ANALYSIS OF MULTIPATH NEURAL SYSTEMS WITH RANDOM MODELS

AN ANALYSIS OF MULTIPATH NEURAL SYSTEMS USING RANDOM PARAMETER MODELS

by

Bernard N. Segal

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Biomedical Engineering Unit,
Department of Electrical Engineering,
McGill University,
Montreal, Quebec

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ABSTRACT

A systematic procedure for statistically analysing deterministic multipath models is presented that can be used to estimate the probable behavior of a secondary neuron in a typical sensory system. A statistical approach is taken since deterministic analysis is physiologically impractical. A class of multipath models is proposed in which each path is viewed as a realization of a random parameter model which is a cascade of random linear and random nonlinear blocks. In the multipath system, waveform variation due to different realizations of this system is approximated. The approach is especially convenient for multipath systems having an infinite number of paths. When finite path systems are considered, numerical methods are required except for simple systems. The approach shows that many multipath models, having nonlinearities in each path, have an expected output that is linearly related to the input under certain conditions. An application to the vestibular system is discussed.

RÉSUMÉ

On introduit une méthode d'analyse statistique de modèles déterministiques à branches multiples applicable à l'étude du comportement d'un neuron secondaire dans un système sensoriel typique. Cette méthode statistique s'impose vu l'irréalisme de l'analyse déterministique en physiologie du système nerveux. On propose une classe de modèles dans lesquels chacune des branches origine d'un modèle à paramètre aléatoire composé d'une suite de blocs linéaires et de blocs à non-linéarité de type statique. On obtient une approximation de la variabilité des signaux chez différentes réalisations du système. La méthode est particulièrement utile pour des systèmes à branches multiples en nombre infini. Pour des systèmes plus limités, des méthodes numériques sont nécessaires sauf pour des cas très simples. Il est démontré que, chez plusieurs modèles à branches multiples possédant une nonlinéarité dans chaque branche, la valeur probable du signal de sortie est, sous certaines conditions, en rapport linéaire avec l'entrée. L'application au système vestibulaire est discuté.

PREFACE

It is hoped that this thesis will interest an audience of mathematicians, engineers and physiologists who are concerned with modelling the nervous system. Due to the wide range of backgrounds of members of this group, it is difficult to adopt a style which is suitable for all readers. As a result, while some sections may be difficult to read for some, others may find it elementary. For example, in Chapter 2, a very basic discussion of part of the vestibular system attempts to provide a minimum background for those unfamiliar with sensory systems. Also, it is hoped that the rather tutorial style of part of Chapter 3 will enable those having only a basic statistics background to follow the essential ideas of the thesis. In order that as large an audience as possible be able to follow the mathematical discussion, mathematical rigor has, for the most part, been dropped. In spite of this, it is hoped that mathematicians will be interested in this application of random processes and in the many mathematical problems that arise.

ACKNOWLEDGEMENTS

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CHAPTER 1

INTRODUCTION

1.0 THE THESIS IN PERSPECTIVE

In the nervous system, it is not uncommon to find that information is transferred between two points along hundreds of nonlinear, parallel paths, each having different characteristics. One way to model this situation is to identify each path and then to simulate all paths acting together. This not only poses many physiological and mathematical problems, but is tedious and without direction. Although other somewhat more efficient approaches are available, they require more effort as the number of parallel paths increase. In addition, the modeller has no way of predicting or analysing results, except empirically, by computer simulation.

This thesis presents a systematic method of approaching this type of problem. Provided that the multipath neural system satisfies some quite general assumptions (Table 2-1), one can, using a statistical approach, analytically predict with relative ease the response of a system having an infinite number of parallel paths. In these cases, two striking results emerge. First, there is a large class of multipath systems that have, under certain conditions, a response which is linearly related to the input despite the presence of nonlinearities in each path. This linear relationship can be markedly different from the input/output relationship of any single path since the system output depends on all the individual input/output relationships acting together. Second, many different multipath systems can have the same input/output relationship. This result implies that it may not be necessary to model each path exactly. Thus, the approach seems to head in a more sensible direction than the alternative method in which each path is modelled more and more exactly with the consequence that multipath simulation may be too complex to consider.

The method presented in this thesis can also be used to systematically analyse models (from the class of Table 2-1) that have a finite number of paths. For any randomly selected system, one can estimate the probable range within which any given system variable will lie. In practice, a numerical solution is often necessary to do this in more complex systems. In addition, the difference between the output of the finite path system and the infinite path one can be estimated.

The method provides a powerful and useful modelling technique for a class of neural systems having a large number of convergent parallel pathways. For these systems, the assumptions used to make the response of the infinite path model approximate the neural system's response often suggest physiological relations to be verified. Also, even when the method is applied to a finite path system for which numerical calculations are required, it is still a more efficient way to approximately describe multipath systems than other approaches available.

1.1 THESIS OUTLINE

To give an example of a multipath neural system, the first part of Chapter 2 will discuss the rotation sensing system of the vestibular organ of man and other vertebrates. A class of multipath models will be proposed in Section 2.3 that, it is felt, can approximate many sensory systems. It will be shown that while it is probably physiologically impossible to use these models deterministically, using them statistically seems feasible.

Chapter 3 will introduce random parameter models and then will show how they can be used to analyse multipath models statistically. In Section 3.3, it will be demonstrated that any multipath model satisfying Table 2-1 can, in principle, be analysed using a systematic procedure which will be presented in Section 3.4.

The first part of Chapter 4 derives the moment relationships for several random parameter models. In the second part, the aspects that introduce difficulties in this systematic procedure will be compared to those in other approaches. Then, the procedure, which employs some of the derived moment relationships, will be used to illustrate the analysis of two multipath models shown in Figures 4-8 and 4-10.

Chapter 5 will discuss the application of the multipath models to the rotation sensing system. In Section 5.2, it will be shown that the approach can be used in systems that have paths with different or random structure. Finally, Section 5.3 will summarize the thesis.

CHAPTER 2

ROTATION SENSING: AN EXAMPLE OF A MULTIPATH NEURAL SYSTEM

2.0 INTRODUCTION

Communication in a typical neural system in vertebrates has evolved in a significantly different direction from that of most modern communication technology. Instead of using one path having a very precise transmission characteristic, a neural system may have possibly hundreds of parallel paths each with different characteristics. This usually results in a communication system which is more reliable since redundancy has minimized the importance of a single path, which has an output with a better signal to distortion ratio than any of its paths due to an increased transmission range and linearization, which has a filtering characteristic, and which minimizes the effect of internal noise sources (Bayly, 1968; Diday, 1971; Laszlo, 1968; Lee, 1969; Maffei, 1968; Williams, 1972).

As an example of a multipath neural system, the part of the vestibular system concerned with rotation sensing is described in Sections 2.1 and 2.2. In Section 2.3, a class of multipath models that may be useful to describe many sensory systems is introduced. Finally, the problem of applying these models in a physiological system is discussed.

2.1 A DESCRIPTION OF ROTATION SENSING IN THE VESTIBULAR SYSTEM

All vertebrate animals from the most primitive to man have a remarkably similar motion sensing apparatus, located in the inner ear, which monitors linear and angular movements of the head (Outerbridge, 1969). Information from these sensors is used to stabilize the head, to reduce "visual slip" of images on the retina, as well as for postural control (Clark, 1970; Bender, 1964; Jones & Milsum 1965;Kearney, 1971; Outerbridge, 1969; Roberts, 1967; Young, 1968).

One angular sensor, called a semicircular canal, is shown in idealized form in Figure 2-1. It can be described as a toroid filled with a viscous fluid that deflects a flap or cupula when the head turns. Since the cupula is elastic, it tends to return to its equilibrium position after deflection (Mayne, 1950; Steer, 1968). The motion of the cupula is sensed by hundreds of tiny haircells of two principle forms (Figure 2-2) that change the firing rate or action potential frequency of primary neurons that synapse with the haircells (Ballentyne & Engstrom, 1968; Engstrom, 1968; Harada, 1972).

Several hundred primary neurons, each receiving inputs from a small but variable number of haircells of both types, carry information to a group of cells, called secondary neurons, located in the brain (Figure 2-3) (Gacek, 1969). In addition, there are a small number of efferent fibers that feed back information from the brain to the haircell region (Gacek, 1967; Smith & Rassmussen, 1967).

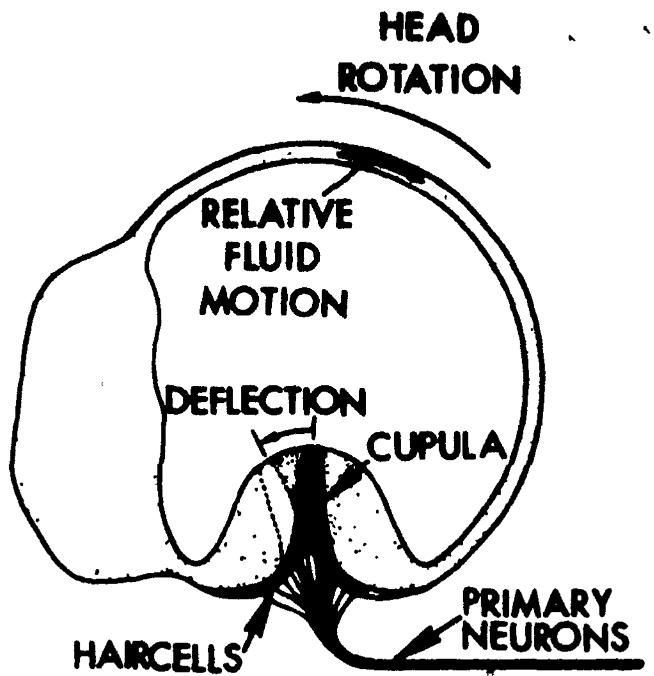

FIGURE 2-1 DIAGRAMATIC REPRESENTATION OF AN IDEALIZED SEMICIRCULAR CANAL (modified from Jones & Milsum, 1969).

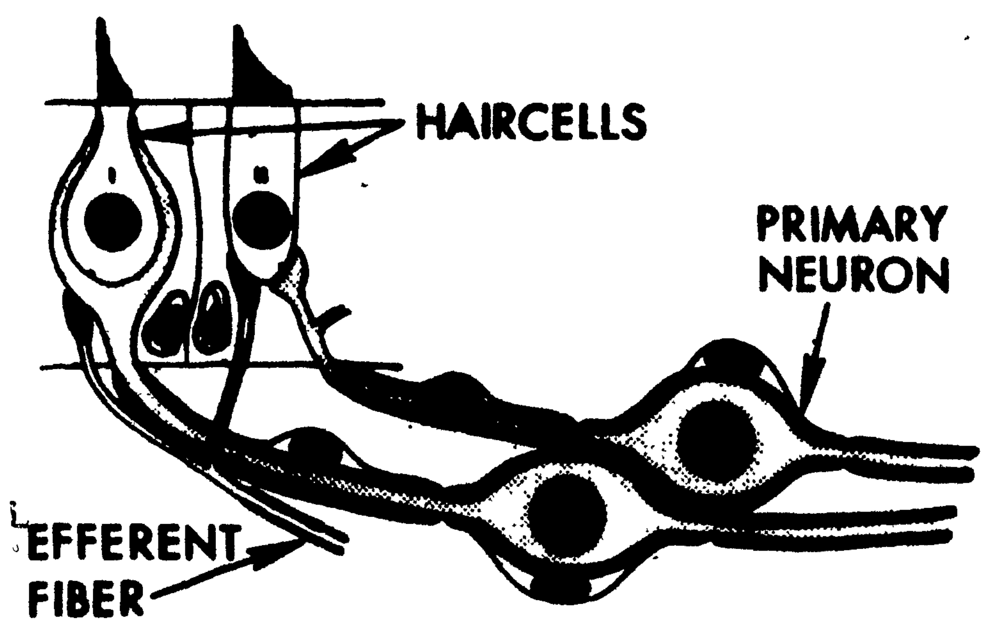

FIGURE 2-2 SCHEMATIC DRAWING OF THE TWO TYPES OF HAIRCELLS AND SOME PRIMARY NEURONS

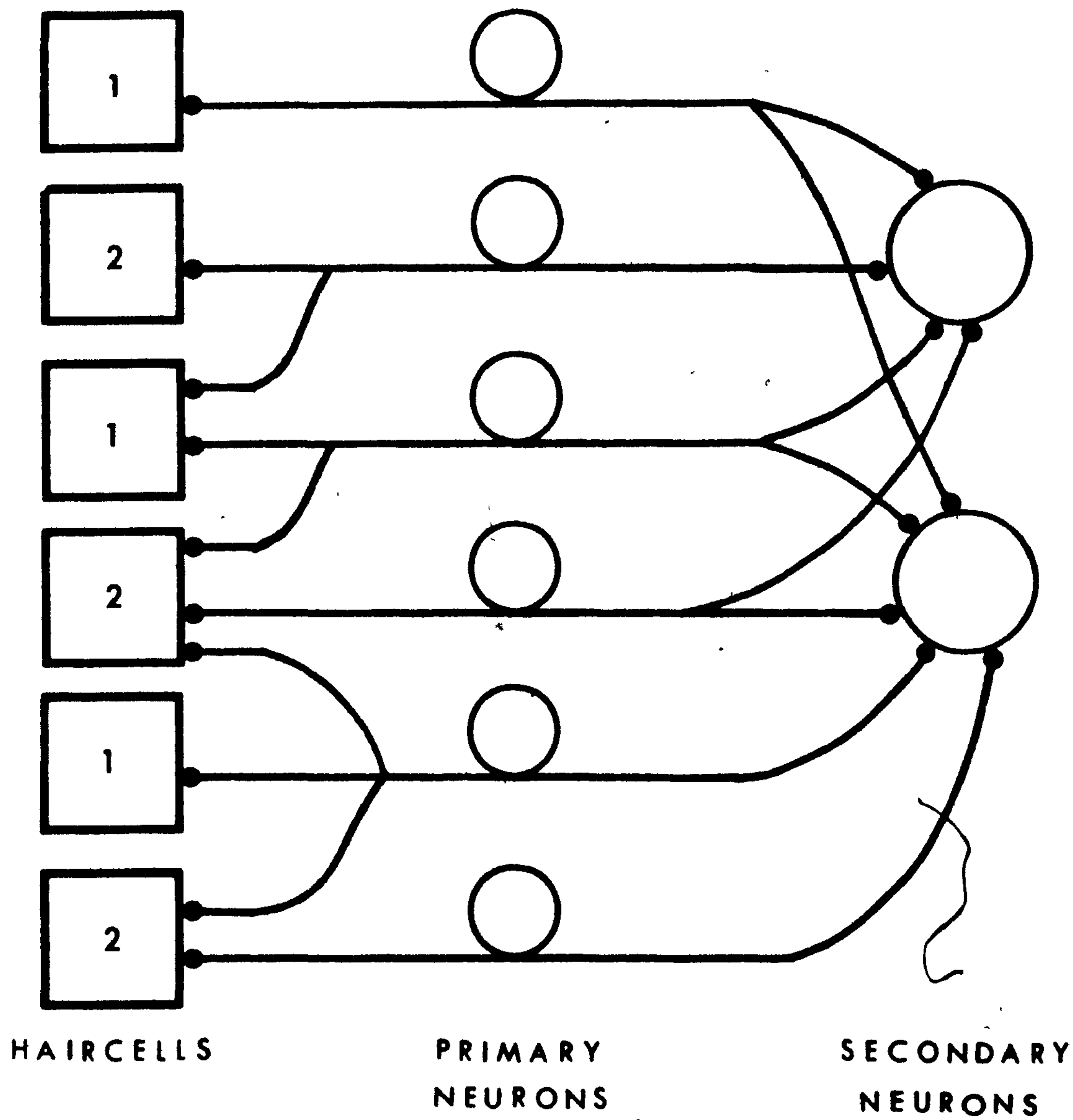

FIGURE 2-3 SCHEMATIC DRAWING OF INTERCONNECTIONS BETWEEN HAIRCELLS, PRIMARY AND SECONDARY NEURONS

2.2 THE RELATIONSHIP BETWEEN THE FIRING FREQUENCY OF PRIMARY NEURONS AND ANGULAR ACCELERATION

A number of models have been developed to describe the relationship between input angular acceleration and the instantaneous firing frequency of the primary neuron (Correia & Landolt, 1973; Goldberg & Fernandez, 1972 a, b; Fernandez & Goldberg, 1972; Lowenstein & Sand, 1940; Precht et al., 1971; Ross, 1936). In most cases, it is found that parameters in these models appear to have different values in different paths. This could be due to several factors such as experimental errors, an inadequate model (Precht et al., 1971) or an actual difference in the parameters in different paths. It is felt that the last two points are the most probable causes of this parameter spread.

The low frequency or static relationship between angular acceleration and frequency for several hypothetical paths is shown in Figure 2-4. Each unit has a threshold and saturates with a sufficiently large input. Units that have a "spontaneous" or resting frequency with no input applied have a negative threshold, while those with positive thresholds are called "silent units". Note the different resting frequencies, thresholds, saturation frequencies, sensitivities and ranges in Figure 2-4. As an actual example of this parameter spread, Figure 2-5 shows the relative number of units in the monkey with specific values of resting frequency, gains and thresholds. No detailed data is available about different values for ranges, saturation frequencies or the shape of the static relationships between threshold and saturation. Since this relationship is not always linear as shown, nonlinear responses are common.

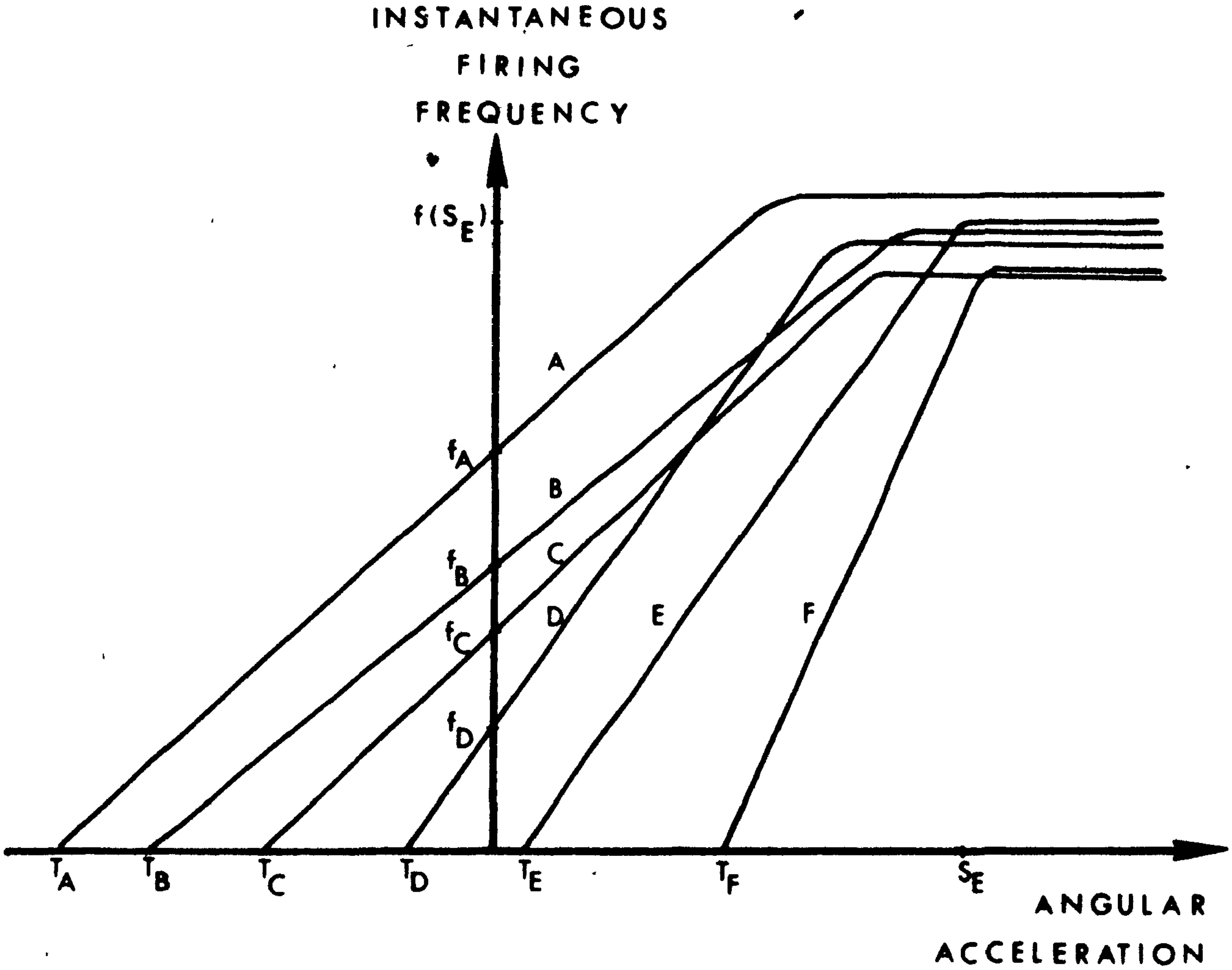

FIGURE 2-4 SOME HYPOTHETICAL STATIC RELATIONSHIPS BETWEEN ANGULAR ACCELERATION AND PRIMARY NEURON FREQUENCY

Units A to D have resting frequencies of f_A, f_B, f_C and f_D. Units E and F are "silent" units with positive thresholds T_E, T_F. All units saturate. Unit E saturates when the input exceeds S_E and has a saturation frequency of $f(S_E)$. The range of unit E is $S_E - T_E$.

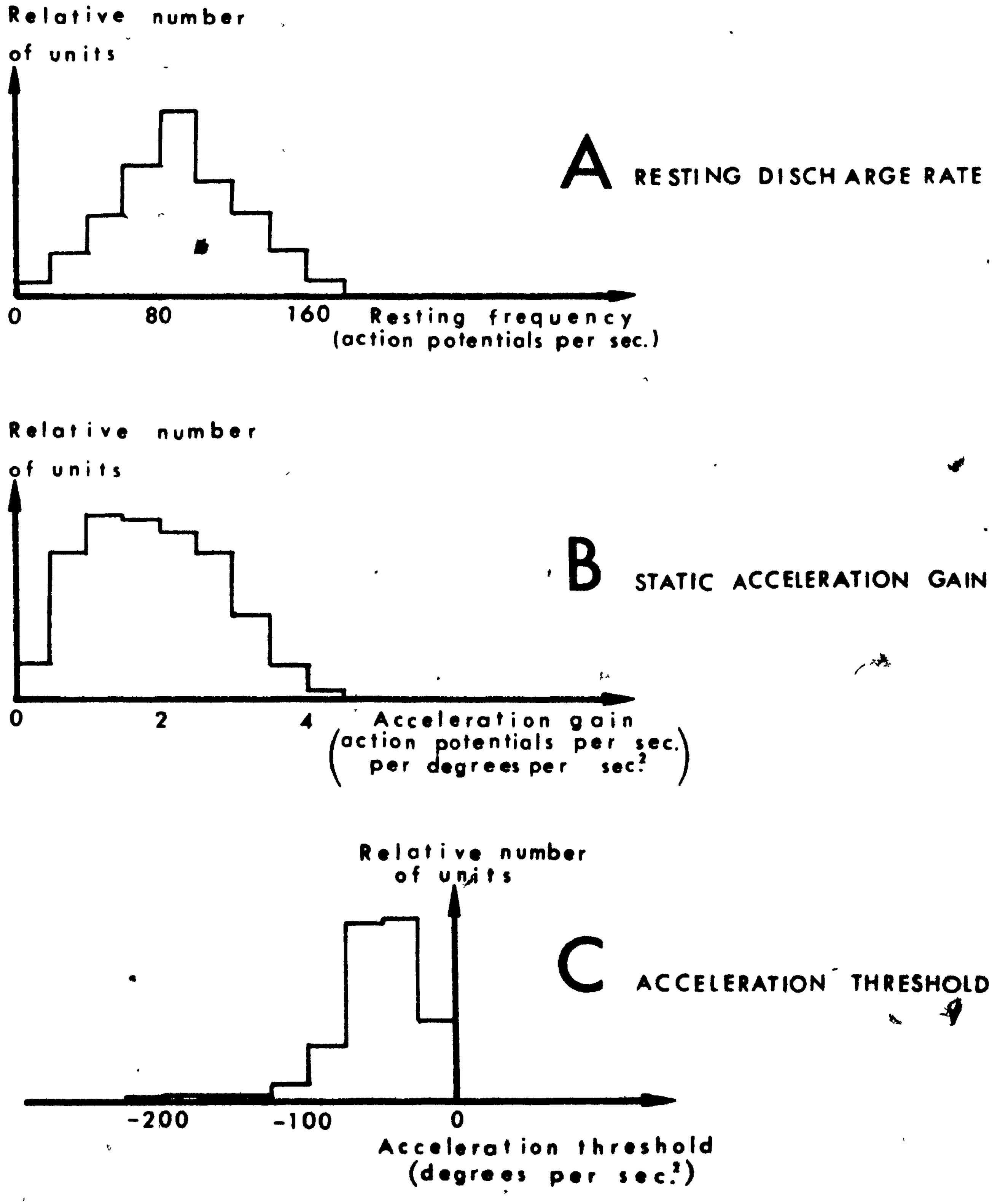

FIGURE 2-5 RELATIVE NUMBERS OF VALUES FOR RESTING FREQUENCY GAIN AND THRESHOLD IN THE MONKEY

Parts (a), (b) are from figures 7, 13 respectively of Goldberg & Fernandez, 1972 a.

Part (c) is based on data from figure 4 of Goldberg & Fernandez, 1972 b.

When modelling the dynamic relationship between angular acceleration and the frequency of the primary neuron, models ranging from transfer functions with one or two poles up to transfer functions with three zeros and four poles have been proposed (Correia & Landolt, 1973; Goldberg & Fernandez, 1972; Fernandez & Goldberg, 1972; Lowenstein & Sand, 1940; Precht et al., 1971; Ross, 1936). In all cases where many careful estimates of these poles or zeros have been made, parameters spread has been found. An example (Fernandez & Goldberg, 1972) of a transfer function obtained is

$$\frac{f(s)}{\ddot{\theta}(s)} = \frac{T_A s\ (1 + T_L s)}{(1 + T_A s)\ (1 + T_1 s)\ (1 + T_2 s)}$$

where $f(s)$ and $\ddot{\theta}(s)$ are the Laplace transforms of firing frequency and angular acceleration respectively, s is the Laplace variable and T_1, T_2, T_A, T_L are fixed constants in each path. Estimates of T_1, T_A, and T_L have been made that lie between 2 - 7 seconds, 30 to an infinite number of seconds and 0.013 - 0.091 seconds respectively. It is felt that an actual difference of these parameters between paths contributes significantly to these parameter spreads. Parameters such as these will be called "path varying parameters".

2.3 MODELLING THE RELATIONSHIP BETWEEN THE FIRING FREQUENCY OF SECONDARY NEURONS AND ANGULAR ACCELERATION

Although there are several empirical models that have been used to describe the relationship between the firing frequency of the secondary neuron and angular acceleration (Crampton, 1965; Jones & Milsum, 1970; Milsum & Jones, 1969), there exist no realistic models. This is because the exact structure of the system converging onto a secondary neuron is unknown.

Table 2-1 describes a class of models that, in this author's opinion, should be tested experimentally to see if they can represent this relationship to a first approximation. An example of a multipath model from this class is shown in Figure 2-6. In this figure, $h_1(t, \vec{a}_1)$ and $h_2(t, \vec{c}_1)$ are the impulse responses of the first and second linear systems, where $\vec{a}_i$ and $\vec{c}_i$ each denotes a set of fixed parameters, $(a_{11}, a_{12}, \dots)$ and $(c_{11}, c_{12}, \dots)$, contained in each impulse response. Thus, for example one can write

$$z_2(t) = \int_0^\infty y_2(t - t_1)\, h_2(t_1, \vec{c}_2)\, dt_1 .$$

Each $g_i(\ )$ represents a static nonlinear relationship between $x_i(t)$ and $y_i(t)$ which can be expressed in several ways. For example, $g_2(\ )$ can be written as:

(1) a <u>power series relationship</u>,

$$y_2(t) = \sum_{j=0}^{K} b_{2j}\, x_2^j(t)$$

where the b_{2j} are parameters of a K^{th} order polynomial,

1. The multipath model has the same known input applied to a parallel combination of N paths and an output that is the average of the outputs of all of the paths.

2. Each path is a cascade of dynamic linear and static nonlinear blocks such that corresponding blocks in different paths have the same mathematical form but may contain different parameters.

3. The relative number of paths having a particular value for a given parameter is known, so that one can estimate the probability of obtaining this value in a randomly chosen path.

4. A particular N-path multipath model can be regarded as having paths that have been randomly selected from an infinite population of paths described by (2) and (3).

5. Parameters in different blocks of one path are independent.

TABLE 2-1 A PROPOSED CLASS OF MODELS FOR REPRESENTING SENSORY SYSTEMS

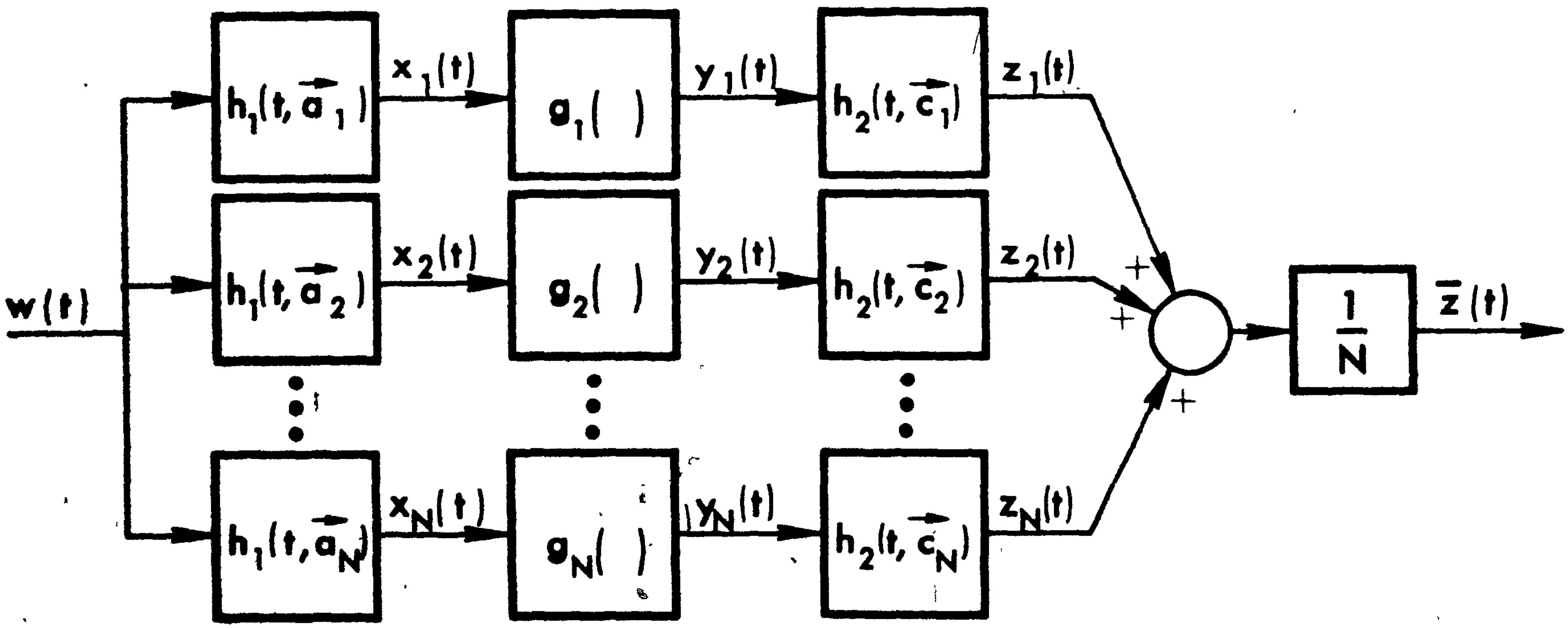

FIGURE 2-6 AN EXAMPLE OF A MULTIPATH MODEL, FROM TABLE 2-1, THAT MAY RELATE ANGULAR ACCELERATION TO THE FREQUENCY OF A SECONDARY NEURON

(2) a <u>functional relationship</u> such as

$$y_2(t) = \begin{cases} 1 - e^{-b_{21} x_2(t)}, & x_2(t) \geq b_{22} \\ 0, & \text{elsewhere} \end{cases}$$

where b_{21}, b_{22} are constants, or as

(3) a <u>graphical or tabular relationship</u>.

Since each path can have different parameters in it, corresponding linear (or static nonlinear) blocks in each path can have a different impulse response (or shape).

There are a number of reasons for the proposal of Table 2-1:
(1) The static nonlinearity is a simple type of nonlinearity, and it is felt that many nonlinear systems that do not generate significant subharmonics can be represented to a first approximation by models consisting of several linear and static nonlinear systems.

(2) A cascade of a linear system, a static nonlinearity and a second linear system has been successfully used in the visual system (Spekreijse, 1969; Spekreijse, 1970; Spekreijse & Van der Tweel, 1972). Further, existing vestibular data relating primary neuron frequency to angular acceleration (Section 2.2) can be adequately approximated by a linear system followed by a static nonlinearity.

(3) Equating the frequency of the secondary neuron to the average of the path's outputs is equivalent to equating this frequency to a weighted sum of the path's outputs. This is assumed due to its simplicity.

(4) A recent identification procedure is available (Korenberg, 1973 a; Korenberg, 1973 b) for identifying any cascade of alternating linear and static nonlinear blocks.

(5) Finally, while the relative number of some parameter values in a multipath model would have to be assumed, others (e.g. Figure 2-5) are readily available.

The question of how to analyse these models naturally arises. Clearly, deterministic modelling would be tedious even if all parameters in all paths were known. In fact, it is physiologically impractical, if not impossible, to obtain all of these parameters since each of the possibly hundreds of paths that converge onto a particular secondary neuron would have to be identified. However, many primary neurons, particularly "silent" ones, would be destroyed during the process of searching for them. Also, the state of the animal would probably change during the time necessary to accumulate enough data to identify all of the paths. Thus, it appears difficult to model the frequency of the secondary neuron deterministically.

An alternate approach, taken in this thesis, is to ask what is the probable firing frequency of any secondary neuron, or equivalently to ask what is the probable output of the multipath model when a known input is applied. This approach is feasible since one can estimate the characteristics of the entire population of primary paths by sampling them randomly. If it is experimentally verified that each path is of the same mathematical form but contains different parameters, then the proportion of paths that have a particular value of a path varying parameter can be found.

Chapter 3 will describe how this information can be used to analyse deterministic multipath systems statistically.

CHAPTER 3

A STATISTICAL APPROACH TO THE ANALYSIS OF DETERMINISTIC MULTIPATH SYSTEMS

3.0 INTRODUCTION

Chapter 2 has described a typical multipath neural system and then proposed a general class of models that could represent many neural systems. In Section 2.3, it was shown that while it is physiologically impossible to use these models deterministically, using them statistically is feasible. Chapter 3 presents this statistical approach.

In Section 3.1, random parameter models are introduced. It is then shown that when these random parameter models are applied to the proposed class of multipath models, there exists a mean and autocorrelation corresponding to each point in the multipath model. Section 3.2 shows how these statistical quantities can be used to approximate the behavior of the corresponding points in the multipath model. Section 3.3 demonstrates that for any model from the proposed class the mean and autocorrelation corresponding to any point can always be found. A systematic method of analysing this class of models is given in Section 3.4.

3.1 RANDOM PARAMETER MODELS: THEIR APPLICATION TO MULTIPATH SYSTEMS

If a random process is the input to a deterministic system, the statistical quantities of the input process can sometimes be used to determine a statistical description of the output process (Barret, 1964; Davenport & Root, 1958; Stratonovitch, 1963). The mean and autocorrelation of the output can easily be found if the system is linear (Papoulis, 1965) and can often be found if the system is nonlinear but static (Harmon, 1963; Thompson, 1955).

Sometimes the system itself varies as, for example, in an electronic circuit that contains a component with a value that changes randomly. Here the system can be represented by a random parameter model described by an equation containing parameters that are random processes (Adomian, 1964; Bharucha-Reid, 1964, 1970; Fuller, 1970; Kozin, 1969). Usually the variation of the system parameter is independent of the input process. In this case, a statistical description of the output process can be obtained in a manner completely analogous to the deterministic system case, except that the output expectations are taken over the additional random process of the system. This procedure will be illustrated in a slightly different physical application that follows but is similar mathematically. The difference arises since the random parameter models to be used in this application have parameters that are random variables instead of random processes.

Instead of having one system with a parameter that varies randomly with time, consider an infinite ensemble of systems, of the same mathematical form, each with a fixed but possibly different parameter. Let a known random process be applied to all systems and assume that the relative number of systems having a particular value of a "system varying parameter" is known. Suppose an experiment is repeatedly performed in which different outputs are randomly selected and observed. Two questions can be asked about this randomly selected output:

(1) What is its expected value? and

(2) What is the expected product of its values at two fixed instants of time?

These questions can be answered by defining a random parameter model that contains a random variable with a probability density consistent with the relative distribution of values of that parameter in the ensemble of systems. Then one can consider each system to be a realization of the random parameter model, and each randomly selected output to be a realization of the output random process of the random parameter model when the known input is applied. By using the definitions of the mean and autocorrelation, the two questions are answered by evaluating these respective statistical quantities of the output process.

A fundamental characteristic of the random process just considered is seen if one considers what happens if a sinusoidal input is applied to an ensemble of linear systems. Since each system output will also be sinusoidal, each realization of the process at the output of the random parameter model just defined will be completely deterministic. While for the random processes normally considered, past values of a realization cannot be used to predict future values of this realization; however, for the process considered above, only three past values of any realization are required to predict all of its future values. Since most of the processes considered in this thesis will have deterministic realizations, many simplifications will result.

As a first example, consider an infinite ensemble of first order low pass filters each of which may have a different angular break frequency $b_1, b_2 \ldots$ in their impulse response $e^{-b_1 t}, e^{-b_2 t}, \ldots$, and let a known, real, random process*, $\underline{x}(t)$, be applied to all systems. In order to describe a randomly selected output, a random parameter model is defined that has a random impulse response, $e^{-\underline{b}t}$, where the angular break frequency, $\underline{b}$, has a probability density, $f(b)$, consistent with the relative distribution of these break frequencies in the ensemble of filters. The output of the random parameter model is

$$\underline{y}(t) = \int_0^\infty e^{-\underline{b}v}\, \underline{x}(t-v)\, dv .$$

The expected output of the random parameter model is

$$\langle \underline{y}(t) \rangle = \left\langle \int_0^\infty e^{-\underline{b}v}\, \underline{x}(t-v)\, dv \right\rangle$$

where $\langle\ \rangle$ denotes expectation. Since the input and system processes are independent, the output can be written as

$$\langle \underline{y}(t) \rangle = \int_0^\infty \langle e^{-\underline{b}v} \rangle \langle \underline{x}(t-v) \rangle\, dv .$$

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* The convention used in this thesis is that an underlined letter denotes a random variable (or random process). When the letter is not underlined, this denotes a realization of that random variable (or process). In addition, all processes (except Fourier transforms) are assumed to be real.

Thus the expected value of a randomly selected output equals the expected input convoluted with the expected impulse response of the ensemble of filters. Note that, in general, the shape of the expected impulse response is different from each impulse in the ensemble.

The autocorrelation of the output of the random parameter model is

$$R_y(t_1, t_2) = \int_0^{\infty}\int_0^{\infty} < e^{-\underline{b}(v_1+v_2)} > < x(t_1-v_1)\; x(t_2-v_2) > dv_1\, dv_2$$

$$= R_h(t_1, t_2) ** R_x(t_1, t_2)$$

where $**$ denotes a double convolution, $R_x(t_1, t_2)$ denotes the autocorrelation of the input process and, analogously, $R_h(t_1, t_2)$ denotes the autocorrelation of the random impulse response.

Completely analogous results are obtained if each impulse response from the ensemble contains many parameters that are different in each system. If the i^{th} impulse response contains K parameters, then it can be written as $h(t, \vec{b}_i)$ where $\vec{b}_i$ is a K-dimensional vector containing the values of these parameters for that path. The random impulse response of the random parameter model can be written as $h(t, \underline{\vec{b}})$ and the joint probability density of the K random variables in $\underline{\vec{b}}$ can be denoted as $f(\vec{b})$.

As a second example, suppose one is interested in the randomly selected output of a new ensemble of static nonlinearities, each of the same mathematical form but with different fixed parameters in each system. Figure 3-1 shows examples of possible relationships between the input,

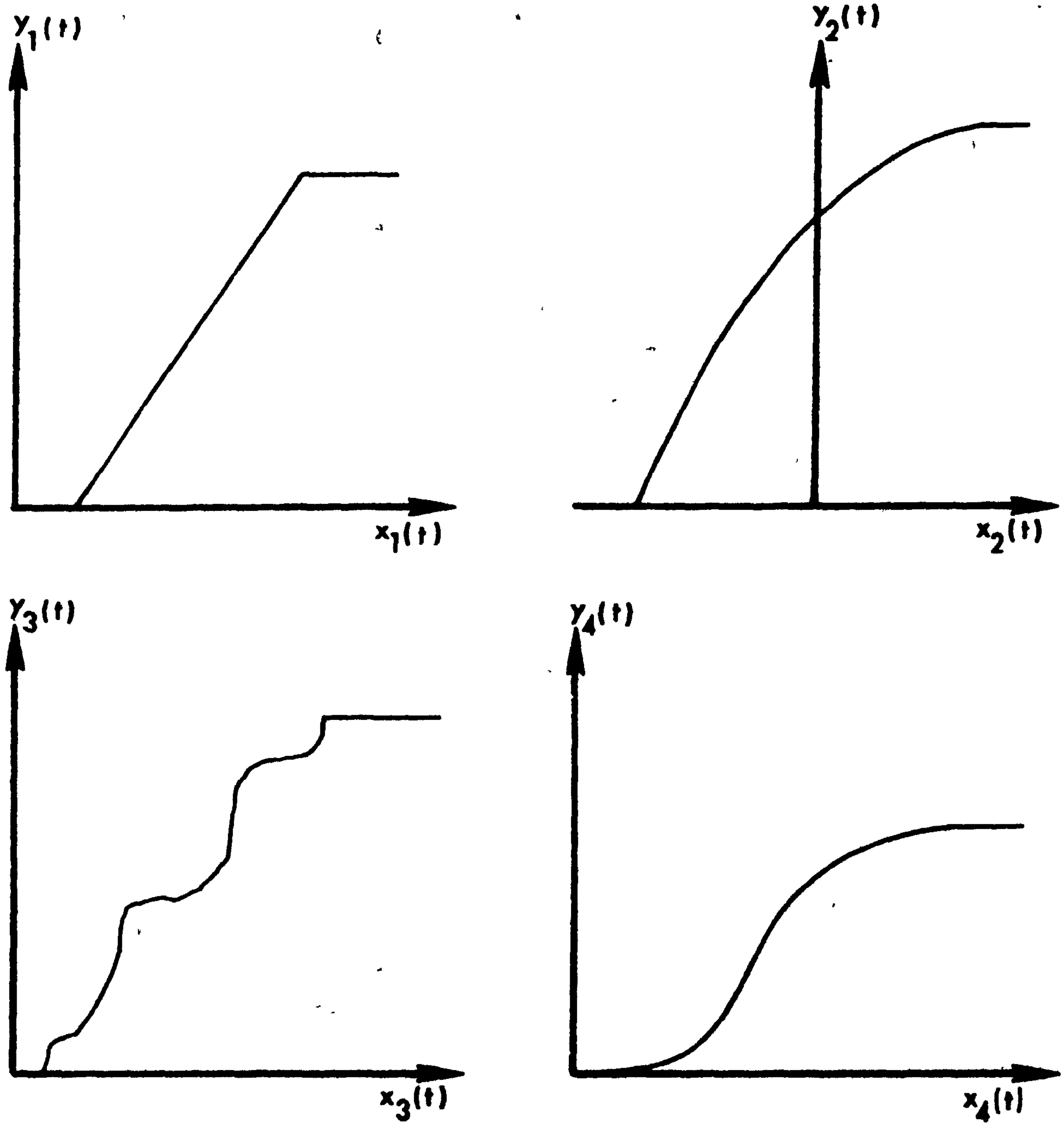

FIGURE 3-1 EXAMPLES OF SOME STATIC NONLINEAR INPUT/OUTPUT RELATIONSHIPS

$x_k(t)$ and the output, $y_k(t)$. Depending on which method (from Section 2.3) is used to represent the static nonlinearity, a different type of random parameter model describes a randomly selected output:

(1) When the <u>power series method</u> is used, the output of each static nonlinearity is represented by a "best fit" K^{th} order polynomial in the input variable. For example, in the k^{th} path

$$y_k(t) = \sum_{j=o}^{K} c_{kj}\, x_k^j\,(t)$$

where $c_{ko}, c_{k1}, \ldots, c_{kK}$ are constants. The relative number of systems with particular values of coefficients (this is assumed to be known) can be used to define a set of random variables $\underline{c}_o, \underline{c}_1, \ldots, \underline{c}_K$ and a joint probability density, $f(c_o, c_1, \ldots, c_K)$, consistent with the relative distributions of these coefficients in the ensemble of nonlinear systems. Note that any order moment of these coefficients can be found. The resulting random parameter model relates the output to the input with a K^{th} order polynomial that has random variables as coefficients.

(2) When the <u>functional relationship method</u> is used, a random parameter model that is a random nonlinear function results. This is handled in the same way as the random linear system already discussed.

(3) When the <u>graphical relationship method</u> is used a "random graphical relationship" random parameter model results which requires all joint probability densities of all adjacent values for a complete statistical description. Denote the value of the graphical relationship at the input

argument, $x(t)$, as $g(x(t))$. The first order probability density, $f(g(x(t)))$, can be estimated, for each $x(t)$, by finding the relative number of systems that have an output between $g(x(t))$ and $g(x(t)) + dg$ (where dg is a small increment about $g(x(t))$ so that*

$$f(g(x(t))) \approx \Pr\{g(x(t)) < \underline{g}(x(t)) < g(x(t)) + dg\}.$$

Similarly, the second order density of adjacent values can be defined as,

$$f(g(x(t_1)), g(x(t_2))) \approx \Pr\left\{\begin{array}{c} g(x(t_1)) < \underline{g}(x(t_1')) < g(x(t_1)) + dg_1 \\ \text{and} \\ g(x(t_2)) < \underline{g}(x(t_2)) < g(x(t_2)) + dg_2 \end{array}\right\}$$

where dg_1 and dg_2 are small increments about $g(x(t_1))$ and $g(x(t_2))$ respectively. Note that these densities are independent of time. This procedure would have to be continued until the resolution of the input variable is reached. It is not known which representation will be more useful experimentally. Regardless of which method is actually used, a static nonlinearity will be denoted as $\underline{g}(\ )$ with a probability density of $f(g(\ ))$.

The previous examples have shown that the output of a randomly selected system chosen from an infinite ensemble of systems with the same input and having the same mathematical form but containing different

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* $\Pr\{A\}$ denotes "the probability of A"

parameters in each system can be described by the output of a random parameter model. This model will be of the same mathematical form as each system but will contain random variables for its parameters.

Reviewing the description of the class of multipath models in Table 2-1, it is clear that the output of a randomly selected path can be described using random parameter models. Since the mathematical form and thus the block diagram of each path and the random parameter model are the same, each block output in any path corresponds to an output random process in the random parameter model. (See Figure 3-2). Thus, corresponding to each block output of the multipath system, there exists a mean and autocorrelation which is the same as that of the corresponding output in the random parameter model that describes all possible outputs at this point in the multipath system.

The next section will show how the mean and autocorrelation can be used to approximately describe the variation, due to different possible parameters in each path, of any waveform in a multipath system.

3.2 THE USE OF THE MEAN AND AUTOCORRELATION OF A SET OF WAVEFORMS

Once one has found the mean and autocorrelation corresponding to a variable in a multipath system, several properties of any realization of this variable can be found. Namely, for any waveform (or realization) of this variable one can estimate confidence intervals for:

(1) its actual value at a particular time,

(2) its Fourier transform (Papoulis, 1962) over an interval of time, and

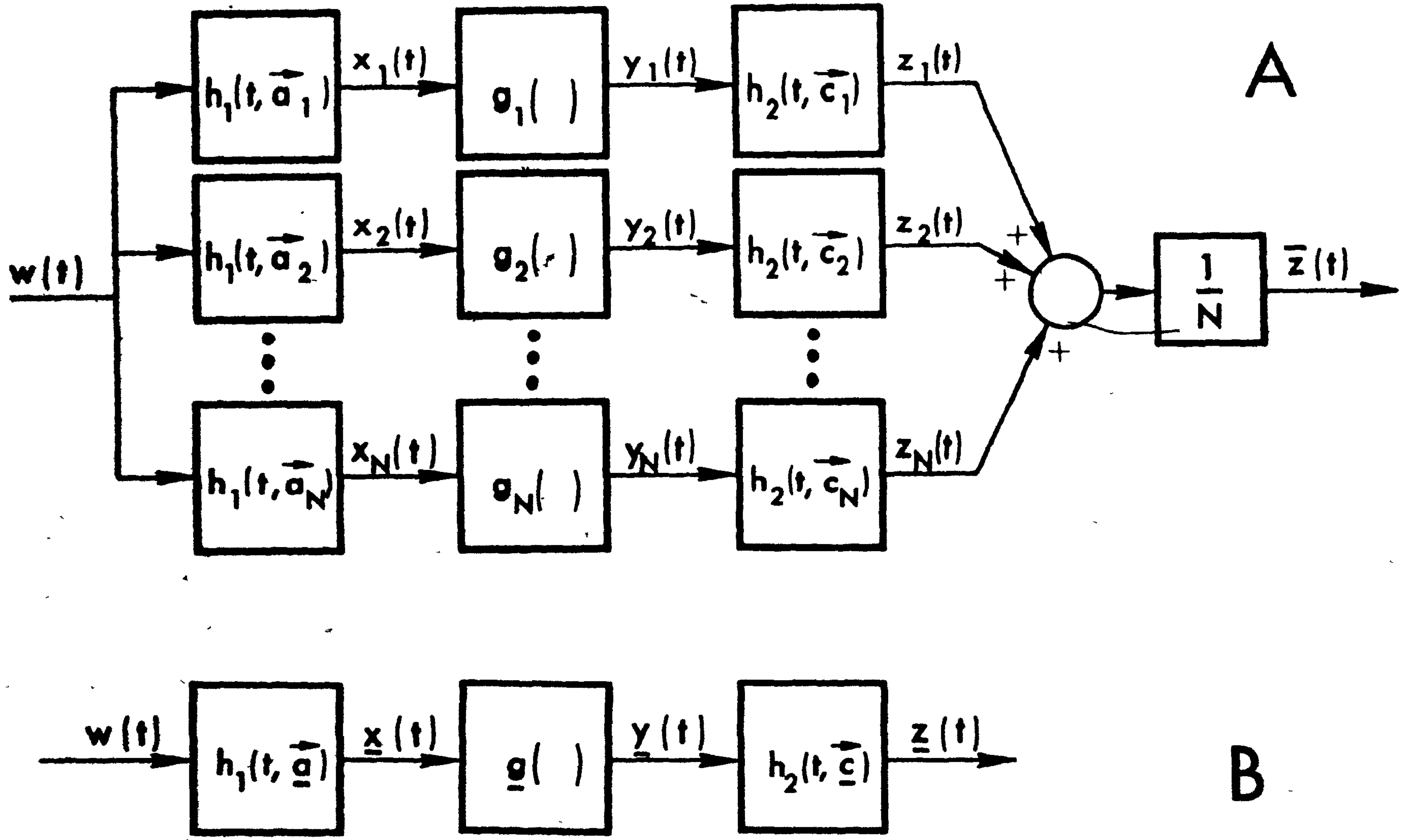

FIGURE 3-2 A MULTIPATH SYSTEM AND THE RANDOM PARAMETER MODEL FOR EACH PATH

Each path of A (Figure 2-6 reproduced) is a realization of the random parameter model in B

(3) its power spectrum (Jenkins & Watts, 1969) over an interval of time.

The derivation of each confidence interval will be discussed.

(1) <u>The Confidence Interval for the Value of a Process Realization at any time</u>.

For any real, random process, $\underline{x}(t)$, having a mean, $\langle \underline{x}(t) \rangle$, and autocorrelation, $R_x(t_1, t_2)$, Chebyshev's Inequality (Freund, 1962); Papoulis, 1965) can be used to estimate the probability that the random process will lie within a specified interval about the mean value of the process (i.e. to establish confidence intervals for the process). The variance of the process, Var $(\underline{x}(t))$, which can be denoted as, σ_t^2, can be written as

$$\sigma_t^2 = R_x(t,t) - \langle \underline{x}(t) \rangle^2 .$$

Chebyshev's Inequality can be used to show that

$$\Pr \{ \langle \underline{x}(t) \rangle - k\,\sigma_t < \underline{x}(t) < \langle \underline{x}(t) \rangle + k\,\sigma_t \} \geq 1 - \frac{1}{k^2}$$

where k is a positive constant. If $\underline{x}(t)$ is known to have a Gaussian probability, at any time, a smaller confidence interval than the one given by Chebyshev's Inequality can be found. For example,

$$\Pr \{ \langle \underline{x}(t) \rangle - k\,\sigma_t < \underline{x}(t) < \langle \underline{x}(t) \rangle + k\,\sigma_t \} \simeq 0.68 .$$

Table 3-1 tabulates the size of an interval about the mean, for 68, 90 and 99% confidence levels, within which an arbitrary or a Gaussian process would lie.

Distribution type	Confidence Level		
	68%	90%	99%
Any	$\pm\ 1.8\ \sigma_t$	$\pm\ 3.2\ \sigma_t$	$\pm\ 10\ \sigma_t$
Gaussian	$\pm\ \sigma_t$	$\pm\ 1.7\ \sigma_t$	$\pm\ 2.5\ \sigma_t$

TABLE 3-1 REQUIRED INTERVAL SIZE ABOUT MEAN TO INCLUDE, AT VARIOUS CONFIDENCE LEVELS, ALL RANDOM PROCESSES

Thus, if the random process, $\underline{x}(t)$, corresponds to a waveform, $x_i(t)$, in the i^{th} path of a multipath system and if $\langle \underline{x}(t) \rangle$ and $R_x(t_1, t_2)$ are known, then one can say that about 90% of all possible systems will have a value, at time t, at this point that lies within

$$\langle \underline{x}(t) \rangle \pm 3.2 \left[R_x(t, t) - \langle \underline{x}(t) \rangle^2 \right]^{\frac{1}{2}}.$$

Alternately, if one looks at the values of corresponding points in all paths of a multipath system, then about 90% of all of the system's paths will have a value at this point within this range.

(2) Confidence Interval for the Fourier Transform of a Process Realization over an Interval of Time.

Any waveform, $x(t)$, which can be considered a realization of the random process, $\underline{x}(t)$, has a Fourier transform*, $X(\omega)$, which is a realization of a random variable, $\underline{X}(\omega)$, where ω is angular frequency. It is assumed that $\langle\underline{x}(t)\rangle$ and $R_x(t_1, t_2)$ are known. The mean, $\langle\underline{X}(\omega)\rangle$, and variance, $Var(\underline{X}(\omega))$, of this random variable will be found in order to approximate $\underline{X}(\omega)$. Let $\underline{X}(\omega) = \underline{A}(\omega) + j\,\underline{B}(\omega)$ where $\underline{A}(\omega)$ and $\underline{B}(\omega)$ are the real and imaginary parts of the complex process, $\underline{X}(\omega)$, and $j = (-1)^{\frac{1}{2}}$. Taking expectations of the definition of the Fourier transform gives

$$\langle\underline{X}(\omega)\rangle = \int_{-\infty}^{\infty} \langle\underline{x}(t)\rangle\, e^{-j\omega t}\, dt$$

$$= \langle\underline{A}(\omega)\rangle + j\langle\underline{B}(\omega)\rangle .$$

Thus $\langle\underline{X}(\omega)\rangle$ can be found from $\langle\underline{x}(t)\rangle$, which is known. Also,

$$\langle\underline{X}(\omega_1)\,\underline{X}^*(\omega_2)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_x(t_1, t_2)\, e^{-j(\omega_1 t_1 - \omega_2 t_2)}\, dt_1 dt_2$$

$$= \Gamma(\omega_1, \omega_2)$$

where $\Gamma(\omega_1, \omega_2)$ is the two dimensional Fourier transform of $R_x(t_1, t_2)$ (Papoulis, 1965). Finally, using the definition (Papoulis, 1965) of the

---

* It is assumed that the Fourier transform exists.

variance of a complex random variable gives

$$\mathrm{Var}(\underline{X}(\omega)) = \langle | \underline{X}(\omega) - \langle \underline{X}(\omega) \rangle |^2 \rangle$$

$$= \langle \underline{X}(\omega)\underline{X}^*(\omega) \rangle - \langle \underline{X}(\omega) \rangle \langle \underline{X}^*(\omega) \rangle$$

which can be evaluated using $\Gamma(\omega,\omega)$, $\langle \underline{X}(\omega) \rangle$ and $\langle \underline{X}^*(\omega) \rangle$. Thus, 90% of all Fourier transforms of any possible waveform in a path that has a corresponding random process, $\underline{x}(t)$, will have a magnitude within

$$| \underline{X}(\omega) | \pm 3.2 \, (\mathrm{Var}(\underline{X}(\omega)))^{\frac{1}{2}}$$

$$= (\langle \underline{A}(\omega) \rangle^2 + \langle \underline{B}(\omega) \rangle^2)^{\frac{1}{2}} \pm 3.2 \, (\mathrm{Var}\,(\underline{X}(\omega)))^{\frac{1}{2}},$$

and a phase within

$$\tan^{-1}\left[\frac{\underline{B}(\omega)}{\underline{A}(\omega)}\right] \pm \tan^{-1}\left[\frac{3.2 \, (\mathrm{Var}(\underline{X}(\omega)))^{\frac{1}{2}}}{(\langle \underline{A}(\omega) \rangle^2 + \langle \underline{B}(\omega) \rangle^2)^{\frac{1}{2}}}\right].$$

This confidence interval is shown in Figure 3-3.

(3) <u>Confidence Interval of the Power Spectrum of a Process Realization over an Interval of Time</u>

Consider again a waveform, x(t), which can be considered a realization of the random process, $\underline{x}(t)$. This waveform has a power spectrum*, S(ω), which can be considered a realization of the process, $\underline{S}(\omega)$.

* It is assumed the power spectrum exists.

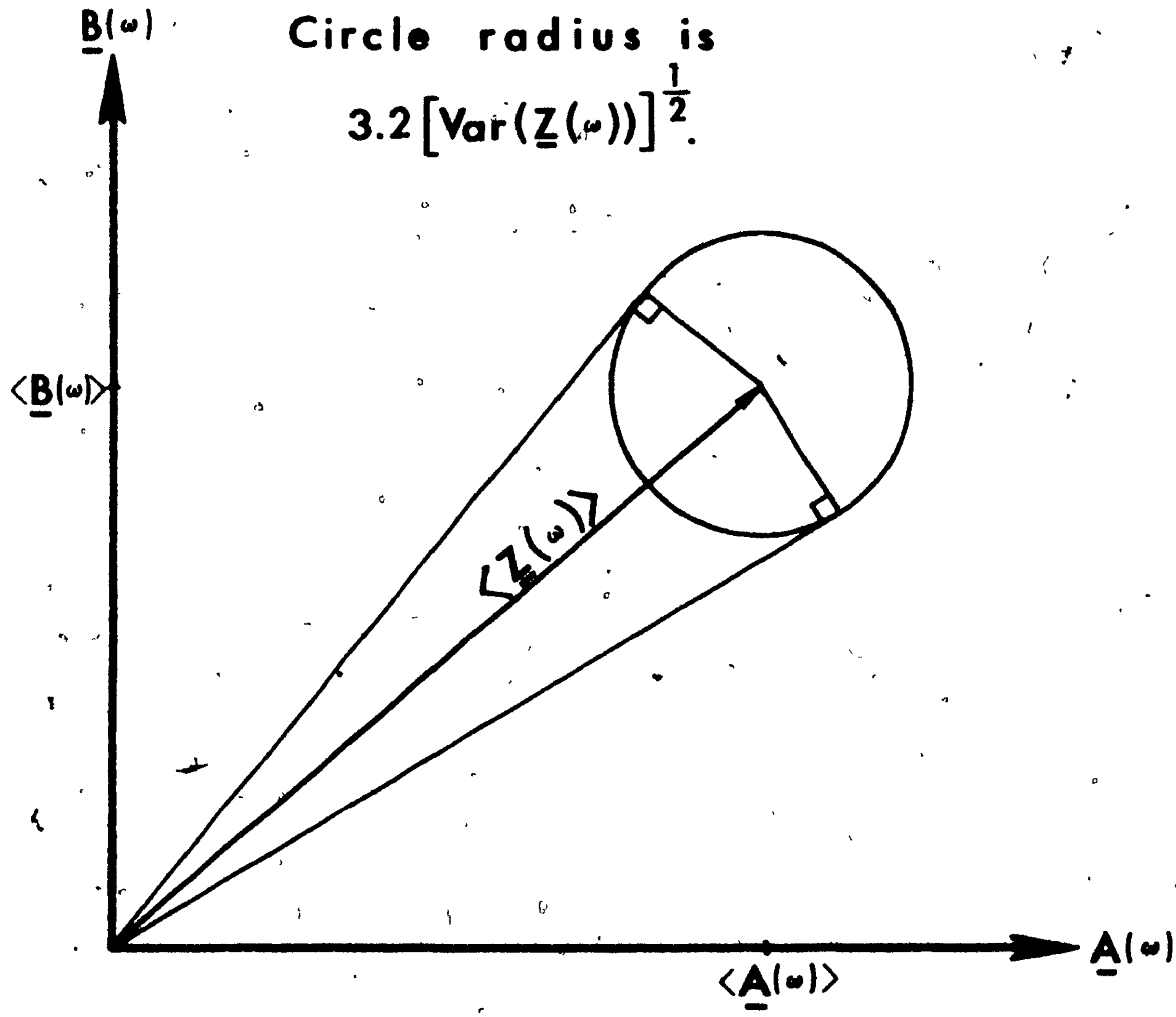

FIGURE 3-3 CONFIDENCE INTERVAL FOR FOURIER TRANSFORM ESTIMATE

Assume that the waveform is zero for $t<o$ and $t>T$. It can be shown (Koles, 1970) that the power spectrum equals

$$\underline{S}(\omega) = \frac{\underline{X}(\omega)\ \underline{X}^*(\omega)}{T} .$$

It will be shown that a confidence interval for this process can be found. Taking expectations gives

$$\langle \underline{S}(\omega) \rangle = \frac{\Gamma\ (\omega,\omega)}{T} .$$

Also, $$\mathrm{Var}(\underline{S}(\omega)) = \langle \underline{S}^2(\omega) \rangle - \langle \underline{S}\ \omega) \rangle^2 .$$

Now

$$\langle \underline{S}^2(\omega) \rangle = \frac{\langle \underline{X}^2(\omega)\ \underline{X}^{*2}(\omega) \rangle}{T^2}$$

$$= \frac{1}{T^2}\int_o^T\int_o^T\int_o^T\int_o^T \langle \underline{x}(t_1)\underline{x}(t_2)\underline{x}(t_3)\underline{x}(t_4) \rangle\, e^{-j\omega(t_1+t_2-t_3-t_4)}\, dt_1 dt_2 dt_3 dt_4$$

and

$$\langle \underline{S}(\omega) \rangle^2 = \frac{1}{T^2}\int_o^T\int_o^T\int_o^T\int_o^T \langle \underline{x}(t_1)\underline{x}(t_2) \rangle \langle \underline{x}(t_3)\underline{x}(t_4) \rangle\, e^{-j\omega(t_1-t_2+t_3-t_4)}\, dt_1 dt_2 dt_3 dt_4 .$$

Thus the variance can be evaluated since the autocorrelation of the input process is known and the fourth moment of the input can be calculated for the processes considered. Confidence intervals can be found as before.

3.3 OBTAINING THE MEANS AND AUTOCORRELATIONS OF VARIABLES IN A MULTIPATH MODEL

This section will demonstrate that for any multipath model from Table 2-1, the mean and autocorrelation of the process associated with any block output or the model output can theoretically be found. In Section 3.4, a systematic method of approximately analysing multipath models is listed. It should be pointed out that for more complex systems, the systematic method requires numerical solutions.

Since each path of a multipath model is considered to be a realization of a known random parameter model, any path in the multipath model can be described once the moments of each variable in the random parameter cascade are found. This is always possible since it will be shown that the input/output moment relationship can theoretically be obtained for any random linear block or any static nonlinear block.

3.3.1 Moment Relationships for a Random Linear Block

Consider the random parameter model of Figure 3-4. $x(t,\underline{a})$ is the input random process and $h(t,\underline{b})$ is the random impulse response of the model, where $\underline{a}$ and $\underline{b}$ are random variables associated with each process. For convenience it is assumed that both $x(t,\underline{a})$ and $h(t,\underline{b})$ each contain only one random variable. If each contains several random variables, the discussion still applies, but $\underline{a}$ and $\underline{b}$ should be replaced by the vectors $\vec{\underline{a}}$ and $\vec{\underline{b}}$ respectively. In addition, $x(t,\underline{a})$ and $h(t,\underline{b})$ will be used interchangeably with $\underline{x}(t)$ and $\underline{h}(t)$.

It is assumed that all n^{th} order moments of the input, $<\underline{x}(t_1)\ \underline{x}(t_2) \ldots \underline{x}(t_n)>$, and of the random impulse response, $<\underline{h}(t_1)\underline{h}(t_2)\ldots\underline{h}(t_n)>$, (where n is an integer) are known. Since the n^{th} order moments contain only one random variable, only the first order density of this random variable is necessary to evaluate the n^{th} order moment. For example,

$$<\underline{x}(t_1) \ldots \underline{x}(t_n)> = \int_{\text{all } a} x(t_1,a) \ldots x(t_n,a)\, f(a)da$$

where $f(a)$ is the first order probability density of $\underline{a}$ which is assumed to be known. It is also assumed that the random variables of the input and the model are independent and that $\underline{x}(t)$ and $\underline{h}(t)$ are real.

The n^{th} order moment of the output, $<\underline{y}(t_1)\ \underline{y}(t_2) \ldots \underline{y}(t_n)>$, is

$$\left\langle \int_0^\infty \underline{h}(v_1)\underline{x}(t_1-v_1)dv_1 \ \ldots \ \int_0^\infty \underline{h}(v_n)\underline{x}(t_n-v_n)dv_N \right\rangle$$

$$= \underbrace{\ldots \int \ldots}_{n \text{ times}} <\underline{x}(t_1-v_1) \ldots \underline{x}(t_n-v_n)> <\underline{h}(v_1) \ldots \underline{h}(v_n)> \underbrace{\ldots dv_1 \ldots}_{n \text{ times}}$$

This will be denoted as

$$<\underline{y}(t_1) \ldots \underline{y}(t_n)> = <\underline{x}(t_1) \ldots \underline{x}(t_n)> \underbrace{* \ldots *}_{n \text{ times}} <\underline{h}(t_1) \ldots \underline{h}(t_n)> \qquad (3\text{-}3\text{-}1)$$

where $*$ denotes convolution.

In particular, the mean output is

$$<\underline{y}(t_1)> = <\underline{x}(t_1)> * <\underline{h}(t_1)> \qquad (3\text{-}3\text{-}2)$$

and the autocorrelation of the output is

$$< \underline{y}(t_1)\underline{y}(t_2) > = < \underline{x}(t_1)\underline{x}(t_2) > * * < \underline{h}(t_1)\underline{h}(t_2) >$$

$$= R_x(t_1, t_2) * * R_h(t_1, t_2) \qquad (3\text{-}3\text{-}3)$$

where $R_x(t_1,t_2)$ and $R_h(t_1,t_2)$ are the input and system autocorrelations respectively. Finally, the joint moment of order $r + s$ is

$$< \underline{y}^r(t_1)\underline{y}^s(t_2) > = < \underline{x}^r(t_1)\underline{x}^s(t_2) > \underbrace{* \cdots *}_{\substack{r+s \\ \text{times}}} < \underline{h}^r(t_1)\underline{h}^s(t_2) > .$$

Note that the mean and autocorrelation of the output of this random linear system require only the first and second moments of the input. When the input is deterministic, these relationships simplify to

$$< \underline{y}(t_1) > = x(t_1) * < \underline{h}(t_1) >$$

and

$$< \underline{y}(t_1)\underline{y}(t_2) > = x(t_1) * x(t_2) * R_h(t_1,t_2) .$$

Since $\underline{h}(t,\underline{s})$ is generally a nonstationary process, $\underline{y}(t)$ is also nonstationary.

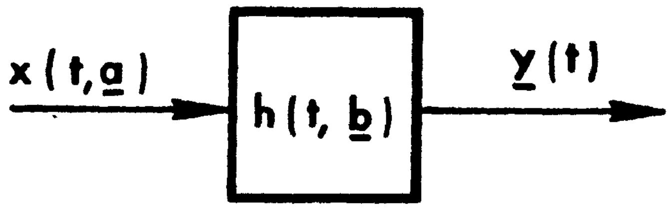

FIGURE 3-4 LINEAR RANDOM PARAMETER MODEL

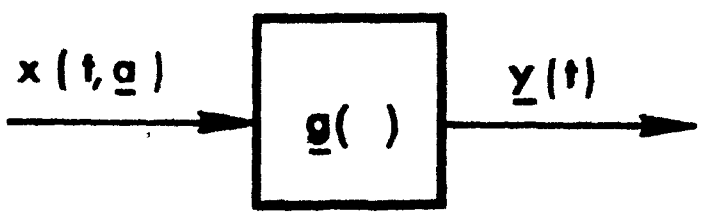

FIGURE 3-5 RANDOM STATIC NONLINEAR MODEL

3.3.2 Moment Relationships for a Random Static Nonlinear Block

Consider the random static nonlinear model in Figure 3-5. $\underline{x}(t)$ is the input process with known n^{th} order moments, $< \underline{x}(t_1) \ldots \underline{x}(t_n) >$, and the random static nonlinearity, $\underline{g}(\ )$, is represented by the power series,

$$\underline{y}(t) = \sum_{i=0}^{K} \underline{b}_i \, \underline{x}^i (t) \ .$$

The coefficients, $\underline{b}_0, \underline{b}_1, \ldots$ are random variables and all of the moments, $< \underline{b}_0^{P_0} \underline{b}_1^{P_1} \cdots \underline{b}_K^{P_K} >$, are known, where $P_0 + P_1 + \ldots + P_K = n$ and $P_0, P_1, \ldots, n$ and K are all integers. The n^{th} order moment of the output is

$$< \underline{y}(t_1) \cdots \underline{y}(t_n) > = < (\sum_{i=0}^{K} \underline{b}_i \, \underline{x}^i(t_1)) \cdots (\sum_{j=0}^{K} \underline{b}_j \, \underline{x}^j(t_n)) >$$

$$= \sum_{\substack{\text{all} \\ \text{products}}} < \underline{b}_0^{P_0} \cdots \underline{b}_K^{P_K} > \ < \underline{x}^{q_1}(t_1) \cdots \underline{x}^{q_n}(t_n) >$$

where $P_0 + \cdots + P_K = n$, and $q_i = 0, 1, \cdots, K$.

In particular, the mean output is

$$< \underline{y}(t) > = \sum_{i=0}^{K} < \underline{b}_i > \ < \underline{x}^i(t) >$$

and the autocorrelation of the output is

$$< \underline{y}(t_1)\underline{y}(t_2) > = \sum_{\substack{\text{all} \\ \text{products}}} < \underline{b}_0^{p_0} \cdots \underline{b}_K^{p_K} > \; < \underline{x}^{g_1}(t_1)\underline{x}^{g_2}(t_2) >$$

where $\sum_{i=0}^{K} p_i = 2$ and $g_1, g_2 = 0, 1, \cdots, K.$

Note that the n^{th} order moment of the output of this K^{th} order random polynomial requires the $(nK)^{th}$ order moment of the input. Finally, the output moment of order $r + s$ is

$$< \underline{y}^r(t_1)\underline{y}^s(t_2) > = \sum_{\substack{\text{all} \\ \text{products}}} < \underline{b}_0^{p_0} \cdots \underline{b}_K^{p_K} > < \underline{x}^{rg_1}(t_1)\underline{x}^{sg_2}(t_2) >$$

where $p_0 + \cdots + p_K = r+s$ and $g_1, g_2 = 0, 1, \cdots, K.$
When the input is deterministic, the expectation over the input can be dropped.

It has been shown that for any random linear or static nonlinear system, the moments of the output can be related to the moments of the input. Thus, the mean and autocorrelation of any variable in any cascade of these random models can theoretically be obtained. In particular, the mean and autocorrelation of the process which is the output of the random cascade can be found. It will now be shown that the mean and autocorrelation of this random process (which corresponds to the output of any path of a multipath model) can be related to the mean and autocorrelation of the random process corresponding to the output of the multipath model.

3.3.3 Obtaining the Mean and Autocorrelation Corresponding to the Output of a Multipath System

For the class of multipath systems considered, all N paths are chosen independently. Let the output of the i^{th} path correspond to a process, $x(t, \underline{a}_i)$. Since the paths are chosen independently, a set of N independent processes result that have the same mean and autocorrelation:

$$< x(t,\underline{a}_i) > = < x(t,\underline{a}) > \tag{3-3-4}$$

$$< x(t_1,\underline{a}_i)x(t_2,\underline{a}_i) > = R_x(t_1,t_2). \tag{3-3-5}$$

Since all of these processes are independent, then

$$< x(t_1,\underline{a}_i)x(t_2,\underline{a}_j) > = < x(t_1,\underline{a}_i) > < x(t_2,\underline{a}_j) > \tag{3-3-6}$$

The process, $\underline{y}(t)$, corresponding to the multipath system's output, is the average of all of the independent processes. That is,

$$\underline{y}(t) = \frac{1}{N} \sum_{i=1}^{N} x(t,\underline{a}_i) .$$

The mean and autocorrelation of $\underline{y}(t)$ will be found. The development is analogous to Theorem 8.2 in Freund, 1962. Figure 3-6 represents the problem schematically.

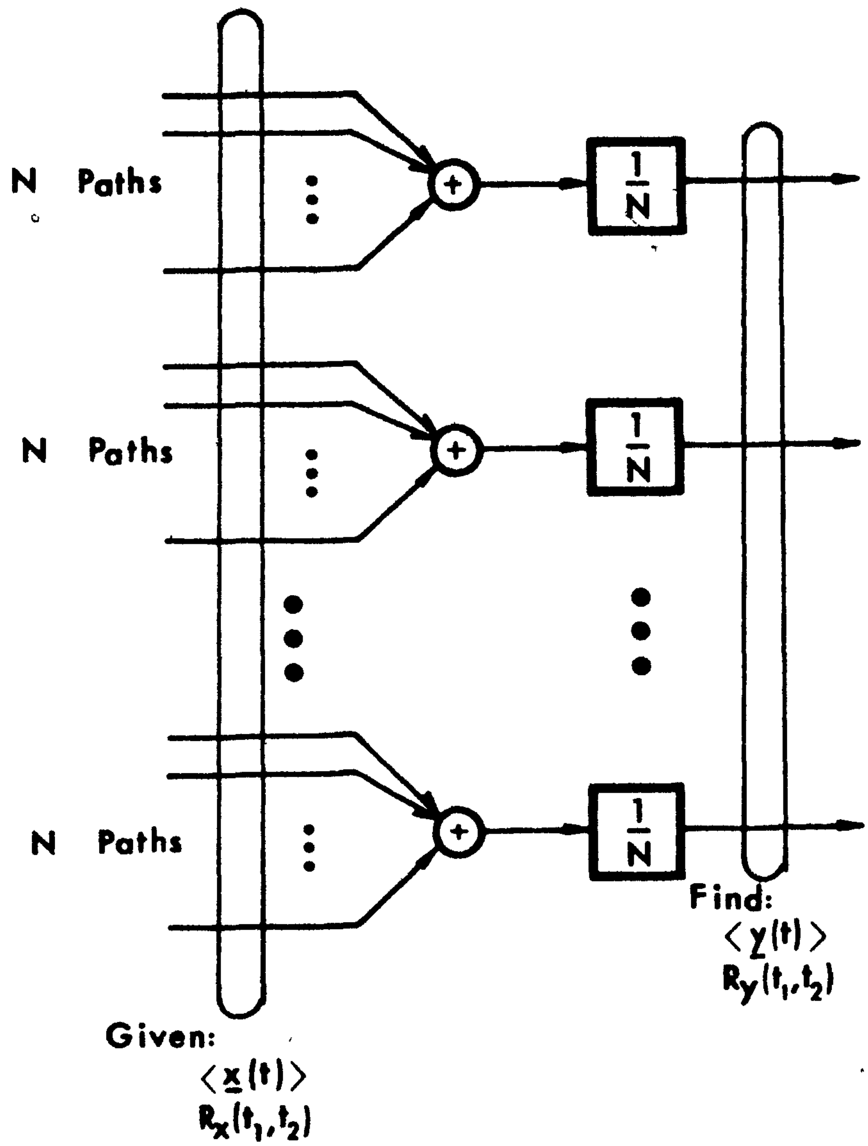

FIGURE 3-6 SCHEMATIC REPRESENTATION OF PROBLEM

The mean of the process corresponding to the output of the multipath system can be written as

$$< \underline{y}(t) > = < \frac{1}{N} \sum_{i=1}^{N} x(t,\underline{a}_i) > .$$

Using (3-3-4) gives

$$< \underline{y}(t) > = < x(t,\underline{a}) > . \tag{3-3-7}$$

The autocorrelation of the process corresponding to the multipath system's output is

$$< \underline{y}(t_1)\underline{y}(t_2) > = \left\langle \frac{1}{N} \sum_{i=1} x(t_1,\underline{a}_i) \frac{1}{N} \sum_{j=1}^{N} x(t_2,\underline{a}_j) \right\rangle$$

$$= \frac{1}{N^2} \left\langle \sum_{i=1}^{N} x(t_1,\underline{a}_i)x(t_2,\underline{a}_i) + \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} x(t_1,\underline{a}_i)x(t_2,\underline{a}_j) \right\rangle ,$$

However, by (3-3-5) and (3-3-6)

$$= \frac{1}{N^2} \left[N\, R_x(t_1,t_2) + \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} < x(t_1,\underline{a}_i) > < x(t_2,\underline{a}_j) > \right] ,$$

which by (3-3-4)

$$= \frac{1}{N^2} \left[N\, R_x(t_1,t_2) + N(N-1) < x(t_1,\underline{a}) > < x(t_2,\underline{a}) > \right] .$$

Therefore,

$$< \underline{y}(t_1)\underline{y}(t_2) > = \frac{R_x(t_1,t_2)}{N} + \frac{(N-1)}{N} < x(t_1,\underline{a}) > < x(t_2,\underline{a}) >. \qquad (3\text{-}3\text{-}8)$$

Thus, the mean and autocorrelation corresponding to the multipath system's output can be related to those of the path's outputs. Therefore, it has been demonstrated that a mean and autocorrelation can be found for all points in any multipath system from the class of Table 2-1.

It is interesting to note that when a sampling viewpoint of this situation is taken, a result consistent with the Central Limit Theorem* (Papoulis, 1965) is obtained. To see this, note that the variance corresponding to the outputs of the paths, at any time t, is

$$\text{Var}\ (\underline{x}(t)) = R_x(t,\ t) - < \underline{x}(t) >^2$$

and that the variance corresponding to the output of the multipath system, at time t, is,

$$\text{Var}\ (\underline{y}(t)) = R_y(t,t) - < \underline{y}(t) >^2$$

$$= \frac{R_x(t,t) + (N-1) < \underline{x}(t) >^2}{N} - < \underline{x}(t) >^2.$$

---

* The Central Limit Theorem says that when groups of N samples are independently drawn from a population with a mean, m, and variance, σ^2, then the averages of the groups of N samples have a density that is approximately Gaussian with a mean, m, and variance, $\frac{\sigma^2}{N}$.

Therefore,

$$\text{Var}\ (\underline{y}(t)) = \frac{\text{Var}\ \underline{x}(t)}{N} . \tag{3-3-9}$$

Thus, at any time t, all the outputs of the paths form a set of N random variables that are considered to have been randomly selected from an infinite population of outputs that has a mean, $\langle \underline{x}(t) \rangle$, and variance, Var $(\underline{x}(t))$. However, the system output is the average of these N independent samples and has a mean value,

$$\langle \underline{y}(t) \rangle = \langle \underline{x}(t) \rangle$$

and a variance,

$$\text{Var}(\underline{y}(t)) = \frac{\text{Var}\ (\underline{x}(t))}{N} ,$$

which is what would be predicted by the Central Limit Theorem. In addition, this theorem says that the probability density of all possible multipath system outputs, at time t, is approximately Gaussianly distributed. The error in this approximation can be found (Papoulis, 1965).

It will be shown in Chapter 4 that under certain conditions $\langle \underline{y}(t) \rangle$ is linearly related to the system input in many multipath systems. For these cases, any difference between the actual system output and the expected system output can be considered a distortion. A convenient output signal-to-distortion ratio* is formed by dividing the output mean

* This is not the usual "power ratio" used in communication textbooks but is the reciprocal of the statistical measure, the coefficient of variation.

by the output standard deviation. The signal-to-distortion ratio of the output is

$$\frac{< \underline{y}(t) >}{[\mathrm{Var}(\underline{y}(t))]^{\frac{1}{2}}} = \frac{\sqrt{N} < \underline{x}(t) >}{[\mathrm{Var}(\underline{x}(t))]^{\frac{1}{2}}} .$$

As N increases, the output signal to distortion ratio increases as $\sqrt{N}$. This is also the improvement of this ratio at the multipath system output over the value of this ratio at the output of a single path.

Equation (3-3-9) shows that as N increases, the output of the multipath system becomes more nearly deterministic or more certain regardless of which paths have been selected. For an infinite path system the output equals the expected output. In this case, the system output, its Fourier transform and its power spectrum are all deterministic and are known exactly.

3.4 THE SYSTEMATIC METHOD OF ANALYSING MULTIPATH SYSTEMS

The previous sections have shown that different realizations of a multipath model, of Table 2-1, will have different waveforms in it with the same input applied. It was shown that the mean and autocorrelation corresponding to this waveform variation can always be found. Section 3.2 showed that these quantities can be used to determine confidence intervals for the possible range of values for any waveform, its Fourier transform or its power spectrum.

This suggests a systematic method of analysing multipath models:

(1) Set up the random parameter model corresponding to a randomly selected path.

(2) Find the relationship between the input and output moments for each block separately in the random parameter model. This is always possible since all blocks are either linear or static nonlinear systems.

(3) Using the relationships in (2) and the known system input, determine the mean and autocorrelation at each point along the cascade in the random parameter model. Note that to find the mean and autocorrelation of the output of the last static nonlinearity in the cascade, in general, all the moments of the output of each preceeding block must be found.

(4) Find the mean and autocorrelation corresponding to the multipath system output from the mean and autocorrelation found in (3).

(5) For any point in the multipath system, use the results of Section 3.2 to approximate its value, at any time, and to approximate its Fourier transform or power spectrum over an interval of time.

When property (5) of Table 2-1 cannot be assumed, the system can still be analysed but means and autocorrelations must be found without breaking up the random parameter model. This is generally much more difficult to do.

It should be pointed out that the multipath system can have more than one input. In particular, independent noise sources can often be added in each path without changing the approach. This will be illustrated in Chapter 4. In addition, Chapter 4 will find several input/output moment relationships for some random parameter models and will discuss the problems involved in applying the systematic method to multipath systems.

CHAPTER 4

THE ANALYSIS OF MULTIPATH MODELS

4.0 INTRODUCTION

The purpose of this chapter is to show how the systematic method described in Section 3.4 is used to analyse multipath systems. To this end, the chapter is divided into two parts. The first part, (Section 4.1) determines some of the moment relationships required in the second stage of the systematic method. This is done for a random low pass filter and for several static nonlinearities. For all random static nonlinear systems considered, outputs are linearly* related to the inputs under certain conditions. Thus, any multipath system employing realizations of these random systems in each path will have, under certain conditions, an expected output that is linearly related to the expected input. Also, if this multipath system has an infinite number of paths, and if the input is deterministic, then the <u>actual</u> output of this system is, under the previous conditions, linearly related to the input.

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* Although the relationships that will be obtained will be linear, the equivalent systems are not. The relation, $y = mx + b$, is an example of a linear function that represents a nonlinear system since the system does not satisfy the Superposition Principle due to the constant, b. (Dorf, 1967).

The second part (Section 4.2) of this chapter illustrates the systematic method, using some of the moment relationships found in the first part, and indicates how the analysis of the two multipath models in Figures 4-8 and 4-10 would proceed. Independent noise sources have been added in Figure 4-8 to show how noise can be handled with this approach.

4.1 MOMENT RELATIONSHIPS OF SOME RANDOM PARAMETER MODELS

4.1.1 Moment Relationships of a Random, First Order Low Pass Filter

Consider the system in Figure 4-1 (a) that has a random impulse response, $e^{-\underline{b}t}$, where the angular break frequency, $\underline{b}$, is a random variable with a probability density given by

$$f(b) = \begin{cases} \frac{1}{S}, & T \leq b \leq T + S \\ 0, & \text{elsewhere}. \end{cases} \qquad (4\text{-}1\text{-}1)$$

The expected impulse response of this system is

$$\langle e^{-\underline{b}t} \rangle = \int_{T}^{T+S} e^{-bt} f(b)\, db .$$

Using (4-1-1) gives

$$\langle e^{-\underline{b}t} \rangle = \frac{e^{-Tt} - e^{-(T+S)t}}{tS} . \qquad (4\text{-}1\text{-}2)$$

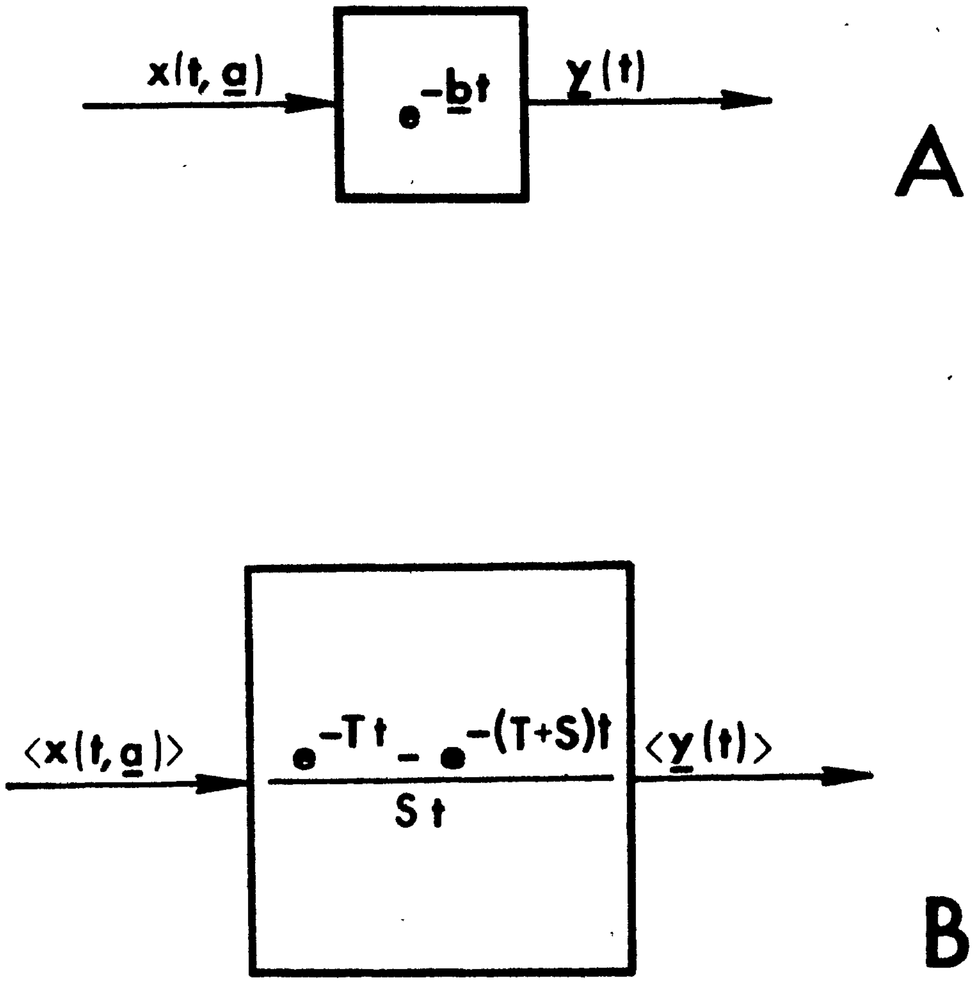

FIGURE 4-1 RANDOM LOW PASS FILTER AND RELATIONSHIP BETWEEN MEANS

(b) shows relationship between input and output means of random parameter model in (a).

Note that the expected impulse response has a different shape than any realization of $e^{-\underline{b}t}$. The autocorrelation of this system is

$$\langle e^{-\underline{b}t_1} e^{-\underline{b}t_2} \rangle = \int_T^{T+S} e^{-b(t_1+t_2)} f(b)\, db.$$

Using (4-1-1) gives

$$\langle e^{-\underline{b}(t_1+t_2)} \rangle = \frac{e^{-T(t_1+t_2)} - e^{-(T+S)(t_1+t_2)}}{tS}. \qquad (4\text{-}1\text{-}3)$$

Employing (3-3-2), the expected output (see Figure 4-1(b)) is

$$< \underline{y}(t) > = \frac{e^{-Tt} - e^{-(T+S)t}}{tS} * < \underline{x}(t) > \qquad (4\text{-}1\text{-}4)$$

and using (3-3-3), the output autocorrelation is

$$R_y(t_1,t_2) = \left(e^{-T(t_1+t_2)} - e^{-(T+S)(t_1+t_2)} \right) ** R_x(t_1,t_2).$$

Similarly, employing (3-3-1)

$$\langle \underline{y}(t_1) \ldots \underline{y}(t_n) \rangle$$

can be found.

4.1.2 Moment Relationships for a Step Element with a Uniformly Distributed Threshold

Figure 4-2 (a) shows a random static nonlinearity that is a step function with a uniformly distributed threshold. The output, $\underline{y}(t,\underline{a},\underline{b})$, which is abbreviated as, $\underline{y}(t)$, is written as

$$\underline{y}(t) = \begin{cases} 0\,, & x(t,\underline{a}) < \underline{b} \\ M\,, & x(t,\underline{a}) \geq \underline{b} \end{cases} .$$

where $x(t,\underline{a})$ is the input process containing, for convenience, a single random variable, $\underline{a}$, with a known probability density, $f(a)$; M is a constant and the threshold, $\underline{b}$, is a random variable having the uniform probability density of (4-1-1). The expected output is

$$\langle \underline{y}(t) \rangle = 0 \cdot \Pr\{ x(t,\underline{a}) < \underline{b} \} + M \cdot \Pr\{ x(t,\underline{a}) \geq \underline{b} \}$$

$$= M \cdot \Pr\{ \underline{b} \leq x(t,\underline{a}) \} .$$

When $x(t,\underline{a}) < T$, then $\Pr\{ \underline{b} \leq x(t,\underline{a}) \} = 0$, so that

$$\langle \underline{y}(t) \rangle = 0, \qquad \text{when} \quad x(t,\underline{a}) < T.$$

When $T \leq x(t,\underline{a}) \leq T + S$,

$$\Pr\{ \underline{b} \leq x(t,\underline{a}) \} = \int_{\text{all } a} \int_{T}^{x(t,\underline{a})} \frac{1}{S}\, db\, f(a)da$$

$$= \frac{\langle x(t,\underline{a}) \rangle - T}{S} .$$

Thus,

$$< \underline{y}(t) > = \frac{M}{S} (< x(t,\underline{a}) > - T), \qquad T \leq x(t,\underline{a}) \leq T + S.$$

Finally, when $x(t,\underline{a}) \geq T + S$, then $\Pr \{ \underline{b} \leq x(t,\underline{a}) \} = 1$, so that

$$< \underline{y}(t) > = M, \qquad x(t,\underline{a}) > T + S .$$

Combining these results, the expected output is

$$<\underline{y}(t)> = \begin{cases} 0, & x(t,\underline{a}) < T \\ \frac{M}{S} (< x(t,\underline{a}) > - T), & T \leq x(t,\underline{a}) \leq T + S \\ M, & x(t,\underline{a}) > T + S \end{cases}$$

Thus when $x(t,\underline{a})$ is restricted to the range of thresholds, the expected output is linearly related to the expected input despite the fact that the random parameter model is nonlinear. If the input is not restricted to this range the expected output requires a numerical solution unless the probability density of $x(t,\underline{a})$ is known. If the input is deterministic, the relationship shown in Figure 4-2(b) results.

The autocorrelation of the output will now be found assuming that the input is restricted to the range of thresholds. It can be written as

$$< \underline{y}(t_1)\, \underline{y}(t_2) > = 0 + M^2 \cdot \Pr \{ x(t_1,\underline{a}) \geq \underline{b} \text{ and } x(t_2,\underline{a}) \geq \underline{b} \} .$$

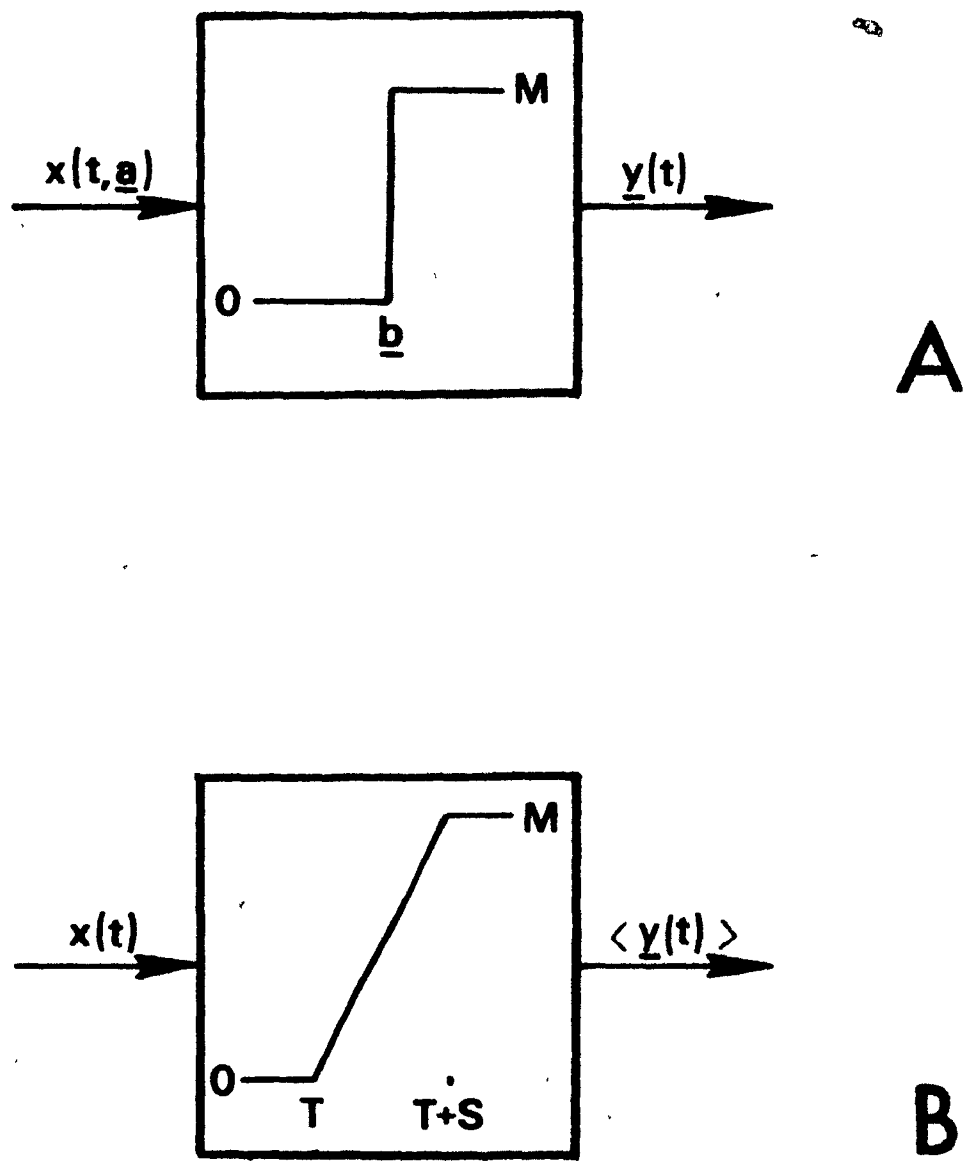

FIGURE 4-2 RANDOM PARAMETER MODEL AND EXPECTED OUTPUT WITH DETERMINISTIC INPUT

(b) shows the static relationship between a deterministic input and the expected output of the random parameter model shown in (a) which is a step function with a uniformly distributed threshold.

Since $\underline{b}$ must be less than or equal to the lesser of $x(t_1,\underline{a})$ and $x(t_2,\underline{a})$, define

$$\underline{q} = \text{Min}\ (x(t_1,\underline{a}),\ x(t_2,\underline{a}))$$

and let $f(q)$ be the probability density of $\underline{q}$.

Then

$$R_y\ (t_1,t_2) = M^2 \cdot \Pr\ \{\ \underline{q} \geq b\ \}$$

$$= M^2 \int_{\text{all } q} \int_T^q \frac{1}{S}\ db\ f(q)\ dq$$

where the integration is over the set of $\underline{q}$ and $\underline{b}$ in the preceeding probability and (4-1-1) has been used. Thus

$$R_y\ (t_1,t_2) = \frac{M^2}{S}\ (\ < \underline{q} > - T\ )\ . \qquad (4\text{-}1\text{-}5)$$

Section 4.2.1 will discuss the problems involved in finding this autocorrelation. Note that when $x(t,\underline{a})$ is a monotonically increasing function, the autocorrelation can be written as

$$R_y\ (t_1,t_2) = \frac{M^2}{S}\ (\ < x(t_2,\underline{a}) > -T\ )$$

Similarly, it can be shown that

$$< \underline{y}(t_1)\ \ldots\ \underline{y}(t_n) > = \frac{M^n}{S}\ (< \underline{q}_n > - T\ )$$

where $\underline{q}_n = \text{Min}\ (x\ (t_1,\underline{a}),\ \ldots,\ x(t_n,\underline{a}))$ and $T \leq \underline{x}(t) \leq T + S$.

Thus, all output moments can theoretically be related to the input moments.

4.1.3 Moment Relationships for a Uniformly Distributed Random Static Nonlinearity which has a Threshold and Saturates

Consider the random parameter model shown in Figure 4-3 (a). The input is a random process having (for convenience) a single random variable, $\underline{a}$, which has a probability density, f(a). The output is

$$\underline{y}(t) = \begin{cases} 0, & x\,(t,\underline{a}) < \underline{b} \\ \underline{g}(x\,(t,\underline{a})-\underline{b}), & \underline{b} \leq x\,(t,\underline{a}) \leq \underline{b} + R \\ M, & x\,(t,\underline{a}) > \underline{b} + R \end{cases} \qquad (4\text{-}1\text{-}6)$$

where the saturation value, M, and range, R, are constants such that R < S, the threshold, $\underline{b}$, is a random variable having the uniform probability density of (4-1-1) and $\underline{g}$ ( ) is a random static nonlinearity whose probability density is denoted as f(g ( )) or f(g). All random variables are assumed to be independent.

If it is assumed that the input is limited to (see Figure 4-3(b))

$$T + R \leq x(t,\underline{a}) \leq T + S \,, \qquad (4\text{-}1\text{-}7)$$

then the expected output is

$$< \underline{y}(t) > = 0 + \int_{\text{all } a} \int_{x-R}^{x} \int_{\text{all } g} g(x-b) f(g) dg \; f(b) db \; f(a) da$$

$$+ \int_{\text{all } a} M \int_{T}^{x-R} f(b) \; db \; f(a) da$$

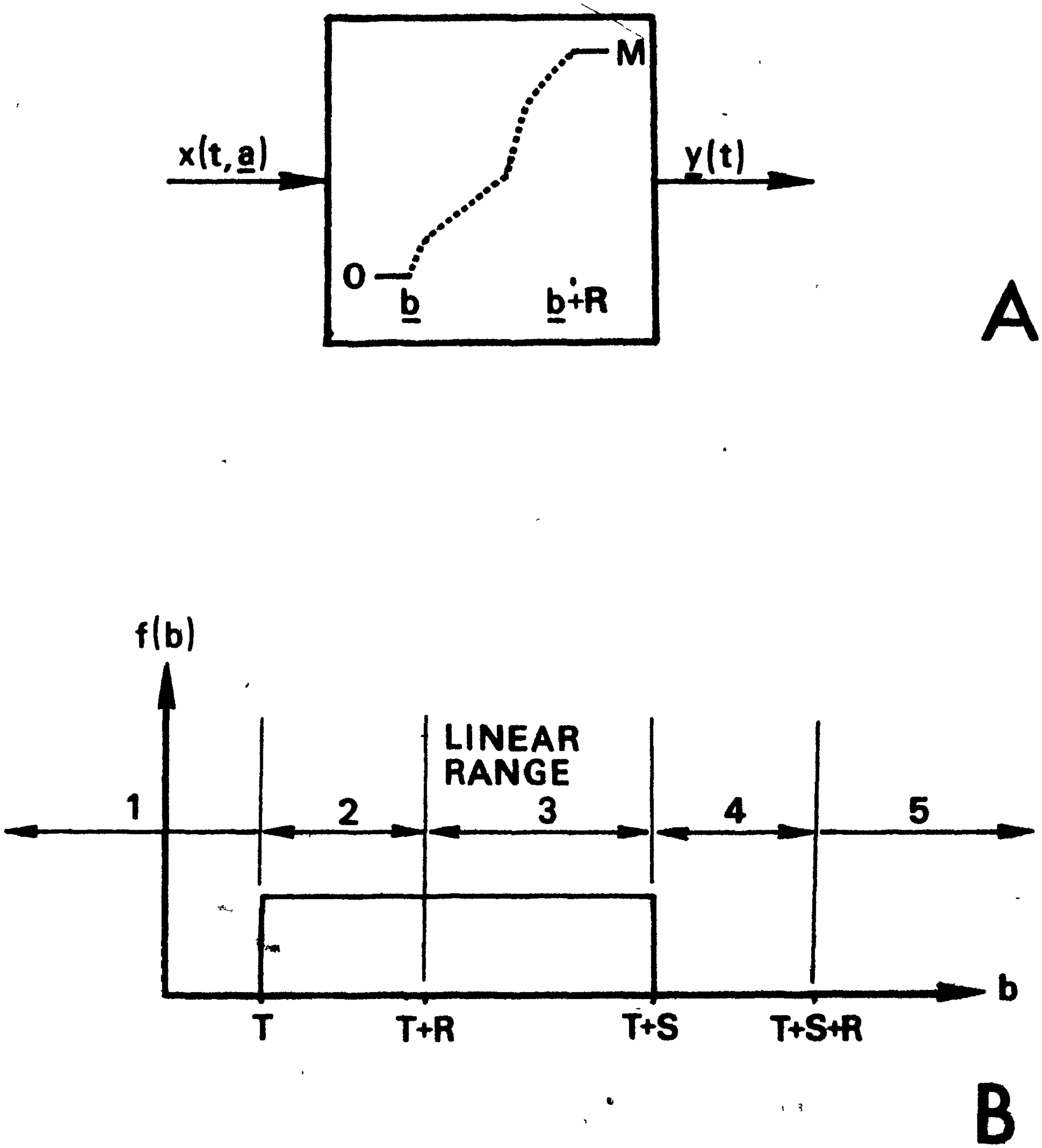

FIGURE 4-3 RANDOM STATIC NONLINEARITY WITH INPUT LINEAR RANGE COMPARED TO THRESHOLD DISTRIBUTION

(a) shows a random static nonlinearity with uniformly distributed threshold and saturation. (b) compares the threshold distribution with the linear input range of the model.

where x(t,a) has been abbreviated as x and the three terms correspond to $\underline{x}(t) < \underline{b}$, $\underline{b} \le \underline{x}(t) \le \underline{b} + R$ and $\underline{x}(t) > \underline{b} + R$ respectively. Define the random variable $\underline{v}(t) = \underline{x}(t) - \underline{b}$. Using (4-1-1) and denoting v(t) as v gives

$$< \underline{y}(t) > = \frac{1}{S} \int_{\text{all } a} \int_0^R \bar{g}(v)dv \; f(a)da + \int_{\text{all } a} \frac{M}{S} (x(t)-(R+T)) \; f(a)da,$$

where
$$\bar{g}(v) = \int_{\text{all } g} g(v)f(g)dg$$

denotes the average input/output relationship for the static nonlinearity. Note that since

$$\int_0^R \bar{g}(v)dv$$

is independent of a, the first term for $< \underline{y}(t) >$ is constant, so that*

$$\left.\begin{aligned} &< \underline{y}(t) > = A < x(t) > + B \\ &\quad \text{where} \quad A = \frac{M}{S}, \quad \text{and} \\ &\quad\quad B = \frac{1}{S}\int_0^R \bar{g}(v)dv - \frac{M}{S}(R+T) \end{aligned}\right\} \quad (4\text{-}1\text{-}8)$$

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* The author originally derived this result using a Fourier series expansion for the static nonlinearity. The derivation above, by M.J. Korenberg, is much simpler and is presented here for this reason.

provided the input is bounded by (4-1-7). Once again the expected output is linearly related to the expected input despite the fact that each realization of the random parameter model has a different threshold, saturates, and can have a different shape between threshold and saturation.

If the input range is not limited by (4-1-7), the relationship between means is complex and requires a numerical solution. If the input to the static nonlinearity is deterministic, the relationship shown in Figure 4-4 (a) results. In order to indicate how this relationship arises, assume for the moment that the shape of the random parameter model, g( ), is deterministic. The expected output is

$$< \underline{y}(t) > = \int g(x(t)-b)\ f(b)db + \int Mf(b)db$$

where the end points of each integral depends on the input. Suppose the input range is divided into five regions, 1 to 5, as shown in Figure 4-3 (b). If the above integrals are denoted as

$$I = \int g(x(t)-b)f(b)db$$

and

$$II = \int Mf(b)db$$

then Figure 4-4(b) can be used to evaluate the end points of I and II. when the input is in each of the five regions. Thus, the expected output can be written as

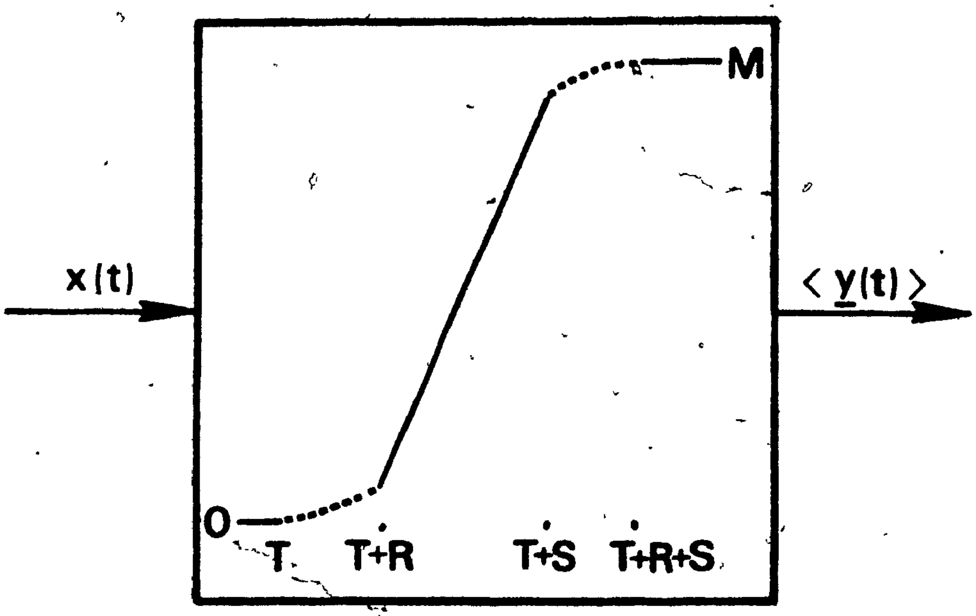
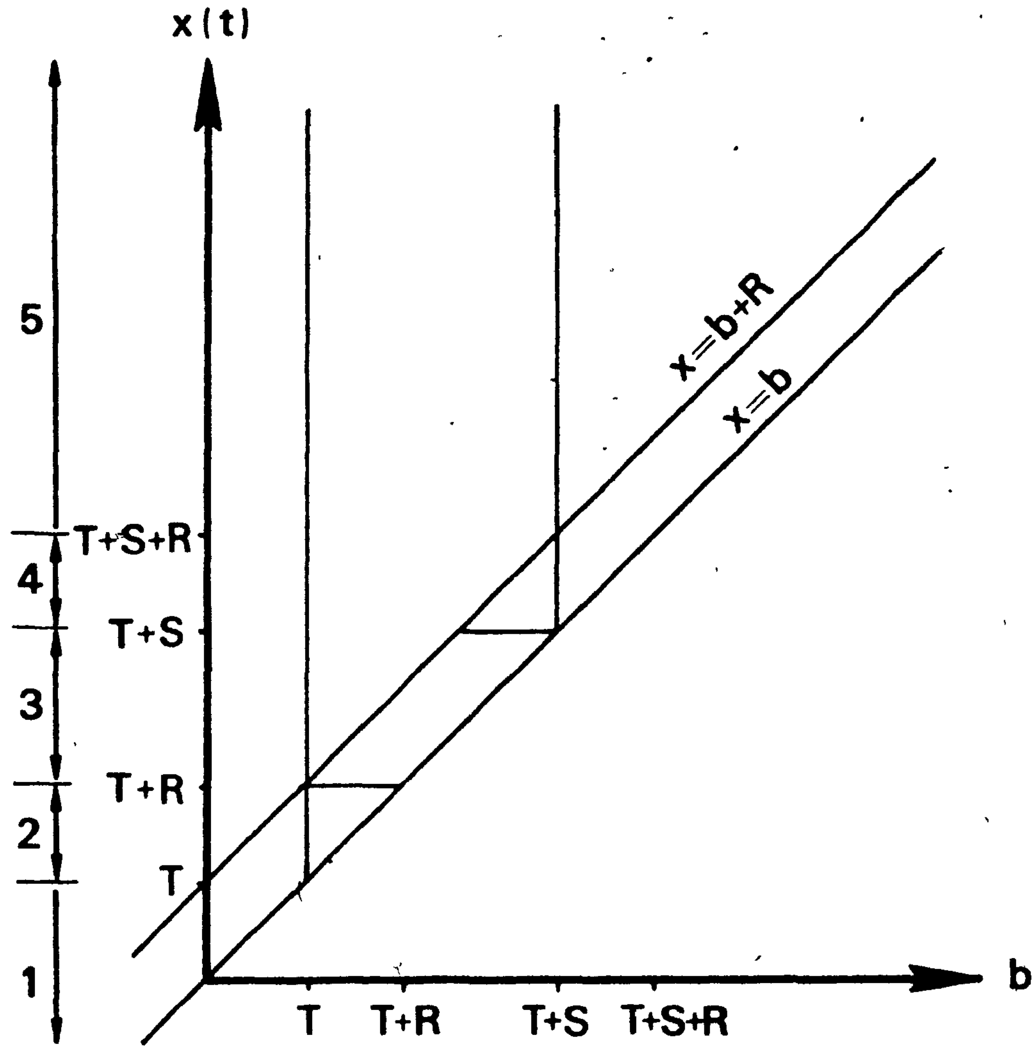


FIGURE 4-4 (a) STATIC RELATIONSHIP BETWEEN OUTPUT MEAN AND DETERMINISTIC INPUT



REGION	LIMITS OF b IN I	LIMITS OF b IN II
1	—	—
2	T, x	—
3	x−R, x	T, x−R
4	x−R, T+S	T, x−R
5	—	T, T+S

$$\langle \underline{y}(t) \rangle = \frac{1}{S}\int g(x-b)\,db + \frac{M}{S}\int db = \mathrm{I} + \mathrm{II}$$

FIGURE 4-4 (b) END POINTS TO EVALUATE EXPECTED OUTPUT

$$\langle \underline{y}(t) \rangle = \begin{cases} 0 & x(t) \le T \\ \frac{1}{S}\int_{T}^{x(t)} g(x-b)db , & T < x(t) < T+R \\ \frac{M}{S}(x(t)-(R+T)) + \frac{1}{S}\int_{0}^{R} g(V)dV , & T+R \le x(t) \le T+S \\ \frac{1}{S}\int_{x(t)-R}^{T+S} g(x(t)-b)db + \frac{M}{S}\int_{T}^{x(t)-R} db , & T+S < x(t) \le T+S+R \\ M, & x(t) > T+S+R \end{cases}$$

If the shape of the static nonlinearity had contained random variables, a similar result would be obtained except that $\bar{g}(\ )$ would appear instead of $g(\ )$.

Note that it is possible for two different random parameter models, each with realizations of different shapes, to have the same expected shape. These models would have the same relationship between input and output means.

One method of obtaining the output autocorrelation employs the Lebesgue Integral (Doob, 1965; Papoulis, 1965). The input process is divided into three regions (as shown in Figure 4-5) which defines three sets of values of $\underline{a}$ and $\underline{b}$: S_1, S_2 and S_3. For convenience, denote $x(t_1,\underline{a})$ and $x(t_2,\underline{a})$ by $\underline{x}_1$ and $\underline{x}_2$ and define

$$\underline{Q} = \text{Max}\ (\underline{x}_1, \underline{x}_2)$$

$$\underline{q} = \text{Min}\ (\underline{x}_1, \underline{x}_2)$$

and $\underline{\Delta} = \underline{Q} - \underline{q}$.

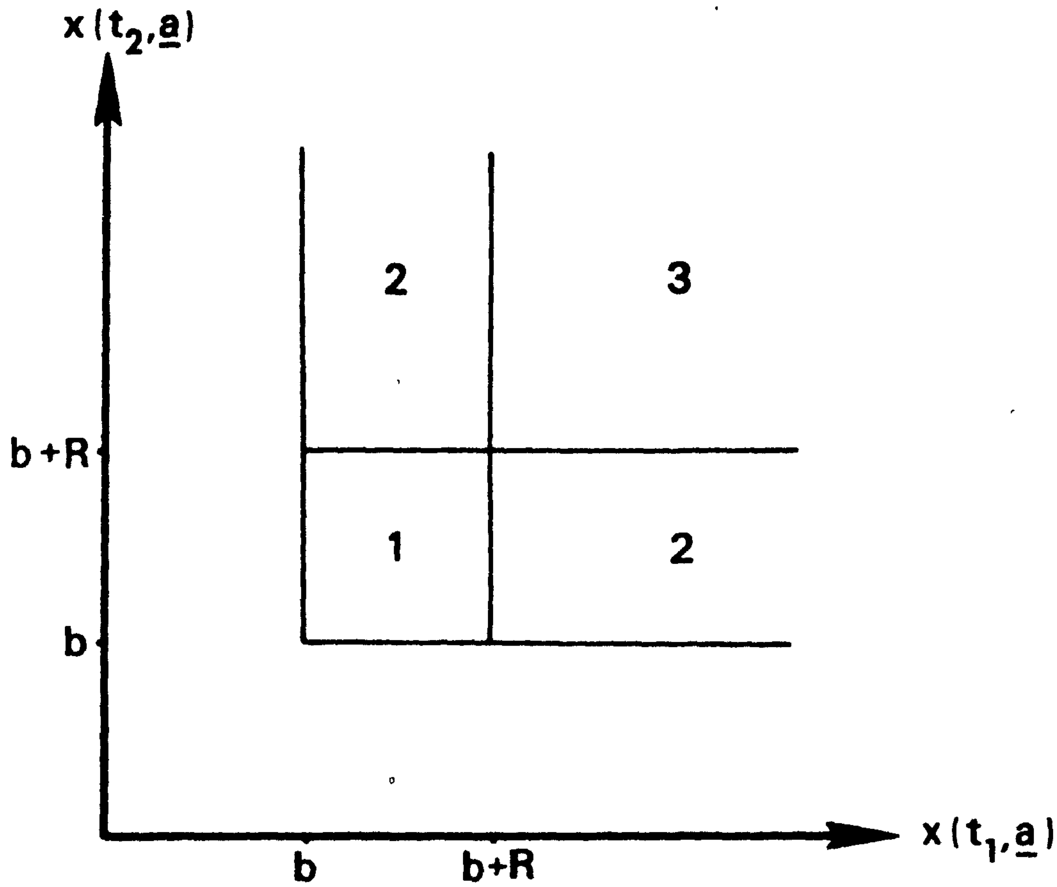


FIGURE 4-5 DIVISION OF INPUT PROCESS IN THREE REGIONS

For any threshold value b, 1,2, and 3 denote the only three regions that contribute to the output autocorrelation. Each region defines a set of values of a and b as follows:

$$S_1 = \{ a,b : b < x(t_1,a) < b + R \text{ and } b < x(t_2,a) < b + R \}$$

$$S_2 = \{ a,b : b < Min(x(t_1,a),\ x(t_2,a)) < b+R < Max(x(t_1,a),\ x(t_2,a)) \}$$

$$S_3 = \{ a,b : b + R < x(t_1,a) \text{ and } b + R < x(t_2,a) \}$$

By integrating in the Lebesgue sense over the sets S_1, S_2 and S_3, the output autocorrelation can be written as

$$\left.\begin{aligned} \langle \underline{y}(t_1)\underline{y}(t_2) \rangle = & \int_{\text{all } a} \int_{Q-R}^{q} \int_{\text{all } g} u_{-1}(R-\Delta) g(x_1-b) g(x_2-b) \frac{f(g)f(a)}{S} \, dg\,db\,da \\ & + \int_{\text{all } a} \int_{q-R}^{q} \int_{\text{all } g} u_{-1}(Q-b-R) M g(q-b) \frac{f(g)f(a)}{S} \, dg\,db\,da \\ & + \int_{\text{all } a} \int_{T}^{q-R} \frac{M^2}{S} f(a) \, db\,da \end{aligned}\right\} \quad (4\text{-}1\text{-}9)$$

where the above three terms correspond to points in S_1, S_2 and S_3 respectively, the threshold density of (4-1-1) has been used and

$$u_{-1}(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 . \end{cases}$$

Although the analytical solution can be continued somewhat, this will not be done since a numerical solution is eventually required. This will be discussed in Section 4.2.1. It should be pointed out that when each realization of the input process to the random model is a monotonically increasing function and limited by (4-1-7) then the output autocorrelation can be written as

$$< \underline{y}(t_1)\underline{y}(t_2) > = \int_{\text{all } a} \int_{x_2-R}^{x_1} \int_{\text{all } g} u_{-1}(R-x_2+x_1)g(x_1-b)g(x_2-b)\frac{f(g)f(a)}{S}\, dg\,db\,da$$

$$+ \frac{M}{S} \int_{\text{all } a} \int_{o}^{R} \bar{g}(w)u_{-1}(w+x_2-x_1-R)dw\ f(a)\ da$$

$$+ \frac{M^2}{S} (< \underline{x}_1 > -(R+T)) \qquad (4\text{-}1\text{-}10)$$

where the substitution $\underline{w} = \underline{x}_1 - \underline{b}$ was used in the second term. This expression can be evaluated analytically.

Note that when the input is limited by (4-1-7), then (4-1-9) can be used to show that*

$$\left. \begin{aligned} &< \underline{y}^2(t) > = C < \underline{x}(t) > + D \\ &\text{where } C = \frac{M^2}{S} \\ &\text{and} \quad D = \frac{1}{S}\int_{o}^{R} \int_{\text{all } g} g^2(v)\ f(g)dg\,dv - \frac{M^2}{S}\ (R+T)\ . \end{aligned} \right\} (4\text{-}1\text{-}11)$$

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* In general, M.J. Korenberg has shown that

$$< \underline{y}^n(t) > = E < \underline{x}(t) > + F$$

where E and F are constants.

A convenient, input independent, upper bound for the output variance can be found as follows:

Since $\langle \underline{y}(t) \rangle = A \langle \underline{x}(t) \rangle + B$

and $\langle \underline{y}^2(t) \rangle = C \langle \underline{x}(t) \rangle + D$

then

$$\mathrm{Var}(\underline{y}(t)) = U + V \langle \underline{x}(t) \rangle + W \langle \underline{x}(t) \rangle^2$$

where
$$U = D - B^2$$
$$V = C - 2AB$$
and
$$W = -A^2 .$$

Maximizing $\mathrm{Var}(\underline{y}(t))$ with respect to $\langle \underline{x}(t) \rangle$ gives a maximum at

$$\langle \underline{x}(t) \rangle = -\frac{V}{2W}$$

so that the variance is

$$\mathrm{Var}(\underline{y}(t)) \le \frac{C^2}{4A^2} + D - \frac{BC}{A} . \qquad (4\text{-}1\text{-}12)$$

To illustrate the use of these equations consider the random parameter model in Figure 4-6(a). The system output is defined as

$$\underline{y}(t) = \begin{cases} 0, & x(t,\underline{a}) < \underline{b} \\ \frac{M}{R}(x(t,\underline{a})-\underline{b}), & \underline{b} \le x(t,\underline{a}) \le \underline{b} + R \\ M, & x(t,\underline{a}) > \underline{b} + R \end{cases}$$

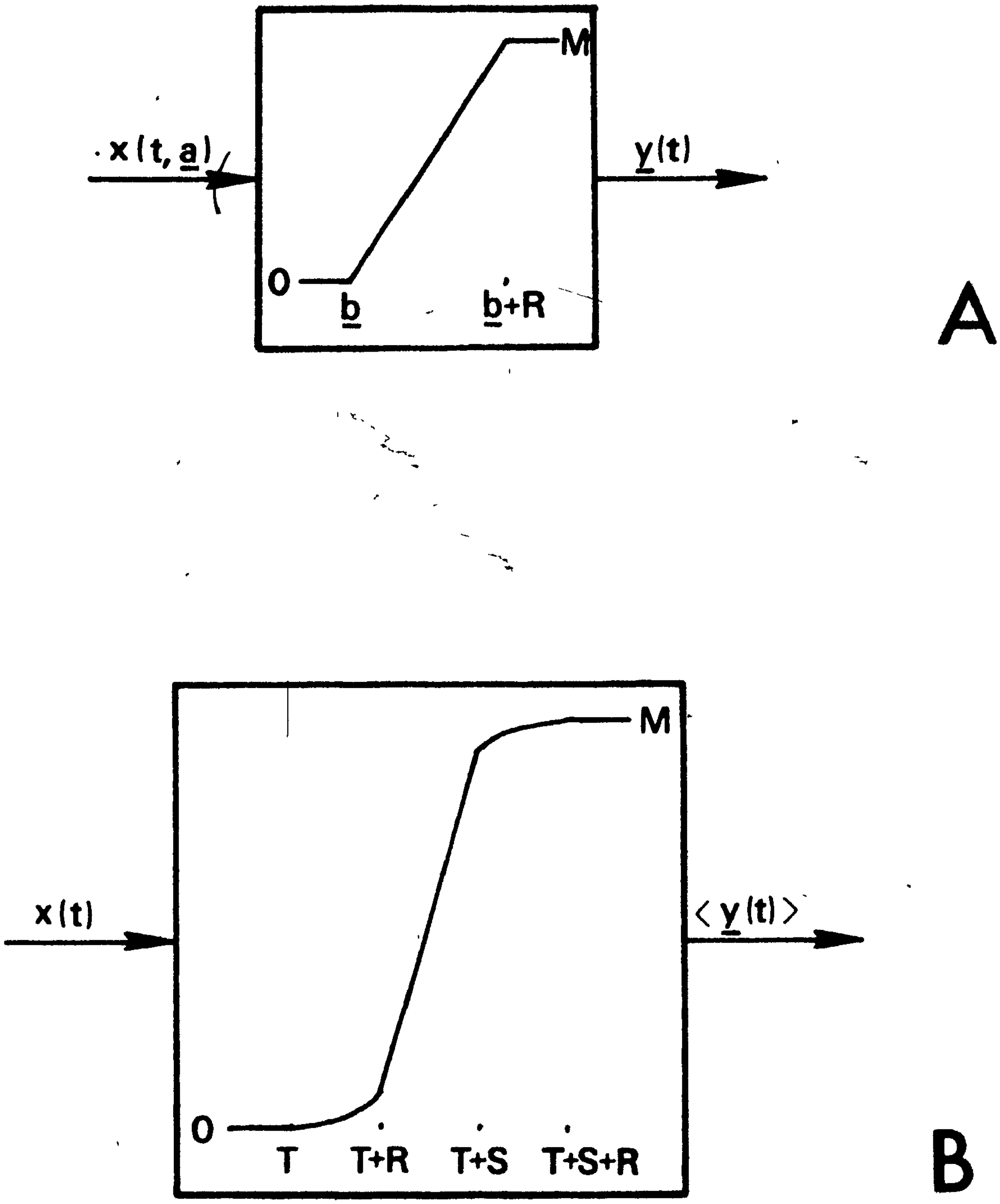

FIGURE 4-6 RANDOM STATIC NONLINEARITY AND RELATION BETWEEN DETERMINISTIC INPUT AND EXPECTED OUTPUT

(b) shows the static relationship between a deterministic input and the expected output of the random parameter model in (a).

where all variables are defined as before. Applying (4-1-8), (4-1-9) and (4-1-12) give an output mean, autocorrelation and maximum variance of:

$$< \underline{y}(t) > \; = \frac{M}{S} < \underline{x}(t) > - \frac{M(R+2T)}{2S} \tag{4-1-13}$$

$$\left.\begin{aligned} R_y(t_1,t_2) &= \frac{M^2}{SR^2} \int_{\text{all } a} \int_{\Delta}^{R} (v^2 - \Delta v)dv \; u_{-1}(R-\Delta) \; f(a)da \\ &+ \frac{M}{SR} \int_{\text{all } a} \int_{o}^{R} w \; u_{-1}(w+\Delta-R)dw \; f(a)da \\ &+ \frac{M^2}{S}\left[\int_{\text{all } a} q \; f(a)da - (R+T) \right] \end{aligned}\right\} \tag{4-1-14}$$

and $$\text{Var}(\underline{y}(t)) \leq M^2 \left[\frac{1}{4} - \frac{R}{6S} \right] \tag{4-1-15}$$

provided the input is limited by (4-1-7).

4.1.4 The Existence of a Class of Nonlinear Systems with Memory that have Linearly Related Means

The previous sections have shown that random linear and random static nonlinear systems can have their output mean linearly related to the input mean provided the input is suitably limited. This section will illustrate that, under certain conditions, many nonlinear random parameter models with memory will have their expected output linearly related to the expected input.

One class of these models can be shown to exist by considering a random static nonlinearity such that

$$< \underline{y}(t) > = A < \underline{x}(t) > .$$

(Put B = 0 in (4-1-8)). Then any linear combination of any other linear system will produce a nonlinear system with memory that has its input and output means linearly related.

Figure 4-7 shows one example* of such a nonlinear random parameter model. It can be shown that

$$< \underline{y}(t) > = \frac{A}{1-AK} < \underline{x}(t) >$$

provided that $R = - T/2$ and the input to the static nonlinearity, $\underline{e}(t)$, is bounded by $T + R < e(t) < T + S$. Higher moments will not be considered.

Another class of random parameter models having linearly related means are those composed of either parallel combinations or cascades of blocks which are either deterministic and linear or are random with linearly related means. This thesis is mainly concerned with the cascade case.

Some of the interesting mathematical problems that are suggested are:

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* Compare with Paynter, 1966.

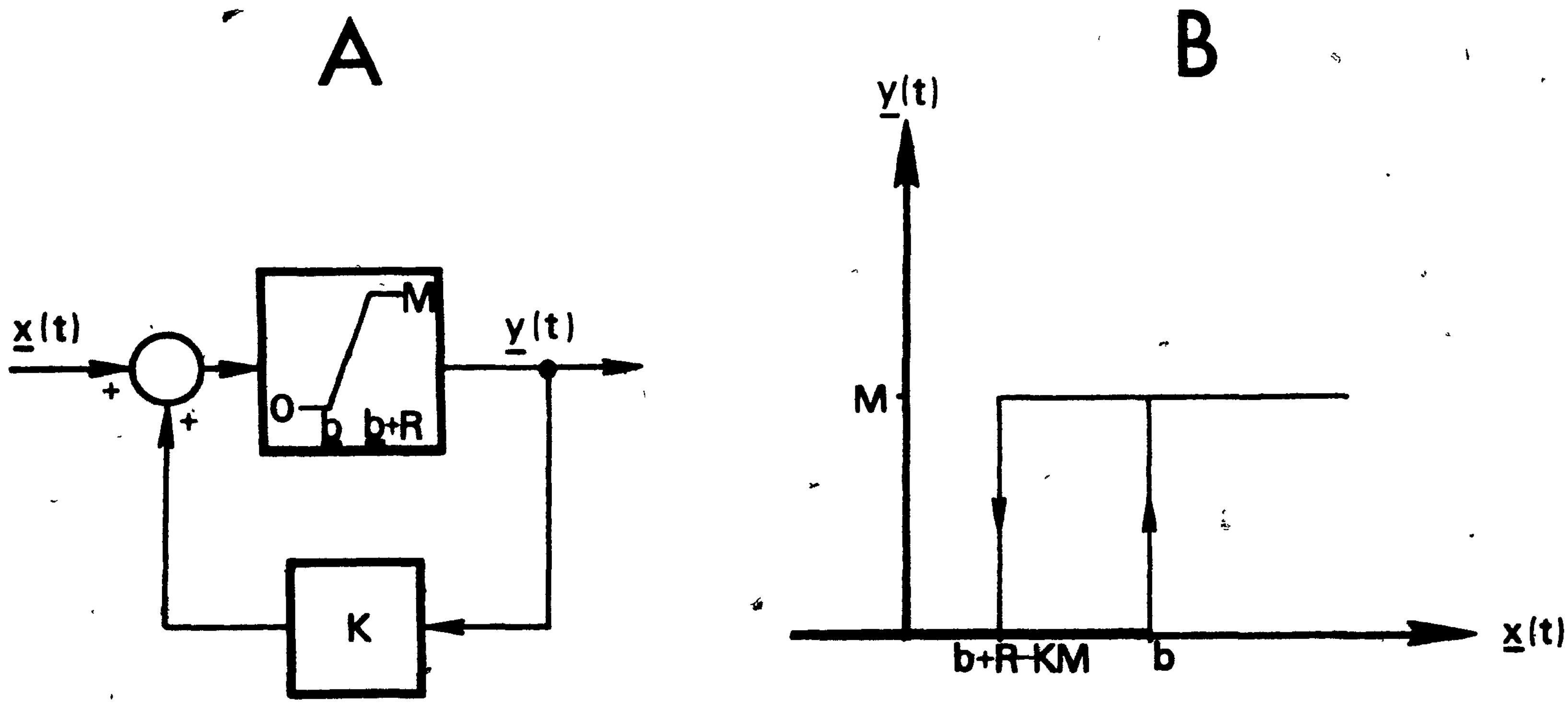


FIGURE 4-7   EXAMPLE OF NONLINEAR RANDOM PARAMETER MODEL WITH MEMORY

The random parameter model in (a) has the static input/output relationship shown in (b) where it is assumed that $KM > R$.

(1) What is the most general nonlinear random parameter model with memory that has linearly related means?

(2) Given a specific mathematical form containing random parameters, how can these parameters be selected so that the resulting random parameter model has linearly related means?

4.2 A DEMONSTRATION OF THE ANALYSIS OF TWO MULTIPATH SYSTEMS

4.2.0 Introduction

Section 3.2 has described a method of finding a confidence interval to estimate the possible waveform variation due to different realizations of a multipath system by using the mean and autocorrelation of the process corresponding to this waveform. However, if the probability density of this process is known, one can specify a confidence interval which is significantly smaller than the larger one which is used for an arbitrary probability density. Section 4.2.1 generally indicates how much effort is required to obtain either the larger confidence interval (for an approximate description of this variation) or the smaller one (for its exact description) which is theoretically available. Sections 4.2.2 and 4.2.3 describe how the models in Figures 4-8 and 4-10 can be approximately described.

4.2.1 An Indication of the Effort Required for Either an Approximate or an Exact Description of Multipath Systems

As discussed at the end of Chapter 3, only first moment relationships are necessary to describe the output of a multipath system with an infinite number of paths exactly since choosing different paths for the system will not change its output. For example, consider an infinite path model from Table 2-1 such that each path is a realization of the random model of Figure 4-1 (a) (or one of Figures 4-2(a), 4-3(a) or 4-6(a)). Thus if the input is deterministic, the relation between the input and output is given by the relationship shown in Figure 4-1(b) (or Figures 4-2(b), 4-4(a) or 4-6(b) respectively). Clearly, analogous results can be found if each path is a cascade of realizations of the previous random parameter models and the input to each static nonlinearity is suitably limited.

In order to approximately describe either waveforms inside an infinite multipath system, or at the output of a finite path system, autocorrelations must be found. In general, finding these autocorrelations analytically is possible only in such simple systems as those with different paths consisting of (1) a linear system, (2) a static nonlinearity, or (3) a static nonlinearity followed by a linear system. For more complicated systems having an arbitrary input, the autocorrelation at the output of each static nonlinearity (e.g. (4-1-5) or (4-1-9)) must be found numerically. However, if the system input is chosen so that the input to each static nonlinearity is changing monotonically, then an analytical expression for the autocorrelation may be written (e.g. (4-1-10)).

Although numerical calculation of autocorrelations is tedious, several facts should be pointed out. First, this calculation is not started until first moment relationships have indicated that the model under consideration is useful. Thus, unnecessary computation is avoided. Second, once all autocorrelations are known, one can immediately approximate confidence limits (using Chebyshev's Inequality) for waveform variation in each path and at the system output as well as for all possible Fourier transforms and power spectra. This should be compared to the alternative of simulating every possible realization of this system, calculating all Fourier transforms and power spectra for each possible waveform and then establishing error bounds for the three sets of waveforms.

It should be pointed out that it is theoretically possible to obtain the probability density of the processes that are used in this thesis so that an exact description of the variation of waveforms in multipath systems is available. This is possible since each realization of the processes considered is deterministic. The next paragraph will show how this can be done despite the presence of linear dynamic systems which normally (Davenport & Root, 1962) make this difficult.

It is well known that the probability density at the output of a static nonlinearity can be related to the input probability density (Papoulis, 1965). This result can be extended to the case of a random static nonlinearity. Further, the probability density of the output of a random linear system can be handled with the same methods since the convolution integral for the output at any particular time is a static function of the random parameters of the input process. Unfortunately,

Number of Paths	Input	Computation	Value Estimate		Fourier Transform Estimate		Power Spectrum Estimate	
			Approx.	Exact	Approx.	Exact	Approx.	Exact
infinite	arbitrary	-		x		x		x
finite	special	-	x		x		x	
finite	arbitrary	1	x		x		x	
finite	arbitrary	1,2		x	x		x	
finite	arbitrary	1,2,3		x		x		x

TABLE 4-1 SUMMARY OF METHODS TO OBTAIN APPROXIMATE AND EXACT DESCRIPTION OF MULTIPATH SYSTEMS

The "special" input is one that produces a monotonically changing input to each static nonlinearity. Computation (1) calculates autocorrelations; (2) calculates the first order probability densities; (3) calculates the probability densities of the Fourier transform and the power spectrum.

finding the output probability density usually requires a numerical solution. Nevertheless, all first order densities are available in principle. In addition, all higher order probability densities can be expressed in terms of the first order density although this again requires a numerical solution. Thus one can exactly describe the variation of waveforms in multipath systems due to different system realizations. Finally, since the Fourier transform and power spectrum functions are again static transformations of the random variables in the random processes, exact descriptions of these processes can be found. Again, determining these last probability densities requires numerical solutions. This discussion is summarized in Table 4-1.

In view of the additional computational requirements for an exact description, it is felt that the approximate methods are more useful. The next two sections will illustrate the approximate analysis.

4.2.2 Multipath System with Paths having a Cascade of a Linear System, a Static Nonlinearity and a Pure Gain Block

This section will illustrate the use of the systematic method by indicating how the multipath model in Figure 4-8 can be analysed. This model is assumed to satisfy all the requirements of Table 2-1 and, in addition, has a noise source in each path which is independent of all other noise sources, the input and all system parameters. It should be noted that noise at this point can correspond to transducer-encoder noise in a physiological system.

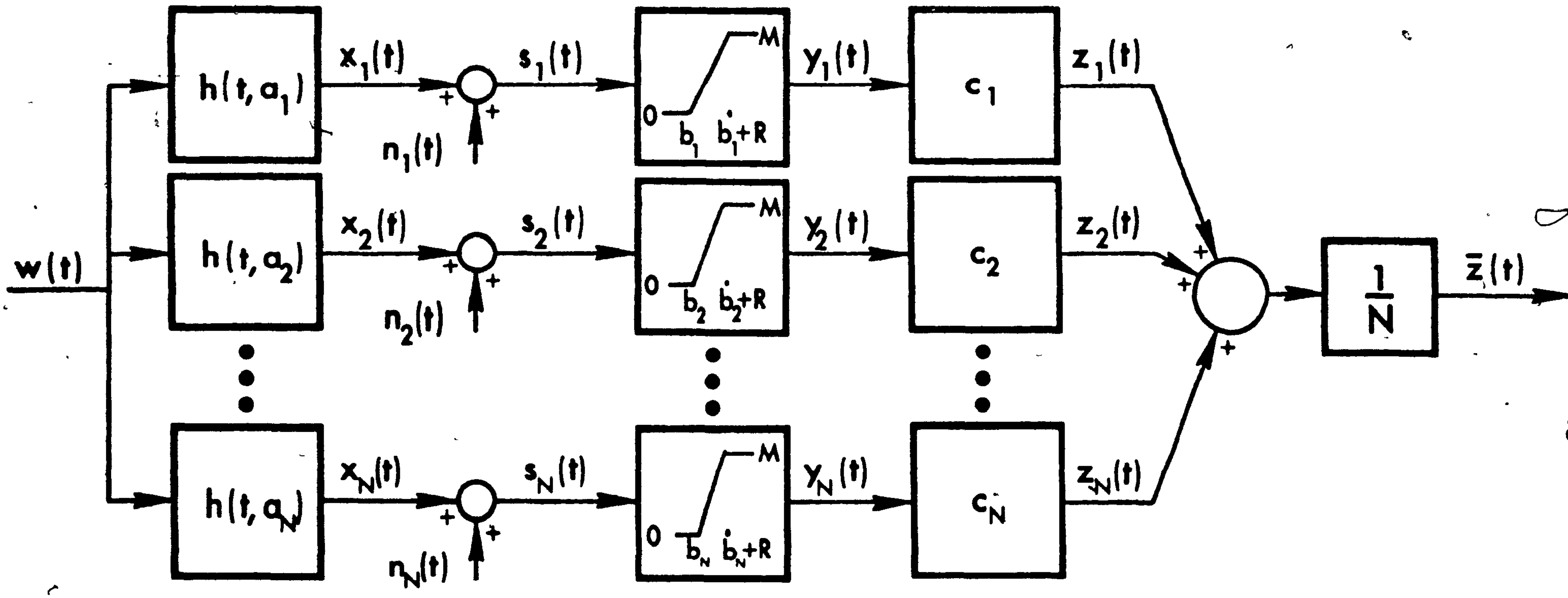


FIGURE 4-8 A POSSIBLE SENSORY SYSTEM MODEL

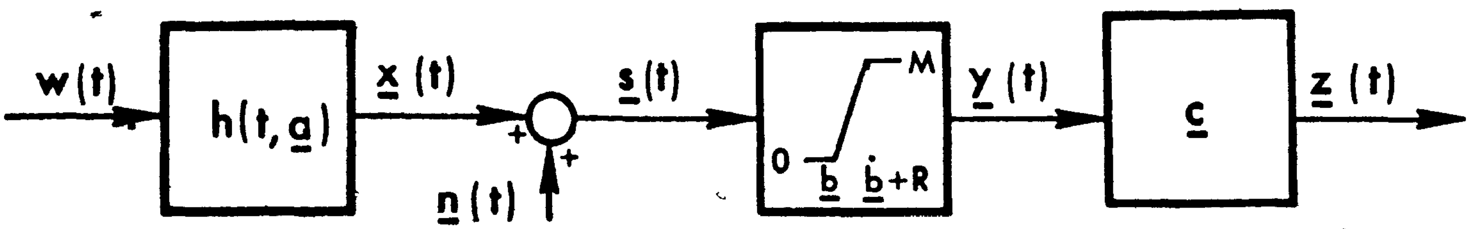


FIGURE 4-9 EACH PATH OF PRECEEDING FIGURE IS A REALIZATION OF THIS RANDOM PARAMETER MODEL

The i^{th} path of the model in Figure 4-8 can be described by

$$x_i(t) = w(t) * h(t,a_i)$$

$$s_i(t) = x_i(t) + n_i(t)$$

$$y_i(t) = \begin{cases} 0, & s_i(t) < b_i \\ \frac{M}{R}(s_i(t)-b_i), & b_i \le s_i(t) \le b_i + R \\ M, & s_i(t) > b_i + R \end{cases}$$

and $z_i(t) = c_i y_i(t)$

where $w(t)$ is the deterministic input,

$h(t,a_i)$ is the real impulse response of the first linear system which contains, for convenience, the single parameter, a_i,

$x_i(t)$ is the output of the first linear system,

$n_i(t)$ is a noise source with a zero mean, a variance, σ^2 and an autocorrelation $R_n(t_2-t_1)$,

$s_i(t)$ is the input to the static nonlinearity,

the saturation value, M, and range, R, are constants,

b_i is the threshold

and c_i is the gain of the last block.

The system output, $\bar{z}(t)$, is

$$\bar{z}(t) = \frac{1}{N}\sum_{i=1}^{N} z_i(t)$$

where N is the number of paths.

Assume that the path varying parameters a_i, b_i and c_i are realizations of the independent random variables $\underline{a}$, $\underline{b}$, and $\underline{c}$ which have probability densities f(a), f(b) given by (4-1-1) with S > R, and f(c) respectively. Thus, the multipath model satisfies the requirements of Table 2-1.

Following the procedure of Section 3.4, each path is considered a realization of the random parameter model in Figure 4-9. The expected value of the process corresponding to the multipath system output is related to the output process of the random parameter model by using (3-3-7) to give

$$< \bar{\underline{z}}(t) > = < \underline{z}(t) >.$$

Since $\underline{c}$ is independent of all other random variables,

$$< \bar{\underline{z}}(t) > = < \underline{c} > < \underline{y}(t) >.$$

It will be assumed that

$$T + R \leq \underline{s}(t) \leq T + S \qquad (4\text{-}2\text{-}1)$$

which, in principle, can be related to conditions on w(t), h(t,$\underline{a}$) and $\underline{n}$(t). Using (4-1-13) gives

$$\left.\begin{aligned} &< \underline{y}(t) > = A < \underline{s}(t) > + B \\ &\text{where } A = \frac{M}{S} \\ &\text{and} \quad B = \frac{-M(R + 2T)}{2S}. \end{aligned}\right\} \qquad (4\text{-}2\text{-}2)$$

Since $< \underline{n}(t) > = 0$, then

$< \underline{s}(t) > = < x(t) >.$

Using (3-3-1)

$$< \underline{x}(t) > = w(t) * < h(t,\underline{a}) >.$$

Therefore, the expected output is

$$< \bar{\underline{z}}(t) > = \frac{< \underline{c} > M}{S} < h(t,\underline{a}) > * w(t) - \frac{< \underline{c} > M(R + 2T)}{2S} . \qquad (4\text{-}2\text{-}3)$$

Thus, if the input and noise are such that (4-2-1) is satisfied, the expected output is linearly related to the input, despite the presence of nonlinearities in each path.

The sequence of operations required to find the correlation corresponding to the output of the multipath system will now be indicated. Equation (3-3-8) gives

$$R_{\bar{z}}(t_1,t_2) = \frac{R_z(t_1,t_2)}{N} + \frac{(N-1)}{N} < \underline{z}(t_1) > < \underline{z}(t_2) >$$

Since $\underline{c}$ is an independent parameter,

$$R_z(t_1,t_2) = < \underline{c}^2 > R_y(t_1,t_2) . \qquad (4\text{-}2\text{-}4)$$

Note that the independent noise source has made it impossible to predict future values of a realization of $\underline{s}(t)$ from its past values. Thus to evaluate $R_y(t_1,t_2)$ an equation analogous to (4-1-14) could be used but its solution would involve simulating all possible noise realizations. Alternately the autocorrelation could be found by finding the second order probability density of $\underline{y}(t)$ which is also lengthy.

An estimate of waveform variance (due to different system realizations) of $\underline{y}(t)$, $\underline{z}(t)$ and $\underline{\bar{z}}(t)$ is easier since $\underline{\Delta} = 0$ (in equation (4-1-14)) in this case. This will be illustrated, assuming (4-2-1), by calculating the system output variance. Using (3-3-9)

$$\mathrm{Var}\ (\underline{\bar{z}}(t)) = \frac{\mathrm{Var}\ (\underline{z}(t))}{N} .$$

The variance of the output of a path is

$$\mathrm{Var.}(\underline{z}(t)) = R_{z}(t,t) - < \underline{z}(t) >^2 .$$

Using (4-1-14), with $\underline{\Delta} = 0$ and taking expectations with respect to all random variables in $\underline{s}(t)$ gives

$$R_y(t,t) = \frac{M^2R}{3S} + \frac{M^2}{S}\ (< \underline{s}(t) > - (R + T)) .$$

Since $< \underline{n}(t) > = 0$, then

$$< \underline{s}(t) > = < \underline{x}(t) >$$

and $< \underline{x}(t) > = w(t) * < h(t,\underline{a}) > .$

Thus

$$\mathrm{Var}(\underline{\bar{z}}(t)) = \frac{1}{N}\Bigg\{ w(t) * < h(t,\underline{a}) > \left[\frac{M^2}{S} < \underline{c}^2 > + \frac{M^2(R+2T)}{S^2} < \underline{c} >^2\right]$$

$$- \left(w(t) * < h(t,\underline{a}) >\right)^2 \frac{< \underline{c} >^2 M^2}{S^2}$$

$$- \frac{M^2}{S}\left[\left(\frac{2R}{3} + T\right) < \underline{c}^2 > + \frac{(R+2T)^2 < \underline{c} >^2}{4S}\right]\Bigg\} . \qquad (4\text{-}2\text{-}5)$$

Note that since both the mean and variance of the system output contain no noise terms, the noise does not affect these aspects of the system response. Physically, this occurs since the noise in each path can be referred to a change in the input for that path, which can either increase <u>or</u> decrease the path output variance (due to different realizations of that path) since (4-2-5) is a quadratic equation in $\langle w(t) \rangle$. For zero mean noise, neither the output mean nor variance is affected by the noise.

A convenient, input independent, upper bound for the variance can be shown to be

$$\mathrm{Var}\ (\underline{z}(t)) \leq \frac{M^2 < \underline{c}^2 >}{N} \left[\frac{< \underline{c}^2 >}{4 < \underline{c} >^2} - \frac{R}{6S} \right] .$$

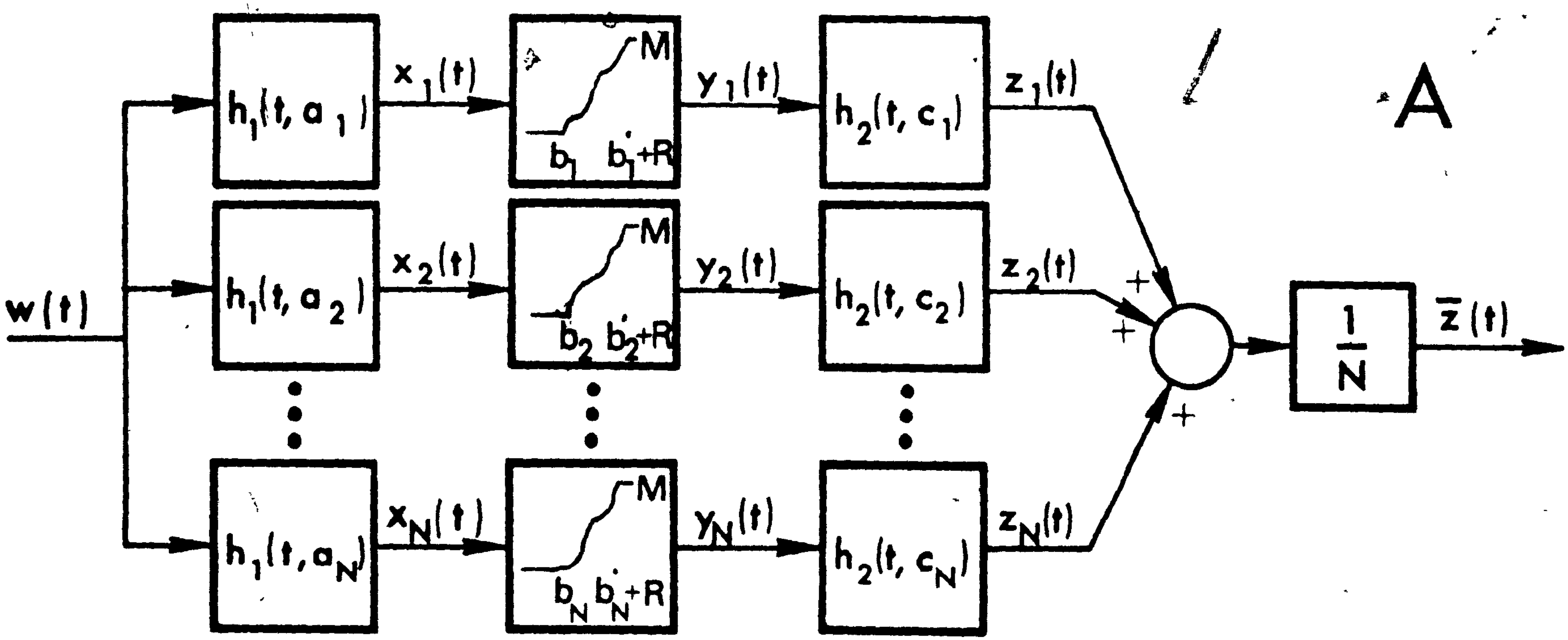
4.2.3 Multipath System with Paths Containing a Cascade of a Linear System, a Random Static Nonlinearity, and a Second Linear System

As a second illustration of the systematic method, the model in Figure 4-10 will be considered. This model is similar to the previous one except that in the i^{th} path

$$z_i(t) = y_i(t) * h_2(t, c_i)$$

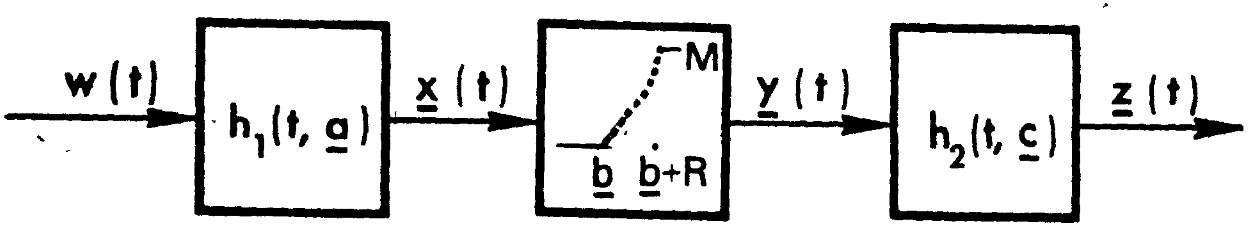
$$y_i(t) = \begin{cases} 0, & x_i(t) < b_i \\ g_i(x_i(t) - b_i), & b_i \leq x_i(t) \leq b_i + R \\ M, & x_i(t) > b_i + R \end{cases}$$

$$x_i(t) = w(t) * h(t, a_i)$$



A

FIGURE 4-10 MULTIPATH SYSTEM WITH CASCADE OF TWO LINEAR SYSTEMS AND A RANDOM STATIC NONLINEARITY IN EACH PATH



B

FIGURE 4-11 RANDOM PARAMETER MODEL FOR EACH PATH OF ABOVE FIGURE and the noise sources in each path have been omitted. All other parameters are defined in the previous section. Here, $g_i(\ )$, denotes the static nonlinearity in the i^{th} path which may be different from those in other paths. $h_1(t,a_i)$ and $h_2(t,c_i)$ are the impulse responses of two different linear systems each containing, for convenience, a single, path-varying parameter.

Following the sequence in Section 3.4, the random parameter model for each path, shown in Figure 4-11, is analysed. Using (3-3-7), the mean of the process corresponding to the multipath system output is

$$< \underline{\bar{z}}(t) > = < \underline{z}(t) > .$$

Using (3-3-2)

$$< \underline{z}(t) > = < \underline{y}(t) > * < h_2(t,\underline{c}) > .$$

From (4-1-8)

$$< \underline{y}(t) > = A < \underline{x}(t) > + B$$

$$\text{where} \qquad A = \frac{M}{S}$$

$$B = \frac{1}{S} \int_0^R \bar{g}(v)dV - \frac{M}{S}(R + T)$$

$$\text{and} \qquad \bar{g}(v) = \int_{\text{all } g} g(v)\ f(g)dg$$

provided that $\underline{x}(t)$ is bounded by

$$T + R \leq \underline{x}(t) \leq T + S$$

which, in principle, can be related to restrictions on $w(t)$ and $h_1(t,\underline{a})$.

Since (3-3-2) gives

$$< \underline{x}(t) > = w(t) * < h_1(t,\underline{a}) >$$

then all the previous equations can be combined to give

$$< \tilde{\underline{z}}(t) > = A < h_1(t,\underline{a}) > * < h_2(t,\underline{c}) > * w(t)$$

$$+ B \int_0^\infty < h_2(t,\underline{c}) > dt$$

where A and B are defined as before. Thus the expected system output is again linearly related to the input despite the presence of nonlinearities in each path.

Finding autocorrelations and variances requires a numerical solution as discussed in Section 4.2.1.

Approximate confidence intervals for waveform variation due to different system realizations can then be found as indicated in Section 3.2.

CHAPTER 5

DISCUSSION AND SUMMARY

5.0 INTRODUCTION

In the first part of this chapter, Section 5.1 briefly discusses the application of the preceeding multipath models to any sensory system. Then it indicates that most data from the rotation sensing system is not inconsistent with these models. Section 5.2 illustrates that multipath systems having paths with different structures in some of them can be handled by the random parameter model approach. In the last part of this chapter, Section 5.3 summarizes the thesis.

5.1 THE APPLICATION OF MULTIPATH MODELS AND THEIR PLAUSIBILITY IN THE VESTIBULAR SYSTEM

In order to establish whether multipath models from the previous chapters can explain the behavior of a secondary neuron in a sensory system, a lengthy series of physiological experiments must be performed. A detailed discussion of these experiments is outside the scope of this thesis, but a brief idea of these experiments will be given. First, a large number of primary paths would have to be identified. Next, the response of a large number of secondary neurons would have to be considered. Then, all the assumptions of Table 2-1 would have to be checked and the threshold density for each static nonlinearity would have to be estimated. Since most sensory systems have a significant number of efferent or feedback pathways, the model of each path might represent an open loop or a closed loop system. Thus, it would be necessary to eliminate, or correct for, the effect of efferents before the model for each path could be used to make anatomical correlations.

Before starting such a lengthy set of experiments, it would be natural to ask whether the presently available data is consistent with the proposed multipath models. The following paragraphs will indicate that this seems true for the rotation sensing system of some animals.

The data describing the rotation sensing system of the frog (Precht et al., 1971) and the pigeon (Correia & Landolt, 1973) seems to be consistent with all of the assumptions in Table 2-1. However, in the monkey some paths may contain dependent parameters (Goldberg & Fernandez, 1971(b)) and others may not all be of the same mathematical form (Fernandez & Goldberg, 1971).

Perhaps surprisingly, there does not seem to be any definite evidence that some of the observed linear behavior of secondary neurons in the rotation sensing system cannot be explained by these models. Some of the reasons for this conclusion will be discussed:

(1) A multipath system with a static nonlinearity in each path can have a linear output range even if the probability density of its thresholds is nonuniform. All that is required for the multipath model to have a linear range is that the threshold density not change significantly over a large enough range of values.

(2) The common observation that estimates of the "path threshold" density are nonuniform (Precht et al., 1971; Figure 2-5(c)) is not inconsistent with a hypothesis that the "static nonlinearity threshold" density is uniform. This could happen if a uniform "static nonlinearity threshold" density were preceeded by elements whose DC gain also had a probability density. In this case, each "path threshold" would depend on both the DC gain preceeding the static nonlinearity and the threshold of this static nonlinearity. The resulting "path threshold" density would probably be nonuniform.

(3) One may object that these models require that a significant proportion of primary paths be saturated when there is no input and that this has not been observed. It should be realized that if these units were observed, they would probably be rejected from most studies on the basis that they "were not responding to the rotational stimulus".

5.2 MULTIPATH SYSTEMS WITH PATHS OF DIFFERENT STRUCTURE

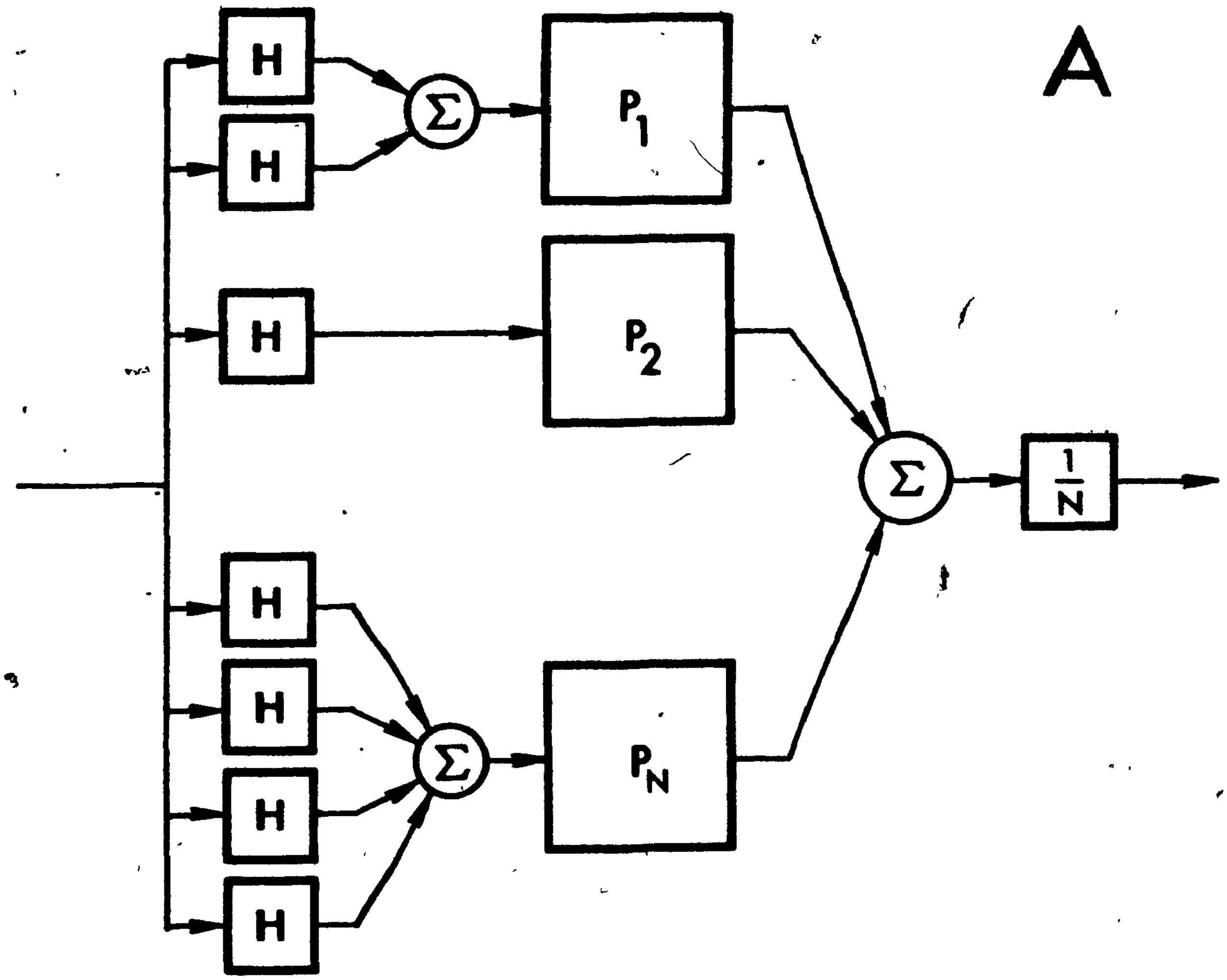
In many multipath sensory systems, it is not unusual for some paths to have a different structure than others. For example, in the rotation sensing system, each primary neuron can have from one to ten (Engstrom, 1968) input haircells of two principle forms. In addition, some paths are affected by efferent feedback pathways while others are not. Thus, one can often consider a randomly selected path from a multipath system to have a <u>random structure</u>. In order for the random parameter model approach to be a meaningful tool for modelling sensory systems they must be able to handle those paths which contain path-varying parameters as well as those having a random structure. The next paragraph will illustrate that random parameter models are naturally suited to do this.

Suppose that the rotation sensing system can be represented by the multipath system shown in Figure 5-1(a), where it is assumed that there is only one type of haircells, labelled H. The random parameter model for each path is shown in Figure 5-1(b) where a random variable, a, has been defined with a probability density that gives the probability of having a given number of haircells in a randomly chosen path. The random parameter model can then be analysed in the usual manner. A procedure similar to this can be used to handle any random structure.

5.3 SUMMARY

Since it is physiologically impossible to observe each part of a large multipath neural system without destroying part of it, a deterministic analysis of these systems is unsuitable. An alternate approach taken in this thesis, is to use random parameter models to describe deterministic systems statistically.

It is proposed that many sensory systems can be represented by the class of models listed in Table 2-1. These models contain a cascade of several linear and static nonlinear systems having the same mathematical form in each path. However, corresponding parameters of the linear



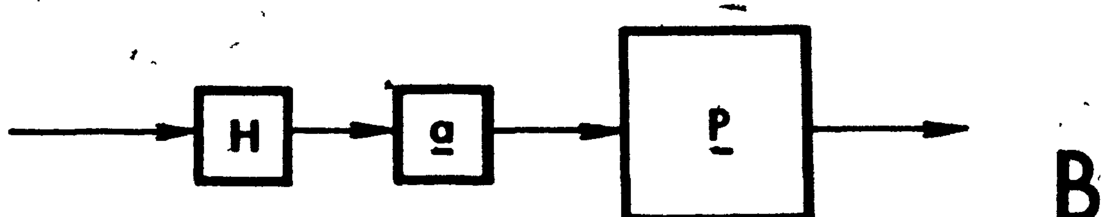


FIGURE 5-1 MULTIPATH MODEL HAVING PATHS OF DIFFERENT STRUCTURE

Each path of (a) is a realization of the random parameter model in (b). H and P_i denote the equations of the haircell and the remaining part of the i^{th} path. P denotes the random equation corresponding to P_i.

systems and the shapes of the static nonlinearities can be different in each path. The model output can be viewed as a weighted sum of the outputs of each path. Any model from this class can be analysed statistically.

The statistical approach is based on the analysis of a random parameter model which is defined such that each path of the multipath system can be considered a realization of this model. Regardless of the complexity of a model chosen from the class of Table 2-1, it is demonstrated that a mean and autocorrelation of any variable in the corresponding random parameter model can theoretically be found. These statistical quantities can then be used to approximate (Section 3.2) the value of each waveform of the multipath system at any time, as well as to approximate its Fourier transform and power spectrum during any time interval. In Section 3.4, a systematic method is listed so that any multipath model from Table 2-1 can be approximately described.

This systematic method is especially convenient when infinite path systems are considered, since the system output can then be described entirely by first moment relationships that are easily determined. The approach shows that many infinite path systems can act linearly despite nonlinearities in each path. As well, since it shows that many different multipath models can behave in the same way, this implies that it may not be necessary to identify each path exactly. Thus, it appears that this approach represents a powerful tool for analysing neural systems having a large number of convergent parallel pathways.

When the number of paths is finite, this method can also be used, but numerical techniques are necessary to evaluate the autocorrelations in more complex systems. In some cases, when special inputs are used, it may be possible to evaluate these autocorrelations analytically. Even when arbitrary inputs are used, one can approximate all possible waveforms in the system, their Fourier transforms and their power spectra with much less effort and computation than would be possible in a straightforward computer simulation of the system.

The systematic method proposed to obtain approximate confidence limits (using Chebyshev's Inequality) on variables in multipath models requires much less effort than the exact description which is theoretically available. The approximate procedure is illustrated by analysing two multipath models which could represent a sensory system.

The application of these models to sensory systems is briefly discussed and it is shown that these models may be useful in the rotation sensing system, since the assumptions made are not inconsistent with some available data. Finally, it is illustrated that random parameter models can be used in systems with paths that have a varying structure.

In view of what has been presented in this thesis, particularly since the method simplifies as the number of paths increases, it is felt that this approach will prove useful in modelling activity at higher levels in the brain. Clearly, many new problems are introduced at these levels, but it is felt that the fields of statistics and random mathematical functions may provide the techniques for solving these problems once they are mathematically formulated.

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