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Purity and Flatness

Doctor of Philosophy

A B S T R A C T

P. M. Cohn calls a submodule P of the left A-module M pure iff $0 \to E \otimes P \to E \otimes M$ is exact for all rt. modules E. Most of the well-known theorems on pure subgroups are valid for pure submodules. Extending a definition of Maranda to arbitrary rings, a module Q is called pure projective iff $\mathrm{Hom}(Q,M) \to \mathrm{Hom}(Q,M/P) \to 0$ is exact whenever P is pure in M. Maranda's results on pure projectivity are extended and a complete structure for pure projective modules is obtained.

Generalizing a known property of regular rings, a (left) A-module is called regular iff all its submodules are pure. The ring A is shown to be regular iff all left (or all rt.) A-modules are regular. A structure theorem for regular projective modules is obtained. A regular socle is defined, analogous to the semi-simple (= usual) socle, and its basic properties established. Several new characterizations of regular rings are given.

It is known that a left module F is flat iff its character module $\mathrm{Hom}_Z(F,Q/Z)$ is injective. For (left) noetherian rings, the dual holds: the left module I is injective iff its character module is flat. It is also shown that the weak (= flat) global dimension of A is equal to: sup weak dimension E, with the sup taken over all left (or rt.) finitely presented cyclic modules E.

Pure simple and indecomposable rings are related to the PP and PF rings of Hattori. The latter are rings in which every principal (left) ideal is projective (or flat). These rings are characterized both in the commutative and non-commutative cases.

Localization theorems for purity, regularity, PP and PF rings are obtained.

Finally, as an application, flat covers of modules are constructed and their basic properties established. They always exist and coincide with the projective cover for the perfect rings of Bass. However, they are not in general unique.

PURITY AND FLATNESS

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PREFACE

Following P. M. Cohn we call a submodule P of the left A-module M pure iff $0 \dashrightarrow E \otimes P \dashrightarrow E \otimes M$ is exact for all rt. modules E. This generalizes the definition of purity for abelian groups, and we prove that most of the well-known theorems on pure subgroups are valid for pure submodules.

For Principal Ideal Domains, Maranda calls a module Q pure projective iff $\mathrm{Hom}(Q,M) \dashrightarrow \mathrm{Hom}(Q,M/P) \dashrightarrow 0$ is exact whenever P is pure in M. Adopting this definition for arbitrary rings, we are able to extend his results on pure projectivity and get a complete structure theorem for pure projective modules.

It is easy to verify that a ring A is (von Neumann) regular iff every left (or every rt.) ideal is pure. Generalizing this idea, we call a (left) A-module regular iff all its submodules are pure. We prove that the ring A is regular iff all left (or all rt.) A-modules are regular. A structure theorem for regular projective module is obtained. We also define a regular socle, analogous to the semi-simple (= usual) socle, and establish its basic properties. Several new characterizations of regular rings are also proved.

Lambek has shown that a left module F is flat iff its character module $Hom_Z(F,Q/Z)$ is injective. For (left) noetherian rings we prove the dual: the left module I is injective iff its character module is flat. We are also able to show that the weak (= flat) global dimension of A is equal to: sup weak dimension E, with the sup taken over all left (or rt.) finitely presented cyclic modules E. This extends and considerably simplifies the proof of a corresponding result of Auslander and Buchsbaum on the global homological (= projective) dimension of A.

Next we relate pure simple and indecomposable rings to the PP and PF rings of Hattori. The latter are rings in which every principal (left) ideal is projective (or flat). We characterize these rings both in the commutative and non-commutative cases.

Our next chapter gives a number of localization theorems for purity, regularity, PP and PF rings.

Finally, as an application, we construct flat covers of modules and establish their basic properties. They always exist and coincide with the projective cover for the perfect rings of Bass. However, they are not in general unique.

Originality can be claimed for the results in this thesis, except in a few places where specific acknowledgement is made in the text. For advice given in conversation I am indebted to members of the McGill staff, in particular Dr. Connell and Dr. Kuyk. Most of all I would like to thank my supervisor, Dr. J. Lambek, for the great deal of time and encouragement he has given me.

CONTENTS

CHAPTER 0: NOTATION, TERMINOLOGY, AND PRELIMINARIES

In general we use the notation and terminology of Bourbaki; divergences are noted explicitly.

Throughout this thesis the word ring will mean associative ring with unit element, but not necessarily commutative. Ring homomorphisms preserve unit elements; the unit element of a subring is the unit element of the overring. Rings are usually denoted by the letters A, B, C, ... and their unit elements by 1_A, 1_B, 1_C, ... respectively.

All modules will be unitary. Unless the contrary is stated, all modules and submodules will be left. Modules are usually denoted by the letters E, F, G, ... M, N, ... and E', E", etc.

The word ideal will mean two-sided ideal. Ideals (both one- and two-sided) are often denoted by the letters I, J, K, ... or m, n, p, ...

Module homomorphisms are denoted: $u:E \dashrightarrow F$ or $f \in \mathrm{Hom}(M,N)$. If u is in Hom(F,G) then for any M, $H(M,u):\mathrm{Hom}(M,F) \dashrightarrow \mathrm{Hom}(M,G)$

denotes the map defined by: $H(M,u)(w) = uw$. If $f: M \dashrightarrow M/\bar{N}$ is the canonical homomorphism, we often set $\hat{m} = f(m)$ for m in M.

Often our considerations involve only one ring, say A. In that case we let 1 be its unit element (instead of 1_A), and write $\otimes$ in place of $\otimes_A$. The word module is then understood to mean left A-module, and the expression "Let $u: E \dashrightarrow F$" to mean: let u be an A-homomorphism from the left A-module E to the left A-module F. If more than one ring is involved, we emphasize the distinction by writing A-flat, B-flat, etc.

As a rule, x, x', x'' denote elements from the sets X, X', X'' respectively. The sets may be rings, modules, ideals, etc.

An <u>exact commutative diagram</u> is a commutative diagram in which all rows and all columns are exact.

Definitions, theorems, etc. are usually given for the "left" side. It is understood, of course, that a corresponding statement holds for the rt. side, although we do not always state it explicitly. In the case of non-symmetric ring properties, the absence of the words left and rt. means that both hold; e.g. by noetherian ring we mean both left and rt. noetherian.

The letters I, J, K, ... are often used for index sets. We use the convention that any summation is over the repeated indices. Thus

$$\sum a_{ij}x_j \text{ will mean } \sum a_{ij}x_j \quad (j \text{ in } J)$$

and

$$\sum a_{ij}b_{jk}c_{kl} \text{ will mean } \sum a_{ij}b_{jk}c_{kl} \quad (j \text{ in } J, \quad k \text{ in } K)$$

Since frequent reference is made to:

(B1) Bourbaki, N. Algèbre Commutative, Ch. I and II,

(B2) Bourbaki, N. Algèbre, Ch. II,

(B3) Bourbaki, N. Algèbre, Ch. VIII,

these three books are referred to as (B1), (B2) and (B3) respectively. Other references are given in the usual way.

The following abbreviations are used: Mono, epi, iso for one-one, onto, one-one and onto homomorphisms, rt., fg, iff for right, finitely generated, if and only if respectively.

We will now recall the definitions and elementary properties of several concepts which will be used frequently in this thesis.

1. Finitely Presented Modules

These modules play an important rôle in our work. A left A-module E is called a fp (= finitely presented) module iff there exists an exact sequence of left A-modules

$$G \dashrightarrow F \dashrightarrow E \dashrightarrow 0$$

with both G and F fg free modules. This definition, as well as the basic properties of fp modules, are given in (Bl, p. 35).

PROPOSITION 1.1. For any module E the following conditions are equivalent:

(1) E is fp.

(2) There exists an exact sequence $0 \dashrightarrow K \dashrightarrow F \dashrightarrow E \dashrightarrow 0$ with F fg free and K fg.

(3) There exists an exact sequence $0 \dashrightarrow H \dashrightarrow P \dashrightarrow E \dashrightarrow 0$ with P fg projective and H fg.

Proof.

(1) $\Rightarrow$ (2): If E is fp, there exists an exact sequence $G \dashrightarrow F \dashrightarrow E \dashrightarrow 0$ with G and F both fg free. Hence we have an

exact sequence $0 \dashrightarrow K \dashrightarrow F \dashrightarrow E \dashrightarrow 0$. By (B1, Lemma 9, p. 37), K is fg.

(2) $\Rightarrow$ (3): is obvious.

(3) $\Rightarrow$ (1): Let p_i generate P, with i in I, a finite index set, and let $u:P \dashrightarrow E$ be the given homomorphism. There exists a free module F with base f_i, i in I, such that $F = P \oplus Q$ for some Q, and $f_i = p_i \oplus q_i$ (B2, Cor. 1, p. 62). Define $v:F \dashrightarrow E$ by $v(f_i) = u(p_i)$. It is routine to verify that $\text{Ker } v = H \oplus Q$. Since H and Q are fg, so is Ker v, with generators k_j say, with j in J, a finite index set. Let G be free with base g_j, j in J, and define $w:G \dashrightarrow F$ by $w(g_j) = k_j$. Clearly we have an exact sequence $G \dashrightarrow F \dashrightarrow E \dashrightarrow 0$ with G and F fg free; hence E is fp.

2. Change of Rings

In this section, we recall some facts about change of rings. For further details and proofs, see the indicated reference in (B2).

Throughout this section let $f: A \dashrightarrow B$ be a ring homomorphism.

(a) Restriction of Scalars from B to A via f (B2, p. 49). Any left B-module E can be canonically made into a left A-module by defining $ae = f(a)e$ for all a in A and e in E.

(b) Extension of Scalars from A to B via f (B2, p. 116). By restricting the scalars from B to A, B can be made into a rt. A-module B_A. With any left A-module E we can canonically associate a left B-module $E_B = B_A \otimes_A E$, and an A-homomorphism $E \dashrightarrow E_B$ given by $e \dashrightarrow 1 \otimes_A e$.

3. Perfect Rings

The definitions and results in this section are almost all due to Bass (4). A left or rt. ideal I of A will be called left T-nilpotent iff for any sequence $a_1, a_2, \ldots$ of elements of I, there exists $n > 0$ such that $a_1 a_2 \ldots a_n = 0$ (right T-nilpotence requires that $a_n a_{n-1} \ldots a_1 = 0$ for some n). A submodule S of E is small in E iff for every submodule F of E such that $S + F = E$, we have $F = E$. An epi $u:P \dashrightarrow E$ is called a projective cover of E iff P is projective and $\operatorname{Ker} u$ is small in P. We call a ring A left perfect iff every left A-module has a projective cover. We quote without proof:

THEOREM 3.1 (Bass). Let N be the Jacobson radical of A. Then the following conditions are equivalent:

(1) N is left T-nilpotent and A/N is semi-simple.

(2) A is left perfect.

(3) Every flat left A-module is projective.

(4) A has no infinite sets of orthogonal idempotents, and every nonzero right A-module has nonzero socle.

CHAPTER 1: PURITY

This chapter will be devoted to defining and establishing the basic properties of pure submodules.

1. Definition and Elementary Properties of Purity

We will adopt a definition of purity due to P. M. Cohn (11). An exact sequence of left A-modules $0 \to E \to F \to G \to 0$ will be called pure exact iff for every rt. A-module D, the induced sequence $0 \to D \otimes E \to D \otimes F \to D \otimes G \to 0$ is exact. Usually we identify E with its image in F and say that E is pure in F. Conversely, a submodule E of F will be called a pure submodule iff the associated exact sequence $0 \to E \to F \to F/E \to 0$ is pure exact. It is easy to verify that for any module E, both 0 and E are pure submodules. Consequently any collection of submodules of E which contains 0 will always contain a pure submodule of E. Since for any rt. A-module D, the functor $D \otimes$ is rt. exact, the condition that E be pure in F is equivalent to the requirement that $D(j) = 1_D \otimes j$

be mono for all rt. A-modules D, where $j: E \dashrightarrow F$ is the canonical injection.

In order to explain the name purity, we quote without proof the following fundamental result of P. M. Cohn (11):

THEOREM 1.1 (P. M. Cohn). Let P be a submodule of M. Then P is pure in M iff given $\sum a_{ij}m_j = q_i$ in P for all i in I, some finite index set, where $a_{ij} \in A$, $m_j \in M$, and j in J, a finite index set, then there exist p_j in P for all j in J, such that

$\sum a_{ij}m_j = \sum a_{ij}p_j$ for all i in I.

Remark 1.1.

(i) This result will sometimes be used as a test for purity. For convenience we shall use the following (equivalent) abbreviated form:

P is pure in M iff $\sum a_{ij}m_j \in P \Rightarrow \sum a_{ij}m_j = \sum a_{ij}p_j$.

(ii) For PID's (Principal Ideal Domains), Kaplansky (25) defines a submodule P of M to be pure iff

(1) am in P $\Rightarrow$ am = ap for all a in A and some p in P.

which is equivalent to:

(2) $aM \cap P = aP$ for all a in A.

It is clear that Cohn's definition of purity is a generalization of (1); in fact, as he remarks, his definition coincides with that of Kaplansky for PID's. Chase (10) has adopted the formulation (2) as a definition of purity for arbitrary rings. In Section 3, we shall examine these two definitions in greater detail, and prove a theorem which extends Cohn's remark to a wider class of rings. However, unless the contrary is stated explicitly, the word pure will always mean pure in the sense of Cohn.

Our next proposition shows that several well-known properties of pure subgroups are also valid for pure submodules.

PROPOSITION 1.2. Suppose $E \subseteq F \subseteq G$ are left A-modules. Then

(1) E pure in F and F pure in $G \Rightarrow E$ pure in G.

(2) E pure in $G \Rightarrow E$ pure in F.

(3) F pure in $G \Rightarrow F/E$ pure in G/E.

(4) If E is pure in G then F/E pure in $G/E \Rightarrow F$ pure in G.

(5) If E is pure in G, then under the one-one correspondence between the submodules of G containing E and the submodules of G/E, pure submodules correspond to pure submodules.

Proof. For any rt. A-module D, we have the following exact commutative diagram:

$$\begin{array}{ccccccc} D\otimes E & \xrightarrow{u} & D\otimes F & \xrightarrow{v} & D\otimes F/E & \longrightarrow & 0 \\ 1\downarrow & & b\downarrow & & \downarrow c & & \\ D\otimes E & \xrightarrow[u']{} & D\otimes G & \xrightarrow[v']{} & D\otimes G/E & \longrightarrow & 0 \end{array}$$

where all the maps are those induced by the canonical maps arising from the inclusions: $E\subseteq F\subseteq G$.

(1) Since $bu = u'$, we have u mono and b mono $\Rightarrow$ u' mono and

(2) u' mono $\Rightarrow$ u mono.

For (3) and (4) we use the Snake Lemma (Bl, Prop. 1, p. 17).

(3) If b is mono then c is mono since 1 and v are epi, by Corollary 2 of the Snake Lemma.

(4) If u' and c are mono then b is mono since 1 is mono, by Corollary 1 of the Snake Lemma.

(5) follows immediately from (3) and (4).

For the most part, the results of the next proposition were given by Cohn (11). For completeness we collect them together here; note that flat modules play the same rôle for pure submodules as torsion free abelian groups do for pure subgroups.

PROPOSITION 1.3. Let $0 \dashrightarrow E \dashrightarrow F \dashrightarrow G \dashrightarrow 0$ be exact. Then:

(1) If G is flat, E is pure in F.

(2) The converse holds if F is flat. In that case E is also flat.

(3) If F is flat, then E is pure in F iff G is flat.

(4) If G is flat, then under the one-one correspondence between submodules of F containing E and submodules of G, pure submodules correspond to pure submodules.

(5) If E is a direct summand of F then E is pure in F.

Proof. Let $j: E \dashrightarrow F$ be the given homomorphism. For any rt. A-module D, the given exact sequence yields the long exact sequence:

$$\mathrm{Tor}(D,F) \dashrightarrow \mathrm{Tor}(D,G) \dashrightarrow D \otimes E \dashrightarrow D \otimes F.$$

(1) If G is flat, Tor(D,G) = 0 and $D(j) = 1_D \otimes j$ is mono. Hence E is pure in F.

(2) If F is flat, Tor(D,F) = 0 and Tor(D,G) = Ker D(j). Hence if E is pure, D(j) is mono and Tor(D,G) = 0 and G is flat. Also E is flat since F and G are flat (B1, Prop. 5, p. 31).

(3) This follows by combining (1) and (2).

(4) If G is flat, E is pure and the result follows from Proposition 1.2.

(5) is immediate since $\otimes$ commutes with $\oplus$ (B2, Cor. 5, p. 93).

The following corollary is the analogue of a well-known proposition for projective modules.

COROLLARY. The left A-module F is flat iff every exact sequence $0 \dashrightarrow D \dashrightarrow E \dashrightarrow F \dashrightarrow 0$ of left A-modules is pure exact.

Proof.

$\Longrightarrow$: is immediate by Part (1).

$\Longleftarrow$: Taking E flat, the result is immediate by Part (2).

Remark 1.3.

(i) The converse of (1) is false in general: let P be a module which is not flat, and let $F = P \oplus P$. Then P is pure in F by Part (5), but $G = F/P \cong P$ is not flat.

(ii) The dual of the situation described in the corollary, i.e. modules D for which every exact sequence $0 \to D \to E \to F \to 0$ is pure exact has been studied by Maddox (31). He calls such modules absolutely pure.

(iii) The converse of (5) is not in general true. In fact we have:

PROPOSITION 1.4. For any ring A the following conditions are equivalent:

(1) A is left perfect.

(2) Pure submodules of flat left A-modules are direct summands.

(3) Every pure exact sequence of left A-modules $0 \to E \to F \to G \to 0$ with F flat is split exact.

Proof.

(1) $\Rightarrow$ (2): If P is a pure submodule of the flat module F, then

F/P is flat by Proposition 1.3. Since A is left perfect, F/P is projective by Theorem 3.1 of Chapter 0, and the exact sequence $0 \to P \to F \to F/P \to 0$ splits (B2, Prop. 4, p. 61), i.e. P is a direct summand.

(2) $\Rightarrow$ (3): is obvious.

(3) $\Rightarrow$ (1): For any flat left A-module F, there exists an exact sequence $0 \to D \to E \to F \to 0$ with E projective. Since F is flat, the sequence is pure exact, hence split exact and F is projective.

COROLLARY 1. If every pure submodule is a direct summand, then A is left perfect.

Proof. Obvious.

COROLLARY 2. There exist pure submodules which are not direct summands.

Proof. There exist rings which are not perfect (the ring of integers for example). See Bass (4).

Remark 1.4. In Chapter 2, we shall examine conditions under which every pure submodule is a direct summand.

PROPOSITION 1.5. Let $0 \to P_i \to M_i \to N_i \to 0$ be exact for all i in I, any index set, and let $P = \oplus P_i$ (i in I) and $M = \oplus M_i$ (i in I). Then P is pure in M iff P_i is pure in M_i for all i in I.

Proof. Let $k_i : P_i \to M_i$ and $k : P \to M$ be the canonical injections. Then $k = \oplus k_i$ (i in I) (B2, Prop. 7, p. 26). For any rt. A-module E, let $E(k) = 1_E \otimes k$, etc. Then $E(k) = \oplus E(k_i)$ since $\otimes$ commutes with $\oplus$ (B2, Prop. 7, p. 90), and the direct sum map $E(k)$ is mono iff each component map $E(k_i)$ is mono (B2, Cor. 1, p. 27). Therefore P is pure in M iff P_i is pure in M_i for all i in I.

THEOREM 1.6. Let I be any directed set, and let (P_i), (M_i), and (N_i) be directed systems of modules with $P = \varinjlim P_i$, $M = \varinjlim M_i$ and $N = \varinjlim N_i$ and suppose $u_i : P_i \to M_i$ and $v_i : M_i \to N_i$

are directed systems of A-homomorphisms with $u = \varinjlim u_i$ and $v = \varinjlim v_i$ such that $0 \to P_i \to M_i \to N_i \to 0$ is exact for all i in I. Then if P_i is pure in M_i for all i in I, P is pure in M.

Proof. For any rt. A-module E let $E(u) = 1_E \otimes u$ etc. Then we have $E(u) = \varinjlim E(u_i)$ since the direct limit commutes with $\otimes$ (B2, Prop. 12, p. 145). Hence if $E(u_i)$ is mono for all i, then $E(u)$ is mono (B2, Prop. 6, p. 134).

COROLLARY 1. The direct limit of any directed system of pure submodules of a given module is pure.

Proof. We apply the theorem with $M = M_i$ for all i.

COROLLARY 2. The union of any chain of pure submodules of a module is pure, i.e. purity is an inductive property.

Proof. Any chain forms a directed system of submodules. Apply Corollary 1.

COROLLARY 3. If P is a submodule of M such that every fg submodule of P is pure in M, then P is pure in M.

Proof. Any submodule P of M is the direct limit of its fg submodules, which are pure in M. Hence P is pure.

COROLLARY 4. If every fg submodule of M is pure, then every submodule of M is pure.

Proof. Obvious using Corollary 3.

THEOREM 1.7. Let P be a submodule of M, and consider the following conditions:

(1) M/P is flat.

(1)' P is pure in M.

(2) $KM \cap P = KP$ for all rt. ideals K.

(2)' $KM \cap P = KP$ for all fg rt. ideals K.

(2)" $KM \cap P = KP$ for all principal rt. ideals K.

(3) $aM \cap P = aP$ for all a in A.

Then we always have the following implications:

$$(1) \Rightarrow (1)' \Rightarrow (2) \Leftrightarrow (2)' \Rightarrow (2)'' \Leftrightarrow (3).$$

If M is flat we have $(1) \Leftrightarrow (1)' \Leftrightarrow (2)$.

Proof.

$(1) \Rightarrow (1)'$: is given in Proposition 1.3.

Since KP is always contained in $KM \cap P$, we need only show the opposite inclusion in each case.

$(1)' \Rightarrow (2)$: If $p = \sum k_j m_j$ (j in J, a finite set) is a typical element of $KM \cap P$ then $p = \sum k_j p_j \in KP$ since P is pure in M (Theorem 1.1). Therefore $KM \cap P$ is contained in KP.

$(2) \Rightarrow (2)'$: is obvious.

$(2)' \Rightarrow (2)$: If $p = \sum k_j m_j$ (j in J, a finite set) is a typical element of $KM \cap P$, let K' be the fg rt. ideal generated by the k_j. Then p is contained in $K'M \cap P = K'P$, which is contained in KP. Therefore $KM \cap P$ is contained in KP.

The remaining implications are obvious.

If M is flat, the equivalences $(1) \Leftrightarrow (1)' \Leftrightarrow (2)$ are given in Proposition 1.3 and (Bl, Cor., p. 33).

2. Pure Left Ideals

Before proceeding to the main theorem of this section, we make a number of important definitions which will be used here and later.

A subset S of A is idempotent iff $S^2 = S$, where S^2 is the collection of all finite sums of elements of the form ss' with s and s' in S.

An element a of A will be called a left zero divisor iff there exists $0 \neq b$ in A so that $ab = 0$. This is equivalent to saying that the homomorphism $f_a: A \dashrightarrow A$ (as left A-modules) defined by $f_a(b) = ab$ is not mono. Similar comments apply for rt. zero divisors. If we set $r(a) = (b \in A \mid ab = 0)$, the rt. annihilator of a, then a is a left zero divisor iff $r(a) \neq 0$. The same comments apply to rt. zero divisors, with $l(a) = (b \in A \mid ba = 0)$, the left annihilator of a. We note that $r(a)$ is a rt. ideal of A and $l(a)$ is a left ideal of A.

Since 0 is both a left and rt. zero divisor, we call a zero divisor proper iff it is nonzero. We say that the ring A has no left zero divisors iff A has no proper left zero divisors; similarily for rt. zero divisors. And we say that A has no zero divisors iff A has no left zero divisors and no rt. zero divisors, i.e. $l(a) = 0 = r(a)$ for all $0 \neq a$ in A.

A left A-module $M \neq 0$ will be called <u>simple</u> (resp. <u>pure simple</u>, <u>indecomposable</u>) iff 0 and M are the only submodules (resp. pure submodules, direct summands) of M. The ring A will be called <u>left simple</u> (resp. <u>left pure simple</u>, <u>left indecomposable</u>) iff it is simple (resp. pure simple, indecomposable) as a left module; and it will be called <u>simple</u> (resp. <u>pure simple</u>, <u>indecomposable</u>) iff it is both left and rt. simple (resp. pure simple, indecomposable). Clearly every simple module or ring is pure simple, and every pure simple module or ring is indecomposable.

THEOREM 2.1. For any left ideal P of A the following conditions are equivalent:

(1) A/P is flat.

(1)' P is pure in A.

(2) $KP = K \cap P$ for all rt. ideals K.

(2)' $KP = K \cap P$ for all fg rt. ideals K.

(2)" $KP = K \cap P$ for all principal rt. ideals K.

(3) $aP = aA \cap P$ for all a in A.

(4) For each p in P, there exists an a in $r(p) = (x \in A \mid px = 0)$, such that $\hat{a} = \hat{1}$ (where $\hat{a}$ is the image of a in A/P). Furthermore $a \neq 0$ unless $P = A$.

Proof. Since A is flat, it suffices by Theorem 1.7 to show (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

(3) $\Rightarrow$ (4): $p \in P \Rightarrow p \in pA \cap P = pP \Rightarrow p = pp'$ for some p' in P. And $a = 1 - p'$ is in $r(p)$ with $\hat{a} = \hat{1}$ since p' is in P. Clearly $a \neq 0$ unless $P = A$.

(4) $\Rightarrow$ (1): To prove that A/P is flat, it suffices to show that $Tor(A/K, A/P) = 0$ for any rt. ideal K (Bl, Prop. 1, p. 55). Now $Tor(A/K, A/P) = (K \cap P)/KP$ by Cartan-Eilenberg ((8), p. 126). If k is in $K \cap P$, there exists a in A such that $ka = 0$ and $\hat{a} = \hat{1}$. Hence $1 - a = p$ is in P. Therefore $kp = k(1-a) = k$ and $K \cap P = KP$. Hence $Tor(A/K, A/P) = 0$.

COROLLARY 1.

(1) If $P \neq A$ is a pure left ideal of A, then all its elements are left zero divisors.

(2) If A has no left zero divisors, A is left pure simple.

(3) If A has no zero divisors, A is pure simple.

(4) Every integral domain is pure simple.

Proof.

(1) By (4) of the theorem, $r(p) \neq 0$ for each p in P since $P \neq A$.

(2), (3) and (4) are obvious.

COROLLARY 2.

(1) If P is a pure left ideal of A, then for each p in P, there exists a sequence p_i in P, $i = 1, 2, \ldots$ such that $p = pp_1p_2 \ldots p_n$ for all $n = 1, 2 \ldots$

(2) 0 is the only left T-nilpotent pure left ideal of A.

Proof.

(1) The sequence can be constructed inductively using the method which was used in proving (3) $\Rightarrow$ (4) in the theorem.

(2) For any sequence p_i in P, there exists an n such that $p_1 \cdots p_n = 0$. (See Section 3 of Chapter 0.)

COROLLARY 3.

(1) Every pure left ideal P is idempotent.

(2) Let P be a left ideal. If $K \cap P$ is idempotent for all principal rt. ideals K, then P is pure in A.

Proof.

(1) Let $P' = PA \supseteq P$. Then $P^2 = P'P = P' \cap P = P$.

(2) $K \cap P = (K \cap P)^2 = (K \cap P)(K \cap P) \subseteq KP$.

Hence $K \cap P = KP$ for all principal rt. ideals K, and P is pure in A.

COROLLARY 4. If P is a principal left ideal, say Ab, then Part (3) of the theorem becomes (3)': $aAb = aA \cap Ab$ for all a in A.

3. Types of Purity

In this section, we study several possible definitions of purity for arbitrary rings, and the relationships between them.

A left module E will be called <u>principal cyclic</u> (resp. <u>fp cyclic</u>) iff it has the form $E = A/I$ with I a principal (resp. fg) left ideal of A. Clearly every principal cyclic module is fp cyclic, and every fp cyclic module is both fp and cyclic.

PROPOSITION 3.1. Let $0 \to E \to F \to G \to 0$ be an exact sequence of left A-modules and $j: E \to F$ the given map.

(1) E is pure in F iff $D(j) = 1_D \otimes j$ is mono for all fp rt. A-modules D.

(2) $0 \to (KF \cap E)/KE \to E/KE \to F/KF \to G/KG \to 0$ is exact, and hence $(KF \cap E)/KE = \mathrm{Ker}(A/K \otimes E \to A/K \otimes F)$ for any rt. ideal K.

Proof.

(1) If E is pure in F then D(j) is mono for all rt. A-modules D by definition. Conversely, any rt. A-module D is the direct limit of fp rt. A-modules D_i (B1, Ex. 10, p. 62). Since the

direct limit commutes with $\otimes$ (B2, Prop. 12, p. 145),

$D(j) = \varinjlim D_i(j)$. Since D_i is fp, $D_i(j)$ is mono for all i, and therefore $D(j)$ is mono (B2, Prop. 6, p. 134). Hence E is pure in F.

(2) Since $A/K \otimes E = E/KE$ etc. (B2, Cor. 2, p. 89), the map $j_K: E/KE \dashrightarrow F/KF$ is defined by $j_K(\hat{e}) = \widehat{(je)}$ ($\hat{}$ denotes the image in the quotient module). Hence $\hat{e}$ is in $\text{Ker } j_K$

iff $\widehat{(je)} = 0$.
iff je is in KF.
iff e is in $KF \cap E$.
iff $\hat{e}$ is in $(KF \cap E)/KE$.

Consequently the given sequence is exact. The other statement is obvious.

In view of Theorem 1.7, one might use any one of the conditions (1)', (2) or (3) as a definition of purity for arbitrary rings. In fact, (1)' is purity in the sense of Cohn (11), and (3) is purity in the sense of Chase (10). Maranda (33) has defined a purity similar to (2), using two-sided ideals instead of rt. ideals.

Let E be a submodule of the left A-module F. For purposes of this section only, let us say that E is I-pure, II-pure, or III-pure in F according as:

(I) E is pure in F in the sense of Cohn.

(II) $KF \cap E = KE$ for all (fg) rt. ideals K.

(III) $KF \cap E = KE$ for all principal rt. ideals K.

By Theorem 1.7, (II) is the same for both rt. ideals and fg rt. ideals, and (III) is the same as: $aF \cap E = aE$ for all a in A.

Let $j : E \to F$ be the canonical injection, and for any rt. A-module D, let $D(j) = 1_D \otimes j$.

THEOREM 3.2. E is I-pure, II-pure, or III-pure in F, according as D(j) is mono for all

(I) fp rt. D,

(II) fp cyclic rt. D or (II)': cyclic rt. D,

(III) principal cyclic rt. D.

Proof. The proof is immediate, using Proposition 3.1. The equivalence of (II) and (II)' follows from Theorem 1.7.

COROLLARY. Let E be a submodule of the left A-module F. Then E is I-pure in F $\Rightarrow$ E is II-pure in F $\Rightarrow$ E is III-pure in F. This may be summarized by saying I $\Rightarrow$ II $\Rightarrow$ III.

Proof. Every principal cyclic rt. module D is fp cyclic, and every fp cyclic module is fp.

Using the corollary of Theorem 3.2, we can deduce that some of the reverse implications hold, provided that there is some connection between fp, (fp) cyclic, and principal cyclic modules.

THEOREM 3.3.

(1) If every fp rt. module D is a direct summand of a direct sum of cyclic (resp. principal cyclic) modules, then II $\Rightarrow$ I (resp. III $\Rightarrow$ I).

(2) If every fp rt. module D is the direct limit of cyclic (resp. principal cyclic) modules, then II $\Rightarrow$ I (resp. III $\Rightarrow$ I).

Proof.

(1) Suppose $D \oplus D' = \oplus N_i$ where the N_i are cyclic (resp. principal cyclic), then $D(j) \oplus D'(j) = \oplus N_i(j)$ since $\otimes$ commutes

with $\oplus$ (B2, Prop. 7, p. 90). If the $N_i(j)$ are all mono, so is $D(j)$ (B2, Cor. 1, p. 27).

(2) Suppose $D = \varinjlim N_i$ with N_i cyclic (resp. principal cyclic), then $D(j) = \varinjlim N_i(j)$, since $\otimes$ commutes with the direct limit (B2, Prop. 12, p. 145) and if the $N_i(j)$ are all mono, so is $D(j)$ (B2, Prop. 6, p. 134).

COROLLARY 1. If A is any one of

(a) PID,

(b) semi-principal (= Bezout) domain (i.e. every fg ideal is principal),

(c) uniserial ring,

then III $\Rightarrow$ I (and hence III $\Rightarrow$ II).

Proof.

(a) If A is a PID, it is well known that every fg module is the direct sum of cyclic modules, which are principal cyclic since the domain is principal.

(b) If A is semi-principal, Chadeyras (9) has shown that every fp module is the direct sum of principal cyclic modules.

(c) Koethe (27) has shown that if A is uniserial, then every left and every rt. module is the direct sum of cyclic modules. Asano (1) and (2) and Faith (15) have shown that A is uniserial iff A is a left and rt. artinian and left and rt. principal ideal ring. Hence every fp module is the direct sum of principal cyclic modules.

COROLLARY 2. If A is an almost maximal valuation ring, then II $\Rightarrow$ I.

Proof. Kaplansky ((24), p. 339) has shown that for such rings every fg module is the direct sum of cyclic modules. Hence II $\Rightarrow$ I.

Remark 3.3.

(i) We note that in this theorem we are deducing some facts about left purity, from the structure of the rt. A-modules. In Chapter 2 we deduce some more facts about left purity using the structure of the left A-modules.

(ii) Since a domain is artinian iff it is a field, Corollary 1 gives us two different types of examples. Consequently this is an extension of Cohn's Remark (see Remark 1.1).

(iii) Since almost maximal valuation rings are not in general PID's (see Kaplansky (25), p. 75), Corollary 2 gives us a third type of example.

4. Change of Rings

THEOREM 4.1. Let $f:A \dashrightarrow B$ be any ring homomorphism. If $0 \dashrightarrow E \dashrightarrow F \dashrightarrow G \dashrightarrow 0$ is a split (resp. pure) exact sequence of left A-modules, then the induced sequence $0 \dashrightarrow E_B \dashrightarrow F_B \dashrightarrow G_B \dashrightarrow 0$ is a split (resp. pure) exact sequence of left B-modules.

Proof. The split exact case is given by (B2, Cor., p. 120, and Prop. 7, p. 90). The pure exact case: Since $E_B = B_A \otimes_A E$ etc., the fact that the given sequence is pure exact implies that the induced sequence is exact. If M is any rt. B-module, then $M \otimes_B B$ is a rt. A-module since B is a rt. A-module. Therefore

$(M \otimes_B B) \otimes_A E \dashrightarrow (M \otimes_B B) \otimes_A F$ is mono.

But $(M \otimes_B B) \otimes_A E = M \otimes_B (B_A \otimes_A E) = M \otimes_B E_B$ etc. (See B2, Prop. 8, p. 94.) Therefore $M \otimes_B E_B \dashrightarrow M \otimes_B F_B$ is mono for all rt. B-modules M and the induced sequence is a pure exact sequence of left B-modules.

COROLLARY. If $0 \dashrightarrow E \dashrightarrow F \dashrightarrow G \dashrightarrow 0$ is an exact sequence of left B-modules which is split (resp. pure) exact as a sequence of left A-modules, then it is pure exact as a sequence of left B-modules.

Proof. Clear since $E = E_B$.

CHAPTER 2: PURE PROJECTIVITY

1. Another Criterion for Purity

For the proof of our main theorem, we need the following technical lemma:

LEMMA 1.1. Let $0 \to G \to F \xrightarrow{u} E \to 0$ be exact with F fg free on base f_j, j in J, a finite index set, and G fg with generators g_i, i in I, a finite index set. And let $g_i = \sum a_{ij} f_j$ with a_{ij} in A and $u(f_j) = e_j$ for all j in J. Then the following are equivalent:

(1) $\sum b_j e_j = 0$, with b_j in A.

(2) $b_j = \sum c_i a_{ij}$ for some c_i in A.

(3) Each b_j is in $\sum Aa_{ij}$ (i in I), the left ideal generated by the a_{ij}, i in I.

Proof. $\sum b_j e_j = 0$ iff $\sum b_j f_j$ is in G.
iff $\sum b_j f_j = \sum c_i g_i = \sum c_i a_{ij} f_j$ for some c_i in A.
iff $b_j = \sum c_i a_{ij}$ (since the f_j are a base).
iff each b_j is in $\sum Aa_{ij}$ (i in I).

We proceed now to our main theorem.

THEOREM 1.2. The exact sequence $0 \longrightarrow E \longrightarrow F \xrightarrow{u} G \longrightarrow 0$ of left A-modules is pure exact iff $H(M,u):\mathrm{Hom}(M,F) \longrightarrow \mathrm{Hom}(M,G)$ is epi for all fp left A-modules M, where $H(M,u)(w) = uw$ for all w in $\mathrm{Hom}(M,F)$.

Proof.

$\Rightarrow$: Let w be in $\mathrm{Hom}(M,G)$ with M any fp module. Then we have an exact sequence $0 \longrightarrow K \longrightarrow N \longrightarrow M \longrightarrow 0$ with N fg free, on base n_j, j in J, a finite index set, and K fg with generators k_i, i in I, a finite index set. For all j in J, let m_j be the image of n_j; and for all i in I, let $k_i = \sum a_{ij}n_j$ with a_{ij} in A. Then we have $\sum a_{ij}m_j = 0$. Since N is projective, we have an exact commutative diagram:

$$\begin{array}{ccccc} \mathrm{Hom}(M,F) & \longrightarrow & \mathrm{Hom}(M,G) & & \\ \downarrow & & \downarrow & & \\ \mathrm{Hom}(N,F) & \longrightarrow & \mathrm{Hom}(N,G) & \longrightarrow & 0 \end{array}$$

Let $\hat{w}$ in $\mathrm{Hom}(N,G)$ be the image of w and let v in $\mathrm{Hom}(N,F)$ be a pre-image of $\hat{w}$. Set $v(n_j) = f_j$ for all j in J.

We have an exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow v & & \downarrow w & & \\ 0 & \longrightarrow & E & \longrightarrow & F & \underset{u}{\longrightarrow} & G & \longrightarrow & 0 \end{array}$$

Now $u(\sum a_{ij}f_j) = uv(\sum a_{ij}n_j) = w(\sum a_{ij}m_j) = w(0) = 0$. Hence $\sum a_{ij}f_j$ is in E for all i in I. And by the purity of E in F $\sum a_{ij}f_j = \sum a_{ij}e_j$ for some e_j in E.

Define w' in $\mathrm{Hom}(M,F)$ by $w'(m_j) = f_j - e_j$ for all j in J. Since the m_j generate M, we need only verify that this is well defined and that $H(M,u)(w') = w$. If $m = \sum b_jm_j = 0$, we have $b_j = \sum c_ia_{ij}$ for some c_i in A, by Lemma 1.1. Therefore $w'(m) = \sum c_ia_{ij}(f_j - e_j) = \sum c_i(\sum a_{ij}f_j - \sum a_{ij}e_j) = \sum c_i0 = 0$. Also for all j in J, $uw'(m_j) = u(f_j - e_j) = u(f_j) = w(m_j)$ by the commutativity of the diagram, and therefore $H(M,u)(w') = uw' = w$.

$\Leftarrow$: Suppose $\sum a_{ij}f_j \in E$ (with finite index sets I and J). Let N be fg free on base n_j, j in J, and let K be the fg submodule generated by k_i, i in I, where $k_i = \sum a_{ij}n_j$. Form the exact sequence $0 \longrightarrow K \longrightarrow N \longrightarrow M \longrightarrow 0$ and let m_j be the image of n_j. Then $\sum a_{ij}m_j = 0$. Let v in $\mathrm{Hom}(N,F)$ be defined by $v(n_j) = f_j$ for j in J. Then $v(k_i) = \sum a_{ij}f_j$ is in E for all i in I and hence $v(K)$ is a submodule of E, so that we may pass to quotient modules to get an exact commutative diagram, which defines w:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow v & & \downarrow w & & \\ 0 & \longrightarrow & E & \longrightarrow & F & \underset{u}{\longrightarrow} & G & \longrightarrow & 0 \end{array}$$

Since M is fp, there exists w' in $\mathrm{Hom}(M,F)$ such that $uw' = w$. Let $w'(m_j) = x_j$ in F, for all j in J. Then for all j in J, $u(x_j) = w(m_j) = u(f_j)$ by the commutativity of the diagram and hence $e_j = f_j - x_j$ is in E for all j in J. Also
$\sum a_{ij}x_j = w'(\sum a_{ij}m_j) = w'(0) = 0$ for all i in I. Hence
$\sum a_{ij}f_j = \sum a_{ij}e_j$ for all i in I, and E is pure in F.

Remark 1.2. Kaplansky ((24), p. 332) remarks in a footnote that "It is conversely true for a module M over an arbitrary ring that purity of a submodule S is implied by the ability to left elements of M/S with preservation of the order ideal. This stronger property should <u>perhaps</u> be used as the definition of purity when working over general rings." (My underlining.) We have shown that rather than lifting elements and preserving the order ideal (which is equivalent to lifting maps from cyclic modules), a better choice is lifting maps from fp modules (Theorem 1.2).

COROLLARY 1. Suppose for all i in I, any index set, that E_i is a submodule of F_i. Then $E = \prod E_i$ (i in I) is pure in $F = \prod F_i$ (i in I) iff E_i is pure in F_i for all i in I.

Proof. Let $0 \dashrightarrow E_i \dashrightarrow F_i \overset{u_i}{\dashrightarrow} G_i \dashrightarrow 0$ and $0 \dashrightarrow E \dashrightarrow F \overset{u}{\dashrightarrow} G \dashrightarrow 0$ be the corresponding exact sequences. Then $u = \prod u_i$ (i in I), and for any M, $H(M,u) = \prod H(M,u_i)$ (i in I) (B2, Cor. 2, p. 27), and hence the product map $H(M,u)$ is epi iff each component map $H(M,u_i)$ is epi for all i in I by (B2, Cor., p. 23).

COROLLARY 2. Every fp flat module is projective.

Proof. Suppose G is fp flat. Let $0 \dashrightarrow E \dashrightarrow F \dashrightarrow G \dashrightarrow 0$ be exact. Then E is pure in F since G is flat by Proposition 1.3 of Chapter 1. Hence $H(G,u):\mathrm{Hom}(G,F) \dashrightarrow \mathrm{Hom}(G,G)$ is epi since G is fp. Therefore the sequence splits, and G is projective.

THEOREM 1.3. The exact sequence $0 \dashrightarrow E \overset{v}{\dashrightarrow} F \overset{u}{\dashrightarrow} G \dashrightarrow 0$ is split exact iff either of the following two equivalent conditions holds:

(1) $H(M,u):Hom(M,F) \dashrightarrow Hom(M,G)$ is epi for all M.

(2) $H(v,M):Hom(F,M) \dashrightarrow Hom(E,M)$ is epi for all M.

Proof. If the given exact sequence splits, then the induced sequence $0 \dashrightarrow Hom(M,E) \dashrightarrow Hom(M,F) \dashrightarrow Hom(M,G) \dashrightarrow 0$ is split exact (B2, Prop. 2, p. 60) for all M. Conversely by putting $M = G$, there exists w in $Hom(G,F)$ such that $H(G,u)(w) = uw = 1_G$ i.e. the sequence splits. Consequently we have the equivalence of splitting and (1). The equivalence of splitting and (2) is established in exactly the same way.

Remark 1.3.

(i) This theorem shows clearly just how much weaker purity is than direct summand.

(ii) It also shows that any attempt to define a concept for $Hom(M,\ )$ or $Hom(\ M)$ which corresponds to the definition of purity for $\otimes M$ or $M \otimes$ does not yield anything new; we just get direct summands.

2. Pure Projectivity

We will call a module P <u>pure projective</u> iff for any pure exact sequence $0 \to E \to F \xrightarrow{u} G \to 0$, the induced sequence $0 \to \mathrm{Hom}(P,E) \to \mathrm{Hom}(P,F) \to \mathrm{Hom}(P,G) \to 0$ is exact. This is, of course, equivalent to requiring that the map $H(P,u):\mathrm{Hom}(P,F) \to \mathrm{Hom}(P,G)$ be epi. Any projective module P is pure projective since $H(P,u)$ is epi for all exact sequences. We shall determine completely the structure of pure projective modules (Theorem 2.4).

PROPOSITION 2.1. Let $P = \bigoplus P_i$ (i in I, any index set). Then P is pure projective iff P_i is pure projective for all i in I.

Proof. Suppose $0 \to E \to F \xrightarrow{u} G \to 0$ is pure exact. Since $H(P,u) = \prod H(P_i,u)$ (i in I) (B2, Cor. 2, p. 27), the product map $H(P,u)$ is epi iff each component $H(P_i,u)$ is epi (B2, Cor., p. 23).

THEOREM 2.2. Every fp module is pure projective.

Proof. Suppose $0 \to E \to F \xrightarrow{u} G \to 0$ is pure exact. If M is any fp module then $H(M,u)$ is epi by Theorem 1.2.

THEOREM 2.3. For any module E there exists a pure exact sequence $0 \dashrightarrow K \dashrightarrow P \dashrightarrow E \dashrightarrow 0$ with P a direct sum of fp modules, and hence pure projective.

Proof. For any fp module M and any h in the <u>set</u> Hom(M,E). Let M_h be a copy of M. Define $\hat{M}$ to be the direct sum of the M_h, with h ranging over Hom(M,E). Define P to be the direct sum of the $\hat{M}$, with M ranging over the <u>set</u> of all fp modules. The class of all fp modules is a set, and hence we can make this construction; the same comment applies to the construction of $\hat{M}$, since Hom(M,E) is a set.

Let $u:P \dashrightarrow E$ be the canonical homomorphism. Then u is epi since every module can be written as the direct limit of fp modules (Bl, Ex. 10, p. 62).

For any fp module M, the map $H(M,u):Hom(M,P) \dashrightarrow Hom(M,E)$ is epi since for any h in Hom(M,E) let $h':M_h \dashrightarrow P$ be the canonical injection. Then $H(M,u)(h') = uh' = h$. Hence $0 \dashrightarrow K \dashrightarrow P \dashrightarrow E \dashrightarrow 0$ is pure exact.

Remark 2.3. Maranda (32) defined and studied pure projectivity for PID's. We have extended his results to arbitrary rings. Our Theorem 1.2 is a generalization of his Lemma 1 for PID's. He remarks "that one need not restrict oneself to the class of all cyclic modules ... but that one may use just certain types of cyclic modules or even more generally, any arbitrary class of modules with suitable properties." We have shown (Theorem 1.2) that the suitable class to choose is the class of fp modules.

THEOREM 2.4 (Structure Theorem for Pure Projective Modules). For any module P the following conditions are equivalent:

(1) P is pure projective.

(2) Every pure exact sequence of the form $0 \to K \to E \to P \to 0$ splits.

(3) P is a direct summand of a direct sum of fp modules.

(4) P is a direct sum of countably generated pure projective modules.

Proof.

(1) ⟹ (2): Since P is pure projective, $H(P,u):Hom(P,E) \dashrightarrow Hom(P,P)$ is epi and there exists an h in Hom(P,E) such that $uh = 1_P$ and the sequence splits.

(2) ⟹ (3): By Theorem 2.3, there exists an exact sequence of the form $0 \dashrightarrow K \dashrightarrow E \dashrightarrow P \dashrightarrow 0$ with E a direct sum of fp modules. This sequence splits by (2) and we have the desired result.

(3) ⟹ (4): By Kaplansky ((26), Thm. 1) P is a direct sum of countably generated modules. Since P is a direct summand of pure projective (since fp) modules, it is pure projective by Proposition 2.1. By the same proposition the countably generated modules are pure projective, since they are direct summands of a pure projective module.

(4) ⟹ (1): Follows immediately from Proposition 2.1.

Remark 2.4. Theorem 2.4 is a generalization of the following well-known result for PID's (see Kaplansky (25), p. 15): If H is a pure submodule of G such that G/H is a direct sum of cyclic modules, then H is a direct summand.

The next theorem shows that any pure submodule is in a certain sense "locally" a direct summand.

THEOREM 2.5. Let $0 \dashrightarrow E \dashrightarrow F \dashrightarrow G \dashrightarrow 0$ be an exact sequence of left A-modules. Then E is pure in F iff E is a direct summand of all modules D such that $E \subseteq D \subseteq F$ and D/E is fp.

Proof.

$\Longrightarrow$: Suppose D is such a module and E is pure in F. Then E is pure in D by Proposition 1.2 of Chapter 1, and therefore we have a pure exact sequence $0 \dashrightarrow E \dashrightarrow D \dashrightarrow D/E \dashrightarrow 0$ which is split exact by Theorem 2.4 since D/E is pure projective by Theorem 2.2.

$\Longleftarrow$: We use Cohn's Criterion (Theorem 1.1 of Chapter 1). Suppose $\sum a_{ij}f_j = x_i$ in E with i in I, j in J, finite index sets. Let D be the submodule of F generated by E and all the f_j. Let M be the free left module with base m_j, j in J, and K the fg submodule generated by $k_i = \sum a_{ij}m_j$. Then M/K is fp by Proposition 1.1 of Chapter 0.

We will show that D/E is fp by showing that $D/E \cong M/K$. Define $u:M \dashrightarrow D/E$ by $u(m_j) = \hat{f}_j$ (where $\hat{d}$ denotes the image of d in D/E). This homomorphism is clearly onto since D is generated by E and the f_j. Since $u(k_i) = \sum a_{ij}\hat{f}_j = \hat{x}_i = 0$, K is contained in the kernel of u. Passing to quotient modules, we have an isomorphism $M/K \cong D/E$.

Therefore E is a direct summand of D, say $D = E \oplus H$. Hence $f_j = e_j + h_j$ and $x_i = \sum a_{ij}f_j = \sum a_{ij}e_j + \sum a_{ij}h_j$, and for all i in I $x_i - \sum a_{ij}e_j \in E \cap H = 0$. Therefore $\sum a_{ij}f_j = \sum a_{ij}e_j$ for all i in I and E is pure in F by Cohn's Criterion.

THEOREM 2.6. Let $f:A \dashrightarrow B$ be any ring homomorphism and E any left A-module.

(1) If E is A-fp then E_B is B-fp.

(2) If E is the direct sum (resp. direct limit) of A-fp modules, then E_B is the direct sum (resp. direct limit) of B-fp modules.

(3) If E is A-pure projective, then E_B is B-pure projective.

Proof.

(1) follows immediately from (Bl, p. 36).

(2) follows immediately from (1) and the facts:

(i) $E_B = B \otimes_A E$ for any E.

(ii) $\otimes_A$ commutes with both the direct sum and the direct limit.

(3) follows from the above and the usual direct sum argument, since pure projectivity is equivalent to being a direct summand of a direct sum of fp modules.

3. Types of Purity

In this section, we add some results to those of Section 3 of Chapter 1. We will again use I-purity and III-purity.

THEOREM 3.1. Let $0 \to E \to F \xrightarrow{u} G \to 0$ be an exact sequence of left A-modules. Then $aF \cap E = aE$ for all a in A (i.e. E is III-pure in F) iff $H(N,u):Hom(N,F) \to Hom(N,G)$ is epi for all principal cyclic modules N.

Proof. The proof is analogous to, but much simpler than the proof of Theorem 1.2.

The following corollaries are obvious:

COROLLARY 1. If $H(N,u)$ is epi for all cyclic modules N then $aF \cap E = aE$ for all a in A.

COROLLARY 2. If A is a left principal ideal ring then $aF \cap E = aE$ iff $H(N,u)$ is epi for all cyclic modules N.

We can now deduce, using the structure of the left A-modules, some facts concerning I- and III-purity.

THEOREM 3.2. If every left fp module M is a direct summand of a direct sum of left principal cyclic modules, then III $\Rightarrow$ I.

Proof. Suppose $M \oplus M' = \oplus N_i$ with N_i principal cyclic. Then $H(M,u) \oplus H(M',u) = \prod H(N_i,u)$. If each $H(N_i,u)$ is epi, so is $H(M,u)$.

COROLLARY 1. If A is any one of

(a) PID,

(b) semi-principal (= Bezout) domain,

(c) uniserial ring,

then III $\Rightarrow$ I (and hence III $\Rightarrow$ II).

Proof. These follow from Corollary 1 of Theorem 3.3 of Chapter 1: (a) and (b) since A is commutative and (c) since uniserial is left-rt. symmetric.

4. Rings for which Pure Submodules are Direct Summands

We know (Proposition 1.3 of Chapter 1) that every direct summand is pure. Here we shall study the class of rings for which the converse holds.

The ring A will be called left PDS iff pure submodules of left A-modules are direct summands; and PDS iff it is both left and rt. PDS.

THEOREM 4.1. For any ring A the following conditions are equivalent:

(1) A is left PDS.

(2) Every left A-module is pure projective.

(3) Every left A-module is the direct sum of countably generated pure projective modules.

(4) Every pure exact sequence is split exact.

Proof.

(1) $\Rightarrow$ (2): For any left A-module E, there exists a pure exact sequence $0 \to K \to P \to E \to 0$ with P pure projective, by Theorem 2.3. Since K is pure in P, K is a direct summand, and E

is a direct summand of a pure projective module, hence pure projective.

The equivalence of (2) and (3) is immediate by Theorem 2.4.

(2) ⇒ (4): If 0 --> D --> E --> F --> 0 is pure exact, then by Theorem 2.4 it is split exact since F is pure projective.

(4) ⇒ (1): If D is pure in E then the exact sequence 0 --> D --> E --> E/D --> 0 is pure exact, hence split exact and D is a direct summand of E.

THEOREM 4.2. Every left PDS ring is left artinian.

Proof. Since pure submodules are direct summands, A is left perfect by Corollary 1 of Proposition 1.4 of Chapter 1. Since every left A-module is the direct sum of countably generated A-modules (Theorem 4.1), A is left noetherian by Faith's Theorem on noetherian rings (see Faith (16), Theorem 1.1). But as Bass ((4), p. 475) has remarked, any left perfect, left noetherian ring is left artinian.

We do not know the complete structure of PDS rings. However, we can show:

THEOREM 4.3. Every uniserial ring A is PDS.

Proof. Every left A-module is the direct sum of cyclic modules. Since A is left artinian, and therefore left noetherian, these cyclic modules are fp modules. (For detailed reasons, see Corollary 1 of Theorem 3.3 of Chapter 1.) Hence every left A-module is pure projective and A is left PDS by Theorem 4.1. Since uniserial is left-rt. symmetric, A is rt. PDS, i.e. PDS.

It is of some interest to classify rings for which every fg flat module is projective. To this end we make the following small contribution:

PROPOSITION 4.4. Let F be a fg flat left A-module. Then the following conditions are equivalent:

(1) F is projective.

(2) F is fp.

(3) F is pure projective.

Proof.

(1) $\Rightarrow$ (2): Every fg projective module is fp (Bl, Lemma 8, p. 36).

(2) $\Rightarrow$ (3): Every fp module is pure projective by Theorem 2.2.

(3) $\Rightarrow$ (1): Let $0 \to D \to E \to F \to 0$ be an exact sequence with E projective. Since F is flat, the sequence is pure exact (Proposition 1.3 of Chapter 1). Since F is pure projective, the sequence splits by Theorem 4.1, and F is projective.

CHAPTER 3: REGULAR MODULES

In this chapter, we define the concept of regular module. We will show that regular modules bear the same relationship to (von Neumann) regular rings as semi-simple modules bear to semi-simple rings.

1. Another Property of Purity

Our main theorem gives a very useful property of purity:

THEOREM 1.1. Suppose we have an exact commutative diagram of left A-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & F & \xrightarrow{u} & G & \longrightarrow & 0 \\
 & & \downarrow & & b\downarrow & & \downarrow c & & \\
0 & \longrightarrow & E' & \longrightarrow & F' & \xrightarrow[v]{} & G' & \longrightarrow & 0
\end{array}$$

with c an isomorphism. If E is pure in F, then E' is pure in F'.

Proof. We have $cu = vb$ and hence for any (fp) M $H(M,c)\, H(M,u) = H(M,v)\, H(M,b)$. Since c is iso, so is $H(M,c)$. Hence $H(M,u)$ epi $\Rightarrow$ $H(M,v)$ epi. And therefore if E is pure in F, then E' is pure in F'.

COROLLARY 1. Let P and Q be two submodules of M. Then

(1) $(P \cap Q)$ pure in $Q \Rightarrow P$ pure in $(P+Q)$.

(2) $(P+Q)$ pure in M and $(P \cap Q)$ pure in $Q \Rightarrow P$ pure in M.

(3) $(P+Q)$ pure in M and $(P \cap Q)$ pure in $M \Rightarrow P$ pure in M and Q pure in M.

(4) $P \cap Q$ pure in $P+Q \Rightarrow P$ and Q are both pure in $P+Q$.

Proof. We have an exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P \cap Q & \longrightarrow & Q & \longrightarrow & Q/(P \cap Q) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow c & & \\
0 & \longrightarrow & P & \longrightarrow & P+Q & \longrightarrow & (P+Q)/P & \longrightarrow & 0
\end{array}$$

where all homomorphisms arise from the natural injections, and c is an isomorphism.

(1) is a straightforward application of the theorem.

(2) By (1), P is pure in $(P+Q)$. But $(P+Q)$ is pure in M and hence P is pure in M by Proposition 1.2 of Chapter 1.

(3) $(P \cap Q)$ pure in $M \Rightarrow (P \cap Q)$ pure in P and $(P \cap Q)$ pure in Q. Apply (2).

(4) Apply (3) with $M = P+Q$.

COROLLARY 2. For all i in I, any index set, let N_i be a submodule of a fixed module M and let $N = \sum N_i$ (i in I). For each k in I define $\hat{N}_k = \sum N_i$ (i in I, $i \neq k$). Then for all k in I, N pure in M and $(N_k \cap \hat{N}_k)$ pure in $\hat{N}_k \Rightarrow N_k$ pure in M.

Proof. Apply Corollary 1 with $P = N_k$ and $Q = \hat{N}_k$.

Remark 1.1. Parts (2) and (3) of Corollary 1 are the most important for us. We give an example to show that the converses are false. Take A = Z, the ring of integers, and $M = Z \oplus Z$. Let P = the subgroup generated by (1,1), and Q = the subgroup generated by (1,-1). Then both P and Q are pure since $n(a,b) = m(1,\pm 1) \Rightarrow na = m = \pm nb$ and therefore $a = \pm b$; hence (a,b) is in P or Q and n(a,b) is in nP or in nQ. (See Remark 1.1 (ii) of Chapter 1.) The elements of P+Q have the form a(1,1) + b(1,-1) = (a+b,a-b). Now 2(1,0) = (1,1) + (1,-1) is in P+Q. But if 2(1,0) = 2(a+b,a-b), then a+b = 1 and a-b = 0, which cannot be solved for integers. Hence P+Q is not pure in M.

2. Regular Rings

In this section, we give some new characterizations of regular rings, which will be used to define regular modules in Section 3.

The ring A will be called (von Neumann) regular iff $a \in aAa$ for all a in A. Bourbaki uses the word regular in (B3, Ex. 15, p. 76) but has changed this to absolutely flat in (B1, Ex. 16, p. 64). We will use the word regular with this single meaning throughout.

We remark that regular is a left-rt. symmetric concept. Therefore, all our results about left ideals or modules will have analogues for rt. ideals and modules, which will be assumed and used, although they have not been stated explicitly.

Before proceeding, we recall that regular rings are characterized by the fact that all modules are flat (B1, Ex. 16, p. 64).

THEOREM 2.1. The following conditions are equivalent for any ring A:

(1) A is regular.

(1)' Every fg left ideal is a direct summand.

(1)" Every principal left ideal is a direct summand.

(2) Every left ideal is pure.

(2)' Every fg left ideal is pure.

(2)" Every principal left ideal is pure.

(3) $K \cap I = KI$ for all rt. ideals K and all left ideals I.

(3)' $K \cap I = KI$ for all fg rt. ideals K and all fg left ideals I.

(3)" $K \cap I = KI$ for all principal rt. ideals K and principal left ideals I.

(4) $aA \cap Ab = aAb$ for all a and b in A.

Proof. We shall give the proof according to the following schema;

$$\begin{array}{ccccccccc}
(1) & \Rightarrow & (1)' & \Rightarrow & (1)'' & & & & \\
\Downarrow & & \Downarrow & & \Downarrow & & & & \\
(2) & \Rightarrow & (2)' & \Rightarrow & (2)'' & & & & \\
\Downarrow & & \Downarrow & & \Downarrow & & & & \\
(3) & \Rightarrow & (3)' & \Rightarrow & (3)'' & \Rightarrow & (4) & \Rightarrow & (1)
\end{array}$$

The equivalence of (1), (1)' and (1)" is well known and given in (B3, Ex. 15, p. 76).

The implications (2) $\Rightarrow$ (2)' $\Rightarrow$ (2)" and (3) $\Rightarrow$ (3)' $\Rightarrow$ (3)" $\Rightarrow$ (4) are obvious.

(1) $\Rightarrow$ (2): For any left ideal I, A/I is flat since all left

A-modules are flat in a regular ring (Bl, Ex. 16, p. 64). Therefore, I is pure in A by Theorem 2.1 of Chapter 1.

(2) $\Rightarrow$ (3): follows immediately from Theorem 2.1 of Chapter 1.

The implications (1)' $\Rightarrow$ (2)' and (1)" $\Rightarrow$ (2)" hold since every direct summand is pure.

The implications (2)' $\Rightarrow$ (3)' and (2)" $\Rightarrow$ (3)" are immediate consequences of Theorem 2.1 of Chapter 1.

(4) $\Rightarrow$ (1): For all a in A, $a \in aA \cap Aa = aAa$ and A is regular.

COROLLARY.

(1) If A is regular, every left (or rt.) ideal is idempotent.

(2) The converse holds if A is commutative.

Proof.

(1) Since A is regular, every left (or rt.) ideal is pure, and therefore idempotent by Corollary 3 of Theorem 2.1 of Chapter 1.

(2) For any ideals P and K, $K \cap P$ will be an idempotent <u>ideal</u> (since A is commutative). Therefore by the same Corollary 3, referred to above, P will be pure in A and A will be regular.

PROPOSITION 2.2. For any element a in A, the following conditions are equivalent:

(1) Aa is a direct summand of A.

(2) Aa is a pure left ideal of A.

(3) $a \in aAa$.

Proof.

(1) $\Rightarrow$ (2): is immediate since every direct summand is pure.

(2) $\Rightarrow$ (3): By Theorem 2.1 of Chapter 1 $aA \cap Aa = aAAa = aAa$ since aA is a rt. ideal. But a is in $aA \cap Aa$.

(3) $\Rightarrow$ (1): If $a = axa$ then $xa = e$ is an idempotent and $Aa = Ae$, which is a direct summand of A.

Following Lambek (30), we will call a ring A <u>semi-primitive</u> iff its Jacobson radical J(A) is zero, and <u>semi-prime</u> iff its prime radical is zero. We remark that since the prime radical of A is contained in J(A), every semi-primitive ring is semi-prime. Bourbaki uses "without radical" (B3, p. 64) for semi-primitive, and "reduced" (B1, Def. 5, p. 97) for semi-prime in the commutative case.

THEOREM 2.3. For any ring A, consider the following conditions:

(1) A is a regular ring.

(2) A/I is a regular ring for every two-sided ideal I of A.

(3) A/I is a semi-primitive ring for every two-sided ideal I of A.

(4) A/I is a semi-prime ring for every two-sided ideal I of A. Then we always have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). If A is commutative, (4) $\Rightarrow$ (1).

Proof.

(1) $\Rightarrow$ (2): is given in (B3, Ex. 15, p. 76).

(2) $\Rightarrow$ (3): Every regular ring is semi-primitive (B3, Ex. 15, p. 76).

(3) $\Rightarrow$ (4): Every semi-primitive ring is semi-prime.

(4) $\Rightarrow$ (1) (A commutative): Suppose K and J are any two ideals of A and let $I = KJ$. Then $\hat{K}\hat{J} = 0$ in the ring $B = A/I$, where $\hat{}$ denotes the image in B. Since B is semi-prime, $\hat{K} \wedge \hat{J} = 0$ (see Lambek (30), p. 56). Hence $K \wedge J$ is contained in KJ. Since we always have the opposite inclusion, A is regular by Theorem 2.1.

3. Regular Modules

A left A-module R will be called (von Neumann) regular iff every submodule is pure. This generalizes the idea of regular ring, as the following theorem shows:

THEOREM 3.1. For any ring A, the following conditions are equivalent:

(1) A is a regular ring.

(2) Every left A-module is regular.

(3) The left A-module A is a regular module.

Proof.

(1) $\Rightarrow$ (2): Since A is a regular ring, every rt. A-module F is flat (Bl, Ex. 16, p. 64). Hence if D is any submodule of the left A-module E, the sequence $0 \to D \otimes F \to E \otimes F \to E/D \otimes F \to 0$ is exact (Bl, Prop. 1, p. 26), and D is pure in E. Therefore E is regular.

(2) $\Rightarrow$ (3): is obvious.

(3) ⟹ (1): If A is a regular left A-module, then all the left ideals of A are pure and A is a regular ring by Theorem 2.1.

Remark 3.1.

(i) This theorem shows that any theorem about regular modules implies a theorem about modules over regular rings.

(ii) It is an easy exercise to see that A is a regular ring iff every left A-module is absolutely pure. See Remark 1.3 of Chapter 1.

PROPOSITION 3.2. R is a regular module iff every fg submodule is pure.

Proof.

⟹ : is clear.

⟸: Every submodule of R is the direct limit of fg submodules of R, i.e. the direct limit of pure submodules, and hence pure, by Corollary 4 of Theorem 1.7 of Chapter 1.

THEOREM 3.3. Let $0 \to R \to S \to T \to 0$ be an exact sequence of left A-modules. Then S is a regular module iff both R and T are regular modules, and R is pure in S.

Proof. In the course of this proof, we shall refer several times to Proposition 1.2 of Chapter 1. For convenience, this will be denoted (1.2) for this proof only.

$\Rightarrow$: R is regular since every submodule of R is a submodule of S, hence pure in S, and therefore pure in R by (1.2). Every submodule of T has the form V/R with $R \subseteq V \subseteq S$. But V is then pure in S and therefore V/R is pure in S/R = T by (1.2). Hence T is regular. And R is pure in S, since S is regular.

$\Leftarrow$: Let V be any submodule of S. Then (V+R)/R is pure in S/R = T, since T is regular. But R is pure in S. Hence V+R is pure in S by (1.2). Also $V \cap R$ is pure in R, since R is regular. Hence $V \cap R$ is pure in S, since R is pure in S, again by (1.2). Therefore, both V+R and $V \cap R$ are pure in S. Hence V is pure in S by Corollary 1 of Theorem 1.1.

THEOREM 3.4. Let $R = \sum R_i$ (i in I, any index set) be left A-modules. Then R is a regular module iff R_i is a regular module for all i in I.

Proof.

$\Rightarrow$: For each i in I, R_i is regular by Theorem 3.3 since it is a submodule of R.

$\Leftarrow$: Since R is a homomorphic image of $S = \bigoplus R_i$ (i in I), it suffices to show that S is regular by Theorem 3.3. We will use Proposition 3.2: Let P be any fg submodule of S. Then P is a submodule of $T = \bigoplus R_i$ (i in J, J some finite subset of I). Since T is a direct summand of S, it is pure in S. Hence if we show that T is regular, we are finished because then P will be pure in T and hence in S. We have reduced the problem to proving the following lemma:

LEMMA 3.5. If $T = \bigoplus_{1}^{n} R_i$, then T is regular if each R_i is regular.

Proof. We use induction. For each $k < n$, let $T(k) = \bigoplus_{1}^{k} R_i$. Clearly $T(1) = R_1$ is regular. Assume $T(k)$ is regular; then $0 \to R_{k+1} \to T(k+1) \to T(k) \to 0$ is exact with $T(k)$ and R_{k+1} regular and R_{k+1} pure in $T(k+1)$ (since it is a direct summand). Hence $T(k+1)$ is regular by Theorem 3.3.

COROLLARY. For any left A-module R, the following conditions are equivalent:

(1) R is a regular module.

(2) Ax is a regular module for all x in R and Ax is pure in R.

(3) Ax is a regular module for all x in R.

Proof. The proof is clear since $R = \sum Ax$ (x in R).

4. Regular Projective Modules

The main result of this section is a structure theorem for regular projective modules, i.e. modules which are both regular and projective.

THEOREM 4.1. Suppose $0 \dashrightarrow P \dashrightarrow F \dashrightarrow E \dashrightarrow 0$ is exact with F free. Then the following conditions are equivalent:

(1) P is pure in F.

(2) E is flat.

(3) Given any x in P, there exists a homomorphism $u:F \dashrightarrow P$ such that $u(x) = x$.

(4) Given any x_i in P, $1 \leq i \leq m$, there exists a homomorphism $u:F \dashrightarrow P$ such that $u(x_i) = x_i$ for all i.

Proof. The equivalence of (1) and (2) has been shown (Proposition 1.3 of Chapter 1). The equivalence of (2), (3) and (4) has been shown by Chase ((10), Prop. 2.2), who attributes the result to Villamayor.

COROLLARY. Suppose $0 \to P \to Q \to E \to 0$ is exact with Q projective and P pure in Q. Then given x_i in P, $1 \leq i \leq m$, there exists a homomorphism $u: Q \to P$ such that $u(x_i) = x_i$ for all i.

Proof. Since Q is projective, there exists $F = Q \oplus Q'$ with F free. And P pure in Q, Q pure in $F \Rightarrow P$ pure in F by Proposition 1.2 of Chapter 1. By the theorem, there exists $w: F \to P$ such that $w(x_i) = x_i$. Let $u = w \mid Q$. Then $u: Q \to P$ and $u(x_i) = w(x_i) = x_i$ for all i, since x_i is in P.

THEOREM 4.2. Suppose $0 \to P \to Q \to F \to 0$ is exact with P fg and Q projective. Then P is pure iff P is a direct summand.

Proof. Since any direct summand is pure, it suffices to show the converse. Suppose then that P is pure and let x_i in P, $(1 \leq i \leq m)$, generate P. Then there exists $u: Q \to P$ such that $u(x_i) = x_i$ for all i. If $j: P \to Q$ is the natural injection, then we have $uj = 1_p$, whence the sequence splits.

COROLLARY 1. If Q is regular projective then every fg submodule is a direct summand.

Proof. Every (fg) submodule is pure.

COROLLARY 2 (Osofsky (38)). A is regular iff every fg submodule of a projective module is a direct summand.

Proof.

$\Rightarrow$: If A is regular, every module is regular (Theorem 3.1), and the result follows from Corollary 1.

$\Leftarrow$: Every fg left ideal is a direct summand, hence pure, and A is regular by Theorem 2.1.

THEOREM 4.3 (Structure Theorem for Regular Projective Modules). A left A-module P is regular projective iff $P = \oplus J_i$ where J_i is a regular projective principal left ideal, which is a direct summand of A.

Remark 4.3. This generalizes and simplifies the proof of a theorem of Kaplansky ((26), Thm. 4).

Proof.

$\Leftarrow$: If each J_i is regular, so is P (Theorem 3.4). If each J_i is projective, so is P. Hence the result in one direction.

$\Rightarrow$: By Kaplansky's Theorem ((26), Thm. 1), every projective module is the direct sum of cg (= countably generated) projective modules; hence we can reduce our problem to this case, and assume that P is cg. Let x_i, $i = 1, 2, 3, \ldots$ generate P. We shall define inductively y_i in P, $i = 1, 2, 3, \ldots$ such that: For all n, the sum $P_n = \sum_1^n Ay_i$ is direct and $P = \bigoplus_1^\infty Ay_i$. Define $y_1 = x_1$ and assume y_i defined for $i \leq n$, so that $P_n = \bigoplus_1^n Ay_i$. Since P_n is fg pure and P is projective, there exists Q so that $P = P_n \oplus Q$. Let $x_{n+1} = p_n + y_{n+1}$ (p_n in P_n, y_{n+1} in Q). Clearly the sum $P_{n+1} = \sum_1^{n+1} Ay_i$ is direct. Since for all n the sum $P_n = \bigoplus_1^n Ay_i$ is direct, so is the sum $P' = \bigoplus_1^\infty Ay_i$. Also for each n, x_{n+1} is in $P_n \oplus Ay_{n+1} = P_{n+1}$. Therefore P is contained in P'. The opposite inclusion holds too since each y_n is in $P = P_n \oplus Q$. Since P is regular projective, so is Ay_n for all n. Since Ay_n is projective, it is isomorphic to a left ideal J_n, which is a direct summand of A, and hence principal.

COROLLARY 1. If P is a regular projective module, every cg submodule is projective.

Proof. Let x_i $i = 1, 2, \ldots$ generate the cg submodule M of P. We define y_i, $i = 1, 2, \ldots$ exactly as we did in the proof of the theorem. Then $M = \bigoplus_1^\infty Ay_i$ is projective since each Ay_i is projective.

COROLLARY 2. If A is regular, every cg submodule of a projective module is projective.

THEOREM 4.4. Let P be a regular projective left A-module. Then

(1) If A is left indecomposable, then P is free.

(2) If A is left pure simple, then either $P = 0$ or A is left simple (i.e. A has no left ideals other than 0 and A).

Proof.

(1) By Theorem 4,3, $P = \bigoplus J_i$ where the J_i are left ideals which are direct summands of A; therefore $J_i = 0$ or A.

(2) Continuing from (1), if $J_i = A$ for some i, then A is regular since the J_i are all regular. But A is pure simple, and therefore A must be left simple. If A is not left simple, then we must have $J_i = 0$ for all i, and therefore $P = 0$.

COROLLARY. If A is an integral domain which is not a field, 0 is the only regular projective A-module.

Proof. A is pure simple, but not simple.

A fg module E will be said to be <u>n-generated</u> (n an integer) iff there exists a finite set of generators with not more than n elements. A module E will be said to be c-generated (c any cardinal) iff there exists a generating set with cardinality c.

PROPOSITION 4.5. Let $D \subseteq P \subseteq Q$ with

(1) D fg.

(2) P pure in Q.

(3) Q an n-generated projective module.

Then there exists an n-generated submodule E of P such that $D \subseteq E \subseteq P$.

Proof. Let x_i, $1 \leq i \leq m$, generate D. Then by the corollary to Theorem 4.1, there exists a homomorphism $u: Q \dashrightarrow P$ such that $u(x_i) = x_i$ for all i. Hence $u(x) = x$ for all x in D. Then $D = u(D) \subseteq u(Q) = E \subseteq P$ and E is n-generated.

THEOREM 4.6. If Q is n-generated projective, then every pure submodule is the direct limit of n-generated submodules.

Proof. Any pure submodule P is the direct limit of its fg submodules. By Proposition 4.5, the n-generated submodules are cofinal and hence P is the direct limit of n-generated submodules.

COROLLARY 1. Let P be an n-generated projective module (n an integer). Then P is regular iff every n-generated submodule is a direct summand (and hence pure) and every submodule is the direct limit of n-generated submodules.

Proof.

$\Rightarrow$: is clear by the definition of regular module and Theorems 4.2 and 4.6.

$\Leftarrow$: Every submodule is the direct limit of pure submodules, hence pure by Corollary 3 of Theorem 1.6 of Chapter 1. Therefore P is regular.

COROLLARY 2. The preceding corollary is true if we replace n by c, any cardinal.

Proof. This is clear since any module is the direct limit of fg modules and hence c-generated modules.

COROLLARY 3. A is a regular ring iff every principal left ideal is a direct summand (and hence pure) and every left ideal is the direct limit of principal ideals.

Proof. A is a 1-generated projective module.

CHAPTER 4: THE REGULAR SOCLE

In this chapter, we define socles which are generalizations of the usual socle (B3, Ex. 9, p. 58). We remark that both Maranda (33) and Dickson (12) have studied radicals, and in doing so, have introduced preradicals which correspond to our socles. However, there is little, if any, overlap with our work.

We also define a regular socle (of a module or ring) and compare it with the semi-simple (= usual) socle and with the regular radical of Brown and McCoy (7).

1. Socles

Let $\underline{C}$ be the category of all left A-modules. A socle is a function T which assigns to each module M of $\underline{C}$ a submodule $T(M)$ of M in such a way that $f:M \dashrightarrow N \Rightarrow f(T(M)) \subseteq T(N)$, i.e. $f(T(M))$ in a submodule of $T(N)$ (or equivalently $f_T = f \mid T(M)$ is a map from $T(M)$ to $T(N)$). In categorical language, a socle is a subfunctor of the identity functor.

Let T be a socle. We make the following definitions:

T is <u>torsion</u> iff $T(N) = N \cap T(M)$ for all submodules N of M.

T is <u>idempotent</u> iff $T^2 = T$, i.e. $T(T(M)) = T(M)$ for all M.

T has <u>radical property</u> iff $T(M/T(M)) = 0$ for all M.

A module M is <u>T-complete</u> iff $T(M) = M$.

If T and T' are socles, $T \leqslant T'$ iff $T(M)$ is a submodule of $T'(M)$ for all modules M. It is easy to verify that if T is torsion, then it is idempotent and $T(M)$ is T-complete.

We now prove a theorem which establishes the basic properties of socles.

THEOREM 1.1. Let T be any socle.

(1) If N is a submodule of M, then $T(N)$ is a submodule of $T(M)$ and $(T(M)+N)/N$ is a submodule of $T(M/N)$.

(2) $T(A)$ is a two-sided ideal of A.

(3) T commutes with direct sums, i.e. $T(\bigoplus M_i) = \bigoplus T(M_i)$.

(4) $T(P) = T(A)P$ for all projective modules P.

(5) $T(A)M$ is a submodule of $T(M)$ for all modules M.

(6) If M is T-complete, so is any image of M.

(7) If $T(M)$ is T-complete, it is the largest T-complete submodule of M.

Proof.

(1) Let $k:N \dashrightarrow M$ and $f:M \dashrightarrow M/N$ be the canonical maps. Since T is a socle, $T(N) = k(T(N)) \subseteq T(M)$ and $(T(M)+N)/N = f(T(M)+N) \subseteq T(M/N)$.

(2) $T(A)$ is a left ideal by definition. For any a in A, define $f_a:A \dashrightarrow A$ (as left A-modules) by $f_a(x) = xa$ for all x in A. Since T is a socle, $(T(A))a = f_a(T(A)) \subseteq T(A)$. Hence $T(A)$ is also a right ideal.

(3) Let $M = \oplus M_i$. By (1) for each i, $T(M_i)$ is a submodule of $T(M)$. Hence $\sum T(M_i) = \oplus T(M_i) \subseteq T(M)$. The sum is direct since for each i, $T(M_i)$ is a submodule of M_i. Let $p_i:M \dashrightarrow M_i$ be the canonical projection. Then $p_i(T(M))$ is a submodule of $T(M_i)$. If x is in $T(M)$ then $x = \sum x_i$ with x_i in M_i. Then $x_i = p_i(x)$ which is in $T(M_i)$. Hence $T(M) = \oplus T(M_i)$.

(4) If F is free, $F = \bigoplus A$ and by (3),

$T(F) = \bigoplus T(A) = \bigoplus (T(A)A) = T(A)(\bigoplus A) = T(A)F$. If P is projective then $F = P \oplus Q$ with F free. Hence

$T(P) \oplus T(Q) = T(F) = T(A)(P \oplus Q) = T(A)P \oplus T(A)Q$. Therefore $T(P) = T(A)P$.

(5) For any M, let $f: F \dashrightarrow M$ be epi with F free. Then $f(T(F)) = f(T(A)F) = T(A)f(F) = T(A)M$. But $f(T(F))$ is a submodule of $T(M)$. Hence the result.

(6) Let $f: M \dashrightarrow N$ be epi and $T(M) = M$. Then $N = f(M) = f(T(M)) \subseteq T(N) \subseteq N$.

(7) If $T(N) = N \subseteq M$ then $N = T(N) \subseteq T(M)$ by (1).

COROLLARY. $T(P) = 0$ for all projective modules P iff $T(A) = 0$.

Proof. Obvious since $T(P) = T(A)P$ by (4).

PROPOSITION 1.2. If the socle T has radical property, then $T(M)$ is the smallest of the submodules N of M such that $T(M/N) = 0$.

Proof. By definition of radical property, $T(M/T(M)) = 0$. If for some N we have $T(M/N) = 0$, then $(T(M)+N)/N \subseteq T(M/N) = 0$ and $T(M)+N = N$. Hence $T(M) \subseteq N$.

2. Semi-Simple and Regular Socles

In this section, we shall define the regular socle of a module which is analogous to the semi-simple (= usual) socle of a module, with purity playing the rôle of direct summand.

A left A-module $M \neq 0$ is called simple iff 0 and M are its only submodules, and semi-simple iff it is the sum of simple modules. For basic facts on (semi-) simple modules and rings, see (B3) or Lambek (30).

For any left A-module M, its ss (= semi-simple) socle $S(M)$ is defined to be the sum of all its simple submodules (= the sum of all its semi-simple submodules). In an analogous way, we will define the regular socle $R(M)$ of a module M to be the sum of all its regular submodules (i.e. submodules which are regular modules). Thus $R(M) = \sum Ax$ (x in M and Ax regular).

THEOREM 2.1. Both the ss socle S and the regular socle R are torsion socles, and hence have all the properties given in Theorem 1.1. A module is S-complete iff it is semi-simple, and R-complete iff it is regular, i.e. $S(M) = M$ iff M is semi-simple and $R(M) = M$ iff M is regular. Also $S \leq R$.

Remark 2.1.

(i) Neither S nor R are radicals. See Proposition 2.3 for examples and discussion.

(ii) By taking A to be a ring which is one of

(a) regular,
(b) not regular,
(c) semi-simple,
(d) not semi-simple,
(e) regular but not semi-simple,

we easily have examples where

(a) $R(M) = M$,
(b) $R(M) \neq M$,
(c) $S(M) = M$,
(d) $S(M) \neq M$,
(e) $S(M) \neq R(M)$ etc.

Proof. Let T be either S or R.

Socle: $T(M)$ was defined to be a submodule. $T(M)$ is the sum of simple (resp. regular) modules. Hence if $f:M \dashrightarrow N$ then $f(T(M))$ is the sum of simple (resp. regular) modules, since the image of a simple module is simple or zero (easy to verify), and the image of a regular module is regular (Theorem 3.3 of Chapter 3).

Torsion: If N is a submodule of M, we know that $T(N)$ is a submodule of $T(M)$, by Theorem 1.1. $T(M)$ is the sum of simple or regular modules and hence $T(M)$ is either semi-simple (well known) or regular (Theorem 3.4 of Chapter 3). Hence the submodule $N \cap T(M)$ of $T(M)$ is either semi-simple (well known) or regular (Theorem 3.3 of Chapter 3), and therefore contained in $T(N)$.

The properties concerning S-complete and R-complete are clear. Also $S \leq R$ since every simple module is regular.

For any socle T, the ring A will be called <u>left T-faithful</u> iff for all left A-modules $M \neq 0$, we have $T(M) \neq 0$.

For example, Bass (4) has shown that a left perfect ring is rt. S-faithful, where S = ss socle. See Theorem 3.1 of Chapter 0.

Clearly if $T \leq T'$ then if A is left T-faithful, it is left T'-faithful. Hence A left perfect $\Rightarrow A$ is rt. S-faithful $\Rightarrow A$ is rt. R-faithful.

PROPOSITION 2.3. If A is left T-faithful for some socle T, then the left A-module M is T-complete iff $T(M/T(M)) = 0$.

Proof.

$\Longrightarrow$: is clear since $T(M) = M$.

$\Longleftarrow$: Since A is T-faithful, we have $M/T(M) = 0$ whence $M = T(M)$ and M is T-complete.

COROLLARY 1. If A is left T-faithful, then the socle T has radical property iff all left A-modules are T-complete.

Proof.

$\Longrightarrow$: For any M, $T(M/T(M)) = 0$ whence $M/T(M) = 0$ and $M = T(M)$.

$\Longleftarrow$: is clear since $T(M) = M$ for all M.

COROLLARY 2. Neither S nor R have radical property.

Proof. Let A be left perfect, but not regular. Then A is rt. T-faithful for $T = R$ or S by Bass (4), but not all rt. modules are regular (= R-complete) and hence not all are semi-simple (= S-complete).

Before stating the next proposition, we recall that the (Johnson) singular submodule K(E) of a left A-module E is the submodule consisting of all x in E such that $O(x) = (a \in A \mid ax = 0)$ is a large left ideal of A. For details see Johnson (23) or Lambek (30). Following Bourbaki (B1, Ex. 24, p. 164), the ring A will be called <u>left neat</u> iff K(A) = 0, regarding A as a left A-module, i.e. $K({}_AA) = 0$, and <u>neat</u> iff it is both left and rt. neat.

PROPOSITION 2.4. The singular submodule functor K is a torsion socle.

Proof. Suppose u:E ⟶ F. Since O(x) is contained in O(ux) for all x in E, x in K(E) ⟹ O(x) is large in A ⟹ O(ux) is large in A ⟹ u(x) is in K(F) and K is a socle.

If E is a submodule of F, then K(E) is contained in $E \cap K(F)$ by Theorem 1.1. Conversely, if x is in $E \cap K(F)$, O(x) is large in A and x is in K(E). Therefore K is a torsion socle.

3. The Brown-McCoy Regular Radical

In this section, we compare our regular socle R(A) with the Brown-McCoy regular radical.

THEOREM 3.1. If A is a left semi-principal ring (i.e. every fg left ideal is principal), then for any a in A, Aa is regular iff Aba is pure in Aa for all b in A.

Proof.

$\Rightarrow$: holds by definition of regular.

$\Leftarrow$: Let $\sum Ab_i a$ be any fg subideal of Aa, then $\sum Ab_i a = (\sum Ab_i)a = Aba$ since A is left semi-principal and Aba is pure in Aa, which is therefore regular.

Brown and McCoy (7) define a regular radical M(A) of the ring A. They call an <u>element</u> a in A regular iff $a \in aAa$, and a two-sided ideal regular iff all its elements are regular. They then prove that the set $M(A) = (a \in A \mid (a) = AaA$ is a regular two-sided ideal) is a two-sided ideal of A, which they call the regular radical. Thus A is a regular ring iff M(A) = A. We note and emphasize that the Brown-McCoy use and our use of the word regular differ. We shall not use regular in the Brown-McCoy sense, except in this paragraph, since we can avoid it in view of Proposition 2.2 of Chapter 3: An element a in A is Brown-McCoy regular iff Aa is pure in A (or equivalently: aA is pure in A). From now on the word regular will have the meaning given in Chapter 3.

We now show the connection between the Brown-McCoy regular radical and our regular socle.

THEOREM 3.1. If A is commutative semi-principal ring, then $M(A) = \sum Aa$ (Aa regular and Aa pure in A).

Proof. a is in M(A)

iff Aba is pure in A for all b in A (definition of M(A)),

iff Aba is pure in Aa for all b in A and Aa pure in A by Proposition 1.2 of Chapter 1,

iff Aa is regular and Aa is pure in A (Theorem 3.3 of Chapter 3).

COROLLARY. $M(A)$ is contained in $R(A)$ for A commutative semi-principal.

Proof. $R(A) = \sum Aa$ (Aa regular).

Remark 3.1. Since A is regular iff $M(A) = A$ iff $R(A) = A$ we have an example where $M(A) = R(A)$. Theorem 3.1 shows that for A commutative semi-principal, $M(A)$ is contained in $R(A)$.

We now give an example to show that this may be a strict containment:

Example 3.1. Let A be a commutative artinian principal ring which is not semi-simple. For example take $A = Z/p^nZ$ p prime, $n \geq 2$. Let P be any pure ideal. Then A/P is flat, hence projective (since an artinian ring is perfect). Therefore P is a direct summand, i.e. every pure ideal is a direct summand. Since A is not semi-simple, there exist ideals which are not direct summands, i.e. not pure. Since A is artinian, we can choose a minimal non-pure ideal I. Hence $I \neq 0$ (since 0 is a pure ideal). If J is an ideal of A, contained in I, then $J = I \Rightarrow J$ is pure in I, and

$J \not\leq I \Rightarrow J$ is pure in A by the minimality of I, and therefore J is pure in I. Hence I is regular and therefore I is contained in $R(A)$ but I is not contained in $M(A)$ since I is not pure in A.

4. The Regular Socle over a Dedekind Domain

This section is devoted to computing the regular socle $R(E)$ of a module E over any Dedekind domain which is not a field. We make use of various well-known properties of Dedekind domains, all of which can be found in Zariski-Samuel (40).

Let A be any commutative ring, $\underline{P}$ the collection of its prime ideals, and E an A-module.

We recall that the ideal Q of A is P-primary (P in $\underline{P}$) iff

(1) Q is contained in P,

(2) a in $P \Longrightarrow a^n$ in Q for some integer n.

(3) ab in Q and a not in $P \Longrightarrow b$ is in Q.

For any x in E, let $O(x) = (a \in A \mid ax = 0)$ be the order ideal. E is a torsion module iff $O(x) \neq 0$ for all $0 \neq x$ in E.
E is P-primary (P in $\underline{P}$) iff $O(x)$ is P-primary for all $0 \neq x$ in E.

If $O(x) = \prod P^{n(P)}$ (P in $\underline{P}$) with $n(P)$ an integer, and $n(P) = 0$ for almost all (i.e. all but a finite number) of P in $\underline{P}$, then we say that x has square free order iff $n(P) \leq 1$ for all P in $\underline{P}$.

We note that if A is a Dedekind domain, then any ideal $I \neq 0$ can be expressed as such a product.

THEOREM 4.1. Let A be a commutative noetherian ring and M a maximal ideal. Then

(1) An ideal $Q \neq A$ is M-primary iff $M^n \subseteq Q$ for some integer n. Also $M^n \subseteq Q \Rightarrow Q \subseteq M$.

(2) For any module E, $E_M = (0$ and $x \in E \mid 0(x)$ is M-primary) is an M-primary submodule of E, called the <u>M-primary component</u> of E.

Proof.

(1) is well known. For a proof see Northcott ((36), Prop. 9, p. 23). Also $M^n \subseteq Q \subseteq N$ for some maximal ideal N. Hence $M \subseteq N$, since N is prime, and therefore $M = N$ since both M and N are maximal ideals.

(2) If x and y are in E_M, then $0(x+y)$ contains $0(x) \cap 0(y) \supseteq M^n \cap M^{n'} = M^{n''}$. Hence $0(x+y)$ is M-primary by (1). Similarily for a in A and x in E_M, $0(ax) \supseteq 0(x) \supseteq M^n$. Therefore E_M is a submodule of E; clearly it is M-primary.

THEOREM 4.2. If A is a Dedekind domain, $\underline{P}$ the collection of its nonzero prime (= maximal) ideals, and T a torsion module, then

(1) $T = \bigoplus T_P$ (P in $\underline{P}$).

(2) If x is in T and $x = \sum x_P$ with $x_P = 0$ for almost all P. Then for $x_P \neq 0$, $O(x_P) = P^{n(P)}$ for some integer $n(P)$ and $O(x) = \prod P^{n(P)}$.

Proof.

(1) has been shown by Matlis (35).

(2) Since T_P is P-primary, $O(x_P) \supseteq P^{n(P)}$. Since A is Dedekind, for all n, there are no ideals between P^{n+1} and P^n. Hence $O(x_P) = P^{n(P)}$ for some integer $n(P)$. Let $I = \prod P^{n(P)}$. Then $Ix_P = 0$ for all P, and therefore $Ix = 0$ and $I \subseteq O(x)$. Since A is Dedekind, $O(x)$ has a factorization $O(x) = \prod P^{n'(P)}$ with $n'(P) \leq n(P)$ since $I \subseteq O(x)$. If for some Q in $\underline{P}$, $n'(Q) \neq n(Q)$, let $I' = \prod P^{m(P)}$ where $m(P) = n(P)$ for all $P \neq Q$ and $m(Q) = n'(Q)$. Then $I \subseteq I' \subseteq O(x)$ and $I'x_P = 0$ for all $P \neq Q$ but $I'x_Q \neq 0$ since $n'(Q) < n(Q)$. Therefore

$0 = I'x = I'x_Q \neq 0$. This contradiction shows that we must have $n(P) = n'(P)$ for all P and therefore $I = O(x)$.

THEOREM 4.3. If A is a semi-hereditary domain (i.e. Pruefer) which is not a field, then

(1) 0 is the only regular torsion free module.

(2) Every regular module is torsion.

Proof.

(1) Suppose F is a regular torsion free module, and take x in F. Then Ax is a fg torsion free module, hence projective (since A is Pruefer), and therefore a regular projective module. By the corollary of Theorem 4.4 of Chapter 3, $Ax = 0$. Therefore $F = 0$.

(2) Let T be the torsion submodule of the regular module R. Then $F = R/T$ is regular torsion free and therefore $F = 0$ by (1). Hence $T = R$.

Since any direct summand is pure, there are many examples of pure submodules. The following proposition shows how to construct non-pure submodules.

PROPOSITION 4.4. Let A be a ring with a left ideal I such $I^2 \neq I \neq A$. Then $E = I/I^2$ is not pure in $M = A/I^2$.

Proof. Let $I' = IA \supseteq I$. Then $I'I = I^2$ and $I'M \cap E = I'/I^2 \cap I/I^2 = I/I^2 \neq 0$, but $I'E = I^2/I^2 = 0$. Therefore E is not pure in M by Theorem 1.7 of Chapter 1.

PROPOSITION 4.5. Let I be any left ideal such that $I^2 \neq I \neq A$. Then A/I^n is regular iff $n = 1$ and A/I is regular.

Proof.

$\Longrightarrow$: By Proposition 4.4, I/I^2 is not pure in A/I^2 and therefore A/I^2 is not regular. For $n > 1$, $I^n \subseteq I^2$, hence A/I^2 is a homomorphic image of A/I^n. If A/I^n were regular, then A/I^2 would be too (Theorem 3.3 of Chapter 3). Therefore if $n > 1$, A/I^n is not regular.

$\Longleftarrow$: is obvious.

Remark 4.5. If A is a Dedekind domain, then for any ideal $I \neq A$ we have $I^2 \neq I$. This follows readily from the unique decomposition of I as a product $\prod P^{n(P)}$.

THEOREM 4.6. Let M be a maximal left ideal of A such that for all integers $n > 0$, there is no proper left ideal between M^{n+1} and M^n. Then the left module A/M^n is pure simple.

Remark 4.6. If A is a noetherian domain, then A is Dedekind iff for all maximal ideals M and all integers $n > 0$, there is no proper ideal between M^n and M^{n+1}. See Bourbaki ((6), Ex. 7, p. 92.)

Proof. The only proper submodules of A/M^n are M^m/M^n with $0 \leq m < n$. If $n = 1$, then A/M is simple and therefore pure simple. Suppose $0 < m < n > 2$. We will show that M^m/M^n is not pure in A/M^n. If it is pure, then $I\, A/M^n \cap M^m/N^n = I(M^m/M^n)$ for all rt. ideals I. Let $k = \mathrm{Max}\,(m, n-m)$ and $I = M^k A$. Then $m \leq k < n$ since $0 \neq m \neq n$. And $I\, A/M^n \cap M^m/M^n = M^k/M^n \cap M^m/M^n = M^k/M^n \neq 0$ since $m \leq k \neq n$. But $I\, M^m/M^n = M^{k+m}/M^n = 0$ since $k+m \geq n$. Therefore by Theorem 1.7 of Chapter 1, M^m/M^n is not pure in A/M^n.

COROLLARY. The quasi-cyclic group $Z(p^\infty)$ and the cyclic groups Z/p^n are all pure simple.

Proof. These groups form a chain

$$0 \subsetneq Z/p \subsetneq Z/p^2 \dots Z/p^n \dots Z(p^\infty)$$

and there are no other subgroups. If Z/p^n were pure in some group containing it, then Z/p^n would be pure in Z/p^{n+1}. But by the theorem Z/p^{n+1} is pure simple for all n, giving us the desired result.

THEOREM 4.7. Let A be a Dedekind domain which is not a field, then

(1) R is a regular A-module iff R is torsion and every element of R has square free order.

(2) For any module E, the regular socle $R(E)$ is the collection of all torsion elements of E with square free order.

Remark 4.7. If A is a field, then A is regular, hence every module is regular and the statement of the theorem does not hold.

Proof.

(1) $\Rightarrow$: By Theorem 4.3, R is torsion, and therefore $R = \bigoplus R_p$ by Theorem 4.2. Hence R_p is regular by Theorem 3.4 of Chapter 3.

For any x in R_P, $O(x) = P^n$ and $A/P^n = Ax$ is regular. Therefore $n = 1$ by Proposition 4.5 and Remark 4.5. Hence every element of R has square free order.

$\Leftarrow$: By Theorem 4.2, $R = \bigoplus R_P$. Since every element of R has square free order, $O(x) = P$ for all x in R_P. But $Ax = A/P$ is simple (since P is maximal), hence regular. Therefore R_P is regular for each P, and so R is regular.

(2) By (1), since $R(E)$ is regular, it is torsion and every element has square free order. Conversely, if x in E is torsion, with square free order, then Ax is torsion and every element of Ax has square free order, since $O(ax) \geq O(x)$ for all a in A. Therefore Ax is regular and $Ax \subseteq R(E)$.

5. Radicals

A socle T with radical property, i.e. $T(M/T(M)) = 0$ for all modules M will be called a <u>radical</u>. In this section, we give a general construction for radicals.

Let $\underline{V}$ be any collection of left A-modules. A submodule N of the left A-module M will be called $\underline{V}$-maximal iff M/N is in $\underline{V}$. For example, if $\underline{V}$ is the collection of all simple modules, then $\underline{V}$-maximal just means maximal. A module M may have no $\underline{V}$-maximal submodules. Clearly if $\underline{V} \subseteq \underline{V}'$ then $\underline{V}$-maximal $\Rightarrow$ $\underline{V}'$-maximal. Note that any module M is $\underline{V}$-maximal in itself iff the 0 module is in $\underline{V}$; and that 0 is $\underline{V}$-maximal in M iff M is in $\underline{V}$. Thus if 0 is in $\underline{V}$, every module M has $\underline{V}$-maximal submodules since M is $\underline{V}$-maximal in M.

PROPOSITION 5.1. If N is a submodule of M, then under the one-one correspondence between submodules of M containing N and submodules of M/N, $\underline{V}$-maximal submodules correspond to $\underline{V}$-maximal submodules.

Proof. Suppose $N \subseteq K \subseteq M$.

Then K is $\underline{V}$-maximal iff M/K is in $\underline{V}$.
iff $(M/N)/(K/N)$ is in $\underline{V}$.
iff K/N is $\underline{V}$-maximal.

THEOREM 5.2. Let $\underline{V}$ be any collection of left A-modules which contains 0 and is closed under submodules. For any left A-module M, define $V(M)$ to be the intersection of all $\underline{V}$-maximal submodules of M. Then

(1) $V(M)$ is the intersection of the kernels of all epis $u:M \longrightarrow V$ with V in $\underline{V}$.

(2) V is a radical, and hence has all the properties of Theorem 1.1.

(3) $V(M/N) = V(M)/N$ for all submodules N of $V(M)$.

(4) $V(\prod M_i)$ is a submodule of $\prod V(M_i)$ for any family M_i of modules.

(5) $V(M) = 0$ iff M is isomorphic to a submodule of a product $\prod V_i$ with V_i in $\underline{V}$.

(6) If $\underline{V}$ is contained in $\underline{V}'$ then $V' \leq V$.

Proof. We note that since 0 is in $\underline{V}$, $V(M)$ is well defined.

(1) This is clear since the $\underline{V}$-maximal submodules are precisely the kernels of epis $u:M \dashrightarrow V$ with V in $\underline{V}$.

(2) Socle: Suppose $f:M \dashrightarrow N$. Then for all epis $u:N \dashrightarrow V$ with V in $\underline{V}$, $uf:M \dashrightarrow V'$ is an epi with $V' = \text{Im } uf$ in $\underline{V}$ since $\underline{V}$ is closed under submodules. Therefore if x is in $V(M)$, $uf(x) = 0$ and fx is in $\text{Ker } u$. Hence $f(V(M))$ is contained in $V(N)$, and V is a socle. The fact that V is a radical follows from (3).

(3) holds because of the one-one correspondence between the $\underline{V}$-maximal submodules of M (containing N) and the $\underline{V}$-maximal submodules of M/N.

(4) If N_k is $\underline{V}$-maximal in M_k then $\hat{M}_k = N_k \times \prod_{i \neq k} M_i$ is $\underline{V}$-maximal in $M = \prod M_i$. Hence $V(M)$ is contained in each such $\hat{M}_k$, hence in the intersection of all such $\hat{M}_k$. Therefore $V(M)$ is contained in $\prod V(M_i)$.

(5) $\Longrightarrow$: If $V(M) = 0$, there exists a family V_i in $\underline{V}$ and a family $u_i: M \dashrightarrow V_i$ of epis such that $\mathrm{Ker}\, u_i = 0$. The canonical map $M \dashrightarrow \prod V_i$ defined by $m \dashrightarrow (u_i(m))$ defines a mono and gives the required result.

$\Longleftarrow$: Suppose $f: M \dashrightarrow \prod V_i = P$ is mono and let $p_i: P \dashrightarrow V_i$ be the projection maps and set $u_i = p_i f$.

Then m in $V(M) \Longrightarrow p_i f(m) = u_i(m) = 0$ for all i.

$\Longrightarrow f(m) = 0$ (since the p_i are projection maps).

$\Longrightarrow m = 0$ since f is mono.

Hence $V(M) = 0$.

(6) Any $\underline{V}$-maximal submodule of M is $\underline{V}'$-maximal.

Remark 5.2. If we take $\underline{V}$ to be the collection of all simple left A-modules together with 0, then V is just the Jacobson radical.

6. Primitivity

In this section, we define and study a general type of primitivity.

Let $\underline{V}$ be any collection of left A-modules. A two-sided ideal P will be called left $\underline{V}$-primitive iff it is the largest two-sided ideal contained in some $\underline{V}$-maximal left ideal M. The ring A will be called left $\underline{V}$-primitive iff it is a left $\underline{V}$-primitive ideal. For example, if $\underline{V}$ is the collection of all simple modules, a left $\underline{V}$-primitive ideal or ring is a left primitive ideal or ring. Bergman (5) has shown that left and rt. $\underline{V}$-primitivity are not equivalent.

PROPOSITION 6.1. Let P be a two-sided ideal and M a left $\underline{V}$-maximal ideal. Then P is left $\underline{V}$-primitive (with $\underline{V}$-maximal left ideal M) iff $P = (a \in A \mid aA \subseteq M)$.

Proof.

$\Rightarrow$: a is in P iff $AaA \subseteq P$ since P is an ideal.

iff $AaA \subseteq M$ since P is the largest ideal contained in M.

iff $aA \subseteq M$ since M is a left ideal.

$\Leftarrow$: P is clearly the largest ideal contained in M.

PROPOSITION 6.2. If P is a two-sided ideal, then P is a left V-primitive ideal iff A/P is a left V-primitive ring.

Proof. This follows immediately from the one-one correspondence between the V-maximal left ideals of A containing P and those of A/P.

THEOREM 6.3. Let V be any collection of left A-modules containing 0 and closed under submodule. Then V(A), the intersection of all V-maximal left ideals, is equal to the intersection of all left V-primitive (two-sided) ideals.

Proof. By Theorem 5.2, V is a radical, hence a socle, and therefore V(A) is a two-sided ideal by Theorem 1.1. Since V(A) is a two-sided ideal,

a is in V(A) iff $aA \subseteq V(A)$.
iff $aA \subseteq M$ for all V-maximal left ideals M.
iff a is in all left V-primitive ideals.

COROLLARY. $V(A) = 0$ iff A is a subdirect product of left <u>V</u>-primitive rings.

Proof. Let P_i be the family of left <u>V</u>-primitive ideals of A. Then $V(A) = \prod P_i = 0$ iff A is the subdirect product of the rings A/P_i which are left <u>V</u>-primitive. If $V(A) = 0$, we call A a <u>V-semi-primitive</u> ring.

For any left A-module N, define $\text{Ann } N = (a \in A \mid aN = 0)$. Then $\text{Ann } N$ is a two-sided ideal of A. N is called <u>faithful</u> iff $\text{Ann } N = 0$.

THEOREM 6.4. A is a left <u>V</u>-primitive ring iff there exists a <u>V</u>-maximal left ideal M such that A/M is a faithful module.

Proof.

$\Rightarrow$: 0 is left <u>V</u>-primitive, and hence the largest ideal in some <u>V</u>-maximal left ideal M. $\text{Ann } A/M$ is a two-sided ideal which is clearly contained in M. Therefore $\text{Ann } A/M = 0$ and A/M is faithful.

$\Leftarrow$: Let P be any ideal contained in M. Then P Ann A/M = 0. Therefore 0 is the largest ideal contained in M, and hence A is left V-primitive.

CHAPTER 5: DIMENSION THEORY

In this chapter, we shall prove a number of results on the weak dimension of modules and weak global dimension of rings. We relate these to the homological (= projective) dimension and global dimension.

1. Weak Dimension

In this section, we adopt the weak dimension definition of Cartan-Eilenberg ((8), p. 122) and prove a number of results which are analogues of results for homological dimension (see Cartan-Eilenberg (8), p. 109 ff.).

A <u>resolution</u> of the left module E is a sequence of modules (F_i), $i = 0, 1, 2, \dots$ which form an exact sequence

$$\longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

A resolution (F_i) of E is <u>free</u> (resp. projective, flat) iff each F_i is free (projective, flat). For any resolution (F_i) of E we let $u_i : F_i \longrightarrow F_{i-1}$ $i = 1, 2, \dots$ be the given maps and $K_i = \operatorname{Ker} u_i = \operatorname{Im} u_{i+1}$ for $i = 1, 2, \dots$ and $K_{-1} = E$.

A resolution (F_i) of E has <u>length n</u> iff

$0 \dashrightarrow F_n \dashrightarrow \cdots \dashrightarrow F_0 \dashrightarrow E \dashrightarrow 0$ is exact. If no such integer exists we say that (F_i) has <u>infinitive length</u>.

Clearly every free resolution (of E) is a projective resolution, and every projective resolution is a flat resolution. It is well known, and easy to show (see Jans (22), p. 33), that every module has a free resolution; hence every module has free, projective and flat resolutions.

THEOREM 1.1. If $0 \dashrightarrow K \dashrightarrow F \dashrightarrow E \dashrightarrow 0$ is an exact sequence of left A-modules with F flat, then for any rt. module X, $\mathrm{Tor}_{n+1}(X,E) = \mathrm{Tor}_n(X,K)$ for all $n > 0$.

Remark 1.1. For simplicity, we have written $=$ for $\cong$.

Proof. For all $n > 0$ and for all rt. A-modules X, we have

$$0 = \mathrm{Tor}_{n+1}(X,F) \dashrightarrow \mathrm{Tor}_{n+1}(X,E) \dashrightarrow \mathrm{Tor}_n(X,K) \dashrightarrow \mathrm{Tor}_n(X,F) = 0$$

since F is flat. Hence the desired result.

COROLLARY. If (F_i) is a flat resolution of E, then for any rt. module X, and for all $n > 0$ we have $\mathrm{Tor}_{n+1}(X,E) = \mathrm{Tor}_1(X,K_{n-1})$.

Proof. For all $i \geq 0$, we have exact sequences $0 \to K_i \to F_i \to K_{i-1} \to 0$ with F_i flat. Hence for all $n > 0$, and for $-1 \leq i \leq n-1$, we have $\mathrm{Tor}_{n+1}(X,K_{i-1}) = \mathrm{Tor}_n(X,K_i)$. Therefore

$$\begin{aligned}\mathrm{Tor}_{n+1}(X,E) = \mathrm{Tor}_{n+1}(X,K_{-1}) &= \mathrm{Tor}_n(X,K_0)\\ &= \mathrm{Tor}_{n-i}(X,K_i) \quad \text{for } -1 \leq i \leq n-1\\ &= \mathrm{Tor}_1(X,K_{n-1}).\end{aligned}$$

For any left A-module $E \neq 0$, we define <u>wd E</u> (= <u>weak dimension</u> of E) to be the largest integer n such that $\mathrm{Tor}_n(X,E) \neq 0$ for some rt. A-module X. If no such integer exists, define $\mathrm{wd}\, E = \infty$. For completeness, we define $\mathrm{wd}\, 0 = -1$. These definitions are all due to Cartan-Eilenberg ((8), p. 122). We remark that $\mathrm{wd}\, E \leq 0$ iff E is flat. Weak dimension for rt. modules is defined similarily.

THEOREM 1.2. For any left module E and any integer $n \geq 0$, the following conditions are equivalent:

(1) Any exact sequence $0 \to K_{n-1} \to F_{n-1} \to \cdots \to F_0 \to E \to 0$ with F_i flat for all $0 \leq i \leq n-1$ has K_{n-1} flat too.

(2) E has a flat resolution of length n.

(3) $\mathrm{wd}\, E \leq n$.

(4) $Tor_k(X,E) = 0$ for all integers $k > n$ and for all rt. modules X.

(4a) $Tor_k(X,E) = 0$ for all integers $k > n$ and for all fg rt. modules X.

(4b) $Tor_k(X,E) = 0$ for all integers $k > n$ and for all fp rt. modules X.

(4c) $Tor_k(X,E) = 0$ for all integers $k > n$ and for all cyclic rt. modules X.

(4d) $Tor_k(X,E) = 0$ for all integers $k > n$ and for all fp cyclic rt. modules X.

(5) $Tor_{n+1}(X,E) = 0$ for all rt. modules X.

(5a) $Tor_{n+1}(X,E) = 0$ for all fg rt. modules X.

(5b) $Tor_{n+1}(X,E) = 0$ for all fp rt. modules X.

(5c) $Tor_{n+1}(X,E) = 0$ for all cyclic rt. modules X.

(5d) $Tor_{n+1}(X,E) = 0$ for all fp cyclic rt. modules X.

Proof.

(1) $\Rightarrow$ (2): Let (F_i) be a flat resolution of E and

$0 \to K_{n-1} \to F_{n-1} \to \cdots \to F_0 \to E \to 0$ be exact.

By (1), K_{n-1} is flat. Hence (2).

(2) $\Rightarrow$ (3): Let $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to E \to 0$

be a flat resolution of E, of length n. Then for all rt. modules X,

$\mathrm{Tor}_{n+1}(X,E) = \mathrm{Tor}_1(X,K_{n-1}) = 0$ since $K_{n-1} = F_n$ is flat. Hence $\mathrm{wd}\ E \leq n$.

(3) $\Rightarrow$ (4): If for some rt. module X and some $k > n$,

$\mathrm{Tor}_k(X,E) \neq 0$, then $\mathrm{wd}\ E \geq k > n$. Contradiction.

Clearly we have the implications (4) $\Rightarrow$ (5) and (4x) $\Rightarrow$ (5x) for

$x = a, b, c, d$ as well as

$$\begin{array}{ccc} (5) \Rightarrow (5a) & \Rightarrow & (5b) \\ \Downarrow & & \Downarrow \\ (5c) & \Rightarrow & (5d) \end{array}$$

To conclude the proof, we will show (5) $\Rightarrow$ (1) and

(5d) $\Rightarrow$ (5c) $\Rightarrow$ (5a) $\Rightarrow$ (5).

(5) $\Rightarrow$ (1): Suppose we have an exact sequence

$0 \to K_{n-1} \to F_{n-1} \to \cdots \to F_0 \to E \to 0$ with F_i flat for

$0 \leq i \leq n-1$. Form a flat resolution of E, the first n terms of which

are F_i, $0 \leq i \leq n-1$. Then for all rt. modules X,

$Tor_1(X,K_{n-1}) = Tor_{n+1}(X,E) = 0$ by the corollary of Theorem 1.1. Hence K_{n-1} is flat.

(5a) $\Rightarrow$ (5): holds since every (rt.) module X is the direct limit of its fg submodules, and $Tor_{n+1}(\ ,E)$ commutes with direct limits.

(5c) $\Rightarrow$ (5a): We show this by induction on the number k of generators of X. The case $k = 1$ holds by assumption; assume that we have established the implication for all modules with not more than k generators, and let X have k+1 generators. Let x in X be one of these generators, $X' = xA$ and $X'' = X/X'$. Then we have an exact sequence $Tor_{n+1}(X',E) \longrightarrow Tor_{n+1}(X,E) \longrightarrow Tor_{n+1}(X'',E)$. We have $Tor_{n+1}(X',E) = 0$ since X' is cyclic, and $Tor_{n+1}(X'',E) = 0$ since X" has not more than k generators. Hence $Tor_{n+1}(X,E) = 0$.

(5d) $\Rightarrow$ (5c): Let I be any rt. ideal of A. Then I is the direct limit of its fg submodules (= rt. ideals) I_k. Therefore $X = A/I$ is the direct limit of A/I_k. Since $Tor_{n+1}(\ ,E)$ commutes with direct limits, we have the desired result.

Remark 1.2.

(i) Clearly a similar theorem holds for rt. modules.

(ii) Parts (1), (3) and (4) of this theorem are in Hillel (21). We have strengthened the theorem by adding Parts (2), (5), as well as the Parts (4a), (4b), (5a), (5b), etc. The significance of these additions (esp. (5d)) is explained in Remark 2.2.

THEOREM 1.3. If $0 \dashrightarrow K \dashrightarrow F \dashrightarrow E \dashrightarrow 0$ is exact with F flat, then $\text{wd } E \leq 1 + \text{wd } K$. Equality holds iff $\text{wd } E \geq 1$, i.e. iff E is <u>not</u> flat. (We assume $K \neq 0 \neq E$.)

Proof. By Theorem 1.1, $\text{Tor}_{n+1}(X,E) = \text{Tor}_n(X,K)$ for all $n > 0$ and for all rt. X. If $\text{wd } K = \infty$, then the inequality always holds. If $\text{wd } K = n < \infty$, then by Theorems 1.1 and 1.2, $\text{Tor}_{n+2}(X,E) = \text{Tor}_{n+1}(X,K) = 0$ for all rt. X. Therefore $\text{wd } E \leq n+1 = \text{wd } K + 1$.

If equality holds, we have $\text{wd } E \geq 1$. Conversely, suppose $\text{wd } E \geq 1$. If $\text{wd } E = \infty$, then we have equality. Assume therefore that $\text{wd } E = m$ with $1 \leq m < \infty$. Then for all rt. X, $\text{Tor}_m(X,K) = \text{Tor}_{m+1}(X,E) = 0$ (since $m \geq 1$). Therefore $\text{wd } K \leq m-1$, and $1 + \text{wd } K \leq m = \text{wd } E$ giving us the desired equality.

We have the following obvious:

COROLLARY. For any nonzero left ideal I of A, $\mathrm{wd}\ A/I \leq 1 + \mathrm{wd}\ I$, with equality holding iff $\mathrm{wd}\ A/I \geq 1$.

THEOREM 1.4. If $0 \to E' \to E \to E'' \to 0$ is exact and any two of the modules have finite weak dimension, then so does the third one.

Proof. For any rt. module X and all $n > 1$ we have an exact sequence

$$\to \mathrm{Tor}_{n+1}(X,E'') \to \mathrm{Tor}_n(X,E') \to \mathrm{Tor}_n(X,E) \to \mathrm{Tor}_n(X,E'') \to \mathrm{Tor}_{n-1}(X,E').$$

For n sufficently large the Tor's of the finite dimensional modules vanish, and the Tor of the other module must be zero too.

2. Weak Global Dimension

In this section, we define and study the weak global dimension of a ring. This is, of course, analogous to the (homological) global dimension of a ring.

We define the left weak global dimension of A to be $\text{lwgl } A = \sup \text{wd } E$, with the sup taken over all left A-modules E. The rt. weak global dimension of A is defined similarily: $\text{rwgl } A = \sup \text{wd } E$, with the sup taken over all rt. A-modules E.

Northcott ((37), p. 150) has shown that $\text{lwgl } A = \text{rwgl } A$. Their common value will be called the weak global dimension of A and written $\text{wgl } A$. Since $\text{wgl } A$ is left-rt. symmetric, every theorem concerning left modules and ideals gives us an "opposite" theorem about rt. modules and ideals. These corresponding theorems will be assumed and used, although they are not always studied explicitly.

PROPOSITION 2.1. The ring A is regular iff $\text{wgl } A \leq 0$.

Proof.

A is regular iff all left A-modules are flat (Bl, Ex. 17, p. 64).
iff $\text{wd } E \leq 0$ for all left A-modules E.
iff $\text{wgl } A \leq 0$.

One of the main results of this section is:

THEOREM 2.2. wgl A = sup wd E with the sup taken over all left fp cyclic modules E.
= sup wd E with the sup taken over all rt. fp cyclic modules E.

Proof. Since wgl A is left-rt. symmetric, it suffices to prove the first equality. Clearly sup wd E $\leq$ wgl A. If sup wd E $= \infty$, we have equality. Assume therefore that sup wd E $= n < \infty$. Hence wd E $\leq n$ for all fp left cyclics E. And $\text{Tor}_{n+1}(X,E) = 0$ for all such E, and for all rt. modules X (Theorem 1.2, Part (5d)). Therefore by Part (5d) of the "rt." version of Theorem 1.2 (see Remark 1.2 (i)), we have wd X $\leq n$ for all rt. modules X, i.e. wgl A $\leq n$ = sup wd E. Hence the desired result.

COROLLARY 1. wgl A $\leq$ 1+ sup wd I with the sup taken over all fg left ideals I.

Proof. For any fg left ideal I, we have $0 \to I \to A \to A/I \to 0$ exact and hence wd A/I $\leq$ 1 + wd I by Theorem 1.3. Therefore wgl A = sup wd A/I $\leq$ 1 + sup wd I with the sup taken over all fg left ideals.

COROLLARY 2. If wgl $A \geq 1$ then wgl $A = 1 + \sup$ wd I where the sup is taken over all fg left ideals I.

Proof. Since wgl $A \geq 1$, there exists a fg left ideal K such that wd $A/K \geq 1$ and hence A/K is not flat. Therefore by Theorem 1.3, $1 +$ wd $K =$ wd $A/K \leq$ wgl A and $1 + \sup$ wd $I \leq$ wgl A. The opposite inequality is given in Corollary 1.

Remark 2.2. This strenghtens a theorem of Hillel (21); he proved the result for all cyclic modules (<u>not</u> fp cyclic). More important than this, we remark that by the addition of condition (5d) to Theorem 1.2, the proof of which is not too difficult, we have very significantly shortened and simplified the original proof given by Hillel (21).

The next theorem characterizes rings for which submodules of flat modules are flat.

THEOREM 2.3. For any ring A, the following conditions are equivalent:

(1) wgl $A \leq 1$.

(2) Every submodule of every flat left module is flat.

(2)' Every fg submodule of every flat left module is flat.

(3) Every left ideal of A is flat.

(3)' Every fg left ideal of A is flat.

Proof.

(1) $\Rightarrow$ (2): Suppose $0 \to K \to F \to E \to 0$ is exact with F flat. If E is flat then K is flat (Bl, Prop. 5, p. 31). If E is not flat, by Theorem 1.3, $1 + \text{wd } K = \text{wd } E$. But $\text{wd } E \leq 1$ by (1), whence $\text{wd } K \leq 0$ and K is flat. The implications

$$\begin{array}{ccc} (2) & \Rightarrow & (2)' \\ \Downarrow & & \Downarrow \\ (3) & \Rightarrow & (3)' \end{array}$$

are obvious.

(3)' $\Rightarrow$ (1): Let $E = A/I$ be a fp left cyclic module. By Theorem 1.3, $\text{wd } E \leq 1 + \text{wd } I = 1$ since I is fg left ideal and therefore flat. By Theorem 2.2, $\text{wgl } A \leq 1$.

COROLLARY. If A is left or rt. semi-hereditary, then $\text{wgl } A \leq 1$.

Proof. If A is left (resp. rt.) semi-hereditary, every fg left (rt.) ideal is projective and therefore flat.

Remark 2.3.

(i) Each of the corresponding "rt." statements of the theorem is also equivalent to $\text{wgl } A \leq 1$.

(ii) Cf. the analogous result of Cartan-Eilenberg ((8), p. 112) for lgl $A \leq 1$.

(iii) The converse of the corollary is false unless A is left (or rt.) coherent. See Chase (10).

THEOREM 2.4. If A is left artinian, then lgl A = wgl A = sup wd S with the sup taken over all simple left modules S. Furthermore if wgl $A \geq 1$ then wgl A = 1 + sup wd M where the sup is taken over all maximal left ideals M.

Proof. If A is left artinian, it is left noetherian and therefore lgl A = wgl A (see Northcott (37), p. 154) and wd S = hd S for all simple left modules S since they are fg (Northcott (37), p. 153). Also if A is left artinian, Jans ((22), p. 56) has shown that lgl A = sup hd S. Therefore, we have the desired result.

If wgl $A \geq 1$, Jans ((22), p. 57) has shown that lgl A = 1 + sup hd M. Since A is left noetherian, every left ideal is fg and therefore wd M = hd M (Northcott (37), p. 153). But wgl A = lgl A, so we are finished.

3. The Dual of Lambek's Theorem

Lambek (28) has shown that a module is flat iff its character module is injective. We now prove the dual for noetherian rings: a module is injective iff its character module is flat.

Let E be a left A-module and $E' = \mathrm{Hom}_Z(E,Q/Z)$ (Q = rationals, Z = integers) be its character module. Then E' is, in a natural way, a rt. A-module. For further details, see Lambek (28) or Northcott ((37), p. 71 ff.). There it is shown that $(\ )'$ is an additive exact contravariant functor from the category of left A-modules to the category of rt. A-modules. Thus if $0 \to E \to F \to G \to 0$ is an exact sequence of left A-modules, then $0 \to G' \to F' \to E' \to 0$ is an exact sequence of rt. A-modules.

We shall also use the fact that for any left A-module E we have $E = 0$ iff $E' = 0$. A proof may be found in Lambek (28).

In order to prove our main result, we need:

PROPOSITION 3.1. Let A be rt. noetherian and M a fg rt. A-module. Then for any rt. A-module E we have a module isomorphism:

$$\mathrm{Tor}(M,E') = (\mathrm{Ext}(M,E))'$$

Proof. This follows immediately from Northcott ((37), p. 152, Lemma 5), since Q/Z is Z-injective.

THEOREM 3.2. Let A be rt. noetherian. Then any rt. A-module E is injective iff the left A-module E' is flat.

Proof. E is injective

iff Ext(M,E) = 0 for all fg rt. M. See Jans ((22), p. 50).

iff (Ext(M,E))' = 0 for all fg rt. M.

iff Tor(M,E') = 0 for all fg rt. M by Proposition 3.1.

iff E' is flat by (Bl, Prop. 1, p. 55).

COROLLARY. If A is left noetherian, then any left A-module E is injective iff the rt. A-module E' is flat.

Proof. Interchange left and rt. throughout.

CHAPTER 6: PURE SIMPLE AND INDECOMPOSABLE RINGS

In this chapter, we study pure simple and indecomposable rings and relate them to the PP and PF rings of Hattori (20).

1. Small Submodules

We recall that a submodule S of E is small in E iff for every submodule F of E such that S+F = E, we have F = E. Clearly any submodule of a small submodule is small.

LEMMA 1.1. Any finite sum of small submodules of E is small in E.

Proof. Use induction on the number of small submodules.

PROPOSITION 1.2. For any x in E, any left A-module, Ax is small in E iff x is in the sum of all small submodules of E.

Proof.

$\Rightarrow$: is clear since x is in Ax.

$\Leftarrow$: We must have $x = \sum x_i$ (finite sum) with x_i in S_i, a small submodule of E. Each Ax_i is a submodule of S_i, and hence small in E. Therefore $S = \sum Ax_i$ is small in E, since the sum is finite. Since x is in S, Ax is a submodule of S and therefore small in E.

THEOREM 1.3. For any module E, J(E) (the Jacobson radical of E) is the sum of all small submodules of E.

Remark 1.3. The statement of this theorem is due to Sandomierski and Kasch and was communicated to me by Prof. J. Lambek. The proof is original.

Proof. By Proposition 1.2, it suffices to show that Ax is small in E iff x is in J(E), or equivalently: x is not in J(E) iff Ax is not small in E. We shall show the latter statement.

$\Rightarrow$: If x is not in J(E), then there exists some maximal submodule M of E such that x is not in M. Therefore $Ax+M = E$, and Ax cannot be small since $M \neq E$.

$\Longleftarrow$: We will call a submodule F of E *proper* iff $F \neq E$. The collection $\underline{C}$ of all proper submodules F of E such that $Ax+F = E$ is non-empty since Ax is not small. Since each F of $\underline{C}$ is a proper submodule, we have $x \notin F$ for each F. It is also clear that any proper submodule of E which contains a member of $\underline{C}$, is itself a member of $\underline{C}$. Therefore if we order $\underline{C}$ by set inclusion, the union F of any chain F_i of members of $\underline{C}$ is a proper submodule, since x is not in F_i for all i. Hence F is a member of $\underline{C}$, since it contains each F_i. Therefore by Zorn's Lemma we can choose a maximal element M of $\underline{C}$. We claim that M is a maximal submodule of E. Since M is in $\underline{C}$, x is not in M and therefore M is proper. Any proper submodule of E containing M is a member of $\underline{C}$, and therefore equal to M by the maximality of M in $\underline{C}$. Therefore M is a maximal submodule of E. Since x is not in M, x is not in $J(E)$.

COROLLARY 1. If $J(E)$ is small in E, then it is the largest small submodule of E.

Proof. Obvious.

COROLLARY 2.

(a) If E is fg, then $J(E)$ is small in E.

(b) $J(A)$ is small in A and therefore all left ideals and all rt. ideals contained in $J(A)$ are small.

(c) If A is a local ring, then all ideals $I \neq A$ (left, rt., and two-sided) are small.

Proof.

(a) Suppose $J(E) + F = E$. If $F \neq E$ then F is contained in some maximal submodule M of E, since E is fg (B3, Prop. 4, p. 30). Therefore $J(E) + F$ is contained in $M \neq E$. This contradiction shows that we must have $F = E$, i.e. $J(E)$ small in E.

(b) A is fg.

(c) All ideals are contained in $J(A)$, which is small.

Remark 1.3.

(i) Corollary 2 is untrue for E non fg. For example, Q the abelian group (= Z-module) of rationals has no maximal subgroups and therefore $J(Q) = Q$.

(ii) Mares (34) has shown that $J(E)$ is small in E if E is semi-perfect.

We now come to one of the main theorems of this section:

THEOREM 1.4. If P is a projective module, then 0 is the only small pure submodule of P.

Proof. Since P is projective, there exists a free module $F = P \oplus Q$. Let S be any small pure submodule of P. Then S is a small pure submodule of F, since S pure in P and P pure in F imply that S is pure in F (Proposition 1.2 of Chapter 1). The smallness is clear. Since S is small in F, $S \subseteq NF = J(F)$, where $N = J(A)$, the Jacobson radical (Theorem 1.3). The fact that $J(P) = NP$ for any projective module has been shown by Mares (34).

Let (x_h) (h in H, an arbitrary index set) be a base for F. Then for any x in $S \subseteq NF$ we have $x = \sum n_i x_i$ (i in I, a finite subset of H) with n_i in N. By Cohn's Theorem (Theorem 1.1 of Chapter 1), since S is pure in F, $x = \sum n_i x_i = \sum n_i s_i$ with s_i in S. For each i in I, $s_i = \sum m_{ij} x_j$ (j in J_i) with m_{ij} in N and J_i a finite subset of H. Then $J = \bigcup J_i$ (i in I) is a finite subset of H. For each i in I, set $m_{ij} = 0$ for each j in J which is not in J_i. Then for each i in I, $s_i = \sum m_{ij} x_j$ (j in J) and $x = \sum n_i x_i = \sum n_i s_i = \sum n_i (\sum m_{ij} x_j) = \sum (\sum n_i m_{ij}) x_j$ since the index sets are finite. Since (x_h) is a base, for all k in I, $n_k = \sum n_i m_{ik}$ (i in I). Let $B_x = \sum n_k A$ (k in I) be the (fg) rt. ideal generated by the n_k. Then each n_k is in $B_x N$ since m_{ik} is in N; therefore $B_x = B_x N$. But since B_x is fg, $B_x = 0$ by Nakayama's Lemma (B3, Thm. 2, p. 68). Hence $n_k = 0$ for all k in I and $x = 0$, i.e. $S = 0$.

COROLLARY.

(1) If J(P) is a small submodule of the projective module P, then 0 is the only pure submodule of P contained in J(P).

(2) J(A) contains no pure left ideals and no pure rt. ideals of A other than 0.

(3) If P is a regular projective module, then 0 is the only small submodule, and therefore $NP = J(P) = 0$.

Proof.

(1) Any pure submodule of P contained in $J(P)$ is a pure small submodule of a projective module and therefore zero.

(2) Since A is projective and $J(A)$ is small in A, we can apply (1).

(3) Since all submodules are pure, 0 is the only small submodule and therefore $NP = J(P) = 0$ since it is the sum of all small submodules (Theorem 1.3).

THEOREM 1.5. Any local ring A is pure simple and hence indecomposable.

Proof. If $P \neq A$ is any pure left or rt. ideal of A, then P is contained in the radical of A. Therefore P is pure small in A and hence $P = 0$ by Corollary 1 of Theorem 1.4. Thus A is both left and rt. pure simple.

COROLLARY. Any regular local ring A is a skewfield.

Proof. Since A is regular, its radical is pure in A. But the radical is always small in A, and therefore must be 0 by Theorem 1.4, and A is a skewfield.

2. PP and PF Rings

Following Hattori (20), we will call a ring A left PP (resp. left PF) iff every principal left ideal of A is projective (resp. flat), and PP (resp. PF) iff it is both left and rt. PP (resp. PF). We recall that the ring A is left (semi-) hereditary iff every (fg) left ideal is projective. See Cartan-Eilenberg ((8), p. 13.)

PROPOSITION 2.1.

(1) Every left PP ring is left PF; every PP ring is PF.

(2) Every left semi-hereditary ring is left PP.

(3) Every regular ring is PP.

(4) If wgl $A \leq 1$, then A is PF.

(5) If lgl $A \leq 1$, then A is left PP.

Proof.

(1) Every projective left ideal is flat.

(2) Every fg left ideal is projective by definition.

(3) Every fg left and every fg rt. ideal is a direct summand, and therefore projective.

(4) Every left and every rt. ideal is flat by Theorem 2.4 of Chapter 5.

(5) Every left ideal is projective since A is left hereditary.

We now characterize both left PP and left PF rings.

THEOREM 2.2. A is left PP (resp. left PF) iff $l(a) = (b \in A \mid ba = 0)$ is a direct summand of (resp. pure in) A for all a in A.

Proof. For any a in A, we have an exact sequence of left A-modules $0 \dashrightarrow l(a) \dashrightarrow A \dashrightarrow Aa \dashrightarrow 0$. And A is left PP (resp. left PF) iff Aa is projective (resp. flat) for all a in A.

iff $l(a)$ is a direct summand of (resp. pure in) A for all a in A (see Proposition 1.3 of Chapter 1).

THEOREM 2.3.

(1) If A has no left zero divisors, then A is left pure simple and rt. PP and hence left indecomposable and rt. PF.

(2) If A is left pure simple and left PF or if A is left indecomposable and left PP then A has no rt. zero divisors.

(3) If A has no left zero divisors and A is left PF then A has no rt. zero divisors.

Proof.

(1) If A has no left zero divisors, then A is left pure simple, and hence left indecomposable by Corollary 1 of Theorem 2.1 of Chapter 1. Also $(b \in A \mid ab = 0) = r(a) = 0$ for all $0 \neq a$ in A, and therefore $r(a)$ is a direct summand of A (as rt. modules) for all a in A, since $r(0) = A$. Hence by Theorem 2.2 A is rt. PP and therefore rt. PF.

(2) If A is left PF (resp. left PP) then $l(a)$ is pure in (resp. a direct summand of) A, as left A-modules. Since A is left pure simple (resp. left indecomposable), $l(a) = 0$ or A. But $l(a) = A$ implies $a = 0$. Therefore $l(a) = 0$ for all $0 \neq a$ in A, and A has no rt. zero divisors.

(3) follows immediately from (1) and (2).

COROLLARY 1. For any ring A, the following conditions are equivalent:

(1) A has no zero divisors.

(2) A is pure simple and PF.

(3) A is indecomposable and PP.

Proof. Follows immediately from the theorem.

COROLLARY 2. For a local ring A, the following conditions are equivalent:

(1) A has no zero divisors.

(2) A is PF.

(3) A is PP.

Proof. Any local ring is pure simple and hence indecomposable (Theorem 1.5).

COROLLARY 3. If A is a commutative local ring then A is an integral domain iff A is PP iff A is PF.

Proof. Obvious using Corollary 2.

Following Bourbaki (B1, Ex. 12, p. 63) we will call the ring A <u>left coherent</u> iff every fg left ideal of A is fp. Chase (10) has shown that A is left coherent iff every product of flat rt. A-modules is flat. It is easy to see that every left noetherian ring is left coherent. Recall that a ring A is left neat iff its left singular ideal is 0. See Chapter 4.

THEOREM 2.4.

(1) Every left PP ring is left neat, and therefore its complete ring of quotients (on the left side) is regular.

(2) Every left coherent left PF ring is left PP, and therefore has all the properties given in (1).

Proof.

(1) For any element a in A we have an exact sequence $0 \dashrightarrow l(a) \dashrightarrow A \dashrightarrow Aa \dashrightarrow 0$ which is split exact since Aa is projective. If a is in $K({}_AA)$, the left Johnson singular ideal of

A, then $l(a)$ is large in A and therefore $l(a) = A$ since $l(a)$ is a direct summand of A. But $l(a) = A$ implies $a = 0$. Therefore $K({}_A A) = 0$ and A is left neat. If A is left neat then its complete ring of quotients (on the left side) is regular. See Lambek ((30), p. 106.)

(2) For any a in A, the left ideal Aa is a fg and therefore a fp flat module, hence projective by Corollary 2 of Theorem 1.2 of Chapter 2. Consequently A is left PP.

Remark 2.4. Cf. the characterizations given for commutative PP and PF in Chapter 7.

CHAPTER 7: LOCALIZATION

We make the convention that wherever localization is discussed, all rings under consideration are commutative; ideals are usually denoted by small letters m, p, etc. Since localization is a special case of change of rings, all our results for change of rings carry over immediately, and in many cases can be extended. We use the notation and terminology of Bourbaki (B1, Ch. 2), with the following minor changes:

When only one base ring A is being considered, we let $\otimes = \otimes_A$. If S is a mult. (= multiplicative) set of A then A_S, E_S, and u_S denote the localization at S of A, E (an A-module) and u (an A-homomorphism) respectively. We write $\mathrm{Hom}_S(E_S, F_S)$ for $\mathrm{Hom}_{A_S}(E_S, F_S)$ and $\otimes_S$ for $\otimes_{A_S}$. If S is the complement of a prime (or maximal) ideal p (or m), then we write p (or m) in place of S everywhere: A_p, A_m, E_m, $\otimes_m$, u_m, etc. We let $\underline{M} = \underline{M}(A)$ denote the collection of all maximal ideals m of A.

1. Purity and Regularity

In this section, we characterize both purity and regularity "locally" and make comparisons with direct summands and semi-simplicity.

THEOREM 1.1. Let E be any A-module, D any submodule of E and S any mult. set of A.

(1) If D is an A-direct summand of E (resp. A-pure in) E, then D_S is an A_S-direct summand of (resp. A_S-pure in) E_S.

(2) If E is A-semi-simple (resp. A-regular), then E_S is A_S-semi-simple (resp. A_S-regular).

(3) $(T(E))_S \subseteq T_S(E_S)$ where T (resp. T_S) is either the ss socle or the regular socle with respect to A (resp. A_S).

(4) If E is A-simple, then E_S is A_S-simple.

Proof.

(1) The direct summand case is given in (B2, Cor., p. 120 and Prop. 7, p. 90). The pure case follows from Theorem 4.1 of Chapter 1.

(2) Any A_S-submodule of E_S has the form D_S where D is an

A-submodule of E (Bl, Prop. 10, p. 89). If E is A-semi-simple (resp. A-regular) then D is an A-direct summand of (resp. A-pure in) E, and the result follows from (1).

(3) $T(E)$ is A-semi-simple (resp. A-regular) (see Chapter 3); therefore by (2) $(T(E))_S$ is an A_S-semi-simple (resp. A-regular) submodule of E_S and hence contained in $T_S(E_S)$.

(4) As in (2), any A_S-submodule of E_S has the form D_S where D is an A-submodule of E. If E is A-simple then $D = 0$ or $D = E$ and $D_S = 0$ or $D_S = E_S$. Hence E_S is A_S-simple.

COROLLARY. Let S be any mult. set of A.

(1) If A is semi-simple (resp. regular), so is A_S.

(2) If A is simple, so is A_S.

Proof. Apply Parts (2) and (4) of the theorem with $E = A$.

The following theorem shows that both purity and regularity are local properties.

THEOREM 1.2. Let E be any A-module and D any submodule of E.

(1) D is A-pure in E iff D_m is A_m-pure in E_m for all m in $\underline{M}$.

(2) E is A-regular iff E_m is A_m-regular for all m in $\underline{M}$.

Proof. By Theorem 1.1, we need only show $\Leftarrow$ in each case.

(1) $\Leftarrow$: Let $j:D \to E$ be the canonical injection. For any A-module F, let $f = 1_F \otimes j$. Then $f_m = 1_{F_m} \otimes_m j_m$. This follows easily from (B2, Sect. 5, p. 116). Since D_m is A_m-pure in E_m, f_m is mono for all m in $\underline{M}$, and therefore f is mono (B1, Thm. 1, p. 111). Hence D is A-pure in E.

(2) $\Leftarrow$: If D is any submodule of E, D_m is A_m-pure in E_m, and therefore D is A-pure in E by (1), and E is A-regular.

In general, the property of being a direct summand is not local. However, we have:

THEOREM 1.3. Let $0 \to D \to E \to F \to 0$ be an exact sequence of A-modules. If F is A-pure projective, then

(1) D is an A-direct summand of E iff D_m is an A_m-direct summand of E_m for all m in $\underline{M}$.

(2) E is A-semi-simple iff E_m is A_m-semi-simple for all m in $\underline{M}$.

Proof. By Theorem 1.1, we need only show $\Leftarrow$ in each case.

(1) $\Leftarrow$: If D_m is A_m-pure in E_m for all m in $\underline{M}$, then D is A-pure in E by Theorem 1.2. Since F is pure projective, the sequence is split exact by Theorem 2.4 of Chapter 2.

(2) $\Leftarrow$: For any submodule D of E, D_m is an A_m-direct summand of E_m for all m in $\underline{M}$. Therefore by (1), D is a direct summand of E.

COROLLARY. If A is PDS, then any exact sequence $0 \to D \to E \to F \to 0$ is split exact iff $0 \to D_m \to E_m \to F_m \to 0$ is split exact for all m in $\underline{M}$.

Proof. If A is PDS, then all A-modules are pure projective by Theorem 4.1 of Chapter 2. Apply the above theorem.

THEOREM 1.4. Let $0 \dashrightarrow D \dashrightarrow E \dashrightarrow F \dashrightarrow 0$ be an exact sequence of A-modules. If F is fg flat, then D_p is an A_p-direct summand of E_p for all prime ideals p of A.

Proof. For all p, F_p is a fg flat A_p-module (Bl, Prop. 13, p. 115) and hence A_p-free since A_p is a local ring (Bl, Ex. 3, p. 167). Since any free module is pure projective (see Chapter 2), the exact sequence $0 \dashrightarrow D_p \dashrightarrow E_p \dashrightarrow F_p \dashrightarrow 0$ splits.

2. PP and PF Rings

In this section, we give a local characterization of PF rings and connect it with known characterizations of related properties (PP, etc.).

THEOREM 2.1. Let A have any one of the following properties:

(1) wgl $A \leq 0$ (i.e. A is regular).

(2) A is semi-hereditary.

(3) wgl $A \leq 1$.

(4) A is PP.

(5) A is PF.

(6) A is semi-prime.

Then the ring A_S has the same property, for any mult. set S.

Proof.

(1) and (3): For any mult. set S, wgl $A_S \leq$ wgl A (see Cartan-Eilenberg (8), p. 123 and p. 142), and these parts are immediate.

(2), (4), (5): Any fg (resp. principal) ideal of A_S has the form I_S where I is a fg (resp. principal) ideal of A. If I is projective (resp. flat) then I_S is projective (resp. flat). See (B2, Cor., p. 120 and Prop. 7, p. 90) for the projective case, and (B1, Prop. 13, p. 115) for the flat case.

(6) is given in (B1, Prop. 17, p. 97).

THEOREM 2.2. Let K be the total quotient ring of A in the sense of Bourbaki (B1, Example 7, p. 77).

(1) wgl $A \leq 0$ (i.e. A is regular) iff A_m is a field for all m in $\underline{M}$.

(2) A is semi-hereditary iff K is a regular ring and A_m is a valuation domain for all m in $\underline{M}$.

(3) wgl $A \leq 1$ iff A_m is a valuation domain for all m in $\underline{M}$.

(4) A is PP iff K is a regular ring, and A_m is a local domain for all m in $\underline{M}$.

(5) A is PF iff A_m is a local domain for all m in $\underline{M}$.

(6) A is semi-prime iff A_m is semi-prime for all m in $\underline{M}$.

Remark 2.2. The first four parts are due to Endo (13) and (14). We have restated them, sometimes in slightly different form, in order to emphasize the relationships between them.

Proof. In view of the remark, we shall only prove (5) and (6):

(5) $\Rightarrow$: By Theorem 2.1, A_m is a local PF ring and therefore an integral domain by Corollary 3 of Theorem 2.3 of Chapter 6.

$\Leftarrow$: If I is any principal ideal of A then for all m in $\underline{M}$, I_m is a principal ideal of A_m, and therefore A_m-flat since A_m is a (local) domain, hence PF. Therefore I is A-flat (Bl, Cor., p. 116), and A is PF.

(6) By Theorem 2.1 we need only show:

$\Leftarrow$: Let N be the nilradical of A. Then for all m in $\underline{M}$, N_m is the nilradical of A_m (Bl, Prop. 17, p. 97). Since each A_m is semi-prime, $N_m = 0$ for all m in $\underline{M}$ and $N = 0$ (Bl, Cor. 2, p. 112).

COROLLARY. Every commutative PF ring is semi-prime and therefore neat.

Proof. Use (5) and (6) of the theorem and the fact that every (local) domain is semi-prime. A commutative ring is semi-prime iff it is neat. See Lambek ((30), p. 108.)

3. Solution of Bass' Conjecture for Commutative Perfect Rings

Bass (4) has conjectured that a ring A is left perfect iff every nonzero left A-module has a maximal submodule and A has no infinite sets of orthogonal idempotents. As he remarks, this is the natural dual to Part (4) of Theorem 3.1 of Chapter 0.

Hamsher (19) has given an affirmative solution for commutative noetherian rings. We shall extend his solution to arbitrary commutative rings. After discovering the following solution, we noticed that Hamsher has announced a complete solution to the conjecture in the commutative case. However, the solution presented below has the advantage of being more direct and less computational than the one of Hamsher.

For the rest of this section, let A be commutative. We quote without proof:

LEMMA 3.1 (Hamsher (19)). If every nonzero module has a maximal submodule, then every prime ideal of A is a maximal ideal.

and add the obvious:

COROLLARY. In this case the Jacobson radical $J = J(A)$ of A coincides with the prime radical of A.

Proof. $J(A)$ is the intersection of all maximal ideals and the prime radical is the intersection of all prime (= maximal) ideals.

For our main theorem we prove:

LEMMA 3.2. If A has the property that every prime ideal is maximal, then

(1) Every quotient ring A/I has the same property.

(2) A_S has the same property for any mult. set S.

Proof.

(1) is an immediate consequence of the one-one correspondence between the prime (resp. maximal) ideals of A and the prime (resp. maximal) ideals of A/I.

(2) Any prime ideal of A_S has the form p_S where p is a prime ideal of A disjoint from S (Bl, Prop. 11, p. 90). But p is maximal and disjoint from S, and therefore a maximal ideal among

ideals disjoint from S. Hence p_S is a maximal ideal of A_S (Bl, Prop. 11, p. 90).

THEOREM 3.3. The ring A is perfect iff every nonzero A-module has a maximal submodule and A has no infinite sets of orthogonal idempotents.

Proof.

$\Longrightarrow$: has been shown by Bass (4).

$\Longleftarrow$: Bass has also shown that under these conditions the Jacobson radical J of A is T-nilpotent. Therefore by Theorem 3.1 of Chapter O, it only remains to show that B = A/J is semi-simple.

Lambek ((30), p. 72) has shown that if J is a nil ideal of A, any countable orthogonal set of nonzero idempotents in B = A/J can be lifted to an orthogonal set of nonzero idempotents of A. Since any T-nilpotent ideal is clearly a nil ideal, this implies that B has no infinite sets of orthogonal idempotents. Osofsky (38) has remarked that any regular ring with no infinite sets of orthogonal idempotents is a semi-simple ring. Therefore to complete the proof it suffices to show that B is a regular ring.

Using Theorem 2.2 we will prove that B is a regular ring by showing that B_n is a field for all maximal ideals n of B.

By the corollary of Lemma 3.1, J = the prime radical of A and hence B = A/J is a semi-prime ring (see Lambek (30), p. 56). Therefore B_n is semi-prime for all maximal ideals n of B by Theorem 2.1. Since in A every prime ideal is maximal, the same is true for B and B_n by Lemma 3.2. Consequently for all n, B_n is a local semi-primitive ring, i.e. a field, and B is a regular ring by Theorem 2.2.

CHAPTER 8: FLAT COVERS

In this chapter, we shall define and study flat covers. We will show that they always exist, and coincide with the projective cover of Bass (4) in the case that A is perfect, but that they are in general not unique.

1. Definition and Existence of Flat Covers

We begin with a few preliminary definitions. A submodule D of E will be called impure in E iff $D \neq E$ and D contains no pure submodule of E other than 0. Thus all proper submodules of a pure simple module are impure.

Any exact sequence $0 \dashrightarrow K \dashrightarrow F \dashrightarrow E \dashrightarrow 0$ of left A-modules will be called a refinement of E; it will be called proper iff $K \neq 0$. Any refinement $0 \dashrightarrow K \dashrightarrow F \dashrightarrow E \dashrightarrow 0$ will be called flat (resp. impure) iff F is flat (resp. K is impure in F), and impure flat iff it is both impure and flat. It is well known that every module has a flat refinement.

A refinement $u:F \dashrightarrow E$ will be called minimal iff for any factorization $u = vw$, w epi $\Longrightarrow$ w iso. A minimal flat refinement (i.e. minimal among the flat refinements) of E will be called a <u>flat cover</u> of E.

PROPOSITION 1.1. The flat refinement $0 \longrightarrow K \longrightarrow F \xrightarrow{u} E \longrightarrow 0$ is minimal flat (i.e. a flat cover) iff it is impure flat.

Proof.

$\Longrightarrow$: We wish to show that K is impure in F. Suppose P is a submodule of K and P is pure in F. Then u has the canonical factorization $u:F \xrightarrow{w} F/P \xrightarrow{v} E$. Since w is epi, it is iso and $P = 0$. Hence K is impure in F.

$\Longleftarrow$: Suppose $u:F \longrightarrow E$ has a factorization $u = vw$ with w epi, say $w:F \longrightarrow F/P$, and F/P flat. By Proposition 1.3 of Chapter 1, P is pure in F. Since P is clearly a submodule of K, we have $P = 0$ and w is iso. Therefore the refinement is minimal flat.

THEOREM 1.2. Every module E has a flat cover.

Proof. Let $0 \longrightarrow K \longrightarrow F \longrightarrow E \longrightarrow 0$ be a flat refinement of E. The collection of all pure submodules of F contained in K is non-empty, since 0 is such a submodule. If we order the collection by set theoretic inclusion, we can choose a maximal element P by Zorn's Lemma, since purity is an inductive property (Corollary 2 of Theorem 1.6 of Chapter 1). Since P is pure submodule of the flat module F, F/P is flat (Proposition 1.3 of Chapter 1) and $0 \to K/P \to F/P \to E \to 0$ is a flat refinement of E. The refinement is impure, since if K'/P were a pure submodule of F/P contained in K/P then K' would be pure in F and therefore K' = P by the maximality of P, i.e. K'/P = 0.

The proof of the theorem also yields the following obvious:

COROLLARY. Every flat refinement can be factored through a minimal flat refinement.

PROPOSITION 1.3. A nonzero module F is flat iff it has no proper impure refinement.

Proof.

$\Longrightarrow$: Let $0 \to K \to G \to F \to 0$ be an impure refinement. Since F is flat, K is pure in G and therefore $K = 0$.

$\Longleftarrow$: Let $0 \to K \to G \to F \to 0$ be a flat cover of F. Since it is an impure refinement we must have $K = 0$ since it is not proper. Hence $F = G$ is flat.

The following corollary shows that every flat module is its own (and therefore unique) flat cover.

COROLLARY. A nonzero module F is flat iff for every flat cover $0 \to K \to G \to F \to 0$ we have $K = 0$.

Proof.

$\Longrightarrow$: Every flat cover is an impure refinement (Proposition 1.1) and therefore $K = 0$ by the proposition.

$\Longleftarrow$: Every F has a flat cover $0 \to K \to G \to F \to 0$ (Theorem 1.2). Since $K = 0$, $F = G$ is flat.

2. Flat Covers over Perfect Rings and PID's

In this section, we show that for perfect rings, the flat cover defined in Section 1 coincides with the projective cover of Bass (4), whenever it exists, i.e. for left perfect rings.

THEOREM 2.1. If A is left perfect, then any refinement is a flat cover iff it is a Bass projective cover (and therefore unique up to isomorphism).

Proof.

$\Longrightarrow$: Let $0 \to K \to F \xrightarrow{u} E \to 0$ be a flat cover of E. Since A is left perfect, E has a Bass projective cover $0 \to S \to P \to E \to 0$ where S is small in P (Theorem 3.1 of Chapter 0). Since A is left perfect, F is projective and u factors through P, and we have an exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & K & \to & F & \xrightarrow{u} & E & \to & 0 \\
 & & \downarrow & w & \downarrow & & \downarrow 1_E & & \\
0 & \to & S & \to & P & \to & E & \to & 0
\end{array}$$

(one readily verifies that $w(K)$ is contained in S).

Now it is easy to verify that P = S+Im w where Im = image. Therefore P = Im w since S is small, i.e. w is epi. But u:F --> E is a flat cover and therefore w is iso, with inverse w' say. Then K = w'w(K) is contained in w'(S) since w(K) is contained in S. But the image of any small submodule is small (see Lambek (30), Ex. 8, p. 93) and therefore w'(S) is small, whence K is small. Hence 0 --> K --> F --> E --> 0 is a projective cover of E.

<==: Let 0 --> S --> P --> E --> 0 be a projective cover of E. If K is pure in P and contained in S then K is a small pure submodule of a projective module, and therefore K = 0 by Theorem 1.4 of Chapter 6.

Bass (4) has shown the uniqueness, up to isomorphism, of the projective cover.

3. Non-Uniqueness of the Flat Cover

We will now give an example of a module with two distinct flat covers. Let $A = Z$, the ring of integers. Then I- and III-purity coincide (see Chapter 1) and torsion-free and flat coincide (Bl, Prop. 3, p. 29). Using III-purity (= I-purity for Z), Banaschewski (3) has shown that

$$0 \dashrightarrow K \dashrightarrow \mathrm{Hom}_Z(Q,Q/Z) \overset{w}{\dashrightarrow} Q/Z \dashrightarrow 0$$

where w is defined by $w(f) = f(1)$, is a minimal torsion-free cover, i.e. a flat cover of Q/Z.

We claim:

LEMMA 3.1. $0 \dashrightarrow Z \dashrightarrow Q \dashrightarrow Q/Z \dashrightarrow 0$ is also a flat cover of Q/Z.

Proof. Since Q is flat, we need only show that Z is impure in Q (Proposition 1.1). Suppose that the subgroup P of Z is pure in Q. Then P is pure in Z. Since Z is pure simple, $P = 0$ or Z. But Z is not pure in Q since Q/Z is not flat

(= torsion-free). Therefore $P = 0$ and the given exact sequence is a flat cover.

PROPOSITION 3.2. Q is not isomorphic to $\mathrm{Hom}(Q,Q/Z)$.

Proof. Let $Q' = \mathrm{Hom}(Q,Q/Z)$. We will show that for any prime p Q' contains a copy of Z_p, the p-adic integers. For this we use three well-known facts:

(1) Q/Z is the direct sum of the groups Q_p/Z where Q_p is the subring of Q consisting of those rationals, denominators of which are powers of the prime p. (See Lambek (30), Ex. 6, p. 19.)

(2) $Z_p = \mathrm{End}_Z(Q_p/Z)$. See Fuchs (17).

(3) Z_p is uncountable. See Fuchs (17).

For any f in Z_p, we have an induced map described by

$$Q \xrightarrow{u} Q/Z \xrightarrow{v} Q_p/Z \xrightarrow{f} Q_p/Z \xrightarrow{k} Q/Z$$

where v and k are the canonical projections and injections arising from the direct sum decomposition of (1), and u is the canonical epi. Define $h: Z_p \longrightarrow Q'$ by $f \longrightarrow kfvu$. It is straightforward to verify that this is a homomorphism. Since k is mono and v and u are epi, the mapping h is mono. Since Q is countable, we have that Q is <u>not</u> isomorphic to Q'.

4. Localization

We can prove one localization theorem:

THEOREM 4.1. Let $0 \dashrightarrow K \dashrightarrow F \dashrightarrow E \dashrightarrow 0$ be any refinement of E. If $0 \dashrightarrow K_m \dashrightarrow F_m \dashrightarrow E_m \dashrightarrow 0$ is an A_m-flat cover of E_m for all m in $\underline{M}$, then the original refinement is a flat cover of E.

Proof. Since F_m is A_m-flat for all m in $\underline{M}$, F is flat (Bl, Cor., p. 116). If the submodule P of K is pure in F, then for all m in $\underline{M}$, the submodule P_m of K_m is pure in F_m (Theorem 1.2 of Chapter 7). Therefore $P_m = 0$ for all m in $\underline{M}$ and $P = 0$ (Bl, Cor. 2, p. 112).

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