

**Radion dynamics in a five-dimensional
brane-world:**
**from the hierarchy problem to a four-dimensional
inflationary universe**

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Abstract

Inspired by the Randall-Sundrum (RS) solution to the weak-scale hierarchy problem we present a new five-dimensional brane-world model. While many cosmological implications of RS-type scenarios have been considered, the question of inflation is still relatively unexplored. In this work we investigate the dynamics induced by fluctuations in the extra dimension and find that the so-called radion can be used for inflation. A class of models are found which simultaneously allow for radius stabilization by utilizing a bulk scalar field with a potential in the bulk and on the branes. The coupled radion-scalar field system is analyzed from the five-dimensional point of view and is seen to consistently reduce to a four-dimensional Friedmann-Robertson-Walker-like cosmology. An analysis of the constraints imposed during inflation fixes the radion mass and we speculate on the possibility that the new physics at work here may be distinguishable from typical four-dimensional models.

Acknowledgments

The Cosmos is about the smallest hole that a man can hide his head in
G. K. Chesterton

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Introduction

In one way or another, every generation may claim to have witnessed a great change in an understanding of the world. This is no less true today. Practitioners and observers of physics alike will undoubtedly agree that if some of the more speculative ideas currently under investigation are proven to be correct then our notions of physical reality will certainly undergo another paradigm shift. One such idea postulates the existence of extra dimensions.

The idea of incorporating extra dimensions into a physical description of nature, although novel, is not entirely a new one. It dates back at least as far as the propositions of Kaluza and Klein [1] in the 1920's. Its reemergence nowadays is chiefly due to theories attempting to reconcile quantum mechanics with relativity, the ever elusive quantum theory of gravity. In fact, there are many instances in physics today where the existence of extra dimensions has become a widely held belief. For instance, the idea finds a natural expression in such theories as supersymmetry, and especially superstring theory where a four-dimensional description of the universe can be reached as a low-energy approximation of the full theory. Obviously, the recovery of four-dimensional physics is paramount to the viability of any proposed theory. Often this has been accomplished by compactifying the higher dimensional space on a manifold which is small, usually much smaller than an atomic nucleus, through Kaluza-Klein reduction.

There have been developments in string theory and its subsequent extension M-theory where another approach to compactifying extra dimensions has arisen [2]. The main idea relies on the fact that the particles of the standard model may be confined to a hypersurface called a brane. Branes are embedded in a higher dimensional space referred to as the bulk where it is assumed that only the graviton or other exotic matter such as the dilaton may propagate there. Originally motivated phenomenologically [3]- [6], it has been suggested that our universe may be such a brane-like object, so standard model particles will be restricted to move in only three spatial dimensions. However gravity in this case, due to its higher dimensional nature, is sensitive to the presence of extra dimensions, suggesting that deviations from Newton's law are to be expected. Needless to say, these developments have been received with a lot of enthusiasm as there is hope that emerging brane-world models will be directly testable.

One such model was proposed by Arkani-Hamed, Dimopoulos and Dvali (ADD) [7]. Their main motivation was to address the longstanding puzzle of the weak scale hierarchy problem. That is, how does one account for the large discrepancy between the electroweak scale at around 1TeV and gravity at the Planck scale? In their scenario, confinement of the standard model to a brane allowed for extra dimensions which were larger than previously anticipated. The new spatial dimensions, transverse to the standard model brane, resulted in a factorizable higher-dimensional spacetime which was essentially flat. It was shown that the fundamental scale of gravity could be close to 1TeV through the large volume of the compactified extra dimensions.

Explicitly, this was given as a relation between the 4D gravity scale, m_{Pl} and the fundamental higher-dimensional mass scale M_d

$$m_{Pl}^2 = M_d^{2+d} R^d \quad (1)$$

where d is the number of extra spatial dimensions and R is the compactification radius. They showed that gravity will deviate from Newton's law on distances smaller than R . However, it is a problem whether or not such a large volume can appear naturally, with R being possibly as big as a tenth of millimeter to solve the hierarchy problem. Even so, with current experimental limits suggesting that gravity behaves in the classical way down to about a millimeter, the possibility of testing their model becomes very real.

The disparity between the fundamental scales of particle physics and gravity has troubled physicists for many years. So the discovery that changing the behaviour of gravity at short distances might naturally explain this hierarchy understandably created intense interest. Following close on the heels of the work of ADD, Randall and Sundrum (RS) considered an alternative involving a non-flat or "warped" bulk geometry which is not factorizable [8]-[9]. They proposed two intriguing variants (RS1,RS2) where the five-dimensional bulk spacetime was taken to be a slice of AdS space. In RS1, the extra dimension is compactified on S^1/Z_2 with positive and negative tension branes located at the orbifold fixed points. It was found that the curvature of the spacetime produces different physical scales on the two branes, where the negative tension brane on which we are supposed to reside is exponentially suppressed relative to the positive tension brane. They found that this exponen-

tial suppression could explain the hierarchy problem naturally without the need for large extra dimensions as in ADD, a hierarchy problem in itself. In the RS model 4D gravity is related to the fundamental gravity scale through the relation

$$\frac{m_{pl}^2}{8\pi} = \frac{M_5^3}{k} (1 - e^{-2kb}) \quad (2)$$

where k determines the curvature of the AdS bulk and b is the radius of the extra dimension. In fact, for all model parameters naturally on the order of the Planck scale, one only requires $kb \sim 37$ to generate the weak scale through the so-called warp factor, $e^{-kb} \simeq \text{TeV}/m_{pl}$.

In their second model, RS proposed that we live on the positive tension brane and that because of the AdS curvature, even if the radius is infinite, 4D gravity could be recovered and localized around the brane. At the expense of relinquishing the solution to the hierarchy problem, by removing the negative tension brane from the setup, they avoided issues raised concerning the viability of the RS1 model. In fact, it was found that the original RS idea had a serious flaw. This was due to the lack of a mechanism for stabilizing the radius of the extra dimension. The problem stemmed from the fact that the radius b , known as the radion, was a modulus and corresponded to a massless particle. The attractive force mediated by the radion in this case would result in an increase in Newton's constant at large distances [10], in obvious conflict with observations, and was therefore ruled out.

A further problem with the RS1 setup, and to a certain extent, brane-world models in general, involved the appearance of unconventional cosmologies on the branes [11]. Certainly, if brane-world models deviate from the plethora of four-

dimensional cosmologies heretofore so well studied, then it could be hoped that this development would point to direct tests of the effects of extra dimensions. While this might be beneficial at the high energies of the early universe where it is expected that such effects are important, it is disastrous for late-time cosmology unless the requirements for a successful Big Bang nucleosynthesis are met. However, as it turns out, this problem too is avoided if the radius of the extra dimension is stabilized [10].

Obviously, stabilization will be an important and necessary feature of our study and various arguments for achieving this in a brane-world setting have been made. For the RS two-brane model, a simple and effective mechanism was provided by Goldberger and Wise (GW) [12] by introducing a bulk scalar field with both bulk and brane potentials into the setup. Fortunately, the presence of the bulk scalar field addressed the issues of stability without requiring any unnatural fine-tuning of the model parameters. The overall effect of introducing a satisfactory stabilization mechanism resulted in an effective potential for the radion. In this context the radion could then reasonably be identified as a massive 4D scalar field. Subsequently, studies were performed which investigated the effects of perturbing the RS solution with a stabilizing bulk scalar field. While it was found that RS1 cosmology may differ significantly from four-dimensional FRW cosmology during the early universe, a cutoff temperature associated with the radion mass, on the order of 1 TeV had to be imposed [13, for example]. In particular, at high energies, the perturbative approach, so often employed in physics in the absence of an exactly solvable theory, necessarily breaks down.

The problem of moduli fixing, particularly in string theory, is an old one. So the connection between the stability of the brane-world and a cure for the moduli problem in the context of RS cosmology highlights the importance of the radion as a scalar degree of freedom. In our study, we will investigate the dynamics of the radion in the early universe. Hence, we will also need to consider a dynamically realized stabilization mechanism. In particular, we would like to know whether the radion can be used for inflation and then decay to its stable value before the onset of nucleosynthesis. This is the equivalent of the graceful exit problem in a higher-dimensional setting. Therefore, a suitable potential for the radion must be generated to ensure a dynamical evolution to its stable minimum.

In general, there have been a number of studies in which inflationary solutions involving brane-world scenarios have been found, to a greater or lesser degree of success¹. For RS-type models though, important already for the solution to the hierarchy problem that they offer, this issue has not been satisfactorily addressed. The ideal situation would involve exact, time-dependent solutions of the higher-dimensional theory incorporating stabilization. Considering that in the very early universe most brane-world models were likely far away from their quasi-static limits, this would certainly facilitate more detailed studies of the dynamics of the brane-world during inflation. Under such circumstances we might even be able to distinguish the effects of the new physics from the variety of 4D inflationary models which have been proposed to date [16, for an overview]. For instance, consider that the increasingly

¹ See [14] and [15] for scenarios which consider inflation due to dynamics on the brane and in the bulk respectively.

precise measurements being made in the cosmological laboratory are improving our ability to investigate the evolution of the universe back to earlier and earlier times, or to higher and higher energies. Consequently, the possibility for testing new physics coming from brane-world scenarios using observations of the CMB, for example, is not so farfetched. Moreover, the vast amount of research being published attests to the interest the brane-world idea is generating as the cosmology and particle physics communities continue to inform one another.

The main motivation for this work is to build on the results of RS1 by investigating the dynamics of the radion during the inflationary era. In the first chapter we outline the developments in cosmology which will be necessary for our study. In particular, we summarily review the important features of standard Big Bang cosmology and its most vaunted extension, inflation. This is followed by a brief sketch of brane-world cosmology, focusing mainly on the RS1 model. In the second chapter we propose a new 5D brane-world model which solves the hierarchy problem through the warping of the spacetime along the extra dimension by utilizing a bulk scalar field. Here the bulk and brane potentials of the scalar field are used in place of the cosmological constants which appear in RS1. The static solution we find is generalized to a time-dependent dilatonic model where the coupled radion-scalar system is analyzed in detail from the five-dimensional point of view. A whole class of possible solutions are found which will facilitate stabilization with a heavy radion. We also find that it is possible to study the resulting 4D FRW-like cosmology without losing any vital information about the bulk energetics. In the third chapter, we present explicit

results from our model which allow for a period of successful inflation. While there are multiple scenarios with more or less particle physics motivation available to us in the setup we propose, emphasis is put on a particular radion potential which allows for sub-Planckian initial field values. Finally, we conclude with a discussion of our results and speculations for future work.

Chapter 1

Cosmology and Particle Physics: The Hierarchy Problem and The Inflationary Universe

The possibility of probing new physics at energy scales beyond those currently accessible in particle accelerators has helped cosmology blossom into a serious physical subject. A new era of precision measurement has compelled physicists to investigate cosmological phenomena with a great deal more scrutiny than in the past, and in the process spawn more sophisticated theoretical analyses to go with ambitious observational programs.

The increasing interaction between the particle physics and cosmology communities has generated a multitude of papers and new research avenues. While the progress made on the observational front is largely responsible, the theoretical motivations for this collaboration are not to be overlooked. One could argue that the successes of the standard model of particle physics and Big Bang cosmology, both being incomplete pictures of physical reality, naturally lead the theorist to look for an underlying theory which will incorporate and hopefully explain their origins. Consider, for example, the emphasis which has been put on research associated with the physics of the early universe. The realization that extensions to the standard model might also provide insights to processes such as inflation and baryogenesis while also

possibly providing candidates for dark matter only further highlights the beneficial overlap of cosmology and particle physics.

In this chapter we will provide some of the necessary background for our work. We begin with a general discussion of standard Big Bang cosmology and the problems associated with it which have led to its most promising extension, inflation. We then establish the necessary criteria for a successful inflationary scenario, focusing our attention on scalar field dynamics. As our main results emerge from a study of a 5D brane-world which addresses the hierarchy problem, we will then give a sketch of brane-world cosmology. In particular, we will review the RS1 model and its subsequent development, wherein the main features of interest will be the stabilization of the extra dimension utilizing a bulk scalar field and the significance of the radion.

1.1 Standard cosmology

The successes of standard Big Bang cosmology are well established [17]-[18]. The general picture of the evolution of the universe from one which was hot and uniform at early times to its subsequent expansion and cooling up to the present has survived a litany of tests. This framework has allowed us to explain, among other things, the observed primordial abundances of elements as well as the redshifts of faraway galaxies. Perhaps most importantly though, for the current progress being made in cosmology, is its allowance for the presence of the cosmic microwave background radiation (CMB) permeating the universe. The increasing precision of the measurements [19]

being made of the CMB has opened up the possibility for testing and constraining the growing number of cosmological models appearing on a regular basis.

However, there are some troubling questions which cannot be addressed by the standard picture of cosmology. For instance, the problem of explaining why the universe is as big, flat, and uniform as it is compels us to extend the standard picture in order to accommodate these concerns. The most lauded extension, and most relevant to our work, is the inflationary universe paradigm. In addition to addressing concerns raised by the Big Bang model, it has the important advantage of providing testable predictions in the cosmological laboratory. Below we will outline some of the prominent issues which plague the Big Bang scenario and lay the necessary groundwork for a discussion of inflation.

1.1.1 Friedmann-Robertson-Walker models and the problems of standard cosmology

A central concept in the development of cosmology is known as the Copernican principle: Earth is not at the center of the universe. Modern cosmology generalizes this idea in its application and states that there is no center of the universe. When viewed on large enough scales, the cosmos looks the same in all directions. This fact leads to one of the most useful tools employed in physics, a symmetry principle. In particular, the distribution of matter in the universe on large scales appears to be homogeneous and isotropic.

Standard Big Bang cosmology begins with the fundamental equations of general relativity, Einstein's equations

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu} \quad (1.1)$$

where $T_{\mu\nu}$ is the stress-energy tensor describing the distribution of matter in the universe and $G_{\mu\nu}$ is the Einstein tensor which is a function of the spacetime metric. Spacetime must also obey the principles of homogeneity and isotropy, and the most general metric consistent with this is the line element

$$ds^2 = dt^2 - a^2(t) d\mathbf{x}^2 \quad (1.2)$$

where the dynamics of the universe are contained in the scale factor $a(t)$, and the vector product $d\mathbf{x}^2$ describes the geometry of space which can be flat, open (negatively curved) or closed (positively curved). Generally the contents of the universe are modelled as a perfect fluid with energy density ρ and pressure p . This leads to the stress energy tensor

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix} \quad (1.3)$$

The description of small scales on the level of planets or even galaxies certainly are not contained in this description, but it is a very reliable approximation for the smooth distribution of matter observed when we average over large scales. Not surprisingly,

if we consider the metric tensor for the Euclidean case, it is also simple:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{pmatrix} \quad (1.4)$$

This is known as a Friedmann-Robertson-Walker (FRW) space. Here, spatial distances are multiplied by the time-dependent scale factor $a(t)$ which describes the contraction or expansion of the spacetime.

With the general metric ansatz (4), Einstein's equations will reduce to the so-called Friedmann equations

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad (1.5)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) = -\frac{4\pi G}{3}\rho(1 + 3\omega) \quad (1.6)$$

where H is the Hubble parameter and k is a constant describing the curvature of space: $k = 0$ (flat), or $k = \pm 1$ (positive or negative curvature). Examining (1.6) we see that the second derivative of the scale factor depends on the fluid's equation of state, $p = \omega\rho$. This is the important relation for describing what epoch the universe is in. For a positive pressure the universe is decelerating due to gravitational interactions slowing the expansion down. For instance, we know that the history of the universe involved a period dominated by radiation ($\omega = 1/3$) which was followed by matter domination ($\omega = 0$) and so $\ddot{a} < 0$. The beauty of this model is its simplicity, and it is not difficult to determine how the scale factor behaves under these conditions. Similarly, a negative pressure will cause the expansion of the universe to accelerate. This is crucial for an era of inflation to be possible. One example where this might occur is if the energy density of the universe is dominated by a cosmological constant

($\omega = -1$). Moreover, suppose that the geometry of the universe is flat. The solution for the scale factor will then be

$$a = a_0 e^{Ht} \tag{1.7}$$

where a_0 (usually normalized to 1) is the initial value of the scale factor when inflation begins. As long as the universe is dominated by vacuum energy it will continue to expand at an accelerated rate without end.

Indirectly we can see that inflation is predicated on a flat geometry. This is just one of the flaws in the standard cosmological scenario which inflation so ably addresses. Before we proceed to consider inflation's other advantages let us summarize some of the issues which plague the Big Bang model of the universe:

1. **The flatness problem.** This refers to the fact that for the total energy density, Ω_{tot} to be nearly 1 today, it would need to be 1 to many significant figures in the early universe. The evidence for this comes from measurements of the first acoustic peak of the CMB [19].
2. **The horizon problem.** Measurements of the CMB support the notion of an isotropic and homogeneous universe on large scales. However, the expansion of the universe as described by standard cosmology leads to a problem of causality. That is, there are points in space that could not have been in causal contact with each other, making a thermal equilibrium impossible without fine-tuning the initial conditions of the Big Bang. This framework can not account for the uniform radiation of the CMB.

3. Anisotropies in the CMB. The temperature fluctuations in measurements made of the CMB require new physics to be explained. This has been a critical observation for studying the dynamics of the early universe.

Another concern is that a universe which started in a hot Big Bang must have had an initial singularity. The problem of singularities is common in physics, and often leads to new innovations. A conceptual innovation which alleviates the aforementioned problems is inflation. We now turn our attention to this subject.

1.1.2 Inflation and the CMB

While it is true that little is known about the history of the observable universe before nucleosynthesis, it is a widely held belief that there was an era of inflation early on [18]. The attractiveness of the inflationary paradigm lies in its ability to account for the initial conditions of the Big Bang, which otherwise must be input directly by hand. We know, for example, that observation requires there be no unwanted relics surviving to the present day. That is, no massive particles or topological defects that might ruin nucleosynthesis predictions. This is often referred to as the monopole problem. A successful inflationary scenario will alleviate this problem in a dynamical way and ultimately lead to a satisfactory late-time cosmology. Furthermore, the dynamics giving rise to inflation can be constrained by observations of the temperature fluctuations in the CMB. Hence, any model automatically has built-in criteria which it must satisfy in order to be physically admissible. Unfortunately there is an

embarrassment of riches, as the vast number of possible models of inflation testifies to the relative ignorance we have of the physics operating at these energy scales.

Inflation, as it is generally believed, is most simply described by the dynamical evolution of a weakly coupled scalar field that was initially displaced from the minimum of its potential. In the present work we will find that scalar field dynamics in an expanding four-dimensional universe will be in one-to-one correspondence with solutions to the five-dimensional model which we propose. Therefore, it will be helpful if we review the necessary requirements for a successful model of inflation involving a scalar field ϕ in some potential $V(\phi)$. Later we will see how a specific model is physically constrained by observations of the CMB.

The general concept of inflation rests on being able to achieve a negative pressure equation of state. In using a scalar field to this end, it is critical that we produce an energy density which mimics that of a cosmological constant. Let us start by writing down the Lagrangian and energy-momentum tensor for the scalar field $\phi(x_\mu)$

$$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V \quad (1.8)$$

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} L \quad (1.9)$$

Also the energy density and pressure can be written as

$$\rho = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla\phi)^2 + V(\phi) \quad (1.10)$$

$$p = \frac{1}{2} \dot{\phi}^2 - \frac{1}{6} (\nabla\phi)^2 - V(\phi) \quad (1.11)$$

These are particular results to be used in Einstein's equations (1.5, 1.6) for a Euclidean space as the k/a^2 term, if present at all, will be diluted by the rapid expansion of

the universe during inflation. Similarly, spatial derivatives are usually taken to be negligible as field gradients are also rapidly stretched by the expansion and so any inhomogeneities are damped away. So from now on we will neglect them.

A classical analysis of the scalar field dynamics begins with the action for the time-dependent inflaton $\phi = \phi(t)$:

$$S = \int d^4x a^3(t) \left(\frac{1}{2} \dot{\phi}^2 - V(\phi) \right) \quad (1.12)$$

This leads to the equation of motion

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi} \quad (1.13)$$

To achieve analytic solutions to (1.13), one generally employs the so-called slow-roll approximation. When ϕ is rolling slowly in a region of its potential which is flat enough to guarantee that time derivatives will be negligible this becomes a tractable problem. That the inflaton should behave in this way can be guessed from the fact that we require a negative pressure equation of state for inflation to proceed. The flatness of the potential is thus a key ingredient in determining whether the inflaton is in the slow-roll regime. The conditions which guarantee this are usually written as

$$\epsilon \equiv \frac{m_{Pl}^2}{16\pi} \left(\frac{V'}{V} \right)^2 \ll 1 \quad (1.14)$$

$$\eta \equiv \frac{m_{Pl}^2}{8\pi} \left(\frac{V''}{V} \right) \ll 1 \quad (1.15)$$

As long as the flatness conditions are satisfied the equation of motion can be approximated by

$$3H\dot{\phi} \simeq -\frac{dV}{d\phi} \quad (1.16)$$

and similarly for the Hubble parameter we find

$$H^2 \simeq \frac{8\pi}{3m_{Pl}^2} V \quad (1.17)$$

When the Hubble parameter is approximately constant during the important (i.e., observable) stage of inflation, the inflaton will mimic a cosmological constant.

It has been established that a successful inflationary scenario needs at least 60 e -foldings of inflation to solve the horizon problem². We can calculate the e -folding number N in a straight forward manner. It is generally given as

$$N = \int_t^{t_f} H dt \quad (1.18)$$

from some time t to t_f when inflation ends. Using the slow-roll equation and substituting $dt = \frac{1}{\dot{\phi}} d\phi$ we may write this as

$$N = -\frac{8\pi}{m_{Pl}^2} \int_{\phi}^{\phi_f} \frac{V}{V'} d\phi \quad (1.19)$$

For a smooth potential, it is possible to approximate $V' \sim V/\phi$ and assuming that $\phi \gg \phi_f$ for the observable part of inflation³ we get the behaviour $N \sim 4\pi (\phi/m_{Pl})^2$. Unfortunately in this case, the initial value of the inflaton needed to meet the requirements of an acceptable model exceeds the Planck scale. Similarly, it can be shown that the slow-roll parameters $\epsilon, \eta \ll 1$ also lead to the same conclusion. This is especially troubling since the treatment we have presented is classical, and any quantum corrections we have neglected will tend to reintroduce the Planck scale. This sug-

² The e -folding number refers to the exponential expansion of the scale factor. Note also that different values are sometimes allowed depending on the scale that inflation occurs (see for example [16]).

³ A model of inflation for us will be taken to mean one which applies after the observable universe exits the horizon. While a given model may allow for a large number of e -foldings, the last 60 or so are the only ones of observational relevance.

gests that if inflation did in fact occur, then a reliable derivation will have to come from a quantum theory of gravity.

The above discussion notwithstanding, inflation is still the soundest explanation for the shortcomings of standard cosmology. Consider the production of anisotropies in the CMB. If we begin with a homogeneous and isotropic universe at the classical level, the expansion due to inflation “freezes in” the vacuum fluctuation of the inflaton and so it becomes essentially a classical quantity. This fluctuation is associated with a primordial energy density perturbation, which not only survives after inflation but may be the origin of structure in the universe. Significantly, the quantum fluctuations of the inflaton field provide a definite mechanism to account for observations of the CMB and so provide models with a direct link to known physics. To see this, let us consider an explicit example. The simplest potential motivated by particle physics which will produce inflation is⁴

$$V = \frac{1}{2}m^2\phi^2 \tag{1.20}$$

The prediction from inflation is that the fluctuations in ϕ will produce a spectrum of horizon scale density perturbations with amplitude⁵

$$\delta_H = \frac{H^2}{2\pi |\dot{\phi}|} \tag{1.21}$$

⁴ This quadratic potential leads to what Linde dubbed chaotic inflation when he invoked a simple $\lambda\phi^4$ potential to study the dynamics of this epoch while inflation was still in its infancy [21]. The term refers to the high temperatures close to the Planck time when the universe would have been dominated by chaotic quantum fluctuations in ϕ .

⁵ Other normalizations are used in the literature, but this will not affect our results. See [16] for example.

Using the slow-roll approximation for the equation of motion (1.16) and the Hubble parameter (1.17) we find

$$\delta_H \simeq 4\sqrt{\frac{8\pi}{3} \frac{V^{3/2}}{m_{Pl}^3 V'}} \quad (1.22)$$

$$= 4\sqrt{\frac{\pi}{3} \frac{m\phi^2}{m_{Pl}^3}} \quad (1.23)$$

Plugging V into (1.19) as well, we find the e -folding number for the simplest model of chaotic inflation as

$$N = \frac{2\pi\phi^2}{m_{Pl}^2} \quad (1.24)$$

Hence

$$\delta_H \simeq 2\sqrt{\frac{1}{3\pi} \frac{mN}{m_{Pl}}} \quad (1.25)$$

Observations taken since the 1990 launch of the cosmic background explorer [20] (COBE) confirm⁶ that when the relevant scales left the horizon, $\delta_H \simeq 2 \times 10^{-5}$. Depending on the exact model, usually one sets $N_{\text{COBE}} \simeq 50$ and so the mass of the inflaton is constrained to be

$$m \simeq 6 \times 10^{-7} m_{Pl} \quad (1.26)$$

We should note that this constraint points to a defect in inflation when it is formulated in this way. That is, it uses the observed fact of δ_H to constrain the theory rather than explaining why the amplitude of the density perturbation spectrum is $\delta_H \sim 10^{-5}$. It is no wonder then that there are so many competing models of inflation which can account for the CMB anisotropies. As inflation is a theory still in search of a derivation

⁶ See [19] for future precision measurements being scheduled.

from some more fundamental theory we must accept this limitation as being beyond the scope of this work.

Another important observation which we need to consider is the spectral index

$$n - 1 = \frac{d \ln (\delta_H^2)}{d \ln k} \simeq -\frac{1}{\delta_H^2} \frac{d (\delta_H^2)}{d N} \quad (1.27)$$

where $k = aH$ is the scale exiting the horizon. This is a measure of the shape of the spectrum of density fluctuations and, as its value becomes more precisely determined, will help further restrict possible models of the early universe. Current observations have constrained this value to be $n = 0.98 \pm 0.02$ [?]. The result for our quadratic potential with $N_{\text{COBE}} = 50$ then gives $n = 1 - \frac{2}{N} \simeq .96$, which is within the limits of observation.

Following the recipe of the above discussion, we are now equipped to initiate the construction of a model of inflation due to scalar field dynamics. Of course, there are other considerations which must be taken into account, but the basic setup is in place. In this work we will not investigate the dynamics of the subsequent cosmology which follows inflation. Nevertheless, any scenario which can be constructed in the five dimensional framework which we propose should also be amenable to an analysis of the transition to FRW cosmology. The most notable study involves the reheating process whereby the rapidly oscillating scalar field is converted into hot radiation through its couplings to matter fields as it decays. In our case, the radion will serve as the inflaton so its phenomenology will be determined by the model parameters. In other words, the way the radion couples to the standard model will also affect the reheating process in a model-dependent (i.e., not arbitrary) way. This observation

suggests that the emerging picture of the universe as being fundamentally higher-dimensional may be able to address many concerns simultaneously.

The fact that the new physics arising out of attempts to extend the standard model of particle physics and big bang cosmology are not mutually exclusive is exciting. Consider inflation, for instance. It is possible to show that any scenario which has at least 140 e-foldings will produce fluctuations today on the order of the visible universe after starting from the Planck length during the inflationary period. This comes from the ratio of the horizon size presently, l_H to that of the Planck length, l_{Pl}

$$\ln \left(\frac{l_H}{l_{Pl}} \right) \sim 140 \quad (1.28)$$

The implication is that Planck-scale physics would be responsible for the generation of quantum modes during inflation [23]. Consequently, remnants of such physics could be imprinted on the CMB. In this way, observations of the cosmos may provide some intriguing opportunities for measuring new physics against observables. Here we are primarily motivated by just such an overlap between particle physics and cosmology. In fact, this dynamic is evident in one of the most vibrant fields of research today: brane-world cosmology.

1.2 Brane-world cosmology

The framework for many of the brane-world models which have appeared have drawn their inspiration from the work of Horava and Witten [2]. They investigated the strong coupling limit of the $E_8 \times E_8$ heterotic string theory which is described by

eleven-dimensional supergravity compactified on an orbifold with Z_2 symmetry at low energies. Significantly, it was found that the E_8 gauge groups could be confined on the ten-dimensional boundaries of the spacetime. Witten further simplified matters by showing that six dimensions could be compactified on a manifold much tinier than the distance between the two branes on the orbifold fixed points, and therefore the spacetime could appear to be five-dimensional with two four-dimensional branes [24]. Along with other compactification schemes which have been investigated, a definite framework has thus emerged in which the standard model can be consistently confined to a brane. Consequently, the possibilities for investigating the higher-dimensional effects of new physics has increased tremendously.

It is not surprising then that brane-world cosmology has attracted an immense amount of interest in recent years [25]. Early on, a great deal of work centered around the suggestion that the 4D cosmology of a brane-world might differ significantly from ordinary FRW cosmology [11]. This stemmed from the higher-dimensional nature of gravity generic to brane-world models. We know from observation thus far that our universe is described by Einstein gravity and so the issue of unconventional cosmologies in such models crucially depends on whether one recovers 4D gravity. This arises in RS1 through the radion which describes the fluctuation in the distance between the two branes. In the absence of a stabilizing potential for the radion, it is found to be massless and so will yield an additional long-range force. In addition, it will contribute to the expansion of the universe giving rise to a cosmology in contradiction with a successful nucleosynthesis (see for example [26]). The importance of stabiliz-

ing the radion before the epoch of nucleosynthesis is thus paramount to building a successful cosmology in a RS-type scenario.

The main focus of this work is motivated primarily by the RS solution to the hierarchy problem. However, in our brane-world model we want to extend previous results to include the dynamics of the radion. In particular, we would like the radion to double as the inflaton. Before we can get there however, we need to first review some of the important results spawned from the original scenario of Randall and Sundrum, highlighting the issues we will need to consider.

1.2.1 The Randall-Sundrum model and beyond

The RS two-brane model which solves the hierarchy problem is both simple and elegant. Here, two 3-branes with opposite tension are taken to sit at the fixed points ($y = 0, 1$) of an S^1/Z_2 orbifold with AdS_5 bulk geometry. Gravity is localized on the Planck brane (hidden sector) and the warping of the spacetime generates the weak scale hierarchy on the TeV brane (visible sector) were we are supposed to reside. Specifically, the action of the five-dimensional model can be written as

$$S = - \int d^5x \sqrt{-g} \left(-\frac{1}{2\kappa^2} R - \Lambda - \delta(y) \Lambda_0 - \delta(y-1) \Lambda_1 \right) \quad (1.29)$$

where R is the Ricci scalar, Λ is a bulk cosmological constant and, Λ_0 and Λ_1 are the tensions on the Planck and TeV branes respectively. A fine-tuning between the bulk cosmological constant and the brane tensions is imposed so that a 4D Poincare invariant solution can be obtained. The metric which solves Einstein's equations

($G_{MN} = \kappa^2 T_{MN}$) can be written as

$$ds^2 = e^{-2kb|y|} \eta_{\mu\nu} dx^\mu dx^\nu - b^2 dy^2 \quad (1.30)$$

where b is the radius of the extra dimension, k is the AdS curvature, and $\eta_{\mu\nu}$ is the 4D Minkowski metric. Furthermore, one must fix the bulk cosmological constant $\Lambda \equiv -6k^2/\kappa^2$ and the brane tensions as $\Lambda_0 = -\Lambda_1 = 6k^2/\kappa^2$ in order to complete the solution⁷. This fine-tuning is equivalent to setting the 4D cosmological constant to zero and so the perpetually difficult conundrum of the cosmological constant problem remains⁸.

The hierarchy problem however, is neatly addressed. With a moderate choice of the value $kb \sim 37$, we find that the large hierarchy between the Planck and TeV scale is generated by the “warping” of the space along the extra dimension from the exponential factor e^{-kby} . This can be seen directly if we consider the following. The action for a scalar field ψ in a potential $V(\psi) = \frac{1}{2}m^2\psi^2$ on the TeV brane, our 4D universe, can be written in this scenario⁹ as

$$S_\psi = \frac{1}{2} \int d^4x e^{-4kb} \left(\dot{\psi}^2 e^{2kb} - m^2 \psi^2 \right) \quad (1.31)$$

After rescaling $\Psi = e^{-2kb}\psi$ we find

$$S_\psi = \frac{1}{2} \int d^4x \left(\dot{\Psi}^2 - e^{-2kb} m^2 \Psi^2 \right) \quad (1.32)$$

⁷ The fundamental 5D mass scale is related to the 5D Newton’s constant by $\kappa^2 = 1/2M^3$. Other normalizations have been used. This will not be important for our model.

⁸ See [27] for related work investigating the cosmological constant problem.

⁹ see, for example, Cline and Firouzjahi in [12].

so that the physical mass as measured by an observer on our brane will be

$$m_{phys} = m e^{-kb} \quad (1.33)$$

Now if $m \sim m_{Pl}$, then the physical mass will be $m_{phys} = 10^{-16}m \sim \text{TeV}$. In fact, an attractive feature in the RS scenario is that all model parameters are naturally on the order of the Planck scale. In contrast to the ADD scenario [7], the warped geometry of the RS model resolves the hierarchy problem without the need for a large extra dimension. Unfortunately though, the possibility of testing deviations from Einstein gravity are negligible. On the other hand, a very discernible spectroscopy is possible at future collider experiments if the RS idea is found to be correct [8].

It turns out that a flaw in the RS idea exists because their solution can be found for any value of the radius, b . In order to ensure that the hierarchy problem is indeed solved, this implies that we need a mechanism which fixes $b \sim 37/k$. At the same time, we must not reintroduce any other large hierarchy of scales and avoid severe fine-tuning. Further complicating matters, there is the concern that a long-range force will be mediated by a massless radion as it will behave like a Brans-Dicke scalar. In other words, the radion must be massive in order to recover 4D Einstein gravity [26].

A mechanism to address these concerns was put forward by GW [12]. They introduced a bulk scalar field Φ with a bulk potential $V(\Phi)$ and potentials on the Planck and TeV branes $V_{0,1}$ to stabilize the interbrane separation. It was shown that a vacuum expectation value for Φ is generated by the competing bulk and brane Lagrangians. This results in a 4D vacuum energy for b , the radion potential, $V_r(b)$. In this scenario, a mild fine-tuning is required to fix b at the proper value of its

stable minimum and the resulting radion mass turns out to be $O(\text{TeV})$ (see also [13]). Fortunately, as the interactions between the radion and standard model particles are $1/\text{TeV}$ suppressed, the radion mass will satisfy the lower bound which is set on the order of $10 - 100\text{GeV}$ [13]. These couplings also ensure that the radion should be able to decay quickly enough to avoid overclosure of the universe before the onset of nucleosynthesis. In other words, there is no cosmological moduli problem associated with the couplings of the radion to the standard model¹⁰.

It turns out also that there is no moduli problem coming from the cosmology of RS1 if the radius is stabilized before nucleosynthesis. That is, a concrete stabilization mechanism alleviates the problem of fine-tuning the energy densities on the two branes. This was a necessary requirement for a static extra dimension with matter present [26]. The origins of this constraint was derived from the extra-dimensional component of Einstein's equations, G_{55} . When a nontrivial radion potential is produced however, it is just the shift¹¹ in the extra dimension δb that is determined by the G_{55} equation and the constraint on the energy densities of the two branes is effectively removed [10]. The main point is that a stabilized radion will result in ordinary FRW cosmology (up to small corrections of $O(\text{TeV}^4)$ [26]).

The obvious benefits of stabilization for the cosmology of the RS1 model are clear. With the addition of a bulk scalar field, the RS solution is put on more solid ground. In general, there has been a great deal of work done in studying the effects

¹⁰ See [13] for details about the phenomenology of a massive radion.

¹¹ Later, when we study our brane world model, we will find that the G_{55} component reduces to the equation of motion for the radion, extending previous results to include the dynamics of a time-dependent scalar field.

of a bulk scalar on the cosmology of brane-world scenarios [28]. Although much progress¹² has been made on this front, relatively little is known about the cosmology of brane-world models during the early universe. This is mainly due to the lack of exact time-dependent solutions to the higher-dimensional theory. Stabilization turns out to be a hindrance in this case, since its natural function is to produce a static brane-world. For instance, in the RS1 model linearized treatments incorporating stabilization have successfully shown the scenario to be robust under small time-dependent perturbations (for e.g. [10], [13], [30]). Once temperatures approach the TeV scale however, these solutions tend to break down. So, for example, when energy scales which drive inflation (typically $\gg \text{TeV}^4$) are reached one can no longer be sure of recovering the stable, static RS configuration. Studying the high-energy regime of RS1 cosmology thus remains a difficult prospect.

There have been efforts made to investigate possible realizations of inflation in both RS models as well as other brane-world scenarios. Whether this occurs through dynamics on the brane [14] or in the bulk [15], there is as yet no satisfactory inflationary solution which simultaneously allows for a successful transition to 4D FRW cosmology, and a dynamically generated stabilization of the extra dimension in the case of two-brane models. It has been suggested that the radion might be used as the inflaton [31]-[32], but the fact that the radius is displaced causes a back-reaction on the metric, thus changing the expansion rate, and introducing time-dependence

¹² See, for example, the self-tuning models attempting to address the cosmological constant problem [29].

into Einstein's equations. At high energies, the problem must be treated as a five-dimensional one [34], further stressing the need for exact solutions.

With the generally nonlinear behaviour of Einstein's equations when time-dependence is introduced, this is obviously a difficult question to probe. For RS-type brane-worlds which address the hierarchy problem at the static level, the central issue as we present it thus becomes: how does one study a 4D inflationary scenario (i.e., the expansion of our brane universe) and simultaneously generate the stabilization of the 5D brane-world without using a low-energy expansion scheme? It is this question we hope to provide some insight into by examining the dynamics of a coupled radion-bulk scalar field system in five dimensions.

Chapter 2

A New Model: Cosmological solutions in a dilatonic brane-world

The difficulty in studying early universe cosmology in brane-world models arises due to the high energies associated with this epoch. For example, in the RS1 model, it is possible to find perturbative solutions to Einstein's equations valid to energies typically of order the TeV scale. However, the lack of exact dynamical solutions prevents one from constructing an inflationary model which simultaneously ensures a stable radius will be recovered. In order to study the early universe in such a setting, it is imperative that we find solutions which are robust enough to handle large departures from the static level. This is a difficult problem given the nonlinear features of Einstein's equations in this regime.

With an infinite number of possible metrics we may write down there is clearly no intuitive way of choosing one which generates exact solutions while meeting other requirements such as stability in a RS-type scenario. However, as we will show, it is possible to find approximate solutions to the global equations of a 5D brane-world model which are extremely precise everywhere and still retain the main features of the RS1 model that have intrigued physicists so much in recent times. This will be the major result of this work, as we will propose a new model in the spirit of RS which addresses the hierarchy problem and issues of stability, while allowing for

the construction of viable models of inflation. The method by which this will be accomplished can be summarized as follows:

1. A new solution which also addresses the hierarchy problem is given at the static level.
2. A specific time-dependent metric modelling a 5D dilatonic brane-world which will allow us to study 4D cosmology is then proposed. This ansatz must contain the static solution.
3. The G_{05} equation is then used to construct an exact time-dependent solution for the stabilizing bulk scalar field.
4. The coupling of the radion to the four-dimensional part of the metric is made small enough to ensure that the rest of Einstein's equations are satisfied to a high degree of precision in the bulk and on the branes.
5. After separating the radion from the graviton, we find that Einstein's equations are reduced to a 4D FRW-like cosmology driven by the dynamics of the renormalized radion.
6. Finally, we impose that the radion potential generated by this solution must vanish at its stable minimum. In this way, the radion acquires a mass and the static solution will be recovered.

Let us begin by introducing the setup for our 5D brane-world model in a very general way. The action for 5D gravity coupled to the stabilizing scalar field, $\Phi =$

$\Phi(y, t)$ with a bulk potential $V(\Phi)$ and potentials on the branes $V_{0,1}(\Phi)$ is

$$S = \int d^5x \sqrt{g} \left(-\frac{1}{2\kappa^2} R + \frac{1}{2} \partial_m \Phi \partial^m \Phi - V(\Phi) \right) - \int d^4x \sqrt{g} V_0|_{y=0} - \int d^4x \sqrt{g} V_1|_{y=1} \quad (2.1)$$

The extra dimension, y is an S^1/Z_2 orbifold and so all functions are symmetric under $y \rightarrow -y$. Thus, as is normally done, we consider the interval $y \in [0, 1]$ for the compact extra dimension with the Planck ($y = 0$) and TeV ($y = 1$) branes located at the orbifold fixed points. The most general metric ansatz which will respect the principle of homogeneity and isotropy of our universe can be written as

$$ds^2 = N^2(t, y) dt^2 - A^2(t, y) d\mathbf{x}^2 - B^2(t, y) dy^2 \quad (2.2)$$

This ansatz, leads to the scalar field equation

$$\frac{d}{dt} \left(N^{-1} B A^3 \dot{\Phi} \right) - \frac{d}{dy} \left(N B^{-1} A^3 \Phi' \right) + N A^3 B \left[\frac{dV}{d\Phi} + \frac{dV_0}{d\Phi} \delta(By) + \frac{dV_1}{d\Phi} \delta(B(y-1)) \right] \quad (2.3)$$

and after substituting into the Einstein equations, $G_{MN} = \kappa^2 T_{MN}$ gives

$$G_0^0 = 3N^{-2} \left(\left(\frac{\dot{A}}{A} \right)^2 + \frac{\dot{A}\dot{B}}{AB} \right) - 3B^{-2} \left(\frac{A''}{A} + \left(\frac{A'}{A} \right)^2 - \frac{A'B'}{AB} \right) \quad (2.4)$$

$$G_i^i = 3N^{-2} \left(2\frac{\ddot{A}}{A} + \left(\frac{\dot{A}}{A} \right)^2 + 2\frac{\dot{A}\dot{B}}{AB} - 2\frac{\dot{A}\dot{N}}{AN} - \frac{\dot{N}\dot{B}}{NB} + \frac{\ddot{B}}{B} \right) - 3B^{-2} \left(2\frac{A''}{A} + \frac{N''}{N} + \left(\frac{A'}{A} \right)^2 + 2\frac{A'N'}{AN} - 2\frac{A'B'}{AB} - \frac{N'B'}{NB} \right) \quad (2.5)$$

$$G_5^5 = 3N^{-2} \left(\frac{\ddot{A}}{A} + \left(\frac{\dot{A}}{A} \right)^2 - \frac{\dot{A}\dot{N}}{AN} \right) - 3B^{-2} \left(\left(\frac{A'}{A} \right)^2 + \frac{A'N'}{AN} \right) \quad (2.6)$$

$$G_{05} = 3 \left(\frac{N'\dot{A}}{NA} + \frac{A'\dot{B}}{AB} - \frac{\dot{A}'}{A} \right) \quad (2.7)$$

in the bulk where the indices are $M, N = 0, 1, 2, 3, 5$. The bulk contribution to the stress energy tensor comes from

$$T_{mn} = g_{mn}V + \partial_m\Phi\partial_n\Phi - \frac{1}{2}(\partial^l\Phi\partial_l\Phi)g_{mn} \quad (2.8)$$

and we have the sources on the Planck and TeV branes:

$$\begin{aligned} T_m^n = & B^{-1}(0, t)\delta(y)\text{diag}(V_0 + \rho_0, V_0 - p_0, V_0 - p_0, V_0 - p_0, 0) \\ & + B^{-1}(1, t)\delta(y-1)\text{diag}(V_1 + \rho_1, V_1 - p_1, V_1 - p_1, V_1 - p_1, 0) \end{aligned} \quad (2.9)$$

where $\rho_{0,1}$ and $p_{0,1}$ correspond to arbitrary distributions of matter. In what follows we will not consider the effects of adding matter to the branes, since we will be primarily interested in the cosmology generated by the dynamical evolution of the radion coupled to the bulk. With the general setup established, we now present our new static solution.

2.1 Static solution

Below we will present the static solution to our brane-world model in order to lay the groundwork for the more complicated dynamical analysis later on. In modifying the usual metric ansatz in the RS1 scenario (1.30), we will see that it is still possible to address the hierarchy problem, while eliminating the need to use cosmological constants to satisfy Einstein's equations. Anticipating the result, our solution is the

conformally flat metric:

$$ds^2 = N^2(y) (dt^2 - d\mathbf{x}^2 - b^2 dy^2) \quad (2.10)$$

$$N(y) = e^{-kby} \quad (2.11)$$

As only y -dependence is present we find that Einstein's equations in the bulk reduce to

$$-3 \left(\frac{N''}{N} \right) = \kappa^2 \left(\frac{1}{2} \Phi'^2 + (bN)^2 V(\Phi) \right) \quad (2.12)$$

$$-6 \left(\frac{N'}{N} \right)^2 = \kappa^2 \left(-\frac{1}{2} \Phi'^2 + (bN)^2 V(\Phi) \right) \quad (2.13)$$

This is satisfied by choosing the bulk potential

$$V = -\frac{3}{2} \Phi'^2 (bN)^{-2} \quad (2.14)$$

in analogy with the work done in [33]. Also, we have the scalar field equation:

$$\Phi'' + 3 \frac{N'}{N} \Phi' - (bN)^2 \frac{dV}{d\Phi} = 0 \quad (2.15)$$

By using the relation

$$\frac{dV}{d\Phi} = \frac{1}{\Phi'} \frac{dV}{dy} \quad (2.16)$$

we find

$$(bN)^2 \frac{dV}{d\Phi} = -3\Phi'' + 3 \frac{N'}{N} \Phi' \quad (2.17)$$

and it immediately follows that the bulk scalar has the solution

$$\Phi' = c_1 \quad (2.18)$$

$$\Phi = c_1 y + c_2 \quad (2.19)$$

In order to satisfy Einstein's equations (2.12, 2.13) we must then fix $c_1 = \frac{\sqrt{3}kb}{\kappa}$. Also, for the sake of brevity, we set $c_2 = 0$ as it is undetermined and arbitrary and so we find

$$\Phi = \frac{\sqrt{3}}{\kappa} kby \quad (2.20)$$

$$V = -\frac{9k^2}{2\kappa^2} e^{\frac{2\kappa}{\sqrt{3}}\Phi} \quad (2.21)$$

This potential is similar to the one found in the Horava-Witten model [2]. Notice also that this solution differs from those usually found in RS1, $\Phi \sim e^{-cy}$ as Φ is linear in y .

Now that the bulk solution has been determined we must examine the boundary terms. The brane potentials are found from the Israel junction conditions [33, for example]:

$$\frac{1}{(bN)} \left[\frac{N'}{N} \right]_{y=0,1} = \pm \frac{\kappa^2 V_{0,1}}{3} \quad (2.22)$$

$$\frac{1}{(bN)} [\Phi']_{y=0,1} = \pm \frac{\partial V_{0,1}}{\partial \Phi} \quad (2.23)$$

A possible choice of potentials is then

$$V_0 = \frac{6k}{\kappa^2} e^{\frac{\kappa}{\sqrt{3}}\Phi} \quad (2.24)$$

$$V_1 = -\frac{6k}{\kappa^2} e^{\frac{\kappa}{\sqrt{3}}\Phi} \quad (2.25)$$

where V_0 and V_1 when evaluated at $y = 0, 1$ respectively result in the necessary tuning of energy densities for a static solution as in RS1, and we are now done¹³.

¹³ As with the original RS1 solution however, there is no guarantee that our solution will result in a stable extra dimension with the branes at the orbifold fixed points $y = 0$ and $y = 1$. This is because we have not yet addressed the dynamical problem that arises when this solution is perturbed. To do this we need to investigate the dynamics of the radion. We will treat this in the time dependent solutions which follow.

2.1.1 Solving the hierarchy problem

Fortunately our static solution can be used to address the hierarchy problem¹⁴. To see this, consider the action for a scalar field $\psi(t)$ with potential $V = \frac{1}{2}m^2\psi^2$ located on the TeV brane where we reside,

$$S_\psi = \frac{1}{2} \int d^4x e^{-5kb} \left(\dot{\psi}^2 e^{2kb} - m^2\psi^2 \right) \quad (2.26)$$

After rescaling $\Psi = e^{-\frac{3kb}{2}}\psi$ we find

$$S_\psi = \frac{1}{2} \int d^4x \left(\dot{\Psi}^2 - e^{-2kb}m^2\Psi^2 \right) \quad (2.27)$$

so that the physical mass as measured by an observer on our brane will be

$$m_{phys} = m e^{-kb} \quad (2.28)$$

So for $m \sim m_{Pl}$, the physical mass will be $m_{phys} = 10^{-16}m \sim \text{TeV}$. Thus, the hierarchy problem can be easily addressed in the same manner as the RS1 model. Furthermore, the 4D mass scale m_{Pl} is found to be related to the fundamental 5D mass scale, $\kappa^{-2/3}$ by

$$\frac{m_p^2}{8\pi} = \frac{2b}{\kappa^2} \int_0^1 e^{-3kby} dy = \frac{2}{3k\kappa^2} (1 - e^{-3kb}) \quad (2.29)$$

and only differs from the RS1 result by a constant factor ~ 1 . In this way, our static solution seemingly offers little improvement to the RS solution. The benefits of the setup we have employed will only become apparent when we incorporate time-dependence.

¹⁴ Here we follow the same procedure which led to (1.33).

2.1.2 Motivating radion dynamics

In order to motivate the approach we will follow, let's trivially modify our solution to include time-dependence. Suppose we have $b = b(t)$, and $\Phi = \Phi(y, t)$ with the metric¹⁵

$$ds^2 = e^{-2kby} (dt^2 - a^2(t) d\mathbf{x}^2 - b^2 dy^2) \quad (2.30)$$

Einstein's equations then become

$$\begin{aligned} G_0^0 &= 3e^{2kby} \left(\left(\frac{\dot{a}}{a} \right)^2 - \frac{\dot{a}}{a} \left(3k\dot{b}y - \frac{\dot{b}}{b} \right) + k\dot{b}y \left(2k\dot{b}y - \frac{\dot{b}}{b} \right) \right) - 3k^2 e^{2kby} \\ &= \kappa^2 \left(e^{2kby} \frac{\dot{\Phi}^2}{2} + b^{-2} e^{2kby} \frac{\Phi'^2}{2} + V \right) \end{aligned} \quad (2.31)$$

$$\begin{aligned} G_i^i &= 3e^{2kby} \left(2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 - 2\frac{\dot{a}}{a} \left(3k\dot{b}y - \frac{\dot{b}}{b} \right) + k\dot{b}y \left(3k\dot{b}y - 2\frac{\dot{b}}{b} \right) - 3k\ddot{b}y + \frac{\ddot{b}}{b} \right) \\ &\quad - 9k^2 e^{2kby} = 3\kappa^2 \left(-e^{2kby} \frac{\dot{\Phi}^2}{2} + b^{-2} e^{2kby} \frac{\Phi'^2}{2} + V \right) \end{aligned} \quad (2.32)$$

$$\begin{aligned} G_5^5 &= 3e^{2kby} \left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 - 3\frac{\dot{a}}{a} k\dot{b}y + (k\dot{b}y)^2 - k\ddot{b}y \right) - 6k^2 e^{2kby} \\ &= \kappa^2 \left(e^{2kby} \frac{\dot{\Phi}^2}{2} - b^{-2} e^{2kby} \frac{\Phi'^2}{2} + V \right) \end{aligned} \quad (2.33)$$

$$G_{05} = 3k^2 b\dot{b}y = \kappa^2 \dot{\Phi}\Phi' \quad (2.34)$$

The G_{05} equation is satisfied by the modified static solution with $b = b(t)$

$$\Phi = \frac{\sqrt{3}}{\kappa} kb(t)y \quad (2.35)$$

¹⁵ A similar metric ansatz was used in Ref. [33] to derive braneworld solutions in the presence of a static bulk scalar field.

Upon substituting the $G_0^0 + G_i^i$ combination into the G_5^5 equation and then plugging in our solution with the bulk potential

$$V = -\frac{9k^2}{2\kappa^2} e^{\frac{2\kappa}{\sqrt{3}}\Phi} \quad (2.21)$$

we can compare it with the scalar field equation of motion (2.3) in the bulk and see that they are equivalent. This will be a consistency requirement for the time-dependent solutions that follow or else the system will be overdetermined.

Now, upon examining the G_0^0 equation we find

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{\dot{a}}{a} \left(3k\dot{b}y - \frac{\dot{b}}{b}\right) + k\dot{b}y \left(\frac{3}{2}k\dot{b}y - \frac{\dot{b}}{b}\right) = 0 \quad (2.36)$$

Since the brane potentials will be cancelled by the nondistributional part of the metric derivatives through the junction conditions, we are left without a potential for the radion. Furthermore, the dynamics of $b(t)$ will not be satisfied everywhere in the bulk because of the y -dependence. Thus, the only possible solution is $\dot{b} = 0$. This illustrates some critical obstacles that must be overcome if we are to investigate radion dynamics in the early universe. Firstly, the most important criteria, is to find a time dependent solution which will generate a radion potential that gives a stable radius. In particular, we will need a potential which vanishes at its minimum, resulting in a heavy radion and the recovery of the static solution. Secondly, although usually all y -dependence is integrated out to study the effective 4D theory, a good solution will have to be valid everywhere in the bulk. This is due to the fact that cosmological backgrounds which describe the evolution of our universe at early times will typically be in a high-energy regime that requires the full theory [34].

From the above discussion we get an indication of the necessary features required for satisfactory global solutions to Einstein's equations with a bulk scalar field. While exact time-dependent solutions to RS-type two-brane models which incorporate stabilization have proven to be elusive¹⁶, we will show that one can find highly accurate approximate solutions and still allow for the investigation of high energy regimes.

2.2 Time dependent solutions

Up until now, any attempts to study radion dynamics in the early universe have been hampered by the lack of global solutions. For example, the RS1 solution has been found to be stable under small time-dependent perturbations and the resulting cosmology of our universe has been fully described when arbitrary matter distributions are included on the branes [30]. However, if we are to investigate the high energy regime associated with inflation, we must be able to get a handle on the time derivatives which will appear, so that larger departures from the static solution can be considered. In other words, we should be able to analyze Einstein's equations without using a linearization procedure which relies only on small perturbations from the static solution.

With the intention of recovering our static solution while satisfying Einstein's equations everywhere, let us start with a general 5D spacetime modelling a dilatonic

¹⁶ In [33] exact time-dependent solutions were explored. However, their treatment required a static bulk scalar field.

brane-world:

$$ds^2 = e^{2\varphi_1(t,y)} (dt^2 - a^2(t) d\mathbf{x}^2) - b^2 e^{2\varphi_2(t,y)} dy^2 \quad (2.37)$$

Notice that we have reparametrized the most general metric (2.2) by choosing

$$A(t, y) = a(t) N(t, y) = a(t) e^{\varphi_1(t,y)} \quad (2.38)$$

Since $a(t)$ is the scale factor associated with our universe, we will more readily be able to study the 4D cosmology. Also, the bulk and brane matter content remains the same, as we still have our stabilizing scalar field $\Phi = \Phi(y, t)$ with bulk and brane potentials $V(\Phi)$, $V_{0,1}(\Phi)$ respectively. So Einstein's equations in the bulk become

$$\begin{aligned} G_0^0 &= 3e^{-2\varphi_1} \left(\left(\frac{\dot{a}}{a} \right)^2 + \frac{\dot{a}}{a} (2\dot{\varphi}_1 + \dot{\varphi}_2) + \dot{\varphi}_1^2 + \dot{\varphi}_1 \dot{\varphi}_2 \right) - 3b^{-2} e^{-2\varphi_2} (\varphi_1'' + 2\varphi_1'^2 - \varphi_1' \varphi_2') \\ &= \kappa^2 \left(e^{-2\varphi_1} \frac{\dot{\Phi}^2}{2} + b^{-2} e^{-2\varphi_2} \frac{\Phi'^2}{2} + V \right) \end{aligned} \quad (2.39)$$

$$\begin{aligned} G_i^i &= 3e^{-2\varphi_1} \left(2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + 2\frac{\dot{a}}{a} (2\dot{\varphi}_1 + \dot{\varphi}_2) + \dot{\varphi}_1^2 + \dot{\varphi}_1 \dot{\varphi}_2 + \dot{\varphi}_2^2 + 2\ddot{\varphi}_1 + \ddot{\varphi}_2 \right) \\ &\quad - 9b^{-2} e^{-2\varphi_2} (\varphi_1'' + 2\varphi_1'^2 - \varphi_1' \varphi_2') = 3\kappa^2 \left(-e^{-2\varphi_1} \frac{\dot{\Phi}^2}{2} + b^{-2} e^{-2\varphi_2} \frac{\Phi'^2}{2} + V \right) \end{aligned} \quad (2.40)$$

$$\begin{aligned} G_5^5 &= 3e^{-2\varphi_1} \left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + 3\frac{\dot{a}}{a} \dot{\varphi}_1 + \dot{\varphi}_1^2 + \ddot{\varphi}_1 \right) - 6b^{-2} e^{-2\varphi_2} \varphi_1'^2 \\ &= \kappa^2 \left(e^{-2\varphi_1} \frac{\dot{\Phi}^2}{2} - b^{-2} e^{-2\varphi_2} \frac{\Phi'^2}{2} + V \right) \end{aligned} \quad (2.41)$$

$$G_{05} = 3(\varphi_1' \dot{\varphi}_2 - \dot{\varphi}_1') = \kappa^2 \dot{\Phi} \Phi' \quad (2.42)$$

Notice that the G_{05} equation is now independent of $a(t)$ and much simpler than the most general case. The G_{05} equation is important since it determines energy conservation conditions for matter on the branes [11]. Although we have not included

matter on the branes in our treatment, the G_{05} equation still plays an important role. That is, we will use it to construct an exact solution for our bulk scalar field, the foundation for our global solutions to Einstein's equations.

2.2.1 Small departures from 4D cosmology

Theories involving extra dimensions should eventually be linked to the known physics of the observable universe. In this way, the low-energy effective 4D theory is invaluable for testing their viability. However, the 4D description of the early universe we hope to achieve will require a solution to the 5D theory. The question we must address is then: under what conditions can a 5D solution also have a 4D description [34, see for example]. The most important criteria for us will be to require that our solution be weakly dependent on the extra dimensional coordinate, so that the dynamics are valid everywhere.

Examining the G_{05} equation, it is obvious that the solution for Φ requires that some further assumptions about the metric must be made. In Refs. [35]-[37], 5D solutions which also include the dynamics of a bulk scalar field with an exponential potential are found for the case of a single brane. One of the techniques employed was the assumption of the separability of the metric functions and the bulk scalar field. We will find that we can apply this criteria to obtain the desired effect of weakening the y -dependent dynamics in our model. To begin with, let's restrict $\varphi_{1,2}$ to being

separable functions in t and y in the following manner:

$$\varphi_1 = n(y) (1 + \tilde{\varphi}_1(t)) \quad (2.43)$$

$$\varphi_2 = n(y) + \tilde{\varphi}_2(t) \quad (2.44)$$

Here, $n(y) = -kby$ is the static result for the warp factor. Then (2.42) becomes

$$n' (\dot{\tilde{\varphi}}_2 (1 + \tilde{\varphi}_1) - \dot{\tilde{\varphi}}_1) = \frac{\kappa^2}{3} \dot{\Phi} \Phi' \quad (2.45)$$

and there is the possibility of recovering the static spacetime when $\tilde{\varphi}_{1,2} \rightarrow 0$. Next, we define the function

$$\tilde{\varphi}_1 = f(\tilde{\varphi}_2) \quad (2.46)$$

in the most general way so that $\tilde{\varphi}_2$ is the radion generating the back reaction on the metric

$$ds^2 = e^{2n(y)} [e^{2(1+f(\tilde{\varphi}_2))} (dt^2 - a^2(t) d\mathbf{x}^2) - b^2 e^{2\tilde{\varphi}_2} dy^2] \quad (2.47)$$

and we only require that

$$f(0) = 0 \quad (2.48)$$

so that the static solution (2.10) is recovered. Now (2.45) can be written as

$$n' \dot{\tilde{\varphi}}_2 \left((1 + f) - \frac{df}{d\tilde{\varphi}_2} \right) = \frac{\kappa^2}{3} \dot{\Phi} \Phi' \quad (2.49)$$

and upon examining the static solution (2.20) for the bulk scalar field we must have

$$\Phi(t, y) = \frac{\sqrt{3}}{\kappa} (\varphi(t) - n(y)) \quad (2.50)$$

where

$$\dot{\varphi} = -\dot{\tilde{\varphi}}_2 \left((1+f) - \frac{df}{d\tilde{\varphi}_2} \right) \quad (2.51)$$

Here we see that $\varphi \rightarrow 0$ as $\tilde{\varphi}_2 \rightarrow 0$, since $f(0) = 0$ and so we will be able to recover the static solution for the bulk scalar field.

The functional form of $f(\tilde{\varphi}_2)$ will be model dependent but we can gain some intuition of how to make this choice by examining the G_0^0 equation:

$$\begin{aligned} & 3e^{-2n(1+f)} \left(\left(\frac{\dot{a}}{a} \right)^2 + \frac{\dot{a}}{a} \dot{\tilde{\varphi}}_2 \left(2n \frac{df}{d\tilde{\varphi}_2} + 1 \right) + \frac{\dot{\tilde{\varphi}}_2^2}{2} \left(2n \frac{df}{d\tilde{\varphi}_2} \left(1 + n \frac{df}{d\tilde{\varphi}_2} \right) - \left((1+f) - \frac{df}{d\tilde{\varphi}_2} \right)^2 \right) \right) \\ &= 3k^2 e^{-2n-2\tilde{\varphi}_2} \left(\frac{3}{2} + 3f + 2f^2 \right) - \frac{9k^2}{2} e^{-2n+2\varphi} \end{aligned} \quad (2.52)$$

where we have inserted equations (2.43)-(2.44), and (2.50)-(2.51) and chosen

$$V = -\frac{9k^2}{2\kappa^2} e^{\frac{2\kappa}{\sqrt{3}}\Phi} \quad (2.21)$$

for the bulk potential as in our static solution¹⁷. The first concern we need to address is that the y -dependence have a negligible effect on the dynamics of $\tilde{\varphi}_2$. An obvious solution to this problem would be $f = 0$, but we will see that this choice does not generate a potential for the radion. Let us now impose the constraint

$$f \ll \tilde{\varphi}_2 \quad \text{and} \quad f \ll 1 \quad (2.53)$$

so that

$$\frac{df}{d\tilde{\varphi}_2} \ll 1 \quad (2.54)$$

¹⁷ There are other instances in the literature where bulk scalar fields with exponential potentials are used in investigating brane-world cosmology (see for example refs. [35]-[37]). Scalar fields with exponential potentials can be found through a sphere reduction in M-theory [38].

We can think of this as being equivalent to a small perturbation of the 4D slices of the full 5D spacetime induced by $\tilde{\varphi}_2$:

$$ds_4^2 \simeq e^{2n(y)} (dt^2 - a^2 d\mathbf{x}^2) (1 + 2nf(\tilde{\varphi}_2) + \dots) \quad (2.55)$$

We are not however restricting $\tilde{\varphi}_2$ to being small and so we will then consider

$$\left(\left(\frac{\dot{a}}{a} \right)^2 + \frac{\dot{a}}{a} \dot{\tilde{\varphi}}_2 - \frac{\dot{\tilde{\varphi}}_2^2}{2} \right) = k^2 e^{-2\tilde{\varphi}_2} \left(\frac{3}{2} + 3f + 2f^2 \right) - \frac{3k^2}{2} e^{2\varphi} \quad (2.56)$$

to be a good approximation of the G_0^0 equation¹⁸. Similar arguments can be applied to the rest of the Einstein equations.

We have now effectively removed the y -dependence from the bulk equations, subject only to the constraint on f . Thus, we should be able to arrive at a 4D description of the 5D theory without losing any vital information about the bulk energetics. However, we have yet to account for the equation of motion of the bulk scalar field and so we must also ensure that our solution for Φ is consistent with this.

2.2.2 The G_5^5 equation is the bulk scalar equation

In a study of the RS two-brane model incorporating radius stabilization Ref. [10] found using a perturbative approach that, due to gauge-invariant degrees of freedom, the G_{55} equation and the bulk scalar equation are equivalent. Although we do not neglect all time derivatives we find that this is also the case here. To see that the G_5^5

¹⁸ The argument that we can neglect terms involving y -dependence will also apply to the $O(f, f^2)$ terms due to the constraint (2.53). We include them here since we do not yet know how $\tilde{\varphi}_2$ and φ are related. We are interested in finding the dominant contributions to Einstein's equations and then checking that the terms we ignore are, in fact negligible. So it could be that the $O(f)$ terms, for example, are the lowest order terms (not involving time derivatives) which dominate.

equation and the scalar equation are equivalent we proceed in the following manner.

Using (2.50) and (2.53), our exact solution for Φ can be recast in the form

$$\Phi = \frac{\sqrt{3}}{\kappa} (-\tilde{\varphi}_2 + \sigma(\tilde{\varphi}_2) - n(y)) \quad (2.57)$$

where

$$\sigma(\tilde{\varphi}_2) \ll \tilde{\varphi}_2 \quad (2.58)$$

$$\sigma(0) = 0 \quad (2.59)$$

The G_{05} equation now imposes the relation

$$\frac{d\sigma}{d\tilde{\varphi}_2} = \frac{df}{d\tilde{\varphi}_2} - f \quad (2.60)$$

for (2.57) to be an exact solution. In addition, from (2.53) we must also have $\sigma \ll 1$. Clearly this is a much simpler constraint as it involves only the small terms $f(\tilde{\varphi}_2)$ and $\sigma(\tilde{\varphi}_2)$, and it will help us to derive suitable potentials for the radion, $\tilde{\varphi}_2$. The exact solution to the G_{05} equation written in this way turns out to be the starting point for our assertion that the bulk equations will be effectively independent of y .

After substituting (2.57), along with our previous results, the G_5^5 equation and the scalar field equation in the bulk now become respectively

$$\begin{aligned} & \frac{3e^{-2n(1+f)}}{2} \left(\ddot{\tilde{\varphi}}_2 + 3\frac{\dot{a}}{a}\dot{\tilde{\varphi}}_2 + \dot{\tilde{\varphi}}_2^2 \left(2n\frac{df}{d\tilde{\varphi}_2} + 1 \right) \right) \\ & = -\frac{9k^2}{2} e^{-2(n+\tilde{\varphi}_2-\sigma)} + \frac{3}{2} k^2 e^{-2n-2\tilde{\varphi}_2} (3 + 4f(1+f)); \end{aligned} \quad (2.61)$$

$$\begin{aligned} & e^{-2n(1+f)} \left((\ddot{\tilde{\varphi}}_2 - \ddot{\sigma}) + 3\frac{\dot{a}}{a}(\dot{\tilde{\varphi}}_2 - \dot{\sigma}) + \left(2n\frac{df}{d\tilde{\varphi}_2} + 1 \right) \dot{\tilde{\varphi}}_2 (\dot{\tilde{\varphi}}_2 - \dot{\sigma}) \right) \\ & = k^2 e^{-2n-2\tilde{\varphi}_2} (3 + 4f) - 3k^2 e^{-2(n+\tilde{\varphi}_2-\sigma)} \end{aligned} \quad (2.62)$$

We only need to drop the small f^2 term in the G_5^5 equation and the small σ time derivatives in the scalar equation to see that they are equivalent. The equations written in this form also reveal once again that y -dependence will have a negligible effect on the radion-scalar dynamics. The important contributions¹⁹ reduce to an equation of motion for $\tilde{\varphi}_2$

$$\left(\ddot{\tilde{\varphi}}_2 + 3\frac{\dot{a}}{a}\dot{\tilde{\varphi}}_2 + \dot{\tilde{\varphi}}_2^2\right) = 6k^2 e^{-2\tilde{\varphi}_2} \left(\frac{2}{3}f - \sigma\right) + O(f^2, \sigma^2) \quad (2.63)$$

and will be true for the bulk if (2.53) and (2.58) are met. We are almost in a position now where we can study the radion dynamics directly from the 5D theory. Next, we must be concerned about what happens with the contributions from the branes.

2.2.3 Boundary terms

Having found that Einstein's equations can be satisfied to a high degree everywhere in the bulk when the deviation in $\varphi_1(t, y)$ is small compared to that in $\varphi_2(t, y)$ from $n(y)$, we know that we will not be limited to small departures from the static solution. However, we must be careful with the boundary terms as they should also be consistent with our solution for the stabilizing field on the branes. In the following analysis we show explicitly that the back-reaction generated by the radion is fixed by our choice of boundary potentials $V_0(\Phi)$ and $V_1(\Phi)$, and by the G_{05} equation which generates an exact solution for the bulk scalar field, $\Phi(y, t)$. In particular, we will see that $f(\phi)$ is not an unphysical free parameter put in by hand into the model.

¹⁹ We see now that the $O(f)$ terms which we kept in (2.56) will contribute to the dynamics and so we cannot ignore them.

Starting with the jump conditions (as in (2.22) and (2.23)) we get

$$b^{-1}e^{-\varphi_2(t,y)} [\varphi'_1]_{y=0,1} = \pm \frac{\kappa^2 V_{0,1}}{3} \quad (2.64)$$

$$b^{-1}e^{-\varphi_2(t,y)} [\Phi']_{y=0,1} = \pm \frac{\partial V_{0,1}}{\partial \Phi} \quad (2.65)$$

Substituting for Φ and $\varphi_2(t, y)$

$$\frac{6k(1+f)}{\kappa^2} e^{-n_{0,1}-\tilde{\varphi}_2} = \pm V_{0,1} \quad (2.66)$$

$$\frac{2\sqrt{3}k}{\kappa} e^{-n_{0,1}-\tilde{\varphi}_2} = \pm \frac{\partial V_{0,1}}{\partial \Phi} \quad (2.67)$$

We already know how the y -dependence is satisfied from our analysis of the static solution, so we focus on the time dependent part to complete the bulk solution. Let's consider an arbitrary brane potential V_0 evaluated at $y = 0$. Using (2.66) and (2.67) we find the exact solution

$$f = \frac{\kappa}{\sqrt{3}} \frac{V_0}{V_0'} - 1 \quad (2.68)$$

where the prime denotes $\frac{d}{d\Phi}$. From (2.68) and (2.57) we have

$$\frac{df}{d\tilde{\varphi}_2} = \left(\frac{d\sigma}{d\tilde{\varphi}_2} - 1 \right) \left(1 - \frac{V_0'' V_0}{V_0'^2} \right) \quad (2.69)$$

Plugging (2.68) and (2.69) into (2.60) and using (2.67) evaluated at $y = 0$ to find

$V_0'' = -\frac{d\tilde{\varphi}_2}{d\Phi} V_0'$, we see that (2.60) reduces to

$$\left(1 - \frac{d\sigma}{d\tilde{\varphi}_2} \right) + \frac{\kappa}{\sqrt{3}} \frac{d\Phi}{d\tilde{\varphi}_2} = 0 \quad (2.70)$$

From (2.57) this is automatically satisfied for a given brane potential V_0 so long as (2.68) and (2.69) are satisfied. Interestingly, the fact that we have guaranteed an exact solution for the bulk scalar field also addresses the issue of the boundary terms affecting the dynamical bulk solutions. Furthermore, the system of equations

(2.60, 2.66, 2.67) is not overdetermined even though we only have the two unknowns f and σ . Similar arguments can be made for the TeV brane potential V_1 evaluated at $y = 1$. In this way a fine-tuning is imposed on the brane potentials which is equivalent to setting the 4D cosmological constant to zero at the static level.

Since we will be primarily interested in the radion potential which is generated from f and σ , our procedure will be to “choose” f and derive σ from (2.60) to see the range of possibilities available to us in this setup. Once we have chosen an explicit model we will make sure that our brane potentials fix f according to (2.68). Further, it will be a requirement that when $\Phi \rightarrow -\frac{\sqrt{3}}{\kappa}n(y)$, our brane potentials evaluated on the branes must become

$$V_0 = \frac{6k}{\kappa^2} \quad (2.71)$$

$$V_1 = -\frac{6ke^{kb}}{\kappa^2} \quad (2.72)$$

so that we can recover the static results of (2.24-2.25). Depending on the model, exponential potentials of the type used for the static solution can be employed to achieve this. Now that we have outlined how Einstein’s equations can be satisfied everywhere, we must demonstrate that the dynamics will conform with the assumptions that got us here.

2.3 The 4D theory

Without evidence to confirm the existence of extra dimensions, any higher-dimensional theory ultimately must make contact with the real world. Generally this is achieved

through the use of a low-energy effective 4D theory. When one considers high energies, as we are doing however, there is a danger in representing intrinsically five-dimensional theories with a four-dimensional description. It is not always possible to distinguish between the true dynamics of the system and its dimensionally reduced equivalent, especially when an attempt is being made to go beyond the moduli-space approximation²⁰. In fact, as we have indicated throughout this work, studying brane-world cosmology in regimes which are far from the static level, in the absence of exact solutions to the full theory, demands a five-dimensional treatment. With this in mind, we now proceed to derive the four-dimensional Friedmann equations from our five-dimensional model.

2.3.1 The Friedmann equations

To study the cosmology of the 4D effective theory one usually integrates out all y dependence to determine the Friedmann equations for the radion. First, we would find the 4D potential for the radion by integrating the G_0^0 equation [26],

$$V_r = \int_0^1 be^{4\varphi_1+\varphi_2} \left(k^2 e^{-2\varphi_2} \left(\frac{3}{2} + 3f + 2f^2 \right) + \kappa^2 V \right) dy \quad (2.73)$$

an analog to Gauss's law in classical electromagnetism. In Einstein's theory we would have

$$\int G_N^M \sqrt{g} dy = \kappa^2 \int T_n^m \sqrt{g} dy \quad (2.74)$$

$$\sqrt{g} = be^{4\varphi_1+\varphi_2} \quad (2.75)$$

²⁰ see [34] for a discussion of this issue involving the radion in a two-brane model.

where the brane potentials V_0, V_1 are implicitly included by recognizing that the metric derivatives on the branes will cancel these sources through the junction conditions. But since we have effectively removed any y -dependence by employing the method in the previous section we need only consider the results given in (2.53) and (2.58) where $f(\tilde{\varphi}_2)$ is used to find Φ exactly. Thus, the following analysis, for the sake of clarity, involves only terms which will make a demonstrable contribution to the radion dynamics.

To arrive at the Hubble equation we write (2.56) as

$$\begin{aligned} \left(\left(\frac{\dot{a}}{a} \right)^2 + \frac{\dot{a}}{a} \dot{\tilde{\varphi}}_2 - \frac{\dot{\tilde{\varphi}}_2^2}{2} \right) &= k^2 e^{-2\tilde{\varphi}_2} \left(\frac{3}{2} + 3f \right) - \frac{3}{2} k^2 e^{2\varphi} + O(f^2) \\ &= k\kappa^2 e^{-\tilde{\varphi}_2} V_r \end{aligned} \quad (2.76)$$

where by keeping terms only up to order f and σ we find

$$V_r = \frac{3k}{\kappa^2} e^{-\tilde{\varphi}_2} (f - \sigma) + O(f^2, \sigma^2) \quad (2.77)$$

Now, in order to separate the massless graviton from the radion, we perform a conformal transformation²¹ on the metric [26]

$$a(t) = \gamma \bar{a}(t) \quad (2.78)$$

$$dt = \gamma \bar{d}t \quad (2.79)$$

²¹ It can be shown that there always exists a transformation allowing this separation for a 5D spacetime with a metric of the form:

$$ds^2 = N^2(t, y) (dt^2 - a^2(t) dx^2) - B^2(t, y)$$

To eliminate the cross terms $\dot{a}\dot{\varphi}_2$ in Einstein's equations γ is fixed to be²²

$$\gamma = e^{-\frac{\dot{\varphi}_2}{2}} \quad (2.80)$$

Explicitly, we find

$$\begin{aligned} \frac{\dot{a}}{a} &= \frac{1}{\gamma} \left(\frac{\dot{\bar{a}}}{\bar{a}} + \frac{\dot{\gamma}}{\gamma} \right) \\ &= \frac{1}{\gamma} \left(\frac{\dot{\bar{a}}}{\bar{a}} - \frac{\dot{\varphi}_2}{2} \right) \end{aligned} \quad (2.81)$$

will reduce the G_0^0 equation to

$$\left(\frac{\dot{\bar{a}}}{\bar{a}} \right)^2 - \frac{3\dot{\varphi}_2^2}{4} = k\kappa^2 \gamma^4 V_r \quad (2.82)$$

Notice also that the difference in the two frames can be quite large for $\frac{\dot{\varphi}_2}{2} > 1$, so it may be possible to see the effects of new physics coming from the 5D brane-world model²³. Using our definition of the fundamental mass scale (2.29) given earlier

$$\frac{m_{Pl}^2}{8\pi} = \frac{2}{3k\kappa^2} (1 - e^{-3kb}) \simeq \frac{2}{3k\kappa^2} \quad (2.83)$$

we can now write the Hubble equation in the Einstein frame as

$$\left(\frac{\dot{\bar{a}}}{\bar{a}} \right)^2 = \frac{16\pi}{3m_{Pl}^2} \left(\frac{1}{2}\dot{\phi}^2 + \bar{V}_r \right) \quad (2.84)$$

²² The exact result found by integrating Einstein's equations over y fixes

$$\gamma = \left[\frac{1 - e^{-3kb}}{1 - e^{-3kb(1 + \frac{2}{3}f)}} \left(1 + \frac{2}{3}f \right) e^{-\dot{\varphi}_2} \right]^{\frac{1}{2}}$$

and so conforms with our assumptions about f required to solve the 5D theory.

²³ Two frames related by a conformal transformation should still describe the same physics. The argument here is that if the scale factor in the conformal frame \bar{a} evolves as in ordinary FRW cosmology then a will exhibit nonstandard behaviour due to its coupling to the radion. We will see how this happens when we study inflation in this model.

where

$$\phi = \frac{3m_{Pl}}{4\sqrt{2\pi}}\tilde{\varphi}_2 = \beta\tilde{\varphi}_2 \quad (2.85)$$

is the renormalized radion field and

$$\bar{V}_r = \gamma^4 V_r = e^{-2\frac{\phi}{\beta}} V_r \quad (2.86)$$

modifies the radion potential by a factor of γ^4 . Similarly, using (2.85) we also find the G_i^r equation reduces to

$$2\frac{\ddot{\bar{a}}}{\bar{a}} + \left(\frac{\dot{\bar{a}}}{\bar{a}}\right)^2 = \frac{16\pi}{m_{Pl}^2} \left(-\frac{1}{2}\dot{\phi}^2 + \bar{V}_r\right) \quad (2.87)$$

and so the Friedmann equations will be analogous to those found in standard 4D cosmology with a scalar field. Given that FRW cosmology involving various potentials has been so extensively studied, we can be confident that our dynamical solutions will be analytically treatable. It now remains to be seen that the G_5^5 equation will produce the equation of motion for the radion, as it did for the bulk scalar field. If this is true, and our assumptions are correct, then we have assured ourselves that the 5D dynamics will have a genuinely 4D description even at high energies. Furthermore, we will also be ready to construct some explicit potentials for the radion with which we can study the inflation of our brane-universe.

2.3.2 The radion potential: explicit models

To arrive at the equation of motion for the radion, we start with (2.63), keeping the highest order terms which will contribute to the dynamics of the system:

$$\ddot{\tilde{\varphi}}_2 + 3\frac{\dot{a}}{a}\dot{\tilde{\varphi}}_2 + \dot{\tilde{\varphi}}_2^2 = 6k^2 e^{-2\tilde{\varphi}_2} \left(\frac{2}{3}f - \sigma \right) \quad (2.88)$$

Transforming to the Einstein frame, and substituting for the renormalized radion we get

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = 6k^2 \beta e^{-3\frac{\phi}{\beta}} \left(\frac{2}{3}f(\phi) - \sigma(\phi) \right) = -\frac{d\bar{V}_r}{d\phi} \quad (2.89)$$

the equivalent FRW equation of motion for a 4D scalar in some potential. Note that the radion potential (2.86) is now written as

$$\bar{V}_r = 2k^2 \beta^2 e^{-3\frac{\phi}{\beta}} (f - \sigma) \quad (2.90)$$

So the last equality is found by evaluating the derivative and substituting our condition for an exact solution of Φ in a given model (2.60):

$$\begin{aligned} \frac{d\bar{V}_r}{d\phi} &= -6k^2 \beta e^{-3\frac{\phi}{\beta}} \left(f - \sigma - \frac{\beta}{3} \left(\frac{df}{d\phi} - \frac{d\sigma}{d\phi} \right) \right) \\ &= -6k^2 \beta e^{-3\frac{\phi}{\beta}} \left(\frac{2}{3}f(\phi) - \sigma(\phi) \right) \end{aligned} \quad (2.91)$$

Hence if our assumptions about $f(\phi)$ are valid then the entire system of equations will be consistent. This also confirms the result that the G_5^5 equation determines the shift in the radion (for e.g. [26] and [10]) due to the cosmological expansion. The obvious advantage we have from our derivation is that the radion dynamics are also included so that our analysis can be extended beyond small departures from the static result.

Next it is also instructive if we calculate the radion mass in general as it will help us further restrict the possible brane potentials we need to fix f :

$$m_r^2 = \left. \frac{d^2 \bar{V}_r}{d\phi^2} \right|_{\phi=0} = 2k^2 \beta \left. \frac{df}{d\phi} \right|_{\phi=0} \quad (2.92)$$

where we have imposed $f(0) = \sigma(0) = 0$. Hence, for a stable radion with $m_r^2 > 0$ we must have $\left. \frac{df}{d\phi} \right|_{\phi=0} > 0$. So the radion mass is determined by the way in which the radion couples to the 4D part of the metric. We now can investigate possible forms $f(\phi)$ may have such that a suitable radion potential is generated which will allow us to explore inflationary scenarios while guaranteeing stability and the recovery of the static results. While there are many existing models of inflation motivated by particle physics [16] which we could explore, only those which conform with the imposed behaviour of f and hence σ will be of interest to us²⁴.

From the above discussion we are obviously restricted to models which are quadratic in ϕ near $\phi = 0$. For example, it is easy to construct the well known chaotic potential $\frac{1}{2}m_r^2\phi^2$ by setting $f(\phi) = \frac{m_r^2}{2k^2\beta^2} e^{3\frac{\phi}{\beta}} \phi \left(\frac{3}{2}\phi + \beta\right)$ and deriving $\sigma(\phi) = f - \frac{m_r^2}{4k^2\beta^2} e^{3\frac{\phi}{\beta}} \phi^2$ from (2.60). However, the problem of super-Planckian field values needed to generate enough inflation will still remain. Therefore our analysis will involve the construction of some new models to study inflation, and the problems this epoch poses. Let us begin with a simple example:

$$f = \alpha \tilde{\varphi}_2 = \alpha \frac{\phi}{\beta} \quad (2.93)$$

²⁴ Exact dynamical solutions to the 5D spacetime would give a specific form of the radion potential. Arguments of naturalness aside, our primary concern here is that $\bar{V}_r \sim m_r^2 \phi^2$ for $\phi \ll m_{Pl}$.

gives us

$$\sigma = \alpha \tilde{\varphi}_2 - \frac{\alpha \tilde{\varphi}_2^2}{2} = \alpha \frac{\phi}{\beta} \left(1 - \frac{\phi}{2\beta} \right) \quad (2.94)$$

where $\alpha \ll 1$ is the necessary assumption. The radion potential now becomes

$$\begin{aligned} \tilde{V}_r &= \alpha k^2 \phi^2 e^{-3\frac{\phi}{\beta}} + O(\alpha^2) \\ &= \frac{m_r^2 \phi^2}{2} e^{-3\frac{\phi}{\beta}} \end{aligned} \quad (2.95)$$

This potential (shown in Fig.2.1) is similar to the simplest chaotic potential, and has a stable minimum at $\phi = 0$. However, the presence of a local maximum and the steepness of \tilde{V}_r for $\phi < 0$ will lead to inflationary results which are either inconsistent with observation or require an unnatural fine tuning of initial conditions²⁵. Next we

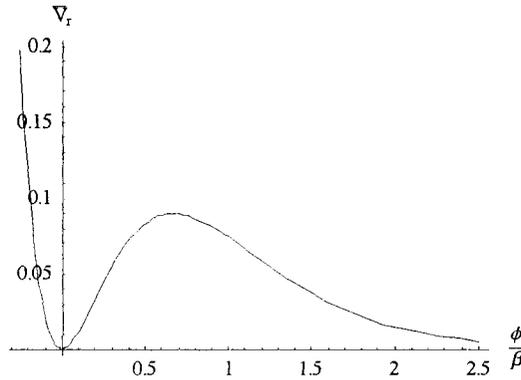


Fig.2.1: $\tilde{V}_r \sim e^{-3\frac{\phi}{\beta}} \frac{\phi^2}{\beta^2}$. For inflation to occur below $\phi = m_p$, $\phi(0)$ must be very close to the local maximum.

²⁵ This is reminiscent of a typical potential for the dilaton field one would expect to find in string theory [39]. The radion potential also has a runaway minimum for which the theory is free and a nontrivial minimum at finite values.

consider the slightly more elaborate model

$$f = \alpha (e^{\omega\tilde{\varphi}_2} - 1) = \alpha \left(e^{\omega\frac{\phi}{\beta}} - 1 \right) \quad (2.96)$$

$$\sigma = f \left(1 - \frac{1}{\omega} \right) + \alpha\tilde{\varphi}_2 = \alpha \left(1 - \frac{1}{\omega} \right) \left(e^{\omega\frac{\phi}{\beta}} - 1 \right) + \alpha\frac{\phi}{\beta} \quad (2.97)$$

where ω is some constant²⁶. This gives a radion potential of the form

$$\tilde{V}_r = 2\alpha k^2 \beta^2 e^{-3\frac{\phi}{\beta}} \left(\frac{1}{\omega} \left(e^{\omega\frac{\phi}{\beta}} - 1 \right) - \frac{\phi}{\beta} \right) \quad (2.98)$$

If we expand for $\frac{\phi}{\beta} \ll 1$ we find

$$\tilde{V}_r \simeq \omega\alpha k^2 \phi^2 = \frac{1}{2} m_r^2 \phi^2 \quad (2.99)$$

which is consistent with (2.92)

$$m_r^2 = 2k^2 \beta \left. \frac{df}{d\phi} \right|_{\phi=0} = 2k^2 \alpha \omega \quad (2.100)$$

Thus, the radion will be heavy and stable provided $\omega\alpha > 0$. As shown in Fig. 2.2, the numerical constant ω determines the curvature of the radion potential and can be restricted by finding the range of parameter space which will allow us to address inflationary constraints.

The optimum choice for a potential which is flat enough to generate a large amount of inflation is $\omega = 3$. In this case

$$\bar{V}_r = \frac{1}{9} m_r^2 \beta^2 \left(1 - e^{-3\frac{\phi}{\beta}} \left(1 + 3\frac{\phi}{\beta} \right) \right) \quad (2.101)$$

²⁶ An exact derivation for $f(\phi)$ using the brane potentials for the special case $\omega = 3$ is given in Appendix A, as we will employ this model in a detailed study of inflation in the next chapter.

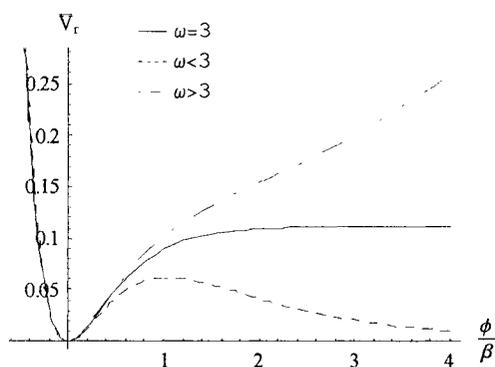


Fig.2.2: \bar{V}_r vs. $\frac{\phi}{\beta}$ showing qualitative behaviour of the radion potential for different choices of ω .

and it will perhaps be the most natural choice as it resembles very closely potentials derived in supergravity models [16], without the additional $\frac{\phi}{\beta}$ term. We also note that steep inflation [40], when the exponential terms become large for $\phi < 0$, is an avenue we can investigate. This corresponds to initial conditions which involve a radius of the extra dimension which is smaller than the stabilized value. Keeping in mind that our approximations require that \tilde{V}_r will be altered for large²⁷ $\tilde{\varphi}_2 = \frac{\phi}{\beta}$, we are now ready to look at inflationary scenarios in detail, and physically constrain the possible models available to us in our five-dimensional brane-world.

²⁷ Regardless, for ϕ much above m_{Pl} , the semiclassical analysis we have employed will likely break down. In particular, if the energy density of the inflaton is above the Planck scale the theory of quantum gravity will be required.

Chapter 3

Inflation and the Radion

The paradigm of inflation has been very successful in addressing many questions left unanswered by Big Bang cosmology. Figuring prominently among these are the horizon problem, the flatness problem, and the lack of explanation for the observed anisotropies in the CMB. In spite of its well known benefits however, this hypothesized epoch of the early universe also has its own set of problems.

To begin with, we know that it is possible to achieve inflation when the energy density of the universe is dominated by a cosmological constant. Such a scenario in the early universe is unacceptable though for the simple reason that inflation must at some point end in order to allow for the emergence of the eras of radiation and matter domination. In other words, if inflation is in fact driven by vacuum energy, it must be dynamic. For this reason, the most likely and most often studied candidate for the inflaton is a time-dependent scalar field, usually considered to be minimally coupled to gravity. In this context the primary concern in model building is the generation of a potential for the scalar driving inflation. Once this is achieved, a mechanism must be in place to allow for a “graceful exit” from the inflationary phase to ordinary FRW cosmology.

The vast array of inflationary scenarios which have been successfully proposed bears witness to the relative ignorance we have of the physics in play at these energy scales. Fortunately, the advent of precision measurement in the cosmos will help to

restrict the myriad of models, and further inspire work towards a derivable theory rather than one whose potential must be put in by hand. Admittedly, our model suffers from the same defect of generic models of inflation. Namely, the freedom to choose the exact form of the metric through $f(\phi)$ (that is, V_0 and V_1) can also lead to a wide variety of potentials which satisfy inflationary constraints. Nevertheless, insight may be gained from the fact that the radion is a natural feature of the brane-world which does not need to be inserted for purely cosmological reasons. That the radion is also a scalar field coupled to gravity suggests that the problem of moduli fixing in higher-dimensional models and justifying its existence for the inflation of our universe may be directly linked. In fact, by demanding that our five-dimensional model have a dynamically realized stabilization mechanism, we have simultaneously allowed for a possible inflationary potential. This result underscores the importance of studying the dynamical evolution of brane-world scenarios in the nonlinear, high-energy regime.

To study inflation in our brane-world model we must determine the equation of motion for the radion, constrain the model parameters according to physical observations by making analytic approximations, and then check our results numerically. The equation of motion for the radion given in (2.89) can also be found from the 4D effective action in the Einstein frame. Typically one finds this by averaging the 5D Einstein action over the bulk after linearizing about the static solution [26]. However, as we have noted, this can lead to the danger of losing information about the bulk at high energies and care must be taken in doing so.

Using our definitions and assumptions about f and σ we may write the effective 4D action after performing the conformal transformation (2.78-2.79) as:

$$\begin{aligned} S &= \int d\bar{t}\bar{a}^3 \left(-\frac{9m_{Pl}^2}{16\pi} \left(\frac{\dot{\bar{a}}^2}{\bar{a}^2} - \frac{1}{2}\dot{\bar{\phi}}^2 \right) - \gamma^4 V_r \right) \\ &= \int d\bar{t}\bar{a}^3 \left(-2\beta^2 \left(\frac{\dot{\bar{a}}^2}{\bar{a}^2} \right) + \frac{1}{2}\dot{\bar{\phi}}^2 - \bar{V}_r \right) \end{aligned} \quad (3.1)$$

Here we have included only the dominant contributions, eliminating the y -dependence, as we did in the more appropriate global treatment. This is again the analogous result to the four-dimensional result for a scalar field in some potential, except that we are working in a different frame. So the equation of motion for the radion is

$$\ddot{\bar{\phi}} + 3\bar{H}\dot{\bar{\phi}} = -\frac{d\bar{V}_r}{d\bar{\phi}} \quad (3.2)$$

where \bar{H} is the Hubble parameter in the Einstein frame, in agreement with the equation of motion more reliably derived from the 5D theory.

During inflation the dynamical evolution of the scalar field proceeds roughly as follows. The inflaton rolls slowly in the flat region of its potential, mimicking a cosmological constant, and so leads to an exponential expansion of the universe. As it rolls towards its minimum, quantum fluctuations in the inflaton produce the primordial density fluctuations which exit the horizon of the universe and generate the observed anisotropies in the CMB. The inflaton then oscillates rapidly about its minimum, and through its couplings to matter lead to the reheating of the universe and particle production. In this work we are not concerned with the details of the transition to the radiation era, but focus solely on the possibility of employing the radion as the inflaton. So the crucial feature of this evolution which will allow for an analytic

treatment of the radion dynamics during inflation is the slow-roll approximation. We now proceed to obtain some analytic results and see how our 5D brane-world model is constrained.

3.1 Analytic results

The slow-roll approximation is more than an analytic tool in investigating scalar field dynamics during inflation. Since the scalar field potential will dominate the energy density during slow-roll it is also a condition allowing for the negative-pressure equation of state required for inflation to proceed. The most important criterion which guarantees that the inflaton is rolling slowly in its potential are the flatness conditions. In our model the conditions given in equations (1.14) and (1.15) are equivalently written as

$$\epsilon \equiv \frac{\beta^2}{9} \left(\frac{V'}{V} \right)^2 \ll 1 \quad (3.3)$$

$$\eta \equiv \frac{2\beta^2}{9} \left(\frac{V''}{V} \right) \ll 1 \quad (3.4)$$

As long as these conditions are satisfied, the time derivatives of the radion will be small. Using the slow-roll approximation the equation of motion for the radion then becomes

$$3\bar{H}\dot{\phi} \simeq -\frac{d\bar{V}_r}{d\phi} \quad (3.5)$$

and for the Hubble parameter in the Einstein frame we get

$$\bar{H}^2 \simeq \frac{16\pi}{3m_{Pl}^2} \bar{V}_r = \frac{3}{2\beta^2} \bar{V}_r \quad (3.6)$$

To determine the number of e -foldings until inflation ends analytically, we must also transform N :

$$\begin{aligned}
N &= \int H dt = \int \bar{H} d\bar{t} + \int \frac{d\gamma}{\gamma} \\
&= \int_t^{t_{\text{end}}} \bar{H} d\bar{t} - \frac{1}{2\beta} \int_{\phi}^{\phi_{\text{end}}} d\phi \\
&= \frac{9}{2\beta^2} \int_{\phi_{\text{end}}}^{\phi} \frac{\bar{V}_r}{\bar{V}'_r} d\phi + \frac{1}{2\beta} (\phi - \phi_{\text{end}})
\end{aligned} \tag{3.7}$$

This result is model independent and only requires that the slow roll conditions be met.

Now that we have arrived at the relevant slow-roll equations in general we can apply them to an explicit model. Here we investigate the potential (2.101) where the radion induces a back reaction on the metric given by $f(\phi) = \alpha \left(e^{\omega \frac{\phi}{\beta}} - 1 \right)$. This choice leads to

$$\frac{\bar{V}_r}{\bar{V}'_r} = \beta \frac{\left(e^{\omega \frac{\phi}{\beta}} - 1 \right) - \omega \frac{\phi}{\beta}}{(\omega - 3) \left(e^{\omega \frac{\phi}{\beta}} - 1 \right) + 3\omega \frac{\phi}{\beta}} \tag{3.8}$$

We will be primarily interested in the region of parameter space²⁸ where $\omega \geq 3$ since $\omega < 3$ requires an extreme fine tuning of the initial conditions near the local maximum of \bar{V}_r (see Fig.2.2).

We know that the minimum amount of inflation required to solve the problems of standard Big Bang cosmology is $N = 60$. Furthermore, we would also like to have field values which are below the Planck scale in order to trust our semiclassical analysis. By numerically integrating (3.7) starting at $\phi(0) = m_{Pl}$, we find there is a

²⁸ We might also consider $\omega < 0$ but a stable radion would require changing the sign of α giving us the same results except with $\phi \rightarrow -\phi$.

narrow range of ω which satisfies both requirements.

$$3 \leq \omega \leq 3.155 \quad (3.9)$$

where the closer we get to $\omega = 3$ the flatter the potential becomes. From (Fig.3.3) we see that the optimally flat case of $\omega = 3$ will enable us to achieve the maximum amount of inflation for initial conditions of the inflaton/radion field below the Planck scale. Although (3.7) is not exactly solvable for ϕ , it is still possible to obtain some

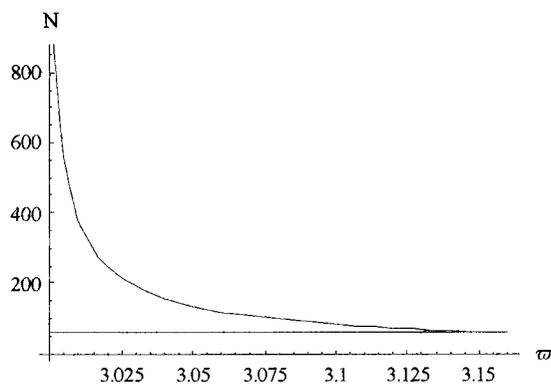


Fig.3.3: N vs. ω , The maximum amount of slow roll inflation possible starting at $\phi(0) = m_p$ for the class of models.

highly accurate analytical results.

Let us now concentrate on the most natural formulation of \bar{V}_r :

$$\bar{V}_r = \frac{1}{9} m_r^2 \beta^2 \left(1 - e^{-3\frac{\phi}{\beta}} \left(1 + 3\frac{\phi}{\beta} \right) \right) \quad (3.10)$$

The e -folding number N is now given by

$$N = \frac{9}{2\beta} \left(\int \frac{\left(1 - e^{-3\frac{\phi}{\beta}} \left(1 + 3\frac{\phi}{\beta} \right) \right)}{\phi} d\phi \right) + \frac{\phi}{2\beta} \quad (3.11)$$

Ignoring the small contributions from the endpoints²⁹ a numerical solution yields the analytic behaviour of ϕ :

$$N_\phi = \frac{1}{2} \left(e^{c \frac{\phi}{\beta}} - 1 \right) \quad (3.12)$$

$$\phi \simeq \frac{\beta}{c} \ln(2N_\phi + 1) \quad (3.13)$$

where $c \simeq 2.23$. This is found to be an accurate approximation in the region of interest before the end of inflation (Fig.3.4). Furthermore, with $N_\phi = 60$ we find that $\phi \simeq .65 m_{Pl}$, confirming that the problem of super-Planckian initial field values is avoided in this model. Now that we know how the radion behaves during the

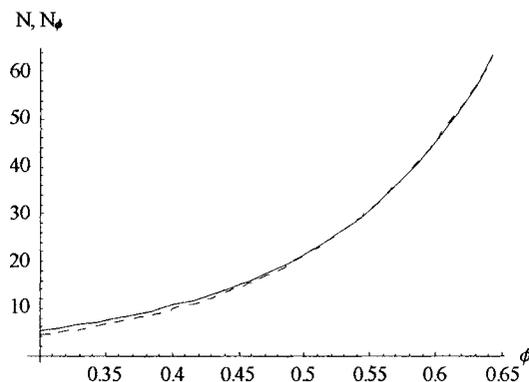


Fig.3.4: N, N_ϕ vs ϕ up to 60 e-folds before the end of inflation. The dotted line is the analytic approximation.

slow-roll regime we must investigate whether we can trust our assumption that $f(\phi)$ is small. In particular, we must determine whether this claim is consistent with the spectrum of energy density perturbations fixed by observations of the CMB. In order to do so the mass of the radion must be constrained.

²⁹ Our results depend very weakly on the precise value of ϕ_{end} . For our purposes, the analytic approximations used here suffice.

3.1.1 Spectrum of density perturbations

The attractiveness of inflation as a viable model of the early universe is its predictive power. The CMB, since its discovery, has provided the most important experimental data constraining possible models. To see how the observational constraint is met in our model we write the perturbation spectrum in the Einstein frame as

$$\begin{aligned}
 \delta_H &= H\delta t = \bar{H}\bar{\delta}t + \frac{\dot{\gamma}}{\gamma}\bar{\delta}t \\
 &= \bar{H}\bar{\delta}t - \frac{1}{2\beta}\delta\phi \\
 &= \frac{\bar{H}^2}{2\pi\dot{\phi}} - \frac{\bar{H}}{4\pi\beta} = \frac{\bar{H}}{2\pi} \left| \frac{\bar{H}}{\dot{\phi}} - \frac{1}{2\beta} \right|
 \end{aligned} \tag{3.14}$$

where the last line follows from the standard derivation³⁰. So, as with the e -folding number N , the spectrum of density perturbations will be affected by the difference in frames describing the evolution of a and \bar{a} . Using the slow roll approximation we find

$$\begin{aligned}
 \delta_H &\simeq \frac{1}{4\pi\beta^2} \sqrt{\frac{3\bar{V}_r}{2}} \left(\left| \frac{9}{\beta} \frac{\bar{V}_r}{\bar{V}'_r} + 1 \right| \right) \\
 &= \frac{1}{2\pi\beta} \sqrt{\frac{3\bar{V}_r}{2}} \left(\frac{dN}{d\phi} \right) \\
 &= \frac{c}{2\pi\beta^2} \sqrt{\frac{3\bar{V}_r}{2}} \left(N_\phi + \frac{1}{2} \right)
 \end{aligned} \tag{3.15}$$

where we have used the analytic approximation in (3.12). Setting $\sqrt{\bar{V}_r} \simeq \frac{m_r\beta}{3}$ for the early part of inflation during which the radion mass must be constrained by

³⁰ The important measurable quantity is actually the amplitude of the spectrum of density perturbations, $|\delta_H|^2$. Hence, we take the absolute value. See [18] for details.

observation we find

$$\delta_H \simeq \frac{cm_r}{2\pi\sqrt{6}\beta} \left(N_\phi + \frac{1}{2} \right) \quad (3.16)$$

for the region of interest. Furthermore, we know that when COBE scales leave the horizon $\delta_H \simeq 2 \times 10^{-5}$ and after setting $N_{\text{COBE}} \simeq 50$, this restricts the radion mass to be

$$\begin{aligned} m_r &= \frac{3\sqrt{6\pi} \times 10^{-5}}{c \left(N_\phi + \frac{1}{2} \right)} m_{Pl} \\ m_r &\simeq 1.15 \times 10^{-6} m_{Pl} \end{aligned} \quad (3.17)$$

This result is not very sensitive to N . Hence, we can approximate

$$\alpha = \frac{m_r^2}{6k^2} \sim 2 \times 10^{-13} \quad (3.18)$$

if we set $k \sim m_{Pl}$ so that all the other input parameters to the model are natural³¹, as it is done in RS1. Also, the maximum value of $f(\phi)$ we would expect in this analysis if we set $\phi = m_{Pl}$ is a very small number:

$$f_{\text{max}}(m_{Pl}) = \alpha \left(e^{3\frac{m_p}{\beta}} - 1 \right) \sim 10^{-10} \quad (3.19)$$

Thus, the assumptions we have made throughout this work are still valid even up to the Planck scale. Although this result gives us hope that our toy model can be used to make some definite physical predictions, it must be noted that we are still required to explain a new mass scale for the radion, as with the $m^2\phi^2$ model of inflation. This is the same deadend which many inflationary models find themselves in, as m_r will be

³¹ The expansion parameter α , written in this way is reminiscent of the small ϵ expansion of the GW stabilization mechanism where the mass of the bulk scalar was required to be much smaller than the curvature scale k [12].

effectively undetectable in the laboratory. Nevertheless, in contrast to the simplest model of chaotic inflation, we have found that we can ameliorate the problem of super-Planckian initial field values. Furthermore, our inflaton has not been introduced into particle physics purely for cosmological purposes as in many of the 4D models, but is a natural outcome of adding dynamics to the brane-world.

Another observational constraint we must test our model against is the spectral index. As we noted in chapter 1, this is a measure of the shape of the spectrum of density perturbations. Using the slow roll approximation, the spectral index is found to be

$$n - 1 \simeq -\frac{\partial \ln \delta_H^2}{\partial N} = -\frac{1}{\delta_H^2} \frac{\partial (\delta_H^2)}{\partial N} \quad (3.20)$$

Substituting (3.16) we get

$$n - 1 \simeq -\frac{2}{N_\phi + \frac{1}{2}} \quad (3.21)$$

Depending on the precise value of N , this can comfortably lie within the bounds set by observation [?], $n = 0.98 \pm 0.02$. We note that this result is similar to non-minimal inflation [41] which has a potential of the form $V = V_0 (1 - e^{-q\phi/m_{Pl}})$, where q and V_0 are constants. However, that model which can be derived from supergravity, suffers from large initial field values [16], and so the semiclassical analysis employed here would become suspect. Now that we have seen that it is possible to construct yet another successful model of inflation, albeit in a higher-dimensional setting, we must check our analytic results numerically.

3.2 Numerical results

Numerical simulations were performed to test the validity of our analytic approximations and were found to be complementary. Below we present some of the results for the $\omega = 3$ scenario. Notice that we have chosen the convenient dimensionless time unit $\bar{H}_0 \bar{t}$, where \bar{H}_0 is the initial value of the Hubble parameter, since during slow roll inflation $\bar{H} \bar{t}$ is approximately constant so that at the end of inflation $\bar{H} \bar{t} \sim 60$, the minimum number of e -foldings required.

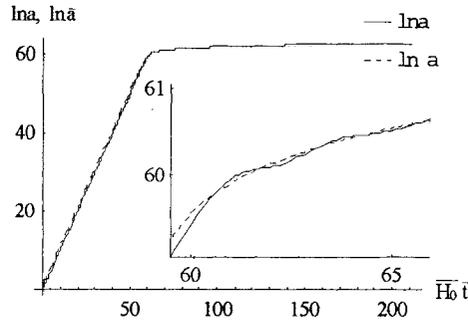


Fig.3.5: $\ln a, \ln \bar{a}$ vs. $\bar{H}_0 \bar{t}$. The minimum amount of inflation required to solve the problems of standard Big Bang cosmology. The inset shows the effect of ϕ oscillating on $\ln a$.

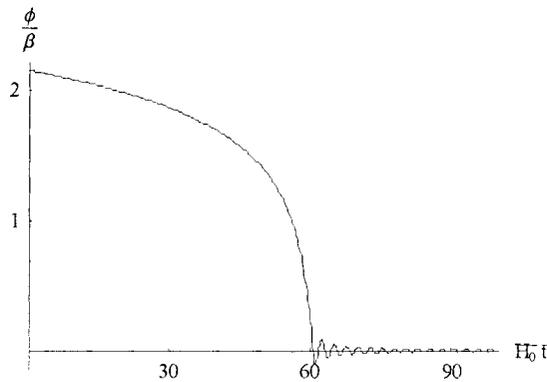


Fig.3.6: $\frac{\phi}{\beta}$ vs. $\bar{H}_0 \bar{t}$. The inflaton/radiation field is in slow-roll during inflation then oscillates about $\phi = 0$. Initially we have $\phi(0) = .645 m_p$.

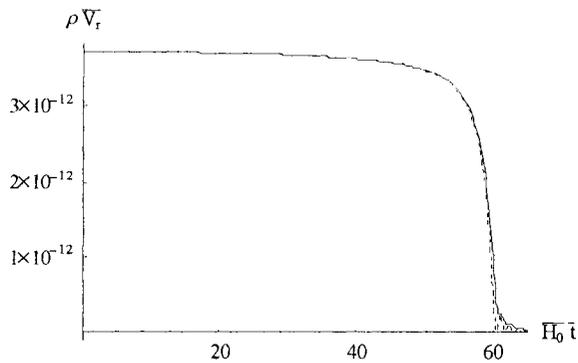


Fig.3.7: ρ, \bar{V}_r vs. $\bar{H}_0 \bar{t}$. The radion potential dominates the total energy density during inflation. The units are m_{Pl}^4 .

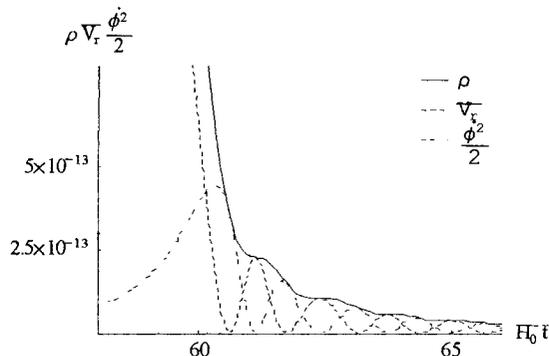


Fig.3.8: $\rho, \bar{V}_r, \frac{\dot{\phi}^2}{2}$ vs. $\bar{H}_0 \bar{t}$. The plot shows the oscillating kinetic and potential energies as inflation ends and the inflaton/radion decays.

Inspecting our results we find that the qualitative dynamical behaviour is similar to that of $m^2 \phi^2$ chaotic inflation. This is not too surprising given that the radion potential becomes quadratic as ϕ approaches zero. The difference is obvious in the early part of inflation where the radion is in an extremely flat region of the potential. Additionally, and this would be generic to any model constructed in this setup, large differences between the evolution of the scale factor of our universe, a and the conformally transformed one, \bar{a} as shown in Fig.3.5, could lead to distinct signatures

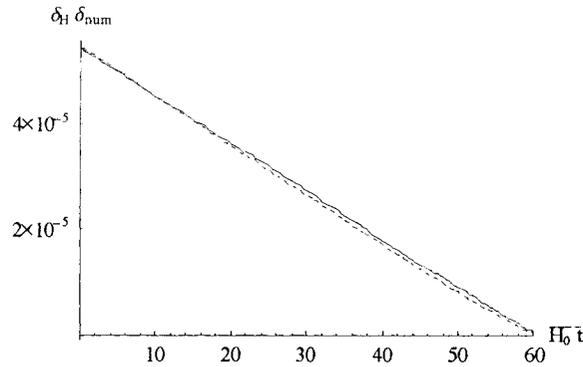


Fig.3.9: $\delta_H, \delta_{\text{num}}$ vs. $\bar{H}_0 \bar{t}$. The analytic approximation and the exact numerical result for the density perturbations are in good agreement for the region of interest.

coming from the higher-dimensional dynamics. Observing the oscillating behaviour of a as inflation ends, this might be especially evident when studying the reheating process in models of this kind.

3.3 Alternative scenarios

As we noted earlier, there a number of possible potentials which would allow for the inflation of our universe. In the absence of a complete set of exact solutions to the Einstein equations, this is inevitable. However, our analysis shows that it is possible to obtain very accurate approximate solutions where ϕ can reach as high as the Planck scale and still be valid. We have raised the cutoff energy applied to studies involving small perturbations by allowing for a dynamical stabilization of the extra dimension. This raises the possibility that there may be other approaches to achieving inflationary solutions which we could employ.

First, we can rule out steep inflation [40] in the radion potential we have studied. Examining (2.101) we see that \bar{V}_r begins to climb very steeply when $\phi < 0$ and blows up as $\phi \rightarrow -\infty$. Physically this amounts to the situation where the radius of the extra dimension approaches zero. With initial conditions in this region of the potential, we will have a case similar to the exponential potentials of steep inflation. The most inflation occurs for ω large (Fig.3.10). So we can approximate

$$N \simeq -\frac{1}{2} \ln \left(\frac{\beta - 3\phi}{\beta} \right) - \frac{\phi}{\beta} \simeq -\frac{\phi}{\beta} \quad (3.22)$$

where an initial value $\phi = -m_{Pl}$ gives at most $N \simeq 2.12$ e -folds of inflation. It could

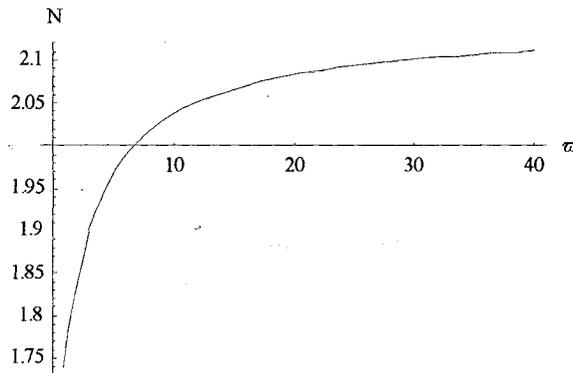


Fig.3.10: N vs. ω , The maximum amount of slow roll inflation possible for $\phi(0) = -m_p$ for the class of models.

be that our universe began as a result of the expansion of an extra dimension, but such an assumption in this scenario will always lead to Planck scale energies and lengths since $b_{phys} \sim m_{Pl}^{-1} e^{\frac{\phi}{2\beta}}$, again pointing to analyses which must come from the quantum theory of gravity.

Using the radion as the inflaton is obviously the cleanest method at our disposal since we are not required to introduce any other entity which would be used solely

for the purposes of inflation. However, given that m_r could be heavier and still allow for $f \ll 1$ if the radion were not used for inflation, there are other scenarios which we could consider. For example, we speculate that assisted inflation [42, see for example] might be possible here. Generally, assisted inflation utilizes the Kaluza-Klein modes of a bulk scalar field to collectively drive this process. The end result is a lowering of the energy scales usually required with distinct signatures from single field models. We might also try to drive inflation by including matter on the brane and then proceeding as the standard lore dictates. In either case, the overall shape of the radion potential need not be significantly altered.

Other than allowing ourselves the freedom to set the radion mass at some other scale by relieving it of its role as the inflaton, there is no clear reason why we would want to explore either approach. However, what our analysis shows is that it is generically possible to study the dynamics of a five-dimensional brane-world in the early universe. Moreover, in spite of (or because of) the wealth of options available to us in this setup, with the radion doubling as the inflaton it is not such a long journey from inflation to a stabilized solution to the hierarchy problem.

Chapter 4

Conclusions: discussion and future work

The cosmos has inspired humanity since time immemorial. In the world of modern theoretical physics however, the study of cosmology has long been a neglected stepchild. Today this is rapidly changing as cosmological parameters are being routinely quoted to many significant figures while cosmology continues to mature as an experimental science. In fact, recent developments suggest that it will be possible to test fundamental theory in the cosmological laboratory in ways which are impossible at our terrestrial laboratories. Many traditional particle physicists are now putting on the hat of the cosmologist and vice versa. This union has been a long time in coming.

In the 1980's two seemingly disparate, albeit fertile ideas emerged in theoretical physics: string theory and inflation. All these years later, legions of papers have appeared in various periodicals, and there is as yet no definite connection established between them. Nevertheless, recent progress seems to suggest that we may be getting close to inflationary scenarios which can be realized by fundamental objects in string theory [43, for example].

The appearance of moduli fields has troubled string theorists for a while. In cosmology their presence is problematic since they can ruin nucleosynthesis predictions or overclose the universe if they acquire a mass [44]. Furthermore, massless moduli, such as the original RS radion can lead to an untenable fifth force. In this

way, the problem of moduli-fixing is generic to studies of higher-dimensional theories. This makes the suggestion that moduli fields may be used for inflation [45] even more attractive as direct tests of fundamental theory will likely be possible.

This obvious overlap between particle physics and cosmology is the main motivation for the work we have presented here. Building on the ideas of RS we have constructed a 5D brane-world which addresses the hierarchy problem at the static level. This result has been achieved in various ways utilizing the brane-world idea inspired by the Horava-Witten model [2], and so adds relatively little to the current discourse. However, the main idea we have exploited in our setup which is novel is that it may be possible to use the radion as the inflaton and dynamically arrive at the stable configuration and the recovery of 4D FRW cosmology. To this end, we have employed a time-dependent bulk scalar field with an exponential potential for which we have analyzed the dynamics of the coupled radion-scalar system in some detail.

Our treatment circumvents the problem of relying on a four-dimensional description of an intrinsically five-dimensional setup. This is necessary given the importance of the nonlinear terms in the Einstein equations once time-dependence is introduced at high energies. We have explicitly shown that, even in the absence of exact solutions to the full system of equations, it is possible to obtain an equivalent 4D representation of the 5D dynamics. This was accomplished by weakening the extra-dimensional dependence on the 4D part of the metric. In particular, the back-reaction of the radion on the metric was made to be small without requiring that it be near its static value. In this manner, we were not restricted to following a typical linearization procedure.

The model in some ways is artificial however, given the freedom we have in determining the back-reaction effect of the radion from our choice of brane potentials. This defect indicates that the dynamics of the radion will be uniquely determined by exact global solutions to Einstein's equations as opposed to the partial solutions we have presented here. On the other hand, the flexibility of our model allows us to construct a variety of suitable potentials for inflation leading to results which can be distinguished from the many existing 4D models. And since our method allows us to obtain Friedmann-like equations even at high energies, we have found that the radion-scalar system and its coupling to gravity is just the setup required for a model of inflation.

We know that observations require that the extra dimension should be static, or very nearly so at least since nucleosynthesis, otherwise we would see variations in fundamental constants [46]. So the fact that we have a variety of potentials which allow for inflation and also radius stabilization is beneficial, but this must be confirmed for explicit models as the transition to ordinary FRW cosmology is made. From a preliminary investigation it appears as though our setup will allow for the radion to decay well before the onset of nucleosynthesis. Detailed work with explicit potentials might even lead to some interesting results during the reheating process as the radion will have a distinct (read: not adhoc) phenomenology associated with it. This is already an improvement on existing 4D models which invoke a scalar field solely for the purpose of powering inflation. Furthermore, we saw that there will be deviations from typical 4D inflationary predictions due to the fact that the radion is coupled to the

graviton in Einstein's equations. Aside from the noted differences from the standard results during inflation, reheating results could strongly differ. This is because our scale factor will fluctuate according to $a = \bar{a} \exp(-\phi/2)$ as the radion oscillates about $\phi = 0$, indicating that our universe will breathe during its transition to the radiation era as the radion decays.

Another interesting line of research would be the nature of cosmological perturbations in this model. There has recently been work done on this subject in brane-world cosmology which could also lead to testable predictions [47]. Given that our analysis allows us to almost breach the Planck scale without violating our consistency requirement that $f(\phi)$ be small, there may also be some hope that Planck scale physics originating early on during inflation could be imprinted on the CMB. This argument follows the model-independent work being done on "trans-Planckian" physics nowadays [48]. Add this to the fact that the phenomenology of our model still needs to be detailed and there is much future work which can be done.

With all the innovative developments in cosmology today, both theoretically and observationally, we can only hope that the universe will yield up a mystery or two. Whether string theory or inflation or any other theory probing deeper into the nature of physical reality turns out to be correct, the envelope will inevitably be pushed further. Needless to say, the prospects for getting something that nobody ordered are as good as they ever have been.

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In the following we show how the brane potentials are used to solve for the back-reaction on the metric in our brane-world model. In particular we show that the particular case which was studied in detail

$$f = \alpha (e^{3\tilde{\varphi}_2} - 1) \quad (2.96)$$

is a derived quantity. Let's begin by choosing a potential on the Planck brane

$$V_0 = \Lambda_0 e^{\frac{\kappa}{\sqrt{3}} a_0 \Phi} + \Delta_0 e^{\frac{\kappa}{\sqrt{3}} c_0 \Phi} \quad (A.1)$$

where Λ_0, Δ_0, a_0 and c_0 are constants. When evaluated at $y = 0$, (2.66, 2.67) imposes

$$\frac{6k(1+f)}{\kappa^2} e^{-\tilde{\varphi}_2} = \Lambda_0 e^{a_0(-\tilde{\varphi}_2+\sigma)} + \Delta_0 e^{c_0(-\tilde{\varphi}_2+\sigma)} \quad (A.2)$$

and

$$\frac{2\sqrt{3}k}{\kappa} e^{-\tilde{\varphi}_2} = \frac{\kappa}{\sqrt{3}} (a_0 \Lambda_0 e^{a_0(-\tilde{\varphi}_2+\sigma)} + c_0 \Delta_0 e^{c_0(-\tilde{\varphi}_2+\sigma)}) \quad (A.2)$$

Anticipating the results given in (2.96) and (2.97), we can fix $a_0 = \frac{1}{1-\alpha}$ and $c_0 = -\frac{2}{1-\alpha}$

and redefine

$$\sigma = -\alpha (\tilde{\varphi}_2 + \epsilon (\tilde{\varphi}_2)) \quad (A.3)$$

where $\sigma \ll \tilde{\varphi}_2$ (eq. 2.58) must still be imposed to protect the bulk solutions. Now we have from (2.69) and (2.70)

$$\frac{6k(1+f)}{\kappa^2} = \Lambda_0 e^{\frac{\alpha}{1-\alpha}\epsilon} + \Delta_0 e^{3\tilde{\varphi}_2 - 2\frac{\alpha}{1-\alpha}\epsilon} \quad (A.4)$$

and

$$\frac{6k(1-\alpha)}{\kappa^2} = \Lambda_0 e^{\frac{\alpha}{1-\alpha}\epsilon} - 2\Delta_0 e^{3\tilde{\varphi}_2 - 2\frac{\alpha}{1-\alpha}\epsilon} \quad (A.5)$$

The recovery of our static result fixes

$$\begin{aligned}\Lambda_0 + \Delta_0 &= \frac{6k}{\kappa^2} \\ \Lambda_0 - 2\Delta_0 &= \frac{6k}{\kappa^2} (1 - \alpha)\end{aligned}$$

from the junction conditions. Solving, we find

$$\Delta_0 = \frac{2k}{\kappa^2} \alpha \quad (\text{A.6})$$

$$\Lambda_0 = \frac{2k}{\kappa^2} (3 - \alpha) \quad (\text{A.7})$$

From equations (A.5)-(A.7) an exact solution is found for ϵ :

$$\epsilon = \left\{ \begin{array}{l} \frac{(1-\alpha)}{\alpha} \ln \left(\frac{1}{\alpha-3} \left(g(\tilde{\varphi}_2) + \frac{(\alpha-1)^2}{g} + \alpha - 1 \right) \right) \\ \frac{(1-\alpha)}{\alpha} \ln \left(\frac{1}{2(\alpha-3)} \left(-g - \frac{(\alpha-1)^2}{g} - 2(\alpha-1) + i\sqrt{3} \left(g - \frac{(\alpha-1)^2}{g} \right) \right) \right) \\ \frac{(1-\alpha)}{\alpha} \ln \left(\frac{1}{2(\alpha-3)} \left(-g - \frac{(\alpha-1)^2}{g} - 2(\alpha-1) + i\sqrt{3} \left(g - \frac{(\alpha-1)^2}{g} \right) \right) \right) \end{array} \right\} \quad (\text{A.8})$$

where

$$g(\tilde{\varphi}_2) = \left(\frac{-\alpha e^{3\tilde{\varphi}_2} (9 - 6\alpha + \alpha^2) - 1 + \alpha (3 - 3\alpha + \alpha^2) + \frac{1}{(\alpha-3) \sqrt{\alpha e^{3\tilde{\varphi}_2} \sqrt{(2-2\alpha(3-3\alpha+\alpha^2) + (9-6\alpha+\alpha^2))}}}}{\alpha-3} \right)^{\frac{1}{3}} \quad (\text{A.9})$$

This is not a very attractive solution for our purported use in the radion potential.

However, using the second solution in (A.8), as it gives real values of ϵ (hence σ), and expanding in powers of α we find

$$\epsilon = \frac{2}{3} (e^{3\tilde{\varphi}_2} - 1) + \frac{2}{3} (1 + 4e^{3\tilde{\varphi}_2} - 5e^{6\tilde{\varphi}_2}) \alpha + O(\alpha^2) \quad (\text{A.10})$$

so that (A.3) becomes

$$\sigma = \left(\tilde{\varphi}_2 + \frac{2}{3} (e^{3\tilde{\varphi}_2} - 1) \right) \alpha + \frac{2}{3} (1 + 4e^{3\tilde{\varphi}_2} - 5e^{6\tilde{\varphi}_2}) \alpha^2 + O(\alpha^3) \quad (\text{A.11})$$

In a similar manner if we expand (A.4) in powers of α we find

$$f = (e^{3\varphi_2} - 1) \alpha + \frac{4}{9} (7e^{3\varphi_2} - 8e^{6\varphi_2} + 1) \alpha^2 + O(\alpha^3) \quad (\text{A.12})$$

We can solve f and σ exactly using (A.8) and (A.9) but it would not affect our results at all since

$$\alpha = \frac{m_r^2}{6k^2} \sim 2 \times 10^{-13} \quad (3.18)$$

from our analysis of inflationary constraints. Also, $\tilde{\varphi}_2 < 1$ ensured we had sub-Planckian field values for the renormalized radion

$$\phi = \frac{3m_{Pl}}{4\sqrt{2\pi}} \tilde{\varphi}_2 = \beta \tilde{\varphi}_2 \quad (2.85)$$

Therefore to an extremely good approximation

$$f = \alpha (e^{3\tilde{\varphi}_2} - 1) \quad (2.96)$$

$$\sigma = \frac{2}{3}f + \alpha \tilde{\varphi}_2 \quad (2.97)$$

as given and are derived quantities from our choice of the brane potential, V_0 .

The potential on the TeV brane must also guarantee our solution for f . If we choose

$$V_1 = \Lambda_1 e^{\frac{\kappa}{\sqrt{3}} a_0 \Phi} + \Delta_1 e^{\frac{\kappa}{\sqrt{3}} c_0 \Phi} \quad (A.13)$$

where

$$\Lambda_1 = -\frac{2ke^{kb}}{\kappa^2} e^{-\frac{kb}{1-\alpha}} (3 - \alpha) \quad (A.14)$$

$$\Delta_1 = -\frac{2ke^{kb}}{\kappa^2} e^{\frac{2kb}{1-\alpha}} \alpha \quad (A.15)$$

is imposed from the boundary conditions on the static results then the same solution for f and σ is also generated. Note again that the fine-tuning of the $\Lambda_{0,1}$ and $\Delta_{0,1}$ constants amounts to setting the four-dimensional cosmological constant to zero at the static level. We note also that the junction conditions are satisfied for Φ and

$V_{0,1}(\Phi)$ only when $y = 0$ and $y = 1$. That is, the branes will remain at the orbifold fixed points even as the extra dimension evolves. So when the radion decays to its minimum, the radius will be stabilized.

This procedure can be followed for any set of brane potentials to derive f and σ in order to arrive at the radion potential. What we have done with our analysis is to “choose” f and use (2.60) to derive σ generally to see what kind of radion potentials can be generated. The jump conditions however must be satisfied at least up to order α for f and σ where we have been working, so as not to introduce dynamics which will make our bulk solutions inconsistent. In this way f is really derived from physical quantities and can be found with as much precision as we want.