecture. *Pour tout graph, $\chi \leq \lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$.*

On appelle cet énoncé la Conjecture Principale, et on propose un Renforcement Local basé sur le voisinage de chaque sommet.

Conjecture. *Pour tout graphe G , $\chi \leq \max_{v \in V(G)} \lceil \frac{1}{2}(d(v) + 1 + \omega(G[\bar{N}(v)]) \rceil$.*

On commence par montrer que la plupart des arguments en faveur de la Conjecture Principale incitent également à croire que le Renforcement Local est vrai. En particulier, la borne donnée par le Renforcement Local vaut pour χ_f et le Renforcement Local peut être montré lorsque le nombre de stabilité vaut deux.

Guidé par ces premiers pas, on s'intéresse aux graphes adjoints, pour lesquels χ_f et χ sont asymptotiquement équivalents. On montre la Conjecture Principale dans le cas des graphes adjoints et on cherche ensuite à généraliser ce résultat.

Pour cela on utilise des résultats récents de Chudnovsky et Seymour, qui ont caractérisé la structure des graphes sans griffes. On affine ces résultats en introduisant la notion de graphes squelettes. Dans les problèmes auxquels on s'intéresse, on peut facilement se ramener au cas des graphes squelettes, et la structure des graphes squelettes sans griffes est plus simple que celle des graphes sans griffes en général.

On étend deux résultats des graphes adjoints aux graphes quasi-adjoints : on montre que la Conjecture Principale est vérifiée pour ces graphes et que leurs nombres chromatique et chromatique fractionnaire sont asymptotiquement équivalents. On considère ensuite l'ensemble des graphes sans griffes, qui sont construits de deux manières différentes par deux opérations de composition, selon que $\chi(\overline{G}) \leq 3$ ou pas. On prouve la Conjecture Principale pour tous les graphes sans griffes et le Renforcement Local pour les graphes sans griffes dont le complémentaire est 3-colorable. Les preuves de ces résultats donnent des algorithmes de coloration en temps polynomial.

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Part I

ω , Δ , and χ

It is well-known and easy to prove that for any graph², the chromatic number χ is at least the clique number ω , and at most one more than the maximum degree Δ . We seek better bounds on the chromatic number in terms of ω and Δ .

Our work originates from a conjecture of Reed, who proposed that χ is essentially closer to ω than to $\Delta+1$. Our main goal is to prove this Main Conjecture for claw-free graphs, but there will be many detours along the way.

In the first chapter we cover the necessary graph colouring preliminaries. We look all the way back to Brooks' Theorem, the first step in bounding the chromatic number away from its trivial upper bound. We also discuss the polyhedral approach to graph colouring and its relationship to the theory of perfect graphs.

In Chapter 2 we trace the origin of the Main Conjecture, beginning with early attempts to strengthen Brooks' Theorem. We also propose a new, stronger variant of the Main Conjecture, which we call the Local Strengthening. We present three early results that support the Main Conjecture, and prove that their local analogues hold, thereby giving evidence in support of the Local Strengthening. The first of these states that the Local Strengthening holds for fixed $\Delta - \omega$ and sufficiently large Δ . The second states that the bound proposed by the Local Strengthening holds for the fractional chromatic number; this is an unpublished result of McDiarmid. The third states that the Local Strengthening holds for graphs containing no stable set of size three. This is the first step towards bounding the chromatic number of claw-free

²We use standard graph theory notation, which can be found in the glossary.

graphs.

Chapter 1

Introduction

This thesis concerns colourings of graphs¹. Graphs arise as a natural model of networks in a variety of fields, including cellular biology, industrial optimization, communications network management, sociology, and of course computer science. In such models the edges represent the connections of the network. Edges can also represent undesirable interaction between elements of a set. For example, the vertices of a graph may represent events, and an edge between two vertices may indicate that the corresponding events cannot be run simultaneously. In such situations it is useful to partition the events into subsets of events that can be run simultaneously. This is one type of problem for which graph colouring is a very natural and practical model.

A *k-colouring* of a graph G is an assignment of k colours to the vertices of G under which no two adjacent vertices receive the same colour. The *chromatic number* of G , denoted $\chi(G)$, is the least k for which there is a k -colouring of G .

We are particularly interested in three graph invariants, $\chi(G)$ being the first. The second is the *maximum degree* of G , denoted by $\Delta(G)$ and equal to $\max_{v \in V(G)} \{d(v)\}$ where $d(v)$ is the degree of the vertex v . A *clique* in G is a set of pairwise adjacent vertices. The third key invariant is the *clique number* of G , denoted by $\omega(G)$ and equal to the size of a largest clique in G .

Computing the chromatic number of a graph is a difficult (i.e. NP-complete) problem. Indeed even approximating the chromatic number is difficult [FK98]. The same is true for the clique number [Hås99], although computing the maximum degree is a simple matter.

¹Throughout the thesis we use standard graph theory notation, which can be found in the glossary.

Bounding the chromatic number is the central concern of this thesis. We investigate two approaches to doing so. The first is to obtain bounds by looking at the local structure of the graph. It is easy to see that the clique number of a graph G gives a lower bound for the chromatic number: there is some set of $\omega(G)$ mutually adjacent vertices, and in any proper colouring of G these vertices must each receive a different colour. Thus at least $\omega(G)$ colours are needed and $\omega(G) \leq \chi(G)$. This is an example of a local bound, i.e. a bound that is determined by the vertices of distance at most k from some fixed vertex, for some fixed k (in this case $k = 1$). A local upper bound for $\chi(G)$ is $\Delta(G) + 1$. To see this we take $\Delta(G) + 1$ colours and attempt to colour the vertices one by one in some arbitrary order. Since no vertex has more than $\Delta(G)$ neighbours, when we come to colour a vertex there will always be a colour not already appearing in its neighbourhood. So $\chi(G) \leq \Delta(G) + 1$.

Our second approach to bounding the chromatic number is to consider the fractional relaxation of the natural integer program that describes the chromatic number (the uninitiated reader will find definitions and some basic results in Section 1.2). The solution to this fractional relaxation is the *fractional chromatic number*, denoted by $\chi_f(G)$. Fractional colourings, interesting in their own right, sometimes give us insight to the chromatic number in the form of approximation results – we will prove such a result in Chapter 7. For some classes of graphs (e.g. line graphs and perfect graphs) it is very fruitful to consider optimization problems over a variety of polyhedra. We will consider this flavour of approach, along with the study of perfect graphs, in Chapter 3.

In Chapter 4 we consider two classes of graphs for which the chromatic number is close to the fractional chromatic number: line graphs and circular interval graphs. In later chapters we will focus on two graph classes which generalize line graphs, using structural properties of graphs in these classes to bound their chromatic number. The first is the class of *quasi-line* graphs, in which the neighbours of any vertex can be covered by two cliques. The second, even more general than quasi-line graphs, is the class of *claw-free* graphs, in which no neighbourhood contains a stable set of size three.

In Chapters 5, 6 and 7 we consider quasi-line graphs. Their structure was described by Chudnovsky and Seymour, who generalized earlier work of Chvátal, Maffray, Reed, and Sbihi on the structure of Berge quasi-line graphs. In studying quasi-

line graphs we will introduce several tools and methods that will help us describe and colour claw-free graphs. In Chapter 7 we prove two bounds on the chromatic number of quasi-line graphs, thereby generalizing known results on line graphs.

In Chapter 8 we consider claw-free graphs with $\alpha \geq 4$ that are not quasi-line. Such a graph contains an odd antihole in some neighbourhood. In fact as Fouquet showed, such a graph contains an induced C_5 in some neighbourhood. By examining the structure around this W_5 we can decompose such graphs and reduce them to quasi-line graphs. In Chapter 9 we use Chudnovsky and Seymour's recent structure theorems for claw-free graphs to give a specific description of all claw-free graphs. There is a unified approach, covering the case $\alpha \leq 3$ and reducing from claw-free graphs and quasi-line graphs to line graphs in one step. They also use a completely different technique for decomposing claw-free graphs with a three-colourable complement. In Chapter 10 we give a new bound on the chromatic number of claw-free graphs, in particular proving a conjecture of Reed on the relationship between ω , Δ , and χ .

The remainder of this chapter gives a gentle introduction to graph colouring and presents the two aforementioned approaches to the problem. We say that a vertex u *sees* a vertex v precisely if u and v are adjacent, and that vertex sets S and T are *complete* or *joined* to one another if every possible edge between them exists. If no edge exists between them they are *anticomplete*. When we say that a graph G *contains* a graph H we mean that H is an induced subgraph of G . A *hole* is a chordless cycle of length ≥ 4 and an *antihole* is the complement of a hole.

1.1 Local bounds on χ

We have already shown that by virtue of easy local bounds on the chromatic number, $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$. Either one of these bounds can be tight. For example if G is a complete graph, i.e. the entire vertex set of G forms a clique, then $\omega(G) = \Delta(G) + 1$ and both bounds are tight. If G is a cycle of odd length then $\omega(G) = 2$ and $\Delta(G) + 1 = 3$, but in any 2-colouring of G we will have two adjacent vertices of the same colour, so $\chi(G) = 3 = \Delta(G) + 1$.

In 1941, Brooks [Bro41] proved that odd cycles and cliques are the only two situations in which a connected graph G will have $\chi(G) = \Delta(G) + 1$. The proof we give is similar to the one given by Lovász in [Lov73].

Theorem 1.1 (Brooks' Theorem). *For any graph G , $\chi(G) \leq \Delta(G)$ unless some component of G is a clique of size $\Delta + 1$, or some component of G is an odd cycle and $\Delta(G) = 2$.*

Proof. Let G be a minimum counterexample. We know that graphs with $\Delta = 2$ are either bipartite or odd cycles, so G must have maximum degree at least three. It is easy to see that G has no cutvertex v , otherwise we could apply induction to $\Delta(G)$ -colour $G' \cup \{v\}$ for each component G' of $G - v$. By permuting the colour classes in the coloured graphs, we can ensure that the same colour appears on v for each G' , so we can paste the coloured subgraphs together on v to complete a $\Delta(G)$ -colouring of G .

So G is 2-connected with maximum degree at least three. We claim that there are three vertices v_1 , v_2 , and v_n such that v_n sees both v_1 and v_2 , which do not see each other, and such that $G - \{v_1, v_2\}$ is connected. We make the easy proof an exercise for the conscientious reader. It appears in the proof of Brooks' Theorem in [Wes00].

Beginning with v_n , we will label the vertices of $G - \{v_1, v_2\}$ from v_{n-1} down to v_3 . To do this, we repeatedly label a vertex which already has a labeled neighbour. This is possible because $G - \{v_1, v_2\}$ is connected. This gives us an ordering v_1, v_2, \dots, v_n of the vertices of G such that for every $i < n$, v_i has a neighbour v_j for some $j > i$. Consequently we can give v_1 and v_2 the same colour, then greedily $\Delta(G)$ -colour the remaining vertices of G in order from v_3 to v_n . Since our ordering ensures that no vertex will ever have $\Delta(G)$ colours already in its neighbourhood when we want to colour it, we can complete the $\Delta(G)$ -colouring safely. \square

This proof introduces two important ideas. First is the idea of repeated colours in a neighbourhood – in this case we begin by insisting on a repeated colour in the neighbourhood of v_n . Second is the idea of ensuring that in a sequential colouring, a vertex has few neighbours appearing before it in the ordering. Both ideas ensure that when we want to colour a vertex, few colours will already appear in its neighbourhood.

In contrast to Brooks' characterization of graphs for which $\chi(G) = \Delta(G) + 1$, the class of graphs for which $\chi(G) = \omega(G)$ is much richer. These graphs include, for example, the class of *perfect graphs* – a graph is perfect precisely if $\chi(H) = \omega(H)$ for every induced subgraph H of G .

Given a multigraph H , the *line graph* of H , denoted $G = L(H)$, is constructed as follows. Let $V(G) = E(H)$, and let two vertices in G be adjacent precisely if their

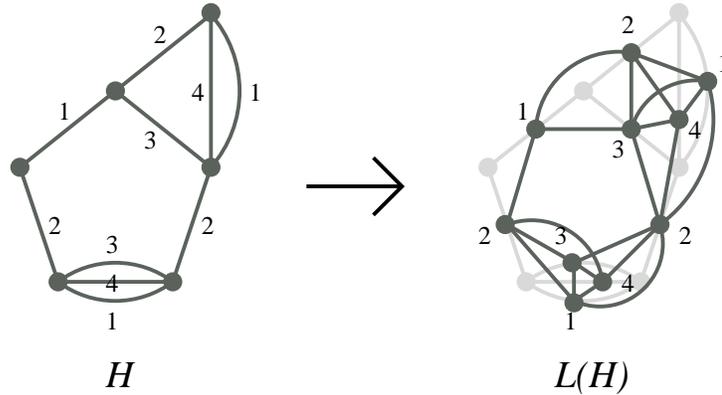


Figure 1.1: A multigraph H and its line graph $G = L(H)$. An edge colouring of H corresponds to a vertex colouring of G .

corresponding edges in H share an endpoint. Note that the matchings of H correspond naturally to the stable sets of G . Likewise an edge colouring of H corresponds naturally to a vertex colouring of G . We say that G is a *line graph* if there is some multigraph H for which $G = L(H)$, and if there is a simple graph $G = L(H)$ we say that G is a *line graph of a simple graph*. We will explore line graphs and their colourings in much more depth later in this thesis, but for now we will just state one of the first fundamental results in the area, Vizing’s Theorem. The *multiplicity* of a multigraph is the maximum number of edges between two vertices.

Theorem 1.2 (Vizing’s Theorem [Viz64]). *If G is the line graph of a multigraph H of multiplicity μ , then $\chi(G) \leq \Delta(G) + \mu$.*

Corollary 1.3. *If G is the line graph of a simple graph H , then $\chi(G) \leq \omega(G) + 1$.*

For perfect graphs and line graphs of simple graphs, $\chi(G)$ is always near $\omega(G)$ but may be far from $\Delta(G)$. Is $\chi(G)$ essentially always as close to $\omega(G)$ as it is to $\Delta(G) + 1$? We explore this question throughout the thesis, particularly in Chapters 2, 4, 7, and 10. Brooks’ Theorem tells us that if $\Delta(G) > 2$, then the bound of $\Delta(G) + 1$ is tight for $\chi(G)$ only if the bound $\omega(G)$ is tight.

1.1.1 Sequential colouring

We proved earlier that any graph G can be coloured using $\Delta(G) + 1$ colours regardless of the order in which we process the vertices. As we showed in the proof of Brooks' Theorem, we can hope to improve the situation by carefully choosing the order in which we colour the vertices. Ideally we would like to ensure that every uncoloured vertex has few coloured neighbours. To this end we define the *colouring number* of a graph G , denoted by $col(G)$, as the maximum of $\delta(H) + 1$ over all subgraphs H of G . We can always colour the vertices of G in order such that no uncoloured vertex has more than $col(G)$ coloured neighbours when we come to colour it.

Lemma 1.4. *For any graph G , $\chi(G) \leq col(G)$.*

Proof. Choose an ordering v_1, \dots, v_n of the vertices as follows. Having chosen v_j for $n \geq j > i$, let v_i be a minimum degree vertex in $G - \{v_{i+1}, v_{i+2}, \dots, v_n\}$. Now $col(G)$ -colour the vertices in order from v_1 onwards, noting that if v_i already has $col(G)$ coloured neighbours then the graph $G - \{v_{i+1}, v_{i+2}, \dots, v_n\}$ has minimum degree at least $col(G)$, contradicting the definition of the colouring number. \square

This early result was proved independently by several people, including Erdős and Hajnal [EH66] and Szekeres and Wilf [SW68]. One special application of this idea is the following: If a graph G has chromatic number k and a vertex v with $d(v) < k - 1$, then $G - v$ also has chromatic number k . This is because a $(k - 1)$ -colouring of $G - v$ will never result in v having $k - 1$ coloured neighbours, so we can always extend such a colouring of $G - v$ to a $(k - 1)$ -colouring of G . We will use this argument repeatedly throughout the thesis. A good example appears in the proof of Theorem III.2, specifically in the proof of Lemma 7.5.

Brooks' Theorem tells us which graphs satisfy $\chi(G) = \Delta(G) + 1$ and which graphs satisfy $\chi(H) = \Delta(H) + 1$ for every induced subgraph H of G (G must be a clique). In contrast it is difficult (i.e. NP-complete) to decide whether or not $\chi(G) = \omega(G)$, but we can determine whether a graph is perfect in polynomial time [CCL⁺05]. This is done through a structural characterization of perfect graphs; we discuss this result in Section 3.6.

It is natural to ask the same question about the colouring number. It is not known whether we can efficiently decide if $\chi(G) = col(G)$ for any graph, and no structural

characterization for such graphs is known. However along these lines Markossian, Gasparian and Reed introduced the class of β -perfect graphs [MGR96]. A graph is β -perfect precisely if $\chi(H) = \text{col}(H)$ for every induced subgraph H of G ($\beta(G)$ is sometimes used to denote the colouring number of G). Can we characterize the structure of β -perfect graphs? It is easy to see that no β -perfect graph contains an even hole. An *even hat* (resp. *odd hat*) is an even (odd) cycle of length ≥ 4 with exactly one chord, which makes a triangle with two edges of the cycle. Markossian, Gasparian and Reed provide two structure theorems [MGR96], although it is still not known whether β -perfect graphs can be recognized in polynomial time.

Theorem 1.5. *If a graph G contains no even hole or even hat, then G is β -perfect.*

Theorem 1.6. *A graph G and its complement are both β -perfect precisely if neither G nor \overline{G} contains an even hole.*

Theorem 1.5 was strengthened by de Figueiredo and Vušković so that only even hats on four or six vertices need to be restricted [dFV00]; Keijsper and Tewes offer more complicated strengthenings [KT02]. Theorem 1.6 is an interesting analogue of the Strong Perfect Graph Theorem, which we present in the next section.

1.2 Fractional colouring and perfect graphs

The chromatic number is the least number of colours needed to properly colour the vertices of a graph. Since each colour class in a proper colouring is a stable set, the chromatic number is also the least number of stable sets needed to cover the vertex set. Therefore we consider the following integer program, where $\mathcal{S}(G)$ denotes the family of stable sets in a graph, and for $S \in \mathcal{S}(G)$, w_S is 1 precisely if S is a colour class. (For definitions and preliminaries on linear programming, see [Chv83].)

$$\begin{aligned}
 & \text{minimize} && \sum_{S \in \mathcal{S}(G)} w_S \\
 & \text{subject to} && \forall v \in V(G), \sum_{S \ni v} w_S = 1 \\
 & && \text{and} \quad \forall S \in \mathcal{S}(G), w_S \in \{0, 1\}
 \end{aligned} \tag{1.1}$$

In this case we insist that we cover each vertex with precisely one stable set. But it is often useful to consider covering a vertex with a combination of stable sets. Thus we consider the fractional relaxation (1.2) of the above integer linear program (1.1).

$$\begin{aligned}
& \text{minimize} && \sum_{S \in \mathcal{S}(G)} w_S \\
& \text{subject to} && \forall v \in V(G), \sum_{S \ni v} w_S \geq 1 \\
& \text{and} && \forall S \in \mathcal{S}(G), w_S \in [0, 1]
\end{aligned} \tag{1.2}$$

A weighting of stable sets that satisfies the conditions of the fractional relaxation is a *fractional colouring* of G , and if $\sum_{S \in \mathcal{S}(G)} w_S = c$ we call it a *fractional c -colouring*. The *fractional chromatic number* of G , denoted by $\chi_f(G)$, is the least c for which there is a fractional c -colouring of G . Since we are dealing with the fractional relaxation of a minimization problem, we always have $\chi_f(G) \leq \chi(G)$. In Chapter 3 we will prove that for any graph, $\chi(G) \leq \lceil \log n \cdot \chi_f(G) \rceil$. These are most definitely not local bounds in any sense. Since $\chi_f(G) \leq \chi(G) \leq \lceil \log n \cdot \chi_f(G) \rceil$ and it is NP-hard to approximate $\chi(G)$ to within $n^{1-\epsilon}$ for any $\epsilon > 0$, we can see that computing the fractional chromatic number of a graph is also hard, since doing so would give us a $(\log n)$ -approximation to $\chi(G)$.

One way to approach (1.2) is to study its dual. In this case the dual linear program maximizes the total weight on vertices so that no stable set has total weight more than 1:

$$\begin{aligned}
& \text{maximize} && \sum_{v \in V} x_v \\
& \text{subject to} && \forall S \in \mathcal{S}(G), \sum_{v \in S} x_v \leq 1 \\
& \text{and} && \forall v \in V, x_v \in [0, 1]
\end{aligned} \tag{1.3}$$

Let $\omega_f(G)$ denote the total weight of an optimum dual solution. LP duality tells us that $\omega_f(G) = \chi_f(G)$. So putting weight 1 on every vertex of a maximum clique and 0 elsewhere gives the bound $\omega(G) \leq \omega_f(G) = \chi_f(G)$. This is a trivial bound and we do not need duality to see it. However, we will later exploit this duality to make nontrivial statements about constructing fractional colourings of claw-free graphs.

There are several important polyhedra related to this linear program – we will discuss them in Chapter 3. Its feasible region is the *fractional clique polytope*, which is the *fractional stable set polytope* of \overline{G} . The fractional stable set polytope of G contains the *stable set polytope*, which is the convex hull of the incidence vectors of the stable sets of G .

One class of graphs for which the stable set polytope is well understood is line graphs, in which stable sets correspond to matchings in the base multigraph – the characterization of the stable set polytope of line graphs is a fundamental result of Edmonds [Edm65a]. We have a good understanding of the stable set polytopes of perfect graphs as well: In Chapter 3 we will show that G is perfect precisely if the fractional stable set polytope is equal to the stable set polytope.

Perfect graphs have been an important area of research from the 1960s onwards, thanks in large part to their connection to communication theory and two conjectures posed by Berge [Ber61] (translated to English in [BR01]):

Conjecture 1.7 (The Weak Perfect Graph Conjecture). *A graph G is perfect precisely if its complement \overline{G} is perfect.*

A graph is called *Berge* if it contains neither an odd hole nor an odd antihole.

Conjecture 1.8 (The Strong Perfect Graph Conjecture). *A graph G is perfect precisely if it is Berge.*

The key to the proof of the Strong Perfect Graph Conjecture by Chudnovsky, Robertson, Seymour, and Thomas [CRST06] is the fact that any Berge graph either lies in some well-understood graph class or is pieced together from smaller graphs in some well-understood way. This is the same angle of attack we use for proving the results on quasi-line and claw-free graphs.

In Chapter 3 we will discuss how both of Berge’s conjectures were proved (thirty years apart) and how the theory of perfect graphs relates to fractional colourings, polyhedra and optimization, structural decompositions, and claw-free graphs.

The Strong Perfect Graph Theorem offers a structural characterization of a class of graphs for which $\chi_f(G) = \chi(G)$. By Vizing’s Theorem, the fractional and integer chromatic numbers must be within 1 of each other for line graphs of simple graphs. The same is not known to be true for line graphs of multigraphs, but would be implied

by the Goldberg-Seymour Conjecture [Gol73]. Kahn [Kah96] proved that they agree asymptotically using properties of the stable set polytope of line graphs.

Chapter 2

ω , Δ , and χ : Conjectures Old and New

We consider the following conjecture of Reed,

Conjecture 2.1 ([Ree98]). *For any graph G and integer $k \geq 1$, if G contains no clique of size greater than $\Delta(G) + 1 - 2k$, then $\chi(G) \leq \Delta(G) + 1 - k$.*

and a strengthening which is proposed for the first time here,

Conjecture 2.2. *For any graph G and integer $c \geq 1$, if every vertex of v satisfies $d(v) + 1 + \omega(v) \leq 2c$, then G is c -colourable.*

where $\omega(v)$ denotes the size of the largest clique containing v . We can see that the second conjecture is stronger by replacing $d(v) + 1 + \omega(v)$ with $\Delta(G) + 1 + \omega(G)$.

The two strongest pieces of evidence for Reed's conjecture are that a fractional variant is true, and for any k there is a Δ_k for which the conjecture holds whenever $\Delta(G) \geq \Delta_k$. Further supporting the conjecture is the fact that when $\omega(G)$ is bounded, the ratio $\chi(G)/\Delta(G)$ approaches zero as $\Delta(G)$ increases.

In this chapter we discuss these results and extend them by proving analogues for the local variant of Reed's conjecture. We then present related results and a further strengthening of the two conjectures, which we prove is false. We begin by presenting the motivation for Conjecture 2.1.

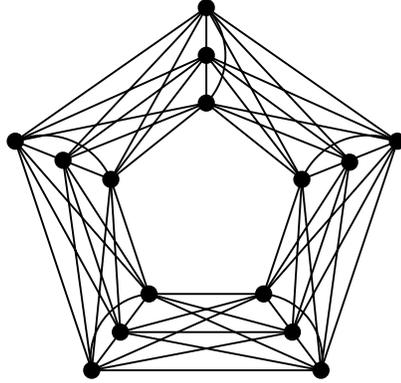


Figure 2.1: A graph G with $\omega(G) = 6$, $\Delta(G) = 8$, and $\chi(G) = \Delta(G)$.

2.1 Beyond Brooks' Theorem

From Brooks' Theorem, we know that we can always colour G using at most $\Delta(G)$ colours as long as no connected component of G is a clique of size $\Delta(G) + 1$ (or a cycle of odd length, if $\Delta(G) = 2$). This gives a complete characterization of graphs for which $\chi(G) = \Delta(G) + 1$. In an attempt to characterize graphs for which $\chi(G) \geq \Delta(G)$, Borodin and Kostochka proved that if $\omega(G) \leq \Delta(G)/2$ then $\chi(G) \leq \Delta(G) - 1$, then made the following conjecture:

Borodin-Kostochka Conjecture ([BK77]). *If a graph G with $\Delta(G) \geq 9$ contains no clique of size $\Delta(G)$, then $\chi(G) \leq \Delta(G) - 1$.*

Figure 2.1 illustrates that the Borodin-Kostochka Conjecture is the strongest variant of this conjecture which is not false. Beutelspacher and Hering independently made the weaker conjecture that there is some Δ_2 for which $\chi(G) \leq \Delta(G) - 1$ provided $\Delta(G) \geq \Delta_2$ and $\omega(G) \leq \Delta(G) - 1$ [BH84]. Reed proved the Beutelspacher-Hering Conjecture for $\Delta_2 = 10^{14}$ [Ree99].

This result suggests the possibility that for all k there may be a Δ_k such that G has $\chi(G) \leq \Delta(G) + 1 - k$ provided that $\Delta(G) \geq \Delta_k$ and $\omega(G) \leq \Delta(G) + 1 - k$. However, this is easily disproved for $k = 3$. To do so, we can join a $(\Delta - 4)$ -clique to a C_5 to get a graph with maximum degree Δ , clique number $\Delta - 2$, and chromatic number $\Delta - 1$. So as we move the upper bound on $\omega(G)$ away from $\Delta(G) + 1$, we cannot hope that $\chi(G)$ will move away from $\Delta(G) + 1$ at the same rate. But we

can hope that it moves away from $\Delta(G) + 1$ at some rate for $k > 2$. An example in [Ree98] shows that a rate of $\frac{1}{2}$ would be best possible, and Reed gave evidence that it may in fact be the case:

Theorem 2.3 ([Ree98]). *For any positive integer k there is some constant Δ_k such that any graph G with $\Delta(G) \geq \Delta_k$ and $\omega(G) \leq \Delta(G) + 1 - 2k$ will have chromatic number at most $\Delta(G) + 1 - k$.*

This result is actually a corollary of a more significant theorem from the same paper, which states that the Conjecture 2.1 holds provided that $\omega(G)$ and $\Delta(G) + 1$ are sufficiently close together:

Theorem 2.4 ([Ree98]). *There is a universal positive constant η such that if $\omega(G) \geq (1 - \eta)(\Delta(G) + 1)$ then $\chi(G) \leq \frac{1}{2}(\Delta(G) + 1 + \omega(G))$.*

This result also tells us that we can find a nontrivial upper bound for $\chi(G)$ by taking a convex combination of $\omega(G)$ and $\Delta(G) + 1$:

Corollary 2.5 (Reed). *There is a universal positive constant a such that $\chi(G) \leq (1 - a)(\Delta(G) + 1) + a\omega(G)$.*

Following this thread, Reed posed Conjecture 2.1, which is equivalent to saying that a can be as high as $\frac{1}{2}$ if we round up. We therefore define $\gamma(G)$ as $\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \rceil$ and restate Conjecture 2.1, which we call the Main Conjecture:

Main Conjecture. *For any graph G , $\chi(G) \leq \gamma(G)$.*

Accordingly, we define $\gamma_l(v)$ as $\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \rceil$ and we define $\gamma_l(G)$ as $\max_{v \in G} \gamma_l(v)$. Equivalently, $\gamma_l(v) = \gamma(G[\bar{N}(v)])$ and $\gamma_l(G) = \max_{v \in G} \gamma(G[\bar{N}(v)])$. Thus we can restate the local variant of the Main Conjecture, which we call the Local Strengthening:

Local Strengthening. *For any graph G , $\chi(G) \leq \gamma_l(G)$.*

Theorem 2.3 is the first major piece of evidence for the Main Conjecture. We now prove that we can strengthen it to find evidence supporting the Local Strengthening.

Theorem 2.6. *For any positive integer k there is some constant Δ'_k such that any graph G with $\Delta(G) \geq \Delta'_k$ in which every vertex v satisfies $d(v) + 1 + \omega(v) \leq 2(\Delta(G) + 1 - k)$ will have chromatic number at most $\Delta(G) + 1 - k$.*

To prove this we use a result of Molloy and Reed that guarantees the existence of a small $\chi(G)$ -chromatic subgraph of G under certain conditions.

Theorem 2.7 (Molloy and Reed [MR01]). *There is an absolute constant Δ_0 such that for any $\Delta \geq \Delta_0$ and $k \leq \sqrt{\Delta} - 3$, if G has maximum degree Δ and $\chi(G) > \Delta - k$ then G contains a subgraph H such that: (i) $|H| \leq \Delta + 1$; (ii) $\chi(H) = \Delta + 1 - k$.*

Corollary 2.8. *There is an absolute constant $\Delta_0 \geq 4$ such that for any $\Delta \geq \Delta_0$ and $k \leq \frac{1}{2}\sqrt{\Delta}$, if G has maximum degree Δ and $\chi(G) > \Delta - k$ then G contains a subgraph H with a universal vertex such that $\chi(H) = \Delta - k + 1$.*

Proof. We claim that given our further constraint on k , there is a subgraph H guaranteed by Theorem 2.7 that contains a universal vertex. To this end, assume that H is minimal and take an optimal colouring of H . The singleton colour classes make up a clique C in H , and since $|H| \leq \Delta + 1$ we can see that $|C| \geq \Delta + 1 - 2k$ and there are at most $2k$ vertices outside C . Since $\chi(H) = \Delta - k + 1$ and by minimality H is vertex-critical, every vertex in $H - C$ has at most k non-neighbours in H . Thus the number of universal vertices in C is at least $\Delta + 1 - 2k - 2k^2 > \Delta + 1 - \sqrt{\Delta} - \frac{1}{2}\Delta > 0$, so there is at least one universal vertex in C . \square

Now we can easily prove Theorem 2.6. The key to the proof is the fact that if v is a universal vertex in a graph G , then $\Delta(G) + 1 + \omega(G) = d(v) + 1 + \omega(v)$.

Proof of 2.6. Let Δ_k and Δ_0 be as in Theorems 2.3 and 2.7 respectively, and let $\Delta'_k = 2 \cdot \max\{\Delta_k, \Delta_0, 2k^2\}$. Suppose G is a counterexample to the theorem, i.e. $\chi(G) > \Delta(G) + 1 - k$, but $\Delta(G) \geq \Delta'_k$ and every vertex v of G satisfies $d(v) + 1 + \omega(v) \leq 2(\Delta(G) + 1 - k)$.

By Corollary 2.8 there is a $\chi(G)$ -chromatic subgraph H of G with a universal vertex v . Let $k' = k + \Delta(H) - \Delta(G)$; then $k' \leq k$ and $\chi(H) > \Delta(H) + 1 - k'$. By our choice of Δ'_k , $\chi(G) > \Delta_k \geq \Delta_{k'}$, therefore H satisfies the conditions of Theorem 2.3. Thus $\omega(H) > \Delta(H) + 1 - 2k'$ because $\chi(H) > \Delta(H) + 1 - k'$. Since v is universal it is in every maximum clique of H , so $d_H(v) + 1 + \omega_H(v) > 2(\Delta(H) + 1 - k')$. Therefore $d_G(v) + 1 + \omega_G(v) > 2(\Delta(G) + 1 - k)$, contradicting our choice of G . \square

Next we will give our second piece of evidence: the fractional analogues of the Main Conjecture and Local Strengthening are true.

2.2 Fractionally, $\chi \leq \gamma$

One of the most encouraging pieces of evidence for the Main Conjecture is the fact that it holds for the fractional chromatic number. We use $\gamma'(G)$ to denote $\frac{1}{2}(\Delta(G) + 1 + \omega(G))$, i.e. the unrounded version of $\gamma(G)$. Similarly we use $\gamma'_l(G)$ to denote $\max_{v \in V} \{\frac{1}{2}(\Delta(G) + 1 + \omega(G))\}$, the unrounded version of $\gamma_l(G)$.

Theorem 2.9 ([MR00]). *For any graph G , $\chi_f(G) \leq \gamma'(G)$.*

One can easily see that $\chi_f(G) \geq \frac{n}{\alpha(G)}$, thus the fractional version of the Main Conjecture implies that every graph G contains a stable set of size at least $\frac{2n}{\Delta(G)+1+\omega(G)}$, i.e. $\frac{n}{\gamma(G)}$. This weaker result was proven much earlier by Fajtlowicz [Faj78, Faj84]. Here we prove that the Local Strengthening also holds fractionally. This is an unpublished result of McDiarmid whose proof closely follows the proof of Theorem 2.9 in Chapter 21 of [MR00]; it is left as an exercise in the book.

Theorem 2.10. *For any graph G , $\chi_f(G) \leq \gamma'_l(G)$.*

The key to this result is the following lemma, which tells us that a random maximum stable set either hits a vertex or two of its neighbours with sufficiently high probability. It strengthens 21.10 from [MR00], replacing $\omega(G)$ with $\omega(v)$.

Lemma 2.11. *Let G be a graph and let S be a maximum stable set of G chosen uniformly at random. Then for every vertex v of G ,*

$$\mathbf{E}(|S \cap N(v)|) \geq 2 - (\omega(v) + 1)\Pr(v \in S). \quad (2.1)$$

We will prove this lemma later in the section. As a warm-up to the proof of Theorem 2.10 we will use Lemma 2.11 to easily prove the natural strengthening of Fajtlowicz's result:

Lemma 2.12. *For any graph G , $\alpha(G) \geq \frac{n}{\gamma'_l(G)}$.*

Proof. Let S be a uniformly random maximum stable set in G . Rearranging (2.1)

and summing over all vertices, we get

$$2n \leq \sum_v (\omega_v + 1) \Pr(v \in S) + \sum_v \mathbf{E}(|S \cap N(v)|) \quad (2.2)$$

$$= \sum_{v \in V} (\omega_v + 1) \Pr(v \in S) + \sum_v \sum_{u \sim v} \Pr(u \in S) \quad (2.3)$$

$$= \sum_{v \in V} (\omega_v + 1) \Pr(v \in S) + \sum_{u \in V} \Pr(u \in S) \quad (2.4)$$

$$= \sum_{v \in V} (\omega_v + 1 + d(v)) \Pr(v \in S) \quad (2.5)$$

$$= \sum_{v \in V} (\omega_v + 1 + d(v)) \Pr(v \in S) \quad (2.6)$$

$$\leq 2\gamma'_i \sum_{v \in V} \Pr(v \in S) \quad (2.7)$$

Since the expected size of a maximum stable set is precisely the sum of the probabilities of each vertex being in the set, we get $\mathbf{E}(|S|) \geq n/\gamma'_i$. The theorem follows. \square

If every vertex was in the same number of maximum stable sets, then this lemma would immediately imply the desired result, Theorem 2.10. We could find a suitable fractional colouring by placing equal weight on every maximum stable set and no weight on smaller stable sets. However, in general we do not have this desirable condition. For example in the star graph $K_{1,k}$ for $k > 1$, one vertex is in no maximum stable sets, and every other vertex is in every maximum stable set (although there is only one). So instead we must use Lemma 2.11 in an iterative fashion. The proof of Theorem 2.9 given in [MR00] uses the same method but uses weaker supporting lemmas.

Proof of Theorem 2.10. We fractionally colour G using the following iterative method.

1. Set $w_S = 0$ for every $S \in \mathcal{S}$. Set $G_0 = G$. Set $i = 0$.

Set $T = 0$. T stands for total weight used.

For each $v \in V$, set $w_{o_v} = 0$ (w_o stands for weight on).

2. If $V(G_i) = \emptyset$ or $T = \gamma'_i$ then stop.

3. For each vertex v of G_i , let $p_i(v)$ be the probability that v is in a uniformly random maximum stable set of G_i . Set $low = \min\{\frac{1-wo_v}{p_i(v)} | v \in V(G_i)\}$. Set $val_i = \min(low, \gamma'_i - T)$.
4. Let \mathcal{S}_i be the set of maximum stable sets of G_i . For each stable set in \mathcal{S}_i , increase w_S by $\frac{val_i}{|\mathcal{S}_i|}$. For each vertex v of G_i , increase wo_v by $p_i(v)val_i$. Increase T by val_i .
5. Let G_{i+1} be the graph induced by those vertices v which satisfy $wo_v < 1$. Increment i and go to Step 2.

Our choice of val_i ensures two things: that T never exceeds γ'_i , and that if the i th iteration is not the last, then $V(G_{i+1})$ is properly contained in $V(G_i)$. Thus the algorithm must terminate.

We claim that at the end of the procedure, the w_S yield a fractional γ'_i -colouring. It is easy to show by induction that at the end of each iteration and for every $v \in V$, $wo_v = \sum_{\{S \in \mathcal{S} | v \in S\}} w_S$ and $T = \sum_{S \in \mathcal{S}} w_S$. The definitions of low and val_i ensure that no wo_v is ever more than 1. We stop if $V(G_i) = \emptyset$ or $T = \gamma'_i$; in the first case we know that we have the desired fractional colouring. We must now show that the same is true in the second case. It suffices to show that in this case, each $wo_v = 1$.

So assume that for some v we have $wo_v < 1$ when we complete the process. For each vertex u and iteration i , denote by $a_i(u)$ the amount by which wo_u was augmented in iteration i , i.e. $a_i(u) = val_i p_i(u)$. By Lemma 2.11, $\mathbf{E}(|S \cap N(v)|) \geq 2 - (\omega_v + 1)\mathbf{Pr}(v \in S)$. It follows that since v was in each G_i , we have, for each i ,

$$\sum_{u \in N(v)} a_i(u) \geq 2(val_i) - (\omega_v + 1)a_i(v). \quad (2.8)$$

Summing over the iterations, we get

$$\sum_{u \in N(v)} wo_u \geq 2T - (\omega_v + 1)wo_v > (\omega_v + d(v) + 1) - (\omega_u + 1) = d(u). \quad (2.9)$$

But since $wo_u \leq 1$ for every u , we cannot have $\sum_{u \in N(v)} wo_u > d(v)$. Thus we obtain a contradiction and so $wo_v = 1$. This completes the proof. \square

It remains for us to prove Lemma 2.11. We will first prove one intermediate result.

Lemma 2.13. *If S is chosen uniformly at random from the maximum stable sets of G , then for every vertex v of G we have*

$$\Pr(|S \cap N(v)| = 1) \leq (\omega_v - 1)\Pr(v \in S). \quad (2.10)$$

Proof. We fix v and condition on our choice of $S' = S - \bar{N}(v)$, showing that the inequality holds no matter what the value of S' is. So fix some $R \in \mathcal{S}(V - \bar{N}(v))$ that can be extended to a maximum stable set of G by adding vertices in $\bar{N}(v)$. To prove the lemma, it is enough to show that for any choice of R ,

$$\Pr(|S \cap N(v)| = 1 | S' = R) \leq (\omega_v - 1)\Pr(v \in S | S' = R). \quad (2.11)$$

So fix a choice of R and let $W = \bar{N}(v) - N(R)$. First suppose that W is not a clique. Then we know that $|R| \leq \alpha(G) - 2$, and since v is universal in R we have $|S \cap \bar{N}(v)| \geq 2$ if $S' = R$, since S is a maximum stable set. Now suppose that W is a clique; we know that $|W| \leq \omega_v$ since $v \in W$. Since R can be extended to a maximum stable set, $|R| = \alpha(G) - 1$ and S is equally likely to be $R + u$ for any vertex $u \in \bar{N}(v)$. If $u = v$ then $|S \cap \bar{N}(v)| = 0$, so it follows that

$$\Pr(v \in S | S' = R) = \frac{1}{|W|} \quad (2.12)$$

and

$$\Pr(|S \cap N(v)| = 1 | S' = R) = \frac{|W| - 1}{|W|} \leq \frac{\omega_v - 1}{|W|}. \quad (2.13)$$

The lemma follows. \square

Proving Lemma 2.11 from here is a simple matter.

Proof of Lemma 2.11. Since S is a maximum (and therefore maximal) stable set, $\Pr(v \in S) = \Pr(S \cap N(v) = \emptyset)$. So since $|S \cap N(v)|$ is a nonnegative integer variable, we have $\mathbf{E}(|S \cap N(v)|) \geq 2 - \Pr(|S \cap N(v)| = 1) - 2\Pr(|S \cap N(v)| = 0)$. The result now follows from Lemma 2.13. \square

2.3 Graphs with small clique number

We have seen that the Main Conjecture holds when $\omega(G)$ is close to $\Delta(G)+1$. We now consider the other end of the spectrum, when $\omega(G)$ is very small. When $\omega \in \{0, 1\}$, $\chi = \omega$ and the result is trivial. However, the conjecture is not known to hold when $\omega = 2$.

This special case is equivalent to the statement that any triangle-free graph has chromatic number at most $\frac{\Delta}{2} + 2$. A classical result of Lovász [Lov66] implies that we can partition the vertices of a graph G into m parts, each inducing a subgraph of maximum degree at most three, if $4m \geq \Delta(G) + 1$. If G is triangle-free, then by Brooks' Theorem each induced subgraph is 3-colourable; this implies that $\chi(G) \leq 3m \leq 3\lceil(\Delta(G)+1)/4\rceil$. Kostochka (see [JT95], Section 4.6) proved that any triangle-free graph G satisfies $\chi(G) \leq \frac{2}{3}\Delta(G) + 2$. He also proved that for every $\Delta > 4$ there is a g_Δ such that any graph with maximum degree Δ and girth at least g_Δ is $(\frac{\Delta}{2} + 2)$ -colourable.

In a stronger asymptotic result, Johansson [Joh96] (also see [MR00]) proved that triangle-free graphs have chromatic number at most $O(\frac{\Delta(G)}{\log \Delta(G)})$, extending a result of Kim [Kim95] for graphs with girth at least five. Asymptotically, this is a much stronger bound than is required by the Main Conjecture. However, the Main Conjecture is still open for triangle-free graphs of small maximum degree greater than four. In unpublished work, Johansson [Joh] claimed a proof that for any fixed k , graphs containing no k -clique satisfy $\chi(G) \leq O(\frac{\Delta(G) \log \log \Delta(G)}{\log \Delta(G)})$. Johansson's results give equal support to the Main Conjecture and Local Strengthening, since as $\Delta(G)$ grows, both $\gamma(G)$ and $\gamma_l(G)$ approach $\frac{1}{2}\Delta(G)$ asymptotically when the clique number is fixed.

In contrast to these upper bounds on χ , we have known for a long time that χ can be arbitrarily high even when ω is bounded. In 1955, Mycielski showed how to construct a sequence of graphs G_i for $i \geq 1$ such that $\chi(G_i) = i$ while $\omega(G_i) \leq 2$ [Myc55]. A few years later, Erdős gave a probabilistic proof of the existence of k -chromatic graphs with girth at least g for any fixed g and k [Erd59].

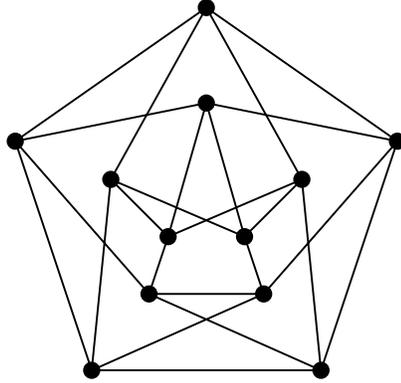


Figure 2.2: The *Chvátal graph*, a triangle-free 4-regular 4-chromatic graph.

2.4 Tightness of both conjectures

We now give some examples to show the Main Conjecture and the Local Strengthening are tight. That is, the round-up in γ and γ_l are necessary. Equivalently, it is necessary to insist that k in Conjecture 2.1 and c in Conjecture 2.2 are integers. Odd cycles are one class of graphs for which the round-up is necessary. It remains necessary even if $\Delta(G) > 2$: The 4-regular, 4-chromatic, triangle-free *Chvátal graph* [Chv70], shown in Figure 2.2, is one example showing this. Kostochka [Kos84] gives an example for arbitrarily high Δ : take the line graph of a 5-cycle in which there are k copies of every edge for some odd k . This graph has $\Delta(G) + 1 = 3k$, $\omega(G) = 2k$, and $\chi(G) = \lceil 5k/2 \rceil = \lceil n/\alpha \rceil$; Figure 2.1 actually shows this graph for $k = 3$. It follows that there is no $a > \frac{1}{2}$ such that $\chi \leq \lceil (1 - a)(\Delta + 1) + a\omega \rceil$ for all graphs. The aforementioned example in [Ree98], which has $\alpha = 2$, gives the same upper bound on a .

2.5 Early work on the Main Conjecture

Attacks on the Main Conjecture tend to fall into two categories: attempts to bound the chromatic number away from $\Delta(G) + 1$ and towards $\omega(G)$ for all graphs (e.g. Theorem 2.4), and attempts to prove the conjecture outright for restricted classes of graphs. In this thesis we present several results of the latter type. In Chapters 4, 7, and 10 we prove that the Main Conjecture holds for three increasingly general classes

of graphs: line graphs (in joint work with Reed and Vetta [KRV07], quasi-line graphs (in joint work with Reed [KR08b]), and claw-free graphs (in joint work with Reed).

Randerath and Schiermeyer proved the conjecture for several classes of graphs, including graphs on at most 12 vertices and graphs for which $\Delta(G) \geq n - \alpha(G)$ [RS06, Sch06, Sch07b]. Rabern improved this latter result [Rab08]:

Theorem 2.14. *Let G be a graph satisfying $\Delta(G) \geq n + 2 - \alpha(G) - \sqrt{n + 5 - \alpha(G)}$. Then $\chi(G) \leq \gamma(G)$.*

He also proved that $\chi(G) \leq \gamma(G)$ for any graph with a disconnected complement and any graph with $\chi(G) > \lceil \frac{n}{2} \rceil$ or $\alpha(G) = 2$. We now prove that the Local Strengthening also holds whenever $\alpha(G) = 2$. Our proof is similar to but independent of Rabern's proof that the Main Conjecture holds when $\alpha(G) = 2$.

Theorem 2.15. *Let G be a graph with $\alpha(G) = 2$. Then $\chi(G) \leq \gamma(G)$.*

Before proving this theorem we need some background on matchings in graphs. If $\alpha(G) = 2$, then an optimal colouring of G corresponds to a maximum packing of vertex-disjoint stable sets of size two, which in turn corresponds to a maximum matching in \overline{G} . We say that a matching *hits* a vertex if the vertex is an endpoint of an edge in the matching, otherwise the matching *misses* the vertex. A *near-perfect matching* is a matching that misses exactly one vertex. The Edmonds-Gallai structure theorem [Edm65b, Gal59] describes the structure of a graph in terms of its maximum matchings (also see [LP86, Pul87]). We say that a connected component of a graph is *odd* if it contains an odd number of vertices, otherwise it is *even*.

Theorem 2.16 (Edmonds-Gallai structure theorem). *For any graph G , let Y be the set of vertices not hit by every maximum matching in G , and let $X = N(Y) \setminus Y$. Then every even component of $G - X$ has a perfect matching, and every odd component has, for each of its vertices v , a near-perfect matching that misses v . Furthermore the odd components of $G - X$ have union Y .*

We now use this theorem to prove Theorem 2.15.

Proof of Theorem 2.15. Let G be a minimum counterexample to the theorem. By minimality of G , removing any vertex will reduce the chromatic number. Therefore

every vertex v is missed by some maximum matching of \overline{G} , otherwise we would have $\chi(G - v) = \chi(G)$.

By the Edmonds-Gallai structure theorem, either \overline{G} is not connected or it has a near-perfect matching. In the first case, $V(G)$ can be partitioned into nonempty V_1 and V_2 such that every possible edge between V_1 and V_2 exists. It is easy to confirm that $\chi(G) = \chi(G[V_1]) + \chi(G[V_2]) \leq \gamma_l(G[V_1]) + \gamma_l(G[V_2]) \leq \gamma_l(G)$, the middle inequality following from the minimality of G .

In the second case, n is odd and $\chi(G) = \lceil \frac{n}{2} \rceil$. Since $\chi_f(G) \geq \frac{n}{\alpha(G)}$, we have $\chi(G) = \lceil \chi_f(G) \rceil$. By Theorem 2.10, $\chi(G) = \lceil \chi_f(G) \rceil \leq \lceil \gamma'_l(G) \rceil = \gamma_l(G)$. This proves the theorem. \square

It is not hard to prove the case $\chi(G) = \lceil \frac{n}{2} \rceil$ without using Theorem 2.10. But applying the fractional result in this way is a useful technique which we will use again in proving the Main Conjecture for line graphs and the Local Strengthening for circular interval graphs in Chapter 4.

We will now provide some insight as to why the Local Strengthening is useful when considering the Main Conjecture, and why it is in a certain sense the best possible strengthening of the Main Conjecture.

2.6 Genesis of the Local Strengthening

The Local Strengthening did not arise in a hunt for a strongest possible conjecture. Rather, it arose naturally because it is sometimes easier to prove than the Main Conjecture. Here we explain why.

A class \mathcal{G} of graphs is *hereditary* if for any graph $G \in \mathcal{G}$, every induced subgraph of G is in \mathcal{G} . One way to prove the Main Conjecture for such a class of graphs is to prove that if $G \in \mathcal{G}$, then G contains a stable set S for which $\gamma(G - S) < \gamma(G)$. Since S can be a colour class in a proper colouring of G , $\chi(G - S) \geq \chi(G) - 1$. Thus we can prove the Main Conjecture for \mathcal{G} by induction. This approach involves finding an S whose removal either lowers $\omega(G)$, or lowers $\Delta(G)$ by two. (Any stable set S in a graph can be extended to a maximal stable set whose removal lowers $\Delta(G)$ by at least one.) However, finding such an S is not always possible.

To make this inductive approach easier, we introduced the invariant $\omega(v)$ for every vertex v ; recall that it is the size of the largest clique containing v . Now we ask: can

we find a stable set S whose removal lowers $d(v) + \omega(v)$ by two for every vertex v maximizing $d(v) + \omega(v)$? In other words, can we rip out a stable set and reduce γ_l , rather than trying to reduce γ ?

This idea arose during efforts to prove the Main Conjecture for *antiprismatic thickenings*, which are an important subclass of claw-free graphs (we will define them in Chapter 9). If G is an antiprismatic thickening with $\alpha(G) > 2$, it is very easy to remove a stable set and lower $\gamma_l(G)$, but it seems much more difficult to do the same for $\gamma(G)$. Thus to prove the Local Strengthening for antiprismatic thickenings we can repeatedly remove a stable set, lowering $\gamma_l(G)$ each time, until we are left with a graph with $\alpha \leq 2$. We then appeal to Theorem 2.15 to complete the proof.

We need more definitions and background before we give a formal proof of the Local Strengthening for antiprismatic thickenings in Chapter 10, but it is important to understand that the Local Strengthening is particularly appealing because it is often much easier to prove than the Main Conjecture. We will exploit this fact repeatedly when we prove the Main Conjecture for claw-free graphs.

2.6.1 Further tightness of the Local Strengthening

The Local Strengthening replaces the global invariant $\gamma(G)$ with the maximum over all local invariants $\gamma_l(v)$. Can we go one step further and formulate a conjecture that imposes a restriction for each local invariant simultaneously? We close the chapter by suggesting a further strengthening of the Main Conjecture in this vein and proving that it is not true.

Proposition 1. *For any graph G there is a proper $\gamma_l(G)$ -colouring of G such that for any vertex v of G , at most $\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \rceil$ colours appear in the closed neighbourhood of v .*

In a proper $\gamma_l(G)$ colouring of a graph G , there is some vertex v such that at most $\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \rceil$ colours appear in the closed neighbourhood of v . The proposition insists that this property is satisfied by every vertex. The following theorem disproves the proposition.

Theorem 2.17. *For any positive integer k , there is a graph G_k such that for any proper colouring of G , there is a vertex v of degree k whose neighbourhood of v induces a k -coloured stable set.*

Proof. We will construct G_k beginning with stable sets V_1, V_2, \dots, V_k, X that partition $V(G)$. Let V_1 be a stable set of size $N = k^3$; N will define the sizes of the other stable sets. We call a vertex set a *transversal* of V_1, \dots, V_i if it has size i and intersects each of V_1, \dots, V_i in exactly one vertex. If it is a stable set, we call it an *independent transversal*.

For $i > 1$, let V_i be a stable set containing exactly one vertex for every transversal of V_1, \dots, V_{i-1} . Accordingly, for every transversal S of V_1, \dots, V_{i-1} there is a vertex of V_i whose neighbourhood in $G_k[\cup_{j=1}^i V_j]$ is S . Finally, let X be a stable set consisting of one vertex for each stable set of size k in $G[\cup_{i=1}^k V_i]$; for every stable set S of size k in $G_k[\cup_{i=1}^k V_i]$ there is a unique vertex in X with neighbourhood S .

We say that a colouring of G_k is *good* if it is proper and there is no vertex in X that sees k different colours. It suffices to prove that there is no good colouring of G . Suppose we have a good colouring of G_k . First observe that each V_i (more so, every subset of it) receives fewer than k colours. We will show that this forces a k -coloured independent transversal, giving us the desired contradiction.

There is some colour c_1 seen on at least N/k vertices of V_1 . Accordingly there are at least N/k vertices of V_2 that cannot receive colour c_1 , hence there is some colour c_2 seen on at least $(N/k)(1/k) = N/k^2$ vertices of V_2 . Moving on to V_3 , there are at least $(N/k)(N/k^2)$ vertices that cannot receive c_1 or c_2 , so there is some third colour c_3 seen on at least $(N/k)(N/k^2)(1/k) = N^2/k^4$ vertices in V_3 .

Continuing this analysis, we find that there are unique colours c_1, c_2, \dots, c_k such that for $i = 2, 3, \dots, k$, the colour c_i appears on at least $(N/k^2)^{i-2}$ vertices of V_i , and c_1 appears on at least N/k vertices of V_1 . In particular, for $i = 1, 2, \dots, k$ there are at least k vertices of colour c_i in V_i .

We will find a k -coloured independent transversal S of V_1, \dots, V_k starting at V_k and working our way back as follows: From V_i , add to S any vertex with colour c_i with no neighbours in S . This is always possible, because at any point each vertex of S will have at most one neighbour in V_i . Thus we can find our k -coloured independent transversal S , proving that the colouring is not good since there is a vertex in X with neighbourhood S . This completes the proof. \square

Notice that the theorem says nothing about the total number of colours used in the colouring; it is easy to see that $\chi(G_k) \leq k + 1$. The proposition may be true for claw-free graphs. The counterexample we just constructed contains a claw for $k \geq 2$.

We conjecture that the fractional weakening of the proposition is also false.

Conjecture 2.18. *There exists a graph G such that in any fractional colouring there is a vertex v for which the stable sets intersecting $N(v)$ have total weight greater than $\frac{1}{2}(d(v) - 1 + \omega(v))$.*

We actually suspect that the graph constructed in the proof of Theorem 2.17 may also prove Conjecture 2.18.

Part II

Fractional and Integer Colourings

Over the next two chapters we explore the connection between fractional and integer colourings of graphs. The next chapter focuses on perfect graphs, for which the fractional and integer chromatic numbers are equal. As we will explain, the relationship between fractional and integer colourings of perfect graphs is actually much deeper than this. Perfect graphs have been the subject of much attention over the past four decades. We trace the development of the field and in doing so gather many useful tools that will help us deal with claw-free graphs in later chapters. The most useful of these are structural decompositions which arose in both the study of restricted subclasses of perfect graphs and the pursuit of the Strong Perfect Graph Conjecture.

In Chapter 4 we investigate two important classes of claw-free graphs for which the fractional and integer chromatic numbers are close together: line graphs and circular interval graphs. As we will see in Chapters 5, 8, and 9, these two classes suggest two composition operations that we will use to construct general claw-free graphs, and they are also used as base classes of these compositions. In a step towards proving the Main Conjecture for claw-free graphs, we will prove the Main Conjecture for line graphs and circular interval graphs. Our proof leads to a polynomial-time γ -colouring algorithm for these graphs.

Chapter 3

Fractional Colouring and Perfect Graphs

Since we are interested in graphs for which $\chi(G)$ is closer to $\omega(G)$ than to $\Delta(G) + 1$, it is natural to consider graphs for which $\chi(G) = \omega(G)$. However, this property does not tell us anything about the structure of the graph other than the fact that it contains a $\chi(G)$ -clique. For example, taking the disjoint union of any graph G and a clique of size $\chi(G)$ gives us a graph for which $\chi(G) = \omega(G)$.

In this chapter we focus on graphs with the stronger property that $\chi(H) = \omega(H)$ for every induced subgraph H of G . These widely studied *perfect graphs* were introduced by Claude Berge [Ber61], who was motivated by a problem of Shannon about the rate at which information can be transmitted across an empty channel without any chance of error [Sha56].

Early interest in perfect graphs was motivated by two conjectures due to Berge. The Weak Perfect Graph Conjecture (WPGC) proposes that G is perfect precisely if \overline{G} is perfect. The Strong Perfect Graph Conjecture (SPGC) proposes that G is perfect precisely if neither G nor \overline{G} contains an odd hole. If neither G nor \overline{G} contains an odd hole we say that G is *Berge*. The strong conjecture obviously implies the weak conjecture.

Since the fractional chromatic number lies between the clique number and the chromatic number, we know that $\chi_f(G) = \chi(G)$ for any perfect graph G . As we discuss later in this chapter, to actually find an optimal colouring we use the stronger fact that when G is perfect the fractional stable set polytope $QSTAB(G)$ has only

integer extreme points, and is therefore equal to the stable set polytope $STAB(G)$; we define these polytopes in the next section. It turns out that in fact the stable set polytopes of perfect graphs can be used to characterize exactly those polytopes of the form $Ax = 1$ which have only integer extreme points and for which A is a $(0, 1)$ matrix. For any such A , we can solve every integer program of the form

$$\max cx \quad \text{s.t. } Ax = 1 \text{ and } x \text{ is integral}$$

by solving the linear program obtained by dropping the integrality constraint. This fact generated considerable interest in perfect graphs amongst combinatorial optimizers.

This second, polyhedral motivation is of more direct interest to us, since we will use the facts that $\chi(G)$ is equal to or near $\chi_f(G)$ and that $QSTAB(G) = STAB(G)$ for various classes of graphs repeatedly when studying claw-free graphs. So we begin the chapter with a discussion of the ratio between $\chi_f(G)$ and $\chi(G)$ for general graphs, a description of the proof that $QSTAB(G) = STAB(G)$ for perfect graphs, and a discussion of algorithms for computing $\chi(G)$ and solving other optimization problems on perfect graphs.

We also discuss the proofs of Berge's two conjectures, which are separated by thirty years. Lovász' 1972 proof of the WPGC is short and elegant, relying on the *Replication Lemma* and homogeneous cliques. We will need to consider homogeneous cliques throughout our work on claw-free graphs in later chapters. In contrast to the proof of the WPGC, the proof of the SPGC due to Chudnovsky, Robertson, Seymour, and Thomas is over 100 pages long and we will not be able to reproduce it. We will however sketch the proof for two reasons. First, it implies the Main Conjecture for Berge graphs. Second, we will introduce structural decompositions and tools that will be of great use to us in our study of claw-free graphs. In particular, we rely heavily on *2-joins* and *homogeneous pairs* for reasons we explain fully in Chapters 5 and 6 respectively.

To round off the chapter, we discuss recognition algorithms for perfect graphs.

3.1 Fractional colouring

In Section 1.2 we defined the fractional chromatic number $\chi_f(G)$ of a graph G as the optimal value of the following linear program (1.2):

$$\begin{aligned} & \text{minimize} && \sum_{S \in \mathcal{S}(G)} w_S \\ & \text{subject to} && \forall v \in V(G), \sum_{S \ni v} w_S \geq 1 \\ & && \text{and } \forall S \in \mathcal{S}(G), w_S \in [0, 1] \end{aligned}$$

The feasible region of this program is $STAB(G)$, the *stable set polytope* of G . As we will show, $\chi_f(G)$ is also equal to the minimum β such that $(\frac{1}{\beta}, \frac{1}{\beta}, \dots, \frac{1}{\beta})$ is in $STAB(G)$ – our result from the previous chapter implies that $(\frac{1}{\gamma_1}, \frac{1}{\gamma_1}, \dots, \frac{1}{\gamma_n})$ is in $STAB(G)$. If we can optimize over the stable set polytope of G efficiently, we can determine the fractional chromatic number of G efficiently.

We consider another polytope, the *fractional stable set polytope* of G , denoted by $QSTAB(G)$. It consists of all nonnegative vertex weightings on G such that no stable set of G has total weight more than 1. For all graphs, $STAB(G) \subseteq QSTAB(G)$. Later in this chapter we will show that a graph is perfect precisely if $STAB(G) = QSTAB(G)$ and furthermore, we can optimize over these polytopes efficiently.

For a probability distribution p on the stable sets of a graph, the value $\Pr(v \in S)$ is called the *marginal* of p at v . We now present the relationship between $STAB(G)$, the fractional chromatic number, and marginals achievable by probability distributions on the stable sets of a graph, which was hinted at in the proof of Theorem 2.10.

Observation 3.1. *The following statements are equivalent:*

- (1) $\chi_f(G) = k$
- (2) $(\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k}) \in STAB(G)$
- (3) *There is a probability distribution p on the stable sets of G such that the marginal of p at each vertex is $\frac{1}{k}$.*

Thus the stable set polytope actually characterizes precisely (and in turn is defined by) the marginal vectors that can be achieved by some such p .

Proof. Suppose for some $k \in \mathbb{R}^+$ the $|V|$ -dimensional vector $(\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k})$ is in the stable set polytope of G . Then there is a convex combination of stable sets of G in which every vertex has weight $\frac{1}{k}$. Taking k times this convex combination gives a combination of stable sets with total weight k in which every vertex has weight 1. This is a fractional k -colouring of G , so $\chi_f(G) \leq k$ and so (2) implies (1). It is easy to see that the reverse implication is also true, i.e. that $(\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k}) \in STAB(G)$ precisely if $\chi_f(G) \leq k$. Furthermore the convex combination of stable sets actually gives a probability distribution on stable sets of G such that if S is drawn from the distribution, then for any $v \in V$, $\Pr(v \in S) = \frac{1}{k}$. Specifically, the probability of S in the distribution is precisely the weight of S in the convex combination. So (2) implies (3) and again it is clear that the converse is also true. \square

We mentioned earlier that any graph G satisfies $\chi(G) \leq \lceil \log n \cdot \chi_f(G) \rceil$. This result is due to Lovász, who gave a different proof of a more general result on set covers [Lov75]. The proof we present uses the relationship between $\chi_f(G)$ and probability distributions on the stable sets of G .

Theorem 3.2. *For any graph G , $\chi(G) \leq \lceil \log n \cdot \chi_f(G) \rceil$.*

Proof. By Observation 3.1 there is some probability distribution on the stable sets of G such that if S is drawn from the distribution, for all $v \in V$ we have $\Pr(v \in S) = \frac{1}{\chi_f(G)}$. We colour G by drawing $\lceil \log n \cdot \chi_f(G) \rceil$ random colour classes from this distribution and arguing that with nonzero probability, each vertex is in at least one colour class. This gives a $\lceil \log n \cdot \chi_f(G) \rceil$ -colouring of G , since if a vertex is in more than one colour class we can just choose one of these colours arbitrarily for the vertex.

Let v be any fixed vertex. Every colour class S has the property that $\Pr(v \in S) = \frac{1}{\chi_f(G)}$. Since the colour classes are drawn independently from the same distribution, the probability that v is in no colour class is

$$\left(1 - \frac{1}{\chi_f(G)}\right)^{\lceil \log n \cdot \chi_f(G) \rceil} \leq \left(\left(1 - \frac{1}{\chi_f(G)}\right)^{\chi_f(G)}\right)^{\frac{\lceil \log n \cdot \chi_f(G) \rceil}{\chi_f(G)}} < \left(\frac{1}{e}\right)^{\log n} = \frac{1}{n}.$$

Thus the probability of every vertex being in at least one colour class is greater than $1 - \frac{1}{n} = 0$. The theorem follows. \square

This proof is an example of the simple but powerful *first moment method* (see, for example, Chapter 3 in [MR00]).

In Chapter 1 we showed that $\omega(G) \leq \chi_f(G) \leq \chi(G)$. Here we give examples that prove $\chi_f(G)$ can be at either end of this range, even if the difference between $\omega(G)$ and $\chi(G)$ is large. This is in contrast to the previous chapter, in which we disproved the analogous statement for $\chi(G)$ and the range $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$. The fractional chromatic number can also be in the middle of the range $[\omega(G), \chi(G)]$.

We first show that $\chi_f(G)$ can be equal to $\omega(G)$ but far from $\chi(G)$. Given positive integers a and b for which $b \leq \frac{a}{2}$, the *Kneser graph* $K_{a,b}$ has order $\binom{a}{b}$. Its vertices correspond to the b -subsets of a base set of size a , and two vertices are adjacent precisely if their corresponding subsets are disjoint. (The famous Petersen graph is actually $K_{5,2}$.) For any integers $k \geq 2$ and $b \geq 2$, the graph $K_{kb,b}$ has clique number k . One can use the well-known Erdős-Ko-Rado Theorem [EKR61] and the vertex-transitive property of Kneser graphs to prove that $\chi_f(K_{kb,b}) = k$; the proof is given in Chapter 3 of [SU97]. Lovász [Lov78] proved that $\chi(K_{a,b}) = a - 2b + 2$, and so $\chi(K_{kb,b}) = (k - 2)b + 2$, far away from the fractional chromatic number.

To see that $\chi_f(G)$ and $\chi(G)$ can be equal and far from $\omega(G)$, let G be the line graph of C_5 with k copies of every edge (as we did in Section 2.1). If k is even, G has clique number $2k$ and independence number 2, thus $\chi_f(G) \geq \frac{5}{2}k$. It is easy to show that $\chi(G) = \frac{5}{2}k$, so $\chi_f(G) = \chi(G)$.

To see that $\chi_f(G)$ can be in the middle of the range $[\omega(G), \chi(G)]$, let G be k disjoint copies of C_5 with all possible edges between them. Then $\omega(G) = 2k$, $\chi_f(G) = \frac{5}{2}k$, and $\chi(G) = 3k$. We will revisit this example later in the thesis. Larsen, Propp, and Ullman [LPU95] proved that Mycielski's sequence of graphs G_k , mentioned in Section 2.1 (see [Myc55]), have $\omega(G_k) = 2$, $\chi(G_k) = k$, and $\chi_f(G_k) = \chi_f(G_{k-1}) + \frac{1}{\chi_f(G_{k-1})}$. Thus $\chi_f(G_k)$ asymptotically approaches $\sqrt{2k - 2}$.

3.1.1 Inapproximability of χ and χ_f

We have shown that $\omega(G)$, $\chi_f(G)$, and $\chi(G)$ can be far apart from one another, even though they are all equal for perfect graphs. It turns out that all three of them are difficult to compute and even approximate in general. Since the fractional chromatic number can be found by solving a linear program, one might make the mistake of thinking that we can find $\chi_f(G)$ in polynomial time. But the linear program contains

a variable for every stable set of G , and can therefore have exponential size compared to the size of G .

Indeed it is NP -hard¹ to approximate the chromatic number of a graph to within a factor of $|V|^{\epsilon_\chi}$ for some positive ϵ_χ . Bellare, Goldreich, and Sudan proved that this ϵ_χ is at least $\frac{1}{7}$ [BGS98]. By Theorem 3.2, $\chi(G)$ exceeds $\chi_f(G)$ by at most a factor of $\log(|V|)$, thus it is hard to compute the fractional chromatic number of a graph in general under the assumption that $P \neq NP$ because $\chi_f(G)$ gives a $\log(|V|)$ -approximation to $\chi(G)$. As pointed out in Exercise 10.8 in [AL95], this implies that approximating $\chi_f(G)$ is also hard.

By results of Håstad [Hås99] and Feige and Killian [FK98], if we make stronger assumptions² than $P \neq NP$, then it is hard to approximate ω , χ , and χ_f to within a factor of $|V|^{1-\epsilon}$ for any $\epsilon > 0$.

In spite of these difficulties it is possible to efficiently compute the chromatic number for a perfect graph and even find an optimal colouring, as we will show later in this chapter. First we must lay out the foundation of these optimization results: the Replication Lemma and the proof of the Weak Perfect Graph Theorem, which have extensive implications in both structural and polyhedral aspects of perfect graph theory.

3.2 The Replication Lemma and the Weak Perfect Graph Theorem

In 1972, Lovász proved the Weak Perfect Graph Conjecture [Lov72]. His proof hinges on the following lemma.

Definition 3.3. *We replicate a vertex v in G by adding a vertex v' to G with neighbourhood $\bar{N}(v)$ (recall that $\bar{N}(v)$ denotes the closed neighbourhood $\{v\} \cup N(v)$ of a vertex v). In other words, we add a twin of v to G .*

¹See [GJ79] for definitions related to NP -hardness.

²Inapproximability of ω relies on the assumption that $NP \neq co-RP$, and inapproximability of χ and χ_f relies on the assumption that $NP \neq ZPP$. We refer the interested reader to [Pap94] for the definitions of these complexity classes. We know that $P \subseteq ZPP \subseteq co-RP \subseteq NP$ but all four classes may be equal, although this is widely disbelieved.

Replication Lemma (Lovász³ [Lov72]). *If G is obtained from a perfect graph by replicating a vertex then G is perfect.*

Proof. We need to show that if G' is obtained by adding a twin v' of a vertex v to a perfect graph G then for every induced subgraph H' of G' , we have $\chi(H') = \omega(H')$.

Since G is perfect we are done if $v' \notin H'$ because in this case H' is an induced subgraph of G . Similarly, we are done if $v' \in H'$ but $v \notin H'$ since in this case H' is isomorphic to $H' + v - v'$

So assume both v and v' are in H' and let $H = H' - v'$. If v is in an $\omega(H)$ clique in H then $\omega(H') = \omega(H) + 1$ and we can $\omega(H')$ -colour H' using an $\omega(H)$ -colouring of H and a new colour for v' . If v is in no $\omega(H)$ clique, we $\omega(H)$ -colour H and let the stable set S be the colour class containing H . Observe that $\omega(H - S) = \omega(H) - 1$. It follows that $H - (S - v)$ has no $\omega(H)$ clique and since G is perfect, $\chi(H - (S - v)) = \omega(H) - 1$. Using $S - v + v'$ as a final colour class gives us the desired $\omega(H')$ -colouring of H' . \square

We immediately get the following corollary.

Corollary 3.4. *If G is obtained from a perfect graph through a sequence of vertex replications then G is perfect.*

This motivates an important definition.

Definition 3.5. *To substitute a graph H for a vertex v in a graph G , we first take the disjoint union of $G - v$ and H , then make each vertex in H adjacent to each vertex in $N(v)$.*

So repeatedly replicating a vertex is equivalent to substituting a clique for the vertex, and this operation preserves perfection. Lovász actually proved that if a perfect graph H is substituted for a vertex v in a perfect graph G , the resulting graph is perfect (see Chapter 2 in [RR01]).

The Replication Lemma is also the key to proving that $QSTAB(G) = STAB(G)$ if G is perfect, as we will show in the next section. We say that a set S of vertices in G is a *homogeneous set* if $|S| \geq 2$, $|V(G) \setminus S| \geq 2$, and every vertex in $V(G) \setminus S$ sees either all or none of S . If S is a clique, we say that S is a *homogeneous clique*.

³Fulkerson had previously noted that the Replication Lemma would be enough to prove the Weak Perfect Graph Conjecture [Ful72].

The notions of replication, substitution, and homogeneous sets are closely related: If we replicate a vertex in a graph on at least three vertices, the resulting graph will contain a homogeneous clique. If we substitute H for v in G , then the vertices of H will be a homogeneous set in the resulting graph.

This idea remains extremely important when we shift our focus from perfect graphs to claw-free graphs in the next chapter. In particular, we will rely heavily on the notion of *thickenings*, which generalize repeated vertex replication, and *homogeneous pairs of cliques*, which generalize homogeneous cliques.

We close this section with a proof the Weak Perfect Graph Conjecture, which is now the Weak Perfect Graph Theorem.

Theorem 3.6 (Lovász [Lov72]). *If G is perfect then \overline{G} is perfect.*

Proof. It suffices to prove that if G is perfect, then $\chi(\overline{G}) = \omega(\overline{G})$, so assume G is a minimal counterexample to this statement. Thus every proper induced subgraph H of G can be covered with $\alpha(H)$ cliques.

Let t be the number of maximum stable sets in G , and for every vertex v of G let t_v be the number of maximum stable sets containing v . Construct the graph G' from G by deleting every vertex v for which $t_v = 0$, then replicating every remaining vertex $t_v - 1$ times. Thus we get G' from G by substituting a clique C_v of size t_v for every vertex v . We can label each vertex of C_v with a unique maximum stable set of G containing v ; it follows that G' is t -colourable.

G' has $t\alpha(G)$ vertices and $\alpha(G') = \alpha(G)$, so $\chi(G') \geq |V(G')|/\alpha(G') = t$, so $\chi(G') = t$. By Corollary 3.4, G' is perfect and therefore contains a clique C' of size t . The maximum stable sets of G give a t -colouring of G' , so C' intersects each of these stable sets. Let $C \subseteq V(G)$ be the set of vertices v for which C' intersects C_v ; C is clearly a clique and it intersects every maximum stable set in G , so $\alpha(G-C) = \alpha(G) - 1$. By minimality of G , $G-C$ can be covered by $\alpha(G) - 1$ cliques. It follows that G can be covered by $\alpha(G)$ cliques, so $\chi(\overline{G}) = \omega(\overline{G})$, a contradiction. \square

Remark: In Section 2.2 we noted that if every vertex in a graph G is contained in the same number of maximum stable sets, then $\chi_f(G) = |V(G)|/\alpha(G)$, and we get an optimal fractional colouring by giving equal weight to every maximum stable set. The above proof follows this line of thinking, as the t -colouring of G' is the result of fractionally colouring G using only maximum stable sets so that every vertex v

receives total weight t_v .

To prove that $QSTAB(G) = STAB(G)$ when G is perfect, we will use a similar construction based on repeated replication. But instead of substituting a clique for each vertex we will substitute a stable set for each vertex, noting that the resulting graph is still perfect.

3.3 Optimizing over perfect graphs

We have established that finding the chromatic number or fractional chromatic number of a graph is hard in general. In this section we will explain how to optimally colour a given perfect graph G in polynomial time. Since G is perfect, computing $\chi(G)$ is equivalent to computing $\chi_f(G)$ and $\omega(G)$. By the Weak Perfect Graph Theorem, if we can compute $\chi(G)$ efficiently we can also compute $\chi(\overline{G})$ and $\alpha(G)$ efficiently. Even more impressive is the fact that we can actually find a maximum stable set and optimally colour G in polynomial time.

Our first step is to calculate $\omega(G)$, which we do by optimizing over $STAB(\overline{G})$ to find a maximum stable set of \overline{G} . Grötschel, Lovász, and Schrijver proved that we can approximately solve a convex optimization problem efficiently if we can solve the associated separation problem approximately. For integer programs in which the vertices of the feasible region are $(0, 1)$ vectors, this implies that we can find an exact solution efficiently if we can separate efficiently. We refer the reader to [GLS93] for further explanation.

To optimize over perfect graphs we exploit the equality of the stable set polytope and fractional stable set polytope. This equality was first proved using antiblocking polyhedra [Ful72]. Chvátal [Chv75] later gave a simple proof that $QSTAB(G) = STAB(G)$ for perfect graphs, based on Lovász' proof of the Weak Perfect Graph Theorem.

Theorem 3.7. *If G is perfect, then $QSTAB(G) = STAB(G)$.*

Proof. We already know that $STAB(G) \subseteq QSTAB(G)$. To show that $QSTAB(G) \subseteq STAB(G)$ it suffices to show that there is no direction in which $QSTAB(G)$ exceeds $STAB(G)$. Thus since the vertices of both polyhedra are rational and in the non-negative orthant, it suffices to show that there is no rational vector contained in

$QSTAB(G) \setminus STAB(G)$. That is for any rational vector c , the maximum of $c \cdot x$ over $QSTAB(G)$ is achieved by a point in $STAB(G)$, i.e. a characteristic vector of a stable set in G . If c is the 1-vector, this follows from the fact that G can be covered by $\alpha(G)$ cliques by the WPGT, so the maximum of $c \cdot x$ over $QSTAB(G)$ is $\alpha(G)$. We need only consider nonnegative integer-valued c because we can rescale c to have integer values without changing its direction.

For arbitrary nonnegative integer-valued c , we construct a graph G_c by substituting a stable set of size c_v for every vertex v of G . This is equivalent to substituting a clique of size c_v for every vertex v of \overline{G} , then taking the complement. So by the Weak Perfect Graph Theorem and the Replication Lemma, G_c is perfect. Every stable set S in G corresponds to a stable set of size $\sum_{v \in S} c_v$ in G_c . Furthermore, just as we bounded the chromatic number of G' in the proof of the WPGT, we can see that $\alpha(G_c)$ is the maximum of $c \cdot x$ such that x is the characteristic vector of a stable set in G .

Since G_c is perfect it can be covered by $\alpha(G_c)$ cliques, so there is no point $y_c \in QSTAB(G_c)$ for which $1 \cdot y_c > \alpha(G_c)$. Suppose there is a point $y \in QSTAB(G)$ for which $c \cdot y > \alpha(G_c)$. This point corresponds to a weighted combination of stable sets S_i in G , and each S_i corresponds to a stable set of size $\sum_{v \in S_i} c_v$ in G_c . Combining these stable sets in G_c gives us a point $y_c \in QSTAB(G_c)$ for which $1 \cdot y_c > \alpha(G_c)$, a contradiction. \square

We mentioned earlier that $QSTAB(G) = STAB(G)$ precisely if G_c is perfect. If G is not perfect then it contains an odd hole or antihole as an induced subgraph by the Strong Perfect Graph Theorem. Knowing this, it is a simple matter to prove that $QSTAB(G) \neq STAB(G)$.

The key to optimizing over perfect graphs is a convex body called the *theta body* of G , denoted $TH(G)$. The theta body, first introduced by Lovász [Lov79], lies between $QSTAB(G)$ and $STAB(G)$ for any graph, so for any perfect graph G the three bodies are equal. Grötschel, Lovász, and Schrijver proved that we can approximately solve convex optimization problems over the theta body efficiently using the *ellipsoid method*, which is a form of geometric binary search [GLS81]. Since the vertices of $TH(G)$ are integral, we can find a maximum stable set in a perfect graph in polynomial time. In fact we can find a maximum weight stable set for any nonnegative integer weighting on the vertices of G .

Since \overline{G} is also perfect by the WPGT, we can also efficiently compute $\omega(G)$, $\chi(G)$ and $\chi_f(G)$. Recall that $\chi_f(G) \leq k$ precisely if $(\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k})$ is in $STAB(G)$, so the vector $(\frac{1}{\omega}, \frac{1}{\omega}, \dots, \frac{1}{\omega})$ is in $STAB(G)$ and it represents a convex combination of stable sets. Equivalently, it can be expressed as a convex combination of vertices of $STAB(G)$. We can find such an expression efficiently [GLS93]; we now show how this fact allows us to find an optimal colouring of G .

If a stable set S is given nonzero weight in a fractional $\omega(G)$ -colouring of a perfect graph G , then S intersects every maximum clique of G . Therefore $\omega(G-S) = \omega(G) - 1$ and by perfection, $\chi(G-S) = \chi(G) - 1$. Thus we can use S as a colour class in an optimal colouring of G and recursively colour $G-S$ using $\chi(G) - 1$ colours. So to efficiently find an optimal colouring of a perfect graph, it suffices to find such an S efficiently. To do this, we first compute $\omega(G)$ by optimizing over the theta body of \overline{G} . We then express $(\frac{1}{\omega}, \frac{1}{\omega}, \dots, \frac{1}{\omega})$ as a convex combination of vertices of $STAB(G)$, and take S to be the stable set represented by some vector given nonzero weight in this expression.

3.4 Combinatorial algorithms and some subclasses of perfect graphs

The decades leading up to the proof of the Strong Perfect Graph Theorem saw a huge amount of work in the theory of perfect graphs. In particular, many classes of Berge graphs were shown to be perfect, and many properties of minimal imperfect graphs⁴ were proved. In this section we discuss some classes of perfect graphs that yield efficient detection algorithms and combinatorial algorithms for solving optimization problems. These algorithms are of interest to us for two reasons. First, the ellipsoid method is not very efficient in practice, and optimizing over the theta body does not give us a strong intuition as to how we are exploiting the structure of the graph. Second, the tools and ideas we introduce in this section are of use to us in more general settings, in particular when dealing with claw-free graphs.

⁴An imperfect graph is *minimal imperfect* if all of its proper induced subgraphs are perfect.

3.4.1 Bipartite graphs and their complements

A graph is bipartite precisely if it is 2-colourable. Bipartite graphs are perfect. It is easy to show that a graph is bipartite precisely if it contains no cycle of odd length.

If G is the complement of a bipartite graph, then G is *cobipartite*. Equivalently, G is cobipartite if its vertices can be covered by two cliques. Clearly an optimal colouring of a cobipartite graph consists of a maximum matching in \overline{G} and a set of singleton colour classes. By König's Theorem [Kön16], the maximum matching in a bipartite graph has size equal to that of a minimum vertex cover, which implies that cobipartite graphs are perfect without using the perfection of bipartite graphs. We will use the perfection of cobipartite graphs repeatedly throughout the remaining chapters, particularly in the context of homogeneous pairs of cliques, which we define later in this section.

3.4.2 Line graphs of bipartite graphs

The edges of a bipartite graph H can be coloured using $\Delta(H)$ colours. This follows easily from the fact that a maximum matching and minimum cover have the same size in a bipartite graph. Since $\omega(L(H)) = \Delta(H)$ for bipartite H , this implies that the line graph of any bipartite graph is perfect. The Replication Lemma extends this result to line graphs of bipartite multigraphs, and the WPGT implies that the complement of the line graph of a bipartite graph is perfect. Line graphs of bipartite graphs make up an important class of claw-free Berge graphs, and as we show in the next section, they also make up an important class of general Berge graphs.

3.4.3 Clique cutsets

A clique C in a graph G is a *clique cutset* if $G - C$ is disconnected. If G_1, \dots, G_l are the connected components of $G - C$, then we can construct G from the graphs $G_i \cup C$ by identifying C in each of the graphs. We call this operation by which we paste together the graphs $G_i \cup C$ a *clique sum*.

If we have a k_i -colouring of each graph $G_i \cup C$, then in each colouring, every vertex of C receives a different colour. Thus by permuting the colour classes we can obtain colourings that agree on C ; together these colourings yield a proper colouring of G using $\max_{1 \leq i \leq l} k_i$ colours.

It follows that if \mathcal{B} is a class of perfect graphs and \mathcal{B}^C is a hereditary class of graphs each of which is in \mathcal{B} or contains a clique cutset, then \mathcal{B}^C is also a class of perfect graphs.

Whitesides [Whi81] proved that we can find a clique cutset in $O(nm)$ time. Tarjan extended this [Tar85], proving that in $O(nm)$ time we can construct a *clique cutset tree* for any graph. A clique cutset tree of a graph G is a rooted tree defined recursively as follows: the root of the tree corresponds to a clique cutset C of G , or to G if G contains no clique cutset. From the root we hang the clique cutset trees of the resulting graphs $G_i \cup C$. Thus if we have a polynomial-time algorithm for colouring graphs in \mathcal{B} , we have a polynomial-time algorithm for colouring graphs in \mathcal{B}^C that involves working our way up the clique cutset tree, starting with the leaves.

We now describe some classes of perfect graphs for which this approach is particularly effective.

Chordal graphs

A graph is *chordal* if it contains no holes. Chordal graphs, also known as triangulated or Gallai graphs, were first identified and characterized by Dirac [Dir61]. They are perfect, so we can colour them in polynomial time using the ellipsoid method. Dirac proved that G is chordal if and only if every induced subgraph of G is a clique or has a clique cutset. Thus if \mathcal{B} is the class of cliques then \mathcal{B}^C is the class of chordal graphs. Therefore we have an algorithm for colouring chordal graphs that is far more efficient and illuminating than the polyhedral approach we use for general perfect graphs.

Fulkerson and Gross proved that G is chordal precisely if every induced subgraph of G has a *simplicial vertex*, i.e. a vertex whose neighbourhood induces a clique [FG65]. Thus any chordal graph G has a *simplicial ordering* of its vertices in which we can remove simplicial vertices one at a time until only a single vertex remains. Unless G is a clique, the neighbourhood of any simplicial vertex is a clique cutset. We can find a simplicial vertex in a chordal graph in $O(m)$ time, which is faster than the best known algorithm for general graphs.

Clique-separable and i -triangulated graphs

Let \mathcal{B} be the class of graphs that are complete multipartite or a clique joined to a bipartite graph. In this case \mathcal{B}^C is the class of *clique-separable* graphs. For $G \in \mathcal{B}$,

if some component of \overline{G} is not a clique, then it is a cobipartite graph and all other components of \overline{G} are isolated vertices. Thus \overline{G} is perfect and it follows that \mathcal{B} is a subclass of perfect graphs. The structure of \overline{G} yields an efficient algorithm for colouring graphs in \mathcal{B} , so we immediately get an efficient combinatorial algorithm for colouring clique-separable graphs.

A graph is *i-triangulated* if every cycle of odd length has at least two noncrossing chords. Gallai [Gal62] proved that these graphs are perfect, and in fact they form a subset of clique-separable graphs. It turns out that we can use the structure of *i-triangulated* graphs to find a maximum-weight induced k -partite subgraph in polynomial time for any fixed k [AKK⁺08]. To do this, we decompose using clique cutsets with special properties which we can show exist because of the special structure of *i-triangulated* graphs.

3.4.4 Star cutsets, even pairs, and weakly triangulated graphs

A *star cutset* A in a graph G is a cutset such that some vertex v in A sees every other vertex in A . We call v the *centre* of the star cutset. Note that every clique cutset is also a star cutset. Chvátal [Chv85] proved that no minimal imperfect graph contains a star cutset, and so by the WPGT its complement does not contain a star cutset.

Lemma 3.8 (Star Cutset Lemma). *No minimal imperfect graph contains a star cutset.*

Proof. Suppose G is minimal imperfect and contains a star cutset A with centre v , and let the components of $G - A$ be G_i for $1 \leq i \leq l$. Then each $G_i \cup A$ is perfect; take an $\omega(G_i \cup A)$ -colouring of each graph and permute the colour classes so that v always receives the same colour. Let the stable set S in G be the union of these colour classes. Every graph $(G_i \cup A) - S$ is $\omega(G) - 1$ colourable and hence has no $\omega(G)$ clique, so $G - S$ has no $\omega(G)$ clique. Since $G - S$ is perfect it is $\omega(G) - 1$ colourable so G is $\omega(G)$ colourable, a contradiction. \square

Given a class \mathcal{B} of perfect graphs, let \mathcal{B}^* be the class of graphs that are in \mathcal{B} or admit a star cutset or admit a star cutset in the complement. The Star Cutset Lemma implies that \mathcal{B}^* is a class of perfect graphs. By properties of star cutsets pointed out by Chvátal [Chv85], it is easy to find a star cutset in polynomial time.

However, unlike the case for clique cutsets, we do not necessarily have an efficient combinatorial algorithm for colouring graphs in \mathcal{B}^* since we do not know how to paste together the colourings of $G_i \cup A$ given a star cutset A .

In certain cases, however, we can further exploit the structure of \mathcal{B}^* to colour graphs efficiently. A graph is *weakly triangulated* if it contains no hole or antihole of length greater than four. The class of weakly triangulated graphs lies between Berge graphs and chordal graphs. Hayward [Hay85] proved that weakly triangulated graphs are the class \mathcal{B}^* when \mathcal{B} is the class of graphs on at most two vertices, so they are perfect. To find a combinatorial colouring algorithm for weakly triangulated graphs we must introduce the notion of *even pairs*.

Two nonadjacent vertices u and v in a graph form an *even pair* if every chordless path between u and v has an even number of edges. For an even pair $\{u, v\}$ in a graph G we contract $\{u, v\}$ by replacing u and v with a new vertex uv adjacent to $N(u) \cup N(v)$. We use G/uv to denote the resulting graph.

Even pairs are of interest to us because contracting on an even pair in a perfect graph always results in a perfect graph, as proved by Fonlupt and Uhry [FU82]. Their result follows from the fact that neither the chromatic number nor the clique number changes:

Theorem 3.9. *Let $\{u, v\}$ be an even pair in a graph G . Then $\omega(G/uv) = \omega(G)$ and $\chi(G/uv) = \chi(G)$.*

Proof. Obviously $\omega(G/uv) \geq \omega(G)$. Suppose $\omega(G/uv) > \omega(G)$. Then there is a maximum clique in G containing vertices w adjacent to v but not u , and x adjacent to u but not v . The four vertices v, w, x, u induce a path on three vertices, contradicting the fact that $\{u, v\}$ is an even pair.

Given a $\chi(G/uv)$ colouring of G/uv we get a $\chi(G/uv)$ colouring of G by giving both u and v the colour appearing on uv , so $\chi(G) \leq \chi(G/uv)$.

If there is a $\chi(G)$ colouring of G such that u and v receive the same colour, we can $\chi(G)$ colour G/uv by giving uv that same colour. Suppose we have a $\chi(G)$ colouring of G such that u and v receive different colours, and let G' be the subgraph of G induced on these two colour classes. If u and v are in the same component of G' then we get an odd chordless path from u to v , contradicting the fact that $\{u, v\}$ is an even pair. Therefore we can take the component of G' containing v and exchange the

two colour classes to reach a proper $\chi(G)$ colouring of G in which u and v receive the same colour. It follows that $\chi(G) = \chi(G/uv)$. \square

This idea of reducing a graph without changing its chromatic number will be of use to us when we colour claw-free graphs, particularly in the setting of homogeneous pairs of cliques. Note, however, that contracting on an even pair can increase the maximum degree of the graph, so even pairs are of little use to us when considering the Main Conjecture and the Local Strengthening.

A graph G is called *even-contractile* if we can reduce G to a clique through a sequence of even pair contractions. If the resulting sequence of graphs is $G = G_0, G_1, \dots, G_k$ where G_k is a clique of size $\chi(G)$, we can optimally colour G as follows. Start with a colouring of G_k . Having optimally coloured G_i for some i , and letting $\{u_i, v_i\}$ be the even pair in G_{i-1} for which $G_i = G_{i-1}/u_i v_i$, we $\chi(G)$ -colour G_{i-1} by giving both u_i and v_i the colour appearing on $u_i v_i$ in the colouring of G_i , and using the same colouring on $G_{i-1} - u_i - v_i$ and $G_i - u_i v_i$.

This approach leads to a polynomial-time colouring algorithm for even-contractile graphs as long as we can find the desired sequence of even pairs in polynomial time. However, the problem of deciding whether or not G admits an even pair is *co-NP*-complete (and thus not believed to be solvable in polynomial time) in general [Bie91].

Hayward, Hoáng, and Maffray proved that every weakly triangulated graph is even-contractile [HHM89]. Since contracting on an even pair in a weakly triangulated graph does not always leave a weakly triangulated graph, they considered a special type of even pairs. A *2-pair* is an even pair $\{u, v\}$ in which every chordless path from u to v has length two. Hayward, Hoáng, and Maffray proved that every weakly triangulated graph is either a clique or contains a 2-pair. Contracting a 2-pair will not create a long hole or antihole, so this easily implies that every weakly triangulated graph is even-contractile. To colour weakly triangulated graphs efficiently it is enough to know that we can find a 2-pair in polynomial time. This is easy, because $\{u, v\}$ is a 2-pair precisely if u and v are in different components of $G - (N(u) \cap N(v))$. Arikati and Pandu Rangan later gave an $O(nm)$ -algorithm for finding a 2-pair in an arbitrary graph if one exists [AP91].

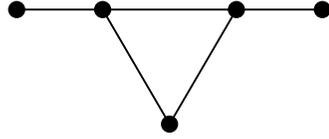


Figure 3.1: A bull.

3.4.5 Bull-free Berge graphs and homogeneous pairs

A *bull* is a triangle along with two pendant vertices that are adjacent to distinct vertices of the triangle (see Figure 3.1). A graph is *bull-free* if it does not contain a bull as an induced subgraph. Bull-free graphs generalize bipartite graphs. Chvátal and Sbihi proved that all bull-free Berge graphs are perfect [CS87].

Their proof uses two important tools: star cutsets and homogeneous pairs. A *homogeneous pair* is a pair (A, B) of disjoint nonempty sets of vertices such that

1. Every vertex outside $A \cup B$ sees either all of A or none of A , and sees either all of B or none of B .
2. At least one of A and B contains at least two vertices.
3. There are at least two vertices outside $A \cup B$.

Homogeneous pairs generalize homogeneous sets, since a homogeneous set is just a homogeneous pair in which one set is empty. If A and B are both cliques then (A, B) is a *homogeneous pair of cliques*; these generalize homogeneous cliques. Homogeneous pairs of cliques are a key idea in the structure theorems for claw-free graphs. Everett, Klein, and Reed gave an $O(mn^3)$ for finding a homogeneous pair in a general graph [EKR97]. We will give faster algorithms for finding specific types of homogeneous pairs of cliques later in this thesis.

Chvátal and Sbihi's work relies on the Star Cutset Lemma, which we already proved, and the *Homogeneous Pair Lemma*, proved by Chvátal in [CS87]:

Lemma 3.10 (Homogeneous Pair Lemma). *No minimal imperfect graph contains a homogeneous pair.*

To prove that bull-free Berge graphs are perfect, Chvátal and Sbihi proved the following structure theorem:

Theorem 3.11. *Let G be a bull-free Berge graph. Then one of G or \overline{G} contains a star cutset or a homogeneous pair, or is bipartite.*

This immediately yields the result that bull-free Berge graphs are perfect, since no minimal imperfect graph is bipartite or the complement of a bipartite graph, thus no minimal imperfect graph is both bull-free and Berge. Reed and Sbihi proved a different structure theorem for bull-free Berge graphs that yields a polynomial-time detection algorithm [RS95]. As we discuss at the end of this chapter, it was not until recently that a polynomial-time detection algorithm was found for general Berge graphs.

The Homogeneous Pair Lemma is also implied by results of Olariu [Ola88] and Conforti, Cornuéjols, Gasparyan, and Vuškovič [CCGV02] (see [PS01]). Later in this thesis we will prove results that easily imply that no minimal imperfect graph contains a homogeneous pair of cliques.

3.5 The Strong Perfect Graph Theorem and a structural characterization of Berge graphs

The Strong Perfect Graph Conjecture was for a long time one of the most significant open problems in the field of graph theory. Chudnovsky, Robertson, Seymour, and Thomas finally announced that the conjecture is true in 2002. The proof, at a length of about 150 pages, was published in 2006 [CRST06]. Chudnovsky and Seymour [CS08c] later shortened it by about 50 pages using an even pairs approach inspired by similar work by Maffray and Trotignon [MT06]. For an idea of how the proof was found, we refer the reader to Seymour's gentle introduction to the work [Sey06]. Since $\chi = \omega$ for Berge graphs, the SPGT gives a strengthening of the Main Conjecture and the Local Strengthening for Berge graphs.

The proof of the SPGT follows from a structure theorem for Berge graphs due to Chudnovsky, Robertson, Seymour, and Thomas [CRST06]. The structure theorem was conjectured to exist by Conforti, Cornuéjols, and Vuškovič (see [Sey06]). It states, essentially, that a Berge graph G must be in one of several well-understood classes of perfect graphs, or else it admits one of several structural decompositions. Thus the structure theorem is of the same flavour as the structure theorems for bull-free Berge

graphs and claw-free graphs, the latter of which we discuss in the next chapter. A minimum counterexample to the SPGT cannot admit any of the decompositions in question, so the theorem follows (since it holds for the base classes of graphs).

It stands to reason, considering the Weak Perfect Graph Theorem, that the base classes of graphs should come in complementary pairs. There are three pairs:

- **Bipartite graphs** and their complements, cobipartite graphs.
- **Line graphs of bipartite graphs** and their complements.
- **Double split graphs** are defined as follows. Let H be a split graph, i.e. a graph whose vertices can be partitioned into a stable set A and a clique B . We double the vertices of H such that the pairs from A are adjacent and the pairs from B are nonadjacent. That is, we let $V(G) = \bigcup_{v \in V(H)} \{v_1, v_2\}$. For $u \in A$ and $v \in B$, we have:

- u_1 sees u_2 if and only $u \in A$
- If $u, v \in A$ then u_i does not see v_j for $\{i, j\} \subseteq \{1, 2\}$.
- If $u, v \in B$ then u_i sees v_j for $\{i, j\} \subseteq \{1, 2\}$.
- If $u \in A$ and $v \in B$, there are two edges between $\{u_1, u_2\}$ and $\{v_1, v_2\}$. They are u_1v_1 and u_2v_2 if u sees v in H . Otherwise they are u_1v_2 and u_2v_1 .

Notice that the complement of a double split graph is again a double split graph, so to show that they are perfect we need only show that they contain no odd holes; this is easy.

If G or its complement is bipartite or the line graph of a bipartite graph or a double split graph, we say that G is *basic* (this definition only applies to this chapter, as we will use different types of basic graphs in the study of claw-free graphs). If a Berge graph is not basic then it yields one of two types of structural decomposition, which we define and discuss now.

3.5.1 2-joins

We already know that no minimal imperfect graph contains a clique cutset. One special type of clique cutset is a *1-join*. A graph G admits a 1-join if we can partition

the vertices of G into A_1, B_1, A_2, B_2 such that $A_1 \cup A_2$ is a clique, $N(B_1) \subseteq A_1 \cup B_1$, and $N(B_2) \subseteq A_2 \cup B_2$ (that is, the only edges between $A_1 \cup B_1$ and $A_2 \cup B_2$ are those between A_1 and A_2). Observe that $A_1 \cup A_2$ is a clique cutset, so it is easy to see that no minimal imperfect graph (and thus no minimum counterexample to the SPGT) contains a 1-join.

Generalizing 1-joins are *2-joins*, which were introduced by Cornuéjols and Cunningham [CC85]. A 2-join is a partitioning of $V(G)$ into V_1 and V_2 with disjoint nonempty $X_i, Y_i \subseteq V_i$ for $i \in \{1, 2\}$ such that X_1 is complete to X_2 and Y_1 is complete to Y_2 , and there are no other edges between V_1 and V_2 .

The decomposition theorem for Berge graphs uses a special type of 2-join called a *proper 2-join*. A proper 2-join is a 2-join with the additional properties that:

- For $i \in \{1, 2\}$, every component of $G[V_i]$ intersects both X_i and Y_i .
- For $i \in \{1, 2\}$, if $|X_i| = |Y_i| = 1$ and $G[V_i]$ is an induced path, then it has odd length ≥ 3 .

2-joins are a special case of *2-amalgams*, in which we allow a homogeneous clique adjacent to $X_1 \cup X_2 \cup Y_1 \cup Y_2$.

To show that 2-joins preserve perfection, as proved by Cornuéjols and Cunningham [CC85], we need to prove that we can find suitable $\omega(G)$ -colourings of $G_1 = G[V_1]$ and $G_2 = G[V_2]$ so that we can paste them together to reach a proper $\omega(G)$ -colouring of G . But observe that G may contain a cycle of odd length even if both G_1 and G_2 are perfect, even if the 2-join is proper. So instead of insisting that G_1 and G_2 are perfect, we insist that two auxiliary graphs G'_1 and G'_2 are perfect. For $i \in \{1, 2\}$ we construct G'_i from G_i by adding a vertex v_1 with neighbourhood X_i , and a vertex v_2 with neighbourhood $Y_i \cup \{v_1\}$ (see Figure 3.2). Denote the clique numbers $\omega(G_i[X_i])$ and $\omega(G_i[Y_i])$ by p_i and q_i respectively. Construct the graph H_i from G'_i by adding a vertex u with neighbourhood $\{v_2\} \cup Y_i$. We can see that H_1 and H_2 are perfect, in particular because $\{v_2\} \cup Y_i$ is a star cutset in H_i .

The following lemma provides us with the desired colourings of G_1 and G_2 .

Lemma 3.12. *Assume H_i is perfect. For any $l \geq \omega(G_i)$ there is a proper l -colouring of G_i in which p_i colours appear on X_i , q_i colours appear on Y_i , and $\max\{0, p_i + q_i - l\}$ colours appear on both X_i and Y_i .*

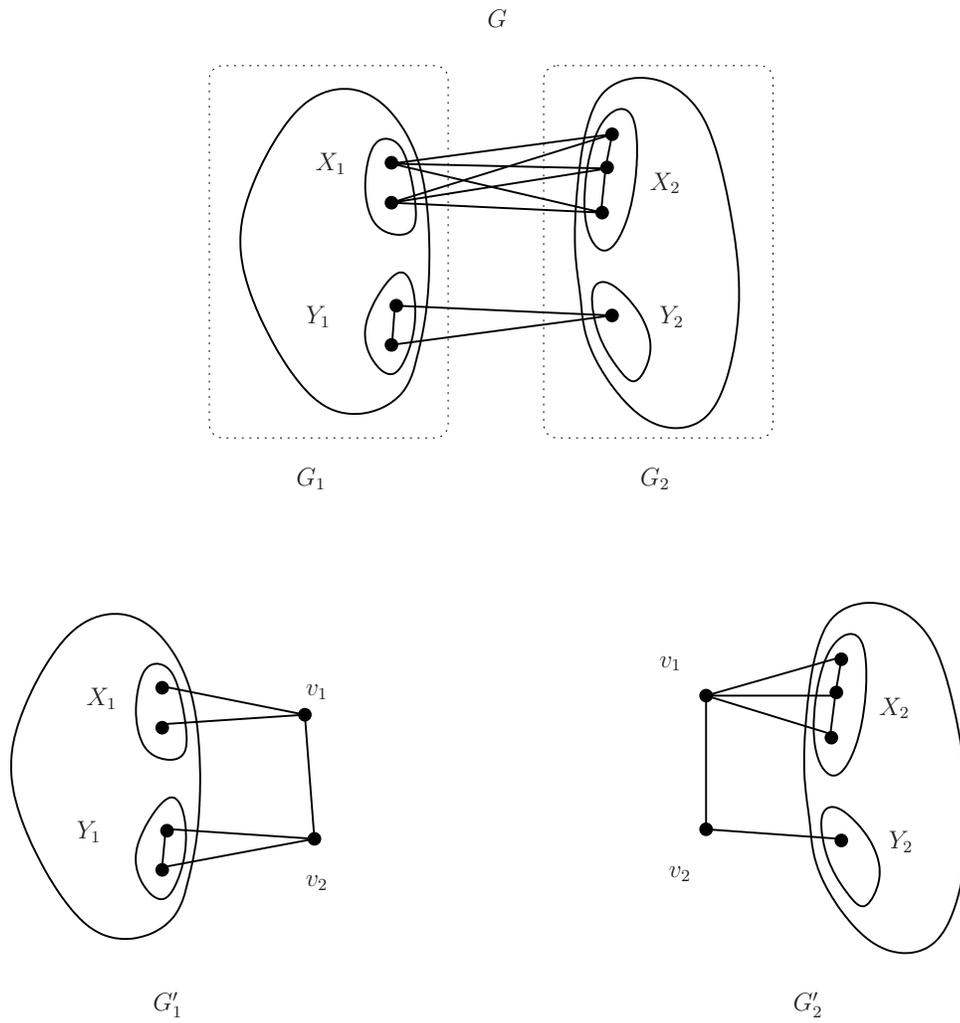


Figure 3.2: A graph G admitting a 2-join into G_1 and G_2 . The 2-join is built from auxiliary graphs G'_1 and G'_2 . If G'_1 and G'_2 are perfect then so is G .

Proof. Beginning with H_i , replace v_1 with a clique C_1 of size $l - p_i$, replace v_2 with a clique C_2 of size $\min\{l - q_i, p_i\}$, and replace u with a clique C_3 of size $\max\{l - p_i - q_i, 0\}$. The resulting graph H'_i is perfect by the Replication Lemma, and it is easy to confirm that $\omega(H'_i) = l$.

Consider an l -colouring of H'_i restricted to G_i (G_i is an induced subgraph of H'_i). Since X_i is joined to a clique of size $l - p_i$ and Y_i is joined to a clique of size $l - q_i$ (i.e. the cliques replacing v_2 and u), p_i colours appear on X_i and q_i colours appear on Y_i . Since we have an l -colouring, at least $p_i + q_i - l$ colours appear on both X_i and Y_i . Furthermore, the colours appearing on neither X_i nor Y_i are precisely the colours appearing on both C_1 and C_3 , of which there must be $\max\{l - p_i - q_i, 0\}$ in a proper l -colouring of H'_i , since $H'_i[C_1 \cup X_i]$ and $H'_i[C_2 \cup C_3 \cup Y_i]$ both have clique number l . The result follows. \square

With this lemma in hand it is now easy to prove that 2-joins preserve perfection: we argue that taking our guaranteed $\omega(G)$ -colourings of G_1 and G_2 , we can simply permute the colour classes and paste the colourings together to get a proper $\omega(G)$ -colouring of G . However, different types of 2-joins are very important to us when we bound the chromatic number of quasi-line and claw-free graphs later in this thesis. With this in mind we present a proof that is less straightforward, but which better motivates our approach to 2-joins in claw-free graphs.

Lemma 3.13. *If G'_1 and G'_2 are perfect, then G is perfect.*

Proof. Let G be a minimum counterexample with clique number ω . We claim that G is minimal imperfect. For if some proper induced subgraph G' of G is imperfect, it admits either a 1-join (if G' contains none of some X_i or Y_i) or a 2-join resulting from perfect induced subgraphs of G'_1 and G'_2 . The 1-join preserves perfection since it is a clique sum, so by minimality of G , G' is perfect and G is minimal imperfect.

It suffices, then, to prove that G is ω -colourable. To do this we find a stable set S such that $G'_1 - S$ and $G'_2 - S$ are both $\omega - 1$ colourable, which proves the lemma by minimality of G .

We begin with an ω -colouring of G_1 and an ω -colouring of G_2 as provided by the previous lemma (setting $l = \omega$). Since $p_1 < \omega$ there is a colour class S_1 in G_1 that intersects Y_1 but not X_1 . Suppose there is a colour class S_2 in G_2 that intersects X_2 but not Y_2 . Observe that $\omega = \max\{\omega(G_1), p_1 + p_2, q_1 + q_2, \omega(G_2)\}$. Thus since $S_1 \cup S_2$

intersects both $X_1 \cup X_2$ and $Y_1 \cup Y_2$, it is easy to see that $\omega(G - (S_1 \cup S_2)) = \omega - 1$ and we are done.

So we can assume that there is no such S_2 in G_2 , i.e. that every colour in X_2 is also in Y_2 . By the previous lemma, $\omega = \omega(G_2) = q_2$. But this is clearly impossible since $Y_1 \cup Y_2$ is a clique and Y_1 is nonempty. \square

We just illustrated a simple example of a very important approach, variations of which we will apply repeatedly throughout this thesis. To prove that G is $\omega(G)$ -colourable, we removed a stable set that lowered $\omega(G)$. In later chapters we will remove stable sets that lower $\gamma(G)$ or $\gamma_l(G)$ in order to prove that $\chi(G) \leq \gamma(G)$ or $\chi(G) \leq \gamma_l(G)$.

3.5.2 Balanced skew-partitions

Proper 2-joins are the first type of decomposition we need for the SPGT. The other is a specific type of *skew-partition*. A skew-partition is a partitioning of V into A and B such that neither $G[A]$ nor $\overline{G}[B]$ is connected. Because $G[A]$ is not connected, skew-partitions are sometimes called *skew-cutsets*. If we split A into A_1 and A_2 with no edges between them and we split B into B_1 and B_2 with all edges between them, we say that (A_1, A_2, B_1, B_2) is a *split* of a skew-partition. Given a split we define the graphs $G_1 = A_1 \cup B$ and $G_2 = A_2 \cup B$.

If a graph G contains a star cutset X with centre v , then $G[X]$ is disconnected in the complement so G admits a skew-partition into X and $G - X$, thus skew-partitions generalize star cutsets (and also clique cutsets). If G contains a homogeneous set H such that the sets N and A of vertices in $G - H$ which see none and all of H respectively are both nonempty, then $(N \cup (H \setminus \{v\}), A \cup \{v\})$ is a skew-partition. Thus skew partitions also generalize homogeneous sets.

Chvátal [Chv85] first introduced skew-partitions upon proving the Star Cutset Lemma. He conjectured that they could not exist in a minimum counterexample to the SPGT; this conjecture remained open until the proof of the SPGT rendered it trivial [Sey06].

A heavy focus on general skew-partitions brought about difficulties in the search for a structure theorem for Berge graphs. As a response, Chudnovsky, Robertson, Seymour, and Thomas introduced *balanced skew-partitions*. A skew-partition (A, B)

is *balanced* if every induced path of length ≥ 2 with endpoints in B and interior in A has even length, and the same property holds for (B, A) in the complement of G . These are the second type of decomposition needed for the structure theorem on Berge graphs.

To prove that no minimum order counterexample to the SPGT contains a balanced skew-partition we must first lay some groundwork.

Lemma 3.14 (The Colouring Lemma). *Suppose (A_1, A_2, B_1, B_2) is a split of a skew-partition in a minimal imperfect graph G . Then there do not exist $\omega(G)$ -colourings C_1 of G_1 and C_2 of G_2 in which B_1 receives the same number of colours.*

Proof. For $i \in \{1, 2\}$ let X_i be the union of the k colour classes intersecting B_1 in the colouring of G_i and let $X = X_1 \cup X_2$. No stable set intersects both B_1 and B_2 , so $X \cap B_2 = \emptyset$. No clique intersects both A_1 and A_2 , and $G - B_2$ is perfect, so $G[X]$ contains no $k + 1$ clique and is therefore k -colourable. Our colouring of G_i further shows that $G_i - X_i$ is $\omega(G) - k$ colourable. Since $G - X$ is perfect and has clique number $\max\{\omega(G_1 - X_1), \omega(G_2 - X_2)\} = \omega(G) - k$, it follows that G is $\omega(G)$ -colourable, contradicting the assertion that it is minimal imperfect. \square

To apply the Colouring Lemma, we use a weaker version of an observation made by Hoàng [Hoà96]:

Observation 3.15. *Construct G_i^* from G_i by adding a vertex v^* with neighbourhood B . If G_i^* is perfect then there is an $\omega(G)$ -colouring of G_i in which precisely $\omega(B_1)$ colours appear on B_1 .*

Proof. At least $\omega(B_1)$ colours appear on B_1 in any colouring. Denote by ω^* the size of the largest clique in G_i^* containing v^* . Replace v^* with a clique C^* of size $\omega(G) - \omega^* + 1$ (this size may be zero, in which case we simply delete v^*). The resulting graph, which we call G'_i , is perfect and has clique number $\omega(G)$. Furthermore, the vertices of B_1 are joined to a clique of size $\omega(G) - \omega(B_1)$ contained in $B_2 \cup C^*$. It follows that in an $\omega(G)$ -colouring of G'_i , precisely $\omega(B_1)$ colours appear on B_1 . Deleting the vertices of C^* gives us the desired colouring of G_i . \square

The reason behind the restriction on path parity in a balanced skew-partition now makes sense, for if the skew-partition is balanced, both G_1^* and G_2^* will be Berge.

Rather than proving that a minimal imperfect graph cannot admit a balanced skew-partition, we prove that a minimum order counterexample to the SPGT cannot admit one.

Lemma 3.16. *If G is a minimum order imperfect Berge graph, it does not admit a balanced skew-partition.*

Proof. First observe that G is minimal imperfect. Suppose it admits a balanced skew-partition with split (A_1, A_2, B_1, B_2) ; we can assume that each of these sets has size at least two, otherwise either G or \overline{G} contains a star cutset. For $i \in \{1, 2\}$ we make G_i^* from G_i as in the previous observation. Both G_1^* and G_2^* have smaller order than G , and by the parity restrictions on a balanced skew-partition, both are Berge and therefore perfect.

By the previous observation, there are $\omega(G)$ -colourings C_1 of G_1 and C_2 of G_2 in which precisely $\omega(B_1)$ colours appear on B_1 . Therefore by the Colouring Lemma, G is not minimal imperfect, a contradiction. \square

We refer the interested reader to Reed’s paper on skew-partitions [Ree08], in which he discusses the original genesis of these ideas, how some related lemmas were used in the proof of the Strong Perfect Graph Theorem, and algorithmic aspects of skew-partitions.

3.5.3 The decomposition theorem

In the original proof of the SPGT, one more decomposition was needed: proper homogeneous pairs. A homogeneous pair (A, B) is proper if no vertex in A sees all or none of B , no vertex in B sees all or none of A , and there is at least one vertex in $V(G) \setminus (A \cup B)$ that sees both A and B (resp. A but not B , B but not A , neither A nor B). The original decomposition theorem is:

Theorem 3.17. *For any Berge graph G that is not basic, either G or \overline{G} admits a 2-join or a balanced skew-partition or contains a proper homogeneous pair.*

However, Chudnovsky proved that proper homogeneous pairs are unnecessary by characterizing the structure of *Berge trigraphs* [Chu06]. Homogeneous pairs and trigraphs are important concepts in the study of claw-free graphs, and we will discuss

them further in Chapter 6. Chudnovsky’s strengthening of the original structure theorem (both appear in [CRST06]) is quite beautiful in its simplicity:

Theorem 3.18. *For any Berge graph G that is not basic, either G or \overline{G} admits a proper 2-join or a balanced skew-partition.*

In light of the results we have discussed in this section, this theorem immediately implies the Strong Perfect Graph Theorem, because no minimum order counterexample admits a 2-join or a balanced skew-partition.

This decomposition theorem is much simpler than the structure theorems for claw-free graphs for several reasons. For claw-free graphs, there are many more types of “basic” graphs, several cases of exceptions, and two separate structure theorems, depending on whether or not \overline{G} is 3-colourable. Furthermore, when we use the structure theorem to prove the Main Conjecture for claw-free graphs, we rely on 2-joins in which we have a very thorough understanding of the structure of either V_1 or V_2 . We would be very interested in a general theorem proving that no minimum counterexample to the Main Conjecture admits a 2-join (or, even better, a 2-amalgam).

The structure theorem for Berge graphs suggests that it may be possible to derive an efficient combinatorial χ -colouring algorithm for perfect graphs. We already showed how to optimize over perfect graphs using the ellipsoid method, but a combinatorial algorithm would be of great interest and none is yet known.

3.6 Recognizing Berge graphs

In spite of everything that was known about perfect graphs, there was a major question that the proof of the Strong Perfect Graph Theorem did not settle: Given a graph G , can we decide whether or not G is Berge in polynomial time?

We have already shown that this is possible for many classes of perfect graphs, for example chordal graphs, in which we forbid all holes. The situation is much more complicated when we only forbid holes of even length. Conforti, Cornuéjols, Kapoor, and Vušković proved a structural decomposition theorem for even-hole-free graphs [CCKV02a] and gave polynomial-time algorithms to decide whether or not a graph contains an even hole, and find an even hole if one exists [CCKV02b]. Chudnovsky, Kawarabayashi, and Seymour improved on this result, giving a simpler and more

general algorithm that does not use the structure theorem [CKS04]. In an analogue of Fulkerson and Gross' characterization of chordal graphs in terms of simplicial vertices, Addario-Berry, Chudnovsky, Havet, Reed, and Seymour proved that every even-hole-free graph contains two nonadjacent bisimplicial vertices (and therefore has a *bisimplicial ordering*) [ACH⁺08].

It is still unknown whether or not the problem of deciding whether or not a graph G is odd-hole-free is NP -complete. However, Chudnovsky, Cornuéjols, Liu, Seymour, and Vušković found a polynomial-time algorithm for recognizing Berge graphs [CCL⁺05]. Surprisingly, their algorithm does not use the structure theorem for Berge graphs. Given a graph G , the algorithm either decides that G is not Berge, or that G contains no odd holes. This does not tell us whether or not G is odd-hole-free, since the algorithm can simply return “not Berge” for a graph regardless of whether or not it contains an odd hole. However, running the algorithm on both G and \overline{G} will tell us whether or not G is Berge.

This algorithm does not seek out proper 2-joins and balanced skew-partitions as per the structure theorem for Berge graphs. In fact, it was not until later that Trotignon [Tro08] proved that balanced skew-partitions can be detected in a Berge graph in polynomial time. This result is based on Cornuéjols and Cunningham's result that proper 2-joins can be found in polynomial time [CC85], and uses the structure theorem for Berge graphs. Trotignon's detection algorithm does not find a specific balanced skew-partition if there is one. In the same paper he proved that the problem of detecting balanced skew-partitions is NP -complete for general graphs. In contrast, de Figueiredo, Klein, Kohayakawa, and Reed gave a polynomial-time ($O(n^{101})$) algorithm for finding a skew-partition in a graph G [dFKKR00]. A much faster ($O(n^4m)$) algorithm was later given by Kennedy and Reed [KR08a].

Chapter 4

Proving the Conjecture When χ and χ_f Are Close

As discussed in Chapter 2, $\lceil \chi_f(G) \rceil \leq \gamma_l(G)$ so if $\chi(G) \leq \lceil \chi_f(G) \rceil$ then the Local Strengthening holds. In this chapter, we use this fact to prove the Main Conjecture for two classes of graphs: circular interval graphs (defined below) and line graphs. Our focus on these graphs is motivated by the fact that these classes are used in the decomposition theorem for claw-free graphs. They are used both as the base classes into which we decompose and in defining some of the operations used to decompose.

Niessen and Kind proved the *round-up property*, that $\chi \leq \lceil \chi_f \rceil$, for circular interval graphs [NK00], and hence proved that the Local Strengthening holds for these graphs. In Section 4.1, we present a corollary of their result which we will use in bounding the chromatic number of quasi-line graphs.

Seymour and Goldberg conjectured that the round-up property holds for a line graph G of H provided the fractional chromatic number of G exceeds $\Delta(H)$. Since $\omega(G) \geq \Delta(H)$, this easily implies the Local Strengthening for line graphs (as a connected graph which satisfies $\gamma_l(G) < \omega(G) + 1$ is a clique and so satisfies the Local Strengthening). However, this famous conjecture has not yet been proved and the Local Strengthening remains open for line graphs.

It is known that the round-up property holds for the line graph G of H provided $\chi(G) > 1.1\Delta(G) + 0.8$. In Section 4.2, we use this partial result towards the Seymour-Goldberg Conjecture to prove the Main Conjecture for line graphs.

4.1 Linear and circular interval graphs

Two well-known classes of claw-free graphs are linear interval graphs and circular interval graphs. Circular interval graphs are sometimes called *proper circular arc graphs* and are not to be confused with circular arc graphs, which are not necessarily claw-free. Linear interval graphs are often known as *unit interval graphs*. As we shall see, these two classes of graphs play an important role in the structure theorems for claw-free graphs.

A *linear interval representation* of a graph $G = (V, E)$ consists of a point on the real number line corresponding to each vertex, along with a set of closed intervals such that two vertices u and v of the graph are adjacent precisely if there is an interval containing the two points corresponding to u and v . Obviously in a representation with the fewest number of intervals, none contains another. A *linear interval graph* is a graph for which there is a linear interval representation. These graphs can be both recognized and represented in linear time [DHH96]. Linear interval graphs are chordal and therefore perfect, and they can be ω -coloured in linear time by colouring the vertices greedily, moving from left to right along the real line.

A *circular interval representation* of a graph $G = (V, E)$ consists of a point on the boundary of the unit circle corresponding to each vertex, along with a set of closed intervals on the boundary of the unit circle such that two vertices of G are adjacent precisely if there is an interval containing both points associated with the vertices (again we assume no interval contains another). A *circular interval graph* is a graph for which there is a circular interval representation. Like linear interval graphs, circular interval graphs can be recognized and represented in linear time [DHH96].

Henceforth, when we are given a circular (resp. linear) interval graph on n vertices we will assume the vertices are labelled v_1, \dots, v_n in clockwise (resp. left-to-right, i.e. ascending) order. Given a linear interval representation of a linear interval graph G in which the points corresponding to v_1, \dots, v_n appear in left-to-right order, we can easily find a circular interval representation of G in which the points corresponding to v_1, \dots, v_n appear in clockwise order, so there is no ambiguity.

As mentioned earlier, Niessen and Kind proved that $\chi = \lceil \chi_f \rceil$ for circular interval graphs, implying the Local Strengthening:

Theorem 4.1. *For any circular interval graph G , $\chi(G) \leq \gamma_l(G)$.*

Teng and Tucker [TT85] gave an algorithm for optimally colouring a circular interval graph in polynomial time; Shih and Hsu [SH89] later improved the complexity analysis:

Theorem 4.2. *Given a circular interval graph G , we can $\chi(G)$ -colour G in $O(n^{3/2})$ time.*

We now use Niessen and Kind’s result to construct an integer colouring of a linear interval graph that emulates a given fractional colouring at the far left and right ends of the graph.

4.1.1 Fractional and integer colourings in linear interval graphs

When we say that a linear interval graph G has *end-cliques* L and R , we mean that G has specified cliques L and R consisting of the $|L|$ leftmost and $|R|$ rightmost vertices in the linear interval representation, respectively. We use the round-up property of circular interval graphs to prove that we can closely emulate any fractional colouring of a linear interval graph with an integer colouring, with respect to L and R . This lemma will be of use to us in relating χ_f and χ for quasi-line graphs.

Lemma 4.3. *Let G be a linear interval graph with end-cliques L and R , and let c and w be nonnegative integers. If there is a fractional c -colouring of G such that the weight of the stable sets intersecting both L and R is precisely w , then there is an integer c -colouring of G such that exactly w colours intersect both L and R .*

Proof. Consider such a fractional c -colouring. Let D be the set of colours appearing in both L and R , let X be the set appearing in L but not R , Y the set appearing in R but not L , and Z the set appearing in neither L nor R .

Observe that since c , w , $|L|$ and $|R|$ are integers, all of $wt(D) = w$, $wt(X) = |L| - w$, $wt(Y) = |R| - w$ and $wt(Z) = c - |L| - |R| + w$ are integers. We construct a circular interval graph G' based on the fractional colouring of G .

Say the n vertices of G are v_1, \dots, v_n , left-to-right – thus in a circular interval representation of G they appear in clockwise order. To construct G' from G , we add cliques V_X , V_Z , and V_Y of size $wt(X)$, $wt(Z)$, and $wt(Y)$ respectively, in clockwise order starting after v_n . To determine their adjacency to each other and the vertices of G , we make three new intervals I_R , I_Z , and I_L . These span, respectively: $\{v_i \mid$

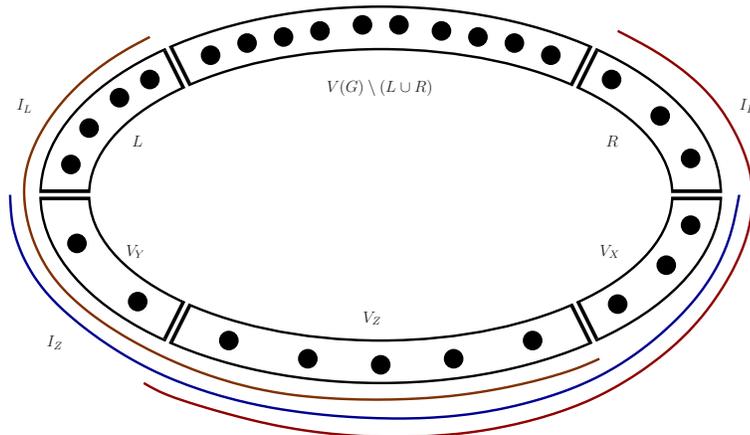


Figure 4.1: G' , with the vertices of G along the top.

$n - |R| < i\} \cup V_X \cup V_Z$, $V_X \cup V_Z \cup V_Y$, and $\{v_i \mid i \leq |L|\} \cup V_Y \cup V_Z$ (see Figure 4.1). Thus V_X is complete to R and V_Z , V_Z is complete to V_X and V_Y , and V_Y is complete to V_Z and L . Since I_L and I_R define c -cliques and I_Z defines a $c - w$ clique, it is easy to see that G' is a circular interval graph with clique number c , hence both the fractional chromatic number and the chromatic number of G' are at least c .

Now we give a fractional c -colouring of G' . On the vertices belonging to G , we keep our initial fractional colouring. We can then cover V_X with the colours in X , V_Z with the colours in Z , and V_Y with the colours in Y . Hence $\chi_f(G') = c$ and so by Niessen and Kind's result that for circular interval graphs, $\chi = \lceil \chi_f \rceil$, we know that $\chi(G') = c$. So consider an integer c -colouring of G' . We claim that on the vertices of G this gives us an integer c -colouring of G containing exactly w colours that appear in both L and R .

Suppose fewer than w colours appear in both L and R . Then there are more than $|L| + |R| - w$ colours that cannot appear in V_Z . But V_Z is a clique of size $c - (|L| + |R| - w)$, contradicting the fact that we have a proper c -colouring. Now suppose more than w colours appear in both L and R . Then none of these colours can appear in $V_X \cup V_Z \cup V_Y$. But $V_X \cup V_Z \cup V_Y$ is a clique of size $c - w$ so again we cannot have a proper c -colouring. Thus exactly w colours appear in both L and R . Restricting this colouring of G' to the vertices of G gives us the desired colouring. \square

4.2 Line graphs

Line graph colouring is of particular interest because a colouring of a line graph corresponds to an edge colouring of the underlying multigraph. Much of the interest in quasi-line graphs and claw-free graphs comes from the fact that they are both natural generalizations of line graphs. This link to edge colouring also makes line graphs easy to colour using techniques from that domain. So it is not surprising that line graphs were the first substantial subclass of claw-free graphs for which the Main Conjecture was proved. This was done by the author with Reed and Vetta [KRV07]. Before we prove this result we need to review some past results on colourings of line graphs, i.e. edge colourings of multigraphs.

4.2.1 Fractional and integer colourings in line graphs

The *chromatic index* of a graph or multigraph H , written $\chi'(H)$, is the chromatic number of $L(H)$. Similarly, the *fractional chromatic index* $\chi'_f(H)$ is equal to the fractional chromatic number of $L(H)$. Holyer proved that determining the chromatic index of an arbitrary graph is *NP*-complete [Hol81], so practically speaking we are bound to the task of approximating the chromatic index of multigraphs and hence the chromatic number of line graphs (and, for that matter, claw-free graphs).

Vizing's Theorem for multigraphs [Viz64] bounds the chromatic index of a multigraph in terms of its maximum degree and multiplicity d , stating that $\Delta(H) \leq \chi'(H) \leq \Delta(H) + d$, where d is the maximum number of edges between any two vertices in H . Both bounds are achievable, but a more meaningful bound should consider other invariants of H .

Of course, $\chi'(H)$ is always bounded below by $\chi'_f(H)$. Let w be a non-negative weighting on the edges of H . Duality tells us that given a non-negative weighting w on the edges of H such that for every matching M in H , $\sum_{e \in M} w(e) \leq 1$, $\chi'_f(H) \geq \sum_{e \in E(H)} w(e)$. Two such weightings give us lower bounds. In the first, we assign a weight of 1 to each edge incident to a given vertex v of maximum degree; every other edge is assigned weight 0. In the second, we take an induced subgraph W of H and assign to each edge of W a weight of $1/\lfloor |V(W)|/2 \rfloor$; other edges of H are assigned weight 0. Edmond's theorem for matching polytopes (presented in [Edm65a], also mentioned in [Kah00]) tells us that the greater of these lower bounds is tight, so

setting

$$\Gamma(H) = \max \left\{ \frac{2|E(W)|}{|V(W)| - 1} : W \subseteq H, |V(W)| \text{ is odd} \right\},$$

we have

$$\chi'_f(H) = \max\{\Delta(H), \Gamma(H)\}. \quad (4.1)$$

The long-standing conjecture posed by Seymour [Sey79] and Goldberg [Gol73] proposes that this gives us a very nearly tight bound on the chromatic index of a multigraph:

Seymour-Goldberg Conjecture. *For a multigraph H for which $\chi'(H) > \Delta(H) + 1$, $\chi'(H) = \lceil \Gamma(H) \rceil$.*

Kahn [Kah96] proved that this bound holds asymptotically, i.e. that $\chi'(H) \leq (1 + o(1))\chi'_f(H)$. He later proved that in fact, the fractional chromatic index asymptotically agrees with the *list chromatic index* [Kah00]. More useful to us in proving the Main Conjecture is the following algorithmic approximation result:

Theorem 4.4 (Nishizeki and Kashiwagi [NK90]). *For any multigraph H , $\chi'(H) \leq \max\{\lfloor 1.1\Delta(H) + 0.8 \rfloor, \lceil \Gamma(H) \rceil\}$. Furthermore in $O(|E(H)| \cdot |V(H)|)$ time we can find an edge colouring of H that achieves this bound.*

Caprara and Rizzi [CR98] later improved the term $\lfloor 1.1\Delta(H) + 0.8 \rfloor$ in the bound to $\lfloor 1.1\Delta(H) + 0.7 \rfloor$, and Scheide [Sch07a] recently improved it to $\lfloor \frac{15}{14}\Delta(H) + \frac{12}{14} \rfloor$. Note that this implies the Seymour-Goldberg Conjecture for any multigraph H with $\Delta(H) \leq 15$, since in this case we have $\lfloor \frac{15}{14}\Delta(H) + \frac{12}{14} \rfloor \leq \Delta(H) + 1$.

With these results in hand we can prove the Main Conjecture for line graphs.

4.2.2 Proving the Main Conjecture for line graphs

In this section we prove the main result of [KRV07]:

Theorem 4.5. *For any line graph $G = L(H)$, $\chi(G) \leq \gamma(G)$.*

To prove this we use the standard approach described in Section 2.6: Assuming G is a minimum counterexample, we find and remove a stable set S such that $\gamma(G - S) < \gamma(G)$. If we cannot find such an S , we somehow exploit this fact to prove $\chi(G) \leq \gamma(G)$ directly.

For line graphs, we can find the desired S unless $\Delta(H)$ and $\Delta(G)$ are far apart, in which case the following easy corollary of Theorem 4.4 allows us to prove directly that the Main Conjecture holds.

Lemma 4.6. *If G is the line graph of a multigraph H and $\Delta(G) \geq \frac{3}{2}\Delta(H) - 1$, then $\chi(G) \leq \gamma(G)$.*

Proof. We know that $\chi_f(G) = \max\{\Delta(H), \Gamma(H)\}$, so Theorem 4.4 tells us that $\chi(G) \leq \max\{\lceil 1.1\Delta(H) + 0.8 \rceil, \lceil \chi_f(G) \rceil\}$. We know that the Main Conjecture holds fractionally for G by Theorem 2.10, so $\lceil \chi_f(G) \rceil \leq \gamma(G)$. Since $\Delta(G) \geq \frac{3}{2}\Delta(H) - 1$, we have $\gamma(G) = \lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \rceil \geq \lceil \frac{1}{2}(\frac{3}{2}\Delta(H) + \Delta(H)) \rceil \geq \lceil \frac{5}{4}\Delta(H) \rceil \geq \lceil 1.1\Delta(H) + 0.8 \rceil$, so $\gamma(G) \geq \chi(G)$ and we are done. \square

We can now proceed to the case in which $\Delta(G) < \frac{3}{2}\Delta(H) - 1$. In this case we will find a stable set S hitting every maximum clique in G . This stable set corresponds to a matching M in H . First we introduce some notation. For vertices u and v in a multigraph, the *multiplicity* of uv , denoted $\mu(u, v)$, is the number of edges between u and v . We denote by $tri(H)$ the maximum number of edges in a triangle in H .

Proof of Theorem 4.5. Let G be a minimum counterexample to the theorem; we know that $\Delta(G) < \frac{3}{2}\Delta(H) - 1$. Note the following facts that relate the invariants of G and H :

Fact 1. $\Delta(G) = \max_{v_1 v_2 \in E_H} \{\deg(v_1) + \deg(v_2) - \mu(v_1, v_2) - 1\}$.

Fact 2. $\omega(G) = \max\{\Delta(H), tri(H)\}$.

We will find a maximal matching M whose removal from H lowers both $\Delta(H)$ and $\max\{\Delta(H), tri(H)\}$. To this end, let S_Δ be the set of vertices of degree $\Delta(H)$ in H and let T be the set of triangles in H that contain $\max\{\Delta(H), tri(H)\}$ edges. It is instructive to consider how the elements of T interact.

Observation 4.7. *If two triangles of T intersect in exactly the vertices a and b then ab has multiplicity greater than $\Delta(H)/2$.*

Proof. For any edge e of H between a and b , the degree of the corresponding vertex of G is at least $2\Delta(H) - \mu(a, b)$. \square

Observation 4.8. *If abc is a triangle of T intersecting another triangle ade of T in exactly the vertex a then $\mu(b, c)$ is greater than $\Delta(H)/2$.*

Proof. The degree of a vertex of G corresponding to an edge between a and d is at least $2\Delta(H) - \mu(b, c) - 1$. \square

Observation 4.9. *If there is an edge of H joining two vertices a and b of S_Δ then $\mu(a, b) > \Delta(H)/2$.*

Guided by these observations, we let T' be those triangles in T that contain no pair of vertices of multiplicity $> \Delta(H)/2$ and S'_Δ be those elements of S_Δ which are in no pair of vertices of multiplicity greater than $\Delta(H)/2$. We treat $T' \cup S'_\Delta$ and $(T \setminus T') \cup (S_\Delta \setminus S'_\Delta)$ separately. A few more observations regarding S'_Δ and T' will serve us well. Recall that for a set S of vertices we denote the union of the vertices' neighbourhoods by $N(S)$.

Observation 4.10. *For any $S \subseteq S'_\Delta$, $|N(S)| \geq |S|$.*

Proof. It follows from Observation 4.9 that S'_Δ is a stable set. This means that S and $N(S)$ are disjoint, and given $S \subseteq S'_\Delta$ there are $|S|\Delta(H)$ edges between S and $N(S)$; the result follows from the fact that no vertex in $N(S)$ has degree $\geq \Delta(H)$. \square

Observation 4.11. *If an edge ab in H has exactly one endpoint in a triangle bcd of T' , then the degree of a is less than $\Delta(H)$.*

Proof. Any vertex in G corresponding to an edge between a and b has degree at least $\deg(a) - 1 + \Delta(H) - \mu(c, d)$, and $\mu(c, d) \leq \Delta(H)/2$. \square

Observation 4.12. *If an edge ab in H has exactly one endpoint in a triangle bcd of T' , then $\mu(a, b) \leq \Delta(H)/2$.*

Proof. The degree of any vertex in G corresponding to an edge between b and c has degree at least $\mu(a, b) + \Delta(H) - 1$. \square

Observation 4.13. *For any vertex v with two neighbours u and w , $\deg(u) + \mu(vw) \leq \frac{3}{2}\Delta(H)$.*

Proof. An edge between u and v is incident to at least $\deg(u) + \mu(vw) - 1$ other edges. \square

Finally, we state Hall's Theorem (see [Hal35]), a fundamental result on matchings in bipartite graphs.

Hall's Theorem. *Let G be a bipartite graph with vertex set $V = (A, B)$. There is a matching that hits every vertex in A precisely if for every $S \subseteq A$, $|N(S)| \geq |S|$.*

It is now straightforward to show that the desired matching exists. We begin with a matching M consisting of one edge between each vertex pair with multiplicity greater than $\Delta(H)/2$ – this hits $S_\Delta \setminus S'_\Delta$ and contains an edge of each triangle in $T \setminus T'$. Observation 4.10 tells us that we can apply Hall's Theorem to get a matching in H that hits S'_Δ ; Observation 4.13 dictates that this matching cannot hit M , so the union M' of these two matchings is a matching in H that hits S_Δ and contains an edge of each triangle in $T \setminus T'$. Every edge in this matching either hits a maximum-degree vertex in H or has endpoints with multiplicity greater than $\Delta(H)/2$.

What, then, can prevent us from extending this M' to contain an edge of every triangle in T' ? Observations 4.7 and 4.8 tell us that any two triangles in T' are vertex-disjoint, so our only worry is that M' already hits two vertices of some triangle in T' . Observations 4.9, 4.11 and 4.12 guarantee that at most one such vertex in a given triangle is hit, and if there is such a vertex, it has degree $\Delta(H)$. We can therefore extend M' to contain an edge of every triangle in T' . The result is a matching that satisfies all of our requirements, so the proof of the theorem is complete. \square

4.2.3 Algorithmic considerations

We can determine whether or not a graph G is quasi-line or claw-free by looking at the structure of each neighbourhood in G , but for line graphs this is not the case. Beineke [Bei70] characterized line graphs of simple graphs in terms of nine forbidden subgraphs, one of which is the claw (see Figure 4.2). This characterization gives us a trivial $O(n^6)$ algorithm for testing whether or not G is the line graph of a simple graph. Lehot [Leh74] and Roussopoulos [Rou73] independently gave $O(m)$ -time algorithms that find a graph H for which $G = L(H)$ if it exists, and determine that G is not a line graph of a simple graph otherwise.

We can assume that if two vertices u and v are together in a homogeneous clique in a line graph G , then they correspond to parallel edges (i.e. edges between the same two endpoints) in the base multigraph H . Thus to find H quickly, we first

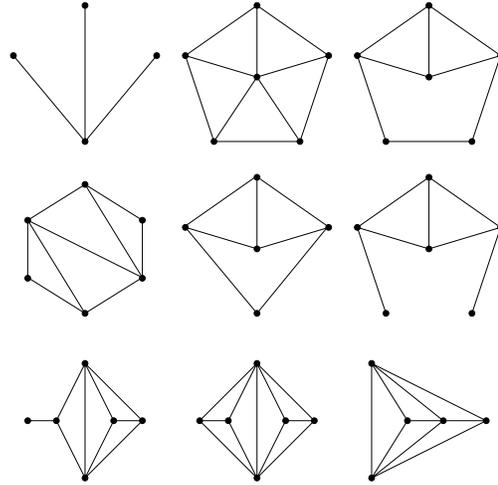


Figure 4.2: The 9 forbidden subgraphs for line graphs of simple graphs.

find the maximal homogeneous cliques in G – they are disjoint. Contracting each homogeneous clique down to a single vertex gives us a graph G' which is a line graph of the simple graph H' underlying H . We can find H' from G' in $O(m)$ time, then construct H from H' using our knowledge of the homogeneous cliques. Thus we can construct H from G in $O(m)$ time if we can find the maximal homogeneous cliques of G in $O(m)$ time. And we can do this using the *modular decomposition tree* of the graph, which can be computed in linear time [CH94, MS94, MdP04].

With H in hand, a polytime algorithm for $\gamma(G)$ -colouring G follows naturally from our proof of Theorem 4.5. Theorem 4.4 implies that using Nishizeki and Kashiwagi's result, we can $\gamma(G)$ colour a line graph $G = L(H)$ in $O(|E(H)| \cdot |V(H)|)$ time if $\Delta(G) \leq \frac{3}{2}\Delta(H) - 1$. If $\Delta(G) > \frac{3}{2}\Delta(H) - 1$ we find a maximal matching M in H whose removal lowers $\omega(G)$ and therefore lowers $\gamma(G)$. To do this we first hit high-multiplicity (i.e. $> \Delta(H)/2$) edges, then we find a matching saturating S'_Δ . Finally we hit the outstanding triangles of weight at least $\max\{\Delta(H), \text{tri}(H)\}$. The difficult part of this process is saturating S'_Δ , but we can reduce this to the problem of finding a maximum matching in a bipartite graph between S'_Δ and its neighbourhood. Using Hopcroft and Karp's $O(\sqrt{nm})$ algorithm for finding a maximum matching in a bipartite graph [HK73], we can find M in $O(|V(H)|^{1/2}|E(H)|) = O(|E(H)|^{3/2})$ time.

Thus we can find a $\gamma(G)$ edge colouring of H in $O(|V(G)|^{5/2})$ time as follows.

1. While $\Delta(L(H)) < \frac{3}{2}\Delta(H) - 1$, remove a matching M from H as in the proof of Theorem 4.5 (and let it be a colour class).
2. Employ Nishizeki and Kashiwagi's algorithm to complete the edge colouring of H .

This gives us the following algorithmic version of Theorem 4.5:

Theorem 4.14. *Let G be a line graph on n vertices. Then in $O(n^{5/2})$ time we can find a proper $\gamma(G)$ -colouring of G .*

Remark: In [KR08b] the complexity of our colouring algorithm was stated as $O(n^{7/2})$. This was obtained by treating the running time of Hopcroft and Karp's matching algorithm as $O(|V(H)^{5/2}|) = O(|V(G)^{5/2}|)$, when in fact the running time of the algorithm is $O(|V(H)|^{1/2}|E(H)|) = O(|V(G)|^{3/2})$.

In the next chapter we will reduce the problem of γ -colouring quasi-line graphs to the problem of γ -colouring line graphs.

Part III

Quasi-line Graphs and Thickenings

If every vertex in a graph G is bisimplicial, i.e. its neighbours can be covered by two cliques, then the graph is clearly claw-free. Such a graph is *quasi-line*. The next three chapters concern these graphs. In the next chapter we describe their structure. In the process we introduce *compositions of strips*, a method of composition which also plays a key role in the description of the structure of claw-free graphs. We also describe two ways to expand vertices and edges in quasi-line graphs: *augmentations*, which are used for Berge quasi-line graphs, and *thickenings*, which are more general and are used for all claw-free graphs.

Thickenings result in homogeneous pairs of cliques. In Chapter 6 we explain how we can manipulate homogeneous pairs of cliques to our advantage. We introduce a reduction that we can apply to certain types of them. This reduction preserves the chromatic number and fractional chromatic number and leaves a graph whose structure is easier to characterize. These irreducible or *skeletal* graphs contain only very simple homogeneous pairs of cliques, and we can restrict our attention to them when proving our colouring results in Chapters 7 and 10.

In Chapter 7 we use a refined structure theorem for skeletal quasi-line graphs to bound the chromatic number of all quasi-line graphs. We extend two bounds from line graphs to quasi-line graphs. First we prove the Main Conjecture for quasi-line graphs, then we prove that χ_f and χ agree asymptotically for these graphs:

Theorem III.1. *For any quasi-line graph G , $\chi(G) \leq \gamma(G)$.*

Theorem III.2. *For any quasi-line graph G , $\chi(G) \leq \chi_f(G) + 3\sqrt{\chi_f(G)}$.*

Chapter 5

The Structure of Quasi-line Graphs

In this chapter we describe the structure of quasi-line graphs, i.e. graphs in which every vertex is bisimplicial. We begin by describing the structure of Berge quasi-line graphs. They arise from two base classes that foreshadow those used in the construction of general quasi-line graphs. Quasi-line graphs are built from circular interval graphs and line graphs via a composition operation that will also be useful for more general classes of claw-free graphs. To expand vertices and edges in quasi-line graphs and claw-free graphs we use *thickenings* and their more specific precursor, *augmentations*.

After introducing the necessary machinery we present a version of Chudnovsky and Seymour's structure theorem for quasi-line graphs that we will sharpen in the next chapter.

5.1 Berge quasi-line graphs

If a claw-free graph G contains a vertex v that is not bisimplicial, then $G[N(v)]$ contains an odd antihole of length at least five, so G is not Berge. Thus the Berge claw-free graphs are precisely the Berge quasi-line graphs. The first step towards characterizing Berge quasi-line graphs was a theorem of Chvátal and Sbihi [CS88], which we state now.

Definition 5.1. *A graph G is elementary if its edges can be 2-coloured such that for distinct $t, u, v \in V$ with t nonadjacent to v but u adjacent to both t and v , the edges tu and uv receive different colours.*

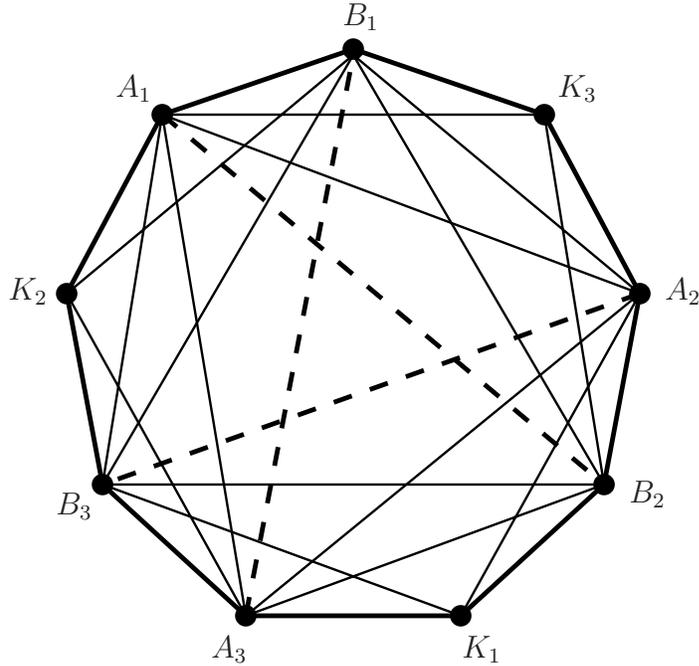


Figure 5.1: A peculiar graph. Circles represent cliques. Two cliques with a solid (resp. dashed) line between them are complete (resp. not complete). Cliques with no line between them are anticomplete.

Definition 5.2. A graph G is peculiar if it can be obtained by the following construction. Begin with three non-complete pairwise disjoint cobipartite graphs (A_1, B_2) , (A_2, B_3) , and (A_3, B_1) , and add all edges between them¹. Now add three pairwise disjoint nonempty cliques K_1, K_2, K_3 , and add all possible edges between K_i and $A_j \cup B_j$ for $i \neq j$ (see Figure 5.1).

Theorem 5.3 ([CS88]). Any Berge quasi-line graph contains a clique cutset or is elementary or peculiar.

This result was refined by Maffray and Reed, who characterized the structure of elementary graphs [MR99]. To do this they introduced *augmentations*, which we describe now.

Let M be a nonempty matching in G , no edge of which is in a triangle. We construct G' as follows. The vertices of G' are partitioned into $|V(G)|$ disjoint nonempty

¹Since none of these cobipartite graphs is a clique, all the cliques A_i and B_i are nonempty.

cliques $\{I(v) \mid v \in V(G)\}$. If $v \in V(G)$ is not the endpoint of an edge in M , then $|I(v)| = 1$. If u and v are nonadjacent then $I(u)$ is anticomplete to $I(v)$. If $uv \in M$ then $I(u)$ is neither complete nor anticomplete to $I(v)$. If $uv \in E(G) \setminus M$ then $I(u)$ is complete to $I(v)$. We say that G' is an *augmentation* of G under M . See Figure 5.2

Maffray and Reed [MR99] proved:

Theorem 5.4. *A graph G is elementary precisely if it is an augmentation of a line graph of a bipartite multigraph.*

Combining this with Theorem 5.3 we obtain:

Theorem 5.5. *Any Berge quasi-line graph contains a clique cutset or is peculiar or is an augmentation of a line graph of a bipartite multigraph.*

Augmentations have many useful properties. Suppose G' is an augmentation of G under M . Then:

- G is a proper induced subgraph of G' .
- G' is quasi-line precisely if G is quasi-line. G' is claw-free precisely if G is claw-free.

This follows easily from the fact that no edge of M is in a triangle.

- For $uv \in M$, $(I(u), I(v))$ is a homogeneous pair of cliques unless both $I(u)$ and $I(v)$ are singletons.
- Suppose G is claw-free. Then for $uv \in M$, G admits a 2-join $((N_G(u) - v, N_G(v) - u), (\{u\}, \{v\}))$ and G' admits a 2-join $((N_G(u) - v, N_G(v) - u), (I(u), I(v)))$.

To see this, note that u and v have no common neighbours, thus $N_G(u) - v$ and $N_G(v) - u$ are disjoint cliques (see Figure 5.2).

With these facts in hand we can easily prove the Strong Perfect Graph Conjecture for quasi-line graphs (and therefore claw-free graphs). Of course, this immediately implies both the Main Conjecture and the Local Strengthening for Berge quasi-line graphs.

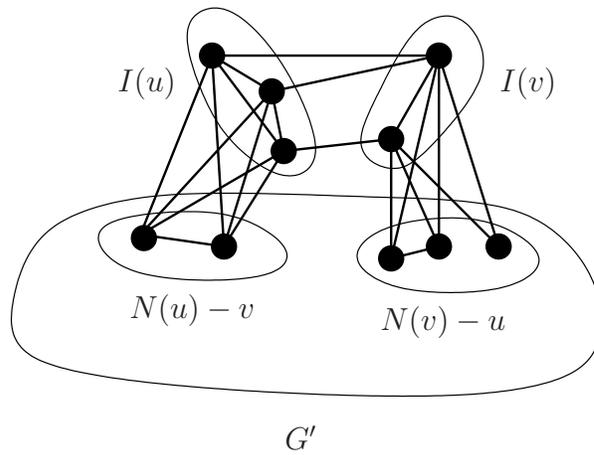
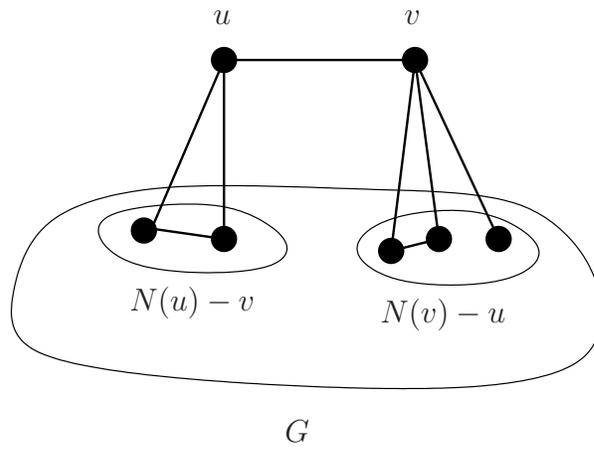


Figure 5.2: The graph G' is an augmentation of G under $M = \{uv\}$. If G is claw-free then both $N(u) - v$ and $N(v) - u$ are cliques and a 2-join arises.

Theorem 5.6. *Every Berge quasi-line graph is perfect.*

Proof. Observe that the class of Berge quasi-line graphs is hereditary. Thus if the theorem is not true there must be a minimal imperfect Berge quasi-line graph G . We know that G cannot contain a clique cutset, so it is elementary or peculiar.

Suppose G is peculiar. Then $A_1 \cup B_1 \cup A_2 \cup B_2$ is a star cutset with a centre in B_1 ; it separates K_3 from the rest of the graph. This contradicts the Star Cutset Lemma, so G cannot be peculiar.

Therefore G is elementary. It cannot be a line graph of a bipartite multigraph, because these graphs are perfect. Thus G arises via augmentation and therefore contains a homogeneous pair of cliques. This contradicts the Homogeneous Pair Lemma. Therefore G cannot exist. \square

Our use of homogeneous pairs of cliques in this proof foreshadows their importance throughout the rest of this thesis. For many classes of claw-free graphs, we can simplify the structure by assuming that no homogeneous pair of cliques exists. For Berge quasi-line graphs, we get the following:

Theorem 5.7. *Let G be a Berge quasi-line graph containing no homogeneous pair of cliques. Then G is either a circular interval graph on nine vertices, or the line graph of a bipartite graph.*

Proof. Suppose that G contains no homogeneous pair of cliques. Observe that since G has at least four vertices, then it cannot contain a homogeneous clique, otherwise it would contain a homogeneous pair of cliques, one of which is a singleton. If G is peculiar, it follows that K_1 , K_2 , and K_3 are singletons. And since none of (A_1, B_2) , (A_2, B_3) , or (A_3, B_1) is a homogeneous pair of cliques, the sets A_i and B_i , $1 \leq i \leq 3$ are also singletons. Thus G must be a circular interval graph on nine vertices (see Figure 5.1) in which dashed lines represent either adjacent or nonadjacent vertices.

If G is an augmentation of a line graph of a bipartite multigraph, then since it contains no homogeneous pair of cliques, $I(v)$ must be a singleton for every vertex v of the base line graph. It follows that G is a line graph of a bipartite multigraph. \square

So the assumption that a graph contains no homogeneous pair of cliques simplifies the description of Berge quasi-line graphs. The same is true for describing quasi-line

graphs and claw-free graphs. We would like to take advantage of this fact by somehow reducing the problem of colouring claw-free graphs to the problem of colouring claw-free graphs containing no homogeneous pair of cliques. We cannot do this, but in the next chapter we will look closely at reducing homogeneous pairs of cliques, and find that we can simplify homogeneous pairs nicely without sacrificing too much information about colourings.

Remark: Theorem 5.6 was first proved in 1976 by Parthasarathy and Ravindra [PR76]. An alternative proof was given by Giles, Trotter, and Tucker in 1984 [GJT84]. Reed presents a simplified version of Parthasarathy and Ravindra's proof in his Ph.D. thesis [Ree86]. These proofs, which came before Chvátal and Sbihi's decomposition theorem, use the Weak Perfect Graph Theorem.

5.2 Compositions of strips

To describe the structure of quasi-line graphs we first need to generalize augmentations of line graphs. We do this using *compositions of strips*. These compositions, introduced by Chudnovsky and Seymour [CS05], are essential to the structure of both quasi-line graphs and claw-free graphs. They generalize augmentations, but more importantly they provide a way to build many claw-free graphs using a beautiful generalization of line graphs.

Consider a multigraph H , possibly containing loops. To find its line graph $L(H)$ we begin with a vertex u_e for each edge of H . For every $v \in V(H)$ we define the set $C_v \subseteq V(L(H))$ as

$$C_v = \{u_e \mid e \text{ is incident to } v \text{ in } H\}.$$

Two vertices u and u' in $L(H)$ are adjacent if and only if they are both in C_v for some $v \in V(H)$. We construct $L(H)$ from our set $\{u_e \mid e \in E(H)\}$ of isolated vertices by adding edges to make each C_v a clique.

This perspective invites a generalization of line graphs. Instead of replacing each edge in H with a vertex, we will replace each edge with a claw-free graph, being careful not to create a claw in the process.

Definition 5.8. A strip (S, X, Y) is a claw-free graph S with two cliques X and Y

such that for any vertex $v \in X$ (resp. Y), the neighbourhood of v outside X (resp. Y) is a clique.

We compose m strips in the following way. First we take a directed multigraph H with m edges corresponding to the m strips. That is, for every edge e of H we have a strip (S_e, X_e, Y_e) . For $v \in V(H)$ we define the set C_v as

$$C_v = \left(\bigcup \{X_e \mid e \text{ is an edge out of } v\} \right) \cup \left(\bigcup \{Y_e \mid e \text{ is an edge into } v\} \right).$$

We construct G from the disjoint union of $\{S_e \mid e \in E(H)\}$ by making each C_v a clique (see Figure 5.3). If G can be built in such a way, we say that G is a *composition of strips*, and that the multigraph H is the *underlying multigraph*. We direct the edges of H to avoid ambiguity between X_e and Y_e : X_e corresponds to the tail of e , and Y_e corresponds to the head.

Note that X_e and Y_e need not be disjoint. If $X_e = Y_e = S_e$ for every strip then G is the line graph of the multigraph obtained by replacing each edge e of H with $|S_e|$ copies of it. Observe that any vertex in X_e or Y_e will be bisimplicial in G , and as a result we can be sure that G is claw-free. Furthermore, G is quasi-line precisely when each S_e is quasi-line. Compositions of strips clearly generalize line graphs, and they also extend augmentations of line graphs. If we have a strip (S, X, Y) in which X and Y partition the vertices of S , then not only is (X, Y) a homogeneous pair of cliques, but G is an augmentation of the graph G' that we get by replacing X and Y with two adjacent vertices. Conversely, every augmentation of a line graph H is a composition of strips for which H is the underlying line graph.

Remark: Chudnovsky and Seymour originally defined a strip (S', a, b) as a claw-free graph in which a and b are nonadjacent simplicial vertices [CS05, KR08b]. We define (S, X, Y) where $S = S' - \{a, b\}$, $X = N(a)$, and $Y = N(b)$. Our definition is basically equivalent and is easier to work with.

5.2.1 2-joins arising from compositions

To solve optimization problems on compositions of strips we need to know how to decompose them. We will do this via *2-joins*. Our definition is slightly different than the definition used by Cornuéjols and Cunningham [CC85].

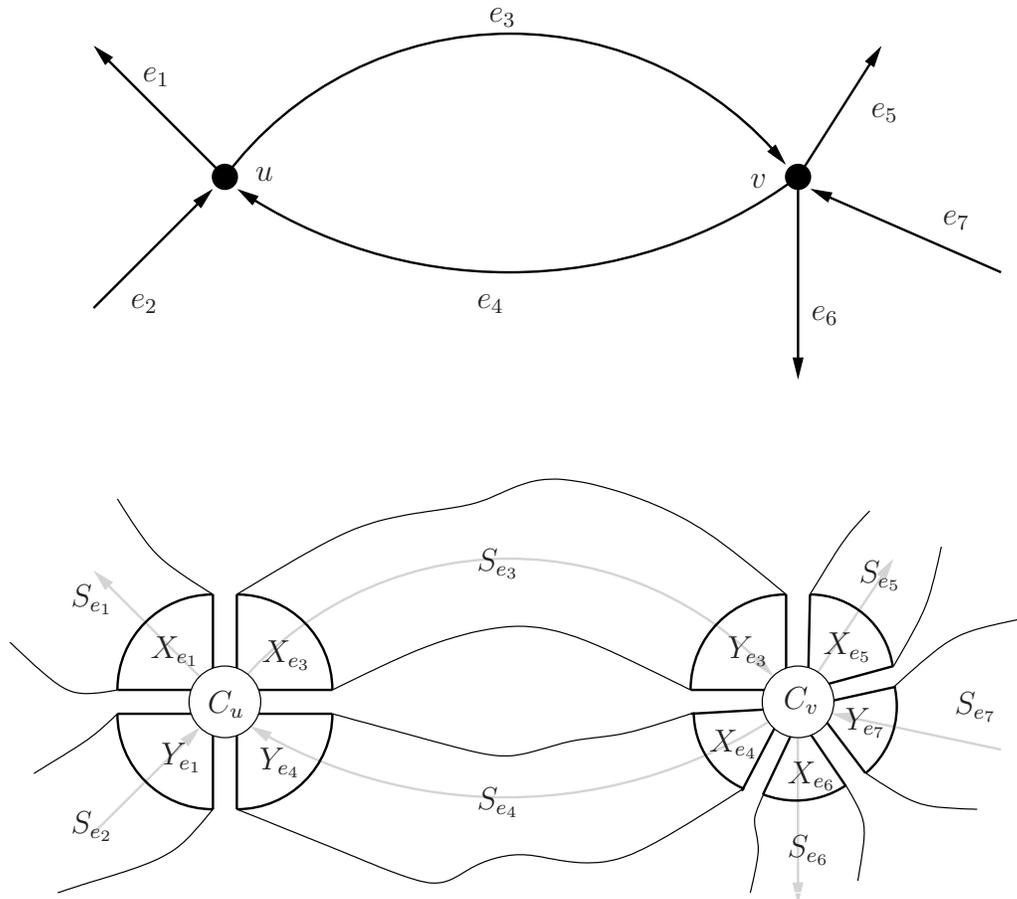


Figure 5.3: We compose a set of strips $\{(S_e, X_e, Y_e) \mid e \in E(H)\}$ by joining them together on their end-cliques. A hub clique C_u will arise for each vertex $u \in V(H)$.

Definition 5.9. *Suppose vertex sets V_1 and V_2 partition $V(G)$ and there are cliques X_i and Y_i in V_i such that $X_1 \cup X_2$ and $Y_1 \cup Y_2$ are cliques, and there are no other edges between V_1 and V_2 . Then we say that $((X_1, Y_1), (X_2, Y_2))$ is a 2-join.*

This is equivalent to a *generalized 2-join* as defined by Chudnovsky and Seymour [CS05].

In a composition of strips we replace each edge xy of the underlying multigraph by a strip S_{xy} and attach it to the graph by joining X_{xy} to C_x and joining Y_{xy} to C_y . Thus it is clear how we should decompose these graphs, because we end up with a 2-join for any strip S_{xy} . The 2-join is

$$((C_x \setminus X_{xy}, C_y \setminus Y_{xy}), (X_{xy}, Y_{xy})).$$

Knowing that a claw-free graph admits such a 2-join is not enough to prove our bounds on the chromatic number. We need to know the structure of S_{xy} so we can exploit properties of restricted colourings in the strips. It turns out that we need five types of strips to describe claw-free graphs, but only one to describe quasi-line graphs.

5.3 The structure of quasi-line graphs

5.3.1 The basics

Chudnovsky and Seymour showed that any quasi-line graph containing no homogeneous pair of cliques is either a circular interval graph or a composition of one special type of strip [CS05], which we define now.

Definition 5.10. *Let S be a linear interval graph with vertices v_1, \dots, v_n in linear order, and let $X = \{v_1, \dots, v_{|X|}\}$ and $Y = \{v_{n-|Y|+1}, \dots, v_n\}$ be cliques in S . Then (S, X, Y) is a linear interval strip.*

Chudnovsky and Seymour [CS05] proved:

Theorem 5.11. *Every quasi-line graph containing no homogeneous pair of cliques is a circular interval graph or a composition of linear interval strips.*

To decompose quasi-line graphs we will exploit the structure of the 2-joins corresponding to linear interval graphs. A 2-join $((X_1, Y_1), (X_2, Y_2))$ separating G_1 and G_2 is an *interval 2-join* if (G_2, X_2, Y_2) is a linear interval strip. We say that an interval 2-join is *trivial* if $V(G_2) = X_2 = Y_2$. A nontrivial interval 2-join is *canonical* if $X_2 \cap Y_2$ is empty. Given a nontrivial interval 2-join $((X_1, Y_1), (X_2, Y_2))$ where $C = X_2 \cap Y_2$, observe that $((X_1 \cup C, Y_1 \cup C), (X_2 \setminus C, Y_2 \setminus C))$ is a canonical interval 2-join.

If every interval 2-join in a composition of linear interval strips is trivial then the graph is a line graph. This implies a useful decomposition result:

Lemma 5.12. *For any quasi-line graph G , one of the following is true:*

- G contains a homogeneous pair of cliques
- G is a circular interval graph
- G is a line graph
- G admits a canonical interval 2-join.

Since we already know a lot about colouring circular interval graphs and line graphs, to prove Theorem III.1 and Theorem III.2 our only concern is how to deal with homogeneous pairs of cliques and canonical interval 2-joins. We now introduce *thickenings*, which generalize augmentations. They help us to understand homogeneous pairs of cliques in quasi-line and claw-free graphs.

5.3.2 Thickenings

Here we present *thickenings*, which broaden the definition of augmentations in two ways. First we allow vertex multiplication, so we can focus on a description of graphs containing no homogeneous clique. Both quasi-line and claw-free graphs are closed under vertex multiplication. Second, we modify the restrictions on the matching M , allowing it to be empty or to contain any edge whose removal does not introduce a claw. This relaxes the restriction we use for augmentations, i.e. that no edge in M is in a triangle.

Definition 5.13. *An edge e in a claw-free graph G is claw-neutral if its removal does not introduce a claw. A matching M in G is claw-neutral if every edge in M is claw-neutral.*

Observation 5.14. *If M is a claw-neutral matching in a claw-free graph G , then $G - M$ is claw-free.*

Let M be a claw-neutral matching in a claw-free graph G . We say that G' is a *thickening of G under M* (or sometimes just a thickening of G) if we can construct it from G in the following way. First we multiply each vertex, substituting a nonempty clique $I(v)$ for every vertex v of G . Then for every $uv \in M$, we remove from G' a nonempty proper subset of the edges between $I(u)$ and $I(v)$. If $M = \emptyset$ we say that G' is a *proper thickening* of G . In this case G' arises from G by a sequence of vertex replications. For a set $S \subseteq V(G)$ we use $I(S)$ to denote $\cup_{v \in S} I(v)$.

We have given theorems describing Berge quasi-line graphs and general quasi-line graphs containing no homogeneous pair of cliques. But these results don't really tell us anything about homogeneous pairs of cliques in a quasi-line graph. Thickenings, on the other hand, tell us how homogeneous pairs of cliques arise: some trivial pairs will arise from vertex replication, and some more interesting pairs will arise from edges in M . If we can describe M , we reach a more useful description of graphs that can be expressed as a thickening of G . This becomes very important when we consider claw-free graphs in later chapters.

To illustrate this point, take the example of peculiar graphs. Looking at Figure 5.1, we can see that there are three special homogeneous pairs of cliques: (A_1, B_2) , (A_2, B_3) , and (A_3, B_1) . Furthermore, any peculiar graph is a thickening of a circular interval graph G on nine vertices under some matching M . The vertices of G are $\{a_i, b_i, k_i \mid i \in \{1, 2, 3\}\}$, corresponding to the cliques $\{A_i, B_i, K_i \mid i \in \{1, 2, 3\}\}$. If two cliques are complete to each other, their corresponding vertices are adjacent. If they are anticomplete, their corresponding vertices are nonadjacent. If they are neither complete nor anticomplete then their corresponding vertices are adjacent, and the edge between them is in M . This description is more specific than the simple statement that a peculiar graph contains a homogeneous pair of cliques or is some circular interval graph on nine vertices. Obviously we do not need this specificity to colour peculiar graphs – they are perfect. But for other classes of claw-free graphs we need a good understanding of the structure of M . Thus the same idea will be very useful when we colour other classes of claw-free graphs in Chapter 10.

Thickenings were inspired by augmentations, and they arise naturally in the study of *trigraphs*, first introduced in Chudnovsky's Ph.D. thesis [Chu03]. Chudnovsky and

Seymour actually characterized the structure of *claw-free trigraphs* [CS08b], in which a claw-neutral matching M of “semi-adjacent” vertices is specified.

5.3.3 Fuzzy linear and circular interval graphs

Thickenings allow us to describe two important types of quasi-line graphs. If a graph is a thickening of a linear interval graph, we say it is a *fuzzy linear interval graph*. If it is a thickening of a circular interval graph, we say it is a *fuzzy circular interval graph*.

Now suppose we have a linear interval strip (S, X, Y) , and S' is a thickening of S . Then defining $X' = \cup\{I(v) \mid v \in X\}$ and $Y' = \cup\{I(v) \mid v \in Y\}$, (S', X', Y') is a strip, and we say it is a *fuzzy linear interval strip*. This gives us a full description of quasi-line graphs due to Chudnovsky and Seymour [CS05]:

Theorem 5.15. *Every quasi-line graph is a fuzzy circular interval graph or a composition of fuzzy linear interval strips.*

To colour a quasi-line graph we must restrict the structure of its homogeneous cliques. In the next chapter we introduce two appropriate restrictions. The first is sufficient for quasi-line graphs. The second one is stronger, and we make use of it when looking at other classes of claw-free graphs.

Chapter 6

Skeletal Graphs

In the previous chapter we introduced thickenings, which result in homogeneous pairs of cliques. In this chapter we consider the structure of these homogeneous pairs. We show that certain types of homogeneous pairs of cliques cannot appear in a minimum counterexample to our theorems bounding χ . The types of homogeneous pairs of cliques that we cannot easily eliminate from a minimum counterexample are *skeletal* homogeneous pairs of cliques, and a graph containing no nonskeletal homogeneous pair of cliques is a *skeletal* graph. As we will show, to prove our bounds on χ we need only prove them for skeletal graphs.

Skeletal quasi-line graphs and skeletal claw-free graphs can be characterized with simpler structure theorems than their unrestricted counterparts. This is the first reason we prefer to deal with them. For quasi-line graphs under a weaker restriction on homogeneous pairs of cliques, a similar result was proved by Chudnovsky and Seymour [CS05] and then applied by Chudnovsky and Fradkin [CO07] to bound the chromatic number of quasi-line graphs. Our stronger restriction simplifies the structure of claw-free graphs, not just quasi-line graphs. The second advantage of insisting that all homogeneous pairs of cliques are skeletal is that skeletal homogeneous pairs of cliques have a very simple structure. This makes it easy to analyze how invariants drop when we remove a stable set from a minimum counterexample when proving the Main Conjecture for quasi-line graphs and claw-free graphs.

6.1 Restricting homogeneous pairs of cliques

Given a homogeneous pair of cliques (A, B) in a graph G we want to remove edges between A and B in G to reach a graph G' such that:

- G' is easier to describe and colour than G
- given a k -colouring of G' we can easily find a k -colouring of G .

We call this action *reducing on* (A, B) . The fact that we can do this rests on our understanding of cobipartite graphs. In a proper colouring of G , the number of colours intersecting both A and B can be any value between 0 and $|A| + |B| - \omega(G[A \cup B])$. Now suppose we construct G' from G by removing edges between A and B without changing $\omega(G[A \cup B])$. Then since (A, B) is a homogeneous pair of cliques in both G and G' , it follows that $\chi(G) = \chi(G')$. We will prove this and give a more detailed explanation in the next section. In this section we will introduce two useful ways to restrict homogeneous pairs of cliques.

Note that one of the most natural ways to eliminate a homogeneous pair of cliques (A, B) is to contract A and B down to single vertices. This is essentially the inverse of the thickening operation, and through repeated application allows us to arrive at a graph containing no homogeneous pair of cliques. However, we lose too much information about colourings when we do this. Specifically, suppose we want to k -colour G and we contract (A, B) down to two vertices to reach G' . A k -colouring of G' doesn't tell us how we might k -colour G , or even whether such a colouring exists. Thus we must use more careful reductions on homogeneous pairs of cliques.

6.1.1 Linear and nonlinear homogeneous pairs of cliques

As we have seen, circular interval graphs and linear interval graphs are fundamental to the structure of quasi-line graphs. Thus one natural restriction on (A, B) is to insist that $G[A \cup B]$ is a linear interval graph.

Definition 6.1. *A homogeneous pair of cliques (A, B) is a linear homogeneous pair of cliques if it induces a linear interval graph. Otherwise it is called a nonlinear homogeneous pair of cliques.*

We can characterize linear homogeneous pairs of cliques in another way:

Lemma 6.2. *A homogeneous pair of cliques (A, B) in a graph G is nonlinear precisely if $G[A \cup B]$ contains an induced C_4 .*

Proof. It is easy to see that if $G[A \cup B]$ is a linear interval graph then it contains no induced C_4 . Now suppose $G[A \cup B]$ contains no induced C_4 . Order the vertices $\{a_1, \dots, a_{|A|}\}$ of A and then $\{b_1, \dots, b_{|B|}\}$ from left to right so that

- For $1 \leq i < j \leq |A|$, a_j has at least as many neighbours in B as a_i does.
- For $1 \leq i < j \leq |B|$, b_i has at least as many neighbours in A as b_j does.

Suppose for some $1 \leq i < j \leq |A|$, $N(a_j) \cap B$ does not contain $N(a_i) \cap B$. Then there must be vertices b_k and b_l such that a_i sees b_k but not b_l , and a_j sees b_l but not b_k , contradicting the fact that $G[A \cup B]$ contains no induced C_4 . Therefore for $1 \leq i < j \leq |A|$, $N(a_j) \cap B$ contains $N(a_i) \cap B$. By symmetry, for $1 \leq i < j \leq |B|$, $N(b_i) \cap A$ contains $N(b_j) \cap A$. It follows that $G[A \cup B]$ is a linear interval graph with vertex ordering $\{a_1, \dots, a_{|A|}, b_1, \dots, b_{|B|}\}$. \square

Linear homogeneous pairs of cliques are extremely useful in characterizing the structure of quasi-line graphs. Chudnovsky and Seymour¹ [CS05] proved the following.

Lemma 6.3. *Any fuzzy linear (resp. fuzzy circular) interval graph containing no nonlinear homogeneous pair of cliques is a linear (resp. circular) interval graph.*

Theorem 6.4. *Let G be a quasi-line graph containing no nonlinear homogeneous pair of cliques. Then G is a circular interval graph or a composition of linear interval strips.*

Chudnovsky and Fradkin later proved that nonlinear homogeneous pairs of cliques can be reduced nicely with respect to colourings [CO07].

Lemma 6.5. *Let G be a quasi-line graph containing a nonlinear homogeneous pair of cliques. Then there is a proper subgraph G' of G such that G' is quasi-line and $\chi(G) = \chi(G')$.*

¹Chudnovsky and Seymour originally called linear homogeneous pair of cliques *trivial* homogeneous pairs of cliques.

The proof of this lemma rests on the simple fact that if (A, B) contains a C_4 then we can remove an edge between A and B without changing the clique number of $G[A \cup B]$. We now introduce a stronger restriction on homogeneous pairs of cliques, which carries this idea through to its logical conclusion.

6.1.2 Skeletal homogeneous pairs of cliques

We can reduce on a nonlinear homogeneous pair of cliques (A, B) by removing edges between A and B in an induced C_4 in $G[A \cup B]$ until no such edge remains. This operation makes $G[A \cup B]$ a linear interval graph and does not change the clique number of $G[A \cup B]$. If we focus on this clique number instead of induced 4-holes, then we can remove even more edges and further simplify the structure of $G[A \cup B]$.

Definition 6.6. *Let (A, B) be a homogeneous pair of cliques in a graph G . We say that (A, B) is skeletal if we cannot remove an edge between A and B without reducing $\omega(G[A \cup B])$.*

If (A, B) is skeletal then the edges between A and B are contained in a single clique $\Omega(A, B)$, which we consider to be empty if there are no edges between A and B (see Figure 6.1). Thus $A \cup B$ is partitioned into the four sets $A \cap \Omega(A, B)$, $B \cap \Omega(A, B)$, $A \setminus \Omega(A, B)$, $B \setminus \Omega(A, B)$, each of which is a homogeneous clique, a singleton, or empty. This is the simple structure we want, and we will exploit it repeatedly when colouring claw-free graphs in Chapter 10. We want to consider graphs in which every homogeneous pair of cliques is skeletal.

Definition 6.7. *A graph is skeletal if it contains no nonskeletal homogeneous pair of cliques.*

Observe that any skeletal homogeneous pair of cliques is linear. Thus no skeletal graph contains a nonlinear homogeneous pair of cliques. This immediately gives us a characterization of skeletal quasi-line graphs as a corollary to Theorem 6.4.

Theorem 6.8. *Every skeletal quasi-line graph is a circular interval graph or a composition of linear interval strips.*

We use the corresponding decomposition theorem to prove our bounds on the chromatic number in the next chapter.

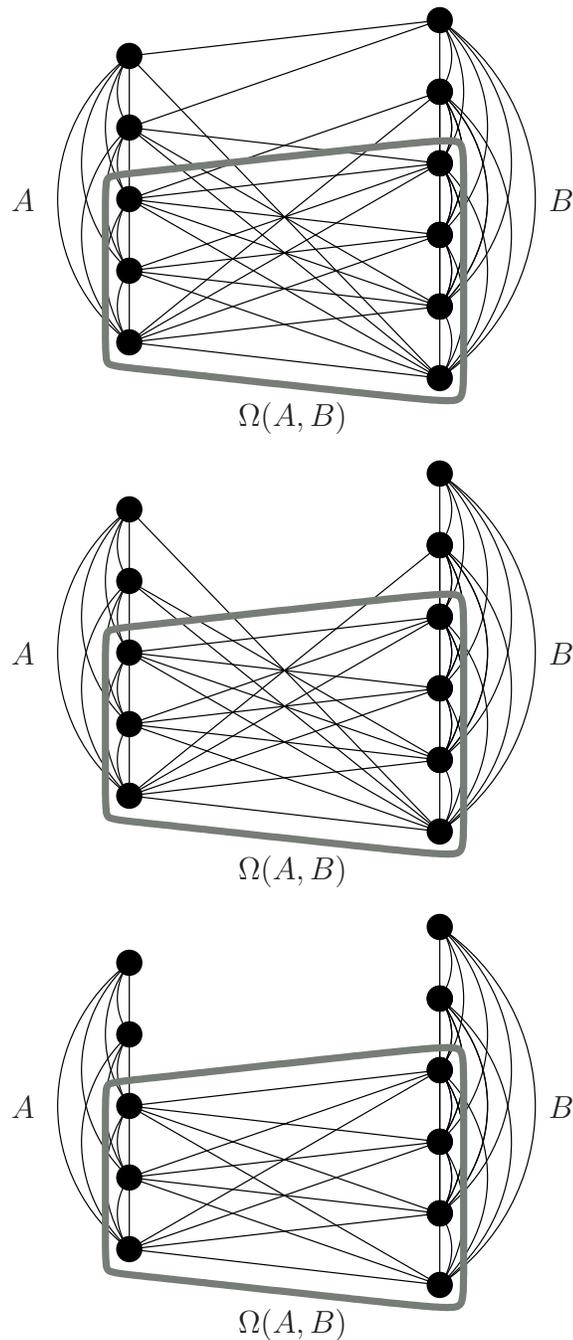


Figure 6.1: Three homogeneous pairs of cliques: one nonlinear (top), one nonskeletal linear (middle), and one skeletal (bottom). We reduce a nonskeletal homogeneous pair of cliques (A, B) by removing edges without changing the size of a maximum clique in $G[A \cup B]$.

Theorem 6.9. *Any quasi-line graph that is not a line graph or a circular interval graph contains a nonskeletal homogeneous pair of cliques or admits a canonical interval 2-join.*

Insisting that the graph is skeletal does not refine the structure theorem. However, this is an anomalous case. For many classes of claw-free graphs, skeletal homogeneous pairs of cliques allow us to simplify structural characterizations in a way that linear homogeneous pairs of cliques do not. We give a simple example now, using a class that we have already seen.

Skeletal Berge quasi-line graphs

To illustrate the usefulness of skeletal graphs in simplifying structure theorems, we rephrase the structure theorem for Berge quasi-line graphs one last time.

Theorem 6.10. *Let G be a skeletal Berge claw-free graph containing no clique cutset. Then G is a circular interval graph or the line graph of a bipartite multigraph.*

Proof. First suppose that G is a skeletal peculiar graph. We claim that G is a proper thickening of the circular interval graph on at most fifteen vertices shown in Figure 6.2. We split A_i and B_i into two cliques each for $i \in \{1, 2, 3\}$. Let A_i^1 (resp. B_{i+1}^1) be the set of vertices in A_i (resp. B_{i+1}) with no neighbour in B_{i+1} (resp. A_i), with indices taken modulo 3. Let A_i^2 and B_i^2 be $A_i \setminus A_i^1$ and $B_i \setminus B_i^1$ respectively. So in the skeletal homogeneous pair of cliques (A_i, B_{i+1}) , the sets A_i^2 and B_{i+1}^2 are $\Omega(A_i, B_{i+1}) \cap A_i$ and $\Omega(A_i, B_{i+1}) \cap B_{i+1}$ respectively. When $|A_i \cup B_{i+1}| = 2$, (A_i, B_{i+1}) is not technically a homogeneous pair of cliques, but in this case our abuse of terminology is harmless.

Now suppose G is a skeletal augmentation of a bipartite line graph H . Take an edge xy along which H was augmented by (X, Y) with $|X \cup Y| > 2$. Then (X, Y) is a skeletal homogeneous pair of cliques. Therefore if $X \cup Y$ is not a clique we can assume there is a vertex $v \in X$ with no neighbour in Y . But $N_H(x) - y$ is a clique, so v is simplicial, contradicting the assumption that G contains no clique cutset. Therefore G is a proper thickening of a line graph of a bipartite graph, so G is a line graph of a bipartite multigraph. \square

Note that in this case we cannot replace “skeletal” with “containing no nonlinear homogeneous pair of cliques.”

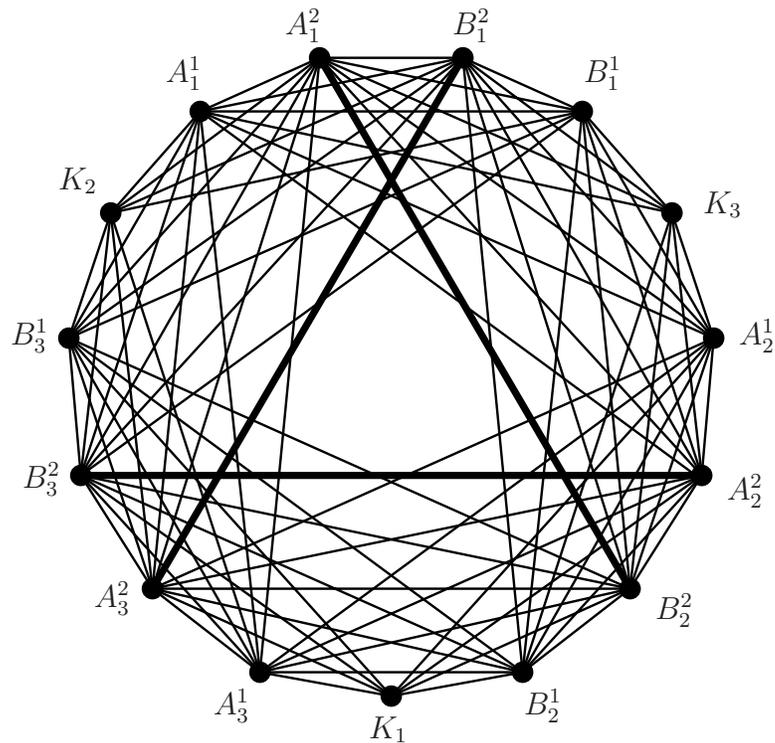


Figure 6.2: A skeletal peculiar graph. Each circle represents a clique; two cliques are complete to each other if they have an edge between them in the figure, otherwise they are anticomplete.

6.2 Reducing nonskeletal graphs

Now that we have explained how to restrict homogeneous pairs of cliques, we will prove that we can reduce nonskeletal graphs efficiently. The main goal of this section is to prove the following:

Theorem 6.11. *Let G be a nonskeletal graph. Then there is a skeletal subgraph G' of G such that:*

1. *If G is quasi-line (resp. claw-free) then G' is also quasi-line (resp. claw-free).*
2. *If G is Berge then G' is also Berge.*
3. *$\chi(G') = \chi(G)$ and $\chi_f(G') = \chi_f(G)$.*

Furthermore we can find G' in $O(n^2m^2)$ time, and given a k -colouring of G' we can construct a k -colouring of G in $O(n^2m^2)$ time.

Crucial to the proof of this result are the following two lemmas, which we apply repeatedly:

Lemma 6.12. *Given a graph G and a nonskeletal homogeneous pair of cliques (A, B) , in $O(n^{5/2})$ time we can find a proper subgraph G' such that:*

1. *If G is quasi-line (resp. claw-free) then G' is also quasi-line (resp. claw-free).*
2. *If G is Berge then G' is also Berge.*
3. *$\chi(G') = \chi(G)$ and $\chi_f(G') = \chi_f(G)$.*

Furthermore given a k -colouring of G' we can construct a k -colouring of G in $O(n^{5/2})$ time.

Lemma 6.13. *For any graph G , we can find a nonskeletal homogeneous pair of cliques, or determine that none exists, in $O(n^2m)$ time.*

Theorem 6.11 follows immediately from these two lemmas, because every time we reduce on a homogeneous pair of cliques we remove an edge, thus we do at most $O(m)$ reductions. We simply reduce on nonskeletal homogeneous pairs of cliques one at a time until none remains.

Reducing on a pair (A, B) (i.e. applying Lemma 6.12) involves finding a maximum clique X in $\omega(G[A \cup B])$ and removing all other edges between A and B . If $|X| = \max\{|A|, |B|\}$ then we simply remove all edges between A and B . Thus (A, B) will become a skeletal homogeneous pair of cliques and X will become $\Omega(A, B)$. See Figure 6.1.

These results rely heavily on the fact that any homogeneous pair of cliques (A, B) induces a cobipartite (and therefore perfect) subgraph $G[A \cup B]$. Results of Hopcroft and Karp provide an $O(n^{5/2})$ -time algorithm for optimally colouring a cobipartite graph and finding a maximum clique in a cobipartite graph [HK73]. Recall that if $\alpha(G) = 2$ then an optimal colouring of G corresponds to a maximum matching in \overline{G} .

6.2.1 Reducing on a nonskeletal homogeneous pair of cliques

We now prove Lemma 6.12. This will tell us exactly how we reduce on a nonskeletal homogeneous pair of cliques (A, B) and how we can manipulate colourings on (A, B) .

Proof of Lemma 6.12. Assume $|A| \geq |B|$. We can find a maximum clique X of $G[A \cup B]$ in $O(n^{5/2})$ time, choosing X to be A if A is a maximum clique. To construct G' from G , we remove precisely the edges between A and B that are not in X . Clearly $\omega(G'[A \cup B]) = \omega(G[A \cup B]) = |X|$. Since (A, B) is not skeletal, G' is a proper subgraph of G . We can find G' in $O(n^{5/2})$ time.

We must prove that G' is claw-free. Suppose there is a vertex v seeing three mutually nonadjacent vertices a, b, c in G' . Then without loss of generality, $a \in A$, $b \in B$, and $c \notin A \cup B$ since G is claw-free. Since c sees neither a nor b in G' , c sees nothing in $A \cup B$ in G . It follows that $v \notin A \cup B$, so v sees all of $A \cup B$ in G . Therefore since A and B are not complete to each other in G , G contains a claw centred at v , a contradiction. So G' is claw-free.

Now suppose G is quasi-line; we must show that G' is quasi-line. Suppose a vertex v is not bisimplicial in G' and let (S, T) be a partitioning of $N_G(v)$ into two cliques. If v has a neighbour $w \in S \setminus (A \cup B)$ that sees A but not B , then $B \subseteq T$ and thus $S \cup A$ and $T \setminus A$ are two cliques covering $N_{G'}(v)$ in G' . By symmetry we can assume that if no such w exists then all of $N_{G'}(v) \setminus (A \cup B)$ sees $A \cup B$, therefore $(S \cup A) \setminus B$ and $(T \cup B) \setminus A$ are two cliques covering $N_{G'}(v)$ in G' . Therefore G' is quasi-line if G is quasi-line. This proves (1).

Suppose G is Berge but G' contains an odd hole or antihole H . H does not contain a homogeneous set, so it must intersect each of A and B exactly once, in vertices a_1 and b_1 . But there are nonadjacent a_i and b_j in G . Replacing a_1 and b_1 with a_i and b_i in H gives us an odd hole or antihole in G , contradicting the assumption that G is Berge. This proves (2).

Let $c_{G'}$ be a proper colouring of G' using $k \geq \chi(G')$ colours. Since (A, B) is a homogeneous pair, to construct a k -colouring of G it is enough to find a colouring of $G[A \cup B]$ that uses the same set of colours as $c_{G'}$ on A and on B . We can do this in $O(n^{5/2})$ time because the number of colours which appear on both A and B in the colouring of G' is at most the maximum size of a matching in $\overline{G'}$, which is the same as the size of a maximum matching in \overline{G} , i.e. $|(A \cup B) - X|$.

Since $G[A \cup B]$ is perfect, this extends to fractional colourings. Specifically, for any $l \geq \omega(G[A \cup B])$ there is a fractional l -colouring of $G[A \cup B]$. Suppose we have a fractional k -colouring of G' . This colouring uses weight $l \geq \omega(G[A \cup B])$ on $A \cup B$, so since (A, B) is a homogeneous pair of cliques we can combine the colouring of $G' - (A \cup B) = G - (A \cup B)$ with a fractional l -colouring of $G[A \cup B]$ to find a fractional k -colouring of G . This proves (3). \square

6.2.2 Finding homogeneous pairs of cliques

In Chapter 3 we mentioned Everett, Klein, and Reed's $O(mn^3)$ algorithm for finding homogeneous pairs, but we have not yet discussed finding homogeneous pairs of cliques.

We begin with an algorithm for finding a nonlinear homogeneous pair of cliques. Then we will give a more efficient algorithm for finding a nonskeletal homogeneous pair of cliques in a graph containing no nonlinear homogeneous pair of cliques.

Observe that if an edge a_1a_2 is contained in A for some nonlinear homogeneous pair of cliques (A, B) , then any edge b_1b_2 such that $\{a_1, b_1, a_2, b_2\}$ induces a C_4 must be contained in B .

Lemma 6.14. *For any graph G we can find a nonlinear homogeneous pair of cliques in G , or determine that none exists, in $O(n^2m)$ time.*

Proof. Since a nonlinear homogeneous pair of cliques contains an induced C_4 , we proceed by checking, for every edge contained in an induced C_4 , whether or not there

is a homogeneous pair of cliques such that this edge is in one of the cliques. Observe that if this is the case, then the other vertices of the C_4 are in the other clique and hence the homogeneous pair of cliques is nonlinear.

For each edge a_1a_2 , we first check that it appears in an induced C_4 ; this can be determined in $O(m)$ time. If it is in a C_4 , we then iteratively grow cliques A_i and B_i such that if there is a homogeneous pair of cliques (A, B) with $a_1, a_2 \in A$, then $A_i \subseteq A$ and $B_i \subseteq B$. As mentioned above, if b_1b_2 is an edge and $G[\{a_1, a_2, b_1, b_2\}]$ is a C_4 then b_1 and b_2 must be in B and so if such an (A, B) exists it is a nonlinear homogeneous pair of cliques. Let $A_0 = \{a_1, a_2\}$ and let $B_0 = \emptyset$. For $t = 1, 2, \dots, n - 2$ we do the following.

1. Search for a vertex v not in $A_{t-1} \cup B_{t-1}$ that sees some but not all of A_{t-1} (resp. B_{t-1}) – it must be in B (resp. A), so let $B_t = B_{t-1} \cup \{v\}$ (resp. $A_t = A_{t-1} \cup \{v\}$) and increment t . If there is no such v then (A_{t-1}, B_{t-1}) form a nonlinear homogeneous pair of cliques; return this fact and terminate.
2. If A_t and B_t are not both cliques or $t = n - 2$, terminate. A and B do not exist.

When building (A, B) we only add a vertex to the homogeneous pair if it cannot be outside the pair, hence we never face the possibility of putting an unnecessary vertex in $A \cup B$. It follows that if our method fails there is no homogeneous pair of cliques containing A_0 . The method is clearly polytime: we can construct (A_t, B_t) from (A_{t-1}, B_{t-1}) in $O(m)$ time, and there are $O(m)$ possible edges to check, so the total running time is at most $O(nm^2)$. However, we can actually do this in $O(n^2m)$ time by maintaining the sets of vertices outside (A, B) that see all of A , none of A , all of B , and none of B – all others must be put in the homogeneous pair. To maintain these lists, when putting a vertex v into A , for example, we check to see if this forces any outside vertex to be put into B . We can check this in $O(n)$ time whenever we put a vertex in the homogeneous pair, so we can find a minimal nonlinear homogeneous pair of cliques, or determine that there is none, in $O(n^2m)$ time. \square

Now we need to find linear nonskeletal homogeneous pairs of cliques. First we prove a structural result that renders the task almost trivial.

Lemma 6.15. *Suppose a graph G contains a nonskeletal linear homogeneous pair of cliques. Then G contains three nonempty disjoint cliques A_1, A_2, B_1 such that*

- $|A_1| \geq |B_1|$.
- Each of A_1 , A_2 , and B_1 is either a singleton or a homogeneous clique.
- $A_1 \cup A_2$ is a clique, $A_2 \cup B_1$ is a clique, and there are no edges between A_1 and B_1 .
- $(A_1 \cup A_2, B_1)$ is a nonskeletal linear homogeneous pair of cliques.

Proof. Suppose the vertices of $G[A \cup B]$ are $a_1, \dots, a_{|A|}, b_1, \dots, b_{|B|}$ in linear order.

By swapping the names of A and B , we can make an important assumption without loss of generality: Either A is a maximum clique in $G[A \cup B]$, or there is a maximum clique X of $G[A \cup B]$ and some vertex in B that sees some but not all of $X \setminus B$. If we cannot assume this, then $\omega(G[A \cup B]) > \max\{|A|, |B|\}$ and there is a unique maximum clique X in $(G[A \cup B])$. Furthermore since $(G[A \cup B])$ is a linear interval graph, no vertex in $A \setminus X$ (resp. $B \setminus X$) has a neighbour in B (resp. A), contradicting the assumption that (A, B) is nonskeletal.

To construct A_1 , A_2 , and B_1 we first select two vertices a_p and a_q in A . Let p be minimum such that a_p is in a maximum clique X of $G[A \cup B]$; note that $p = 1$ if $\omega(G[A \cup B]) = |A|$. We claim that there is some minimum $q > p$ such that $\bar{N}(a_p) \subset \bar{N}(a_q)$, i.e. a_p and a_q are not twins. If q does not exist then by our above assumption either (i) $X = A$ and there are no edges between A and B , a contradiction since (A, B) is nonskeletal, or (ii) $|X| > |A|$ and no vertex in B sees some but not all of $X \setminus B$, a contradiction since in this case X must be the unique maximum clique of $G[A \cup B]$.

Let A_1 be a_p along with its twins, and let B_1 be the set of vertices that see that see a_p but not a_q . Clearly $B_1 \subseteq B$, and observe that $|A_1| \geq |B_1|$, otherwise a_p would not be in a maximum clique in $G[A \cup B]$, whereas a_q would. So let A_2 be a_q along with its twins. An example is shown in Figure 6.3.

To see that $(A_1 \cup A_2, B_1)$ is a homogeneous pair of cliques, it is enough to show that $(\{a_p, a_q\}, B_1)$ is a homogeneous pair of cliques. By the structure of linear interval graphs, every vertex in $A \setminus (A_1 \cup A_2)$ sees either all of B_1 or none of B_1 , so B_1 is a singleton or a homogeneous clique. Therefore $(\{a_p, a_q\}, B_1)$ is a homogeneous pair of cliques, following from the fact that (A, B) is a homogeneous pair of cliques. Furthermore since B_1 is complete to A_2 and anticomplete to A_1 , and $|A_1| \geq |B_1|$, it

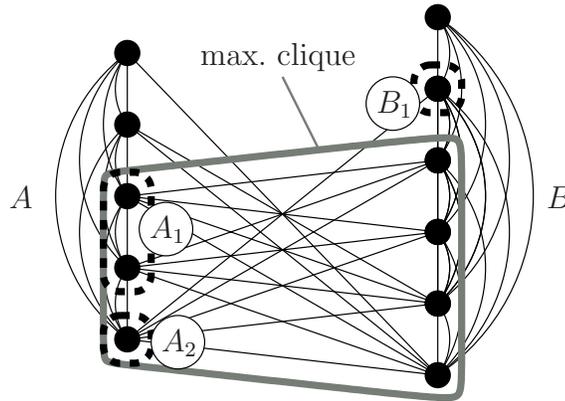


Figure 6.3: If a linear homogeneous pair of cliques is not skeletal, we can find within it a homogeneous pair of cliques with a very specific structure.

is easy to see that $(A_1 \cup A_2, B_1)$ is a nonskeletal linear homogeneous pair of cliques (in particular, $A_1 \cup A_2$ is a maximum clique in $G[A_1 \cup A_2 \cup B_1]$). \square

Thus when searching for a linear nonskeletal homogeneous pair of cliques, we can focus on this specific structure.

Lemma 6.16. *Let G be a graph containing no nonlinear homogeneous pair of cliques. Then in $O(nm)$ time we can find some nonskeletal linear homogeneous pair of cliques (A, B) in G , or determine that G is skeletal.*

Observe that Lemma 6.13 follows immediately from this lemma and Lemma 6.14.

Proof. We find a nonskeletal homogeneous pair of cliques (A, B) by finding the cliques A_1, A_2 , and B_1 guaranteed by the previous lemma, as follows. First we partition the vertices of G into maximal homogeneous cliques in $O(m)$ time as discussed in Section 4.2.3. After that we just need to find three vertices a_1, a_2 , and b_1 inducing a path such that a_1 has at least as many twins as b_1 , no vertex sees a_1 but not a_2 , and b_1 and its twins are the only vertices that see a_2 but not a_1 . We can easily do this in $O(nm)$ time by first guessing b_1 , then deleting b_1 and checking for the appropriate resulting twins in $O(m)$ time. \square

Finally, we remark that we can find a skeletal homogeneous pair of cliques in $O(m)$ time. First we search for twins in time $O(m)$ – twins immediately lead to a

homogeneous pair of cliques if the graph has at least four vertices. But the existence of a skeletal homogeneous pair (A, B) implies the existence of twins: Either $(A \cap \Omega(A, B), B \cap \Omega(A, B))$ is a homogeneous pair of cliques with all edges between them, or (A, B) is a homogeneous pair of cliques with no edges between them. Either case leads to twins. With the results of this section, this implies the following:

Theorem 6.17. *In $O(n^2m)$ we can find a homogeneous pair of cliques in a graph or determine that none exists.*

6.3 Decomposing skeletal quasi-line graphs

We conclude the chapter with an algorithm to decompose a skeletal composition of linear interval strips. Specifically, for such a graph G we will give an efficient method for determining strips (S_e, X_e, Y_e) and an underlying graph H such that G is a composition of the strips under H . We call this a *strip representation* of G . This strip representation is by no means unique, as one can easily see by considering a cycle.

6.3.1 Finding a canonical interval 2-join

To find a strip representation, we first need to find a canonical interval 2-join efficiently.

Lemma 6.18. *Let G be a connected quasi-line graph. In time $O(n^2m)$ we can find a canonical interval 2-join in G or determine that none exists.*

Proof. Suppose a canonical interval 2-join $((X_1, Y_1), (X_2, Y_2))$ exists such that X_2 and Y_2 are nonempty, and G_2 is connected. Then there are non-simplicial vertices x and y in G such that G_2 has a linear interval representation with x and y at the extreme left and right. We proceed by guessing x and y , then checking to see if they yield a desired join.

Since x is not simplicial both X_1 and $N(x) \setminus (X_1 \cup X_2)$ are nonempty. Thus $\bar{N}(x)$ has exactly two maximal cliques, namely $X_1 \cup X_2$ and $\bar{N}(x) \setminus X_1$. We can find one maximal clique greedily in linear time, and having generated one of them, C , we can find the other by generating a maximal clique in $\bar{N}(x)$ containing some

arbitrarily chosen element of $\bar{N}(x) \setminus C$. Thus, X_2 is the intersection of these two maximal cliques and there are two choices for X_1 . Similarly we can deduce Y_2 and two possible choices for Y_1 . For each of these four possible choices of (X_1, Y_1) we first check if we indeed have a 2-join. We check that it is an interval 2-join by adding to G_2 vertices x' and y' with neighbourhoods X_2 and Y_2 respectively along with a vertex z with neighbourhood $\{x', y'\}$, then checking to see if the result is a circular interval graph. All of this can be done in $O(m)$ time, so checking every possible x and y takes $O(n^2m)$ time. Furthermore, any simplicial vertex will result in a canonical interval 2-join in a trivial way, and we can find such vertices in $O(nm)$ time. \square

6.3.2 Decomposing a composition of linear interval strips

We are almost ready to find a strip representation of G . We will do this by finding strips one at a time and replacing X_e and Y_e with nonadjacent vertices. This ensures that we will never introduce a nonlinear homogeneous pair of cliques, so we can repeatedly apply Theorem 6.4. We continue until we have a line graph, from which we determine a strip representation of G . However, we must address one case in which this may introduce a claw. This occurs when S_e is a clique. We now show that if each S_e is a clique then G is a line graph.

One case of this is when $S_e = X_e \cup Y_e$ for every strip, in which case we already know G is a line graph. Suppose that every S_e is a strip. To show that G is a line graph, take H and instead of replacing each edge $e = \{x_e, y_e\}$ with a strip, add a vertex v_e . We deal with two cases: the case $V(S_e) = X_e \cup Y_e$ and the case $V(S_e) \setminus (X_e \cup Y_e) \neq \emptyset$. First suppose $V(S_e) = X_e \cup Y_e$. Put $|X_e \setminus Y_e|$ edges between v_e and x_e , put $|Y_e \setminus X_e|$ edges between v_e and y_e , and put $|X_e \cap Y_e|$ edges between x_e and y_e . If $V(S_e)$ contains a vertex outside $X_e \cup Y_e$ but is still a clique, then by the definition of a strip we can see that X_e and Y_e are disjoint (otherwise adding nonadjacent vertices with neighbourhoods X_e and Y_e would result in a claw). Thus as in the previous case we put $|X_e|$ edges between v_e and x , $|Y_e|$ edges between v_e and y , but now we put $|V(S_e) \setminus (X_e \cup Y_e)|$ edges between v_e and a new pendant vertex z_e . If we construct H' from H by doing this for each strip (S_e, X_e, Y_e) , then G will be the line graph of H' .

Thus we will disregard canonical interval 2-joins for which G_2 is a clique, allowing us to proceed safely and still arrive at a line graph.

Theorem 6.19. *Let G be a connected quasi-line graph containing no nonlinear homogeneous pair of cliques. Then in $O(n^2m)$ time we can either determine that G is a circular interval graph or find a strip representation of G .*

Proof. If G is a circular interval graph, we can determine this in $O(m)$ time. If G is a line graph, then in $O(m)$ time we can determine this and find a multigraph H such that $G = L(H)$, as discussed in Section 4.2.3. Note that this gives us a strip representation of G , setting each strip to be a singleton.

So assume that G is neither a line graph nor a circular interval graph. We begin by finding a strongly canonical interval 2-join $((X_1, Y_1), (X_2, Y_2))$ such that G_2 is connected and is not a clique; we are guaranteed that this exists because G is not a line graph, as we discussed above. We proceed by insisting that G_2 is a strip and recursing in the following way. We construct G_S from G by deleting G_2 and adding nonadjacent vertices x_2 and y_2 with neighbourhoods X_1 and Y_1 respectively. It is easy to see that G_S is an induced subgraph of G , so it is quasi-line.

We claim that G_S contains no nonlinear homogeneous pair of cliques, so by Theorem 6.4 it is a circular interval graph or a composition of linear interval strips. To see this, suppose G_S contains a nonlinear homogeneous pair of cliques (A, B) . If neither x_2 nor y_2 is in $A \cup B$, then each of A and B is either in X_1 or Y_1 , or completely outside $X_1 \cup Y_1$. In this case it is easy to confirm that (A, B) is a nontrivial homogeneous pair of cliques in G , a contradiction. So assume at least one of x_2 and y_2 , say x_2 , is in A . Thus $A \subseteq X_1$.

Clearly in this case B is not in X_1 . If $B \cap X_1 = \emptyset$, then it is straightforward to confirm that $(A \setminus \{x_2\}, B \setminus \{y_2\})$ is a nonlinear homogeneous pair of cliques since neither x_2 nor y_2 is in an induced C_4 (but note that y_2 is not necessarily in B). So assume that B is partially in X_1 . Since $X_1 \cap B$ is complete to $A \cap B$, $(A, B \setminus X_1)$ is a nonlinear homogeneous pair of cliques in G_S . Thus so is $(A \setminus \{x_2\}, B \setminus (X_1 \cup \{y_2\}))$. But as in the previous paragraph, this implies the existence of a nonlinear homogeneous pair of cliques in G , a contradiction. Therefore G_S contains no nonlinear homogeneous pair of cliques and is either a circular interval graph or a composition of linear interval strips.

Observe that if G_S is a circular interval graph, then it is also a composition of linear interval strips. Four strips, to be exact, two of which are singletons $(\{x_2\}, \{x_2\}, \{x_2\})$ and $(\{y_2\}, \{y_2\}, \{y_2\})$. Thus it follows easily that G is a composition of three linear

interval strips, and the underlying multigraph H consists of three parallel edges. We can easily find this strip representation in $O(m)$ time by first finding a circular interval representation of G_S .

If G_S is a composition of linear interval strips, then we find a strip representation of it recursively in $O(n^2m)$ time. If x_2 and y_2 are two singleton strips then we can proceed as before. Otherwise we can modify the strips to find a strip representation in which x_2 and y_2 are singleton strips. If x_2 is not in a singleton strip but is rather in an end-clique of another strip, say X_e , then we can simply remove x_2 from X_e and insist that x_2 is a singleton strip because it is simplicial and therefore sees nothing outside its own hub clique. If x_2 is in the middle of a strip S_e , i.e. not in $X_e \cup Y_e$, then since x_2 is simplicial we can break S_e into three smaller strips: one is x_2 , and the other two consist of the vertices to the left and to the right of x_2 , respectively, in a linear interval representation of (S_e, X_e, Y_e) . We can easily do the extra work for this step in $O(nm)$ time.

The only thing to make clear is how we find our 2-joins – we do not want to spend $O(n^2m)$ time finding one at each step. Our method for finding canonical interval 2-joins actually finds all possible canonical interval 2-joins, i.e. $O(n^2)$ of them. We must maintain our set of possible subgraphs G_2 – in a strip representation they will partition the graph, so we never want to take a 2-join giving us a G_2 that intersects a vertex already selected to be in a strip. Each time we make a recursion step, we just take the first canonical interval 2-join such that G_2 is connected and X_2 and Y_2 are both nonempty, and such that G_2 contains no vertex already assigned to be in a strip. We can do this in $O(n^2m)$ time overall, and this method naturally limits us to making $O(n)$ iterations. The result follows. \square

We remark that this algorithm gives us a strip decomposition with a nice property: Every strip (S_e, X_e, Y_e) is either a singleton or has S_e connected and not a clique, and X_e and Y_e disjoint. We call such a strip representation a *canonical strip representation*.

Chapter 7

Colouring Quasi-line Graphs

Now we can apply the tools that we have gathered to bound the chromatic number of quasi-line graphs. Given any quasi-line graph G we can find a skeletal quasi-line subgraph G' with equal chromatic number, so any minimum counterexample to Theorem III.1 or Theorem III.2 must be skeletal. Thus by Theorem 6.8 we need only prove these results for circular interval graphs and compositions of linear interval strips.

We begin by proving the Main Conjecture. Our approach is to decompose a minimum counterexample on a canonical interval 2-join, colour what remains, then complete the colouring of the entire graph. This is a very straightforward approach. To prove Theorem III.2 we need to be a bit more clever. As we will show, both proofs lead to polynomial-time algorithms that find a colouring satisfying their respective bounds.

7.1 Proving the Main Conjecture

Theorem 4.1 implies that the Main Conjecture holds for circular interval graphs. Theorem 4.5 tells us that the Main Conjecture holds for line graphs. Therefore by Theorems 6.9 and 6.11 we only need to prove that a minimum counterexample to Theorem III.1 cannot admit a canonical interval 2-join. We do this now.

7.1.1 Dealing with canonical interval 2-joins

In Chapter 3 we showed that clique cutsets are easy to deal with when colouring graphs. This idea is relevant to the 2-joins arising from compositions of strips in two ways. First, if some X_i or Y_i is empty then our 2-join amounts to a clique cutset and G_2 is a linear interval graph. So if we have a $\gamma(G_1)$ -colouring of G_1 then we can easily extend it to a $\gamma(G)$ -colouring of G . Second, we can paste together colourings on 2-joins provided we have suitable colourings of G_1 and G_2 – this is what we did when proving that 2-joins preserve perfection in Chapter 3.

Before stating Lemma 7.1, which implies that no minimum counterexample to the Main Conjecture or Local Strengthening admits a canonical interval 2-join, we give some notation. Given G admitting a canonical interval 2-join $((X_1, Y_1), (X_2, Y_2))$ let H_2 denote $G[V_2 \cup X_1 \cup Y_1]$.

For $v \in H_2$ we define $\omega'(v)$ as the size of the largest clique in H_2 containing v and not intersecting both $X_1 \setminus Y_1$ and $Y_1 \setminus X_1$, and we define $\gamma_l^i(H_2)$ as $\max_{v \in H_2} \lceil d(v) + 1 + \omega'(v) \rceil$ (here the superscript i denotes *interval*). Observe that $\gamma_l^i(H_2) \leq \gamma_l(G)$. If $v \in X_1 \cup Y_1$, then $\omega'(v)$ is $|X_1| + |X_2|$, $|Y_1| + |Y_2|$, or $|X_1 \cap Y_1| + \omega(G[X_2 \cup Y_2])$.

Lemma 7.1. *Let G be a graph on n vertices and suppose G admits a canonical interval 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper l -colouring of G_1 for any $l \geq \gamma_l^i(H_2)$, we can find a proper l -colouring of G in $O(nm)$ time.*

Since $\gamma_l^i(H_2) \leq \gamma_l(G) \leq \gamma(G)$ this lemma implies the Main Conjecture for quasi-line graphs. Furthermore it implies that a proof of the Local Strengthening for line graphs would imply the Local Strengthening for quasi-line graphs.

Proof. We proceed by induction on l , observing that the case $l = 1$ is trivial. We begin by modifying the colouring so that the number k of colours used in both X_1 and Y_1 in the l -colouring of G_1 is maximal. That is, if a vertex $v \in X_1$ gets a colour that is not seen in Y_1 , then every colour appearing in Y_1 appears in $N(v)$. This can be done in $O(n^2)$ time. If l exceeds $\gamma_l^i(H_2)$ we can just remove a colour class in G_1 and apply induction on what remains. Thus we can assume that $l = \gamma_l^i(H_2)$ and so if we apply induction we must remove a stable set whose removal lowers both l and $\gamma_l^i(H_2)$.

We use case analysis; when considering a case we may assume no previous case applies. In some cases we extend the colouring of G_1 to an l -colouring of G in one

step. In other cases we remove a colour class in G_1 together with vertices in G_2 such that everything we remove is a stable set, and when we remove it we reduce $\gamma_l^i(v)$ for every $v \in H_2$; after doing this we apply induction on l . Notice that if $X_1 \cap Y_1 \neq \emptyset$ and there are edges between X_2 and Y_2 we may have a large clique in H_2 which contains some but not all of X_1 and some but not all of Y_1 ; this is not necessarily obvious but we deal with it in every applicable case.

Case 1. $Y_1 \subseteq X_1$.

H_2 is a circular interval graph and X_1 is a clique cutset. We can $\gamma_l(H_2)$ -colour H_2 in $O(n^{3/2})$ time using Theorem 4.2. By permuting the colour classes we can ensure that this colouring agrees with the colouring of G_1 . In this case $\gamma_l(H_2) \leq \gamma_l^i(H_2) \leq l$ so we are done. By symmetry, this covers the case in which $X_1 \subseteq Y_1$.

Case 2. $k = 0$ and $l > |X_1| + |Y_1|$.

Here X_1 and Y_1 are disjoint. Take a stable set S greedily from left to right in G_2 . By this we mean that we start with $S = \{v_1\}$, the leftmost vertex of X_2 , and we move along the vertices of G_2 in linear order, adding a vertex to S whenever doing so will leave S a stable set. So S hits X_2 . If it hits Y_2 , remove S along with a colour class in G_1 not intersecting $X_1 \cup Y_1$; these vertices together make a stable set. If $v \in G_2$ it is easy to see that $\gamma_l^i(v)$ will drop: every remaining vertex in G_2 either loses two neighbours or is in Y_2 , in which case S intersects every maximal clique containing v . If $v \in X_1 \cup Y_1$ then since X_1 and Y_1 are disjoint, $\omega'(v)$ is either $|X_1| + |X_2|$ or $|Y_1| + |Y_2|$; in either case $\omega'(v)$, and therefore $\gamma_l^i(v)$, drops when S and the colour class are removed. Therefore $\gamma_l^i(H_2)$ drops, and we can proceed by induction.

If S does not hit Y_2 we remove S along with a colour class from G_1 that hits Y_1 (and therefore not X_1). Since $S \cap Y_2 = \emptyset$ the vertices together make a stable set. Using the same argument as before we can see that removing these vertices drops both l and $\gamma_l^i(H_2)$, so we can proceed by induction.

Case 3. $k = 0$ and $l = |X_1| + |Y_1|$.

Again, X_1 and Y_1 are disjoint. By maximality of k , every vertex in $X_1 \cup Y_1$ has at least $l - 1$ neighbours in G_1 . Since $l = |X_1| + |Y_1|$ we know that $\omega'(X_1) \leq$

$|X_1| + |Y_1| - |X_2|$ and $\omega'(Y_1) \leq |X_1| + |Y_1| - |Y_2|$. Thus $|Y_1| \geq 2|X_2|$ and similarly $|X_1| \geq 2|Y_2|$. Assume without loss of generality that $|Y_2| \leq |X_2|$.

We first attempt to l -colour $H_2 - Y_1$, which we denote by H_3 , such that every colour in Y_2 appears in X_1 – this is clearly sufficient to prove the lemma since we can permute the colour classes and paste this colouring onto the colouring of G_1 to get a proper l -colouring of G . If $\omega(H_3) \leq l - |Y_2|$ then this is easy: we can $\omega(H_3)$ -colour the vertices of H_3 , then use $|Y_2|$ new colours to recolour Y_2 and $|Y_2|$ vertices of X_1 . This is possible since Y_2 and X_1 have no edges between them.

Define b as $l - \omega(H_3)$; we can assume that $b < |Y_2|$. We want an $\omega(H_3)$ -colouring of H_3 such that at most b colours appear in Y_2 but not X_1 . There is some clique $C = \{v_i, \dots, v_{i+\omega(H_3)-1}\}$ in H_3 ; this clique does not intersect X_1 because $|X_1 \cup X_2| \leq l - \frac{1}{2}|Y_1| \leq l - |Y_2| < l - b$. Denote by v_j the leftmost neighbour of v_i . Since $\gamma_i^j(v_i) \leq l$, it is clear that v_i has at most $2b$ neighbours outside C , and since $b < |Y_2| \leq \frac{1}{2}|X_1|$ we can be assured that $v_i \notin X_2$. Since $\omega(H_3) > |Y_2|$, $v_i \notin Y_2$.

We now colour H_3 from left to right, modulo $\omega(H_3)$. If at most b colours appear in Y_2 but not X_1 then we are done, otherwise we will “roll back” the colouring, starting at v_i . That is, for every $p \geq i$, we modify the colouring of H_3 by giving v_p the colour after the one that it currently has, modulo $\omega(H_3)$. Since v_i has at most $2b$ neighbours behind it, we can roll back the colouring at least $\omega(H_3) - 2b - 1$ times for a total of $\omega(H_3) - 2b$ proper colourings of H_3 .

Since $v_i \notin Y_2$ the colours on Y_2 will appear in order modulo $\omega(H_3)$. Thus there are $\omega(H_3)$ possible sets of colours appearing on Y_2 , and in $2b + 1$ of them there are at most b colours appearing in Y_2 but not X_1 . It follows that as we roll back the colouring of H_3 we will find an acceptable colouring.

Henceforth we will assume that $|X_1| \geq |Y_1|$.

Case 4. $0 < k < |X_1|$.

Take a stable set S in $G_2 - X_2$ greedily from left to right. If S hits Y_2 , we remove S from G , along with a colour class from G_1 intersecting X_1 but not Y_1 . Otherwise, we remove S along with a colour class from G_1 intersecting both X_1

and Y_1 . In either case it is a simple matter to confirm that $\gamma_l^i(v)$ drops for every $v \in H_2$ as we did in Case 2. We proceed by induction.

Case 5. $k = |Y_1| = |X_1| = 1$.

In this case $|X_1| = k = 1$. If G_2 is not connected then X_1 and Y_1 are both clique cutsets and we can proceed as in Case 1. If G_2 is connected and contains an l -clique, then there is some $v \in V_2$ of degree at least l in the l -clique. Thus $\gamma_l^i(H_2) > l$, contradicting our assumption that $l \geq \gamma_l^i(H_2)$. So $\omega(G_2) < l$. We can $\omega(G_2)$ -colour G_2 in linear time using only colours not appearing in $X_1 \cup Y_1$, thus extending the l -colouring of G_1 to a proper l -colouring of G .

Case 6. $k = |Y_1| = |X_1| > 1$.

Suppose that k is not minimal. That is, suppose there is a vertex $v \in X_1 \cup Y_1$ whose closed neighbourhood does not contain all l colours in the colouring of G_1 . Then we can change the colour of v and apply Case 4. So assume k is minimal.

Therefore every vertex in X_1 has degree at least $l + |X_2| - 1$. Since $X_1 \cup X_2$ is a clique, $\gamma_l^i(H_2) \geq l \geq \frac{1}{2}(l + |X_2| + |X_1| + |X_2|)$, so $2|X_2| \leq l - k$. Similarly, $2|Y_2| \leq l - k$, so $|X_2| + |Y_2| \leq l - k$. Since there are $l - k$ colours not appearing in $X_1 \cup Y_1$, we can $\omega(G_2)$ -colour G_2 , then permute the colour classes so that no colour appears in both $X_1 \cup Y_1$ and $X_2 \cup Y_2$. Thus we can extend the l -colouring of G_1 to an l -colouring of G .

These cases cover every possibility, so we need only prove that the colouring can be found in $O(nm)$ time. If k has been maximized and we apply induction, k will stay maximized: every vertex in $X_1 \cup Y_1$ will have every remaining colour in its closed neighbourhood except possibly if we recolour a vertex in Case 6. In this case the overlap in what remains is $k - 1$, which is the most possible since we remove a vertex from X_1 or Y_1 , each of which has size k . Hence we only need to maximize k once. We can determine which case applies in $O(m)$ time, and it is not hard to confirm that whenever we extend the colouring in one step our work can be done in $O(nm)$ time. When we apply induction, i.e. in Cases 2, 4, and possibly 6, all our work can be done in $O(m)$ time. Since $l < n$ it follows that the entire l -colouring can be completed in $O(nm)$ time. \square

Chudnovsky and Ovetsky Fradkin [CO07] proved that $\chi(G) \leq \frac{3}{2}\omega(G)$ for any quasi-line graph G , extending a result of Shannon for line graphs [Sha49]. Our theorem strengthens these results, since for any quasi-line graph $\Delta(G) \leq 2\omega(G) - 1$ and consequently $\gamma(G) \leq \frac{3}{2}\omega(G)$.

7.2 Asymptotics of the chromatic number for quasi-line graphs

In Chapter 4 we presented two important classes of graphs for which χ and χ_f are close: line graphs and circular interval graphs. In this section we prove that χ and χ_f are close together for quasi-line graphs. We do this by combining the structure theorem for quasi-line graphs with colouring results on line graphs and circular interval graphs.

As we mentioned in Chapter 4, Kahn first proved that χ and χ_f agree asymptotically for line graphs. Sanders and Steurer followed this with a more precise result [SS05]:

Theorem 7.2. *Any line graph G can be coloured in polynomial time using $\chi_f(G) + \sqrt{\frac{9}{2}\chi_f(G)}$ colours.*

We now extend this result, which gives a strengthening of the main result of [KR07]:

Theorem 7.3. *Any quasi-line graph G can be coloured in polynomial time using $\chi_f(G) + 3\sqrt{\chi_f(G)}$ colours.*

We first prove that $\chi(G) \leq \chi_f(G) + 3\sqrt{\chi_f(G)}$, leaving algorithmic considerations until later.

Let G be a minimum counterexample to the theorem. Theorem 6.11 implies that G must be skeletal. Theorem 4.2 tells us that G cannot be a circular interval graph, and Theorem 7.2 tells us that G cannot be a line graph. Therefore G is some composition of linear interval strips (S_e, X_e, Y_e) with underlying directed multigraph H , and it must admit a canonical interval 2-join.

We deal with a 2-join by emulating a fractional colouring on the corresponding linear interval strip with an integer colouring. However, we cannot deal with these 2-joins one at a time as we did in the previous section. Each time we turn a fractional

colouring of a linear interval strip (S, X, Y) into an integer colouring, we need to use an extra colour in case the total weight of stable sets intersecting both X and Y in the fractional colouring is not an integer. Thus we must deal with all 2-joins simultaneously, emulating an optimal fractional colouring as closely as possible. Our approach is as follows:

We begin by finding a near-optimal fractional colouring of G in which the total weight of stable sets intersecting both X_e and Y_e is an integer, for every e . We then use this fractional colouring to construct a line graph G' by contracting each strip into a carefully chosen clique. Using Theorem 7.2, we find a $(1+o(1))\chi_f(G)$ -colouring of G' . This colouring gives us a partial colouring of G on the end-cliques. Finally we complete the colouring of G by emulating our near-optimal fractional colouring on by integer colourings on the strips.

We now address the specifics of this approach.

7.2.1 Finding a good fractional colouring

Given a fractional colouring of G , we use w_e to denote the total weight of stable sets in the colouring that intersect both X_e and Y_e . Our first job is to find a near-optimal fractional colouring of G in which w_e is an integer for every edge e of the underlying multigraph H . We will prove:

Lemma 7.4. *There is a fractional $(\chi_f(G) + \frac{1}{3}\sqrt{\omega(G)})$ -colouring of G such that w_e is an integer for all $e \in E(H)$.*

To find this fractional colouring will take a fractional $\chi_f(G)$ -colouring and modify it on the cliques X_e . We can assume that for all $e \in E(G)$, $|X_e| \leq |Y_e|$ – if $|X_e| > |Y_e|$, simply change the direction of e in H and swap the names of X_e and Y_e . Note that the case $X_e = \emptyset$ will result in a simplicial vertex, which cannot exist in a minimum counterexample.

We begin by proving that we don't need to modify too many cliques X_e which have edges between them. To this end, we say that an edge e of H is *trivial* if $X_e = Y_e$. For a trivial edge e , w_e is always equal to $|X_e|$ and is therefore an integer. For $v \in V(H)$, let $D(v)$ denote the number of nontrivial edges out of v . Let $D(H)$ be the maximum of $D(v)$ over all $v \in V(H)$. We bound $D(H)$ using the following result:

Lemma 7.5. *If e is a nontrivial edge of H , then $|X_e| > 3\sqrt{\chi_f(v)}$.*

Proof. By the minimality of G , every vertex in G must have degree at least $\chi(G) - 1 > \omega(G) + 3\sqrt{\chi(G)} - 1$.

Let the vertices of S_e be $\{u_1, \dots, u_{V(S_e)}\}$ in linear order, such that $X_e = \{u_1, \dots, u_{|X_e|}\}$. Because e is nontrivial and $|Y_e| \geq |X_e|$, the vertex $u_{|X_e|+1}$ exists. By the structure of linear interval graphs, its closed neighbourhood outside X_e is a clique if $u_{|X_e|+1}$ is not in Y_e . If $u_{|X_e|+1} \in Y_e$ then its neighbourhood outside X_e is contained in the hub clique containing Y_e . Therefore $\omega(G) + |X_e| \geq d(u_{|X_e|+1}) + 1 > \omega(G) + 3\sqrt{\chi(G)}$. \square

This immediately gives us a bound on $D(H)$, because every hub clique C_v has size at most $\omega(G) \leq \chi_f(G)$.

Corollary 7.6. $D(H) < \frac{1}{3}\sqrt{\chi_f(G)}$.

We use our bound on $D(H)$ to prove the existence of the desired fractional colouring. We omit the simple proof of the following lemma, which reduces to finding an optimal edge colouring of a set of disjoint stars:

Lemma 7.7. *We can colour the nontrivial edges of H with $D(H) < \frac{1}{3}\sqrt{\omega(G)}$ colours such that no two edges out of the same vertex get the same colour.*

We deal with one of these colour classes at a time:

Lemma 7.8. *Let E_1 be a set of nontrivial edges in H , no two of which go out of the same vertex. Given a fractional k -colouring of G with overlaps $\{w_e \mid e \in E\}$, there is a fractional $(k+1)$ -colouring of G with overlaps $\{w'_e \mid e \in E\}$ such that $w'_e = \lfloor w_e \rfloor$ for $e \in E_1$, and $w'_e = w_e$ for $e \notin E_1$.*

Proof. Take some single edge e of E_1 and an optimal fractional colouring of G . We claim that we can make w'_e an integer by adding $w_e - \lfloor w_e \rfloor$ extra weight to the fractional colouring. To do this, take a collection of stable sets, each intersecting both X_e and Y_e , of total weight $w_e - \lfloor w_e \rfloor$ in the colouring (it may be necessary to split one stable set into two identical stable sets of lesser weight to do this). Now remove the vertex in X_e from each of these stable sets, and fill the missing weight in X_e (i.e. $w_e - \lfloor w_e \rfloor$) with singleton stable sets. This gives us the desired fractional colouring in which w'_e is an integer. Note that we did not change the colouring outside X_e , so every other overlap is unchanged.

To see that we can “integralize” each w_e using extra weight less than 1, note that for $e, e' \in E_1$, there are no edges between X_e and $X_{e'}$. Thus we can integralize X_e for every $e \in E_1$ using $|E_1|$ extra weight, and replace the resulting singleton colour classes with stable sets of size up to $|E_1|$, with total weight less than 1. \square

This gives us an easy proof of Lemma 7.4:

Proof of Lemma 7.4. Begin with a fractional $\chi_f(G)$ -colouring of G and a colouring of the edges of H guaranteed by Lemma 7.7. For each matching in this edge colouring, apply Lemma 7.8. The result is a fractional $(\chi_f(G) + D(H))$ -colouring of G for which each overlap w_e is an integer. \square

7.2.2 Completing the proof

We are now ready to prove our bound on $\chi(G)$.

Proof of Theorem 7.3. Let G be a minimum counterexample to the theorem. We know that it is a composition of strips (S_e, X_e, Y_e) with underlying multigraph H . We construct a line graph G' from G as follows. Take a fractional $(\chi_f(G) + \frac{1}{3}\sqrt{\omega(G)})$ -colouring of G such that every w_e is an integer. We want to replace every strip (S_e, X_e, Y_e) with a new strip (S'_e, X'_e, Y'_e) such that (i) the resulting composition G' of these strips is a line graph, (ii) $|X'_e| = |X_e|$ and $|Y'_e| = |Y_e|$, and (iii) in any colouring of G' , precisely w_e colours intersect both X'_e and Y'_e .

With this in mind, for each e we define a new linear interval strip (S'_e, X'_e, Y'_e) on $|X_e| + |Y_e| - w_e$ vertices such that X'_e and Y'_e are cliques of size $|X'_e|$ and $|Y'_e|$ respectively, and $|X'_e \cap Y'_e| = w_e$. Furthermore S'_e itself is a clique. Let G' be the composition of the strips (S'_e, X'_e, Y'_e) with H as the underlying multigraph (see Figure 7.1). Equivalently, we construct a multigraph H' from H by replacing every edge e of H with a triangle on $|X_e| + |Y_e| - w_e$ edges, exactly w_e of which are between the original endpoints of e . We then set $G' = L(H')$.

We can see that $\chi_f(G') \leq \chi_f(G) + \frac{1}{3}\sqrt{\omega(G)}$: Because $X'_e \setminus Y'_e$, $Y'_e \setminus X'_e$, and $X'_e \cap Y'_e$ are all homogeneous cliques in G' , we can safely map the colours in G that hit X_e (resp. Y_e) into stable sets that hit X'_e (resp. Y'_e). Thus Theorem 7.2 gives us a bound on $\chi(G')$:

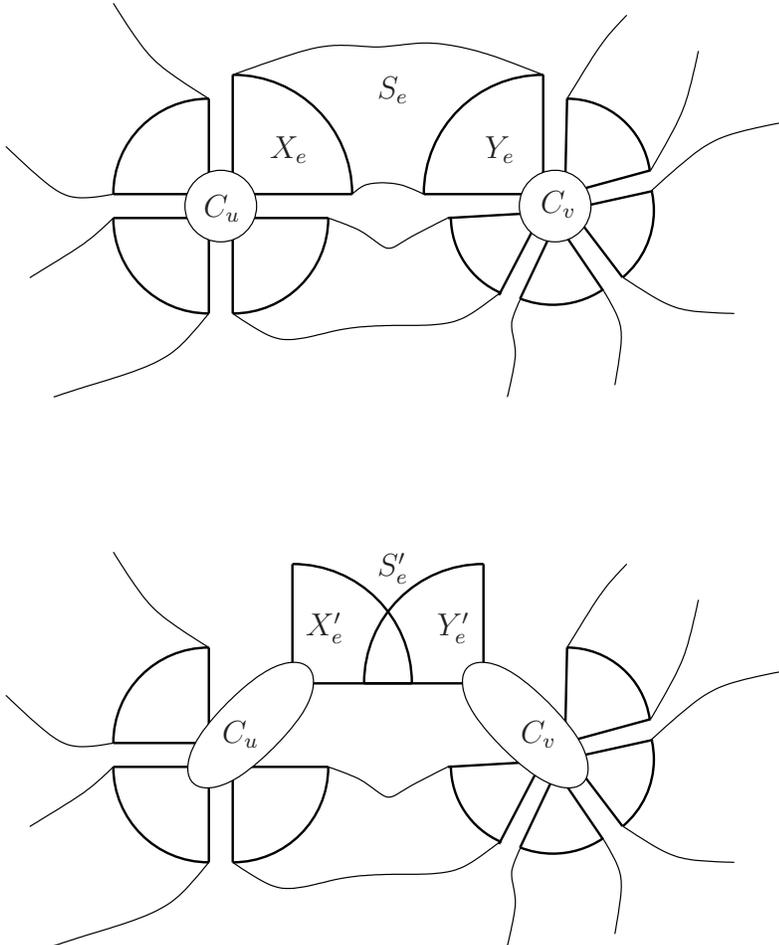


Figure 7.1: We construct a line graph G' from G by contracting each strip (S_e, X_e, Y_e) into a (S'_e, X'_e, Y'_e) where S'_e is a clique, $X'_e \cup Y'_e$ together cover S'_e , and $|X'_e \cap Y'_e| = w_e$.

$$\chi(G') \leq \chi_f(G') + \sqrt{\frac{9}{2}\chi_f(G')} \quad (7.1)$$

$$\leq \chi_f(G) + \frac{1}{3}\sqrt{\chi_f(G)} + \sqrt{\frac{9}{2}\left(\chi_f(G) + \frac{1}{3}\sqrt{\chi_f(G)}\right)} \quad (7.2)$$

$$\leq \chi_f(G) + \frac{1}{3}\sqrt{\chi_f(G)} + \sqrt{\frac{9}{2}\left(\frac{4}{3}\chi_f(G)\right)} \quad (7.3)$$

$$\leq \chi_f(G) + 3\sqrt{\chi_f(G)}. \quad (7.4)$$

Consider a colouring of G' using $\lfloor \chi_f(G) + 3\sqrt{\chi_f(G)} \rfloor$ colours. Since S'_e is a clique for every edge e of H , the number of colours appearing in both X'_e and Y'_e is precisely w_e . So from this colouring of G' we can construct a partial colouring of G' , colouring $\cup\{X_e \cup Y_e \mid e \in E(H)\}$ with $\lfloor \chi_f(G) + 3\sqrt{\chi_f(G)} \rfloor$ colours so that precisely w_e colours appear in both X_e and Y_e .

The original fractional $(\chi_f(G) + \frac{1}{3}\sqrt{\omega(G)})$ -colouring of G gives us a fractional colouring of each S_e such that the overlap between X_e and Y_e is precisely w_e , so Lemma 4.3 tells us that we can $\lceil \chi_f(G) + \frac{1}{3}\sqrt{\omega(G)} \rceil$ -colour each S_e such that precisely w_e colours appear in both X_e and Y_e . Thus by permuting colour classes, we can complete the partial colouring of G and find a proper $\lfloor \chi_f(G) + 3\sqrt{\chi_f(G)} \rfloor$ -colouring of G , proving the theorem. \square

7.3 Algorithmic considerations

In this section we prove that given a quasi-line graph G , in polynomial time we can colour G using $\min\{\gamma(G), \chi_f(G) + 3\sqrt{\chi_f(G)}\}$ colours.

7.3.1 The Main Conjecture

Our approach to colouring quasi-line graphs as in the proof of Lemma 7.1 suggests an algorithm for $\gamma(G)$ -colouring these graphs in polynomial time.

Theorem 7.9. *Let G be a quasi-line graph on n vertices. Then in $O(n^2m^2)$ time we can find a proper $\gamma(G)$ -colouring of G .*

Proof. We first apply Theorem 6.11 to construct a skeletal quasi-line subgraph G' of G such that $\chi(G) = \chi(G')$ and $\chi_f(G) = \chi_f(G')$. We can do this in $O(n^2m^2)$ time. We then $\gamma(G')$ -colour G' . If G' is a circular interval graph then we can γ -colour it in $O(n^{3/2})$ time by Theorem 4.2. If G' is a line graph we can γ -colour it in $O(n^{5/2})$ time by Theorem 4.14.

Otherwise G' is a composition of linear interval strips containing no nonlinear homogeneous pair of cliques, so we first find a canonical strip representation of G' as described in the previous chapter. Take a non-singleton strip (S_e, X_e, Y_e) . We can recursively $\gamma(G')$ -colour $G - S_e$ in $O(n^2m)$ time, then apply Lemma 7.1 on the canonical interval 2-join $((X_1, Y_1), (X_e, Y_e))$ associated with (S_e, X_e, Y_e) . This gives us a $\gamma(G')$ -colouring of G' in $O(n^2m^2)$ time.

The final step is to convert this into a $\gamma(G')$ -colouring of G . But Theorem 6.11 tells us that we can do this in $O(n^2m^2)$ time, so we are done. \square

7.3.2 Achieving the asymptotic bound

We now prove that we can $\lfloor \chi_f(G) + 3\sqrt{\chi_f(G)} \rfloor$ -colour a quasi-line graph G in polynomial time. Just as with γ -colouring, if G is not a composition of linear interval strips then we can reduce the problem in polynomial time. If G is a composition of linear interval strips but the underlying directed multigraph H has $D(H) \geq \frac{1}{3}\sqrt{\chi_f(G)}$, then G has a vertex v of degree at most $\chi_f(G) + 3\sqrt{\chi_f(G)} - 1$. We can reduce this case by colouring $G - v$ and giving v a colour not appearing in its neighbourhood. Thus we can assume that G is a composition of linear interval strips and $D(H) < \frac{1}{3}\sqrt{\chi_f(G)}$, as in the proof of Theorem 7.3.

As with the Main Conjecture, we begin by determining a canonical strip representation of G . Once we have constructed the line graph G' from G as in the proof of Theorem 7.3, it is straightforward to show that we can find a proper $\lfloor \chi_f(G) + 3\sqrt{\chi_f(G)} \rfloor$ -colouring of G in polynomial time. To see this, first note that Theorem 7.2 tells us that we can find the desired colouring of G' in polynomial time. This immediately gives us a partial colouring of G . The only remaining step is to find the restricted colourings of the strips. But the proof of Lemma 4.3 tells us that this can be done by colouring an auxiliary circular interval graph for each strip. We can do this in polynomial time by Theorem 4.2.

The only remaining job is to construct the line graph G' from G .

Constructing the line graph

We must now show that we can indeed construct G' from G in polynomial time. Minty [Min80], Nakamura and Tamura [NT01], and recently Oriolo, Pietropaoli, and Stauffer [OPS08] give polynomial-time algorithms for finding a maximum-weight stable set in any claw-free graph. By polynomial equivalence results of Grötschel, Lovász, and Schrijver (see [GLS81] §7), this implies that we can compute the fractional chromatic number of any claw-free graph in polynomial time. We can deal with weights as well: Given positive integer weights $\beta(v)$ for each vertex v of the graph G , let G_β be the graph obtained from G by substituting a clique of size $\beta(v)$ for each vertex v of G . We can find $\chi_f(G_\beta)$ in polynomial time in terms of $|V(G)| + \log(\max_{v \in V(G)} \beta(v))$.

But in fact the equivalence results in [GLS81] tell us that for claw-free graphs, we can efficiently optimize over the feasible region of the linear program (1.3). That is, the dual of the linear program (1.2), which we can solve to compute the fractional chromatic number. Theorem 6.5.14 in [GLS93] states:

Theorem 7.10. *There exists an oracle-polynomial time algorithm that, for any well-described polyhedron $(P; n, \phi)$ given by a strong separation oracle and for any $c \in \mathbb{Q}^n$, either*

1. *finds a basic optimum standard solution, or*
2. *asserts that the dual problem is unbounded or has no solution.*

We refer the reader to [GLS93] for the formal definitions of a *well-described polyhedron* and an *oracle-polynomial time algorithm*. In our specific situation, this theorem states that since we can optimize over the dual feasible region (1.3), we can find a *basic solution* to (1.2) in polynomial time for any claw-free graph. A basic solution to this linear program is a fractional colouring in which the incidence vectors of the stable sets with nonzero weight are linearly independent. Thus by the dimension of the stable set polytope, there are at most n nonzero stable sets in the fractional colouring.

Once we have a basic fractional colouring of G , it is a simple matter to compute the overlaps for the strips and construct the line graph G' in polynomial time.

Part IV

Claw-free Graphs

We now move on from quasi-line graphs to the more general class of claw-free graphs, with the eventual goal of proving new bounds on the chromatic number.

We note that a claw-free graph G is not quasi-line precisely if there is an odd antihole in the neighbourhood of some vertex. Fouquet has shown that in fact if $\alpha(G) \geq 3$ then G is not quasi-line precisely if no vertex contains an induced C_5 in its neighbourhood, i.e. precisely if G contains no W_5 . The next chapter begins with a proof of this result. Then, by characterizing the first and second neighbourhoods of such a C_5 , we generalize the structure theorem for quasi-line graphs to obtain a simple structure theorem for claw-free graphs with stability number at least four.

In Chapter 9 we complete the description of skeletal claw-free graphs using Chudnovsky and Seymour's characterization of all claw-free graphs. This includes a characterization of claw-free graphs with stability number at most three, as well as a more rigorous treatment of claw-free graphs with stability number at least four. This description will give us enough knowledge about the structure of claw-free graphs to prove our bounds on the chromatic number. In Chapter 10 we consider the Main Conjecture once again. We prove that it holds for all claw-free graphs, and furthermore we prove that the Local Strengthening holds for all three-cliqued claw-free graphs.

Chapter 8

An Invitation to the General Structure Theorems

In this chapter we examine the structure of connected claw-free graphs. First we prove a result of Fouquet, which implies that a connected claw-free graph G with $\alpha(G) \geq 3$ is quasi-line precisely if it contains no W_5 or *5-wheel* (see Figure 8.1). We then show that when $\alpha(G) \geq 4$, every W_5 in the graph is separated from its non-neighbourhood by a 1-join or a 2-join. Using this result, we easily extend the structure theorem for quasi-line graphs to a structure theorem for claw-free graphs with $\alpha(G) \geq 4$. These results were obtained independently and slightly earlier by Oriolo, Pietropaoli, and Stauffer [OPS08], and should be credited to them. We obtained the result in August of 2008, while they had presented it at a conference that May.

These results can easily be obtained as a corollary of Chudnovsky and Seymour's much more detailed decomposition and structure theorems for claw-free graphs [CS07c, CS08b]. Furthermore, both we and Oriolo, Pietropaoli, and Stauffer rely on Chudnovsky and Seymour's work, since the only known proof of the structure theorem for quasi-line graphs is also as a corollary to the claw-free structure theorem. Given this fact, one might be tempted to consider these results useless.

However, this would be incorrect for two reasons. First, if a short proof of Theorem 5.11 were to be found, this would lead to a short and very natural proof of the decomposition theorem for claw-free graphs with $\alpha \geq 4$. This would be very attractive, particularly as Chudnovsky and Seymour's proof of the decomposition theorem is very technical and over eighty pages long. Second, the decomposition given here

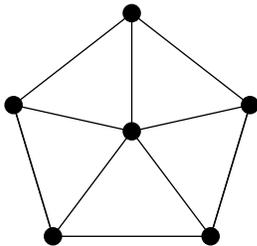


Figure 8.1: W_5 , the smallest claw-free graph that is not quasi-line.

is more intuitive and easier to work with, and has been used to prove several interesting results. For example, Oriolo, Pietropaoli, and Stauffer first used it in a new algorithm for finding a maximum weight stable set in a claw-free graph [OPS08]. Galluccio, Gentile, and Ventura have used the distinction between quasi-line graphs and claw-free graphs to make progress in characterizing the stable set polytope of claw-free graphs [GGV08]. We describe their work briefly in Section 8.4.

Unfortunately, we were unable to prove the Main Conjecture for claw-free graphs using this decomposition; we needed the extra detail provided by Chudnovsky and Seymour’s theorems. We present our colouring results in Chapter 10, after presenting their structure theorems in Chapter 9. It would be of interest to find a shorter, simpler proof of their result based on the decomposition we present in this chapter.

8.1 Graphs with no C_5 in a neighbourhood

Ben Rebea’s Lemma [Ben81], proved in [CS88], states that if a claw-free graph with stability number at least three contains an odd antihole, then it contains an antihole of length 5, i.e. C_5 . We are interested in a C_5 contained in the neighbourhood of some vertex. Fouquet [Fou93] provided a result analogous to Ben Rebea’s Lemma; we prove it now.

Theorem 8.1 (Fouquet). *Let G be a claw-free graph with $\alpha(G) \geq 3$. Then for every vertex v of G , either v is bisimplicial or the neighbourhood of v contains an induced C_5 .*

Proof. Assume for a contradiction that G is a counterexample containing a stable set of size three. Then G contains a non-bisimplicial vertex v whose neighbourhood

contains an odd antihole H of length $k \geq 7$, and no shorter odd antihole. Denote the vertices of H v_1, \dots, v_k with indices taken modulo k such that v_i misses v_{i+1} for $1 \leq i \leq k$.

The key to the proof is to analyze the possible neighbourhoods in H of the vertices in $V - H$. We begin with the following:

Observation 8.2. *Every vertex of $V - H$ either sees none of H or misses exactly a clique of H .*

Proof. If $u \in V - H$ misses v_i and v_{i+1} for some i , then u must miss $H - \{v_{i-1}, v_{i+2}\}$. Since v_{i+2} is adjacent to v_{i+4} and v_{i+5} , u cannot see v_{i+2} or there would be a claw. By symmetry, u cannot see v_{i-1} , thus u sees none of H . \square

From this we easily obtain the next observation:

Observation 8.3. *Every vertex of $V - H$ misses exactly a clique of H , and hence sees more than half of the vertices of H .*

Proof. By the last observation and the fact that \overline{H} induces an odd cycle, it is enough to show that no vertex misses all of H . Suppose the contrary. Since G is connected, there are two vertices u and w in $V - H$ such that u misses H and w sees some of H . But w misses a clique of H , so it sees two nonadjacent vertices in H and there is a claw, a contradiction. \square

Observation 8.4. *If a vertex u of $V - H$ sees v then it misses at most two vertices of H . Furthermore if u misses exactly two vertices of H then they are v_i and v_{i+2} for some i .*

Proof. Suppose $u \in V - H$ sees v , and there exists some i and some odd j with $3 \leq j < k - 2$ such that u misses v_i and v_{i+j} but sees $\{v_{i+l} \mid 1 \leq l < j\}$, with indices modulo k . Then $\{u\} \cup \{v_{i+l} \mid 0 \leq l \leq j\}$ induces an odd antihole of length less than k (see Figure 8.2), and this antihole is in the neighbourhood of v , contradicting the minimality of k .

Suppose u misses exactly two vertices of H . Since k is odd and u misses no two consecutive vertices of H , it follows that these i and j exist unless u misses v_i and v_{i+2} for some i : there are two paths between the non-neighbours of u in \overline{H} , one of which is even and one of which is odd. Similarly, if u misses at least three vertices of

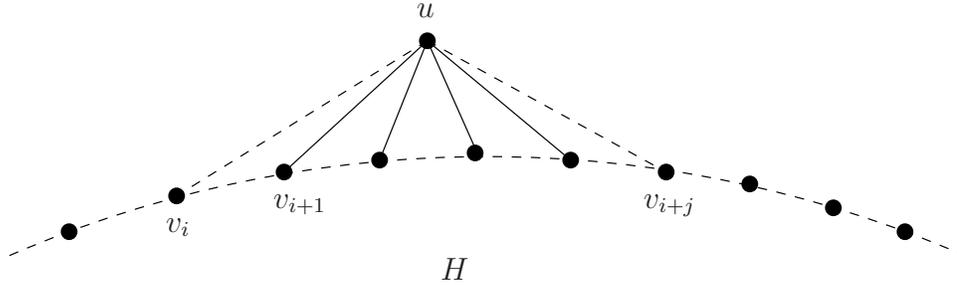


Figure 8.2: Any $u \in W$ is contained in an odd antihole of length less than k , passing through u and a contiguous portion of H .

H then i and j exist since k is odd: the gaps between the non-neighbours of u in H have odd sum and are all at least two. \square

Now let T be a stable set of size three in G .

Observation 8.5. T cannot contain two vertices of H .

Proof. This follows immediately from the fact that $\alpha(H) = 2$ and every vertex of $V - H$ misses a clique in H . \square

Observation 8.6. T cannot contain exactly one vertex of H .

Proof. Suppose T contains v_i and vertices u and u' in $V - H$. Since there is no claw in G , no vertex in $H - \{v_{i-1}, v_{i+1}\}$ sees both u and u' . Thus since $k \geq 7$ and neither u nor u' can miss two consecutive vertices of H , both u and u' miss at least three vertices of H . By Observation 8.4, neither u nor u' sees v . But by Observation 8.3 both u and u' see v_{i+1} since they miss v_i . Therefore $\{u, u', v, v_{i+1}\}$ induces a claw, a contradiction. \square

Observation 8.7. T cannot be disjoint from H .

Proof. Suppose $T = \{t_1, t_2, t_3\}$ is contained in $V - H$. Observation 8.3 tells us that v cannot be in a stable set of size three with two vertices of T , otherwise this stable set would be in the neighbourhood of some vertex of H . On the other hand v cannot see all of H as G is claw-free. Thus v sees two vertices t_1 and t_2 of T and misses the third, t_3 . By Observation 8.4, both t_1 and t_2 miss at most two vertices of H .

Now, no vertex of H sees all of T or there would be a claw. Thus t_3 must miss at least $k - 4$ vertices of H , and since t_3 misses a clique (of size at most $\frac{k-1}{2}$) in H , we can see that $k = 7$ and t_3 misses three vertices of H . Thus t_1 and t_2 each miss two vertices of H , and since they see v Observation 8.4 tells us that there are i_1 and i_2 such that t_1 misses $\{v_{i_1}, v_{i_1+2}\}$ and t_2 misses $\{v_{i_2}, v_{i_2+2}\}$.

There is only one possible configuration. Without loss of generality we can assume that the non-neighbourhoods in H of t_1, t_2 , and t_3 are $\{v_1, v_3\}$, $\{v_5, v_7\}$, and $\{v_2, v_4, v_6\}$ respectively. Since v sees t_1 and t_2 , $\{t_1, v_4, t_2, v_3, v_5\}$ induces a C_5 in the neighbourhood of v , a contradiction. \square

Observations 8.5, 8.6, and 8.7 together imply Theorem 8.1. \square

This theorem immediately implies that if $\alpha(G) \geq 3$, G is quasi-line or contains a W_5 .

Corollary 8.8. *Let G be a connected claw-free graph in which no neighbourhood contains an induced C_5 . Then $\alpha(G) \leq 2$ or G is quasi-line.*

We now examine how a W_5 and its neighbourhood can be connected to the rest of a claw-free graph.

8.2 Investigating the attachments of a W_5

In this section we will prove that if $\alpha(G) \geq 4$, then any W_5 and its neighbourhood are separated from the rest of the graph by a special type of 1-join or 2-join. The main results in this section were first proved by Oriolo, Pietropaoli, and Stauffer [OPS08].

8.2.1 The neighbourhood of a W_5

Let H be a C_5 in the neighbourhood of some vertex v . To prove our results we partition the vertices into sets depending on their neighbourhood in H . We then examine how these sets interact with one another. Beginning with the vertices v_1, \dots, v_5 of H in cyclic order (note that in the previous section we used cyclic order in the complement) we define our sets as follows, with indices modulo 5:

- A is the set of vertices seeing all of H ; A contains v .

- N is the set of vertices with no neighbour in H .
- For $i = 1, \dots, 5$, B_i is the set of vertices outside H whose neighbourhood in H is precisely $\{v_i, v_{i+1}\}$ (with indices taken modulo 5). $B = \cup_i B_i$.
- U_i is the set of vertices in B_i with a neighbour in N . $U = \cup_i U_i$.
- For $i = 1, \dots, 5$, C_i is the set of vertices whose neighbourhood in H is precisely $\{v_i, v_{i+1}, v_{i+2}\}$. Denote $C_i \cup \{v_{i+1}\}$ by C'_i and define $C = \cup_i C_i$ and $C' = \cup_i C'_i = C \cup H$.
- For $i = 1, \dots, 5$, D_i is the set of vertices whose neighbourhood in H is precisely $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$. $D = \cup_i D_i$.

Since G is claw-free, if a vertex outside H sees v_i , then it also sees v_{i-1} or v_{i+1} . Thus we can see that the sets A, B, C', D, N partition $V(G)$.

We can begin with some simple observations. Again and throughout this chapter, indices are modulo 5.

- For $i = 1, \dots, 5$, $B_i \cup C'_i \cup D_i$ is a clique. This follows from the fact that G is claw-free, every vertex in $B_i \cup C'_i \cup D_i$ sees v_i , and none sees v_{i-1} .
- There is no edge between B and A , since a vertex in B_i is nonadjacent to v_{i-1} and v_{i+2} , which are nonadjacent to each other but both adjacent to every vertex in A . Thus an edge between B and A would imply a claw in G .

By the same reasoning, for $i = 1, \dots, 5$ there is no edge between B_i and $D_{i-1} \cup D_{i+1} \cup D_{i+2} \cup C'_{i+2}$.

- For $i = 1, \dots, 5$, $B_i \cup B_{i+1}$ is a clique, since v_{i+1} sees $B_i \cup B_{i+1}$ and v .

These facts are symmetric in the sense that just as $B_i \cup C'_i \cup D_i$ is a clique, so is $B_i \cup C'_{i-1} \cup D_{i-2}$.

8.2.2 A dominating W_5

If G is connected and U is empty, then $N(H)$ is the entire graph and H is dominating. We now prove that in this situation, $\alpha(G) \leq 3$. This was proved by Lovász and Plummer, appearing as Claim 5 in the proof of 12.4.3 in [LP86].

Theorem 8.9. *If a claw-free graph G contains a dominating W_5 then $\alpha(G) \leq 3$.*

Proof. Let H be a dominating 5-hole in the neighbourhood of some vertex v . We partition the vertices of G as described above. In this case N is empty and A is nonempty. Suppose G contains a stable set X of size four.

If $|H \cap X| = 2$, then the other two vertices of X are in the clique B_i for some i , a contradiction. Suppose then that $H \cap X = v_1$. Then $X - H$ is contained in $B \cup C' \cup D$. More specifically, it is contained in $B_2 \cup B_3 \cup B_4 \cup C'_2 \cup C'_3 \cup D_2$. Since G is claw-free, v_2 and v_5 each have at most one neighbour in $X - v_1$ while v_3 and v_4 each have at most two neighbours in $X - v_1$. Since each vertex of $X - v_1$ has at least two neighbours in H , it follows that $X - v_1$ consists of one vertex from each of B_2 , B_3 , and B_4 . This contradicts the fact that $B_2 \cup B_3$ and $B_3 \cup B_4$ are cliques. Therefore X and H are disjoint.

There cannot be more than ten edges between H and X , otherwise some vertex of H has three neighbours in X , giving a claw. Since $\alpha(G[B]) \leq 2$, it follows that $X \subseteq B \cup C'$, and $|X \cap B| = 2$. Assume by symmetry that X contains a vertex of B_1 and a vertex of B_3 . But B_i is complete to both C'_i and C'_{i-1} , so X contains two vertices in C'_4 , a clique. Therefore X cannot exist and $\alpha(G) \leq 3$. \square

8.2.3 The second neighbourhood of a W_5

Every vertex with a neighbour in H and a neighbour in N must see a clique in H . Therefore the only vertices with a neighbour in H and a neighbour in N are those vertices in U .

Suppose a vertex $w \in N$ has a neighbour in U_i . Observe that if B_{i+1} is nonempty, then w must see all of $U_i \cup B_{i+1}$, otherwise there would be a claw. Furthermore the set of vertices of N with a neighbour in U_i form a clique. This simple fact suggests something more significant: By characterizing U based on which sets U_i are empty and nonempty, we can determine how $N(H)$ is attached to the rest of the graph via the second neighbourhood of H . Specifically, we deal with two separate cases, depending on whether or not the nonempty U_i sets appear in contiguous order around the cycle. Specifically:

Definition 8.10. *The set U is contiguous if there is no i such that U_i and U_{i+2} are nonempty but U_{i-1} and U_{i+1} are empty.*

Because two consecutive nonempty U_i and U_{i+1} imply homogeneity of U_i and U_{i+1} outside the neighbourhood of the W_5 , the way in which H and its neighbourhood are connected to the rest of the graph is strongly influenced by whether or not U is contiguous. In the remainder of this section we will show:

- If U is contiguous, then either $\alpha(G) \leq 3$ or $N(H)$ is separated from the rest of the graph by a 1-join.
- If U is not contiguous, then $N(H)$ is separated from the rest of the graph by one of two specific types of 2-join.

As the decomposition that we find depends on the structure of U , our first job is to prove that $\alpha(G) \leq 3$ whenever U is empty.

8.2.4 Decomposing around a non-dominating W_5

Now suppose U is nonempty, which is equivalent to saying that our W_5 is non-dominating. We examine the structure around H in two cases, depending on whether or not U is contiguous. When U is contiguous we get a 1-join or $\alpha(G) = 3$. When U is not contiguous we get one of two types of 2-join.

Case 1: U is contiguous

Note that U is a cutset, so if U is a clique it is obviously a clique cutset. Proving that it is one side of a 1-join takes a little more work. We will additionally show that the 1-join has a specific structure with respect to H .

Definition 8.11. *A 1-join (X, Y) is a W_5 1-join if $X = U$, and $N(H)$ is equal to either G_2 or $G_2 \cup U$.*

Lemma 8.12. *If U is a clique, then G admits a W_5 1-join (U, X) for some set X .*

Proof. Assume U is a clique. First suppose $U = U_i$ for some i ; assume $i = 1$. Since every vertex in U_1 has a neighbour in N , we can see that U_1 is complete to $B_5 \cup B_1 \cup B_2$, $C'_1 \cup C'_5$, and $D_1 \cup D_4$. Likewise U_1 is anticomplete to A , $D_2 \cup D_3 \cup D_5$, $C'_2 \cup C'_3 \cup C'_4$, and $B_3 \cup B_4$. Letting X be the set of vertices in $V(G) \setminus (N \cup U)$ that are complete to U , it is easy to see that X is a clique and so (U, X) is a 1-join.

Now suppose U has vertices $x \in U_i$ and $y \in U_j$ for some nonequal i and j . Since any vertex in U_i has neighbourhood $\{v_i, v_{i+1}\}$ in H , x does not see every neighbour of y in H , and vice-versa. Thus x and y have the same neighbourhood in N since x sees y and G is claw-free. It follows that every vertex in U has the same neighbourhood X in N , so U is complete to X . Since G is claw-free X is clearly a clique, thus (U, X) is a 1-join. \square

We now prove, in several steps, that if U is contiguous but not a clique, then $\alpha(G) = 3$.

Lemma 8.13. *Suppose U is contiguous but is not a clique. Then $U = B$ and N is a homogeneous clique.*

Proof. Since U is contiguous, we can assume by symmetry that there is a $j \leq 5$ such that $U_i = \emptyset$ precisely if $i > j$. Since U is not a clique, $j \geq 3$.

Suppose that there is a vertex x in B_i for $i \leq j - 1$ and a vertex $y \in U_{i+1}$ such that $N(x) \cap N \subset N(y) \cap N$. Then there is a vertex $z \in N$ that sees y but not x , so x, y , and z form a claw with v_{i+1} , a contradiction. Therefore for $1 \leq i < j$, every vertex in N seen by a vertex in U_{i+1} is seen by every vertex in B_i . By symmetry, for $1 < i \leq j$ every vertex in N seen by a vertex in U_i is seen by every vertex in B_{i-1} . It follows that $U = B$ and every vertex in U has the same neighbourhood in N . Since G is claw-free, $N \cap N(U)$ is a clique. And since U is not a clique, $N \subseteq N(U)$. \square

With this result in hand, we can prove that if U_i is nonempty for all $1 \leq i \leq 5$, then $\alpha(G) \leq 3$ when U is not a clique.

Lemma 8.14. *Suppose U_i is nonempty for $i = 1, \dots, 5$. If U is not a clique, then $\alpha(G) \leq 3$.*

Proof. Assume that U is not a clique and there is a stable set X of size four. Theorem 8.9 tells us that $\alpha(G - N) \leq 3$, so X must contain a vertex of N . Thus X does not intersect B since by the previous lemma $B = U$.

If X contains two vertices of H , then every vertex in $C' \cup D \cup A$ sees at least one of these vertices, so X cannot be a stable set of size four. Suppose then that $X \cap H = \{v_1\}$. There are two vertices of X in $C' \cup D \cup A$, and they must both be in $C'_2 \cup C'_3 \cup D_2$. We already know that $C'_2 \cup D_2$ and $C'_3 \cup D_2$ are cliques, so X

must contain two vertices of $C'_2 \cup C'_3$. But B_2 is nonempty and sees $C'_2 \cup C'_3 \cup N$, contradicting the fact that G is claw-free. Thus $X \cap H = \emptyset$.

Since B_i is nonempty for every i , we can see that $C'_i \cup C'_{i+1}$ is a clique for every i , otherwise there would be a claw containing a vertex of B_i . It follows that $\alpha(C') \leq 2$, so $|X \cap C'| \leq 2$. And since there are at most ten edges between $X \cap (C' \cup D \cup A)$ and H , we see that $X \cap A = \emptyset$ and $|X \cap C'| \geq 2$, so $|X \cap C'| = 2$ and $|X \cap D| = 1$. Thus we can assume that X contains a vertex of C'_1 and a vertex of C'_3 , and since G is claw-free the vertex in $X \cap D$ must be in D_4 . But every vertex of B_1 is complete to $C'_1 \cup D_4 \cup U$, giving us a claw and a contradiction. Therefore X cannot exist. \square

Lemma 8.15. *If U is contiguous and is not a clique, then $\alpha(G) \leq 3$.*

Proof. By the previous lemma, assume that U is not a clique, $j \in \{3, 4\}$, and U_i is nonempty precisely when $i \leq j$. We claim that in this case G has a dominating W_5 , which implies the result by Theorem 8.9.

Take $v_0 \in A$, $u_2 \in U_2$, and $u_3 \in U_3$. The vertices $v_2 u_2 u_3 v_4 v_0$ induce a C_5 in the neighbourhood of v_3 . It is simple to confirm that this 5-wheel is dominating if B_5 is empty. Lemma 8.13 implies $B_5 = U_5 = \emptyset$, so we are done. \square

We have just proved that $\alpha(G) = 3$ if G has a nondominating 5-wheel for which U_i is nonempty precisely when $i \leq j$ for some $j \in \{3, 4, 5\}$. In these cases G contains, as an induced subgraph, a large (i.e. at least ten vertices) induced subgraph of the icosahedron. It is possible to prove that in this situation G is very close to being a thickening of this induced subgraph of the icosahedron. More specifically, G is an *icosahedral thickening*; we define these graphs in the next chapter. We omit the straightforward details of this characterization, which was done by Chudnovsky and Seymour in Section 5 of [CS07c].

Case 2: U is not contiguous

If U is not contiguous there are two cases to consider, by symmetry. In the first case, $U_i \neq \emptyset$ if and only if $i \in \{1, 3\}$. In the second, $U_i \neq \emptyset$ if and only if $i \in \{1, 3, 4\}$. We will prove that these two cases result in two types of 2-join, which we define now.

Definition 8.16. *A 2-join $((X_1, Y_1), (X_2, Y_2))$ is a W_5 2-join if there is an induced 5-hole H in the neighbourhood of some vertex v such that $G_2 \cup X_1 \cup Y_1 = N(H)$ and with respect to H , $U = U_1 \cup U_3$ where U_1 and U_3 are nonempty.*

Definition 8.17. A 2-join $((X_1, Y_1), (X_2, Y_2))$ is a gear 2-join if there is an induced 5-hole H in the neighbourhood of some vertex v such that $G_2 \cup Y_1 = N(H)$ and with respect to H , $U = U_1 \cup U_3 \cup U_4$ where U_1 , U_3 , and U_4 are nonempty.

Galluccio, Gentile, and Ventura [GGV08] investigated gear 2-joins and W_5 2-joins (and introduced the term *gear*), describing facets of the stable set polytope of claw-free graphs with $\alpha(G) \geq 4$.

Denote by N_1 those vertices of N with a neighbour in U_1 , and denote by N_3 those vertices of N with a neighbour in $U_3 \cup U_4$. Denote by U'_1 those vertices of B_1 with a neighbour in $B_3 \cup B_4$.

Lemma 8.18. Suppose U_i is nonempty precisely for $i \in \{1, 3, 4\}$. Then G admits a gear 2-join.

Proof. We claim that

$$((U_1 \cup U'_1, N_3 \cup U'_1), ((B_1 \setminus (U_1 \cup U'_1)) \cup C'_5 \cup C'_1, U_3 \cup U_4))$$

is a 2-join; it is easy to confirm that it is a gear 2-join. Since B_3 is nonempty, D_1 must be complete to A , but since B_1 and B_3 are nonempty there is a claw at D_1 if D_1 is nonempty. Thus by symmetry D_1 and D_4 are both empty. We already know that B_1 is complete to $C'_5 \cup C'_1$. If $v \in U'_1$ then v is complete to $N_3 \cup B_3 \cup B_4$ otherwise there would be a claw at $B_3 \cup B_4$. Furthermore N_3 is complete to $U_3 \cup U_4$. Since no vertex outside B sees a vertex in N , we have the desired gear 2-join. \square

Lemma 8.19. Suppose U_i is nonempty precisely for $i \in \{1, 3\}$. Then G admits a W_5 2-join.

Proof. Now we assume U_4 is empty. In this case, U_1 is complete to B'_1, C'_1, C'_5, D_1 , and D_4 , and anticomplete to $A, C'_2, C'_3, C'_4, D_2, D_3$, and D_5 . Similarly, U_3 is complete to B'_3, C'_3, C'_2, D_3 , and D_1 , and anticomplete to $A, C'_4, C'_5, C'_1, D_4, D_5$, and D_2 . Thus again we have a 2-join:

$$((U_1, U_3), (B'_1 \cup C'_1 \cup C'_5 \cup D_1 \cup D_4, B'_3 \cup C'_2 \cup C'_3 \cup D_3 \cup D_1)).$$

It is easy to confirm that it is a W_5 2-join. \square

Combining the results from this chapter gives us a decomposition theorem, which corresponds to Lemma 3.6 in [OPS08].

Theorem 8.20. *Let G be a connected claw-free graph with $\alpha(G) \geq 4$. If G is not quasi-line then G admits a W_5 1-join or a W_5 2-join or a gear 2-join.*

Proof of Theorem 8.20. Suppose G is claw-free but not quasi-line, and suppose $\alpha(G) \geq 4$. If U is contiguous we know it must be a clique, but then G admits a W_5 1-join. If U is not contiguous then it admits either a W_5 2-join or a gear 2-join. \square

We have just shown that if $\alpha(G) \geq 4$, the 5-wheels of G lie in clusters that can be separated from the rest of the graph by 1-joins and 2-joins. In the next section we will turn this decomposition theorem into a structure theorem.

8.3 Reducing on a W_5 2-join

In this section we extend Theorem 5.15, the structure theorem for quasi-line graphs. We use Theorem 5.15 to prove the following: By allowing “ W_5 strips” and “gear strips” (which we define shortly) as well as fuzzy linear interval strips, we can construct any claw-free graph with $\alpha(G) \geq 4$ that does not admit a 1-join and is not a fuzzy circular interval graph.

Theorem 8.21. *Let G be a claw-free graph with $\alpha(G) \geq 4$. Then one of the following applies:*

- G admits a 1-join
- G is a fuzzy circular interval graph
- G is a composition of fuzzy linear interval strips, W_5 strips, and gear strips.

First we must define W_5 strips and gear strips, which are the simplest strips that result in W_5 2-joins and gear 2-joins under the composition operation.

Let H_1 be the graph on vertices $v_0, v_1, \dots, v_5, u_1, u_3$ such that v_1, \dots, v_5 induce, in order, a C_5 in the neighbourhood of v_0 . Let u_1 and u_3 have neighbourhoods $\{v_1, v_2\}$ and $\{v_3, v_4\}$ respectively (see Figure 8.3). Now suppose G is a claw-free graph containing H_1 as an induced subgraph, such that u_1 and u_3 are simplicial in G and

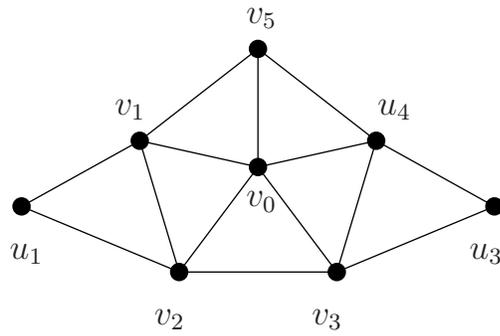


Figure 8.3: The graph H_1 from which W_5 strips are built.

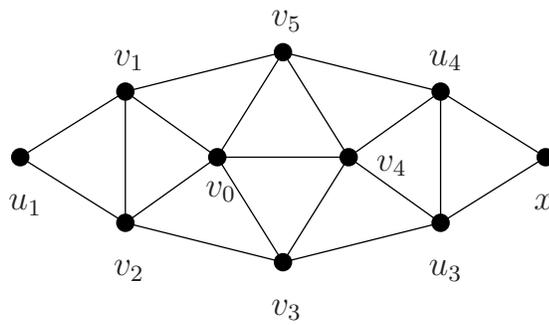


Figure 8.4: The graph H_2 from which gear strips are built.

every vertex in G has a neighbour in $\{v_1, \dots, v_5\}$. Then $(G - \{u_1, u_3\}, N(u_1), N(u_3))$ is a strip and we say it is a W_5 strip.

Construct H_2 from H_1 by adding vertices u_3 and x with neighbourhoods $\{v_4, v_5, u_3\}$ and $\{u_3, u_4\}$ respectively (see Figure 8.4). Suppose G is a claw-free graph containing H_2 as an induced subgraph, such that u_1 and x are simplicial in G and every vertex in G has a neighbour in $\{v_1, \dots, v_5, u_3, u_4\}$. Then $(G - \{u_1, x\}, N(u_1), N(x))$ is a strip and we say it is a *gear strip*.

We are now ready to prove the structure theorem.

Proof of Theorem 8.21. Let G be a minimum counterexample to the theorem. We know that G is not a quasi-line graph, therefore it contains a W_5 which we call H . By the results of the previous section, we know that H is nondominating and that G admits either a W_5 2-join or a gear 2-join, $((X_1, Y_1), (X_2, Y_2))$, separating G_1 from G_2 and such that H is contained in G_2 . We will prove that G is a composition of strips in which the vertices of G_2 arise from a single W_2 strip or gear strip.

To do this we first construct a graph G' from G_1 by adding a path G'_2 on five vertices p_1, \dots, p_5 in order such that p_1 sees all of X_1 , p_5 sees all of Y_1 , and there are no other edges between $G_1 = G'_1$ and G'_2 . Observe first that $\alpha(G') \geq 4$ since H is nondominating in G , and second that $|V(G')| < |V(G)|$. Finally, observe that since G admits no 1-join, G' admits no 1-join: a 1-join in G' would have to be contained in G'_2 , but this is clearly impossible.

Thus by the minimality of G , G' is either a fuzzy circular interval graph or a composition of fuzzy linear interval strips, W_5 strips, and gear strips.

Suppose first that G' is a fuzzy circular interval graph. We make two observations. First, G' contains a hole of length at least five, so it cannot be a fuzzy linear interval graph. Second, no vertex in G'_2 is in a homogeneous pair of cliques, so since G'_2 is a path, the vertices p_1, \dots, p_5 are in consecutive circular order in any circular interval representation of G' . It follows that G_1 is a fuzzy linear interval graph, and it has a linear interval representation with X_1 at the far left and Y_1 at the far right. Therefore G' is a composition of two fuzzy linear interval strips: (G_1, X_1, Y_1) and $(G'_2, \{p_1\}, \{p_5\})$. Thus G is a composition of two strips: (G_1, X_1, Y_1) and (G_2, X_2, Y_2) .

We can therefore assume that G' is not a fuzzy circular interval graph, so it is a composition of fuzzy linear interval strips, W_5 strips, and gear strips. It is straightforward to confirm that no vertex in a W_5 strip or a gear strip has a disconnected

neighbourhood, so no vertex of G'_2 is in a W_5 strip or a gear strip.

Now we claim that there is a strip representation for G' in which one of the strips is $(G'_2, \{p_1\}, \{p_2\})$. To see this, consider a strip representation for G' in which p_i and p_{i+1} are in different fuzzy linear interval strips $(S_1, \{a\}, \{b\})$ and $(S_2, \{c\}, \{d\})$ respectively. Since p_i and p_{i+1} are adjacent, we can assume that $b = p_i$ and $c = p_{i+1}$. Observe that removing $(S_1, \{a\}, \{b\})$ and $(S_2, \{c\}, \{d\})$ from the strip representation and replacing it with $(S_1 \cup S_2, a, d)$ again yields a strip representation for G' . By repeating this argument we can find a strip representation for G' in which every vertex of G'_2 is in the same fuzzy linear interval strip.

So consider this strip representation and let (S, X, Y) be the fuzzy linear interval strip from which the vertices of G'_2 arise. We will break the strip into three smaller strips to get the desired strip representation of G' . There is a fuzzy linear interval representation of S in which p_1, \dots, p_5 appear in consecutive linear order; let S_L and S_R be the vertices of S to the left of p_1 and the right of p_5 respectively. Then it is simple to confirm that replacing (S, X, Y) with the three strips (S_L, X, X_1) , $(G'_2, \{p_1\}, \{p_2\})$, (S_R, Y_1, Y) (omitting any empty strips) gives us a strip representation for G' .

From this strip representation for G' we now construct a strip representation for G . We need only replace $(G'_2, \{p_1\}, \{p_2\})$ with (G_2, X_2, Y_2) . Observe that this is equivalent to replacing G'_2 with G_2 in the graph. Thus G is a composition of fuzzy linear interval strips, W_5 strips, and gear strips. This proves the theorem. \square

Remark: We replace G_2 with a path on five vertices to ensure that $\alpha(G') \geq 4$. The natural thing to do is to replace G_2 with a path on three vertices, in which case G' is actually an induced subgraph of G whether we have a W_5 2-join or a gear 2-join. This would require more careful consideration of compositions of two strips.

8.4 Applying the decomposition

We conclude the chapter with a brief description of two recent applications of the decomposition described in this chapter. The first is an $O(n^6)$ algorithm for finding a maximum weight stable set in a claw-free graph due to Oriolo, Pietropaoli, and Stauffer [OPS08]; the decomposition first appeared in this paper.

Their approach is straightforward, and similar to our approach to fractionally colouring quasi-line graphs. Suppose G is a composition of strips, one of which is (G_s, X_s, Y_s) . A stable set in G intersects G_s in X_s , or Y_s , or both, or neither. For each of these intersection cases, find a maximum weight stable set in G_s subject to the intersection restriction. Call the weights $w_{X,Y}$, $w_{\bar{X},Y}$, $w_{X,\bar{Y}}$, and $w_{\bar{X},\bar{Y}}$, labelled according to whether or not the stable set intersects X_s or Y_s (e.g. the heaviest stable set in G_s intersecting both X_s and Y_s has weight $w_{X,Y}$).

Having computed these values for G_s , they construct G' from G by replacing G_s with a three vertex strip $(G'_s, \{x, z\}, \{y, z\})$, weighting the vertices x , y , and z as $w_{X,\bar{Y}} - w_{\bar{X},\bar{Y}}$, $w_{X,Y} - w_{\bar{X},\bar{Y}}$, and $w_{\bar{X},Y} - w_{\bar{X},\bar{Y}}$, respectively. It is not hard to see that $\alpha_w(G')$, the maximum weight of a stable set in G' , is exactly $\alpha_w(G) - w_{\bar{X},\bar{Y}}$.

It is easy to solve the maximum weight stable set problem for graphs with stability number at most three in $O(n^3)$ time. Thus this approach of strip replacement can be used to reduce the maximum weight stable set problem on claw-free graphs to that on quasi-line graphs.

To solve the problem on quasi-line graphs first note the following. Given a non-linear homogeneous pair of cliques in a quasi-line graph, we can remove an edge from the graph without changing the fact that it is quasi-line or without changing the maximum weight of a stable set. Thus by the structure theorem for quasi-line graphs, only circular interval graphs and compositions of linear interval strips remain. Second, circular interval graphs belong in a class called “distance claw-free”, for which Pulleyblank and Shepherd provided an $O(n^3)$ algorithm for the maximum weighted stable set problem [PS93]. So we are left to deal with compositions of linear interval strips. Given a strip decomposition, we can reduce each strip, thereby reducing the problem to finding a maximum weight stable set in a line graph. This problem is equivalent to finding a maximum weight matching in a graph, so we can apply Edmonds’ matching algorithm [Edm65b] to complete the solution.

In [GGV08], Galluccio, Gentile, and Ventura investigated polyhedral properties of claw-free graphs, in particular the effect of gear strips. The stable set polytope of quasi-line graphs was recently described in two cases by Chudnovsky and Seymour [CS05] and Eisenbrand, Oriolo, Stauffer, and Ventura [EOSV05]. Galluccio, Gentile, and Ventura conjectured that the stable set polytope of a claw-free graph G that

has $\alpha \geq 4$ and is not quasi-line can be described by nonnegativity inequalities, rank inequalities, and two separate types of inequalities that they call *lifted 5-wheel inequalities* and *lifted geared inequalities*. They arise in compositions of strips containing W_5 strips and gear strips, respectively, and it was proved that lifted W_5 inequalities alone are not sufficient [GGV08].

Chapter 9

The structure of claw-free graphs

In this chapter we refine the results of the previous chapter, giving a detailed characterization of skeletal claw-free graphs. We do this by adapting Chudnovsky and Seymour's more general characterization of claw-free trigraphs [CS08b].

We have already shown that compositions of W_5 strips, gear strips, and linear interval strips provide a natural way to decompose claw-free graphs when $\alpha \geq 4$. However, our description of W_5 strips and gear strips is not very detailed, and in particular it is not detailed enough to let us prove our colouring results in the next chapter. Therefore we will discuss Chudnovsky and Seymour's more thorough investigation of the structure of these strips, and that will allow us to bound the chromatic number as desired.

There is one other composition that we will use to build claw-free graphs with a three-colourable complement. These compositions are called *worn hex-joins*, and require several base classes of claw-free graphs that we will define in Section 9.3.2.

We begin by describing some fundamental types of claw-free graphs. These let us characterize W_5 strips, gear strips, and the base classes of worn hex-joins, but they also cover skeletal claw-free graphs that do not arise as a composition of strips or a worn hex-join.

In the next chapter we will use these refined structural results to bound the chromatic number of claw-free graphs.

9.1 Some important types of claw-free graphs

Here we present several fundamental classes of claw-free graphs. We will use them to describe the base classes for our two composition operations.

9.1.1 Long circular interval graphs

We discussed circular interval graphs in Chapter 4. A circular interval graph is a *long circular interval graph* if it has a circular interval representation in which no three intervals cover the entire circle. Note that it is still possible for three intervals to cover all vertices. Just as a fuzzy circular interval graph is a thickening of a circular interval graph, a *long fuzzy circular interval graph* is a thickening of a long circular interval graph. It is not hard to show that if G is a thickening of a long circular interval graph G' under a claw-neutral matching M , then we can assume that each edge of M corresponds to the two extreme vertices of an interval. Using this fact, one can easily see that a skeletal long fuzzy circular interval graph is a long circular interval graph. Chudnovsky and Seymour described the structure of these graphs in depth and provided a forbidden subgraph characterization [CS08a].

9.1.2 Nearly-line thickenings

We now define a slight generalization of line graphs. They are *nearly-line thickenings*, which are special thickenings of line graphs.

Suppose $G = L(H)$ for some graph H . Let M be a matching of G such that for any $uv \in M$, the edges of H corresponding to u and v share an endpoint of degree two in H . In this case M is a claw-neutral matching, so any thickening G' of G under M is claw-free. We call such a graph G' a *nearly-line thickening*.

If M is empty then G' is a line graph, and if M is nonempty then G' admits a linear interval 2-join.

Lemma 9.1. *Every skeletal nearly-line thickening is a line graph.*

This result is based on an important property of skeletal homogeneous cliques. For a thickening of H under a matching M , if $v_1v_2 \in M$ and $(I(v_1), I(v_2))$ is skeletal, then we can simply replace v_1 and v_2 in H by four vertices corresponding to $I(v_1) \cap \Omega(v_1v_2)$,

$I(v_1) \setminus \Omega(v_1v_2)$, $I(v_2) \cap \Omega(v_1v_2)$, and $I(v_1) \setminus \Omega(v_1v_2)$. The resulting graph will be a line graph, just as H is.

Proof. Suppose G , a skeletal nearly-line thickening, is a thickening of a line graph H under a claw-neutral matching M . We choose H to be minimal, so we can assume that H is the line graph of a simple graph J . We proceed by induction on $|M|$. If $M = \emptyset$ then G is clearly a line graph.

Let e_1, e_2 be edges of J corresponding to vertices v_1, v_2 of H with v_1v_2 in M . We construct the graph J' from J as follows. Make two new pendant edges e'_1, e'_2 of J , such that e'_1 (resp. e'_2) is incident to the endpoint of e_1 (resp. e_2) not shared by e_2 (resp. e_1). If $I(v_1) \subseteq \Omega(v_1v_2)$ then delete e'_1 , and if $I(v_2) \subseteq \Omega(v_1v_2)$ then delete e'_2 . Denote $L(J')$ by H' and call the vertices corresponding to e'_1 and e'_2 , v'_1 and v'_2 respectively.

We claim that G is a thickening of H' under $M - \{v_1v_2\}$, which implies the lemma by induction on $|M|$. To see this observe that the thickening will remain the same except on $I(v_1)$, $I(v'_1)$, $I(v_2)$, and $I(v'_2)$. Simply let $I(v'_1) = I(v_1) \setminus \Omega(v_1v_2)$ and let $I(v'_2) = I(v_2) \setminus \Omega(v_1v_2)$. \square

9.1.3 Antiprismatic thickenings

A *triad* is a stable set of size three. A graph G is *antiprismatic* if every triad T contains exactly two neighbours of every vertex in $G - T$. Such graphs are clearly claw-free. Let M be a matching in G such that $G - M$ is also antiprismatic. Then M is claw-neutral in G . If G' is a thickening of an antiprismatic graph G under such a matching M , then we say that G' is an *antiprismatic thickening*.

Just like skeletal nearly-line thickenings, skeletal antiprismatic thickenings behave nicely:

Lemma 9.2. *Every skeletal antiprismatic thickening is a proper thickening of an antiprismatic graph.*

The proof of this result uses the same idea as the proof for nearly-line thickenings.

Proof. Suppose G , a skeletal antiprismatic thickening, is a thickening of an antiprismatic graph H under a matching M . Again we proceed by induction on $|M|$, so

assume M is minimum. Clearly if $M = \emptyset$ then G is a proper thickening of H and we are done.

Take vertices v_1 and v_2 of H such that $v_1v_2 \in M$. Then $(I(v_1), I(v_2))$ is a skeletal homogeneous pair of cliques. Construct H' from H by adding vertices v'_1 and v'_2 with neighbourhoods $\bar{N}(v_1) \setminus \{v_2\}$ and $\bar{N}(v_2) \setminus \{v_1\}$ respectively. We claim that G is a thickening of H' under $M - \{v_1v_2\}$ and that H' is antiprismatic.

To see that H' is antiprismatic, we recall that by the definition of an antiprismatic thickening, $H - v_1v_2$ is antiprismatic. Chudnovsky and Seymour characterized “changeable edges” of this type (§16 in [CS07a]), proving that in this case neither v_1 nor v_2 is in a triad. It easily follows that H' is antiprismatic.

As in the previous proof, to see that G is a thickening of H' under $M - \{v_1v_2\}$ we use the same thickening, except we set $I(v'_1) = I(v_1) \setminus \Omega(v_1v_2)$ and $I(v'_2) = I(v_2) \setminus \Omega(v_1v_2)$. \square

This proof, along with Lemma 6.12, provides a useful corollary:

Corollary 9.3. *Suppose G is an antiprismatic thickening of an antihat H under a matching M of H . Then there is a subgraph G' of G such that G' is a proper thickening of an antihat graph H' , and $\chi(G) = \chi(G')$.*

Antiprismatic thickenings are very easy to define, but describing their structure is a different matter altogether. Chudnovsky and Seymour provided a lengthy and difficult characterization of their structure [CS07a, CS07b], but we do not need to use it in this thesis. In fact our desire to avoid it inspired us to consider both skeletal graphs and the Local Strengthening. Proving the Local Strengthening for these graphs is very easy given Lemma 9.2.

It is important to bear in mind that any graph G with $\alpha(G) \leq 2$ is trivially antiprismatic, just as it is trivially claw-free.

9.1.4 Antihat thickenings

We need to consider certain thickenings of graphs that are nearly antiprismatic. Let $k \geq 2$. We first define a graph H with vertex set $A \cup B \cup C$ as follows. Let $A = \{a_0, a_1, \dots, a_k\}$, $B = \{b_0, b_1, \dots, b_k\}$, and $C = \{c_1, \dots, c_k\}$ be disjoint cliques. Adjacency between the cliques is as follows:

- a_0 has no neighbour outside $A \cup \{b_0\}$, and b_0 has no neighbour outside $B \cup \{a_0\}$.
- For $1 \leq i, j \leq k$, a_i and b_j are nonadjacent if $i \neq j$ and adjacent if $i = j$.
- For $1 \leq i, j \leq k$, a_i and b_i are adjacent to c_j if $i \neq j$, and nonadjacent to c_j if $i = 0$ or if $i = j$.

Let $X \subset A \cup B \cup C \setminus \{a_0, b_0\}$ such that $|C \setminus X| \geq 2$, and let $G = H - X$. We say that G is an *antihat graph*. To define antihat thickenings, we first define a set $M \in V(G)^2$ as follows:

- M is a matching in $G \cup M$ containing no edge of $G[A]$, $G[B]$, or $G[C]$.
- a_0b_0 is in M if a_0 and b_0 are adjacent in G .
- If $1 \leq i, j$ and $a_ib_j \in M$ then $i = j$ and $c_i \in X$.
- If $1 \leq i, j$ and $b_ic_j \in M$ then $i = j$ and $a_i \in X$.
- If $1 \leq i, j$ and $a_ic_j \in M$ then $i = j$ and $b_i \in X$.

In this case $G \cup M$ is claw-free and M is a claw-neutral matching in $G \cup M$. If G' is a thickening of $G \cup M$ under M then we say that it is an *antihat thickening*.

Observe that given an antihat graph G , adding an edge between a_0 and b_0 gives us an antiprismatic graph, as does deleting one or both of a_0 and b_0 .

9.1.5 Icosahedral thickenings

The icosahedron is the unique vertex-transitive graph on twelve vertices in which the neighbourhood of every vertex induces a C_5 . Theorem 8.1 tells us that a claw-free graph with $\alpha \geq 3$ is quasi-line precisely if no neighbourhood contains an induced C_5 , so the icosahedron is the epitome of a claw-free graph that is not quasi-line.

There are several graphs related to the icosahedron that we must treat as a structural exception. These were hinted at in Section 8.2.4. The first is the icosahedron itself, which we define explicitly. Let the graph G_0 have vertices v_0, v_1, \dots, v_{11} . For $i = 1, \dots, 10$, v_i is adjacent to v_{i+1} and v_{i+2} with indices modulo 10. The neighbourhood of v_0 is $\{v_i : 1 \leq i \leq 10, i \text{ is odd}\}$, and the neighbourhood of v_{11} is $\{v_i : 1 \leq i \leq 10, i \text{ is even}\}$. G_0 is the icosahedron (see Figure 9.1).

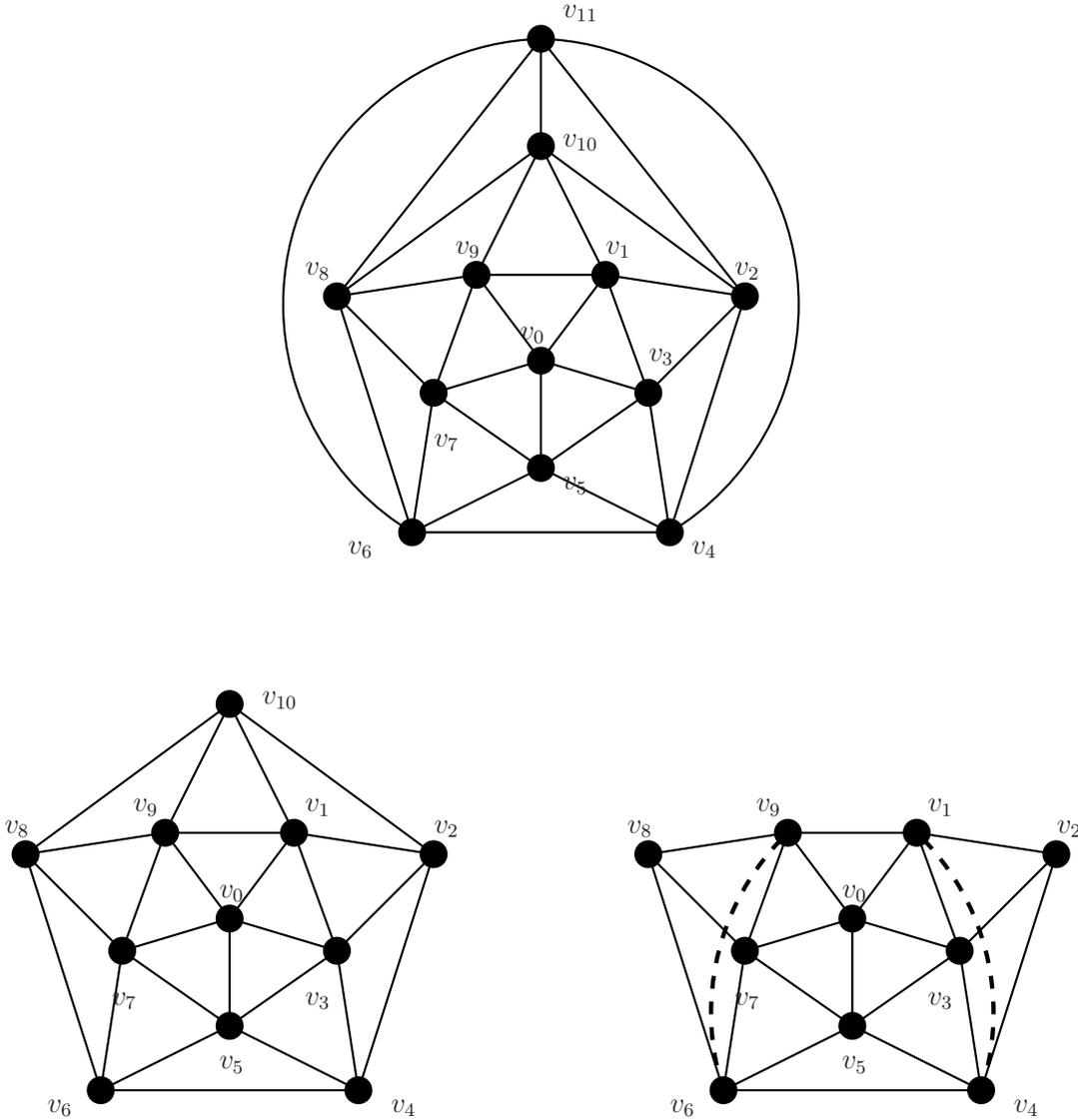


Figure 9.1: The icosahedron G_0 (top), with its derivative graphs G_1 (left) and $G_2 \cup M$ (right). In $G_2 \cup M$, each of $\{v_1, v_4\}$ and $\{v_6, v_9\}$ is a nonadjacent pair or is in M .

We obtain G_1 from G_0 by deleting v_{11} , and we obtain G_2 from G_1 by deleting v_{10} . Note that if $M \in V(G_2)^2$ is a subset of $\{v_1v_4, v_6v_9\}$, then M is a claw-neutral matching in $G_2 \cup M$. We say that G' is an *icosahedral thickening* if it is a proper thickening of G_0 or G_1 , or is a thickening of $G_2 \cup M$ under some $M \subseteq \{v_1v_4, v_6v_9\}$. Any icosahedral thickening G' has $\alpha(G') = 3$ and $\chi(\overline{G'}) = 4$.

9.2 Claw-free graphs with $\alpha \geq 4$

In the previous chapter we proved that any claw-free graph containing a stable set of size four is quasi-line or admits a 1-join or is a composition of fuzzy linear interval strips, gear strips, and W_5 strips. To prove our colouring results we need a better description of W_5 strips and gear strips. We now present such a description that follows from Chudnovsky and Seymour's work [CS08b].

9.2.1 Five types of strips

In total we need to consider five types of strips. The first is linear interval strips, which we already defined and used when considering quasi-line graphs. Now we define the other four. Their structure will be enough to let us prove the Main Conjecture for compositions of strips.

Antihat strips

Let G be an antihat graph, and let G' be an antihat thickening, i.e. a thickening of $G \cup M$ under M as defined in Section 9.1.4. We specify two cliques of G' : $A' = I(A \setminus (X \cup \{a_0\}))$ and $B' = I(B \setminus (X \cup \{b_0\}))$. Then $(G' - I(a_0) - I(b_0), A', B')$ is a strip and if it contains a W_5 we say that it is an *antihat strip*. These antihat strips are a slight generalization of the antihat strips used in Chudnovsky and Seymour's survey [CS05].

Pseudo-line strips

We will define a type of line graph and modify it slightly. Let J be a graph containing a path on vertices j_1, j_2, j_3 in order such that every edge of J is incident to at least one of j_1, j_2, j_3 . Let $H = L(J)$, and for $i \in \{1, 3\}$ let X_i be the set of vertices of H

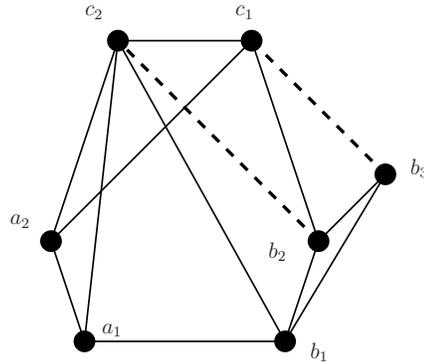


Figure 9.2: The graph underlying strange strips. Dashed lines represent edges in M .

corresponding to edges incident to j_i in J . Both X_1 and X_3 are cliques. Let v_1 and v_2 be the vertices of H corresponding to the edges j_1j_2 and j_2j_3 respectively. Let G be a thickening of H under $M = \{v_1v_2\}$. Then (G, X_1, X_3) is a strip and if it contains a W_5 we say it is a *pseudo-line strip*.

These strips correspond to thickenings of the class \mathcal{Z}_3 defined in [CS08b]. We call the vertices of J other than $\{j_i \mid 1 \leq i \leq 3\}$ *centres* of J .

Strange strips

Let H be a claw-free graph on cliques $A = \{a_1, a_2\}$, $B = \{b_1, b_2, b_3\}$, and $C = \{c_1, c_2\}$ with adjacency as follows: a_1, b_1 are adjacent; c_1 is adjacent to a_2 and b_2 and b_3 ; c_2 is adjacent to a_1, a_2, b_1 , and b_2 . All other pairs are nonadjacent. Let G be a thickening of H under $M = \{b_3c_1, b_2c_2\}$ (see Figure 9.2). Then $(G, I(A), I(B))$ is a strip and we say that it is a *strange strip*.

Gear strips

Now we give a full description of the structure of gear strips – there is only one type. We begin with a graph H on vertices $\{v_1, \dots, v_{10}\}$ with adjacency as follows. The vertices v_1, \dots, v_6 are a 6-hole in order. Next, v_7 is adjacent to v_1, v_2, v_3, v_6 ; v_8 is adjacent to v_3, v_4, v_5, v_6, v_7 ; v_9 is adjacent to $v_3, v_4, v_6, v_1, v_7, v_8$; v_{10} is adjacent to $v_2, v_3, v_5, v_6, v_7, v_8$. There are no other edges in H . Let $X \subseteq \{v_9, v_{10}\}$. See Figure 9.3.

If G is a thickening of $H \setminus X$ under a matching $M \subseteq \{v_7v_8\}$, then $(G, I(v_1) \cup I(v_2), I(v_4) \cup I(v_5))$ is a strip, and we say that it is a *gear strip*. These correspond to

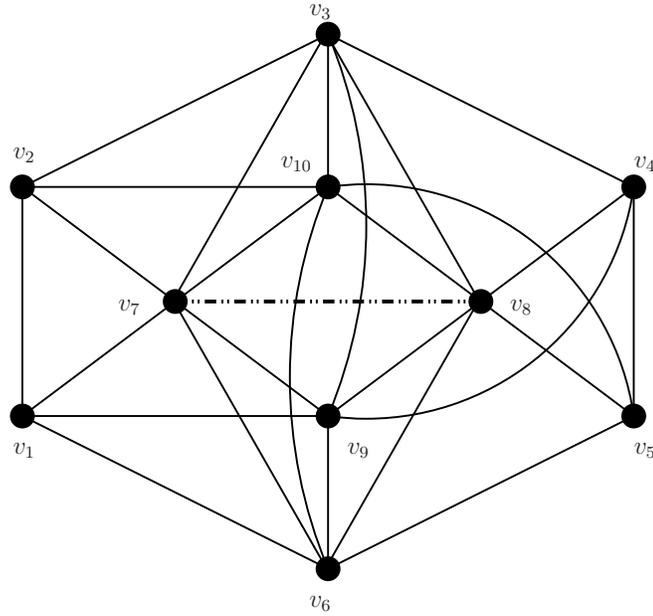


Figure 9.3: The graph underlying gear strips.

thickenings of the class \mathcal{Z}_5 in [CS08b] and are a slight generalization of thickenings of XX-strips as defined in [CS05]. In the definition of \mathcal{Z}_5 there is another vertex with neighbourhood $\{v_1, v_2, v_4, v_5\}$, but we can consider this vertex to be a part of a separate strip.

9.2.2 Five types of 2-joins

Just as we defined interval 2-joins using linear interval strips in Chapter 5, we will define 2-joins corresponding to our other strips. These will allow us to prove the Main Conjecture for these graphs using a minimum counterexample approach, just as we did for quasi-line graphs in Section 7.1.1.

Suppose in our claw-free graph G there is a 2-join $((X_1, Y_1), (X_2, Y_2))$ separating G_1 and G_2 , such that X_1, X_2, Y_1 , and Y_2 are cliques and are pairwise disjoint except for possibly X_1 and Y_1 .

- **Antihat 2-joins.** If (G_2, X_2, Y_2) is an antihat strip then we say that $((X_1, Y_1), (X_2, Y_2))$ is an *antihat 2-join*.

- **Pseudo-line 2-joins.** If (G_2, X_2, Y_2) is a pseudo-line strip then we say that $((X_1, Y_1), (X_2, Y_2))$ is a *pseudo-line 2-join*.
- **Strange 2-joins.** If (G_2, X_2, Y_2) is a strange strip then we say that $((X_1, Y_1), (X_2, Y_2))$ is a *strange 2-join*.
- **Gear 2-joins.** If (G_2, X_2, Y_2) is a gear strip then we say that $((X_1, Y_1), (X_2, Y_2))$ is a *gear 2-join*.

If G arises as a composition of strips, one of which is a pseudo-line strip (G_2, X_2, Y_2) , G will admit a pseudo-line 2-join with X_2 and Y_2 disjoint – we can simply consider their intersection to be in $X_1 \cap Y_1$ instead, as we did with canonical interval 2-joins.

These types of 2-joins allow us to use a decomposition theorem analogous to the structure theorem. We now present these theorems to close the section.

9.2.3 Stating the theorems

Here we state the structure theorem and the decomposition theorem for skeletal claw-free graphs with $\alpha \geq 4$. First is the refinement of Theorem 8.21 with the new characterizations of strips.

Theorem 9.4. *Let G be a skeletal claw-free graph with $\alpha(G) \geq 4$. Then one of the following applies:*

- G admits a 1-join
- G is a circular interval graph
- G is a composition of linear interval strips, antihat strips, pseudo-line strips, strange strips, and gear strips.

This theorem is a corollary of Theorem 7.2 from [CS08b], stated in the terms we have defined in this chapter. Of the fifteen types of strips defined in [CS08b], ten correspond to 1-joins and five correspond to the five types of strips we have just defined.

To attack optimization problems on these graphs we will need a corresponding decomposition theorem:

Theorem 9.5. *Let G be a skeletal claw-free graph with $\alpha(G) \geq 4$. Then one of the following applies:*

- G admits a 1-join
- G is a circular interval graph or a line graph
- G admits a canonical interval 2-join, an antihat 2-join, a pseudo-line 2-join, a strange 2-join, or a gear 2-join.

9.3 Claw-free graphs with $\alpha \leq 3$

We now describe the structure of skeletal claw-free graphs containing no stable set of size four. We must consider two separate cases, depending on whether or not $\chi(\overline{G}) \leq 3$.

9.3.1 Graphs not covered by three cliques

We already have all the tools we need to describe claw-free graphs with $\alpha(G) \leq 3$ and $\chi(\overline{G}) \geq 4$.

Theorem 9.6. *Let G be a skeletal claw-free graph with $\alpha(G) \leq 3$ and $\chi(\overline{G}) \geq 4$. Then one of the following applies:*

- G admits a 1-join
- G is a circular interval graph
- G is a composition of linear interval strips, antihat strips, pseudo-line strips, strange strips, and gear strips
- G is an antiprismatic thickening
- G is an icosahedral thickening.

From this structure theorem we get a corresponding decomposition theorem.

Theorem 9.7. *Let G be a skeletal claw-free graph with $\alpha(G) \leq 3$ and $\chi(\overline{G}) \geq 4$. Then one of the following applies:*

- G admits a 1-join
- G is a circular interval graph or a line graph
- G admits a canonical interval 2-join, an antihat 2-join, a pseudo-line 2-join, a strange 2-join, or a gear 2-join
- G is an antiprismatic thickening
- G is an icosahedral thickening.

9.3.2 Three-cliqued graphs

We now turn our attention to claw-free graphs with $\chi(\overline{G}) = 3$. Given cliques A , B , and C that partition the vertices of G , we say that (G, A, B, C) is a *three-cliqued claw-free graph*. We also sometimes just call G a three-cliqued claw-free graph without specifying a 3-colouring of \overline{G} .

A skeletal three-cliqued claw-free graph either admits a *worn hex-join* or belongs to one of several base classes. We first define these base classes, then we explain how to compose them. For a worn hex-join to result in a skeletal graph, most base graphs must be *weakly skeletal*:

Definition 9.8. *Let (X, Y) be a nonskeletal homogeneous pair of cliques in a three-cliqued graph (G, A, B, C) . Then (X, Y) is weakly skeletal if at least one of X or Y intersects more than one of A , B , and C . We say that (G, A, B, C) is weakly skeletal if every nonskeletal homogeneous pair of cliques is weakly skeletal.*

The following fact justifies our focus on weakly skeletal base graphs; we leave the simple proof to the reader:

Observation 9.9. *Let (X, Y) be a non-weakly-skeletal homogeneous pair of cliques in a three-cliqued graph (G_1, A_1, B_1, C_1) . If (G, A, B, C) is a worn hex-join of (G_1, A_1, B_1, C_1) and any three-cliqued graph (G_2, A_2, B_2, C_2) , then (X, Y) is not skeletal in (G, A, B, C) . In particular, (G, A, B, C) is not skeletal.*

Base classes of three-cliqued claw-free graphs

The first four classes we define correspond to thickenings of members of the classes $\mathcal{TC}_1, \dots, \mathcal{TC}_4$ as defined by Chudnovsky and Seymour [CS08b]. Recall once again that a triad is a stable set of size three.

- **A type of line graph.** Let H be a multigraph with pairwise nonadjacent vertices a, b, c such that each of a, b, c has at least three neighbours, and such that every edge of H has an endpoint in $\{a, b, c\}$. We further insist that for distinct $u, v \in \{a, b, c\}$ there is at most one vertex outside $\{a, b, c\}$ that sees u but not v . Let $G = L(H)$, and let cliques A, B , and C in G correspond to the edges incident to a, b , and c respectively in H . Then (G, A, B, C) is a three-cliqued claw-free graph. Let \mathcal{TTC}_1 be the set of all such three-cliqued graphs such that every vertex is in a triad, with the added condition of being weakly skeletal.
- **Long circular interval graphs.** Let (G, A, B, C) be a three-cliqued long circular interval graph with a circular interval representation such that each of A, B, C is a set of consecutive vertices in circular order. Let \mathcal{TTC}_2 be the set of all such graphs such that every vertex is in a triad.
- **Antihat thickenings.** Let G be an antihat thickening, and let A, B, C , and X be as they are in the definition of G . Let $A' = A \setminus X$ and define B' and C' accordingly. Then $(G - X, I(A'), I(B'), I(C'))$ is a three-cliqued claw-free graph. Let \mathcal{TTC}_3 be the class of all such three-cliqued graphs with the added condition of being weakly skeletal.
- **Antiprismatic thickenings.** Let (G, A, B, C) be a three-cliqued antiprismatic graph, and let $(G', I(A), I(B), I(C))$ be a proper thickening of G . Let \mathcal{TTC}_4 be the class of all such graphs $(G', I(A), I(B), I(C))$.

The final two exceptional cases correspond to thickenings of graphs in Chudnovsky and Seymour's class \mathcal{TC}_5 [CS08b].

- **Exception I.** Let G be a graph on vertices v_1, \dots, v_8 with adjacency as follows: v_1 is adjacent to v_2, v_3, v_6, v_7 ; v_2 is adjacent to v_3, v_4 ; v_3 is adjacent to v_4, v_5 ;

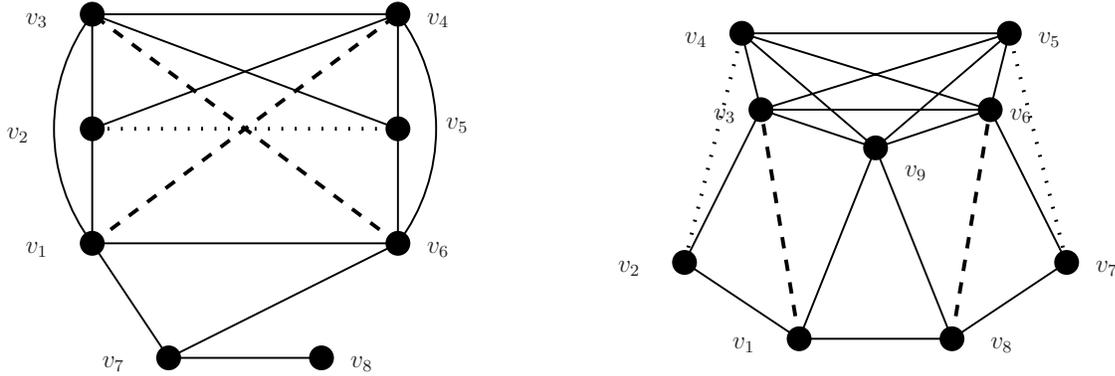


Figure 9.4: The graphs underlying exceptional thickenings in \mathcal{TC}'_5 (left) and \mathcal{TC}''_5 (right). Solid, dashed, and dotted lines represent adjacent vertices, edges in M , and unspecified adjacency respectively. All other pairs are nonadjacent.

v_4 is adjacent to v_5, v_6 ; v_5 is adjacent to v_6 ; v_6 and v_8 are adjacent to v_7 ; v_2 may or may not be adjacent to v_5 . There are no other edges. Now let M be a matching containing v_1v_4 , v_3v_6 , and possibly v_2v_5 . Let $X \subseteq \{v_3, v_4\}$. Let G' be a thickening of $(G \cup M) - X$ under M (see Figure 9.4). Then $(G', I(\{v_1, v_2, v_3\}), I(\{v_4, v_5, v_6\}), I(\{v_7, v_8\}))$ is a three-cliqued claw-free graph. Let \mathcal{TTC}_5 be the set of all such graphs with the added condition of being weakly skeletal.

- **Exception II.** Let G be a graph on vertices v_1, \dots, v_9 with the following structure. Let $A = \{v_1, v_2\}$, $B = \{v_7, v_8\}$, and $C = \{v_3, v_4, v_5, v_6, v_9\}$ be cliques. Let v_1 be adjacent to v_3, v_8 , and v_9 . Let v_8 be adjacent to v_6 and v_9 . Let v_2 be adjacent to v_3 and possibly v_4 . Let v_7 be adjacent to v_6 and possibly v_5 . Now let M be a matching in G containing v_1v_3 and v_6v_8 , as well as possibly v_2v_4 and v_5v_7 (see Figure 9.4). Let X be a subset of $\{v_3, v_4, v_5, v_6\}$ such that:

- v_2 and v_7 each have a neighbour in $C \setminus X$.
- If X contains neither v_4 nor v_5 then v_2 is adjacent to v_4 and v_7 is adjacent to v_5 .

Again we insist that every vertex of $(G - M) - X$ is in a triad. Let G' be a thickening of $G - X$ under M . Then $(G', I(A), I(B), I(C \setminus X))$ is a three-cliqued claw-free graph. Let \mathcal{TTC}_6 be the set of all such graphs with the added

condition of being weakly skeletal.

We allow permutations of the sets A, B, C for any of these classes, so if (G, A, B, C) is in \mathcal{TTC}_i for some $1 \leq i \leq 5$ and $\{A', B', C'\} = \{A, B, C\}$, then (G, A', B', C') is also in \mathcal{TTC}_i .

Having described the building blocks for three-cliqued claw-free graphs, we now move on to how they can be combined.

Worn hex-chains and worn hex-joins

We build three-cliqued claw-free graphs from the base classes we just defined, using a single composition operation: worn hex-chains. First we define a simpler version, *hex-chains*, that we prefer to use whenever possible. These composition operations are closely related to circular interval graphs. For $k \geq 1$ and $1 \leq i \leq k$ let (G_i, A_i, B_i, C_i) be a three-cliqued graph. Suppose we construct G from the disjoint union of G_i , $1 \leq i \leq k$ by adding edges as follows:

- Let $A = \cup_{i=1}^k A_i$, $B = \cup_{i=1}^k B_i$, $C = \cup_{i=1}^k C_i$ be cliques.
- For $1 \leq i, j \leq k$, make $A_i \cup B_j$, $B_i \cup C_j$, and $C_i \cup A_j$ cliques precisely if $j < i$.

Then we call the sequence (G_i, A_i, B_i, C_i) a *hex-chain* for (G, A, B, C) . If $k = 2$, we say that (G, A, B, C) is a *hex-join* of (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2) .

Suppose (G, A, B, C) admits a hex-chain into (G_i, A_i, B_i, C_i) for $1 \leq i \leq k$, and let $J \subseteq \{1, \dots, k\}$ be the set of terms in \mathcal{TTC}_4 . Further suppose that (G', A, B, C) is constructed from (G, A, B, C) by adding edges such that (i) every added edge is between a vertex in G_i and a vertex in G_j for $i \neq j$ both in J , and (ii) no endpoint of an added edge is in a triad in G . Then we say that (G', A, B, C) is a *worn hex-chain* of (G_i, A_i, B_i, C_i) for $1 \leq i \leq k$. If $k = 2$ we call it a *worn hex-join*.

To see why hex-chains generalize circular interval graphs, consider the three-cliqued circular interval graph $(G_c, \{a_1, \dots, a_k\}, \{b_1, \dots, b_k\}, \{c_1, \dots, c_k\})$ such that a_i and b_j (resp. b_i and c_j ; c_i and a_j) are adjacent precisely if $j < i$. Then the hex-chain (G, A, B, C) of (G_i, A_i, B_i, C_i) is the result of taking the disjoint union of these graphs and joining those cliques whose corresponding vertices in G_c are adjacent.

Thus (G, A, B, C) is also a generalization of a fuzzy circular interval graph, where instead of allowing homogeneous pairs of cliques, we allow certain homogeneous triples of cliques that induce claw-free graphs.

Our structure theorem for skeletal three-cliqued claw-free graphs tells us that aside from the complication of worn hex-chains, the minimal homogeneous triples of cliques are in the base classes we just defined:

Theorem 9.10. *Let (G, A, B, C) be a skeletal three-cliqued claw-free graph. Then for some $k \geq 1$, (G, A, B, C) admits a worn hex-chain into terms (G_i, A_i, B_i, C_i) for $1 \leq i \leq k$, each of which is in one of $\mathcal{TTC}_1, \dots, \mathcal{TTC}_6$.*

This theorem is a straightforward weakening of Chudnovsky and Seymour's more precise structure theorem for three-cliqued claw-free trigraphs (4.1 in [CS08b]). Notice that if (G, A, B, C) is a worn hex-join of (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2) and T is a triad in G_1 , then every vertex in G_2 has exactly two neighbours in T . This leads to a useful observation about graphs in \mathcal{TTC}_4 .

Lemma 9.11. *If (G, A, B, C) is a worn hex-join of (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2) , both of which are in \mathcal{TTC}_4 , then (G, A, B, C) is in \mathcal{TTC}_4 .*

Observe that for the worn hex-chain in the previous structure theorem, $(G \setminus G_1, A \setminus A_1, B \setminus B_1, C \setminus C_1)$ admits a worn hex-chain into (G_i, A_i, B_i, C_i) for $2 \leq i \leq k$. If every term is in \mathcal{TTC}_4 then (G, A, B, C) is in \mathcal{TTC}_4 , otherwise we can assume that (G_1, A_1, B_1, C_1) is in one of $\mathcal{TTC}_1, \mathcal{TTC}_2, \mathcal{TTC}_3, \mathcal{TTC}_5, \mathcal{TTC}_6$. This leads us to the desired decomposition theorem:

Theorem 9.12. *Any skeletal three-cliqued claw-free graph (G, A, B, C) not in \mathcal{TTC}_4 admits a hex-join into terms (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2) , where (G_1, A_1, B_1, C_1) is in one of $\mathcal{TTC}_1, \mathcal{TTC}_2, \mathcal{TTC}_3, \mathcal{TTC}_5, \mathcal{TTC}_6$.*

Now that we know the structure of every skeletal claw-free graph, we look at how we can colour them.

Chapter 10

Proving the Main Conjecture for Claw-free Graphs

In this chapter we will prove two results:

Theorem 10.1. *For any claw-free graph G , $\chi(G) \leq \gamma(G)$.*

Theorem 10.2. *For any three-cliqued claw-free graph G , $\chi(G) \leq \gamma_l(G)$.*

That is, the Main Conjecture holds for all claw-free graphs, and the Local Strengthening holds for all three-cliqued claw-free graphs. Our proofs lead to polynomial-time algorithms for finding a colouring that satisfies the bound.

The proof of the Main Conjecture for line graphs, which we presented in Chapter 4, illustrates our general approach to these two theorems. We assume that G is a minimum counterexample and use the decomposition theorems from the previous chapter. If possible, we remove a stable set S from G such that $\gamma_l(G - S) < \gamma_l(G)$ or $\gamma(G - S) < \gamma(G)$. This contradicts the minimality of G . If this is not possible, we use the structure of the graph to prove that $\chi(G) \leq \gamma_l(G)$ or $\chi(G) \leq \gamma(G)$. Actually the situation is sometimes slightly more complicated; when dealing with strips we will ensure that a carefully chosen invariant related to $\gamma_l(G)$ and $\gamma(G)$ behaves appropriately.

10.1 The importance of being skeletal

In Chapter 6 we explained that skeletal claw-free graphs are easier to describe and colour than general claw-free graphs. In the previous chapter we gave a simplified version of Chudnovsky and Seymour's structure theorems by describing only skeletal claw-free graphs. The ease with which we can describe skeletal claw-free graphs certainly helps us with the colouring, but skeletal graphs are also simple to colour for a very specific reason which we now explain.

Our approach to colouring often involves removing a stable set S from a supposedly minimum counterexample G and looking at what happens to a set C of vertices. We generally need to confirm that when we remove S from G , $\max_{v \in C} (d(v) + \omega(v))$ drops by two. We can easily insist that S be a maximal stable set, so $d(v) + \omega(v)$ drops by at least one for every vertex in $C - S$. Thus we only need to worry about vertices in C maximizing $d(v) + \omega(v)$. In particular, if there is a vertex v in C whose closed neighbourhood properly contains the closed neighbourhood of v' , we can safely disregard v' in our analysis. In this case we say that v *trumps* v' .

Now consider the vertices in a homogeneous pair of cliques (A, B) . If (A, B) is skeletal, then all edges between A and B are contained in a clique $\Omega(A, B)$. We can make several simple observations:

1. Any vertex in $A \setminus \Omega(A, B)$ is trumped by any vertex in $A \cap \Omega(A, B)$.
2. Removing a vertex from $A \cap \Omega(A, B)$ lowers $d(v)$ for any $v \in A \cup \Omega(A, B)$.
3. Removing a vertex from $A \cap \Omega(A, B)$ lowers $\omega(v)$ for any $v \in A$.
4. Removing a vertex from $A \cap \Omega(A, B)$ and a vertex from $B \setminus \Omega(A, B)$ lowers $d(v)$ by two for any $v \in B \cap \Omega(A, B)$, and lowers $\omega(v)$ for any $v \in B \setminus \Omega(A, B)$.

These facts will be of great use to us throughout this chapter.

Before moving on we present some terminology and notation that will aid us. For a set of vertices S we define $\Delta(S)$ as $\max_{v \in S} d(v)$. Likewise we define $\omega(S)$ as $\max_{v \in S} \omega(v)$ and $\gamma_l(S)$ as $\max_{v \in S} \gamma_l(v)$. For convenience, when talking about a thickening we often use $\Omega(v_i v_j)$ to denote $\Omega(I(v_i), I(v_j))$.

We begin by proving the Local Strengthening for antiprismatic thickenings. This serves as an illustrative warm-up.

10.2 A warm-up: Antiprismatic thickenings

In Section 2.6 we explained that antiprismatic thickenings motivated the Local Strengthening of the Main Conjecture. The easiest way to prove the Main Conjecture for antiprismatic thickenings is by proving the Local Strengthening. Recall Lemma 9.2, which states that skeletal antiprismatic thickenings are precisely the proper thickenings of antiprismatic graphs. Also note that antiprismatic thickenings are a hereditary class of graphs.

Theorem 10.3. *Let G be an antiprismatic thickening. Then $\chi(G) \leq \gamma_l(G)$.*

Proof of Theorem 10.3. Let G be a minimum counterexample to the theorem. By Corollary 9.3, we can assume that G is a proper thickening of an antihat graph H . By Theorem 2.15 we know that $\alpha(G) = 3$. Let T be any triad in G ; we claim that $\gamma_l(G - T) < \gamma_l(G)$, contradicting the minimality of G .

Suppose $v \notin T$ has only one neighbour in T . Lemma 9.2 tells us that G is a proper thickening of some antiprismatic graph H . By the definition of antiprismatic graphs, v has some twin $u \in T$. Thus T intersects every maximal clique containing v , so if $d(v)$ does not drop by two when T is removed, $\omega(v)$ drops by one and $d(v)$ drops by one. Therefore $\gamma_l(G - T) < \gamma_l(G)$, contradicting the minimality of G . The theorem follows. \square

10.3 Three-cliqued claw-free graphs

To bound the chromatic number of three-cliqued claw-free graphs, we generalize our approach of removing triads, which we just used to deal with antiprismatic thickenings. Three-cliqued claw-free graphs have stability number three unless they are antiprismatic. We will proceed by removing a triad with nice properties whenever possible.

Definition 10.4. *Let T be a triad in a graph G . If every vertex v in $G - T$ has two neighbours in T or a twin in T or is trumped by a vertex in T , then we say that T is a good triad.*

Good triads make very good candidates for a colour class when we are trying to γ_l -colour a graph:

Lemma 10.5. *Let T be a good triad in a graph G . Then $\gamma_l(G - T) < \gamma_l(G)$.*

Corollary 10.6. *No minimum counterexample to Theorem 10.2 contains a good triad.*

Furthermore, good triads behave nicely with respect to worn hex-joins:

Observation 10.7. *Suppose that a three-cliqued claw-free graph (G, A, B, C) admits a worn hex-join into (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2) . If T is a good triad in G_1 , then it is also a good triad in G .*

Let G be a minimum counterexample to Theorem 10.2. Then G is skeletal and is not an antiprismatic thickening. So Theorem 9.12 implies that G admits a worn hex-join into (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2) , where (G_1, A_1, B_1, C_1) is in \mathcal{TTC}_1 , \mathcal{TTC}_2 , \mathcal{TTC}_3 , \mathcal{TTC}_5 , or \mathcal{TTC}_6 . We deal with these three possibilities individually.

First note that G_2 may be empty, but this does not affect our approach. Also note that since G is skeletal, (G_1, A_1, B_1, C_1) is weakly skeletal.

10.3.1 Five classes to consider

We now prove a set of lemmas that together imply Theorem 10.2, dealing with the easier cases first.

Long circular interval graphs (\mathcal{TTC}_2)

Lemma 10.8. *Any three-cliqued graph (G_1, A_1, B_1, C_1) in \mathcal{TTC}_2 contains a good triad.*

Proof. Suppose to the contrary that (G_1, A_1, B_1, C_1) is in \mathcal{TTC}_2 , and call the vertices of G $\{a_1, \dots, a_i, b_1, \dots, b_j, c_i, \dots, c_k\}$ in circular order.

We can find a triad T containing a_1 by adding b_p for the minimum p such that b_p does not see a_1 , then adding c_q for the minimum q such that b_p does not see c_q . The triad T exists since a_1 is in a triad and it follows from the structure of circular interval graphs that a_1 and b_p are in a triad together. If some vertex in $(A \setminus \{a_1\}) \cup \{b_x \mid x < p\}$ does not see both a_1 and b_p , then we are in a degenerate case where G_1 is a linear interval graph, and the vertex in question is a twin of a_1 or b_p , or it is trumped by a_1 or b_p . The same applies to every vertex in $\{b_x \mid x > p\} \cup \{c_y \mid y < q\}$: each vertex has two neighbours in T or a twin in T or is trumped by a vertex in T . Similarly, if

some vertex v in $\{c_l \mid l > q\}$ has only one neighbour in T then it has no neighbours in A , hence it is trumped by or is a twin of c_q . Thus T is a good triad. \square

Antihat thickenings (TTC_3)

Lemma 10.9. *Any three-cliqued graph (G_1, A_1, B_1, C_1) in TTC_3 contains a good triad.*

Proof. Let T be a triad consisting of a vertex a of $I(a_0)$ and vertices b in $I(B \setminus \{b_0\})$ and c in $I(C)$ respectively, following the definition of an antihat thickening. Furthermore if b and c are in $I(b_i)$ and $I(c_i)$ respectively, we insist that T intersects $\Omega(b_i c_i)$. We insist that if $I(a_0) \cap \Omega(a_0 b_0)$ is nonempty, then T intersects it. It is easy to confirm from the structure of an antihat thickening that T exists and is a good triad. \square

Exception I (TTC_5)

Lemma 10.10. *Any three-cliqued graph (G_1, A_1, B_1, C_1) in TTC_5 contains a good triad.*

Proof. Let T be a triad including one vertex in each of $I(v_7)$, $I(v_2)$, and $I(v_5)$, such that T intersects $\Omega(v_2 v_5)$ if it is not empty. It is easy to confirm that T is a good triad, since any vertex in $I(v_2) \setminus \Omega(v_2 v_5)$ is trumped by a vertex in $I(v_3)$ if $v_3 \notin X$, and is trumped by a vertex in $I(v_6)$ if $v_3 \in X$. Similarly any vertex in $I(v_2) \setminus \Omega(v_2 v_5)$ is trumped (see Figure 9.4). \square

Exception II (TTC_6)

Lemma 10.11. *Any three-cliqued graph (G_1, A_1, B_1, C_1) in TTC_6 contains a good triad.*

Proof. Let T be a triad including one vertex in each of $I(v_2)$, $I(v_7)$, and $I(v_9)$, such that T intersects $\Omega(v_2 v_4)$ if it is not empty, and intersects $\Omega(v_5 v_7)$ if it is not empty. It is easy to confirm that T is a good triad (see Figure 9.4). \square

A type of line graph (TTC_1)

We now prove the necessary lemma for G_1 in TTC_1 . This is by far the most difficult case. We make extensive use of the fact that line graphs of bipartite multigraphs are perfect.

Lemma 10.12. *Let (G, A, B, C) be a minimum counterexample to Theorem 10.2 and suppose it admits a worn hex-join into (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2) . Then (G_1, A_1, B_1, C_1) is not in \mathcal{TTC}_1 .*

Proof. Suppose (G_1, A_1, B_1, C_1) is in \mathcal{TTC}_1 . Then G_1 is the line graph of some bipartite multigraph H which has a stable set $\{a, b, c\}$ corresponding to A_1, B_1 , and C_1 . Assume without loss of generality that $|C_1| \leq |B_1| \leq |A_1|$. We call the other vertices of H *centres*. Depending on the structure of G_1 we will take one of two actions:

1. Remove a triad from G_1 , lowering $\gamma_l(G)$.
2. Remove edges from G_1 without changing $\chi(G)$.

Every vertex of G_1 is in a triad. If there are only three centres then removing any triad T will lower $\gamma_l(G)$ since every vertex in $G_1 - T$ will have two neighbours or a twin in T – this can be confirmed easily since the graph underlying H will be a subgraph of $K_{3,3}$. So there are at least four centres. Call the four centres of highest degree w, x, y , and z such that $d(w) \geq d(x) \geq d(y) \geq d(z)$.

For any centre s , denote by A_s the clique corresponding to the edges of H between a and s . Define B_s and C_s accordingly. Denote $A_s \cup B_s \cup C_s$ by X_s . We now consider, for some vertex $v \in A_s$, what cliques of size $\omega(v)$ can contain v . By the structure of a hex-join¹, observe that such a clique must be one of:

- A clique in G_1 intersecting all of A_1, B_1, C_1 . Specifically, $A_s \cup B_s \cup C_s = X_s$.
- A clique in $A_1 \cup B_1 \cup A_2$ containing all of A_2 . Specifically, $A_2 \cup A_s \cup B_s$.
- A clique in $A_1 \cup C_1 \cup C_2$ containing all of C_2 . Specifically, $C_2 \cup A_s \cup C_s$.
- A clique in $A_1 \cup A_2 \cup C_2$ containing all of A_1 . Such a clique has size at least $|A_1| + \max\{|A_2|, |C_2|\}$.

Note also that the closed neighbourhood of v is $A_1 \cup X_s \cup A_2 \cup C_2$. We can make similar observations about the cliques of size $\omega(v)$ when v is in $A_1 \setminus A_s$ or B_1 or C_1 . These observations tell us when removing a triad T lowers $\omega(v)$ and therefore $\gamma_l(v)$.

¹As we noted at the end of the previous chapter, the worn hex-join is a hex-join because every vertex of G_1 is in a triad.

Note that at most two centres have degree $\geq d(a)$, since there are at least four centres and the sum of their degrees is $d(a) + d(b) + d(c)$. Suppose there are at most three centres with degree $\geq d(c)$. Then we can hit them all with a triad T by the definition of \mathcal{TTC}_1 . We will now show that removing T will lower $\gamma_l(v)$ for all $v \in G_1$. This triad T will hit A_1 , B_1 , and C_1 . Any vertex v in G_1 without two neighbours or a twin in T will correspond to an edge in H incident to some centre s , where $d(s) < d(c)$. By our above observations about cliques of size $\omega(v)$, we can see that since $|X_s| < |C_1| \leq |B_1| \leq |A_1|$, any clique of size $\omega(v)$ containing v must contain one of C_1 , B_1 , or A_1 . Therefore such a clique intersects T , so removing T lowers $\omega(v)$ and also $\gamma_l(v)$. This contradicts the minimality of G , so we can assume that there are at least four centres of degree $\geq |C|$, i.e. $d(z) \geq d(c)$. We now consider several cases.

Case 1: $d(w) \geq d(a)$ and c sees w .

Since $d(w) \geq d(a)$ it follows that $d(x) + d(y) + d(z) \leq |B_1| + |C_1|$, and so $d(x) + d(y) \leq |B_1|$ and $|C_1| \leq d(z) \leq \frac{1}{3}(|B_1| + |C_1|)$. Therefore $2|C_1| \leq 2d(z) \leq d(x) + d(y) \leq |B_1|$. Take a triad T that hits X_w , X_x , and X_y , and consider a vertex v for which $\omega(v)$ does not drop when T is removed. Clearly v is not in $X_w \cup X_x \cup X_y$, so it is in X_s for some centre s with $d(s) \leq d(z) \leq \frac{1}{2}|B_1|$. Since $|X_s| < |B_1|$ and $\omega(v)$ does not drop, v must be in C_s . Take some $u \in C_w$. We will show that $d(u) + \omega(u) > d(v) + \omega(v)$, which implies that $\gamma_l(G - T) < \gamma_l(T)$.

Clearly u has at least $|A_1| - |C_1|$ neighbours in $G_1 - C_1$. But v has at most $\frac{1}{2}|B_1| - 1$ neighbours in $G_1 - C_1$. Therefore $d(u) > d(v) + \frac{1}{2}|A_1| - |C_1|$. Recall the structure of maximal cliques containing u and v . If $\omega(v) > \omega(u)$ then either $|C_s| + |A_s| > \max\{|C_w| + |A_w|, |C_1|\}$ or $|C_s| + |B_s| > \max\{|C_w| + |B_w|, |C_1|\}$. But in this case $\omega(v) \leq \omega(u) + d(s) - |C_1| \leq \omega(u) + \frac{1}{2}|A_1| - |C_1|$. It follows that $d(v) + \omega(v) < d(u) + \omega(u)$, completing the case.

Case 2: $d(w) \geq d(a)$ and c does not see w .

Make the subgraph G' of G by removing all edges between C_1 and $G_1 \setminus C_1$ – observe that G' is claw-free and three-cliqued. Further observe that because H has at least four centres, if $G' = G$ then (A_1, B_1) is a nonskeletal homogeneous pair of cliques in G , a contradiction. Thus G' is a proper subgraph of G . We claim that $\chi(G') = \chi(G)$, contradicting the minimality of G . Denote by G'_1 the subgraph of G'

induced on the vertices of G_1 .

Take a $\chi(G')$ -colouring \mathcal{C}' of G' . We will rearrange the colour classes of \mathcal{C}' on G'_1 to reach a proper colouring of G_1 . Denote by t the number of triad colour classes in \mathcal{C}' restricted to G'_1 . Denote by d_{AB} , d_{AC} , and d_{BC} the number of diads (i.e. colour classes of size two) in \mathcal{C}' restricted to G'_1 intersecting A_1 and B_1 , A_1 and C_1 , and B_1 and C_1 respectively. It suffices to show that we can pack the appropriate disjoint stable sets into G_1 . That is, we want to find t triads in G_1 , d_{AB} diads intersecting A_1 and B_1 , d_{AC} diads intersecting A_1 and C_1 , and d_{BC} diads intersecting B_1 and C_1 , such that all of these stable sets are disjoint.

We begin with $|A_1| + |B_1| - d(w)$ diads intersecting A_1 and B_1 . Since $G[A_1 \cup B_1]$ is cobipartite, these diads hit every vertex of $(A_1 \cup B_1) \setminus X_w$. Observe that $|A_1| + |B_1| - d(w) \geq t + d_{AB}$. So we want to extend some of the diads to triads. We can actually extend $|C_1|$ of them. To see this, note that there are at least three triads of degree $\geq |C_1|$ other than w , so every vertex in C_1 has at least $|C_1|$ non-neighbours in $(A_1 \cup B_1) \setminus X_w$. So we have $|A_1| + |B_1| - d(w) - |C_1|$ disjoint diads intersecting A and B and a further $|C_1|$ disjoint triads.

Thus it is clear that we can find the desired disjoint stable sets, beginning with the diads intersecting A and B . When picking our $d_{AC} + d_{BC}$ remaining diads we take a vertex in X_w not intersecting an AB diad whenever possible. Once we have found the necessary diads, we have enough AB diads remaining so that we can extend them to triads. These stable sets give us a $\chi(G')$ -colouring of G , contradicting the minimality of G .

Case 3: $d(w) < d(a)$.

As in the previous case, we remove edges from G_1 without introducing a claw or changing the chromatic number of G . There is at most one clique X in $G[B_1 \cup C_1]$ of size greater than $|B_1|$. If X exists, construct G' from G by removing all edges from G_1 except those within A_1 , B_1 , C_1 , and X . If such an X does not exist, set X as B_1 and construct G' from G by removing all edges from G_1 except those within A_1 , B_1 , and C_1 . It is easy to confirm that G' is claw-free and a proper subgraph of G . We will show that $\chi(G') = \chi(G)$, contradicting the minimality of G .

We claim that there is an $\omega(G_1)$ -colouring of G_1 using $|B_1| + |C_1| - |X|$ triads. To see this, we remove $|A_1| - |X|$ vertices from A_1 one at a time, always taking one

from the largest clique X_s that still has a vertex in A_1 . If after removing k vertices we have disjoint X_s and $X_{s'}$ of size $|A_1| - k$, then we have $|A_1| + |B_1| + |C_1| \geq k + 2(|A_1| - k) + 2|C_1|$, contradicting the fact that there are at least four centres of degree $\geq |C_1|$ in H and $|B_1| \geq |A_1| - k$. Thus we can see that we reach a perfect graph on $|X| + |B_1| + |C_1|$ vertices with clique number $|X|$. In an $|X|$ -colouring of this graph every colour class intersects both A_1 and X , thus the colouring uses exactly $|B_1| + |C_1| - |X|$ triads. The other colour classes are diads intersecting X and A_1 . Thus as in the previous case, we can rearrange the colour classes of a $\chi(G')$ -colouring \mathcal{C}' of G' to construct a $\chi(G')$ -colouring of G . \square

10.3.2 Completing the proof

We now combine our lemmas to prove Theorem 10.2.

Proof of Theorem 10.2. Let (G, A, B, C) be a minimum counterexample to the theorem. Then G is skeletal and is not an antiprismatic thickening. Combining Theorem 9.12 with Lemma 9.11 tells us that (G, A, B, C) admits a worn hex-join into (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2) such that (G_1, A_1, B_1, C_1) is in one of TTC_1 , TTC_2 , or TTC_4 . Lemmas 10.12, 10.8, 10.9, 10.10, and 10.11 tell us that (G_1, A_1, B_1, C_1) cannot be in TTC_1 , TTC_2 , TTC_3 , TTC_5 or TTC_6 respectively. Thus G cannot exist, proving the theorem. \square

10.4 Icosahedral thickenings

Now that we have proved the Local Strengthening for three-cliqued claw-free graphs, we can extend the result to icosahedral thickenings. We do this by removing triads once again, so first we need to consider induced subgraphs of icosahedral thickenings.

Lemma 10.13. *Let G be an icosahedral thickening. Then any induced subgraph G' of G is an icosahedral thickening or contains a clique cutset or admits a canonical linear interval 2-join.*

The proof of this lemma is straightforward. We leave it to the end of this section. This lemma allows us to prove the desired result:

Theorem 10.14. *Suppose G is an induced subgraph of an icosahedral thickening. Then $\chi(G) \leq \gamma_l(G)$.*

Proof. Let G be a minimum counterexample to the theorem. By Lemma 10.13 we know G is an icosahedral thickening or contains a clique cutset or is three-cliqued or admits a canonical interval 2-join. But G is vertex-critical so it cannot contain a clique cutset. Lemma 7.1 and Theorem 10.2 tell us that G is in fact an icosahedral thickening.

First suppose that G is a proper thickening of G_0 , the icosahedron. We remark that the icosahedron is 4-colourable, so we remove four stable sets with union X containing exactly one vertex in $I(v_i)$ for every vertex v_i of G_0 . When X is removed, every remaining vertex v in G loses six neighbours (one of which is a twin), and since every maximal clique in G corresponds to a triangle in G_0 , $\omega(v)$ drops by three. Thus $d(v) + \omega(v)$ drops by nine and it follows that $\gamma_l(G)$ drops by at least four, contradicting the minimality of G .

Now suppose that G is a proper thickening of G_1 (see Figure 9.1). Again we remove one vertex from each $I(v_i)$, this time for $0 \leq i \leq 10$, again using four stable sets. When we remove the vertices, every remaining vertex loses five neighbours, one of which is a twin. And as with G_0 , every vertex v of G has $\omega(v)$ drop by three. Thus $\gamma_l(G)$ drops by at least four, contradicting the minimality of G .

Finally suppose that G is a thickening of $G_2 \cup M$ under a matching M ; we know that $M \subseteq \{v_1v_4, v_6v_9\}$. By minimality of G , $(I(v_1), I(v_4))$ and $(I(v_6), I(v_9))$ are skeletal homogeneous pairs of cliques. We remove two stable sets with union X . One intersects $I(v_1)$, $I(v_4)$, and $I(v_7)$ and intersects $\Omega(v_1v_4)$ if it is not empty. The other intersects $I(v_3)$, $I(v_6)$, and $I(v_9)$ and intersects $\Omega(v_6v_9)$ if it is not empty. These stable sets must exist because neither $I(v_1) \cup I(v_4)$ nor $I(v_6) \cup I(v_9)$ is a clique.

It is straightforward to confirm that X intersects every maximal clique in G , so $\omega(v)$ drops by at least one for every $v \in G - X$, thus $\gamma_l(v)$ drops by at least two for any vertex with three neighbours in X . Observe that any vertex in $G - X$ with only two neighbours in X must be in $(I(v_1) \cup I(v_4)) \setminus \Omega(v_1v_4)$ or $(I(v_6) \cup I(v_9)) \setminus \Omega(v_6v_9)$. Furthermore, every such vertex has a twin in X . Thus we can easily confirm that $\omega(v)$ drops by two for every such vertex. So for any v with only two neighbours in X , $\omega(v)$ drops by two. Therefore $\gamma_l(G - X) \leq \gamma_l(G) - 2$, contradicting the minimality of G . This completes the proof. \square

We now prove Lemma 10.13.

Proof of Lemma 10.13. Suppose first that G is a thickening of $G_2 \cup M$ (see Figure 9.1). If G' has $I(v_i)$ nonempty for all $0 \leq i \leq 9$ then clearly G_2 is an icosahedral thickening. If $I(v_i)$ is empty for some $i \in \{0, 2, 5, 8\}$ then it is not hard to check that G' is three-cliqued. If $I(v_i)$ is empty for some $i \in \{1, 4, 6, 9\}$ then G' contains a clique cutset. If none of these aforementioned sets $I(v_i)$ is empty but one of $I(v_3)$ and $I(v_7)$ is empty, then G' admits a canonical interval 2-join. For example, if G' is reached from G by deleting $I(v_3)$, then $((I(v_0) \cup I(v_9), I(v_5 \cup I(v_6))), (I(v_1), I(v_4)))$ is a canonical interval 2-join.

Now suppose that G is a thickening of G_1 . Obviously G' is an icosahedral thickening if $I(v_i)$ is nonempty for all $0 \leq i \leq 10$. If $I(v_i)$ is empty for any $i \in \{2, 4, 6, 8, 10\}$ then the desired result follows from the previous paragraph. If $I(v_0)$ is empty then G' is a circular interval graph. If $I(v_i)$ is empty for some $i \in \{1, 3, 5, 7, 9\}$ then it is easy to see from Figure 9.1 that G' admits a canonical interval 2-join or a clique cutset.

Finally, suppose that G is a thickening of G_0 . If G' has any $I(v_i)$ empty for $0 \leq i \leq 11$ then the desired result follows from the previous two paragraphs. Otherwise G' is clearly a thickening of G_0 . This completes the proof. \square

10.5 Compositions of strips

In Section 7.1.1 we proved Lemma 7.1, which implies that no minimum counterexample to the Local Strengthening or the Main Conjecture admits a canonical interval 2-join. We now extend this approach, proving that no minimum counterexample to the Local Strengthening or the Main Conjecture admits an antihat, strange, or gear 2-join. Like line graphs, pseudo-line 2-joins present some difficulty with the Local Strengthening. However, later in this section we will prove that a minimum counterexample to the Main Conjecture cannot admit a pseudo-line 2-join. This allows us to prove Theorem 10.1 using our decomposition result, Theorem 9.5.

As with the proof for canonical interval 2-joins, we consider a 2-join $((X_1, Y_1), (X_2, Y_2))$ joining G_1 and G_2 , and we let H_2 denote $G[V_2 \cup X_1 \cup Y_1]$. For $v \in H_2$ we define $\omega'(v)$ as the size of the largest clique in H_2 containing v and not intersecting both $X_1 \setminus Y_1$ and $Y_1 \setminus X_1$, and we define $\gamma_l^j(H_2)$ as $\max_{v \in H_2} [d_G(v) + 1 + \omega'(v)]$ (here the super-

script j denotes 2-join). Observe that $\gamma_l^j(H_2) \leq \gamma_l(G)$. If $v \in X_1 \cup Y_1$, then $\omega'(v)$ is $|X_1| + |X_2|$, $|Y_1| + |Y_2|$, or $|X_1 \cap Y_1| + \omega(G[X_2 \cup Y_2])$.

In light of what we have already done, the following lemma, which is a generalization of Lemma 7.1, deals with antihat 2-joins, strange 2-joins, and gear 2-joins.

Lemma 10.15. *Suppose a skeletal claw-free graph G admits a canonical interval 2-join or an antihat 2-join or a strange 2-join or a gear 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper l -colouring of G_1 for any $l \geq \gamma_l^j(H_2)$, we can find a proper l -colouring of G .*

We split the proof up into three lemmas corresponding to antihat 2-joins, strange 2-joins, and gear 2-joins. Our approach in each case is to set up the colouring of G_1 so that we can do one of two things. When possible, we colour G_2 directly by constructing an auxiliary graph from G_2 and appealing to perfection or Theorem 10.2. If that is not possible then we remove stable sets, reducing $\gamma_l^j(H_2)$ each time, until G_2 becomes degenerate and we can appeal to a previous result.

Lemma 10.16. *Suppose a skeletal claw-free graph G admits an antihat 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper l -colouring of G_1 for any $l \geq \gamma_l^j(H_2)$, we can find a proper l -colouring of G .*

Proof. Consider a minimum counterexample for some fixed l . As in the proof of Lemma 7.1, we can assume that $l = \gamma_l^j(H_2)$. Furthermore if G_2 contains a skeletal homogeneous pair of cliques (A, B) then one of A and B is partially but not completely contained in one of X_2 or Y_2 . We denote $G_2 - X_2 - Y_2$ by Z_2 .

Let k be the number of colours appearing in both X_1 and Y_1 . We begin by making k minimal, as we did in Case 6 of the proof of Lemma 7.1. This minimality of k ensures a bound on l , as long as $k \geq 1$. Let vertices $u \in X_1$ and $v \in Y_1$ have the same colour. Then $d(u) + 1 \geq |X_2| + (l - |Y_1| + k)$, since minimality ensures that u has a neighbour in G_1 of every colour except possibly those in Y_1 not appearing in X_1 . Similarly, $d(v) + 1 \geq |Y_2| + (l - |X_1| + k)$. Therefore since $\omega'(u)$ and $\omega'(v)$ are at least $|X_1| + |X_2|$ and $|Y_1| + |Y_2|$ respectively, $l \geq |X_2| + \frac{1}{2}(l + k + |X_1| - |Y_1|)$ and $l \geq |Y_2| + \frac{1}{2}(l + k + |Y_1| - |X_1|)$. Consequently $l \geq |X_2| + |Y_2| + k$ if $k > 0$.

Suppose there is a colour class S in G_1 hitting X_1 but not Y_1 . Then add to this colour class a stable set S' of size two intersecting Y_2 and Z_2 . By the structure of

antihat thickenings, we can assume that S' intersects $I(b_1)$ and $I(c_1)$ without loss of generality. If $\Omega(b_1c_1)$ is nonempty, we insist that S' intersect it.

Note first that every vertex in $I(b_1) \cup I(c_1)$ is trumped or has a twin in S' or has two neighbours in S' . Every vertex in $Z_2 \setminus I(c_1)$ has two neighbours in S' , as does every vertex in $Y_2 \setminus I(b_1)$. Every vertex in X_2 has a neighbour in S and a neighbour in S' – this neighbour will be in Y_2 for vertices in $I(a_1)$, and in Z_2 for all other vertices in X_2 (recall from the definition of an antihat 2-join that since $I(b_1)$ and $I(c_1)$ are both nonempty, $I(a_1)$ is complete to $I(b_1)$ and anticomplete to $I(c_1)$). Thus $\gamma_l^j(v)$ drops for any $v \in G_2$. Since $S \cup S'$ intersects both $X_1 \cup X_2$ and $Y_1 \cup Y_2$, $\gamma_l^j(v)$ drops for any $v \in X_1 \cup Y_1$. Therefore we remove $S \cup S'$ and lower $\gamma_l^j(H_2)$.

We repeat this approach until either $Y_2 \cup Z_2$ is a clique, or all colours in X_1 appear in Y_1 . Suppose we remove t_1 stable sets in this way. We then take colour classes of G_1 hitting Y_1 but not X_1 , and remove them along with stable sets of size two in $X_2 \cup Z_2$, using the symmetric argument to show that $\gamma_l^j(H_2)$ drops each time. We do this until either all colours appearing in Y_1 are in X_1 , or until $X_2 \cup Z_2$ is a clique. Let t_2 be the number of stable sets we remove in this way, let S_1 be the set of all vertices we have removed from G , and let $t = t_1 + t_2$. Notice that $\gamma_l^j(H_2 - S_1) \leq \gamma_l^j(H_2) - t$.

Suppose $X_1 \setminus S_1$ is empty. Then we can colour $G_2 - S_1$ using $l - t$ colours by Theorem 10.2, since G_2 is three-cliqued. Since $Y_1 \setminus S_1$ is a clique cutset in $G - S_1$, this immediately gives us an $(l - t)$ -colouring of $G - S_1$ and therefore an l -colouring of G . So we can assume $X_1 \setminus S_1$ and symmetrically $Y_1 \setminus S_1$ are nonempty.

Now suppose every colour hitting $Y_1 \setminus S_1$ also hits $X_1 \setminus S_1$. Again we $(l - t)$ -colour $G_2 - S_1$, noting that at most $|X_2| + |Y_2| - t$ colours appear on $(X_2 \cup Y_2) \setminus S_2$ because $|(X_2 \cup Y_2) \setminus S_2| = |X_2| + |Y_2| - t$. We ensure that no colour hits both X_1 and X_2 , and that no colour hits both Y_1 and Y_2 . This is possible because $l - t > |(X_1 \cup X_2) \setminus S_1|$ and $l - t \geq |X_2| + |Y_2| + k - t$, as we proved above. This gives us a proper $(l - t)$ -colouring of $G - S_1$, and therefore an l -colouring of G .

By symmetry, this covers the case in which every colour hitting $X_1 \setminus S_1$ also hits $Y_1 \setminus S_1$. Thus there is a colour in X_1 but not Y_1 , and one in Y_1 but not X_1 . So our method stopped because both $(Y_2 \cup Z_2) \setminus S_1$ and $(X_2 \cup Z_2) \setminus S_1$ are cliques.

In this final case, we $(l - t)$ -colour $G_2 - S_1$ by applying Lemma 7.1 as follows. Notice that $(X_2 \setminus S_1, Y_2 \setminus S_1)$ is a homogeneous pair of cliques in $G - S_1$. We reduce it to a skeletal homogeneous pair of cliques without changing the chromatic number

using Lemma 6.12; the result is a graph G' in which $((X_1 \setminus S_1, Y_1 \setminus S_1), (X_2 \setminus S_1, Y_2 \setminus S_1))$ is a canonical interval 2-join. We can therefore apply Lemma 7.1 to find an $(l - t)$ -colouring of G' . Again using Lemma 6.12, we can construct an $(l - t)$ -colouring of $G - S_1$. This immediately gives us an l -colouring of G , proving the lemma. \square

The next case is strange 2-joins; we use a similar approach.

Lemma 10.17. *Suppose a skeletal claw-free graph G admits a strange 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper l -colouring of G_1 for any $l \geq \gamma_l^j(H_2)$, we can find a proper l -colouring of G .*

Proof. As in the proof of the previous lemma, assume $\gamma_l^j(H_2) = l$ and let k denote the number of colours appearing in both X_1 and Y_1 . We begin by modifying the colouring of G_1 so that k is minimal, so again we can assume that either $k = 0$ or $l \geq |X_2| + |Y_2| + k$. Denote $G_2 - X_2 - Y_2$ by Z_2 .

Let $t = \min\{|I(a_1)|, |I(c_1) \cap \Omega(c_1, b_3)|, |Y_2| - k\}$. We remove t colours hitting Y_1 but not X_1 . With each colour class we remove a vertex of $I(a_1)$ and a vertex of $I(c_1) \cap \Omega(c_1, b_3)$. Together these vertices form t stable sets; call their union S_1 . As in the proof of the previous lemma, we now consider our situation depending on the value of t . Note that each time we remove a stable set, every vertex in G_2 is either trumped or loses two neighbours or loses a twin. It is therefore easy to see that $\gamma_l^j(H_2 - S_1) \leq \gamma_l^j(H_2) - t$.

Suppose $I(a_1)$ is empty. We apply Lemma 7.1 to $(l - t)$ -colour $G - S_1$ as follows. First observe that removing S_1 turns $((X_1 \setminus S_1, Y_1 \setminus S_1), (X_2 \setminus S_1, Y_2 \setminus S_1))$ into a fuzzy linear interval 2-join, in which $(Z_2 \setminus S_1, Y_2 \setminus S_1)$ is a homogeneous pair of cliques. We reduce this homogeneous pair to a skeletal homogeneous pair of cliques using Lemma 6.12, at which point $((X_1 \setminus S_1, Y_1 \setminus S_1), (X_2 \setminus S_1, Y_2 \setminus S_1))$ becomes a canonical linear interval 2-join in a graph G' . We can therefore apply Lemma 7.1 to G' , since we already have an $(l - t)$ -colouring of $G_1 - S_1$, to find an $(l - t)$ -colouring of G' . Lemma 6.12 tells us that we can use this colouring to construct an $(l - t)$ -colouring of $G - S_1$. Combining this with a t -colouring of $G[S_1]$ gives us an l -colouring of G .

Now suppose $I(c_1) \cap \Omega(c_1, b_3)$ is empty but $I(c_1)$ is not empty. To $(l - t)$ -colour $G - S_1$, we first remove the vertices of $I(b_3)$, which have become simplicial. Now observe that $((X_1 \setminus S_1, Y_1 \setminus S_1), (X_2 \setminus S_1, Y_2 \setminus (I(b_3) \cup S_1)))$ is an antihat 2-join. The remaining sets of G_2 are $I(a_1)$, $I(a_2)$, $I(b_1)$, $I(b_2)$, $I(c_1)$, and $I(c_2)$. To see the antihat

2-join, we relabel these sets as in the definition of an antihat thickening. Respectively, these sets become $I(a_1)$, $I(a_2)$, $I(b_1)$, $I(b_3)$, $I(c_3)$, and $I(c_1)$ (see Section 9.1.4). We can therefore apply Lemma 10.16 to find an $(l-t)$ -colouring of $G - (S_1 \cup I(b_0))$, then replace and colour the simplicial vertices in $I(b_0)$ to get an $(l-t)$ -colouring of $G - S_1$. This gives us an l -colouring of G , completing the case of strange 2-joins. \square

The final and most difficult case is that of gear 2-joins.

Lemma 10.18. *Suppose a skeletal claw-free graph G admits a gear 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper l -colouring of G_1 for any $l \geq \gamma_l^j(H_2)$, we can find a proper l -colouring of G .*

Proof. We proceed by induction on $|G|$, taking as our basis the trivial case in which $\min\{|X_1|, |Y_1|\} = 0$; in this case we have a 1-join and the result follows from Theorem 10.2 since gear strips are three-cliqued. So assume both X_1 and Y_1 are nonempty. Let Z_2 denote $G_2 \setminus (X_2 \cup Y_2)$. Again we can let G be a minimum counterexample and assume that $l = \gamma_l^j(H_2)$.

In this case we make k , the overlap between X_1 and Y_1 in the colouring of G_1 , maximal.

Case 1: $k > 0$.

If $k > 0$, we remove a colour class hitting both X_1 and Y_1 , along with one vertex each of $I(v_9)$ and $I(v_{10})$, if they are both nonempty. In this case every vertex of G_2 loses a twin or two neighbours. Since we remove a vertex in both X_1 and Y_1 , it is easy to see that $\gamma_l^j(H_2)$ drops. Since removing vertices from $I(v_9)$ and $I(v_{10})$ will not change the fact that we have a gear 2-join, we can proceed by induction, having reduced both $\gamma_l^j(H_2)$ and l .

So assume that $I(v_9) \cup I(v_{10})$ is a clique, i.e. one of $I(v_9)$ and $I(v_{10})$ is empty. We do the same thing, but instead we remove a colour class hitting both X_1 and Y_1 , along with a vertex of $I(v_3)$ and a vertex of $I(v_6)$. Clearly $\gamma_l^j(H_2)$ drops as before and we can proceed by induction, since as long as neither $I(v_3)$ nor $I(v_6)$ becomes empty we will still have a gear 2-join.

Suppose $I(v_6)$ becomes empty, and one of $I(v_9)$ and $I(v_{10})$ is empty. By symmetry we can assume that $I(v_9)$ is empty. We are now left with a fuzzy linear interval 2-join. The vertices, in linear order, are $I(v_1)$, $I(v_2)$, $I(v_7)$, $I(v_3)$, $I(v_{10})$, $I(v_8)$, $I(v_4)$,

$I(v_5)$, and the possibly nonlinear homogeneous pairs of cliques are $(I(v_7), I(v_8))$ and $(I(v_3) \cup I(v_{10}), I(v_4) \cup I(v_5))$. The reader can confirm this, along with symmetry between v_9 and v_{10} , by consulting Figure 9.3.

So, as in the proof of the previous two lemmas, we can find our l -colouring of G by reducing on these two homogeneous pairs of cliques and invoking Lemma 7.1.

This completes the proof of the lemma when $k > 0$.

Case 2: $k = 0$; $l > |X_1| + |Y_1|$.

In this case we remove a colour class hitting neither X_1 nor Y_1 , along with a stable set of size three in G_2 . Call their union S . If $I(v_{10})$ is nonempty, we remove a vertex of $I(v_{10})$ along with one vertex each of $I(v_1)$ and $I(v_4)$. Every vertex in G_2 loses a twin or two neighbours, so it is easy to confirm that $\gamma_l^j(H_2)$ drops. Thus we can proceed by induction, provided that both $I(v_1)$ and $I(v_4)$ are still nonempty.

If $I(v_1)$ and $I(v_4)$ are both empty, then we extend the colouring of G_1 to an l -colouring of $G_1 \cup I(v_2) \cup I(v_5)$. We then note that $((I(v_2) \cup I(v_{10}) \setminus S, I(v_5) \cup I(v_{10}) \setminus S), (I(v_3) \cup I(v_7), I(v_6) \cup I(v_8)))$ is a fuzzy linear interval 2-join, in which $(I(v_3) \cup I(v_7), I(v_6) \cup I(v_8))$ is the only possible nonlinear homogeneous pair of cliques. So we can construct an $(l - 1)$ -colouring of $G - S$ by Lemma 7.1 as in the previous two proofs. This gives us an l -colouring of S .

So assume $I(v_1)$ is now empty but $I(v_4)$ is not. Clearly we can extend the $(l - 1)$ -colouring of $G_1 - S$ to a proper $(l - 1)$ -colouring of $(G_1 - S) \cup I(v_2)$. We claim that we now have an antihat 2-join and we can find an $(l - 1)$ -colouring of $G - S$ using Lemma 10.16.

The 2-join in $G - S$ is $((I(v_2), Y_1 \setminus S), ((I(v_3) \cup I(v_7)) \setminus S, Y_2 \setminus S))$. To see that $(G_2 - S) - (I(v_1) \cup I(v_2))$ is an antihat strip, we will relabel the vertices to conform with the definition of an antihat thickening. We relabel the sets $I(v_3)$, $I(v_{10})$, and $I(v_7)$ as $I(a_1)$, $I(a_2)$, and $I(a_3)$ respectively. We relabel $I(v_4)$ and $I(v_5)$ as $I(b_1)$ and $I(b_2)$ respectively. Finally, we relabel $I(v_6)$, $I(v_9)$, and $I(v_8)$ as $I(c_1)$, $I(c_2)$, and $I(c_3)$ (or $I(c_4)$ if $I(v_7) \cup I(v_8)$ is a clique) respectively. It is straightforward to confirm that this is an antihat strip. We therefore have an antihat 2-join in $G - S$, so by Lemma 10.16 we can find an $(l - 1)$ -colouring of $G - S$ and an l -colouring of G .

If $I(v_{10})$ is empty, then instead of taking vertices from $I(v_{10})$, $I(v_1)$ and $I(v_4)$, we take vertices from $I(v_1)$, $I(v_3)$ and $I(v_5)$, and proceed symmetrically. This time, we

may worry that $I(v_3)$ will become empty, but in this case, since $I(v_{10})$ is also empty, we get a fuzzy linear interval 2-join exactly as in Case 1.

Case 3: $k = 0$; $l = |X_1| + |Y_1|$.

In this final case, every colour appears in $X_1 \cup Y_1$, and no colour appears twice. Therefore X_2 and Y_2 must receive colours appearing in Y_1 and X_1 respectively. Since k is maximal, $l \geq |X_2| + |X_1| + \frac{1}{2}|Y_1|$ (from a vertex in X_1), and $l \geq |Y_2| + |Y_1| + \frac{1}{2}|X_1|$ (from a vertex in Y_1). It follows that $2l \geq \frac{3}{2}(|X_1| + |Y_1|) + |X_2| + |Y_2|$, so $|X_2| + |Y_2| \leq \frac{1}{2}l$.

Notice that Z_2 is cobipartite, and that the only non-edges in Z_2 are in $I(v_3) \cup I(v_6)$, $I(v_7) \cup I(v_8)$, and $I(v_9) \cup I(v_{10})$. We begin with an optimal colouring of Z_2 , removing the colour classes of size two. Let t_1 be the number of such colour classes in $I(v_3) \cup I(v_6)$, and let t be the total number of such colour classes. Denote these $2t$ vertices by S , noting that $Z_2 - S$ is a clique.

We construct an auxiliary graph G' from $G_2 - S$ by adding all possible edges between X_2 and Y_2 . Now G' is cobipartite and perfect, and since a proper colouring of G' will give vertices in X_2 and Y_2 distinct colours, it suffices to prove that $\omega(G') \leq l - t$. This gives us an l -colouring of G_2 in which no colour appears twice on $X_2 \cup Y_2$, so we can use it to extend the l -colouring of G_1 to an l -colouring of G .

Suppose there is a clique W of size greater than $l - t$ in G' . We will now prove that $l - |X_2| - |Y_2| \geq \frac{1}{2}|Z_2| \geq t$, which implies that W cannot be $X_2 \cup Y_2$. Consider vertices u, v, x, y in $I(v_1)$, $I(v_2)$, $I(v_4)$, and $I(v_5)$ respectively. Since every vertex in Z_2 has two neighbours in this set, the sum of the four degrees is at least $2(|X_1| + |X_2| + |Y_1| + |Y_2| + |Z_2|) - 4$. Therefore the sum $\gamma_l^j(u) + \gamma_l^j(v) + \gamma_l^j(x) + \gamma_l^j(y)$ is at least $4l \geq 2(|X_1| + |X_2| + |Y_1| + |Y_2|) + |Z_2|$. Thus $2l \geq |Z_2| + 2(|X_2| + |Y_2|)$, so $\frac{1}{2}|Z_2| + |X_2| + |Y_2| \leq l$.

A maximal clique W in G' intersecting both $I(v_1)$ and $I(v_2)$ as well as Z_2 must be $(I(v_1) \cup I(v_2) \cup I(v_7)) \setminus S$. But a vertex v in $(I(v_7) \cap \Omega(v_7, v_8)) \setminus S$ (this set is nonempty because $(I(v_7), I(v_8))$ is a skeletal homogeneous pair) has either two neighbours or a twin in each stable set of size two in S . This means that if $|W| > l - t$, then $\gamma_l^j(v) > l$, a contradiction. So W is not such a clique, and by symmetry W does not intersect all three of $I(v_4)$, $I(v_5)$, and $I(v_8)$. A similar argument implies that W cannot intersect only one of $I(v_1)$, $I(v_2)$, $I(v_4)$, and $I(v_5)$. Since $|X_2| + |Y_2| \leq l - t$ we can see that W cannot intersect three of these sets. Furthermore $|Z_2 - S| = \omega(Z_2) - t \leq l - t$, so

W cannot be contained in $Z_2 - S$. Therefore W intersects all three of X_2 , Y_2 , and Z_2 , and we can assume by symmetry that W is $I(v_4) \setminus S$ and its neighbourhood in $X_2 \cup Y_2$, i.e. $(I(v_2) \cup I(v_3) \cup I(v_4)) \setminus S$.

Suppose that $|W| > l - t$. This inequality will provide us with new bounds on l , giving us a contradiction and completing the proof of the lemma. Let u and v be vertices in $I(v_2)$ and $I(v_4)$ respectively. Observe that $d(u) + 1 \geq |X_1| + |X_2| + |I(v_3) \setminus S| + t$, since u sees one vertex in every stable set in S . Thus $d(u) + 1 \geq |X_1| + |X_2| + |I(v_3)| + (t - t_1)$, and likewise $d(v) + 1 \geq |Y_1| + |Y_2| + |I(v_3)| + (t - t_1)$. Since $I(v_2) \cup I(v_3) \cup I(v_{10}) \cup I(v_7)$ is a clique, it follows that $\omega'(u) \geq |I(v_2)| + |I(v_3)| + (t - t_1)$, because every stable set of S hits $I(v_3) \cup I(v_7) \cup I(v_{10})$ exactly once. Likewise, $\omega'(v) \geq |I(v_4)| + |I(v_3)| + (t - t_1)$. The sum of these figures is at most $2\gamma_l^j(u) + 2\gamma_l^j(v)$, which is at most $4l$. This implies:

$$4l \geq (|X_1| + |Y_1|) + (|X_2| + |Y_2|) + 4(t - t_1) + 4|I(v_3)| + |I(v_2)| + |I(v_4)|.$$

We know that $|X_1| + |Y_1| = l$, $|X_2| + |Y_2| > |I(v_2)| + |I(v_4)|$, and by assumption, $|I(v_2)| + |I(v_4)| + |I(v_3)| - t_1 > l - t$. Therefore,

$$\begin{aligned} 3l &\geq 2(|I(v_2)| + |I(v_4)| + |I(v_3)|) + 2|I(v_3)| + 4(t - t_1) \\ &\geq 2l + 2|I(v_3)| + 2(t - t_1) \end{aligned}$$

Thus $|I(v_3)| - t_1 \leq \frac{l}{2} - t$. And since $|X_2| + |Y_2| \leq \frac{l}{2}$, we get $|W| = |I(v_2)| + |I(v_3)| + |I(v_4)| - t_1 \leq l - t$, contrary to our assumption.

It follows that $\omega(G') \leq l - t$, so we can indeed complete the l -colouring of G_2 that is compatible with the colouring of G_1 . This proves the lemma. \square

Lemmas 10.16, 10.17, and 10.18 together immediately imply Lemma 10.15.

To deal with pseudo-line 2-joins we use another invariant, which is essentially a global analogue to the local γ_l^j . Define $\omega'(H_2)$ as $\max_{v \in H_2} \omega'(v)$, and define $\gamma_g^j(H_2)$ as $\lceil \frac{1}{2}(\Delta_G(H_2) + 1 + \omega'(H_2)) \rceil$. Note that $\Delta_G(H_2)$ is the maximum degree in G over all vertices in H_2 . Clearly $\gamma_l^j(H_2) \leq \gamma_g^j(H_2) \leq \gamma(G)$.

Lemma 10.19. *Suppose a skeletal claw-free graph G admits a canonical interval 2-join or an antihat 2-join or a strange 2-join or a gear 2-join or a pseudo-line 2-join*

$((X_1, Y_1), (X_2, Y_2))$. Then given a proper l -colouring of G_1 for any $l \geq \gamma_g^j(H_2)$, we can find a proper l -colouring of G .

Proof. We prove the lemma by induction on l . We let G be a minimum counterexample, noting that $l = \gamma_g^j(H_2)$. Assume that $|X_1| \geq |Y_1|$.

If $((X_1, Y_1), (X_2, Y_2))$ is a canonical interval 2-join or an antihat 2-join or a strange 2-join or a gear 2-join, then the lemma is immediately implied by Lemma 10.15 given the observation that $\gamma_i^j(H_2) \leq \gamma_g^j(H_2)$. So we can assume that we have a pseudo-line 2-join.

Recall that G_2 is based on the line graph of a graph J , and the vertices of J other than j_1, j_2 , and j_3 are called *centres*. For a centre t in J , we call the corresponding clique C_t . That is, $C_t = \cup I(e)$ over all vertices e of H whose corresponding edge in J is incident to t . Let the edges j_1j_2 and j_2j_3 be e_1 and e_2 respectively. Note that Z_2 is a clique and so is $Z_2 \cup \Omega(e_1, e_2)$.

We begin by making the number k of colours in G_1 that hit both X_1 and Y_1 maximal. First suppose that there is no colour class appearing in neither X_1 nor Y_1 . As in the previous case, $l > |X_1|$. Since k is maximal, there is a vertex $v \in X_1$ with a colour not appearing in Y_1 , and it must have at least $l - 1$ neighbours in G_1 . This vertex is in $X_1 \cup X_2$, so $l = \gamma_g^j(H_2) \geq \frac{1}{2}l + \frac{1}{2}|X_1| + |X_2|$. Hence $l \geq |X_1| + 2|X_2|$. Since $l = |X_1| + |Y_1| - k$, we have $|X_2| \leq \frac{1}{2}|Y_1| - \frac{1}{2}k$. Now since $|X_2|$ is nonempty, $|Y_1| > k$ and there is a vertex in Y_1 with a colour not appearing in X_1 . We can therefore apply the symmetric argument to prove that $l \geq \frac{1}{2}l + \frac{1}{2}|Y_1| + |Y_2|$, and consequently $|Y_2| \leq \frac{1}{2}|X_1| - \frac{1}{2}k$.

Observe that if $|Z_2| \leq \frac{1}{2}(|X_1| + |Y_1|)$ we can easily finish the colouring by giving X_2 colours appearing in Y_1 but not X_1 , Y_2 colours appearing in X_1 but not Y_1 , and Z_2 colours appearing in both X_1 and Y_1 , and any leftover colours. In fact we can do this whenever $|Z_2| \leq l - |X_2| - |Y_2|$. So assume $|Z_2| > l - |X_2| - |Y_2|$. Let A be a maximum clique in $G[X_2 \cup Z_2]$. Since $G[X_2 \cup Z_2]$ is cobipartite, we can colour it with $|A|$ colours, $|X_2|$ of which intersect X_2 . Therefore if $|A| \leq l - |Y_2|$ we can colour Y_2 using colours that appear in X_1 but not in Y_1 , then colour X_2 and Z_2 using $|A|$ colours such that those colours appearing in X_2 do not appear in X_1 .

To see that $|A| \leq l - |Y_2|$, note that $\omega'(H_2) \geq |A|$ and since the degree of any vertex in $I(e_1)$ is at least $|X_1| + |X_2| + |Z_2| - 1$, $l = \gamma_g^j(H_2) \geq \frac{1}{2}(|A| + |Z_2| + |X_2| + |X_1|)$. Since $|Z_2| > l - |X_2| - |Y_2|$, this implies that $l > |A| + |X_1| - |Y_2| \geq |A| + \frac{1}{2}|Y_2|$.

Therefore $|A| \leq l - |Y_2|$ and we can complete the $\gamma_g^j(G)$ -colouring of G .

We can now assume that there is a colour class S in G_1 that appears in neither X_1 nor Y_1 . We will find a stable set S_2 in G_2 such that removing $S \cup S_2$ lowers $\gamma_g^j(H_2)$; this will imply that $\chi(G) \leq l$ by induction.

First note that if there are at most two centres then we actually have an antihat 2-join – this is straightforward to confirm as there are only five vertices in J . So we can assume that there are at least three centres.

Suppose we set S_2 to be a diad (i.e. a stable set of size two) in $G[I(e_1) \cup I(e_2)]$ such that S_2 intersects $\Omega(e_1, e_2)$ if it is nonempty. S_2 exists because $G[I(e_1) \cup I(e_2)]$ is not a clique. If removing $S \cup S_2$ does not lower $\omega_j(G)$, then there must be a maximal clique in G_2 disjoint from S_2 . Such a clique must be C_t for some centre t that sees j_1, j_2 , and j_3 in J .

The size of C_t must be at least $\max\{|X_1 \cup X_2|, |Y_1 \cup Y_2|, |Z_2|\} > \frac{1}{3}|V(G_2)|$, so by the number of vertices in G_2 there can be at most two such “centre cliques” of size $\omega'(H_2)$, since they must be disjoint – call the other one $C_{t'}$ if it exists. If we let S_2 be a stable set corresponding to a matching in J that hits three centres and in particular hits t and (if it exists) t' , we can see that removing $S \cup S_2$ lowers $\omega_j(G)$ so we are done. This S_2 must exist because C_t intersects all of X_2, Y_2 , and Z_2 , so we can find S_2 unless every other centre has neighbourhood j_2 in J . If this is the case we can again easily confirm that we have an antihat 2-join, so we are done. \square

We can now prove Theorem 10.1:

Proof of Theorem 10.1. Let G be a minimum counterexample. Then it is skeletal and contains no clique cutset. Furthermore, G is not three-cliqued or an icosahedral thickening or an antihat thickening or a quasi-line graph. Therefore by Theorem 9.5, G admits an antihat 2-join or a pseudo-line 2-join or a strange 2-join or a gear 2-join. Thus Lemma 10.19 contradicts the minimality of G , proving Theorem 10.1. \square

10.6 Algorithmic Considerations

We now show that our proofs of Theorems 10.1 and 10.2 yield polynomial time algorithms for $\gamma(G)$ - and $\gamma_l(G)$ -colouring G , respectively.

By Theorem 6.11 we can assume that G is skeletal. Furthermore we can identify sets of twin vertices in G in polynomial time. This immediately implies that we can recognize skeletal antiprismatic thickenings and skeletal icosahedral thickenings in polynomial time. We deal with these two simple cases first.

If G is an antiprismatic thickening, we can repeatedly remove triads from G , each time lowering $\gamma_l(G)$. These triads will be colour classes. What remains is a graph containing no triad; we can colour these optimally in polynomial time, since an optimal colouring corresponds to a maximum matching in the complement.

If G is an icosahedral thickening, then observe that since G is skeletal there are at most 12 equivalence classes of twin vertices; therefore there are at most 12^3 different types of stable sets. We can formulate the problem of colouring G as an integer program in which each variable represents the number of stable sets of a given type we use in the colouring. Each variable has size at most n , so we can exhaustively solve the problem in $O(n^{12^3})$ time. This is clearly not optimal.

We now consider the problem of colouring three-cliqued claw-free graphs.

10.6.1 Three-cliqued graphs

Maffray and Preissmann proved that it is NP -complete to decide whether or not a triangle-free graph is three-colourable [MP96]. Consequently it is NP -complete to decide whether or not a claw-free graph is three-cliqued. This makes dealing with three-cliqued claw-free graphs a slightly delicate issue. However, consider a claw-free graph G . If $\alpha(G) \leq 2$ we know we can optimally colour it in polynomial time. We will show that if $\alpha(G) = 3$, then in polynomial time we can either $\gamma_l(G)$ -colour G , or determine that G is not three-cliqued.

To prove that we can $\gamma_l(G)$ -colour skeletal three-cliqued claw-free graphs in polynomial time, we first note that we can find a good triad efficiently:

Lemma 10.20. *Let G be a skeletal claw-free graph with $\alpha(G) \leq 3$. If G contains a good triad then we can find one in polynomial time.*

Proof. To find T we check every set of three vertices, of which there are $O(n^3)$, to see if it satisfies the requirements. Since we can efficiently determine which vertices are twins and which vertices trump which vertices, we can do this in polynomial time. \square

Lemma 10.21. *Let G be a skeletal claw-free graph with $\alpha(G) = 3$, and suppose G contains no good triad. Then in polynomial time we can $\gamma_l(G)$ -colour G or determine that G is not three-cliqued.*

Proof. We define the *triad graph* $t(G)$ of G . We let $V(t(G)) = V(G)$, and two vertices are adjacent in $t(G)$ precisely if some triad in G contains both of them. We can easily find the components of $t(G)$ in polynomial time; there is at least one which is not a singleton.

Suppose that G is three-cliqued. Then it admits a hex-join into terms (G_1, A_1, B_1, C_1) and (possibly empty) (G_2, A_2, B_2, C_2) such that G_1 is minimal and contains a triad. Since G contains no good triad, it follows from the proofs of Lemmas 10.8, 10.9, 10.10, and 10.11 that (G_1, A_1, B_1, C_1) is in \mathcal{TTC}_1 . Furthermore the graph from which G_1 arises, i.e. H such that $G_1 = L(H)$, has more than three centres and hence more than six vertices, otherwise G would contain a good triad.

It is therefore easy to confirm that there is a component X of $t(G)$ such that $X = V(G_1)$. We can find such a component X in polynomial time, because any graph in \mathcal{TTC}_1 is a proper thickening of a line graph of a specific bipartite graph H . In particular we can find (G_1, A_1, B_1, C_1) efficiently, because we can find H efficiently and the definition of \mathcal{TTC}_1 implies that the choice of vertices a , b , and c of H is unique. Thus since G_1 is a term in a hex-join, we can determine A_2 , B_2 , and C_2 by taking a vertex in G_2 and looking at its neighbourhood in G_1 , assuming that G is three-cliqued.

We now proceed as in the proof of Lemma 10.12. With our base graph H of (G_1, A_1, B_1, C_1) in hand, it is not hard to see that we can decide which action is necessary in polynomial time. In each case we find a triad whose removal is guaranteed to lower $\gamma_l(G)$ or we remove edges from G to reach a graph G' such that $\chi(G') = \chi(G)$. From the proof of Lemma 10.12 it is clear that we can find G' in polynomial time, and given a k -colouring of G' we can find a k -colouring of G in polynomial time. We can recursively $\gamma_l(G)$ -colour G' in polynomial time, possibly appealing to Lemma 10.20, since it is a proper subgraph of G .

Now suppose G is not three-cliqued. If there is a component X of $t(G)$ such that $G[X]$ is in \mathcal{TTC}_1 , then again we have a unique choice of a , b , and c in H and a unique expression of $G[X]$ as (G_1, A_1, B_1, C_1) . Let A_2 be the set of vertices in $G - X$ with a neighbour in A_1 and a neighbour in B_1 ; we define B_2 and C_2 accordingly. Since G

is not three-cliqued, either A_2 , B_2 , and C_2 do not partition the vertices of $G - X$, or they are not all cliques. Either way we can determine this in polynomial time. \square

Using these two lemmas we can prove the desired result:

Theorem 10.22. *Let G be a claw-free graph with $\alpha(G) \geq 3$. Then in polynomial time we can either $\gamma_l(G)$ -colour G or determine that $\chi(\overline{G}) \geq 4$.*

Proof. By Theorem 6.11 we can assume G is skeletal. If G contains a good triad T , we can find T in polynomial time and recursively $\gamma_l(G) - 1$ colour $G - T$. If G does not contain a good triad, then the result follows immediately from Lemma 10.21. \square

10.6.2 Graphs that are not three-cliqued

By Theorem 9.5 and Theorem 9.7, if G is a skeletal claw-free graph that is not three-cliqued, then one of the following applies:

1. G is an antiprismatic thickening
2. G is an icosahedral thickening
3. G is quasi-line
4. G contains a clique cutset
5. G admits an antihat 2-join or a pseudo-line 2-join or a strange 2-join or a gear 2-join.

We already know how to deal with all of these cases efficiently, either by colouring in polynomial time or reducing to a smaller colouring problem, except in the case of 2-joins. To see that we can find an antihat 2-join or a pseudo-line 2-join or a gear 2-join efficiently, recall Chapter 8, in which we describe the structure of such a 2-join relative to a W_5 . Given the correct choice of a W_5 in G , we can easily find a W_5 2-join or a gear 2-join $((X_1, Y_1), (X_2, Y_2))$ separating G_1 from G_2 . There are $O(n^6)$ 5-wheels in G , so we can find such a 2-join in polynomial time.

Since G is skeletal, we can easily check whether or not G_2 is a gear strip in polynomial time: a skeletal gear strip has at most twelve equivalence classes of twin vertices. So assume that we have an antihat 2-join or a pseudo-line 2-join or a

strange 2-join. Checking if we have a strange 2-join is trivial. Checking if we have an antihat 2-join is straightforward once we determine the adjacency between X_2 and Y_2 . Otherwise we have a pseudo-line 2-join. In this case, $I(e_1)$ and $I(e_2)$ are precisely those vertices in X_2 and Y_2 respectively that are complete to $G_2 - X_2 - Y_2$. Furthermore, adding all edges between $I(e_1)$ and $I(e_2)$ leaves us with a line graph, the structure of which we can easily determine. Thus we can find these desired 2-joins in polynomial time.

To reduce on these 2-joins, we now consider the proof of Lemmas 10.16, 10.17, 10.18, and 10.19. We do one of two things: reduce the size of the graph and apply induction, or complete the l -colouring of G in one step. Just as with Lemma 7.1, showing that we can do this in polynomial time is straightforward given the proof of the lemma. Thus we get the desired algorithmic result:

Theorem 10.23. *For any claw-free graph G , we can $\gamma(G)$ -colour G in polynomial time.*

Chapter 11

Future Directions

We now conclude the thesis by describing some related questions that remain unanswered. They lie mainly in the two general areas of the Main Conjecture and claw-free graphs.

Problems related to the Main Conjecture

The most prominent problem that remains open is Reed's conjecture on ω , Δ , and χ , i.e. the Main Conjecture. An outright proof would represent an enormously significant result in the field of graph colouring. However, at this moment it seems quite far out of reach. Here we describe some more accessible steps towards a solution.

Triangle-free graphs

The Main Conjecture is still open for triangle-free graphs, i.e. the case $\omega = 2$. In this case we get a special case of the Main Conjecture:

Conjecture 11.1. *For any triangle free graph G , $\chi(G) \leq \frac{1}{2}\Delta(G) + 2$.*

This conjecture is implied by Brooks' Theorem when $\Delta = 4$, and when $\Delta > 10^4$ it is implied by Johansson's result that triangle-free graphs have chromatic number at most $9\frac{\Delta}{\log \Delta}$ (see [MR00]). It is striking that the conjecture is open even for $\Delta \in \{5, 6, 7\}$, and although they are attractive cases to work on they should not be taken lightly. The case $\Delta = 6$ should be the easiest of the three, and would easily imply the

case $\Delta = 5$. The triangle-free case of the Main Conjecture is of significant interest in its own right, and of further interest because it is a prerequisite to attacking the Main Conjecture for bull-free graphs using Chudnovsky's recent structure theorem [Chu08]. Since we know that the Main Conjecture holds for graphs containing no triad, this would give us the first nontrivial self-complementary class of graphs, other than perfect graphs, for which the Main Conjecture is known to be true (the class would be all graphs for which $\max\{\alpha, \omega\} = 2$).

Some steps towards the triangle-free case of the Main Conjecture have been made. Kostochka [Kos78] proved that the conjectured bound holds for graphs containing no cycle of length at most $4(\Delta + 2) \log \Delta$. And as mentioned in Chapter 2, he also proved that triangle free graphs have chromatic number at most $\frac{2\Delta}{3} + 2$.

Let k_Δ be the largest possible chromatic number of a triangle-free graph with maximum degree Δ . Suppose we can prove that $k_5 \leq 4$, $k_7 \leq 5$, $k_{10} \leq 6$, $k_{13} \leq 7$, $k_{17} \leq 10$, and $k_{19} \leq 11$. Then we could combine these results using a classical result of Lovász to prove the Main Conjecture for all triangle-free graphs. Observe that the Main Conjecture requires all of these partial results except $k_{10} \leq 6$ and $k_{13} \leq 7$, which are stronger than the Main Conjecture's bound by one.

Asymptotic approaches

Very little is known about asymptotic weakenings of the Main Conjecture. In the original paper, Reed made two weaker conjectures: $\chi \leq \frac{1}{2}(\Delta + \omega) + o(\Delta)$ and $\chi \leq \frac{1}{2}(\Delta + \omega) + o(\omega)$ [Ree98]. Both seem difficult.

If we insist that $\omega = o(\Delta)$, can we prove that $\chi \leq (\frac{1}{2} + o(1))\Delta$? With or without this restriction, can we prove that $\chi \leq (\frac{1}{2} + a)(\Delta + \omega)$ for some value of a much smaller than $\frac{1}{2}$? The most obvious approach to this problem is the *Rödl Nibble*, in which groups of randomized stable sets are removed as colour classes and invariants related to Δ and ω are proven to decay at some desirable rate (see [MR00]). However, maintaining any kind of independence among randomly chosen stable sets (which hopefully allows us to apply the Lovász Local Lemma) poses a challenge when we need γ to drop sufficiently between nibbles.

Problems related to claw-free graphs

The work in this thesis has brought up several questions about fractional and integer colourings of claw-free graphs. We know that χ can be as high as $\frac{6}{5}\chi_f$ for claw-free graphs, and we know that χ and χ_f agree asymptotically for quasi-line graphs. When $\alpha \leq 2$ it is well-known that $\chi \leq \frac{6}{5}\chi_f$, and we think this extends to all claw-free graphs with stability number at most three. In contrast, we can generalize our approach to quasi-line graphs to show that the fractional and integer chromatic numbers agree asymptotically for claw-free graphs with $\alpha \geq 4$. The only remaining details concern the ten types of 1-joins defined by Chudnovsky and Seymour [CS08b]. These must be treated with some care because they force us to consider claw-free graphs with $\alpha(G) \leq 3$ containing a simplicial vertex. Our current case analysis is somewhat lacking in elegance – a deeper investigation into fractional colourings of three-cliqued claw-free graphs may yield a nicer solution.

Many other questions related to claw-free graphs remain. In Chapter 8 we gave a natural and easy reduction from claw-free graphs with $\alpha \geq 4$ to quasi-line graphs. However, at this time the only known proof of the structure theorems for quasi-line graphs are as a special case of claw-free graphs. A direct proof of the structure theorems would be a very nice result.

Finally, in Chapter 6 we introduced skeletal graphs, which have never been used before. Our work in this thesis tells us that homogeneous pairs of cliques should be a point of interest outside the area of perfect graph theory. Are there other classes of graphs that can be better described and manipulated using nonskeletal homogeneous pairs of cliques? Can skeletal homogeneous pairs be generalized in a useful way to homogeneous k -tuples of cliques? It would be very interesting to see whether or not skeletal graphs have useful applications outside the domain of this thesis.

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Glossary

Here we use G to denote a graph, u and v to denote vertices, e to denote an edge, and S and T to denote vertex sets.

Basic graph terminology

Term	Symbol	Meaning
graph	$G = (V, E)$	a set V of vertices and a set E of unordered pairs of vertices
multigraph	$H = (V, E)$	a set V of vertices and a multiset E of unordered pairs of vertices
vertex set of G	$V(G)$	the set of vertices of G
edge set of G	$E(G)$	the set of edges of G
clique		a set of mutually adjacent vertices
stable set		a set of mutually nonadjacent vertices
u sees v		u is adjacent to v there is an edge between u and v $\{u, v\} \in E(G)$
neighbour of v		a vertex adjacent to v
neighbourhood of v	$N(v)$	the set of neighbours of v
closed neighbourhood of v	$\bar{N}(v)$	$N(v) \cup \{v\}$
matching		a set of edges, no two of which share an endpoint
subgraph of G induced on S	$G[S]$	$V(G[S]) = S,$ $E(G[S]) = \{uv \in E(G) \mid \{u, v\} \subseteq S\}$
complement of G	\bar{G}	$V(\bar{G}) = V(G),$ $E(\bar{G}) = \{uv \mid \{u, v\} \in V(G), uv \notin E(G)\}$

Graph invariants

Term	Symbol	Meaning
clique number of G	$\omega(G)$	size of the largest clique in G
stability number of G	$\alpha(G)$	size of the largest stable set in G
degree of v	$d(v)$	size of $N(v)$
maximum degree of G	$\Delta(G)$	$\max_{v \in V(G)} d(v)$
minimum degree of G	$\delta(G)$	$\min_{v \in V(G)} d(v)$
clique number of G	$\omega(v)$	$\omega(G[\bar{N}(v)])$
chromatic number of G	$\chi(G)$	
gamma of G	$\gamma(G)$	$\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \rceil$
local gamma of v	$\gamma_l(v)$	$\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \rceil$
local gamma of G	$\gamma_l(G)$	$\max_{v \in V(G)} \gamma_l(v)$
Main Conjecture		for all G , $\chi(G) \leq \gamma(G)$
Local Strengthening		for all G , $\chi(G) \leq \gamma_l(G)$

Further terminology

Term	Symbol	Meaning
claw		a vertex v and three mutually nonadjacent neighbours of v
triad		a stable set of size three
S is complete to T		every possible edge between S and T exists
S is anticomplete to T		no edges between S and T exists
hole		an induced cycle of length ≥ 4 in G
antihole		induced cycle of length ≥ 4 in \bar{G}