QUANTIFIED DOXASTIC LOGIC AND THE PROBLEM OF DEDUCTION

Brian MacPherson
Department of Philosophy
McGill University, Montreal
July, 1990

A Thesis
submitted to the Faculty of Graduate Studies
and Research in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

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Abstract – "Quantified Doxastic Logic and the Problem of Deduction"

In the first chapter, the reader is introduced to the 'problem of deduction'. I.e., any doxastic logic that is a normal modal system containing D where the *necessity* operator is construed as 'x believes that' will presuppose that believers are 'ideal' in the sense that their beliefs are consistent and closed under classical conjunction and implication.

Chapters two through four are devoted to a discussion of *quantified* doxastic normal systems. In chapter four, a set of axiom systems is proposed such that the substitution of co-referentials is restricted for doxastic contexts although 'quantifying in' is permitted provided that the quantifiers are read substitutionally in the semantics.

The systems proposed in chapter four inherit the problem of deduction and so possible solutions are considered in chapters five and six. The partial solution to the problem of deduction ultimately endorsed involves construing the *possibility* operator as 'x (non-ideally) believes that'.

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Sommaire – "La logique doxastique quantifiée et le problème de la déduction"

Dans le premier chapitre, nous présentons au lecteur le "problème de la déduction", i.e., toute logique doxastique constituant un système modal normal contenant D, où l'opérateur *nécessité* accepté "x croit que" présuppose que les partisans sont "idéaux" en ce sens que leurs croyances sont uniformes et fermées selon la conjonction et l'implication classiques.

Les chapitres deux à quatre comportent une discussion sur les systèmes doxastiques normaux *quantifiés*. Le chapitre quatre suggère une série de systèmes d'axiomes de sorte que la substitution de co-référentiels est limitée aux contextes doxastiques, bien que la "quantification" soit permise si les termes quantitatifs sont lus en remplacement dans la sémantique.

Les systèmes proposés au chapitre quatre héritent du problème de la déduction; ainsi, les chapitres cinq et six se consacrent à des solutions possibles. La solution partielle au problème de la déduction ultimement appuyée implique l'acceptation l'opérateur *possibilité* comme étant "x (non-idéalement) croit que".

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Preface

This thesis is in part an attempt to salvage the enterprise of adopting normal modal logics as logics of the epistemic modalities such as knowledge and belief. In particular, I shall be concerned with (quantified) doxastic logic - or the logic of belief.

The strategy of adopting normal modal systems as logics of the so-called epistemic modalities is not a new idea. For example, von Wright in *An Essay in Modal Logic* (1951) suggests that by replacing the necessity operator '**N**' for his system M_1 (which is a 'normal' system in the sense defined in section 1 of chapter one) with the epistemic operator '**V**' (i.e., 'it is known or verified that') we attain a logic of knowledge. He also discusses quantificational logics of knowledge such as his system 'VE'. However, von Wright did not have at his disposal relational semantics - but merely 'normal forms'. The advent of relational semantics for logics of the *alethic* modalities developed by Kripke (in for example Kripke (1963)) and for logics of the epistemic modalities developed by Hintikka (in Hintikka (1962, 1969)) paved the way for a more extensive treatment in the literature of the supposed analogy between the alethic modalities and the epistemic modalities.

The tradition in the literature with respect to doxastic logic has been to treat belief as analogous to alethic necessity and hence to simply replace the necessity operator '**L**' for sentential and first-order normal alethic systems with '**B**' (it is believed that). See for example Hintikka (1962), Sleigh (1972), Eberle (1974), Rescher (1968, 1974) and Rantala (1982, 1983). In fact certain authors such as Lewis (in Lewis (1986)) refer to belief as 'doxastic necessity'. In addition, it is supposed that the alethic possibility operator '**M**' can be replaced for doxastic logics by a 'doxastic' possibility operator such as Hintikka's '$\mathbf{P_B}$' which is informally interpreted as 'it is possible for all x believes that'.

However, normal modal systems adopted as doxastic logics where the necessity operator '**L**' is replaced by the belief operator '**B**' (and where '**M**' is replaced by '$\mathbf{P_B}$') presuppose that believers are 'ideal' in the sense that their beliefs are consistent and closed under conjunction and implication. But these principles are unacceptable qua principles of belief attribution in the light of various counterexamples discussed in the literature. For example, see Makinson (1965), Kyburg (1971), Marcus (1981), Stalnaker (1976,1984) and Lewis (1986). Using Stalnaker's turn of phrase, I call this the 'problem of deduction' for sentential and first-order belief logic.

The moral to be drawn from these counterexamples to the supposed deductive closure and consistency of belief is that there is a need for a logic of the 'non-ideal' believer, i.e., of the believer whose beliefs are not necessarily consistent and deductively closed. Normal modal logics where belief is taken to be analogous to alethic necessity do not appear to be suited to the task. Further, I shall argue that the attempt by Veikko Rantala to salvage the apparent analogy between necessity and belief (or knowledge) by suggesting alterations to normal axiom systems and their semantics in Rantala (1982, 1983) ultimately does not work because his solution involves not only an equivocation with respect to the connectives in his impossible worlds *semantics* (as would be suggested by Cresswell - see Cresswell (1973, 1982, 1985)) but also in the corresponding axiom systems (which is a result I attempt to establish).

However, it is hasty to conclude from these considerations that normal modal logics will only provide us with logics of the 'ideal' believer. The solution - albeit a partial one - which I propose in this thesis to the problem of deduction for normal doxastic logics is the following: I suggest that if we wish to salvage the tradition of treating belief logics as variants of normal alethic modal logics, then the more fruitful strategy is to treat the alethic *possibility* operator rather than the necessity operator as 'x believes that'. In treating belief as analogous to possibility rather than necessity, we end up with doxastic logics which do not presuppose that agents always conjoin their beliefs and which do not presuppose that agents' beliefs are always consistent. Granted, these logics retain the feature that agents believe all the logical consequences of what they believe although I shall argue that this feature is mitigated.

The solution herein proposed to the problem of deduction for normal sentential and first-order doxastic logic was prompted by a remark made by Marcus in "A Proposed Solution to a Puzzle About Belief" (1981): 507, where she notes that belief *like possibility* does not always factor out of conjunction as is evident in Kripke's puzzling Pierre case. In pursuing this idea, it became clear to me that belief is like possibility in another respect, viz., a proposition and its negation can both be possible - can both be believed. Further, given that an agent can believe both that α and that $\sim\alpha$ without thereby conjoining them, it is not a consequence of treating belief as analogous to possibility that agents who have contradictory beliefs thereby end up believing everything.

Finally, there is the question as to how to reconstrue the necessity operator for normal modal logics adopted as doxastic logics. I suggest that the necessity operator can be construed as 'x *ideally* believes that'. Thus, if we adopt normal modal systems (containing the schema D) as logics of belief, then the resulting logics can be taken as characterizing both the ideal believer (viz., one whose beliefs are consistent and closed under deduction) as well as the non-ideal believer (viz., one whose beliefs are not always consistent or closed under deduction).

A semantics is then required for our doxastic systems which gives some sort of intuitive content to the notion that belief is like possibility in the respect that agents can fail to conjoin beliefs and are capable of having contradictory (though not self-contradictory) beliefs. One of the semantics which I propose is a formalization (within the context of a relational semantics) of Stalnaker's suggestion that an agent is capable of being in more than one 'belief state'. To my knowledge, no-one has yet formalized this idea in terms of a relational semantics for first-order belief logic.

A belief state is defined as a set of worlds such that all the contents of a subset of the agent's beliefs obtain at each world in the set. Stalnaker develops this idea in chapter five of *Inquiry* (1984). If agents are capable of being in more than one belief state, then x could believe that α and x could also believe that $\sim\alpha$ provided these two contents obtain at all members of distinct belief states which are non-overlapping. Further, the agent will not conjoin these beliefs since at no belief state will it be the case that α and its negation will both obtain at all worlds in the state. Further, since belief states are sets of *possible* worlds, agents will nonetheless believe the consequences of what they believe. Thus, a semantics of belief states the central idea of which being that x believes that α at w_1 just in case there is at least one state s such that α obtains at all members of s) seems to match up with the idea that belief (as in 'non-ideal' belief) is analogous to alethic possibility.

The systems of belief logic which are ultimately endorsed in this thesis - such that the possibility operator is construed as x believes that - are normal *quantificational* systems with *identity*. Thus a further task which I herein undertake is to propose a set of logics of the non-ideal believer which are relatively unproblematic with respect to two of the more prominent difficulties arising from combining the epistemic modalities with quantifiers and identity. The first such difficulty is the issue of 'quantifying into' doxastic constructions. For example, under what conditions (if any) are we allowed to infer from Jones' belief that the next Prime Minister will abolish Free Trade, that *there is someone* such that Jones believes of that person that he/she will abolish Free Trade? The issue of 'quantifying in' gained attention in the literature following Quine's 1956 article "Quantifiers and Propositional Attitudes". For example, see Hintikka (1962, 1969), Kaplan (1969), Sosa (1970), Burge (1977) and Stich (1983).

A difficulty arising from combining the epistemic modalities with identity is that the principle that co-referential terms are intersubstitutible *salve veritate* breaks down for doxastic and epistemic contexts. Thus, even though Jones believes that Kripke is a gifted logician, he may fail to believe that the author of *Wittgenstein On Rules And Private Language* is a gifted logician. The failure of the substitutivity principle for belief contexts was first discussed by Frege in "On Sense and Reference" (1892). This issue has been discussed over the years in for instance Quine (1960), Hintikka (1962, 1967, 1969), Sellars (1969) and more recently in Kripke (1979) as well as in Barwise and Perry (1983).

The various logics which I propose for non-ideal belief in chapter six handle the problems of quantifying in and the failure of the substitutivity principle for belief contexts as follows: Although the substitutivity principle is restricted to non-modal contexts in which case belief contexts are treated as unambiguously 'oblique', there are no strictures on quantifying into doxastic constructions given that the quantifiers are construed substitutionally in the semantics. As Kripke has remarked, there is no problem of quantifying in for modal contexts if the quantifiers are substitutional. (Kripke (1976): 375) A result which I have hopefully established in chapters four and six is that a *truth-value semantics* where the identity statements of the language can take on different values at distinct indices and where the truth-conditions for the quantifiers are substitutional, will provide a characteristic semantics for logics which restrict the substitutivity principle while allowing unrestricted quantifying in.

By way of some closing remarks, I developed an interest in doxastic logic (and more generally in propositional attitude logics) in a roundabout way as a result of my readings in Action Theory. I became interested in the logic of action and subsequently in the logic and semantics of the belief-desire 'premises' of practical syllogisms explaining action. It was at that point that I read Stalnaker's article "Propositions" (1976) where he suggests that belief and desire qua 'functional' states (explaining behaviour) involve a partitioning of possible alternative situations (to the actual world) into those compatible with the given attitudes and those which are not. Thus, in terms of the practical syllogism, Jones' doing X is explained by his desire that Y obtains and by his belief that by doing X, Y will obtain, such that Jones' desire that Y obtains involves his partitioning the set of alternatives to the actual course of events into Y alternatives and into not-Y alternatives. The role which his belief plays in explaining his doing X is that it determines which of those Y-alternatives to be sought is most likely to become actual - and in this case, the prime candidate is an X & Y alternative.

I then read Hintikka's "Semantics for Propositional Attitudes" (1969) in which he proposes a semantics (though not a logic) of belief which is similar to Stalnaker's possible worlds analysis of belief and desire as follows: According to Hintikka, to say that x believes that α at w_i means that α obtains at all those worlds in the set ϕ_B which intutively is the set of all those worlds 'compatible' with the attitude in question. I.e., if x believes that α at w_i then α will obtain at each and every world at which all the other contents of his beliefs obtain. This led me to consider the analogy between alethic modal logics and their semantics on the one hand and propositional attitude logics and their semantics on the other.

Hintikka's semantics was proposed with the aim of resolving two of the problems associated with possible worlds semantics of the attitudes, viz., the issue of quantifying in and the failure of the substitutivity principle for doxastic contexts. It was at that point that I began thinking about and reading the literature on these issues. In delving into the literature I also became interested in the problem of deduction for doxastic logics. I came to realize that Hintikka's semantics for belief logic (as Hintikka himself notes in Hintikka (1975)) in terms of possible worlds presupposes that agents believe all the consequences of what they believe. Thus, I began looking at various proposed solutions to the 'logical omniscience' problem which led me to consider Rantala's proposals for an 'impossible worlds' semantics for belief logic - and which was the beginning of this work.

By way of acknowledgements, I wish to thank the Philosophy Department at McGill University for their continued financial support while I was in residence in the PhD program. I am also indebted to a friend and former Professor, R.C. Pinto for a discussion which led to my investigating the merits of substitutional quantification as a way of handling the problem of quantifying in. Further, I am indebted to Storrs McCall for his support, encouragement and his insightful suggestions vis a vis my work over the years. Finally, I wish to thank Tony Lariviere, Bill Massicotte and James Pettit for numerous philosophical discussions and for their friendship.

Introduction

The aim of this work is to develop a first-order logic and semantics of belief within the tradition which treats doxastic logics as *normal* modal logics[1], while avoiding at least some of the more serious objections generally raised against such a program. At least two sets of objections have been raised against this enterprise in the literature. The first set of objections can be categorized as the 'problem of deduction'[2] which arises from construing the alethic necessity operator '**L**' as 'x believes that'. The resulting logics and their semantics presuppose that believers are 'ideal' in the sense that they conjoin whatever they believe, that they believe all the (classical) logical consequences of what they believe and finally that agents always hold consistent beliefs. However, there are counterexamples in the literature to each of these principles qua principles of belief attribution. The logic of belief which will be proposed in chapter six avoids the consistency and the adjunctive components of the problem of deduction by construing the *possibility* operator '**M**' as 'x (non-ideally) believes that'.

Further, the second set of objections to the tradition of treating first-order belief logics as quantified normal modal logics concerns those theses having to do with the behaviour of the identity symbol and the quantifiers in belief contexts. At least for systems containing the Barcan Formula[3] which are characterized by invariant domain semantics, it is a thesis that co-referential terms are intersubstitutible in doxastic contexts. But this is

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[1] The term 'normal' is defined in the first chapter, p. 12.

[2] As is noted in chapter one, this is Stalanker's phraseology.

[3] The Barcan Formula is discussed in the second chapter, pp. 68–69

wrong since if for example Jones believes that Tully was bald, it does not follow that Jones thereby believes that Cicero was bald, given that he may fail to recognize that the terms 'Tully' and 'Cicero' are co-referentials. It will be argued that the best way of handling this difficulty is to adopt the Fregean tact of treating belief contexts as unambiguously 'oblique' in the sense that it is not permissible to unrestrictedly substitute co-referentials in doxastic contexts.

It is also a thesis of quantified systems with the Barcan Formula that existential generalization with respect to a term occurring in the scope of a belief operator is permissible not only inside but also outside the belief operator. If we are construing the quantifiers standardly (as in 'objectually') this thesis is wrong since from Jones' belief that the next Prime Minister (whoever he/she turns out to be) will attempt to balance the budget, it does not follow that there is someone such that Jones believes of that person that he/she will attempt to balance the budget. It will be argued that the best way of dealing with this type of situation is to interpret the quantifiers substitutionally without appeal to domains of so-called individuals.

In order to provide the reader with a kind of map or guide through this work, the remainder of these introductory remarks will be devoted to outlining what will be discussed in each chapter as well as to indicating how the various chapters connect up.

In chapter one, the reader is provided with a brief introduction to normal modal propositional calculi and their corresponding relational semantics. It is then noted that one of the traditions in the literature has been to regard normal systems as providing *doxastic* or *epistemic* logics where the alethic necessity operator is construed as 'x believes (knows) that'. Doxastic logics are distinguished from epistemic logics by suggesting that the former should not contain T, ($\mathbf{B}\alpha \supset \alpha$ for *doxastic* systems, which can be read as 'if x believes that α then α obtains') whereas the latter should (if we are traditionalists in our analysis of knowledge) contain T ($\mathbf{K}\alpha \supset \alpha$ for *epistemic* systems). As is mentioned, the focus of discussion will be with doxastic rather than epistemic systems.

It is then noted that normal systems of doxastic logic containing D, $\mathbf{B}\alpha \supset \mathbf{P_B}\alpha$ which result from construing the necessity operator as 'x believes that' provide us with logics of the 'ideal believer' in the sense that the following are thesis-schemata/derived inference rules for these systems:

T1: $(\mathbf{B}\alpha \ \& \ \mathbf{B}\beta) \supset \mathbf{B}(\alpha \ \& \ \beta)$	adjunction schema
T2: $\sim(\mathbf{B}\alpha \ \& \ \mathbf{B}{\sim}\alpha)$	consistency schema
DR 1: $\vdash \alpha \supset \beta \longrightarrow \vdash \mathbf{B}\alpha \supset \mathbf{B}\beta$	omnidoxasticity schema

T1 says that agents always conjoin beliefs and T2 asserts that agents' beliefs are always consistent. It will be argued that Kripke's puzzling Pierre case and his Paderewski example could be regarded as hypothetical cases where an agent fails to conjoin inconsistent beliefs held in different contexts, which thereby casts doubt on T1, T2 qua principles of belief attribution. DR 1 informally says that agents believe whatever is logically classically implied by what they believe - they are logically omnidoxastic. But several counterexamples to DR 1 are then presented such as Stalnaker's William III case, which calls DR 1 into question qua principle of belief attribution. In the light of the various counterexamples to T1, T2 and DR 1, it is suggested that on the assumption that the principles of belief attribution employed in setting up these examples are sound, there appears to be a need for a logic of the *non-ideal* believer. The need for such a logic can be seen as a challenge to the tradition of adopting normal modal logics as doxastic logics, which has been called the 'problem of deduction'.

The task of developing a logic (and corresponding semantics) of the *non-ideal* believer within the parameters of normal modal logic (and its corresponding Kripkean semantics) is then deferred to chapters five and six. In confining our discussion of doxastic logics to purely *propositional* calculi in the first chapter, we will have isolated the 'problem of deduction' in abstraction from the second set of problems mentioned above with respect to normal doxastic logics, viz., those difficulties having to do with the behaviour of the quantifiers and the identity symbol in belief contexts. This set of objections to the program of adopting normal logics as *quantified* doxastic logics will be discussed in chapters three and four.

The main purpose of chapter two is to provide the reader with a technical introduction to *quantified* doxastic logic and its semantics. The set of axiom systems which are considered, the $SQC^{=}$ systems of doxastic quantificational calculi, are 'normal' and therefore inherit the problem of deduction discussed in the first chapter. Two types of characteristic semantics are then considered, viz., an invariant domain semantics which lends itself to an objectual interpretation of the quantifiers and a truth-value semantics which involves assigning truth-values to atomic wffs 'directly', thereby lending itself to a substitutional interpretation of the quantifiers.

Finally, it is argued that what is problematic about the metaphysics of the invariant domain semantics, viz. that individuals are transindexical even though they vary in their properties from index to index, is avoided in the truth-value semantics since the latter dispenses with domains of individuals. This foreshadows the move in chapter four to ultimately adopt a truth-value semantics for quantified doxastic logic where the quantifiers are read substitutionally, partly on the grounds that the metaphysics of this type of semantics is relatively unproblematic.

Having introduced the reader to the technical aspects of quantified belief logic in chapter two, in chapter three the reader is then introduced to two of the problems associated with the behaviour of the quantifiers and the identity symbol in belief contexts. First, it is noted that difficulties arise from the feature that co-referentials are unrestrictedly intersubstitutible in belief contexts for the $SQC^{=}$ axiom systems. Various counterexamples to the substitutivity feature such as the Tully/Cicero case are discussed. It is further contended that contrary to Kripke's arguments in "A Puzzle About Belief" (1979), it is fair to assume that the problem in such cases rests with the substitutivity principle.

Next, it is argued that the feature of the $SQC^{=}$ axiom systems that quantifying into belief constructions is unrestricted leads to such counter-intuitive results as the one mentioned above viz., the 'next Prime Minister' case - at least if adopt an invariant domain semantics for these systems where the quantifiers are read objectually. Even if we were to adopt a truth-value semantics for the $SQC^{=}$ systems, there is still the problem that co-referentials are intersubstitutible for belief contexts.

Hintikka's proposed solutions on the syntactic front to both the difficulties just mentioned are then discussed. Informally, his suggestion for dealing with the substitutivity problem is to stipulate that the agent must recognize that the relevant identity obtains or that he/she is somehow 'acquainted' with the given individual (under both names). In terms of the issue of quantifying in, Hintikka restricts generalization with respect to constants occuring in the scope of belief operators to those constants which denote individuals with whom the individual is 'acquainted'.

In chapter four, Hintikka's syntactic suggestions for dealing with the problems of quantifying in and the apparent failure of substitutivity for doxastic contexts are incorporated into the amended versions of the $SQC^=$ axiom systems which we call the Hin-$SQC^=$ systems. A corresponding semantics is developed for these axiom-systems based on Hintikka's remarks in "Semantics for Propositional Attitudes" (1969). It is argued that this semantics is problematic since it employs the dubious notion of an individual's having 'correlates' across indices. The truth-value semantics for the alternative axiom-systems to the Hin-$SQC^=$ systems proposed in the final section avoid this difficulty by dispensing with domains of individuals. These alternative axiom-systems which are called the Sub-$SQC^=$ systems treat belief contexts as unambiguously oblique in the sense that substitution of co-referentials is still restricted for belief contexts. However, since the quantifiers are given a substitutional reading in their semantics, existential generalization into doxastic constructions is unrestricted. I.e., the Sub-$SQC^=$ systems circumvent the problem of quantifying in entirely.

Thus, by the end of chapter four a set of normal axiom-systems will have been proposed, the Sub-$SQC^=$ systems, which can be regarded as at least involving partial solutions to the problems of quantifying in and the failure of substitutivity for belief contexts. However, because these systems are 'normal' they have inherited the problem of deduction and hence they are not logics characterizing the 'non-ideal' believer. Thus, in the final two chapters, we shall discuss ways in which the Sub-SQC$^=$ systems and their characteristic semantics can be amended to accommodate the problem of deduction.

Rantala's syntactic solution to the problem of logical omnidoxasticity which involves restricting the applicability of the doxastic version of the rule of necessitation ($\vdash\alpha \longrightarrow \vdash\mathbf{B}\alpha$) is discussed in chapter five. His suggestions are then extended to rendering various instances of the adjunction and consistency schemata underivable. The resulting logics, the Sub-SQC$^=_\Omega$ systems therefore provide us with logics of the non-ideal believer, or so it seems until the corresponding semantics which employs non-standard indices is considered. It is argued that this semantics equivocates with respect to the connectives ~, v, &, ⊃ and ≡ thereby rendering it beside the point. It is further argued that the tact of defining these connectives in terms of their roles in inference does not avoid this charge if it is not assumed that only non-doxastic contexts are relevant in determining their roles in inference. Thus, it is concluded that the Sub-SQC$^=_\Omega$ systems do not provide us with logics of the non-ideal believer after all.

However, another strategy for altering the Sub-SQC$^=$ systems to provide us with logics of the non-ideal believer is considered in the sixth (and final) chapter. It is argued that if we construe the alethic *possibility* operator as 'x non-ideally believes that' then since possibiltiy does not 'factor out of conjunction' (as Marcus notes[4]) and since it is not a thesis of any normal system (with D) that $\sim(\mathbf{M}\alpha \ \& \ \mathbf{M}{\sim}\alpha)$ then the resulting logics provide us with logics of the non-ideal believer who does not always conjoin his/her beliefs and who does not always hold consistent beliefs. How-

[4] See Marcus (1981), p. 507.

ever, it is noted that even if we construe the possibility operator as 'x (non-ideally) believes that', it is still the case that non-ideal agents are logically omnidoxastic. It is further noted that for systems containing D, if we construe the alethic *necessity* operator as 'x *ideally* believes that' then combining this tact with the move of construing the possibility operator as 'x non-ideally believes that' provides us with a set of logics characterizing both the ideal and the non-ideal believer.

Two types of characteristic semantics are then considered for the Sub-$SQC^{=}$ systems (where the necessity operator is construed as ideal belief and the possibility operator is construed as non-ideal belief) which attempt to make sense out of the idea that non-ideal believers can hold inconsistent beliefs in different contexts - such as in the puzzling Pierre case. One of these semantics is based on Stalnaker's proposal that agents are capable of being in more than one 'belief state' at the same time (where a belief state is defined as the set of worlds such that all the contents of a subset of the agent's beliefs are true at each member of the set). The other type of semantics developed is based on Rescher's notion that belief is a relation obtaining between an agent at a world and a special sort of non-standard world.

And so, the result of our work will be a set of first-order logics of belief which characterize the ideal and the non-ideal believer and which treat belief contexts as unambiguously oblique while allowing unrestricted quantification into doxastic contexts given a substitutional reading of the quantifiers. Although these logics presuppose that agents are logically omnidoxastic, this feature is mitigated in the case of non-ideal belief. Then perhaps it is hasty to abandon the tradition of basing first-order doxastic logics on normal modal systems, if we are willing to drop that part of the tradition which construes the necessity operator rather than the possibility operator as 'x non-ideally believes that'.

Chapter One

The Problem of Finding a Propositional Logic of Belief

1. Treating Propositional Belief Logic as Modal Logic

In *Topics in Philosophical Logic* (1968), Nicholas Rescher observes that operators representing *modalities* qualify the truth or falsity of propositional expressions in such a way that the resulting qualified complex is itself a propositional expression.[1] More precisely, if α is some propositional expression of an arbitrary formal (or natural) language then so is $\psi\alpha$ where ψ is some modal operator. For example, the qualifier ψ could represent a so-called alethic modality such as *necessity* in which case if α is an expression of some formal or natural language then so is 'it is necessary that α'. And of course, much work has been done in the area of propositional and quantificational alethic modal logics. If the qualifier ψ represents an *epistemic* modality such as knowledge, acceptance or belief then if α is a propositional expression, so is 'it is believed that (known that, accepted that) α' or 'x believes (knows, accepts) that α'.

Several attempts at formulating elementary logics for the so-called *epistemic* modalities have consisted in adopting normal alethic systems[2] and informally construing the necessity operator as 'it is believed (known, accepted) that'.[3] In sections 2, 3 and 4 we shall explore various 'normal'

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1 Rescher (1968), p. 24

2 The phrase 'normal system' is defined on page 3 below.

3 For example, see Hintikka (1962), Binkley (1968), Harrison (1969), Cresswell (1970), Rescher (1968, 1974) and also Lenzen (1981).

systems of modal logic as well as their semantics without suggesting any sort of informal interpretation of the modal operator ψ (which for alethic systems is construed as 'it is necessary that'). This is to ensure generality in the sense that our results can be applied to logics where ψ is construed as an epistemic modality such as knowledge or belief, the logics of which will be introduced in section 5.

As will be argued in sections 6 and 7, when we construe the qualifier ψ as representing an epistemic modality, any normal modal system will give us a logic for 'ideal' believers or knowers in the following sense: For any normal system S regarded as a system of epistemic or doxastic logic, if any material conditional is an S-thesis then it is also an S-thesis that some agent x's believing (knowing, accepting) this conditional's antecedent logically implies x's believing (knowing, accepting) the conditional's consequent. In short, agents are regarded as believing all the logical consequences of what they believe. Hintikka amongst others calls this the problem of logical omniscience.[4] Further, it is also a thesis-schema of all normal systems (construed as systems of epistemic or doxastic logic) that agents believe the conjunction of any two propositions which they believe separately. Finally, in some normal systems, it is a thesis-schema that agents do not believe self-contradictions or contradictories at the same time.

In both chapters five and six, after having discussed some of the problems peculiar to *quantified* doxastic logics, we shall then consider several attempts at altering the semantics and the axiomatics of normal systems in order to obtain logics of the epistemic modalities (or more specifically, belief) which do not assume that agents are 'ideal' in the sense specified above. It will be argued that normal modal logics do provide us with logics

[4] Hintikka (1975), p. 475 and later, Rantala (1982), p. 106.

of believers who are not ideal.

2. An Excursus into Normal Modal Axiom Systems

Before explaining what a normal modal system is, a few syntactical preliminaries are in order. The language L for a *classical* propositional modal logic is a triple $<U, O, F>$ consisting of a set U of sentence variables, a set O of primitive connectives such as ~ and & (and such that v, ⊃, ≡ are definable in terms of the primitive ones) as well as the modal operator ψ and finally the set F of well-formed formulae (wffs) which are either variables or are constructed out of members of the sets U and O. The set F of wffs can be defined recursively in the same way as the set of wffs for the classical propositional calculus with the additional proviso that if α is a wff then so is $\psi\alpha$. In addition, we can introduce by definition a second modal operator Δ as follows: $\Delta\alpha =_{df.} \sim\psi\sim\alpha$. If we were to construe ψ as the necessity operator then $\Delta\alpha$ would read 'it is possible that α'. Or if ψ represents an epistemic modality such as 'x believes that' then we might read $\Delta\alpha$ as 'it is consistent with everything x believes that α'.[5]

A modal axiom system is *normal* if in addition to containing every thesis of PC, it contains every instance of the schema K, $(\psi\alpha \ \& \ \psi(\alpha \supset \beta)) \supset \psi\beta$. Further, any normal system will have as rules of inference the following: If α is a thesis of the normal system S then so is $\psi\alpha$ (which we shall call Rψ: $\vdash_S \alpha \rightarrow \vdash_S \psi\alpha$) as well as modus ponens, viz., α, $(\alpha \supset \beta) \longrightarrow \beta$.[6] The system K is the 'weakest' such system meeting these

5 As an example of this treatment of the possibility operator as an epistemic operator, see Hintikka (1962), pp. 10-11.

6 See Lemmon and Scott (1977), section 2 as well as Chellas (1980), ch. 4 and Hughes and Cresswell (1984), ch. 1.

minimal requirements in the sense that it is *properly* contained in every other normal modal system.[7] K can be 'strengthened' or extended by adding to it any of a number of schemata including the following:[8]

D: $\psi\alpha \supset \Delta\alpha$

T: $\psi\alpha \supset \alpha$

4: $\psi\alpha \supset \psi\psi\alpha$

5: $\Delta\alpha \supset \psi\Delta\alpha$

Consider the following series of strengthenings of K. The system D (=KD) obtained by adding the schema D to the system K contains all theses of K. Or, more succinctly, K $\subseteq$ D. And by adding the schema T to the system K we obtain the system T (=KT) where K $\subseteq$ D $\subseteq$ T. Further, by adding the schema 4 to the system T we obtain the Lewis system S4 (=KT4) such that K $\subseteq$ D $\subseteq$ T $\subseteq$ S4. And finally, when we add to the system T the schema 5 we obtain the Lewis system S5 (=KT5) such that K $\subseteq$ D $\subseteq$ T $\subseteq$ S4 $\subseteq$ S5. An alternative axiomatic base for S5 proposed by Lemmon[9] is to add to the system T the schema E, $\Delta\psi\alpha \supset \psi\alpha$ (or equivalently by contraposition and by the definition of Δ in terms of ~ and ψ, $\sim\psi\alpha \supset \psi\sim\psi\alpha$). The schema E is the *dual* of the schema 5. It is a metatheorem of normal systems that an axiomatic system S contains α iff it contains α's dual.[10] As we shall later see, all systems in this series of extensions of K *from T onwards* could with some plausibility be regarded as systems of epistemic logic (or the logic of knowledge) of varying strengths.

Alternatively, we could strengthen the system K without the addition

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[7] See Hughes and Cresswell (1968), pp. 29-30.

[8] See Lemmon and Scott (1977), section 4 and also Chellas (1980), ch. 4. Note further that strictly speaking, in 'adding' a schema to a particular system S, we are saying that S contains all instances of that schema as well as all of its deductive consequences.

[9] Lemmon and Scott (1977), section 4.

[10] This is proven in Chellas (1980), pp. 128-129.

of T but with the addition of D to obtain the following series of K-extensions: $K \subseteq D \subseteq KD4 \subseteq KD45$. Or, we could obtain a series of strengthenings of K without either of the schemata D or T: $K \subseteq K4 \subseteq K45$. As we shall later see, both these series of K-extensions (including K itself) could be regarded as systems of belief logic of varying strengths.

The containment relations which were exhibited in the three series of extensions of the normal system K considered above are well established in the literature.[11] (In the case of the second and third series, $K \subseteq D \subseteq KD4 \subseteq KD45$ and $K \subseteq K4 \subseteq K45$, the containment relations are obvious.) By way of illustration, for one of the containment relations exhibited in the first series viz., that $D \subseteq T$, we merely need to show that all of the instances of the schema D are provable in the system T given that the systems D and T are both K-extensions. We shall call the following sequences of schemata 'derivations' although strictly speaking a derivation is a finite sequence of wffs. Perhaps we could call the following sequences 'derivation schemata'. To avoid unduly lengthy derivations, we shall also take the liberty of using implicational thesis-schemata of the Propositional Calculus. We first need to show that $\vdash_T \alpha \supset \Delta\alpha$:

1.	$\psi{\sim}\alpha \supset {\sim}\alpha$	Schema T
2.	${\sim}{\sim}\alpha \supset {\sim}\psi{\sim}\alpha$	1, Transposition
3.	${\sim}{\sim}\alpha \supset \Delta\alpha$	2, Df. Δ
4.	$\vdash_T \alpha \supset {\sim}{\sim}\alpha$	
5.	$\alpha \supset \Delta\alpha$	3,4 Propositional Calculus and Modus Ponens

We are now in a position to show that $\vdash_T \psi\alpha \supset \Delta\alpha$:

1.	$\psi\alpha \supset \alpha$	Schema T
2.	$\alpha \supset \Delta\alpha$	Theorem Schema 1

[11] A detailed treatment of these inclusion relations is presented in chapters 4 and 5 in Chellas (1980).

3. $\Downarrow\alpha \supset \Delta\alpha$ 1,2 Propositional Calculus and Modus Ponens

Thus, the system D is contained in the system T. Also, D is *properly* contained in T since the system D does not contain T as a thesis-schema.

3: Normal Modal Systems - Semantic Considerations

So far, we have been discussing normal systems of modal logic apart from any kind of semantic considerations. The usual kind of semantics proposed for normal modal systems is based on Kripke's work in this area.[12] In a Kripkean semantics, a *model structure* for a normal system S is an ordered pair $<W,R,>$[13] where W is a non-empty set. As Kripke notes, we can informally regard the members of W as 'possible worlds'[14]. For the purpose of developing plausible semantics for modal belief logics, we shall simply treat these 'worlds' as primitives in our formal semantic theory just as 'individuals' are treated as primitives in the formal semantics for standard first-order logic. All questions concerning the nature of these so-called possible worlds as well as their ontological status will be deferred to a subsequent chapter once the formal semantics has been developed. In fact, to avoid any charges that the members of W are more than just formal constructs of our semantic theory we shall from now on call any w_i in W (in a normal model) an *index*. R is a two-place relation defined over members of W such that $R \subseteq W \times W$. Informally, for alethic systems, w_iRw_j can be read as 'w_j is accessible from w_i'.[15] Because of the

[12] For example, see Kripke (1963).

[13] In fact, in Kripke's semantics a normal model structure also contains a designated member of W which informally might be regarded as the 'real world.'

[14] Kripke (1963), p. 64

[15] Hughes and Cresswell (1984), p. 7.

element R in a normal model structure, a Kripkean semantics for normal modal logic is called 'relational'.

An S *model* is a triple <W,R,V> where <W,R> is an S model structure and where V is an assignment function which to each atomic wff of L at each member of W assigns either 0 or 1. I.e., $V\colon U \times W \longrightarrow \{0,1\}$. We could regard the function V as determining all the indices at which a given atomic wff is true in a given model M. Following the leads of both Stalnaker[16] and Lewis[17], we could say that V determines the *proposition* which a given atomic sentence expresses.

Finally, a *valuation over a model* is a function from wffs and indices into truth-values. I.e., $V_M\colon F \times W \longrightarrow \{0,1\}$. We can define V_M inductively as follows (for all w_i, $w_j \in W$):

1) $V_M(p,w_i) = V(p,w_i)$.

Supposing $V_M(\alpha,w_i)$ and $V_M(\beta,w_i)$ are defined for any $w_i \in W$ then:

2) $V_M(\sim\alpha,w_i) = 1$ iff $V_M(\alpha,w_i) = 0$.

3) $V_M(\alpha \,\&\, \beta,w_i) = 1$ iff $V_M(\alpha,w_i) = V_M(\beta,w_i) = 1$.

4) $V_M(\alpha \vee \beta,w_i) = 1$ iff $V_M(\alpha,w_i) = 1$ or $V_M(\beta,w_i) = 1$.

5) $V_M(\alpha \supset \beta,w_i) = 1$ iff $V_M(\alpha,w_i) = 0$ or $V_M(\beta,w_i) = 1$.

6) $V_M(\alpha \equiv \beta,w_i) = 1$ iff $V_M(\alpha,w_i) = V_M(\beta,w_i)$.

7) $V_M(\psi\alpha,w_i) = 1$ iff for all w_j such that w_iRw_j, $V_M(\alpha,w_j) = 1$.

8) $V_M(\Delta\alpha,w_i) = 1$ iff for at least one w_j where w_iRw_j,
$V_M(\alpha,w_j) = 1$.

Further, validity in an S model is truth at all members of W and validity in the relevant class C of S models is validity in all models in that class. The relevance of a class C of models to a particular normal system S is related to the restrictions placed on the *accessibility relation* R for all

[16] Robert Stalnaker (1976).

[17] Lewis (1973).

models in the class. For any given normal system S, those restrictions are imposed on R which ensure that the schemata constituting the axiomatic base are valid in that class of models. For our purposes, the following are some of the restrictions which the relation R may be required to satisfy:

1) R is ***serial*** in an S-model iff for all $w_i \in W$, there is at least one $w_j \in W$ such that w_iRw_j.
2) R is ***reflexive*** in an S-model iff for all $w_i \in W$, w_iRw_i.
3) R is ***symmetric*** in an S-model iff for all w_i, $w_j \in W$, if w_iRw_j then w_jRw_i.
4) R is ***transitive*** in an S-model iff for all w_i, w_j, $w_k \in W$, if w_iRw_j and w_jRw_k then w_iRw_k.
5) R is ***euclidean*** in an S-model iff for all w_i, w_j, $w_k \in W$, if w_iRw_j and w_iRw_k then w_jRw_k.

A few comments are in order here. First of all, if R in an S-model is reflexive, symmetric and transitive then we say that R is an 'equivalence relation'. What this means is that every member of W is related to every other member of W. (As we shall later see, R is an equivalence relation for S5 (=KT5) models.) Further, a relation R which is euclidean as well as reflexive is also an equivalence relation given the following proposition: If R is reflexive then R is euclidean iff it is both symmetric and transitive. For a proof of this proposition, see Lemmon and Scott (1977).[18] Finally, if a relation R is reflexive in an S-model then R is also serial since at the very least, every member of W will be related to itself.

We now present a list of the schemata mentioned above in connection with forming a fragment of all possible extensions of the system K (as well as the schema K itself). Following each schema is a specification of what restrictions the dyadic relation R must meet in every model in a class of

[18] Lemmon and Scott (1977), p. 56.

models to render that schema $\mathcal{C}$-valid. (A schema is $\mathcal{C}$-valid iff every instance of that schema is valid in every model in the class $\mathcal{C}$.):[19]

K:	$(\psi\alpha \ \& \ \psi(\alpha \supset \beta)) \supset \psi\beta$	R is *unrestricted*.
D:	$\psi\alpha \supset \Delta\alpha$	R is *serial*.
T:	$\psi\alpha \supset \alpha$	R is *reflexive*.
4:	$\psi\alpha \supset \psi\psi\alpha$	R is *transitive*.
5:	$\Delta\alpha \supset \psi\Delta\alpha$	R is *euclidean*.

Again, these results are well established in the literature, but for the purposes of illustration, it will be shown that 4 is valid in the class of all transitive normal models.

The proof of this will have the structure of a reductio ad absurdum and runs as follows: Suppose that 4 is invalid in a model M with a transitive relation R, in which case for some $w_i \in W$, $V_M(\psi\alpha, w_i)$ is 1 but $V_M(\psi\psi\alpha, w_i) = 0$. Then for some $w_j \in W$ where w_iRw_j, $V_M(\psi\alpha, w_j) = 0$. Since $V_M(\psi\alpha, w_j) = 0$ there is at least one $w_k \in W$ such that w_jRw_k and $V_M(\alpha, w_k) = 0$. But since w_iRw_j and w_jRw_k and given that R is transitive, it follows that w_iRw_k. But since w_iRw_k and since $V_M(\psi\alpha, w_i) = 1$, then it follows that $V_M(\alpha, w_k) = 1$. But we have already shown that $V_M(\alpha, w_k) = 0$ on the supposition that $V_M(\psi\psi\alpha, w_i) = V_M(\psi\alpha, w_j) = 0$ (such that w_iRw_j). Therefore, our original supposition is false. Q.E.D.

By way of a second example, we shall show that the schema D is valid in the class of all *serial* models. Suppose that for some $w_i \in W$ in some model M where R is serial, $V_M(\psi\alpha, w_i) = 1$ but $V_M(\Delta\alpha, w_i) = 0$. Given R's seriality we are guaranteed that there is at least one other index w_j (or perhaps w_i itself) in W such that w_iRw_j. Since $V_M(\psi\alpha, w_i) = 1$ it therefore follows that there is at least one w_j in W such that w_iRw_j and

[19] For a more detailed treatment of this, see Chellas (1980), p. 80

$V_M(\alpha, w_j) = 1$. But our intial supposition is also that $V_M(\Delta\alpha, w_i) = 0$ in which case for any w_j such that w_iRw_j, $V_M(\alpha, w_j) = 0$. Therefore our initial supposition is false. Q.E.D.

Notice that any instance of the schema D would be invalid in the class of *all* normal models (which validates the schema K) since there could be models such that for some w_i in W, there is no w_j such that w_iRw_j. This member of W is what Hughes and Cresswell call a 'dead end'.[20] At any such index, every wff of the form $\psi\alpha$ will be true since trivially, for all w_j *such that w_iRw_j* $V_M(\alpha, w_j) = 1$. Also, since there will be no w_j such that w_iRw_j, it follows that $V_M(\Delta\alpha, w_i) = 0$ for any wff α. (For that matter, there can be K-models such that every index is a dead end if $R = \emptyset$.) Thus, it is the seriality restriction on R which rules out this type of model.

4. Soundness and Completeness Results for Normal Systems

A normal system S is *sound* relative to a class C of normal models iff for every wff α, if α is an S-thesis then α will be valid in all models in the class C. I.e., If $\vdash_S \alpha$ then $\models_C \alpha$. Soundness of a system S relative to a class C of models is established by proving that all the axiom schemata of S are C-valid and also that our two rules of inference , modus ponens and $\vdash_S \alpha \longrightarrow \vdash_S \psi\alpha$ preserve validity.

We can now sketch a soundness proof for K as well as for the various K-extensions we have considered along the following lines: It is already established that the schema K is valid in the class of all models, that D is valid in the class of all serial models, that T is valid in the class of all reflexive models, that 4 is valid in the class of all transitive models and

[20] See Hughes and Cresswell (1984), pp. 33-38.

finally that 5 is valid in the class of all euclidean models. Let S be either K or any extension of K so far considered in which case the axiomatic base for S will consist of some combination of the schema K (minimally), D, T, 4 and 5 each of which is valid in a certain class of models. Then any schema constituting S's axiomatic base will be valid in the *intersection* of these classes.[21]

The following list illustrates for K and each of its extensions we've considered, the class of models with respect to which each schema in the axiomatic base of the system is valid:

K: The class of *all* models.

D(=KD): The class of *serial* models.

T(=KT): The class of *reflexive* models.

K4: The class of *transitive* models.

KD4: The class of *serial and transitive* models.

K45: The class of *transitive and euclidean* models.

KD45: The class of *serial, transitive and euclidean* models.

S4(=KT4): The class of *reflexive and transitive* models.

S5(=KT5): The class of *reflexive and euclidean* models.

So far, we know that for K and each of the K-extensions considered above, the axiom-schemata constituting their bases are valid in a certain class of models. We now need to show that Rψ and modus ponens preserve the validity of the axiom-schemata (which is to say that these rules preserve the validity of all instances of these schemata) for each of the above-mentioned systems. From this it will follow that each of these systems is *sound* with respect to a certain relevant class of models.

In order to show that Rψ and modus ponens preserve $\mathcal{C}$-validity it is

[21] This method of proof is used in Chellas (1980), section 5.1 in chapter 5.

sufficient to show that these rules preserve validity for the class of *all* normal models of which $\mathcal{C}$ will be a subclass. For example, the class of all models where R is reflexive will be a subclass of the class of all models. First then, to show that modus ponens preserves validity, suppose that $\models \alpha$ and $\models \alpha \supset \beta$ in which case, for every w_i in W in every normal model, $V_M(\alpha, w_i) = V_M(\alpha \supset \beta, w_i) = 1$. But given the truth conditions for wffs of the form of $\alpha \supset \beta$, $V_M(\beta, w_i) = 1$. Q.E.D. Further, to show that $R\psi$ preserves validity, suppose that $\models \alpha$. Then for each $w_i \in W$ in each model M, $V_M(\alpha, w_i) = 1$. But α will also be true at any index w_j such that $w_i R w_j$ and hence by the truth-conditions for wffs of the form $\psi\alpha$, $V_M(\psi\alpha, w_i) = 1$ for any such w_i. Q.E.D.

So far, we have merely established that K and the various extensions of K considered above are sound relative to certain classes of models. For any such normal system S, these results guarantee that any S-thesis will be valid relative to a certain class of models. However, this relevant class of models will be said to 'characterize' the normal system S if in addition to soundness, S is *complete* relative to this class of models.[22] A system S is complete relative to a certain class $\mathcal{C}$ of models just in case for every wff α if α is $\mathcal{C}$-valid then α is a thesis of S. I.e., for every wff α, if $\models_{\mathcal{C}} \alpha$ then $\vdash_S \alpha$.

A method that is frequently used in proving completeness for normal systems is the method of *canonical models*.[23] Just what canonical models are will become clear in the course of our exposition. The reader will recall that a normal system S is complete relative to a class $\mathcal{C}$ of models just in case for every wff α, if $\models_{\mathcal{C}} \alpha$ then $\vdash_S \alpha$. Taking the contrapositive

22 Hughes and Cresswell (1984), p. 12.

23 For a more detailed treatment of the canonical model method of proving completeness, see Hughes and Cresswell (1984), chapters 2 and 9.

of this, proving completeness amounts to proving that if α is not an S-thesis then α will be invalid with respect to $\mathcal{C}$. I.e., proving completeness amounts to proving that for every wff α, if $\neg\vdash_S\alpha$ then $\neq\!\mid_{\mathcal{C}}\alpha$ or equivalently that if $\neg\vdash_S\alpha$ then $V_M(\alpha, w_i) = 0$ for some $w_i \in W$ in some $\mathcal{C}$-model.[24] What follows is a description of how such a proof generally proceeds.

We shall say that a wff is S-consistent just in case its negation is not an S-thesis. A *set* of wffs $\{\alpha_1, \ldots, \alpha_n\}$ is S-consistent just in case the wff $\sim(\alpha_1 \;\&\; \ldots \;\&\; \alpha_n)$ is not an S-thesis. Thus, for any wff α such that $\neg\vdash_S\alpha$ we know that $\sim\alpha$ will be S-consistent. According to Lindenbaum's lemma, every S-consistent set of wffs Λ (which of course includes sets consisting of just one wff) has an extension Γ which is also S-consistent as well as maximal and such that $\Lambda \subseteq \Gamma$.[25] A set Γ of wffs is *maximal* just in case for every wff α either it or its negation is in Γ. The following lemmas (which we shall not bother to prove here[26]) illustrate properties which any maximal consistent set will possess. Any *maximal consistent* set Γ will be such that:

1) For every wff α, either α or its negation but not both will be in Γ.

2) $\alpha \vee \beta$ is in Γ iff α is in Γ or β.

3) $\alpha \;\&\; \beta$ is in Γ iff α is in Γ and β is in Γ.

4) Any S-thesis is in Γ.

5) If α is in Γ and $\alpha \supset \beta$ is in Γ then β is in Γ.

Also, it follows from lemmas 4 and 5 that if α is in Γ and $\alpha \supset \beta$ is an S-thesis then β is in Γ. These lemmas will be crucial in proving the so-called fundamental theorem of canonical models which will be described

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24 Hughes and Cresswell (1984), p. 17.

25 For a proof of this lemma, the reader is referred to Hughes and Cresswell (1984), pp. 19-20.

26 For proofs of these lemmas, the reader is referred to Hughes and Cresswell (1984), pp. 18-19.

presently.

Recall that it is being supposed that some wff α is such that $-|_S \alpha$ from which it follows that $\sim\alpha$ is S-consistent. Given Lindenbaum's lemma there is a maximal consistent extension of $\sim\alpha$, $[\sim\alpha]$ such that $\sim\alpha \in [\sim\alpha]$. The *canonical model* $\mathcal{M}$ for S is a triple $\langle W,R,V\rangle$ such that W is the set of all maximal consistent sets of wffs (and hence $[\sim\alpha] = w_i$ (for some i) is in W). Also, for any w_i, w_j in W, $w_i R w_j$ iff $(\alpha)(\psi\alpha \in w_i \longrightarrow \alpha \in w_j)$. Further, for any sentential variable p, $V(p,w_i) = 1$ iff $p \in w_i$. A valuation over S's canonical model $\mathcal{M}$ for sentential variables is defined as follows: $V_{\mathcal{M}}(p,w_i) = V(p,w_i)$. What remains to be proved is the so-called fundamental theorem for canonical models:

For any wff α, $V_{\mathcal{M}}(\alpha,w_i) = 1$ iff $\alpha \in w_i$.

This theorem's proof is a crucial step in the completeness proof for the following reason: Recalling once again the supposition concerning some arbitrary wff α (viz., that it is not a thesis of S), its negation $\sim\alpha$ will be a member of $\sim\alpha$'s maximal consistent extension (its m.c.e.), i.e., $\sim\alpha \in w_i$ such that w_i is in W in S's canonical model $\mathcal{M}$. But by the fundamental theorem of canonical models, it follows that $V_{\mathcal{M}}(\sim\alpha,w_i) = 1$ and hence $V_{\mathcal{M}}(\alpha,w_i) = 0$. Now assuming that $\mathcal{M}$ is in fact a model in the class $\mathcal{C}$ of models with respect to which S is sound then α is $\mathcal{C}$-invalid, which is what we wanted to show. I.e., we will have shown that if $-|_S \alpha$ then $=|_{\mathcal{C}} \alpha$ for any α supposing in addition that the canonical model $\mathcal{M}$ is in $\mathcal{C}$ (which of course is the class of models with respect to which S is sound).

The fundamental theorem of canonical models is proven by induction on the complexity of wffs. We shall consider the cases where α is atomic, is of the form $\sim\beta$, $\beta \,\&\, \gamma$ and $\psi\beta$.

Base Clause: Suppose that α is a sentential variable. Then given the definition of V for $\mathcal{M}$ and given that $V(p, w_i) = V_{\mathcal{M}}(p, w_i)$, the theorem holds for the case where α is a sentential variable.

Inductive hypothesis: Suppose that the theorem holds for wffs of degree of complexity n. Then show that it holds for wffs of degree of complexity n + 1.

Case 1: α is of the form $\sim\beta$.

$\sim\beta \in w_i$ iff $\beta \notin w_i$ (since w_i is maximal consistent.)
$\beta \notin w_i$ iff $V_{\mathcal{M}}(\beta, w_i) = 0$ (by the inductive hypothesis.)
$V_{\mathcal{M}}(\beta, w_i) = 0$ iff $V_{\mathcal{M}}(\sim\beta, w_i) = 1$. Q.E.D.

Case 2: α is of the from β & γ.

$\beta \,\&\, \gamma \in w_i$ iff $\beta, \gamma \in w_i$ (since w_i is max. cons.)
$\beta, \gamma \in w_i$ iff $V_{\mathcal{M}}(\beta, w_i) = V_{\mathcal{M}}(\gamma, w_i) = 1$. (by ind. hyp.)
$V_{\mathcal{M}}(\beta, w_i) = V_{\mathcal{M}}(\gamma, w_i) = 1$ iff $V_{\mathcal{M}}(\beta \,\&\, \gamma, w_i) = 1$. Q.E.D.

Case 3: α is of the form $\psi\beta$.

i) Suppose that $\psi\beta \in w_i$.
$\beta \in w_j$ for every w_j such that $w_i R w_j$. (by def. of R for $\mathcal{M}$.)
$V_{\mathcal{M}}(\beta, w_j) = 1$ for every w_j such that $w_i R w_j$. (by ind. hyp.)
Then $V_{\mathcal{M}}(\psi\beta, w_i) = 1$. Q.E.D.

ii) Suppose that $\psi\beta$ is not in w_i.[27]
Then $\sim\psi\beta \in w_i$. (since w_j is max. cons.)
Let $\omega = \{\gamma \mid \psi\gamma \in w_i\}$.
Then $\omega \cup \{\sim\beta\}$ is S-consistent.[28]
Then $[\omega \cup \{\sim\beta\}] = w_j$ in W is $\omega \cup \{\sim\beta\}$'s m.c.e. (Lind.'s lemma)
$\sim\beta \in [\omega \cup \{\sim\beta\}] = w_j$ since $\sim\beta \in \omega \cup \{\sim\beta\}$.
$\beta \notin w_j$ since w_j is max. cons. and so $V_{\mathcal{M}}(\beta, w_j) = 0$. (ind. hyp.)
$w_i R w_j$ since $\omega \subseteq w_j$ (given that $\omega \subseteq \omega \cup \{\sim\beta\}$) and since

[27] This proof can be found in Hughes and Cresswell (1984), pp. 24-25.

[28] See Hughes and Cresswell (1984), p.21 for the proof of this lemma.

$\omega = \{\gamma \mid \psi\gamma \in w_i\}$.
Then $V_{\mathcal{M}}(\psi\beta, w_i) = 0$. Q.E.D.

In outline form, this is how the proof of the fundamental theorem of canonical models would proceed. And so it has been established that for any wff α of any degree of complexity, $V_{\mathcal{M}}(\alpha, w_i) = 1$ iff $\alpha \in w_i$ where $\mathcal{M}$ is S's canonical model.

Recall once more the initial supposition concerning some arbitrary wff α, viz., that $-\!\!\mid_{S}\alpha$ in which case $\sim\alpha$ is S-consistent. Since $\sim\alpha$'s m.c.e., $[\sim\alpha] = w_i$ is in W in S's canonical model it follows by the fundamental theorem of canonical models that $V_{\mathcal{M}}(\sim\alpha, w_i) = 1$ and hence that $V_{\mathcal{M}}(\alpha, w_i)$ is 0. Then on the supposition that some arbitrary wff α is not a thesis of S, it follows that in S's canonical model α is false at a member of W (and in fact this member of W is $\sim\alpha$'s m.c.e.). However, we cannot yet conclude that α is invalid in the class of models $\mathcal{C}$ with respect to which S is sound until we have shown that the canonical model $\mathcal{M}$ is indeed a member of $\mathcal{C}$. Now in the case of the minimal normal system K, the completeness result follows immediately since K is sound with respect to the class of *all* normal models. However, in the case of the various K-extensions, it is the restrictions imposed on R for every model M in $\mathcal{C}$ that distinguishes one class of models from another. Therefore, for these K-extensions, showing that $\mathcal{M}$ is in $\mathcal{C}$ amounts to showing that R as it is defined for $\mathcal{M}$ meets the appropriate restrictions. Once again these results are well established in the literature, although for purposes of illustration we shall prove completeness for D, T, K4, KD4 and S4.

In the case of D's canonical model, we know that each instance of the schema D, $\psi\alpha \supset \Delta\alpha$ is in every w_i in W given that each w_i is maximal consistent. Consider any w_i in W. Suppose for some wff α that $\psi\alpha$ is in w_i. Then since w_i is maximal consistent, it follows that $\Delta\alpha$ is in w_i given that all instances of the schema D are in w_i. Further, by the fundamental theorem of canonical models it follows that $V_{\mu}(\Delta\alpha, w_i) = 1$ in which case there must be some w_j in W such that w_iRw_j and where $V_{\mu}(\alpha, w_j)$ is 1. It once again follows by the fundamental theorem that α is in w_j. But then, whenever $\psi\alpha$ is in w_i there will be a w_j such that α is in w_j. In other words, for any w_i there will always be a w_j such that w_iRw_j given the definition of R for D's canonical model. Therefore, R is serial for D's canonical model. Q.E.D.

Consider T's canonical model, μ. Each w_i in W will contain every instance of the schema T, $\psi\alpha \supset \alpha$ since each w_i is maximal consistent. Suppose for some wff α, $\psi\alpha$ is in w_i. Then given one of the lemmas for maximal consistent sets, α is also in w_i. But then for any w_i in W, whenever $\psi\alpha$ is in w_i so is α. So by the definition of R for T's canonical model, w_iRw_i for any w_i in W. Then R in T's canonical model is reflexive. Q.E.D.

Consider the canonical model μ for the system K4. Suppose for any w_i, w_j, w_k in W that w_iRw_j and w_jRw_k. Then we must show that w_iRw_k. Given the definition of R for K4's canonical model, if w_iRw_j then $(\alpha)(\psi\alpha \in w_i \longrightarrow \alpha \in w_j)$ and if w_jRw_k then $(\alpha)(\psi\alpha \in w_j \longrightarrow \alpha \in w_k)$. Each member of W in K4's canonical model will contain every instance of the schema 4, $\psi\alpha \supset \psi\psi\alpha$ given that each member of W is maximal consistent. Therefore, if $\psi\alpha$ is in w_i then so is $\psi\psi\alpha$. But if $\psi\psi\alpha$ is in w_i then by the supposition that $(\alpha)(\psi\alpha \in w_i \longrightarrow \alpha \in w_j)$ it follows that $\psi\alpha$ will be in w_j. But if $\psi\alpha$ is in w_j then by the supposition that $(\alpha)(\psi\alpha \in w_j \longrightarrow$

$\alpha \in w_k$) it follows that α is in w_k. But then $(\alpha)(\psi\alpha \in w_i \longrightarrow \alpha \in w_k)$. I.e., w_iRw_k on the supposition that w_iRw_j and w_jRw_k for any w_i, w_k and w_k in W. Therefore, R is transitive for K4's canonical model. Q.E.D.

The proof that R in the canonical model for KD4 is serial and transitive is immediate given our proof that R is serial for D and transitive for K4. Further, the proof that R in the canonical model for S4 is reflexive and transitive follows from our proof that R is reflexive for T and that R is transitive for K4.

And so, using the method of canonical models, it can be established that K and its various extensions are *complete* with respect to the classes of models which validate all their theses. I.e., K is characterized by (is both sound and complete with respect to) the class of all models, D is characterized by the class of serial models, K4 is characterized by the class of transitive models and so on. A less formal way of expressing these results is to say that the Kripkean 'possible world' semantics for systems of normal modal logic is *adequate* in the sense that there is a match-up or correspondence between the various normal systems and their semantics. And this 'correspondence' consists in the fact that each restriction on the relation R in a class of models can be regarded as the semantic counterpart of the axiom-schema which that restriction validates. Consequently, any results in the axiom-system will be mirrored in the semantics and vice-versa.

And so, having placed the minimal normal modal system K as well as a fragment of its extensions into perspective as it were, we are now in a position to consider how K and its extensions can be construed as systems of doxastic (and epistemic) logic.

5. Epistemic Logic Contrasted with Doxastic Logic

If we are adopting some normal system S as a system of doxastic logic, the tradition has been to construe ψ (which is the necessity operator for alethic normal systems) as 'it is believed that' or as 'x believes that'.[29] Also, its dual Δ can be construed as 'it is possible for all x believes that'. In addition, to make all this more conspicuous, we shall use '$\mathbf{B}$' instead of ψ and we shall use '$\mathbf{P_B}$' instead of Δ. So for any wff α, $\mathbf{P_B}\alpha$ =df. $\sim\mathbf{B}\sim\alpha$. Informally, this says that it is possible for all x believes that α is by deffinition it is not the case that x believes that $\sim\alpha$.[30] In terms of the Kripkean semantics for wffs of the form $\mathbf{B}\alpha$, $\mathbf{B}\alpha$ is true at an index w_i just in case α is true at all indices w_j *doxastically* accessible from w_i. Any wff of the form $\mathbf{P_B}\alpha$ is true at an index w_i in a normal model just in case α is true in at least one index w_j such that w_j is *doxastically* accessible from w_i. In this semantics, the belief operator $\mathbf{B}$ functions as kind of a doxastic necessity operator and $\mathbf{P_B}$ functions as a kind of doxastic possibility operator. And in fact, doxastic necessity and possibility coincide with logical necessity and possibility although this will not be the case for Rantala's non-standard index semantics which will be considered in chapter five.

We shall first of all make a few remarks concerning the *semantics* of normal doxastic (and epistemic) systems. As was noted, the relation R in the semantics for a normal belief logic is informally construed as a *doxastic* accessibility relation. The intuitive idea behind this construal of R is that for any index w_i (and for any agent x 'inhabiting' w_i), R divides all

[29] See Hintikka (1962, 1969), Rescher (1974) and Rantala (1982, 1983).

[30] For this type of treatment of the belief operator see Hintikka (1962), pp. 10-11.

the members of W relative to w_i into those which are 'alternatives' to w_i and those which are not. The alternatives to w_i determined by R are all those members of W at which all the content wffs α such that $\mathbf{B}\alpha$ is true at w_i are true. It follows from this that if we were to take the *conjunction* of all the wffs α such that $\mathbf{B}\alpha$ is true at w_i then this conjunction will be true at each w_j such that $w_i R w_j$. Then what R does is to determine the set of alternatives to w_i at which the agent's beliefs will *all* be true. Some authors such as Rescher and Hintikka call these alternatives to w_i agents' belief worlds or belief alternatives to w_i.[31] The notion that the belief alternatives to an index w_i are those indices at which agents' beliefs are *all* true appears for example in Hintikka's 'Semantics for Propositional Attitudes' (1969):

> My basic assumption ... is that an attribution of any propositional attitude to the person in question involves a division of all the possible worlds ... into two classes: into those possible worlds which are in accordance with the attitude in question and into those which are incompatible with it.[32]

Instead of a doxastic accessibility relation R, Hintikka in his semantics for first-order belief logic introduces a two-place function Φ_B which to an individual **a** at a 'world' w_i assigns a set of alternatives to w_i such that all of **a**'s beliefs (or more accurately, **a**'s believed statements) are true at each of these alternatives.

Although in a relational model in the semantics for normal propositional systems there is no domain D of individuals, we could replace the relation R with a one-place function f which to each member of W assigns a set of doxastic alternatives. We could then impose the same kinds of restrictions

[31] Rescher (1979), p.104 and Hintikka (1969), p. 28.

[32] Hintikka (1969), p. 25.

on f that we place on R depending on what system we are considering, and hence the two kinds of semantics would amount to the same thing in terms of what they validate.[33] For example, suppose we are considering the normal system D. Then if we define a D-model as a triple $\langle W, f, V\rangle$ (as opposed to $\langle W, R, V\rangle$), we would require for any f in a model in this class of models and for any w_i in W, $f(w_i) \neq \emptyset$. In other words, in a D-model, every index is such that at least one index is assigned to it by f. This is equivalent to requiring that R is *serial* in the relational semantics. Now consider the doxastic version of D, viz., $\mathbf{B}\alpha \supset \mathbf{P_B}\alpha$. Suppose there is a D-model $\langle W, f, V\rangle$ and a member of W, w_i such that $V_M(\mathbf{B}\alpha, w_i) = 1$ but $V_M(\mathbf{P_B}\alpha, w_i) = 0$. So, for all $w_j \in f(w_i)$, $V_M(\alpha, w_j) = 1$ on the supposition that $\mathbf{B}\alpha$ is 1 (or 'true') at w_i. But since f is such that $f(w_i) \neq \emptyset$, it follows that there is at least one w_j in W such that $w_j \in f(w_i)$ and given that $V_M(\mathbf{B}\alpha, w_i) = 1$ it immediately follows that $V_M(\mathbf{P_B}\alpha, w_i) = 1$. Thus, a semantics which requires that for any w_i in W and any f in a D-model, $f(w_i) \neq \emptyset$ will validate the schema D comparable to its relational counterpart.

Given our characterization of R for models in the semantics of normal doxastic logics, it is a feature of this type of semantics that belief is a relation between the 'typical' believer x at an index w_i and a set of indices assigned to w_i by R. The set of indices determined by R with respect to any given index w_i constitutes the *intersection* of all the propositions expressed by the contents of the agent's beliefs at that index. The concept of proposition operative here is the following: Propositions are sets of indices such that for any given wff α, the proposition which α expresses is the set of indices such that α is true at all and only these indices.[34] So the *inter*–

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33 See Chellas (1980), p. 74.

34 This concept of proposition for natural language is found in Stalnaker (1976, 1984) and in Lewis (1979).

section of all the propositions expressed by the contents of agents' beliefs at some index w_i (determined by R) will be the set of indices common to all these propositions. We could call this set the 'intersection proposition' and say that belief in the sort of semantics we are considering is a relation between a typical believer x (at an index) and the intersection proposition.

There has been for a number of years a debate in the literature concerning the objects of the attitudes for *natural* language. Two of the most popular candidates for the objects of belief and other attitudes are *propositions* and *sentences*. Russell of course coined the term 'propositional attitudes' and there have been several recent defenders of the claim that propositions (as sets of indices) are the objects of the attitudes.[35] On the other hand, Carnap in *Meaning and Necessity* seems to have held that attitudes are relations between agents and sentences. This position has recently regained some popularity in the 'mental representation' camp. For example, Fodor in 'Propositional Attitudes' wants to defend the claim that the objects of the attitudes are so-called internal representations which can be thought of as "sentences of a *non*-natural language".[36]

In any case, the debate in the semantics of natural language discussed in the previous paragraph is circumvented for the simple formal languages we are considering. That the objects of the attitudes for these formal languages are sets of indices or intersection propositions (and not linguistic entities such as sentences) is a feature of the semantics. This is just the way the semantics is set up.

The remarks which we have made concerning the semantics of normal systems of doxastic propositional logic also apply to normal systems of

[35] For example, Stalnaker defends this position in Stalnaker (1976).

[36] Fodor (1981), p.194.

epistemic logic. For epistemic logics, we can replace the operator ψ with the more conspicuous operator $\mathbf{K}$ such that $\mathbf{K}\alpha$ can informally be read as 'x knows that α'. Further, the dual of ψ can be replaced by the operator $\mathbf{P_K}$ such that $\mathbf{P_K}\alpha$ reads 'It is possible for all x knows that α'. And of course $\mathbf{P_K}$ is definable in terms of $\mathbf{K}$ for any wff α as follows: $\mathbf{P_K}\alpha$ =df. $\sim\mathbf{K}\sim\alpha$.

Since knowledge and belief are different sorts of epistemic modalities or attitudes, one might expect that their *logics* should in some way reflect this difference. Presumably, a key difference between the attitude of believing and the attitude of knowing is that it is possible to have false beliefs but it is not possible to know things that are false.[37] This distinction is regarded as crucial in the traditional analysis of knowledge in terms of justified true belief (and some additional fourth condition given the Gettier paradox). In traditional epistemology, a necessary condition for an agent x's knowing that α is that x's belief that α be true. And informally, this is just what the schema T, $\mathbf{K}\alpha \supset \alpha$ says, viz., if x knows that α then α obtains. So, if we adhere to the traditional analysis of knowledge then we would want our logic of *knowledge* based on a normal modal system to contain as theses all instances of the schema T. Also, any logic of *belief* based on normal systems should *not* contain the schema T, $\mathbf{B}\alpha \supset \alpha$ since we would not want any theses to the effect that if x believes that α then α is true. A brief scan of the literature on the subject of belief and epistemic logic will show that this has in fact been the general tradition.[38]

In section 2, we considered three possible series of strengthenings of the normal system K:

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[37] For example, see Hintikka's comments with regards to this issue in Hintikka (1962), p. 48 as well as Marcus (1981), p. 504.

[38] See Hintikka (1962), pp. 48-49, Harrison (1969), Rescher (1973), p. 104, Eberle (1974), p. 361 and more recently Rantala (1982).

Series 1: $K \subseteq D \subseteq T \subseteq S4 \subseteq S5$

Series 2: $K \subseteq D \subseteq KD4 \subseteq KD45$

Series 3: $K \subseteq K4 \subseteq K45$

Supposing that any normal system of epistemic logic should contain T, then any member of series 1 *from T onwards* could be adopted as a system of epistemic logic whereas any member of either series 2 or series 3 (but not series 1 from T onwards) could be adopted as a system of doxastic logic. Which member of series 1 from T onwards we choose as our system of epistemic logic and which member of series 2 or 3 we choose as our system of doxastic logic will depend on our philosophical biases.

For example, if we maintain that belief and knowledge are iterated in the sense that if x believes (knows) that α then x believes (knows) that he believes (knows) that α, then we would chose as our logic of belief or knowledge any system containing the schema 4. The doxastic version of 4 is $B\alpha \supset BB\alpha$ and its epistemic version is $K\alpha \supset KK\alpha$. These schemata have been the objects of contention in the literature with respect to their philosophical plausibility. Eberle for example maintains that 4 is unacceptable for either epistemic or doxastic logics since in the case of belief, an agent may believe some claim on the basis of certain evidence and yet "he may not believe himself to be in possession of such sufficient evidence".[39] Other logicians such as Hintikka uphold 4 for doxastic and epistemic logics.[40]

It is not our purpose here to engage in these debates. Our concern will be with the fact that whatever normal system we adopt as our logic of belief (or knowledge), any such system will presuppose that agents are ideal (or at least partially ideal) in the sense defined above. Our focus of

[39] Eberle (1974), p. 362.

[40] Hintikka (1962), p. 105.

attention will be *doxastic* logics for the remainder of this work and presumably any results attained can be generalized to epistemic logics.

6. Believing Contradictions

It is our task in this and the next two sections to consider some of the schemata derivable in the doxastic systems in series 2 and 3 of the K-extensions discussed above. We shall in fact focus on the schemata and rules which are the formal counterparts of the conditions for a believer's being *ideal* discussed above on page 2. As we shall see, these schemata and rules are all derivable in the systems of Series 2 from D onwards. Hence, these systems could be said to provide us with logics of the 'ideal believer'. However, as we shall see, there are ordinary language 'counterexamples' to the principles of belief attribution informally expressed by these schemata. If these examples are sound then it follows that believers are not ideal and this in turn points to a need for a logic of the non-ideal believer.

We shall first of all consider two of the more interesting thesis-schemata derivable in any doxastic system in series 2 *containing D*. Any K-extension in series 2 (excluding the system K itself) will contain as theses all instances of the schema $\sim\mathbf{B}(\alpha \ \& \sim\alpha)$ which informally says that it is not the case that x believes a contradiction of the form α and not-α. We shall call this the self-consistency schema. The reader will note that this is a formal counterpart of the condition mentioned on page 2 that the ideal believer is incapable of believing self-contradictions. That any wff of the form $\sim\mathbf{B}(\alpha \ \& \sim\alpha)$ is provable in any K-extension containing the schema D can be shown as follows:

1. $\mathbf{B}\sim(\alpha \;\&\; \sim\alpha) \supset \mathbf{P_B}\sim(\alpha \;\&\; \sim\alpha)$ axiom-schema D

2. $\sim(\alpha \;\&\; \sim\alpha)$ PC thesis-schema

3. $\mathbf{B}\sim(\alpha \;\&\; \sim\alpha)$ doxastic version of Rψ (i.e., RB), 2

4. $\mathbf{P_B}\sim(\alpha \;\&\; \sim\alpha)$ Modus Ponens 1, 3

5. $\sim\mathbf{B}(\alpha \;\&\; \sim\alpha)$ given that $\vdash_D \sim\mathbf{B}\alpha \equiv \mathbf{P_B}\sim\alpha$

Q.E.D.

From the point of view of the semantics for systems in series 2 containing D, any wff of the form $\mathbf{B}(\alpha \;\&\; \sim\alpha)$ will be unsatisfiable in any model for such a system and hence $\models_{\mathcal{C}} \sim\mathbf{B}(\alpha \;\&\; \sim\alpha)$. This is because of the seriality restriction imposed on R for all models validating the schema D. We can prove that $\mathbf{B}(\alpha \;\&\; \sim\alpha)$ will be unsatisfiable in any class of models where R is serial as follows: Suppose that $\mathbf{B}(\alpha \;\&\; \sim\alpha)$ is satisfiable in some serial model M. I.e., suppose that $V_M(\mathbf{B}(\alpha \;\&\; \sim\alpha), w_i) = 1$ for some w_i in W. Then since R is serial we are guaranteed that there is at least one w_j in W such that $w_i R w_j$. Therefore, there is at least one w_j in W such that $V_M(\alpha \;\& \sim\alpha, w_j) = 1$, which is impossible. Q.E.D.

Related to the self-consistency schema is $\sim(\mathbf{B}\alpha \;\&\; \mathbf{B}\sim\alpha)$. Informally, this schema says that it is never the case that agents believe contradictories. I.e., it is never the case that any agent x believes that α and that x believes that $\sim\alpha$. This schema will be derivable in any system of doxastic logic containing D (and hence the self-consistency schema $\sim\mathbf{B}(\alpha \;\&\; \sim\alpha)$) for the following reason: As we shall see below, every S system contains as a thesis-schema $(\mathbf{B}\alpha \;\&\; \mathbf{B}\beta) \supset \mathbf{B}(\alpha \;\&\; \beta)$ which we shall call the *adjunction schema*. Informally, this schema says that agents believe the conjunction of what they believe. Then for any doxastic system containing D and hence the consistency schema, we can derive $\sim(\mathbf{B}\alpha \;\&\; \mathbf{B}\sim\alpha)$ from the contrapositive of the adjunction schema, the self-consistency schema and modus ponens. In terms of the semantics for any S system containing D, if $\mathbf{B}\alpha$ and $\mathbf{B}{\sim}\alpha$ were both true at an index w_i, then for all w_j such that w_iRw_j, both α and ${\sim}\alpha$ are true at each such w_j which is impossible. And seriality guarantees that there will be at least one such alternative to w_i.

It is worth noting that no system in our series 3 of K-extensions, viz., $K \subsetneq K4 \subsetneq K45$ contains D or T which therefore effectively blocks the proof of any instance of ${\sim}\mathbf{B}(\alpha \ \& \ {\sim}\alpha)$. Further, in terms of the semantics, K-, K4- and K45-models are *neither reflexive nor serial* which as we shall see invalidates ${\sim}\mathbf{B}(\alpha \ \& \ {\sim}\alpha)$. Any of these classes of models will contain models such that some member w_i of W is a so-called dead end which means that for any such w_i, ${\sim}(\exists w_j)w_iRw_j$ which of course includes w_i itself.

To show that a wff of the form $\mathbf{B}(\alpha \ \& \ {\sim}\alpha)$ is satisfiable in a model where at least one of its members is a dead end, consider the following instance of $\mathbf{B}(\alpha \ \& \ {\sim}\alpha)$, $\mathbf{B}(p \ \& \ {\sim}p)$. The following is an admissible model in the class of K, K4 and K45 models: $W = \{w_1\}$, $R = \emptyset$ and $V(p,w_1) = V_M(p,w_1) = 1$ though the assignment which V gives to p is immaterial. Since $R = \emptyset$ then trivially, $V_M(\mathbf{B}(p \ \& \ {\sim}p),w_1) = 1$. And in fact, for any wff α, $V_M(\mathbf{B}\alpha,w_1)$ will be 1.

It is a peculiar feature of this sort of model, viz., that agents at dead end indices will for any wff α believe it and its negation whether α is valid, contradictory or contingent. In short, at dead ends agents will believe everything.

Also, the consistency schema ${\sim}(\mathbf{B}\alpha \ \& \ \mathbf{B}{\sim}\alpha)$ is not derivable in any system not containing D since its proof is blocked by the fact that ${\sim}\mathbf{B}(\alpha \ \& \ {\sim}\alpha)$

is not derivable in any such system. Further, in terms of the semantics of such systems, there will be instances of $B\alpha$ & $B{\sim}\alpha$ which are satisfiable in models with dead ends. For example, consider Bp & $B{\sim}p$ and the model $W = \{w_1\}$, $R = \emptyset$ and $V(p,w_1) = 1$. Then $V_M({\sim}p,w_1) = 0$. Since $R = \emptyset$ it follows that $V_M(Bp,w_1) = V_M(B{\sim}p,w_1) = 1$. Q.E.D.

It is worth noting that the *epistemic* systems in series 1, viz., T, S4 and S5 will contain the schemata ${\sim}K(\alpha$ & ${\sim}\alpha)$ and ${\sim}(K\alpha$ & $K{\sim}\alpha)$ since the schema D is provable in all these systems. And, in terms of the semantics of these systems all models for T and its extensions will be reflexive and therefore serial in which case the schemata ${\sim}K(\alpha$ & ${\sim}\alpha)$ and ${\sim}(K\alpha$ & $K{\sim}\alpha)$ will be valid relative to their semantics. These are presumably desirable thesis-schemata for a logic of *knowledge* since at least in traditional epistemology a necessary condition for an agent's knowing that α is that α be true. Then in the case of the schema ${\sim}K(\alpha$ & ${\sim}\alpha)$, if it were 'allowed' that agents can know contradictions then it would seem to follow that contradictions can be true which is absurd, at least if we construe ~ and & as *classical* negation and conjunction respectively. A similar argument would establish that the schema ${\sim}(K\alpha$ & $K{\sim}\alpha)$ is desirable for epistemic logic since if x knows that α and x knows that ${\sim}\alpha$, it would follow that α and ${\sim}\alpha$ are both true at the same indices.

However, in the case of doxastic logic, it is not so clear that either ${\sim}B(\alpha$ & ${\sim}\alpha)$ or ${\sim}(B\alpha$ & $B{\sim}\alpha)$ are desirable thesis-schemata. As we shall see, examples can be constructed where apparently, agents hold contradictory beliefs in different contexts thus violating ${\sim}(B\alpha$ & $B{\sim}\alpha)$ or they hold self-contradictory beliefs thereby violating ${\sim}B(\alpha$ & ${\sim}\alpha)$. The cases where agents apparently believe self-contradictory statements have not received much sympathy in the literature.[41]

First of all, it could be argued that agents will sometimes assent to the negations of logical or mathematical truths. For example, an agent who is not well versed in classical logic may assent to and hence believe that the negation of some instance of Pierce's law, viz., $((\alpha \supset \beta) \supset \alpha) \supset \alpha$ is false for classical two-valued logic. But the negation of any PC tautology will of course be self-contradictory in which case, it will be logically equivalent to some instance of $\alpha \mathbin{\&} {\sim}\alpha$ relative to this sort of semantics. Then it would follow that the agent has a belief that can be represented as $\mathbf{B}(\alpha \mathbin{\&} {\sim}\alpha)$, on the assumption that agents believe whatever is logically equivalent to what they believe, which is a derivable principle for any K-extension. This principle is discussed below. Given that agents can and do have false or mistaken beliefs, then there seems to be no reason why some of an agent's false beliefs can't be *logically* false as opposed to merely contingently so.

However, in believing that some logical truth does not obtain, the agent will thereby end up believing everything since a self-contradiction logically implies everything. But this is an absurd consequence of the supposition that agents can believe logical truths to be false. Therefore, there is good reason after all for wanting ${\sim}\mathbf{B}(\alpha \mathbin{\&} {\sim}\alpha)$ as a thesis-schema for any doxastic logic. This reductio-style argument rests on the assumption that agents believe whatever is logically classically implied by what they believe. This principle holds for any K-extension and it is represented in the K thesis-schema $(\mathbf{B}\alpha \mathbin{\&} \vdash \alpha \supset \beta) \supset \mathbf{B}\beta$. Then one way of countering this reductio argument is to question the assumption that agents are 'omniscient' or more precisely, 'omnidoxastic' with respect to the consequences of what

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[41] For example, see Dummett (1973) and Marcus (1981). On the other hand, Lewis does not discount the possibility of self-contradictory beliefs. See Lewis (1986), p. 36.

they believe.

But the only way to get around this assumption is to alter the semantics in such a way that there are models with *non-standard* as well as standard indices where the connectives are defined non-classically. Suppose non-standard indices are admissible as doxastic alternatives such that even though α logically implies β, α may be true at some such index while β is false. In such a case, an agent may believe that α and fail to believe that β. However, it will be argued in chapter five that such tactics ultimately do not succeed.

Nevertheless, suppose for the sake of argument that the strategy of allowing doxastic alternatives to be both classical and non-classical will get rid of the unpalatable result that an agent who believes that some truth of classical logic is false (relative to the appropriate semantics) thereby believes everything. There is another problem with our example in which an agent allegedly believes that the negation of (some instance of) Pierce's law is classically false, viz., it is not clear what sorts of doxastic alternatives will be such that negated tautologies can be true. This problem will now be discussed in more detail vis a vis Marcus' comments concerning the supposed impossibility of agents' having self-contradictory beliefs.

First of all, we are assuming in the above example some sort of attributive principle along the following lines: Sincere assent is at the very least *sufficient* for belief. Thus, in our example, the agent has *sincerely* assented to the claim that Pierce's law is false for classical two-valued logic and on this basis we would attribute to this agent a belief whose content is logically false. The principle that sincere assent is sufficient for belief attribution has been called the 'disquotation' principle by Kripke. Ruth

Barcan Marcus has argued that Kripke's disquotation principle needs to be bolstered by an additional condition, viz., that what is assented to describes a logically possible state of affairs.[42] She also holds that this condition is a *necessary* one.[43] So for Marcus, sincere assent does not carry over into belief unless the state of affairs assented to is logically possible - i.e., it is realizable in some possible world or other. Thus, it is not possible that an agent believes that some truth of *classical* logic fails since there is no logically possible world where its negation obtains, if by logically possible we mean that the connectives ~, &, etc. are defined classically for such worlds.

Presumably, Marcus wants to claim that analogous to maintaining that a necessary condition for attributing knowledge to an individual is that the claim to which he assents is *true*, a necessary condition for attributing belief to an individual is that the claim to which he assents is *possible*. In the case of knowledge, to require that what is known is true is a kind of 'reality' restriction in the sense that what is known must obtain in the 'actual' world (or more neutrally, in the world or index at which the knowledge claim is being evaluated.) Thus, she is attempting to impose some sort of 'reality' restriction on belief in the sense that what is believed must be realizable at *some* logically possible world though not necessarily the actual one.[44]

However, Marcus does not really offer any arguments in favour of her reality restriction for belief. In effect, Marcus' restriction simply reflects the feature of a (minimally serial) relational semantics for belief logic, that an agent has beliefs at an index just in case there are *logically possible* alternatives to that index such that what is believed obtains at these alter-

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[42] Marcus (1981), p. 505.

[43] Marcus (1981), p. 505.

[44] ibid, p. 507.

natives. So employing Marcus' strategy of introducing a reality restriction to rule out cases where agents hold logically false (as in self-contradictory) beliefs amounts to claiming that a serial relational semantics disallows such cases - but this is exactly what is at issue. I.e., is there anything that can and should be done to a serial relational semantics for belief (and the corresponding logic) to accommodate cases where agents hold logically false beliefs?

Nonetheless, Marcus' suggestion for a reality restriction on belief does raise an important issue, viz., in a relational semantics for doxastic logic where R is minimally serial, if we allow models where agents can believe the negations of logical truths, then it is not clear what sorts of indices would constitute doxastic alternatives for such agents. The alternatives to indices where agents believe that the negations of logical truths obtain cannot be logically possible in the sense that the connectives are defined classically since the negation of a classical logical truth could not turn out to be true at such indices.

And so, doxastic alternatives where self-contradictions can turn out to be true must be logically impossible in the sense that the connectives are interpreted non-standardly. However, as was already noted, and as will be argued in chapter six, such tactics ultimately do not succeed owing to the fact that it involves an equivocation with respect to the connectives. They mean one thing for standard indices and something else for non-standard indices. Then since there is no way of making model-theoretic sense of an agent's believing that the negations of logical truths hold within the parameters of a minimally serial relational semantics, it must be concluded that this is a feature of the semantics that is intractable and which can at best only be made more palatable. I.e., even though there is some sort of case to be made for agents believing self-contradictions, a relational semantics standard or otherwise will not be able to accommodate this.

On the other hand, there has been a certain amount of sympathy in the literature for the view that an individual in *different contexts* can believe that α and can also believe that $\sim\alpha$, thus casting doubt on the plausibiltiy of the schema $\sim(\mathbf{B}\alpha \ \& \ \mathbf{B}\sim\alpha)$. This position has been espoused by Dummett,[45] Stalnaker,[46] Rescher[47] and Barcan Marcus. We shall now consider an apparent case where an agent holds contradictory beliefs, though in different contexts.

Saul Kripke has proposed two cases in 'A Puzzle About Belief' which can be interpreted as cases where an agent holds contradictory beliefs in different contexts (although Kripke himself does not endorse this construal). Before describing one of these cases, it is necessary to allude to two principles which Kripke uses in its construction: The first principle which he appeals to is the *disquotation principle* alluded to above, viz., that if an agent S (upon reflection) sincerely assents to a claim p then S believes that p. And the second principle which he employs is the *translation principle*, viz., that if a sentence p expresses a truth in language L_1 then its translation p' in language L_2 expresses a truth in L_2. We shall in the next paragraph briefly describe Kripke's 'puzzling Pierre' case.

Pierre, a monolingual French speaker living in Paris has never been to London and knows of it only through pictures and verbal descriptions. Sup-

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45 Dummett (1973), p. 288

46 Stalnaker (1984), p. 83.

47 Rescher and Brandon (1980).

pose further that he sincerely assents to and hence by the disquotation principle believes the claim 'Londres est jolie'. In short, the sentence 'Pierre croit que Londres est jolie' is true in French. But given the translation principle it follows that 'Pierre believes that London is pretty' is true in English. Now suppose that Pierre moves to a rather shabby part of London where he acquires a spoken knowledge of English. He does not make an association between what he calls 'Londres' in French and what he calls 'London' in English. In his new environment, Pierre speaks only English. He soon gives sincere assent to and hence believes the claim 'London is not pretty' and hence the sentence 'Pierre believes that London is not pretty' is true in English. Finally, Pierre does not withdraw his assent to what he believed as a monolingual French speaker, viz., that London is pretty. Then what does Pierre believe? And this, says Kripke is the puzzle. He maintains that any answer to this question leads to an absurdity which therefore renders the puzzling Pierre case paradoxical.[48]

Kripke wants to claim that in this case it is unfair to accuse Pierre of holding contradictory beliefs since on the basis of his logical acumen alone he could not detect the inconsistency in the contents of his alleged beliefs, even if he were a brilliant logician. It is only if he had the additional information that 'London' and 'Londres' name the same place that he would be in a position to see that these contents are contradictory, thus abandoning assent to one or the other.[49] And it would only be at this point that Pierre could rightly be charged with inconsistency if he failed to abandon assent to one or the other content. However, the assumption which Kripke employs in his argument against this construal of the puzzling Pierre case, viz., that an agent can be charged with inconsistencies in his beliefs only if

[48] See Kripke (1979), pp. 257-259.

[49] Kripke (1979), p. 257.

he is in a position to detect these inconsistencies without appeal to additional information, is open to doubt.

It could be countered that it is in just those sorts of cases where the contradictory nature of the contents of an agent's alleged beliefs is *logically* undetectable in the sense that only by acquiring additional information could the inconsistency be detected, that we would be *most* inclined to attribute to the agent contradictory beliefs. Presumably, an agent with a requisite degree of logical acumen would not hold contradictory beliefs *unless* he/she failed to recognize that their contents were contradictory. Then a situation where even the most brilliant logician is unable to detect an inconsistency such as in the puzzling Pierre case is a kind of *limiting* situation where the agent would fail to recognize the inconsistency in his alleged beliefs barring the addition of relevant information. By the principle of charity, we may even refrain from attributing to a logically astute agent contradictory beliefs if he *is* in a position to detect the inconsistency without recourse to additional information, the idea being that he will eventually withdraw assent to one content or the other. Then it is only if he is unable to detect the inconsistency without recourse to additional information that it would be fair to attribute to him contradictory beliefs.

The idea here is that agents can hold distinct and possibly incompatible sets of beliefs in different contexts, without necessarily being in a position to integrate these sets of beliefs. A 'context' can simply be a time or as in the puzzling Pierre case, it could be a language. Lewis provides the example of a hypochondriac who at certain times believes that he is healthy and at other times believes that he is ill. As Lewis suggests, in such a case, the agent holds contradictory beliefs though at different times.[50] As will be

[50] Lewis (1986), p. 31.

argued in chapter six, we can make sense of this type of situation as well as the puzzling Pierre case via Stalnaker's notion that agents are capable of being in more than one 'belief state' at the same time (or perhaps at different times as in Lewis' 'hypochondriac' example). A belief state is a set of worlds such that all the contents of a subset of an agent's beliefs obtain at each world in the set. And so, it is not patently absurd after all to attribute to puzzling Pierre contradictory beliefs. Or is it?

It could still be argued along the following lines that attributing to puzzling Pierre contradictory beliefs involves an absurdity: Does Pierre in the above example also believe that London is pretty and London is not pretty? Pierre would presumably not give sincere assent to this claim. Then by a *strengthened* version of Kripke's disquotation principle, viz., that x's sincere assent to α is both sufficient *and* necessary for ascribing the belief that α to x, it would follow that Pierre does not believe this self-contradictory claim. But the adjunction schema to be discussed below, $(\mathbf{B}\alpha \ \& \ \mathbf{B}\beta) \supset \mathbf{B}(\alpha \ \& \ \beta)$ which says that agents believe the conjunction of what they believe expresses a valid principle for any logic of belief based on a normal system where the alethic necessity operator is construed as 'it is believed that'. Thus, given that Pierre believes that London is pretty and given that he believes that London is not pretty, then even though Pierre would not give assent to the self-contradictory claim that London is pretty and that it is not pretty, he would nonetheless believe this claim - assuming that the adjunction principle is sound.

And so, if we attribute to Pierre the belief that London is pretty and the belief that London is not pretty then by the adjunction principle (and contrary to the strengthened disquotation principle), we are forced to conclude that Pierre believes that London is pretty and that London is not pretty. But if we are forced to conclude that Pierre holds a belief whose content is of the form α & $\sim\alpha$ then, given the 'omnidoxasticity' principle (corresponding to the schema $(\mathbf{B}\alpha \ \& \vdash_S(\alpha \supset \beta)) \supset \mathbf{B}\beta)$ discussed earlier, viz., that agents believe the logical consequences of what they believe, it follows that Pierre believes everything, which is absurd. However, it is somewhat hasty to lay the blame for this generated absurdity on our attribution to Pierre a set of contradictory beliefs. The absurdity generated above is avoidable *either* by rejecting the claim that Pierre holds contradictory beliefs *or* by abandoning the adjunction principle *or* by abandoning the omnidoxasticity principle. Our tact will be to abandon the adjunction principle since as will be argued in chapter six, the omnidoxasticity feature of a relational semantics for belief logic is intractable, although there are moves that can be made to rid doxastic logic of the adjunction feature.

It will therefore be argued in chapter six that there is a way of accommodating the sort of situation where an agent has contradictory beliefs without thereby believing their conjunction for a two-place relational semantics of belief (where R is serial), although it involves interpreting the alethic *possibility* operator as 'it is believed that'. In interpreting the possibility operator as 'it is believed that' rather than the necessity operator, we avoid (on both the syntactic and the semantic fronts) the consequence of x's believing that α and x's believing that $\sim\alpha$, that x thereby believes their conjunction - and hence everything. This approach is hinted at though not developed by Marcus in a recent article.[51] We shall discuss this approach in the sixth chapter in conjunction with a resolution to this prob-

[51] See Marcus (1981).

lem discussed by Stalnaker in his book *Inquiry*, which we shall describe briefly in section 9 and in more detail in chapter six.

As a concluding note on doxastic theses involving the negation operator '~', any doxastic K-extension will contain all instances of the schemata $B{\sim\sim}\alpha \supset B\alpha$ and its converse $B\alpha \supset B{\sim\sim}\alpha$ as theses. These schemata as we shall see are more palatable as principles of belief attribution for the non-ideal believer than the consistency schemata. Both are derivable by applying RB to the PC schemata ${\sim\sim}\alpha \supset \alpha$ and $\alpha \supset {\sim\sim}\alpha$ respectively along with K and modus ponens. The former expresses the principle that if an agent x believes that not-not α then x also believes that α and the latter expresses the principle that x will believe that not-not α if x believes that α. For example, if x believes that every natural number has a successor then x believes that it is false that not every natural number has a successor. Or conversely, if x believes that it is false that not every natural number has a successor then x believes that every natural number has a successor. However, the following variants of these principles involving four or more iterations of the negation operator become a bit harder to swallow, viz., $B\alpha \supset B{\sim\sim\sim\sim}\alpha$ or its converse $B{\sim\sim\sim\sim}\alpha \supset B\alpha$. These are derivable given the PC schemata $\alpha \supset {\sim\sim\sim\sim}\alpha$ and ${\sim\sim\sim\sim}\alpha \supset \alpha$ respectively.

The only way of mitigating this situation is to interpret '~' in some non-standard way, though if we are concerned with '~' interptreted *classically* then this strategy will not work. We shall discuss the shortcomings of the non-standard worlds approach (or at least Rantala's version of it) further in chapter five. More immediately, in the next two sections, we shall discuss the adjunction and omnidoxasticity schemata which are shared by all normal doxastic systems.

7. The Apparent Failure to Conjoin Beliefs

To date we have determined that if we adopt a K-extension containing D as representing a system of doxastic logic then informally, we are committed to the claim that it is never the case that agents believe self-contradictions since any such system will contain all instances of $\sim \mathbf{B}(\alpha \ \& \ \sim\alpha)$. We are also committed to the claim that agents cannot believe contradictories separately since any such system will contain all instances of $\sim(\mathbf{B}\alpha \ \& \ \mathbf{B}\sim\alpha)$. But no matter which K-extension in either series 2 or 3 we consider as a system of doxastic logic, all instances of the following schema are derivable/valid in K and its extensions:

$$(\mathbf{B}\alpha \ \& \ \mathbf{B}\beta) \supset \mathbf{B}(\alpha \ \& \ \beta)$$

As was noted above, this schema which we have called the adjunction schema says that any agent believing that α and believing that β separately will believe thier conjunction. Or more succinctly, agents believe the conjunction of what they believe. We shall consider the philosophical ramifications of this schema presently, but first it will be demonstrated that the adjunction schema is derivable/valid for all K-extensions. To show this, it will be sufficient if we demonstrate that any instances of the adjunction schema will be derivable in K and that any instance of it will be valid in the class of all K models.

First of all, to show that any instance of the adjunction schema is derivable in the system K, consider the following abbreviated sequence.[52] I.e., any instance of the adjunction schema would be derived in this way:

1. $\vdash_K \alpha \supset (\beta \supset (\alpha \ \& \ \beta))$

[52] This derivation sequence appears in Hughes and Cresswell (1968), p. 34.

2. $\mathbf{B}(\alpha \supset (\beta \supset (\alpha \ \& \ \beta)))$ RB, 1.

3. $\mathbf{B}(\alpha \supset (\beta \supset (\alpha \ \& \ \beta))) \supset (\mathbf{B}\alpha \supset \mathbf{B}(\beta \supset (\alpha \ \& \ \beta)))$ instance of K.

4. $\mathbf{B}\alpha \supset \mathbf{B}(\beta \supset (\alpha \ \& \ \beta))$ Modus Ponens, 2,3.

5. $\mathbf{B}(\beta \supset (\alpha \ \& \ \beta)) \supset (\mathbf{B}\beta \supset \mathbf{B}(\alpha \ \& \ \beta))$ instance of K.

6. $\mathbf{B}\alpha \supset (\mathbf{B}\beta \supset \mathbf{B}(\alpha \ \& \ \beta))$ PC[53], 4,5.

7. $(\mathbf{B}\alpha \ \& \ \mathbf{B}\beta) \supset \mathbf{B}(\alpha \ \& \ \beta)$ PC[54], 6.

Q.E.D.

Thus, the adjunction schema is a K thesis-schema and hence it will be a thesis-schema of any K-extension. And in fact, given soundness it also follows that all instances of this schema will be valid in the class of all normal models. In order to see how the semantics for K works, however, we shall verify that the adjunction schema is valid with respect to the class of K models and hence for all of its extensions.

Suppose that some instance of the adjunction schema is invalid in some K model. Then there will be some w_i in W in this model such that $V_M(\mathbf{B}\alpha \ \& \ \mathbf{B}\beta, w_i) = V_M(\mathbf{B}\alpha, w_i) = V_M(\mathbf{B}\beta, w_i) = 1$ but $V_M(\mathbf{B}(\alpha \ \& \ \beta), w_i) = 0$. Then for all w_j such that w_iRw_j, $V_M(\alpha, w_j) = V_M(\beta, w_j) = 1$. But also, it must be the case that there is some w_k in W such that w_iRw_k and $V_M(\alpha \ \& \ \beta, w_k) = 0$ which is impossible. Q.E.D.

And so, any normal system construed as a system of doxastic logic involves the claim that agents believe the conjunction of what they believe separately given that each normal system contains the adjunction schema. However, when we consider examples such as the puzzling Pierre case discussed in the previous section, the desirability of having a system of doxastic logic for non-ideal believers (i.e., believers who don't always have consistent sets of beliefs) which contains the adjunction schema becomes

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[53] The relevant PC thesis-schemata are $((\alpha \supset \beta) \ \& \ (\beta \supset \gamma)) \supset (\alpha \supset \gamma)$, $\alpha \supset (\beta \supset (\alpha \ \& \ \beta))$.

[54] The relevant PC thesis-schema here is $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \ \& \ \beta) \supset \gamma)$.

doubtful. In the puzzling Pierre case we wish to avoid the consequence that in believing that London is pretty and in believing that London is not pretty (in different linguistic contexts), Pierre thereby believes a content of the form α & $\sim\alpha$ since he would end up believing *everything*.

There are cases where the adjunction schema seems undesirable even where x believes that α and x believes that β and α and β are not contradictories. In chapter five of *Inquiry* (1984), in connection with the adjunction principle, Stalnaker presents the 'paradox of the preface'.[55] This so-called paradox of the preface was first formulated by Makinson in Analysis (1965) and it has been discussed by a number of other authors through the years.[56] Suppose that a certain history book contains a disclaimer in the preface stating that there will be certain sentences in the book which are false. The author admits that he will most certainly be mistaken in one or more of his assertions though he does not know *which* of his assertions are false. But as Stalnaker notes, the author "... continues to *believe* everything he wrote..."[57] although he believes that there will be some sentences constituting his book which are false. Thus, the author believes each sentence in the book individually but he believes that the *conjunction* of these sentences is false.

Then the paradox of the preface is not *immediately* relevant to the adjunction schema but rather it bears directly on the plausibility of a closely related schema, $(\mathbf{B}\alpha \ \& \ \mathbf{B}\beta) \supset \sim\mathbf{B}\sim(\alpha \ \& \ \beta)$. Informally this schema says that if an agent x believes that α and that β individually then it is not the case that x believes that their conjunction is false. This schema is

[55] R. Stalnaker (1984), p. 92.

[56] For example, see R. Hoffman (1968, 1973), A. R. Lacey (1970) and C. New (1978).

[57] ibid., p. 92.

valid and hence provable in doxastic K-extensions containing all instances of the schema D. In fact, any instance of the above schema is derivable from the appropriate instances of the schema D in conjunction with the adjunction schema.

However, in any normal doxastic system containing D, it is a thesis-schema that $\sim(\mathbf{B}\alpha \;\&\; \mathbf{B}{\sim}\alpha)$ which is truth-functionally equivalent to $\mathbf{B}{\sim}\alpha \supset {\sim}\mathbf{B}\alpha$. I.e., it is a thesis of systems containing D that if an agent believes that α is false then he does not believe that α is true. Thus, the preface paradox is relevant to the adjunction schema for systems containing D since if this is a case where an agent believes α, β, etc. while believing that their conjunction is false then it is also a case where the agent does not believe that their conjunction is true.

Stalnaker claims that what is peculiar about the preface paradox is that it not only shows that agents are non-ideal in the sense that they do not always conjoin beliefs, but that it is *rational* in some cases not to conjoin belief. I.e., Stalnaker wants to question the adjunction principle as a 'rationality' condition for belief.[58]

Related to the preface paradox is the lottery paradox first discussed by Kyburg.[59] Several authors including Stalnaker have alluded to Kyburg's lottery paradox which bears directly on the weaker version of the adjunction schema, $(\mathbf{B}\alpha \;\&\; \mathbf{B}\beta) \supset {\sim}\mathbf{B}{\sim}(\alpha \;\&\; \beta)$. In the lottery example, suppose there is some arbitrary number of tickets, say 1,000,000. Then each ticket relative to all the others has a 999,999/1,000,000 probability of losing. Thus, it is 'rational' to believe of each and every ticket that it will lose. However, it is *not* 'rational' to believe that no ticket will win.

Also, the lottery example like the preface paradox is directly relevant

58 Stalnaker (1984), p. 88.

59 See Kyburg (1971).

to the weaker version of the adjunction schema, viz., $(B\alpha \;\&\; B\beta) \supset \sim B\sim(\alpha \;\&\; \beta)$ since any 'rational' agent would be disposed to assent to the claim that not all the tickets will lose. But as we have noted, for systems containing D, if x believes that α is false then x also fails to believe that α is true. Thus, the lottery paradox would also be a case (for systems containing D) where the agent does not believe that the appropriate conjunction obtains. In any case, as Stalnaker notes, the paradox of the preface is the better of the two counterexamples to the principle that agents believe the conjunction of what they believe since it does not rely on the notion of probability.[60] Stalnaker finds it questionable that we can say that the agent believes of any one ticket *without* reservation that it will lose. I.e., he questions the assumption "that a probability of .999999 is sufficient for acceptance".[61]

And so presumably, the preface and lottery paradoxes not only indicate that agents are non-ideal but they serve to impugn the adjunction principle qua 'rationality' principle. However, even if this is the correct conclusion to be drawn from these paradoxes, it does not follow that the 'ideal' believer in the sense defined in section 1 does not conjoin his beliefs. Such an inference could be made only if we wrongly conflate the terms 'ideality' and 'rationality'. Our sense of 'ideal' is stipulative - i.e., ideality is stipulated to be tied up with the adjunction, consistency and omnidoxasticiy principles. There is no claim being made in defining ideality in this way that agents *ought* to conjoin their beliefs. The claim that agents *ought* to conjoin their beliefs (that their beliefs ought to be consistent, etc.) is the sense of 'rationality' which Stalnaker is employing in his discussion of the

[60] See Stalnaker (1984), p. 91. Stalnaker also maintains that belief and acceptance generally is not a matter of degree.

[61] ibid, p. 91.

adjunction principle as a 'rationality' condition for belief.

In chapters five and six we shall critically examine ways of altering the semantics and corresponding axiomatics of normal doxastic systems in order to accommodate the puzzling Pierre case as well as the preface (and lottery) paradoxes. As will be noted in chapter six, Stalnaker suggests that the preface paradox can be explained by claiming that the historian merely *accepts* in some sense other than believes the statements of his narrative. Even if Stalnaker is correct, there is still a need for a logic of belief which does not suppose that agents conjoin their beliefs in order to be able to accommodate cases such as the puzzling Pierre case.

By way of some concluding remarks concerning the relation between the belief operator '**B**' and '&' *classically* construed, the converse of the adjunction schema, viz., $\mathbf{B}(\alpha \ \& \ \beta) \supset (\mathbf{B}\alpha \ \& \ \mathbf{B}\beta)$ is more palatable qua principle of belief attribution (for the non-ideal believer) than its close cousin. This schema expresses the principle that if x believes that α and β both obtain then x believes that α obtains and x believes that β obtains. For example, if x believes that the natural numbers and the integers are both denumerably infinite sets then x also believes that the set of natural numbers is denumerably infinite and x believes that the set of integers is denumerably infinite. This schema is derivable given the following two thesis schemata,

$$\mathbf{B}(\alpha \ \& \ \beta) \supset \mathbf{B}\alpha$$

$$\mathbf{B}(\alpha \ \& \ \beta) \supset \mathbf{B}\beta$$

These two schemata which we might wish to call 'doxastic simplification' express the principle that if x believes that $\alpha \ \& \ \beta$ then x believes that either conjunct obtains. Once again, these schemata are acceptable as principles of belief attribution, at least in comparison to the adjunction schema and at least if we are only considering two-termed conjunctions.

In the next section, we shall examine two additional thesis-schemata and their associated rules of inference contained in all normal systems. These rules and schemata are the formal counterparts of the principles (qua principles of attribution) that agents believe whatever is logically equivalent to what they believe and whatever is logically classically implied by what they believe.

8: Are Agents Logically Omnidoxastic?

Given an unrestricted use of **RB**, we can derive the following rules of inference for any normal doxastic modal system:

DR 1: $\vdash_S (\alpha \supset \beta) \longrightarrow \vdash_S (\mathbf{B}\alpha \supset \mathbf{B}\beta)$

DR 2: $\vdash_S (\alpha \equiv \beta) \longrightarrow \vdash_S (\mathbf{B}\alpha \equiv \mathbf{B}\beta)$

What these rules of inference amount to qua principles of belief attribution and how they are 'violated' will be discussed presently. First of all, we shall illustrate the role which the rule **RB** plays in their derivation. Consider the following sequence:

1. $\alpha \supset \beta$ assumption
2. $\mathbf{B}(\alpha \supset \beta)$ RB, 1
3. $\mathbf{B}(\alpha \supset \beta) \supset (\mathbf{B}\alpha \supset \mathbf{B}\beta)$ K schema
4. $\mathbf{B}\alpha \supset \mathbf{B}\beta$ Modus Ponens 2,3

Further, any instance of the equivalential version of this schema will be derivable as follows:

1. $\alpha \equiv \beta$ assumption

2. $(\alpha \supset \beta)$ & $(\beta \supset \alpha)$ df '≡'.

3. $\vdash((\alpha \supset \beta) \,\&\, (\beta \supset \alpha)) \supset (\alpha \supset \beta)$

4. $\alpha \supset \beta$ Modus Ponens 2,3

5. $\mathbf{B}(\alpha \supset \beta)$ RB, 4

6. $\mathbf{B}\alpha \supset \mathbf{B}\beta$ DR 1, 5

7. $\vdash((\alpha \supset \beta) \,\&\, (\beta \supset \alpha)) \supset (\beta \supset \alpha)$

8. $\beta \supset \alpha$ Modus Ponens 5,6

9. $\mathbf{B}(\beta \supset \alpha)$ RB, 8

10. $\mathbf{B}\beta \supset \mathbf{B}\alpha$ DR1, 9

11. $\vdash(\mathbf{B}\alpha \supset \mathbf{B}\beta) \supset ((\mathbf{B}\beta \supset \mathbf{B}\alpha) \supset ((\mathbf{B}\alpha \supset \mathbf{B}\beta) \,\&\, (\mathbf{B}\beta \supset \mathbf{B}\alpha)))$

12. $(\mathbf{B}\alpha \supset \mathbf{B}\beta) \,\&\, (\mathbf{B}\beta \supset \mathbf{B}\alpha)$ Modus Ponens, 2 X using 6, 10, 11.

12. $\mathbf{B}\alpha \equiv \mathbf{B}\beta$ df '≡', 12. Q.E.D.

Further, the semantic counterparts of these two derived rules can be established immediately given the soundness of any normal system relative to its semantics:

$$\models_{\mathcal{C}}(\alpha \supset \beta) \longrightarrow \models_{\mathcal{C}}(\mathbf{B}\alpha \supset \mathbf{B}\beta)$$
$$\models_{\mathcal{C}}(\alpha \equiv \beta) \longrightarrow \models_{\mathcal{C}}(\mathbf{B}\alpha \equiv \mathbf{B}\beta)$$

Now that we have shown that DR 1 and DR 2 are rules of inference of any normal doxastic system, we shall next consider their intuitive import. DR 1 informally says that any agent x will believe the logical consequences of what he believes. We could express DR 1 as an implicational schema as follows: $(\mathbf{B}\alpha \,\&\, \vdash_{S}(\alpha \supset \beta)) \supset \mathbf{B}\beta$. In the literature, this schema is known as the logical omniscience thesis.[62] Further, the equivalential version of DR 1 informally says that agents believe any wff logically equivalent to what they believe. DR 2 can also be expressed as an implicational schema: $(\mathbf{B}\alpha \,\&\, \vdash_{S}(\alpha \equiv \beta)) \supset \mathbf{B}\beta$. These logical omniscience or more appropriately

[62] For example, see Hintikka (1975) and Rantala (1982).

logical omnidoxasticity rules of inference and their corresponding schemata are the formal counterparts of the condition for ideal believers that such agents believe the logical consequences of what they believe.

In chapter five of *Inquiry*, Stalnaker formulates the following informal counterexample to the logical omniscience principle (qua principle of belief attribution):[63]

> William III of England believed, in 1700, that England could avoid a war with France. But avoiding a war with France entails avoiding a nuclear war with France. Did William III believe England could avoid a nuclear war? It would surely be strange to say that he did.

Given the omnidoxasticity principles, even though William III would not sincerely assent to the claim that England could avoid a nuclear war with France, we are committed to saying that he held this belief. Yet, there is something wrong in attributing to good King William this belief.

An even stronger counterexample to both DR 1 and DR 2 runs as follows: Suppose that agents believe all the logical consequences of what they believe or whatever is logically equivalent to what they believe. Then if any agent believes one logical truth, he believes all logical truths because any truth of logic (classically) entails and is entailed by every other logical truth. But this seems absurd in the case of 'non-ideal' believers.

Stalnaker has alluded to this situation in his 1972 article 'Propositions' as well as in *Inquiry* in attempting to vindicate his characterization of propositions as sets of 'worlds'. This metaphysics of propositions is implicit in the relational semantics for normal doxastic systems. The assignment function V in a model determines for any atomic wff α all those indices at which α is true, viz., the 'proposition' which that wff expresses. And in

[63] R. Stalnaker (1984), p. 88.

general, the proposition which any wff expresses in this type of semantics will be the set of indices at which that wff is true. But if any two wffs are logically equivalent then it follows that any given agent will believe the one if he believes the other since both will be true at all his doxastic alternatives if the one is. Or if x believes that α and α logically implies β then since there are no alternatives at which α is true but β is false, it follows that x also believes that β.

In fact, in the type of semantics we have been considering for normal doxastic modal systems, in any given model (including models where R is empty) agents at any index will believe all valid wffs since these wffs will be true at all doxastic alternatives. In short, the following expresses a classical entailment relationship for all normal systems: $\models_{\mathcal{C}}\alpha \longrightarrow \models_{\mathcal{C}}\mathbf{B}\alpha$. Thus, any agent x at some index will believe all valid material conditionals from which it follows that if x believes that the antecedent of any such conditional obtains, then x will believe that its consequent obtains. And this is the omnidoxasticity feature of the semantics discussed above. The syntactic counterpart of $\models_{\mathcal{C}}\alpha \longrightarrow \models_{\mathcal{C}}\mathbf{B}\alpha$ is the doxastic version of the rule of necessitation, RB, viz., $\vdash_{S}\alpha \longrightarrow \vdash_{S}\mathbf{B}\alpha$. Thus, if α is any conditional thesis then it is also a thesis that $\vdash_{S}\mathbf{B}\alpha$ so that if x believes that the antecedent of α obtains then by the schema K and modus ponens it follows that x believes that α's consequent obtains. And this is the omnidoxasticity feature on the syntactic front. As we shall see in chapter five, Rantala suggests that we can block this feature of the axiom systems by restricting the application of RB.

On the semantic front, allowing logically impossible indices to be belief alternatives would be one way of solving the so-called problem of logical omnidoxasticity since at impossible indices (such that the connectives are not defined classically), logically equivalent wffs can differ in their truth-values. However, as logicians such as Bigelow have pointed out, such tactics involve the rather cumbersome task of "reworking the semantics of extensional and intensional operators."[64] These tactics will be discussed, criticized and ultimately rejected in chapter five. In chapter six, we shall then argue that although there is no way of ridding the relational semantics for doxastic logics of the omnidoxasticity feature, there are ways of mitigating this feature.

It may be worth noting that Stalnaker's attempted ad hoc solution to the problem of omnidoxasticity with respect to belief in logical or mathematical truths is to say that agents can sometimes have mistaken beliefs about which propositions various sentences of mathematics expresses.[65] Hence, cases of mathematical or logical ignorance can be explained in terms of ignorance of the relationship between a sentence expressing the 'necessary proposition' and the necessary proposition. Another consequence of Stalnaker's proposal is that an agent who believies that a truth of mathematics or logic is false (such as in our example of the agent who believes that Pierce's law is false) may simply have a mistaken belief concerning the relationship between the sentence expressing the necessary truth and the necessary truth. In our earlier example, the agent may have a false belief concerning the relationship between the expression $((\alpha \supset \beta) \supset \alpha) \supset \alpha$ and the necessary proposition. Then perhaps our agent does not believe that a logical truth is a falsehood after all. We shall return to Stalnaker's proposal in more detail in chapter six.

64 John Bigelow (1978), p. 105.

65 R. Stalnaker (1976), p. 87

9. The Non-Ideal Believer and the Problem of Deduction

And so, in summarizing our discussion to date, it would seem that if we adopt a normal system of modal logic in series 2 containing D we are committed to the consistency schema, the adjunction schema and the logical omniscience schema (as well as its equivalential version). A system of logic containing all three schemata could be said to capture the notion of the *ideal* believer. As noted earlier, we could define an *ideal* believer as one who is incapable of believing any contradictions, who believes the conjunction of everything he believes and finally who is logically omnidoxastic. We also noted that any system in Series 3 without D will not contain the consistency schemata (though at the price of allowing indices where agents have maximally inconsistent sets of beliefs - viz., at dead ends).

In the preceding sections, we considered in relation to the consistency, adjunction and omnidoxasticity schemata a number of supposed 'counterexamples' which are possible situations where the principles of belief attribution asserted by these schemata break down. Thus, a counterexample to the omnidoxasticity schema, viz., a case where the principle of attribution it asserts breaks down, would be a case where an agent x believes that α and even though α logically implies β, it is somehow wrong to attribute to x the belief that β. If these counterexamples are not spurious, then it would seem that agents can believe contradictions (at least separately), that there can be cases where agents fail to believe the conjunction of what they believe and that it isn't always the case that agents believe the consequences of what they believe - agents are not logically omnidoxastic. In short, it would seem that believers are (or at least can be) non-ideal.

The moral to be drawn from these alleged counterexamples to the principles expressed by the omnidoxasticity, consistency and adjunction schemata is that a logic embodying principles of belief attribution for the *non-ideal* believer is needed. The normal systems of doxastic logic in series 2 and 3 are perfectly adequate qua logics of the 'ideal' believer, but they do not provide us with logics of non-ideal believers. Authors such as Dummett claim that there is no logic of belief. The line we are adopting here is that there *is* a logic of belief - i.e., of ideal belief but there is to date no logic characterizing believers who for example do not always conjoin their beliefs.

Admittedly, the alleged counterexamples to the consistency, adjunction and omnidoxasticity schemata all rely on some version of Kripke's disquotation principle in terms of sincere assent. Or they rely on some sort of principle of belief attribution or other (such as a dispositional account). Further, these cases can be regarded as involving a conflict between either the disquotational or dispositional principle on the one hand and one of these three principles (adjunction, consistency or omnidoxasticity) on the other. For example, in the William III example, William III's probable lack of assent to the claim that England will avoid a *nuclear* war with France seems to imply that he would not believe that this claim is true assuming Kripke's strengthened disquotation principle. Yet according to the omnidoxasticity principle, viz., that agents believe the consequences of what they believe, we are forced to conclude that William III *does* believe that England can avoid a *nuclear* war with France (if he believes that England can avoid a war with France). So there is a clash here between two principles of belief attribution, viz., disquotational vs. omnidoxasticity.

But then as Kripke would be quick to point out, we may be too hasty in indicting the claim that agents are ideal believers. Perhaps our disquotational or dispositional principles are at fault such that amending them would block these alleged counterexamples to the ideality of believers. And so, perhaps what the various cases we considered in the previous three sections have established is not that agents are non-ideal (and hence there is a need for a logic of non-ideal believers) but rather the weaker claim that *if* some version of the disquotation principle (or for that matter *any* principle of belief attribution such as a dispositional principle) used in constructing these cases are sound *then* believers are non-ideal. So we shall characterize our task more humbly as follows: *Supposing* that the disquotation priniciple or some analogous principle of belief attribution are sound and hence believers are non-ideal, we shall want to develop a logic of non-ideal beleivers.

By way of introducing some terminology, the problem that believers are (at least *apparently*) non-ideal and that this 'fact' is not taken into account by standard 'possible worlds' or indexical semantics, Stalnaker calls the 'problem of deduction'.[66]

In chapters five and six we shall attempt to develop a logic of belief within the parameters of an indexical (relational) semantics which does not assume that believers are ideal. In chapter five, Rantala's proposals for both a logic and semantics of the non-ideal believer will be considered. His proposal on the syntactic front involves restricting the doxastic variant of the rule of necessitation. On the semantic front, he proposes an 'impossible worlds' semantics for normal belief logics which allows the

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[66] R. Stalnaker (1984), p. 81.

doxastic accessiblity relation R to range over both normal and non-normal indices. At non-normal indices, the connectives of the language are defined non-classically - they misbehave as it were at such indices. And by virtue of the fact that the connectives can misbehave at non-normal indices, agents can hold contradictory beliefs both separately and conjointly without believing everything and also agents will sometimes fail to conjoin beliefs and fail to believe all the consequences of what they believe.

But as promising as this approach seems to be, it will be argued that this solution to the problem of deduction for belief logic is ultimately beside the point since it does not explain how *classical* conjunction, negation and implication misbehave at non-normal indices. The connectives ~, &, v, ⊃ and ≡ represent classical negation, conjunction, etc. for normal indices but they represent non-classical negation, conjunction, etc. at non-normal indices. Further, it will be argued that opting for defining the connectives solely in terms of their roles in inference does not sidestep the problem that there is an equivocation in the semantics with respect to ~, &, v, ⊃ and ≡, since this equivocation is also mirrored in the syntax.

In the sixth chapter, we shall then explore ways of altering the semantics and axiomatics of normal systems of belief logic which do not involve a non-classical construal of the connectives. The approach we shall develop is motivated by Marcus' suggestion that (on the syntactic front), the belief operator is more like the alethic possibility operator than the necessity operator. As we shall see, on this approach agents are capable of having contradictory beliefs separately (but not conjointly) and agents need not always conjoin their beliefs although we still have the result that agents are logically omnidoxastic. However, treating belief as analogous to alethic possibility mitigates the omnidoxasticity feature.

The corresponding semantics for these logics will be a formalization of Stalnaker's suggestion that an agent can be in more than one belief state at the same time. A belief state is a set of indices such that some of an agent's belief contents obtain at each of these indices. If an agent x can be in more than one belief state then x can hold contradictory beliefs in distinct states as well as fail to conjoin beliefs which are believed in different states.

In the next three chapters we shall see what happens when we introduce quantification and identity into normal doxastic systems. It will be argued that many of the major problems that are peculiar to *quantified* doxastic logic can be adequately dealt with on the semantic front within the framework of a relational semantics with or without domains of so-called individuals although we shall opt for the domainless semantics because it is metaphysically less problematic. The problem of deduction is inherited by quantified systems and it is to this problem that we shall return in the final two chapters.

Chapter Two

Quantificational Belief Logic

1. Doxastic Quantificational Calculi with Identity

The reader will recall that in the last chapter it was remarked that any of the following normal modal systems (amongst others) can be adopted as systems of doxastic logic where it is assumed that agents are 'ideal believers': KD, KD4 and KD45. Normal systems without D (and T) such as K, K4 and K45 do not presuppose that agents always have consistent beliefs. This is by no means an exhaustive list of doxastic logics although any normal system containing the schema T, $B\alpha \supset \alpha$ will not qualify for membership in this list since it is assumed that (even 'ideal') agents are capable of having false beliefs.

Despite Quine's warnings concerning the evils of Aristotelian essentialism, much work has been done in the area of quantified normal alethic modal logics especially following Kripke's proposed semantics for quantified normal systems in a famous 1963 article entitled 'Semantical Considerations on Modal Logic'. And parallel developments have occurred in the area of normal *doxastic* and *epistemic* quantified modal logics starting with Hintikka's work in *Knowledge and Belief* in 1962.

In terms of the language of any normal system of *quantified* doxastic logic we need to add to our 'logical' symbols a denumerably infinite set of so-called individual variables, x, y, z, x_1, x_2, ... as well as the quantifier symbols ∃ and ∀ (where the former is called the 'existential' quantifier and the latter is called the 'universal' quantifier) and finally the identity symbol '='. In addition, we shall also add to the list of primitive symbols sets of 'non-logical' symbols which will include a set of individual constants, a, b, c, a_1, a_2, ... as well as a set of predicate variables F, G, H, F_1, F_2, ... Intuitively, the constant symbols are the formal language counterparts of proper names for natural languages and predicate variables are the counterparts of class terms. We may also wish to add a list of so-called function symbols, f, g, h, f_1, f_2, ... although for our purposes this will not be necessary.

The union of the set of constants and the set of individual variables will be called the set of 'individual terms'. The notion of well-formed formula (wff) can be defined recursively as follows: The base clause is that any predicate variable followed by a finite string of *individual terms* is a wff and and so is $t_1 = t_2$ where t_1 and t_2 are terms. Wffs of either of these types will be called *atomic*. If both α and β are wffs then so is $\sim\alpha$, $\alpha \ \& \ \beta$, $\alpha \vee \beta$, $\alpha \supset \beta$, $\alpha \equiv \beta$ and $\mathbf{B}\alpha$. (Recall that $\mathbf{B}\alpha$ informally can be read as 'x believes that α'.) Finally, if A is any wff then so are $(\forall v)\alpha$ and $(\exists v)\alpha$ where v is a metasymbol ranging over variables. The wff α is said to be the *scope* of the quantifiers ∀ and ∃. Any variable v occuring in the scope of a quantifier that contains it is said to be *bound*. Otherwise, v is *free*. Notice also that only *individual* variables are said to be bound by quantifiers. Systems of quantified logic where this is the case are often called 'first-order'. If quantifiers are also allowed to bind predicate variables then the system is said to be 'second-order'. For our purposes we shall only be concerned with first-order systems. Finally, we shall call wffs containing no free variables *closed*. Wffs which are not closed are open.[1] This distinction will be important later on when we come to discuss the corresponding semantics for the $SQC^{=}$ systems since only closed wffs will be assigned truth-values.

There are a few things to be noted concerning the so-called universal and existential quantifiers. First of all, the quantifiers are interdefinable as follows: $(\exists v)\alpha$ =df. $\sim(\forall v)\sim\alpha$ and $(\forall v)\alpha$ =df. $\sim(\exists v)\sim\alpha$. For the set of axiom systems which we shall propose, the existential quantifier will be taken as primitive.

Also, for any system of first-order logic (standard, modal, doxastic) there are at least two possible ways of informally reading the quantifiers $\forall$ and $\exists$. If we provide a so-called 'substitutional' reading of these quantifiers, then any wff of the form $(\forall v)\alpha$ will be read as 'all substitution instances of α with respect to free v are true' and $(\exists v)\alpha$ will be read as 'some substitution instance of α with respect to free v is true'. This is not a formal definition but intuitively, a *substitution instance* of any wff α with respect to all free occurrences in α of some variable v is the result of uniformly replacing these occurrences of v by some constant t. We can denote any such substitution instance of α with respect to free v as $\alpha(t/v)$. The substitution approach to quantification dates back to Frege's 'Begriffsschrift' as well as Russell's 'On Denoting' and it was revived by Ruth Barcan Marcus[2] and later endorsed by such logicians as Dunn and Belnap as well as Stine[3].

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[1] See for example Hunter (1969), p. 139 for this distinction.

[2] See Marcus (1961, 1962).

Parallel to Marcus' attempt to vindicate the substitutional reading of the quantifiers was Quine's defense of the so-called *objectual* or *referential* interpretation of the quantifiers[4]. The referential reading usurped Frege's and Russell's attempts to construe quantification substitutionally (in part due to the efforts of Quine) and it is regarded as the 'standard' way of informally reading the quantifiers[5]. We can characterize the objectual approach to quantification roughly as follows: $(\forall v)\alpha$ is read as 'Every object is such that α' and $(\exists v)\alpha$ is read as 'There exists at least one object such that α'. As we shall see below in section 4, this way of interpreting quantification is on shakier metaphysical and ontological grounds than is the substitutional approach because it appeals to the problematic notion of 'object'. It is for this reason that a semantics exclusively supporting the latter approach will be endorsed for first-order belief logic.

As Quine and others have pointed out, the substitutional and the objectual interpretations of the quantifiers are by no means equivalent for reasons that will be discussed in the next section.[6] Nonetheless, it will be shown in this chapter that both a domain semantics supporting an objectual reading of the quantifiers and a truth-value semantics supporting a substitutional reading both characterize the first-order doxastic systems to be discussed below.

In terms of the axiomatics for normal first-order belief logic, the most straightforward way to proceed is to simply add to the axiomatic base for the system K or any K-extension one of a number of possible axiomatic

[3] See Dunn and Belnap (1968) and Gail Stine (1976).

[4] For example, see Quine's reply to Marcus (1963).

[5] See Kripke (1979).

[6] See Quine (1969), p. 106. Also, see Haack (1978), p. 51.

bases for standard non-modal first-order logic with identity. Thus, the axiomatic base for the minimal system K with quantification and identity (which we shall call $KQC^=$ [7]) would look something like this, where it is to be understood that 't', t_1 and t_2 occurring in any of the following schemata are metasymbols ranging over *constants*:

AS 1: α (where α has the form of any PC thesis-schema)[8]

AS 2: $(\mathbf{B}\alpha \;\&\; \mathbf{B}(\alpha \supset \beta)) \supset \mathbf{B}\beta$

AS 3: $\alpha(t/v) \supset (\exists v)\alpha$

AS 4: $t = t$

AS 5: $(\alpha(t_1/v) \;\&\; t_1 = t_2) \supset \alpha(t_2/v)$

AS 6: $\sim(\exists v)\sim\mathbf{B}\alpha \supset \mathbf{B}\sim(\exists v)\sim\alpha$ (Barcan Formula)

The primitive rules of inference will be:

MP: $\alpha,\ \alpha \supset \beta \longrightarrow \beta$

R$\exists$: $\vdash\alpha(t/v) \supset \beta \longrightarrow \vdash(\exists v)\alpha \supset \beta$ provided t is foreign to $(\exists x)\alpha \supset \beta$.

RB: $\vdash\alpha \longrightarrow \vdash\mathbf{B}\alpha$ [9]

In a similar fashion, we can suggest axiomatic bases for any *extension* of $KQC^=$ to obtain the systems $DQC^=$, $K4QC^=$, $K45QC^=$, $KD4QC^=$, $KD45QC^=$, and so on. We shall hereafter call any system in this set an $SQC^=$ system. The reader will note that any $SQC^=$ system will contain the so-called *Barcan Formula* as an axiom-schema. As Hughes and Cresswell note, the Barcan Formula is not derivable in any normal quantified modal system weaker than S5.[10] However, it can be consistently added to any quantified system

[7] I.e., The system K plus the Quantificational Calculus with Identity.

[8] For example, $(\forall x)Fx \supset (\forall x)Fx$ would have the form of the PC thesis-schema $\alpha \supset \alpha$.

[9] To avoid the rather cumbersome notation $\vdash_{SQC^=}\alpha$ which signifies that the wff α is a thesis of some $SQC^=$ axiom system, we shall simply write $\vdash\alpha$, it being understood that thesishood is relative to some system or other.

[10] Hughes and Cresswell (1968), pp. 170-171.

weaker than S5 as an axiom schema. The rationale for including the Barcan Formula as an axiom schema in the various quantified doxastic systems of logic is that the corresponding semantics is made simpler.[11]

Given that each of the $SQC^=$ systems have as part of their axiomatic base the schema K, $(\mathbf{B}\alpha \ \& \ \mathbf{B}(\alpha \supset \beta)) \supset \mathbf{B}\beta$ as well as containing the rule of inference **RB**, $\vdash\alpha \longrightarrow \vdash\mathbf{B}\alpha$ then the so-called adjunction schema and the following two rules of inference are derivable in each of these systems:

$(\mathbf{B}\alpha \ \& \ \mathbf{B}\beta) \supset \mathbf{B}(\alpha \ \& \ \beta)$	[adjunction schema]
$\vdash\alpha \supset \beta \longrightarrow \vdash\mathbf{B}\alpha \supset \mathbf{B}\beta$	[omnidoxasticity]
$\vdash\alpha \equiv \beta \longrightarrow \vdash\mathbf{B}\alpha \equiv \mathbf{B}\beta$	[strong omnidoxasticity]

Infomally, the adjunction schema says that agents believe the conjunction of what they believe. The two derived rules of inference informally assert that agents beleive the logical consequences of what they believe and that agents believe whatever is logically equivalent to what they believe. These three principles asserted by the adjunction schema and the two rules of inference are (as we have seen) open to counterexamples. Thus, the quantified $SQC^=$ systems inherit from their sentential counterparts the *problem of deduction* .

Further, any $SQC^=$ system containing the schema D, $\mathbf{B}\alpha \supset \mathbf{P_B}\alpha$ also contains all instances of the schema $\sim\mathbf{B}(\alpha \ \& \sim\alpha)$ which says that it is not the case that agents believe self-contradictions and all instances of $\sim(\mathbf{B}\alpha \ \& \ \mathbf{B}\sim\alpha)$ which says that agents never believe contradictories separately. Once again, the principles asserted by these schemata are open to counterexamples.

Given the soundness and completeness results for the $SQC^=$ systems (to be discussed in the next two sections), all instances of the adjunction

[11] For details about this, see chapter ten of Hughes and Cresswell (1968).

schema will be valid in the classes of relational models characterizing the various $SQC^=$ systems. Further, the semantic counterparts of the two derived rules of inference mentioned above hold for these systems. Finally, the non-contradiction schemata will be valid in the classes of minimally *serial* models characterizing those $SQC^=$ systems containing D. Our task in the remainder of this dissertation following the fourth chapter will be to critically examine various attempts at amending the relational semantics for doxastic logic (both quantified and sentential) in order to accommodate these difficulties.

We shall now consider a few of the theses derivable in the *quantified* versions of the doxastic systems we considered in the previous chapter. It is noteworthy that all instances of the following schema are derivable in any of the $SQC^=$ systems: $t_1 = t_2 \supset \mathbf{B}(t_1 = t_2)$. A proof of any instance of this schema in any $SQC^=$ system would look something like this:

1.	$(\mathbf{B}(t_1 = t_1) \,\&\, t_1 = t_2) \supset \mathbf{B}(t_1 = t_2)$	version of AS 5
2.	$\mathbf{B}(t_1 = t_1) \supset (t_1 = t_2 \supset \mathbf{B}(t_1 = t_2))$	1, PC
3.	$t_1 = t_1$	version of AS 4
4.	$\mathbf{B}(t_1 = t_1)$	3, RB
5.	$t_1 = t_2 \supset \mathbf{B}(t_1 = t_2)$	2,4 MP

Intuitively, this schema says that agents are omnidoxastic with respect to identities. This is plausible for identities of the sort $t = t$ but not necessarily for so-called contingent identities. Other $SQC^=$ theses which will spell trouble philosophically are the doxastic versions of AS 3 and AS 5 above, $\mathbf{B}\alpha(t/v) \supset (\exists v)\mathbf{B}\alpha$ and $(\mathbf{B}\alpha(t_1/v) \,\&\, t_1 = t_2) \supset \mathbf{B}\alpha(t_2/v)$ respectively. Informally, the former says that we are allowed to existentially generalize across belief operators and the latter says that co-referentials are inter-substitutible in belief contexts. As we shall see in the next chapter, both of these assertions are highly problematic and have been discussed in the literature for a number of years.

However, the set of axiom systems {KQC$^=$, K4QC$^=$, ...} which we are now considering is not meant to be our final word on quantified doxastic logic. We shall suggest modifications to these axiom systems and their characteristic semantics in chapter four which will best be able to accommodate the philosophical difficulties connected with quantificational doxastic systems. The reader can consider these axiom systems and their characteristic semantics as a kind of dry run as well as a framework in which to discuss (in chapter three) some of the major difficulties associated with quantified belief logic.

Another thesis-schema contained in each SQC$^=$ system which we shall discuss in chapter three and which is also of philosophical interest is the schema $(\exists v)\mathbf{B}\alpha \supset \mathbf{B}(\exists v)\alpha$.[12] A proof sequence of any instance of this schema would look something like this:

1.	$\alpha(t/v) \supset (\exists v)\alpha$	AS 3
2.	$\mathbf{B}(\alpha(t/v) \supset (\exists v)\alpha)$	1, RB
3.	$\mathbf{B}\alpha(t/v) \supset \mathbf{B}(\exists v)\alpha$	from 2, using AS 2 and MP
4.	$(\exists v)\mathbf{B}\alpha \supset \mathbf{B}(\exists v)\alpha$	3, R$\exists$

Informally, what this schema says is that belief *about* some thing (a res) implies belief *that* such and such is the case. Stated another way, belief

[12] This is a close cousin of the Barcan Formula, $\sim(\exists v)\sim\mathbf{B}\alpha \supset \mathbf{B}\sim(\exists v)\sim\alpha$. The reader should also not the converse of $(\exists v)\mathbf{B}\alpha \supset \mathbf{B}(\exists v)\alpha$, $\mathbf{B}(\exists v)\alpha \supset (\exists v)\mathbf{B}\alpha$ is not a thesis schema of any SQC$^=$ system.

de re implies belief *de dicto*. The reader may wish to skip ahead to chapter three, section 2 for a more detailed explanation of this distinction.

In the next two sections, we shall consider semantics for the $SQC^{=}$ systems which support both the objectual and the substitutional interpretation of the quantifiers. It will be argued in the final section that the semantics supporting the substitutional interpretation of the quantifiers is preferable because it presupposes a less problematic metaphysics than its objectual counterpart. This latter sort of semantics will also be endorsed vis a vis some of the problems peculiar to *quantified* doxastic logic, which will be discussed in some detail in chapter three.

2. Domain Semantics for the $SQC^{=}$ Axiom Systems

There are two types of relational semantics which we shall consider for the $SQC^{=}$ axiom sets. In the first type of semantics, a model structure for a normal doxastic system with quantification and identity ($SQC^{=}$) will be a triple, $\langle W,R,D \rangle$ where W and R are defined as for S model structures. I.e., W is a non-empty set of indices and R is a 2-place relation ranging over members of W. Depending on what sort of axiom set we want, it is as usual possible to impose various restrictions on the relation R. For *doxastic* logic one restriction we would not impose on R is reflexivity since this would validate the schema $B\alpha \supset \alpha$. Finally, D is a non-empty set of 'individuals' or 'objects' which may be finite, denumerably infinite or non-denumerably infinite.

This first type of semantics which we are considering will be called a 'domain semantics' (DS). In this type of semantics, an $SQC^{=}$ model is an ordered 4-tuple <W,R,D,V> where <W,R,D> is an $SQC^{=}$ model structure and where V is an assignment function defined as follows:

1) V: Ind. Cons. $\longrightarrow$ D

2) V: Pred. Var. $\longrightarrow$ PD^n X W

In simple language, V assigns to individual constants members of D. And to predicate variables, V assigns sets of ordered n + 1- tuples whose first n members is an ordered n-tuple of members of D and whose n + 1st member is an index chosen from W. In other words, V relativizes the extension of predicate variables to indices so that in a given model the function V can assign different extensions to the same 'class term' from index to index. In addition, it is stipulated that V is not a partial function.

A *valuation* over an $SQC^{=}$ model, V_M is a function from *closed* wffs and indices into truth-values. For the sake of notational simplicity, we shall use 'Wffs' to denote the set of well-formed *closed* formulae. And so, V_M: Wffs X W $\longrightarrow \{0,1\}$. As usual, V_M can be defined inductively as follows (for all w_i, $w_j \in W$):

Basis: i. $V_M(Pt_1 \ldots t_n, w_i) = 1$ iff $\langle V(t_1), \ldots, V(t_n), w_i \rangle \in V(P)$

ii. $V_M(t_1 = t_2, w_i) = 1$ iff $V(t_1) = V(t_2)$.

Supposing that $V_M(\alpha, w_i)$ and $V_M(\beta, w_i)$ are defined for any $w_i \in W$ then: $V_M(\sim\alpha, w_i)$, $V_M(\alpha \ \& \ \beta, w_i)$, $V_M(\alpha \vee \beta, w_i)$, $V_M(\alpha \supset \beta, w_i)$, $V_M(\alpha \equiv \beta, w_i)$ and $V_M(B\alpha, w_i)$ are defined as for the sentential normal systems. Recall that $V_M(B\alpha, w_i)$ is defined as follows:

$V_M(B\alpha, w_i) = 1$ iff $V_M(\alpha, w_j) = 1$ for all w_j in W such that w_iRw_j.

Less formally, 'x believes that α' is true at an index w_i just in case the content wff α is true at all doxastic alternatives to w_i. These alternatives to w_i are all those indices at which all content sentences of all wffs of the form $B\alpha$ true at w_i are true. In section 4 we shall have more to say concerning the metaphysical status of indices. For now, we shall treat W as simply a set of unanalysed 'points' or indices as in chapter one. Finally, V_M for *quantified* wffs will be defined as follows:

$V_M((\forall v)\alpha, w_i) = 1$ iff $V_{M'}(\alpha(t/v), w_i) = 1$ for all M' based on the same model *structure* as M and differing from M if at all only in terms of what V assigns to t, which is an arbitrarily chosen constant foreign to $(\forall v)\alpha$.

$V_M((\exists v)\alpha, w_i) = 1$ iff $V_{M'}(\alpha(t/v), w_i) = 1$ for at least one M' based on the same model structure as M and differing from M if at all only in terms of what V assigns to t, which is an arbitrarily chosen constant foreign to $(\exists v)\alpha$.

Intuitively, what these truth-conditions assert is that a universally quantified wff $(\forall v)\alpha$ is true at some index w_i just in case the arbitrarily chosen substitution instance of the scope α, $\alpha(t/v)$ is true at w_i no matter what member of D is assigned to t. I.e., $\alpha(t/v)$ must be true at w_i for all members of D. Further, an existentially quantified wff $(\exists v)\alpha$ is true at w_i just in case the arbitrarily chosen substitution instance of the scope α, $\alpha(t/v)$ is true at w_i for at least one member of D assigned to t. This reading of the quantifiers is therefore 'referential' or 'objectual' in the sense defined above in section 1.

The strategy of spelling out the truth-conditions of quantified wffs in terms of what is assigned to an arbitrary t in some substitution instance $\alpha(t/v)$ of the scope α across all models (which differ only in terms of what V assigns to t), has been suggested by Leblanc.[13] The rationale behind his proposal is that it ensures that all members of the domain D in the appropriate model structure will be taken into account in evaluating quantified wffs.[14] If we were to provide a *substitutional* reading of the quantifiers in our domain semantics, then for models with domains containing individuals that are not assigned to any constant t, these individuals would be left out of consideration. Thus, suppose as an alternative to the above characterization of V_M for quantified wffs, we instead stipulated that $V_M((\forall v)\alpha, w_i) = 1$ iff $V_M(\alpha(t/v), w_i) = 1$ for all constants t where it is understood for any such t, $V(t) \in D$. Unless it is assumed that each and every member of D will be assigned to some constant or other (and that V is not partial) then these truth-conditions will leave 'unnamed' individuals (if there are any in the given model) out of the account.

The moral to drawn here is that the substitutional interpretation and the objectual interpretation of the quantifiers are not equivalent.[15] For example, the following infinite set is 'semantically consistent' on an objectual reading of the quantifiers in the semantics just considered, viz., $\{\sim Fa, \sim Fb, \sim Fc, \ldots, \sim Fa_n, \ldots (\exists x)Fx\}$. I.e., there will be an $SQC^=$ model M and a w_i in W such that all members of this set will be true at w_i - given an objectual reading of the existential quantifier. This model would be such that for some index w_i, even though no member d of D *assigned* by V to any of the constants is such that $\langle d, w_i\rangle \in V(F)$, if we consider an alternate model M' which differs from M only in what V assigns to some designated cons-

[13] See Leblanc (1976a), p. 307 and Leblanc (1976b), chs. 1 and 4.

[14] See Leblanc (1976a), p. 307.

[15] See Quine (1969), p. 106 and Van Frassen (1971), p. 127.

tant a_m, then it may be the case that $V(a_m) = d'$ such that $<d',w_i> \in V(F)$. I.e., $V_{M'}(Fa_m,w_i) = 1$. Therefore, it will be the case that for M, $V_M((\exists x)Fx,w_i) = 1$ even though $V_M(\sim Fa_k,w_i) = 1$ for any constant a_k. On the other hand, if we were to read the existential quantifier in $(\exists x)Fx$ *substitutionally* then the set $\{\sim Fa, \sim Fb, \sim Fc, \ldots, \sim Fa_n, \ldots (\exists x)Fx\}$ would be semantically *inconsistent*. I.e., if for every constant a_k, $V_M(\sim Fa_k,w_i)$ is 1 and hence $V_M(Fa_k,w_i) = 0$ for some w_i in an SQC$^=$ model M, then by the substitutional truth-conditions for wffs of the form $(\exists v)\alpha$, it will be the case that $V_M((\exists x)Fx,w_i) = 0$.

In the next section a so-called truth-value semantics will be developed for the SQC$^=$ systems which dispenses with domains of individuals and which involves the assignment of truth-values directly to the atomic wffs of the language. Quantified wffs are therefore naturally read substitutionally in this sort of semantics. Thus, a wff of the form $(\forall v)\alpha$ is true at an index w_i just in case $\alpha(t/v)$ is true at w_i for all constants t. However, the substitutional interpretation of the quantifiers for this truth-value semantics (to be discussed in the next section) and the objectual interpretation of the quantifiers for the domain semantics just described will not be equivalent as just illustrated. (For example, certain infinite sets of wffs semantically consistent in the domain semantics will be inconsistent in the truth-value semantics.) Nonetheless, this has no bearing on the fact that both types of semantics characterize the SQC$^=$ axiom systems. It will be shown in this and the next section that for any SQC$^=$ axiom system, both its corresponding domain semantics (with an objecutal interpretation of the quantifiers) and its corresponding truth-value semantics (with a substitutional interpretation of the quantifiers) will validate all and only those wffs which are theses of the appropriate SQC$^=$ system.

Validity in an SQC$^=$ model is truth at all members of W. And validity in the class of SQC$^=$ models (determined by the restrictions imposed on R) is validity in all models in the class.[16]

Soundness of the various SQC$^=$ axiom sets is easily established by showing that all instances of the axiom schemata are valid and that the rules of inference preserve validity in the appropriate class of models. For example, consider the axiom-schema $\alpha(t/v) \supset (\exists v)\alpha$ common to all the SQC$^=$ systems which is the dual of the so-called particularity schema, viz., $(\forall v)\alpha \supset \alpha(t/v)$. Informally, suppose that for some SQC$^=$ model $V_M(\alpha(t/v), w_i) = 1$ for some w_i in W. Then there is some model M' like M such that $V_{M'}(\alpha(t''/v), w_i) = 1$ where $V(t'') = V(t)$. Therefore, $V_M((\exists v)\alpha, w_i) = 1$. Q.E.D.

By way of another example, consider the Barcan Formula, $\sim(\exists v)\sim \mathbf{B}\alpha \supset \mathbf{B}\sim(\exists v)\sim\alpha$. Suppose for some SQC$^=$ model, $V_M(\sim(\exists v)\sim\mathbf{B}\alpha, w_i) = 1$ but that $V_M(\mathbf{B}\sim(\exists v)\sim\alpha, w_i) = 0$. I.e., $V_M((\forall v)\mathbf{B}\alpha, w_i) = 1$ and $V_M(\mathbf{B}(\forall v)\alpha, w_i) = 0$. If w_i is a dead end (for KQC$^=$) then this set of assignments is inadmissible. If w_i is not a dead end, then there will be some w_j such that w_iRw_j and such that $V_M((\forall v)\alpha, w_j) = 0$ and hence $V_{M'}(\alpha(t/v), w_j) = 0$ for at least one M' like M. However, given that $V_M((\forall v)\mathbf{B}\alpha, w_i) = 1$ then it is the case that $V_{M'}(\mathbf{B}\alpha(t/v), w_i) = 1$ for all M' like M. But then for all w_j such that w_iRw_j, $V_{M'}(\alpha(t/v), w_j) = 1$ for any such M', including the M' such that $V_{M'}(\alpha(t/v), w_j) = 0$, which is a contradiction. Q.E.D.

[16] The symbol $\models_{\mathcal{C}} \alpha$ indicates that the wff α is valid in a class of models, $\mathcal{C}$ although for the remainder of this chapter we shall simply use $\models\alpha$ with it being understood that α is valid in an appropriate class of models, membership in a class being a matter of the restriction(s) placed on R.

As a final example, we shall consider the axiom-schema $(\alpha(t_1/v)\ \&\ t_1 = t_2) \supset \alpha(t_2/v)$ which again is common to all the SQC$^=$ axiom systems. The proof that all instances of this schema are valid would proceed by induction on the complexity of $\alpha(t_1/v)$ and its substitutional variant $\alpha(t_2/v)$. The *basis* of the induction is where $\alpha(t_1/v)$ and its variant $\alpha(t_2/v)$ are atomic. Then $\alpha(t_1/v)$ is either $t_1 = t_n$ or $P\ldots t_1\ldots t_m$. Since $\alpha(t_2/v)$ is $\alpha[(t_2/t_1)(t_1/v)]$ then $\alpha(t_2/v)$ will be $t_2 = t_n$ if $\alpha(t_1/v)$ is $t_1 = t_n$ or $\alpha(t_2/v)$ will be $P\ldots t_2\ldots t_m$ if $\alpha(t_1/v)$ is $P\ldots t_1\ldots t_m$. Further, suppose that for some w_i in W in an SQC$^=$ model M, $V_M(\alpha(t_1/v), w_i) = V_M(t_1 = t_2, w_i) = 1$. Then $V(t_1) = V(t_2)$. If $\alpha(t_1/v)$ is of the form $t_1 = t_n$ then $V(t_1) = V(t_n)$ on the supposition that $\alpha(t_1/v)$ is true at w_i and so it follows immediately that $V(t_2) = V(t_n)$ and hence, $V_M(t_2 = t_n, w_i) = 1$. Or, if $\alpha(t_1/v)$ is of the form $P\ldots t_1\ldots t_m$ then since $<\ldots V(t_1)\ldots V(t_m)> \in V(P)$ and since $V(t_1) = V(t_2)$ it follows that $<\ldots V(t_2)\ldots V(t_m)> \in V(P)$ and so $V_M(P\ldots t_1\ldots t_m, w_i) = 1$. In either case, $V_M(\alpha(t_2/v), w_i) = 1$. This proves the basis of the induction.

The *inductive hypothesis* is that whenever $\alpha(t_1/v)$ and $t_1 = t_2$ are true at an index w_i in an SQC$^=$ model, then $\alpha(t_2/v)$ is true, where $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are of degree of complexity n. Then, it must be shown that the this charateristic holds where $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are of degree of complexity n + 1. Cases to be considered are where $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are negations, conjunctive expressions, existentially quantified expressions and finally of the form $B\beta(t_1/v)$ and $B\beta(t_2/v)$ respectively. We shall consider the last case only since the other cases are trivial. Suppose then that $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are of the form $B\beta(t_1/v)$ and $B\beta(t_2/v)$ respectively.

Suppose further that there is a model M and index w_i where $V_M(B\beta(t_1/v), w_i) = V_M(t_1 = t_2, w_i) = 1$. Then for any w_j such that w_iRw_j, $V_M(\beta(t_1/v), w_j) = V_M(t_1 = t_2, w_j) = 1$. Then by the *inductive hypothesis*, it follows that $V_M(\beta(t_2/v), w_j) = 1$ for all w_j such that w_iRw_j and therefore, $V_M(B\beta(t_2/v), w_i) = 1$. Q.E.D.

For a thorough discussion of *completeness* results, the reader is referred to Hughes and Cresswell (1968, 1984) for their remarks concerning normal alethic modal systems with quantification. We shall here mention some important features of these results. The canonical model $\mathcal{M}$ for any $SQC^=$ system is a 4-tuple $\langle W, R, D, V\rangle$ where W is a set of maximal consistent sets of wffs with the $\exists$-property. The $\exists$-property can be characterized as follows: If $(\exists v)\alpha$ is in w_i (where w_i is a maximal consistent set of $SQC^=$ wffs) then so is $\alpha(t/v)$ for at least one constant t. Hughes and Cresswell[17] show how to extend any consistent set of wffs to a maximal consistent set with the $\exists$-property. The trick as it were is to ensure that every maximal consistent set has the so-called $\exists'$-property. A set of wffs has the $\exists'$-property just in case for every wff of the form $(\exists v)\alpha$, the set contains the implicational wff $(\exists v)\alpha \supset \alpha(t/v)$ for at least one constant t. Then any set with the $\exists'$-property will also have the $\exists$-property since if any such set contains a wff of the form $(\exists v)\alpha$ then given that it contains $(\exists v)\alpha \supset \alpha(t/v)$ for at least one constant t, it will also contain $\alpha(t/v)$ for at least one constant t. As we shall see in a few paragraphs, this result is important for the case where α is of the form $(\exists v)\alpha$ in the inductive proof of the fundamental theorem for canonical models, viz., that for any $SQC^=$ wff α, $V_{\mathcal{M}}(\alpha, w_i) = 1$ iff $\alpha \in w_i$. The relation R ranging over members of the

[17] See Hughes and Cresswell (1968).

set of maximal consistent sets of wffs with the $\exists$-property is defined as it was for the canonical model for sentential normal systems: $w_i R w_j$ iff $(\alpha)(B\alpha \in w_i \longrightarrow \alpha \in w_j)$. Further, D is the set of all *constants*.

Defining V is a bit tricky beccause we want to prevent the situation where any identity wff is false at every member of W in $\mathcal{M}$. If D for the canonical model for any $SQC^{=}$ system is simply the set of constants, then allowing V to be the identity function such that for any constant t, $V(t) = t$ will have as a consequence that $V_{\mathcal{M}}(t_1 = t_2, w_i) = 0$ for any two distinct constants t_1, t_2 for any w_i in W since $V(t_1) \neq V(t_2)$. Thus, we cannot stipulate that V is the identity function from constants into constants. In *An Introduction to Modal Logic*, Hughes and Cresswell offer the following strategy for defining V for the canonical model for normal modal identity systems: First of all, we suppose that there is an ordering of all the constants of the language. We can then define V for constants as follows:

$V(t_i) =$ $V(t_j)$ if t_j occurs earlier than t_i in the ordering such that $t_j = t_i$ is in some w_i in W.
t_i otherwise.

As Hughes and Cresswell argue, this way of defining V for constants will ensure that whenever $t_1 = t_2 \in w_i$, $V_{\mathcal{M}}(t_1 = t_2, w_i) = 1$ and vice-versa. How this is so will be explained shortly when we discuss the base clause of the proof of the fundamental theorem of canonical models. Finally, in terms of defining V for predicate variables we shall stipulate that for any constants $t_1, \ldots, t_n$, for any predicate variable P and for any $w_i \in W$, $\langle t_1, \ldots, t_n, w_i \rangle \in V(P)$ iff $Pt_1 \ldots t_n \in w_i$.

We shall now outline the proof of the fundamental theorem of canonical models, viz., $V_{\mathcal{M}}(\alpha, w_i) = 1$ iff $\alpha \in w_i$ for all SQC$^=$ wffs α and for all w_i in W in $\mathcal{M}$, which proceeds by mathematical induction.

Base Clause: Suppose α is atomic. Then either α is of the form $Pt_1 \ldots t_n$ or $t_1 = t_2$.

Suppose α is of the form $t_1 = t_2$:[18]

i) If $t_1 = t_2 \in w_i$ then if t_1 occurs earlier in the ordering, $V(t_2)$ is $V(t_1)$. Hence, $V_{\mathcal{M}}(t_1 = t_2, w_i) = 1$. For the next part of the proof, the reader should keep in mind that $\vdash t_1 = t_2 \supset t_2 = t_1$ and hence that $t_2 = t_1$ is in w_i if $t_1 = t_2$ is. Now if t_1 does not occur earlier in the ordering than t_2 (in which case t_2 occurs earlier in the ordering than t_1) and given that $t_2 = t_1 \in w_i$ (since $t_1 = t_2 \in w_i$) it follows that $V(t_1)$ is $V(t_2)$. Then $V_{\mathcal{M}}(t_1 = t_2, w_i) = 1$.

ii) If $V_{\mathcal{M}}(t_1 = t_2, w_i) = 1$ then $V(t_1) = V(t_2)$. Supposing t_1 and t_2 are not distinct (and hence $V(t_1) = t_1$) then $t_1 = t_2$ will be of the form $c = c$ which is of course an SQC$^=$ axiom schema and hence $t_1 = t_2$ is in w_i. Or, if t_1, t_2 are distinct constants and given that $V(t_1) = V(t_2)$ then there are two possibilities: First, $V(t_1)$ and $V(t_2)$ are assigned either t_1 or t_2. But then by definition of V for constants, this assumes that either $t_1 = t_2$ is in w_i or that $t_2 = t_1$ is in w_i. If $t_2 = t_1$ is in w_i and given that $\vdash t_2 = t_1 \supset t_1 = t_2$ it follows that $t_1 = t_2$ is in w_i. Second, $V(t_1)$ and $V(t_2)$ are assigned some constant distinct from t_1 and t_2, say t_3. Then $V(t_1) = t_3$ and hence $t_3 = t_1$ is in w_i. Also, $V(t_2) = t_3$ and hence $t_3 = t_2$ is in w_i. Since $\vdash ((t_3 = t_1) \,\&\, (t_3 = t_2)) \supset t_1 = t_2$ it follows that $t_1 = t_2$ is in w_i.[19] Q.E.D.

Suppose A is of the form $Pt_1 \ldots t_n$:

$V_{\mathcal{M}}(Pt_1 \ldots t_n, w_i) = 1$ iff $\langle V(t_1), \ldots, V(t_n), w_i \rangle \in V(P)$

iff $\langle u_1, \ldots, u_n, w_i \rangle \in V(P)$ (where the u_i's may be distinct from the t_i's given our earlier definition of V for constants.)

iff $Pu_1 \ldots u_n \in w_i$. (given our definition of

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[18] If $t_1 = t_2$ is of the form $c = c$ then the result is immediate.

[19] I owe the reasoning here in ii) to Hughes and Cresswell (1968), pp. 193–194.

V for predicate variables.)

Given that $V(t_1) = u_1, \ldots, V(t_n) = u_n$ it follows that $u_1 = t_1$, $\ldots u_n = t_n$ are all in w_i. Hence, so is their conjunction. Also, as a kind of generalization of the SQC$^=$ axiom schema $(A(t_1/v)$ & $t_1 = t_2) \supset A(t_2/v)$, we have $\vdash (u_1 = t_1 \,\&\, \ldots \,\&\, u_n = t_n) \supset (Pt_1 \ldots t_n \equiv Pu_1 \ldots u_n)$ and hence $Pt_1 \ldots t_n \equiv Pu_1 \ldots u_n$ is in w_i. But then $Pt_1 \ldots t_n$ is in w_i if $Pu_1 \ldots u_n$ is. Hence, it follows that $V(Pt_1 \ldots t_n, w_i) = 1$ iff $Pt_1 \ldots t_n \in w_i$.[20]

This completes the rather cumbersome proof of the base clause for the fundamental theorem. The so-called inductive hypothesis is that we suppose the fundamental theorem holds for SQC$^=$ wffs of degree of complexity n. It must then be shown that the theorem holds for wffs of degree of complexity n + 1. The cases where α is of the form $\sim\beta$, $\beta \,\&\, \gamma$ as well as $B\beta$ are proven as before for non-quantified doxastic K-extensions. We come now to consider the case where α is of the form $(\exists v)\beta$:[21]

i) Suppose $(\exists v)\beta \in w_i$.

$\beta(t/v) \in w_i$ for some constant t. ($\exists$-property)

$V_{\mathcal{M}}(\beta(t/v), w_i) = 1$ by the *inductive hypothesis*.

$V_{\mathcal{M}'}(\beta(t''/v), w_i) = 1$ for some $\mathcal{M}'$ like $\mathcal{M}$ except that $V(t'')$ for $\mathcal{M}'$ is $V(t)$ for $\mathcal{M}$.[22]

$V_{\mathcal{M}}((\exists v)\beta, w_i) = 1$ by the truth conditions for wffs of the form $(\exists v)\beta$.

ii) Suppose $V_{\mathcal{M}}((\exists v)\beta, w_i) = 1$

$V_{\mathcal{M}'}(\beta(t/v), w_i) = 1$ for at least one $\mathcal{M}'$ like $\mathcal{M}$ where t is foreign to $(\exists v)\beta$.

$\beta(t/v) \in w_i$ by the inductive hypothesis.

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[20] Once again, I owe the reasoning here to Hughes and Cresswell (1968), p. 194.

[21] Since the truth-functional operators are definable in terms of ~ and &, since $\forall$ is definable in terms of $\exists$ and since P_B is definable in terms of B, it will be sufficient to only consider the cases where α is of the form $\sim\beta$, $\beta \,\&\, \gamma$, $B\beta$ and $(\exists v)\beta$.

[22] As Hughes and Cresswell point out, proof of this would proceed by induction on the complexity of wffs. See Hughes and Cresswell (1984), p. 168.

$\vdash \beta(t/v) \supset (\exists v)\beta$

$\beta(t/v) \supset (\exists v)\beta \in w_i$ since w_i is maximal consistent.

$(\exists v)\beta \in w_i$ since w_i is maximal consistent.

Q.E.D.

This completes the proof of the fundamental theorem of canonical models.

And so, we have established that for any SQC$^=$ wff α and for any $w_i \in W$ in SQC$^=$'s canonical model $\mathcal{M}$, $V_{\mathcal{M}}(\alpha, w_i) = 1$ iff $\alpha \in w_i$. Now, consider any SQC$^=$ wff α such that α is not a theorem. Then $\sim\alpha$ is syntactically consistent from which it follows that there is an m.c.e. of $\sim\alpha$ with the $\exists$-property, w_i such that $\sim\alpha$ is in w_i and such that w_i is in the set of maximal consistent sets, W in the canonical model $\mathcal{M}$. Then by the fundamental theorem, it follows that $V_{\mathcal{M}}(\sim\alpha, w_i) = 1$ and hence that $V_{\mathcal{M}}(\alpha, w_i) = 0$. Thus, all SQC$^=$ non-theorems are invalid in SQC$^=$'s canonical model and therefore all valid SQC$^=$ wffs are SQC$^=$ theorems, provided that it can be shown that the canonical model for any SQC$^=$ system is in the class of models which validates all of its theorems. And the proof of this consists in showing that R in the canonical model for the SQC$^=$ system satisfies the appropriate restriction(s) if any which are imposed on R for any model in the class of SQC$^=$ models. For example, if the SQC$^=$ system is K4QC$^=$ then R in the canonical model must be shown to be *transitive*. The reader is referred back to chapter one to see just how such a proof is carried out.

We shall now consider a simpler type of semantics for the set of SQC$^=$ axiom systems which dispenses with domains of objects or individuals. It will be argued in the next section that the SQC$^=$ axiom systems are both sound and complete with respect to this semantics.

3. Truth-Value Semantics for the $SQC^{=}$ Axiom Systems

An alternative to the domain semantics for the $SQC^{=}$ axiom systems is what Leblanc has called 'truth-value semantics' or simply TVS. As we shall see, this semantics dispenses with domains of individuals although it still makes use of indices in the characterization of models. A TV semantics naturally lends itself to a substitutional reading of the quantifiers since as we have seen, such a reading of the quantifiers makes no explicit reference to domains of individuals but simply to substitution instances.

Let us say that in a TV semantics, an $SQC^{=}$ model structure is simply an S model structure $\langle W, R \rangle$ where $W \neq \emptyset$ and $R \subseteq W \times W$ (where R may or may not have certain restrictions imposed on it). An $SQC^{=}$ TV model will be a triple, $\langle W, R, V \rangle$ where $\langle W, R \rangle$ is an S model structure and such that V is a so-called truth-value assignment[23] which to all *closed* atomic wffs of the language assigns at an index either 0 or 1: $V\colon \text{Atom.Wffs} \times W \longrightarrow \{0, 1\}$, where 'Atom. Wffs' is the set of all *closed* atomic wffs. We impose the following restrictions on V in order to guarantee that the $SQC^{=}$ axiom schemata $t = t$ and $(\alpha(t_1/v) \,\&\, t_1 = t_2) \supset \alpha(t_2/v)$ are validated in this semantics:

Restriction 1: If α is of the form $t = t$ then $V(\alpha, w_i) = 1$ for all w_i in W.

Restriction 2: If $V(t_1 = t_2, w_i) = 1$ then for all w_j, $V(t_1 = t_2, w_j) = 1$.

Restriction 3: If $V(t_1 = t_2, w_i) = 1$ for any w_i in W then $V(\alpha(t_1/v), w_i) = V(\alpha(t_2/v), w_i)$ for any atomic wff α.

Just how these restrictions ensure validity of the above-mentioned axiom-

[23] This phrase has been coined by Leblanc in a number of places including Leblanc (1976a, 1976b).

schemata will become evident in our discussion of *soundness*.

A valuation over an SQC$^{=}$ TV model, V_M is a function from *closed* wffs and indices into truth values. I.e., V_M: Wffs $\longrightarrow \{0,1\}$, where 'Wffs' is set of all *closed* wffs. The function V_M can be defined inductively as follows for all w_i, $w_j \in W$:

Basis: $V_M(\alpha, w_i) = V(\alpha, w_i)$ where α is atomic and such that V satisfies restrictions 1, 2 and 3 outlined above.

Inductive Step: Suppose that $V_M(\alpha, w_i)$, $V_M(\beta, w_i)$ are defined. Then $V_M(\sim\alpha, w_i)$, $V_M(\alpha \;\&\; \beta, w_i)$, $V_M(\alpha \vee \beta, w_i)$, $V_M(\alpha \supset \beta, w_i)$, $V_M(\alpha \equiv \beta, w_i)$ and $V_M(B\alpha, w_i)$ are defined as for the domain semantics. We now come to consider the cases where α is of the form $(\exists v)\beta$ and where α is of the form $(\forall v)\beta$:

$V_M((\forall v)\beta, w_i) = 1$ iff $V_M(\beta(t/v), w_i) = 1$ for all constants t.

$V_M((\exists v)\beta, w_i) = 1$ iff $V_M(\beta(t/v), w_i) = 1$ for at least one constant t.

In short, a universally quantified wff is true at an index w_i just in case all of its substitution instances are true at w_i and an existentially quantified wff is true at w_i just in case at least one of its substitution instances is true at w_i. No mention is made of a set of 'individuals' in these truth-conditions. The reading of the quantifiers here is strictly substitutional.

Finally, *validity* in an SQC$^{=}$ model is truth at all members of W and validity in a class of SQC$^{=}$ models is validity in all models in the class.

What is distinctive about this semantics is the simplicity of the truth conditions for atomic and quantified wffs in comparison to the domain semantics. It is because of this theoretical simplicity along with a sounder metaphysics that we shall eventually adopt a TV semantics for quantified doxastic logic. An argument for this claim will be developed in section 5.

It should be noted that the idea for this type of semantics for quantified modal logic dates back to Carnap's notion of 'state description' in *Meaning and Necessity*. A state description is defined as a set S of wffs such that for any atomic wff α, either α is in S or $\sim\alpha$ is in S but not both - and nothing else is in S.[24] According to Carnap, to say that a wff α 'holds' in S means that if all the wffs in S were 'true' i.e., if S were actual, α would be true. It can be defined inductively what it is for any wff α to 'hold' in S. Any atomic wff α 'holds' in S iff α is in S, and for any wffs β, γ, $\sim\beta$ holds in S iff β doesn't, $\beta \vee \gamma$ holds in S iff either β or γ hold in S, etc. and $(\forall v)\alpha$ is in S iff "all substitution instances of its scope ... hold in it".[25] Notice that Carnap treats '$\forall$' substitutionally and that atomic wffs 'hold' in S by virtue of membership in S without appeal to 'individuals'. In these respects, indices in TV semantics are like state descriptions except that Carnap didn't have the additional machinery of an accessibility relation.

So given these similarities between indices in a TV semantics and Carnapian state descriptions, we could regard indices as kinds of state descriptions. I.e., we could regard TV indices as sets of wffs such that for any atomic wff α, either it or its negation is in the set, but not both. And of course truth at indices so conceived is as usual defined inductively. Then with respect to atomic wffs and their negations such sets are maximal and they are consistent. Thus, an advantage of a TV semantics is that it provides us with a framework for a plausible metaphysics of indices.

We shall now end this digression into the history of truth-value semantics for quantified alethic and doxastic modal logics by noting that Ruth

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24 Carnap (1947), p. 9.

25 Carnap (1947), p.9.

Barcan Marcus in 'Dispensing With Possibilia' (1976) suggests that in the semantics for quantified modal logic we can dispense with domains of individuals and instead associate with each world or index in a model a set of constants. She also notes that such a semantics lends itself to a substitutional reading of the quantifiers. And of course, Leblanc as well as Dunn and Belnap have done work in truth-value semantics for various first-order logics.[26]

Soundness as usual can easily be established by showing validity of the axiom-schemata as well as the validity preservingness of the rules of inference in the appropriate class of $SQC^{=}$ models. Again, consider as an example the $SQC^{=}$ axiom schema $\alpha(t/v) \supset (\exists v)\alpha$. Suppose for some $SQC^{=}$ TV model M that $V_M(\alpha(t/v), w_i) = 1$ but that $V_M((\exists v)\alpha, w_i) = 0$. Then $V_M(\alpha(t/v), w_i) = 0$ for all constants t. But this contradicts our supposition that $V_M(\alpha(t/v), w_i)$ is 1 for some constant t. Q.E.D.

Soundness of any $SQC^{=}$ system relative to the appropriate class of TV models is in part guaranteed by the restrictions we have placed on the indexed truth-value assignment V. To see how these restrictions help to guarantee soundness, consider for example the axiom-schema $(\alpha(t_1/v)\ \&\ t_1 = t_2) \supset \alpha(t_2/v)$. Suppose there is a model M such that $V_M(t_1 = t_2, w_i) = V_M(\alpha(t_1/v), w_i) = 1$ but $V_M(\alpha(t/v), w_i) = 0$. But given Restriction 3 for V mentioned above, if $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are atomic then given that 4 $\alpha(t_1/v)$ and $t_1 = t_2$ are true at w_i, $\alpha(t_2/v)$ must also be true at w_i.

We must now consider what happens when $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are non-atomic in $(\alpha(t_1/v)\ \&\ t_1 = t_2) \supset \alpha(t_2/v)$. In order to consider this, we shall introduce the notion of 'subformula'.[27] First of all, any wff α atomic

[26] See Leblanc (1976b, 1982) as well as Dunn and Belnap (1968).

or otherwise is a subformula of itself. We next appeal to the notion of 'immediate subformula'. If α is of the form $\sim\beta$ then β is an immediate subformula of α. If α is of the form $\beta \,\&\, \gamma$, $\beta \vee \gamma$, $\beta \supset \gamma$ or $\beta \equiv \gamma$ then both β and γ are immediate subformulas of α. If α is of the form $(\exists v)\beta$ or $(\forall v)\beta$ then all substitution instances of β will be immediate subformulas of α. Finally, if α is of the form $\mathbf{B}\beta$ then β is an immediate subformula of α. All subformulas of subformulas of α are subformulas of α. And finally, an 'atomic subformula' of a wff α is a subformula of α which is atomic.

In our TV semantics V_M is defined inductively with $V(\alpha, w_i) = V_M(\alpha, w_i)$ as the basis of the induction for atomic α and such that V is an indexed truth-value assignment to the atomic wffs of the language. Therefore, in this type of semantics, the truth-value assigned to a non-atomic wff α at an index w_i will be determined by what V assigns to α's atomic subformulas at that index unless α is of the form $\mathbf{B}\beta$. Then, the value V_M gives to $\mathbf{B}\beta$ at w_i will be determined by what V assigns to the content wff β's atomic subformulas at all w_j such that w_iRw_j (assuming that the content wff β itself is not of the form $\mathbf{B}\gamma$).

For example, suppose α is $\mathbf{B}(Fa \vee (\forall x)Gx)$. Then for some TV model M $V_M(\mathbf{B}(Fa \vee (\forall x)Gx), w_i)$ will be determined by what V (and hence V_M) will assign to Fa as well as Ga, Gb, Gc, ... at all w_j such that w_iRw_j.

Given this brief digression into the notion of subformulas (atomic or otherwise) we are now in a position to consider what happens when $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are non-atomic in $(\alpha(t_1/v) \,\&\, t_1 = t_2) \supset \alpha(t_2/v)$. Suppose there is a TV model M such that $V_M(\alpha(t_1/v), w_i) = V_M(t_1 = t_2, w_i) = 1$ but $V_M(\alpha(t_2/v), w_i) = 0$. We shall now present an informal argument that

[27] See Leblanc (1976b), section 1.1. Leblanc attributes the notion of subformula to Gentzen.

shows any such assignment for any w_i in some arbitrary TV model M to be inadmissible: If $V_M(t_1 = t_2, w_i) = 1$ (and hence $V(t_1 = t_2, w_i) = 1$) and given that the atomic subformulas of $\alpha(t_1/v)$ are the same as those of $\alpha(t_2/v)$ except for the occurrence of t_1 and t_2 then V will assign the same values at w_i to $\alpha(t_1/v)$'s atomic subformulas containing t_1 that it does to $\alpha(t_2/v)$'s atomic subformulas containing t_2 given Restriction 3 for V.

If any of $\alpha(t_1/v)$ and $\alpha(t_2/v)$'s atomic subformulas occur in the scope of a doxastic operator, then the value of $\alpha(t_1/v)$ and $\alpha(t_2/v)$ will depend in part on what V assigns to these atomic subformulas at all w_j such that w_iRw_j. And once again, V will assign the same values at each of these w_j's to $\alpha(t_1/v)$'s atomic subformulas containing t_1 that it does to $\alpha(t_2/v)$'s atomic subformulas containing t_2 given Restriction 2 for V. (Recall that Restriction 2 for our TV semantics stipulates that if $V(t_1 = t_2, w_i) = 1$ then for any other w_j in W, $V(t_1 = t_2, w_j) = 1$.) Therefore, $V_M(\alpha(t_1/v), w_i) = V_M(\alpha(t_2/v), w_i)$ and since by supposition, $V_M(\alpha(t_1/v), w_i) = 1$ then $V_M(\alpha(t_2/v), w_i) = 1$. Q.E.D.

As an alternative to this informal proof using the notion of subformula, we could have used an inductive-style proof in the same manner as for the domain semantics for the $SQC^=$ systems. The basis would simply be that whenever $V_M(\alpha(t_1/v), w_i) = V_M(t_1 = t_2, w_i) = 1$, $V_M(\alpha(t_2/v), w_i) = 1$ for $\alpha(t_1/v)$, $\alpha(t_2/v)$ atomic by Restriction 3.

Also with respect to soundness, consider the Barcan Formula which as noted earlier is an axiom schema of any $SQC^=$ system. We shall show that all instances of this schema are valid in a TV semantics. Suppose that for some TV model, $V_M((\forall v)B\alpha, w_i) = 1$ but $V_M((B(\forall v)\alpha, w_i) = 0$. Then $V_M((\forall v)\alpha, w_j) = 0$ for at least one w_j such that w_iRw_j. Hence, for this

w_j, $V_M(\alpha(t/v),w_j) = 0$ for at least one constant t. However, on the supposition that $V_M((\forall v)B\alpha,w_i) = 1$ then $V_M(B\alpha(t/v),w_i) = 1$ for all constants t. Then for any w_j such that w_iRw_j, $V_M(\alpha(t/v),w_j) = 1$ for all constants t. Q.E.D.

Completeness of any of the SQC$^=$ systems with respect to the appropriate class of TV models is established as usual by the method of canonical models. A canonical model $\mathcal{M}$ for any SQC$^=$ system will be a triple $\langle W,R,V\rangle$ such that W is a maximal consistent set of wffs with the $\exists'$ and $\exists$-properties. R for a TV canonical model is defined as usual: w_iRw_j iff $(\alpha)(B\alpha \in w_i \longrightarrow \alpha \in w_j)$.

In addition, each member of W will also have the following properties: Since $t = t$ is an axiom schema and since each w_i in W is maximal consistent, it follows that each m.c. set in W will contain every wff of the form $t = t$. Further, since $\vdash t_1 = t_2 \supset B(t_1 = t_2)$ it follows that for any m.c. set w_i in W, if $t_1 = t_2$ is in w_i then so is $B(t_1 = t_2)$. And since R is defined as usual such that w_iRw_j iff $(\alpha)(B\alpha \in w_i \longrightarrow \alpha \in w_j)$ it will follow that if $t_1 = t_2$ is in w_i in which case so is $B(t_1 = t_2)$ then for any w_j such that w_iRw_j, $t_1 = t_2$ is in every such w_j. Finally, given that $\vdash t_1 = t_2 \supset (\alpha(t_1/v) \supset \alpha(t_2/v))$ and $\vdash t_1 = t_2 \supset (\alpha(t_2/v) \supset \alpha(t_1/v))$ it follows that $\vdash t_1 = t_2 \supset (\alpha(t_1/v) \equiv \alpha(t_2/v))$. Thus, if $t_1 = t_2$ is in w_i then so is $\alpha(t_1/v) \equiv \alpha(t_2/v)$ and hence if $\alpha(t_1/v)$ is in w_i then so is $\alpha(t_2/v)$ and if $\alpha(t_1/v)$ is not in w_i then neither is $\alpha(t_2/v)$.

The function V in the SQC$^=$ canonical model $\mathcal{M}$ will be defined for atomic wffs as follows: For any wff α of the form $Pt_1 \ldots t_n$ or $t_1 = t_2$, $V(\alpha,w_i) = 1$ iff $\alpha \in w_i$. Because V is defined in this way for the canonical model, it is redundant to mention the restrictions 1 and 3 on V mentioned a few pages back given our characterization of members of W for the canonical model. For example, since any w_i in W in the canonical model is such that any wff of the form $t = t$ is in w_i and given that $t = t \in w_i$ iff $V(t = t, w_i) = 1$ then it follows that $V(t = t, w_i) = 1$ for all wffs of the form $t = t$.

However, the fact that each member of W contains every wff of the form $t_1 = t_2 \supset B(t_1 = t_2)$ merely assures us that if $t_1 = t_2$ is a member of w_i then $t_1 = t_2$ is also a member of any w_j *such that w_iRw_j*. But this property of members of W does not guarantee that if $t_1 = t_2$ is contained in w_i then $t_1 = t_2$ is in *every w_j in W* regardless of whether or not w_iRw_j. But we need such a guarantee if we are to mirror restriction 2 on V, viz., that if $V(t_1 = t_2, w_i) = 1$ then for all w_j in W, $V(t_1 = t_2, w_j) = 1$. Thus, it will be necessary to impose the following restriciton on V for the canonical model, viz., if $V(t_1 = t_2, w_i) = 1$ then $V(t_1 = t_2, w_j) = 1$ for all w_j in W.

The fundamental theorem of canonical models, viz., for any SQC$^{=}$ wff α and for any w_i in W, $V_{\mathcal{M}}(\alpha, w_i) = 1$ iff $\alpha \in w_i$ is proven as usual by mathematical induction:

Base Clause: α is of the form $Pt_1 \ldots t_n$ or $t_1 = t_2$.
$V_{\mathcal{M}}(\alpha, w_i) = V(\alpha, w_i)$ and so the theorem holds by definition.

Inductive Hypothesis: Suppose the fundamental theorem holds for wffs of degree of complexity n. Show that it holds for wffs of degree n+1.

Once again, the cases where α is of the form $\sim\beta$, β & γ and $B\beta$ are handled in much the same way as they were for the sentential normal systems.

See section 4 of chapter one for details. We shall now consider the case where α is of the form $(\exists v)\beta$.

i) Suppose: $(\exists v)\beta \in w_i$
$\beta(t/v) \in w_i$ for some constant t by the $\exists$-property.
$V_\mu(\beta(t/v), w_i) = 1$ for some t by the *inductive hypothesis*.
$V_\mu((\exists v)\beta, w_i) = 1$

ii) Suppose: $V_\mu((\exists v)\beta, w_i) = 1$
$V_\mu(\beta(t/v), w_i) = 1$ for some t.
$\beta(t/v) \in w_i$ by the *inductive hypothesis*.
$\vdash \beta(t/v) \supset (\exists v)\beta$
$\beta(t/v) \supset (\exists v)\beta \in w_i$ since w_i is maximal consistent.
$(\exists v)\beta \in w_i$ since w_i is maximal consistent.
Q.E.D.

This completes the proof of the fundamental theorem of canonical models for the $SQC^=$ systems. As remarked earlier, all that needs to be shown is that the canonical model is in the class of models with respect to which the particular $SQC^=$ axiom system is sound. The reader is once again referred to section 4, chapter one for details of how such a proof is carried out.

Now that we have outlined a set of axiom systems for first-order belief logic with identity as well as two types of semantics which characterize these systems, we shall in the next chapter consider some of the *philosophical* difficulties associated with quantified belief logic. However, for the remainder of this chapter, we shall examine the metaphysical underpinnings of the two types of semantics which we have considered. It will be argued that the metaphysics of the TV semantics just developed is simpler and hence less problematic than the metaphysics of the domain semantics discussed in section 2.

4. An Excursus into the Metaphysics of the Semantics for the $SQC^=$ Axiom Systems

Technically, both the invariant domain semantics and the truth-value semantics which we have just outlined in the previous two sections *characterize* the $SQC^=$ systems. On purely technical grounds, either type of semantics will do. However, it will be argued in this section that the metaphysics of the truth-value semantics is less problematic than the metaphysics of the domain semantics. And from this it follows that there is some presumption in favour of adopting the former type of semantics rather than the latter for the $SQC^=$ systems.

The *domain* semantics which we have discussed for the $SQC^=$ sytems presents the following metaphysical picture: The set D in a model will consist of a set of so-called individuals to which various properties are ascribed at each index. The set of properties and relations ascribed to an individual at an index is determined by the assignment function V: A property or relation in this type of semantics is simply a set of n + 1-tuples, each n + 1-tuple being an n-tuple of members of D and an index. Thus, in the case of a property P, a member *x of D has P at an index* w_i just in case the 2-tuple $\langle x, w_i \rangle$ is in P's extension, V(P). Then members of D can be individuated from one another *at an index* by considering the properties which each individual 'possesses' at that index. One such individuating principle known as Leibniz's *principle of the indiscernibility of identicals* says that if x and y are identical then they will have all their properties in common. Otherwise, x and y are distinct. This principle is expressible in second-order logic as follows:

$$(\forall x)(\forall y)(x = y \supset (\forall F)(Fx \equiv Fy))$$

However, as Loux notes[28] things go awry when we consider the identity of members of D across indices. If we appeal to Leibniz's priniciple to determine 'transworld' (or in our parlance, 'transindex') identitiy of individuals then we are faced with the problem that an individual may vary in its properties from index to index. Yet, we cannot say that this isn't the 'same' transindexical individual given the way the semantics is set up such that members of D are invariant across indices. Therefore, Leibniz's principle is inadequate as a criterion of *transindexical* individuation of individuals for the type of semantics under consideration.

But then, what is it that accounts for transworld identity if not the properties and relations ascribed to things? It would seem that we are forced into the position that members of D are 'bare (transindexical) particulars' whose individuation across indices is property-independent. Some philosophers such as Kripke[29] and Kaplan have objected to the so-called bare particular metaphysics implicit in standard modal semantics. Kaplan rejects the metaphysical assumption implicit in the notion of a model that individuals "have an existence which is quite independent of whatever properties the model happens to tack onto them".[30]

However, there are alternatives to the bare particular approach of handling transindexical identity of individuals. One such alternative which

[28] See Loux (1979), p. 37.

[29] Kripke (1980), p. 52.

[30] Kaplan (1979), p. 97.

Loux discusses is to treat properties and relations as index-bound. Hence, there would be no discrepancy or incompatibility between a thing's having *P at w_i* and its not having *P at w_j* where $w_i \neq w_j$.[31] Then individuals are individuated across indices according to the set of *indexed* properties and relations they possess. This amended version of Leibniz's principle would be that x and y are transindex identical (in a model) just in case they possess all the same *indexed* properties and relations. This is expressible in second-order logic as follows, where x and y are transindex individuals:

$$(\forall x)(\forall y)(x = y \supset (\forall P_i)(P_i x \equiv P_i y))$$

But this way of accounting for transindex identity of individuals in a model is simply a restatement of a feature of the invariant domain semantics, viz., that a transindex individual in D can have different properties from index to index. Then we are still left the problem of determining how an individual can have different attributes from index to index and yet remain the 'same' transindex individual.

So it would seem that we cannot make good metaphysical sense of a semantics where individuals remain invariant across indices and yet can vary in their properties from index to index. It also appears that we are forced to accept some sort of bare particular metaphysics in order to account for transindexical identity of individuals. However, there is at least one further move we could make here. We could say that an individual x at w_i is identical to an individual y at w_j only if x and y share the same *essential* properties. But what is it for a property to be 'essential'?

Both Kripke and Plantinga endorse the following definition of a property P's being essential for an object x, although ultimately neither author

31 Loux (1979), p. 42.

appeals to the notion of 'essential property' as a solution to the supposed problem of transindex identity:[32]

> A property P is essential for an object (individual) x just in case x has P at all indices at which x exists.

Plantinga regards the rider "at which x exists" as important for the following reason: If it is assumed that objects exist at all worlds or indices then supposing existence to be a predicate, it follows that any object exists essentially and hence necessarily.

Of course the rider 'at which x exists' is superfluous for the type of semantics under consideration since D is shared by all members of W. I.e., in the semantics for the $SQC^{=}$ systems, existence of an individual at an index can be understood in terms of membership in D as follows:

x exists at w_i =df. $x \in D$.

It then follows from this definition of 'existence at w_i' that any individual existing at one index will exist at all indices in which case all existents are *necessary* existents (independently of the assumption that existence is a predicate). This is clearly an unpalatable consequence of the metaphysics of an invariant domain semantics for belief logic. But such a consequence can be avoided in a semantics which allows domains to vary across indices.

In a varying domain semantics, an individual x may be a member of the domain of individuals D_i associated with the index w_i and yet x may fail to be a member of D_j associated with the index w_j. Then existence for a varying domain semantics can be defined as follows:

x exists at w_i =df. $x \in D_i$

From this definition of 'existence at w_i' it does *not* follow that if x exists at w_i then x exists *necessarily* (i.e., at all indices) since there could be an

[32] See Kripke (1980), p. 48 and Plantinga (1974), p. 60.

index w_j distinct from w_i such that x is not in D_j. In fact, there are counterpart semantics such that no individual exists at more than one index. We shall consider varying domain semantics for belief logic in the fourth chapter.

In any case, the notion of essential property will not on its own do the work we want it to, viz., to serve as a criterion of individuation of transindexical individuals. This is becuase there will be what Plantinga calls 'trivial' essential properties such as *being self-identical* or *being red or something else* which all objects will possess at all indices.[33] Any object at any index will be red or something else, and hence such a property is essential to every object. Therefore, such shared essential properties will not serve to individuate transindexical individuals. Something more is required.

As Plantinga argues, the something more which is required for an essential property P of an object x to be an *individuating* property is P's being unique to x in the folowing sense: There is no index w_i such that some object y distinct from x (at w_i) has P at w_i.[34] Such an individuating essential property is sometimes in the literature called an 'essence' or an 'individual essence'. More formally,

> A property P is an essence for an object x just in case 1) P is essential for x and 2) for no index w_i is it the case that some object y distinct from x (at w_i) has P at w_i.

A metaphysics of individuals and properties which appeals to the notion of essences may be called an *essentialist* metaphysics. (As we shall see below, although Plantinga seems to subscribe to an essentialist metaphysics, he does not appeal to the doctrine of essences as a solution to the 'problem'

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[33] Plantinga (1974), pp. 60–62.

[34] Plantinga (1974), p. 70.

of transindexical individuation of individuals.) Plantinga suggests that an example of an essence for any object x would be *x's being x* since for no index w_i will it be the case that some object y distinct from x at w_i will possess this property at w_i.[35] More concretely, an essence of Socrates would be *Socraeity* or his *being Socrates*. There will be no w_i where some individual distinct from Socrates at w_i will possess the property of *Socraeity* at w_i.

And so, as an alternative to a bare particular metaphysics we could adopt an essentialist metaphysics for the purpose of accounting for the individuation of individuals across indices. However, philosophers such as Kripke and Plantinga think it is wrongheaded to appeal to essences as providing criteria of individuation given their conceptions of what possible worlds are. We shall first of all consider Plantinga's views on the 'problem' of transindexical identity of individuals.

In *The Nature of Necessity* Plantinga claims that it is mistaken to view possible worlds (or in our parlance, indices) as if they were like the 'actual' world although occupying a different position in logical space. Extending the spatial metaphor further, it is tempting to think that it is possible to locate each world in logical space and then to inspect its inhabitants. The idea that we can inspect the inhabitants of each possible world is consistent with the view that each inhabitant possesses 'empirically manifest' essences by means of which we can distinguish it from other inhabitants. Both Plantinga and Kripke reject the view that empirically manifest essences can serve to individuate transindexical objects given their alternative accounts of what possible worlds are.

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35 Plantinga (1974), pp. 71-72.

Plantinga proposes the following theory of possible worlds: A possible world is a possible state of affairs S which is maximal.[36] We shall now describe what Plantinga means by 'state of affairs' and a state of affair's being 'possible' as well as 'maximal'. Although Plantinga does not provide the reader with a hard and fast definition of 'state of affairs', we could say that a *description* of a state of affairs S will have one of the following forms: either *x's being P* or *x's being* $Ry_1 \ldots y_n$ where P and R are variables ranging over properties and relations respectively. Thus, the following would constitute descriptions of states of affairs: *Nixon's being the former President of the U.S*. and *Agnew's being the President of Yale University*. The former state of affairs is actual because it obtains whereas the latter state of affairs is possible because it currently does not obtain. Thus, a state of affairs S is actual iff it obtains and it is possible otherwise.

Also, Plantinga leaves open the question as to whether propositions and states of affairs can be identified although he does claim that they bear the following intimate relation to one another: S obtains iff *that* S is true. Thus, *Nixon's being a former President* obtains iff *that Nixon is a former President* is true.[37]

Further, a state of affairs S (actual or possible) is maximal just in case for any other state of affairs S', S includes S' or S precludes S'.[38] And, S includes S' just in case it is impossible that S obtains but S' fails to obtain. S precludes S' just in case it is impossible that S and S' both obtain. And so, for Plantinga, a possible world is a possible state of affairs S such that for any other state of affairs S', S either includes or precludes S'. The so-

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[36] Plantinga, pp. 44-45.

[37] Plantinga (1974), pp. 45-46.

[38] Ibid, p. 45.

called actual world is then a maximal actual state of affairs, i.e., a maximal state of affairs which obtains. Now that we have set up Plantinga's metaphysics of possible worlds, we are in a position to examine his way of handling the problem of transindexical identity.

According to Plantinga, if we regard possible worlds not as co-obtaining simultaneously with the 'actual' world in logical space but rather as maximal states of affairs which do *not* obtain, then we shall not be tempted to adopt the picture that inhabitants of possible worlds are on an equal ontological footing with actual individuals such that they can be inspected and distinguished from one another in terms of unique essential properties. Rather, an individual x exists in a maximal possible state of affairs S just in case it is impossible that if S were to obtain, this very individual x would fail to exist.

Then, there is no problem of transworld identity because the existence of an individual x at a possible world is simply a matter of whether or not that very individual x would exist if that particular world were actual. Thus, to borrow Kripke's example, a possible world 'at which' Nixon exists and such that he is a gardener would be a maximal possible state of affairs such that this very individual Nixon would not fail to exist if this maximal state of affairs were to obtain. It is, according to Plantinga, mistaken to think that somehow we can 'inspect' Nixon's essence at the possible world in question to see whether or not this gardener really is Nixon. It is mistaken to think that we can somehow inspect Socrates' essence at a possible world 'where' he is a carpenter to determine whether or not he really is Socrates. Rather, our carpenter is this very person Socrates who would exist if the state of affairs including his being a carpenter were to obtain:

> ...consider the state of affairs consisting in Socrates' being a carpenter, and call this state of affairs 'S'. Does Socrates exist in S? Obviously: had this state of affairs been actual, he would have existed. But is there a problem of identitifying him, picking him out, in S—that is, must we look into S to see which thing is Socrates? Must there be or must we know of some empirically manifest property he has in this and every other state of affairs in which he exists? Surely not.[39]

Thus, Plantinga regards the so-called problem of transworld identity as a pseudo-problem arising from a mistaken view of what possible worlds and their inhabitants are.

As elegant as Plantinga's attempted resolution to the problem of transworld identity seems to be, it does not address this problem for the semantics characterizing the $SQC^{=}$ systems. For one thing, indices in the semantics for the $SQC^{=}$ systems come as it were ready-made such that no one index is designated. I.e., indices are on an equal ontological footing in the sense that no one index is designated as 'actual'. Further, the existence of an individual at an index in the formal semantics for any $SQC^{=}$ system is unproblematic and it need not be spelled out in terms of what would happen if that particular index were 'actual' or were to obtain (whatever that means). Rather, an individual x exists at a member of W in an $SQC^{=}$ model just in case $x \in D$. Granted, there is the unpalatable consequence of the way this semantics is set up that individuals exist necessarily in the sense that they exist at every index.

However, we are still left with the residual problem that x in D could vary in its attributes from index to index and hence we still must make

39 Plantinga (1974), p. 96.

sense of how x can be the 'same' individual at all these indices. And this is where either a bare particular metaphysics or an essentialist metaphysics come into play. I.e., we resort to one of two ploys, viz., we claim that x is a bare particular to which we can tack on any property at any index or we claim that x has an individual essence (or essences) that we use to pick it out from index to index. The upshot of these remarks is that Plantinga's ultimate resolution to the problem of transindexical identity is beside the point in terms of the formal semantics characterizing the $SQC^=$ systems. In *Naming and Necessity*, Saul Kripke offers a solution to the problem of identifying individuals across indices or 'worlds' which is similar to Plantinga's in the sense that Kripke also maintains that the problem of transindex identity is a pseudo-problem based on a misconception of what possible worlds are.[40] According to Kripke, *possible worlds* are not to be thought of as points in logical space which (to carry the spatial analogy further) we locate and subsequently attempt to identify transworld individuals inhabiting these worlds by means of certain uniquely identifying essential properties. He maintains that this 'confused' way of thinking about possible worlds and their inhabitants has its origins in the model theory (which he helped to develop) for quantified modal logic.[41]

Kripke's alternative metaphysical account of possible worlds (indices) is that they are partial counterfactual situations which are stipulated at the so-called actual world - or more neutrally, at some designated world.[42] At the actual or designated world we are given a set of identifiable objects.

[41] Kripke (1980), p. 48.

[41] Kripke (1980), p. 48.

[42] Kripke (1980), p. 44.

In stipulating counterfactual situations, we ask what some of these objects we are given in the actual world would be like if they had different properties.[43]

Thus, there is no need to appeal to unique essential properties or 'essences' in order to identify and individuate objects in these counterfactual situations because we are stipulating situations where these *very objects* in the actual world have different sorts of attributes from the ones they have at the actual world:

> Some properties of an object may be essential to it, that it could not have failed to have them. But these properties are not used to identify the object in another possible world, for such an identification is not needed.
> ...on the contrary, we begin with the objects which we *have*, and can identify, in the actual world. We can then ask whether certain things might have been true of the objects.[44]

For example, one might ask what would have happend to Nixon if he had been a gardener. There is no question of whether Nixon in this counterfactual situation (where he is a gardener) is the same as Nixon in the actual world because we are *stipulating* that this is a situation where this very man (Nixon) is a gardener. And so, there is no problem of transworld identification of objects. This is only a problem if we base our metaphysics of possible worlds on the model theory for quantified modal logic. (But of course, this is exactly the sort of semantics we are working with - and so we are stuck with its metaphysics, so to speak. We shall say more about this below.)

[43] Kripke (1980), p. 53.
[44] ibid., p. 53.

This is not to say that essential properties of individuals play no role in Kripke's metaphysics. They simply don't serve as individuating criteria for transindexical individuals. Presumably, the role which essential properties play in Kripke's metaphysics is that of restricting the kinds of questions we can intelligibly ask about objects at counterfactual situations. For example, it makes no sense to ask what Nixon would have done if he were an automaton. In such a case, we would no longer be talking about Nixon.[45] For Kripke, what counts as a (unique) essential property for at least material objects is their origin.[46] Then in all possible worlds at which Nixon exists, he will have had the same biological parents. However, it *would* make sense to ask what would have happened if the individual Nixon who had such and such parents had been a gardener, a poet, etc. rather than a crooked politician.

Kripke's way of handling the problem of transindex identity, viz., by claiming that partial counterfactual situations are stipulated at the actual world such that 'actual' individuals are said to have alternative sets of properties at these various situations, does not apply to the domain semantics for the $SQC^=$ systems. As was noted in arguing the irrelevance of Plantinga's solution to the problem of transindex identity for the semantics of the $SQC^=$ systems, a model comes *ready-made* with a set W of indices such that no member of W is 'actual' or designated - no member of W is stipulated relative to any other. The members of W are on an equal ontological footing. Further, each member of D exists at every member of W

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[45] Kripke (1980), p. 112.

[46] ibid (1980), pp. 114-115. Another type of essential property which Kripke appeals to in the case of material objects is that such objects are made from the same sort of substance. Thus, it would make no sense to ask whether or not this very wooden table could have been made of ice.

by virtue of its membership in D. This sort of metaphysics is implicit in the semantics we are working with.

Then even if Kripke is correct in his claim that there is no problem of transindexical individuation of individuals in the metaphysics of the semantics of ordinary language, it does not follow that this is still not a problem in the metaphysics of the formal domain semantics which characterizes the various $SQC^{=}$ systems of doxastic logic. Once again, for the domain semantics of the $SQC^{=}$ systems, the question naturally arises as to how a member of D can be identified from one index to the next even though it could vary in many (if not all) of its properties. Then it would seem that one answer to this question is that members of D are bare particulars which are numerically distinct from one another independently of the properties they possess at a given index. Or, another avenue open to us with respect to identifying transindexical individuals is to adopt an essentialist metaphysics where individuals are identified across indices in terms of unique essential properties they possess.

And so, it would seem that the solutions of both Kripke and Plantinga to the problem of transindexical identity are entirely beside the point with respect to the metaphysics of the domain semantics for the $SQC^{=}$ axiom systems.

If we find a domain semantics for the $SQC^{=}$ systems to be metaphysically problematic on the grounds that we are forced to adopt either a bare particular metaphysics or an essentialist metaphysics in order to account for transindexical identity then we may wish to consider the alternative domainless truth-value semantics which also characterizes these systems. The reader will recall that in a TV model, the indexed assignment function

V simply assigns truth values to all the atomic wffs of the language at every member of W thus obviating the need for a domain D of 'individuals'. But since TV models do not contain a set D of individuals, there is no problem of transindex identification of individuals. Thus, we are not forced into accepting an inelegant metaphysics of bare particulars or an equally inelegant essentialist metaphysics, not to mention an ontology which posits only necessary existents.

Furthermore, a truth-value semantics lends itself to a plausible metaphysics of indices. Since the introduction of the notion of 'index' in the characterization of a model for normal doxastic logic in chapter one, we have been mute concerning the *metaphysical* status of indices. As was noted in section 3 of this chapter, an index in a truth-value semantics for any of the $SQC^{=}$ systems can be regarded as a kind of Carnapian state description. I.e *an index in a TV model can be thought of as a set of atomic wffs or their negations such that for any atomic wff α, either α or $\sim\alpha$ is in the set, but not both.* Thus, any TV index is consistent (in the sense of negation consistent) and maximal with respect to atomic wffs and their negations. So in attempting to make some sort of intuitive (and not just model-theoretic) sense out of TV indices, we have exploited the close analogies between these and Carnapian state descriptions, viz., that atomic wffs are true at Carnapian state descriptions by virtue of membership and not by appeal to individuals. And further, quantified wffs are understood substitutionally for Carnapian descriptions. Finally, just as truth at a state description is defined inductively so is truth at an index defined inductively.

This metaphysics of indices can also be applied to the semantics for the sentential doxastic systems discussed in the previous chapter: Given that in the semantics for the sentential normal doxastic systems, the function V is also an indexed assignment to the atomic wffs of the language, indices in models for the sentential systems can also be regarded as sets of atomic wffs or their negations such that for any atomic wff α either it or $\sim\alpha$ is in the set. And these sets are maximal consistent with respect to atomic wffs and their negations.

The metaphysical picture of indices as kinds of state descriptions or as maximal consistent sets of atomic wffs or their negations seems to be relatively unproblematic ontologically speaking, at least if one is not concerned with the status of sentences or sets. However, there is one problem on the ontological front with our Carnapian metaphysics of indices, viz., that in a given model one would expect that at least one index must be 'actual' in the sense that it is not simply a set of wffs. I.e., what sense does it make to say that the 'actual' world is a set of sentences? When an agent holds a set of beliefs at the actual world which is not itself merely a set of sentences, he considers a set of alternative *descriptions* to the world he inhabits and these (since they are not 'actual') will merely be sets of atomic sentences or their negations on the basis of which his contents are true or false at that set. However, the way our semantics is set up, no index in a model is designated as actual and so in Lewis' parlance, actuality is treated as a kind of indexical like 'here' or 'now'. Then there is no reason to exempt any one index in a model from being a state description.

There are two ways out of this bind, viz., we could revoke our identification of indices with state descriptions and instead treat them as semantic primitives in our model theory. I.e., indices are simply undefined elements in a set with respect to which truth in a model is relativized. But then we are no further ahead than we were in the first chapter in terms of having a metaphsyics of indices. Or, we can still put the notion of state description to work by regarding any index in a model as a semantic primitive but at the same time stipulating that with each such primitive is associated a state description in the following sense: The given state description will consist of every atomic wff or its negation assigned '1' by V_M at the index.

Concluding Remarks

And so, having examined the metaphsycial underpinnings of the two types of characteristic semantics for the $SQC^=$ systems, it is apparent that a TV semantics supporting a substitutional reading of the quantifiers is preferable to the domain semantics. This is because the former type of semantics avoids the metaphysical (as well as the ontological) difficulties of the latter by dispensing with domains of so-called individuals.

In the next chapter, we shall consider some of the philosophical difficulties associated with *quantified* belief logic rooted in ordinary language. And in the fourth chapter, we shall consider how the $SQC^=$ systems can be altered on the axiomatic and semantic fronts to accomodate these difficulties, where ultimately a truth-value semantics will be endorsed.

Chapter Three

Problems Associated with Quantified Belief Logic

1. Identifying the Source of Trouble: Substitutivity or Disquotation?

As was noted in the previous chapter, any quantified SQC$^{=}$ system of belief logic has as part of its axiomatic base the schema K and the rule of inference RB from which it follows that any such system inherits the problem of deduction. In addition, certain SQC$^{=}$ thesis-schemata and corresponding rules of inference concerned with the connection between the *identity symbol* and the belief operator on the one hand and the connection between the *existential quantifier* and the belief operator on the other have been some cause for consternation in the literature when we consider them qua principles of belief attribution. In this section, we shall consider various schemata and rules of inference of the former sort and in the next section we shall consider various schemata and rules of inference of the latter sort.

First, we shall consider the schema $(\mathbf{B}\alpha(t_1/v) \;\&\; t_1 = t_2) \supset \mathbf{B}\alpha(t_2/v)$ which is simply the doxastic version of the SQC$^{=}$ axiom schema $(\alpha(t_1/v) \;\&\; t_1 = t_2) \supset \alpha(t_2/v)$. Using this schema as well as modus ponens, we can derive the following rule of inference: $\mathbf{B}\alpha(t_1/v),\ t_1 = t_2 \longrightarrow \mathbf{B}\alpha(t_2/v)$. More general versions of this schema and its corresponding inference rule would simply be $(\alpha(t_1/v) \;\&\; t_1 = t_2) \supset \alpha(t_2/v)$ and $\alpha(t_1/v),\ t_1 = t_2 \longrightarrow \alpha(t_2/v)$ where it is stipulated that $\alpha(t_1/v)$ and $\alpha(t_2/v)$ may contain the occurrence of a belief operator '**B**' within whose scope t_1 in and t_2 occur. For simplicity of exposition, we shall focus on the less general versions of this thesis-schema and inference rule. These rules of inference and the related schemata assert the principle that co-referential terms are inter-substitutible in belief contexts, which we shall hereafter call the 'substitutivity principle'.

When we come to consider ordinary language examples of the derived inference rule, the principle of belief attribution it asserts seems to break down. For example, from Jones' believing that Cicero was an orator and given the truth of the identity sentence 'Tully is Cicero', it is, according to the substitutivity principle permissible to infer that Jones believes that Tully was an orator. But it could be objected that Jones may assent to the claim that Cicero was an orator while witholding assent from the claim that Tully was an orator regardless of the truth of the identity sentence 'Tully is Cicero'.[1] Then assuming the *strengthened* diquotation principle discussed in the first chapter, viz., that x's sincere assent to p is necessary and sufficient for x's believing that p, it would follow that Jones believes that Cicero was an orator while not believing that Tully was an orator. Thus, we have constructed an apparent counterexample to the substitutivity principle assuming the soundness of the strengthened disquotation principle.

A rule of inference derivable in any $SQC^{=}$ system containing the schema D which is related to the above mentioned rule is $\mathbf{B}\alpha\,(t_1/v),\ t_1 = t_2 \longrightarrow {\sim}\mathbf{B}{\sim}\alpha\,(t_2/v)$. This rule is derivable using the schema $(\mathbf{B}\alpha\,(t_1/v)\ \&\ t_1 = t_2)$

[1] One of the first to impugn the substitutivity principle was Frege in 'Sense and Reference'. And over the years the apparent failure of this principle has been discussed by a number of philosophers including Quine (1960), Hintikka (1962, 1969), Sellars (1969) and as we shall soon see, Kripke (1979).

$\supset \sim\mathbf{B}\sim\alpha\,(t_2/v)$ and modus ponens. In turn, any instance of this schema is derivable in any SQC$^=$ system containing D as follows:

1. $\vdash (\mathbf{B}\alpha\,(t_1/v)\ \&\ t_1 = t_2) \supset \mathbf{B}\alpha\,(t_2/v)$
2. $\mathbf{B}\alpha\,(t_2/v) \supset \mathbf{P_B}\alpha\,(t_2/v)$ — instance of D.
3. $(\mathbf{B}\alpha\,(t_1/v)\ \&\ t_1 = t_2) \supset \mathbf{P_B}\alpha\,(t_2/v)$ — PC[2]
4. $(\mathbf{B}\alpha\,(t_1/v)\ \&\ t_1 = t_2) \supset \sim\mathbf{B}\sim\alpha\,(t_2/v)$ — df. $\mathbf{P_B}$ in terms of $\mathbf{B}$

Intuitively this derived thesis-schema and the related rule of inference assert the principle that it is impossible that x believes that $\alpha\,(t_1/v)$ and that the identity $t_1 = t_2$ is true and yet x believes that $\alpha\,(t_2/v)$ is false. A more concrete example of what this rule of inference permits is the following: Suppose that Jones believes that Cicero was an orator. Given that Tully is Cicero, it is false that Jones believes that Tully was not an orator.

However, using the disquotation principle, we can construct an informal counterexample to this variant of the substitutivity principle asserted by the SQC$^=$ + D derived rule $\mathbf{B}\alpha\,(t_1/v),\ t_1 = t_2 \longrightarrow \sim\mathbf{B}\sim\alpha\,(t_2/v)$ as follows: Suppose Jones sincerely assents to the claim that Cicero was an orator, but not realizing that Cicero and Tully are one and the same person he also sincerely assents to the claim that Tully was not an orator. Hence, by the disquotation principle, it follows that Jones believes that Cicero was an orator although he also believes that Tully was not an orator.

In fact, our example is apparently a case where Jones holds contradictory beliefs, supposing that our original version of the substitutivity principle, $\mathbf{B}\alpha\,(t_1/v),\ t_1 = t_2 \longrightarrow \mathbf{B}\alpha\,(t_2/v)$ applies. I.e., given the disquotation principle, since Jones has assented to the claim that Tully was not an orator we can conclude that Jones *believes* that Tully was not an

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[2] The PC schema used here is $((\alpha \supset \beta)\ \&\ (\beta \supset \gamma)) \supset (\alpha \supset \gamma)$ which is the implicational version of the hypothetical syllogism, along with modus ponens.

orator. Further, given our original version of the substitutivity principle, we can also attribute to Jones the belief that *Cicero* was not an orator. Then since we have also concluded that Jones believes that Cicero was an orator, it follows that Jones believes that Cicero was an orator and he believes that Cicero was not an orator. However, since the substitutivity principle $\mathbf{B}\alpha\,(t_1/v),\ t_1 = t_2 \longrightarrow \mathbf{B}\alpha\,(t_2/v)$ has been called into question in our first counterexample a few pages back, it is hasty in this case to infer that Jones has contradictory beliefs. Thus, minimally, we shall construe this second case merely as a situation where the substituvity principle $\mathbf{B}\alpha\,(t_1/v),\ t_1 = t_2 \longrightarrow {\sim}\mathbf{B}{\sim}\alpha\,(t_2/v)$ for D systems has been violated but not necessarily as a case where an agent has contradictory beliefs.

For this second case just considered, Kripke would maintain that it can be construed as a sort of *reductio* argument against the substitutivity principle as expressed by the SQC$^=$ rule $\mathbf{B}\alpha\,(t_1/v),\ t_1 = t_2 \longrightarrow \mathbf{B}\alpha\,(t_2/v)$[3]. Kripke argues that it would be unfair to attribute contradictory beliefs in this case to Jones since "even if he is a brilliant logician, he need not be able to deduce that at least one of his beliefs must be in error."[4] Hence, by assuming the substitutivity priniciple and the disquotation principle, we have constructed a case where we are forced to conclude that an agent holds apparently contradictory beliefs. But this is absurd since we would not want to attribute contradictory beliefs to Jones in this example.

However, Kripke's claim that it is an *absurd* consequence of assuming the disquotation principle and the substitutivity principle represented by the rule $\mathbf{B}\alpha\,(t_1/v),\ t_1 = t_2 \longrightarrow \mathbf{B}\alpha\,(t_2/v)$ (or its more general version, $\alpha\,(t_1/v),\ t_1 = t_2 \longrightarrow \alpha\,(t_2/v)$) that Jones holds contradictory beliefs is

[3] Kripke (1979), p. 251.

[4] Kripke (1979), p. 251.

open to question. As we have claimed in the first chapter, to say that Jones in such a case holds contradictory beliefs is *not* being unfair to the facts of the situation *even if* we assume that Jones' possessing the requisite logical acumen does not enable him to detect the inconsistency of his beliefs by logic alone (but only with the additional information that Tully and Cicero are one and the same person). If anything, the undetectability of inconsistencies in one's beliefs constitutes good grounds for saying that agents *can* sometimes hold contradictory beliefs because an agent with a high degree of logical acumen would not believe a pair of outright contradictory statements *unless* the contradictoriness of this pair is in some sense 'hidden'. Hinitikka for example has tried to make model-theoretic sense of this type of situation by allowing as epistemically accessible to a given world worlds whose descriptions bear hidden inconsistencies.[5]

If we are right here, then Kripke's claim that our second example constitutes a reductio ad absurdum to the substitutivity principle as represented by the SQC$^=$ rule $\mathbf{B}\alpha\,(t_1/v),\ t_1 = t_2 \longrightarrow \mathbf{B}\alpha\,(t_2/v)$ misses the mark. However, the second case which we described at the beginning of this section does seem to constitute a *counterexample* to the SQC$^=$+D rule $\mathbf{B}\alpha\,(t_1/v)$, $t_1 = t_2 \longrightarrow {\sim}\mathbf{B}{\sim}\alpha\,(t_2/v)$. I.e., it is a case where the principle of belief attribution expressed by this inference rule is violated. As we have argued, we shall refrain from labelling it a situation where an agent holds contradictory beliefs since the other version of the substitutivity principle which itself has been called into question would need to be assumed.

At this point, Kripke would be quick to point out that what *may* be at fault in our apparent counterexamples to the substitutivity principles is the disquotation principle. I.e., we are being too hasty in indicting the two

[5] See Hintikka (1975).

substitutivity principles.

To support this claim, Kripke attempts to construct two cases which assume the disquotation principle but which do not assume the substitutivity principle (unlike his earlier example) and such that an absurd consequence is generated. However, we shall argue that there are *no* absurdities generated by his example and hence there is no reason to suspect the plausibility of the disquotation principle. To be fair, Kripke makes it clear that he is not in these cases trying to vindicate the substitutivity principle nor is he trying to indict the disquotation principle. He merely wishes to show that it is hasty to indict the former principle in apparent counterexamples to it which make use of the disquotation principle.[6]

Suppose for the sake of argument that Kripke's construal of the case where Jones believes that Cicero was an orator and he believes that Tully, i.e., Cicero was not an orator *on the strength of the disquotation principle* constitutes an apparent reductio argument against the substitutivity principle as asserted by the rule $B\alpha(t_1/v),\ t_1 = t_2 \longrightarrow B\alpha(t_2/v)$. Then, says Kripke, it is hasty to conclude that it is the substitutivity principle which is at fault. An absurdity has been generated by assuming the truth of the substitutivity principle *and* the disquotation principle. Thus, the most we can conclude is that *either* the substitutivity principle *or* the disquotation principle *or both* are at fault. Kripke likens this situation to the situation in topology where from a given hypothesis we derive an absurdity but only with the help of some set-theoretic axiom-schema. Then all we can conclude in this case is that either the initial hypothesis is at fault or that the set-theoretic axiom schema is at fault.[7]

Kripke first of all proposes the 'puzzling Pierre' case which we have

[6] Kripke (1979), p. 269.

[7] Kripke (1979), pp. 253-254.

already discussed in the first chapter in connection with the schema $\sim(B\alpha \,\&\, B\sim\alpha)$ which says that agents do not hold contradictory beliefs. The reader will recall that in the puzzling Pierre example, Pierre who is a monolingual Parisian assents to the claim that Londres est jolie and given the disquotation principle the sentence "Pierre croit que Londres est jolie" is true in French. Then, applying the translation principle (viz., that if p expresses a truth in L then its translation p' expresses a truth in L') we can conclude that the sentence "Pierre believes that London is pretty" is true in English. Further, suppose that Pierre ends up in some shabby section of London where he learns to speak English. He retains his assent to the claims he assented to while in Paris including the claim that Londres est jolie. Seeing the shabbiness of his new surroundings and not realizing that his new environment is the Londres of his dreams, he assents to the claim that London is not pretty from which it follows that he believes that London is not pretty. Then it follows that Pierre believes that London is pretty and he believes that London is not pretty. Or does he?

Kripke then argues that no matter how we construe this situation, we are led to an absurdity. For example, on one construal we could claim that Pierre did not have contradictory beliefs on the grounds that once he moved to London and learned to speak English he gave up the belief that London is pretty.[8] But, says Kripke, this is unacceptable because part of the story is that Pierre still assents to every claim he assented to in French. Then we have no grounds for saying that Pierre gave up his belief that London is pretty. Also, supposing that we did not know about Pierre's move to London and his acquisition of English then "on the basis of his normal command of French we would be *forced* to conclude that he *still* believes that London is

[8] Kripke (1979), p. 256.

pretty."[9] On the other hand, if we are forced to conclude that Pierre holds contradictory beliefs then we are being unfair to the facts of the situation since Pierre "lacks information, not logical acumen. He cannot be convicted of inconsistency: to do so is incorrect."[10]

And so, concludes Kripke, the puzzling Pierre case is a paradox because there is simply no way of telling just what Pierre believes. Any answer to this question leads to an absurdity.[11] However, as was argued in the first chapter, Kripke can be taken to task on his claim that attributing contradictory beliefs to Pierre is an unacceptable construal of the situation. He seems to assume without argument that an agent cannot be charged with holding contradictory beliefs unless he is, by means of his logical acumen alone, able to discover this. And this assumption is at best dubious. Then perhaps the puzzling Pierre case is not paradoxical after all because there is no reason why we cannot attribute to Pierre contradictory beliefs which at the very least he assents to in different linguistic contexts.

The 'puzzling Pierre' case admittedly makes use of not just the disquotation principle but also the translation principle. Thus, Kripke constructs a second case which does not depend on the translation principle but merely on the disquotation principle alone.[12] This second case can be characterized as follows: Peter learns about a famous pianist (who unbeknownst to Peter was also a famous politician) named 'Paderewski'. Peter then assents to the claim that Paderewski had musical talent and from the disquotational principle it follows that Peter *believes* that Paderewski had musical talent. In another context (but in the same language) Peter who assumes

[9] Kripke (1979), p. 256.

[10] ibid, p. 257.

[11] ibid., p. 259.

[12] ibid., pp. 265-266.

that no politicians are musically talented hears about a famous politician who was also named Paderewski and who unbeknownst to Peter was one and the same person as Paderewski the pianist. Given his belief that no politicians are musically talented, Peter assents to the claim that Paderewski had no musical talent and hence by the disquotation principle it follows that Peter *believes* that Paderewski had no musical talent. Then by two applications of the disquotation principle it would seem that Peter in one context believes that Paderewski had musical talent and in another context Peter believes that Paderewski had no musical talent. Does Peter in this case hold contradictory beliefs?

Kripke wants to argue that parallel to the puzzling Pierre case, no matter how we construe this situation we are led into absurdities. For example, one may wish to argue that once Peter has learnt about Paderewski the politician who (Peter assumes) had no musical talent then Peter will no longer believe that Paderewski had musical talent. But, as with the puzzling Pierre case, Peter presumably would not abandon assent to the claim that Paderewski had musical talent supposing that he does not realize that Paderewski the pianist and Paderewski the politician were one and the same person. Thus he still believes that Paderewski had musical talent. But, Kripke would then argue that if we are forced to admit that Peter holds contradictory beliefs then this is unfair to the facts of the situation since Peter is unable to determine that the contents of his alleged beliefs are contradictory *without* the additional information that Paderewski the pianist and Paderewski the politician were one and the same person. Kripke is assuming here that an agent can be charged with inconsistencies in his/ her beliefs only if it is possible for that agent to notice said inconsistencies without any additional information (such as the knowledge that some contingent identity obtains). In short, the agent in such a case cannot be held logically responsible for what he has assented to.

However, as was suggested with reference to the puzzling Pierre case, this assumption is dubious. It was argued in chapter one that it is in just those sorts of cases where the agent cannot determine without additional information that two sentences to which he has assented are contradictories that we would be most likely to attribute to the agent contradictory beliefs. Thus, suppose Peter had somehow found out that Paderewski the pianist and Paderewski the politician were one and the same person. Then given as premises the contents of his beliefs that Paderewski the politician had no musical talent and that Paderewski the pianist had musical talent, he would be in a position to infer both that Paderewski the pianist (and politician) had musical talent and that Paderewski the pianist (and politician) had no musical talent. He would thus be in a position to see that his beliefs were contradictory. At this point, if we are charitable, we would suppose that Peter will come to abandon assent to the claim that Paderewski had no musical talent or to the claim that he had talent.

On the other hand, if Peter did not have access to the information that Paderewski the pianist and Paderewski the politician were one and the same person, then he would not be in a position to draw the inference just alluded to. In such a case, it would seem natural to attribute to Peter contradictory beliefs, viz., that Paderewski had musical talent and that Paderewski did not have musical talent, since only in the presence of the requisite information would he abandon assent to one or the other claim.

And so, the upshot of our discussion of Kripke's puzzling Pierre case and of his Paderewski example is that neither of these so-called puzzles are paradoxical because in both cases there is a construal which does *not* lead to absurdities, viz., that the agent in question holds contradictory beliefs. It could be argued that if we were to attribute to agents contradictory beliefs then we would be forced to attribute to such agents *every* belief since a contradiction logically implies everything. However, it was suggested in chapter one that this reductio against the intelligibility of attributing to agents contradictory beliefs relies on two principles of belief attribution. The first such principle is that agents always conjoin their beliefs (so that if Pierre believes that London is pretty and Pierre believes that London is not pretty then he believes that London is pretty and not pretty). The second principle is that agents are logically omnidoxastic (so that if Pierre believes that London is pretty and not pretty then he believes that Q, where Q is any proposition whatsoever). If either of these principles are abandoned, then the reductio just outlined does not go through. Our ultimate strategy in chapter six will be to abandon the 'adjunction principle' (the first of the two principles).[13]

Supposing the intelligibility of agents being able to hold contradictory beliefs without thereby conjoining them, it would seem that Kripke's disquotation principle has survived his two 'puzzles' and so he has given us no grounds for doubting this principle after all. Then in our alleged counterexamples to the substitutivity principle discussed above, it is not at all unreasonable to suspect that the substitutivity principle is at fault.

To recap our discussion, Kripke's main argument for his claim that it

[13] This line of reasoning is pursued by Ruth Marcus in Marcus (1981).

is hasty to indict the substitutivity principle in apparent counterexamples to this principle (which also assume some principle of belief attribution or other, such as the disquotation principle) can be roughly characterized as follows:

Premise 1: By assuming the disquotation principle alone the same kind of absurdities can be 'derived' that are derived by assuming the disquotation and substitutivity principles together.

Premise 2: If Premise 1 is true then it is hasty to conclude that the substitutivity principle is at fault in the alleged counterexamples to it.

Conclusion: It is hasty to conclude that the substitutivity principle is at fault in the allieged counterexamples proposed against it.

As we have argued in this section, the first premise in this argument is questionable. Kripke's two puzzles which employ the disquotation principle and which are not obviously relevant to the truth of the substitutivity principle *do not* generate any absurdities, supposing that it is intelligible that agents are capable of holding contradictory beliefs in different contexts (where a 'context' can be temporal, linguistic or locational). We shall make sense of this claim in chapter six vis a vis Stalnaker's notion that agents can be in more than one 'belief state'.

And so, Kripke's argument in favour of the claim that it is hasty to indict the substitutivity principle such as in the alleged counterexamples we considered earlier does not seem terribly compelling. In any case, what these alleged counterexamples do show is the following: Supposing that the disquotation principle (or any principle of belief attribution we employ) is sound - and Kripke's puzzling Pierre and his Paderewski cases give us no reasons for thinking that it is unsound - then we have constructed cases where the substitutivity principle fails. These are cases where there is a clash between two sorts of principles of belief attribution, viz., behavioural principles on the one hand (i.e., disquotation) and logical principles on the other (i.e., substitutivity).

2. Is Existential Generalization Believable?

Another SQC$^=$ schema which we shall discuss in this section and which has generated a fair amount of contraversy in the literature qua principle of belief attribution is the *doxastic* version of the axiom-schema $\alpha(t/v) \supset (\exists v)\alpha$, viz., $B\alpha(t/v) \supset (\exists v)B\alpha$. Given this schema and modus ponens the rule of inference, $B\alpha(t/v) \longrightarrow (\exists v)B\alpha$ can be derived in any SQC$^=$ system. More general versions of this schema and its corresponding rule of inference are $\alpha(t/v) \supset (\exists v)\alpha$ and $\alpha(t/v) \longrightarrow (\exists v)\alpha$ where it is stipulated that t occuring in $\alpha(t/v)$ occurs in the scope of a belief operator. Once more, for simplicity of exposition, we shall primarily concern ourselves with the less general version of this schema and inference rule.

These schemata and the corresponding derived rules of inference express the principle that it is permissible to existentially generalize with respect to the occurrence of a constant t in the scope of a belief operator *outside of* that operator. In short, it is permissible to existentially quantify *into* belief contexts.[14] An ordinary language example of an inference which accords with the derived rule permitting quantification into belief constructions is as follows: Suppose that Jones believes that Tully was a Roman orator. Then from this we can infer that *there is* some person

[14] The term 'quantifying in' was first coined by Quine in Quine (1956), p. 103 appearing in Linsky (ed.) 1979.

such that Jones believed that this very person was a Roman orator - assuming an objectual reading of the existential quantifier. Assuming a substitutional reading of the quantifiers, we can infer from Jones' belief that Tully was a Roman orator that *some* substitution instance of 'Jones believes that v was a Roman orator' is true. More conspicuously, we can infer that $(\exists v)$(Jones believes that v is a Roman orator.) However, if we treat definite descriptions as names or singular terms then we run into trouble as we shall see in the next paragraph, but *only if* we assume an objectual rather than a substitutional interpretation of the quantifiers in the corresponding semantics.

The following constitutes an informal counterexample to the generalization rule mentioned above: Suppose that Jones believes that the next Prime Minister of Canada (whoever he/she is) will attempt to balance the budget. But there may be no individual that Jones has in mind in the sense that if questioned he could name no specific person. And so it seems odd to say that *there is* some person such that Jones believes that that person will be either attempt to balance the budget. More conspicuously, it is wrong to infer from Jones' belief that the next Prime Minister of Canada (whoever he/she is) will attempt to balance the budget that $(\exists v)$(Jones believes that v will attempt to balance the budget). And so, it would seem that we have a case where a sentence of the form x believes that $\alpha(t/v)$ is true and yet it is false that $(\exists v)$(x believes that α), or symbolically, $(\exists v)B\alpha$ - supposing once more an objectual reading of '$\exists$'. This sort of counterexample to any generalization rule allowing quantification into belief constructions is discussed by a number of authors including Quine[15], Hintikka[16], Kaplan[17]

[15] Quine (1956, 1960).

[16] Hintikka (1962).

and Stich[18].

The above counterexample to the generalization rule assumes that the 'the next P.M. of Canada' can for the purposes of the SQC= language be treated as a constant and for the purposes of ordinary language as a proper name. However, this assumption could be criticized on the grounds that the expression 'the next P.M. of Canada' is more aptly treated as a *definite description*. Thus, its rough translation in the language of the SQC= systems would be $(\exists x)(Px \,\&\, (\forall y)(Py \supset y = x))$ rather than simply treating it as a constant, c. However, even if we grant that the expression 'the next P.M. of Canada' is best treated as a definite description, we are still faced with an ambiguity much as we would be if we were to treat it in ordinary language as a proper name (and hence as a constant with respect to the language for the SQC= systems).[19] Thus, as we shall argue, treating the expression 'the next P.M. of Canada' as a definite description does not circumvent the problem that we cannot infer that there is a particular person whom Jones believes will attempt to balance the budget from his belief that the next Prime Minister whoever he/she may be will attempt to balance the budget.

The sort of ambiguity which we have in mind was first alluded to by Keith Donnellan in his 1966 article 'Reference and Definite Descriptions'. Donnellan in this article argues that definite descriptions are ambiguous in the sense that they can be used by a speaker (or for that matter a believer) either *referentially* or *attributively*.

To illustrate Donnellan's notion of the referential use of definite descriptions, if we were using the expression 'the next P.M. of Canada' *refer-*

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[17] Kaplan (1969).

[18] Stich (1983).

[19] This point is made by Hintikka in Hintikka (1967), p. 47.

entially then we would have a particular individual in mind although success in referring to the individual in question does not depend on his/her uniquely satisfying the description. For example, at the 1990 Liberal Leadership convention, the newly appointed leader who is a hopeful for the position of Prime Minister may be introduced as 'the next Prime Minister of Canada'. Then even though the new leader of the Liberal party may in fact lose the election, the announcer has succeeded in picking out or referring to the individual in question. Thus, what is characteristic of the speaker's using a definite description referentially is that it will "enable his audience to pick out whom or what he is talking about and states something about that person or thing."[20]

On the other hand, in our counterexample to the rule permitting generalization *into* belief constructions, if we treat the expression 'the next P.M. of Canada' as a definite description then this description is being used *attributively* in the sense that Jones intends that there is exactly one individual who fits that description although he *may not* have the slightest idea who that individual might be. Thus, Jones believes that there is exactly one individual *whoever he may be* who is such that he/she will be the next Prime Minister of Canada and such that he/she will attempt to balance the budget. Thus, what is characteristic of a description being used by a speaker (or believer) attributively is that the speaker or believer "states something about whoever or whatever is the so-and-so."[21]

And so, even if we wish to treat the expression 'the next P.M. of Canada' as a definite description in our alleged counterexample to the $SQC^{=}$ rule permitting quantification into belief contexts, it is clear that Jones is

[20] Donnellan (1966), reprinted in Schwartz (1977), p. 46.

[21] ibid, p. 46.

using this description attributively rather than referentially. This is because he believes that the next P.M. *whoever he/she may be* will attempt to balance the budget.[22] And of course there will only be one such person whoever he may be that satisfies his description. But then from Jones' belief as so characterized, it would be wrong to infer that there is some particular individual such that Jones believes that that person will attempt to balance the budget. And hence, treating 'the next P.M. of Canada' as a definite description rather than as a name does not circumvent our apparent counterexample to the rule permitting existential generalization across belief constructions.

It is worth noting that the $SQC^{=}$ rule allowing quantification *into* belief constructions, viz., $\mathbf{B}\alpha\,(t/v) \longrightarrow (\exists v)\mathbf{B}\alpha$ is not to be confused with the $SQC^{=}$ rule $\mathbf{B}\alpha\,(t/v) \longrightarrow \mathbf{B}(\exists v)\alpha$, which informally says that it is permitted to existentially generalize with respect to the occurrence of a term t in the scope of a belief operator *inside* the belief operator. In the literature this rule is qua principle of belief attribution is regarded as relatively unproblematic.[23] It is derivable in any $SQC^{=}$ system as follows:

1.	$\mathbf{B}\alpha\,(t/v)$	hyp.
2.	$\vdash \alpha\,(t/v) \supset (\exists v)\alpha$	
3.	$\mathbf{B}(\alpha\,(t/v) \supset (\exists v)\alpha)$	2, RB
4.	$\mathbf{B}\alpha\,(t/v) \supset \mathbf{B}(\exists v)\alpha$	3, K, modus ponens
5.	$\mathbf{B}(\exists v)\alpha$	1,4 modus ponens

To see that this rule is relatively unproblematic - even if we construe the quantifiers objectually - consider once again our example of Jones who believes that the next P.M. of Canada, whoever he/she may be will attempt

22 This is not to say that Jones has no individual in mind. He may or he may not, but nonetheless in this case he is using the description attributively.

23 For example, see Hintikka's comments in Hintikka (1962), p. 141.

to balance the budget. Although it is wrong to infer that there is some individual such that Jones believes that that individual will attempt to balance the budget, we *can* infer that even though Jones may have no one person in mind, he believes that some person or other is such that he/she will attempt to balance the budget.

It would seem then that there is an ordinary language distinction between constructions of the form 'x believes that $(\exists v)\alpha$' and $(\exists v)$(x believes that α). In our examples discussed above, there is a distinction between Jones' believing that there is at least one person who will attempt to balance the budget on the one hand and there being some person such that Jones believes this very person will attempt to balance the budget. The former construction (unlike the latter) is inferable from the sentence 'Jones believes that the next P.M. of Canada (whoever he/she is) will attempt to balance the budget' because it is not necessary that Jones have anyone in mind for him to believe that there is some person or other who will attempt to balance the budget.

Assuming an analogy between belief (or 'doxastic necessity') and alethic necessity, the distinction which we are discussing is analogous to the de re/de dicto distinction sometimes made for *alethic* necessity contexts. Thus, the construction 'it is necessary that $(\exists v)\alpha$' is said to be *de dicto* since the necessity operator has as its scope the content sentence '$(\exists v)\alpha$'. An example of this construction would be 'it is necessary that primes exist' which asserts that the content sentence (i.e., the 'dictum') 'Primes exist' is necessary, which is to say that some number system is such that at least one of its elements *must be* prime. On the other hand, the construction '$(\exists v)$(it is necessary that α)' is said to be *de re* since the necessity operator occurs *within* the scope of the quantifier. An example of this would be 'there exists at least one number such that *necessarily* it is prime'. This sentence could also be read as saying that there is at least one number that is *essentially* prime and clearly this is distinct from the claim that *necessarily*, primes exist.

Because allowing quantification into necessity contexts seems to commit us to some sort of essentialist metaphysics, Quine has taken exception to this sort of construction.[24] We are of course explaining this distinction in terms of an objectual reading of the quantifiers. This distinction can also be made sense of substitutionally as follows: To assert that at least one substitution instance of '*necessarily* x is a prime' is true is not the same thing as asserting that *necessarily* at least one substitution instance of 'x is a prime' is true. And in fact, Quine's charge of essentialism would not apply to the distinction made in these terms - since no mention is made of any objects possessing 'essential' properties.

Analogous to the de re/de dicto distincition discussed above for alethic modal contexts, there seems to be grounds for making this distinction for propositional attitude contexts as we have seen. Thus, if we treat 'x believes that' as analogous to 'it is necessary that' then the construction 'x believes that $(\exists v)\alpha$' is *de dicto* and '$(\exists v)$(x believes that α)' is *de re*. And so, if we adapt our example for alethic modal contexts to belief contexts, the sentence 'Jones believes that primes exist' would be de dicto and the sentence 'There exists at least one number such that Jones believes that that number is prime' is de re.[25] And intuitively, Jones may believe that

24 For example, see Quine's discussion of this point in Quine (1960, 1961).

25 This apparent distinction has been alluded to by Quine (1956), p. 102, Hintikka (1962), p. 142 and by Kaplan (1977), p. 116. It has been questioned by Stich (1983). We shall discuss Stich's views

there are primes without believing of any particular number that it is prime. He may not even be able to identify primes. Perhaps he has simply accepted on authority that such numbers exist without knowing what they are.

If this intuitive distinction between the de dicto construction 'x believes that $(\exists v)\alpha$' and the de re locution '$(\exists v)$(x believes that α)' is correct for ordinary language then it would seem desirable that the two constructions be distinct for the $SQC^{=}$ formal systems of doxastic logic. And this is tantamount to saying that we would not want the following biconditional to hold for the $SQC^{=}$ systems, viz., $B(\exists v)\alpha \equiv (\exists v)B\alpha$ such that the existential quantifier is non-vacuous and where '$B(\exists v)\alpha$' represents the *de dicto* locution 'x believes that $(\exists v)\alpha$' and where $(\exists v)B\alpha$ represents the *de re* locution '$(\exists v)$(x believes that α)'. And in fact, this equivalence does *not* hold for the $SQC^{=}$ systems as will be shown presently.

It will be shown that the biconditional schema $B(\exists v)\alpha \equiv (\exists v)B\alpha$ does not hold for the $SQC^{=}$ systems of doxastic logic by showing that one half of this biconditional schema, viz., $B(\exists v)\alpha \supset (\exists v)B\alpha$ does not hold for any of these systems. Intuitively, this conditional says that belief de dicto logically implies belief de re. This seems intuitively unacceptable if we consider the following simple instance of this conditional schema, $B(\exists x)Fx \supset (\exists x)BFx$. Informally, if we let 'F' stand for 'prime' then this schema says that if x believes that primes exist then there is some number such that x believes that it is prime. But as we have seen in our earlier example, it is possible that Jones believes that primes exist without it being the case that there is any one number such that he believes that it is prime, especially in the case where he accepts the claim that primes exist on authority alone.

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on the matter below.

Further, even if we make the de re/de dicto distinction substitutionally, it is still not correct to endorse as a thesis any instance of $B(\exists x)Fx \supset (\exists x)BFx$. Thus, if Jones believes that *at least one substitution instance of v is a prime is true*, it presumably would not follow that at least one substitution instance of "Jones believes that *v is a prime*" is true. (I.e., he may still fail to believe that 'v is a prime' is true for some one value of v.) However, although the schemata $B(\exists v)\alpha$ and $(\exists v)B\alpha$ are distinct even on a substitutional reading of the existential quantifier, it is still the case that we are warranted on such a reading in inferring both from $B\alpha\,(t/v)$. Thus, from 'Jones believes that 3 is prime' it follows both that 'Some substitution instance of "Jones believes that *v is prime*" is true' and also that 'Jones believes that *some substitution instance of v is prime is true*'.

We shall now construct a formal $SQC^{=}$ countermodel to $B(\exists x)Fx \supset (\exists x)BFx$ thus invalidating the schema $B(\exists v)\alpha \supset (\exists v)B\alpha$. As was noted in the previous chapter, there are two sorts of characteristic semantics for the $SQC^{=}$ systems, viz., a domain semantics which lends itself to an objectual reading of the quantifiers and a truth-value semantics which dispenses with individuals and which lends itself to a substitutional reading of the quantifiers. We shall show that the schema $B(\exists v)\alpha \supset (\exists v)B\alpha$ is invalid in both types of semantics and given our completeness results it follows that not all instances of this schema are provabvle in the $SQC^{=}$ systems.

The reader will recall from chapter two that an $SQC^{=}$ *model structure* for a domain semantics is a triple $\langle W,R,D\rangle$ where W is a non-empty set of indices, R is a 2-place 'accessibility' relation ranging over members of W and D is a non-empty set of so-called individuals. An $SQC^{=}$ model is a 4-tuple $\langle W,R,D,V\rangle$ such that $\langle W,R,D\rangle$ is an $SQC^{=}$ model structure and V is an assignment function which to each constant and to each variable assigns exactly one member of D and which to each n-place predicate variable assigns a set of n + 1-tuples, the first n members being elements of D and the n + 1st member being an index in W.

Then consider the $SQC^=$ model $M = \langle W,R,D,V \rangle$ such that $W = \{w_1, w_2, w_3\}$, $D = \{d_1,d_2\}$, $\{\langle w_1,w_2 \rangle, \langle w_1,w_3 \rangle\} \subseteq R$ and $V(t) = d_1$, $V(F) = \{\langle d_1, w_2 \rangle, \langle d_2,w_3 \rangle\}$. Let M' and M" be models based on the same model structure as M such that $V(t) = d_1$ and $V(t) = d_2$. So $V_{M'}(Ft,w_2) = 1$, $V_{M'}(Ft, w_3) = 0$, $V_{M''}(Ft, w_2) = 0$ and $V_{M''}(Ft, w_3) = 1$. It follows that $V_M((\exists x)Fx, w_2) = V_M((\exists x)Fx, w_3) = 1$ and hence $V_M(\mathbf{B}(\exists x)Fx, w_1) = 1$. Further, since there are two individuals in D, then there will be no model M' such that $V_{M'}(Ft, w_2) = V_{M'}(Ft, w_3)$ and therefore $V_M(\mathbf{B}Ft,w_1)$ will be 0 for any model M' based on the same model structure as M and differing from M (if at all) in terms of what V assigns to arbitrary t. It therefore follows that $V_M((\exists x)\mathbf{B}Fx,w_1) = 0$. Then we have constructed a model M such that $V_M(\mathbf{B}(\exists x)Fx,w_1)$ is 1 but $V_M((\exists x)\mathbf{B}Fx,w_1)$ is 0 which therefore invalidates the conditional $\mathbf{B}(\exists x)Fx \supset (\exists x)\mathbf{B}Fx$. We shall next show that this conditional is invalidated in a truth-value (TV) semantics.

An $SQC^=$ model for a TV semantics is an ordered triple $\langle W,R,V \rangle$ such that W and R are defined as for an $SQC^=$ model for the domain semantics. The assignment function V assigns to each *atomic* wff '1' or '0'. And a valuation over a TV model is defined inductively with $V(\alpha,w_i) = V_M(\alpha,w_i)$ as the basis. The reader is referred to chapter two for a description of this type of semantics. The following is a TV countermodel to the $SQC^=$ conditional $\mathbf{B}(\exists x)Fx \supset (\exists x)\mathbf{B}Fx$: $M = \langle W,R,V \rangle$ such that $W = \{w_1, w_2, w_3\}$, $\{\langle w_1,w_2 \rangle, \langle w_1,w_3 \rangle\} \subseteq R$ and $V(Fa,w_2) = V(Fb,w_3) = 1$, $V(Fa,w_3) =$

$V(Fb,w_2) = 0$. Further, we shall stipulate that for this model no constant t will be such that $V(Ft,w_2) = V(Ft,w_3)$. Given our characterization of V, it follows that $V_M(Fa,w_2) = V_M(Fb,w_3) = 1$ and $V_M(Fa,w_3) = V_M(Fb,w_3) = 0$. Hence, $V_M((\exists x)Fx,w_2) = V_M((\exists x)Fx,w_3) = 1$ and thus $V_M(B(\exists x)Fx, w_1) = 1$. However, since we have stipulated that for this model no constant t will be such that $V(Ft,w_2) = V(Ft,w_3)$ it follows that for no constant t will it be the case that $V_M(Ft,w_2) = V_M(Ft,w_3)$. Then it will be the case that for any constant t, $V_M(BFt,w_1) = 0$ from which it follows that $V_M((\exists x)BFx,w_1) = 0$. Thus, we have constructed an SQC$^=$ (TV) model such that $V_M(B(\exists x)Fx,w_1)$ is 1 but $V_M((\exists x)BFx,w_1)$ is 0, which therefore invalidates the conditional $B(\exists x)Fx \supset (\exists x)BFx$.

And so, since we have invalidated the conditional $B(\exists x)Fx \supset (\exists x)BFx$ in the two types of characteristic semantics for the SQC$^=$ systems and given soundness it follows that $B(\exists x)Fx \supset (\exists x)BFx$ is not a thesis of any of these systems. Therefore, $B(\exists v)\alpha \supset (\exists v)B\alpha$ is not an SQC$^=$ thesis schema which is just what we want, supposing that we maintain that belief de dicto does not logically imply belief de re. Also, since this conditional schema is not an SQC$^=$ thesis schema it follows that the biconditional $B(\exists v)\alpha \equiv (\exists v)B\alpha$ is not an SQC$^=$ thesis schema.

Although one half of the biconditional $B(\exists v)\alpha \equiv (\exists v)B\alpha$ is not an SQC$^=$ thesis schema, the other half of the biconditional, viz., $(\exists v)B\alpha \supset B(\exists v)\alpha$ is an SQC$^=$ thesis schema as we shall show presently. Intuitively, this conditional schema says that belief de re logically implies belief de dicto. This seems to be intuitively plausible if we once more consider the case of Jones and his beliefs concerning prime numbers. Suppose that there is some particular number such that Jones believes that that number is a prime. Then presumably it would also be the case that Jones believes that primes exist. And in general, if there is some t such that x believes that t is F, then it would seem to follow that x believes that there are F's. Thus, the conditional schema $(\exists v)B\alpha \supset B(\exists v)\alpha$ is desirable for quantified doxastic logic at least if we construe the quantifiers objectually. (Similar remarks could also be made in terms of a substitutional construal of the quantifiers.) It will now be demonstrated that any instance of this schema is derivable (and given soundness, any such intance is valid) in any SQC$^=$ system.

A proof sequence of any instance of the schema $(\exists v)B\alpha \supset B(\exists v)\alpha$ for an SQC$^=$ system would look something like this:

1. $\vdash \alpha(t/v) \supset (\exists v)\alpha$	
2. $B(\alpha(t/v) \supset (\exists v)\alpha)$	1, RB
3. $B\alpha(t/v) \supset B(\exists v)\alpha$	schema K, detachment.
4. $(\exists v)B\alpha \supset B(\exists v)\alpha$	supposing t is foreign to $B(\exists v)\alpha$ and given the rule $\alpha(t/v) \supset \beta \longrightarrow (\exists v)\alpha \supset \beta$ where t is foreign to $(\exists v)\alpha \supset \beta$.

Thus, any instance of the schema $(\exists v)B\alpha \supset B(\exists v)\alpha$ which says that belief de re implies belief de dicto will be derivable in the SQC$^=$ systems of doxastic logic. Given our soundness results, it then follows that any instance of this schema is valid in the two types of characteristic semantics for the SQC$^=$ systems, viz., the invariant domain semantics and the truth-value semantics.

By way of some closing observations concerning the problem of quantifying in, it is worth noting that quantifying in is also unrestricted for *doxastic possibility* contexts for the SQC$^=$ axiom systems given that the following schema is a variant of the axiom-schema $\alpha(t/v) \supset (\exists v)\alpha$:

$\mathbf{P_B}\alpha(t/v) \supset (\exists v)\mathbf{P_B}\alpha$

Intuitively, this schema says that it is permissible to existentially generalize with respect to a term t occurring in the scope of a doxastic possibility operator *outside of* that operator. (And of course, this schema can be used to derive the rule of inference $\mathbf{P_B}\alpha(t/v) \longrightarrow (\exists v)\mathbf{P_B}\alpha$.) An ordinary language example of this schema would be the following: If it is possible for all x believes that Pegasus is a winged horse, then *there is* something such that it is possible for all x believes that it is a winged horse.

In considering the above instance of $\mathbf{P_B}\alpha(t/v) \supset (\exists v)\mathbf{P_B}\alpha$, what is objectionable is that there seems to be some sort of commitment to what Marcus and others have called *possibilia* or fictional entities. Thus, in our example, it would seem that an existence claim is being made concerning doxastically possible winged horses. If one finds possibilia as unpalatable as Quine finds essential properties, then unrestricted quantifying in for contexts of doxastic possibility is a serious matter. However, similar to the case of quantifying into doxastic necessity contexts, it is only on an *objectual* reading of the quantifiers that this type of situation is problematic. If we were to read the existential quantifier occurring in any instance of $\mathbf{P_B}\alpha(t/v) \supset (\exists v)\mathbf{P_B}\alpha$ *substitutionally* then the above ordinary language instance of this schema would read "If it is possible for all x believes that Pegasus is a winged horse, then at least one substitution instance of 'it is possible for all x believes that v is a winged horse' is true". And this latter reading does not suggest any sort of ontological commitment to possibilia.

Further, as we shall see, it is a thesis-schema of the $SQC^{=}$ systems that $\mathbf{P_B}(\exists v)\alpha \equiv (\exists v)\mathbf{P_B}\alpha$, which could be taken as asserting that any wff of the form $P_B(\exists v)\alpha$ is logically equivalent to the appropriate instance of $(\exists v)P_B\alpha$. This schema is significant in that it can be regarded as an *elimination* schema. I.e., if $\vdash P_B(\exists v)\alpha \equiv (\exists v)P_B\alpha$, then if α is a wff containing one or more occurrences of the locution $(\exists v)P_B\alpha$ (which on an objectual reading of the quantifiers seems to involve a commitment to doxastic possibilia) then β which results from replacing one or more occurrences of the locution $(\exists v)P_B\alpha$ with its logical equivalent $P_B(\exists v)\alpha$ (which as will be noted does not involve any sort of commitment to possibilia) will be such that $\vdash\beta \equiv \alpha$. Then all occurrences of the locution $(\exists v)P_B\alpha$ in a wff α are *eliminable* in the sense that they can be replaced by $P_B(\exists v)\alpha$ resulting in a wff β such that $\vdash\beta \equiv \alpha$.[26] Thus, any instance of the schema $P_B\alpha\,(t/v) \supset (\exists v)P_B\alpha$ permitting quantifying in for doxastic possibility constructions is logically equivalent to the appropriate instance of $P_B\alpha\,(t/v) \supset P_B(\exists v)\alpha$ such that the occurrence of the locution $(\exists v)P_B\alpha$ has been eliminated. This therefore can be regarded as a solution to the problem of quantifying in for doxastic possibility constructions, since all locutions of the form $(\exists v)P_B\alpha$ are eliminable. As Marus notes, the solution here amounts to claiming that fictional entities can be 'analysed away'.[27]

Then it is worth proving that the following are $SQC^=$ thesis-schemata:

1) $P_B\alpha\,(t/v) \supset P_B(\exists v)\alpha$

2) $P_B(\exists v)\alpha \equiv (\exists v)P_B\alpha$

The first of these schemata is the doxastic possibility counterpart of the innocuous schema $B\alpha\,(t/v) \supset B(\exists v)\alpha$. An ordinary language instance of 1) would be "If it is possible for all x believes that Pegasus is a winged horse then it is possible for all x believes that *there are* winged horses". Unlike

26 See Hughes and Cresswell (1968), pp. 183-188.

27 Marcus (1976), p. 42. Her solution to the problem of possibilia involves adopting a TV semantics.

its cousin $P_B\alpha(t/v) \supset (\exists v)P_B\alpha$, this instance of 1) does not suggest that *there are* possible winged horses but merely that it is consistent with (or possible for) everything x believes that there are such things as winged horses. In short, there is no suggestion here of a commitment to possibilia, even on an objectual reading of the existential quantifier in 1). Any instance of schema 1), $P_B\alpha(t/v) \supset P_B(\exists v)\alpha$ will be derivable for any SQC$^=$ system by using the appropriate instance of $\alpha(t/v) \supset (\exists v)\alpha$ well as the derived rule $\vdash\alpha \supset \beta \longrightarrow \vdash P_B\alpha \supset P_B\beta$. Further, by applying R3 to the appropriate instance of $P_B\alpha(t/v) \supset P_B(\exists v)\alpha$ we can derive any instance of $(\exists v)P_B\alpha \supset P_B(\exists v)\alpha$ which is one half of the biconditional schema $P_B(\exists v)\alpha \equiv (\exists v)P_B\alpha$. The remaining half, $P_B(\exists v)\alpha \supset (\exists v)P_B\alpha$ is derivable by contraposing the Barcan Formula, $\sim(\exists v)\sim B\alpha \supset B\sim(\exists v)\sim\alpha$ and by employing the fact that 'P_B' and 'B' are interdefinable. The elimination schema $P_B(\exists v)\alpha \equiv (\exists v)P_B\alpha$ could also be considered as asserting the principle that all locutions of *de re* doxastic possibility are eliminable, where the de re/de dicto distinction for doxastic possibility contexts is made in terms of the scope of the quantifier as it was for belief contexts.

However, the problem of possibilia reappears for *belief* contexts. The following is an ordinary language instance of the schema $B\alpha(t/v) \supset (\exists v)B\alpha$: If Jones believes that Pegasus is a winged horse, then *there is* an x such that Jones believes that x is a winged horse. In short, if Jones believes that Pegasus is a winged horse, then he has a *de re* belief concerning a fictional entity, viz., a winged horse, which again seems to suggest an ontological commitment to possibilia in the sense of 'fictional entities'. Since there is apparently no reduction schema which will allow us to eliminate de re *belief* constructions for the SQC$^=$ systems (nor is such a schema even desirable) then if we find talk of 'possibilia' distasteful, the option is once more open to us to adopt a substitutional interpretation of the quantifiers.

As a final perspective on the problem of possibilia for the $SQC^=$ systems, any constant in the language is assigned a member of D for every model in the domain semantics. Then even a term t whose ordinary language construal is 'Pegasus' will denote some member of D for any $SQC^=$ model. Also, there are no 'fictional' entities (existing at merely 'possible' worlds) in an *invariant* domain semantics since for a model M, x exists at w_i =df $x \in D$ and hence any such x is a *necessary* existent given that D is shared by all indices in the model. So the individual denoted by 'Pegasus' is not a fictional entity after all, but rather a necessary existent, which can be taken as a reductio against an invariant domain semantics for quantified belief logic.

3. The Myth of 'The Myth of Ambiguity'

It was noted in the previous section that the de re/de dicto distinction can be made for propositional attitude contexts analogous to alethic modal contexts in terms of whether or not the belief operator occurs within the *scope* of the existential quantifier. Thus, if the belief operator occurs *inside* the scope of the quantifier, in which case the entire locution has the schematic form $(\exists v)\mathbf{B}\alpha$, then the resulting locution is de re. On the other hand, if the belief operator occurs *outside* the scope of the quantifier (and hence the sentence has the form $\mathbf{B}(\exists v)\alpha$), then this locution is de dicto.

As we shall see, Stephen Stich has claimed that the de re/ de dicto distinction as just described is a 'myth' for propositional attitude contexts. His alternative account of a situation such as Jones' believing that primes exist (de dicto) vs. Jones' believing that a particular natural number n is prime (de re) is that the ambiguity here is not inherent to Jones' belief states, but is instead traceable to the *contents* of Jone's beliefs. In short, Stich wants to claim that belief is never ambiguous - only contents are. If Stich is right, then the de re/de dicto distinction is spurious and this in turn calls into question any logic of belief which mirrors such a distinction in terms of quantifier scope.

We shall now critically examine Stich's argument for his claim that belief is not fundamentally ambiguous. It will be argued that in the final analysis, Stich's argument is not persuasive. Before appraising Stich's argument, it will be necessary to briefly describe his account of belief constructions and belief states.

Stich applies Davidson's remarks concerning indirect quotation contexts in 'On Saying That' to propositional attitude constructions by maintaining that any belief construction of the form 'x believes that α' can be paraphrased as 'α. x believes that.' where α is an utterance token or a speech act and where 'that' functions as a demonstrative referring to α. Actually, Stich paraphrases the construction 'x believes that' (where 'that' is a demonstrative referring to an utterance token) further as follows: 'x is in a *similar* belief state to the one which "would play the typical causal role if my utterance of *that* had had a typical causal history".[28] The italicized 'that'[29] in Stich's paraphrase of 'x believes that' is the demonstrative referring to α. Intuitively, what this paraphrase means is that the agent x to whom we are attributing the belief must be in a state which is *similar* to the state which would lead us (the belief ascribers) to utter α. And the

[28] Stich (1983), p. 88.

[29] The italics in the previous quotation are our own.

sorts of similarity which Stich has in mind here are 'functional' or 'causal pattern' similarity along with other types such as 'ideological' similarity. Two belief states S_1 and S_2 are *functionally* similar just in case they play similar causal roles with respect to *many* of the same types of behaviours and dispositions to behave and with respect to *many* of the same types of causal interactions amongst belief states. It follows from this definition of functional similarity that this concept admits of degrees given the quantity term 'many'. It is also worth noting that the central notion of similarity upon which the other notions depend is this notion of *functional* similarity which we have just defined.[30]

To see briefly how Stich's story actually works, consider the case where after hearing Trudeau during an interview speaking about the evils of the Meech Lake Accord, someone ascribes to him the belief that the Meech Lake Accord ought to be rejected. What attributing this belief to Trudeau amounts to on Stich's account is the claim that the belief state playing a central causal role in the "causal history leading up to" Trudeau's utterance of 'The Meech Lake Accord ought to be rejected' is *similar* to the state of the ascriber which would play a central causal role in the history leading up to the ascriber's utterance of 'The Meech Lake Accord ought to be rejected'. Stich makes it clear that he is breaking here with the so-called functionalist account of belief states[31] presumably because the 'sameness'

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[30] Stich also takes into account similarity of two belief states with respect to other parameters such as the network of belief states in which each state occurs. Thus, state S_1 is *ideologically* similar to a distinct state S_2 just in case S_1 and S_2 occur in networks of belief such that a significant number of the belief states in S_1's network are causally or functionally similar to a significant number of the states in S_2's network of belief. It follows from this definition that 'ideological similarity' admits of degrees. For details of Stich's account of similarity of belief states, see his discussion of this in Stich (1982), pp. 180-203 and in Stich (1983), pp. 88-90.

[31] Stich (1983), pp. 6-7. What Stich takes a 'functionalist' theory of mental states to be and how this

of belief states is connected not just with functional similarity but also with other types of similarity conceptually parasitic upon this type.

It is not our concern here to determine the plausibility of Stich's neo-functionalist account of belief ascription nor to attempt to deal with any problems which Stich's theory may inherit from Davidson's account of indirect quotation contexts. *For the sake of argument* we shall grant that Stich's account of belief ascription and belief constructions for ordinary language can survive criticisms. Then the question we shall address in the remainder of this section is the following: Supposing that Stich's account of belief ascription and of ordinary language belief constructions is sound, does it (in conjunction with a certain view of indefinite descriptions which he holds and which will be discussed presently) pose a threat to the tenability of the de re/ de dicto distinction?

As an example of the position which Stich is attacking, Quine in 'Quantifiers and Propositional Attitudes' espouses the view that the de re construction '($\exists$v)(x believes y(y is an F) of v)' (where y(y is an F) is an expression denoting an attribute) and the de dicto construction 'x believes that ($\exists$v)(v is an F)' are both paraphrases of the ambiguous sentence schema 'x believes that someone is an F'. For example, the supposedly ambiguous sentence 'Jones believes that someone is a Liberal' can be paraphrased as either '($\exists$v)(Jones believes y(y is a Liberal) of v)' if Jones' belief is *de re* (i.e., he bears a primitive relation to an attribute and to an individual) or as 'Jones believes that ($\exists$v)(v is a Liberal)' if Jones' belief is *de dicto* (i.e., he bears a two-termed relation to a proposition).

Hence, what Quine is assuming in the above example is that the ambiguity of the above sentence can be traced to whether or not Jones' belief is

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is connected with his definition of functional similarity is explained in Stich (1982), pp. 181–4.

de re or de dicto. In other words, Quine is assuming that the source of ambiguity in sentences of the form 'x believes that someone is an F' is the 'believes that' construction since belief can be either de re or de dicto. And these assumptions, viz., 1) that the ambiguity of the above sentence is tied up with the 'believes that' construction and 2) that there are two types of belief, de re and de dicto which explains this ambiguity are what Stich wants to call into question.

Stich's way of handling the ambiguity of the sentence 'Jones believes that someone is a Liberal' would be to apply his Davidsonian (as well as neo-functionalist) method of paraphrasing belief constructions as follows: The sentence 'Jones believes that someone is a Liberal' is analysable as 'Someone is a Liberal. Jones believes that.' where the pronoun 'that' functions as a demonstrative referring to the utterance 'Someone is a Liberal'.[32] Furthermore, the sentence 'Jones believes that' could itself be paraphrased according to Stich's neo-functionalist analysis of such constructions discussed above.[33] What is important about Stich's analysis is that it involves (a la Davidson) the isolation or separation of the content 'Someone is a Liberal' from the 'believes that' construction. Then assuming we admit that the original belief sentence is ambiguous, this move sets the stage for Stich's argument that the ambiguity has as its source an ambiguity in the separable *content* sentence and not the 'believes that' construction. We shall now briefly describe this argument.

According to Stich, the content 'Someone is a Liberal' which has been isolated from the 'believes that' construction in his analysis of the above

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[32] Stich (1983), p. 121.

[33] Stich (1983), p. 121.

sentence is ambiguous, which is evidenced by the fact that it can be paraphrased in one of two ways: The first way of paraphrasing the content utterance 'Someone is a Liberal' is to treat the term 'someone' as an ordinary language analogue of the existential quantifier. Thus, we could paraphrase the content as '$(\exists v)$(v is a Liberal)' and hence we are treating it as a so-called 'indefinite description'.[34] And this sort of paraphrase of the content as an indefinite description is appropriate in cases where the agent has no particular individual in mind in the sense that when questioned as to whom in particular he is referring, he is hard pressed to name any specific individual.[35] In short, he merely believes that there are Liberals, which is what Quine and others would call a *notional* belief.

On the other hand, the content 'Someone is a Liberal' is analysable as a kind of definite description in which case the term 'someone' does not function as the ordinary language analogue of a quantifier occurring in an indefinite description.[36] In other words, the term 'someone' functions as a kind of uniqueness operator (i.e., $\exists!$). For example, if the believer has Trudeau in mind, then this content might be analysed roughly as 'Someone is a Liberal. He is a former Prime Minister of Canada. He is from an influential Canadian family, he is a lawyer, and so on.'[37] This sort of analysis of the content utterance is appropriate, according to Stich in cases where the person to whom we are ascribing the belief has someone in particular in mind in the sense that when questioned as to whom he is referring, he might give a name or a series of definite descriptions. And this sort of case is what traditionally has been called by Quine and others a

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[34] Stich (1983), p. 120.

[35] Ibid, p. 119.

[36] Ibid, p. 120.

[37] Ibid., p. 119.

case of de re or relational belief.

The special twist to Stich's analysis of the ambiguous construction 'Jones believes that someone is a Liberal' is that the ambiguity does not rest with the 'believes that' construction but rather with the content utterance. From this, Stich concludes that there is no such thing as de re and de dicto *belief states*. There is only one kind of belief which can be accounted for (at least as a first-stab) within a neo-functionalist theory, although there are two ways of analysing the *contents* of beliefs or belief states. We shall now present an objection to Stich's handling of the de re/de dicto distinction.

Although the content 'Someone is a Liberal' when interpreted as an indefinite description, viz. as '$(\exists v)(v$ is a Liberal$)$' seems to wear its logical form on its sleeve (to borrow a turn of phrase from Davidson), it is not in any way obvious how we can sometimes construe this as a kind of extended definite description. What is the relation between the analysandum, viz., 'Someone is a Liberal' and the analysans such as 'Someone is a Liberal. He is a member of an influential Canadian family, and so on . . .'? The relation is certainly not one of making the logical form of the analysandum apparent, especially since the analysans will presumably vary from believer to believer. I.e., two different believers may have different sets of descriptions by means of which they pick out the relevant individual(s) they have in mind. They may have in Kaplan's parlance distinct 'inner stories'.

Stich's claim that in some cases a content of the form 'Someone is an F' can be read as an indefinite description and sometimes as an extended definite description is on the same footing as Quine's claim that 'Jones believes that x is an F' can sometimes be given a notional reading and sometimes a relational reading. In neither case is it in any way evident what interpretation we should give to the appropriate locution, at least given the locution alone. However, if we are just concerned with logical form then the 'correct' construal would seem to be that of an indefinite description in the case of 'Someone is an F'. Presumably, contextual considerations such as the agent's other beliefs (i.e., his 'inner story') would have to be taken into account in order to decide how to interpret the content 'Someone is an F'. But if we need to appeal to the agent's beliefs in order to determine what his/her beliefs are, then this amounts to circularity. Thus, at the very best, Stich's alternative account of the apparent amibiguity of belief constructions is subject to the same sorts of difficulties as the view (such as Quine's) which it is replacing.

4. Interlude

And so, to summarize our discussion to date, in the first two sections we have identified three problematic $SQC^{=}$ rules of inference concerned with the connection between either the identity symbol and the belief operator or between the existential quantifier and the belief operator. These three problematic rules are as follows:

R1: $\mathbf{B}\alpha\,(t_1/v),\ t_1 = t_2 \longrightarrow \mathbf{B}\alpha\,(t_2/v)$ - or its more general version.

R2: $\mathbf{B}\alpha\,(t_1/v),\ t_1 = t_2 \longrightarrow {\sim}\mathbf{B}{\sim}\alpha\,(t_2/v)$

R3: $\mathbf{B}\alpha\,(t/v) \longrightarrow (\exists v)\mathbf{B}\alpha$ - or its more general version.

Actually, the second rule of inference, $\mathbf{B}\alpha\,(t_1/v),\ t_1 = t_2 \longrightarrow {\sim}\mathbf{B}{\sim}\alpha\,(t_2/v)$ is derivable only in the $SQC^{=}$ + D systems of doxastic logic. The first two rules express the so-called substitutivity principle, viz., that co-referential terms are intersubstitutible in belief contexts. The third rule expresses the principle that it is permissible to existenially generalize with respect to the occurrence of a constant t in the scope of a belief operator outside of the belief operator. In other words, the third rule permits unrestricted quantification *into* belief constructions. As we have seen, there are reasons for suspecting the plausibility of all three rules given various ordinary language 'counterexamples' which we have constructed. However, the counterexamples to R3 allowing unrestricted quantification into belief contexts are relevant only if the existential quantifier is given an *objectual* reading in the corresponding domain semantics. Thus, R3 is unproblematic if the existential quantifier is read substitutionally in the corresponding TV semantics.

It was also noted that the counterpart of R3 for doxastic *possibility* viz., $P_B\alpha(t/v) \longrightarrow (\exists v)P_B\alpha$ which allows unrestricted quantifying into doxastic possibility constructions is derivable for the SQC$^=$ axiom-systems. This schema is philosophically objectionable on the grounds that it at least seems to involve a commitment to possibilia - assuming '$\exists$' is read objectually. Like R3, this rule (and the corresponding schema) is unproblematic if '$\exists$' is read substitutionally. Further, given the SQC$^=$ elimination schema $P_B(\exists v)\alpha \equiv (\exists v)P_B\alpha$, the problem of quantifying in for doxastic possibility constructions is resolvable, even given an objectual reading of the quantifiers.

On the other hand, the following rule of inference and schema which hold for the SQC$^=$ systems are intuitively plausible and hence desirable for any system of doxastic logic:

R4: $B\alpha\,(t/v) \longrightarrow B(\exists v)\alpha$

S1: $(\exists v)B\alpha \supset B(\exists v)\alpha$

The rule of inference R4 permits existential generalization with respect to the occurrence of a term t in the scope of a belief operator *inside* the belief operator. And as we have just seen, the schema S1 says that belief de re implies belief de dicto.

Apparently, what is needed if we adopt a domain semantics for the $SQC^{=}$ systems are modifications to the rules of inference R1, R2 and R3 and hence to the $SQC^{=}$ axiom systems which will accommodate the counter-examples we have constructed. To ensure soundness and completeness of the resulting systems, corresponding changes will need to be made to their domain semantics. (Of course, we shall also want to retain the rule R4 and the schema S1 mentioned above, if possible.) So in sections 5 and 6, we shall consider Hintikka's suggestions for modifying the axiomatics of quantified belief logic *where the quantifiers are read objectually* to accommodate these counterexamples. He discusses these proposals in a number of places including *Knowledge and Belief* (1962).

As we shall see, Hintikka's suggestions for a quantified logic of belief involves treating belief constructions as ambiguous in the sense that some constructions are 'notional' and others are 'relational' (to borrow Quine's phraseiology). He then restricts quantifying in (i.e., the inferring of *de re* constructions) and the substitution of co-referentials to what we shall call relational constructions. And so treating belief contexts as ambiguous is integral to Hintikka's solutions to the problems of failure of substitutivity of co-referentials and to the problem of quantifying in. (It will be noted in chapter four that his solution to the problem of quanfifying in applies a fortiori to this problem for contexts of doxastic possibility.)

Although Hintikka's suggestions for a quantified logic of belief are able to accommodate the various informal counterexamples we have considered in the first two sections for logics where the quantifiers are read objectually, it will be argued in the fourth chapter that Hintikka's corresponding *semantics* presupposes a problematic 'counterpart' metaphysics. This is owing to the fact that the semantics he proposes for quantified belief logic is a domain semantics such that with each index in a model is associated a unique domain of individuals. Then although there is no problem of trans-index identity since all individuals are index-bound, there is a problem connected with identifying individuals' counterparts across indices.

On the other hand, if we adopt a truth-value semantics for the $SQC^{=}$ systems such that the quantifiers are given a substitutional reading, then R3 (and its counterpart for doxastic possibility) is unproblematic. Then emendations must be made in the logic and in the semantics to eliminate R1 and R2 allowing unrestricted substitution of co-identicals although we can can get by without dispensing with or modifying the rule R3 permitting existential quantification *into* belief contexts. We still of course retain the distinction between the de re construction $(\exists v)\mathbf{B}\alpha$ and the de dicto construction $\mathbf{B}(\exists v)\alpha$ in the semantics in the sense that the latter does not entail the former. And, in chapter four we shall propose a quantified logic of belief (or to be more precise, a collection of belief logics) which eliminates R1 and R2 (for systems containing 4) and yet retains R3. In proposing these logics we dispense with the relational/notional distinction and follow the Fregean path of treating belief contexts as always oblique - i.e., substitution of co-referentials is never permitted in belief contexts.

Finally, it will be argued in the fourth chapter that although both Hintikka's logic of belief (and its objectual semantics) and our own logic of belief (where the quantifiers are read substitutionally) are able to handle the counterexamples discussed in the first two sections, the latter is the preferable of the two. For one thing, the truth-value semantics characterizing our own proposed quantified doxastic systems dispenses with domains of individuals and hence the 'counterpart' problem encountered in Hintikka's varying domain semantics is avoided. Further, there is no need to invoke the controversial relational/notional distinction in order to solve the problem of 'quantifying in' since there is no such problem for a logic where the quantifiers are construed substitutionally in the corresponding semantics.

Section 5. Hintikka's Solution to the Problem of 'Quantifying In'

According to Hintikka, existential generalization with respect to the occurrence of a constant t outside a belief construction of the form $\mathbf{B}\alpha(t/v)$ is permissible only if the agent x has an opinion as to who (or what) t is. I.e., *there is some individual v* such that x believes that this individual is t.[38] Hintikka renders the locution 'there is some individual v such that x believes that v is t' for first-order belief logic as '$(\exists v)\mathbf{B}(v = t)$' such that the quantifier is to be read objectually. This locution signifies that x is

[38] See Hintikka (1962), pp. 144-146; Hintikka (1969) reprinted in Linsky (1971), p. 156. There is as Hintikka notes himself in Hintikka (1962), p. 145 an analogy between the locution ⌜$(\exists v)\mathbf{B}(v = t)$⌝ and the locution ⌜$(\exists v)(v = t)$⌝ which asserts that there is some individual denoted by t – or more simply that t exists since the former locution asserts that there is some individual denoted by t such that x has an opinion concerning who t is.

'acquainted' with the individual denoted by t (and we shall have more to say about the concept of 'acquaintance' below). Then Hintikka's proposal for a logic of belief which avoids the rule R3, $\mathbf{B}\alpha\,(t/v) \longrightarrow (\exists v)\mathbf{B}\alpha$ permitting unrestricted quantification into belief constructions with respect to a constant t is to only allow quantication into belief constructions with respect to t which are *conjoined* with the appropriate 'acquaintance' locution. More formally, existential generalization with respect to a constant t occurring in a locution of the form $\mathbf{B}\alpha\,(t/v)$ is permitted only if $\mathbf{B}\alpha\,(t/v)$ is conjoined with $(\exists v)\mathbf{B}(v = t)$.[39] Or in terms of inferential contexts, Hintikka's stricture is that generalization with respect to t occurring in a locution of the form $\mathbf{B}\alpha\,(t/v)$ is permitted only if $(\exists v)\mathbf{B}(v = t)$ is added as a premise. This stricture on existential generalization for belief contexts is imposed for the reason that x may believe that $\alpha\,(t/v)$ and yet he may have no idea who or what t is, in which case it would not be appropriate to infer that *there is some individual v* such that x believes that α.[40]

Consider the example discussed in the second section, viz., Jones' belief that the next P.M. of Canada (whoever he/she is) will attempt to balance the budget. This would be symbolized in the language of the $SQC^{=}$ systems simply as BFp. Then Hintikka would not permit existential generaliztion with respect to t outside the belief operator. I.e., we could not infer from BFp the de re locution $(\exists x)\mathbf{B}x$. This inference would only go through if it were added as a premise that Jones has some individual in mind - that he is 'acquainted' with someone or other whom he thinks will fit the description of attempting to balance the federal budget. I.e., only if we add as a premise $(\exists x)\mathbf{B}(x = p)$ can we infer from BFp that $(\exists x)\mathbf{B}Fx$.

[39] "If we are right, "(Ex)Bap" is implied by "Bap(b/x) & (Ex)Ba(b = x)" but not by "Bap(b/x)" alone." This appears in Hintikka (1962), p. 149.

[40] Hintikka (1962), p. 143-4, p. 149.

In the light of Hintikka's remarks concerning existential generalization into belief constructions, we could restrict the SQC$^=$ axiom-schema $\alpha(t/v) \supset (\exists v)\alpha$ to cases where t does not occur in the scope of a belief operator and we could add as an axiom-schema (and thus derive as a rule of inference):

S2 $(\alpha(t/v) \mathbin{\&} (\exists v)(v = t \mathbin{\&} \mathbf{B}(v = t)) \supset (\exists v)\alpha$ where t may occur in the scope of a belief operator(s).

R3* $(\alpha(t/v) \mathbin{\&} (\exists v)(v = t \mathbin{\&} \mathbf{B}(v = t)) \longrightarrow (\exists v)\alpha$ where t may occur in the scope of a belief operator(s).

Instances of S2 would be the following:

1) $((\mathbf{B}(Fa \vee Gb) \vee Ha) \mathbin{\&} (\exists x)(x = a) \mathbin{\&} \mathbf{B}(x = a)) \supset (\exists x)\mathbf{B}((Fx \vee Gb) \vee Hx)$

2) $((\mathbf{P_B}Fb \mathbin{\&} Hb) \mathbin{\&} (\exists x)(x = b \mathbin{\&} \mathbf{B}(x = b)) \supset (\exists x)((\mathbf{P_B}Fx \mathbin{\&} Hx)$

2) is an instance of S2 since '$\mathbf{P_B}Fb$' is definable as $\sim\mathbf{B}\sim Fb$ in which case, 'b' does occur in the scope of a belief operator. We shall discuss the philosophical significance of 2) in chapter four. Also, given the definability of '$\mathbf{P_B}$' in terms of '$\mathbf{B}$', the restrictions for S2 and R3* can be made more general. I.e., it can be required that t may occur in the scope of a *doxastic* operator(s). For S2 and R3*, restricting quantifying in to contexts involving 'acquaintance', the singular term t may also occur outside the scope of the doxastic operator as in 1) and 2). Then in such cases, Hintikka argues that the 'actual world' must be taken into account (or more neutrally, the index at which the wff is being evaluated).[41] This is the significance of adding '$v = t$' to the acquaintance locution resulting in '$(\exists v)(v = t \mathbin{\&} \mathbf{B}(v = t)$'. As we shall see in discussing Hintikka's solution to the problem of the failure of substitutivity of co-referentials for belief contexts, this locution can informally be construed as saying that x has a *true* opinion as to who t is.

41 Hintikka (1969) reprinted in Linsky (1971), p. 157.

x's having a *true* opinion as to who some individual t is will be regarded as a special sort of relational context which guarantees subsitutivity.

Further, the locution $(\exists v)(v = t \;\&\; \mathbf{B}(v = t))$ is needed to guarantee the validity of the more general schema S2 with respect to the semantics to be discussed in chapter four. For reasons which will be discussed in the next chapter, for systems not containing the schema 4, $\mathbf{B}\alpha \supset \mathbf{BB}\alpha$ it will be necessary to impose the proviso on the above schema and rule of inference that there is no iteration of any belief operators in $\alpha(t/v)$ within whose scope t lies.

In order to make some sort of philosophical sense out of Hintikka's proposal for dealing with the problem of quantifying in for doxastic logic, we shall compare his proposal to those of both Quine and Kaplan who try to resolve this problem on the ordinary language front. As we shall see, Hintikka's resolution to the the problem of quantifying in resembles the solution which Kaplan offers in spirit if not in detail. Further, Kaplan provides an analysis of the notion of 'representation' which at least gives some intuitive content to Hintikka's notion of 'having an opinion as to who t is' which is symbolized in terms of what we have called the 'acquaintance' locution. These brief digressions will therefore help us to put into perspective Hintikka's formal solution to the problem of quantifying in.

We shall first of all compare Hintikka and Quine on the issue of quantifying in. The position of Quine's which we are about to examine is not his final word on the subject of quantification in propositional attitude contexts, although it is the position which is most widely discussed in the literature.

In 'Quantifiers and Propositional Attitudes' (1956) and in *Word and Ob-*

ject (1960), Quine developed the view that there are two types of belief construction, viz., relational and notional. In the simplest sort of case, a *relational* belief construction is such that at least one singular term t occurs within the scope of the belief operator 'purely referentially' or 'transparently' in the sense that the believer bears some sort of primitive relation R to the denotatum of t and to an atttibute.[42] Schematically, any relational belief construction attributing a property P to some individual t can be represented as 'x believes y(y is a P) of t' where the locution 'y(y is a P)' is an expression for an attribute. Quine treats attributes as *intensions* of degree 1 such that the degree of the intension is determined by the number of free variables occurring in the intensional idiom.[43] An instance of this schema would be 'Jones believes that y(y is a Liberal) of Trudeau' where this construction depicts a three termed relation R between Jones, the attribute denoted by 'y(y is a Liberal)' and the individual denoted by 'Trudeau'.

Quine's treatment of relational constructions can easily be generalized. For example, a relational belief construction expressing a four termed relation between a believer, a 2nd-degree intension (asserting a relationship between two individuals y and z) and two individuals t_1 and t_2 will have the schematic form 'x believes yz(y R'd z) of t_1 and t_2' where 'yz(y R'd z)' denotes a 2nd degree intension - a two-termed relation. An instance of this schema would be 'Jones believes that yz(y denounced z) of Cicero and Cataline' where this construction depicts a four-termed relation between Jones, Cicero, Cataline and the 2nd-degree intension 'yz(y denounced z). In

[42] Quine (1960), p. 145.

[43] Quine (1960), p. 104. Quine does not ultimately commit himself ontologically to intensional entities although he finds them useful for elucidating the relational/notional distinction. He also uses this tact in an earlier article, viz., Quine (1956).

general, *relational* belief for Quine will be an $n \geq 3$-termed relation which obtains between a believer, some nth-degree intension ($n \geq 1$) and $n \geq 1$ individuals. It follows, at least prima facie, that there will be an infinite number of irreducibly primitive senses of relational belief.

Relating Quine's notion of relational belief to the problem of quantifying in, Quine restricts quantification into belief constructions with respect to singular terms occurring *transparently* within the scope of the belief operator, because in such constructions the believer bears a primitive relation (call it 'acquaintance' or whatever) to the individuals denoted by these terms. Hence, we can existentially generalize into belief constructions only with respect to terms denoting individuals to whom the believer is related.

For example, in the case of the *relational* construction 'Jones believes y(y is a Liberal) of Trudeau', Jones bears a three-termed relation to the attribute denoted by 'y(y is a Democrat)' and the individual denoted by 'Trudeau'. Then we can infer that *there is some individual v* such that Jones believes y(y is a Liberal) of v, or more conspicuously we can infer that $(\exists v)$(Jones believes y(y is a Liberal) of v). Notice then the similarity between Quine's solution to quantifying in and Hintikka's, viz., that like Quine, Hintikka only allows existential generalization with respect to terms denoting indivduals with whom the believer is 'acquainted'. Quine simply treats this 'acquaintance' relation as a primitive.

It was noted earlier that Quine posited two senses of belief and therefore two types of belief constructions, viz., relational and notional. In contrast to relational constructions, singular terms occurring withing the scope of the belief operator in a *notional* belief construction occur 'opaquely'. A singular term in the scope of a belief operator occurs *opaquely* if it does not occur transparently or purely referentially, i.e., if it is such that the individual it denotes is not an individual to which the believer is related. Since the believer is not primitively related to the appropriate individual, then Quine prohibits existential generalization with respect to the term in the belief construction which would normally denote this individual. In short, we cannot make de re generalizations from purely notional constructions. Once again, Quine's prohibiting quantifying into notional constructions is similar to Hintikka's strictures against quantifying into constructions which are not conjoined with locutions indicating that the believer is 'acquainted' with the appropriate individual.

More formally, Quine treats notional belief as a two-termed relation between a believer and a *proposition*. (Quine regards propositions as 0th degree intensions. And of course, Quine ultimately disavows any commitment to propositions in favour of 'eternal sentences'.) Thus, the schematic form of a notional belief construction is 'x believes that p' where p is a 0th-degree intension, i.e., a proposition. Further, Quine then prohibits existential generalization with respect to any variables occurring in expressions denoting intensions occurring within the scope of a belief operator - thereby not granting intensions the same ontological status as individuals denoted by singular terms. Since the expression 'p' in a notional construction of the form 'x believes that p' denotes a 0th degree intension, i.e., a proposition, then it is not permissible to existentially generalize with respect to any variables or constants occurring in p.[44] However, although we cannot infer de re constructions from notional locutions, Quine does permit de dicto inferences from notional locutions - i.e., we can existentially

[44] Quine (1956), p. 189.

generalize with respect to t occurring in the content proposition p if the quantifier occurs as part of the scope of the belief operator, as we shall next see.

To see more clearly what Quine's proposal amounts to with respect to notional constructions, consider once more our example of Jones' belief concerning the next P.M. of Canada. Since there is no particular individual such that Jones believes of that very individual that he/she will attempt to balance the budget, then there is no individual to whom Jones is related. Thus, we would render this belief notionally, i.e., as a two-termed relation between Jones and the proposition 'that the next P.M. of Canada will attempt to balance the budget'. I.e., the appropriate construction in this case would be 'Jones believes that the next P.M. of Canada (whoever he/she is) will attempt to balance the budget' which is an instance of the notional schema 'x believes that p'. Therefore, it is not permissible in this case to existentially generalize (outside the belief operator) with respect to the singular term 'the next P.M. of Canada (whoever he/she is)'. However, since Jones believes that individuals who would attempt to balance the budget exist similar to our case where the agent believes that primes exist, then from the above notional construction we *are* permitted to infer that 'Jones believes that $(\exists x)$(x will attempt to balance the budget)', which is a *de dicto* construction.

And so, Quine and Hintikka appear to both agree that quantification into 'notional' belief constructions where the believer is not related to the appropriate individual should be prohibited but that quantifying in is permitted in purely relational contexts where such a relation does obtain. If the belief context is relational then in quantifying in, the scope of the de re quantifier is the *relational* construction for Quine whereas for Hintikka, the scope of the de re quantifier is the notional locution – i.e., the inferred de re construction will simply have the form $(\exists v)B\alpha$ where the scope of the quantifier does not include what we have called the 'acquaintance' locution. Further, they disagree on the issue of whether or not there are two 'irreducible' senses of belief and corresponding belief constructions. (Or at least Quine in one of his philosophical moments disagrees with Hintikka on this issue.) For Quine, a relational construction is not *partially* re-expressible as a notional locution conjoined with something else. This is because notional belief is a primitive two-termed relation between a believer and a proposition whereas relational belief is an $n \geq 3$-termed relation between a believer, an intension and an individual. In fact, there are an infinite number of irreducible types of relational belief.

Hintikka, on the other hand, seems to allow that any relational belief construction involving the claim that the believer is 'acquainted' with the appropriate individual is expressible as the conjunction of a *notional* construction (which by itself does not assert any such relationship) and what we might call an 'acqaintance' locution. I.e,. the *relational* schema $B\alpha(t/v)$ & $(\exists v)B(v = t)$ is the conjunction of the *notional* schema $B\alpha(t/v)$ and the 'acquaintance' schema $(\exists v)B(v = t)$. Then for Hintikka, relational belief is reducible to notional belief in the sense that relational *locutions* are partially expressible in terms of notional *locutions*. It is this sense of reduction that has been the subject of much discussion in the literature.[45] The strategy that Hintikka is employing concerning the issue of quantifying in, viz., that there are types of belief constructions which we shall herein call notional and relational, the latter being 'reducible' to the

[45] For example, see Hintikka (1967), Kaplan (1969), Sosa (1970) and Burge (1977).

former is similar to the strategies employed by Kaplan and Sosa. In particular, we shall compare Hintikka and Kaplan on this score.

Kaplan presents his characterization of the notional/relational distinction in his article 'Quantifying In' (1969). In our example of Jones and his belief concerning the next Prime Minister of Canada, assuming his belief is *notional* then the sentence attributing to him this belief would be characterized by Kaplan as 'Jones B <the next Prime Minister of Canada (whoever he/she might be) will attempt to balance the budget>'. The symbol 'B' is of course the belief operator and the idiosyncratic quotation marks serve to indicate that anything occurring within these marks (and hence the entire content sentence itself) occurs *referentially*, though the reference of the terms and the entire content sentence will not be their 'usual' references (such as individuals in the case of singular terms) but rather themselves. In short, the entire content sentence within the peculiar quotation marks refers to an expression, viz., itself.[46] (Kaplan bases his characterization of notional belief sentences at least in part on Frege's remarks concerning indirect quotation and propositional attitude constructions.) So schematically, any notional belief sentence will have the form 'x B <S>' where the expression variable S occurring within the idiosyncratic marks ranges over names for sentences.

Kaplan then prohibits existential generalization with respect to any singular terms occurring in the content S of a notional construction since although the terms are referential in the sense that they denote themselves, they are not *customarily* referential and so they do not denote any individuals with whom the individual is in some sense acquainted. So we cannot generalize with repect to the term 'the next Prime Minister of

[46] See Kaplan (1969) reprinted in Linsky (1971), p. 122.

Canada' in the above example, since this expression does not refer to any particular individual with whom the believer is 'acquainted' (or in Kaplan's parlance, the individual is not *represented* to the believer by the expression which customarily denotes it).

Notice that there are three senses of the term 'referential' being used in the above paragraph. First, there is a distinction between *customary* and *indirect* reference implicit in Kaplan's treatment of notional constructions, a distinction which has its origins in the writings of Frege. The customary referent of a singular term t will be an individual and its indirect referent will be itself. Further, for both Quine and Kaplan, the third sense of 'referential' which is crucial for determining whether or not we can generalize with respect to a term t occurring in the scope of a belief operator is that the term not only denotes its 'customary' referent but also that the believer is somehow 'acquainted' with this individual. I.e., a singular term t occurs referentially$_3$ in the scope of a belief operator just in case 1) t denotes its customary referent and 2) the believer x is 'acquainted' with t. This third sense of 'referential' or at least the second condition (referring to the notion of 'acquaintance') is similar to Hintikka's notion of a believer's having an opinion as to who t is, or more precisely, it is similar to Hintikka's notion of having a *true* opinion as to who some individual is. As we shall see presently, Kaplan attempts to provide an analysis of this third sense of 'referential' in terms of the notion of 'representation'.

According to Kaplan, a *relational* belief construction is partially expressible as a notional construction in the following sense: In the simplest sort of case, a relational belief construction is a two-termed conjunction, one of whose conjuncts is a *notional* construction and whose other conjunct is a 'representation' locution. The representation locution asserts that the relevant singular term t occurring in the notional conjunct 'represents' an individual y for the believer x in the sense that t denotes an individual with whom the individual is 'acquainted' - or in Kaplan's parlance, t is a vivid name of y for the believer x. (We shall presently discuss precisely what Kaplan means by 't is a vivid name of y for x'.) The schematic form of a representation locution is R(t, y, x) which can be read as 't represents the individual y to the believer x'. Then, the schematic form of a *relational* belief construction (in the simplest sort of case where there is only one singular term t) is '$R(t, y, x)\ \&\ x\ B\ {}^{<}S(t/v)^{>}$' such that the first conjunct is the representation locution and the second conjunct is a notional construction. Further, we can make a *de re* generalization with respect to t occurring in this construction since the individual it customarily denotes is one with whom the believer is 'acquainted'. Thus, it is possible to quantify into any such construction (with respect to t) resulting in $(\exists \$)[R(\$, y, x)\ \&\ x\ B\ {}^{<}S(\$/v)^{>}]$.

An instance of the above relational schema would be 'R(*Trudeau,* Trudeau, Jones) & Jones B <*Trudeau* is a Liberal> where the first conjunct asserts that the individual Trudeau which is the customary referent of the term *Trudeau* occurring in the notional construction is such that Jones is 'acquainted' with this individual. Further, quantification into this relational construction is permitted with respect to the term *Trudeau* since it 'represents' the individual Trudeau to the believer Jones. I.e., the following de re generalization is permissible, viz., $(\exists \$)[R(\$, \text{Trudeau}, \text{Jones})\ \&\ \text{Jones}\ B\ {}^{<}\$ \text{ is a Liberal}^{>}]$.

And so for Kaplan, relational belief is *reducible* to notional belief in the sense that relational constructions are partially expressible in terms of notional constructions. And on this score, Kaplan and Hintikka are in agreement. Recall that for Hintikka, a relational locution (in the simplest sort of case where only one singular term is under consideration) is a two-termed conjunction consisting of a *notional* locution whose schematic form is '$B\alpha(t/v)$' and whose second conjunct is an 'acquaintance' locution, the schematic form of which is '$(\exists v)B(v = t)$'.

On the other hand, Hintikka and Kaplan differ with respect to their characterization of quantifying into relational contexts in the following sense: Whereas Hintikka stipulates that the scope of the quantifier in a *de re* construction will be a notional locution only (as was already noted), Kaplan stipulates that the scope of the quantifier in a de re construction will be a relational construction and hence, the schematic form of a de re construction will be $(\exists\delta)[R(\delta, y, x) \,\&\, x\, B^{<}S(\delta/v)^{>}]$. In short, Hintikka allows quantifying into notional locutions occurring in relational contexts whereas Kaplan allows quantifying into relational locutions.

We shall now consider Kaplan's notion of 'representation' since this may shed some light on the intuitive construal of Hintikka's 'acquaintance' locution '$(\exists v)B(v = t)$' as well as the locution '$(\exists v)(v = t \,\&\, B(v = t))$'.

According to Kaplan, a singular term t represents its customary referent y to the believer x just in case 1) t denotes y which is tantamount to saying that t must have y as its customary referent, 2) t is a name of y for the believer x and finally 3) t is vivid.[47] Kaplan claims that a term t *denotes* an individual by virture of its descriptive content and hence that denotation is "the analogue for names to resemblance for pictures"[48] Then

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[47] Kaplan (1969) reprinted in Linsky (1971), p. 138.

[48] ibid., p. 136.

presumably a term, if not itself a definite description must have certain descriptions associated with it if it is to denote an individual. Further, Kaplan provides a so-called genetic account of how a singular term comes to be a *name of something for someone*. For example, Jones may read in a magazine article about a famous political figure Jean Chretien who is described in the article as quite possibly the next Prime Minister of Canada, a loyal Liberal and so on. If he had not heard others mention this individual beforehand then Jones would, on the basis of the descriptions in the article dub him 'Chretien'. Then the expression 'Chretien' has become for Jones a name of the individual Kennedy. Or, if he has heard others speak of this individual Chretien using the same descriptions he would have read in the article then once again the expression 'Chretien' will become a name of the individual Chretien for Jones. In short the term 'Chretien' (whose initial reference has been established by descriptions or perception or whatever) has been passed on from speaker to speaker which is what Kripke calls a 'causal' theory of naming.[49] And so, as Kaplan notes, there is no one way by which an expresssion δ must come to be a name of some object y for a believer x. In some cases, the so-called dubbing of the object may come about through direct perceptual means, in other cases through descriptive means and in still other cases by being passed on from speaker to speaker.[50]

According to Kaplan, a further condition which a term must satisfy for it to represent some individual y to a believer x is that in addition to its

[49] Kripke develops a so-called causal account of names in Kripke (1980) which bears certain similarities to the account which Kaplan is providing. However, Kripke only allows that proper names can serve as names of objects since they, unlike descriptions, are rigid designators. A rigid designator is an expression which denotes the same object in all the worlds at which it exists.

[50] Kaplan (1969) reprinted in Linsky (1971), p. 135.

denoting y and its being a name of y for x, it must also be *vivid*. A name's vividness is tied up with the role its denotatum plays in a believer's so-called inner story which is the set of contents the agent believes.[51] Thus, Jones may believe that some individual is a loyal Liberal, is a former Minister of Finance, is the new leader of the Liberal Party and so on and perhaps the individual he has dubbed 'Chretien' satisfies these descriptions. Thus, when he considers these descriptions, the individual he has dubbed 'Chretien' comes readily to mind. Kaplan also admits of degrees of vividness of a term and the degree of vividness will depend upon the role the denotatum of the term plays in the agent's inner story.[52] Thus, in the case of Jones, if he has many beliefs about the life and accomplishments of a certain former Minister of Finance then the name 'Chretien' will for him be of a high degree of vividness. Thus, Kaplan stipulates that for an expression t to represent y to x, it is a necessary condition that t be a *sufficiently* vivid name of y for x.

Kaplan notes that his notion of '*vivid* name' and Hintikka's notion of 'having an opinion (belief) as to who t is' more or less amount to the same thing.[53] I.e., to have a vivid name of some individual x implies that this individual has a central role to play in the agent's 'inner story' and hence, it would seem to follow that the believer will have an opinion as to who this individual is. Thus, we could say that a believer x's having a (sufficiently) vivid name t of an object y is a *sufficient* condition for x's having an opinion as to who (or what) y is. However, if a term t *represents* an object y to x in the sense that not only is t a vivid name of y for x but

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51 Kaplan in Linsky (1971), p. 136.

52 Ibid., p. 136.

53 Kaplan in Linsky (1971), pp. 136-7.

also that t 'actually' denotes an object y, this is still not *sufficient* for x's having a *true* opinion as to who some individual y (denoted by t) is, as will be argued below.

Whether or not we agree with Kaplan's story of what it is for an expression t to represent an individual y to a believer x, at least his account of representation gives some substance or flesh to the notion of an agent's having an opinion as to who y is.[54] I.e., we can make some sort of intuitive sense of Hintikka's 'aquaintance' locution whose schematic form is $(\exists v)B(t = v)$ which occurs as a conjunct in relational construtions (or as a premise in a relational inferential context).

Before discussing Hintikka's solution to the problem of the failure of substitution of co-referentials in belief contexts, there is a possible objection to his way of handling the problem of quantifying in. As was noted, Hintikka's solution to the problem of quantifying in relies on the distinction between *relational* and *notional* constructions such that sentences of the former type are partially expressible in terms of sentences of the latter type. However, there has been much controversy in the literature concerning the tenability of the thesis that relational belief is *reducible* to notional belief in Hintikka's (and Kaplan's) sense. Tyler Burge, for example, has called the reducibility thesis into question.[55]

Burge maintains that Kaplan's thesis that relational belief constructions are partially expressible in terms of notional belief constructions amounts to the position that relational belief is a 'mere species' of notional belief.[56]

[54] Ernest Sosa in Sosa (1970) argues that an unpalatable consequence of Kaplan's notion of 'vividness' is that one could not have relational beliefs concerning individuals that play only a minor role in the believer's so-called inner story. Thus, Sosa replaces the notion of 'vivid name' with the notion of 'distinguished term' in his characterization of relational belief which supposedly avoids this consequence. His remarks on this matter can be found on pp. 889-891.

[55] See Burge (1977).

According to Burge, this characterization of relational beliefs ignores their fundamentality and hence it is wrongheaded. Burge has argued at length that relational beliefs are more fundamental than notional ones in the sense that having beliefs of the former type "is a necessary condition for using and understanding language - and in fact for any any propositional understanding - and for acquiring empirical knowledge".[57] Burge defines *relational* beliefs as involving some sort of non-conceptual relation between the believer and the appropriate individuals or objects, whereas *notional* beliefs involve a relation between the believer and a dictum - such as a proposition.[58] (In fact, Burge uses the terms de re/de dicto to make this distinction since he first of all alludes to it in terms of the scope of the existential quantifier.) Thus, Burge seems to be claiming that we could not even have notional beliefs (defined as above) without having at least some relational beliefs. In this sense, relational beliefs are more fundamental than notional beliefs and hence the former is not reducible to the latter.

As support for his fundamentality thesis, Burge cites the case of a computer which has perfect mastery of the syntax of some mathematical language. In such a case, it would presumably be unwarranted to claim that the computer has any *understanding* of this language since this would require an ability on the computer's part to interpret its symbols. But the ability to interpret the symbols of a language would, claims Burge, require the ability to make non-linguistic or non-conceptual correlations between the symbols and what they denote.[59] Further, Burge claims that the ability to make non-conceptual correlations between symbol and referent

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56 Burge (1977), p. 350.

57 Burge (1977), p. 349.

58 Burge (1977), pp. 345-6.

59 Burge (1977), p. 347.

presumably presupposes that the believer has relational beliefs. Therefore, we could not even attribute to the computer notional beliefs - since the ability to understand dicta such as propositions presumably requires the ability to have relational beliefs. And so Burge concludes that in general, a necessary condition for attributing beliefs (including notional ones) to an agent is that we can also attribute to the agent (irreducibly) relational beliefs.[60]

Burge then sees Kaplan's reducibility thesis as posing a potential threat to his fundamentality thesis and so he attempts to refute Kaplan's position. This of course bears directly on Hintikka's treatment of belief since he too subscribes to the reducibility thesis. To counter Kaplan's position, Burge argues that there can be cases where an agent has a relational belief (in Burge's sense of the term) and yet, we would not say in such a case that the agent has a vivid name that denotes the individual who is the object of his belief. One example which Burge cites is that we may see someone walking in the distance and not see him clearly enough to individuate him. Thus, he does not play a central role in our 'inner story', meaning that we do not possess a vivid name of this individual and yet, we may believe *of this individual* that he has a red cap - which is a relational belief. Therefore, there is something irreducibly non-conceptual about relational belief (in the sense that we can have such beliefs without possessing vivid names) and hence, they are not mere species of notional beliefs. The reducibility thesis is therefore wrong.

Ernest Sosa has suggested adopting a more modest version of Kaplan's reducibility thesis, viz., that to have a relational belief, it is not necessary that the agent have a vivid name of the relevant individual(s) in his pos-

[60] Burge (1977), p. 348.

session, but merely a 'distinguished term'.[61] As Sosa notes, a distinguished term can vary from context to context - sometimes it may be associated with a complex of intricate descriptions and sometimes it may simply be a name in the absence of descriptions.[62] Then this would allow for cases where an indivdiual does not play a central role in the agent's inner story and yet, the agent may still have a relational belief concerning this individual. Thus, in Burge's case of the man with the red cap, Sosa would say that the believer has a distinguished term for this individual, viz., 'the man in the distance with the red cap' although this may be the only description of this individual the believer possesses. In response to this, Burge would retort that even though an agent will often or even always have a distinguised term (in Sosa's sense) in his/her possession, the fact that distinguised terms are contextually dependent indicates that there is something 'irredicibly' non-linguistic or conceptual about relational belief.[63]

It is not initially clear that Burge's indictment of Kaplan's vivid name criterion for having relational beliefs is thereby an indictment of Hintikka's notion of 'having an opinion as to who (or what) t is'. This is because having a vivid name for an object y is (as we have suggested) merely a *sufficient* and not a neccessry condition for having an opinion as to who or what some individual is. There is no reason why a believer x couldn't have an opinion as to who or what some individual t is even though x doesn't have an elaborate set of descriptions or a vivid name by means of which to pick out or identify t.[64] Therefore, it would not be inconsistent with Hintikka's characterization of *relational* belief (in some sense of the term)

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[61] Sosa (1970), p. 890.

[62] ibid, pp. 890-1.

[63] Burge (1977), p. 352.

[64] For example, see Hintikka (1962), p. 149.

that one could have a relational belief without thereby having any set of descriptions associated with it.

However, extending Burge's indictment of Kaplan's vivid name critierion to Sosa's 'distinguisded term' criterion for attributing to an agent relational beliefs does thereby call into question Hintikka's 'acquaintance' condition for attributing relational beliefs to agents. This is because minimally, having an opinion as to who t is requires that the agent possesses a singular term (name or description) of the individual, viz., t. (I.e., the locution $(\exists v)B(v = \text{Tully})$ says that there is some individual v such that x beleves that v is Tully.) But distinguished terms in certain types of contexts can turn out to be merely singular terms - names or definite descriptions.

Nonetheless, although Hintikka like Kaplan subscribes to the reducibility thesis and is thereby open to Burge's objection just discussed, there is a certain sense in which for Hintikka, relational beliefs are irreducible. I.e., the acquaintance locution whose schematic form is '$(\exists v)B(t = v)$' which is conjoined with a notional locution in a *relational* construction of the form '$B\alpha(t/v)$ & $(\exists v)B(t = v)$' is itself a *de re* construction which is inferable from a relational construction. I.e., one of the components of a relational construction is itself relational. As Hintikka himself has noted, the de re locution '$(\exists v)B(t = v)$' is not inferable from the notional locution $B(t = t')$ unless this in turn is conjoined with the de re locution $(\exists v)B(t = v)$. Also, this stricture prevents the sort of case where from Jones' belief that Tully is (identical with) Tully we infer that Jones has an opinion as to who Tully is. I.e., it prevents all inferences of the form $B(t = t) \longrightarrow (\exists v)B(t = v)$ since the conclusion $(\exists v)B(t = v)$ must also be a premise of the argument.[65] In short, we must beg the question for any such inference to go through.[66]

Now that we have considered in some detail Hintikka's views on quantifying in, which relies on making a distinction between relational and notional belief (although Hintikka himself does not employ this terminology) such that the former is 'reducible' to the latter, we shall next consider his suggestions for dealing with the problem of the failure of the substitutivity principle for first-order belief logic. As we shall see, Hintikka's solution to this problem also relies on distinguishing between notional and relational constructions - although the relational constructions are of a special type.

6. Hintikka's Treatment of the Apparent Failure of Substitutivity of Co-referentials for Belief Contexts

Vis a vis the sorts of examples we considered in section 1, the following schemata which allow unrestricted substitution of co-referentials in doxastic contexts were called into question qua principles of belief attribution:

i) $(\alpha(t_1/v) \,\&\, t_1 = t_2) \supset \alpha(t_2/v)$ where t_1 and t_2 occurring in $\alpha(t_1/v)$ and $\alpha(t_2/v)$ respectively, may occur in the scope of a doxastic operator(s).

ii) $(\mathbf{B}\alpha(t_1/v) \,\&\, t_1 = t_2) \supset \sim\mathbf{B}\sim\alpha(t_2/v)$ for $SQC^{=} + D$ systems only.

Corresponding to these schemata are the following rules of inference:

iii) $\alpha(t_1/v),\ t_1 = t_2 \longrightarrow \alpha(t_2/v)$ where t_1 and t_2 occurring in $\alpha(t_1/v)$ and $\alpha(t_2/v)$ respectively may occur in

[65] Hintikka (1962), p. 145-6.

[66] ibid, p. 145. As Hintikka further notes, this type of stricture is similar to the stricture imposed in free logics which would prevent the inference $t = t \longrightarrow (\exists v)(v = t)$ since in free logic, an additional required premise is the conclusion $(\exists v)(v = t)$.

the scope of a doxastic operator(s).

iv) $B\alpha(t_1/v),\ t_1 = t_2 \longrightarrow {\sim}B{\sim}\alpha(t_2/v)$ for $SQC^= + D$ systems only.

Notice that i) and iii) are more general versions of $(B\alpha(t_1/v)\ \&\ t_1 = t_2) \supset B\alpha(t_2/v)$ and $B\alpha(t_1/v),\ t_1 = t_2 \longrightarrow B\alpha(t_2/v)$ respectively. Thus, we have stipulated in the case of i) and its corresponding inferential version iii) that t_1 and t_2 may occur in the scope of *doxastic* operators (which includes both 'B' and 'P_B'). The following would both be instances of i):

i)$_1$ $(BFa\ \&\ a = b) \supset BFb$

ii)$_2$ $((Gc\ \&\ (BFa \vee P_BHa))\ \&\ a = b) \supset (Gc\ \&\ (BFb \vee P_BHb))$

Supposing the undesirability of i) - iv), then we would not want any of their instances as theses/inference rules for a first-order logic of belief. To this end, all instances of the schema i) can be blocked (as theses) for any system of quantified belief logic by restricting the $SQC^=$ axiom-schema $(\alpha(t_1/v)\ \&\ t_1 = t_2) \supset \alpha(t_2/v)$ to cases where t_1, t_2 do not occur within the scope of a doxastic operator in $\alpha(t_1/v)$, $\alpha(t_2/v)$. Or, we could simply stipulate that $\alpha(t_1/v)$, $\alpha(t_2/v)$ are wffs of standard first-order logic. Further, in restricting this axiom-schema in this way, we thereby render all instances of ii) - iv) underivable for our quantified doxastic logic since they depend on the appropriate instances of i) for their derivation. (In the case of the derivation of any instance of ii), see section 1.)

Also, by restricting the axiom-schema $(\alpha(t_1/v)\ \&\ t_1 = t_2) \supset \alpha(t_2/v)$ to cases where t_1, t_2 do not occur within the scope of a doxastic operator in $\alpha(t_1/v)$, $\alpha(t_2/v)$ we also block the derivation of any instance of the $SQC^=$ thesis-schema $t_1 = t_2 \supset B(t_1 = t_2)$ which as was noted in section 1 of chapter two is derivable by appealing to the unrestricted version of this axiom-schema. It was also noted that the schema $t_1 = t_2 \supset B(t_1 = t_2)$ asserts the somewhat implausible principle that agents are omnidoxastic with respect to contingent identities. Thus, by restricting $(\alpha(t_1/v) \& t_1 = t_2) \supset \alpha(t_2/v)$ in the way suggested above, we shall thereby rid our logic of the undesirable feature that agents are omnidoxastic with respect to identities.

Since we shall want to eliminate i) - iv) as thesis-schemata/inference rules of our quantified logic of belief, we shall consider alternative schemata/rules to replace them, such that these schemata/rules do not allow unrestricted substitution of co-referentials in belief (or more generally, doxastic) contexts. Hintikka has two suggestions along these lines.

As a first stab, Hintikka's way of handling the counterexamples to the schemata and rules of inference i) - iv) is to simply require that the relevant identity must occur within the scope of a belief operator.[67] I.e., in the case of the schemata i) and ii), we add to the antecedent '$\mathbf{B}(t_1 = t_2)$' which says that x *believes* that the identitiy $t_1 = t_2$ obtains and in the case of the rules of inference we make an analogous move. So, the emendated versions of i) - iv) will be:

i)* $(\alpha(t_1/v) \& t_1 = t_2 \& \mathbf{B}(t_1 = t_2)) \supset \alpha(t_2/v)$ where t_1 and t_2 occurring in $\alpha(t_1/v)$ and $\alpha(t_2/v)$ respectively, may occur in the scope of a doxastic operator(s).

ii)* $(\mathbf{B}\alpha(t_1/v) \& t_1 = t_2 \& \mathbf{B}(t_1 = t_2)) \supset {\sim}\mathbf{B}{\sim}\alpha(t_2/v)$ for SQC$^=$ + D systems.

iii)* $\alpha(t_1/v),\ t_1 = t_2,\ \mathbf{B}(t_1 = t_2) \longrightarrow \alpha(t_2/v)$ where t_1 and t_2 occurring in $\alpha(t_1/v)$ and $\alpha(t_2/v)$ respectively may occur in the scope of a doxastic operator(s).

iv)* $\mathbf{B}\alpha(t_1/v),\ t_1 = t_2,\ \mathbf{B}(t_1 = t_2) \longrightarrow {\sim}\mathbf{B}{\sim}\alpha(t_2/v)$ for SQC$^=$ + D systems

In the case of i)* and iii)*, the conjunct '$t_1 = t_2$' in the antecedent of i)* and the premise '$t_1 = t_2$' in iii)* are both superfluous if $\alpha(t_1/v)$ and

[67] See Hintikka (1967), p. 55 and Hintikka (1969) reprinted in Linsky (1971), p. 155.

$\alpha(t_2/v)$ are $B\beta(t_1/v)$, $B\beta(t_2/v)$ or are $P_B\beta(t_1/v)$, $P_B\beta(t_2/v)$. Also, $t_1 = t_2$ could be deleted from ii)* and iv)* without any loss of plausibility. This is owing to the fact that in purely doxastic constructions all that matters is that the agent *believes* that $t_1 = t_2$ for substitution to go through. This will become evident when we consider Hintikka's semantic proposals.

Note that the locution $B(t_1 = t_2)$ in the above constructions is notional - it is not assumed that the agent has any opinions as to who the individual denoted by t_1 and t_2 is. The agent x merely believes that the identity $t_1 = t_2$ obtains. And this would seem to guarantee that substitution will go through, as we shall now demonstrate.

Consider the counterexample outlined in section 1 to i) and iii), viz., the case where Jones believes that Cicero was an orator and yet, given the *strengthened* disquotation principle, Jones may withold assent to the claim that Tully was an orator in which case, Jones does not believe that Tully was an orator (even though exactly one person is denoted by 'Cicero' and 'Tully'). However, if it is stipulated beforehand that Jones believes that the identity 'Tully = Cicero' obtains, then presumably this situation would not arise. Similar remarks apply to the counterexample to ii) and iv). Further, since it is no longer assumed that agents are omnidoxastic with respect to contingent identities if we block all instances of $t_1 = t_2 \supset B(t_1 = t_2)$, then from the mere fact that an identity actually obtains, it does not follow that the agent believes that it does. Thus in the Tully/Cicero example, the reason that Jones does not believe that Tully is an orator even though he believes that Cicero is an orator is that he is not logically omnidoxastic with respect to contingent identities.

A second suggestion which Hintikka proposes for dealing with the failure of substitutivity of co-referentials for belief contexts is the following: Suppose that the agent x has an opinion as to who some individual denoted by t_1 is and that x also has an opinion as to who the individual denoted by t_2 is. Suppose further that the agent's opinions as to who t_1 and t_2 are, are *true* - i.e., his opinions hold sway in the actual world. Then according to Hintikka, we could render this situation symbolically as '$(\exists v)(v = t_1 \& \mathbf{B}(v = t_1)) \& (\exists v)(v = t_2 \& \mathbf{B}(v = t_2))$' which intuitively says that x has *true* opinions as to who the individuals denoted by t_1 and t_2 are. Further, suppose that the identity $t_1 = t_2$ actually obtains, or more neutrally, that this identity obtains at the index which the agent 'inhabits'. Then since x's opinions (as to who t_1 and t_2 are) are true and given that t_1 and t_2 are 'in fact' identical, it would seem to follow that x will recognize or believe that t_1 and t_2 are identical. I.e., Hintikka is here suggesting that from $(\exists v)(v = t_1 \& \mathbf{B}(v = t_1))$, $(\exists v)(v = t_1 \& \mathbf{B}(v = t_1))$ and $t_1 = t_2$ we are warranted in inferring that $\mathbf{B}(t_1 = t_2)$.[68] Notice that both $(\exists v)(v = t_1 \& \mathbf{B}(v = t_1))$ and $(\exists v)(v = t_1 \& \mathbf{B}(v = t_1))$ are special sorts of 'acquaintance' locutions - such that the opinions that x has about t_1 and t_2 are in fact true.

Vis a vis Hintikka's remarks here, one of the emendations we shall propose to the $SQC^{=}$ systems will be to add as an axiom-schema,

v) $((\exists v)(v = t_1 \& \mathbf{B}(v = t_1)) \& (\exists v)(v = t_2 \& \mathbf{B}(v = t_2)) \& t_1 = t_2) \supset \mathbf{B}(t_1 = t_2)$

Given v) as an axiom-schema, we could obtain its inferential version, viz.,

vi) $((\exists v)(v = t_1 \& \mathbf{B}(v = t_1)), (\exists v)(v = t_2 \& \mathbf{B}(v = t_2)), t_1 = t_2 \longrightarrow (\mathbf{B}(t_1 = t_2)$

Once again, v) and vi) both express the attributive principle that if x has true opinions as to who two individuals are, and these two individuals are

68 See Hintikka (1967), pp. 55-56. Hintikka makes this suggetion for epistemic contexts.

'in fact' identical then x will believe (notionally) that this identity obtains.

It is also worth noting that if an agent x has a true opinion as to who some individual denoted by t_1 is and if the identity $t_1 = t_2$ obtains then presumably x will thereby have a true opinion as to who the individual denoted by t_2 is (viz., the same person who is denoted by t_1). Thus, Hintikka would supposedly endorse the schema $((\exists v)(v = t_1 \& B(v = t_1)) \& t_1 = t_2) \supset (\exists v)(v = t_2 \& B(v = t_2)$.

We shall presently show how Hintikka's principle can be used to explain why the substitutivity of identicals sometimes fails for belief contexts. First, however, we shall consider a possible objection to the principle expressed by v) and vi). It could be objected that in a case such as Kripke's Paderewski example discussed in section 1, Jones has true opinions as to who Paderewski is, viz., both a politician and a pianist. Yet Jones does not recognize that Paderewski the pianist and Paderewski the politician are one and the same person - under different descriptions. Then this is a counterexample to Hintikka's principle that if an agent x has true opinions as to t_1 and t_2 are and if t_1 and t_2 are one and the same person then x will recognize that this identity obtains.

We shall now consider a possible response which Hintikka could make to this objection. It could be countered that in the Paderewski example, although Jones has opinions as to who Paderewski is - under certain descriptions - and although these descriptions are true, it is hasty to conclude that Jones *knows* (or even has a *true* opinion as to) who Paderewski is. I.e., it could be claimed that although having true descriptions of someone may often play a role in *knowing* who that person is - or having *true* opinions as to who x is - the possession of said descriptions is by no means *sufficient* (or even necessary) for knowing who someone is. Perhaps having a true opinion as to who someone is, is in part, or in some cases such as the ones Burge has outlined even wholly a matter of some sort of non-conceptual 'acquaintance' with the individual as Hintikka himself has suggested. Thus, in the Tully/Cicero case, Jones may have opinions as to who the individual named 'Tully' is and he may have opinions as to who the individual named 'Cicero' is, by virtue of the possession of a set of (true) descriptions. Yet, Jones may fail to make the connection that Tully and Cicero are one and the same person. Then in such a case one would be inclined to say that Jones does not know (nor does he have a *true* opinion as to) who Tully, i.e., Cicero is.

To continue with our exposition of Hintikka's second way of handling the failure of the substitutivity of co-referentials for belief contexts, he uses the principle (right or wrong) expressed by v) and vi) to explain why this failure sometimes occurs and to show that in certain types of relational contexts, substitution will go through.

Suppose that x believes that $\alpha\,(t_1/v)$ at w_i. Suppose further that x has true opinions as to who t_1 and t_2 are and that the identity '$t_1 = t_2$' obtains at the particular index. Then it follows by vi) above that x *believes* that $t_1 = t_2$ obtains. But by a more specific version of iii)*, viz., $\mathbf{B}\alpha\,(t_1/v)$, $t_1 = t_2$, $\mathbf{B}(t_1 = t_2) \longrightarrow \mathbf{B}\alpha\,(t_2/v)$, it follows that x also believes that $\alpha\,(t_2/v)$ at w_i. However, if x fails to have a true opinion as to who the individual denoted by t_1 and t_2 is then this sort of situation is not sufficient for inferrring that x believes that $\alpha\,(t_2/v)$.

To see how this way of handling co-referentials for belief contexts works, consider once more our example where Jones believes that Cicero was an orator and yet he does not believe that Tully was an orator. Hintikka's first explanation of why this sort of situation is possible is that Jones may not believe that Tully is identical to Cicero. Otherwise, he would (if he is in some sense of the term 'rational') also believe that Tully was an orator. Notice that in this example, since no assumptions are made to the effect that Jones is 'acquainted' with the individual Cicero (i.e., Tully) then it follows that Jones' belief that Cicero is an orator is best treated as being *notional*. However, what Hintikka's second proposal for handling co-referentials in belief contexts amounts to is that if Jones' belief that Cicero is an orator is *relational* in a special sense, viz., that he has a *true* opinion as to who Cicero is in which case given that Tully = Cicero, Jones has a *true* opinion as to who Tully is then it follows that Jones believes that Tully is identical with Cicero.[69] And given that Jones believes that Cicero was an orator, it follows (by iii)*) that Jones also believes that Tully was an orator. If Jones fails to have true opinions as to who the indivdual denoted by both 'Tully' and 'Cicero' is then we are not warranted in inferring from his belief that Cicero was an orator that Tully was an orator.

Thus, Hintikka's explanations of the Tully/Cicero example are 1) Jones does not believe (notionally) that the same indivdual is denoted by these terms or 2) Jones does not have *true* opinions as to who Tully, i.e., Cicero is. Further, Hintikka's two explanations are linked as follows: If Jones does have a true opinion as to who Tully, i.e., Cicero is then he will thereby believe (notionally) that Tully is identical to Cicero from which it follows that if he believes that Tully was an orator he believes that Cicero was an orator. On the other hand, Jones' believing notionally that Tully is identical to Cicero will not be sufficient for claiming that he has true

[69] See once again Hintikka (1967), pp. 55–6.

opinions as to who Tully, i.e. Cicero is. He may believe that the identity 'Tully = Cicero' obtains without his having the slightest idea as to who this individual is.

Finally, in terms of adopting Hintikka's second way of dealing with co-referentials in belief contexts (in addition to his first way), if we have already revised the SQC* systems in such a way that i)* and v) discussed above are added as axiom-schemata, then any instance of the following schema can be easily derived for these emendated systems (as we shall demonstrate in the next chapter):

vii) $(\alpha(t_1/v)\ \&\ (\exists v)(v = t_1\ \&\ B(v = t_1))\ \&\ (\exists v)(v = t_2\ \&\ B(v = t_2))\ \&$
$t_1 = t_2) \supset \alpha(t_2/v)$ - where t_1 and t_2 occurring in $\alpha(t_1/v)$, $\alpha(t_2/v)$ respectively may occur within the scope of a doxastic operator(s).

Given vii), the following will be a derived rule of inference:

viii) $\alpha(t_1/v),\ (\exists v)(v = t_1\ \&\ B(v = t_1)),\ (\exists v)(v = t_2\ \&\ B(v = t_2)),\ t_1 = t_2$
$\longrightarrow \alpha(t_2/v)$ - where t_1 and t_2 occurring in $\alpha(t_1/v)$, $\alpha(t_2/v)$ respectively may occur within the scope of a doxastic operator(s).

Although both vii) and viii) are rather horrendous-looking, they merely express the principle that co-referentials are intersubstitutible in belief contexts provided that the agent has true opinions as to who the referents of these terms are.

Concluding Remarks:

Now that Hintikka's proposals for dealing with 1) the problem of quantifying in and 2) the problem of the failure of substitutivity of co-referentials for belief contexts have been discussed, we shall in the next chaper systematize his proposals on the axiomatic front vis a vis emendations to the SQC$^{=}$ systems discussed in chapter two. Also, we shall discuss Hintikka's proposals on the *semantic* front for dealing with the two problems just mentioned and a characteristic semantics for the emendated SQC$^{=}$ systems based on his proposals will be developed.

Finally, since quantifying in is only problematic for systems where the quantifiers are construed objectually, we shall propose a collection of logics where the quantifiers are interpreted substitutionally and such that quantification into belief contexts is unrestricted and yet substitution of co-referentials in belief contexts is prohibited. These logics will also be emendated versions of the SQC$^{=}$ systems discussed in chapter two, and their semantics will be emendated versions of the TV semantics also discussed in that chapter. It will then be argued that we should adopt the substitutional SQC$^{=}$ systems rather than the emendated systems based on Hintikka's proposals since the former has a less problematic semantics than the latter and because the former does not posit two types of belief.

Chapter Four

Some Proposals for a Quantified Logic of Belief

1. The Hin-SQC= Systems of Doxastic Quantificational Calculi

To bring together Hintikka's various suggestions for a logic of belief which were discussed in the previous chapter into some kind of coherent whole, we shall now propose a set of alternative axiom systems to the $SQC^=$ axiom systems for belief logic. These alternative axiom systems are based on Hintikka's proposals for dealing with the apparent failure of the substitutivity principle for belief contexts as well as his proposed stricture with respect to quantifying into such contexts. We shall call the following set of axiom schemata and rules of inference the system Hin-$KQC^=$ such that any Hin-$SQC^=$ system *not* containing 4, $\mathbf{B}\alpha \supset \mathbf{BB}\alpha$ can be obtained by 'extending' the doxastic sentential fragment of Hin-$KQC^=$ in the way described in the first chapter. (For example, by adding the schema D, viz., $\mathbf{B}\alpha \supset \mathbf{P_B}\alpha$ to Hin-$KQC^=$ we would obtain the system Hin-$KDQC^=$, and so on.):

AS 1: α (where α has the form of any PC thesis-schema)

AS 2: $(\mathbf{B}\alpha \;\&\; \mathbf{B}(\alpha \supset \beta)) \supset \mathbf{B}\beta$

AS 3: $\alpha(t/v) \supset (\exists v)\alpha$ (provided that t does not occur within the scope of a doxastic operator)

AS 4: $(\alpha(t/v) \;\&\; (\exists v)(v = t \;\&\; \mathbf{B}(v = t)) \supset (\exists v)\alpha$ (where t may occur within the scope of a doxastic operator(s) and where there is no iteration of any doxastic operator.)

AS 5: $t = t$

AS 6: $(\alpha(t_1/v) \,\&\, t_1 = t_2) \supset \alpha(t_2/v)$ (provided that t_1, t_2 do not occur in the scope of a doxastic operator)

AS 7: $(\alpha(t_1/v) \,\&\, t_1 = t_2 \,\&\, B(t_1 = t_2)) \supset \alpha(t_2/v)$ (provided that t_1, t_2 may occur in the scope of a doxastic operator or possibly several and where there is no iteration of any such operators.)

AS 8: $(t_1 = t_2 \,\&\, (\exists v)(v = t_1 \,\&\, B(v = t_1)) \,\&\, (\exists v)(v = t_2 \,\&\, B(v = t_2)) \supset B(t_1 = t_2)$

The primitive rules of inference will be:

MP: $\alpha,\ \alpha \supset \beta \longrightarrow \beta$

R$\exists$: $\vdash \alpha(t/v) \supset \beta \longrightarrow \vdash (\exists v)\alpha \supset \beta$ (for any constant t foreign to $(\exists v)\alpha \supset \beta$ and provided that t does not occur in the scope of a doxastic operator.)

RB: $\vdash \alpha \longrightarrow \vdash B\alpha$

AS 3 and R$\exists$ prohibit unrestricted quantification into belief contexts as well as quantification into contexts of doxastic possibility. Thus for example, neither of the following are instances of AS 3, viz., $BFa \supset (\exists x)BFx$ and $P_BFa \supset (\exists x)P_BFx$ (the latter being equivalent to $\sim B\sim Fa \supset (\exists x)\sim B\sim Fx$). AS 4 in effect restricts quantifying in (for both doxastic necessity and possibility) to *relational* contexts.

The following is a version of AS 4 for doxastic possibility, $(P_B\alpha(t/v) \,\&\, ((\exists v)(v = t \,\&\, B(v = t)) \supset (\exists v)P_B\alpha$. It could be objected that in the Hintikka systems, we are committed ontologically to possibilia (and in this case there is supposedly no reduction schema to mitigate the situation - although we shall leave this question open), provided that the agent is 'truly acquainted' with the appropriate individual. For example, an ordinary language instance of this schema might be "If it is true for all Jones believes that Pegasus is a winged horse and he has a true opinion as to who Pegasus is, then there is something such that it is possible for all Jones believes that it is a winged horse". However, it can be countered that we are not here committed ontologically to (doxastically) possible winged horses since it must be the case that (from a semantic point of view) 'Pegasus' denotes something existing at Jones' world (because his opinion is 'true') as well as at all the doxastic alternatives to his world for it to be the case that *there is* something such that it is *possible* for all he believes that it is a winged horse. This will be discussed in further detail in the next section once the semantics for our Hin-SQC$^{=}$ systems has been developed.

To continue, AS 6 prohibits unrestricted substitution of co-referentials in contexts of doxastic necessity (and possibility) and AS 7 restricts substitution of co-referentials in doxastic contexts to cases where the agent believes (notionally) that the relevant identity obtains. AS 8 states that having true opinions as to who t_1 and t_2 are, viz., one and the same person is a sufficient condition for x's believing that $t_1 = t_2$. AS 8 in conjunction with AS 7 can be used to derive TS 1 (described below) which restricts substitution of co-referentials to special sorts of relational contexts. Also, for Hin-SQC$^{=}$ systems containing D, AS 8 in conjunction with D, $\mathbf{B}\alpha \supset \mathbf{P}_{\mathbf{B}}\alpha$ ensures the derivability of the following variant of AS 8 for doxastic possibility:

$$(t_1 = t_2 \;\&\; (\exists v)(v = t_1 \;\&\; \mathbf{B}(v = t_1)) \;\&\; (\exists v)(v = t_2 \;\&\; \mathbf{B}(v = t_2)) \supset \mathbf{P}_{\mathbf{B}}(t_1 = t_2)$$

This schema says that having true opinions as to who t_1 and t_2 are, viz., one and the same person is a sufficient condition for it being consistent with all x believes that $t_1 = t_2$.

Note that instead of the schema $(B\alpha(t/v) \;\&\; (\exists v)B(v = t)) \supset (\exists v)B\alpha$ where $\alpha(t/v)$ contains no doxastic operators within whose scope t lies, we introduced the more general schema $(\alpha(t/v) \;\&\; (\exists v)(v = t \;\&\; B(v = t)) \supset (\exists v)\alpha$ as an axiom-schema (AS 4) for the Hintikka systems (where $\alpha(t/v)$ contains no iterated belief (and for that matter no iterated doxastic possibility) operators within whose scope the constant t lies). The reason for making this axiom-schema more general, in the sense that $\alpha(t/v)$ may involve doxastic wffs but may not itself be a doxastic wff, is in order to ensure completeness of the Hin-SQC$^=$ systems with respect to the semantics we shall consider in the next section. Two instances of AS 4 would be $(BFa \;\&\; (\exists x)(x = a \;\&\; B(x = a)) \supset (\exists x)BFx$ as well as $((BGa \vee BFa) \;\&\; (\exists x)(x = a \;\&\; B(x = a)) \supset (\exists x)(BGx \vee BFx)$. Similar remarks apply to the axiom-schema AS 7, $(\alpha(t_1/v) \;\&\; t_1 = t_2 \;\&\; B(t_1 = t_2)) \supset \alpha(t_2/v)$ since it is a more general version of $(B\alpha(t_1/v) \;\&\; t_1 = t_2 \;\&\; B(t_1 = t_2)) \supset B\alpha(t_2/v)$.

If we were to include in the axiom-set the schema 4 then the resulting Hin-KQC$^=$ + 4 system (or more generally any Hin-SQC$^=$ + 4 system) would differ from the Hin-SQC$^=$ systems in the following respect: The proviso for AS 4, AS 7 and $R\exists_2$ viz., that there is no iteration of the relevant doxastic operator(s), would be lifted for reasons to be discussed in the next section when we come to consider the semantics for these axiom sets.

What is noteworthy about the above set of axiom-schemata both containing and not containing 4 is that it does not contain the Barcan Formula $(\forall v)B\alpha \supset B(\forall v)\alpha$. The reasons for not including the Barcan Formula in the axiomatic base is that this base would not be *sound* relative to the semantics which Hintikka proposes. This will be discussed in section 3.

Also, a somewhat counterintuitive result of restricting the rule of inference R$\exists$, viz., $\alpha(t/v) \supset \beta \longrightarrow (\exists v)\alpha \supset \beta$ such that t is foreign to $(\exists v)\alpha \supset \beta$ and provided that t is not in the scope of a doxastic operator is that the proof of the implicational schema $(\exists v)\mathbf{B}\alpha \supset \mathbf{B}(\exists v)\alpha$ asserting that belief de re entails belief de dicto is effectively blocked. As was discussed in section 1 of chapter two, the unrestricted rule R$\exists$ where it is *not* required that t is not in the scope of a belief operator is integral to the derivation of any instance of $(\exists v)\mathbf{B}\alpha \supset \mathbf{B}(\exists v)\alpha$. The underivability of this schema in any of the Hin-SQC$^{=}$ systems turns out to be a necessary evil to guarantee soundness of the axiom system relative to the semantics discussed in the next section. Further, the reduction schema for doxastic possibility, viz., $(\exists v)\mathbf{P_B}\alpha \equiv \mathbf{P_B}(\exists v)\alpha$ is not a thesis-schema for the the Hin-SQC$^{=}$ systems since AS 3 mentioned above is restricted and given that the Barcan Formula is not a thesis-schema. However, this reduction schema is not needed for the purpose of eliminating all instances of quantifying into doxastic possibility constructions since the schema $\mathbf{P_B}\alpha(t/v) \supset (\exists v)\mathbf{P_B}\alpha$ is not a version of AS 3, viz., $\alpha(t/v) \supset (\exists v)\alpha$ and if we are right, since wffs such as $(\mathbf{P_B}\alpha(t/v)\ \&\ ((\exists v)(v = t\ \&\ \mathbf{B}(v = t)) \supset (\exists v)\mathbf{P_B}\alpha$ are innocuous from an ontological point of view.

The following is a theorem-schema for the Hin-SQC$^{=}$ systems:

TS 1: $(\alpha(t_1/v)\ \&\ t_1 = t_2\ \&\ (\exists v)(v = t_1\ \&\ \mathbf{B}(v = t_1)\ \&\ (\exists v)(v = t_2\ \&\ \mathbf{B}(v = t_2))$
$\supset \alpha(t_2/v)$ (where t_1, t_2 may occur in the scope of doxasti operator(s) and such that there is no iteration of said belief operator(s) - for systems not containing 4.)

As was noted in the previous chapter, TS 1 restricts substitution of co-referentials to special sorts of relational contexts, viz., contexts where the agent has true opinions as to who t_1 and t_2 are, i.e., one and the same person. Further, the following are rules of inference which are derivable for any of the Hin-SQC$^=$ systems:

DR 1: $\alpha(t/v),\ (\exists v)(v = t \,\&\, \mathbf{B}(v = t)) \longrightarrow (\exists v)\alpha$ (with the same provisos as for AS 4.)

DR 2: $\alpha(t_1/v),\ t_1 = t_2,\ \mathbf{B}(t_1 = t_2) \longrightarrow \alpha(t_2/v)$ (with the same provisos as for AS 7.)

DR 3: $(\alpha(t_1/v) \,\&\, t_1 = t_2 \,\&\, (\exists v)(v = t_1 \,\&\, \mathbf{B}(v = t_1) \,\&\, (\exists v)(v = t_2 \,\&\, \mathbf{B}(v = t_2)) \longrightarrow \alpha(t_2/v)$ (with the same provisos as for TS 1.)

First of all, any instance of TS 1 is derivable in any Hin-SQC$^=$ system using AS 7 and AS 8 as follows (where the provisos mentioned above are understood and where it is also understood that there is no iteration of the belief operator(s) within whose scope any of the relevant constants occur for systems not containing 4):

1. $(t_1 = t_2 \,\&\, (\exists v)(v = t_1 \,\&\, \mathbf{B}(v = t_1)) \,\&\, (\exists v)(v = t_2 \,\&\, \mathbf{B}(v = t_2)) \supset \mathbf{B}(t_1 = t_2)$ AS 8
2. $(t_1 = t_2 \,\&\, (\exists v)(v = t_1 \,\&\, \mathbf{B}(v = t_1)) \,\&\, (\exists v)(v = t_2 \,\&\, \mathbf{B}(v = t_2)) \supset (\mathbf{B}(t_1 = t_2) \,\&\, t_1 = t_2)$ 1, P.C.[1]
3. $\mathbf{B}(t_1 = t_2) \,\&\, t_1 = t_2) \supset (\alpha(t_1/v) \supset \alpha(t_2/v))$ AS 7
4. $(t_1 = t_2 \,\&\, (\exists v)(v = t_1 \,\&\, \mathbf{B}(v = t_1)) \,\&\, (\exists v)(v = t_2 \,\&\, \mathbf{B}(v = t_2)) \supset (\alpha(t_1/v) \supset \alpha(t_2/v))$ 1,2 P.C.
5. $(\alpha(t_1/v) \,\&\, t_1 = t_2 \,\&\, (\exists v)(v = t_1 \,\&\, \mathbf{B}(v = t_1) \,\&\, (\exists v)(v = t_2 \,\&\, \mathbf{B}(v = t_2)) \supset \alpha(t_2/v)$ 3, P.C.

Q.E.D.

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[1] I.e., $\vdash[(\alpha \,\&\, \beta \,\&\, \gamma) \supset \delta] \equiv [(\alpha \,\&\, \beta \,\&\, \gamma) \supset (\delta \,\&\, \alpha)]$.

Finally, given TS 1, DR 3 is derivable using modus ponens for the Hin-$SQC^=$ systems. Also, given the appropriate versions of AS 4 and AS 7 along with modus ponens, both DR 1 and DR 2 respectively are derivable for any Hin-$SQC^=$ system.

By way of some final remarks, the Hin-$SQC^=$ systems inherit the problem of deduction as discussed in chapter one. This is owing to the fact that these systems like their $SQC^=$ counterparts are normal modal systems in the sense that they contain the schema K and have RB as a rule of inference. Thus, the logical omnidoxasticity inference rule $\vdash \alpha \supset \beta \longrightarrow B\alpha \supset B\beta$ and the adjunction schema $(B\alpha \ \& \ B\beta) \supset B(\alpha \ \& \ \beta)$ are derivable in the Hin-$SQC^=$ systems and the consistency schemata $\sim B(\alpha \ \& \ \sim\alpha)$ and $\sim(B\alpha \ \& \ B\sim\alpha)$ are derivable in any Hin-$SQC^=$ + D system. Further, since as we have seen, there is no iteration of the belief operator for various thesis-schemata allowing quantification into notional locutions occurring in relational contexts (or allowing substitution of co-referentials for special sorts of relational contexts) for Hin-$SQC^=$ systems not containing 4, there is some presumption in favour of adopting as a system of doxastic logic any Hin-$SQC^=$ + 4 system, although we shall not push this point.

In the next section, we shall examine Hintikka's proposals on the semantic front for dealing with the failure of the substitutivity principle and for dealing with the problem of quantifying in. Based on his proposals, we shall attempt to develop a semantics with respect to which the Hin-$SQC^=$ systems are sound as well as complete. I.e., we shall argue that this semantics (which is a 'varying domain semantics) is characteristic for the Hin-$SQC^=$ systems just discussed.

2. Hintikka's Suggestions for a Semantics of Belief

As we shall see, a distinctive feature of Hintikka's semantics for belief logic is that each index in the set W of indices in a model has associated with it its own set of individuals (such that these indexed domains are non-overlapping), in distinction to the domain semantics for the $SQC^=$ systems where each member of W in a model shares the same set of individuals. This distinctive feature of Hinitkka's semantics is tied up with his solution to the problem that co-referentials are not unrestrictedly intersubstitutible in belief contexts. How this is so, we shall describe below.

Recall that in order to deny thesis-hood to any wff of the form $(\alpha(t_1/v) \mathbin{\&} t_1 = t_2) \supset \alpha(t_2/v)$ where t_1, t_2 may occur in the scope of a doxastic operator, it was necessary to restrict the axiom-schema $(\alpha(t_1/v) \mathbin{\&} t_1 = t_2) \supset \alpha(t_2/v)$ to cases where t_1 and t_2 do not occur in the scope of any doxastic operators for the Hin-$SQC^=$ axiom systems. To deny thesis-hood to the doxastic version of $(\alpha(t_1/v) \mathbin{\&} t_1 = t_2) \supset \alpha(t_2/v)$ is tantamount to denying that co-referentials are intersubstitutible for notional belief constructions. But what does this proposal amount to on the semantic front and more specifically within the framework of an indexical or possible worlds semantics for belief? To answer this, it will first be helpful by way of review to outline just what an indexical or possible worlds semantics amounts to for a quantified doxastic logic.

In an indexical or possible worlds semantics for belief logic, to say that an agent x believes that $\alpha(t/v)$ at an index w_i in a model (or more formally, for $V_M(\mathbf{B}\alpha(t/v), w_i)$ to take the value '1') it must be the case that the content $\alpha(t/v)$ is true at all the *doxastic alternatives* to w_i. As Hintikka notes, the set of doxastic alternatives to a given index will be the set of indices at which all the content wffs of agents' beliefs will be true.[2] Further, we cashed out the notion of 'doxastic alternative' in terms of a so-called doxastic accessibility relation R which may or may not have various restrictions imposed on it depending on the axiom system we are considering. Thus, if our doxastic system is K then R is unrestricted, if it is D then R is serial and so on. For *belief* models, R must never be reflexive since agents can have false beliefs although for *epistemic* models, R must always be reflexive since, presumably, what one knows *is* the case at the 'actual' world.[3] In effect, what R does is to determine for any given index w_i at what indices all the content wffs of belief wffs true at w_i are true. This idea is central to the syntactic definition of R for the canonical model of any normal system, i.e., w_iRw_j iff $(\forall\alpha)(B\alpha \in w_i \longrightarrow \alpha \in w_j)$.

Hintikka in 'Semantics for Propositional Attitudes' (1969) cashes out the notion of 'doxastic alternative' in terms of a two-place function ϕ_B which to each 'world' (in Hintikka's parlance) w_i and to each individual assigns a set of worlds where any member w_j of this set will be such that all the contents of the beliefs which the individual holds at w_i will be true at w_j. Thus, syntactically we could define $\phi_B(t, w_i)$ for any $t \in D_i$ (where D_i is the set of individuals associated with w_i) and for any $w_i \in W$ as a function determining the set S where for any w_j in W, w_j is a member of S just in case $(\forall\alpha)$('t believes that α' $\in w_i \longrightarrow \alpha \in w_j$). Then '$t_1$ believes that $\alpha(t_2/v)$' is true at a world w_i just in case $\alpha(t_2/v)$ is true at all w_j such that $w_j \in$ (the set S determined by) $\phi_B(t_1, w_i)$.

Although Hintikka does not mention this issue, it is not clear how (sim-

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2 Hintikka (1969) reprinted in Linsky (1971), pp. 150–151.

3 By 'actual' world here we mean the world or index at which the belief wff is being evaluated.

ilar to the doxastic accessibility relation R) Φ_B could be appropriately restricted to fit a given *normal* axiom system. Thus, for a system containing K and 4 (i.e., $B\alpha \supset BB\alpha$) R must be transitive. But how could such a restriction be imposed on Φ_B? The function Φ_B is relativized not only to worlds but also to individuals and since (as we shall presently see) individuals are 'world-bound' in this type of semantics in the sense that the domains associated with each world are non-overlapping, we cannot characterize the transitivity requirement for Φ_B as follows: Φ_B for K4 systems must be such that for any individual t in D and any worlds w_i, w_j and w_k if $w_j \in \Phi_B(t,w_i)$, $w_k \in \Phi_B(t,w_j)$ then $w_k \in \Phi_B(t,w_i)$. This characterization is ill conceived because it assumes that t is a transworld individual. But the possibility of transworld individuals is strictly speaking disallowed given that world-associated domains are non-overlapping. The reason for this feature of Hintikka's semantics is (as we shall presently see) tied up with his attempt to deal with the failure of the substitutivity principle.

Granted, we could introduce as Lewis does the notion of 'counterpart', but at the very least this would make the characterization of transitivity, etc. for Φ_B somewhat messy because as Lewis notes, it is possible for an individual at a world to have more than one counterpart at another world.[4] (This is because if we cash out the notion of counterpart in terms of the notion of similarity then it is possible for two or more individuals at w_j to be equally similar to t at w_i.) We shall have more to say concerning Lewis' notion of counterpart below. Hintikka sidesteps this difficulty by proposing the introduction into his semantics a 'family' of functions F such that each member of this set takes as a value exactly one individual for each world. Then although there are *strictly speaking* no transworld indi-

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[4] See Lewis (1983), p. 29.

viduals, *loosely speaking* the individuals which are the values of any f_i in F at various worlds amount to one transworld individual.[5] This presumably prevents the problem of an individual's 'splitting' across worlds.[6] But as we shall argue below, we still do not obtain transworld individuals in this way, even loosely speaking.

In the light of the above-mentioned difficulties with Hintikka's function Φ_B it is perhaps best to stay with our doxastic accessibility relation R in developing the characteristic semantics for the Hin-SQC$^=$ systems. If there is any moral to be drawn from the above remarks concerning the function Φ_B which in effect relativizes doxastic alternatives to worlds *and* to individuals, it is that such a device will generate a semantics for belief but not a semantics characterizing any corresponding normal axiom system. In any case, the special twist to Hintikka's semantics is his relativization of domains of individuals to indices or worlds such that these domains are non-overlapping. We shall now discuss this feature of Hintikka's semantics with respect to the problem of the apparent failure of the substitutivity principle for belief contexts.

And so, if we want a characteristic semantics for the Hin-SQC$^=$ systems, what is needed is an alteration to the semantics characterizing the SQC$^=$ systems which invalidates the *doxastic* version of the schema $(\alpha(t_1/v) \,\&\, t_1 = t_2) \supset \alpha(t_2/v)$, where t_1, t_2 may occur in the scope of the belief operator. The semantical 'sleight of hand' which Hintikka employs towards this end in 'Semantics for Propositional Attitudes' is to construct belief models in such a way that associated with each index or world is its own domain of individuals such that there is no overlapping. In terms of

[5] Hintikka (1969), p. 160.

[6] ibid, p. 159.

the machinery required for this sort of move, Hintikka simply replaces the set D of individuals for belief models with a set $\{D_i\}$ of sets of individuals[7] where each of the subscripts of the D_i's corresponds to the appropriate subscripts of the w_i's. For example, D_1 would be the domain of individuals associated with w_1. Further, it would be required that for any two D_i's in $\{D_i\}$, if $i \neq j$ then $D_i \cap D_j = \emptyset$. Equivalently, we could introduce into the definition of a Hin-SQC$^{=}$ model a function Q [8] which to each world assigns a subset of the set D of 'individuals' where it is specified that for any two of these relativized domains, i.e., for any two of the $Q(w_i)$'s, their intersection is the empty set. I.e., for any $Q(w_i)$ and for any $Q(w_j)$ where $i \neq j$, $Q(w_i) \cap Q(w_j) = \emptyset$.

If domains of individuals are relativized to indices with no overlapping then naturally the assignment function V would also be relativized to indices for terms and for predicate variables. And this is just what Hintikka proposes in 'Semantics for Propositional Attitudes'.[9] In short, for any constant t and for any index w_i, $V(t,w_i) \in D_i$ where D_i is the domain of individuals associated with w_i and for any predicate variable P, $V(P,w_i) \subseteq D_i^n$ where D_i^n is the set of all n-tuples of individuals in D_i. In short, V assigns to a constant at an index some member of that index's domain and to each predicate variable and index, V assigns to this pair a subset of the set of all n-tuples of members of that index's domain of individuals. If we relativize the assignment function V to indices for constants then it is possible for distinct constants t_1 and t_2 to be assigned the *same* individual at one index w_i for a model M in which case the identity $t_1 = t_2$ holds for w_i

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7 See Hinitikka (1969) reprinted in Linsky (1971), p. 151. Hintikka uses the locution $I(\mathcal{M})$ to represent the phrase 'the domain of individuals I associated with the world $\mathcal{M}$'.

8 See the discussion of the 'world-associating' function Q in Hughes and Cresswell (1968), p. 171.

9 See Hintikka (1969) reprinted in Linsky (1971), pp. 151-152.

while V assigns to these constants *different* individuals at another index w_j where $i \neq j$ in which case, the identity $t_1 = t_2$ is false for w_j.[10]

This feature of Hintikka's semantics puts it at odds with the domain semantics for the $SQC^=$ systems since in the latter type of semantics, if an identity $t_1 = t_2$ is true at any index in the model it is true at all indices in the model. This is owing to the fact that the assignment function V for the $SQC^=$ models is not relativized to indices given that each index in a model draws on the same domain of individuals. Thus, the schema $t_1 = t_2 \supset B(t_1 = t_2)$ which intuitively says that agents are omnidoxastic with respect to contingent identities is valid in the $SQC^=$ semantics but it is *invalidated* with respect to the Hin-$SQC^=$ semantics since although $t_1 = t_2$ may be true at some w_i in W it could be false at some doxastically accessible index w_j if V assigns distinct individuals to t_1 and t_2 at w_j.

In the light of Hintikka's suggested semantics for belief logic, consider a simple Hin-$SQC^=$ model M[11] consisting of two indices, w_1 and w_2 such that $D_1 = \{d\}$ and $D_2 = \{e,f\}$ and such that $V(a,w_1) = V(b,w_1) = d$ but $V(a,w_2) = e$ whereas $V(b,w_2) = f$. Further, suppose that $V(F,w_2) = \{e\}$. And finally, suppose that w_2 is *doxastically accessible* from w_1, i.e., suppose that $\{<w_1,w_2>\} \subseteq R$. This model will serve to invalidate the following instance of the $SQC^=$ thesis-schema $(A(t_1/v) \,\&\, t_1 = t_2) \supset A(t_2/v)$ where t_1, t_2 may occur in the scope of the belief operator, $(BFa \,\&\, a = b) \supset BFb$. More concretely, we could think of a, b as 'Tully' and 'Cicero' repectively and let F be 'Roman orator'. The schema in question asserts the principle that co-referentials are without restriction intersubstitutible in belief constructions and the instance we are considering says that if x believes that

[10] Hintikka (1969) reprinted in Linsky (1971), p. 155.

[11] To be technically precise, we have not yet established that this is a Hin-$SQC^=$ model, although we shall have something to say concerning soundness and completeness below.

Tully is a Roman orator and given that Tully is Cicero then x also believes that Cicero is an orator.

Informally, we can see how the (partial) model described above could serve to invalidate (BFa & a = b) ⊃ BFb: Even though a and b are assigned by V the same individual d for w_1 they are assigned distinct individuals, viz., e and f respectively for w_2 (which is doxastically accessible from w_1) and hence a = b is false for w_2. Further, since e assigned to the constant a is in the extension of F for w_2 and f assigned to b for w_2 is not in the extension of F for w_2 it follows that Fa is true for this index but that Fb is false. Then although Fa is true for all doxastic alternatives to w_1 in this model, Fb is false for some doxastic alternative to w_1 in the model. Hence, although it is true at w_1 that BFa and that a = b, it is also false that BFb at this index. Q.E.D.

The upshot of these remarks is that in the sort of semantics which Hintikka has proposed for belief logic, there can be models where x believes that $\alpha(t_1/v)$ at some w_i and the identity $t_1 = t_2$ holds at w_i and yet x may fail to believe that $\alpha(t_2/v)$ at this index since there is some alternative w_j to w_i such that the identity $t_1 = t_2$ does not hold at this alternative. In short, co-referentials are not intersubstitutible in notional constructions for this sort of semantics which jibes with Hintikka's syntactic proposal that for us to infer a wff of the form $\alpha(t_2/v)$ from $\alpha(t_1/v)$ (such that t_1, t_2 occur in the scope of a belief operator) and $t_1 = t_2$ we must add as a premise $B(t_1 = t_2)$.[12] From a semantic point of view, this guarantees that the identity $t_1 = t_2$ holds for all the doxastic alternatives to the index at which $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are being evaluated.[13] This is because if $B(t_1 = t_2)$ is

[12] In fact, for the inference of $B\alpha(t_2/v)$ from $B\alpha(t_1/v)$ to go through it is sufficient that $B(t_1 = t_2)$ is an additional premise rendering $t_1 = t_2$ superfluous.

[13] Thus, in Hinitikka (1967), p. 55 Hintikka states with respect to epistemic logic that "Substitutivity

true at the index at which $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are being evaluated then the identity $t_1 = t_2$ holds for all the doxastic alternatives to this index in which case it is impossible for the contents of the belief wffs containing t_1, t_2 to differ in their truth-values at any of these alternatives. Otherwise, there is no guarantee that t_1 and t_2 won't refer to distinct individuals in some alternative to the index at which $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are being evaluated.[14]

As was noted in the previous section, any wff of the form $(\alpha(t_1/v)\ \&\ t_1 = t_2\ \&\ \mathbf{B}(t_1 = t_2)) \supset \alpha(t_2/v)$ where t_1, t_2 may occur within the scope of a doxastic operator is a Hin-SQC$^=$ thesis *provided* that the contents of any doxastic wffs in which t_1, t_2 occur are not themselves doxastic wffs. This restriction holds for systems *not* containing the schema $\mathbf{B}\alpha \supset \mathbf{BB}\alpha$. This provision was introduced so as to ensure soundness of the Hin-SQC$^=$ systems not containing $\mathbf{B}\alpha \supset \mathbf{BB}\alpha$ with respect to the type of semantics we are now considering. To illustrate why the introduction of this provision was necessary, consider the following instance of the schema $(\alpha(t_1/v)\ \&\ t_1 = t_2\ \&\ \mathbf{B}(t_1 = t_2)) \supset \alpha(t_2/v)$ where t_1, t_2 may occur in the scope of a doxastic operator viz., $(\mathbf{BB}Fa\ \&\ a = b\ \&\ \mathbf{B}(a = b)) \supset \mathbf{BB}Fb$ which involves an *iteration* of the belief operator. (A parallel example for doxastic possibility would be $(\mathbf{P_B P_B}Fa\ \&\ a = b\ \&\ \mathbf{B}(a = b)) \supset \mathbf{P_B P_B}Fb$.)

Now, the following simple model based on Hintikka's proposal for a semantics of belief will invalidate the formula $(\mathbf{BB}Fa\ \&\ a = b\ \&\ \mathbf{B}(a = b)) \supset \mathbf{BB}Fb$: Suppose that this model consists of exactly three indices, w_1, w_2 and w_3. Let the domain of individuals associated with w_1, viz., $D_1 = \{d\}$,

everywhere presupposes that two terms refer to the same individual in *each* epistemically possible world we have to consider. If we are talking of what a knows or does not know, this is guaranteed only by the sentence (26) 'Ka(b = c)'."

14 Hintikka (1969) reprinted in Linsky (1971), p. 155.

and let $D_2 = \{e\}$ and $D_3 = \{f,g\}$. Further, suppose that w_2 is doxastically accessible from w_1 and that w_3 is doxastically accessible from w_2 but that w_3 is not accessible from w_1. I.e., $\{\langle w_1,w_2\rangle, \langle w_2,w_3\rangle\} \subseteq R$. R's intransitivity with respect to this model is permissible if this is a model for a Hin-SQC$^=$ system *without* 4. Also, let $V(a,w_1) = V(b,w_1) = d$, $V(a,w_2) = V(b,w_2) = e$ but $V(a,w_3) = f$ and $V(b,w_3) = g$. Finally, let the extension of F for w_3 be f. More formally, $V(F,w_3) = \{f\}$. Then Fa will be true at w_3 and $a = b$ and Fb will both be false at w_3. Further, $a = b$ and BFa are both true at w_2 whereas BFb is false at w_2. Then it follows that BBFa and $B(a = b)$ and $a = b$ are all true at w_1 but BBFb is false at w_1. This (partial) model therefore invalidates the formula $(BBFa \mathbin{\&} a = b \mathbin{\&} B(a = b)) \supset BBFb$ which is an instance of the schema $(\alpha(t_1/v) \mathbin{\&} t_1 = t_2 \mathbin{\&} B(t_1 = t_2)) \supset \alpha(t_2/v)$ such that t_1, t_2 may occur in the scope of a doxastic operator.

However, the model described above would not be admissible as a model for any Hin-SQC$^=$ + 4 system since any model in the class of models characterizing these systems would have to be *transitive*. And so, to guarantee soundness of any Hin-SQC$^=$ system not containing 4 with respect to some class of models where it is not required that R be transitive, it is necessary to restrict the schema $(\alpha(t_1/v) \mathbin{\&} t_1 = t_2 \mathbin{\&} B(t_1 = t_2)) \supset \alpha(t_2/v)$ where t_1, t_2 may occur in the scope of a belief operator to instances where no iteration of this belief operator is involved. So this explains the rationale behind our proviso. Similar remarks apply to the restrictions we imposed on the *doxastic* axiom-schema $(\alpha(t/v) \mathbin{\&} (\exists v)(v = t \mathbin{\&} B(v = t)) \supset (\exists v)\alpha$ and the theorem-schema $(\alpha(t_1/v) \mathbin{\&} t_1 = t_2 \mathbin{\&} (\exists v)(v = t_1 \mathbin{\&} B(v = t_1) \mathbin{\&} (\exists v)(v = t_2 \mathbin{\&} B(v = t_2)) \supset \alpha(t_2/v)$ and their corresponding rules of inference for the

Hin–SQC$^{=}$ systems not containing 4. These restrictions on AS 4 and AS 7 can be dropped for the Hin–SQC$^{=}$ + 4 systems given the transitivity requirement on R in the semantics, since this gurantees that if $\mathbf{B}(t_1 = t_2)$ is true at some index w_i then so is $\mathbf{BB}(t_1 = t_2)$.

Now that we have discussed Hintikka's solution to the apparent failure of the substitutivity principle on the semantic front, we still need to consider how his semantics deals with *relational* constructions with respect to the substitution of co-referentials and with respect to the issue of existential generalization into such constructions.

Hintikka's relativization of sets of individuals to indices (and hence his relativization of V to indices for terms and predicate variables) besides preventing *unrestricted* substitution of co-referentials in belief constructions also disallows existential quantification into *notional* constructions not occurring in relational contexts. That is, the sort of semantics Hintikka is proposing would invalidate the schema $\alpha\,(t/v) \supset (\exists v)\alpha$ and the inference rule $\alpha\,(t/v) \longrightarrow (\exists v)\alpha$ where in both cases, t occurs in the scope of a doxastic operator. To illustrate that this is the case, consider a more specific version of the above schema, viz., $\mathbf{B}\alpha\,(t/v) \supset (\exists v)\mathbf{B}\alpha$. If $\mathbf{B}\alpha\,(t/v)$ is true at an index w_i then $\alpha\,(t/v)$ must be true at all w_j such that w_iRw_j. But given that indices have associated with them their own domain of individuals and hence that V is relativized to indices, it follows that 't' occurring in the content $\alpha\,(t/v)$ will denote a distinct individual in every w_j such that w_iRw_j where $\alpha\,(t/v)$ holds. Then it is false to say that *there is some individual v* such that x believes of v that α. In sum, from $\mathbf{B}\alpha\,(t/v)$ we cannot infer $(\exists v)\mathbf{B}\alpha$ because in the type of semantics we are considering, the existence of transindexical individuals is disallowed. And all of this jibes with Hintikka's syntactic stricture against quantifying into notional constructions.

However, as we noted, Hintikka does allow quantification into relational contexts (as defined in the third chapter, section 5) and also substitution of co-referentials with respect to special sorts of relational contexts (where the individual has *true* opinions as to who t_1 and t_2 are). And in terms of relational contexts where quantifying in is permissible, Hintikka portrays any such context (similar to Kaplan) in terms of a notional construction conjoined with any 'acquaintance' construction of the form $(\exists v)\mathbf{B}(v = t)$ which intutively says that x has an opinion as to who t is. In terms of the special sorts of relational contexts where substituion of co-referentials is permissible, the relevant sorts of contexts must be such that x has *true* opinions as to who t_1 and t_2 are, which are represented respectively as $(\exists v)(v = t_1 \& \mathbf{B}(v = t_1))$ and as $(\exists v)(v = t_2 \& \mathbf{B}(v = t_2))$. These locutions are also relevant for quantifying into constructions which are not purely doxastic.

But given that in the sort of semantics which Hintikka is proposing there are no transindexical individuals, it is not clear how it is possible for any locutions of the form $(\exists v)\mathbf{B}(v = t)$ or of the form $(\exists v)(v = t \& \mathbf{B}(v = t))$ to be true at any index in any *belief* model not containing dead ends.[15] I.e., how can there be some indivdual v such that 'v = t' is true at every doxastic alternative to the index in question if all individuals are index (or world) bound? And this is tantamount to asking how it is possible for bel-

[15] The reader may recall that so-called dead ends as Hughes and Cresswell call them are such that no indices are accessible from them and hence trivially, any wff of the form $\mathbf{B}\alpha$ will be true at any such index given the truth conditions for wffs of this form. Then at a dead end, any wff of the form $(\exists v)\mathbf{B}\alpha$ will also be true since '$\exists$' will be vacuous or if not, then its scope will be true for at least one world-bound indivdual which is denoted by v in $\mathbf{B}\alpha$.

lievers to have opinions as to who individuals are in this type of semantics. More generally, it is not clear how any relational locution of the form $(\exists v)B\alpha$ could be true at any index (which is not a dead end) in a model in this type of semantics, which amounts to asking how Hintikka can make semantic sense of quantifying into belief constructions (and by extension into constructions of doxastic possibility). It would therefore seem that more semantic machinery has to be added to Hintikka's semantics to allow for relational contexts where agents have opinions as to who certain relevant individuals are and more generally to allow for quantifying into belief constructions even though this semantics does not allow for transindexical individuals.[16]

In order to allow in his semantics for relational contexts where agents have opinions as to who certain relevant individuals are and in order to make sense of quantifying into belief constructions, Hintikka introduces into the definition of a belief model a 'family' of functions $F = \{f_1, f_2, \ldots, f_n, \ldots\}$ such that each member of F is assigned for every index in the model exactly one member of the domain of individuals associated with that index.[17] In other words, each f_k in F is a function from indices into index-bound individuals. More formally, for any w_i in W, $f_k(w_i) \in D_i$ where f_k may be undefined for some of the indices in W, in which case f_k would be partial.[18] According to Hintikka, the set of world-bound individuals determined by some f_k in F (where we could define this set by 'abstraction' as $\{f_k(w_i)\colon f_k(w_i) \in D_i\}$) is what we normally mean by the 'same' individual

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[16] This point is made by Hintikka in Hintikka (1969), p. 159.

[17] Hintikka makes this proposal in Hintikka (1969), pp. 159–162.

[18] Hintikka imposes the requirement for membership in F that if $f_1(w_i) = f_2(w_i)$ then $f_1(w_j) = f_2(w_j)$ for any doxastic alternative w_j to w_i.

existing at different indices or worlds. Thus, he is claiming that "...the apparently different individuals which are correlated by one of the functions $f \in F$ is just what we ordinarily mean by one and the same individual".[19]

And we can see how the f_k's in F are intended to sidestep the problem that any locution of the form $(\exists v)B\alpha$ (or more generally $(\exists v)\alpha$ where free v occurs in the scope of a belief operator) will be false for any index which is not a dead end. It can be stipulated that any wff of the form $(\exists v)\alpha$ (where free v occurs in the scope of a belief operator) is true at an index w_i in a model just in case $\alpha(t/v)$ is true at w_i for at least one individual d in D_i such that for some f_k in F, $f_k(w_i) = d$ and such that f_k is defined for all doxastic alternatives to w_i. The proviso that d in D_i is the value of $f_k(w_i)$ where this f_k is defined for all alternatives to w_i ensures that there will be a set of index-bound individuals which includes d where intuitively this set can be regarded as the 'same' individual (loosely speaking, of course) existing at w_i and at all of its alternatives. Then loosely speaking, if any wff of the form $(\exists v)B\alpha$ (which is a more specific version of $(\exists v)\alpha$ where free v is in the scope of a belief operator) is true at w_i, we can say that *there is some individual v* such that x believes that A of v where the description *the individual v* is cryptic for a set of individuals defined by '$\{f_k(w_i): f_k(w_i) \in D_i\}$'.

More concretely, suppose that there is some individual v such that Jones believes of this individual that he was a Roman orator. I.e., his de re belief can be represented as '$(\exists v)$(Jones believes that v was an orator)' such that this is inferable from a relational context. Then the individual that Jones has in mind, i.e., the 'individual' with whom Jones is acquainted, will not be a transworld individual existing at all the dox-

[19] Hintikka (1969), p. 160.

astic alternatives to the world he inhabits. Rather, the 'individual' with whom Jones is acquainted is really a collection of index-bound individuals who are 'correlated' via some sort of 'correlating' function f_k.

And so, we can make sense in Hintikka's semantics of existential quantification *outside* the belief operator after all. This paves the way for the allowance of there being relational contexts where the believer has an opinion as to who some relevant individual is. I.e., what having an opinion as to who some individual is amounts to is the believer's considering this 'same' individual to speak loosely - since we are really speaking here of a collection of correlated index-bound individuals - at all the alternatives to the world he (the believer) inhabits. More formally, any locution of the form $(\exists v)\mathbf{B}(v = t)$ is true at an index w_i just in case $\mathbf{B}(t' = t)$ is true for at least one individual d in D_i such that d is the value of an f_k in F and such that f_k is defined for all doxastic alternatives to w_i. And to say that $\mathbf{B}(t' = t)$ is true at w_i for at least one individual d in D_i such that d is the value of an f_k in F and such that f_k is defined for all doxastic alternatives to w_i is to say that $t' = t$ is true at any alternative w_j to w_i *where* $t' = t$ *is true at any such* w_j *for exactly one member of the set* $\{f_k(w_i): f_k(w_i) \in D_i\})$, viz., for the indivdual in D_j which is the value of $f_k(w_j)$.

We shall now consider how this rather complicated semantic machinery parallels Hintikka's proposal on the syntactic front of restricting quantifying in to *relational* contexts. I.e., we can infer $(\exists v)\mathbf{B}\alpha$ from $\mathbf{B}\alpha\,(t/v)$ only if we add as a premise $(\exists v)\mathbf{B}(v = t)$. (Or if $\alpha\,(t/v)$ is a construction which is not purely doxastic such that t may also occur outside the scope of a belief operator, then we shall need to add '$(\exists v)(v = t \;\&\; \mathbf{B}(v = t))$' as will be explained presently.) From a semantic point of view, what the premise $(\exists v)\mathbf{B}(v = t)$ guarantees is that the individual denoted by t in $\mathbf{B}\alpha(t/v)$ is the 'same' individual at all the doxastic alternatives to the world at which $\mathbf{B}\alpha(t/v)$ is being evaluated. I.e., if $(\exists v)\mathbf{B}(v = t)$ is true then there is some individual (in the domain D_i associated with the index at which it is true) denoted by t and which is such that its correlates are denoted by t at all alternatives to the index in question. Given this guarantee, then it is true that *there is some individual v* such that x believes that α of v. Thus, if Jones believes that Tully was bald and further, if Jones has an opinion as to who Tully is - which semantically means that the term 'Tully' denotes the 'same' individual at all the alternatives which Jones considers - then *there is some individual v* such that Jones believes of v that he was bald.

Suppose that we wish to quantify into a construction such as **BFa** & Ga which could be a symbolization of for example, "Jones believes that Tully was bald, and Tully in fact was an orator". Then in order to quantify into this construction which is not purely doxastic, it would not be sufficient to add as a premise '$(\exists x)\mathbf{B}(x = a)$' (which for our example reads "Jones has an opinion as to who Tully is"). From a semantic point of view, this is owing to the fact that we need to guarantee that the individual denoted by the constant 'a' at the index at which **BFa** & Ga is being evaluated, w_i, is the same individual (strictly speaking) who is denoted by some t" *at* w_i such that $V(t'',w_i) = f_k(w_i)$ for some f_k in F and such that $t'' = a$ is true at all doxastic alternatives to w_i. All that $(\exists x)\mathbf{B}(x = a)$ guarantees is that 'a' denotes the 'same' individual (loosely speaking) at all the *alternatives* to w_i. Thus, if instead of $(\exists x)\mathbf{B}(x = a)$, suppose we add as a premise to **BFa** & Ga the construction $(\exists x)(x = a \;\&\; \mathbf{B}(x = a))$ which for our example reads

"Jones has a *true* opinion as to who Tully is". Then if both premises are true, there will be an M' like M such that t" = a is true at w_i and $\mathbf{B}$(t" = a) is true at w_i such that $V(t", w_i) = V(a, w_i) = f_k(w_i)$. Then 'a' will denote the 'same' individual at w_i which is denoted by 'a' at all the alternatives to w_i and hence, we can quantify with respect to 'a' occurring in 'Ga' as well as with respect to 'a' occurring in $\mathbf{B}$Fa – they denote the 'same' individual (loosely speaking). Thus, in our example, we can conclude that *there is* some individual x such that Jones believes that x is bald and such that x was an orator. I.e., from $\mathbf{B}$Fa & Ga in conjunction with $(\exists x)(x = a \,\&\, \mathbf{B}(x = a))$ we can infer that $(\exists x)(\mathbf{B}Fx \,\&\, Gx)$.

As was stipulated in section 1 in AS 4, $(\alpha(t/v) \,\&\, (\exists v)(v = t \,\&\, \mathbf{B}(v = t)) \supset (\exists v)\alpha$ and its inferential counterpart, DR 1, we shall in general require that to infer $(\exists v)\alpha$ from $\alpha(t/v)$ where t's occurrence *has not* been restricted to non-doxastic wffs, then we must add $(\exists v)(v = t \,\&\, \mathbf{B}(v = t))$ as a premise. This will certainly handle cases such as the one discussed in the previous paragraph. Further, requiring $(\exists v)(v = t \,\&\, \mathbf{B}(v = t))$ rather than just $(\exists v)(\mathbf{B}(v = t))$ as an additional premise avoids a commitment to possibilia, if by a 'possibilium' we mean a purely fictional entity – fictional relative to some index that is[20] – for versions of AS 4 such as $\mathbf{P_B}\alpha(t/v) \,\&\, ((\exists v)(v = t \,\&\, \mathbf{B}(v = t)) \supset (\exists v)\mathbf{P_B}\alpha$, which was alluded to briefly in section 1. Thus, suppose that at some index w_i it is true that it is possible for all Jones believes that Pegasus is a winged horse. Then for there *to be* something such that it is possible for all Jones believes that it is a winged horse, the term 'Pegasus' must denote not only the 'same' individual at all alternatives to Jones' world w_i but also this 'same' individual denoted by 'Pega-

[20] We could say that in Hintikka's semantics, x exists at $w_i =_{df.} x \in D_i$, from which it follows that if x is not in D_i then x is *fictional* relative to w_i.

sus' must exist at Jones' world. (I.e., Jones must have a true opinion as to who Pegasus is.) But if Pegasus is 'fictional' (relative to w_i) then we cannot infer that *there is* some entity v such that it is possible for all Jones believes that it is a winged horse. (There will be a commitment to correlates at other indices if the entity is non-fictional and so in this sense, there is still a commitment to possibilia in Hintikka's semantics.[21])

Finally, given AS 4 and DR 1 such that to infer $(\exists v)\alpha$ from $\alpha(t/v)$ where t's occurrence *has not* been restricted to non-doxastic wffs, we must add $(\exists v)(v = t \,\&\, \mathbf{B}(v = t))$ as a premise, then we can never *infer* that agents have *de re* beliefs concerning fictional entities. Thus, if Jones inhabits a world where there are no winged horses, then even though he may believe notionally that Pegasus is a winged horse and even though he may have an opinion as to who Pegasus is (or symbolically, $(\exists x)\mathbf{B}(x = \text{Pegasus})$ is true at Jones' world) he cannot have a *true* opinion as to Pegasus is. I.e., the the locution '$(\exists x)(x = \text{Pegasus} \,\&\, \mathbf{B}(x = \text{Pegasus})$' will not be true at the index which Jones inhabits given that Pegasus is not contained in the domain of individuals associated with that index. Thus, we cannot infer from Jones' notional belief that Pegasus is a winged horse in conjunction with his opinion as to who Pegasus is that *there is* an x such that Jones believes that x is a winged horse.

Nonetheless, even in the light of these remarks, it would seem that in the final analyis Hintikka's semantics for belief does involve an ontological commitment to *possibilia* in the sense of 'fictional entities' as we shall now argue. Since each index w_i in a model has associated with it its own domain of individuals D_i, then to say that x *exists at* w_i amounts to requir-

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[21] Marcus seems to identify possibilia with purely fictional entities. See Marcus (1976). Then what we have argued is that AS 4 does not commit us ontologically to possibilia – in Marcus' sense of the term.

ing that $x \in D_i$. This is presumably why quantifiers whose scopes do not involve doxastic expressions range only over members of D_i. Then an entity x is *fictional relative to an index w_i* just in case 1) x is not a member of D_i although 2) x is a member of at least one D_k associated with some w_k such that $i \neq k$. (Thus, so-called impossible objects or 'impossibilia' such as square triangles would not qualify as fictional entities relative to any index since they do not satisfy condition 2).) However, since no domains overlap in Hintikka's semantics, it follows that relative to any index w_i, any object not in D_i but in any other D_j will be fictional. Therefore, any object 'existing' at w_i will be fictional relative to any other index. Thus, there seems to be a proliferation of (relative) fictional entities or *possibilia* in Hintikka's semantics just as there is a proliferation of *necessary* existents for the $SQC^{=}$ domain semantics developed in chapter two.

However, the set F discussed above seems to mitigate the situation just alluded to, viz., that there is a proliferation of fictional entities in Hintikka's semantics for belief logic. I.e., an individual at w_i may have 'correlates' at other indices in which case, we are considering the 'same' individual at different indices rather than a whole set of fictional entities relative to w_i. We shall now consider whether or not the set F does what it is supposed to, viz., providing a way of making sense of talking about the 'same' transindexical individual for a varying domain semantics.

Although Hintikka's somewhat ad hoc maneuvre of introducing the set F of functions (where each member of F is assigned for each index in the model exactly one individual) seems to provide us with a way of making sense of relational contexts in a semantics which relativizes domains of individuals to indices, there are some problems which need to be dealt with in relation to this maneuvre. First of all, Hintikka would have to admit (as he in fact does[22]) that *strictly speaking*, the set of individuals defined by a set such as $\{f_k(w_i): f_k(w_i) \in D_i\}$ does not constitute a transworld individual. And philosophers such as Kripke and Plantinga would be quick to find fault with this. They would point out that what we ordinarily mean when we talk about a counterfactual such as 'If Humphrey had won the 1968 election, he would have done such and such' is that in some possible world this *very same person* Humphrey (and not some 'correlate') won the election and in such a world he did such and such.[23] (In the case of Kripke, his criticism is tied up with his view that possible worlds are stipulated as situations in which some individual existing at the actual world has different non-essential attributes.) Their comments concerning counterfactuals could also be extended to the case of propositional attitudes such as belief since they could claim that when x *believes* that Humphrey did such and such, what x is doing is to consider all the doxastic alternatives to the 'actual' world where this *very same person* Humphrey (and not some set of 'correlated' individuals determined by an f_k in F) did such and such.

Nonetheless, it could be noted that if there is no problem in analysing counterfactual locutions (and the same could be said of propositional attitude locutions) in terms of such abstract entities as 'possible worlds' then what is wrong with also introducing the notion of 'correlate' (or in Lewis' jargon, 'counterpart') into the analysis. However, one response to this is that the notion of 'correlate' (or 'counterpart') perhaps unnecessarily complicates the analysis of counterfactual and propositional attitude locutions. On the other hand, as we have seen, the notion of 'transworld individual'

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22 Hintikka (1969), p. 160.

23 See Kripke (1980), p. 45 and also see Plantinga (1979) in Loux (1979), pp. 162-3.

is itself problematic, at least for indexical semantics of formal systems.[24] In any case, even if Hintikka is able to answer the objections of Kripke and Plantinga concering his set F of 'correlating' functions, he is faced with the further difficulty that even *loosely speaking* a set such as $\{f_k(w_i): f_k(w_i) \in D_i\}$ does not amount to a transworld individual since it is possible that any two 'correlates' who are the values of an f_k at different indices share *no* properties in common. More to the point, it is possible that at the index w_i where a belief wff $\mathbf{B}\alpha\,(t/v)$ is being evaluated, the individual denoted by t who is the value of $f_k(w_i)$ may have no properties in common with one of its correlates at some w_j such that w_iRw_j and where this correlate is the value of $f_k(w_j)$. In short, Hintikka has placed no strictures on the members of F which would prevent this sort of situation from arising. But then in what sense would such a set of individuals determined by a 'correlating' function f_k where possibly some of the members of this set share no properties in common constitute even loosely speaking the 'same' individual existing at different indices? Clearly, what is needed is the introduction of appropriate strictures for members of the set F of 'correlating' functions that would prevent this sort of situation from arising or at least further strictures on the truth-conditons for wffs of the form $(\exists v)\mathbf{B}\alpha$ with respect to the function f_k.

At least as a rough beginning, we could require that for any locution of the form $(\exists v)\mathbf{B}\alpha$ to be true at an index, not only must it be the case that $\mathbf{B}\alpha\,(t/v)$ is true at w_i for at least one individual d in D_i such that for some f_k in F, $f_k(w_i) = d$ and such that f_k is defined for all doxastic alternatives to w_i, it must *also* be the case that all the individuals who are the values of f_k at the doxastic alternatives to w_i must be *counter-*

[24] See chapter two, section 4.

parts of d at w_i in the sense that they are more similar to d than any other member of their respective indices. David Lewis has been the recent champion of counterpart theory.[25] Lewis cashes out the notion of 'counter-parthood' in terms of the notion of 'comparative similarity' (which as Quine has pointed out is itself a somewhat problematic notion[26]) as follows:

> y at w_j is a counterpart of x at w_i =df. y at w_j resembles x at w_i more closely (with respect to certain relevant properties) than any other individual z at w_j.

As Lewis notes, it is possible that an individual x at w_i will have more than one counterpart at w_j since there could be two or more individuals at w_j who equally resemble x at w_i and such that no other individuals resemble x more closely than they do.[27] Also, as Lewis admits, the notion of comparative similarity in terms of which the counterpart relation is defined is itself problematic in the sense that determining comparative similarity is a matter of contextual considerations meaning that it has to be determined what sorts of properties are important or relevant.[28]

Unfortunately, Lewis' counterpart relation is faced with the same sorts of difficulties that Hintikka's notion of 'correlate' was, viz., it could be objected that when we speak counterfactually or in the context of propositional attitudes we are consdering what the *very same* individual is doing in alternatives to the 'actual' world and not what his counterparts are doing. Further, as was noted, an individual x at w_i can have more than

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[25] For example, see Lewis (1968) reprinted in Lewis (1983).

[26] Quine (1969), pp. 118 – 9. Quine in this article seems to be sceptical that any general definition of comparative similarity apart from various branches of the theoretical sciences can be formulated (eg., in terms of 'kinds').

[27] Lewis (1983), p. 29.

[28] ibid, p. 28

one counterpart at some alternative w_j in which case it seems somewhat arbitrary that f_k takes as its value only one of these individuals at w_j to serve as the 'correlate' of x at w_i. But the most serious problem with appealing to Lewis' notion of counterpart as a way of preventing the sort of case where an individual's correlates at other indices have nothing in common with it (and perhaps with each other) is that there is nothing to prevent the sort of situation where *trivially* every indvidual at w_j is a counterpart of x at w_i since no individual at w_j shares any properties in common with x at w_i. In such a case, since nothing at w_j shares anything in common with x at w_i (except perhaps self-identity) then all individuals at w_j equally resemble x and hence every individual at w_j is trivially a counterpart of x at w_i.

Perhaps in the final analysis, Hintikka does not need Lewis' notion of counterpart since he could argue that the notion of correlate is no more problematic than the notion of 'transworld' individual given that in a bare particular metaphsyics, an individual conceived as a 'bare particular' may share no properties in common with itself existing at other indices.[29]

Given our informal remarks concerning Hintikka's semantics for belief logic, we shall now attempt to make his proposals more precise by describing what a Hin-SQC$^=$ model would look like. In the next section, we shall attempt to show that the Hin-SQC$^=$ systems not containing 4 are sound and complete with respect to the type of semantics we are about to describe.

Any Hin-SQC$^=$ model M is a 5-tuple $\langle W, R, \{D_i\}, F, V \rangle$ where W is a nonempty countable set of indices, R is a 2-place relation ranging over members of W (i.e., $R \subseteq W \times W$) where various restrictions may be imposed on

[29] See our remarks concerning the problems associated with the notion of transworld individual in the semantics for the SQC$^=$ systems where all indices share the same domain of individuals.

R such as transitivity if M is a model for a Hin-SQC$^=$ + 4 system. Further, $\{D_i\}$ is a set of sets of individuals such that the subscript of each of the D_i's 'corresponds' with the subsrcipt of the appropriate member of W and such that for any D_i and D_j such that $i \neq j$, $D_i \cap D_j = \emptyset$. Intuitively, D_i is the domain of individuals associated with the index w_i in W. The set $F = \{f_1, f_2, \ldots, f_n, \ldots\}$ is a possibly non-empty set of so-called correlating partial functions such that for any f_k in F and for every w_i in W, $f_k(w_i)$ has as its value at most one member of D_i or is undefined. We shall have more to say concerning the component F below when we discuss the truth-conditions of wffs of the form $(\exists v)\alpha$ and $(\forall v)\alpha$ where free v occurs in the scope of a belief operator. Finally, V is an assignment function defined for terms and for predicate variables as follows:

1) For any constant t and for any w_i in W, $V(t,w_i) \in D_i$

2) For any predicate variable P and for any w_i in W, $V(P,w_i) \subseteq {D_i}^n$

In order to be able to evaluate wffs of the form $(\exists v)\alpha$ and $(\forall v)\alpha$ where free v occurs in the scope of a belief operator and in order to guarantee soundness of the Hin-SQC$^=$ systems with respect to the type of semantics under consideration, it will be necessary to impose the following restriction on V for constants:

For any consant t and for any w_i in W, if $V(t,w_i) = f_k(w_i)$ for some f_k in F then for any w_j such that w_iRw_j, $V(t,w_j) = f_k(w_j)$.

The import of this stricture will become clear when we consider the truth conditions for quantified belief wffs of the form $(\exists v)\mathbf{B}\alpha$ and $(\forall v)\mathbf{B}\alpha$ or for wffs where the scope of the quantifier involves a doxastic operator(s) within whose scope free v lies.

For simplicity of exposition we shall stipulate that V for constants is always defined which guarantees that if $V(t,w_i) = f_k(w_i)$ for some f_k in F then by the above restriction on V, $f_k(w_j)$ will be defined for any w_j such that w_iRw_j. Stipulating that V is always defined for constants also avoids having a logic with truth-value gaps on the semantic front and on the syntactic front, we can get by with $\alpha(t/v) \supset (\exists v)\alpha$ (where $\alpha(t/v)$ is not a belief wff) as an axiom schema rather than its 'free logic' variant, $(\alpha(t/v) \,\&\, (\exists v)(v = t)) \supset (\exists v)\alpha)$. Further, Hintikka suggests that one index or 'world' in W be regarded as 'distinguished' in the sense that it is singled out as the so-called *actual* world.[30] However, as is well established, the two sorts of indexical semantics (viz., our version where no member of W is designated and Hintikka's version where one member of W *is* designated) validate the same sets of wffs.[31]

A valuation over a Hin-SQC$^=$ model, V_M is a function from wffs into truth-values, i.e., V_M:Wffs $\longrightarrow \{0,1\}$ defined inductively as follows for all w_i, w_j in W:

Basis i. $V_M(Pt_1\ldots t_n,w_i) = 1$ iff $\langle V(t_1,w_i),\ldots,V(t_n,w_i)\rangle \in V(P,w_i)$

ii. $V_M(t_1 = t_2,w_i) = 1$ iff $V(t_1,w_i) = V(t_2,w_i)$

Supposing that $V_M(\alpha,w_i)$ and $V_M(\beta,w_i)$ are defined for any w_i in W then: $V_M(\sim\alpha,w_i)$, $V_M(\alpha \,\&\, \beta,w_i)$, $V_M(\alpha \vee \beta,w_i)$, $V_M(\alpha \supset \beta,w_i)$ and $V_M(\alpha \equiv \beta,w_i)$ are defined as for the sentential systems. Also, $V_M(B\alpha,w_i)$ is defined as for the sentential doxastic systems as follows:

$V_M(B\alpha,w_i) = 1$ iff for all w_j such that w_iRw_j, $V_M(\alpha,w_j) = 1$.

If A is the scope of a quantiifer such that free v does not occur in the scope of a belief operator then:

$V_M((\exists v)\alpha,w_i) = 1$ iff $V_{M'}(\alpha(t/v),w_i) = 1$ for at least one M' based on

30 Hintikka (1969), p. 152.

31 Hughes and Cresswell (1968), p. 351.

the same model structure as M and differing from M (if at all) only in terms of what V assigns to arbitary t foreign to $(\exists v)\alpha$ from D_i.

$V_M((\forall v)\alpha, w_i) = 1$ iff $V_{M'}(\alpha(t/v), w_i) = 1$ for every M' based on the same model structure as M and differing from M (if at all) only in terms of what V assigns to t foreign to $(\forall v)\alpha$ from D_i.

Notice that the truth of a quantified wff at an index depends on what V assigns to some arbitrary constant t occurring in the scope α (and which is foreign to the quantified wff) from the domain of individuals *associated with that index* as opposed to the semantics for the $SQC^=$ systems where the truth of such wffs was tied up with what V assigned to contants from the *shared* domain D. In other words, in this type of semantics, the quantifiers range over individuals in the domain associated with the index at which the quantified wff is being evaluated.

Further, as is the case for the domain semantics for the $SQC^=$ systems, in the current type of semantics we are considering, it is not being assumed that each member of a domain in a given model is assigned to a constant. The only assumption that is being made is that each constant denotes exactly one member of any index-associated domain. This allows for the possibility that there will be (indexed) domains with 'more' individuals than denoting constants. So, to ensure that the scope of a universally quantified wff is true for *all* the individuals in the appropriate indexed domain, it is stipulated that the scope α must be true over all possible assignments of individuals to some arbitrary constant t replacing free v in α.

Finally, in order to evaluate quantified wffs where the scope is doxastic (where for example the scope may be 'Fx & BGx'), it is necessary to consider the set $F = \{f_1, f_2, \ldots, f_n, \ldots\}$ of 'correlating' partial functions mentioned above where each f_k in F is defined as follows: For any w_i in W, $f_k(w_i) \in D_i$ or is undefined. The following is a necessary condition for membership in the set F, viz. that for any f_k, f_m in F, if $f_k(w_i) = f_m(w_i)$ then for any w_j such that w_iRw_j, $f_k(w_j) = f_m(w_j)$. Intuitively, each f_k in F is a function which for any w_i in W takes as its value exactly one member of the domain of individuals D_i associated with w_i or is undefined. As we have seen, Hintikka's reason for introducing the set F of 'correlating' functions is to enable us to make sense of quantifying across belief operators by allowing us to speak loosely of the 'same' individual existing at the alternatives to a given index. Whether or not F accomplishes this task is as we have seen open to doubt. In any case, given this definition of the set F of 'correlating' functions, we can characterize the truth-conditions of quantified wffs whose scopes are doxastic as follows:

Suppose free v occurring in the scope α of $(\exists v)\alpha$ occurs in the scope of at least one 'B' or 'P_B' operator. Then,

$V_M((\exists v)\alpha, w_i) = 1$ iff $V_{M'}(\alpha(t/v), w_i) = 1$ for at least one M' based on the same structure as M and differing from M (if at all) in terms of what V assigns to t foreign to $(\exists v)\alpha$ from D_i and where $V(t, w_i)$ for any such M' must be the value of an f_k in F.[32,33]

[32] Since $V(v,w_i)$ must be the value of an f_k in F and since $V(v,w_j)$ for any w_iRw_j is defined, it follows from our restriction on V mentioned above that for any such w_j, $f_k(w_j)$ must also be defined.

[33] These truth-conditions are more or less in the spirit of the truth conditions for wffs of the form $(\exists v)\alpha$ (where v is in the scope of a belief operator) appearing in Hintikka (1969), p. 161.

Further, it is possible that M' will differ from M in what V (for M') assigns to t at any w_j such that $w_i R w_j$.

Suppose free v occurring in the scope α of $(\forall v)\alpha$ occurs in the scope of at least one 'B' or 'P_B' operator. Then,

$V_M((\forall v)\alpha, w_i) = 1$ iff $V_{M'}(\alpha(t/v), w_i) = 1$ for every M' based on the same model structure as M and differing from M (if at all) only in terms of what V assigns to t foreign to $(\forall v)\alpha$ from D_i and where $V(t, w_i)$ for each such M' must be the value of an f_k in F. Further, it is possible that M' may differ from M in terms of what V (for M') assigns to t at any w_j such that $w_i R w_j$.

In short, if a universally or existentially quantified wff's scope is such that free v occurs in the scope of a doxastic operator, then it must be added to the truth-conditions that the individuals assigned to arbitrary t from D_i for any M' based on M such that $V_M(\alpha(t/v), w_i)$ is true must be the value of an f_k such that f_k is defined for all alternatives to w_i. In short, non-vacuous quantifiers occurring outside a belief operator range over special kinds of individuals in the appropriate D_i (i.e., in the domain of indviduals associated with the index w_i at which the quantified wff is being evaluated). These individuals are special in the sense that they have associated with them (via some f_k in F) 'correlates' for all indices which are doxastic alternatives to the index in question. Loosely speaking, the individuals over which the quantifiers outside a belief operator range are 'transindexical' or more precisely 'transalternative'. Consequently, the rider that M' may differ from M not only in terms of what V assigns to arbitrary t at w_i but in terms of what V assigns to t at any w_j such that w_iRw_j is that V must assign to t for M' the 'same' individual (loosely speaking) at *all* the alternatives to w_i that it assigns to t at w_i.

Now that we have laid out formally Hintikka's proposed semantics for quantified belief logic, we shall show in the next section that in fact the Hin-SQC$^=$ systems are sound (and complete) with respect to this semantics.

3. The Hin-SQC$^=$ Systems - Soundness and Completeness Results

Soundness of any of the Hin-SQC$^=$ systems relative to the appropriate class of *belief* models, each model in the class being a 5-tuple $\langle W, R, \{D_i\}, F, V\rangle$ such that the elements of this 5-tuple are defined as above and where R is restricted depending on the Hin-SQC$^=$ system under consideration, is proven in the usual manner. For example, if the system under consideration is Hin-KDQC$^=$ then soundness of this system with respect to the class of *serial* belief models is proven by showing that the axiom-schemata are valid in this class of models and that the rules of inference preserve validity. And in general, for any Hin-SQC$^=$ system, soundness is proven by showing that the axioms are valid with respect to the appropriate class of belief models and by showing that the rules of inference preserve validity.

We shall show below that the following crucial axiom-schemata which are common to all the Hin-SQC$^=$ systems and which distinguish these systems from the SQC$^=$ systems discussed in the second chapter are valid with respect to any $\langle W, R, \{D_i\}, F, V\rangle$ model:

AS 3: $\alpha(t/v) \supset (\exists v)\alpha$ provided that t does not occur in the scope of a doxastic operator.

AS 4: $(\alpha(t/v) \,\&\, (\exists v)(v = t \,\&\, \mathbf{B}(v = t)) \supset (\exists v)\alpha$ where t may occur in the scope of doxastic operator(s) and provided that there is no iteration of said operator(s) for systems not containing 4.

AS 6: $(\alpha(t_1/v) \,\&\, t_1 = t_2) \supset \alpha(t_2/v)$ provided that t_1, t_2 do not occur in the scope of a doxastic operator.

AS 7: $(\alpha(t_1/v) \,\&\, t_1 = t_2 \,\&\, \mathbf{B}(t_1 = t_2)) \supset \alpha(t_2/v)$ where t occurs in the scope of doxastic operator(s) which in the case of systems not containing 4 are uniterated.

AS 8: $(t_1 = t_2 \,\&\, (\exists v)(v = t_1 \,\&\, \mathbf{B}(v = t_1)) \,\&\, (\exists v)(v = t_1 \,\&\, \mathbf{B}(v = t_1))) \supset$
$\mathbf{B}(t_1 = t_2)$.

Further, we shall show that the following restricted rule of inference for all Hin-SQC$^=$ systems preserves validity:

R$\exists$: $\alpha(t/v) \supset \beta \longrightarrow (\exists v)\alpha \supset \beta$ for any constant t foreign to $(\exists v)\alpha \supset \beta$ and provided that t does not occur in the scope of a doxastic operator.

Before proving soundness, it was mentioned in section 1, the Barcan Formula $(\forall v)\mathbf{B}\alpha \supset \mathbf{B}(\forall v)\alpha$ was not included as an axiom-schema (or more generally as a thesis schema) for any of the Hin-SQC$^=$ systems in order to *ensure* soundness since as we shall now show, this schema is invalid in any class of 'Hintikka' models - in any model of the sort $\langle W, R, \{D_i\}, F, V \rangle$.

To show that the Barcan Formula is invalid in any class of 'Hintikka' models, we shall construct a countermodel to the following simple instance of this schema, viz., $(\forall x)BFx \supset B(\forall x)Fx$. First of all, let $W = \{w_1, w_2\}$ and let $\{<w_1, w_2>\} \subseteq R$.[34] Further, suppose that $D_1 = \{c, d\}$ and that $D_2 = \{e, f\}$ and we shall introduce the set $F = \{f_1\}$ such that $f_1(w_1) = c$ and $f_1(w_2) = e$. Then although the individuals c and e are both index-bound, we can think of the set $\{c, e\}$ determined by f_1 as the 'same' individual existing at different indices. So far we have described a Hin-SQC$^=$ model *structure*. Then let the model M based on this structure be described by $V(t, w_1) = d$ and $V(t, w_2) = f$ and such that $V(F, w_2) = \{e\}$. Let M' be a model based on this structure and differing from M only in that $V(t, w_1) = c$ and in that $V(t, w_2) = e$. It should be obvious that any other model based on the above structure and differing from M if at all only in terms of what V assigns to arbitrary t for w_1 (and for w_2) *such that* $V(t, w_1) = f_1(w_1)$ will be M' as described above.

Now that we have constructed the models M and M' we shall next show that M is a *countermodel* to $(\forall x)BFx \supset B(\forall x)Fx$ by showing that $V_M((\forall x)BFx, w_1) = 1$ but that $V_M(B(\forall x)Fx, w_1) = 0$. First, $V_{M'}(Ft, w_2) = 1$ from which it follows that $V_{M'}(BFt, w_1) = 1$. Since for every M' like M such that $V(t, w_1) = f_1(w_1)$ – which in this case does not include M – $V_{M'}(BFt, w_1) = 1$, it follows that $V_M((\forall x)BFx,\ w_1) = 1$. On the other hand, although $V_{M'}(Ft, w_2) = 1$, $V_M(Ft, w_2) = 0$ and hence there is at least one model like M, viz., M itself such that Ft at w_2 is false. Thus, it is the case that $V_M((\forall x)Fx, w_2) = 0$. Then $V_M(B(\forall x)Fx, w_1) = 0$. Q.E.D.

Hence, the sort of semantics under consideration invalidates the Barcan Formula and so to preserve soundness of the Hin-SQC$^=$ axiom systems with respect to this type of semantics, we have not added BF as an axiom-schema to the Hin-SQC$^=$ systems.

[34] For the sake of generality, it would be better to say that $\{<w_1, w_2>\} \subseteq R$.

We shall now show that Hintikka's proposed semantics validates the crucial schemata AS 3, 4, 7, 8 which are axiom-schemata shared by all the Hin-SQC$^{=}$ systems. In addition, it will be shown that in Hintikka's semantics R$\exists$ mentioned above which are inference rules shared by all the Hintikka systems, preserve validity. These soundness results will apply to Hin-SQC$^{=}$ systems not containing 4, although they can be generalized to the Hin-SQC$^{=}$ + 4 systems.

It will first of all be demonstrated that all instances of AS 3 and AS 6 which prohibit quantifying in and substitution of co-referentials for notional contexts respectively are valid. In addition, the inferential version of AS 3, R$\exists$ will be shown to be validity-preserving and it will also be demonstrated that all instances of AS 8 (which involves the claim that having true opinions as to who t_1 and t_2 are, viz., one and the same individual logically implies that x will recognize their identity) are validated in the semantics under consideration.

First of all, consider the schema AS 3, viz., $\alpha(t/v) \supset (\exists v)\alpha$ such that t in $\alpha(t/v)$ does not occur in the scope of a belief operator. Suppose that there is a Hintikka model $M = \langle W,R,\{D_i\},F,V\rangle$ such that at least one w_i in W, $V_M(\alpha(t/v),w_i) = 1$. Then there is an M' like M such that for some arbitrary t", $V(t",w_i)$ for M' is $V(t,w_i)$ for M. Then $V_{M'}(\alpha(t/v),w_i) = 1$ and hence, $V_M((\exists v)\alpha,w_i) = 1$. Q.E.D.

We shall next consider AS 6, $(\alpha(t_1/v)\ \&\ t_1 = t_2) \supset \alpha(t_2/v)$ such that t_1 and t_2 do not occur in the scope of a belief operator. The proof of AS 6 will be by induction on the complexity of $\alpha(t_1/v)$ and $\alpha(t_2/v)$. The *basis* of the induction is where $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are atomic. The proof of the basis, viz., that all instances of $(\alpha(t_1/v) \mathbin{\&} t_1 = t_2) \supset \alpha(t_2/v)$ where $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are atomic proceeds in much the same way as it did for the SQC$^=$ systems, although V for t_1 and t_2 is relativized to the appropriate index. The *inductive hypothesis* is that all instances of $(\alpha(t_1/v) \mathbin{\&} t_1 = t_2) \supset \alpha(t_2/v)$ are valid for cases where $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are of degree of complexity n. I.e., for any w_i in any model, whenever $V_M(t_1 = t_2, w_i) = 1$, $V_M(\alpha(t_1/v), w_i) = V_M(\alpha(t_2/v), w_i)$ where $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are of degree of complexity n. It must then be shown that this characteristic holds for cases where $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are of degree of complexity n + 1.

The cases that need to be considered are where $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are 1) of the forms $\sim\beta(t_1/v)$ and $\sim\beta(t_2/v)$, 2) of the forms $[\beta \mathbin{\&} \gamma](t_1/v)$ and $[\beta \mathbin{\&} \gamma](t_2/v)$ and finally 3) of the forms $(\exists v'')\beta(t_1/v)$ and $(\exists v'')\beta(t_2/v)$. The case where $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are of the forms $B\beta(t_1/v)$ and $B\beta(t_2/v)$ need not be considered since it has been stipulated that t_1, t_2 do not occur in the scope of any doxastic operators.

Case 1: $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are of the forms $\sim\beta(t_1/v)$ and $\sim\beta(t_2/v)$.

Suppose that $V_M(\sim\beta(t_1/v), w_i) = V_M(t_1 = t_2, w_i) = 1$.

Then $V_M(\beta(t_1/v), w_i) = 0$ and given that $V_M(t_1 = t_2, w_i) = 1$, it follows that $V_M(\beta(t_2/v), w_i) = 0$ by the *inductive hypothesis*.

Then $V_M(\sim\beta(t_2/v), w_i) = 1$.

Case 2: $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are of the forms $[\beta \mathbin{\&} \gamma](t_1/v)$, $[\beta \mathbin{\&} \gamma](t_2/v)$.

Suppose that $V_M([\beta \mathbin{\&} \gamma](t_1/v), w_i) = V_M(t_1 = t_2, w_i) = 1$.

So $V_M(\beta(t_1/v), w_i) = V_M(\gamma(t_1/v), w_i) = 1$ and since $V_M(t_1 = t_2, w_i) = 1$, $V_M(\beta(t_2/v), w_i) = V_M(\gamma(t_2/v), w_i) = 1$ by the *inductive hyp.*

Then $V_M([\beta \mathbin{\&} \gamma](t_2/v), w_i) = 1$.

Case 3: $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are of the forms $(\exists v")\beta(t_1/v)$ and $(\exists v")\beta(t_2/v)$.

Suppose that $V_M((\exists v")\beta(t_1/v), w_i) = V_M(t_1 = t_2, w_i) = 1$.

So there is an M' like M such that $V_{M'}(\beta[(t_3/v"), (t_1/v)], w_i) = 1$

and since $V_{M'}(t_1 = t_2, w_i) = 1$ then by the *inductive hyp.*,

$V_{M'}(\beta[(t_3/v"), (t_2/v)], w_i) = 1$.

So, $V_M((\exists v")\beta(t_2/v), w_i) = 1$.

Q.E.D.

This completes the proof that all instances of AS 6 are valid.

Another schema which we shall consider is AS 8, $(t_1 = t_2 \& (\exists v)(v = t_1 \& B(v = t_1)) \& (\exists v)(v = t_2 \& B(v = t_2))) \supset B(t_1 = t_2)$. Suppose that there is a Hintikka model M and an index w_i in W such that $V_M(t_1 = t_2, w_i) = V_M((\exists v)(v = t_1 \& B(v = t_1)), w_i) = V_M((\exists v)(v = t_2 \& B(v = t_2)), w_i) = 1$ but $V_M(B(t_1 = t_2), w_i) = 0$. Since $V_M(t_1 = t_2, w_i) = 1$ then $V(t_1, w_i) = V(t_2, w_i)$. Since $V_M((\exists v)(v = t_1 \& B(v = t_1)), w_i) = 1$ then there is at least one M' like M such that $V_{M'}(t_3 = t_1, w_i) = V_{M'}(B(t_3 = t_1)), w_i) = 1$ for an arbitrary t_3 and where $V(t_3, w_i) = f_k(w_i)$ for some f_k in F. So $V(t_3, w_i) = V(t_1, w_i)$ and for all w_j such that w_iRw_j $V_{M'}(t_3 = t_1, w_j) = 1$ and hence $V(t_1, w_j) = V(t_3, w_j) = f_k(w_j)$. (Recall our condition on V that if $V(t, w_i) = f_k(w_i)$ then for all w_j such that w_iRw_j, $V(t, w_j) = f_k(w_j)$.) And given that $V_M((\exists v)(v = t_2 \& B(v = t_2)), w_i) = 1$ then then there is at least one M" like M such that $V_{M"}(t_3 = t_2, w_i) = V_{M"}(B(t_3 = t_2)), w_i) = 1$ and where $V(t_3, w_i) = f_n(w_i)$ for some f_n in F. So $V(v, w_i) = V(t_2, w_i)$ and for all w_j such that w_iRw_j $V_{M'}(t_3 = t_2, w_j) = 1$ and hence $V(t_2, w_j) = V(t_3, w_j) = f_n(w_j)$.

Given that for M, M' and M", $V(t_1, w_i) = V(t_2, w_i)$ (this is because M' and M" will differ from M if at all only in what V assigns to t_3 for w_i and possibly for any w_j such that w_iRw_j) then for M, $V(t_1, w_i) = V(t_3, w_i)$

$= V(t_2,w_i) = f_k(w_i) = f_n(w_i)$. Since $f_k(w_i) = f_n(w_i)$ then for any w_j such that w_iRw_j, $f_k(w_j) = f_n(w_j)$. But since for any w_j such that w_iRw_j, $V(t_1,w_j) = f_k(w_j)$ and $V(t_2,w_j) = f_n(w_j)$ and further, since for any such w_j, $f_k(w_j) = f_n(w_j)$ it follows that for any w_j such that w_iRw_j, $V(t_1,w_j) = V(t_2,w_j)$. Therefore, for any w_j such that w_iRw_j, $V_M(t_1 = t_2,w_j) = 1$ and hence $V_M(\mathbf{B}(t_1 = t_2),w_i) = 1$ which contradicts our initial supposition that $V_M(\mathbf{B}(t_1 = t_2),w_i) = 0$. Q.E.D.

Finally, we shall show that the restricted rule $\alpha(t/v) \supset \beta \longrightarrow (\exists v)\alpha \supset \beta$ for any constant t (provided that t is foreign to $(\exists v)\alpha \supset \beta$ and that t occurring in $\alpha(t/v)$ does not occur in the scope of a doxastic operator) preserves validity. (The proof that modus ponens and the doxastic counterpart of the rule of necessitation preserve validity is similar to the proof for the sentential systems.) Thus, our hypothesis will be that $\models \alpha(t/v) \supset \beta$ for any constant t such that t does not occur in β. Then it will be shown that $\models (\exists v)\alpha \supset \beta$. The proof proceeds as follows:

Suppose that for some Hintikka model $M = \langle W,R,\{D_i\},F,V\rangle$,

$V_M(\beta,w_i) = 0$.

Then for any M' like M except in V's assignment to t" foreign to β some member of D_i,

$V_{M'}(\beta,w_i) = 0$ - since t" is foreign to β.

By hypothesis,

$V_{M'}(\alpha(t''/v) \supset \beta,w_i) = 1$ for any M' like M.

$V_{M'}(\alpha(t''/v),w_i) = 0$ for any M' like M.

$V_M((\exists v)\alpha,w_i) = 0$.

Thus, for any index w_i in any Hintikka model, whenever $V_M(\beta,w_i) = 0$, $V_M((\exists v)\alpha,w_i) = 0$ on the assumption that $\models \alpha(t/v) \supset \beta$. I.e., $\models (\exists v)\alpha \supset \beta$

if $\models \alpha(t/v) \supset \beta$.[35] Q.E.D.

It will now be shown that all instances of AS 4 which restricts quantifying in to relational contexts and that AS 7 which restricts substitution of co-referentials to contexts where the agent believes that the relevant identity holds are valid.

First, consider AS 7, $(\alpha(t_1/v)\ \&\ t_1 = t_2\ \&\ B(t_1 = t_2)) \supset \alpha(t_2/v)$ where it is allowed that t_1, t_2 may occur in the scope of a belief operator. The proof that all instances of AS 7 are valid will (as for its restrictive 'cousin' AS 6), proceed by *induction* on the complexity of $\alpha(t_1/v)$ and $\alpha(t_2/v)$. The *basis* is where $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are atomic. Then for any index w_i in a model, the truth of the identity '$t_1 = t_2$' at said index is sufficient to guarantee that if $V_M(\alpha(t_1/v), w_i) = 1$ then $V_M(\alpha(t_2/v), w_i) = 1$. The *inductive hypothesis* is that all instances of AS 7 are valid where $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are of degree of complexity n. What needs to be shown is that all instances of AS 7 are valid for cases where $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are of degree of complexity n + 1. The cases to be considered are where $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are of the forms 1) $\sim\beta(t_1/v)$ and $\sim\beta(t_2/v)$, 2) $[\beta\ \&\ \gamma](t_1/v)$ and $[\beta\ \&\ \gamma](t_2/v)$, 3) $(\exists v'')\beta(t_1/v)$ and $(\exists v'')\beta(t_2/v)$ and finally, 4) $B\beta(t_1/v)$ and $B\beta(t_2/v)$. Cases 1) - 3) proceed in much the same way that they did for AS 6 although we shall illustrate case 3). Finally, we shall consider case 4, which involves considering the purely doxastic version of AS 7.

Case 3: $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are of the forms $(\exists v'')\beta(t_1/v)$ and $(\exists v'')\beta(t_2/v)$

> Suppose that $V_M((\exists v'')\beta(t_1/v), w_i) = V_M(t_1 = t_2, w_i) = V_M(B(t_1 = t_2), w_i) = 1$.
>
> So there is an M' like M such that $V_{M'}(B[(t_3/v''), (t_1/v)], w_i) = 1$ (and if free v" occurs in the scope of a doxastic operator then

[35] The author has modelled the reasoning here after Hughes and Creswell (1968), pp. 140-141.

$V(t'', w_i) = f_k(w_i)$ for some $f_k \in F$).

Then since $V_{M'}(t_1 = t_2, w_i) = V_{M'}(B(t_1 = t_2), w_i) = 1$, by the *inductive hyp.*, $V_{M'}(B[(t_3/v''), (t_2/v)], w_i) = 1$.

So, $V_M((\exists v'')B(t_2/v), w_i) = 1$.

Q.E.D.

Case 4: $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are of the forms $B\beta(t_1/v)$ and $B\beta(t_2/v)$ respectively.

Suppose that for some Hin-SQC$^=$ model M and for some w_i in W, $V_M(B\alpha(t_1/v), w_i) = V_M(t_1 = t_2, w_i) = V_M(B(t_1 = t_2), w_i) = 1$.

Since $V_M(B\alpha(t_1/v), w_i) = 1$, then for any w_j such that w_iRw_j, $V_M(\alpha(t_1/v), w_i) = 1$. Further, since $V_M(B(t_1 = t_2), w_i) = 1$ then for any w_j such that w_iRw_j, $V_M(t_1 = t_2, w_i) = 1$.

It can then be shown by induction on the complexity of the contents $\alpha(t_1/v)$ and $\alpha(t_2/v)$ where t_1, t_2 do not occur in the scope of doxastic operators for systems not containing 4, that $V_M(\alpha(t_2/v), w_j)$ is 1 for all w_j such that w_iRw_j. The proof here is as for AS 6. (For systems containing 4, it would also be the case that $V_M(B(t_1 = t_2, w_j) = 1$ and so we could simply appeal to the inductive hypothesis.)

And so for any w_j such that w_iRw_j, $V_M(\alpha(t_2/v), w_j) = 1$ on the suppostion that $V_M(B\alpha(t_1/v), w_i) = V_M(t_1 = t_2, w_i) = V_M(B(t_1 = t_2), w_i) = 1$ and therefore, $V_M(B\alpha(t_2/v), w_i) = 1$.

Q.E.D.

Next, consider AS 4, $\alpha(t/v)$ & $(\exists v)(v = t$ & $B(v = t)) \supset (\exists v)\alpha$. Suppose there is some w_i in W in a Hintikka model M such that $V_M(\alpha(t/v), w_i) =$

$V_M((\exists v)(v = t \,\&\, \mathbf{B}(v = t), w_i) = 1$. Then it will be shown that $V_M((\exists v)\alpha, w_i) = 1$. Supposing that $V_M((\exists v)(v = t \,\&\, \mathbf{B}(v = t), w_i) = 1$ then there will be at least one M' like M except possibly in V's assignment to some arbitrary t" (which must be foreign to $(\exists v)\alpha$) at w_i some member of D_i such that $V_{M'}(t" = t, w_i) = 1 = V_{M'}(\mathbf{B}(t" = t), w_i) = 1$ and such that $V(t", w_i) = f_k(w_i)$. Further, since $V_M(\alpha\,(t/v), w_i) = 1$ then $V_{M'}(\alpha\,(t/v), w_i) = 1$ (given that t" is foreign to $(\exists v)\alpha$ - and given that t replaces all occurrences of free v is $\alpha\,(t/v)$). So, it is the case that $V_{M'}(t" = t, w_i) = 1 = V_{M'}(\mathbf{B}(t" = t), w_i) = V_{M'}(\alpha\,(t/v), w_i) = 1$. Then it will be shown that $V_{M'}(\alpha\,(t"/v), w_i) = 1$. This can be proven by induction on the complexity of $\alpha\,(t/v)$, $\alpha\,(t"/v)$.

The basis of the induction is where $\alpha\,(t/v)$, $\alpha\,(t"/v)$ are atomic and the proof of the basis proceeds as it did for AS 7. The inductive hypothesis is that for $\alpha\,(t/v)$ of degree of complexity n, whenever $V_{M'}(t" = t, w_i) = 1 = V_{M'}(\mathbf{B}(t" = t), w_i) = V_{M'}(\alpha\,(t/v), w_i)$ then $V_{M'}(\alpha\,(t"/v), w_i) = 1$. We then need to show that this will hold true for $\alpha\,(t/v)$, $\alpha\,(t"/v)$ of degree of complexity n + 1. The cases to be considered are where $\alpha\,(t/v)$, $\alpha\,(t"/v)$ are 1) negations, 2) conjunctions, 3) existentially quantified wffs and 4) of the form $\mathbf{B}\beta(t/v)$, $\mathbf{B}\beta(t"/v)$ respectively. These cases are proven exactly as they were for AS 7.

Then by induction on the complexity of wffs, we can conclude that whenever $V_{M'}(t" = t, w_i) = V_{M'}(\mathbf{B}(t" = t), w_i) = V_{M'}(\alpha\,(t/v), w_i) = 1$ then $V_{M'}(\alpha\,(t"/v), w_i) = 1$ where $\alpha\,(t/v)$, $\alpha\,(t"/v)$ are of any degree of complexity. Since $V(t", w_i) = f_k(w_i)$ for some member of F and since $V_{M'}(\alpha\,(t"/v), w_i) = 1$ (which in turn relied on the suppositions that $V_M(\alpha\,(t/v), w_i) = V_M((\exists v)(v = t \,\&\, \mathbf{B}(v = t), w_i) = 1)$, it follows that $V_M((\exists v)\alpha, w_i) = 1$.

Q.E.D.

Now that we have illustrated some results concerning *soundness* of the Hin-SQC$^=$ systems (at least those not containing 4) with respect to the appropriate classes of Hintikka models, we shall in the remainder of this section sketch a Henkin-style proof of *completeness* of these systems with respect to this type of semantics. As usual we shall sketch a Henkin-style completeness proof for the Hin-SQC$^=$ systems. What we shall want to show for any Hin-SQC$^=$ system is that any non-theorem will be invalid in the appropriate Hin-SQC$^=$ canonical model. I.e., for any non-theorem α, $V_{\mathcal{M}}(\alpha, w_i) = 0$ for some maximal consistent set w_i (of a special sort) in the system's canonical model, $\mathcal{M}$. The characteristics of a Hin-SQC$^=$ canonical model will now be described.

Any Hin-SQC$^=$ canonical model $\mathcal{M}$ is a 5-tuple, $\langle W, R, \{D_i\}, F, V \rangle$ such that W is a set of maximal consistent sets of wffs with the $\exists$-property and with the $\exists_B$-property. I.e., $W = \{w_i : w_i$ is a maximal consistent set with the $\exists$-property and with the $\exists_B$-property$\}$. As was discussed in chapter two, a maximal consistent set w_i has the $\exists$-property just in case for any wff of the form $(\exists v)\alpha$, if $(\exists v)\alpha$ is in w_i then so is $\alpha(t/v)$ for some constant t. In the case of the Hin-SQC$^=$ systems we shall add the rider that the scope of the quantifier is such that free v does not occur in the scope of a belief operator. We can guarantee that any maximal consistent set has the $\exists$-property by (consistently) adding for any wff of the form $(\exists v)\alpha$ in w_i an implicational wff having the form $(\exists v)\alpha \supset \alpha(t/v)$ for at least one constant t. What it is for a maximal consistent set w_i to possess the $\exists_B$-property will be defined once we have defined the element F for the canonical model.

Given our definition of W for the canonical model, R is a two-place relation ranging over members of W which can be defined as follows: For any w_i, w_j in W, w_iRw_j iff $(\forall\alpha)(B\alpha \in w_i \longrightarrow \alpha \in w_j)$. Further, $\{D_i\}$ is a set of sets of constants, each set of constants being associated with the appropriate w_i in W (where the subscripts 'match up') where D_1 is the set CONS which is the set of all constants which can occur in any well-formmed formula of the language L for the Hin-SQC$^=$ systems. We shall assign the subscript '1' to this set, i.e., we shall call the set CONS which is D_1 the set $CONS_1$. Also, we shall construct a set of sets of constants, $\{CONS_i\}$ where $CONS_1 \in \{CONS_i\}$ (each set being denumerably infinite like $CONS_1$) such that no member of any of these sets if $i \neq 1$ can occur in any wff of L for the Hin-SQC$^=$ systems. Each of these sets is assigned a subscript such that if a set in this series is $CONS_i$ then we shall say that this set is D_i. A requirement for membership in $\{CONS_i\}$ is that for any i, j where $i \neq j$, $CONS_i \cap CONS_j$ is $\emptyset$. In short, the members of $\{CONS_i\}$ are non-overlapping.

Then we can define $\{D_i\}$ as follows: For any D_i, D_j in $\{D_i\}$ such that $i \neq j$, $D_i = \{u| \ u \in CONS_i\}$ and $D_j = \{p| \ p \in CONS_j\}$ such that $D_i \cap D_j = \emptyset$. Also, $D_1 = \{t| \ t \in CONS_1\}$ where $CONS_1$ is simply the set CONS which is the set of all constants which can occur in any wff of L for the Hin-SQC$^=$ systems. For any constant in $CONS_i$ where $i \neq 1$, we shall introduce the function g which to each constant in $CONS_i$ assigns exactly one constant from $CONS_1$ (= CONS). We shall call the value of g(u) where u is a constant in $CONS_{i(\neq 1)}$ u^* where u^* is in $CONS_1$ (=CONS).

The set F is a set $\{f_1, f_2, \ldots, f_n, \ldots\}$ of 'correlating' functions where each member of this set, f_k for each w_i in W takes as its value exactly one constant in the D_i associated with w_i or is undefined. Then for any f_k in F and for any w_i in W, $f_k(w_i) \in D_i$ or is undefined. Further, for any f_k, f_n in F, if $f_k(w_i) = f_n(w_i)$ then for any w_j such that w_iRw_j, $f_k(w_j) = f_n(w_j)$. Given our definition of F for the canonical model we can now present the characteristics of any maximal consistent set w_i possessing the so-called $\exists_B$-property. Any w_i in W possesses the $\exists_B$-property just in case for any wff of the form $(\exists v)\alpha$ where $\exists$ is non-vacuous and such that α is either a belief wff or involves a belief wff in whose scope v occurs, if $(\exists v)\alpha$ is in w_i then $\alpha(t/v)$ is in w_i for at least one constant t such that t denotes (where 'denotes' will be spelled out in terms of V) a constant in D_i which is the value of an f_k in F. We can ensure that any w_i in W has the $\exists_B$-property by adding for every wff of the form $(\exists v)\alpha$ in w_i (where $\exists$ is non-vacuous and such that A is either a belief wff or involves a belief wff in whose scope v occurs) the implicational wff $(\exists v)\alpha \supset \alpha(t/v)$ for at least one constant t such that t denotes a term in D_i which is the value of an f_k in F (on the condition that we can preserve consistency).

Finally, we can define V for constants for $w_1 \in W$ similarly to how we defined V for constants for $\mathcal{M}$ for the $SQC^=$ systems by first of all supposing that the members of CONS can be ordered. Then for any constant t_i, and for $w_1 \in W$, $V(t_i,w_1) = V(t_j,w_1)$ if t_j occurs earlier in the ordering such that $t_j = t_i \in w_1$ or $V(t_i,w_1) = t_i$ (where t_i is in CONS) otherwise. For any constant t_i and for any $w_{j(\neq 1)}$ in W, $V(t_i,w_j) = u_n$ such that $u_n \in D_j$ and such that $t_i = u_n{}^* \in w_j$.[36] V for predicate variables for $w_1 \in W$ is defined as for the $SQC^=$ systems as follows: For any $P \in$ PRED, for $w_1 \in W$, $\langle t_1,\ldots,t_n\rangle \in V(P,w_1)$ iff $Pt_1\ldots t_n \in w_1$ where the t_i's in the n-tuple

[36] We shall also impose the same restriction on V for terms for the canonical model as for any model in the appropriate class, viz., that if $V(t,w_i) = f_k(w_i)$ for some f_k then for all w_j such that w_iRw_j, $V(t,w_j) = f_k(w_i)$.

$<t_1, \ldots, t_n>$ are constants in D_1. V for predicate variables for any $w_{i(\neq 1)}$ in W is defined as follows: $<u_1, \ldots u_n> \in V(P, w_i)$ iff $Pu_1^* \ldots u_n^* \in V(P, w_i)$ where the u_i's in the n-tuple $<u_1, \ldots u_n>$ are constants in D_i.

Now that we have characterized the canonical model for any Hin-SQC$^=$ system, we shall in outline form provide a proof of the fundamental theorem of canonical models, viz., $V_{\mu}(\alpha, w_i) = 1$ iff $\alpha \in w_i$ for any w_i in W. The proof proceeds by induction on the complexity of formulae:

Base Clause: Suppose α is atomic in which case α is i) of the form $t_1 = t_2$ or is ii) of the form $Pt_1 \ldots t_n$.

i) α is of the form $t_1 = t_2$.
The proof that $V_{\mu}(t_1 = t_2, w_1) = 1$ iff $t_1 = t_2 \in w_1$ proceeds as it did for the SQC$^=$ systems. We shall now prove that $V_{\mu}(t_1 = t_2, w_i) = 1$ iff $t_1 = t_2 \in w_i$ for w_i in W where $i \neq 1$.

Suppose that $V_{\mu}(t_1 = t_2, w_i) = 1$.
Then $V(t_1, w_i) = V(t_2, w_i)$
Then $V(t_1, w_i) = u_n = V(t_2, w_i)$ where u_n is a constant in D_i.
Then $t_1 = u_n^* \in w_i$ and $t_2 = u_n^* \in w_i$ given the definition of V for constants for $w_{i(\neq 1)}$ in W.
Then $t_1 = u_n^* \ \& \ t_2 = u_n^* \in w_i$ given that w_i is maximal consistent
$\vdash (t_1 = u_n^* \ \& \ t_2 = u_n^*) \supset t_1 = t_2$
Hence, $(t_1 = u_n^* \ \& \ t_2 = u_n^*) \supset t_1 = t_2 \in w_i$ since w_i is max. con.
Hence, $t_1 = t_2 \in w_i$ since w_i is max. con.

Suppose that $t_1 = t_2 \in w_i$.
Then $t_1 = u_n^* \in w_i$ such that $u_n \in D_i$
Then $t_1 = t_2 \ \& \ t_1 = u_n^* \in w_i$ since w_i is maximal consistent.
$\vdash (t_1 = t_2 \ \& \ t_1 = u_n^*) \supset t_2 = u_n^*$
Hence, $(t_1 = t_2 \ \& \ t_1 = u_n^*) \supset t_2 = u_n^* \in w_i$ since w_i is max. con.
$t_2 = u_n^* \in w_i$ since w_i is max. con. where $u_n \in D_i$.
$V(t_2, w_i) = u_n = V(t_1, w_i)$ given the definition of V for constants for any $w_{i(\neq 1)}$ in W.
$V_{\mu}(t_1 = t_2, w_i) = 1$.

ii) A is of the form $Pt_1 \ldots t_n$.

We can prove $V_{\mu}(Pt_1 \ldots t_n, w_1) = 1$ iff $Pt_1 \ldots t_n \in w_1$ in the same way we proved this for the SQC$^=$ systems.

The proof of $V_{\mu}(Pt_1 \ldots t_n, w_i) = 1$ iff $Pt_1 \ldots t_n \in w_i$ for any $w_{i(\neq 1)}$ in W proceeds as follows:

$V_{\mu}(Pt_1 \ldots t_n, w_i) = 1$ iff $\langle V(t_1, w_i), \ldots, V(t_n, w_i)\rangle \in V(P, w_i)$
iff $\langle u_1, \ldots, u_n\rangle \in V(P, w_i)$ where each $u_i \in D_i$.
iff $Pu_1{}^* \ldots u_n{}^* \in w_i$. (given the definition of V for predicate variables.)

Now if $V(t_1, w_i) = u_1, \ldots, V(t_n, w_i) = u_n$ where $u_1, \ldots, u_n$ are all in D_i then $t_1 = u_1{}^*, \ldots, t_n = u_n{}^*$ are all in w_i given the definition of V for constants for any $w_{i(\neq 1)}$ in W and hence $t_1 = u_1{}^* \&, \ldots, \& \ t_n = u_n{}^* \in w_i$. Since $\vdash (t_1 = u_1{}^* \&, \ldots, \& \ t_n = u_n{}^*) \supset (Pu_1{}^* \ldots u_n{}^* \equiv Pt_1 \ldots t_n)$[37] then $(t_1 = u_1{}^* \&, \ldots, \& \ t_n = u_n{}^*) \supset (Pu_1{}^* \ldots u_n{}^* \equiv Pt_1 \ldots t_n) \in w_i$. Therefore $Pu_1{}^* \ldots u_n{}^* \equiv Pt_1 \ldots t_n \in w_i$ in which case $Pt_1 \ldots t_n \in w_i$ if $Pu_1{}^* \ldots u_n{}^*$ is. Therefore, $V_{\mu}(Pt_1 \ldots t_n, w_i) = 1$ iff $Pt_1 \ldots t_n \in w_i$. Q.E.D.

Now that we have proven the base clause, we appeal to the *inductive hypothesis*, viz., that the fundamental theorem of canonical models holds for wffs of degree of complexity n. What we must now show is that the fundamental theorem holds for wffs of degree of complexity n + 1. The proof of this for the cases where α is of the form $\sim\beta$, $\beta \ \& \ \gamma$ and $\mathbf{B}\beta$ proceeds as it did for the sentential doxastic systems and for the SQC$^=$ systems. What needs to be considered is the case where α is of the form $(\exists v)\beta$. The proof of the case where the scope β of the quantifier is simply a wff where free v is not in the scope of a doxastic operator proceeds roughly along the same lines as the proof of the case where α is of the form $(\exists v)\beta$ for the SQC$^=$ systems. What remains to be shown is that the fundamental theorem holds for the case where free v in the scope β of $(\exists v)\beta$ occurs in the scope of a doxastic operator.

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37 See the reasoning behind this on pages 67-68 in chapter two.

Case: α is of the form $(\exists v)\beta$ where the scope β is such that free v occurs in the scope of a doxastic operator.

Suppose that $(\exists v)\beta \in w_i$.

Then $\beta(t/v) \in w_i$ for at least one cons. t which denotes some cons. in D_i which is the value of an f_k in F. I.e., $V(t,w_i) = f_k$ for some f_k in F. This is guaranteed by the $\exists_B$-property.

$V_{\mathcal{M}}(\beta(t/v),w_i) = 1$ for at least one constant t such that $V(t,w_i) = f_k$ by the *inductive hypothesis*.

$V_{\mathcal{M}'}(\beta(t''/v),w_i) = 1$ for an $\mathcal{M}'$ like $\mathcal{M}$ and such that $V(t'',w_i) = V(t,w_i) = f_k(w_i)$.

$V_{\mathcal{M}}((\exists v)\beta,w_i) = 1$. Q.E.D.

Suppose that $V_{\mathcal{M}}((\exists v)\beta,w_i) = 1$.

Then $V_{\mathcal{M}'}(\beta(t''/v),w_i) = 1$ for at least one $\mathcal{M}'$ like $\mathcal{M}$ and such that $V(t'',w_i) = f_k(w_i)$.

Then $V_{\mathcal{M}}(\beta(t/v),w_i) = 1$ for some constant t such that $V(t,w_i) = V(t'',w_i) = f_k(w_i)$.

Since $\mathcal{M}'$ differs from $\mathcal{M}$ if at all only in what V assigns to t" at w_i then $V(t.w_i) = f_k(w_i)$ for $\mathcal{M}'$. But $V(t'',w_i) = f_k(w_i)$ for $\mathcal{M}'$.

So, $V_{\mathcal{M}'}(t = t'',w_i) = 1$.

For for any w_j in W such that w_iRw_j, $V(t,w_j) = V(t'',w_j) = f_k(w_j)$ assuming that the same restriction applies to V for terms for the canonical model as for any other Hintikka model, viz., that if $V(t,w_i) = f_k(w_i)$ for some f_k then for all w_j such that w_iRw_j, $V(t,w_j) = f_k(w_j)$.

Then $V_{\mathcal{M}'}(t'' = t,w_j) = 1$ for all w_j such that w_iRw_j (since $\mathcal{M}'$ is like $\mathcal{M}$ except in V's assignment to t".)

Thus, $V_{\mathcal{M}'}(\mathbf{B}(t'' = t),w_i) = 1$.

Thus, $V_{\mathcal{M}'}(t'' = t \,\&\, \mathbf{B}(t'' = t),w_i) = 1$

Thus, $V_{\mathcal{M}}((\exists v)(v = t \,\&\, \mathbf{B}(v = t),w_i) = 1$.

Then $V_{\mathcal{M}}(B(t/v),w_i) = V_{\mathcal{M}}((\exists v)(v =t \,\&\, \mathbf{B}(v = t),w_i) = 1$ and hence:

$V_{\mathcal{M}}(B(t/v) \,\&\, (\exists v)(v = t \,\&\, \mathbf{B}(v = t),w_i) = 1$.

Given that the fundamental theorem has been proven for wffs of the form β & γ then:

$\beta(t/v)$ & $(\exists v)(v = t$ & $B(v = t)$ $\in w_i$

$\vdash(\beta(t/v)$ & $(\exists v)(v = t$ & $B(v = t)) \supset (\exists v)\beta$

$(\beta(t/v)$ & $(\exists v)(v = t$ & $B(v = t)) \supset (\exists v)\beta \in w_i$ since w_i is max. con.

$(\exists v)\beta \in w_i$ since w_i is max. con.

Q.E.D.

This completes the proof of the fundamental theorem of canonical models.

By induction on the complexity of wffs, we have proven that $V_{\mathcal{M}}(\alpha, w_i) = 1$ iff $\alpha \in w_i$ for any wff α and for any maximal consistent set of wffs, w_i. Now suppose α is a non-theorem for some Hin-SQC$^=$ system. Then $\sim\alpha$ will be syntactically consistent and hence by Lindenbaum's lemma we can construct $\sim\alpha$'s maximal consistent extension w_i which is in the set of maximal consistent sets, W in the canonical model, $\mathcal{M}$ such that $\sim\alpha \in w_i$. Then by the fundamental theorem of canonical models, $V_{\mathcal{M}}(\sim\alpha, w_i) = 1$ and hence $V_{\mathcal{M}}(\alpha, w_i) = 0$. In short, any non-theorem will be invalid in the appropriate system's canonical model. What remains to be shown is that the relevant Hin-SQC$^=$ system's canonical model $\mathcal{M}$ is in the class of models with respect to which that system is sound. And as we outlined in chapter one this can be proven by showing that R has the appropriate characteristics. For example, if we are to prove that Hin-KDQC$^=$'s canonical model is in the class C of models with respect to which this system is sound, it must be shown that R in the canonical model is serial. And how this can be done is discussed in the first chapter.

4. An Alternative to Hintikka's Logic and Semantics for Belief

As we have just seen, the Hintikka axiom systems for belief and their characteristic semantics enable us to make sense of quantification into belief constructions as well as substitution of co-referentials in such constructions. This is achieved by making a distinction between relational and notional belief contexts which has been the time-honoured tradition on the ordinary language front with such philosophers as Kaplan and Sosa.

In particular, the problem of quantifying into ordinary language belief contexts is a problem for a belief logic mirroring such contexts only if we construe the existential quantifier in our logic *objectually*. Thus, from 'Jones believes that the next Prime Minister of Canada will attempt to balance the budget' we would not infer that *there is* some individual v such that Jones believes of v that he/she will attempt to balance the budget'. And so, in any logic of belief where the existential quantifier is construed objectually, it would be undesirable to have a rule of inference permitting unrestricted quantification across propositional attitude operators. However, if we read the quantifiers *substitutionally* in the semantics for our belief logic then there is no problem with respect to 'quantifying in'. I.e., if '$\exists$' is given a substitutional reading in the semantics, then from $\mathbf{B}\alpha\,(t/v)$ we can infer $(\exists v)\mathbf{B}\alpha$ since this says intuitively that there is a substitution instance of $\mathbf{B}\alpha$ (which in this case is $\mathbf{B}\alpha\,(t/v)$) which is true. No mention is made of there being any individual (in the appropriate domain of individuals) v such that $\mathbf{B}\alpha\,(t/v)$ is true.

Then a logic of belief in which the quantifiers are construed substitutionally does not inherit the ordinary language problem of quantifying into belief constructions. This point has been made by Kripke in a paper dealing with substitutional quantification for first-order logic:

> ...the intelligibility of substitutional quantification into a belief or a

> modal context is guaranteed provided the belief or modality is intelligible when applied to a closed sentence ... As Quine has pointed out, even for a context as opaque as quotation, where no-one thinks that satisfaction for referential variables makes any immediate sense, substitutional quantification is immediately intelligible.[38]

Then it would seem reasonable to adopt a logic of belief or for that matter of any modality where the quantifiers are read substitutionally. However, Kripke claims that it is best to have first-order modal or belief logics which contain *both* kinds of quantifiers since "substitutional quantification is here, as always, not a *rival* theory to referential quantification".[39] We shall now offer an argument against Kripke's claim and in favour of the claim that at least for a logic of propositional attitude modalities such as *belief* it is best to adopt a logic of belief where the quantifiers are read substitutionally only in the semantics. The obvious advantage of adopting such a logic is that we are able to sidestep the problem of quantifying in entirely. But there are other reasons for adopting a belief logic where the quantifiers are read solely substitutionally (in conjunction with a truth-value semantics), as we shall presently see.

An obvious candidate for a semantics of a logic of belief where the quantifiers are read substitutionally would be a *truth-value* semantics. As the reader may recall, a TV semantics dispenses with domains of individuals for TV models. In fact, the atomic wffs of the language for the appropriate system are assigned truth-values at any given index in the model without appeal to individuals in the same way that the atomic wffs are assigned truth-values at indices for sentential systems. And employing $V(\alpha, w_i) = V_M(\alpha, w_i)$ where α is atomic as the basis, we can define the

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38 Kripke (1976), p. 375.

39 ibid, p. 375.

truth-conditions for the different sorts of wffs of the language inductively.

The advantage which a TV semantics supporting a substitutional reading of the quantifiers has over a domain semantics (such as for the SQC$^=$ systems where the indices in a model share the same domain of individuals or such as for the Hin-SQC$^=$ systems where the indices in a model have their own non-overlapping domains) is that it has none of the difficulties associated with the metaphysics of the domain semantics.

As was argued in chapter two, a problem with the domain semantics for the SQC$^=$ systems is that it is not clear what the criteria for transindex individuation would be other than regarding members of D in an SQC$^=$ model as so-called bare particulars, which is a notion that philosophers such as Kaplan find objectionable. In addition, it was noted that individuals in SQC$^=$ models are *necessary* existents given that for any such model, the domain D is shared by all indices. On the other hand, if we opt for a Hintikka-type semantics for belief logic where each index has associated with it its own non-overlapping domain of individuals then we are faced with the problem of making sense of an index-bound individual's having 'correlates' across doxastic alternatives which is necessary for dealing with quantification across propositional attitude operators. Further, there is the objection raised by both Kripke and Plantinga that names denoting individuals in propositional attitude constructions (as well as in counterfactual conditional constructions) denote the *same* individual existing at various alternatives and not a series of 'correlates'. Finally, whereas the invariant domain semantics involves a proliferation of necessary existents, the varying domain semantics for the Hin-SQC$^=$ systems involves a proliferation of possibilia - given that indexed domains of individuals are non-overlapping.

A TV semantics for belief logic avoids the metaphysical and ontological embarassments associated with the different types of domain semantics by simply dispensing with domains of individuals. The solution to the difficulties outlined above is simple enough: If there are no domains of individuals for models, then there is no question of transindexical identity of individuals or of transindexical similarity (as in the case of determining conditions for counterparthood)[40]. Then there is some presumption in favour of adopting a TV semantics for belief logic.

Another presumption in favour of adopting a TV semantics for first-order belief logic is that it obviates the need for relational contexts with respect to which existential generalization would normally be restricted. In short, from the point of view of quantifying across propositional attitude operators, it is unnecessary to appeal to the notional/relational distinction or the de re/de dicto distinction (although this distinction still applies when it is characterized in terms of the quantifiers) in Kaplan's or Hintikka's sense where 'acquaintance' with the relevant individual(s) distinguishes a relational context from a notional one.

However, the TV semantics we have outlined for the SQC$^{=}$ systems while rightly allowing for unrestricted *substitutional* quantification across propositional attitude operators also allows for unrestricted substitution of co-referential terms for propositional attitude constructions. But this latter feature is undesirable in the light of the schema $t_1 = t_2 \supset \mathbf{B}(t_1 = t_2)$ that is validated in the SQC$^{=}$ TV semantics and which intuitively says that agents are omnidoxastic with respect to contingent identities. This schema is un-

[40] As with the TV semantics for the SQC$^{=}$ systems, the TV semantics for the set of axiom-systems we shall presently develop does not involve any 'assignments' of individuals to constants. The question of what constants denote is sidestepped.

tenable since presumably agents can fail to ***believe*** that contingent identities obtain - as in the Tully/Cicero case discussed in the third chapter. Then in the light of these considerations, it is undesirable to have a semantics (and a logic) of belief which allows for unrestricted substitution of co-referentials (or to be unprejudiced, co-identicals) for belief constructions. Since agents are presumably not omnidoxastic with respect to contingent identities then if x believes that $\alpha(t_1/v)$ at an index w_i and if the identity $t_1 = t_2$ holds at this index, it does not follow that the agent also believes that $\alpha(t_2/v)$. (Once again, consider the Tully/Cicero case.) The agent may fail to ***believe*** that the identity $t_1 = t_2$ holds. The obvious way of guaranteeing that this identity holds at all alternatives is to require that the agent ***believes*** that the identity holds which was one of Hintikka's ways of dealing with this problem. However, we shall not use Hintikka's second strategy of allowing substitution to go through for special sorts of relational contexts since our logic and semantics does not assume a relational/notional distinction.

Notice that this diagnosis of the problem of the failure of substitutivity of co-referentials for belief constructions makes no appeal to the notional vs. relational distinction for belief contexts. In fact, this diagnosis of the problem lends itself to the view that belief contexts are fundamentally unambiguous and in fact they are, to borrow Frege's phrase, unambiguously 'oblique' in the sense that co-referentials (or more appropriately, co-identicals) are never substitutible in such contexts, given that agents are not omnidoxastic with respect to contingent identities. On the other hand, contexts not involving propositional attitude modalities are 'transparent' with respect to the substitution of co-referentials in which case there is an ambiguity that does not lie in the belief operator itself but rather in the distinction between non-doxastic and doxastic contexts (or more generally between modal vs. non-modal contexts).[41]

What is here being proposed is that we adopt a TV semantics for belief logic which lends itself solely to a substitutional reading of the quantifiers thus allowing unrestricted quantification across doxastic operators and yet which treats belief contexts as unambiguously oblique with respect to the substitution of co-referentials. As we shall see, this type of semantics can be developed by simply adopting the TV semantics for the $SQC^{=}$ systems and lifting the restriction on V that if $V(t_1 = t_2, w_i) = 1$ then $V(t_1 = t_2, w_j) = 1$ for all w_j in W. This allows that even though $t_1 = t_2$ holds at w_i, it may fail to hold at some w_j in W such that $w_i R w_j$ in which case an agent at w_i will fail to believe that this identity holds thus invalidating $t_1 = t_2 \supset B(t_1 = t_2)$. Then even though x at w_i believes $\alpha(t_1/v)$ and even though $t_1 = t_2$ holds at w_i, x may fail to believe that $\alpha(t_2/v)$ at w_i since $\alpha(t_1/v)$ and $\alpha(t_2/v)$ may differ in their truth-values at some alternative w_j (since $t_1 = t_2$ may fail to hold at this w_j). Further, since in this type of semantics the quantifiers are interpreted substitutionally then unrestricted quantification into belief constructions is permitted thus obviating the need for the relational/notional distinction. Nor does this distinction rear its inelegant head in discussing the failure of substitutivity for belief contexts since Hintikka's AS 8 which allows subsitution of identicals to go through for special sorts of relational contexts turns out to be invalid in this semantics - as will be shown presently.

It should be apparent that the advantage of a TV semantics of belief

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41 This view of things is in the spirit of Frege's treatment of belief contexts as 'oblique' and non-belief contexts as transparent with respect to the issue of substitution in 'Sense and Reference'.

logic which lends itself to a solely substitutional reading of the quantifiers *while restricting substitution of co-identicals* vs. the type of semantics which Kripke is advocating which allows for both kinds of quantifiers or vs. a Hintikka-type semantics where the quantifiers are read solely objectually is that the former type does not treat belief contexts as ambiguous (thus for example sidestepping Stich's objections concerning the so-called myth of ambiguity of belief contexts). It is presumably better in the sense of 'theoretically simpler' to get by with one rather than two senses of belief. Also, the purely substitutional TV semantics avoids the difficulties that are associated with the *metaphysics* of any semantics appealing to so-called domains of indviduals.

The appropriate set of axiom-systems for the kind of semantics we are proposing will simply be the $SQC^{=}$ axiom systems each of which will include as an axiom schema $\alpha(t/v) \supset (\exists v)\alpha$ which allows for unrestricted *substitutional* quantification across propositional attitude operators (if t occurring in $\alpha(t/v)$ occurs in the scope of a doxastic operator[42]) with the following emendation: Any such system will have as an axiom-schema a *restricted* version of $(\alpha(t_1/v) \& t_1 = t_2) \supset \alpha(t_2/v)$ where it is stipulated that t_1, t_2 in $\alpha(t_1/v)$, $\alpha(t_2/v)$ do not occur in the scope of doxastic operators. The restriction on this schema effectively blocks the proof of the schema $t_1 = t_2 \supset \mathbf{B}(t_1 = t_2)$ which says that agents are omnidoxastic with respect to contingent identities since the unrestricted version is integral to the proof of this schema. More importantly, this restriction disallows as a specific version of this schema $(\mathbf{B}\alpha(t_1/v) \& t_1 = t_2) \supset \mathbf{B}\alpha(t_2/v)$ which says that co-identicals are freely substitutible in belief constructions. Also, the above-mentioned restriction disallows as instances of this schema wffs

[42] This would allow as instances of this schema such wffs as $\mathbf{B}Fa \supset (\exists x)\mathbf{B}Fx$ as well as $(Fa \& \mathbf{B}Fa) \supset (\exists x)(Fx \& \mathbf{B}Fx)$.

such as $((Fa \mathbin{\&} BGa) \mathbin{\&} a = b) \supset (Fb \mathbin{\&} BGb)$ or $((Fa \mathbin{\&} P_B Ga) \mathbin{\&} a = b) \supset (Fb \mathbin{\&} P_B Gb)$.

Further, we shall add as an axiom-schema $(\alpha(t_1/v) \mathbin{\&} t_1 = t_2 \mathbin{\&} B(t_1 = t_2)) \supset \alpha(t_2/v)$ where t_1 and t_2 may occur in the scope of doxastic operators which in the case of systems without 4 are uniterated for the same reasons as discussed in the case of the Hin-SQC$^=$ systems. Intuitively, this schema asserts that the substitutivity of identicals is permissible for belief contexts *provided* that the agent believes that the relevant identity obtains. That all instances of this schema are valid/provable for the systems we shall presently discuss and that Hintikka's AS 8 is not valid/derivable for these same systems reflects our Fregean position that belief contexts are unambiguously oblique.

And so, we shall call the following axiom-system Sub-KQC$^=$ such that any Sub-SQC$^=$ system can be obtained by extending the doxastic sentential fragment in the way discussed in the first chapter:

AS 1: α where α has the form of a PC thesis.

AS 2: $(B\alpha \mathbin{\&} B(\alpha \supset \beta)) \supset B\beta$

AS 3: $\alpha(t/v) \supset (\exists v)\alpha$

AS 4: $t = t$

AS 5: $(\alpha(t_1/v) \mathbin{\&} t_1 = t_2) \supset \alpha(t_2/v)$ (provided that t_1 and t_2 do not occur in the scope of a doxastic operator.)

AS 6: $(\alpha(t_1/v) \mathbin{\&} t_1 = t_2 \mathbin{\&} B(t_1 = t_2)) \supset \alpha(t_2/v)$ (where t_1, t_2 may occur in the scope of doxastic operator(s) which for systems without 4 are uniterated.)

AS 7: $(\forall v)\mathbf{B}\alpha \supset \mathbf{B}(\forall v)\alpha$ (Barcan Formula)

The rules of inference will simply be *modus ponens* as well as:

RB $\vdash \alpha \longrightarrow \vdash \mathbf{B}\alpha$

R∃ $\vdash \alpha(t/v) \supset \beta \longrightarrow \vdash (\exists v)\alpha \supset \beta$ (for any t foreign to $(\exists v)\alpha \supset \beta$)

Before discussing the corresponding semantics, we shall show that belief de re logically (classically) implies belief de dicto for the Sub-SQC$^{=}$ systems, which as discussed in chapter three also makes sense if we construe the quantifiers substitutionally. Thus, suppose that some substitution instance of 'Jones believes that v is prime' is true. Then it follows that Jones believes that some substitution instance of 'v is a prime' is true. So, it is desirable that $\vdash (\exists v)\mathbf{B}\alpha \supset \mathbf{B}(\exists v)\alpha$ for the Sub-SQC$^{=}$ systems. A derivation sequence of any instance of $(\exists v)\mathbf{B}\alpha \supset \mathbf{B}(\exists v)\alpha$ for the Sub-SQC$^{=}$ systems will look like this:

1. $\vdash \alpha(t/v) \supset (\exists v)\alpha$
2. $\mathbf{B}(\alpha(t/v) \supset (\exists v)\alpha)$ 1, RB
3. $\mathbf{B}\alpha(t/v) \supset \mathbf{B}(\exists v)\alpha$ 2, K and modus ponens.
4. $(\exists v)\mathbf{B}\alpha \supset \mathbf{B}(\exists v)\alpha$ 3, R∃

Q.E.D.

Further, given that our Sub-SQC$^{=}$ axiom systems contain the Barcan Formula, and given that '**B**' and '$\mathbf{P_B}$' are interdefinable, then any instance of the following schema for doxastic possibility is derivable (by contraposing the appropriate instance of the Barcan Formula), viz., $\mathbf{P_B}(\exists v)\alpha \supset$

$(\exists v)P_B\alpha$. And given AS 3, $\alpha(t/v) \supset (\exists v)\alpha$ along with the derived rule of inference $\vdash \alpha \supset \beta \longrightarrow \vdash P_B\alpha \supset P_B\beta$ as well as R∃, any instance of the schema $(\exists v)P_B\alpha \supset (\exists v)P_B\alpha$ is derivable for the Sub-SQC$^=$ systems. It therefore follows that $\vdash (\exists v)P_B\alpha \equiv (\exists v)P_B\alpha$ which thereby ensures the elimination of all de re construtions for doxastic possibility for these systems. However, nothing hangs on this reduction schema from the point of view of the TV semantics for the systems proposed in this section, since questions of ontology (including the problem of 'possibilia') have been sidestepped or at the very least, deferred given that models do not contain as elements domains of individuals.

We shall now provide a somewhat more formal presentation of the TV semantics for these Sub-SQC$^=$ axiom systems which lends itself to a substitutional reading of the quantifiers. What follows is a desription of this semantics, with remarks on what it does and does not validate as well as remarks concerning soundness and completeness.

A Sub-SQC$^=$ TV model will be a triple $\langle W, R, V \rangle$ such that $W \neq \emptyset$, $R \subseteq W \times W$ with the appropriate restrictions placed on R depending on the system under consideration. V is an indexed truth-value assignment to the atomic wffs of the language. I.e., V: {Atomic Wffs} X W $\longrightarrow$ {0.1}. We have obviated the need for domains of individuals in our characterization of a Sub-SQC$^=$ TV model. For each member of W, the function V simply assigns to the atomic wffs of the language truth-values.

Further, to guarantee soundness, we shall impose the following restrictions on V for any w_i in W:

Restriction 1: If α is of the form $t = t$ then $V(\alpha, w_i) = 1$ for all w_i in W.

Restriction 2: If $V(t_1 = t_2, w_i) = 1$ then $V(\alpha(t_1/v), w_i) = V(\alpha(t_2/v), w_i)$ for any w_i in W and where $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are atomic.

Restriction 1 bears directly on the validity of AS 4, $t = t$ and of course the second restriction bears on the validity of AS 5, $(\alpha(t_1/v) \& t_1 = t_2) \supset \alpha(t_2/v)$ such that t_1 and t_2 do not occur in the scope of a belief operator. Also, notice that there is no restriction on V to the effect that if $V(t_1 = t_2, w_i) = 1$ then $V(t_1 = t_2, w_j) = 1$ for all w_j in W. Hence, there is nothing preventing V from assigning to a contingent identity wff different truth-values at different indices. And this feature of the semantics is what invalidates the schema $(B\alpha(t_1/v) \& t_1 = t_2) \supset B\alpha(t_2/v)$ allowing substitution of identicals in belief constructions as well as the schema $t_1 = t_2 \supset B(t_1 = t_2)$ which asserts that agents are omnidoxastic with respect to contingent identities.

Finally, a *valuation* over a Sub-SQC$^=$ TV model is a function from wffs and indices into truth-values. I.e., V_M: Wffs X W $\longrightarrow \{0,1\}$. And V_M can be defined inductively with the following as the basis:

$V_M(\alpha, w_i) = V(\alpha, w_i)$ where α is atomic (either of the form $t_1 = t_2$ or of the form $Pt_1 \ldots t_n$) where V has imposed on the two restrictions mentioned above.

Inductive Step: Suppose that $V_M(\alpha, w_i)$ and $V_M(\beta, w_i)$ are defined.

1) $V_M(\sim\alpha, w_i) = 1$ iff $V_M(\alpha, w_i) = 0$.

2) $V_M(\alpha \& \beta, w_i) = 1$ iff $V_M(\alpha, w_i) = V_M(\beta, w_i) = 1$.

3) $V_M(B\alpha, w_i) = 1$ iff $V_M(\alpha, w_j) = 1$ for all w_j such that w_iRw_j.

4) $V_M((\exists v)\alpha, w_i) = 1$ iff $V_M(\alpha(t/v), w_i) = 1$ for at least one constant t.

Notice that the existential quantifier is treated *substitutionally* in the definition of V_M for wffs of the form $(\exists v)\alpha$. This makes the Sub-SQC$^=$ axiom schema $\alpha(t/v) \supset (\exists v)\alpha$ more palatable for cases where t in $\alpha(t/v)$ occurs in the scope of a doxastic operator.

Validity of a wff A in a Sub-SQC$^=$ model is truth of that wff at all w_i in W for that model and validity of a wff A with respect to a class of such models is validity of that wff in each model in the class.

Before discussing our soundness results, we shall verify that the schemata $(\mathbf{B}\alpha(t_1/v)\ \&\ t_1 = t_2) \supset \mathbf{B}\alpha(t_2/v)$ and $t_1 = t_2 \supset \mathbf{B}(t_1 = t_2)$ as well as Hintikka's AS 8, viz., $(t_1 = t_2\ \&\ (\exists v)(v = t_1\ \&\ \mathbf{B}(v = t_1))\ \&\ (\exists v)(v = t_2\ \&\ \mathbf{B}(v = t_2)) \supset (\mathbf{B}(t_1 = t_2)\ \&\ t_1 = t_2)$ are invalidated for any Sub-SQC$^=$ TV model. First of all, consider the following instance of the schema $(\mathbf{B}\alpha(t_1/v)\ \&\ t_1 = t_2) \supset \mathbf{B}\alpha(t_2/v)$, viz., $(\mathbf{B}Fa\ \&\ a = b) \supset \mathbf{B}Fb$. The following is a countermodel to this wff, viz., $W = \{w_1, w_2\}$ and further, for the sake of generality we shall say that $\{<w_1, w_2>\} \subseteq R$. Let $V(a = b, w_1) = 1$ in which case $V(Fa, w_1) = V(Fb, w_1)$ given restriction 2 for V. Further, let $V(Fa, w_2) = 1$ and $V(a = b, w_2) = V(Fb, w_2) = 0$ and hence, $V_M(Fa, w_2) = 1$ and $V_M(a = b, w_2) = V_M(Fb, w_2) = 0$. Then $V_M(\mathbf{B}Fa, w_1) = 1$. Also, even though $V_M(a = b, w_1) = 1$, $V_M(\mathbf{B}(a = b), w_1) = 0$. Finally, $V_M(\mathbf{B}Fb, w_1) = 0$. Q.E.D.

Next, consider the following instance of $t_1 = t_2 \supset \mathbf{B}(t_1 = t_2)$, viz., $a = b \supset \mathbf{B}(a = b)$. The countermodel to $(\mathbf{B}Fa\ \&\ a = b) \supset \mathbf{B}Fb$ will serve as a countermodel to $a = b \supset \mathbf{B}(a = b)$ Q.E.D.

Finally, consider the following instance of Hintikka's AS 8, $(a = b\ \&\ (\exists x)(x = a\ \&\ \mathbf{B}(x = a))\ \&\ (\exists x)(x = b\ \&\ \mathbf{B}(x = b)) \supset (\mathbf{B}(a = b)\ \&\ a = b)$. Let M be such that $W = \{w_1, w_2\}$, $<w_1, w_2> \subseteq R$ and $V(a = b, w_1) = V(c = a, w_1) = V(d = b, w_1) = 1$ and hence $V_M((a = b, w_1) = V_M(c = a, w_1) = V_M(d = b, w_1) = 1$. Suppose further that $V(a = b, w_2) = V_M(a = b, w_2) = 0$ but that $V(c = a, w_2) = V(d = b, w_2) = V_M(c = a, w_2) = V_M(d = b, w_2) = 1$.

Then $V_M(\mathbf{B}(c = a), w_1) = V_M(\mathbf{B}(d = b), w_1) = 1$ and hence, $V_M((\exists x)(x = a \,\&\, \mathbf{B}(x = a)), w_1) = V_M((\exists x)(x = b \,\&\, \mathbf{B}(x = b), w_1) = 1$. But, $V_M(\mathbf{B}(a = b), w_1) = 0$ and hence $V_M(\mathbf{B}(a = b) \,\&\, a = b, w_1) = 0$. Q.E.D.

Soundness of any Sub-SQC$^=$ system with respect to the appropriate class of TV models is established in the usual manner, viz., by showing that the axiom-schemata are valid and that the rules of inference preserve validity. We shall consider four crucial axiom schemata with respect to validity, viz., AS 3, 5, 6 and 7, viz., $\alpha(t/v) \supset (\exists v)\alpha$, $(\alpha(t_1/v) \,\&\, t_1 = t_2) \supset \alpha(t_2/v)$, $(\alpha(t_1/v) \,\&\, t_1 = t_2 \,\&\, \mathbf{B}(t_1 = t_2)) \supset \alpha(t_2/v)$ and the Barcan Formula respectively.

First of all, suppose that M is a TV model of the sort specified above such that for some w_i in W, $V_M(\alpha(t/v), w_i) = 1$ but $V_M((\exists v)\alpha, w_i) = 0$ for some instance of $\alpha(t/v) \supset (\exists v)\alpha$. But since a substitution instance of $(\exists v)\alpha$, viz., $\alpha(t/v)$ is true at w_i in W by supposition, it must also be the case that $(\exists v)\alpha$ is true at w_i which contradicts our initial supposition that $V_M((\exists v)\alpha, w_i) = 0$. Q.E.D.

Next, suppose that $V_M(\alpha(t_1/v), w_i) = V_M(t_1 = t_2, w_i) = 1$, $V_M(\alpha(t_2/v), w_i) = 0$ for some instance of $(\alpha(t_1/v) \,\&\, t_1 = t_2) \supset \alpha(t_2/v)$ where t_1 and t_2 do not occur in the scope of a doxastic operator. If $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are atomic then this is inadmissible as a Sub-SQC$^=$ TV model because of Restriction 2 for V cited above. Where $\alpha(t_1/v)$ and $\alpha(t_2/v)$ are not atomic, then the values assigned to them by V_M for w_i are determined by what V (and hence V_M) assign to their atomic subformulas at this index. (Or if they contain any subformulas with doxastic operators such that t_1 and t_2 *do not* occur in their scopes then the values of these subformulas will be determined by what V assigns to their atomic subformulas at all doxastic alternatives to w_i.) And so, given Restriction 2 for V cited above, whatever value V assigns to any atomic subformula of $\alpha(t_1/v)$ at w_i containing occurrences of t_1, it must assign the same value to the corresponding atomic subformulas of $\alpha(t_2/v)$ which differ only in that t_2 occurs wherever t_1 occurs in $\alpha(t_1/v)$'s atomic subformulas. Thus, $V_M(\alpha(t_1/v), w_i) = 1 = V_M(\alpha(t_2/v), w_i)$ given that $V(t_1 = t_2, w_i) = V_M(t_1 = t_2, w_i) = 1$. And so the set of valuations described above are inadmissible. Q.E.D.

To show that all instances of the axiom-schema AS 6, $(\alpha(t_1/v) \& t_1 = t_2 \& \mathbf{B}(t_1 = t_2)) \supset \alpha(t_2/v)$ are valid, suppose that for some w_i in a TV model M, $V_M(\alpha(t_1/v), w_i) = V_M(t_1 = t_2, w_i) = V_M(\mathbf{B}(t_1 = t_2), w_i) = 1$. Then for all w_j such that w_iRw_j, $V_M(t_1 = t_2, w_j) = 1$. $\alpha(t_2/v)$ is simply $\alpha[(t_2/t_1)(t_1/v)]$. Then all those atomic subformulas of $A(t_2/v)$ containing occurrences of t_2 which are not atomic subformulas of contents of doxastic operators will be assigned the same value by V at w_i that V assigns to the corresponding atomic subformulas of $\alpha(t_1/v)$ by Restriction 2 for V (given that $V_M(t_1 = t_2, w_i) = 1$). Further, all those atomic subformulas of $\alpha(t_1/v)$ containing occurrences of t_1 which are atomic subformulas of contents of (uniterated) doxastic operators will be assigned the same value by V at all w_j such that w_iRw_j which V assigns to the 'corresponding' atomic subformulas of $\alpha(t_1/v)$ at each such w_j – and this is guaranteed by Restriction 2 since for all w_j such that w_iRw_j, $V_M(t_1 = t_2, w_j) = 1$. So in either case, $V_M(\alpha(t_2/v), w_i) = V_M(\alpha(t_1/v), w_i) = 1$. Q.E.D.

Finally, to show that the Barcan Formula is validated in this type of semantics, suppose that $V_M((\forall v)\mathbf{B}\alpha, w_i) = 1$ and $V_M(\mathbf{B}(\forall v)\alpha, w_i) = 0$ for a TV model M for some w_i in W for some instance of the Barcan Formula. Then for every term t, $V_M(\mathbf{B}\alpha(t/v), w_i) = 1$ and hence for any w_j in W such that w_iRw_j, $V_M(\alpha(t/v),w_j) = 1$ for every constant t. Then for any w_j where w_iRw_j, $V_M((\forall v)\alpha, w_j) = 1$ and hence $V_M(B(\forall v)\alpha, w_i) = 1$ which contradicts our initial supposition that $V_M(B(\forall v)\alpha, w_i) = 0$. Q.E.D.

To conclude our remarks concerning soundness, we shall show that the rule of inference $\alpha(t/v) \supset \beta \longrightarrow (\exists v)\alpha \supset \beta$ for any t foreign to $(\exists v)\alpha \supset \beta$ preserves validity. Suppose that $\models \alpha(t/v) \supset \beta$ for any such t in which case for any TV model M, whenever $V_M(\beta, w_i) = 0$, $V_M(\alpha(t/v), w_i) = 0$. But in such a case, $V_M((\exists v)\alpha, w_i) = 0$ where free v in α replaces t in $\alpha(t/v)$ foreign to $(\exists v)\alpha \supset \beta$ and so whenever $V_M(\beta, w_i) = 0$, then $V_M((\exists v)\alpha, w_i) = 0$, i.e., $\models (\exists v)\alpha \supset \beta$ on the supposition that $\models \alpha(t/v) \supset \beta$. Q.E.D.

Finally, *completeness* for the various Sub-SQC$^=$ systems with respect to the appropriate class of TV models can be established by the method of canonical models. The canonical model $\mathcal{M}$ for a Sub-SQC$^=$ system will be a triple $\langle W,R,V\rangle$ where $W = \{w_i | w_i$ is a maximal consistent set with the $\exists$-property$\}$. As usual we can guarantee that any maximal consistent set w_i in W has the $\exists$-property if for any wff of the form $(\exists v)\alpha$ we can consistently add $(\exists v)\alpha \supset \alpha(t/v)$ for at least one constant t. R is defined in the usual manner, i.e., w_iRw_j iff $(\forall\alpha)(B\alpha \in w_i \longrightarrow \alpha \in w_j)$. Every member of W will have the following properties: Since any w_i in W is maximal consistent then it contains every wff of the form t = t given that $\vdash t = t$. Further, $\vdash (\alpha(t_1/v) \,\&\, t_1 = t_2) \supset \alpha(t_2/v)$ where t_1, t_2 do not occur in the scope of a doxastic operator, in which case any wff of the form $(\alpha(t_1/v) \,\&\, t_1 = t_2) \supset \alpha(t_2/v) \in w_i$ for any w_i in W. So if $t_1 = t_2$ is in any such w_i then if $\alpha(t_1/v)$ is in w_i so is $\alpha(t_2/v)$. Notice that these properties which any member of W will possess in the canonical model are the syntactic counterparts of the two restrictions we imposed on V in the semantics.

And given the way V is defined for the canonical model, it is redundant to impose these restrictions on V for this sort of model.

Since $t_1 = t_2 \supset \mathbf{B}(t_1 = t_2)$ is not a thesis-schema of the Sub-SQC$^=$ systems (owing to the fact that AS 5, $(\alpha(t_1/v) \; \& \; t_1 = t_2) \supset \alpha(t_2/v)$ is restricted to instances where t_1 and t_2 do not occur in the scope of a doxastic operator), then there is no guarantee that any of its instances will be a member of each and every maximal consistent set of wffs in W. And this in turn means that there is no guarantee that if $t_1 = t_2 \in w_i$ in W then for all w_j such that w_iRw_j, $t_1 = t_2 \in w_j$. (Thus, the canonical model for the Sub-SQC$^=$ systems differs from the canonical model for in the TV semantics for the SQC$^=$ semantics in this respect.) This feature of the canoncal model for any of the Sub-SQC$^=$ systems reflects the fact that in the semantics, the restrition on V which stipulates that an identity holds at *all* indices in the model if it holds in at least one, is lifted.

In the canonical model for any Sub-SQC$^=$ system, V is defined as $V(\alpha, w_i) = 1$ iff $\alpha \in w_i$. A *valuation* over the canonical model, V_μ is defined as follows for atomic wffs: $V_\mu(\alpha, w_i) = V(\alpha, w_i)$ from which it will follow that $V_\mu(\alpha, w_i) = 1$ iff $\alpha \in w_i$. This in fact is the *basis* of the inductive proof of the fundamental theorem of canonical models which states that $V(\alpha, w_i) = 1$ iff $\alpha \in w_i$ for any wff α. The inductive hypothesis in the proof of the fundamental theorem is that the theorem holds for wffs of degree of complexity n. What needs to be shown is that the theorem holds for wffs of degree of complexity n + 1.

The cases where α is of the form $\sim\beta$, $\beta \; \& \; \gamma$ and $\mathbf{B}\beta$ are proven in roughly the same manner that they were proven for the sentential systems in the first chapter. The case where α is of the form $(\exists v)\beta$ is proven as follows:

i) Suppose that $(\exists v)\beta \in w_i$.

Then $\beta(t/v) \in w_i$ for at least one constant t by the $\exists$-property.

$V_\mu(\beta(t/v), w_i) = 1$ for any such t by the inductive hyp.

$V_\mu((\exists v)\beta, w_i) = 1$. Q.E.D.

ii) Suppose that $V_\mu((\exists v)\beta, w_i) = 1$.

Then $V_\mu(\beta(t/v), w_i) = 1$ for at least one constant t.

$\beta(t/v) \in w_i$ by the *inductive hypothesis*.

$\vdash \beta(t/v) \supset (\exists v)\beta$.

$\beta(t/v) \supset (\exists v)\beta \in w_i$ since w_i is maximal consistent.

$(\exists v)\beta \in w_i$ since w_i is maximal consistent.

Q.E.D.

Given the fundamental theorem of canonical models, all that needs to be shown is that the canonical model for any Sub-SQC$^=$ system is in that system's class of models with respect to which it is sound. And this is proven by showing that R has the requisite properties. The proof of this proceeds in the usual manner.

Concluding Remarks:

And so, in our discussion of systems of quantified belief logic where to any normal system not containing T we add axiom-schemata concerned with quantification and identity (and the relation between these and the belief operator) we have seen that two problems are associated with first order belief logic, i.e., the problem of quantifying in and the problem of the failure of substitutivity of co-referentials. As we have seen, we could adopt Hinitikka's proposals for a logic and semantics of quantified belief logic which restrict quantification into belief constructions to relational contexts and substitution of co-referentials to special sorts of contexts. Or we could adopt as systems of quantified belief logic (for the purposes of dealing with the above-mentioned problems) the Sub-$SQC^{=}$ systems and their characteristic TV semantics. These systems sidestep the problem of quantifying in by providing a substitutional reading of the quantifiers and also, they treat all belief contexts as unambiguously oblique with respect to the substitution of co-referentials.

We have argued that there is some presumption in favour of adopting the Sub-$SQC^{=}$ systems as opposed to the Hintikka systems since the semantics for the former set of systems is on metaphysically more solid ground than the semantics for the latter type of systems. Further, the Sub-$SQC^{=}$ systems posit only one sense of belief as opposed to the Hin-$SQC^{=}$ systems which posit two senses of belief which is also a reason for preferring the former set of systems to the latter.

Since both the Sub-$SQC^{=}$ and the Hin-$SQC^{=}$ systems are normal modal systems (with quantification) they inherit the problem of deduction discussed in the first chapter. It is now to this problem which we shall return in the next chapter.

Chapter Five

Non-normal Indices and the Problem of Deduction

1. Introductory Remarks

In the previous chapter we argued that there is some presumption in favour of adopting the Sub-SQC$^{=}$ systems rather than the Hin-SQC$^{=}$ systems as quantificational doxastic logics since the characteristic semantics of the former is less problematic than that of the latter. Both types of axiom-systems share the feature that strictures are imposed on substitution of co-referentials in doxastic contexts although they differ with respect to the issue of quantification into such contexts. For any Hin-SQC$^{=}$ system, quantification into doxastic constructions is restricted to so-called relational contexts because the quantifiers are given an objectual reading whereas for the Sub-SQC$^{=}$ systems quantification into doxastic constructions is unrestricted given that the quantifiers are read substitutionally.

However, both types of quantificational doxastic systems inherit what Stalnaker has called the problem of deduction since all instances of the following schemata are derivable (and thus valid) in each of these systems:

$(\mathbf{B}\alpha \ \& \ \mathbf{B}\beta) \supset \mathbf{B}(\alpha \ \& \ \beta)$ adjunction schema

$(\mathbf{B}\alpha \ \& \vdash \alpha \supset \beta) \supset \mathbf{B}\beta$ omnidoxasticity schema

$(\mathbf{B}\alpha \ \& \vdash \alpha \equiv \beta) \supset \mathbf{B}\beta$ omnidoxasticity schema (equivalential version)

The omnidoxasticity schemata could also be characterized as the following rules of inference respectively:

$$\vdash(\alpha \supset \beta) \longrightarrow \vdash(B\alpha \supset B\beta)$$

$$\vdash(\alpha \equiv \beta) \longrightarrow \vdash(B\alpha \equiv B\beta)$$

All instances of the three schemata mentioned above are theses of any Hin-SQC$^{=}$ or Sub-SQC$^{=}$ system and the two rules are derivable in any of these systems owing to the fact that both types of systems are normal in the sense defined in chapter one. I.e., any such system contains the propositional calculus, the schema K, $(B\alpha \;\&\; B(\alpha \supset \beta)) \supset B\beta$ as well as the doxastic variant of the rule of necessitation, viz., $\vdash\alpha \longrightarrow \vdash B\alpha$. It was then demonstrated how the above schemata and rules of inference are derivable in any normal doxastic system.

Informally, the adjunction schema says that agents believe the conjunction of what they believe separately (i.e., that belief is 'closed' under conjunction) and the omnidoxasticity schemata can be read as saying that agents believe all logical consequences of (or what is logically equivalent to) whatever they believe. I.e., the omnidoxasticity schemata assert the principle that belief is closed under logical implication or under logical equivalence. As was suggested in chapter one, the tenability of these principles qua principles of belief attribution is questionable in the light of certain ordinary language counterinstances, which in turn render them doubtful as thesis-schemata for any modal logic construed as a logic of non-ideal belief. The fact that relative to a possible worlds semantics for belief, belief is closed under conjunction as well as as under both logical implication and logical equivalence is part of what Stalnaker calls the problem of deduction in the light of these various counterinstances (such as the paradox of the preface with respect to the adjunction principle).

Another principle of belief attribution which can be called into question in the light of such examples as the puzzling Pierre case (discussed in chapters one and three) is the principle that agents are incapable of believing both a statement and its negation. The formal counterpart of this principle in the language of either the Hin-SQC$^=$ or the Sub-SQC$^=$ systems is the schema $\sim(\mathbf{B}\alpha \ \& \ \mathbf{B}\sim\alpha)$. Stalnaker regards this principle as the remaining part of the problem of deduction. I.e., qua principle of belief attribution it is problematic since it would seem that there can be cases where an agent will believe both a statement and its negation (perhaps in different 'contexts'). A close cousin of this principle is the claim that agents cannot believe self-contradictory statements, which in the language of our formal systems is expressible as the schema $\sim\mathbf{B}(\alpha \ \& \sim\alpha)$. There appears to be more sympathy in the literature for the principle expressed by this schema than for the former principle.[1] As was noted in chapter one, both of these schemata, viz., $\sim(\mathbf{B}\alpha \ \& \ \mathbf{B}\sim\alpha)$ and $\sim\mathbf{B}(\alpha \ \& \sim\alpha)$ are derivable in any Hin-SQC$^=$ or Sub-SQC$^=$ system containing the schema D, $\mathbf{B}\alpha \supset \mathbf{P_B}\alpha$.

What will concern us in this and the next chapter is whether or not there is any way of altering the axiomatics as well as the corresponding *relational* semantics (i.e., a semantics where models have as elements a set W of indices and a two-place relation R such that R $\subseteq$ W X W) of the Sub-SQC$^=$ systems of quantificational doxastic logic (with identity) in such a way that at least not all instances of the adjunction and omnidoxasticity schemata are derivable/valid in these systems. The reason we shall confine our attention to the Sub-SQC$^=$ systems as opposed to the Hin-SQC$^=$ systems is that the characteristic semantics of the former is simpler than that of the latter in the sense that the semantics of the former class of systems does not appeal to domains of individuals. Thus, in considering emendations

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1 For example, see Marcus (1981) as well as Dummett (1980).

to the semantics of the Sub-SQC$^=$ systems we do not need to factor into these emendations domains of individuals.

Further, we shall consider what can be done in terms of altering the axiomatics of those Sub-SQC$^=$ systems containing the schema D so that not all instances of the schema $\sim(\mathbf{B}\alpha \ \& \ \mathbf{B}{\sim}\alpha)$ are theses for these systems. Or from the point of view of the semantics of these systems, is there any way of rendering at least some instances of $\mathbf{B}\alpha \ \& \ \mathbf{B}{\sim}\alpha$ satisfiable in certain Sub-SQC$^=$ models?

More immediately, in the next two sections, we shall be concerned with critically examining Rantala's proposal for altering the semantics of normal doxastic quantificational calculi and the corresponding syntactic alterations to rid them of the problem of deduction. On the syntactic front Rantala suggests restricting the doxastic variant of the rule of necessitation to some pre-selected subset of the set of wffs. Depending on which set to which this rule is applicable we select, this move can effectively block the derivation of certain or all instances of the adjunction schema, our variant of the consistency schema and most instances of the omnidoxasticity schemata.

On the semantic front, Rantala's proposal for dealing with the problem of deduction is to allow the doxastic accessibility relation R to range over not only normal indices (such that wffs are evaluated at these indices in the 'standard' way) but also over 'non-normal' indices. As we shall see presently, non-normal indices in such a semantics turn out to be indices where in terms of the truth-functional connectives and the belief operator, almost anything goes. I.e., wffs are evaluated non-standardly at such indices and hence, it is possible that at these indices so-called logical truths could turn out to be false or logical falsehoods could turn out to be true.

Vis a vis Cresswell's comments concerning impossible worlds semantics for belief logic, it will be argued in section 5 that this type of semantics ultimately does not succeed in freeing our systems of doxastic logic from the closure of belief under conjunction and logical implication (and from the consistency stricture on belief) - at least if we are speaking of 'classical' conjunction and implictation. The type of semantics which Rantala has proposed for his restricted axiom systems equivocates with respect to the connectives &, $\supset$ and ~ since they are defined inductively for normal worlds and non-inductively for non-normal worlds. Therefore, all that he has shown on the semantic front is that for example, agents may fail to conjoin contents of beliefs in some *non-classical* sense of 'conjoin'. It will further be argued that the response open to Rantala, viz., that the connectives are defined in terms of their roles in inference does not mitigate the charge of equivocation.

In the next chapter, a less extreme alternative to Rantala's semantics for belief developed by Rescher will be critically discussed. Rescher's semantics involves the assumption that belief is a relation between a believer and a non-standard world although even so-called non-standard worlds are such that all the truths of classical two-valued logic hold. Nonetheless, at non-standard worlds, α and β may obtain without their conjunction thereby obtaining. Thus, agents may fail to conjoin their beliefs - or believe that α and that $\sim\alpha$ without thereby believing their conjunction - although agents are still omnidoxastic in this type of semantics. Further, it will be argued that Rescher's semantics can be vindicated of the charge that Rantala's is open to, viz., equivocation with respect to ~ and &.

2. Rantala's Syntactic Proposals for a Logic of Belief not Presupposing Omniscience

In two recent articles Veikko Rantala has suggested a number of alterations to the axiomatics and corresponding semantics of both sentential and quantificational normal systems construed as logics of propositional attitudes.[2] The purpose of these alterations is to obtain logics which do not presuppose that agents are logically omniscient in the case of epistemic logics and which do not presuppose that agents are logically omnidoxastic in the case of belief logics. We shall consider his suggested modifications with respect to the omnidoxasticity schemata and their associated rules of inference for the Sub-SQC$^{=}$ systems of belief logic proposed in section 4 of the last chapter. Rantala's suggested changes to propositional attitude logics and their semantics can also be used to rid the Sub-SQC$^{=}$ systems of the adjunction schema as well as the close cousin of the consistency schema, $\sim(\mathbf{B}\alpha \;\&\; \mathbf{B}{\sim}\alpha)$ as thesis-schemata, although he does not explicitly discuss these particular applications of his proposals. In short, Rantala provides us with the syntactic and semantic machinery to rid the Sub-SQC$^{=}$ systems of the problem of deduction. In this section, we shall examine in detail exactly how Rantala's proposed changes work with reference to the problem of deduction for the Sub-SQC$^{=}$ (and Sub-SQC$^{=}$ + D) systems although as we shall see in section 5 his suggestions on the semantic front are at the very least philosophically objectionable.

On the syntactic front, Rantala's proposal which though simple is effective is to restrict the applicability of the doxastic variant of the rule of

[2] Rantala (1982), Rantala (1983).

necessitation. The details of his proposal are as follows: Given the set S of wffs of the appropriate language L, he defines the set Ω as any possibly non-empty (recursive) subset of S. I.e., $\Omega \subseteq S$. It is arbitrary as to what Ω is. Our choice of this set depends on what sorts of derivations we wish to block. The applicability of our doxastic variant of the rule of necessitation RB, $\vdash\alpha \longrightarrow \vdash\mathbf{B}\alpha$ is then restricted to the set Ω as just defined.[3] Thus, the restricted version of RB which following Rantala we shall call RB_Ω can be characterized as follows:

$\vdash\alpha \longrightarrow \vdash\mathbf{B}\alpha$ where $\alpha \in \Omega$.

Note that for our unrestricted Sub-SQC$^=$ systems, $\Omega = S$. It is evident that given the restriction that α is in Ω, we cannot unrestrictedly substitute for the scope α of the belief operator any wff β logically equivalent to α thus preserving the thesishood of $\mathbf{B}\alpha$, *unless* β is itself a member of Ω. I.e., if $\vdash\alpha \equiv \beta$, and given that from α (where $\vdash\alpha$ and $\alpha \in \Omega$) we can infer $\mathbf{B}\alpha$ such that $\vdash\mathbf{B}\alpha$, the substitution of β for α in $\mathbf{B}\alpha$ resulting in $\mathbf{B}\beta$ preserves theoremhood only if β is also in Ω.

For example, suppose that from a wff of the form $\alpha \vee \sim\alpha$ (the 'law of the excluded middle') which is by stipulation in Ω we infer that $\mathbf{B}(\alpha \vee \sim\alpha)$ by RB_Ω where $\vdash\mathbf{B}(\alpha \vee \sim\alpha)$ since RB_Ω preserves theoremhood provided that the scope of the belief operator is in Ω. Suppose further that $\sim(\alpha \mathbin{\&} \sim\alpha)$ is not in Ω. Then even though $\vdash(\alpha \vee \sim\alpha) \equiv \sim(\alpha \mathbin{\&} \sim\alpha)$, we cannot substitute $\sim(\alpha \mathbin{\&} \sim\alpha)$ for $\alpha \vee \sim\alpha$ in the wff $\mathbf{B}(\alpha \vee \sim\alpha)$ to obtain $\mathbf{B}\sim(\alpha \mathbin{\&} \sim\alpha)$ as a theorem since $\sim(\alpha \mathbin{\&} \sim\alpha)$ is not in Ω.

As Rantala notes, Ω can be a logic (such as the intuitionistic calculus) although he does not make this a requirement given that "it is hardly adequate to suppose that a person's attitudes are necessarily guided by a

[3] Rantala (1982), p. 108 and Rantala (1983), pp. 56-57.

logic".[4] Thus, his *only* stricture on Ω for any restricted doxastic or epistemic logic is that Ω be recursive. However, if we wish to block certain select instances of the adjunction schema or the consistency schema (and perhaps not all instances of the omnidoxasticity schema) given the idiosyncracity of agents' belief systems, then an additional requirement is needed for Ω. The significance of this requirement will be discussed when we come to consider the corresponding *semantics* for the Sub-$SQC^{=}_{\Omega}$ axiom systems in the next section. Thus, in addition to Rantala's minimal recursivity requirement, we shall impose the following stricture on the set Ω:

RΩ: *If Ω is not a 'calculus' then $\alpha \notin \Omega$ only if $\beta \supset \alpha \notin \Omega$.*

For simplicity of exposition, we shall primarily concern ourselves with restricted doxastic systems where Ω is not a calculus. Thus, we can simplify RΩ as follows:

RΩ*: *$\alpha \notin \Omega$ only if $\beta \supset \alpha \notin \Omega$.*

As will be argued in the next section, this stricture will help to ensure completeness of the Rantala systems (where Ω is not a calculus) with respect to his impossible worlds semantics by ensuring that any wff α rendered underivable by exlcuding the appropriate wffs from Ω will also be invalidated in the semantics.

We have already seen how *axiom-schemata* can be restricted as for example in the case of the schema $(\alpha(t_1/v) \& t_1 = t_2) \supset \alpha(t_2/v)$ which for both the Hin-$SQC^{=}$ and Sub-$SQC^{=}$ systems is restricted to cases where $\alpha(t_1/v)$ and its substitutional variant $\alpha(t_2/v)$ are such that t_1, t_2 do not occur in the scope of a doxastic operator. This stricture imposes limits on what counts as an instance of the schema $(\alpha(t_1/v) \& t_1 = t_2) \supset \alpha(t_2/v)$. Similarly, Rantala's restriction on the rule RB such that it is applicable

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[4] Rantala (1982), p. 108.

only to a recursive subset Ω of the set of wffs imposes limits on which instances of $\mathbf{B}\alpha$ such that $\vdash\alpha$ count as theses. This stricture on RB can in turn be used to block the derivation of select instances of the omnidoxasticity schemata, the adjunction schema or the schema $\sim(\mathbf{B}\alpha\ \&\ \mathbf{B}{\sim}\alpha)$ for systems containing D. The details of how this is so will now be discussed.

First of all, we shall focus on the omnidoxasticity schemata and their associated rules of inference and in particular on their implicational versions such that our remarks concerning them can easily be extended to the equivalential versions. Consider the *simplest* sort of case where we wish to block the derivation of $(\mathbf{B}\alpha\ \&\ \vdash\alpha \supset \beta) \supset \mathbf{B}\beta$ and its associated rule where $\alpha \supset \beta$ is the thesis-schema $\alpha \supset (\alpha \vee \beta)$ for exactly *one instance* of this schema, viz., $Fa \supset (Fa \vee Gb)$. Suppose for the sake of exposition that we are not concerned with blocking the derivation of any instances of any other problematic schemata such as the adjunction schema. Thus, our goal is to block the derivation of $(\mathbf{B}Fa\ \&\ (Fa \supset (Fa \vee Gb)) \supset \mathbf{B}(Fa \vee Gb)$ and we shall want to restrict the derived rule $\vdash(\alpha \supset \beta) \longrightarrow \vdash(\mathbf{B}\alpha \supset \mathbf{B}\beta)$ to cases where $\alpha \supset \beta$ is not the thesis $Fa \supset (Fa \vee Gb)$. However, given the restriction $R\Omega^*$, viz., *$\alpha \in \Omega$ only if $\beta \supset \alpha \in \Omega$* we shall end up blocking the derivation of much 'more' than we bargained for as will be demonstrated presently.

In terms of the schema $(\mathbf{B}Fa\ \&\ (Fa \supset (Fa \vee Gb)) \supset \mathbf{B}(Fa \vee Gb)$, its derivation would proceed along the following lines for the unaltered Sub-SQC$^=$ systems:

1. $\vdash Fa \supset (Fa \vee Gb)$
2. $\vdash \mathbf{B}(Fa \supset (Fa \vee Gb))$ 1, RB

3. $\vdash BFa \supset B(Fa \vee Gb)$ 2, K and MP

4. $\vdash \sim(Fa \supset (Fa \vee Gb)) \vee (BFa \supset B(Fa \vee Gb))$ 3, PC[5]

5. $\vdash (Fa \supset (Fa \vee Gb)) \supset (BFa \supset B(Fa \vee Gb))$ 4, PC

6. $(BFa \ \& \ (Fa \supset (Fa \vee Gb)) \supset B(Fa \vee Gb)$ 5, PC

The first three lines of this derivation constitutes the derivation of the rule of inference $\vdash(\alpha \supset \beta) \longrightarrow \vdash(B\alpha \supset B\beta)$ for the case where $\alpha \supset \beta$ is the thesis $Fa \supset (Fa \vee Gb)$. And more generally, any instance of this schema and rule would be derived in the same way. Notice that what is crucial to the derivation of this (and for that matter any) instance of the omnidoxasticity schema and its associated rule is the application of RB to the thesis $Fa \supset (Fa \vee Gb)$ (or more generally, to whatever wff is under consideration).

So applying Rantala's suggestion of restricting RB to the set Ω where $\Omega \subseteq S$ and where Ω is recursive to the particular case we are considering, we would initially stipulate that Ω excludes the thesis $Fa \supset (Fa \vee Gb)$. However, our additional proposed stricture on Ω, viz., $R\Omega^*$ requires us to exclude from Ω an infinite number of wffs. I.e., Ω will not contain $Fa \supset (Fa \vee Gb)$ as well as any wff of the form $\beta \supset (Fa \supset (Fa \vee Gb))$. Further, since any wff of the form $\beta \supset (Fa \supset (Fa \vee Gb))$ is excluded from Ω by $R\Omega^*$ then for any instance of $\beta \supset (Fa \supset (Fa \vee Gb))$, each wff of the form $\gamma \supset (\beta \supset (Fa \supset (Fa \vee Gb)))$ will also be excluded from Ω - and so on ad infinitum. Since all of the wffs being excluded from Ω are implicational (and in fact they are also *theses*[6]) then we shall in effect end up blocking the derivation of an infinite number of instances of the omnidoxasticity schema.

Of course, there will be implicational theses which survive exclusion

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[5] We are appealing here to the PC schema $\alpha \supset (\beta \vee \alpha)$ and detachment.

[6] These implicational wffs will be theses since if $\vdash\alpha$ then $\vdash\beta \supset \alpha$ for any wff α.

from Ω. In particular, any substitutional variant of (Fa ⊃ (Fa v Gb)) as for example Gb ⊃ (Hc v (Gb & (∃x)Rxc)) will survive exclusion (i.e., any wff of the form $\alpha \supset (\alpha \vee \beta)$). Granted, any thesis of the form $\alpha \supset (\alpha \vee \beta)$ will be logically equivalent to Fa ⊃ (Fa v Gb) as well as to any other thesis excluded from Ω, although exclusion from Ω is not closed under detachment - the stricture RΩ* is not a 'closure under detachment' condition for exclusion from Ω.

Thus, if we wish to block the derivation of (BFa & (Fa ⊃ (Fa v Gb)) ⊃ B(Fa v Gb) without in general blocking the derivation of *every* instance of $(B\alpha \ \& \ (\alpha \supset (\alpha \vee \beta)) \supset B(\alpha \vee \beta)$ then as a way of meeting the recursivity requirement on Ω we can stipulate that the set $\Omega = S - \{\delta \mid \delta$ is Fa ⊃ (Fa v Gb) or δ is an instance of $\beta_1 \supset (\ldots(\beta_n \supset$ (Fa ⊃ (Fa v Gb))$\ldots$).} where any of the β_i's are wffs of any degree of complexity. It should be noted that each instance of $\beta_1 \supset (\ldots(\beta_n \supset$ (Fa ⊃ (Fa v Gb))...) will be a thesis since ⊢Fa ⊃ (Fa v Gb). Finally, as was noted, in blocking the derivation of this one instance, (BFa & (Fa ⊃ (Fa v Gb)) ⊃ B(Fa v Gb) of one version $(B\alpha \ \& \ (\alpha \supset (\alpha \vee \beta)) \supset B(\alpha \vee \beta)$ of the omnidoxasticity schema we thereby block the derivation of an infinite number of instances of other versions of the omnidoxasticity schema given our adherence to RΩ*.

Suppose that we wished to block the derivation of *several* instances of $(B\alpha \ \& \ (\alpha \supset (\alpha \vee \beta)) \supset B(\alpha \vee \beta)$ such as (BGb & (Gb ⊃ (Hc v (Gb & (∃x)Rxc)))) ⊃ B(Hc v (Gb & (∃x)Rxc)) as well as (BFa & (Fa ⊃ (Fa v Gb)) ⊃ B(Fa v Gb) without blocking the derivation of any other instances of this version of the schema. Then in accordance with Rantala's recursivity requirment for Ω and given our stricture RΩ* we can let $\Omega = S - \{\delta \mid \delta$ is

$Fa \supset (Fa \vee Gb)$ or δ is $Gb \supset (Hc \vee (Gb \,\&\, (\exists x)Rxc))$ or δ is an instance of $\beta_1 \supset (\ldots (\beta_n \supset (Fa \supset (Fa \vee Gb)) \ldots)$ or δ is an instance of $\beta_1 \supset (\ldots (\beta_n \supset (Gb \supset (Hc \vee (Gb \,\&\, (\exists x)Rxc)))) \ldots).\}$ The point being made here is that despite our restriction $R\Omega^*$ on the set Ω, it is possible to block the derivation of select instances of at least version of the omnidoxasticity schema. It would of course be possible to block the derivation of *all* instances of all versions of this schema by simply excluding from Ω all implicational theses. However, as we shall next see, such a radical move would not be desirable.

In the light of the following considerations, there is a version of the omnidoxasticity schema none of whose instances should be blocked for the restricted doxastic systems being considered in this section, viz., $(B\alpha(t/v) \,\&\, \vdash \alpha(t/v) \supset (\exists v)\alpha) \supset B(\exists v)\alpha$: Some system where any or all instances of $\alpha(t/v) \supset (\exists v)\alpha$ are excluded from Ω will be such that $-\vdash (\exists v)B\alpha \supset B(\exists v)\alpha$ (which says that belief de re logically implies belief de dicto). But it was suggested in the previous chapter that this schema is desirable even for a doxastic logic where the quantifiers are given a substitutional reading in the semantics. More importantly, completeness of the restricted Sub-SQC* systems relative to the sort of semantics to be considered in the next section is guaranteed only if $\alpha(t/v) \supset (\exists v)\alpha \in \Omega$ for all instances of $\alpha(t/v) \supset (\exists v)\alpha$. As a way of ensuring that no instance of $\alpha(t/v) \supset (\exists v)\alpha$ will be excluded from Ω, we could impose the following additional stricture on Ω:

$R\exists\Omega_1$: *Any wff of of the form* $\alpha(t/v) \supset (\exists v)\alpha \in \Omega$.

Further, we need to ensure that $R\exists\Omega_1$ does not conflict with our other stricture on Ω, viz., $R\Omega^*$. Thus, a final stricture which we shall impose on Ω is the following:

$R\exists\Omega_2$: *Any wff of of the form $(\exists v)\alpha \in \Omega$*

This stricture will prevent the sort of case where a wff of the form $(\exists v)\alpha$ has been excluded from Ω, in which case by $R\Omega^*$, it would follow that no instances of $\alpha(t/v) \supset (\exists v)\alpha$ for some scope α are in Ω - which would conflict with our requirement that every instance of $\alpha(t/v) \supset (\exists v)\alpha$ is in Ω.[7] However, as will become apparent, if we are adopting the Rantala systems solely for the purpose of blocking various instances (or perhaps all instances) of the omnidoxasticity, consistency and adjunction schemata (in their present forms) then the only wffs that will be excluded from Ω will be *implicational* wffs. Then this renders $R\exists\Omega_2$ superfluous.

It is worth noting that even if we were to block the applicability of RB_Ω to all implicational theses by excluding them from Ω, it is nonetheless a rule of the Rantala systems that $\vdash\alpha \supset \alpha \longrightarrow \vdash B\alpha \supset B\alpha$. This rule does not require for its derivation the rule RB_Ω since it is in fact derivable simply by substituting '$B\alpha$' for 'α' in $\alpha \supset \alpha$. It would seem odd if this were not a rule even for a logic which does not assume that agents' beliefs are consistent and deductively closed.

As was noted, the substitution of logically equivalent scopes does not preserve theoremhood of theses of the form $B\alpha$ where α itself is a thesis in Ω *unless* the substituens is itself in Ω. Hence, we cannot sidestep the blockage of the derivation of some instance of $(B\alpha \;\&\; \vdash\alpha \supset \beta) \supset B\beta$ - or the corresponding inferential version $\vdash\alpha \supset \beta \longrightarrow \vdash B\alpha \supset B\beta$ by applying RB to the appropriate equivalent of $\alpha \supset \beta$ (eg., $\sim\alpha \vee \beta$ or $\sim(\alpha \;\&\; \sim\beta)$) and then

[7] For proving completeness, it is sufficent that Ω includes every instance of $\alpha(t/v) \supset (\exists v)\alpha$ though it need not include every instance of $(\forall v)\alpha \supset \alpha(t/v)$. What is crucial for completeness of the Rantala systems with respect to his proposed non-normal index semantics is that every instance of $B\alpha(t/v) \supset B(\exists v)\alpha$ is derivable, which depends for its derivation on the given instance of $\alpha(t/v) \supset (\exists v)\alpha$.

by substituting $\alpha \supset \beta$ for the scope of $\mathbf{B}(\sim\alpha \vee \beta)$ or $\mathbf{B}\sim(\alpha \,\&\, \sim\beta)$. Similar remarks apply to the equivalential version of the omnidoxasticity schema and inference rule.

What is excluded from Ω to which the restricted rule RB_Ω applies for the purpose of blocking the derivation of select instances of some implicational version of the omnidoxasticity schema *could* affect the status of at least some instances of the adjunction schema $(\mathbf{B}\alpha \,\&\, \mathbf{B}\beta) \supset \mathbf{B}(\alpha \,\&\, \beta)$ as well as the status of at least some instances of the consistency schema $\sim(\mathbf{B}\alpha \,\&\, \mathbf{B}\sim\alpha)$ for Sub-SQC$^=$ systems containing D. Recall that the proof of any instance of $(\mathbf{B}\alpha \,\&\, \mathbf{B}\beta) \supset \mathbf{B}(\alpha \,\&\, \beta)$ depends on the appropriate instance of the thesis-schema $\alpha \supset (\beta \supset (\alpha \,\&\, \beta))$ as follows:

1. $\vdash \alpha \supset (\beta \supset (\alpha \,\&\, \beta))$
2. $\mathbf{B}(\alpha \supset (\beta \supset (\alpha \,\&\, \beta)))$ 1, RB
3. $\vdash \mathbf{B}(\alpha \supset (\beta \supset (\alpha \,\&\, \beta))) \supset (\mathbf{B}\alpha \supset \mathbf{B}(\beta \supset (\alpha \,\&\, \beta)))$
4. $\mathbf{B}\alpha \supset \mathbf{B}(\beta \supset (\alpha \,\&\, \beta))$ 2,3 MP
5. $\vdash \mathbf{B}(\beta \supset (\alpha \,\&\, \beta)) \supset (\mathbf{B}\beta \supset \mathbf{B}(\alpha \,\&\, \beta))$
6. $\mathbf{B}\alpha \supset (\mathbf{B}\beta \supset \mathbf{B}(\alpha \,\&\, \beta))$ 4,5 PC[8]
7. $(\mathbf{B}\alpha \,\&\, \mathbf{B}\beta) \supset \mathbf{B}(\alpha \,\&\, \beta)$ 6, PC

In short, the proof of any instance of the adjunction schema depends on the application of RB to the appropriate instance of the thesis-schema $\alpha \supset (\beta \supset (\alpha \,\&\, \beta))$. But if certain instances of $\alpha \supset (\beta \supset (\alpha \,\&\, \beta))$ is not in the set Ω to which Rantala's restricted version of RB, viz., RB_Ω applies (for the purpose of blocking some instance of the omnidoxasticity schema) then the derivation of the appropriate instances of the adjunction schema will be effectively blocked. Appealing to the non-implicational versions of the appropriate instance of the schema $\alpha \supset (\beta \supset (\alpha \,\&\, \beta))$ will not help matters

[8] To be more precise, we would have to appeal to the appropriate instances of the thesis-schemata $((\alpha \supset \beta) \,\&\, (\beta \supset \gamma)) \supset (\alpha \supset \gamma)$ as well as $\alpha \supset (\beta \supset (\alpha \,\&\, \beta))$.

since once again, substitution of logically equivalent scopes of belief theses (where the scopes are themselves theses) does not preserve theoremhood of the belief wff unless the substituens is itself in Ω.

As was noted, despite the stricture RΩ*, *$\alpha \notin \Omega$ only if $\beta \supset \alpha \notin \Omega$*, it was possible to block the derivation of select instances of at least one version of the omnidoxasticity schema - although at the price of rendering underivable an infinity of instances of other versions of the schema - not to mention instances of other schemata such as the adjunction schema. Similarly, it is possible to block select instances of the adjucntion schema without thereby blocking the derivation of every instance of this schema - provided that nothing else has been excluded from Ω for the purpose of rendering instances of other sorts of schemata underivable, and provided we are willing to pay the price of rendering underivable an infinity of other wffs.

Suppose for example that we wish to block the derivation of $(B(\forall x)Fxa \,\&\, BGbc) \supset B((\forall x)Fxa \,\&\, Gbc))$ as well as $(BFa \,\&\, BGb) \supset B(Fa \,\&\, Gb))$ but no other instances of the adjunction schema. Then provided that nothing else has been excluded from Ω for the purpose of rendering instances of other sorts of schemata underivable, we can effectively block their derivation by stipulating that $\Omega = \{\delta \mid \delta$ is $(\forall x)Fxa \supset (Gbc \supset ((\forall x)Fxa \,\&\, Gbc))$ or δ is $Fa \supset (Gb \supset (Fa \,\&\, Gb))$ or δ is an instance of $\beta_1 \supset (\ldots(\beta_n \supset (Gbc \supset ((\forall x)Fxa \supset (Gbc \supset ((\forall x)Fxa \,\&\, Gbc))))\ldots)$ or δ is an instance of $\beta_1 \supset (\ldots(\beta_n \supset (Fa \supset (Gb \supset (Fa \,\&\, Gb))))\ldots)\}$. Further, even though any instance of the schema $\alpha \supset (\beta \supset (\alpha \,\&\, \beta))$ is logically equivalent to the wffs excluded from Ω, none of these instances of $\alpha \supset (\beta \supset (\alpha \,\&\, \beta))$ are thereby themselves excluded from Ω. This is owing to the fact that exclusion from Ω is not closed under detachment - as was noted earlier, the stricture $R\Omega^*$ is not a 'closure under detachment' condition for exclusion from Ω.

More generally, if Ω excludes *all* instances of the schema $\alpha \supset (\beta \supset (\alpha \,\&\, \beta))$ then *no* instance of the adjunction schema will be derivable in the appropriate Sub-SQC$^=$ system and of course no instance of the omnidoxasticity schema with respect to any instance of $\alpha \supset (\beta \supset (\alpha \,\&\, \beta))$ will be provable - along with an infinity of other instances of the omnidoxasticity schema. Presumably, one version of the adjunction schema, the derivation of whose instances we would not want to block, is $(\mathbf{B}\alpha \,\&\, \mathbf{B}\alpha) \supset \mathbf{B}(\alpha \,\&\, \alpha))$ all of whose instances will be derivable using $R\mathbf{B}_\Omega$ and assuming that Ω includes every instance of $\alpha \supset (\alpha \supset (\alpha \,\&\, \alpha))$. (We shall not, however, impose any hard and fast stricture on Ω to ensure that all instances of this version of the adjunction schema are derivable since neither soundness nor completeness depend on this.) Further, for Sub-SQC$^=$ systems containing the schema D, if we stipulate that Ω excludes *all* instances of the schema $\alpha \supset (\beta \supset (\alpha \,\&\, \beta))$ (and therefore an infinity of other wffs), then we also end up blocking the derivation of all instances of the schema $\sim(\mathbf{B}\alpha \,\&\, \mathbf{B}\sim\alpha)$ which is the other third of the so-called problem of deduction. It will now be explained why this is so.

For Sub-SQC$^=$ systems containing D, the derivation of any instance of the schema $\sim(\mathbf{B}\alpha \,\&\, \mathbf{B}\sim\alpha)$ depends on the appropriate instance of the following version of the adjunction schema, viz., $(\mathbf{B}\alpha \,\&\, \mathbf{B}\sim\alpha) \supset \mathbf{B}(\alpha \,\&\, \sim\alpha)$ which in turn depends for its derivation on the schema $\alpha \supset (\sim\alpha \supset (\alpha \,\&\, \sim\alpha))$:

1. $\vdash \sim(\alpha \,\&\, \sim\alpha)$
2. $\mathbf{B}\sim(\alpha \,\&\, \sim\alpha)$ 1, $R\mathbf{B}$ (unrestricted version)

3. $\vdash B{\sim}(\alpha\ \&\ {\sim}\alpha) \supset {\sim}B(\alpha\ \&\ {\sim}\alpha)$ (a version of D)

4. ${\sim}B(\alpha\ \&\ {\sim}\alpha)$ 2,3 detachment

5. $\vdash {\sim}{\sim}B(\alpha\ \&\ {\sim}\alpha) \supset {\sim}(B\alpha\ \&\ B{\sim}\alpha)$ contrapositive of adjunction schema

6. ${\sim}(B\alpha\ \&\ B{\sim}\alpha)$ 4,5 detachment

If we were to restrict the rule RB to the set Ω where it is stipulated that Ω does not include some instance of $\alpha \supset ({\sim}\alpha \supset (\alpha\ \&\ {\sim}\alpha))$ (as well as an infinity of other wffs in accordance with $R\Omega^*$) then this would effectively block the derivation of the appropriate instance of $(B\alpha\ \&\ B{\sim}\alpha) \supset B(\alpha\ \&\ {\sim}\alpha)$ as well as an infinity of instances of the omnidoxasticity schema. In turn, the underivability of this instance of $(B\alpha\ \&\ B{\sim}\alpha) \supset B(\alpha\ \&\ {\sim}\alpha)$ blocks the derivation of the appropriate instance of ${\sim}(B\alpha\ \&\ B{\sim}\alpha)$.

And so, if we were to adopt Rantala's proposal for handling the logical omniscience problem for modal logics construed as propositional attitude logics and by extension for dealing with the more general 'problem of deduction', then we would replace the Sub-SQC$^=$ systems with the Sub-SQC$^=_\Omega$ systems as embodying principles of belief attribution. The Sub-SQC$^=_\Omega$ systems have the same set of axiom-schemata as the Sub-SQC$^=$ systems and the same rules of inference except that in the former case, the doxastic variant of the rule of necessitation is restricted in its application to members of an arbitrary recursive subset of the set S of wffs, Ω. What Ω is depends on our purposes as well as on the strictures $R\Omega^*$ and $R\exists\Omega_1$. Thus, for some $\Omega \subseteq S$, the axiom system Sub-KQC$^=_\Omega$ would be characterized as follows:

Axiom-schemata: AS 1 - AS 7 as for Sub-KQC$^=$

AS 1 - AS 7

Inference Rules: Modus Ponens, R∃.

And in place of RB, RB_Ω:

$\vdash \alpha \longrightarrow \vdash B\alpha$, provided that $\alpha \in \Omega$

where Ω is *recursive* and is a subset of S, the set of wffs. Further, Ω must meet the following additional strictures:

$R\Omega$*: *$\alpha \notin \Omega$ only if $\beta \supset \alpha \notin \Omega$.*

$R\exists\Omega_1$: *Any wff of of the form $\alpha(t/v) \supset (\exists v)\alpha \in \Omega$.*

Notice finally that for any particular normal quantificational doxastic system such as Sub-KQC$^=$, there will be a whole set of logics, viz., {Sub-KQC$^=_\Omega$ | $\Omega \subseteq$ S} where S is the set of wffs and where the rule RB is restricted to the set Ω. In the limiting case where Ω = S, we simply have the system Sub-KQC$^=$ and in the other direction, the limiting case where $\Omega = \emptyset$ would result in a system where there are no theses of the form $B\alpha$ since RB_Ω is inapplicable and such that no instances of the adjunction, consistency and omnidoxasticity schemata are theses (with the exception of the omnidoxasticity schemata and rules with respect to wffs of the form $\alpha \supset \alpha$ and $\alpha \equiv \alpha$). Further, there will be an infinite number of logics 'in between' these two limiting systems where $\Omega \subset$ S (i.e., $\Omega \subseteq$ S, $\Omega \neq$ S) and such that S $\neq \emptyset$ where any such system is properly contained in Sub-KQC$^=$. Any such system is properly contained in Sub-KQC$^=$ since every thesis of any such Sub-KQC$^=_\Omega$ system is also a thesis of Sub-KQC$^=$ but not vice-versa.

3. Non-normal Index Semantics for Quantified Belief Logic

Having seen on the syntactic front how the derivation of select instances of some version of the omnidoxasticity schemata, and certain or all instances of both the adjunction and the consistency schemata can be blocked - at the price in all cases of their being an infinity of wffs whose derivation will be blocked - we shall now investigate Rantala's corresponding proposals on the *semantic* front for invalidating various instances of these schemata. As we shall see presently, the general semantic sleight of hand which Rantala employs is to allow the relation R in a model to range not only over normal but also over so-called non-normal indices where the connectives are defined non-standardly.

The reader will recall that a Sub-SQC$^=$ model is an ordered triple, $\langle W, R, V \rangle$ such that W is a non-empty set of indices and R is a two-place 'doxastic accessibility' relation ranging over members of W such that R is serial if the system is Sub-KDQC$^=$, transitive if the system is Sub-K4QC$^=$ and so on. Further, V is a function which to each *atomic* wff assigns either '1' or '0' with the two restrictions mentioned in the previous chapter, viz., that for any w_i in W, $V(t = t, w_i) = 1$ and if $V(t_1 = t_2, w_i) = 1$ then $V(\alpha\,(t_1/v), w_i) = V(\alpha\,(t_2/v), w_i)$. Further, a valuation over a model V_M is defined inductively with $V(\alpha, w_i) = V_M(\alpha, w_i)$ as the basis of the induction. Finally, the truth-conditions for quantified wffs are substitutional rather than objectual. For example, $V_M((\exists v)\alpha, w_i) = 1$ iff $V_M(\alpha\,(t/v), w_i) = 1$ for at least one constant t. In the previous chapter we tried to show that this type of semantics characterizes the Sub-SQC$^=$ systems of doxastic logic.

What is now needed is a type of semantics which characterizes the various Sub-SQC$^=_\Omega$ systems. The type of semantics described in the previous paragraph validates all instances of the omnidoxasticity schemata and the adjunction schema as well as the consistency schema whenever R is serial. (This is assuming soundness results.) Hence, this sort of semantics will not do for systems for which certain or all instances of these crucial schemata are underivable. For example, suppose we wish to set up a *characteristic* semantics for the system Sub-$KDQC^{=}_{\Omega}$ where $\Omega = S - \{\delta \mid \delta$ is an instance of $\alpha \supset (\beta \supset (\alpha \ \& \ \beta))$ or δ is an instance of $\beta_1 \supset (\ldots(\beta_n \supset (\alpha \supset (\beta \supset (\alpha \ \& \ \beta)))) \ldots).\}$. Then all instances of the adjunction schema, our variant of the consistency schema and an infinity of instances of the omnidoxasticity schema (as well as other schemata) are underivable in this system. So what is needed is a semantics which invalidates whatever is rendered underivable by restricting RB_{Ω} to Ω as just specified and of course which validates whatever remains derivable for Sub-$KDQC^{=}_{\Omega}$ (soundness) as well as validating *only* that which is derivable for Sub-$KDQC^{=}_{\Omega}$ (completeness). Such a characteristic semantics is needed for any Sub-$SQC^{=}_{\Omega}$ system where $\Omega \neq S$.

Rantala's suggestion for a type of semantics which would characterize the Sub-$SQC^{=}_{\Omega}$ systems and hence which would invalidate all wffs rendered underivable by virtue of how Ω is set up runs roughly as follows:[9] A Sub-$SQC^{=}_{\Omega}$ model is a 4-tuple $\langle W, W^{*}, R, V\rangle$ such that $W \neq \emptyset$ and W^{*} is a possibly non-empty set of 'non-normal' indices[10] such that $W \cap W^{*} = \emptyset$. Further, R ranges over members of $W \cup W^{*}$ or more formally, $R \subseteq (W \cup W^{*}) \times (W \cup W^{*})$. The assignment function V is defined for members of $W \cup W^{*}$ as follows: V: Atomic Wffs $\times$ $(W \cup W^{*}) \longrightarrow \{0,1\}$. In addition, the two restrictions concerning the behaviour of the identity symbol which applied to V for members of W in the semantics for the Sub-$SQC^{=}$ systems,

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[9] See Rantala (1982), p. 109; Rantala (1983), pp. 46–47.

[10] The significance of the appellation 'non-normal' will be explained presently.

apply to V for members of $W \cup W^*$. We shall now put the set W^* to work in dealing with the 'problem of deduction' because as we shall now see, the special twist to the sort of semantics we are currently considering is that the *valuation* function V_M is not defined inductively for members of W^*.

The valuation V_M as usual takes wffs at indices into truth-values, although what is distinctive about V_M in Rantala's semantics is that it assigns values to wffs at both normal and 'non-normal' indices. I.e., V_M: Wffs X $(W \cup W^*) \longrightarrow \{0,1\}$. V_M for members of W is defined as usual by induction with $V(\alpha, w_i) = V_M(\alpha, w_i)$ as the basis. V_M for members of W^* for *atomic* wffs is of course defined as it is for members of W, i.e., $V(\alpha, w_i) = V_M(\alpha, w_i)$ since a *function* can only assign to a wff '1' or '0' at the same index but not both. However, V_M for non-atomic wffs is *not* defined for members of W^* by induction using $V(\alpha, w_i) = V_M(\alpha, w_i)$ as the basis. In effect, the standard conditions for the connectives, the belief operator and the quantifiers are at least initially lifted for non-normal indices. Finally, validity in a model is defined as truth at all normal indices and validity in a class of models is of course validity in all models in the class.[11]

Since validity in any Sub-$SQC^{=}_{\Omega}$ model is defined as truth at all normal indices and validity in a class of models is truth at all normal indices in all models in the class, it is sufficient that R has imposed on it the requisite strictures (for validating all instances of an appropriate axiom-schema such as D, 4 or 5) for members of W only. For example, for any Sub-$KDQC^{=}_{\Omega}$ system containing D, $B\alpha \supset P_B\alpha$ it will be sufficient to require that if w_i is in W then there is at least one w_j in W such that w_iRw_j. Or if we are considering the system Sub-$KD4QC^{=}_{\Omega}$ then it is sufficient to

[11] See Rantala (1982), p. 109. To be more precise, Rantala uses 'true in M' instead of 'valid in M'.

require that R is serial and transitive for members of W.[12]

And so the reason that the members of W^* (such that $W \cap W^* = \emptyset$) are called non-normal (or non-standard or impossible) is that V_M is not defined inductively for such indices and hence if the connectives, the belief operator and the quantifiers 'misbehave' then theses of the particular unrestricted system could turn out to be false at these indices, a situation which Nicholas Rescher calls "logical anarchy"[13]. However, as will be argued, the phrase 'logical anarchy' is a misnomer with respect to Rantala's impossible worlds.

Although the standard truth-conditions for non-atomic wffs which hold for the normal (or 'classical'[14]) indices are lifted for non-normal (or 'non-classical') indices, it is not the case that 'anything goes' at impossible indices. This is because a number of strictures need to be imposed on V_M for impossible indices in order to ensure soundness. One such stricture which Rantala discusses is the following:

1) For any w_i in W^*, if $V_M(\alpha, w_i) = V_M(\alpha \supset \beta, w_i) = 1$ then $V_M(\beta, w_i) = 1$.[15]

In other words, even impossible indices are closed under detachment although this is *not* equivalent to re-introducing the standard truth-conditions for material implication. This is because there is nothing in the above restriction which prevents any implicational thesis from being false at a non-normal index - provided that as we shall presently see, this thesis is not in the set Ω described above. All that this restriction pre-

12 Rantala (1982), p. 109.

13 Rescher and Brandom (1980), p. 21.

14 The appellations 'classical' vs. 'non-classical' for the kind of semantics we are now discussing have been used by Cresswell in a number of places including Cresswell (1970) and Cresswell (1973).

15 This is discussed in Rantala (1982), p. 109 and in Rantala (1983), p. 61.

cludes is the sort of case where for some non-normal index w_i and for any wffs α, β, $V_M(\alpha, w_i) = V_M(\alpha \supset \beta, w_i) = 1$ but $V_M(\beta, w_i) = 0$.

What this stricture imposed on implicational wffs for members of W^* guarantees is the validity of the Sub-SQC$^=_\Omega$ axiom-schema $(\mathbf{B}\alpha \ \& \ \mathbf{B}(\alpha \supset \beta)) \supset \mathbf{B}\beta$ for the following reason. Suppose that for some Sub-SQC$^=_\Omega$ model $\langle W, W^*, R, V\rangle$, such that for some wffs α, β, $V_M(\mathbf{B}\alpha, w_i) = V_M(\mathbf{B}(\alpha \supset \beta), w_i) = 1$ but that $V_M(\mathbf{B}\beta, w_i) = 0$. Then there is some $w_j \in W \cup W^*$ such that $w_i R w_j$ and where although $V_M(\alpha, w_j) = V_M(\alpha \supset \beta, w_j) = 1$, $V_M(\beta, w_j)$ is 0. But if w_j is in W this set of valuations is inadmissible by virtue of the truth-conditions for '$\supset$' and if w_j is in W^* then this set of valuations is inadmissible by virtue of the above mentioned 'closure under detachment' stricture.

An additional stricture which Rantala imposes on members of W^* for *quantificational* doxastic systems is that for any universally quantified wff of the form $(\forall v)\alpha$ and for any $w_i \in W^*$:

2) If $V_M(\alpha(t/v), w_i) = 1$ for all constants t then $V_M((\forall v)\alpha, w_i) = 1$.[16]

A similar stricture (also in conditional form) would be imposed on V_M for members of W^* for existentially quantified wffs.[17] Of course these strictures are equivalent to re-introducing one half of the classical or standard truth-conditions for quantified wffs. As Rantala notes, these restrictions are needed for propositional attitude logics which contain as an axiom-schema the Barcan Formula (BF), $(\forall v)\mathbf{B}\alpha \supset \mathbf{B}(\forall v)\alpha$.[18] In short, the reintroduction of one half of the classical truth-conditions for the quantifiers via the above mentioned strictures is necessary to guarantee the

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[16] Rantala makes this stricture a biconditional although rendering it as a condtional does the same work and greatly simplifies the completeness proof.

[17] I.e., for any w_i in W^*, if $V_M(\alpha(t/v), w_i) = 1$ for at least one term t then $V_M((\exists v)\alpha, w_i) = 1$.

[18] See Rantala (1983), p. 61.

soundness of the Sub-$SQC^{=}_{\Omega}$ systems all of which contain BF relative to the type of semantics we are considering as will now be shown.

Suppose that there is a Sub-$SQC^{=}_{\Omega}$ model $M = \langle W, W^{*}, R, V \rangle$ such that $V_M((\forall v)B\alpha, w_i) = 1$ for some w_i in W. For every constant t, $V_M(B\alpha(t/v), w_i) = 1$. So, for every constant t, $V_M(\alpha(t/v), w_j) = 1$ for any $w_j \in W \cup W^{*}$ such that w_iRw_j. If w_j is in W then by the standard truth-conditions for univerally quantified wffs, $V_M((\forall v)\alpha, w_j) = 1$ and if w_j is in W^{*} then by the above stricture imposed on V_M for non-normal indices, it also follows that $V_M((\forall v)\alpha, w_j) = 1$. Thus, since for every $w_j \in W \cup W^{*}$ such that w_iRw_j, $V_M((\forall v)\alpha, w_j) = 1$ it follows that $V_M(B(\forall v)\alpha, w_i) = 1$. Q.E.D.

It should be noted that even though the quantifiers behave standardly at all non-normal indices, it is still possible for theses whose major connectives are truth-functional or which are belief wffs but which involve quantifiers to be false at such indices (provided they are not in Ω). They can turn out to be false by virtue of the non-standard behaviour of the major connective or the belief operator at impossible worlds.

There is one further stricture which Rantala imposes on V_M for the members of W^{*} in order to ensure soundness, viz., that wffs which are valid in the model in the sense that they are true at all the normal indices and which are such that they are in the set Ω (viz., the recursive subset of the set S of wffs to which the rule RB_{Ω} applies) must also be true at all non-normal indices in the model. Expressed more formally,

3) For any w_i in W^{*} and for any wff α such that $\alpha \in \Omega$,

if $V_M(\alpha, w_j) = 1$ for all w_j in W then $V_M(\alpha, w_i) = 1$.

It will now be shown how this condition helps to guarantee soundness of any Sub-$SQC^{=}_{\Omega}$ system relative to this sort of semantics.

Suppose that for some wff α, $\vdash\alpha$ and suppose further that α is in Ω. Then by RB_Ω it follows that $\vdash B\alpha$. Now suppose also that $\models\alpha$, i.e., that α is valid. What we need to show is that $\models B\alpha$, from which it follows that the rule RB_Ω preserves validity. Since by supposition α is valid, it follows that for any model $M = \langle W, W^*, R, V \rangle$ in the appropriate class of models, $V_M(\alpha, w_i) = 1$ for all w_i in W. Further, since by supposition α is in the set Ω and given our stricture on V_M for members of W^* it follows that for any non-normal index w_j in W^* in any model, $V_M(\alpha, w_j) = 1$. In other words, the validity of α coupled with its membership in the set Ω ensures that α is true at *all* indices both normal and non-normal in any model. And this in turn guarantees that for any normal index w_i in any model (in the relevant class of models), $V_M(B\alpha, w_i) = 1$. And so the rule of inference RB_Ω preserves validity by virtue of Rantala's stricture that valid wffs in Ω are true at all non-normal indices in any model. Q.E.D.

By way of clarification, Rantala's 'non-normal' or 'impossible' indices are not to be confused with the so-called *dead ends* for classes of models for K-extensions where R is not serial. As described in chapter one, at any such index, all wffs of the form $B\alpha$ are true and all wffs of the form $P_B\alpha$ are false, thus invalidating D, since no world is accessible from a dead end including itself. However, the connectives and the belief operator 'behave' standardly at such indices. Hence, all theses of the appropriate system are validated at dead ends. And these two characteristics, viz., that the connectives are defined standardly and that theses remain valid at dead ends distinguishes them from Rantala's non-normal indices. Finally, at a dead end although agents believe anything including α & $\sim\alpha$, α & $\sim\alpha$ could never be true at any dead end whereas α & $\sim\alpha$ *could* turn out to be true at a Rantalian impossible world since '~' and '&' are not defined inductively.

Nor are the impossible indices of Rantala to be confused with Kripkean non-normal indices[19] which can be used in setting up the characteristic semantics for the (non-normal[20]) modal systems S2 and S3.[21] Like dead ends for K-extensions not containing D or T, any non-normal index for S2 and S3 models is such that no index is accessible from it including itself although it must be accessible from some other index. I.e., if w_i in W in an S2 or S3 model is 'non-normal' then $\sim(\exists w_j)(w_j \in W \cup W^{*}$[22] $\& \; w_i R w_j)$ & $(\exists w_k)(w_k \in W \; \& \; w_k R w_i)$. However, what distinguishes non-normal indices for S2 and S3 models from dead ends is that in the former case any wff of the form $\mathbf{B}\alpha$ is false at such indices and hence given the interdefinability of '**B**' in terms of '$\mathbf{P_B}$' any wff of the form $\mathbf{P_B}\alpha$ will be true at such indices, which of course is the reverse of the situation for dead ends.

The reason that this 'reverse' situation obtains for Kripkean non-normal indices has to do with the truth-conditions for belief wffs in this type of semantics. Given that an S2, S3 model is a triple <W,R,V> where W is a non-empty set of at least one normal and possibly some non-normal worlds, and where R is quasi-reflexive[23] (for S2) or R is quasi-reflexive and transitive (for S3), $V_M(\mathbf{B}\alpha, w_i) = 1$ iff $w_i R w_i$ and for all w_j such that $w_i R w_j$, $V_M(\alpha, w_j) = 1$. The proviso that $w_i R w_i$ in the truth-conditions for belief wffs gives us a semantics that validates the schema T, $\mathbf{B}\alpha \supset \alpha$ which

19 See Kripke (1965).

20 By 'non-normal' here, we mean that S2 and S3 do not have an unrestricted rule of necessitation. We have not considered S2 and S3 as potential logics of belief since both contain $\mathbf{B}\alpha \supset \alpha$. See Hughes and Cresswell (1968), pp. 246-253 for a detailed discussion of these axiom systems.

21 See Hughes and Cresswell (1968), pp. 274-278.

22 W^* is a set of Kripkean non-normal indices.

23 R is a quasi-relexive relation ranging over the members of W =df. for any w_i in W, if there is at least one w_j in W such that if $w_i R w_j$ then $w_i R w_i$.

therefore rules out S2, S3 as viable logics of belief. Since any Kripkean non-normal world is such that no world *including itself* is accessible from it, it follows that the 'w_iRw_i' proviso is never satisfied and hence for any wff of the form $\mathbf{B}\alpha$, $\mathbf{B}\alpha$ will be false at such an index. And by the interdefinabitlity of $\mathbf{P_B}$ in terms of $\mathbf{B}$, all wffs of the form $\mathbf{P_B}\alpha$ will be true at this sort of index.

Finally, validity in an S2, S3 model is truth at all *normal* worlds and hence certain modal theses of S2, S3 such as $\vdash_{S2,3}\mathbf{B}\sim(\alpha \ \& \ \sim\alpha)$ where α is a wff of PC, can turn out to be false at Kripkean non-normal worlds. In fact, all instances of $\mathbf{B}\sim(\alpha \ \& \ \sim\alpha)$ will be false at non-normal indices even though $\vdash\mathbf{B}\sim(\alpha \ \& \ \sim\alpha)$. And this is so by virtue of the 'w_iRw_i' proviso in the truth conditions for belief wffs coupled with the inaccessibilty of non-normal worlds. Nonetheless, we still could never have the type of situation where $\alpha \ \& \ \sim\alpha$ is true at Kripkean non-normal indices since once again, the connectives '~' and '&' are defined classically. And this distinguishes Kripkean non-normal indices from Rantalian impossible indices since in the latter case, wffs of the form $\alpha \ \& \ \sim\alpha$ could turn out to be true.

To discern for any given Sub-$SQC^{=}_{\Omega}$ system whether or not Rantala's proposed semantics validates all and only what is derivable in the system, we must prove soundness and completeness. How such proofs might proceed will be discussed in outline fashion. However, before addressing these questions, more needs to be said concerning the exact 'mechanics' of Rantala's proposed semantics for 'restricted' propositional attitude logics. On the syntactic front, we saw that what blocks the derivation of certain philosophically objectionable wffs such as certain instances of the adjunction schema was to set up the set Ω in such a way that a key application of RB_Ω in the derivation of that wff is blocked. In the case of the adjunction schema, we set up Ω in such a way that it excludes the appropriate instance of $\alpha \supset (\beta \supset (\alpha \,\&\, \beta))$ as well as any instance of $\beta_1 \supset (\ldots(\beta_n \supset (\alpha \supset (\beta \supset (\alpha \,\&\, \beta))))$. But what we want to determine is what aspects of the 'corresponding' (if we are not question begging) semantics *invalidates* the instance of the adjunction schema whose derivation has been blocked in the appropriate axiom system.

What provides the answer to this question is the role which the set Ω plays in the *semantics* intended to characterize the restricted Sub-$SQC^{=}_{\Omega}$ systems. This arbitrarily selected subset of the set of wffs is the crucial link between the syntactic move of blocking the derivation of a certain objectional wff and the semantic move of invalidating this wff. This set Ω plays a role in invalidating those schemata which it renders underivable in the 'corresponding' axiom system via one of the strictures that Rantala imposes on the valuation function for impossible indices in a model, i.e., if α is true at all normal indices in the model (i.e., if α is valid or 'true' in the model) then α is also true at all non-normal indices in the model *provided* that α is in the set Ω. If α is not in Ω then it could turn out to be false at some non-normal index even though $\models_M \alpha$ or $\models \alpha$.

In order to illustrate how this stricture on V_M for non-normal indices invalidates objectionable wffs whose derivations are blocked in the syntax, we shall consider the following instance of the adjunction schema, viz., $(BFa \,\&\, B{\sim}Fa) \supset B(Fa \,\&\, {\sim}Fa)$. The derivation of this wff can be blocked in any Sub-$SQC^{=}_{\Omega}$ system by stipulating that $\Omega = S - \{\delta \mid \delta$ is $Fa \supset ({\sim}Fa \supset (Fa \,\&\, {\sim}Fa))$ or δ is an instance of $\beta_1 \supset (\ldots(\beta_n \supset (Fa \supset ({\sim}Fa \supset (Fa \,\&\, {\sim}Fa))))\}$.

In short, by excluding the wff Fa ⊃ (~Fa ⊃ (Fa & ~Fa)) and any instance of β_1 ⊃ (...(β_n ⊃ (Fa ⊃ (~Fa ⊃ (Fa & ~Fa)))) from the set Ω to which the rule RB_Ω applies, we effectively block the derivation of (BFa & B~Fa) ⊃ B(Fa & ~Fa) since a crucial step in this derivation involves applying RB_Ω to the wff Fa ⊃ (~Fa ⊃ (Fa & ~Fa)). Further, the derivation of the following instances of the omnidoxasticity and consistency schemata (for systems containing D), viz., (BFa & ⊢Fa ⊃ (~Fa ⊃ (Fa & ~Fa)) ⊃ B(~Fa ⊃ (Fa & ~Fa)) and ~(BFa & B~Fa) respectively are blocked for reasons discussed in the previous section. And of course there is an infinity of other wffs whose derivation is effectively blocked including an infinity of instances of the omnidoxasticity schema. We shall now see on the semantic front exactly how setting up Ω in the way we have *invalidates* the above mentioned instances of the adjunction and omnidoxasticity schemata.

The following will constitute a Sub-$SQC^{=}_{\Omega}$ countermodel to the following instances of the adjunction, omnidoxasticity and consistency schemata, viz., (BFa & B~Fa) ⊃ B(Fa & ~Fa), (BFa & ⊢Fa ⊃ (~Fa ⊃ (Fa & ~Fa)) ⊃ B(~Fa ⊃ (Fa & ~Fa)) and ~(BFa & B~Fa) respectively. Let M be such that W = $\{w_1\}$, $W^* = \{w_2\}$, $\{\langle w_1, w_2\rangle\} \subseteq R$. Let Ω = S − {δ | δ is Fa ⊃ (~Fa ⊃ (Fa & ~Fa)) or δ is an instance of β_1 ⊃ (...(β_n ⊃ (Fa ⊃ (~Fa ⊃ (Fa & ~Fa))))} Let $V(Fa,w_1) = V_M(Fa,w_1) = 1$ and since w_1 is normal, $V_M(\sim Fa,w_1) = 0$. Also, since $V_M(Fa,w_1) = 0$ it follows that V_M(Fa ⊃ (~Fa ⊃ (Fa & ~Fa)),w_1) = 1 given once again that w_1 is normal. Suppose further that $V(Fa,w_2) = V_M(Fa,w_2) = 1$. We shall set up $V_M(\sim Fa,w_2)$ as 1, which is admissible because w_2 is non-normal and hence '~' is not defined inductively.

The index w_2 is non-normal and in addition, Fa ⊃ (~Fa ⊃ (Fa & ~Fa)) *is not in Ω.* Further, given our restriction RΩ* for Ω, viz., that α ∉ Ω only if $\alpha \supset \beta \notin \Omega$, then any wff of the form $\beta_1 \supset (\ldots(\beta_n \supset (Fa \supset ({\sim}Fa \supset (Fa \& {\sim}Fa))))$ will be excluded from Ω. Then the following valuation is admissible, viz., $V_M(Fa \supset ({\sim}Fa \supset (Fa \& {\sim}Fa)), w_2) = 0$ as will now be shown.

Our stipulating that $V_M(Fa \supset ({\sim}Fa \supset (Fa \& {\sim}Fa)), w_2) = 0$, Rantala's closure restriction on members of W^*, viz., *that for any w_i in W^*, if $V_M(\alpha, w_i) = V_M(\alpha \supset \beta, w_i) = 1$ then $V_M(\beta, w_i) = 1$* is not violated since any wff of the form $\beta_1 \supset (\ldots(\beta_n \supset (Fa \supset ({\sim}Fa \supset (Fa \& {\sim}Fa))))$ is excluded from Ω. Thus, we can stipulate that for any wff of the form $\beta_1 \supset (\ldots(\beta_n \supset (Fa \supset ({\sim}Fa \supset (Fa \& {\sim}Fa))))$, $V_M(\beta_1 \supset (\ldots(\beta_n \supset (Fa \supset ({\sim}Fa \supset (Fa \& {\sim}Fa)))), w_2) = 0$. [Note also that this stipulation will not violate the restriction for any member of W^*, that *if $\alpha \in \Omega$* and if $V_M(\alpha, w_i) = 1$ for all w_i in W then $V_M(\alpha, w_k) = 1$ for all w_k in W^*.]

To illustrate that the closure restriction on members of W^* will not be violated in letting $V_M(Fa \supset ({\sim}Fa \supset (Fa \& {\sim}Fa)), w_2) = 0$, suppose that some wff β is true at w_2 in W^* such that $\vdash \beta \supset (Fa \supset ({\sim}Fa \supset (Fa \& {\sim}Fa)))$. Since $\beta \supset (Fa \supset ({\sim}Fa \supset (Fa \& {\sim}Fa)))$ has been excluded from Ω by our stricture $R\Omega^*$ in which case our stipulating that $V_M(\beta \supset (Fa \supset ({\sim}Fa \supset (Fa \& {\sim}Fa))), w_2) = 0$ is admissible, then the closure restriction on members of W^* has not been violated. [For example, β might be ${\sim}Fa \supset (Fa \supset (Fa \& {\sim}Fa))$ in which case, $V_M({\sim}Fa \supset (Fa \supset (Fa \& {\sim}Fa)), w_2) = 1$. Nonetheless, given $R\Omega^*$, the wff $({\sim}Fa \supset (Fa \supset ({\sim}Fa \& Fa))) \supset (Fa \supset ({\sim}Fa \supset (Fa \& {\sim}Fa)))$ is not in Ω and hence it is admissible to stipulate that it is false at w_2.] But, it could be countered that $V_M(\beta \supset (Fa \supset ({\sim}Fa \supset (Fa \& {\sim}Fa))), w_2) = 0$ is *not* admissible since there could be some wff γ such that γ is true at w_2 and such that $\vdash \gamma \supset (\beta \supset (Fa \supset ({\sim}Fa \supset (Fa \& {\sim}Fa))))$ – which could violate the closure restriction on members of W^*. However, the wff $\gamma \supset (\beta \supset (Fa \supset$

$(\sim Fa \supset (Fa \,\&\, \sim Fa))))$ is itself excluded from Ω by $R\Omega^*$ and so, it is admissible to stipulate that $V_M(\gamma \supset (\beta \supset (Fa \supset (\sim Fa \supset (Fa \,\&\, \sim Fa)))), w_2) = 0$.

And in general, for any wff of the form $\beta_2 \supset (\ldots(\beta_n \supset (Fa \supset (\sim Fa \supset (Fa \,\&\, \sim Fa))))\ldots)$, if there is a wff β_1 true at w_2 such that $\vdash \beta_1 \supset (\beta_2 \supset (\ldots(\beta_n \supset (Fa \supset (\sim Fa \supset (Fa \,\&\, \sim Fa))))\ldots))$ then since $\beta_1 \supset (\beta_2 \supset (\ldots(\beta_n \supset (Fa \supset (\sim Fa \supset (Fa \,\&\, \sim Fa))))\ldots))$ is not in Ω by $R\Omega^*$, it is admissible to stipulate that this wff is false at w_2 thereby not violating the closure restriction on members of W^* - if it has also been stipulated that the consequent $\beta_2 \supset (\ldots(\beta_n \supset (Fa \supset (\sim Fa \supset (Fa \,\&\, \sim Fa))))\ldots)$ is false at w_2. Q.E.D.

Given that $V_M(Fa \supset (\sim Fa \supset (Fa \,\&\, \sim Fa)), w_2) = 0$ then the closure restriction on implication is not violated for w_2 in letting $V_M(\sim Fa \supset (Fa \,\&\, \sim Fa), w_2) = 0$ even though $V_M(Fa, w_2) = 1$. Thus, $V_M(BFa, w_1) = 1$ although $V_M(B(\sim Fa \supset (Fa \,\&\, \sim Fa)), w_1) = 0$ which therefore invalidates the above instance of the omnidoxasticity schema.

Also, since '&' is not defined inductively, it is admissible to let $V_M(Fa \,\&\, \sim Fa, w_2) = 0$ even though we have stipulated that $V_M(Fa, w_2) = V_M(\sim Fa, w_2) = 1$. Further, we shall not be violating the 'closure' restriction on implication for non-normal worlds in stipulating that $V_M(Fa \,\&\, \sim Fa, w_2) = 0$ even though $V_M(Fa, w_2) = V_M(\sim Fa, w_2) = 1$ since $V_M(Fa \supset (\sim Fa \supset (Fa \,\&\, \sim Fa)), w_2) = V_M(\sim Fa \supset (Fa \,\&\, \sim Fa), w_2) = 0$. Then even though $V_M(BFa, w_1) = V_M(B{\sim}Fa, w_1) = V_M(BFa \,\&\, B{\sim}Fa, w_1) = 1$ (which therefore invalidates the consistency schema for systems containing D), $V_M(B(Fa \,\&\, \sim Fa), w_1) = 0$ thereby invalidating the above mentioned instance of the adjunction schema. Q.E.D.

This example was in part intended to illustrate the significance behind the restriction $R\Omega^*$ on the set Ω. As was noted in the previous section, it is possible to construct Ω in such a way that the resulting Sub-$SQC^=_\Omega$ axiom system will not contain certain select instances of some schema such as the adjunction, consistency or omnidoxasticity schema - provided that we are willing to pay the price of having a logic such that an infinity of other wffs derivable in the corresponding unrestricted Sub-$SQC^=_\Omega$ system are thereby rendered underivable. The cost which is exacted by employing the restriction $R\Omega^*$ in the axiom system is paid back in the semantics since we are ensured that the select instance of the crucial schema (such as the adjucntion schema) which is rendered underivable in the syntax is invalidated in the semantics.

With respect to invalidating select instances of the omnidoxasticity, adjunction or consistency schemata rendered underivable in the given axiom system, the rule $R\Omega^*$ serves the function of ensuring that the closure restriction on V_M for non-normal indices and the restriction that if $\alpha \in \Omega$ and if $V_M(\alpha, w_i) = 1$ for all w_i in W then $V_M(\alpha, w_j) = 1$ for all w_j in W^*, do not conflict. For example, suppose that γ is a wff to which the rule RB_Ω must be applied for the appropriate instance of the adjunction schema to be derived. In such a case, γ will be an instance of $\alpha \supset (\beta \supset (\alpha \,\&\, \beta))$. Then excluding γ from Ω as well as all instances of $\beta_1 \supset (\ldots(\beta_n \supset \alpha)\ldots)$ given $R\Omega^*$ results in the appropriate instance of the given schema being rendered underivable. The parallel situation in the semantics is that in excluding α from Ω, α can take on the value 'false' at some non-normal alternative w_i to the index at which the given instance of the schema is being evaluated. However, if there is some wff β such that $\vdash \beta \supset \alpha$ and such that $V_M(\beta, w_i) = 1$ then $\beta \supset \alpha$'s being excluded from Ω given $R\Omega^*$

allows the assignment of the value 'false' to $\beta \supset \alpha$ at w_i thereby avoiding a violation of the closure restriction if γ has been assigned 'false' at w_i. Further, assigning 'false' to $\beta \supset \alpha$ at w_i will not itself involve a violation of the closure restriction since any wff of the form $\delta \supset (\beta \supset \alpha)$ is also excluded from Ω given $R\Omega^*$.

What is now needed are general completeness results to show that for *any* wff α, if α is not a thesis of the given Sub-$SQC^=_\Omega$ system then α is invalid in the appropriate class of models. In the next section, we shall therefore address ourselves to the question of soundness and completeness of the Sub-$SQC^=_\Omega$ systems relative to the sort of semantics just presented.

4. Soundness and Completeness Results for the Sub-$SQC^=_\Omega$ Systems

We have already seen how the three strictures which Rantala imposes on V_M for non-normal indices helps to ensure soundness of the various Sub-$SQC^=_\Omega$ systems relative to his proposed impossible worlds semantics. These strictures ensure that the schema K and the Barcan Formula are valid in the appropriate class of models and that the restricted rule RB_Ω preserves validity. The proofs that the remaining axiom-schemata such as $t = t$, $\alpha(t/v) \supset (\exists v)\alpha$, $\alpha(t_1/v) \,\&\, t_1 = t_2) \supset \alpha(t_2/v)$ where t_1, t_2 do not occur in the scope of doxastic operators and α where α has the form of a PC thesis, are valid and that the other rules of inference, MP and restricted $R\exists$ preserve validity are straightforward enough. They proceed roughly along the same lines as the proofs for the unrestricted Sub-$SQC^=$ systems.

However, a fourth stricture is needed for V_M for non-normal indices in Sub-$SQC^=_\Omega$ models which will ensure the validity of all instances of the following Sub-SQC$^{=}{}_{\Omega}$ axiom-schema, viz., $(\alpha(t_1/v) \& t_1 = t_2 \& \mathbf{B}(t_1 = t_2)) \supset \alpha(t_2/v)$ - where it is stipulated that t_1, t_2 may occur in the scope of doxastic operators. This schema restricts substitution of identicals to cases where the agent believes that the relevant identity obtains. However, as it stands, the following simple instance of this schema viz., $(\mathbf{B}(Fa \vee Gc) \& a = b \& \mathbf{B}(a = b)) \supset \mathbf{B}(Fb \vee Gc)$ is invalidated in the following Sub-SQC$^{=}{}_{\Omega}$ (partial) model: $W = \{w_1\}$, $W^* = \{w_2\}$, $\langle w_1, w_2\rangle \subseteq R$ and $V(Fa, w_2) = 1$ and $V(a = b, w_2) = 1$. Then by one of the restrictions applying to V for members of $W \cup W^*$, $V(Fb, w_2)$ must also be 1. Then $V_M(a = b, w_2) = V_M(Fa, w_2) = V_M(Fb, w_2) = 1$. But since w_2 is non-normal in which case 'V_M' is not defined inductively, the following are admissible valuations: $V_M(Fa \vee Gc, w_2) = 1$ and $V_M(Fb \vee Gc, w_2) = 0$. Then $V_M(\mathbf{B}(Fa \vee Gc), w_1) = 1$ but $V_M(\mathbf{B}(Fb \vee Gc), w_1) = 0$.

Thus, what is needed to avoid the above kind of situation is the introduction into the semantics of a fourth restriction, which merely stipulates that V_M for members of W^* is such that for any wffs $\alpha(t_1/v)$ and $\alpha(t_2/v)$ of any degree of complexity,

4) If $V_M(t_1 = t_2, w_i) = 1$ then $V_M(\alpha(t_1/v), w_i) = V_M(\alpha(t_2/v), w_i)$.

In particular, this restriction would disallow the above (partial) counter-model to $(\mathbf{B}(Fa \vee Gc) \& a = b \& \mathbf{B}(a = b)) \supset \mathbf{B}(Fb \vee Gc)$ since given that $V_M(Fa \vee Gc, w_2) = V_M(a = b, w_2) = 1$, then it must also be the case (vis a vis Restriction 4 on V_M for members of W^*) that $V_M(Fa \vee Gc, w_2) = 1$. Generally speaking, Restriction 4 on V_M for members of W^* would ensure that the restriction on V for members of $W \cup W^*$ viz., that if $V(t_1 = t_2, w_i) = 1$ then $V(\alpha(t_1/v), w_i) = V(\alpha(t_2/v), w_i)$, can be extended to V_M for members of W^* where the connectives and the belief operator are defined non-classically.

Completeness is a little tricky to establish for the Sub-$SQC^=_\Omega$ systems since it is not immediately obvious how we can characterize the set W^* for any Sub-$SQC^=_\Omega$ system's canonical model. In his 1982 article, Rantala suggests characterizing W^* for a system's canonical model roughly as follows: Given that W for the particular system's canonical model $\mathcal{M} = \langle W, W^*, R, V\rangle$ is a set of maximal consistent sets of wffs each set having the ∃-property, we define the set W^* as itself a set of sets of wffs where for any member of this set, w_j there is exactly one member of W, w_i such that $w_j = \{\alpha \in Wffs \mid B\alpha \in w_i\}$.[24] In other words, any member of W^*, w_j is a set of wffs where for exactly one member of W, w_i, each wff α in the set w_j will be such that $B\alpha$ is in w_i. Given that R for $\mathcal{M}$ is defined as usual, viz., w_iRw_j iff $(\forall\alpha)(B\alpha \in w_i \longrightarrow \alpha \in w_j)$, it follows that for any member w_j of W^* there is exactly one member of W, w_i such that w_iRw_j. Further, for any member of W, w_i there is exactly one member of W^*, w_j such that w_iRw_j[25] provided that w_i contains at least one wff of the form $B\alpha$. What this amounts to intuitively is that for a Sub-$SQC^=_\Omega$ canonical model, for each maximal consistent set with the ∃-property in W, w_i we construct a set of wffs w_j consisting of all and only the content wffs of all the belief wffs contained in w_i. Whether or not the members of W^* themselves have the ∃-property is immaterial given - as we shall see - that the fundamental theorem of canonical models for members of W^* is not proven inductively.

To summarize, a Sub-$SQC^=_\Omega$ canonical model $\mathcal{M}$ is a 4-tuple $\langle W, W^*, R, V\rangle$ where W is a set of maximal consistent sets of wffs with the ∃-proper-

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24 Rantala (1982), p. 110.

25 Rantala (1982), p. 110 and Rantala (1983), p. 53.

ty and W^* is a set of sets of wffs where for each w_j in W there is a w_i in W^* which is such that for any wff α in w_i, **B**α is in w_j. The element R is defined as w_iRw_j iff $(\forall\alpha)(\mathbf{B}\alpha \in w_i \longrightarrow \alpha \in w_j)$ although as we have not yet noted, Rantala requires that for any ordered pair $\langle w_i, w_j \rangle$ in R, $w_i \in W$ and $w_j \in W \cup W^*$. I.e., Rantala stipulates that for any Sub-$SQC^=_\Omega$ canonical model, $R \subseteq W \times (W \cup W^*)$.[26] This in effect means that no index is accessible from any non-normal index including itself for the canonical model although given the definition of W^* for $\mathcal{M}$, it follows that each w_j in W^* is such that w_iRw_j for exactly one w_i in W. Given the 'inaccessibility' of members of W^* for the canonical model plus the fact that every w_i in W^* is 'accessible' from some member of W, it would seem that in these very respects, non-normal indices in the *canonical model* for any Sub-$SQC^=_\Omega$ system are similar to Kripkean non-normal indices which are used in setting up the semantics for S2 and S3. There is also a fundamental difference between Kripkean non-normal indices and members of W^* for $\mathcal{M}$ as will be argued below.

We can define the element V in the canonical model $\mathcal{M}$ for members of $W \cup W^*$ as follows: Where α is atomic and for any w_i in $W \cup W^*$, $V(\alpha, w_i) = 1$ iff $\alpha \in w_i$. For members of W^*, we let $V_{\mathcal{M}}(\alpha, w_i) = V(\alpha, w_i)$ and hence $V_{\mathcal{M}}(\alpha, w_i) = 1$ iff $\alpha \in w_i$ for any atomic wff α although we do not prove the fundamental theorem of canonical models for members of W^* using induction (presumably since V_M is not defined inductively for members of W^* in any Sub-$SQC^=_\Omega$ model, M). Thus, it is simply stipulated that for any wff α and for any w_i in W^*, $V_{\mathcal{M}}(\alpha, w_i) = 1$ iff $\alpha \in w_i$.[27] And so, if a wff of the form **BB**α is in some member of W, w_i then it follows that

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[26] See Rantala (1982), p. 110.

[27] Rantala (1982), p. 111.

$\mathbf{B}\alpha$ is in some w_j in W^* such that w_iRw_j. Then by the fundamental theorem of canonical models for non-normal indices, $V_M(\mathbf{B}\alpha, w_i) = 1$ and this shows that unlike Kripkean non-normal indices in the semantics for S2 and S3, belief wffs can be true at Rantalian non-normal indices in any Sub-SQC$^=_\Omega$ canonical model.

Further, for the valuation over any Sub-SQC$^=_\Omega$ canonical model, V_μ, the fundamental theorem of canonical models is proven for members of W by induction with $V_\mu(\alpha, w_i) = V(\alpha, w_i)$ and so $V_\mu(\alpha, w_i) = 1$ iff $\alpha \in w_i$ for any atomic wff α as the basis of the induction. And the inductive proof would proceed along the same lines as it did for the unrestricted Sub-SQC$^=$ systems (using the inductive hypothesis that the theorem holds for wffs of degree of complexity n) except for the case where α is of the form $\mathbf{B}\alpha$. In this case, the proof of the subcase that if $V_\mu(\mathbf{B}\alpha, w_i) = 1$ then $\mathbf{B}\alpha \in w_i$ is considerably simplifed by appeal to the fact that every non-normal index in W^* is accessible from exactly one normal index in W in the canonical model:

Subcase a): Suppose $V_\mu(\mathbf{B}\alpha, w_i) = 1$

then $V_\mu(\alpha, w_j) = 1$ such that $w_j \in W^*$ where w_iRw_j.

$\alpha \in w_j$ since w_j is non-normal and given that for any such index, $V_\mu(\alpha, w_j) = 1$ iff $\alpha \in w_j$.

$\mathbf{B}\alpha \in w_i$ given that $w_j = \{\alpha \mid \mathbf{B}\alpha \in w_i\}$.[28]

The remaining subcase, viz., that if $\mathbf{B}\alpha \in w_i$ then $V_\mu(\mathbf{B}\alpha, w_i) = 1$ is proven in basically the same way as it is for the Sub-SQC$^=$ systems:

Subcase b): Suppose $\mathbf{B}\alpha \in w_i$

then $\alpha \in w_j$ for any w_j in $W \cup W^*$ such that w_iRw_j.

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[28] Rantala (1982), p. 111. Our proof of subcase b) differs from Rantala's since his proof relies on the the system's containing the schema T. His proposals are primarily for epistemic logic.

$V_{\mathcal{M}}(\alpha, w_j) = 1$ by the inductive hypothesis if $w_j \in W$ or immediately if $w_j \in W^*$.

$V_{\mathcal{M}}(\mathbf{B}\alpha, w_i) = 1$ by the truth-conditions for $\mathbf{B}\alpha$.

Q.E.D.

Now that the fundamental theorem of canonical models has been proven for normal indices (and is stipulated to hold for non-normal ones), all that remains to be shown is that the canonical model is a model for the particular Sub-$SQC^{=}_{\Omega}$ system under consideration.

Showing that the canonical model $\mathcal{M}$ is in the class of models with respect to which the particular Sub-$SQC^{=}_{\Omega}$ system is sound involves two steps. First, as before, we must show that the element R in $\mathcal{M}$ has the requisite characteristics. For example, for any Sub-$KDQC^{=}_{\Omega}$ system it must be proven that R in its canonical model is serial for W in the sense defined above, viz., for any normal index w_i in W there is at least one w_j in W such that w_iRw_j.

Once we have established that R in the particular Sub-$SQC^{=}_{\Omega}$ system's canonical model meets the appropriate constraints, it must next be shown that $V_{\mathcal{M}}$ meets the strictures imposed on V_M for non-normal indices in any Sub-$SQC^{=}_{\Omega}$ model.[29] First of all, it must be shown that for any w_i in W^* in $\mathcal{M}$ and for any wffs α, β, if $V_{\mathcal{M}}(\alpha, w_i) = V_{\mathcal{M}}(\alpha \supset \beta, w_i) = 1$ then $V_{\mathcal{M}}(\beta, w_i) = 1$. The reader is referred to Rantala's 1982 article for details of this proof although in outline fashion, it involves supposing that $V_{\mathcal{M}}(\alpha, w_i) = V_{\mathcal{M}}(\alpha \supset \beta, w_i) = 1$. Then $\mathbf{B}\alpha \in w_j$ and $\mathbf{B}(\alpha \supset \beta) \in w_j$ since $w_i = \{\alpha \mid \mathbf{B}\alpha \in w_j\}$ for exactly one w_j in W. From this it follows that $\mathbf{B}\alpha \ \& \ \mathbf{B}(\alpha \supset \beta) \in w_j$ and since $\vdash (\mathbf{B}\alpha \ \& \ \mathbf{B}(\alpha \supset \beta)) \supset \mathbf{B}\beta$ it further follows that $\mathbf{B}\beta \in w_j$. And given that $w_i = \{\alpha \mid \mathbf{B}\alpha \in w_j\}$, $\beta \in w_i$ and so $V_{\mathcal{M}}(\beta, w_i) = 1$. Q.E.D.

29 Rantala (1982), pp. 111-112.

Next, we need to show that the second set of strictures which Rantala imposes on V_M for members of W^*, viz., that for any w_i in W^*, if $V_M(\alpha(t/v), w_i) = 1$ for all constants t then $V_M((\forall v)\alpha, w_i) = 1$ and if $V_M(\alpha(t/v), w_i) = 1$ for at least one constant t then $V_M((\exists v)\alpha, w_i) = 1$ applies to V_μ for the canonical model. We shall prove that these two strictures apply to V_μ for any w_i in W^* as follows:

1) For any w_i in W^*, suppose $V_\mu(\alpha(t/v), w_i) = 1$ for all constants t.

thus, $\alpha(t/v) \in w_i$ for all t by def. of V_μ for members of W^*.

thus, $B\alpha(t/v) \in w_j$ for all t such that $w_j \in W$ where $w_i = \{\alpha \mid B\alpha \in w_j\}$

$V_\mu(B\alpha(t/v), w_j) = 1$ for all t.

thus, $V_\mu((\forall v)B\alpha, w_j) = 1$.

$(\forall v)B\alpha \in w_j$ by the fundamental theorem.

$\vdash (\forall v)B\alpha \supset B(\forall v)\alpha$ BF

$(\forall v)B\alpha \supset B(\forall v)\alpha \in w_j$ since w_j is maximal consistent

$B(\forall v)\alpha \in w_j$ since w_j is max. con.

$(\forall v)\alpha \in w_i$ since $w_i = \{\alpha \mid B\alpha \in w_j\}$.

$V_\mu((\forall v)\alpha, w_i) = 1$ by df. V_μ for members of W^*.

2) For any w_i in W^*, suppose $V_\mu(\alpha(t/v), w_i) = 1$ for some constant t.

then $\alpha(t/v) \in w_i$ for some constant t by the def. of V_μ for members of W^*.

$B\alpha(t/v) \in w_j$ for some cons. t such that w_j in W and $w_i = \{\alpha \mid B\alpha \in w_j\}$.

$\vdash B\alpha(t/v) \supset B(\exists v)\alpha$ which is a thesis for any Sub-SQC$^{=}_{\Omega}$ system such that Ω includes all instances of $\alpha(t/v) \supset (\exists v)\alpha$.

(Note: The upshot of this step in the proof is that the completeness result goes through only for systems such that RB_{Ω} is applicable to all instances of the axiom-schema $\alpha(t/v) \supset (\exists v)\alpha$. This is because the derivation of any instance of $B\alpha(t/v) \supset B(\exists v)\alpha$ depends on the application of RB_{Ω} to the appropriate instance of $\alpha(t/v) \supset (\exists v)\alpha$ along with the schema K and modus ponens. In short, for the sake of guaranteeing completeness, it should be stipulated that any Sub-SQC$^{=}_{\Omega}$ system we chose as our system of quantified doxastic logic should be such that Ω includes all instances of $\alpha(t/v) \supset (\exists v)\alpha$. Therefore, the type of semantics we have discussed in this section only chatacterizes those Sub-SQC$^{=}_{\Omega}$ systems such that Ω includes all instances of $\alpha(t/v) \supset (\exists v)\alpha$. We shall now proceed with the proof.)

$B\alpha(t/v) \supset B(\exists v)\alpha \in w_j$ (w_j is max. con.)

$B(\exists v)\alpha \in w_j$ since w_j is max. con.

$(\exists v)\alpha \in w_i$ since $w_i = \{\alpha \mid B\alpha \in w_j\}$.

$V_{\mathcal{M}}((\exists v)\alpha, w_i) = 1$ by df. of $V_{\mathcal{M}}$ for non-normal indices.

Q.E.D.

Further, we shall need to show that the third stricture which Rantala imposes on V_M for members of W^* for Sub-SQC$^{=}_{\Omega}$ models, viz., that for all

w_i in W^*, if $V_M(\alpha, w_j) = 1$ for all w_j in W and if α is in Ω then $V_M(\alpha, w_i) = 1$ applies to V_μ for members of W^* in the appropriate canonical model. The proof of this can be found in Rantala's 1982 article and so we shall reproduce it here only in outline form.[30] In essence, the proof involves showing that if for some α in Ω, $V_\mu(\alpha, w_j) = 1$ for all w_j in W then α is in each such w_j in W from which it follows that $\vdash\alpha$ and by RB_Ω that $\vdash B\alpha$. Then $B\alpha$ is in every w_j in W. So for every w_i in W^* it follows that α is in w_i and hence that $V_\mu(\alpha, w_i) = 1$. Q.E.D.

Finally, the fourth stricture which was imposed on V_M for members of W^* viz., that if $V_M(t_1 = t_2, w_i) = 1$ then $V_M(\alpha(t_1/v), w_i) = V_M(\alpha(t_2/v), w_i)$ (for wffs $\alpha(t_1/v)$, $\alpha(t_2/v)$ of any degree of complexity), is proven to hold for V_μ for members of W^* as follows:

Suppose: $V_\mu(t_1 = t_2, w_i) = V_\mu(\alpha(t_1/v), w_i) = 1$ but $V_\mu(\alpha(t_2/v), w_i) = 0$ for some w_i in W^* in μ. I.e., we are supposing that even though $V_\mu(t_1 = t_2, w_i) = 1$, $V_\mu(\alpha(t_1/v), w_i) \neq V_\mu(\alpha(t/v), w_i)$.

thus, $t_1 = t_2 \in w_i$ and $\alpha(t_1/v) \in w_i$

thus, $B(t_1 = t_2) \in w_j$ and $B\alpha(t_1/v) \in w_j$ such that $w_j \in W$ and where $w_i = \{\alpha \mid B\alpha \in w_j\}$ and hence, w_jRw_i.

thus, $B\alpha(t_1/v)$ & $B(t_1 = t_2) \in w_j$ given that w_j is max. cons.

$\vdash(B\alpha(t_1/v) \,\&\, B(t_1 = t_2)) \supset B\alpha(t_2/v)$[31]

" $(B\alpha(t_1/v) \,\&\, B(t_1 = t_2)) \supset B\alpha(t_2/v) \in w_j$ since w_j is max. cons.

" $B\alpha(t_2/v) \in w_j$ since w_j is max. cons.

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30 See Rantala (1982), p. 112.

31 This schema can be regarded as the 'purely doxastic' variant of the axiom-schema $(\alpha(t_1/v) \,\&\, t_1 = t_2 \,\&\, B(t_1 = t_2)) \supset \alpha(t_2/v)$ where t_1 and t_2 may occur in the scope of doxastic operators. The conjunct in the antecedent, $t_1 = t_2$ is rendered superfluous. To avoid any technical objections, we could simply add this variant of the above schema to any Sub-SQC^*_Ω system as an axiom-schema.

$\alpha(t_2/v) \in w_i$ since $w_i = \{\alpha \mid B\alpha \in w_j\}$.

$V_{\mathcal{M}}(\alpha(t_2/v), w_i) = 1$ by the fundamental theorem which conradicts our earlier supposition that $V_{\mathcal{M}}(\alpha(t_2/v), w_i) = 0$.

Q.E.D.

And so once we have proven that a given system's canonical model is such that the four strictures on V_M for non-normal indices apply to $V_{\mathcal{M}}$ for members of W^*, then this in conjunction with showing that R in $\mathcal{M}$ meets the relevant constraints establishes that $\mathcal{M}$ is in the appropriate class of models. And this in turn would complete the completeness proof. In the next section, we shall see that although Rantala's proposals for a logic of propositional attitudes not presupposing omniscience and for that matter (although he does not discuss this) not presupposing the adjunction or the consistency of attitudes seems to work, the characteristic semantics which he proposes is objectionable which effectively puts us back to square one. I.e., we are in need of an unobjectionable characteristic semantics for the restricted Sub-$SQC^{=}_{\Omega}$ systems.

5. Non-Standard Indices and Equivocation

In the previous section, it was argued that Rantala's semantics is efficacious in invalidating select instances of the omnidoxasticity schemata (and their corresponding rules of inference), the adjunction schema as well as the consistency schemata. This is owing to the feature of his semantics that any thesis not in Ω can turn out to be false at non-normal indices - provided that we impose the stricture $R\Omega^*$ on Ω for cases where Ω is not a calculus.

One objection which could be made to Rantala's semantics is that we cannot make good *intuitive* sense of non-normal indices, even if we can make good model-theoretic sense of them. In chapter two it was argued that intuitively, (normal) indices can be regarded as Carapian state descriptions. I.e., for any atomic wff α and for any index w_i, either α is in w_i or $\sim\alpha$ is but not both. Thus, indices conceived in this way are consistent (and maximal) sets of atomic wffs or their negations. It was also remarked that if we find objectionable the view that (normal) indices are sets of wffs, then we could treat indices as primitives in our semantics, although we could associate with each index a state description in the following manner: The associated state description consists of all those atomic wffs or their negations assigned '1' at the index by V_M in the model.

However, a Rantalian non-normal index is such that V_M may assign to any atomic wff α *and* its negation '1' or it may assign to α *and* its negation '0' or it may differ in its assignment of a member of $\{1,0\}$ to α and its negation. This is owing to the feature of the semantics that V_M is defined non-inductively for non-normal indices (while being subject to the four strictures discussed in the previous section). Suppose we were to associate with each non-normal index a set S of atomic sentences or their negations such that membership in the set is determined by what V_M assigns to any atomic wff α and its negation at the index. Then the resulting set may not be a Carnapian state description since it may be both negation inconsistent as well as non-maximal - for some atomic wff α and its negation, it is possible that both it and $\sim\alpha$ are in S or that neither α nor $\sim\alpha$ are in S. Hence, in the general case, we cannot conceive of non-normal indices either as state descriptions or as being associated with state descriptions. Then it would seem to follow that we cannot make intuitive sense out of non-normal indices.

But even though in the general case we cannot conceive of non-normal indices either as state descriptions or as being associated with state descriptions, it could be argued that non-normal indices are either identifiable with or can be associated with what we shall call 'quasi-state descriptions'. Like state descriptions, a quasi state description Q is a set of wffs, particularly either atomic wffs or their negations although for any atomic wff α and its negation $\sim\alpha$, either both are in S, neither are in S or one or the other is in S. Therefore, unlike state descriptions, quasi state descriptions may fail to be consistent or maximal. Then we could say that a non-normal index *is* a quasi-state description or that associated with each non-normal index is a quasi-state description S such that for any atomic wff α or its negation $\sim\alpha$, α or $\sim\alpha$ is in S just in case V_M assigns '1' to either or both of these wffs at the appropriate non-normal index.

And so it would seem that we can make some sort of intuitive sense out of Rantalian non-normal indices if we think of them as being associated with what we have called quasi state descriptions. Further, there is no reason why any non-normal index conceived as being associated with some quasi state description cannot be a respectable belief alternative to the index which an agent inhabits, since its associated description will by definition contain no self-contradictory wffs of the form α & $\sim\alpha$ (where α is atomic) even if the description contains both α and $\sim\alpha$. And since non-normal indices are not closed under conjunction, then even though both α and $\sim\alpha$ may be true at such an index, it does not follow that their conjunction is unless the index contains the appropriate instance of $\alpha \supset (\sim\alpha \supset (\alpha \; \& \sim\alpha))$.

Hence any inconsistencies obtaining at non-normal indices may be 'hidden' in the sense that their conjunction may fail to obtain.

However, there is an additional problem alluded to by Max Cresswell concerning impossible index semantics. Supposing that the connectives are not defined inductively at non-normal indices, then there is no way to determine what any of these connectives represent - they collapse into one another.[32] Anything goes for all of the connectives with the exception of the closure stricture for '$\supset$' discussed above. Thus, there is no difference truth-conditionally speaking between any wff of the form $\alpha \vee \beta$, $\alpha \& \beta$ and $\alpha \equiv \beta$. At non-normal indices, $\vee$, & and $\equiv$ are semantically indistinguishable - they cannot be individuated.

However, even if Cresswell's charge that there is no way of individuating logical connectives (with the exception of the strictures imposed on V_M for members of W^*) for non-normal indices can somehow be answered, there is a more serious objection which he levels against impossible index semantics for belief: If the connectives $\sim$, $\vee$, & , $\supset$, and $\equiv$ are defined non-inductively for non-normal indices then we are not showing how they misbehave if they are *classically* construed. All that we are showing is that $\sim$, $\vee$, & , $\supset$, and $\equiv$ do not represent *classical* negation, disjunction, conjunction, implication and equivalence for non-normal indices.[33] For example, if & and $\supset$ represented classical conjunction and implication at non-normal indices then any thesis containing these connectives would be true at any such index. But this need not be the case if the thesis is not in Ω. Therefore, & and $\supset$ are not classical conjunction and implication which happen to misbehave at non-normal indices - classical conjunction and

[32] Cresswell (1982), pp. 74-75.

[33] See Cresswell (1973), p. 41 and Cresswell (1982), p. 74.

implication cannot misbehave and still be classical.

In sum, Cresswell's charge here seems to be that we are equivocating with respect to the connectives ~, v, & , ⊃, ≡ in a non-normal index semantics. Cresswell's objection is itself less objectionable if it is not assumed that the classical interpretations of ~, &, v, ⊃ and ≡ are in any sense privileged.[34] I.e., the objection is not that ~, &, v, ⊃ and ≡ do not represent 'real' negation, conjunction, etc. for non-normal indices but simply that we are equivocating with respect to these connectives. They mean one thing for impossible indices and they mean something else for normal indices. (Stating the objection in this way avoids any rejoinders to the effect that there is no priveleged interpretation of ~, &, v, ⊃ and ≡.) This equivocation is not benign for the reason that Rantala's impossible index semantics is supposed to explain for example how agents can fail to *classically* conjoin believed contents which obtain at non-standard alternatives. But if a content of the form α & β is false at some impossible alternative to an index w_i even though the 'conjuncts' α and β are true, then '&' in α & β is not *classical* conjunction. So, it has not been demonstrated how some instance of the adjunction schema, $(B\alpha \ \& \ B\beta) \supset B(\alpha \ \& \ \beta)$ is invalid if '&' in the scope of the belief operator in the consequent is classical conjunction.

The 'classical' rejoinder to Cresswell's second criticism of an impossible worlds semantics for belief logic is to first of all claim that the connectives of a formal logic are definable solely in terms of their role in inference - or in terms of certain characteristic axioms.[35] For example, the axiom-

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[34] Cresswell seems to suggest that the problem with impossible worlds semantics is that for example '~' is not 'real' negation at impossible indices. See Cresswell (1973), ch. 3. He also seems to assume that '~' qua 'real' negation is truth-functional. See Cresswell (1985), p. 74.

[35] This stance has received support in the literature including Belnap (1961), Rescher (1980), and Read (1988), to name a few.

schema '$\alpha \supset (\beta \supset (\alpha \,\&\, \beta))$' could be regarded as a kind of syntactic definition of '&' since it asserts that if α obtains and then if β obtains, $\alpha \,\&\, \beta$ obtains. On the other hand, if $\alpha \,\&\, \beta$ fails to obtain then it follows that either $\sim\alpha$ obtains or that $\sim\beta$ obtains or both. Then this is the syntactic counterpart of the characteristic two-valued matrix for '&' classically construed. Similar remarks apply for example to '~' since $\sim(\alpha \,\&\, \sim\alpha)$ could be regarded as the syntactic counterpart of the characteristic matrix for '~' classically interpreted. Also, the two 'paradoxes of material implication', $\alpha \supset (\beta \supset \alpha)$ and $\sim\alpha \supset (\alpha \supset \beta)$ can be regarded as characterizing '$\supset$' construed as material implication.

Suppose for the sake of argument that the connectives of a formal language really are definable in terms of their role in inference or in terms of certain 'characteristic' axioms - as illustrated above. Then it can be further argued that even though Rantala's semantics equivocates with respect to the interpretation of the connectives, there is no corresponding equivocation in the axiom-systems which are sound and complete with respect to this semantics. I.e., like our Sub-SQC$^{=}$ systems, any of the restricted Sub-SQC$^{=}{}_{\Omega}$ doxastic systems contain all the thesis-schemata (as well as material detachment) of the classical propositional calculus. So in the Sub-SQC$^{=}{}_{\Omega}$ axiom-systems, the connectives ~, &, v, $\supset$ and $\equiv$ 'behave' inferentially as they would for the unrestricted systems. Since by supposition the connectives are definable purely syntactically, then we could opt for defining them in this way rather than truth-conditionally thereby circumventing Cresswell's charge of equivocation. This line of reasoning is in fact taken up by Rescher in defense of a less extreme version of a non-standard worlds semantics for belief logic, which will be discussed briefly in the next chapter.[36]

As might be suspected, the Achilles' heel in this line of reasoning is the crucial supposition that the connectives of a language are definable solely in terms of their roles in inference. Prior has called this supposition into question by proposing the following reductio argument against it: Suppose that we wish to introduce into the language of some formal system the connective 'tonk'. Then an additional clause is added to the formation rules to the effect that if α, β are wffs then 'α tonk β' is a wff. We might then define the connective 'tonk' proof-theoretically in any number of ways including the following:

1) $\alpha \vdash \alpha$ tonk β

2) α tonk $\beta \vdash \beta$

1) says that α tonk β is a deductive consequence of α and 2) says that β is a deductive consequence of α tonk β. But by the transitivity of the deductive consequence relation, we obtain:

3) $\alpha \vdash \beta$

3) says that from any wff α we can deduce any wff β, which is absurd. Therefore, connectives cannot be defined solely in terms of their role in inference.[37]

A similar reductio-style argument could be offered for the claim that the connectives of a formal system cannot be defined solely in terms of certain characteristic axioms. I.e., suppose that we introduce the following axiom-schemata characterizing 'tonk':

4) $\alpha \supset (\alpha$ tonk $\beta)$

5) $(\alpha$ tonk $\beta) \supset \beta$

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36 Rescher (1980), pp. 22-23.

37 See Prior (1961, 1964).

Then by the transitivity of material implication, we can derive from 4) and 5):

6) $\vdash \alpha \supset \beta$

6), which is also attainable from 3) by the *deduction theorem* says that any wff logically implies any other wff, which is absurd. So, connectives cannot be defined solely in terms of characteristic axiom-schemata.

The obvious counter-move at this point is to argue that Prior has not shown that the connectives such as $\sim$, $\vee$, &, $\supset$ and $\equiv$ cannot be defined proof-theoretically, but merely that certain strictures need to be imposed regarding the introduction of new connectives into a formal system.[38] (The existing connectives would also need to satisfy these strictures.) For example, Belnap has suggested that any new connective must be a so-called *conservative extension* of an existing axiom-system.[39] A connective such as 'tonk' is an *extension* of an existing system in the sense that 1) a new clause must be added to the current formation rules and 2) additional axiom-schemata or inference rules are introduced. However, 'tonk' is not a *conservative* extension of the existing system since new inference rules or axiom-schemata which characterize it result in the derivation of wffs not involving 'tonk'. The conservativeness requirement therefore blocks the derivation of $\alpha \supset \beta$ from 4) and 5). Belnap regards the conservativeness requirement as an 'existence' condition for any new connective.

The point being made is that by imposing the right sorts of strictures as to what counts as a connective of a formal system, we can avoid Prior's objection that the connectives cannot be defined proof-theoretically. However, even if this is the case, Rantala's impossible index semantics is not vindicated of Cresswell's charge of equivocation since as will now be shown

38 Read (1988), p. 169.

39 See Belnap (1961), reprinted in Strawson (1967).

this equivocation is mirrored in the corresponding axiom-systems.

The response to Cresswell's charge of equivocation with respect to the connectives in an impossible worlds semantics such as Rantala's was that these connectives can be defined proof-theoretically - although the lesson to be drawn from Prior's 'tonk' example is that the defined connectives must meet certain requirements. But the axioms (or rules) in terms of which $\sim$, &, v, $\supset$ and $\equiv$ are definable involve no apparent equivocation with respect to them - the connectives behave *classically* in inferential contexts for the Sub-$SQC^{=}_{\Omega}$ systems. But in fact, the fallacy in this line of reasoning is the assumption that to determine how $\sim$, &, v, $\supset$ and $\equiv$ behave, we merely need to take into account various non-doxastic or non-modal thesis-schemata or inference rules. This view is somewhat myopic. In order to fully characterize the connectives $\sim$, &, v, $\supset$ and $\equiv$ for a modal or doxastic logic, presumably we must also take into account how they behave in modal or doxastic contexts.

If it is granted that to characterize the connectives $\sim$, &, v, $\supset$ and $\equiv$, we must take into account their behaviour in doxastic as well as non-doxastic contexts, then for example the adjunction schema, $(\mathbf{B}\alpha \ \& \ \mathbf{B}\beta) \supset \mathbf{B}(\alpha \ \& \ \beta)$ could be regarded as expressing the principle that belief factors out of '&' if it is *classical* conjunction. The conjunction in the *scope* of the consequent, $\mathbf{B}(\alpha \ \& \ \beta)$ is classical since any instance of this schema is derivable (for the unrestricted Sub-$SQC^{=}$ systems) by applying RB to the appropriate instance of $\alpha \supset (\beta \supset (\alpha \ \& \ \beta))$ which charcterizes '&' for non-doxastic contexts. (The reader should note that all 'classical' non-modal theses for '&' such as commutativity and associativity hold for the Rantala systems.) Now suppose that there is a Sub-SQC$^{=}_{\Omega}$ system such that some or all instances of $(B\alpha \ \& \ B\beta) \supset B(\alpha \ \& \ \beta)$ are *not* theses of the system by virtue of the appropriate wffs being excluded from Ω to which RB_{Ω} applies - in accordance with the stricture $R\Omega$*. Then '&' occurring in the consequent $B(\alpha \ \& \ \beta)$ of any instances of the adjunction schema which are not theses is not *classical* conjunction but some sort of 'hyperintensional' (to coin a phrase of Cresswell's) conjunction since in such cases, belief does not factor out of it. Therefore, in the syntax, there is an equivocation with respect to '&'.

Further, this equivocation in the axiom-system mirrors the situation in the semantics that for impossible indices, '&' and the other connectives are defined non-inductively. Thus, if some instance of $(B\alpha \ \& \ B\beta) \supset B(\alpha \ \& \ \beta)$ is not a thesis of a given Sub-SQC$^{=}_{\Omega}$ system it is (given completeness) also invalid in the semantics. And this invalidity implies the existence of at least one index in a model which is assigned at least one impossible alternative such that '$\alpha \ \& \ \beta$' is false at this alternative even though both 'conjuncts' α and β are true. Therefore, '&' at this alternative does not represent classical conjunction, which means that we are equivocating with respect to '&'. And so, Rantala's impossible index semantics cannot be vindicated by opting for defining the connectives of the language proof-theoretically since the same charge of equivocation applies to the proof-theoretic definition of the connectives.

Concluding Remarks

Although the Sub-SQC$^{=}$ systems can be altered in such a way that RB is restricted to some recursive subset of the set of wffs, thereby rendering certain instances of the omnidoxasticity, adjunction and consistency schemata underivable, the corresponding semantics which allows impossible indices to serve as doxastic alternatives involves an equivocation with respect to the connectives ~, &, v, ⊃ and ≡. This equivocation is not benign since for example, the semantics does not explain how agents can fail to *classically* conjoin beliefs. Further, defining the connectives proof-theoretically (rather than truth-conditionally) does not help matters since in taking into account how the above-mentioned connectives behave in doxastic contexts, there is an equivocation with respect to ~, &, v, ⊃ and ≡.

Then we are back to square one since we still have not shown how agents can fail to *classically* conjoin beliefs or can fail to believe all the *classical* logical consequences of what they believe. In fact, it would seem that any attempt at such an explanation will be entirely beside the point. The alternative is to accept the adjunction, consistency and omnidoxasticity schemata as features of logics of belief which involve construing the alethic necessity operator as 'x believes that'.

There is however another alternative to the one just mentioned. Perhaps the long-standing tradition of construing the necessity operator for alethic systems as 'x believes that' is best seen as a degenerating research program. It will be argued in the next chapter that if we wish to treat doxastic logics as variants of normal alethic modal logics, the more fruitful tact is to treat belief as *possibility* rather than necessity. Since for normal systems possibility does not factor out of conjunction and since $\sim(\mathbf{M}\alpha \ \& \ \mathbf{M}{\sim}\alpha)$ is not even a thesis of normal systems with D, then treating belief as a kind of possibility avoids the result that agents conjoin their

Chapter Six

The Intractable Problem of Logical Omnidoxasticity

Section 1: The Possibility of Belief

In the previous chapter, an attempt by Rantala to modify the axiomatics and semantics of normal systems of doxastic logics in order to deal with the problem of deduction was critically discussed. Any restricted normal logic based on Rantala's suggestions will render certain instances of the following schemata invalid/underivable:

$(\mathbf{B}\alpha \ \& \ \mathbf{B}\beta) \supset \mathbf{B}(\alpha \ \& \ \beta)$ adjunction schema

$(\mathbf{B}\alpha \ \& \ \vdash \alpha \supset \beta) \supset \mathbf{B}\beta$ omnidoxasticity schema

Further, if our particular axiom system contains D then Rantala's suggestions will give us a logic and semantics which renders any or all instances of the following invalid/underivable:

$\sim(\mathbf{B}\alpha \ \& \ \mathbf{B}{\sim}\alpha)$ consistency schema

$(\mathbf{B}\alpha \ \& \ \mathbf{B}\beta) \supset {\sim}\mathbf{B}{\sim}(\alpha \ \& \ \beta)$ weakened adjunction schema

The so-called weakened adjunction schema says that an agent will never believe that any conjunction of whatever he believes will fail to obtain.

As was explained in chapter one, if we regard these schemata informally as embodying principles of belief attribution, then there are ordinary language counterexamples to these principles which make it undesirable to have a logic of belief containing the 'corresponding' schemata. The Kripke puzzle discussed in chapters one and three can be regarded as a case not only where an agent has inconsistent beliefs but also (and arguably) as a case where the agent fails to conjoin these contradictory beliefs, which bears directly on the intuitive plausibility of both the consistency and the adjunction schemata. I.e., these schemata qua principles of belief attribution seem to conflict with Kripke's disquotation principle of belief attribution. Further, the omnidoxasticity principle conflicts with something like a Kripkean disquotation principle of belief attribution since even though an agent may assent to and hence believe that some truth of logic obtains, he may fail to assent to some other logical truth. Yet by the omnidoxasticity principle, we would be forced to attribute to the agent belief in both truths.

Although Rantala's suggestions seem to rid the Sub-SQC$^{=}$ systems of the problem of deduction by restricting RB to some arbitrary set Ω, which in the semantics also plays a role in invalidating various instances of the above-mentioned schemata, his semantics equivocates with respect to the connectives $\sim$, &, $\vee$, $\supset$ and $\equiv$. Further, the tact of defining the connectives proof-theoretically does not escape this difficulty since the equivocation with respect to $\sim$, &, $\vee$, $\supset$ and $\equiv$ is mirrored in the corresponding axiom-systems.

More generally, from a syntactic perspective, *any* alteration to a Sub-SQC$^{=}$ normal system which renders some instance of the omnidoxasticity, adjunction or consistency schemata underivable involves a (proof-theoretic) redefinition of, and hence an equivocation with respect to one of $\sim$, &, $\vee$, $\supset$ and $\equiv$. For example, suppose that we wish to block the derivation of some instance of the adjunction schema, $(B\alpha \ \& \ B\beta) \supset B(\alpha \supset \beta)$. As we have illustrated elsewhere[1], the derivation of any instance of this schema pro-

[1] See chapter one, section 6.

ceeds as follows:

1. $\vdash \alpha \supset (\beta \supset (\alpha \ \& \ \beta))$
2. $\mathbf{B}(\alpha \supset (\beta \supset (\alpha \ \& \ \beta)))$ 1, RB
3. $\vdash \mathbf{B}(\alpha \supset (\beta \supset (\alpha \ \& \ \beta))) \supset (\mathbf{B}\alpha \supset \mathbf{B}(\beta \supset (\alpha \ \& \ \beta)))$
4. $\mathbf{B}\alpha \supset \mathbf{B}(\beta \supset (\alpha \ \& \ \beta))$ 2,3 MP
5. $\vdash \mathbf{B}(\beta \supset (\alpha \ \& \ \beta)) \supset (\mathbf{B}\beta \supset \mathbf{B}(\alpha \ \& \ \beta))$
6. $\mathbf{B}\alpha \supset (\mathbf{B}\beta \supset \mathbf{B}(\alpha \ \& \ \beta))$ 4,5 PC
7. $(\mathbf{B}\alpha \ \& \ \mathbf{B}\beta) \supset \mathbf{B}(\alpha \ \& \ \beta)$ 6, PC

In order to block the derivation of any instance of $(\mathbf{B}\alpha \ \& \ \mathbf{B}\beta) \supset \mathbf{B}(\alpha \ \& \ \beta)$ there are a number of possible moves that could be made. First, we could deny thesishood to the appropriate instance of $\alpha \supset (\beta \supset (\alpha \ \& \ \beta))$ or to any PC thesis used in the derivation, although this would involve a redefinition of the 'classical' connectives, if they are being defined proof-theoretically. If we arbitrarily block any instance of modus ponens then we are equivocating with respect to '$\supset$' since in some instances, '$\supset$' detaches and in others it does not. Or, if we deny thesishood to the appropriate instance of K then we are redefining '$\supset$' (and '$\mathbf{B}$') since for normal systems, belief always distributes into '$\supset$' if '$\supset$' is classical. Finally, if RB is arbitrarily restricted such that it does not apply to some instance of $\alpha \supset (\beta \supset (\alpha \ \& \ \beta))$ then once again, we are redefining '&' (and possibly '$\supset$').

And so, either on the semantic front (using a non-standard worlds semantics) or on the syntactic front, any alteration such that some or all instances of the omnidoxasticity, adjunction or consistency schemata are rendered invalid/underivable will not show how 'classical' negation or conjunction or implication misbehave for modal or doxastic contexts. If either negation or conjunction or implication misbehave for modal or doxastic contexts, then they are no longer classical. Therefore, the very enterprise of attempting to alter normal doxastic systems in such a way that some or all instances of the omnidoxasticity, adjunction or consistency schemata are rendered invalid/underivable, is ill-conceived. Any such effort would be beside the point. If our logic of belief is based on a normal system of modal logic such that the necessity operator is informally construed as 'x believes that', then the omnidoxasticity, adjunction and (for systems containing D) consistency features are intractable.

Given the above considerations, if we find any of the omnidoxasticity, adjunction or consistency schemata objectionable qua principles of belief attribution then the tact of adopting normal systems where the *necessity* operator is construed as 'x believes that' ought to be abandoned. However, it does not follow from this that normal logics (with corresponding relational semantics) cannot serve as logics characterizing the 'non-ideal' believer, viz., one who for example does not always conjoin his/her beliefs.

In discussing the Kripke puzzle about belief, Marcus argues that puzzling Pierre does not believe a self-contradictory state of affairs, viz., London's being both pretty and not pretty, given her reality restriction on belief. The moral that she draws from this is that " ... belief, like possibility, does not always factor out of a conjunction".[2] Perhaps the moral to be drawn from her remark is that in drawing an analogy between alethic modal logic and doxastic logic, rather than construing the necessity operator as 'x believes that', it may be more instructive to treat the belief operator as a kind of *possibility* operator.

Further, we have argued elsewhere that a case can be made for the claim that puzzling Pierre holds contradictory beliefs in different 'contexts'.

[2] Marcus (1979), p. 507.

I.e., he believes that London is pretty and he believes that London is not pretty. Then for cases such as this, belief once more closely resembles the alethic *possibility* operator rather than the necessity operator since $\sim(\mathbf{M}\alpha \ \& \ \mathbf{M}{\sim}\alpha)$ is not a thesis schema for normal systems (where '**M**' here is Polish notation for the posibility operator).

To summarize, hypothetical situations such as the puzzling Pierre case suggest that belief is analogous to alethic possibility rather than to necessity since belief in such cases does not factor out of classical conjunction. Further, it is apparent in such cases that agents are capable of holding contradictory beliefs in different 'contexts'. It is established in the literature that for normal alethic systems (where '**M**' here is Polish notation for the possibility operator), the following are *not* thesis-schemata[3]:

i) $(\mathbf{M}\alpha \ \& \ \mathbf{M}\beta) \supset \mathbf{M}(\alpha \ \& \ \beta)$

ii) $\sim(\mathbf{M}\alpha \ \& \ \mathbf{M}{\sim}\alpha)$

However, the following alethic variant of the omnidoxasticity rule of inference is derivable in any normal alethic system:

iii) $\vdash \alpha \supset \beta \longrightarrow \vdash \mathbf{M}\alpha \supset \mathbf{M}\beta$

The derivation of iii) would proceed as follows (where **L** is Polish notation for 'it is necessary that'):

1. $\vdash \alpha \supset \beta$	hyp.
2. $\mathbf{L}(\alpha \supset \beta)$	1, RL
3. $\vdash \mathbf{L}(\alpha \supset \beta) \supset (\mathbf{M}\alpha \supset \mathbf{M}\beta)$	
4. $\mathbf{M}\alpha \supset \mathbf{M}\beta$	2,3 Modus Ponens.

Further, any instance of the corresponding schema is derivable for any normal system, viz.,

iv) $(\mathbf{M}\alpha \ \& \vdash \alpha \supset \beta) \supset \mathbf{M}\beta$

[3] See Hughes and Cresswell (1968), ch. 2.

The derivation of any instance of this schema would consist of lines 1) - 4) above as well as:

5) $\sim(\alpha \supset \beta) \vee (\mathbf{M}\alpha \supset \mathbf{M}\beta)$ PC, 4

6) $(\alpha \supset \beta) \supset (\mathbf{M}\alpha \supset \mathbf{M}\beta)$ PC, 5

7) $(\mathbf{M}\alpha \ \& \ \alpha \supset \beta) \supset \mathbf{M}\beta$ PC, 6 (where $\vdash \alpha \supset \beta$)

Also, if it is the case that $\vdash \alpha \equiv \beta$ in which case $\vdash \alpha \supset \beta$ and $\vdash \beta \supset \alpha$ it can easily be shown that $\vdash \mathbf{M}\alpha \equiv \mathbf{M}\beta$. Thus, the *equivalential* versions of iii) and iv) are derivable for any normal system.

Suppose that we construe the *possibility* operator **M** for alethic normal systems as 'x (non-ideally) believes that', thereby replacing every occurrence of **M** in the above schemata by **B**. Then we would obtain doxastic logics which though presupposing that agents are logically omnidoxastic, do not presuppose that agents always conjoin what they believe and which do not assume that agents are incapable of having contradictory beliefs. Therefore, normal logics will provide us with logics characterizing the non-ideal believer, supposing that the possibility operator (rather than the necessity operator) is construed as 'x believes that'.

Further, the alethic *necessity* operator can be reinterpreted as 'x *ideally* believes that' since all instances of the following schemata/rules of inference involving the necessity operator are derivable for any normal system:

v) $(\mathbf{L}\alpha \ \& \ \mathbf{L}\beta) \supset \mathbf{L}(\alpha \ \& \ \beta)$

vi) $\vdash \alpha \supset \beta \longrightarrow \vdash \mathbf{L}\alpha \supset \mathbf{L}\beta$

vii) $(\mathbf{L}\alpha \ \& \vdash \alpha \supset \beta) \supset \mathbf{L}\beta$

In addition, all instances of the following schema are derivable for any normal system containing D:

viii) $\sim(L\alpha \ \& \ L{\sim}\alpha)$

Replacing each occurrence of the 'L' operator in each of these schemata (and for the given set of axiom-schemata) with the operator 'B_I' which reads 'x ideally believes that' gives us a set of logics characterizing the 'ideal believer' - assuming that the normal systems we are working with contain D. This is because it was stipulated in chapter one that the ideal believer always conjoins his/her beliefs, does not hold contradictory beliefs and always believes the consequences of what he/she believes.[4]

And so, combining the proposal to construe the possibility operator as 'x believes that' and the necessity operator as 'x ideally believes that' for normal systems containing D, we obtain logics which characterize both the ideal and the non-ideal believer. Further, this tact does not involve any sort of alteration to the syntax of the given normal system and thus, there is no redefinition of the connectives ~, &, v, ⊃ and ≡ for modal or doxastic contexts. I.e., it cannot be charged that there is any sort of equivocation with respect to these connectives such that they behave in one way in non-modal contexts and another way in modal contexts.

What we are here proposing is to adopt the Sub-SQC$^=$ + D systems developed in the fourth chapter as doxastic logics where the necessity operator is construed as 'x ideally believes that' and where the possibility operator is construed as 'x (non-ideally) believes that'. The resulting logics will be called the Stal-SQC$^=$ + D systems since their truth-value semantics will be based on Stalnaker's informal solution on the semantic front to the problem of deduction. The reason for specifying that these systems contain D is to ensure that they also characterize the 'ideal' believer as defined in the first chapter. I.e., logics with D will be such that an agent does not

[4] See section 1 of chapter one.

hold contradictory beliefs if we are discussing 'ideal' belief. Notice further that for all these systems, $B\alpha =_{df.} \sim B_I \sim\alpha$ and hence it is a simple matter to show that $\vdash B\alpha \equiv \sim B_I \sim \alpha$ and that $\vdash B_I\alpha \equiv \sim B \sim \alpha$. What follows is a description of the Stal-KDQC$^=$ system of doxastic logic such that any KD extension (not containing T) can also serve as a doxastic logic depending on what our philosophical biases are:

AS 1: α where α has the form of a PC thesis

AS 2: $(B_I\alpha \;\&\; B_I(\alpha \supset \beta)) \supset B_I\beta$

AS 3: $B_I\alpha \supset B\alpha$

AS 4: $\alpha\,(t/v) \supset (\exists v)\alpha$

AS 5: $t = t$

AS 6: $(\alpha\,(t_1/v) \;\&\; t_1 = t_2) \supset \alpha\,(t_2/v)$ provided t_1, t_2 do not occur in the scope of any doxastic operators.

AS 7: $(\alpha\,(t_1/v) \;\&\; t_1 = t_2 \;\&\; B_I(t_1 = t_2)) \supset \alpha\,(t_2/v)$ where t_1, t_2 may occur in the scope of doxastic operators.

AS 8: $(\forall v)B_I\alpha \supset B_I(\forall v)\alpha$

Rules of inference:

RB_I $\vdash\alpha \longrightarrow \vdash B_I\alpha$

MP $\alpha,\ \alpha \supset \beta \longrightarrow \beta$

$\exists I$ $\alpha\,(t/v) \supset \beta \longrightarrow (\exists v)\alpha \supset \beta$ for any t foreign to $(\exists v)\alpha \supset \beta$.

The following schemata and rules are all those derivable for the corresponding Sub-KDQC$^=$ system except that we have replaced all occurrences of the operator 'B' with the operator 'B_I' to signify that these schemata and rules characterize the *ideal* believer:

T1 $(B_I\alpha \;\&\; B_I\beta) \supset B_I(\alpha \;\&\; \beta)$ ideal adjunction schema

T2 $\sim(B_I\alpha \;\&\; B_I\sim\alpha)$ ideal consistency schema

T3 $(B_I\alpha \ \& \vdash\alpha \supset \beta) \supset B_I\beta$ ideal omnidoxasticity schema

T4 $(B_I\alpha \ \& \vdash\alpha \equiv \beta) \supset B_I\beta$ ideal omnidoxasticity schema - equivalential version

R1 $\vdash\alpha \supset \beta \longrightarrow \vdash B_I\alpha \supset B_I\beta$ ideal omnidoxasticity - implicational

R2 $\vdash\alpha \equiv \beta \longrightarrow \vdash B_I\alpha \equiv B_I\beta$ ideal omnidoxasticity - equivalential

Thus, the Stal-KDQC$^=$ system and its extensions will provide us with logics characterizing the *ideal* believer. Further, the following are instances of AS 4, $\alpha(t/v) \supset (\exists v)\alpha$ and of AS 7, $(\alpha(t_1/v) \ \& \ t_1 = t_2 \ \& \ B_I(t_1 = t_2)) \supset \alpha(t_2/v)$ respectively:

AS 4": $B_I\alpha(t/v) \supset (\exists v)B_I\alpha$ ideal doxastic generalization schema

AS 7": $(B_I\alpha(t_1/v) \ \& \ t_1 = t_2 \ \& \ B_I(t_1 = t_2)) \supset B_I\alpha(t_2/v)$

ideal doxastic substitution schema

What these indicate is that quantification into the *ideal* belief operator is unrestricted and that substitution of co-referentials in *ideal* belief contexts is restricted to cases where the agent ideally believes that the relevant identity holds. It could be objected that a so-called ideal believer would be omnidoxastic with respect to contingent identities. However, to reiterate, our definition of what constitutes the 'ideal believer' is purely stipulative - the ideal believer conjoins his/her beliefs, does not hold contradictory beliefs and believes all the classical logical consequences of what he/she believes. Then whether or not the agent is onmnidoxastic with respect to identities has no bearing on his/her ideality. The ideality criteria just mentioned can be regarded as purely *deductive* constraints on belief.

It was earlier claimed that the Stal-SQC$^=$ + D systems can also be regarded as logics which characterize the *non-ideal* believer who nonetheless believes all the logical consequences of what he/she believes. And in fact, not all instances of the adjunction and consistency schemata are derivable in any Stal-SQC$^{=}$ + D system for non-ideal belief:

$-\!| \ (\mathbf{B}\alpha \ \& \ \mathbf{B}\beta) \supset \mathbf{B}(\alpha \ \& \ \beta)$ (non-ideal) adjunction schema

$-\!| \ \sim(\mathbf{B}\alpha \ \& \ \mathbf{B}{\sim}\alpha)$ (non-ideal) consistency schema

That the non-ideal adjunction and consistency schemata are not thesis-schemata for the Stal-SQC$^{=}$ systems is owing to the fact that their alethic counterparts, $(\mathbf{M}\alpha \ \& \ \mathbf{M}\beta) \supset \mathbf{M}(\alpha \ \& \ \beta)$ and $\sim(\mathbf{M}\alpha \ \& \ \mathbf{M}{\sim}\alpha)$ are not thesis-schemata for normal alethic modal systems.

For alethic systems, $(\mathbf{M}\alpha \ \& \ \mathbf{M}\beta) \supset \mathbf{M}(\alpha \ \& \ \beta)$ is not a thesis schema since the derivation of any instance of this schema depends upon **M**'s distributing into '$\supset$' in the appropriate instance of $\alpha \supset (\beta \supset (\alpha \ \& \ \beta))$. However, in general **M** does not distribute into '$\supset$' for normal systems since the possibilitation version of the schema K is not a thesis-schema for normal systems. I.e., $-\!| \ \mathbf{M}(\alpha \supset \beta) \supset (\mathbf{M}\alpha \supset \mathbf{M}\beta)$. On the other hand, the following is a theorem-schema for any alethic modal system:

$\vdash \mathbf{L}(\alpha \supset \beta) \supset (\mathbf{M}\alpha \supset \mathbf{M}\beta)$

This follows directly from K.[5] In order to derive some instance of $\mathbf{M}(\alpha \supset \beta) \supset (\mathbf{M}\alpha \supset \mathbf{M}\beta)$, we would need $\mathbf{L}(\alpha \supset \beta) \supset (\mathbf{M}\alpha \supset \mathbf{M}\beta)$ as well as the appropriate instance of $\mathbf{M}(\alpha \supset \beta) \supset \mathbf{L}(\alpha \supset \beta)$ which is not a thesis of any normal system. Thus, $\mathbf{M}(\alpha \supset \beta) \supset (\mathbf{M}\alpha \supset \mathbf{M}\beta)$ is not a thesis-schema for any normal alethic system and hence neither is $(\mathbf{MA} \ \& \ \mathbf{MB}) \supset \mathbf{M}(\alpha \ \& \ \beta)$. But then neither is $\sim(\mathbf{M}\alpha \ \& \ \mathbf{M}{\sim}\alpha)$ a thesis-schema of any alethic K-extension since the derivation of any of its instances depends on the appropriate version of $(\mathbf{M}\alpha \ \& \ \mathbf{M}\beta) \supset \mathbf{M}(\alpha \ \& \ \beta)$.

Further, since $\mathbf{M}(\alpha \supset \beta) \supset (\mathbf{M}\alpha \supset \mathbf{M}\beta)$ is not a thesis-schema for alethic

[5] See Hughes and Cresswell (1968), ch. 2, p. 37.

normal systems, then neither is its doxastic counterpart a thesis-schema for the Stal-SQC$^{=}$ systems. I.e., $\dashv B(\alpha \supset \beta) \supset (B\alpha \supset B\beta)$. Thus, unlike ideal belief, non-ideal belief does not distribute into '$\supset$'. However, this is not necessarily an undesirable state of affairs in the light of the following variant of the puzzling Pierre case, which in fact is suggested by Kripke.[6] Suppose that while in France, puzzling Pierre assents to "Si Londres n'est pas jolie, New York n'est pas jolie". Then by the disquotation and the translation principles, Pierre believes that if London is not pretty then New York is not pretty. Suppose further that Pierre after having moved to London assents to "London is not pretty". Then by the disquotation principle he believes that London is not pretty. So, Pierre believes that London is not pretty and Pierre believes that if London is not pretty then New York is not pretty. Yet, Pierre may not assent to the claim that New York is not pretty (while living in London) even though he believes that London is not pretty. Thus, by the disquotation principle, it is false that Pierre believes that New York is not pretty. Further, if we are speaking of *non-ideal* belief where for example the agent can fail to conjoin beliefs (and we have good reason to suspect that Pierre does not always conjoin his beliefs), then the disquotation principle does not in this particular case conflict with any other principle of belief attribution, such as the non-ideal variant of K. I.e., for *non-ideal* belief, it is not assumed that agents will always make modus ponens inferences from contents of the forms α and $\alpha \supset \beta$ to β.

One possible explanation of the above case is that Pierre holds two separate beliefs (whose contents are of the forms α and $\alpha \supset \beta$) in different linguistic contexts and hence he fails to make the inference to the claim that New York is not pretty. If our explanation of this situation is correct,

[6] Kripke (1979), pp. 257-8.

then it is a desirable feature of our doxastic logic that the non-ideal variant of K, $(\mathbf{B}\alpha \ \& \ \mathbf{B}(\alpha \supset \beta)) \supset \mathbf{B}\beta$ is *not* a thesis-schemata. In other words, it is being suggested that Kripke's variant of the puzzling Pierre case is a hypothetical situation where an agent x believes that α and believes that α classically implies β and yet x fails to believe that β.

Although the Stal-SQC$^{=}$ systems provide us with logics which characterize the believer who is non-ideal in the sense that he/she may fail to conjoin beliefs and/or may hold contradictory beliefs (though separately), agents are nonetheless assumed to be logically omnidoxastic. The omnidoxasticity schemata and rules of inference (both the implicational and the equivalential versions) are derivable for these systems as was explained above with reference to the **M** operator. I.e., all instances of the following are derivable for any Stal-SQC$^{=}$ + D system:

T5	$(\mathbf{B}\alpha \ \& \vdash \alpha \supset \beta) \supset \mathbf{B}\beta$	non-ideal omnidoxasticity schema
T6	$(\mathbf{B}\alpha \ \& \vdash \alpha \equiv \beta) \supset \mathbf{B}\beta$	non-ideal equivalential omnidoxasticity schema
R3	$\vdash \alpha \supset \beta \longrightarrow \vdash \mathbf{B}\alpha \supset \mathbf{B}\beta$	non-ideal omnidoxasticity - implicational
R4	$\vdash \alpha \equiv \beta \longrightarrow \vdash \mathbf{B}\alpha \equiv \mathbf{B}\beta$	non-ideal omnidoxasticity - equivalential

Further, the non-ideal variant of $\mathbf{RB_I}$ is derivable for any Stal-SQC$^{=}$ + D system by virtue of D. I.e., the following is a rule of any such system:

RB $\vdash \alpha \longrightarrow \vdash \mathbf{B}\alpha$

The rule RB is derivable (using D) as follows:

1.	α	hyp.
2.	$\mathbf{B_I}\alpha$	1, $\mathbf{RB_I}$
3.	$\vdash \mathbf{B_I}\alpha \supset \mathbf{B}\alpha$	D
4.	$\mathbf{B}\alpha$	2,3 MP

Q.E.D.

Finally, we shall consider the relation between non-ideal belief and the existential quantifier on the one hand and the relation between non-ideal belief and the identity symbol on the other. First, since the following is an instance of AS 4, $\alpha(t/v) \supset (\exists v)\alpha$, it follows that quantification into non-ideal belief contexts is unrestricted:

AS 4"" $\mathbf{B}\alpha(t/v) \supset (\exists v)\mathbf{B}\alpha$

It comes as no surprise that quantification into non-ideal doxastic contexts is unrestricted given that the quantifieres are construed substitutionally in the corresponding truth-value semantics to be discussed below. Also, as with *ideal* belief, non-ideal belief de re implies non-ideal belief de dicto since the following is a thesis-schema for any Stal-SQC$^{=}$ + D system:

T7: $(\exists v)\mathbf{B}\alpha \supset \mathbf{B}(\exists v)\alpha$

Any instance of T7 is easily derivable as follows:

1. $\vdash \alpha(t/v) \supset (\exists v)\alpha$
2. $\mathbf{B}\alpha(t/v) \supset \mathbf{B}(\exists v)\alpha$ 1, R3
3. $(\exists v)\mathbf{B}\alpha \supset \mathbf{B}(\exists v)\alpha$ 2, R∃

The substitution of co-referentials in non-ideal doxastic contexts is restricted to cases where the agent *ideally* believes that the relevant identity obtains, given that the following is an instance of AS 7, $(\alpha(t_1/v)\ \&\ t_1 = t_2\ \&\ \mathbf{B_I}(t_1 = t_2)) \supset \alpha(t_2/v)$, viz.,

AS 7"" $(\mathbf{B}\alpha(t_1/v)\ \&\ t_1 = t_2\ \&\ \mathbf{B_I}(t_1 = t_2)) \supset \mathbf{B}\alpha(t_2/v)$

The reason why substitution of co-referentials is restricted to cases where the agent *ideally* believes that the relevant identity obtains will become evident when we consider the semantics for the Stal-SQC$^{=}$ + D systems below.

And so, the Stal-SQC$^{=}$ + D systems are simply the Sub-SQC$^{=}$ + D systems such that all occurrences of **B** and $\mathbf{P_B}$ in theses of the latter are replaced by $\mathbf{B_I}$ and **B** respectively in theses of the former. For example, D for the Sub-SQC$^=$ + D systems, $\mathbf{B}\alpha \supset \mathbf{P_B}\alpha$ is the counterpart of $\mathbf{B_I}\alpha \supset \mathbf{B}\alpha$ for the Stal-SQC$^=$ + D systems. Thus, the characteristic semantics for the Sub-SQC$^=$ + D systems, viz., the truth-value semantics described in chapter four, also characterizes the Stal-SQC$^=$ + D systems although we replace $V_M(\mathbf{B}\alpha, w_i)$ with $V_M(\mathbf{B_I}\alpha, w_i)$ and we further replace $V_M(\mathbf{P_B}\alpha, w_i)$ with $V_M(\mathbf{B}\alpha, w_i)$. Thus, where a truth-value model is a triple $\langle W, R, V \rangle$ such that minimally R is serial and such that W and V are defined as they were for the Sub-SQC$^=$ + D systems, the truth-conditions for ideal and non-ideal belief are as follows:

$V_M(\mathbf{B_I}\alpha, w_i) = 1$ iff for all w_j such that w_iRw_j, $V_M(\alpha, w_j) = 1$.
$V_M(\mathbf{B}\alpha, w_i) = 1$ iff for at least one w_j such that w_iRw_j, $V_M(\alpha, w_j) = 1$.
Thus, in the semantics for the Stal-SQC$^=$ + D systems, ideal belief is treated like alethic necessity and non-ideal belief is treated like alethic possibility.

Although we have a 'ready-made' semantics for the Stal-SQC$^=$ + D systems as illustrated above, it would be preferable to have a characteristic semantics which makes some sort of intuitive sense of situations such as the puzzling Pierre case. It has been suggested that in the puzzling Pierre case, Pierre holds imcompatible beliefs in distinct *contexts*. Thus, what is needed is a semantics which gives some content to the notion that agents are capable of holding distinct sets of beliefs in different contexts.

Robert Stalnaker in *Inquiry* has argued that agents can be in more than one 'belief state' and that this accounts for why agents can sometimes fail to conjoin beliefs or hold contradictory beliefs. A belief state s_i is the set of possible situations such that all the contents of a subset of an agent's beliefs obtain at each situation in the set. Stalnaker suggests that an agent can be in more than one belief state at the same time.[7] Thus, x believes that α at w_i iff for at least one belief state, α obtains at every member of the state. So, puzzling Pierre can believe that London is pretty and he can also believe that London is not pretty if he is in at least two distinct belief states such that the former content obtains at all members of one state and the latter content obtains at all members of the other state. Also, Pierre does not conjoin these beliefs since in neither state is α & $\sim\alpha$ true at any member. The notion that Pierre is in more than one belief state can be regarded as explicating what it means to say that Pierre holds separate sets of beliefs in distinct *contexts*. We shall provide a more detailed exposition of Stalnaker's solution to the problem of deduction in terms of 'belief states' in the next section.

In the third section, an attempt will be made to make model-theoretic sense out of Stalnaker's informal semantic proposal that agents can be in two or more distinct belief states. I.e., a formal relational semantics will be developed for the Stal-SQC$^=$ + D systems which incorporates Stalnaker's idea that agents can be in more than one belief state. It will be argued that this semantics in fact characterizes the Stal-SQC$^=$ + D doxastic systems.

An alternative to Stalnaker's semantics which also makes sense of the notion of holding separate beliefs in distinct 'contexts' will be developed in section 4. This alternative semantics will be based on Rescher's proposal that belief can be treated as a relation between a believer and a non-standard world. However, it will be argued that unlike Rantala's non-standard worlds semantics, Rescher's semantics avoids the charge of equivocation

[7] Stalnaker (1984), pp. 82-4.

with respect ~, &, v, ⊃ and ≡.

Superposed worlds are constructed out of normal worlds as follows: α obtains at a superposed world w^+ constructed out of w_1 and w_2 just in case α obtains at *either* w_1 or w_2. Then even though α is true at w_1 and is false at w_2 and even though β is false at w_1 but true at w_2, both α and β will obtain at the superposed world w^+. Yet their conjunction will fail to obtain since α & β is false at both component worlds. Now if w^+ is the non-standard world to which the believer is related, then α and β will both be true at this superposed world and yet their conjunction α & β is false and hence, x will believe that α and that β without believing that α & β. If β happens to be $\sim\alpha$ then this sort of situation would also be a case where an agent holds contradictory beliefs (but without believing their conjunction). It will be suggested how Rescher's solution to the problem of deduction in terms of non-standard worlds can be adapted to provide a characteristic relational semantics for the Stal-SQC$^=$ + D systems.

Finally, in section 5 it will argued that in general, the problem of logical omnidoxasticity is intractable for a normal logic of belief since any alteration to the logic and semantics in order to avoid the omnidoxasticity feature will result in an equivocation with respect to the connectives ~, &, v, ⊃ and ≡. Thus, any such solution to the omnidoxasticity element of the problem of deduction will be beside the point. However, it will be argued that the features of our Stal-SQC$^=$ + D systems and their corresponding semantics that agents do not always conjoin beliefs and are capable of having inconsistent beliefs mitigates the omnidoxasticity feature of these logics.

2. Belief States and the Problem of Deduction

In *Inquiry*, Stalnaker attempts to solve the problem of deduction in part by appealing to the notion that agents are capable of being in several distinct belief states or more generally, acceptance states. Before discussing what he means by a 'state', we shall first clarify what he means by the notion of 'acceptance'. Stalnaker classifies the attitude of belief as belonging to a genus or class of propositional attitudes which he calls attitudes of acceptance. He further claims that what is involved in *accepting* a proposition P is to regard P, even if only tacitly, as true. I.e., in accepting a proposition , the agent is disposed to act in at least some of the ways he would act if he (without reservation) were to believe P to be true. Thus, a criterion of something's being an acceptance attitude with respect to some content proposition P is that "the attitude is said to be *correct* whenever the proposition is true"[8] Then other types of attitudes such as desires or hopes differ from acceptance attitudes in that correctness of the former is not judged in terms of whether or not the content proposition is true.[9]

Attitudes of acceptance include such diverse attitudes as tacitly presupposing, assuming, supposing and believing.[10] According to Stalnaker, belief differs from these other sorts of acceptance attitudes in several ways. For one thing, belief supposedly requires some sort of 'entertaining' of a proposition whereas an acceptance attitude such as presupposing does not. Thus, one may simply take for granted that a certain proposition P obtains (thereby acting in ways consistent with P's being true) without

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[8] Stalnaker (1984), p. 80.

[9] ibid, p. 80.

[10] Stalnaker (1984), p. 79.

ever really having considered or entertained this proposition in the process of inquiry or deliberation. Or, a person may tacitly presuppose some proposition for the sake of inquiry or investigation without committing himself to its being true in the long run. As we shall see below, the 'cash value' of Stalnaker's classifying belief as a kind of acceptance attitude is the efficacy of this move in helping to resolve the problem of deduction. A few more preliminaries are in order before shall be in a position to explain exactly what Stalnaker means by belief or acceptance *states*.

Stalnaker adopts what he calls a 'causal-pragmatic' account of the acceptance attitude of belief: The *pragmatic* element of this account is that an agent x believes that P only if x is disposed to act in ways which "will tend to serve his interests and desires in situations in which P is true".[11] This dispositional account (or for that matter, functionalist account[12]) of belief assumes that there is an intimate connection between beliefs and desires, a theme which is also present in Stalnaker's 1976 article 'Propositions'. I.e., the agent's desires and beliefs function as premises in Aristotelian practical syllogisms in the sense that for any given action α, that action is explained by x's wanting or desiring that α will obtain and by the agent's believing that by doing β, α will obtain.[13]

The *causal* element of Stalnaker's account of belief is that x believes that P only if x's belief 'indicates' that P. And x's belief indicates that P means that under 'optimal conditions' x's belief that P is caused by some state s of the environment such that the proposition Q asserting that s obtains, entails P.[14] Further, Stalnaker regards this causal account of belief

[11] ibid, p. 82. See also ch. 1 and in particular, p. 15.

[12] Stalnaker regards the pragmatic account as functionalist since beliefs (and desires) are understood in terms of their role in determining and rationalizing action. See Stalnaker (1976), p. 80.

[13] Stalnaker (1972), p. 81.

as supplementary to the pragmatic account (rather than as an alternative to it) for the following reason: Many types of 'representational' states of an agent are causally related to the environment and so in order to distinguish belief from other types of representational states, it is necessary to recognize that beliefs are connected with dispositions to act (in the manner specified above).[15] I.e., the pragmatic account of belief provides us with a criterion for distinguishing belief states from other types of so-called mental representations which are also caused by states of the environment.

Stalnaker's causal-pragmatic account of belief can serve as a principle of belief attribution. I.e., x believes that P just in case 1) x is disposed to act in ways which tend to realize his desires in all those situations in which P obtains and 2) x's belief indicates that P. Thus, if either of these two conditions fail to obtain, viz., that the agent does not have the requisite dispositions (or does not exhibit the appropriate behaviours, verbal or otherwise) or if the appropriate causal circumstances are absent (whatever they may be) then we would not attribute to the agent the appropriate attitude. At least the pragmatic aspect of Stalnaker's account of belief qua principle of belief attribution is consistent with Kripke's disquotation principle since in the latter case, a belief that P is ascribed to an agent on

[14] Stalnaker (1984), p. 18. Stalnaker offers a more detailed account of the 'indication' relation in Stalnaker (1984), pp. 12-13. I.e., if a is the state of an object and f is a one-one function taking each state a of the object into exactly one state of the environment, f(a), then says Stalnaker, this correlation is explained by f(a)'s causing a in the object - assuming 'normal' or optimal conditions of the environment. Then a state a of an object x indicates that P just in case the proposition that the environment is in state f(a) entails that P. Thus, x believes that P only if his belief indicates that P, i.e., only if P is entailed by a proposition asserting that some state of the world causing x's belief state obtains.

[15] Ibid, pp. 18-19. By 'representational state', Stalnaker is referring to Fodor's notion of mental representation.

the basis of his verbal behaviours, viz., his sincere assent to P.

More importantly, Stalnaker claims that the pragmatic element of his account of belief naturally lends itself to the view that the appropriate objects of belief (and of the attitudes generally) are non-empty sets of possible situations.[16] I.e., because a necessary condition of x's believing that P on the pragmatic approach is that x is disposed to act in ways which will tend to satisfy his desires *in possible situations* where P obtains, then it would seem that belief is a relation between an agent and a set of worlds. We shall now describe briefly what Stalnaker means by possible world or situation. (For the sake of exposition we shall use the terms 'world' and 'situation' interchangeably.)

Stalnaker maintains a so-called moderate realism with respect to possible worlds, viz., that *there are* alternative ways the 'actual' world could have been and that these alternative possible worlds are 'respectable entities in their own right'[17] in the sense that they are not reducible to other sorts of things such as sets of sentences. (This latter part of his moderate realist thesis does not put his view of possible worlds at odds with the view we espoused earlier since it was maintained that an index is 'associated' with a state description.) Further, like extreme modal realists such as Lewis, Stalnaker maintains that 'actual' functions as an indexical. On the other hand, Stalnaker parts company with the extreme modal realists since he holds that possible worlds are not on an equal ontological footing with the so-called actual world in the sense of being things of the same sort as the actual world.[18]

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16 Stalnaker (1972), p. 81.

17 See Stalnaker (1984), p. 50.

18 Ibid, p. 47.

One of Stalnaker's grounds for holding that an indexical analysis of 'actual' is not incompatible with the view that possible worlds are not on an equal ontological footing with the 'actual' world is that just as an indexical analysis of tense does not entail a commitment to the existence of concrete 'times', so an indexical analysis of 'actual' does not commit one to there being concrete worlds like our own.[19] It is beyond the scope of this discussion to evaluate Stalnaker's views here. We are merely attempting to explain roughly what he means by 'possible worlds' or 'possible situation', which is important since as we shall next see, the notion of possible situation is integral to his characterization of what belief and acceptance *states* are.

If we view belief and acceptance generally as any relation between an agent and an appropriate set of possible situations where the content proposition obtains (or for that matter we could say that this set of possible situations simply *is* the proposition which the agent accepts), then a belief or acceptance *state* can be defined as a set of possible situations where all the contents of the agent's beliefs or acceptance attitudes obtain. Further, since all those propositions which an agent accepts obtain at all worlds in the corresponding acceptance state[20] it follows that x accepts that P just in case P obtains at all those worlds contained in the agent's acceptance state.[21] If we regard propositions as sets of worlds (or in our parlance, indices) then we can think of an acceptance state as the *intersection* of all the propositions which the agent accepts. This feature of belief and acceptance states is consistent with the view espoused in chapter one that the objects of beliefs in a relational semantics are partial propositions, viz., the

[19] ibid, p. 47.

[20] ibid, p. 81.

[21] ibid, p. 69.

set of indices where all the contents of agents' beliefs are true, such that this set of indices relative to an index is determined by the accessibility relation R.

Given Stalnaker's definition of an acceptance (and belief) state as a set of worlds such that all the contents of what the agent accepts (or believes) obtain at each of these worlds, it follows that these states are deductively closed. I.e., for any contents P, Q and for any acceptance state s:

1) If P obtains at every world in s then if P entails Q, Q obtains in every world in s.

2) If P and Q both obtain at every world in s then P & Q obtains at every world of s.

3) If P obtains at every world in s then ~P obtains at no world in s.

From now on, we shall refer to P, Q neutrally as contents rather than as content *propositions* since if we regard propositions as sets of worlds, it is not clear in what sense a set of worlds can be said to obtain at (or be true at) each world in an acceptance state. Stalnaker regards these three deductive constraints for acceptance states as 'defining conditions' on what an acceptance state is. I.e., any acceptance state must satisfy these three deductive conditions.

The above deductive constraints on any acceptance state are explained by the fact that each world w in a state s is 'normal' in the sense that the connectives are defined standardly. Thus, if P is true at a world w in s and given that $\vdash P \supset Q$, it follows that Q will also obtain at w in s since '$\supset$' is interpreted classically for every w in s. Further, if P and Q both obtain at a world w in s, it follows that P & Q will also obtain at w given that '&' is interpreted classically for members of s. Finally, if P obtains at some w in s and given that '~' is interpreted classically at members of s, then ~P will not obtain at w.

Since x believes (or more generally, accepts) that P iff P obtains at each world in the agent's acceptance state, then the following *qua* principles of belief or acceptance attribution 'correspond' to (in the sense of being inferable from) the three deductive constraints on acceptance states:

1)* If x believes that P and P entails that Q then x believes that Q.

2)* If x believes that P and x believes that Q then x believes that P & Q.

3)* If x believes that P then x does not believe that ~P.

1)* corresponds to the deductive constraint 1), 2)* corresponds to 2) and finally, 3)* corresponds to the deductive constraint 3). Further, the principles of belief attribution 1)*, 2)* and 3)* have as their formal counterparts the omnidoxasticity, adjunction and consistency schemata respectively for the Sub-SQC$^{=}$ systems of doxastic logic. I.e., 1)*, 2)* and 3)* have as their formal counterparts the following:

1)** $(B\alpha \ \& \vdash \alpha \supset \beta) \supset B\beta$ omnidoxasticity schema

2)** $(B\alpha \ \& \ B\beta) \supset B(\alpha \ \& \ \beta)$ adjunction schema

3)** $\sim(B\alpha \ \& \ B{\sim}\alpha)$ consistency schema

It is at this stage of the dialectic that Stalnaker alludes to the 'problem of deduction'.

As Stalnaker notes, the conditions 1)* through 3)* inferable from the deductive constraints 1) - 3), when applied to the 'totality' of an agent's beliefs admit of at least apparent counterexamples. These counterexamples are hypothetical cases where agents fail to believe the consequences of what they believe, thus impugning 1)*, or cases where agents fail to conjoin beliefs thereby impugning 2)* or cases where agents seem to believe contradictories thereby impugning 3)*. As we have argued in the first chapter, the Kripke puzzle can be regarded as a counterexample to the adjunction and consistency conditions 2)* and 3)* and the William III case (as well as examples involving mathematical beliefs) can be regarded as counterexamples to the omnidoxasticity condition 1)*. As was also noted in chapter one, the principles of belief attribution used in setting up these apparent counterexamples could themselves be called into question. However, if for the sake of argument we assume that the principles used are sound, then the problem of deduction is the problem that the three deductive constraints on acceptance states seem to break down in the light of these examples. I.e., it would seem that belief states are not deductively closed which means that the view that possible worlds are the relata of beliefs does not take account of the 'facts'.

Stalnaker's diagnosis of and his solution to the problem of deduction involves two approaches. The first approach is that the alleged counterexamples to 2)* - 3)* and hence to 2) - 3) do not impugn these constraints qua defining conditions of belief *states*. Rather, what these examples do show is that we cannot apply 2) - 3) to the *totality* of agents' beliefs, at least if agents are not ideally rational.[22] What Stalnaker's diagnosis here implies, is the possibility that the totality of an agent's beliefs is not necessarily exhausted by just one belief state in which case it is possible that agents can be in more than one belief state at the same time. Thus, an agent x believes that P just in case for at least one belief state amongst possibly several, P obtains at each world in that state.

This approach explains why an agent who (for simplicity of exposition) is in two belief states s_1 and s_2 can fail to conjoin his belief that P and his

[22] Stalnaker (1984), p. 83.

belief that Q. I.e., P may obtain at all the worlds in s_1 while Q obtains at all the worlds in s_2 and yet there may be at least one world in s_1 where Q does not obtain and there may be at least one world in s_2 such that P does not obtain. In such a case as this, x will not believe that P & Q because this conjunction fails to obtain in at least one world in s_1 and in at least one world in s_2. These remarks can be extended to the case where Q is simply ~P, i.e., an agent x may believe that P in s_1 and he may believe that ~P in s_2 and yet in neither state does x believe that P & ~P.

Or less formally, as Stalnaker notes, we can make sense of an agent's being in more than one belief state on the *pragmatic* account of belief as follows: In one type of context, x may be disposed to act in ways that will satisfy his desires in P-worlds which is explained by belief state s_1[23] and in a different context, x may be disposed to act in ways that would satisfy his desires in Q-worlds which is explained by belief state s_2 (or in ~P-worlds if Q is simply the negation of P).[24] And yet x may not be disposed in either context to act in ways that would satisfy his desires in P & Q-worlds explainable by either belief state. This is because in the first context where he is disposed to act in accordance with state s_1 he may not be disposed to act in ways that bring about his desires in Q-worlds and in the second context where he is disposed to act in accordance with state s_2 he may not be disposed to act in ways to satisfy his desires in P-worlds.

This part of Stalnaker's solution to the problem of deduction can also be regarded as a kind of solution to the Kripke puzzle discussed in chapters one and three. In the puzzling Pierre case, assuming the disquotation and translation principles, it would seem that we have a situation where the

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23 The belief state explains this action since P obtains at every world in that state.

24 Stalnaker (1984), p. 83.

agent Pierre has inconsistent beliefs, viz., that London is pretty and that London is not pretty, at least if we maintain that the indetectability of certain inconsistencies is a reason for (rather than against) holding that agents can sometimes hold inconsistent beliefs. Further, if we agree with Marcus' reality restriction discussed in chapter one, then Pierre would presumably *not* believe that London is pretty and London is not pretty. I.e., this is a case where an agent x believes that α and x believes that $\sim\alpha$ thereby violating the consistency condition for belief states. This is also a case where the agent fails to conjoin these contradictory beliefs thereby violating the conjunction condition for belief states.

So what does puzzling Pierre believe if we were to adopt Stalnaker's view that agents can be in more than one belief state? He believes that London is pretty and he believes that London is not pretty. The content 'London is pretty' obtains at all the members of one belief state s_1 and the content 'London is not pretty' obtains at all the members of a different belief state, s_2. The self-contradictory state of affairs that London is both pretty and not pretty obtains at no member of either s_1 or s_2 and hence Pierre does not believe that London is both pretty and not pretty. Or given a pragmatic account of belief, Pierre is disposed to act (verbally) in two different ways in two different contexts (the one context being when he is speaking French and the other being when he is an English speaker in London) as a result of being in incompatible belief states. Further, Pierre does not conjoin his beliefs as long as he remains in these separate belief states. He cannot be disposed to act in accordance with some belief state such that his actions would tend to satisfy his desires in 'impossible' worlds where a self-contradictory state of affairs obtains.

Stalnaker would argue here that the Kripke puzzle and any supposed counterexample to the deductive constraints 2) and 3) do not impugn these constraints qua defining conditions on belief *states*. What for example the puzzling Pierre case shows is that if we try to impose these two deductive constraints on the *totality* of the agent's beliefs, then we run into trouble. In the puzzling Pierre case, his two states explaining his dispositions to act in different sorts of contexts are internally consistent and presumably closed under conjunction. But because they are incompatible with one another then when we come to consider the totality of Pierre's beliefs, certain inconsistencies arise, such as his believing that London is pretty and also believing that London is not pretty.

We have so far discussed the first part of Stalnaker's strategy of dealing with apparent counterexamples to the second and third deductive constraints on belief and acceptance states. I.e., he stipulates that agents can be in more than one belief state and then argues that these constraints break down when we misapply them to the totality of agents' beliefs. So, qua defining conditions of belief *states*, conditions 2) and 3) which are the claims that belief states are internally consistent and are closed under conjunction remain intact.

However, the strategy of allowing agents to be in more than one belief state will not answer the various counterexamples to the first deductive constraint on belief states, viz., that states are closed under logical consequence. This is because if x believes that P then for some state s_i, P will be true at every world in s_i. Further, if $\vdash P \supset Q$ then the $P \supset Q$ will be true at every world in every belief state. Then for any state s_i such that P is true at every world in s_1 (and hence x believes that P) it will also be the case that Q is true at every world in s_1. Thus, x also believes that Q. So clearly, another approach is needed for dealing with the alleged counterexamples to the first deductive condition on belief states. We shall defer discussion of the second part of Stalnaker's solution to the problem of deduction vis a vis the closure under logical consequence condition for belief states until the final section. Suffice it to say that his solution simply involves making this condition more palatable.

Finally, before attempting to formalize Stalnaker's notion that agents can be in more than one belief state, it is important to note that he does not think that this strategy will adequately answer the preface paradox discussed in chapter one. The preface paradox was a case where the author of a narrative believed each statement in the narrative separately although he believed that their conjunction was false. The preface case bears directly on the plausibility of the schema $(B\alpha \ \& \ B\beta) \supset \sim B\sim(\alpha \ \& \ \beta)$ which is contained in any normal doxastic system containing D. Further, this case is also relevant to the adjunction schema for systems containing D since for such systems, $\vdash B\sim\alpha \supset \sim B\alpha$. However, Stalnaker does not see the preface paradox as bearing merely on the plausibility of condition 2) for belief states qua defining condition.

Rather, he maintains that even if a non-ideal agent is aspiring towards possessing an integrated system of beliefs (in which case he/she would be in one belief state), it may not be warranted that he always conjoin his beliefs. The apparent moral to be drawn from the preface paradox seems to be that it would be unwarranted for the agent to conjoin his beliefs in this case, if he has reason to believe that there will be at least one false statement in his narrative. So the preface paradox does not merely impugn the conjunctive closure constraint qua defining condition for belief states. In addition, it apparently impugns the conjunctive closure constraint qua 'rationality condition'.[25]

What is implied in Stalnaker's remarks just discussed is that there are two ways in which the deductive constraints on belief states, 2) - 3) can be regarded. They can first of all be thought of as defining conditions for belief states. Qua defining conditions of belief states they cannot be applied to the totality of an agent's beliefs assuming that the agent in question may not have integrated his various belief states. Thus, a way of resolving the Kripke puzzle is to resist applying the deductive constraints on belief states to the totality of Pierre's beliefs. Second, the deductive constraints 2) - 3) can be regarded as 'rationality conditions' for potentially integrated systems of beliefs[26]. I.e., the 'ideal' believer that we discussed in chapter one would have an integrated system of beliefs - he would be in one integrated belief state. Hence, the ideal believer would always conjoin his beliefs and finally he would not hold inconsistent beliefs. (As we noted in chapter one, the ideal believer will also be regarded as an agent who believes the logical consequences of what he believes.) So anyone aspiring to ideality as a believer will aspire to integrate his belief states into one system such that the deductive constraints 2) - 3) on belief states also apply to his integrated system. These constraints qua rationality conditions can be regarded as goals. However, the preface paradox seems to pose a case where even the non-ideal believer aspiring to ideality would be wise not to conjoin his/her beliefs, and so this impugns the conjunctive closure constraint qua rationality condition for non-ideal believers.

[25] Stalnaker (1984), p. 88.

[26] He alludes to this distinction in Stalnaker (1984), p. 84.

The solution which Stalnaker offers to the paradox of the preface exploits his earlier distincition between belief vs. acceptance (and belief states vs. acceptance states). Given this distinction, an agent x may accept that P without believing that P. He may tacitly presuppose that P is true for the sake of inquiry or he may simply take it for granted that P is true. But we would not be inclined to say in such a case that x *believes* that P. And, claims Stalnaker, this is what happens in the case of the preface. The author accepts (in the sense of tacitly presupposes) that the entire narrative is true and hence he merely *accepts* that any given statement in the narrative is true. But he does not *believe* of the whole story, or of any one statement, that it is true, since he is ready to abandon any of these in the light of new evidence:

> The explanation of the preface phenomenon that I am suggesting requires that we say that the historian does not, without qualification *believe* that the story he accepts is correct; nor does he believe without qualification, all of the individual statements he makes in telling the story . . .[27]

Thus, the preface paradox is not really a case where the conjunctive constraint qua rationality condition for integrated systems of beliefs fails since the author of the narrative does not really *believe* any of the statements in his narrative.

And so, in this section, we have discussed Stalnaker's solution to the problem of deduction for the conjunctive and consistency constraints on belief states qua defining conditions, which involves allowing agents to be in more than one belief state. Further, we have seen that he makes a distinction between defining conditions and rationality conditions which are two ways in which the deductive constraints discussed above can be re-

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[27] Stalnaker (1984), p. 94.

an agent can be in more than one belief state at the same time. Thus, what is needed is an alternative (truth-value) semantics for the Stal-SQC$^{=}$ + D systems which formalizes Stalnaker's notion that agents can be in several distinct belief states.

The intuitive idea behind this alternate semantics is this: We define a model in the usual way as consisting in part of a set W of indices. Since ideal belief is 'integrated', the ideal believer will be in one belief state consisting of *all* the doxastic alternatives to the world he inhabits. Thus, x *ideally* believes that α at w_i just in case α is true at all the alternatives to w_i. For technical purposes, instead of defining 'alternativeness' in terms of a two place relation R such that w_j is an alternative to w_i just in case $\langle w_i, w_j \rangle \in R$, we shall define it equivalently in terms of a function f from W into **P**W.[28] I.e., f will assign to each index a set of indices which can be thought of as the alternatives to that index. These two ways of defining 'alternativeness' are equivalent since for any index w_i, we can define f in terms of R as follows: $f(w_i) = \{w_j \mid w_i R w_j\}$.[29] Thus, the semantics in which we use f instead of R to define 'alternativeness' can still be regarded as a relational semantics since at any time we could dispense with f in favour of R. Finally, the restrictions we would impose on R such as seriality can also be mirrored by f. Thus, we could represent the seriality restriction for f as follows: For any w_i in W, $f(w_i) \neq \emptyset$.

A third element in a Stal-SQC$^{=}$ model in this alternate semantics will be the set S where the members of S are sets of 'belief states'. Each belief state is itself a set of indices. For each member S_j of S (i.e., for each set of belief states) there will correspond exactly one member of W, w_j. The

[28] See chapter one, p. 30 of this dissertation.

[29] See Chellas (1980), p. 74.

members of each belief state in S_j will be drawn from the alternatives assigned to the corresponding index w_j by f. I.e., each S_j will be the set of all subsets (i.e., the 'power set') of $f(w_j)$ in which case, each state s in S_j will be a set of alternatives to w_j. (i.e., $s \subseteq f(w_j)$.) Further, since for *doxastic* logics, indices are not necessarily alternatives to themselves, then it is not a requirement that $\{w_j\} \in S_j$. We shall use two subscripts for any member s of each set of states S_j, where the first subscript j will denote from which set of states s was drawn and the second subscript k will simply number the state (in the same way that the indices in W are numbered). Thus, 's_{jk}' can be read as 'the kth member of S_j'.

The purpose of restricting the members of each state in any S_j to the alternatives of the corresponding index, w_j is to ensure an interdependence between ideal belief and non-ideal belief - intuitively, the non-ideal believer will partition the set of alternatives to the index he inhabits into distinct states whereas the ideal believer will integrate all the alternatives into one system. Also, this ensures the validity of the equivalence $\mathbf{B}\alpha \equiv {\sim}\mathbf{B}_I{\sim}\alpha$ which states that non-ideal belief is definable in terms of ideal belief. The definability of non-ideal belief in terms of ideal belief is important for establishing completeness results, as will soon become evident.

The set S of sets of belief states, S_j, S_k, etc. will figure into the truth-conditions for non-ideal belief as follows:

> x *non-ideally* believes that α at w_i just in case for at least one non-empty belief state s_{ik} such that $s_{ik} \subseteq f(w_i)$, α is true at every index in that state.

I.e., x non-ideally believes that α at w_i just in case for at least one non-empty belief state whose members are all alternatives to w_i, α is true at each member of that state. Thus, in terms of the semantics, non-ideal belief is still treated analogously to alethic possibility since x non-ideally believes that α at w_i only if there is *at least one* state such that α is true at all members of that state. Therefore, it is not required that the content of non-ideal belief is true at *every* alternative to the index in question but merely that it be true at *every* alternative in some state. Further, the truth-conditions for non-ideal belief jibe with the truth-conditions for non-ideal belief in Stalnaker's informal semantics and this was our aim - to make the notion of belief state and its role in defining non-ideal belief more conspicuous.

Finally, the fourth element of a Stal-SQC$^=$ model in this alternate semantics will be the assignment function V which as usual assigns to atomic wffs independently of any domain of 'individuals' either '1' or '0'. And V will have the two restrictions imposed on it as for the standard semantics for the Stal-SQC$^=$ systems. As usual, a valuation over a model V_M will be defined inductively with $V(\alpha, w_i) = V_M(\alpha, w_i)$ for α atomic as the basis. We shall now provide a more formal characterization of this semantics.

A Stal-SQC$^=$ model will be defined as a 4-tuple, $\langle W, f, S, V \rangle$ such that the elements W, f, S and V are defined as follows:

1) $W \neq \emptyset$ (i.e., W is a non-empty set of indices.)

2) $f: W \longrightarrow PW$ (i.e., for each w_i in W, $f(w_i) \subseteq W$)

3) S is a set of 'belief states' where each $S_j \in S = Pf(w_j)$ for exactly one $w_j \in W$. (Then each $s_{jk} \in S_j$ is such that $s_{jk} \subseteq f(w_j)$.)[30]

4) V: Atomic Wffs X W $\longrightarrow \{0,1\}$ such that:

i) If α is $t = t$ then for all $w_i \in W$, $V(\alpha, w_i) = 1$.

ii) For all $w_i \in W$, if $V(t_1 = t_2, w_i) = 1$ then $V(\alpha(t_1/v), w_i) =$

[30] Stalnaker does not require that belief states are sets of *alternatives* to some index. We are here attempting to formalize his semantics for belief with the context of a relational semantics.

$V(\alpha\,(t_2/v), w_i)$.

A valuation over any such model, V_M:Wffs X W $\longrightarrow \{0,1\}$ is defined inductively as follows:

Basis: $V(\alpha, w_i) = V_M(\alpha, w_i)$ for any α atomic.

Suppose that $V_M(\alpha, w_i)$ and $V_M(\beta, w_i)$ are both defined. Then,

a) $V_M(\sim\alpha, w_i) = 1$ iff $V_M(\alpha, w_i) = 0$.

b) $V_M(\alpha \;\&\; \beta, w_i) = 1$ iff $V_M(\alpha, w_i) = V_M(\beta, w_i) = 1$.

c) $V_M((\forall v)\alpha, w_i) = 1$ iff $V_M(\alpha\,(t/v), w_i) = 1$ for all $t \in$ CONS

d) $V_M(B_I\alpha, w_i) = 1$ iff for all $w_j \in W$ such that $w_j \in f(w_i)$,

$$V_M(\alpha, w_j) = 1.$$

e) $V_M(B\alpha, w_i) = 1$ iff for at least one non-empty $s_{ik} \in S_i$ such that $s_{ik} \subseteq f(w_i)$, $V_M(\alpha, w_j) = 1$ for all $w_j \in s_{ik}$.

Further, *validity in a model* of the sort described above will be truth at all indices in the model and *validity in the appropriate class of models* (determined by the restrictions imposed on f) is validity in all models in the class.

Now that we have provided a somewhat formal description of our alternate semantics for the Stal-SQC$^=$ systems, we shall see whether or not this semantics *characterizes* the Stal-SQC$^=$ systems.

First of all, it needs to be shown that any given Stal-SQC$^=$ system is *sound* relative to the appropriate class of Stal-SQC$^=$ models of the sort just described. And as usual, soundness is established by demonstrating that all instances of the axiom-schemata are valid and that the rules of inference preserve validity in the appropriate class of models. We shall not set out to prove this here since it parallels the proof of soundness for the Sub-SQC$^=$ systems.

However, we shall consider the status of the following crucial schemata and rule of inference with respect to the type of semantics we are proposing:

i) $\mathbf{B}\alpha \equiv \sim\mathbf{B_I}\sim\alpha$

ii) $\mathbf{B_I}\alpha \equiv \sim\mathbf{B}\sim\alpha$

iii) $(\mathbf{B}\alpha \ \& \ \mathbf{B}\beta) \supset \mathbf{B}(\alpha \ \& \ \beta)$

iv) $\sim(\mathbf{B}\alpha \ \& \ \mathbf{B}\sim\alpha)$

v) $(\mathbf{B}\alpha\,(t_1/v) \ \& \ t_1 = t_2 \ \& \ \mathbf{B}(t_1 = t_2)) \supset \mathbf{B}\alpha\,(t_2/v)$

vi) $\vdash\alpha \supset \beta \longrightarrow \vdash\mathbf{B}\alpha \supset \mathbf{B}\beta$

If the two equivalences i) and ii) are both valid then **B** and $\mathbf{B_I}$ are interdefinable. (We could consistently add i) and ii) as axiom-schemata to the Stal-SQC$^=$ systems.) The interdefinability **B** and $\mathbf{B_I}$ will be important later on in terms of establishing completeness.

To establish the validity of all instances of i), $\mathbf{B}\alpha \equiv \sim\mathbf{B_I}\sim\alpha$, suppose that for some Stal-SQC$^=$ + D model M and for some index w_i, $V_M(\mathbf{B}\alpha, w_i) = V_M(\mathbf{B_I}\sim\alpha, w_i) = 1$. Then for some non-empty s_{ik} in S_i such that $s_{ik} \subseteq f(w_i)$, $V_M(\alpha, w_j) = 1$ for all w_j in s_{ik}. Supposing further that $V_M(\mathbf{B_I}\sim\alpha, w_i) = 1$, then for *all* w_j in W such that $w_j \in f(w_i)$, $V_M(\sim\alpha, w_j) = 1$ which contradicts one of the consequences of our supposition that $V_M(\mathbf{B}\alpha, w_i) = 1$. Or, on the other hand, suppose that $V_M(\mathbf{B_I}\sim\alpha, w_i) = V_M(\mathbf{B}\alpha, w_i) = 0$. Then there is at least one w_j in $f(w_i)$ such that $V_M(\alpha, w_j) = 1$. But since w_j is in $f(w_i)$, then $\{w_j\} \in Pf(w_i)$. And since each $s_{ik} \in Pf(w_i)$ then there will be some s_{ik} in S_i such that $s_{ik} = \{w_j\}$ in which case there is an $s_{ik} \subseteq f(w_i)$ such that $V_M(\alpha, w_j) = 1$ for all members of s_{ik}. Hence, $V_M(\mathbf{B}\alpha, w_i) = 1$ which contradicts our earlier supposition that $V_M(\mathbf{B}\alpha, w_i) = 0$. Q.E.D.

To establish the validity of all instances of ii), $\mathbf{B_I}\alpha \equiv \sim\mathbf{B}\sim\alpha$, suppose that there is a Stal-SQC$^{=}$ + D model M such that $V_M(B_I\alpha, w_i) = V_M(B{\sim}\alpha, w_i) = 1$. Then for all w_j in W such that $w_j \in f(w_i)$, $V_M(\alpha, w_j) = 1$. If $V_M(B{\sim}\alpha, w_i)$ is 1 then there is at least one $s_{ik} \subseteq f(w_i)$ such that $V_M({\sim}\alpha, w_j) = 1$ for all w_j in s_{ik}. But this contradicts one of the consequences of our supposition that $V_M(B_I\alpha, w_i) = 1$. On the other hand, suppose that $V_M(B{\sim}\alpha, w_i) = 0 = V_M(B_I\alpha, w_i)$. If $V_M(B{\sim}\alpha, w_i) = 0$, then there is no $s_{ik} \subseteq f(w_i)$ such that $V_M({\sim}\alpha, w_j) = 1$ for all w_j in s_{ik}. But if $V_M(B_I\alpha, w_i) = 0$ then there is at least one w_j in $f(w_i)$ such that $V_M(\alpha, w_j) = 0$ and hence $V_M({\sim}\alpha, w_j) = 1$. But then there is some s_{ik} in S_i such that $s_{ik} = \{w_j\}$ in which case there is some s_{ik} such that $V_M({\sim}\alpha, w_j) = 1$ for all w_j in s_{ik}. And from this it follows that $V_M(B{\sim}\alpha, w_i) = 1$ which contradicts our earlier assumption that $V_M(B{\sim}\alpha, w_i) = 0$. Q.E.D.

Now that we have established that the equivalences i) and ii) are both valid, it follows that the operators B_I and B are interdefinable for the Stal-SQC$^{=}$ + D systems. We shall next show that the schemata iii) and iv) have invalid instances, thus showing that this semantics does not presuppose that non-ideal believers always conjoin their beliefs nor presupposing that agents always have consistent beliefs.

Consider the following instance of iii), viz., $(BFa \,\&\, BGb) \supset B(Fa \,\&\, Gb)$. Consider the Stal-SQC$^{=}$ + D model M such that $W = \{w_1, w_2, w_3\}$ and such that $f(w_1) = \{w_2, w_3\}$. Then, $S_1 = Pf(w_1) = \{\emptyset, \{w_2\}, \{w_3\}, \{w_2, w_3\}\}$. Let $s_{11} = \{w_2\}$ and let $s_{12} = \{w_3\}$ and let $s_{13} = \{w_2, w_3\}$. Let $V(Fa, w_2) = 1$, $V(Fa, w_3) = 0$, $V(Gb, w_2) = 0$ and $V(Gb, w_3) = 1$. Since V_M is V for atomic wffs, then these assignments also hold for V_M. Then $V_M(Fa \,\&\, Gb, w_2) = V_M(Fa \,\&\, Gb, w_3) = 0$. Thus, there is no non-empty $s_{1k} \subseteq f(w_1)$ such that for every member w_j of s_{1k}, $V_M(Fa \& Gb, w_j) = 1$. Thus, $V_M(\mathbf{B}(Fa \& Gb), w_1) = 0$. However, $V_M(\mathbf{B}Fa, w_1) = 1$ since $V_M(Fa, w_2) = 1$ such that $s_{11} = \{w_2\}$ and also, $V_M(\mathbf{B}Gb, w_1) = 1$ since $V_M(Gb, w_3) = 1$ such that $s_{12} = \{w_3\}$. Q.E.D. Notice that this model is also a countermodel to $\sim(\mathbf{B}Fa \& \mathbf{B}{\sim}Fa)$ which is an instance of the consistency schema iv).

As was noted in section 1, substitution of co-referentials for the Stal-SQC$^=$ + D systems for non-ideal doxastic contexts is restricted to cases such that the agent x *ideally* believes that the relevant identity obtains. It was also promised that some sort of explanation would be offered on the semantic front as to why it is not sufficient that the agent merely *non-ideally* believes that the relevant identity obtains for substitution to go through for non-ideal doxastic contexts. This amounts to showing that some instance of $(\mathbf{B}\alpha(t_1/v) \& t_1 = t_2 \& \mathbf{B}(t_1 = t_2)) \supset \mathbf{B}\alpha(t_2/v)$, which says that substitution of co-referentials goes through for non-ideal doxastic contexts provided the agent *non-ideally* believes that the relevant identity obtains, is invalid in the sort of semantics being considered.

Consider the following instance of $(\mathbf{B}\alpha(t_1/v) \& t_1 = t_2 \& \mathbf{B}(t_1 = t_2)) \supset \mathbf{B}\alpha(t_2/v)$, viz., $(\mathbf{B}Fa \& a = b \& \mathbf{B}(a = b)) \supset \mathbf{B}Fb$. It will be shown that there is a Stal-SQC$^=$ + D model such that this wff is invalid. In fact, we shall employ the same model-structure employed above in invalidating the instances of the non-ideal adjunction and consistency schemata. I.e., $W = \{w_1, w_2, w_3\}$, $f(w_1) = \{w_2, w_3\}$ and $S_1 = Pf(w_1) = \{\emptyset, \{w_2\}, \{w_3\}, \{w_2, w_3\}\}$. As before, let $s_{11} = \{w_2\}$ and let $s_{12} = \{w_3\}$ and let $s_{13} = \{w_2, w_3\}$. Let $V(Fa, w_2) = 1$ and $V(Fb, w_2) = V(a = b, w_1) = 0$. Then $V_M(Fa, w_2) = 1$ and $V_M(a = b, w_1) = V_M(Fb, w_2) = 0$. Further, let $V(Fa, w_3) = V(Fb, w_3) = 0$ and $V(a = b, w_3) = 1$. Then $V_M(Fa, w_3) = V_M(Fb, w_3) = 0$ and $V_M(a = b, w_1)$

= 1. Since for all members of s_{11}, Fa is assigned '1' by V_M it follows that $V_M(\mathbf{B}Fa, w_1) = 1$. Since for all members of s_{12}, a = b is assigned '1' by V_M it follows that $V_M(\mathbf{B}(a = b), w_1) = 1$. Finally, because there is no non-empty s_{1k} in S_1 such that Fb is assigned '1' by V_M for all members of s_{1k} then it follows that $V_M(\mathbf{B}Fb, w_1) = 0$. Q.E.D.

Informally, what this countermodel suggests is that it is on the condition that the relevant identity $t_1 = t_2$ obtains at *all* alternatives to an index w_i in a model that we are guaranteed that if $\alpha\,(t_1/v)$ is true at all members of some belief state then so is $\alpha\,(t_2/v)$. In other words, provided the agent x *ideally* believes that $t_1 = t_2$ obtains then if x believes that $\alpha\,(t_1/v)$, x also believes that $\alpha\,(t_2/v)$. The agent's merely *non-ideally* believing that $t_1 = t_2$ obtains leaves open the possibility that $t_1 = t_2$ fails to obtain at any alternative at which $\alpha\,(t_1/v)$ obtains. Then this situation in turn leaves open the possibility that there is no alternative at which $\alpha\,(t_2/v)$ obtains.

What remains to be discussed is the implicational version of the logical omnidoxasticity rule vi), i.e., $\vdash \alpha \supset \beta \longrightarrow \vdash \mathbf{B}\alpha \supset \mathbf{B}\beta$. It will be shown that the semantics we are considering for the Stal-SQC* + D systems presupposes that non-ideal believers are logically omnidoxastic by establishing the validity preservingness of the above rule. Suppose that $\models \alpha \supset \beta$. Then for any model M and for every index w_i in that model, $V_M(\alpha \supset \beta, w_i) = 1$. Suppose however that $V_M(\mathbf{B}\alpha \supset \mathbf{B}\beta, w_i) = 0$ for at least one w_i for some model M. Then $V_M(\mathbf{B}\alpha, w_i) = 1$ but $V_M(\mathbf{B}\beta, w_i) = 0$. If $V_M(\mathbf{B}\alpha, w_i) = 1$ then for at least one non-empty $s_{ik} \subseteq f(w_i)$, $V_M(\alpha, w_j) = 1$ for all $w_j \in s_{ik}$. And if $V_M(\mathbf{B}\beta, w_i) = 0$ then for every non-empty $s_{ik} \subseteq f(w_i)$, $V_M(\beta, w_j) = 0$ for at least one w_j in s_{ik}. However, by supposition that $V_M(\mathbf{B}\alpha, w_i) = 1$, it was concluded that there is at least one s_{ik} such that $V_M(\alpha, w_j) = 1$ for every w_j in s_{ik}. But by supposing that $V_M(B\beta, w_i) = 0$, this s_{ik} will be such that for at least one w_j in s_{ik}, $V_M(\beta, w_j) = 0$. But $V_M(\alpha, w_j) = 1$ at this same member of s_{ik} and by another supposition, $V_M(\alpha \supset \beta, w_j) = 1$. And this leads to a contradiction. Q.E.D.

Completeness of the Stal-SQC$^=$ systems relative to the type of semantics we have proposed is established in the usual way by the method of canonical models. A Stal-SQC$^=$ canonical model $\mathcal{M}$ is a 4-tuple, $\langle W, f, S, V \rangle$ such that:

a) $W = \{w_i \mid w_i$ is a maximal consistent set of wffs with the $\exists$-property$\}$

b) For any w_i in W, $f(w_i) = \{w_j \in W \mid (\forall A)(B_I\alpha \in w_i \longrightarrow \alpha \in w_j)\}$.

c) $S = \{S_j \mid (\exists! w_i)(w_i \in W \;\&\; S_j = Pf(w_i))\}$

d) For any atomic wff α, $V(\alpha, w_i) = 1$ iff $\alpha \in w_i$.

Using as the basis $V(\alpha, w_i) = V_{\mathcal{M}}(\alpha, w_i)$ from which it follows given d) that $V_{\mathcal{M}}(\alpha, w_i) = 1$ iff $\alpha \in w_i$ for α atomic, we prove the *fundamental theorem* of canonical models, i.e., for any wff α, $V_{\mathcal{M}}(\alpha, w_i) = 1$ iff $\alpha \in w_i$, by induction on the complexity of α. The inductive proof proceeds along the same lines as it did for the Sub-SQC$^=$ systems.

Further, the case where α is of the form $B_I\beta$ proceeds in the same way as the case where α is of the form $B\beta$ for the Sub-SQC$^=$ systems given that the truth-conditions for wffs of the form $B_I\beta$ for the Stal-SQC$^=$ systems are identical to the truth-conditions for wffs of the form $B\beta$ for the Sub-SQC$^=$ systems. Finally, it was shown that for the Stal-SQC$^=$ systems, the non-ideal belief operator **B** can be defined in terms of the ideal belief operator B_I with the latter taken as primitive. Thus, once it has been established that the fundamental theorem holds for wffs of the form $B_I\beta$ and given that it holds for the primitive connectives and a primitive quantifer, it follows that the fundamental theorem holds for *all* wffs including wffs of the form $\mathbf{B}\beta$.

Finally, once the fundamental theorem has been established it remains to be shown that the canonical model for the given Stal-SQC$^{=}$ system is in the class of models with respect to which that system is sound. And this is shown by proving that f meets the appropriate restrictions.

And so, we have attempted to formalize Stalnaker's model that agents can be in more than one belief state (thereby explaining why belief is not deductively closed) on the semantic front. To make Stalnaker's notion of belief states more conspicuous, we considered an alternative semantics for the Stal-SQC$^{=}$ systems which factored into the truth-conditions for non-ideal belief wffs, non-empty sets of indices - intuitively, belief states - such that these indices are amongst the alternatives to the given index, thus ensuring an interdependence between ideal and non-ideal belief.

Although the Stal-SQC$^{=}$ + D systems of doxastic logic are the closest we have come to modal logics capturing principles of belief attribution for the non-ideal believer, we are still left with the presupposition that agents are logically omnidoxastic. In the final section, we shall reconsider the problem of logical omnidoxasticity arguing that it is intractable for logics of belief with relational semantics. However, in the next section, we shall briefly consider as a plausible alternative to Stalnaker's informal proposal that agents can be in more than one belief state, a formal proposal of Rescher's which involves the supposition that non-ideal belief is a relation between an agent and a so-called superposed world.

4. When Worlds Collide

In defining belief as a relation between an agent and possibly many sets of worlds or belief states, Stalnaker is able to construct a possible worlds semantics of belief which does not presuppose that agents always conjoin their beliefs or that their beliefs are always consistent. Linked to his view that the objects of belief are (possibly disjoint) sets of worlds is his causal-pragmatic account of belief and belief attribution. I.e., a necessary condition for attributing to x the belief that α is that x be disposed in one sort of context to act in ways that will tend to satisfy his desires at all α-worlds - which suggests that the objects of beliefs are sets of worlds. Then if an agent acts in incompatible ways in two different contexts (for example he may act in ways that satisfy his desires in α worlds and in another context, he may act in ways that satisfy his desires in $\sim\alpha$ worlds - as in the puzzling Pierre case), this is explainable by his being related to two disjoint sets of worlds or belief states.

In *The Logic of Inconsistency*, Nicholas Rescher et al offer a solution to the problem of deduction (but not to the problem of logical omnidoxasticity) in the same vein as Stalnaker's solution. For Rescher, belief involves a dispostion to assent to a statement. Further, it is possible that an agent can have various dispositions to assent in *different* contexts. His account of belief attribution is not incompatible with Stalanker's since he gives a dispositional analysis, thus making it a 'pragmatic' account, although the relevant types of actions are speech acts, viz., verbal assent (thus bearing similarities to Kripke's disquotation principle). So, if an agent can be disposed to act in different ways in different situations, then this leaves open the possibility that the agent may be disposed to assent to α in one context (thereby believing that α) and that he may be disposed to assent to $\sim\alpha$ in another context (thereby believing that $\sim\alpha$), without the agent thereby believing that α & $\sim\alpha$:

> At a more mundane level, it is common for an individual to be simultaneously disposed to assent to a statement if queried and to be disposed to assent to the denial of that statement if queried in some variant context (once again, this does not mean that one is ever disposed to assent to the conjunction of a statement and its denial).[31]

Thus, he might explain the Kripke puzzle along Stalnakerian lines by arguing that puzzling Pierre was disposed to assent to 'Londres est jolie' in one context and he was disposed to assent to 'London is not pretty' in another, without his thereby believing that London is both pretty and not pretty.

To avoid the sort of situation where an agent who has incompatible beliefs in different contexts ends up believing a contradiction, Rescher imposes the following restriction on any agent's set of beliefs: The set of believed statements must be 'minimally consistent' in the sense that it is at most *weakly* inconsistent.[32] A set of statements is *weakly* inconsistent just in case for some wff α and for some world w, α is true at w and $\sim\alpha$ is true at w, but α & $\sim\alpha$ is true at no w.[33] *Strong* inconsistency is defined as follows: For some wff α, and for some world w, α & $\sim\alpha$ is true at w - hyperinconsistency occurs when for *every* wff α and for some world w, α & $\sim\alpha$ is true at w.[34]

According to Rescher (and as argued in chapter one), if the 'minimal

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31 Rescher et al (1980), p.101.

32 Rescher et al (1980), p. 100.

33 ibid, p. 25.

34 ibid, p. 24.

consistency' restriction were not imposed on an agent's set of beliefs, then since a contradiction logically implies everything, any agent believing that α & $\sim\alpha$ would thereby believe everything.[35] So, puzzling Pierre's set of beliefs must satisfy the minimal consistency requirement to avoid the consequence that he believes everything. Thus, if he believes that London is pretty and he believes that London is not pretty, he does not thereby believe that London is both pretty and not pretty. So for Rescher (as for Marcus), assent carries over into belief only if what is assented to is *self-consistent*.

It is our task in the remainder of this section to examine Rescher's formal proposal for making sense out of the above type of situation. I.e., we shall consider his suggestions for a semantics of belief which allows that agents can have contradictory beliefs (whose contents are assented to in different contexts) and which allows that agents can fail to conjoin beliefs, but which requires that an agent's system of beliefs be minimally consistent. In short, Rescher is offering an alternative account as to how it is possible that agents can hold incompatible sets of beliefs in different contexts. Finally, it will then be argued that Rescher's semantic proposals can be adopted to provide a characteristic semantics for the Stal-$SQC^{=}$ + D systems.

The formal semantics of belief which Rescher develops rests on the assumption that belief is a relation between a believer (at a world) and a non-standard world.[36] The contents of the agent's beliefs would all hold at this world. Depending on the type of non-standardness we are considering, the contents true at this world may include for some wff α, both α and

[35] ibid, p. 102.

[36] Rescher et al (1980), p. 105.

$\sim\alpha$, but not $\alpha \;\&\; \sim\alpha$ thus meeting his minimal consistency requirement. Further, the non-standard world to which the believer at a world is related may be such that even though α and β both obtain, it is possible that $\alpha \;\&\; \beta$ fails to obtain. Then in this type of semantics, if what we have called non-ideal belief is a relation between a believer at a world and the right sort of non-standard world, the following schemata (couched in the language of our Stal-SQC$^{=}$ + D systems) would be invalid:

i) $(\mathbf{B}\alpha \;\&\; \mathbf{B}\beta) \supset \mathbf{B}(\alpha \;\&\; \beta)$ non-ideal adjunction

ii) $\sim(\mathbf{B}\alpha \;\&\; \mathbf{B}\sim\alpha)$ non-ideal consistency

And the following would be valid in this type of semantics:

iv) $\sim\mathbf{B}(\alpha \;\&\; \sim\alpha)$ non-ideal self-consistency

We shall now see exactly how Rescher spells out 'non-standardness'.

In Rescher's semantics, there are two types of non-standard worlds, viz., 'schematic' and 'superposed'. Both types of non-standard worlds are constructed by means of 'world-fusion' with standard worlds as the initial basis of the fusion. Thus, a belief model would consist of a set of standard worlds and a set of non-standard worlds ultimately constructable from the standard ones by means of world-fusion. Consider the simplest sort of case where a non-standard world is constructed out of two standard ones. Then the two types of world-fusion are as follows:[37]

1) *World conjunction* - Given two standard worlds w_i and w_j, $w_i \Cap w_j$ is the world such that for any proposition α, α is true at $w_i \Cap w_j$ just in case α is true at both w_i and w_j.

2) *World disjunction* - Given two standard worlds w_i and w_j, $w_i \Cup w_j$ is the world such that for any proposition α, α is true at $w_i \Cup w_j$ iff α is true at either w_i or w_j.

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[37] Rescher et al (1980), pp. 9 - 11.

Non-standard worlds formed by the operation of world conjunction are called 'schematic' and non-standard worlds formed by the operation of world disjunction are 'superposed'.

As Rescher notes, *schematic* worlds will be incomplete in the sense that for some proposition α and its negation $\sim\alpha$, it is possible that neither α nor $\sim\alpha$ will obtain.[38] Thus, in the simplest sort of case, if α is true at w_i and false at w_j and if $\sim\alpha$ is false at w_i but true at w_j then neither α nor $\sim\alpha$ will obtain at the schematic world $w_i \cap w_j$. What is non-standard about so-called *schematic* worlds is their incompleteness. Standard possible worlds will be such that for any proposition α, either α or $\sim\alpha$ obtains (but not both). However, for a schematic world $w_i \cap w_j$, it is never the case that α and $\sim\alpha$ can both obtain at such a world since this would mean that α and $\sim\alpha$ are both true at both w_i and w_j. But this is impossible since schematic worlds are ultimately constructed out of standard worlds where the connectives are defined classically. Further, for any schematic world $w_i \cap w_j$, if α and β obtain then so must their conjunction $\alpha \ \& \ \beta$, since α and β must each obtain at both w_i and at w_j.

Since *ideal* believers cannot hold contradictory beliefs and do not necessarily conjoin their beliefs, then ideal belief in Rescher's semantics could be regarded as a relation between a believer at a world and a schematic world (constructed out of the doxastic alternatives to the world the agent inhabits). Thus, ideal belief is a relation to non-standard worlds which are schematic. This idea will be explored further below when we come to consider Rescher's proposals vis a vis a characteristic semantics for the Stal-SQC$^{=}$ + D systems.

On the other hand, so-called *superposed* worlds will be weakly in-

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[38] Ibid, p. 9.

consistent in the sense that it is possible that for some proposition α, both α and $\sim\alpha$ may obtain. Thus, if α is true at w_i and if $\sim\alpha$ is true at w_j then both α and $\sim\alpha$ will be true at $w_i \uplus w_j$. And generally, for any wffs α and β, if α is true at w_i but false at w_j and β is false at w_i but true at w_j then α & β is false at both w_i and w_j and hence α & β is false at $w_i \uplus w_j$. However, α and β are individually (though not conjointly) true at $w_i \cup w_j$ since α is true at w_i and β is true at w_j.

Then the kind of non-standard world that would best serve as the relatum of *non-ideal* belief is the superposed world. What is non-standard about superposed worlds is that unlike standard worlds, they are weakly (though not strongly) inconsistent. Further, the conjunctions of propositions or more neutrally statements true at superposed worlds need not themselves be true. And this makes them suitable relata of belief if we wish to allow for non-ideality. What is being suggested is that if x believes that α is true (at some world - perhaps the actual world) then α must be true at a non-standard world formed by *superposition*. Thus, it is possible that x beleives that α and that x believes that $\sim\alpha$ (since α and $\sim\alpha$ may both obtain at the appropriate superposed world) without thereby believing that α & $\sim\alpha$ (since superposed worlds are minimally consistent).

However, superposed worlds are not non-standard in the sense that Rantala's non-normal indices are. For one thing, superposed worlds although weakly inconsistent are not *strongly* inconsistent since ultimately they are formed from standard worlds as their basis, and so there will never be a standard world where α & $\sim\alpha$ obtains. The same can be said of schematic worlds. Also, if the relata of non-ideal belief are superposed worlds then since α & $\sim\alpha$ obtains at no superposed world, it will never be the case in this type of semantics that x (non-ideally) believes that α & $\sim\alpha$ – i.e., there will be no superposed world where this content obtains. Thus, no instance of the schema $B(\alpha \ \& \sim\alpha)$ is satisfiable and hence $\sim B(\alpha \ \& \sim\alpha)$ will be valid in this type of semantics.

Another reason that superposed worlds are not non-standard in the sense that Rantala's non-normal worlds are is that since superposed worlds are constructed ultimately from standard ones and since every thesis of the appropriate system of logic will be true at each standard world, then every thesis will be true at every superposed world. And the same can be said of schematic worlds. But given this feature of superposed worlds, it follows that agents will be omnidoxastic with respect to all theses of the appropriate system. I.e., if belief is a relation between a believer and a non-standard possible world, then since all theses are true at any non-standard world, every agent will believe all theses. This incidentally is another one of the intuitive requirements which Rescher imposes on agents' systems of beliefs in addition to the minimal consistency condition.[39]

The following will be 'semantic' prinicples which hold for standard or normal worlds but which fail for superposed worlds: (We shall let M designate some arbitrary model consisting of a set of standard and non-standard worlds, and V_M will be a valuation over any such model.)[40]

1) If $V_M(\alpha, w_i) = 1$ then it is not the case that $V_M(\sim\alpha, w_i) = 1$.

2) If $V_M(\alpha, w_i) = V_M(\beta, w_i) = 1$ then $V_M(\alpha \ \& \ \beta, w_i) = 1$.

It was explained earlier why these semantic principles break down for

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39 ibid, p. 100.

40 Rescher uses different notation. He discusses the status of these principles amongst others in section 5, pp. 15–20.

superposed worlds. Given that they do, it is evident how such a semantics can refute the *non-ideal* adjunction and consistency schemata. I.e., if a superposed world can be weakly inconsistent, then if there is some believer x related to such a world where α and $\sim\alpha$ both obtain at this world then x will believe that α and x will believe that $\sim\alpha$ (at some world). Thus, the non-ideal consistency schema $\sim(\mathbf{B}\alpha \ \& \ \mathbf{B}\sim\alpha)$ is refuted in this type of semantics. Further, given that condition 2) is violated for superposed worlds, it follows that if some believer x is related to a superposed world where α and β both obtain but $\alpha \ \& \ \beta$ fails to obtain then x will believe that α and x will believe that β but x will not believe that $\alpha \ \& \ \beta$ (at some world). Thus, the non-ideal adjunction schema $(\mathbf{B}\alpha \ \& \ \mathbf{B}\beta) \supset \mathbf{B}(\alpha \ \& \ \beta)$ is refuted in this type of semantics.

Similar to our Stalnakerian semantics for the Stal-SQC$^{=}$ + D systems, Rescher's proposed semantics for belief logic presupposes that agents non-ideally believe all the logical consequences of what they believe. The reason that the semantic principle that if $V_M(\alpha, w_i) = V_M(\alpha \supset \beta, w_i) = 1$ then $V_M(\beta, w_i) = 1$ fails for superposed worlds is that there could be two worlds w_i and w_j such that α is true at w_i, $\alpha \supset \alpha$ and β are false at w_i and such that $\alpha \supset \beta$ is true at w_j, α and β are both false at w_j. It follows from this that α and $\alpha \supset \beta$ are both true at the superposed world $w_i \cup w_j$ but β will be false at $w_i \cup w_j$.[41] However, the failure of this semantic principle for non-standard indices does not alter the fact that whenever $\vdash \alpha \supset B$, this principle holds. This is because $\alpha \supset \beta$ obtains for *every* world if $\vdash \alpha \supset \beta$ and so the type of situation mentioned above where the implication principle breaks down by virtue of $\alpha \supset \beta$ being false at a component world could not arise (even if the component world is non-standard).

[41] Rescher et al (1980), p. 19.

This situation of logical omnidoxasticity is stonger than Rescher's requirement for belief systems (in addition to the minimal consistency requirement) that "all sentences deducible from belief sentences by one-premise inferences of first-order logic is believed"[42] since in the type of semantics he is proposing, agents would believe *any* consequence deducible from believed sentences given the omnidoxasticity feature of his semantics.

And so, it would appear that Rescher has developed a semantics for belief logic which invalidates the non-ideal adjunction and consistency schemata, which does not allow agents to hold self-contradictory beliefs and which presupposes that agents believe (both ideally and non-ideally) all logical truths. But although Rescher's superposed worlds are not like Rantala's non-normal worlds in certain respects, they also resemble Rantala's non-normal worlds in the respect that '~' and '&' behave non-standardly for superposed worlds (as does '⊃'[43]). But then this feature of Rescher's semantics is open to the same objection levelled by Cresswell against a Rantala-type semantics, viz., that '~' and '&' do not represent *classical* negation and conjunction respectively for superposed worlds. So for example, this semantics does not illustrate how it is possible for agents to believe both that α and its *classical* negation, $\sim\alpha$. Rather, all that his semantics shows is how it is possible for an agent to believe both that α and its *paraconsistent* negation.

Rescher responds to this objection to his proposed semantics for belief by emphasizing that he is not proposing a deviant *logic*, but merely a deviant semantics.[44] For example, although the *semantic* principle that if

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[42] ibid, p. 100.

[43] ibid, p. 19. The principle that if $V_M(\alpha, w_i)$ and $V_M(\alpha \supset \beta, w_i) = 1$ then $V_M(\beta, w_i) = 1$ breaks down for superposed worlds.

[44] Rescher et al (1980), p. 18.

$V_M(\alpha, w_i) = V_M(\beta, w_i) = 1$ then $V_M(\alpha \;\&\; \beta, w_i) = 1$, fails for superposed worlds, the corresponding *syntactic* principle $\alpha, \beta \vdash \alpha \;\&\; \beta$ is retained for the appropriate axiom-system.[45] And so, Rescher's response to the charge of equivocation with respect to the connectives '~', '&' (and '⊃') in his semantics is that he admits the charge but claims that what is important is how the connectives behave inferentially:

> Our own choice here is clear - we follow the mainstream of logical tradition in giving priority to the inferential aspect, taking the stance that what a logical connective "really is" is to be determined in terms of what it *does* in inferential situations.[46]

We have already considered this response to the charge of equivocation with respect to the connectives in relation to Rantala's non-standard index semantics for belief logic discussed in the previous chapter. It was suggested that this response may not avoid the charge of equivocation with respect to the connectives ~, &, v, ⊃ and ≡ if it rests on the dubious assumption that ~, &, v, ⊃ and ≡ are definable solely in terms of their behaviour in non-modal or non-doxastic inferential contexts. As was suggested in the previous chapter and in section 1, it is also necessary to take into account how the connectives behave in doxastic or modal contexts in order to discern their roles in inference for doxastic logics.

Thus, Rescher's semantics can escape the charge of equivocation only if the corresponding logic is not deviant - not only for non-modal but also for *modal* contexts. Recall that the Stal-SQC$^=$ + D systems are logics such that the connectives behave non-deviantly not only for non-modal but also for modal contexts. They are simply normal systems of alethic modal logic

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45 ibid, p. 18.

46 ibid, p. 23.

such that the necessity operator is construed as 'x ideally believes that' and the possibility operator is construed as 'x non-ideally believes that'. Then for example, instances of the adjunction schema for *non-ideal* belief are underivable for these systems not because the connectives behave non-standardly in doxastic contexts, but owing to the fact that the alethic possibility operator is construed as 'x ideally believes that'. Now, suppose that Rescher's semantic proposals are adopted as providing an account of both the *non-ideal* and the *ideal* believer. I.e., non-ideal belief involves a relation to a superposed world and ideal belief involves a relation to a schematic world. These characterizations of both ideal and non-ideal belief will be incorporated into a semantics characterizing the Stal-SQC= + D systems - which can be regarded as logics of the ideal and non-ideal believer.

Then even though the connectives ~, & and ⊃ are defined non-standardly for superposed worlds, this sort of semantics can escape the charge of equivocation with respect to ~, & and ⊃ since the corresponding logics are non-deviant. (Note also that in this semantics, the connective 'v' behaves non-classically for *schematic* worlds since it is possible that $V_M(\alpha \vee \beta, w^+) = 1$ for some schematic world w^+, and yet $V_M(\alpha, w^+) = V_M(\beta, w^+) = 0$.[47]) We shall now consider in detail just what this Rescherian semantics for the Stal-SQC= + D systems will look like.

A Rescherian TV semantics for the Stal-SQC= + D systems will involve defining a Stal-SQC= + D model as a 7-tuple $\langle W, R, W^+, W^*, f, g, V\rangle$ where W and R are defined as usual as a non-empty set of indices and a two-place (minimally serial) relation ranging over members of W respectively. Further, the sets W^+ and W^* are non-empty sets of *superposed* and

[47] Rescher et al (1980), p. 15.

schematic indices respectively. Also, f is a one-one onto function which associates with each member of W, exactly one member of W^+ and g is a one-one onto function which associates with each member of W exactly one member of W^*. The idea here is that with each index in W is associated exactly one superposed index and exactly one schematic index. Then, a non-ideal belief wff, $B\alpha$ will be true at w_i in W just in case the content α is true at the *superposed* index $f(w_i)$ and an ideal belief wff, $B_I\alpha$ will be true at w_i in W just in case the content α is true at the *schematic* index $g(w_i)$. The truth-conditions for wffs at superposed and schematic indices will be outlined in the next paragraph when we discuss the valuation function V_M.

The assignment function V as ususal assigns to atomic wffs at members of W, either '0' or '1' where '1' is the designated value, with the same two restrictions imposed on V (in relation to the identity symbol '=') as for the Stalnakerian TV semantics for the Stal-$SQC^=$ + D systems. Further, for members of W, the valuation function V_M is defined inductively in the usual manner such that the truth-conditions for quantified wffs are substitutional. The truth-conditions for non-ideal belief and for ideal belief will *initially* be the truth-conditions for possibility and necessity respectively for alethic normal systems, viz., in terms of the alternativeness relation R.

Once V_M is defined for members of W, it can be defined for the *schematic* indices in W^* as follows: For any wff α, $V_M(\alpha, g(w_i)) = 1$ iff for all w_j such that w_iRw_j, $V_M(\alpha, w_j) = 1$. I.e., a wff α is true at a schematic world $g(w_i)$ assigned to a member of W, w_i just in case α is true at *all* the alternatives to w_i determined by R. But if α is true at all alternatives to w_i then it follows that $\mathbf{B_I}\alpha$ is true at w_i. This leads to the specification of truth-conditions for ideal belief in terms of schematic indices as follows: $V_M(\mathbf{B_I}\alpha, w_i) = 1$ iff $V_M(\alpha, w_j) = 1$ for all w_j such that $w_i R w_j$ iff $V_M(\alpha, g(w_i)) = 1$. These truth-conditions for ideal belief intuitively jibe with Rescher's injunction that the non-standard world to which a believer at a world is related "must satisfy all and only the statements of a language L which the individual believes".[48] I.e., the wffs true at a schematic index $g(w_i)$ associated with a member of W, will be the contents of all wffs of the form $\mathbf{B_I}\alpha$ true at w_i.

Further, V_M can be defined for the *superposed* indices in W^+ as follows: For any wff α, $V_M(\alpha, f(w_i)) = 1$ iff for at least one w_j such that $w_i R w_j$, $V_M(\alpha, w_j) = 1$. I.e., a wff α is true at a superposed world $f(w_i)$ assigned to a member of W, w_i just in case α is true at *at least one* of the alternatives to w_i determined by R. But if α is true at some alternative to w_i then it follows that $\mathbf{B}\alpha$ is true at w_i. This leads to the specification of truth-conditions for non-ideal belief in terms of superposed indices as follows: $V_M(\mathbf{B}\alpha, w_i) = 1$ iff $V_M(\alpha, w_j) = 1$ for at least one w_j such that $w_i R w_j$ iff $V_M(\alpha, f(w_i)) = 1$. Finally, all those wffs true at a superposed index $f(w_i)$ will be all and only those contents of non-ideal belief wffs of the form $\mathbf{B}\alpha$ true at the associated index w_i.

We shall now provide a somewhat more formal characterization of the TV semantics for the Stal-SQC$^=$ + D systems described above:

A Stal-SQC$^=$ + D serial model M = $\langle W, R, W^+, W^*, f, g, V \rangle$ such that

1) $W \neq \emptyset$.

2) $R \subseteq W \times W$ where R is minimally serial.

3) $W^+ \neq \emptyset$.

48 Rescher et al (1980), p. 105.

4) $W^{\#} \neq \emptyset$.

5) $f: W \longrightarrow W^{*}$ such that f is 1-1 and onto.

6) $g: W \longrightarrow W^{\#}$ such that g is 1-1 and onto.

7) V: Atomic Wffs X W $\longrightarrow \{0,1\}$ with the following restrictions:

a) If α is of the form t =t then $V(\alpha, w_i) = 1$ for all $w_i \in W$.

b) For all $w_i \in W$, if $V(t_1 = t_2, w_i) = 1$ then $V(\alpha\,(t_1/v), w_i) =$ $V(\alpha\,(t_2/v), w_i)$.

A ***valuation*** over a Stal-SQC$^{=}$ + D model, V_M is such that:

V_M: Wffs X $(W \cup W^{*} \cup W^{\#}) \longrightarrow \{0,1\}$.

V_M is defined for all members of W inductively as follows:

Basis: $V_M(\alpha, w_i) = V(\alpha, w_i)$ for α atomic.

Inductive Hypothesis: Suppose that $V_M(A, w_i)$, $V_M(B, w_i)$ are defined. Then:

i) $V_M(\sim\alpha, w_i) = 1$ iff $V_M(\alpha, w_i) = 0$.

ii) $V_M(\alpha \supset \beta, w_i) = 1$ iff either $V_M(\alpha, w_i) = 0$ or $V_M(\beta, w_i) = 1$.

iii) $V_M((\exists v)\alpha, w_i) = 1$ iff $V_M(\alpha\,(t/v), w_i) = 1$ for at least one constant t.

iv) $V_M(B\alpha, w_i) = 1$ iff $V_M(\alpha, w_i) = 1$ for at least one w_j in W such that $w_i R w_j$.

v) $V_M(B_I\alpha, w_i) = 1$ iff $V_M(\alpha, w_i) = 1$ for all w_j in W such that $w_i R w_j$.

Notice that the truth-conditions for ideal and non-ideal belief are specified initially by appeal to the doxastic accessibility relation R. In defining the valuation function for members of W^{*} and $W^{\#}$ it will be possible to restate the truth-conditions for ideal and non-ideal belief by appeal to the notion of non-standard indices.

Given the definition of V_M for members of W, we can define V_M for members of W^{*} (i.e., the set of 'superposed' indices) as follows:

For any wff α and for any $f(w_i)$ in W^*:

$V_M(\alpha, f(w_i)) = 1$ iff $V_M(\alpha, w_j) = 1$ for at least one w_j in W such that w_iRw_j.

It follows from this definition of the valuation function V_M for superposed indices and from the definition of V_M for members of W that:

$V_M(B\alpha, w_i) = 1$ iff $V_M(\alpha, f(w_i)) = 1$.

In other words, the truth-conditions for non-ideal belief in terms of the serial relation R are equivalent to the truth-conditions for non-ideal belief in terms of superposed indices.

Further, given the definition of V_M for members of W, we can define V_M for members of W^* (i.e., the set of 'schematic' indices) as follows: For any wff α and for any $g(w_i)$ in W^*:

$V_M(\alpha, g(w_i)) = 1$ iff $V_M(\alpha, w_j) = 1$ for all w_j in W such that w_iRw_j.

It follows from this definition of the valuation function V_M for schematic indices and from the definition of V_M for members of W that:

$V_M(B_I\alpha, w_i) = 1$ iff $V_M(\alpha, g(w_i)) = 1$.

Thus, once again, it is possible to restate the truth-conditions for ideal belief in terms of schematic indices.

Finally, as usual, validity in a model of the type just described will be truth at all members of W and validity in a class of models will be validity in all models in the class.

Since the truth-conditions for both ideal and non-ideal belief wffs are stateable solely in terms of R, without appeal to non-standard indices, the elements of a Stal-$SQC^=$ + D model, W^+ and W^* are from a purely technical point of view dispensable. I.e., the elements W, R and V are sufficient to give us a characteristic semantics for the Stal-$SQC^=$ + D systems as was previously mentioned. However, the elements W^+ and W^* serve the purpose of formalizing on the model-theoretic front Rescher's suggestion that belief can be regarded as involving a relation between a believer at a world and a non-standard world such that if the believer is related to the right sort of non-standard world (viz., a superposed world), he may fail to conjoin beliefs or he may hold contradictory beliefs. Further, because the truth-conditions for belief wffs in terms of R which treat the connectives classically are more fundamental than and equivalent to the Rescherian truth-conditions for belief wffs in terms of non-standard worlds, any charge of equivocation with respect to the connectives for the non-standard worlds can be avoided. And in any case, the corresponding axiom-systems are non-deviant.

Since from a technical point of view, the elements W^+ and W^* are dispensable, soundness and completeness results are immediate. However, we shall illustrate how the Rescherian element of the semantics validates the axiom-schema D, $\mathbf{B_I}\alpha \supset \mathbf{B}\alpha$ for the Stal-SQC$^=$ + D systems. Suppose that there is a Stal-SQC$^=$ + D model, $\langle W, R, W^+, W^*, f, g, V\rangle$ such that for some member of W, w_i and for some instance of $\mathbf{B_I}\alpha \supset \mathbf{B}\alpha$, $V_M(\mathbf{B_I}\alpha, w_i) = 1$ but $V_M(\mathbf{B}\alpha, w_i) = 0$. On the supposition that $V_M(\mathbf{B_I}\alpha, w_i) = 1$ then $V_M(\alpha, g(w_i)) = 1$. Then for all w_j such that w_iRw_j, $V_M(\alpha, w_j) = 1$. But if by supposition $V_M(\mathbf{B}\alpha, w_i) = 0$ then $V_M(\alpha, f(w_i)) = 0$ and hence, for any w_j such that w_iRw_j, $V_M(\alpha, w_j) = 0$ which contradicts our earlier result that for all w_j such that w_iRw_j, $V_M(\alpha, w_j) = 1$, given that R is serial. Q.E.D.

In terms of completeness, for the canonical model $\mathcal{M}$, W would as usual be a set of maximal consistent sets of wffs with the $\exists$-property, w_iRw_j iff

$(\alpha)(B_I\alpha \in w_i \supset \alpha \in w_i)$ and $V(\alpha, w_i) = 1$ iff $\alpha \in w_i$. The fundamental theorem of canoncial models with $V(\alpha, w_i) = 1$ iff $\alpha \in w_i$ as the basis is proven inductively for members of W. Further, any member of the set W^+ $f(w_i)$ could be defined as $\{\alpha \mid B\alpha \in w_i\}$ and any member of W^*, $g(w_i)$ could be defined as $\{\alpha \mid B_I\alpha \in w_i\}$. Further, the fundamental theorem could be proven for members of W^* along the following lines: $V_M(\alpha, f(w_i)) = 1$ iff $V_M(B\alpha, w_i) = 1$ iff $B\alpha \in w_i$ (given the fundamental theorem for members of W) iff $\alpha \in f(w_i)$. Similar remarks apply to members of W^*. Finally, the given canonical model is proven to be in the relevant class of models by showing the R is serial.

And so, it has been argued that the Stal-SQC$^=$ + D systems provide us with logics of both the ideal and the non-ideal believer by construing the possibility operator as 'x non-ideally believes that'. Further, we developed two types of characteristic semantics for these systems, both of which are attempts to make some sort of model-theoretic sense out of the notion that agents can hold contradictory beliefs in different contexts without thereby conjoining these beliefs. The semantics based on Stalnaker's suggestion attempted to make sense out of this sort of situation in terms of the idea that agents are capable of being in more than one belief state. On the other hand, the semantics based on Rescher's suggestions attempted to make sense out of agents holding incompatible beliefs in different contexts by claiming that agents' 'belief worlds' (i.e. the worlds at which all and only the contents of the agents beliefs obtain) can be non-standard.

However, the Stal-SQC$^=$ + D systems and their characteristic semantics, although providing us with logics of the non-ideal believer are such that even the non-ideal believer accepts all the logical consequences of what he believes. In short, such logics still assume that agents are logically omnidoxastic. It will be argued in the next section that this is the best we can do within the parameters of doxastic logics based on normal systems since any alterations to rid these logics of the omnidoxasticity feature will result in an equivocation with respect to the connectives.

5. The Intractable Feature of Logical Omnidoxasticity

If our semantics for belief logic rests on the assumption that belief is a relation between a believer at an index and a standard 'alternative' index or a set of standard alternative indices (determined by some two place relation or function) such that the contents of one's beliefs are true at these alternatives, then this semantics will presuppose that any agent x will believe the logical consequences of (or whatever is logically equivalent to) what he believes. Thus, in this type of semantics all instances of the following are valid for either ideal or non-ideal belief (and we shall use '**B**' here to denote both interchangeably):

i) $(\mathbf{B}\alpha \ \& \vdash \alpha \supset \beta) \supset \mathbf{B}\beta$

ii) $(\mathbf{B}\alpha \ \& \vdash \alpha \equiv \beta) \supset \mathbf{B}\beta$

And the corresponding derived rules of inference preserve validity:

iii) $\vdash \alpha \supset \beta \longrightarrow \vdash \mathbf{B}\alpha \supset \mathbf{B}\beta$

iv) $\vdash \alpha \equiv \beta \longrightarrow \vdash \mathbf{B}\alpha \equiv \mathbf{B}\beta$

The explanation of this runs as follows: Any index which is logically possible or 'standard' or 'normal' will be such that it is closed under implication. Further, (supposing soundness), any thesis - implicational or otherwise will be true at any standard index. Thus, any belief alternative or any member of a belief state (which can be regarded as a set of alternatives to a given index) will be such that it is closed under implication and will be such that every thesis is true. Now, suppose some agent x believes that α at an index w_i in which case α will be true at the appropriate alternatives - or at all members of some belief state. Then since every alternative is closed under implication and given that all implicational theses are true at each alternative, it follows that for any wff β such that $\vdash \alpha \supset \beta$ or $\vdash \alpha \equiv \beta$, β will also be true at each relevant alternative. Then x will also believe that β. This is so whether we treat belief as analogous to necessity or to possibilty.

The problem of logical omnidoxasticity for a relational semantics for belief becomes even more acute if we consider the case of belief with respect to logical truths or truths of mathematics. Since any logical truth or truth of mathematics will be true at every normal index in a model, then all these truths will obtain at any alternative assigned to (a typical believer at) an index. Then if belief is understood as a relation between a believer at an index and an alternative or set of alternatives, every believer will believe all the truths of logic and mathematics, whether or not the agent has entertained any such truth. Further, the agent will believe all the consequences of some truth of logic which he believes, since they themselves will be logical truths and therefore true at all the same indices.

If logical omnidoxasticity is found to be objectionable then we may wish to rethink our views concerning the relata of belief along the following lines: We could regard belief as a relation between the 'typical' agent at an index and some *non-standard* possible world or index, or a set of such indices or a set of both normal and non-normal indices. These non-standard alternatives would involve a redefinition of '$\supset$' since either detachment would not hold or the relevant implicational thesis would turn out to be false at such indices. But as was argued in chapter five, this move invites disaster since we can be charged with equivocation with respect to the connective '$\supset$' given that it is defined non-classically for non-standard indices.

Further, the ploy of opting for defining '$\supset$' in terms of its role in inference does not escape the charge of equivocation since its behaivour will be deviant - at least for *doxastic* contexts, if the appropriate instance of $(B\alpha \;\&\; \vdash \alpha \supset \beta) \supset B\beta$ is rendered underivable. In short, any alteration to the axiom-system such that some instance of the omnidoxasticity schema is underivable simply mirrors the equivocation with respect to '$\supset$' in the semantics. If '$\supset$' is classical, then it can misbehave neither for doxastic nor for non-doxastic contexts. Therefore, the enterprise of altering the axiom-system such that some instance of the omnidoxasticity schema is rendered underivable is beside the point.

Granted, the *non-ideal* adjunction and consistency schemata can be rendered underivable for doxastic logics without risk of equivocation only because this tact involves reconstruing alethic possibility as 'x non-ideally believes that'. There are no alterations made to the given axiom-system such that any of the connectives $\sim$, $\vee$, $\&$, $\supset$ and $\equiv$ misbehave in doxastic contexts. The resulting logic is non-deviant. Further, even if we construe alethic possibility as non-ideal belief, the resulting system retains the omnidoxasticity feature with respect to non-ideal belief. Thus, the only way to rid the axiom-system of the omnidoxasticity feature would be to redefine the inferential role of '$\supset$' for doxastic contexts, which once more leaves us open to the charge of equivocation

And so, if we wish to avoid the charge of equivocation with respect to '$\supset$' while retaining a normal system (and its corresponding relational semantics) as our quantified doxastic logic then the best we can do, it would seem, is to attempt to make the omnidoxasticity feature of the semantics more palatable, philosophically.

For example, Stalnaker in a number of places[49] has tried to make omnidoxasticity with respect to mathematical or logical truths more palatable as follows: He claims that if an agent apparently fails to believe some mathematical truth which is logically implied by (and logically implies) any mathematical truth he already believes, what is really going on is that the agent simply does not recognize that the sentence he is considering expresses a mathematical truth. Thus, if he believes one mathematical truth he believes them all, but he may fail to believe that some sentence or other expresses any given truth:

> The apparent failure to see that a proposition is necessarily true or that propositions are necessarily equivalent, is to be explained as the failure to see what propositions are expressed by the expression in question.[50]

A consequence of this view is that the objects of mathematical investigation are twofold, viz., the *necessary* proposition expressed by all true mathematical expressions and secondly the propositions having to do with the relationship between mathematical expressions and the proposition they express.[51] When agents fail to recognize that two mathematical truths are equivalent, the source of this failure will be the latter objects of study.

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49 For example, see Stalnaker (1976) and Stalnaker (1984), the end of chapter four.

50 Stalnaker (1984), p. 84.

51 Stalnaker (1984), pp. 84–85.

As Stalnaker recognizes, a feature of this view is that the objects of mathematical inquiry turn out to be contingently true propositions, viz., those propositions expressing the connections between mathematical expressions and the necessary proposition - and these relations may vary from world to world or from context to context.
In the case of omnidoxasticity with respect to non-mathematical belief, such as the William III case mentioned in chapter one, Stalnaker's strategy is to admit that William III does not believe that England could avoid a nuclear war with France, although in some sense of *acceptance*, he accepts this. (This is how Stalnaker exploits his distinction between belief and acceptance.) Thus, it may not be a defining condition of belief states that they are closed under logical consequence - i.e., we cannot characterize belief states simply as sets of worlds. For example, we may require that x believes that α if α is true at all worlds in some belief state and if the agent has entertained this content - or understood it.[52] The problem with this move is that Stalnaker is departing from a possible worlds approach to belief rather than making the omnidoxasticity feature of such an approach more palatable.

Finally, although the problem of logical omnidoxasticity is intractable for relational indexical semantics of belief, there is one advantage which our Stalnakerian semantics for the Stal-SQC$^{=}$ + D systems in terms of belief states has over the other approaches, viz., that if an agent is in more than one belief state then he may fail to conjoin his beliefs. Thus, suppose that $\vdash(\alpha \ \& \ \beta) \supset Q$, where Q is any statement. In a Stalnakerian belief state semantics, x may believe that α and x may believe that β and yet he may not believe their conjunction. And so x may fail to believe that Q. But in a

52 ibid, p. 89.

semantics such that agents believe the conjunction of what they believe, the agent in believing that α and that β would also end up believing that Q. Thus, even though agents are omnidoxastic they in some sense believe 'less'. Also, in this type of semantics although $\vdash(\alpha \;\&\; \sim\alpha) \supset Q$ where Q is any statement, if x believes that α and that $\sim\alpha$, he will not thereby believe everything. Similar remarks apply to our Rescherian semantics for the Stal-$SQC^{=}$ + D systems.

Conclusion

An attempt has been made in this dissertation to salvage the enterprise of basing first-order doxastic logic on normal alethic modal logic by countering one of the major objections to this program. I.e., if we construe the alethic necessity operator as 'x believes that' then the resulting logics characterize the 'ideal' believer who believes all the consequences of what he/she believes, who conjoins his/her beliefs and who never holds inconsistent beliefs. But various counterexamples cited in the literature indicate the need for a logic embodying principles of belief attribution for the believer who is *non-ideal*. We have argued that normal doxastic logics do just that if we construe the alethic *possibility* operator as 'x non-ideally believes that' since possibility does not factor out of conjunction for normal systems and further given that $\sim(\mathbf{M}\alpha \ \& \ \mathbf{M}{\sim}\alpha)$ is not a thesis-schema for normal systems. The omnidoxasticity feature is retained for non-ideal belief although this feature is mitigated given the failure of non-ideal belief to factor out of conjunction.

More specifically, our final proposal for a set of first-order logics of the non-ideal believer are the Stal-SQC$^=$ systems of doxastic logic. These logics can be thought of as embodying principles of belief attribution for the non-ideal as well as the ideal believer by construing the necessity operator as 'x *ideally* believes that' and such that the possibility operator is construed as 'x *non-ideally* believes that'. Two types of characteristic semantics were considered, viz., a relational semantics formalizing Stalnaker's notion of agents' being capable of being in more than one belief state and a relational semantics based on Rescher's proposal that belief is a relation obtaining between a believer at an index and some non-standard index.

A more promising set of emendations to the Sub-SQC$^{=}$ systems discussed in chapter four resulting in logics characterizing the *non-ideal* believer is the set of non-normal logics called the Sub-SQC$^{=}{}_{\Omega}$ systems. These systems not only get rid of the adjunction and consistency features of the Sub-SQC$^{=}$ + D systems, but in addition they render underivable an infinity of instances of the omnidoxasticity schemata. These results are achieved vis a vis Rantala's proposal to restrict for normal systems the doxastic version of the rule of necessitation to some recursive subset of the set of wffs. However, it is when we come to consider the corresponding impossble index semantics for these systems that the following difficulty becomes evident: At non-normal indices, the connectives behave non-standardly in which case, we are equivocating with respect to these connectives in the semantics. For example, classical conjunction cannot misbehave and remain classical.

Further, the strategy of defining the connectives in terms of their roles in inference in the corresponding axiom-systems does not circumvent the charge of equivocation - assuming that *doxastic* contexts are also relevant in determining the roles of ~, v, &, ⊃ and ≡ in inference. If we are right here, then any attempt at altering normal axiom systems where the necessity operator is construed as 'x (non-ideally) believes that' such that any instances of the adjunction, consistency or omnidoxasticity schemata are rendered underivable will involve an equivocation with respect to one or more of the connectives, thereby being entirely beside the point with respect to the problem of deduction.

The Stal-SQC$^{=}$ (+ D) systems do not equivocate with respect to the connectives since the elimination of the adjunction and consistency features for non-ideal belief is not achieved by altering the role of the connectives in doxastic contexts. Further, in the Stalnakerian semantics proposed for these systems, the connectives behave classically at indices which are the elements of belief states. The Rescherian semantics in terms of superposed and schematic indices could initially be charged with equivocation with respect to the connectives ~, & and ⊃ for *superposed* indices. However, on closer inspection, superposed indices are formed by 'world-fusion' on standard indices where the connectives behave classically. Thus, it could be countered that in terms of the semantics, the connectives are definable in terms of their behaviour at standard indices out of which superposed indices are 'fused'.

The Stal-SQC$^{=}$ axiom-systems are therefore our final word on the problem of deduction. The omnidoxasticity feature is intractable for these systems since any attempt to rid them of this characteristic would involve altering the role of '⊃' in non-ideal belief contexts, thereby resulting in an equivocation with respect to these connectives. The corresponding move in the semantics would involve the introduction of Rantalian non-normal indices - such that '⊃' behaves non-standardly. However, the omnidoxasticity feature of our Stal-SQC$^{=}$ systems will not have as a consequence that an agent who believes that α and who also believes that $\sim\alpha$ thereby ends up believing everything, even though α & $\sim\alpha$ logically implies Q such that Q is any statement. This is owing to the fact that these logics do not presuppose that agents always conjoin their beliefs. And in fact, the Stal-$SQC^=$ systems of doxastic logic can be regarded as accomodating the Kripke puzzle along the following lines: Although puzzling Pierre believes that London is pretty and he believes that London is not pretty, he does not thereby end up believing everything since he presumably does not conjoin these beliefs.

Finally, the Stal-$SQC^=$ systems of doxastic logic also provide elegant ways of handling some of the problems associated with *quantified* doxastic systems, viz., the problem of quantifying in and the apparent failure of the substitutivity of co-referentials for belief contexts. First, the quantifiers are construed *substitutionally* in the corresponding semantics, which therefore circumvents the problem of quantifying into doxastic constructions. For example, if Jones believes (ideally or non-ideally) that the next Liberal leader will be in favour of balancing the budget, we are warranted in inferring that some substitution instance of 'Jones believes that x will be in favour of balancing the budget' is true. Thus, unlike Hintikka's proposed solution to the problem of quantifying in which appeals to the traditional relational/notional distinction, no such distinction is necessary for the Stal-$SQC^=$ systems.

Along Fregean lines, all belief constructions (ideal or non-ideal) are unambiguously oblique for these systems in the sense that co-referentials are not unrestrictedly substitutible for belief contexts. Only if the agent believes that the relevant identity obtains, is subtitution warranted. And this is the solution to the substitutivity problem afforded by the Stal-$SQC^=$ systems. What is elegant about the solutions to the substitutivity problem and to the problem of quantifying in provided by the Stal-SQC$^=$ systems as opposed to the Hin-SQC$^=$ systems is that in the former case, only one type of belief context for both ideal and non-ideal belief is posited. Granted, we have embraced our own dichotomy between ideal and non-ideal belief although the payoff of making such a distinction is a partial solution to the problem of deduction for first-order belief logic.

In addition, the semantics characterizing the Stal-SQC$^=$ systems is metaphysically less problematic than the 'correlate' semantics characterizing the Hin-SQC$^=$ systems. Domains of individuals are dispensed with in the truth-value semantics characterizing the Stal-SQC$^=$ systems thereby avoiding the problem of having to make intuitive (and not just model-theoretic) sense of the notion of 'correlates'.

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