Omnipotence of surface groups

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Dedicated to my beloved grand parents

Shri Vidya Sagar Tiwari

Smt. Shanti Tiwari

and

Smt. Satyawati Awasthi

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ABSTRACT

Roughly speaking, a group G is omnipotent if orders of finitely many elements can be controlled independently in some finite quotients of G. We proved that $\pi_1(S)$ is omnipotent when S is a surface other than $\mathbb{P}^2, \mathbb{T}^2$ or $\mathbb{K}^2$. This generalizes the fact, previously known, that free groups are omnipotent. The proofs primarily utilize geometric techniques involving graphs of spaces with the aim of retracting certain spaces onto graphs.

RÉSUMÉ

Approximativement, on peut dire qu' un groupe G est omnipotent si les ordres d'une quantité finie d'élements peuvent être contrôlés indépendamment dans un quotient fini de G. Nous avons prouvé que $\pi_1(S)$ est omnipotent quand S est une surface autre que $\mathbb{P}^2, \mathbb{T}^2$ ou $\mathbb{K}^2$. Cela généralise le fait, déjà connu, que les groupes libres sont omnipotents. La preuve utilise principalement des techniques géométriques impliquant des graphiques d'espaces ayant pour but de retracter certains espaces en graphiques.

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1. Introduction

The notion of *omnipotent groups* was introduced by Wise in [15]. There are not many known examples of omnipotent groups. The first example found by Wise in [15] are free groups.

In this article, we show that surface groups are omnipotent which gives the first examples of omnipotent groups beyond free groups. More importantly the proof should generalize to many other examples.

Recall that a group G is *potent* if for each $g \in G$, and each $n \in \mathbb{Z}$, there exists a finite quotient $G \longrightarrow \bar{G}$ such that $\bar{g}$ has order n. For example: Free abelian groups are potent. The potency of a group sometimes plays a role in subgroup separability. For more discussion on potent groups see ([1],[7],[13],[18],[17]).

The notion of potency has been generalized to *quasi-potency*, namely a group G is called *quasi-potent* if for each element $g \in G$, there exist a positive integer n, which depends on g, and for each $n \geq 1$ there exists a finite quotient $G \longrightarrow \bar{G}_k$ such that image of g has order $k \cdot n$. It is obvious that a potent group is quasi-potent. For more discussion on quasi-potent groups see [4]. We note that Wise had termed "quasi-potency" as *1-omnipotence* in [15].

Omnipotence of a group is a generalized variation of potency of a group.

We say that an ordered set S of distinct nontrivial elements of a group G is *independent* provided that no two elements of S have conjugate non-trivial powers.

A group G is *omnipotent* if for any finite independent ordered set $S = \{g_1, g_2, \ldots g_r\}$ of elements of G, there exists a constant $k = k(S)$ such that for any ordered set of positive natural numbers $A = \{n_1, n_2, \dots n_r\}$, there exists a finite quotient $G \longrightarrow \bar{G}_A$ such that the order of $\bar{g}_i$ in $\bar{G}_A$ is $k \cdot n_i$ for every i.

Note that $\mathbb{Z} \times \mathbb{Z}$ is potent but not omnipotent, for more explanation see Theorem 7.4 in section 7.

With this notion of omnipotence we show that most "**Surface groups are omnipotent**". This is the main result of this paper, which is discussed in section 7:

Theorem 7.2. Let M be a closed surface other than $\mathbb{P}^2, \mathbb{T}^2$ or $\mathbb{K}^2$ then $\pi_1(M)$ is omnipotent.

The most important section of this paper is section 5 where all the essential blocks are assembled to achieve the goal of this paper. The main technical results of this paper are the following three statements:

Lemma 5.5. Let $\sigma_1, ..., \sigma_k$ be immersed closed paths with a minimal number of horizontal edges in their homotopy classes, representing independent conjugacy classes of $\pi_1(X)$. Suppose σ_1 is not vertically elliptic, then there exists a finite cover $\hat{X}$ of X such that letting $\hat{p} : \hat{X} \longrightarrow \Gamma^v_{\hat{X}}$ be the projection map, the following properties hold:

1. $\hat{p}(\hat{\sigma}_1)$ is a simple cycle, where $\hat{\sigma}_1$ is an elevation of σ_1 on $\hat{X}$.
2. for each $j \neq 1$ and each elevation $\hat{\sigma}_j$ of σ_j, $\hat{p}(\hat{\sigma}_j)$ is either a point or a simple cycle different from $\hat{p}(\hat{\sigma}_1)$.

Theorem 5.6. Let $G = \pi_1 X$ where X is graph of spaces whose each vertex space is a graph and each edge space is a cylinder and $\pi_1 X$ is word hyperbolic. Let $\sigma_1, \sigma_2, ..., \sigma_k$ be closed paths representing independent conjugacy classes of $\pi_1 X$. Suppose σ_1 is not elliptic, then there exists a finite index subgroup H of G such that $\sigma_i^{n_i} \in H$ for some $n_i > 0$ and a homomorphism $\phi : H \longrightarrow \mathbb{Z}$ such that $\phi(\sigma_1^{n_1}) \neq 0$ but $\phi(g\sigma_j^{n_j}g^{-1}) = 0$ for all $j \neq 1$ and all $g \in G$.

In section 6 we discussed the examples and non-examples of omnipotent groups with an interesting result obtained in Theorem 6.3 which is our main criteria for proving omnipotence:

Theorem 6.3. Let $\{g_1, g_2, \ldots, g_r\}$ be the independent set of elements of group G. Suppose for each i, there exists a finite index subgroup H_i of G and a quotient $\phi_i : H_i \longrightarrow \mathbb{Z}$ such that $\phi_i(g_i^{m_i}) \neq 0$ for some m_i, with $g_i^{m_i} \in H_i$, but $\phi_i(fg_j^{m_j}f^{-1}) = 0$ for each m_j, f such that $fg_j^{m_j}f^{-1} \in H_i$ and $j \neq i$. Then G is omnipotent.

2. Covering Spaces of Graphs

The main goal for this section to build the basic notion of canonical completion and retraction of a combinatorial map of graphs. This canonical way of defining completion and retraction of a combinatorial map was introduced by Wise in [15].

The material presented in this section and more is given by Wise in [15].

Definition 2.1. A combinatorial map of graphs $\phi : \Lambda \longrightarrow \Gamma$ is said to be an *immersion* if it is locally injective.

Remark. Immersions have some interesting properties of covering maps. Precisely speaking, a lift of an immersion may not always exist but in case it exist then they are unique. The characterization of a subgroup is more efficient through immersions than covering maps.

Fact 2.2. *The composition of two immersions is also an immersion.*

This is an easy exercise. The following Lemma which says that immersions preserve the reduced paths in graphs, is a special case of this fact.

Lemma 2.3. *If $\phi : \Lambda \longrightarrow \Gamma$ is an* immersion *of graphs, and λ is a reduced path in Λ, then $\phi(\lambda)$ is a reduced path in Γ.*

Lemma 2.4. *If $\phi : \Lambda \longrightarrow \Gamma$ is an* immersion *of graphs, and λ and λ' are paths in Λ having the same initial vertex, and if $\phi(\lambda) = \phi(\lambda')$ then $\lambda = \lambda'$.*

Remark. Its clear from the above Lemma that immersions preserve the uniqueness of path-lifting and similarly its also easy to see that they preserve the uniqueness of lifting also.

Lemma 2.5. *If Λ is any graph, and if λ and λ' are reduced and homotopic paths in λ, then $\lambda = \lambda'$.*

Theorem 2.6. *Suppose that $\phi : \Lambda \longrightarrow \Gamma$ is an immersion of graphs, then $\phi_* : \pi_1(\Lambda) \longrightarrow \pi_1(\Gamma)$ is an injection.*

For complete proof of above Lemmas and Theorem see [12].

Definition 2.7. By a labeled oriented graph with a certain set of labels and orientations such that each edge is oriented and possesses a label. Such graph can be drawn by placing an arrow on each-edge. We say an edge is *incoming* at its target vertex and *outgoing* at its origin.

A bouquet of circles is a special graph consisting of a single-vertex and various labeled-oriented edges. We usually denote bouquet of circles by B.

Many interesting results regarding covering spaces etc. can be proved easily by the help of dealing with bouquet of circles rather than graph with several vertices.

We can easily see that a map from graph Γ to the B can be determined by the labeling and orientation of the edges of a graph.

Definition 2.8. Given a graph Γ, we define Γ^* to be the quotient of Γ obtained by identifying its vertices. thus, there is a quotient map $q : \Gamma \longrightarrow \Gamma/\Gamma^{(0)} = \Gamma^*$. Obviously q is an immersion and so the image $q_*(\pi_1(\Gamma)) \hookrightarrow \pi_1(\Gamma^*)$ is a free factor.

Definition 2.9. A complete labeled oriented graph with respect to its set of labels is a graph of which each vertex of the graph has an incoming and an outgoing edge of each label.

Lemma 2.10. *If a graph is complete relative to its oriented labeling then the induced map to the bouquet of circles is a covering space projection.*

Definition 2.11. Let $\Gamma \longrightarrow B$ be an immersion where B is a bouquet of circles. By a *completion* $\Gamma^\bullet$ of Γ we mean an embedding $\Gamma \hookrightarrow \Gamma^\bullet$, such that $\Gamma^\bullet \longrightarrow B$ is a covering space and the following diagram is commutative.

$$\begin{array}{ccc} \Gamma & \hookrightarrow & \Gamma^\bullet \\ & \searrow & \downarrow \\ & & B \end{array}$$

Note. *If* Γ *has finitely many* 0*-cells then we can choose a completion* $\Gamma^\bullet$ *with the same* 0*-cells as* Γ.

Remark. An infinite ray corresponding to one half (cut at 0-cell) of the universal cover of a circle together with the associated labeling does not have a completion without additional vertices.

Corollary 2.12. *Given an immersed map of connected graphs* $\psi : \Lambda \to \Gamma$ *if* Λ *is compact, then we can complete* Λ *to a finite covering space of* Γ. *More precisely, we can extend* ψ *to a finite covering space map.*

It is nice to have a canonical way to complete an immersion of graphs.

Definition 2.13 (Canonical Completion). Given an immersion $\Gamma \to B$ we define its *canonical completion* $\Gamma^\bullet$ as follows: For each labeled edge c in B, we consider the subgraph Γ_c which is the union of the vertices and the $c-$edges of Γ, that is Γ_c is the preimage of the $c-$circle of B in Γ. Now, we complete Γ_c to a cover $\Gamma_c^\bullet$ of the $c-$circle of B. We assume that there are finitely many vertices in Γ. We consider each component of Γ_c. It is clear that each component can be completed in a unique way without adding any new vertices. In fact, each component gets completed to a cover of the $c-$circle, with the same degree as the number of vertices in that component. It may happen that some components may already be complete.

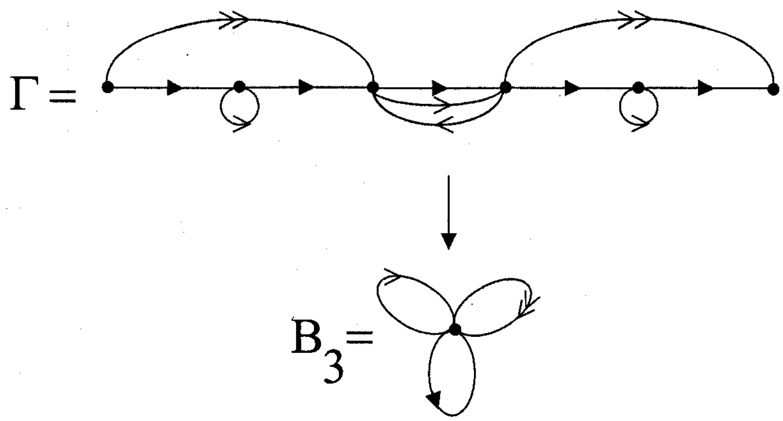

Figure 2–1: Immersion

After performing this process for each circle of B, we define $\Gamma^\bullet$ to be the union of the $\Gamma_c^\bullet$ amalgamated along their common vertices that is $\Gamma^0 = (\Gamma^\bullet)^0$. It is easy to see that Γ embeds in $\Gamma_c^\bullet$.

For more explanation consider the following example:

Example Consider the immersion of graphs $\Gamma \to B_3$, as shown in the figure 2–1. B_3 represents a bouquet of 3-circles labeled by a, b and c corresponding to single, double and solid arrows respectively. The three graphs Γ_a, Γ_b and Γ_c as shown in figure 2–2 are constructed as explained in the definition above corresponding to the circles a, b and c. The three graphs $\Gamma_a^\bullet, \Gamma_b^\bullet$ and $\Gamma_c^\bullet$ as shown in figure 2–3 are complete covers of Γ_a, Γ_b and Γ_c. Now to construct the completion $\Gamma^\bullet$, shown in figure 2–4, of the immersion $\Gamma \to B_3$, we take the union of $\Gamma_a^\bullet, \Gamma_b^\bullet$ and $\Gamma_c^\bullet$ with amalgamated along Γ^0.

Definition 2.14 (Retraction map). We define *retraction* map from $\Gamma^\bullet \to \Gamma$ as the union of retraction maps from $\Gamma_c{}^\bullet \to \Gamma_c$. These maps fix the vertices so the union is well defined. They simply map the added edge of each component of $\Gamma_c{}^\bullet$ to the

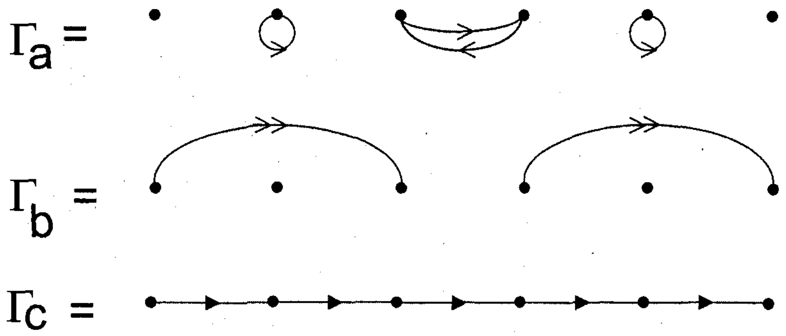

Figure 2–2: Γ_a, Γ_b and Γ_c

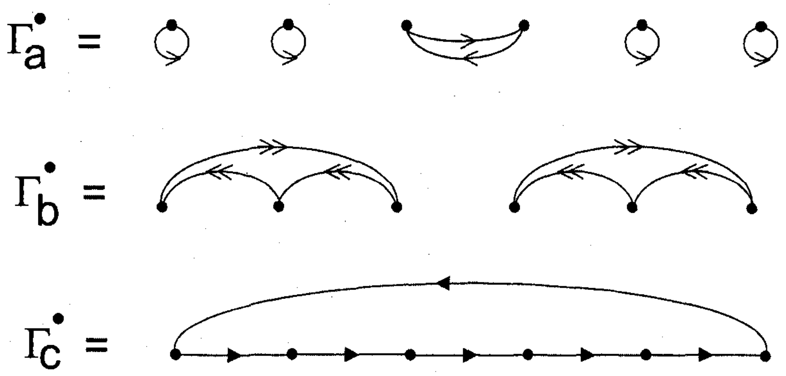

Figure 2–3: Covers of Γ_a, Γ_b and Γ_c

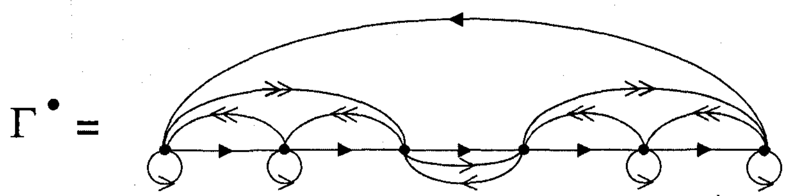

Figure 2–4: Canonical completion of an immersion

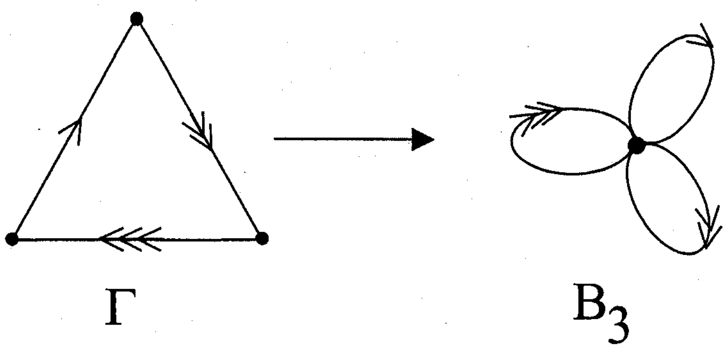

Figure 2–5: Immersion

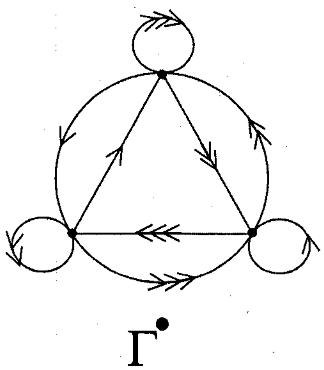

Figure 2–6: Retraction map

union of the other edges. This retraction map preserves the labeling. We note that this map is not a combinatorial map of graphs without first subdividing.

Example: Consider an immersion of graphs $\Gamma \to B_3$, as shown in the figure 2–5. We can construct its completion $\Gamma^{\bullet}$ canonically as explained in previous example and its retraction according to the above definition, shown in figure 2–6.

3. Subgroup Separability

The property of subgroup separability of groups has been an interesting topic of study, for a long time. Historically, subgroup separable groups are also called as LERF (locally extended residually finite) groups. Roughly speaking, the study of subgroup separable groups begin with the study of *polycyclic groups.* In 1938, K. A. Hirsch initiated the study of polycyclic groups and Mal'cev gave an interesting property of subgroups of polycyclic groups that is: Let H be a subgroup of a polycyclic group G. Then H equals the intersection of all the subgroups of finite index in G that contain H. This result can also be reformulated by saying that every subgroup of a polycyclic group is closed into the profinite topology. Later on, Hirsch proved that every polycyclic group is *residually finite.* In a way, subgroup separability is an extended version of residual finiteness property. Later on, Mal'cev proved that *every polycyclic group is subgroup separable*, see [8]. This property came into light after Marshall Hall's result that free groups are subgroup separable, for details [6]. The family of subgroup separable groups was later enriched in 1978 by the result of Peter Scott that every surface group and fuchsian group is subgroup separable (see [9], [10]), in this paper Scott made a beautiful connection between the subgroup separability property of groups and geometric topology.

From a group theory point of view, the study of subgroup separable groups is very important because finitely presented subgroup separable groups have decidable membership problem or generalized word problem : given a finite subset of finitely presented such groups, there is an algorithm to determine if a given element belongs to the subgroup generated by that set.

The concept of subgroup separability plays an interesting role in three-dimensional topology. Subgroup separability plays an important role in this paper in proving the omnipotence of surface groups. We now review the basic notions and terminology regarding subgroup separability and mention the results which I will be using later in this paper.

Definition 3.1. A group G is said to be *LERF* or *subgroup separable* if every finitely generated subgroup of G is closed in the profinite topology of G. In other words, A group G is said to be *subgroup separable*, if each finitely generated subgroup H of G is the intersection of finite index subgroups of G.

Note. *The profinite topology on a group G is the topology whose closed basis consists of the cosets of finite index subgroups of the group. Sometimes it is convenient to express the argument in terms of the profinite topology on a group.*

Equivalently G is subgroup separable if any of the following conditions hold:

1. For any finitely generated subgroup H of G and any element $g \in G \setminus H$, there exists a subgroup of finite index K, such that $H \subset K$ but $g \notin K$.
2. For any finitely generated subgroup $H \subset G$, and any element $g \in G \setminus H$, there is a finite-index subgroup K of G, such that $gK \not\subseteq HK$.
3. For any finitely generated subgroup $H \subset G$ and any element $g \in G \setminus H$, there is a finite quotient $\bar{G}$ of G such that $\bar{g} \notin \bar{H}$.

It is easy to show the equivalence of the above three conditions.

Lemma 3.2. *If H is a finite index subgroup of G then G is subgroup separable if and only if H is subgroup separable.*

For the proof of this Lemma see [15]

Lemma 3.3 (Peter Scott's Geometric characterization). *Let X be a connected cell complex. Let $\hat{X}$ be a based cover of X, then the following are equivalent :*

1. *$\pi_1(\hat{X})$ is separable in $\pi_1(X)$.*
2. *For each compact subspace $C \subset \hat{X}$ there exists a finite cover $\bar{X}$ of X such that $\hat{X} \longrightarrow X$ factors through $\bar{X}$ with the property that C embeds in $\bar{X}$.*

Proof. (1) $\Rightarrow$ (2) : For each 0-cell $p \in C^{(0)}$, let σ_p denote a path from the base point $b \in C$ to p. For each pair of points $p, q \in C$ which are mapped to the same point of X, $\sigma_p \cdot {\sigma_q}^{-1}$ is a path in C which projects to a path in X which does not lift to a closed path in $\hat{X}$.

Since C is compact, there are finitely many such pairs. Consequently, since $\pi_1(\hat{X})$ is separable, there is a finite cover $\bar{X}$ such that $\pi_1(\bar{X}) \subset \pi_1(\hat{X})$, but for each p, q as above, $\sigma_p \cdot {\sigma_q}^{-1} \in \pi_1(\hat{X})$. Finally, since for each p, q the path $\sigma_p \cdot {\sigma_q}^{-1}$ does not lift to a closed path in $\hat{X}$, we see that the lift $C \longrightarrow \hat{X}$ is an embedding.

(1) $\Rightarrow$ (2) : Let $\sigma \in \pi_1(X) - \pi_1(\hat{X})$. The result follows by letting C denote an injective path in $\hat{X}$ with the same endpoints as the lift of σ. ☐

Definition 3.4 (Elevation of a map). Consider the following commutative diagaram:

$$\begin{tikzcd} (\widetilde{X}, \tilde{x}) \arrow[r, "\widetilde{\phi}"] \arrow[d, "\tau"] & (\hat{B}, \hat{b}) \arrow[d, "p"] \\ (X, x) \arrow[r, "\phi"] & (B, b) \end{tikzcd}$$

where $p : (\hat{B}, \hat{b}) \to (B, b)$ is a covering map, $\phi : (X, x) \to (B, b)$ be a map, where X is connected complex and $\tau : (\widetilde{X}, \tilde{x}) \to (X, x)$ is a smallest cover of (X, x) such that $\phi \cdot \tau$ has a lift $\widetilde{\phi}$ then we say that $\widetilde{\phi} : (\widetilde{X}, \tilde{x}) \to (\hat{B}, \hat{b})$ is an *elevation* of ϕ.

Remark. ϕ above is elevated to an embedding if $\widetilde{\phi}$ above is injective. Elevation is a generalization of a lift of a map. We regard $\widetilde{X}$ as a subspace of $\hat{B}$ and we say this subspace as *the elevation* of X.

Lemma 3.5. *Suppose that the elevation of $\phi : X \to B$ to the covering space $p : \hat{B} \to B$ is injective. If q is a covering that factors through p then the elevation of ϕ to q is also injective.*

Lemma 3.6. *Let $\phi_i : \Gamma_i \to B$ be a set of immersions of finite graphs. There exists a finite regular covering $p : \hat{B} \to B$ so that for each i, every elevation of ϕ_i is an embedding.*

Theorem 3.7. *If Γ is a graph and a $\phi : \lambda \hookrightarrow \Gamma$ is any simple cycle, then there exists a finite cover $\hat{\Gamma}$ such that any elevation of λ in $\hat{\Gamma}$ is linearly independent in $H_1(\hat{\Gamma})$ from the full set of elevations of all other simple cycles of Γ.*

Note. *Proof of the above theorem involves the concept of canonical completion, retraction and the above two lemmas. For complete proof see [15]. This theorem was the main tool to proof the omnipotence of free groups in [15].*

Corollary 3.8. *If $\hat{\Gamma} \to \Gamma$ is a finite cover of graph Γ then there exists a homomorphism ${\pi_1}^* : \pi_1(\hat{\Gamma}) \to \mathbb{Z}$ such that ${\pi_1}^*(\tilde{\lambda}) \neq 0$ but ${\pi_1}^*(\tilde{\tau}) = 0$ for every $\tilde{\tau}$, where $\tilde{\tau}$ denotes the choice of elevation of a simple cycle τ in Γ distinct from λ.*

Corollary 3.9. *If $\hat{\Gamma} \to \Gamma$ is a finite cover of graph Γ then there exists a map $\phi : \hat{\Gamma} \to S^1$ such that $\phi(\tilde{\lambda})$ is not null homotopic, but for each $\tilde{\tau} \neq \tilde{\lambda}$, $\phi(\tilde{\tau})$ is null homotopic in S^1.*

Theorem 3.10. *Let G be a word hyperbolic group that splits as a graph of free groups with cyclic edge groups then G is subgroup separable.*

Proof. This is a special case of the main Theorem in [15] where this is proven when G has the property that $g^n \sim g^m \implies n = \pm m$. This is well known for torsion free word hyperbolic groups. For more discussion on word hyperbolic groups see [5], [2], [3]. □

4. VH-complexes

A *square complex* is a 2-complex whose 2-cells are attached by combinatorial paths of length 4.

A square complex X is said to be a VH*-complex* if 1-cells of X are partitioned into two classes V and H called vertical and horizontal edges respectively, and the attaching map of each 2-cell of X alternates between edges in V and H.(See Figure 4–1)

The link of each 0-cell v of X is a graph whose vertices corresponds to 1-cells adjoin with v and whose edges corresponds to corners of 2-cells adjoin with v. In other words we can say that the link of each 0-cell v is a sphere of δ-radius about v in X.

A square complex X is *non-positively curved* when the link of each 0-cell has girth ≥ 4.(See Figure 4–2)

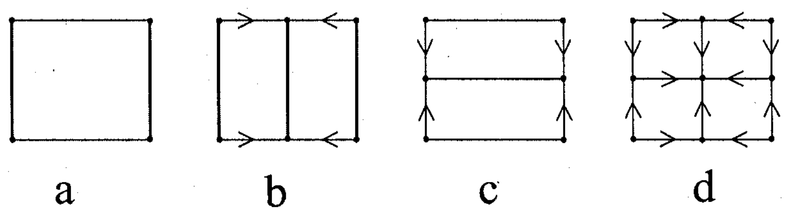

Figure 4–1: Few Examples of VH-complexes
(a) a VH-complex whose vertical edges are bold, (b) a vertically subdivided VH-complex whose horizontal edges are directed, (c) a horizontally subdivided VH-complex whose vertical edges are directed and (d) a fully subdivided directed VH-complex since its all edges are directed.

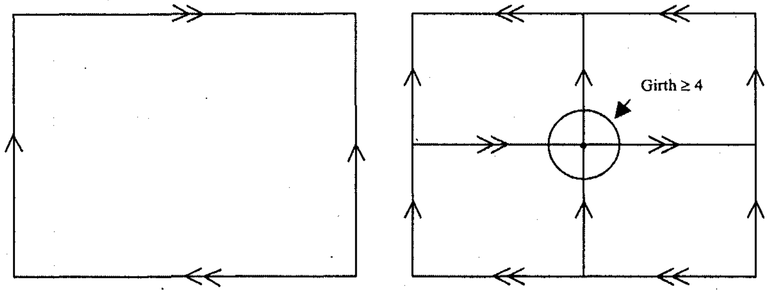

Figure 4–2: The Klein Bottle
The Klein Bottle has a cell-structure which is a non-positively curved square complex. To see this we can subdivide (as shown on the left in the figure) and observe that link(v) has girth ≥ 4 as illustrated on right.

We let $V_X = V \cup X^0$ denote the vertical 1-skeleton and $H_X = H \cup X^0$ denote the horizontal 1-skeleton. Given a 0-cell $x \in X^0$, we let V_x denote the component of V_X containing x. We define H_x similarly.

A combinatorial map $\Phi : Y \to X$ between VH-complexes is called a *VH-map* provided that vertical and horizontal edges of Y are mapped respectively to vertical and horizontal edges of X.

A square complex X is said to be *directed VH-complex* if its vertical and horizontal edges are directed with a choice of orientation on X such that opposite edges of each square are of the same orientation. It is said to be *vertically directed* or *horizontally directed* if only its all vertical edges or horizontal edges of X directed respectively.

Definition 4.1. Let Γ be a directed graph of spaces. A graph of spaces associated with Γ contains the following data : For each vertex v of Γ there is a topological space X_v called vertex space, and for each edge e of Γ, there is a topological space $X_e \times [-1, 1]$ called an edge space. For each edge e there are attaching maps to its initial and terminal vertices $\iota(v)$ and $\tau(v)$, there are monomorphisms $G_e \to G_{\iota(v)}$ and $G_v \to G_{\tau(v)}$.

Note. *The most familiar examples of graphs of spaces are constructed as Eilenberg-Maclane spaces associated to graphs of groups. For more details about graphs of spaces see [11] and [3].*

Remark. A horizontally or vertically directed VH-complex has a decomposition as a graph of spaces. Hence X can be decomposed into vertically directed graph of space Γ^v_X or horizontally directed graph of space Γ^h_X. Subdivisions are always directed.

Definition 4.2. An element g of $\pi_1(X)$ is *vertically elliptic* if $gu = u$ for some $u \in \Gamma^v_{\tilde{X}}$ otherwise we say g is *vertically non-elliptic.* We can define *horizontally elliptic* and *non-elliptic* in a very similar manner.

Lemma 4.3. *Let X be a non-positively curved VH-complex. A closed path represents an elliptic element relative to the horizontal decomposition iff it is homotopic to a vertical path.*

Lemma 4.4. *Let X be a non-positively curved VH-complex. A closed nontrivial immersed path that is entirely horizontal is non-elliptic relative to the horizontal decomposition.*

Remark. Lemma 4.3 and Lemma 4.4 hold with both horizontal and vertical decomposition.

Theorem 4.5. *Let X be a non-positively curved VH-complex then every nontrivial element of $\pi_1(X)$ is either vertically non-elliptic or horizontally non-elliptic .*

Proof. Consider a nontrivial element $g \in \pi_1(X)$ which is elliptic in $\Gamma^h_{\tilde{X}}$ then by Lemma 4.3 it is homotopic to a vertical path and by Lemma 4.4 it is non-elliptic with respect to $\Gamma^v_{\tilde{X}}$. $\square$

Remark. Every horizontally directed VH-complex X has a natural structure as a graph of spaces. The vertex spaces are the connected components of V. The open edge spaces $X_e \times (-1, 1)$ correspond to connected components of $X - V$. Let X have a structure as a graph of spaces then there is an induced structure as a graph of spaces for $\hat{X}$ whenever $\Phi : \hat{X} \longrightarrow X$ is a covering space. The vertex spaces of $\hat{X}$ are connected components of $\Phi^{-1}(X_v)$ where v varies over the vertices of Γ. Similarly, the edge spaces of $\hat{X}$ correspond to components of $\Phi^{-1}(X_e \times (-1, 1))$. There is also an induced map $\Gamma_{\hat{X}} \longrightarrow \Gamma_X$ between the underlying graphs of $\hat{X}$ and X. See Figure 4–3.

(a)

(b)

(c)

(d)

(e)

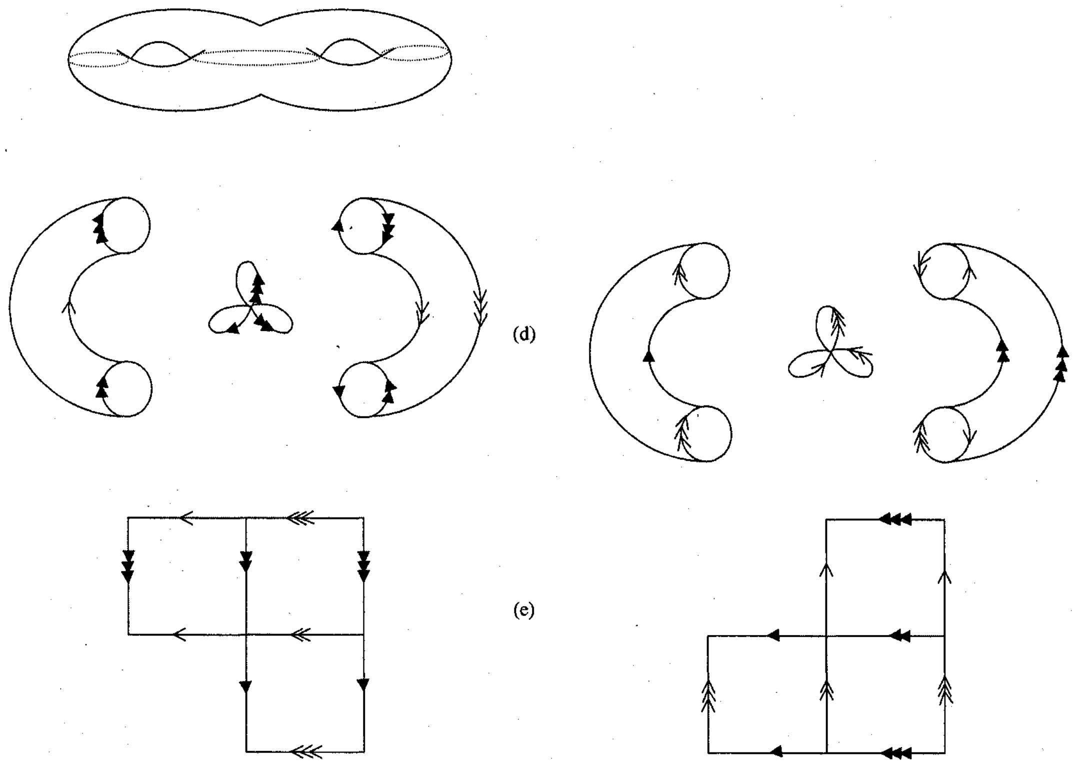

Figure 4–3: Decomposition

The vertical and horizontal decomposition of a directed $VH-$complex homeomorphic to genus 2 orientable surface: (a) genus 2 orientable surface, (b) a graphical decomposition of (a) corresponding to its VH-complex in (c) similarly (d) a graphical decomposition of (a) corresponding to its VH-complex shown in (e), in both cases all attaching maps are immersions and clearly edge spaces are cylinders.

5. Separating Cycles

Definition 5.1 (Missing Corner). let $\Phi : Y \to X$ be an immersion of graphs of spaces. Φ is said to have *no missing corner* if for every distinct pair of edges v, h that are adjacent to y in Y^0 and if $\Phi(v), \Phi(h)$ form the corner of a 2-cell C in X then there is a 2-cell $\hat{C}$ in Y such that v, h form the corner of $\hat{C}$ at y and $\Phi(\hat{C}) = C$. See Figure 5–1

Lemma 5.2. *Let X and C be graphs of spaces whose vertex spaces are graphs and whose edge spaces are cylinders. Let $\Phi : C \longrightarrow X$ be an immersion of graphs of spaces with no missing corners then $\tilde{\Phi} : \tilde{C} \longrightarrow \tilde{X}$ is an embedding.*

Proof. The proof will follow from the following two claims:

claim 1: $\tilde{\Phi}$ restricts to an embedding on each vertex and each edge space.

Proof of claim 1: $\tilde{\Phi}$ is an immersion since Φ is an immersion and hence $\tilde{\Phi}$ restricts to an immersion between vertex and edge spaces. An immersion between trees $\tilde{C}$ and $\tilde{X}$ is clearly an embedding and likewise an immersion from $\tilde{C} \times I$ to $\tilde{X} \times I$ is an embedding. Therefore, $\tilde{\Phi}$ restricts to an embedding on each vertex space and each edge space of $\tilde{C}$.

claim 2: $\tilde{\Phi}_\Gamma : \Gamma_{\tilde{C}} \longrightarrow \Gamma_{\tilde{X}}$ is an immersion.

Proof of claim 2: Suppose $\tilde{\Phi}_\Gamma$ is not an immersion. Then there is a vertex space V of $\tilde{C}$ and two distinct edge spaces A and B with A and B both adjacent to V such that $\Phi(A)$ and $\Phi(B)$ lie in the same edge space of $\tilde{X}$. Let a, b be horizontal edges in A and B respectively. Consider $\Phi(a)$ and $\Phi(b)$. Since $\Phi(a)$,$\Phi(b)$ lie in the same edge space, there is a sequence of rectangles between them. Since Φ has no missing

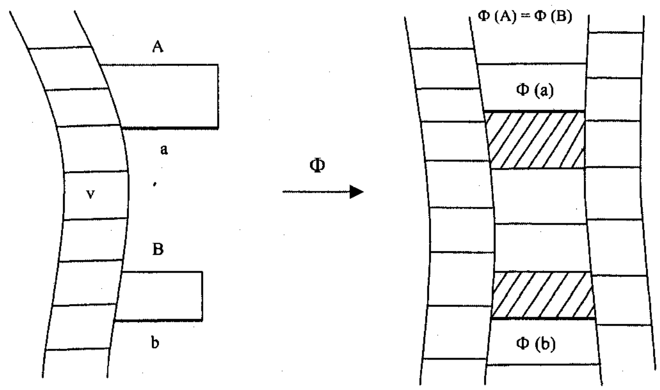

Figure 5–1: Missing corner
The map Φ above has two missing corners shown by shaded region in this figure

corners, this lifts to a sequence of rectangles in $\tilde{C}$ at V between a and b, but then a and b will lie in the same edge space which is a contradiction since $A \neq B$. So, $\tilde{\Phi}_\Gamma : \Gamma_{\tilde{C}} \longrightarrow \Gamma_{\tilde{X}}$ is an immersion.

Now, we can complete the proof of lemma 5.2 as follows:

If p, q are two distinct points of $\tilde{C}$ lying in the same vertex space or edge space then $\tilde{\Phi}(p) \neq \tilde{\Phi}(q)$ by claim 1 and if p, q lie in the interior of distinct vertex space or edge space then $\tilde{\Phi}(p) \neq \tilde{\Phi}(q)$ by claim 2 and the following commutative diagram.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \tilde{X} \arrow[r] \arrow[d] & \Gamma_{\tilde{X}} \arrow[d] \\ C \arrow[r] & \tilde{C} \arrow[r] & \Gamma_{\tilde{C}} \end{tikzcd}$$

□

Corollary 5.3. *If $\Phi : C \longrightarrow X$ is an immersion of graphs of spaces with no missing corners then $\Phi_* : \pi_1(C, c) \longrightarrow \pi_1(X, x)$ is injective.*

Corollary 5.4. *Let $\Phi : C \longrightarrow X$ be the immersion of graphs of spaces with no missing corners and $\hat{X}$ be the based covering space of X corresponding to $\Phi_*(\pi_1(C,c))$ then the following lift is injective.*

$$\begin{tikzcd} & \hat{X} \arrow[d] \\ C \arrow[r] \arrow[ur, hook] & X \end{tikzcd}$$

The following is the main technical result in this paper:

Lemma 5.5. *Let $\sigma_1, ..., \sigma_k$ be immersed closed paths with minimal number of horizontal edges in their homotopy classes, representing independent conjugacy classes of $\pi_1 X$. Suppose σ_1 is not vertically elliptic, then there exists a finite cover $\hat{X}$ of X such that letting $\hat{p} : \hat{X} \longrightarrow \Gamma^v_{\hat{X}}$ be the projection map, the following properties hold:*

1. *$\hat{p}(\hat{\sigma}_1)$ is a simple cycle, where $\hat{\sigma}_1$ is an elevation of σ_1 on $\hat{X}$.*
2. *for each $j \neq 1$ and each elevation $\hat{\sigma}_j$ of σ_j, $\hat{p}(\hat{\sigma}_j)$ is either a point or a simple cycle different from $\hat{p}(\hat{\sigma}_1)$.*

Proof. Let $\widetilde{X}$ be the universal cover of X. Since X is a graph of spaces, $\widetilde{X}$ is a tree of spaces whose vertex spaces are trees and edge spaces are infinite-strips.

Let $\widetilde{\sigma_1} \hookrightarrow \widetilde{X}$ be a lift of $\widetilde{\sigma_1} \longrightarrow X$, so that we obtain the following commutative diagram:

$$\begin{tikzcd} \widetilde{\sigma_1} \arrow[r, hook] \arrow[d] & \widetilde{X} \arrow[r] \arrow[d] & \Gamma_{\widetilde{X}} \arrow[d] \\ \sigma_1 \arrow[r] & X \arrow[r] & \Gamma_X \end{tikzcd}$$

where $X \longrightarrow \Gamma^v_X$ and $\tilde{p} : \widetilde{X} \longrightarrow \Gamma^v_{\widetilde{X}}$ are the maps to the underlying graphs of spaces. We consider the subspace of $\widetilde{X}$ consisting of all vertex spaces and edge spaces that $\widetilde{\sigma_1}$ passes through (see Figure 5–2). Specifically let $V_1 = \tilde{p}^{-1}(\tilde{p}(\widetilde{\sigma_1}))$. By

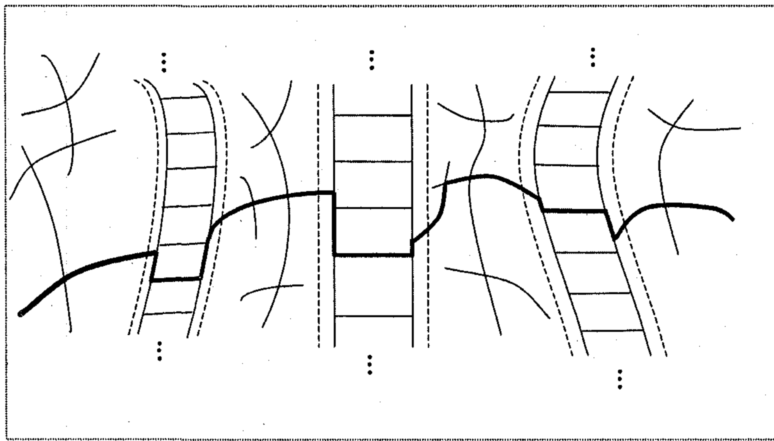

Figure 5–2: V_1

our assumption that σ_1 has a minimal number of horizontal edges,we see that $\widetilde{\sigma_1}$ has no backtracks, i.e. $\tilde{p}(\widetilde{\sigma_1})$ is homeomorphic to $\mathbb{R}$.

Indeed, if $\widetilde{\sigma_1}$ has a backtrack (that is $\widetilde{\sigma_1}$ is conjugate to some $\widetilde{\sigma_1}'$, See Figure 5–3) then after cyclic permutation, $\sigma_1 = \alpha h v h^{-1} \beta$ where h is a horizontal edge and v is a vertical path in the edge space of h and $hvh^{-1} = v'$ for some vertical path on the other side of this edge space. Thus σ_1 is homotopic to $\alpha v' \beta$ which has fewer horizontal edges.

Now, we obtain a subspace W_1 from V_1 by taking the smallest connected subcomplex of V_1 containing $\widetilde{\sigma_1}$ and each edge space of V_1. See Figure 5–4. Let Z_1 be the quotient of W_1 that is obtained by quotienting each edge space to a cylinder of sufficiently large size (to be chosen later), so that the map $Z_1 \longrightarrow X$ is locally the same as the map $W_1 \longrightarrow X$ and consequently the map $Z_1 \longrightarrow X$ is an immersion and has no missing corners. Moreover, we do this in such a way that each cylinder is of the same size and so the group generated by $\langle \sigma_1 \rangle$ acts on Z_1 cocompactly. See Figure 5–5. We define A_1 to be the quotient of Z_1 that is

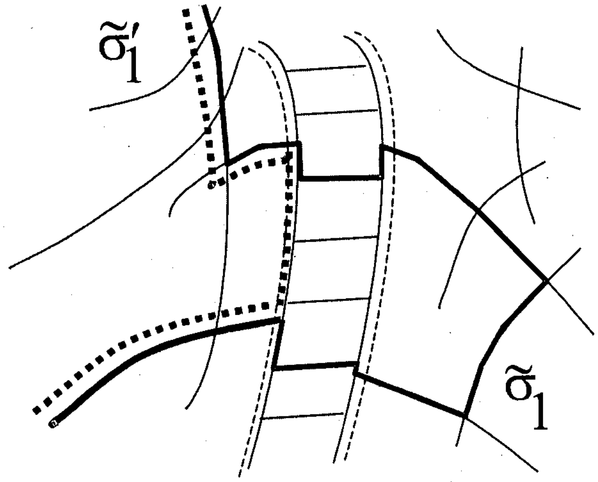

Figure 5–3: $\tilde{\sigma_1}'$ over $\tilde{\sigma_1}$

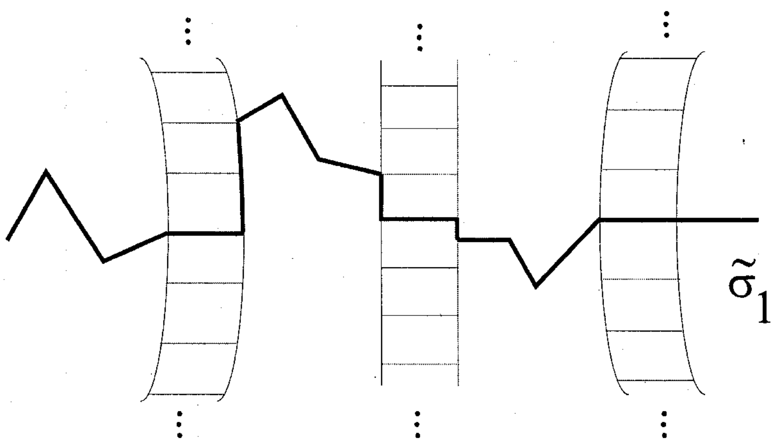

Figure 5–4: W_1
$\tilde{\sigma}_1$ is shown by the black line in W_1

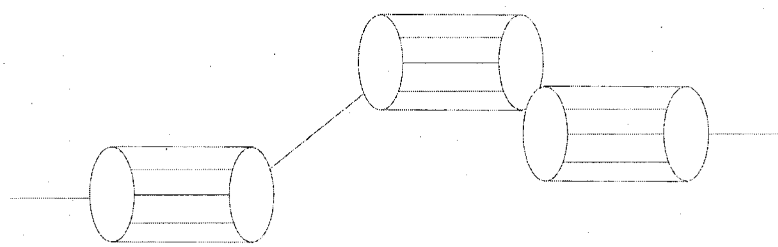

Figure 5–5: Z_1

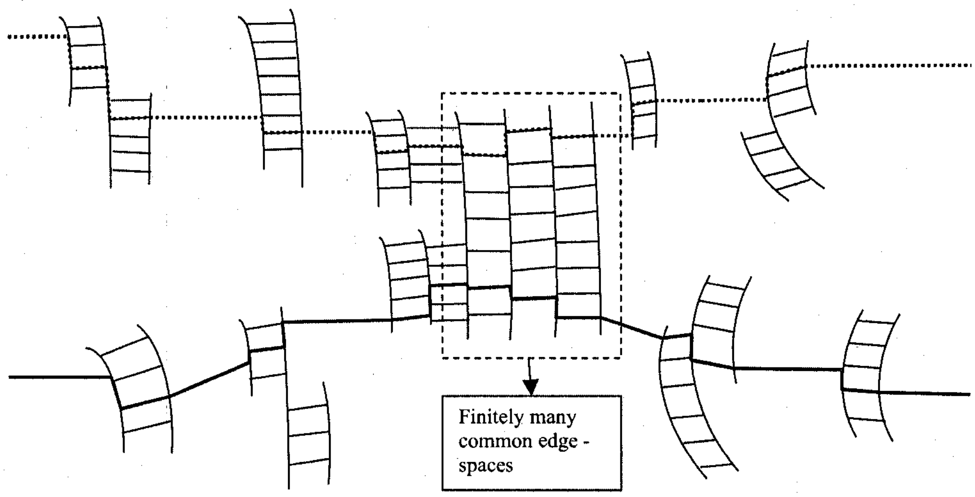

Figure 5–6: $W_1 \cup W_{j(E,g)}$

obtained by the group action of $\langle \sigma_1^m \rangle$ on Z_1, where m is a sufficiently large positive integer to be chosen later.

By a similar construction, for each σ_j we obtain W_j followed by Z_j and then A_j for each $j \neq 1$.

For each j we have chosen a $W_j \subset \widetilde{X}$ such that a fixed lift of $\widetilde{\sigma_j}$ is contained in W_j. We shall now consider the various different ways that translates gW_j overlap with W_1 along some edge spaces. In fact up to certain group actions, there are only finitely many types of such overlaps.

Now, note that $W_1 \cap gW_j$ contains an edge space E or in other words $\tilde{p}(\widetilde{\sigma_1}) \cap \tilde{p}(g\widetilde{\sigma_j})$ is a nontrivial path in $\Gamma^v_{\widetilde{X}}$ and observe that there are only finitely many possible edge spaces E up to the action of $stab(\widetilde{\sigma_1})$ and for each E there are only finitely many translates $g\widetilde{\sigma_j}$ of $\widetilde{\sigma_j}$ that pass through the common edge space E up to the action of $stab(E)$. Let (E, g) denote the choice of one such edge space E and $g \in \pi_1(X)$. Consider, $W_1 \cup W_{j(E,g)}$, where $W_{j(E,g)} = gW_j$. See Figure 5–6. The quotients $W_1 \longrightarrow Z_1$ and $W_j \longrightarrow Z_j$ induce a quotient $W_1 \cup W_{j(E,g)} \longrightarrow Z_1 \cup Z_{j(E,g)}$

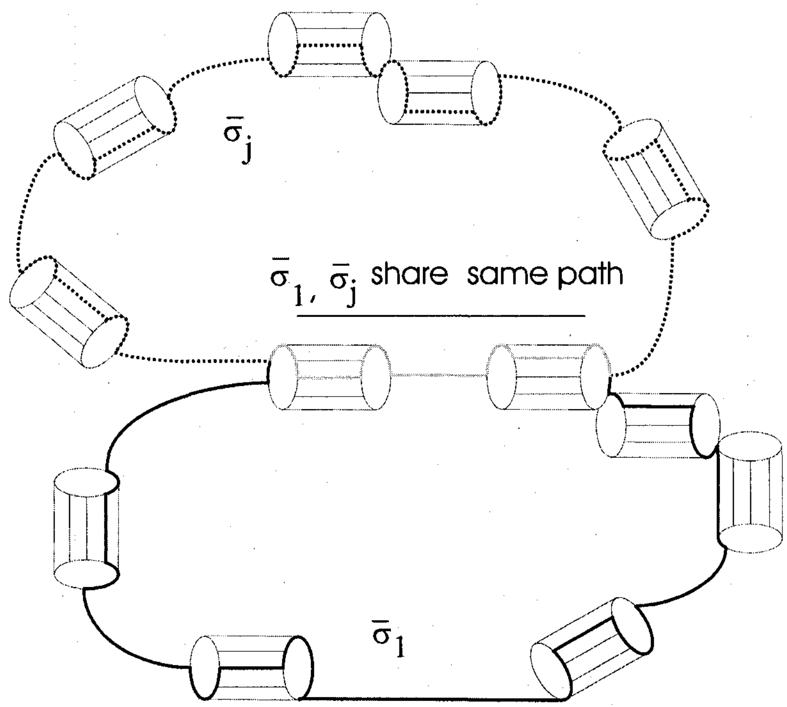

Figure 5–7: $Y_{j(E,g)} = A_1 \cup A_{j(E,g)}$

and the quotients $Z_1 \longrightarrow A_1$ and $Z_j \longrightarrow A_j$ induce a quotient $Z_1 \cup Z_{j(E,g)} \longrightarrow A_1 \cup A_{j(E,g)}$. We define $Y_{j(E,g)} = A_1 \cup A_{j(E,g)}$. See Figure 5–7. Let $\bar{\sigma}_1$,$\bar{\sigma}_j$ denote the elevations of σ_1, σ_j to $Y_{j(E,g)}$ corresponding to $\tilde{\sigma}_1/\langle\sigma_1^m\rangle$, $\tilde{\sigma}_j/\langle\sigma_j^m\rangle$. Observe that $\bar{p}(\bar{\sigma}_1)$, $\bar{p}(\bar{\sigma}_j)$ are distinct cycles in $\Gamma_{Y_{j(E,g)}}$ by construction. See Figure 5–8. The map $\phi : Y_{j(E,g)} \longrightarrow X$ is locally the same as $W_1 \cup W_{j(E,g)} \longrightarrow X$ and so ϕ is an immersion with no missing corners. Let $\hat{X}_{j(e,g)}$ be the finite based cover of X corresponding to $\phi_*(\pi_1(Y_{j(E,g)}))$. Observe that $Y_{j(E,g)} \longrightarrow \hat{X}_{j(E,g)}$ is injective by Corollary 5.4, and π_1-injective by Corollary 5.3, so $\pi_1(Y_{j(E,g)}) = \pi_1(\hat{X}_{j(E,g)})$. By Theorem 3.10, $\pi_1(\hat{X}_{j(E,g)})$ is separable in $\pi_1(X)$, and thus by Lemma 3.3 there exists an intermediate finite cover $\bar{X}_{j(E,g)}$ of X such that $\hat{X}_{j(E,g)} \longrightarrow X$ factors through $\bar{X}_{j(E,g)}$ with the property that $Y_{j(E,g)}$ embeds in $\bar{X}_{j(E,g)}$.

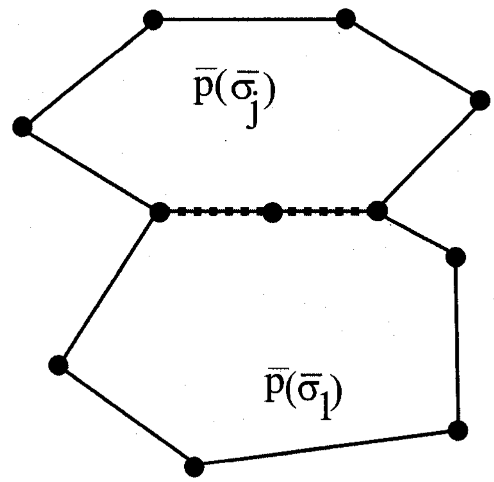

Figure 5–8: $\Gamma_{Y_{j(E,g)}}$

The elevations $\bar{\sigma}_1$, $\bar{\sigma}_j$ in $Y_{j(E,g)}$ lie in $\bar{X}_{j(E,g)}$ and it remains true that $\bar{p}(\bar{\sigma}_1)$, $\bar{p}(\bar{\sigma}_j)$ are distinct cycles in $\Gamma_{Y_{j(E,g)}}$ because of the following commutative diagram:

$$\begin{tikzcd} Y_{j(E,g)} \arrow[r, hook] \arrow[d] & \bar{X}_{j(E,g)} \arrow[d] \\ \Gamma^v_{Y_{j(E,g)}} \arrow[r, hook] & \Gamma^v_{\bar{X}_{j(E,g)}} \end{tikzcd}$$

Let $\hat{X}$ be a finite regular cover factoring through each $\bar{X}_{j(E,g)}$. Let $\hat{\sigma}_1$ and $\hat{\sigma}_j$ be elevations of σ_1 and σ_j to $\hat{X}$. If $\hat{\sigma}_1$ and $\hat{\sigma}_j$ do not have horizontal edges in the same edge space, then obviously $\hat{p}(\hat{\sigma}_1)$ and $\hat{p}(\hat{\sigma}_j)$ are distinct in $\Gamma_{\hat{X}}$. So assume that there is a common edge space in the sense that $\hat{\sigma}_1$ and $\hat{\sigma}_j$ contain horizontal edges f_1, f_j lying in the same edge space F of $\hat{X}$. Let γ be a path in F from f_1 to f_j. See Figure 5–9. Consider the based lift $\widetilde{\sigma_1}$ of $\hat{\sigma}_1$ to $\widetilde{X}$. Let $\widetilde{\gamma}$ be the lift of γ starting at $\widetilde{f_1} \subset \widetilde{\sigma_1}$. Let $\widetilde{f_j}$ be the lift of f_j at the end point of $\widetilde{\gamma}$. This determines a lift of $\widetilde{\sigma_j}$ at $\widetilde{f_j}$ that corresponds to some translate $h\widetilde{\sigma_j}$ of $\widetilde{\sigma_j}$. Earlier we found that there are only finitely many $\langle\sigma_1\rangle$ orbits of edges of E, so $\widetilde{F} = E$ for one such choice. There are finitely many $stab(E)$-orbits of translates of $\widetilde{\sigma_j}$ passes through E. In particular,

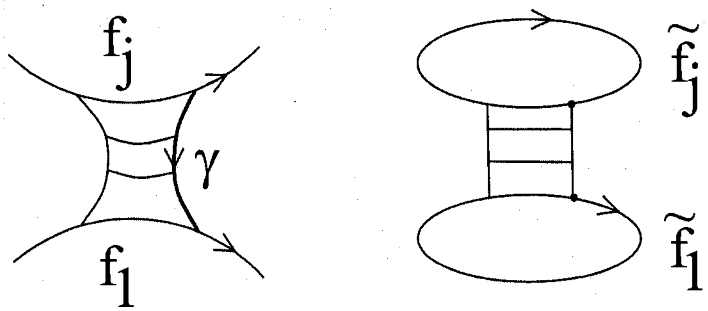

Figure 5–9: a path γ in F

$h\widetilde{\sigma}_j$ lies in one such orbit, represented by $g\widetilde{\sigma}_j$. The paths $\hat{\sigma}_1$ and $\hat{\sigma}_j$ map to $\bar{\sigma}_1$ and $\bar{\sigma}_j$ respectively under the map $\hat{X} \longrightarrow \bar{X}_{j(E,g)}$. By construction $\bar{p}(\bar{\sigma}_1)$ and $\bar{p}(\bar{\sigma}_j)$ are distinct in $\Gamma^v_{\bar{Y}_{j(E,g)}}$ and hence in $\Gamma^v_{\bar{X}_{j(E,g)}}$.

$$\begin{tikzcd}
\widetilde{\sigma}_1 \arrow[r, hook] \arrow[d] & \widetilde{X} \arrow[r, "\tilde{p}"] \arrow[d] & \Gamma^v_{\widetilde{X}} \arrow[d] \\
\hat{\sigma}_1, \hat{\sigma}_{j(E,g)} \arrow[r, hook] \arrow[d] & \hat{X}_{j(E,g)} \arrow[r, "\hat{p}"] \arrow[d] & \Gamma^v_{\hat{X}_{j(E,g)}} \arrow[d] \\
\bar{\sigma}_1, \bar{\sigma}_{j(E,g)} \arrow[r, hook] \arrow[d] & \bar{X}_{j(E,g)} \arrow[r, "\bar{p}"] \arrow[d] & \Gamma^v_{\bar{X}_{j(E,g)}} \arrow[d] \\
\sigma_1, \sigma_j \arrow[r, hook] & X \arrow[r, "p"] & \Gamma^v_X
\end{tikzcd}$$

$\square$

Theorem 5.6. *Let $G = \pi_1 X$ where X is graph of spaces whose each vertex space is a graph and each edge space is a cylinder and $\pi_1 X$ is word hyperbolic. Let $\sigma_1, \sigma_2, ..., \sigma_k$ be closed paths representing independent conjugacy classes of $\pi_1 X$. Suppose σ_1 is not elliptic, then there exists a finite index subgroup H of G such that*

$\sigma_i^{n_i} \in H$ for some $n_i > 0$ and a homomorphism $\phi : H \longrightarrow \mathbb{Z}$ such that $\phi(\sigma_1^{n_1}) \neq 0$ but $\phi(g\sigma_j^{n_j}g^{-1}) = 0$ for all $j \neq 1$ and all $g \in G$.

Proof. By Lemma 5.5, there is finite index subgroup $K \subset G$ where $K = \pi_1\hat{X}$ and a homomorphism $\psi : K \longrightarrow F$ where $F = \pi_1\Gamma^v_{\hat{X}}$ is a free group and $\psi : K \longrightarrow F$ is map induced by the map $\hat{p} : \hat{X} \longrightarrow \Gamma^v_{\hat{X}}$. By Corollary 3.8, there exists $F' \subset F$ such that $[F : F'] < \infty$ and $\varphi : F' \longrightarrow \mathbb{Z}$ with $\varphi(\psi(\sigma_1^{n_1})) \neq 0$ and $\varphi(\psi(g\sigma_j^{n_j}g^{-1})) = 0$, for all j and g. Now let $H = \psi^{-1}(F')$ and $\phi = \varphi \circ \psi$. $\square$

6. Omnipotent Groups

Definition 6.1. We say that an ordered set of distinct non-trivial elements $g_1, g_2, \ldots, g_r$ of a group G is *independent* provided that nontrivial powers of distinct elements are not conjugate. More precisely, if $i \neq j$ and m, n are nonzero integers, then $g_i^m \nsim g_j^n$.

Definition 6.2. A group G is *omnipotent*, if for any r-tuple of independent elements $S = \{g_1, g_2, \ldots, g_r\}$, there exists a constant $k = k(S)$, such that for any r-tuple of positive integers $A = \{n_1, n_2, \ldots, n_r\}$ there exists a finite quotient $G \longrightarrow \bar{G_A}$ so that for each i, the image $\bar{g_i}$ has order $k \cdot n_i$ in $\bar{G_A}$.

Examples: Free Groups. See [15].

Non-Examples: $\mathbb{Z} \times \mathbb{Z}$, Any non-cyclic nilpotent group.

Theorem 6.3. *Let $\{g_1, g_2, \ldots, g_r\}$ be the independent set of elements of group G. Suppose for each i, there exists a finite index subgroup H_i of G and a quotient $\phi_i : H_i \longrightarrow \mathbb{Z}$ such that $\phi_i(g_i^{m_i}) \neq 0$ for some m_i, with $g_i^{m_i} \in H_i$, but $\phi_i(f g_j^{m_j} f^{-1}) = 0$ for each m_j, f such that $f g_j^{m_j} f^{-1} \in H_i$ and $j \neq i$. Then G is omnipotent.*

Proof. Let $\{n_1', \ldots, n_r'\}$ be any r-tuple of positive integers.

For a fixed i, consider

$$H_i \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_{n_i'}.$$

Let

$$K_{(i,n_i')} = Ker(H_i \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_{n_i'}).$$

Let

$$N_{(i,n_i')} = \bigcap_{g \in G} K^g_{(i,n_i')}.$$

Consider

$$N = \bigcap_{i=1}^{r} N_{(i,n_i')}$$

Claim: G/N is finite.

we have

$$N \hookrightarrow N_{(i,n_i')} \hookrightarrow K_{(i,n_i')} \hookrightarrow H_i \hookrightarrow G \to \mathbb{Z} \to \mathbb{Z}_{n_i'}$$

By hypothesis, $[G : H_i] < \infty$ and $[H_i : K_{(i,n_i')}] = n_i' < \infty$ and so $[G : K_{(i,n_i')}] < \infty$. Thus $[G : N_{(i,n_i')}] < \infty$, since G contains finitely many conjugates of $K_{(i,n_i')}$ and the intersection of finite index subgroups is of finite index.

This implies that $[G : N] < \infty$, since each $[G : N_{(i,n_i')}] < \infty$ and again the intersection of finite index subgroups is of finite index.

Consider $G \longrightarrow G/N$. We now compute the order of $\bar{g}_i$ for each i in G/N. Observe that the order of $\bar{g}_i$ in G/N is equal to LCM of the order of the image of g_i in $G/N_{(j,n_j')}$, where $j \in \{1, 2, \ldots, r\}$. Thus, the order of the image of g_i in G/N is equal to LCM of the order of the image of g_i in $G/N_{(j,n_j')}$, for $j \neq i$ and order of the image of g_i in $G/N_{(i,n_i')}$.

Let k_{ij} be the order of the image of g_i in $G/N_{(j,n_j')}$. We shall now prove that for $i \neq j$, k_{ij} is a constant independent of n_j'.

For any $g \in G$, the order of the image of g in G/N is $\big[\langle g \rangle : N \cap \langle g \rangle\big]$.

Thus, the order $o(\bar{g}_i)$ of $\bar{g}_i$ in $G/N_{(j,n_j')} = \big[\langle g_i \rangle : N_{(j,n_j')} \cap \langle g_i \rangle\big]$.

By the definition of $N_{(j,n_j')}$, the order $o(\bar{g}_i)$ of $\bar{g}_i$ in $G/N_{(j,n_j')}$ equals to

$$\Big[\langle g_i \rangle : \langle g_i \rangle \cap \Big(\bigcap_{g \in G} K^g_{(j,n_j')}\Big)\Big].$$

Now, conjugating by g^{-1}, $o(\bar{g}_i)$ in $G/N_{(j,n'_j)}$ equals to

$$\left[\langle g_i^{g^{-1}}\rangle : \bigcap_{g\in G}\left(\langle g_i^{g^{-1}}\rangle \cap K_{(j,n'_j)}\right)\right].$$

Observe that for $i \neq j$, $\langle g_i^{g^{-1}}\rangle \cap K_{(j,n'_j)} = \langle g_i^{g^{-1}}\rangle \cap H_j$. Indeed, $\langle g_i^{g^{-1}}\rangle \cap K_{(j,n'_j)} \subset \langle g_i^{g^{-1}}\rangle \cap H_j$ since $K_{(j,n'_j)} \subset H_j$. On the other hand, $\langle g_i^{g^{-1}}\rangle \cap H_j \subset \langle g_i^{g^{-1}}\rangle \cap K_{(j,n'_j)}$ since for each $m_i \in \mathbb{Z}$, $(g_i^{m_i})^{g^{-1}} \in H_j \Longrightarrow (g_i^{m_i})^{g^{-1}} \in K_{(j,n'_j)}$, by hypothesis.

Therefore, $o(\bar{g}_i)$ in $G/N_{(j,n'_j)}$

$$= \left[\langle g_i^{g^{-1}}\rangle : \bigcap_{g\in G}\left(\langle g_i^{g^{-1}}\rangle \cap H_j\right)\right].$$

Conjugating by g, we get $o(\bar{g}_i)$ in $G/N_{(j,n'_j)}$

$$= \left[\langle g_i\rangle : \bigcap_{g\in G}\left(\langle g_i\rangle \cap H_j^g\right)\right].$$

We have thus determined that

$$k_{ij} = \left[\langle g_i\rangle : \bigcap_{g\in G}\left(\langle g_i\rangle \cap H_j^g\right)\right].$$

We shall now compute k_{ii}, the order $o(\bar{g}_i)$ of the image $\bar{g}_i$ of g_i in $G/N_{(i,n'_i)}$.

The order $o(\bar{g}_i)$ of $\bar{g}_i$in $G/N_{(i,n'_i)} is \left[\langle g_i\rangle : \langle g_i\rangle \cap N_{(i,n'_i)}\right]$.

By the definition of $N_{(i,n'_i)}$, we get $o(\bar{g}_i)$ in $G/N_{(i,n'_i)}$ equals to

$$\left[\langle g_i\rangle : \langle g_i\rangle \cap \left(\bigcap_{g\in G} K^g_{(i,n'_i)}\right)\right].$$

equivalently, $o(\bar{g}_i)$ in $G/N_{(i,n'_i)}$ equals to

$$\left[\langle g_i\rangle : \bigcap_{g\in G}\left(\langle g_i\rangle \cap K^g_{(i,n'_i)}\right)\right].$$

Conjugating by g^{-1}, we get $o(\bar{g}_i)$ in $G/N_{(i,n_i')}$ equals to

$$\left[\langle g_i^{g^{-1}}\rangle : \bigcap_{g\in G}\left(\langle g_i^{g^{-1}}\rangle \cap K_{(i,n_i')}\right)\right].$$

Now, there are two cases here to consider:

1. $(g_i^{g^{-1}})^{m_i} \in K_{(i,n_i')}$.
2. $(g_i^{g^{-1}})^{m_i} \notin K_{(i,n_i')}$ (this occurs, in particular, when $g = 1$ by hypothesis).

In case 1, we get $\langle g_i^{g^{-1}}\rangle \cap K_{(i,n_i')} = \langle g_i^{g^{-1}}\rangle \cap H_i$.

Therefore,

$$\left[\langle g_i^{g^{-1}}\rangle : \bigcap_{g\in G}\left(\langle g_i^{g^{-1}}\rangle \cap K_{(i,n_i')}\right)\right] = \left[\langle g_i^{g^{-1}}\rangle : \bigcap_{g\in G}\left(\langle g_i^{g^{-1}}\rangle \cap H_i\right)\right].$$

Note that n_i' doesn't yield extra choices of $g \in G$.

We define the constant β_i which is independent of the n_i' as follows :

$$\beta_i = \left[\langle g_i^{g^{-1}}\rangle : \bigcap_{g\in G}\left(\langle g_i^{g^{-1}}\rangle \cap H_i\right)\right]$$

be the constant.

In case 2, we have $\langle g_i^{g^{-1}}\rangle \cap K_{(i,n_i')} = n_i'\left[\langle g_i^{g^{-1}}\rangle \cap H_i\right]$. Therefore,

$$\left[\langle g_i^{g^{-1}}\rangle : \bigcap_{g\in G}\left(\langle g_i^{g^{-1}}\rangle \cap K_{(i,n_i')}\right)\right] = n_i' \cdot \left[\langle g_i^{g^{-1}}\rangle : \bigcap_{g\in G}\left(\langle g_i^{g^{-1}}\rangle \cap H_i\right)\right].$$

We define a constant α_i which is independent of the n_i' as follows.

Let

$$\alpha_i = \left[\langle g_i^{g^{-1}}\rangle : \left(\langle g_i^{g^{-1}}\rangle \cap H_i\right)\right].$$

Note that,

$$\left[\langle g_i^{g^{-1}}\rangle : \bigcap_{g\in G}\left(\langle g_i^{g^{-1}}\rangle \cap K_{(i,n_i')}\right)\right] = \alpha_i \cdot n_i'.$$

In $G/N_{(i,n_i')}$

$$o(\bar{g}_i) = \left[\langle g_i \rangle : \bigcap_{g \in G} \left(\langle g_i \rangle \cap K^g_{(i,n_i')}\right)\right].$$

$$o(\bar{g}_i) = lcm\left(\beta_i, \alpha_i \cdot n_i'\right).$$

Now, $o(\bar{g}_i)$ in G/N is LCM $\left(K_{ij} : j \neq i, lcm(\beta_i, \alpha_i \cdot n_i')\right)$

Let

$$K = LCM\left(K_{ij} : j \neq i, \beta_i, \alpha_i\right)$$

Let

$$(n_1', n_2', \ldots, n_r') = \left(\frac{K}{\alpha_1} \cdot n_1, \frac{K}{\alpha_2} \cdot n_2, \ldots, \frac{K}{\alpha_r} \cdot n_r\right)$$

Then $o(\bar{g}_i)$ in G/N is LCM $\left(K_{ij} : j \neq i, lcm(\beta_i, \alpha_i \cdot \frac{K}{\alpha_i} \cdot n_i\right)$

Hence, in G/N

$$o(\bar{g}_i) = K \cdot n_i$$

$\square$

7. Omnipotence of Surface Groups

The main goal of this section is to show that surface groups are omnipotent.

Theorem 7.1. *Every closed surface other than $\mathbb{P}^2$,$\mathbb{S}^2$ is homeomorphic to a non-positively curved directed VH-complex. Moreover, since X is a surface, each edge space of X is a cylinder relative to vertical or horizontal decomposition.*

Proof. Figure 7-1(a) is a non-positively curved directed VH-complex for genus 2 non-orientable surface and Figure 7-1(b) is a non-positively curved VH-complex for genus 3 non-orientable surface.

Any closed surface other than $\mathbb{P}^2$ or $\mathbb{S}^2$ is a finite cover of one of the above two non-positively curved VH-complexes and the class of non-positively curved directed VH-complexes is closed under taking covers. See [3], [14], [16]. □

Remark. There are many non-positively curved VH-complexes homeomorphic to a genus 2 aspherical surface. For example see Figure 7–2.

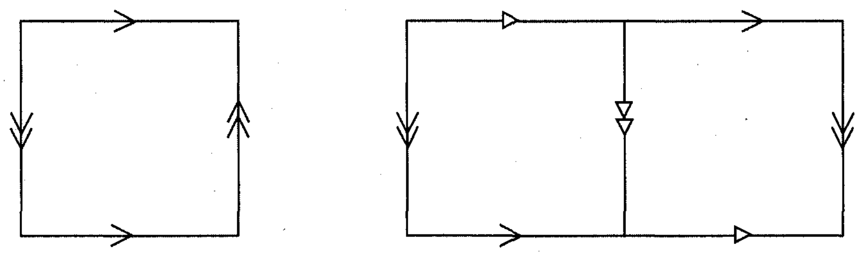

Figure 7–1: (a)$\mathbb{P}^2\sharp\mathbb{P}^2$ (b)$\mathbb{P}^2\sharp\mathbb{P}^2\sharp\mathbb{P}^2$

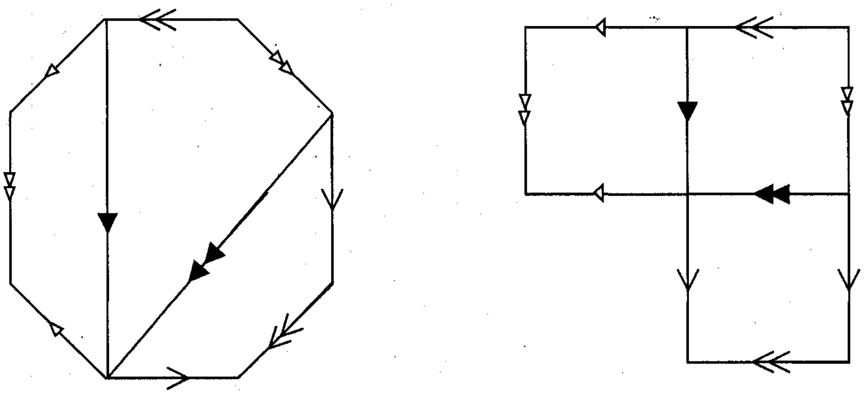

Figure 7–2: genus 2 aspherical surface
genus 2 aspherical surface (as shown on the left)and its corresponding homeomorphic directed VH-complex(as shown on the right)

Theorem 7.2. *Let M be a closed surface other than $\mathbb{T}^2$, $\mathbb{K}^2$ or $\mathbb{P}^2$ then $\pi_1(M)$ is omnipotent.*

Proof. This follows by combining Lemma 5.5 and Theorem 6.3. □

Theorem 7.3. *Let G be free product of aspherical surface groups then G is omnipotent.*

Proof. Let $G = G_1 * G_2 * \ldots G_n$ then by Theorem 7.1, $G_i = \pi_1(M_i)$, where M_i is a non-positively curved directed VH-complex . G is isomorphic to $\pi_1(M)$ where

$$M = \vee_{i=1}^{n} M_i$$

Note that M is a non-positively curved directed VH-complex whose edge spaces are cylinders in both Γ_X^v and Γ_X^h, therefore $\pi_1(M)$ is omnipotent by Theorem 7.2.

□

Theorem 7.4. *Not every potent group is omnipotent.*

For example consider the group $\mathbb{Z} \times \mathbb{Z}$. This is potent since it is a direct product of two potent groups (see [7]) but with respect to the independent set $\{(1,0),(1,1),(0,1)\}$, $\mathbb{Z} \times \mathbb{Z}$ is not omnipotent. However, in a way we can say free-abelian groups are 2-omnipotent but not omnipotent. More precisely, we can define similarly the notion of r-omnipotence of a group G that is a group G is omnipotent with respect to r-tuple of independent elements.

Also, Its clear that 3-manifold groups are not omnipotent as $\mathbb{Z} \times \mathbb{Z}$ embeds inside many 3-manifold groups.

8. Suggested Problems

We conclude this paper with the following open problems.

1. Is every subgroup of an omnipotent group omnipotent?
2. Is the free product of omnipotent groups omnipotent?
3. Suppose H is a finite index omnipotent subgroup of a torsion free group G. Is G also omnipotent?
4. Is every omnipotent group subgroup separable?

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