APPLICATIONS OF STABILITY-THEORETICAL METHODS TO THEORIES OF MODULES

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ABSTRACT:

I introduce a class of totally transcendental (tt) theories called basic and prove a structure theorem for the models of such a theory. Every tt module is basic. The model-complete theories of modules introduced by Eklof and Sabbagh provide some strong analogies between the ideal theory of a Noetherian ring and the stability-theoretic treatment of types. I give purely model-theoretic proofs of Matlis' theorem on injective modules and the Lasker-Noether decomposition of ideals. I generalize these results to arbitrary tt theories of modules and give a purely model-theoretic proof of Garavaglia's theorem on the decomposition of tt modules. I extend these methods to topological modules in the topological logic L_t. I generalize Deissler's rank for minimal models in order to analyze the complexity of the injective envelope of a module over a commutative Noetherian ring in terms of Krull dimension.

RÉSUMÉ:

J'introduis une classe de théories totalement transcendantes (tt) dites "basic" et je demontre une théorème concernant la structure des modèles de ces théories. Tout module tt est "basic". Les théories modèle-complètes des modules introduites par Eklof et Sabbagh produisent des analogies puissantes entre la théorie des idéaux d'un anneau Noetherien et le traitement des types par stabilité. Je donne des démonstrations purement modèle-théoriques du théorème de Matlis sur les modules injectifs et de Lasker-Noether sur la décomposition des idéaux. Je généralise ces resultats pour les théories tt de modules quelconque et donne une demonstration purement modele-théorique du théorème de Garavaglia sur la decomposition des modules tt. J'étends ces méthods aux modules topologique dans la logique topologique L_t. Je généralise le rang de Deissler pour les modèles minimaux afin d'analyser la complexité de l'enveloppe injective d'un module sur un anneau commutative Noetherien en fonction de la dimension de Krull.

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0. INTRODUCTION

0.0 HISTORICAL SURVEY:

In this thesis I investigate some abstract stability theory, then consider the application of this theory to the study of modules. In the course of this study, many interesting and useful parallels are drawn between the type structure of a complete theory and the classical theory of ideals in rings. These parallels prove useful in different ways, first when I prove several classical theorems about rings and modules from abstract model theory, then later on when I use algebraic concepts such as Krull dimension as part of the solution of model-theoretic problems.

Stability theory originated in the work of Morley [Mr, 1965] and afterwards, Baldwin and Lachlan [BL, 1971]. Here already the ideas of rank of types, equal-rank extensions, and independence of elements were central to the theory. The full theory was developed by S. Shelah in a series of articles dating from the early 1970's and culminating in his book, Classification Theory of 1978 [Sh]. Here for the first time the important concepts of orthogonality (a sort of independence relation for types) and regularity of types were studied. Regularity is a central concept of this thesis.

In recent years the treatment of the elements of stability theory has been considerably refined and simplified; and my treatment and terminology follows more closely the recent survey article of Makkai [M] and is strongly influenced by the work of Lascar and Poizat [LP], [L].

The study of the model theory of modules originated in a paper of Eklof and Sabbagh [ES, 1971]. Here they introduced the theory T_Λ of unitary left Λ-modules over a fixed ring Λ and determined conditions on Λ so that T_Λ should have a model completion T_Λ^*. The whole theory was put on firm foundations by the work of Baur [Ba] and Monk [Mo] (1975) who proved the basic elimination of quantifiers result for T_Λ. My own work was inspired in part by the paper of Eklof and Sabbagh, in part by an example (vector spaces with a linear operator) studied in Poizat's thesis [Po], and at first involved extending Poizat's ideas to T_Λ^*. Later on I did much more, finding the theories T_Λ^* a fruitful source of examples of many concepts of stability theory beyond those considered by Poizat. Afterwards I turned the whole process around, studying general totally transcendental theories of modules by using abstract stability theory to prove things about them analogous to the properties of the earlier examples.

The model theory of modules has become an intense field of study since 1979 when Garavaglia began publishing a series of articles on the topic [G2-G5]. Other work followed rapidly from authors including E. Bouscaren, A. Pillay, M. Prest, P. Rothmaler and M. Ziegler (1980-1984). Some of this work, as will be noted later, overlaps with my own.

In 1975-76, T. McKee, S. Garavaglia and M. Ziegler independently introduced a logic suitable for the study of topological structures and Garavaglia [G1] laid the foundations of the model theory of topological modules. As a result I was inspired to extend some of my own work on modules to the topological context. Work in the field has continued, with Cherlin and Schmidt considering decision problems and elementary invariants for various theories of topological abelian groups.

In 1977, R. Deissler introduced a rank suitable for the study of minimal models (with respect to $\prec$) of a complete theory. I was able to generalize Deissler's rank, in particular in a way that allowed me to study models of some of the theories T_Λ^* . Woodrow and Knight have also studied Deissler's rank.

The balance of this section contains an outline of this thesis. In the next section of this chapter, I give a fairly detailed outline of the necessary elements of stability theory from the literature. The ring theory and module theory will be introduced in each chapter as needed.

0.1 OUTLINE OF RESULTS:

Chapter I studies basic theories. A totally transcendental basic theory is one in which every type is non-orthogonal to a strongly regular 1-type over $\emptyset$. Examples are the theories T_Λ^* where Λ is a Noetherian ring. As a direct consequence of elementary properties of strongly regular types as expounded in Shelah's book [Sh, Chapter V.1] I prove the following Basis Theorem for such theories T (see I, 2.1):

Let P be a maximal set of pairwise orthogonal strongly regular 1-types over $\emptyset$. A basis $\mathcal{B}$ for $M \models T$ is a maximal independent set of elements of M realizing types in P . Then every model M of T is determined up to isomorphism by its basis, in fact M is prime and minimal over its basis. Equivalently, M is determined by its dimension, the vector $\langle\delta_p\rangle_{p \in P}$ where δ_p is the maximum cardinality of an independent set of elements of M realizing p .

This chapter represents work done mostly in 1979, although the material in section 4 is more recent.

Chapter II studies the model complete theories of modules T_Λ^* introduced by Eklof and Sabbagh in 1971. I show how all of the basic concepts of stability theory manifest themselves in these theories: I characterize independence, the Lascar-Poizat fundamental order, orthogonality, regularity, and so on. From these characterizations it follows that T^*_Λ is basic if Λ is Noetherian. The most important outcome is an intimate correspondence between the stability-theoretic type theory on the one hand and the methods and concepts of classical ideal theory on the other. As a result I am able to give purely model-theoretic proofs of two important theorems of classical ring and module theory: Matlis' theorem [Ma, 1958] that every injective module over a Noetherian ring can be written uniquely as a direct sum of indecomposable modules, and the generalization by Lesieur and Croisot [LC 1958] to the noncommutative case of the Lasker-Noether normal decomposition of an ideal of a Noetherian ring. In actual fact the proof of the Lesieur-Croisot result is delayed until Chapter III where an entirely new and more general version is presented.

This chapter represents work done mostly in 1979-1980, while the normal decomposition theorem was derived in 1982.

Chapter III extends the ideas and methods of Chapter II to arbitrary totally transcendental theories of modules. Here I am inspired by a result of Garavaglia [G4] that every tt module can be written uniquely as a direct sum of indecomposable modules (extending the result of Matlis mentioned above). Garavaglia's proof is partly algebraic in nature. As a result of the Basis Theorem I am able to give a purely model-theoretic proof using characterizations that I develop of independence, regularity and weight. This represents work done mostly in 1980-1981, but I have modified my presentation somewhat after receiving preprints of articles of Pillay and Prest [P1, P2, PP] in the spring of 1981, along with a very helpful letter from Mike Prest criticizing an early draft of the results of Chapter II and III.

Later I realized that the work on the model theory of modules could be put in a much broader context: the abelian structures of Fisher [F]. This context ties in naturally with the work of Chapter IV. For any tt theory of abelian structures closed under products, the set of 1-types over $\emptyset$ shares many characteristics in common with the set of left ideals of a Noetherian ring Λ. For such a theory I give a normal decomposition theorem for $S_1(\emptyset)$ which generalizes the Lesieur-Croisot decomposition of ideals in a Noetherian ring. This result dates from 1982-1983.

Chapter IV studies the topological logic introduced by McKee [Mc, 1976], Ziegler, and Garavaglia. I first show how the basics of stability theory can be brought in to topological model theory by providing a translation of the topological logic into ordinary first order logic. I then concentrate on topological modules. By means of the aforementioned translation, they provide a new example of Fisher's abelian structures, and all of the results of Chapter III apply. In addition I show that the complete theory of a compact topological module has elimination of topological quantifiers.

This chapter represents work done mostly in 1981-1982.

Chapter V considers generalizations of Deissler's rank rk introduced in [D, 1977], and their applications to modules. A model M of T is minimal with respect to $\prec$ iff every element of M has a rank. In some sense rk measures the difficulty of defining an element by choosing elements from definable sets.

The generalization is accomplished by considering various restrictions Φ on the concept of definable set. Associated with each Φ are a Deissler type rank rk^{Φ} and a relation $\prec_{\Phi}$ between structures. If certain elementary properties are satisfied, rk_{Φ} and $\prec_{\Phi}$ have the same relationship as rk and $\prec$. I consider various Φ's and their relationship with each other and with some standard model-theoretic and algebraic concepts. I solve a problem of Deissler on the relation between rk and Morley rank (actually I get a sharper result in terms of Lascar's rank U).

The specific application is to injective modules over a commutative Noetherian ring with Φ being the set of positive primitive formulas and the corresponding rank being rk^+, positive Deissler rank. Analyzing injective modules in terms of their direct sum decompositions into indecomposables, I obtain upper bounds on rk^+ in terms related to the Krull dimension of the underlying ring.

This chapter represents work done mostly in 1982-1983.

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0 1. SUMMARY OF THE ELEMENTS OF STABILITY THEORY

1.0 SOURCES: The prime source for the work on stability theory in this thesis is the book of Shelah [Sh]. The work of Shelah has been so important that it has already inspired several articles whose main purpose is to refine and clarify the ideas and techniques of Shelah. The most important of these for my purposes are the paper of Lascar and Poizat [LP] on the elements of forking theory, the paper of Lascar on orthogonality and regularity [L], (both of which contribute major new insights to their subjects) and the survey article of Makkai [M] which synthesizes the various approaches. Wherever possible I will refer each result below to each of these sources. The articles mentioned adequately cover the history of the individual results, some of which is quite involved. I also refer the reader to the article of Harnik and Harrington [HH].

1.1 NOTATION AND CONVENTIONS:

My notation and terminology are fairly standard and much of it can already be found in [CK]. L, L', etc. denote finitary first order languages; $\phi(\vec{x})$, $\psi(\vec{x})$ are formulas of L with free variables included in the possibly infinite sequence of variables $\vec{x}$; $|L|$ is the cardinality of the set of all formulas of $L_{\omega\omega}$; Σ, $\Sigma(\vec{x})$ are arbitrary sets of sentences or formulas of L (with free variables incluced in $\vec{x}$); T, T' are generally used to denote complete theories in L ; $p(\vec{x})$, $q(\vec{x})$, etc. denote complete types of a theory T , with free variables included in $\vec{x}$.

The equality sign "=" is the predicate symbol for equality in L and is also used informally for the predicate of equality. The symbol "::=" is used for definitions and is read "is defined to be". The symbol ":=" assigns a temporary value to a variable symbol. The symbol ":=:" compares two strings of symbols:

$$"\phi :=: \phi(x, y) :=: 'x + y = 0' "$$

means that ϕ is the formula $\phi(x, y)$ with free variables x and y, and ϕ is specifically the formula $'x + y = 0'$. The symbol "::" compares the length of strings: "$\vec{a} :: \vec{x} :: n$" means that the sequence of constants $\vec{a}$ has the same length as the sequence of variables $\vec{x}$, and this length is $n \in \omega$. In many-sorted logic, "::" means that the sorts match as well.

The model-theoretic symbols "$\equiv$", "$\cong$", "$\prec$", "$\models$", and "$\vdash$" all have their standard meanings. If parameters from a set A are to be considered, I write "$M \equiv_A N$" and "$M \cong_A N$" or "$M \equiv N\ (A)$" and "$M \cong N\ (A)$".

I adopt the convention of working entirely in some saturated model $\mathfrak{C}$ of T of large cardinality Θ (say Θ strongly inaccessible). $\mathfrak{C}$ is called the <u>Shelah Universe</u> or <u>Monster Model</u>. The important feature of $\mathfrak{C}$ is that saturation implies that $\mathfrak{C}$ is universal and homogeneous, so the subsets and submodels of $\mathfrak{C}$ adequately represent the models of T. Unless explicitly stated otherwise, all sets, models, sequences are assumed to be subsets of $\mathfrak{C}$ of cardinality $< \Theta$, in particular, types are over sets of cardinality $< \Theta$ and have fewer than Θ variables. Sometimes for emphasis such sets are referred to as <u>small</u> sets, but usually I just say set with the restriction understood. By the saturation of $\mathfrak{C}$ every type, in fact every type in $\leq \Theta$ variables over a small set, is realized in $\mathfrak{C}$. Furthermore if $\vec{b}, \vec{c}, A$ are small, $\vec{b} \equiv_A \vec{c}$, then there is an automorphism of $\mathfrak{C}$ fixing A and sending $\vec{b}$ to $\vec{c}$. That is, if $\vec{b}$ and $\vec{c}$ have the same elementary type over A, $t(\vec{b}, A) = t(\vec{c}, A)$, then they have the same automorphism type in $\mathfrak{C}$ over A (and of course, conversely). If A is a small set, $L(A)$ is the language extending L with new constant symbols for each $a \in A$ (also denoted by a) and $T(A)$ is $\{\sigma | \sigma$ a sentence of $L(A)$, $\mathfrak{C} \models \sigma\}$.

When I want to make the distinction, script letters $\mathcal{M}, \mathcal{N}$ denote models, M, N denote the corresponding underlying sets. Since everything is contained in $\mathfrak{C}$ and in particular $\mathcal{M}, \mathcal{N} \prec \mathfrak{C}$, mostly the distinction doesn't matter, although in Chapter V where I am not always dealing with complete theories it does. A, B, C, etc. denote arbitrary sets, lower case letters a, b, c denote elements and an arrow $\vec{\ }$ is used to denote possibly infinite sequences: $\vec{a}, \vec{b}, \vec{c}$. $\vec{a} \in A$ means that each element of the sequence $\vec{a}$ is an element of A, and $\vec{A}$ denotes some enumeration of the set A as a sequence.

If $\vec{a} \in A$ and $\phi(x, \vec{a})$ is a formula, $\phi[A, \vec{a}) ::= \{b \in A | \mathfrak{C} \models \phi[b, \vec{a}]\}$. Similar definitions are made for formulas $\phi(\vec{x}, \vec{a})$ and types $p(\vec{x})$. If the formula is $\phi(\vec{c}, \vec{x}, \vec{a})$ I write $\phi(\vec{c}, [\vec{A}], \vec{a})$.

$S_n(A)$ is the set of all complete n-types over A; if I need to specify the formal variables appearing as the free variables of the types I write $S_{\vec{x}}(A)$. n, $\vec{x}$ may be infinite: $S_*(A)$ denotes the class of all types in $\leq \Theta$ variables over A , S(A) denotes the set of all finitary types over A , $S(A) ::= \bigcup_{n<\omega} S_n(A)$.

A useful device introduced in [M] is that of an <u>ideal type</u>: a complete type $\bar{p}$ over the universe $\mathfrak{C}$; by the conventions adopted such a $\bar{p}$ is not, properly speaking, a type. "Choose an ideal type $\bar{p}$ extending p" stands in place of "choose an arbitrary extension p' of p to some set B (to some sufficiently saturated model M)"; choosing ideal extensions means that we do not have to keep track of the domains of the extensions, and the saturation of $\mathfrak{C}$ allows us to talk (in a useful way) about automorphic copies of $\bar{p}$.

<u>1.1 DEFINITION</u>: [Shelah]

Let λ be an infinite cardinal, T a complete theory in L .

(i) T is <u>stable in λ</u> if for all A , $|A| \leq \lambda$ implies that $|S_1(A)| \leq \lambda$ (equivalently we can write $|S(A)| \leq \lambda$).

(ii) T is <u>stable</u> if it is stable in some λ .

(iii) T is <u>superstable</u> (<u>ss</u>) if for some λ_0, T is stable in all $\lambda \geq \lambda_0$.

(iv) T is <u>totally transcendental</u> (<u>tt</u>) if for every countable sublanguage L' of L , $T \upharpoonright L'$, the restriction of T to L' , is stable in all infinite powers.

<u>1.2 THEOREM</u>: Every totally transcendental theory is superstable with $\lambda_0 \leq |L|$; every superstable theory is stable. The converses do not hold. For countable L , T is tt iff T is $\aleph_0$-stable; there are uncountable superstable theories T with $\lambda_0 = |T|$ but T not tt.

<u>1.3 DEFINITION</u>: [LP 2.1] Let $p \in S_{\vec{x}}(M)$, $q \in S_{\vec{x}}(N)$. A formula $\phi(\vec{x}, \vec{v})$ is <u>represented</u> in p if for some $\vec{m} \in M$, $\phi(\vec{x}, \vec{m}) \in p$. "$p \geq q$" if every formula that is represented in p is represented in q. "$\geq$" is a pre-order on the class of types over models of T with induced equivalence relation $\sim$. The equivalence classes of $\sim$ are called <u>classes of types</u>; the partially ordered set (by $\leq$) of the classes of types of T is called the <u>fundamental order</u> (of n-types) of T. If $p \subset q$ and $p \sim q$ then q is called an heir of p.

<u>1.4 THEOREM</u>: [LP, 4.8, 5.1, 5.10] [M, "First Basic Fact"] Let $p \in S(A)$, T stable. The automorphisms of $\mathfrak{C}$ fixing A ($\mathrm{Aut}_A(\mathfrak{C})$) induce an equivalence relation on the ideal type extensions of p: $\overline{p}_0$ and $\overline{p}_1$ are in the same <u>orbit</u> if for some $\alpha \in \mathrm{Aut}_A(\mathfrak{C})$, $\alpha(\overline{p}_0) = \overline{p}_1$.

(i) There is exactly one orbit of ideal type extensions of p under automorphisms of $\mathfrak{C}$ fixing A with cardinality $< \Theta$ (in fact $\leq 2^{|T|}$). All other orbits have cardinality $\geq \Theta$.

(ii) The ideal types in the small orbit are all equivalent ($\sim$). Furthermore, their class, called the <u>bound of p</u>, $\beta(p)$, is the unique class maximal among the classes of extensions of p to models, so if A was a model, p is in this class, and so are all the heirs of p.

<u>1.5 DEFINITION</u>: (Continue the notation and terminology of (1.4)).

(i) The ideal types in the small orbit are called <u>non-forking extensions</u> of p. All the other ideal type extensions are called <u>forking</u> extensions of p. If $p \subset q$, q is a non-forking extension of p iff q is contained in some ideal non-forking extension of p. If $q \in S(B)$, $B \supset A$, and q is a non-forking extension of $q \restriction A$, then we say that q <u>does not fork</u> over A. (Abbreviations: "q is a nf ext. of p", "q dnf over A").

(ii) Sets B, C are <u>independent</u> over A, $B \underset{A}{\downarrow} C$, if for any $\vec{b} \in B$, $t(\vec{b}, A \cup C)$ does not fork over A. A family $(B_i)_{i \in I}$ is independent over A iff for each $i \in I$, $B_i \underset{A}{\downarrow} \bigcup (B_j \mid i \neq j \in J)$.

(iii) The cardinality of the small orbit is called the <u>multiplicity</u> of p. A type is <u>stationary</u> if it has multiplicty 1. If p is stationary, it has a unique non-forking extension to any set B containing the domain of p, denoted by $p|B$. If p, q are stationary types over A, $p \otimes q$ is the (unique) type of the sequence $\vec{a} \frown \vec{b}$ over A, where $\vec{a}$ and $\vec{b}$ are independent over A, $\vec{a}$ satisfies p, $\vec{b}$ satisfies q. The notation is extended in the obvious way to $\otimes_{i \in I} p_i$ and to $p^{(\kappa)} ::= \otimes_{i \in \kappa} p$. If p and q are stationary types over arbitrary sets, p and q are <u>parallel</u>, written $p \parallel q$, if $p|\mathfrak{C} = q|\mathfrak{C}$.

<u>1.6 THEOREM</u>: [Sh, III, Cor 3.2], [LP 5.9] [M, "Second Basic Fact"]. Let T be stable.

There is an infinite cardinal $\kappa \leq |T|$ such that every type p is the unique non-forking extension of its restriction to some set A with $|A| < \kappa$. The least possible κ satisfying this condition is denoted by $\kappa(T)$, and $\kappa_r(T)$ is the least regular cardinal $\geq \kappa(T)$. In this situation (p dnf over A, $p \restriction A$ stationary) we say that p is <u>based on A</u>, so the theorem says that every type is based on a set of cardinality $< \kappa(T)$.

<u>1.7 THEOREM</u>: Let T be stable.

(i) $\downarrow$ is invariant under automorphisms of $\mathfrak{C}$.

(ii) For any A, B, C there is an automorphism α of $\mathfrak{C}$ fixing A such that $\alpha[B] \underset{A}{\downarrow} C$.

Thus for every $p \in S(A)$, $A \subset C$, there is $q \in S(C)$, $q \supset p$, and q is a non-forking extension of p , q dnf over A .

(iii) $B \underset{A}{\downarrow} C$, $A \subset A' \subset A \cup C$, $C' \subset A \cup C \Rightarrow B \underset{A'}{\downarrow} C'$

(iv) $A \subset A'$, $B \underset{A'}{\downarrow} C$, $B \underset{A}{\downarrow} A' \Rightarrow B \underset{A}{\downarrow} C$

(v) $B \underset{A}{\downarrow} C \Rightarrow C \underset{A}{\downarrow} B$

(vi) Suppose that $\langle A_\alpha \rangle_{\alpha \leq \beta}$, $\langle B_\alpha \rangle_{\alpha \leq \beta}$ and $\langle C_\alpha \rangle_{\alpha \leq \beta}$ are increasing chains of sets with $A_\beta = \bigcup_{\alpha < \beta} A_\alpha$ and similarly for B_β, C_β . Then $B_\alpha \underset{A_\alpha}{\downarrow} C_\alpha$ for all $\alpha < \beta$ implies that $B_\beta \underset{A_\beta}{\downarrow} C_\beta$.

(vii) Suppose that $p \in S(B)$ forks over $A \subset B$. Then there is a finite $B_0 \subset B$ and a formula $\psi(\vec{x}) \in p$ over B_0 such that for any q , $\mathrm{dom}(q) \supset B_0 \cup A$, $\psi(\vec{x}) \in q$ implies that q forks over A .

REMARKS: These are the basic properties of $\downarrow$ (i.e. of non-forking) in the form as summarized in [M, A1-A7]. They can all be found in various forms in [Sh, Chapter II, III] and in [LP, section V].

1.7a ASSUMPTION: From now on I assume without stating it that T is a complete stable theory.

1.8 THEOREM: (i) The Open Mapping Theorem [LP, 5.12], see also [M, A8]. Let $A \subset B$, $\phi(\vec{x})$ a formula over B . Then there is a formula $\psi(\vec{x})$ over A such that for any $p \in S(A)$, $\psi \in p$ iff there is $q \in S(B)$, q a non-forking extension of p , with $\phi \in q$.

As a consequence if q is isolated and q dnf over A , then $q \restriction A$ is isolated.

(ii) The Finite Equivalence Relation Theorem [SH, III, Theorem 2.8, Corollary 2.9]. See also [LP, 5.13] and [M, B3]. If $p \neq q \in S(B)$ do not fork over $A \subset B$, then there is a finite equivalence relation E definable by a formula of $L(A)$ such that $p(\vec{x}) \cup q(\vec{y}) \vdash \neg E(\vec{x}, \vec{y})$.

<u>1.9 THEOREM</u>: [LP, Section 8] T is superstable iff the fundamental order of T is well-founded.

The foundation rank of the fundamental order is called <u>Lascar's rank U</u>, $U(p)$ is the rank of $\beta(p)$ in the fundamental order.

<u>1.10 THEOREM</u>: Let T be tt. Then:

(i) [Mr] Every type has finite multiplicity.

(ii) [Sh, IV, Theorem 4.18] Let A be any set. Then there is a prime model M over A. M is unique up to isomorphism over A and is characterized as being atomic over A with no non-trivial uncountable sets of indiscernibles.

(iii) [Sh, IV, Theorem 4.21] Suppose M is atomic over A. Then M is minimal over A with respect to $\prec$ iff there are no infinite non-trivial sets of indiscernibles over A in M.

(iv) (Notation) If M is a model, p a type over M, $\vec{a}$ realizes p, then $M(\vec{a})$ or $M(p)$ denotes the unique prime model over $M \cup \{\vec{a}\}$.

(v) [Sh III, 3.8] See also [M, A9, A14, A16]. Every type is based on a finite set. That is $\kappa(T) = \aleph_0$. (In fact a stable theory T' is superstable iff $\kappa(T') = \aleph_0$.)

<u>1.11 DEFINITION</u>: [Sh, V, Definition 1.1] See also [L, 2.2, 6.1] and [M, C.1]. Let p, q be *-types over arbitrary sets. p and q are <u>orthogonal</u>, $p \perp q$, if for every p', q' non-forking extensions of p, q respectively to the same set A, and for every $\vec{b}, \vec{c}$ realizing p', q' respectively, $\vec{b} \underset{A}{\downarrow} \vec{c}$.

1.12 THEOREM: [Sh, V, Theorem 1.2, 1.4]. See also [L, comments after 6.1] and [M, C4, C5].

(i) $\perp$ is invariant under parallelism of stationary types.

(ii) Suppose $\{\vec{c}_i \mid i < n\}$ is independent over A. Then $t(\vec{b}, A) \perp t(\vec{c}_i, A)$ for all $i < n$ iff $t(\vec{b}, A) \perp t(\vec{c}_0 \frown \vec{c}_1 \frown \dots \frown \vec{c}_{n-1}, A)$.

(iii) Suppose $(q_i)_{i \in I}$ are stationary. Then $p \perp q_i$ for all $i \in I$ iff $p \perp \otimes_{i \in I} q_i$.

(iv) Suppose B, C are independent sets over A and for each $b \in B$, $c \in C$, $t(b, A)$, $t(c, A)$ are stationary and $t(b, A) \perp t(c, A)$. Then $B \cup C$ is independent over A.

REMARKS: The proofs of (i)-(iii) are elementary from the definition of $\perp$ and the properties of $\downarrow$. (iv) is an immediate consequence of (ii) (i.e. of [Sh, V, 1.4(1)]) as follows:

Without loss of generality B, C are finite. Fix $b \in B$. Then C is independent over A and for each $c \in C$, $t(b, A) \perp t(c, A)$ so by (ii) $t(b, A) \perp t(C, A)$. Now B is independent over A and for each $b \in B$, $t(C, A) \perp t(b, A)$ as I have just shown, so again by (ii), $t(C, A) \perp t(B, A)$. Thus by the definition of $\perp$, $B \underset{A}{\downarrow} C$. Since B, C are each independent over A, $B \cup C$ is independent over A.

1.13 DEFINITION: (The ideas here ultimately have their origin in [Sh, V, 2] where orders $\leqq_w$, $\leqq_s$ between indiscernible sets are considered.)

(i) [L, 5.1] See also [M, C.10] Let A, B, C be sets. B is dominated by C over A iff for any set X,

$$X \underset{A}{\downarrow} C \Rightarrow X \underset{A}{\downarrow} B$$

(ii) [L, 5.2], [M, C.13] Let $p \in S_*(A)$, $q \in S_*(B)$ be stationary. "$p \lhd q$" (p is eventually dominated by q) if for some $E \supset A \cup B$ there are realizations $\vec{C}, \vec{D}$ of $p|E$, $q|E$ respectively such that C is dominated by D over E.

$$p \trianglelefteq q \text{ means } p \lhd q \text{ and } q \lhd p$$

1.14 THEOREM: [L, 2.3] [M, C.12(i)] Let T be tt. Then for any model M and set C, $M(C)$ is dominated by C over M.

1.15 THEOREM: [M, C.13‴], see also [L, section 5, especially 5.4].

(i) $\lhd$ is invariant under parallelism.

(ii) $\lhd$ is a pre-order, $\underset{\sim}{\lhd}$ is an equivalence relation.

(iii) $p \lhd q$, $q \perp r \Rightarrow p \perp r$

(iv) $p \lhd q \Rightarrow p \otimes r \lhd q \otimes r$

1.16 DEFINITION: [Sh, V, Definition 3.2] See also [L, 7.2] and for the most general version, [M, D.1] on which this is based.

(i) $pw(B/A) ::= \sup\{|I| \mid$ there is $(C_i)_{i \in I}$, independent over A, $B \underset{A}{\not\downarrow} C_i$ for all $i \in I\}$.

(ii) The weight of B over A is

$w(B/A) ::= \sup\{pw(B/A') \mid A \subset A', B \underset{A}{\downarrow} A'\}$.

(iii) If $p \in S_*(A)$, B realizes p, then $w(p) ::= w(B/A)$.

1.17 THEOREM: [M, D.2] See [Sh, V, 3.10, 3.12] and [L, Section 7]

(o) $w(B/A) \leq (\kappa_r(T))^- + |B|$, where for a cardinal κ, $\kappa^- = \kappa$ if κ is a limit cardinal, $\kappa^- = \lambda$ if $\kappa = \lambda^+$.

(i) $B \subset B' \Rightarrow w(B/A) \leq w(B'/A)$

$A \subset A' \Rightarrow w(B/A') \leq w(B/A)$

(ii) $A \subset A'$, $B \underset{A}{\downarrow} A' \Rightarrow w(B/A) = w(B/A')$

(iii) $w(B/A) = 0$ iff $B \subset acl(A)$, i.e. iff every element of B is algebraic over A .

(iv) If p is stationary, $p \not\perp q_i$ for each $i \in I$, $q_i \perp q_j$ for $i \neq j \in I$, then $|I| \leq w(p)$.

(v) B is dominated by C over A implies that $w(B/A) \leq w(C/A)$. Hence $p \lhd q$ implies that $w(p) \leq w(q)$.

(vi) Suppose $\langle C_i \rangle_{i \in I}$ is independent over B . Then there is $I' \subset I$, $|I - I'| \leq w(A/B)$, such that $A \underset{B}{\downarrow} \cup \langle C_i \rangle_{i \in I'}$.

<u>1.18 THEOREM</u>: [Sh, V3.11, 1.15, exercise 3.12 (in part)]. [M, D4, D5].

(i) If $\langle B_\alpha \rangle_{\alpha < \kappa}$ is independent over A then $w(\bigcup_{\alpha < \kappa} B_\alpha \,/\, A) = \Sigma_{\alpha < \kappa}\, w(B_\alpha \,/\, A)$.

(ii) Suppose that $B_i, C_j \subset D$ for $i \in I$, $j \in J$, $w(B_i/A) = w(C_j/A) = 1$ for all i, j , and $\langle B_i \rangle_{i \in I}$, $\langle C_j \rangle_{j \in J}$ are maximal independent families over A in D . Then $|I| = |J|$.

<u>1.19 DEFINITION</u>: [Sh, III, Definition 4.5] Let $p \in S(A)$ have weight 1, $A \subset M$. Let $\mathcal{B}$ be a maximal independent family of realizations of p in M . Then $\mathcal{B}$ is called a <u>p-basis</u> of M , and $\dim(p, M) ::= |\mathcal{B}|$ is well defined by the preceding theorem. If $C \subset p[\mathfrak{C}]$, $\dim(p, C)$ is defined similarly.

<u>1.20 THEOREM</u>: [M, D5'] See also [Sh, V3.1 and V, 1.13(1)].

(i) For p, q stationary, $w(p) = 1$, we have $p \not\perp q$ iff $p \lhd q$.

(ii) If $w(\vec{b}, A) = 1$ then $\vec{a} \underset{A}{\not\downarrow} \vec{b}$, $\vec{b} \underset{A}{\not\downarrow} \vec{c} \Rightarrow \vec{a} \underset{A}{\not\downarrow} \vec{c}$.

(iii) If p is stationary, $w(p) = 1$ then $q \not\perp p$, $p \not\perp r \Rightarrow q \not\perp r$.

Hence $\not\perp$ is an equivalence relation on the stationary weight 1 types and in fact, for such types p, q, $p \not\perp q$ iff $p \trianglelefteq q$.

1.21 DEFINITION:

(i) [Sh, V, Definition 1.2] See also [L, 6.2] and [M, D.6]. $p \in S(A)$ is regular if it is stationary, non-algebraic and orthogonal to every forking extension of itself.

(ii) [Sh, V, Definition 3.5, Exercise 3.10]. See also [L, 2.6] and [M, D.13]. $p \in S(A)$ is strongly regular (sr for short) via ϕ if it is stationary, non-algebraic, $\phi \in p$ and for any q with $\phi \in q$ either $p \perp q$ or $p \parallel q$.

1.22 THEOREM:

(i) Every regular type has weight 1.

(ii) Every strongly regular type is regular.

(iii) Regularity is parallelism invariant. If p is sr via $\phi(\vec{x}, \vec{a})$, $\mathrm{dom}(q) \supset \vec{a}$, and $q \parallel p$, then q is sr via ϕ .

(iv) (T tt) Let p be a stationary type over A , $\phi(\vec{x}) \in p$. Then p is sr via ϕ iff there are models $M \prec N$, $A \subset M$ such that for every $\vec{a} \in \phi[N \setminus M]$, $t(\vec{a}, M)$ is a non-forking extension of p .

(v) (T tt). If $M \subsetneqq N$ then there is $a \in N \setminus M$ such that $t(a, M)$ is sr.

REMARKS: (i) is [Sh, V, Lemma 3.10(2)]. See also [M, D.8]. (ii) is [Sh, V, Theorem 3.18]. See also [L, 6.3] and [M, D.14]. (iii) is [Sh V, 1.8]. See also [L, 6.5] and [M, D.7]. (iv) is [Sh, V, Exercise 3.18]. See also [M, D.15]. (v) is [Sh, V, Exercise 3.11]. See also [L, 2.8] and [M, D.16].

<u>1.23 THEOREM</u>: [Sh, V, Theorem 3.9]. See also [L, 7.1] and [M, D.10, D.11]. Let T be superstable.

(i) Every stationary type q is equivalent to a finite produce of regular types $(p_i)_{i<n}$. $q \trianglelefteq\!\!\trianglerighteq \otimes_{i<n} p_i$ is called a <u>regular decomposition</u> of p, and $n = w(p)$, so every type has finite weight.

(ii) If $q' \trianglelefteq\!\!\trianglerighteq \otimes_{i<n'} p_i'$ is a regular decomposition then $q' \triangleleft q$ iff for some 1-1 map $f: n' \to n$, $p_i' \not\perp p_{f(i)}$ for all $i < n'$.

(iii) If q, q' are stationary then $q \not\perp q'$ iff there is a regular p such that $p \not\perp q$, $p \not\perp q'$.

(iv) If T is tt we may take all the regular types mentioned in (i)-(iii) to be strongly regular.

<u>1.24 THEOREM</u>: [Sh, V, Exercise 3.16], [L, Section 2]. See also [M, D19']. Let T be tt, $M \models T$. Let $p \in S(M)$ be strongly regular. Then for every type q, p is realized in $M(q)$ iff $p \not\perp q$.

<u>1.25 THEOREM</u>: [Sh, V, Exercise 3.14]. See also [L, 4.2] and [M, D.19]. Let T be tt, $M \models T$, $p \in S(M)$. Then there is a sequence of models $M = M_0 \prec M_1 \prec \ldots M_n = M(p)$ such that for each $i < n$, $M_{i+1} = M_i(p_i)$ for some sr type p_i over M_i (and $(p_i)_{i<n}$ is a regular decomposition of p).

I. BASIC THEORIES

I 0. INTRODUCTION

The main content of this chapter is a Basis Theorem [K1] for certain kinds of totally transcendental theories which I call basic. The Basis Theorem provides a partial description of the structure of the models of a basic theory.

The original motivation for the Basis Theorem was twofold. The first aspect was an attempt to understand the stability theory underlying the structure of the models of T_Λ^* , the model completion of the theory of modules over a commutative Noetherian ring Λ as introduced by Eklof and Sabbagh [ES]; and more particularly to understand the model theory behind the theorem of Matlis [Ma] on the existence and uniqueness of a direct sum decomposition for an injective module over Λ . The second aspect was simply the desire to understand fully the meaning and significance of the concept of "strongly regular type", particularly in such theories as T_Λ^* . In fact, this program was successful, and it applied in a more general setting, namely to the work of Garavaglia [G4] on totally transcendental modules, discussed in full in Chapter III. Several examples illustrating the ideas of the present chapter will have to be delayed until the necessary background has been developed in Chapters II and III.

The material necessary to treat these goals is discussed in sections 1 and 2.

Later I realized that the Basis Theorem was contained (although not explicitly so stated) as a special case of the proof of the "Main Theorem" of Shelah's book [Sh, IX, Theorems 2.3, 2.4]. The Basis Theorem has a simple proof, depending only on the elementary properties of strongly regular types given in Chapter V.3 of [Sh]. (That is, principally 1.12, 1.18, 1.20, 1.22, 1.24 of the introduction to this thesis.) The reader should take note that the treatment of orthogonality and strong regularity in this thesis has been influenced strongly by papers appearing after my abstract [K1] reporting the Basis Theorem was published, in particular Lascar's paper [L] on the subject and Makkai's survey article [M]. Therefore I hoped that the Basis Theorem could provide some illumination of the proof of [Sh, IX, 2.4], which in its original form is very difficult. The relationship between the two theorems is not directly apparent: the Basis Theorem talks about the structure of the models of T, whereas [Sh, IX, 2.4] talks about the "spectrum functions" $I^t(\aleph_\alpha, \aleph_\beta, T)$ of the theory T, the number of non-isomorphic models of certain kinds.

In 1982 A. Pillay published an article [Pi] giving a more straightforward version of the proof of [Sh IX 2.4] for countable T and $I^t(\aleph_\alpha, \aleph_0, T)$ only. In section 3, I present the entirely combinatorial arguments necessary to calculate the spectrum functions of an arbitrary basic theory T. Then, as a sort of an appendix, I give in section 4 a fairly detailed outline of how to prove the full version of [Sh, IX, 2.4], working from the Basis Theorem and the calculations of section 3. The details of the proofs all come from Pillay's article (and of course ultimately have their source in Shelah's original proof), but the approach and emphasis is quite different. I am very careful to separate important structural questions ("models of T look like ...", "two models of T are isomorphic when ...") from the purely combinatorial problem of counting the number of models. Neither Shelah nor Pillay emphasize structure, and I feel that failing to do so obscures some important facts about the models of non-multidimensional totally transcendental theories.

In section 1, (1.0-1.2) are included to put the material on basic theories following into the proper broader context, and to prepare for the application of the Basis Theorem in section 4, but (1.1) and (1.2) are not actually applied in sections 1 and 2, and in section 3 all that is used is $\mu(T) \leq |T|$ which is obvious anyways.

I 1. BASIC THEORIES

1.0 REMARKS: Most of Definition 1.1 and Theorem 1.2 below are contained at various places in Shelah's book [Sh], to which the following references are made. $\mu(T)$ is not introduced until [IX, Theorem 2.3], where it is called ND(T), although properties of it are dealt with extensively in Chapter V. "Multidimensional" is developed mostly in [V.5] although it begins in [V.2]. See in particular [V definition 2.2, definition 5.3, and theorem 5.8].

For the simpler version given here, see [Bo L, Definition 3.2 and following] and [Pi, III.1]. I restrict my attention to totally transcendental theories, although of course the definition can be made more general. So throughout the balance of this chapter T represents a complete totally transcendental theory and M_0 denotes the prime model of T.

1.1 DEFINITION: Consider the equivalence relation $\not\perp$ defined on the class of all weight one types of T. Let $\mu(T)$ be the number of equivalence classes of $\not\perp$; I write $\mu(T) = \infty$ if this is $\geq |\mathfrak{C}|$

(i) $\mu(T)$ is called the number of dimensions of T.

(ii) T is multidimensional if $\mu(T) = \infty$.

(iii) T is non-multidimensional (nmd) if $\mu(T) < \infty$.

(iv) T is unidimensional if $\mu(T) = 1$.

1.2 THEOREM: (See [Sh] as referenced in (1.0), also [Bo L 3.2-3.4], [Pi, III.2, III.7]).

The following are equivalent:

(i) $\mu(T) \leq |T|$

(ii) $\mu(T) < \infty$

(iii) Every non-algebraic type is non-orthogonal to some strongly regular 1-type over the prime model M_0 .

<u>1.3 DEFINITION</u>: (i) T is <u>basic</u> if there is a complete set P of representatives of the $\not\perp$-equivalence classes of weight one types in the form of sr 1-types over $\emptyset$. Such a set P is called a <u>set of primes for T</u> and each $p \in P$ is called a <u>prime</u>.

(ii) If $M \models T$, <u>$\dim(M)$</u> $::= \langle \dim(p, M) | p \in P \rangle$ is the <u>dimension vector</u> of M .

In what follows, δ, δ', etc. represent vectors of finite or infinite cardinals $\langle \delta_p \rangle_{p \in P}$, $\leqq$ is the pointwise partial order on such vectors, and $\delta_0 ::= \dim(M_0)$. If κ is an infinite cardinal, δ_κ is the dimension vector satisyfing $\delta_\kappa(p) = \kappa$ for all $p \in P$.

(iii) A <u>basis</u> for $M \models T$ is the union of p-bases for M , one for each $p \in P$. B, B', etc. denote bases. If B is a basis for M and $p \in P$ then $B_p ::= B \cap p[M]$.

<u>1.4 REMARKS</u>: Clearly a basic tt theory is nmd. In fact, by (1.2), T is nmd iff $T(M_0)$ is basic.

The choice of a particular set of representatives P may be important: dimension is not an invariant of the $\not\perp$ -classes of weight one types, as example (1.6) shows (among other things).

<u>1.5 EXAMPLES</u>:

(i) The characteristic example of a multidimensional tt theory T is the theory of one equivalence relation E with infinitely many equivalence classes, each one infinite. In this case $\mathfrak{C}$ has $\Theta = |\mathfrak{C}|$ classes, each of cardinality Θ, and each class corresponds to a weight one type, any two distinct classes being orthogonal. So $\mu(T) = \infty$.

(ii) The theory of one equivalence relation E with exactly n classes, each one infinite $1 \leq n < \omega$, on the other hand, is nmd with $\mu(T) = n$. (As in (i), the classes of E are in 1-1 correspondence with the classes of $\mathfrak{C}$). However this theory is not basic: the sr types are of the form $E(x, a)$ where a is a parameter.

(iii) Any complete theory of modules is nmd. In fact, any tt theory of modules is basic. I prove this in Chapter III only for the special theories of modules which provide the proper context for the results of that chapter, but afterwards I make some remarks on the general problem (III, 3.3, 3.4).

(iv) In particular, Chapter II is an extended study of a family which includes many basic tt theories of modules. Let Λ be a commutative Noetherian ring, T_Λ^* the theory of existentially closed Λ-modules as put forth by Eklof and Sabbagh [ES]. Spec(Λ) denotes the set of prime ideals of Λ. In Chapter II I establish a 1-1 correspondence between the strongly regular 1-types over $\emptyset$ of T_Λ^* and Spec (Λ), and as a consequence $\mu(T_\Lambda^*) = |\mathrm{Spec}(\Lambda)|$.

(v) Let T be a complete theory in a countable language. Then T is unidimensional tt iff T is ω_1-categorical.

Some ω_1-categorical basic theories are the theory of $\mathbb{Z}$ with the successor function and the theory of algebraically closed fields of characteristic 0.

1.6 EXAMPLE: Section 4 of Baldwin and Lachlan's classic paper [BL] contains examples which may be adapted to show that two naive intuitions about the dimension of a model of a basic theory are in fact false. The examples are quite strong, being of basic ω_1-categorical theories, so $\mu = 1$ and up to $\not\perp$ there is a unique sr 1-type p over $\emptyset$.

One intuition is that the dimension δ_0 of the prime model M_0 should be "simple": p is not isolated ($\delta_0 = 0$), p is realized once ($\delta_0 = 1$) or p is realized infinitely often ($\delta_0 = \aleph_0$) . Possibilities such as, say, $\delta_0 = 17$ seem unnatural. The first example has $\delta_0 = 2$.

The second example shows that dimension is not $\not\perp$ -invariant, i.e. it is not a property of equivalence classes of weight one types over $\emptyset$. The example has prime model N', $p_1 \not\perp p_2$ sr 1-types over $\emptyset$, $\dim(p_1, N') = 1$ $\dim(p_2, N') = 2$. Thus the dimension of a model is sensitive to the choice of a set of primes P .

The second example of course also illustrates the same point as the first; both must be presented since the second is constructed from the first. Furthermore the first example has a unique sr 1-type over $\emptyset$ so it is impossible to make the dimension of the prime model be anything else but 2.

(i) Let $M ::= \langle \mathbb{Q}, T\rangle$ with $T(x, y, z) = -x + y + z$. Baldwin and Lachlan prove the following:

Every n-ary relation on $\mathbb{Q}$ definable by a formula of $L = \{T\}$ is a Boolean combination of relations R of the form

$$R(x_1, \ldots, x_n) \Leftrightarrow \sum_{i=1}^{n} q_i x_i = 0$$

where $q_1, \ldots, q_n \in \mathbb{Q}$ and $\sum_{i=1}^{n} q_i = 0$ (of course $q_1, \ldots, q_n$ can be taken in $\mathbb{Z}$). In fact, for any n-ary term t of L, there are $a_1, \ldots, a_n \in \mathbb{Z}$, $\sum_{i=1}^{n} a_i = 1$, such that for any $x_1, \ldots, x_n \in \mathbb{Q}$, $t[x_1, \ldots, x_n] = \sum_{i=1}^{n} a_i x_i$. It is easy to see that any such function occurs as the evaluation of some term t of L, and as a consequence any relation R of the form above is definable by a formula of L. (Hint: in fact we can take t to be "special" where the set of special terms is defined inductively as follows: All variables are special, and if x, z are variables and t is special then so is $T(x, t, z)$.)

Baldwin and Lachlan then show that it easily follows from the above elimination result that the formula "$x = x$" is strongly minimal and so $Th(M)$ is $\aleph_1$-categorical.

Now, the elimination result shows that every consistent formula in one or two variables x, y is equivalent to one of "$x = x$", "$x = y$", "$x \neq y$" and so there is a unique 1-type $p(x)$ over $\emptyset$ and a unique 2-type $q(x, y)$ over $\emptyset$ of distinct elements. Since "$x = x$" is strongly minimal, p is strongly regular. Also, $q(x, y)$ must be the type $p \otimes p$, i.e. every pair of distinct elements is an independent set over $\emptyset$ in $p[M]$.

It is clear that M is the prime model of its theory, in fact M is minimal. Thus since every pair of distinct elements is independent, $\dim(p, M) \geq 2$. On the other hand, given a, b, c distinct elements of $\mathbb{Q}$, the system of linear equations

$$\left\{ \begin{array}{l} aq_1 + bq_2 = c \\ q_1 + q_2 = 1 \end{array} \right\}$$

is consistent, with a unique solution q_1, q_2, because $a \neq b$. Since

$q_1 + q_2 = 1$, there is a formula of L defining the relation $R(x, y, z) \Leftrightarrow$ $\Leftrightarrow q_1x + q_2y = z$. Thus c is definable from $\{a, b\}$, hence c depends on $\{a, b\}$. Hence $\dim(p, M) = 2$.

(ii) Note that in the structure M above, for any $b, c \in \mathbb{Q}$ the function $(\lambda x)T(b, c, x)$ defines an automorphism of M sending b to c , and that if the automorphisms $(\lambda x)T(b, c, x)$ and $(\lambda x)T(b', c', x)$ agree at one point, then they are identical.

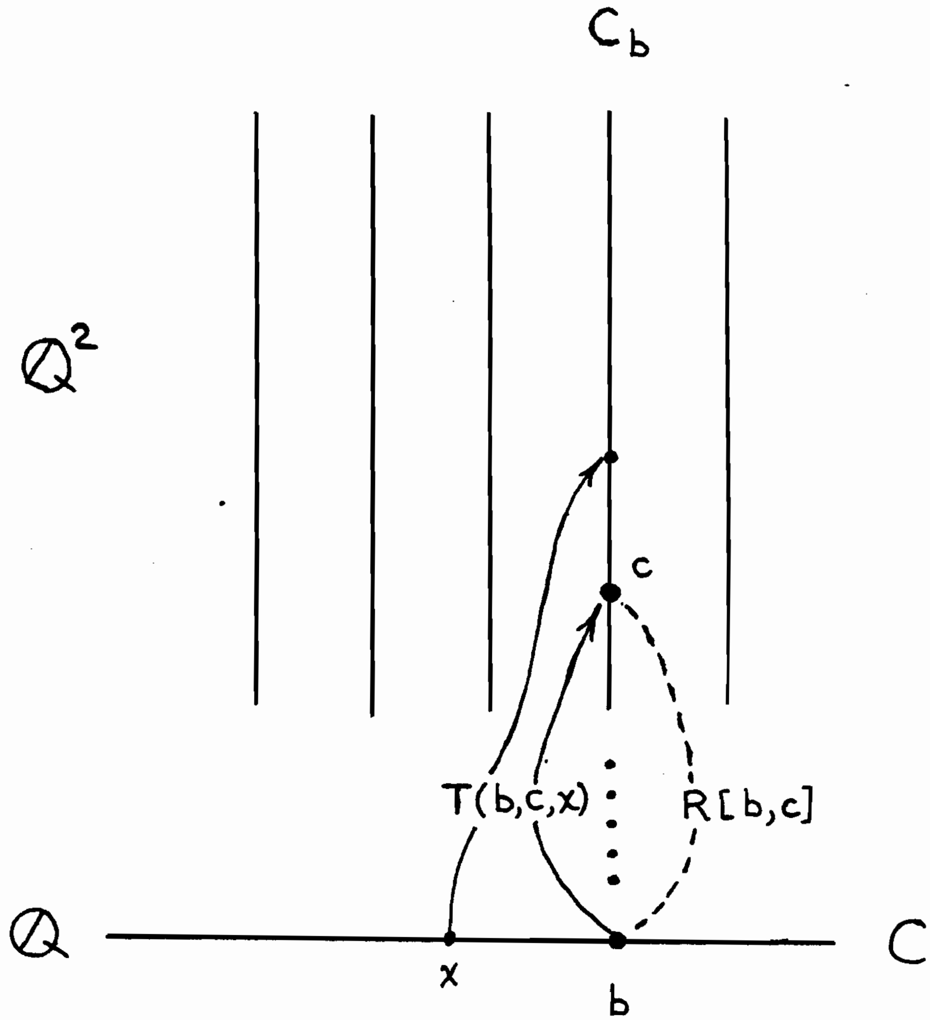
Example (ii) is constructed from M by pasting copies of M as "fibres" over each element of a base copy of M by a binary relation R: $R[b, c]$ holds iff b is in the base copy and c is in the fibre over b . T is a ternary partial function which, restricted to each copy of M , gives it the structure of example (i). T is extended using the remark of the first paragraph to provide isomorphisms between the base copy and each of the fibres. Thus any model of the resulting theory is constructed from some $M' \equiv M$ in the manner described. The main points of the construction are illustrated in figure 1.6a. Formally: Let $N = \langle N, T, R\rangle$ where $N = \mathbb{Q} \oplus \mathbb{Q}^2$ and $R(x, y)$ holds iff $x \in \mathbb{Q}$, $y \in \mathbb{Q}^2$ and $y = \langle x, z\rangle$ for some z . Let C denote the base copy $\mathbb{Q}$. C is definable over $\emptyset$ by the formula $(\exists y)R(x, y)$. For each $b \in C$, let C_b denote the fibre over $\{b\}$. C_b is not definable over $\emptyset$, but is definable over $\{b\}$ by the formula $R(b, y)$. $T(x, y, z)$ is defined just in case

(a) $x, y, z \in \mathbb{Q}$ in which case it is $-x + y + z \in \mathbb{Q}$ as before.

(b) $x = \langle b, x_0\rangle$, $y = \langle b, y_0\rangle$, $z = \langle c, z_0\rangle$ in $\mathbb{Q}^2$ for some b , x_0, y_0, and z_0 , in which case it is $\langle b, -x_0 + y_0 + z_0\rangle$.

(c) $x, z \in \mathbb{Q}$, $y = \langle x, y_0\rangle \in \mathbb{Q}^2$, in which case it is $\langle x, -x + y_0 + z\rangle$.

FIGURE 1.6a:



Now (a) and (b) establish that $<C, T>$ and each $<C_b, T>$ are isomorphic to M . (c) provides a definable-with-parameters isomorphism between $<C, T>$ and $<C_b, T>$, in fact, for each $b \in C$ and $c \in C_b$, $(\lambda x)T(b, c, x)$ is such an isomorphism. Notice that the definable isomorphisms on C_b "commute" with this isomorphism: if $c' \in C_b$ then for all $x \in C$, $T(b, c', x) = T(c, c', T(b, c, x))$.

Clearly then every model of $Th(N)$ is uniquely determined (by means of this construction) by some $M' \equiv M$. Thus N is the prime model of its theory which is $\aleph_1$-categorical. Furthermore, since "$x = x$" was strongly minimal in $Th(M)$, the formulas $(\exists y)R(x, y)$ and $R(b, y)$ defining the various isomorphic copies of M are strongly minimal in $Th(N)$.

Now pick any $b \in C$ and let $N' = <N, b>$, $T_0 = Th(N')$. T_0 has (at least) two strongly minimal formulas over $\emptyset$: $(\exists y)R(x, y)$ and $R(b, y)$, with corresponding sr 1-types over $\emptyset$ p_1 and p_2 respectively. $\dim(p_1, N') = \dim(p_1, <C, b>) = 1$ (since $\dim(p_1, C) = 2$ by part (i)) and $\dim(p_2, N') = \dim(p_2, C_b) = 2$, since no element of C_b is named in N' . Since T_0 is $\aleph_1$-categorical, hence uni-dimensional, $p_1 \not\perp p_2$.

I 2. THE BASIS THEOREM

2.0 THEOREM: Let T be a tt theory, P a set of primes for T, M_0 the prime model of T, $\kappa_0 = |M_0|$, $\delta_0 = \dim(M_0)$. Let $M, N \models T$, $|M| = \kappa$ (so $\kappa \geq \kappa_0$). Let δ be a dimension vector and λ an infinite cardinal.

(a) on bases:

(i) any basis of M is independent over $\emptyset$.

(ii) any two bases of M are isomorphic over $\emptyset$.

(iii) any independent set of elements in M realizing types $p \in P$ can be extended to a basis of M.

(iv) M is prime and minimal over any basis $\mathcal{B}$ of M.

(b) on dimensions:

(v) $\dim(M) = \dim(N) \Leftrightarrow M \cong N$.

(vi) $\dim(M) \leq \dim(N) \Leftrightarrow M \precsim N$.

(vii) $\delta_0 \leq \dim(M) \leq \delta_\kappa$.

(viii) $\delta_0 \leq \delta_{\aleph_0}$.

(c) on the existence of models and saturation:

(ix) $\delta_0 \leq \delta \Leftrightarrow (\exists N \models T)[\dim(N) = \delta]$

In such a case, $|N| = \kappa_0 + \Sigma_{p \in P}\delta_p$.

(x) M is λ-saturated $\Leftrightarrow$ $\delta_\lambda \leq \dim(M)$.

In particular, every $N \models T$ may be extended to a λ-saturated model of T.

(xi) M is the saturated model of power κ iff $\dim(M) = \delta_\kappa$.

PROOF: (i), (ii) and (iii) are all immediate: (i) by (0, 1.12(iv)), (ii) follows from (i) since for any two bases $\mathcal{B}, \mathcal{B}'$, $|\mathcal{B}_p| = |\mathcal{B}'_p|$ for each $p \in P$, and each p is stationary, and (iii) is obvious from the definition of basis (0, 1.19).

(iv): Let $\mathcal{B} \subset M$ be a basis for M. Let M_1 be prime over $\mathcal{B}$, $\mathcal{B} \subset M_1 \prec M$. Suppose $M_1 \neq M$. Then by (0, 1.22(v)) there is $a \in M \setminus M_1$ such that $t(a, M_1)$ is sr. Hence there is $p \in P$, $p \not\perp t(a, M_1)$. By (0, 1.12(i)), $p|M_1 \not\perp t(a, M_1)$ and by (0, 1.22(iii)) $p|M_1$ is sr. Hence by (0, 1.24) $p|M_1$ is realized in $M_1(a) \subset M$, say by b. Thus in particular, $b \downarrow \mathcal{B}_p$, contradicting that $\mathcal{B}_p$ is a p-basis for M.

(v) Since $\dim(M) = \dim(M')$ it is easy to see that a basis $\mathcal{B}$ of M is isomorphic to a basis $\mathcal{B}'$ of M'. Since M is prime over $\mathcal{B}$ and M' is prime over $\mathcal{B}'$ by (iv), $M \cong M'$ follows from the uniqueness of prime models (0, 1.10(ii)).

(vi)($\Rightarrow$) Let $\mathcal{B}' \subset N$ be an independent set of elements realizing types $p \in P$ such that for each $p \in P$, $|p[\mathcal{B}']| = \dim(p, M)$. This is possible since $\dim(p, M) \leq \dim(p, N)$. $\mathcal{B}'$ is isomorphic to a basis $\mathcal{B}$ of M, and by (iv) M is prime over $\mathcal{B}$, so there is an elementary embedding $f\colon M \precsim N$.

($\Leftarrow$) Without loss of generality, $M \prec N$. Let $\mathcal{B}$ be a basis for M. By (iii) $\mathcal{B}$ may be extended to a basis $\mathcal{B}'$ for N, i.e. $\mathcal{B} \subset \mathcal{B}'$. Hence $\dim(M) \leq \dim(N)$.

(vii) $M_0 \precsim M$, so $\delta_0 \leq \dim(M)$ by (vi). Clearly for each $p \in P$, $\dim(p, M) \leq |M|$, so $\dim(M) \leq \delta_\kappa$.

(viii) Immediate by the characterization of prime models (0, 1.10(ii)).

(ix) Let $\mathcal{B}_0$ be a basis for M_0. Each $p \in P$ is non-algebraic, so has arbitrarily large independent families in $\mathfrak{C}$. In $\mathfrak{C}$, choose $\mathcal{B} \supset \mathcal{B}_0$, an independent set of elements satisfying types $p \in P$, such that for each $p \in P$, $|p[\mathcal{B}]| = \delta_p$. This is possible by the above remark and the fact that $\delta_p \geq (\delta_0)_p$.

Let N be prime over $\mathcal{B}$. Clearly $|N| = \kappa_0 + \Sigma_{p\in P}\,\delta_p$. By the maximality of $\mathcal{B}_0$, $M_0 \cap \mathcal{B} = \mathcal{B}_0$. Suppose $\mathcal{B}$ is not a basis of N . Then there is $b \in N$, $t(b) \in P$, and $\mathcal{B} \cup \{b\}$ is independent. Thus $b \underset{\mathcal{B}_0}{\perp} \mathcal{B}$. Now T is tt, so N is atomic over $\mathcal{B}$ by (0, 1.10(ii)) and thus $t(b, \mathcal{B})$ is isolated. Hence by the Open Mapping Theorem (1.8(i)) $t(b, \mathcal{B}_0)$ is isolated. Thus there is $b' \in M_0$, $b' \equiv_{\mathcal{B}_0} b$. But $b \perp \mathcal{B}_0$, so $b' \perp \mathcal{B}_0$, contradicting that $\mathcal{B}_0$ is a basis of M_0 .

(x)($\Rightarrow$) Let $M \models T$ be λ-saturated, $p \in P$. Suppose B is an independent set of elements of M realizing p , $|B| < \lambda$. By the λ-saturation of M , $p|B$ is realized in M , hence B is not a p-basis. Thus $\lambda \leq \dim(p, M)$.

($\Leftarrow$) Suppose $\delta_\lambda \leq \dim(M)$, $A \subset M$, $|A| < \lambda$. Let $M' = \langle M, a\rangle_{a\in A}$, a model of $T' = T(A)$. Clearly $T(A)$ is basic with set of primes $P' = \{p|A \;|p \in P\}$.

<u>Claim</u>: $\dim(M') \geq \delta_\lambda$ (the dimension being taken with respect to P' of course).

<u>Proof</u>: Let $\mathcal{B}$ be a basis for M . $|\mathcal{B}_p| \geq \lambda$ for all $p \in P$. If $\lambda > \aleph_0$ by (0, 1.17(o)) $w(A) \leq |A| + \aleph_0 < \lambda$ and if $\lambda = \aleph_0$ (so A is finite) by (0,1.23(i)) w(A) is finite, i.e. $w(A) < \lambda$. Fix $p \in P$, choose $C_p \subset \mathcal{B}_p$ maximal such that $A \perp C_p$. Then for each $b \in \mathcal{B}_p \setminus C_p$, $A \not\perp C_p \cup \{b\}$, thus $A \underset{C_p}{\not\perp} b$. But $\mathcal{B}_p \setminus C_p$ is independent over C_p so by the definition of weight $|\mathcal{B}_p \setminus C_p| \leq w(A, C_p) \leq w(A) < \lambda$. Thus $|C_p| = |\mathcal{B}_p|$, and, working in M' , C_p is an independent set of

elements realizing $p|A$. Thus $\dim(p|A, M') = \dim(p, M) \geq \lambda$.

As a consequence of the claim it suffices to prove that if $\dim(M) \geq \delta_\lambda$ and $p \in S(\emptyset)$ then p is realized in M , for if $p \in S(A)$, $|A| < \lambda$, $A \subseteq M$, we may pass to $\langle M, a\rangle_{a \in A}$ in $T(A)$. So let $p \in S(\emptyset)$, let $\mathcal{B}$ be a basis of M and let $\mathcal{B} \uplus \mathcal{B}'$ be a basis of $M(p|M)$. $\mathcal{B}'$ is finite (since $w(p)$ is finite), and so since each $\mathcal{B}_p$ is infinite, I can find $\mathcal{B}_1 \uplus \mathcal{B}_1' \subseteq \mathcal{B}$ such that $\dim(\mathcal{B}_1) = \dim(\mathcal{B})$, $\dim(\mathcal{B}_1') = \dim(\mathcal{B}')$. Thus I can find $M_1 \prec M_1' \prec M$, M_1 prime over $\mathcal{B}_1$ and isomorphic to M , the isomorphism extending to an isomorphism of M_1' to $M(p|M)$, M_1' prime over $\mathcal{B}_1 \uplus \mathcal{B}_1'$. In particular, p is realized in $M_1' \setminus M$.

(xi) Immediate by (vii) and (x).

2.1 REMARKS: Clearly this theorem generalizes the "easy" version of the Baldwin-Lachlan Theorem [BL, Theorem 2] in which the strongly minimal formula is assumed to be over $\emptyset$. See for instance, the presentation in G.E. Sacks, Saturated Model Theory, W.A. Benjamin 1972, sections 38, 39. For further comments on this relationship, see I4. One consequence of the Baldwin-Lachlan theorem is the homogeneity of the countable models of an $\aleph_1$-categorical theory. In [Bo L], Bouscaren and Lascar show that every countable model M of a non-multidimensional tt theory is almost homogeneous in the sense that if $\vec{a}$ and $\vec{b}$ have the same strong type in M then there is an automorphism of M taking $\vec{a}$ to $\vec{b}$. Thus, if T is stationary (i.e. every type is stationary) then every countable model of T is homogeneous.

Examples of tt basic theories which are not stationary are easy to find, for instance the theory of the abelian group $\mathbb{Z}_2^{(\aleph_0)} \oplus \mathbb{Z}_3$ is one such. (It is easy to see, based on some of the results quoted in Chapter III, that this group is totally categorical with strongly minimal formula "$2x = 0 \wedge x \neq 0$" , but the complete formula "$3x = 0 \wedge x \neq 0$" is algebraic with multiplicity 2.) However, I do not know any example of a tt basic theory with a non-homogeneous countable model.

2.2 REMARKS AND EXAMPLE: The proof of the basis theorem depends on several different things, of which perhaps the two most important are the existence of a prime model over any set, and the fact that if $M \prec N$, there is a strongly regular type over M realized in $N \setminus M$. Both of these fail for superstable theories, even (in the second case) if we only look for a regular type. Thus it would seem unlikely that there could be any sort of natural or direct generalization of the Basis Theorem to superstable theories.

Nonetheless, the theory typically offered as an example of a theory which is superstable but not tt satisfies a suitably modified version of the Basis Theorem:

Let κ be an infinite cardinal, $(P_i)_{i<\kappa}$ unary predicate symbols and for every finite $I, J \subseteq \kappa$, $I \cap J = \emptyset$,

$$T \vdash (\exists x)[\bigwedge_{i\in I} P_i(x) \wedge \bigwedge_{j\in J} \neg P_j(x)] .$$

(T is the theory of "κ independent unary relations"). Clearly T is superstable, in fact for any A , $|S_1(A)| = 2^\kappa + |A|$, the types over A being exactly those of the form "$x = a$" for some $a \in A$ and those of the form

$$p_\zeta^A = \{x \neq a \mid a \in A\} \cup \{P_i(x) \mid i \in \zeta\} \cup \{\neg P_i(x) \mid i \notin \zeta\}$$

for some $\zeta \subseteq \kappa$.

The description of the forking relation is trivial: If $A \subset B$ the unique non-algebraic and hence the unique non-forking extension of p_ζ^A to B is p_ζ^B. Thus every p_ζ^A has Lascar rank $U(p_\zeta^A) = 1$, and $P = \{p_\zeta^\emptyset | \zeta \subset \kappa\}$ is a complete set of pairwise orthogonal regular types over $\emptyset$. (Clearly there are no strongly regular types). Notice that if $|A| < \kappa$ there is no prime model over A: A is not a model, but the only isolated types over A are the realized ones. But contrariwise, every element of a model realizes a regular type in P and by the description of forking, every set is independent. Thus every model is its own basis, and so is trivially prime and minimal over its basis and determined uniquely by its dimension.

<u>2.2a PROBLEM</u>: Define a superstable theory T to be <u>quasi-basic</u> if there is a complete set P of representatives of the $\not\perp$-classes of weight one types in the form of regular 1-types over $\emptyset$. Is there a natural condition on T which forces the Basis Theorem to hold for T, in particular which forces every model to be prime over its basis? (Note that by considering the unified proof given by Shelah of [Sh IX, 2.3, 2.4] a natural theory can be developed for the a-models of such a theory).

<u>2.3 PROPOSITION</u>: Let T be basic, $M \models T$, $p \in S_*(M)$, let $\vec{a} \in M(p)$ realize p.

(i) $M(p)$ is minimal over $M \cup \{\vec{a}\}$.

(ii) Suppose $w(p) = 1$. Then $M(p)$ is a minimal proper prime extension of M.

<u>REMARK</u>: If fact it suffices that T be tt nmd.

<u>PROOF</u>: Suppose that $M \precneqq N \precneqq M(p)$ and in addition for part (i) only suppose $M \cup \{\vec{a}\} \subset N$. By (0, 1.20(v)) there are $b_0 \in N \setminus M$ and $b_1 \in M(p) \setminus N$ such that $t(b_0, M)$, $t(b_1, N)$ are strongly regular. Since T is nmd, by (1.2) $t(b_1, N) \not\perp q$, some sr type q over M, and by (0, 1.24) $q|N$ is realized in $N(b_1) \subset M(p)$. So wlog $b_1 \underset{M}{\downarrow} N$. By (0, 1.14), since $b_0, b_1 \in M(\vec{a})$, $b_i \underset{M}{\not\downarrow} \vec{a}$ for $i \in 2$. In part (i), $\vec{a} \in N \setminus M$ so $b_1 \underset{M}{\downarrow} N$ implies $b_1 \underset{M}{\downarrow} \vec{a}$, contradiction. In part (ii), $b_0 \in N \setminus M$ so $b_0 \underset{M}{\downarrow} b_1$, contradicting that $w(\vec{a}) = 1$.

I 3. THE SPECTRUM FUNCTION FOR BASIC THEORIES

3.0 DEFINITION: [Sh VII, Definition 1.1, Definition 2.1]

$I(\kappa, T)$ is the number of isomorphism classes of models of T of cardinality κ.

For $x :=: s, t$, $I^x(\kappa, \lambda, T)$ is the number of isomorphism classes of F^x_λ-saturated models of T of cardinality κ. (See below.)

3.1 REMARKS: For F^s_λ, F^t_λ see [Sh, Ch IV, Definition 1.1, Definition 2.1]. For the following remarks see [Sh, Ch IV, Lemma 2.2, 2.18]. The notation is introduced solely to connect the present results with a theorem of Shelah's and is not used in any essential way.

For totally transcendental T, $F^t_{\aleph_0}$-saturated models are just models, so $I^t(\kappa, \aleph_0, T) = I(\kappa, T)$. $M \models T$ is F^s_λ-saturated iff M is λ-saturated. For $\lambda > |T|$, $M \models T$ is F^t_λ-saturated iff M is F^s_λ-saturated. Thus $I^s(\kappa, \lambda, T)$ is the number of isomorphism classes of λ-saturated models of T of power κ, and for $\lambda > |T|$, $I^t(\kappa, \lambda, T)$ is the same thing.

$I(\kappa, T)$ is called the spectrum of T. It was a central concern of Shelah's book to find various spectrum functions explicitly. As a result of the Basis Theorem, I can find the spectrum functions $I(\kappa, T)$ and $I^s(\kappa, \lambda, T)$ (and so $I^t(\kappa, \lambda, T)$ for $\lambda > |T|$) for basic theories T. This result is seen to be a special case of Shelah's "Main Theorem" [Sh, IX, Theorem 2.4]. It is, however, an important special case for two reasons: the proof of the Basis Theorem is easy and depends only on elementary properties of strongly regular types, whereas the "Main Theorem" of Shelah is the culmination of a very long and difficult theory; and as will be seen in Chapters II-IV, many interesting and important theories are basic tt theories.

I will not present the computations for $I^t(\kappa, \lambda, T)$, $\lambda \geq |T|$, since the computations are identical to those for $I(\kappa, T)$ but based on (x) of the Basis Theorem rather than on (ix) alone. I will make a further comment to this effect at the end of the proof.

3.2 COROLLARY: (to the Basis Theorem 2.0) Let T be a tt basic theory, with prime model of cardinality $\aleph_\alpha \leq |T|$, $\mu = \mu(T)$. Then the spectrum function $I(\aleph_\beta, T)$ $(\aleph_\beta \geq \aleph_\alpha)$ is given by the following table:

	$I(\aleph_\alpha, T)$	$I(\aleph_\beta, T)$, $\beta > \alpha$
(i) $\mu = 1$		
(a) $\delta_0 = \aleph_0$	$\lvert\alpha + 1\rvert$	1
(b) $\delta_0 < \aleph_0$	$\lvert\alpha + \aleph_0\rvert$	
(ii) $1 < \mu < \aleph_0$		
(a) $\delta_0 = \delta_{\aleph_0}$	$\lvert\alpha + 1\rvert^\mu$	$\lvert{}^\mu(\beta + 1) \setminus {}^\mu\beta\rvert$
(b) $\delta_0 < \delta_{\aleph_0}$	$\lvert\alpha + \aleph_0\rvert$	$\lvert\beta + \aleph_0\rvert$
(iii) $\aleph_0 \leq \mu \leq \lvert T\rvert$		
(a) $\delta_0 = \delta_{\aleph_0}$	$\lvert\alpha + 1\rvert^\mu$	
(b) $\delta_0(p) < \aleph_0$ for at least one but only finitely many $p \in P$	$\aleph_0$ $(\alpha = 0)$; $\lvert\alpha + 1\rvert^\mu$ $(\alpha > 0)$	$\lvert\beta + 1\rvert^\mu$
(c) $\delta_0(p) < \aleph_0$ for infinitely many $p \in P$	$\lvert\alpha + \aleph_0\rvert^\mu$	

3.2': For the case T countable of common interest, we have $\alpha = 0$, $\mu \leq \aleph_0$ in (3.2), so the table reduces to:

	$I(\aleph_0, T)$	$I(\aleph_\beta, T)$, $\beta > 0$
(i) $\mu = 1$		
(a) $\delta_0 = \aleph_0$	1	1
(b) $\delta_0 < \aleph_0$	$\aleph_0$	
(ii) $1 < \mu < \aleph_0$		
(a) $\delta_0 = \delta_{\aleph_0}$	1	$\lvert {}^{\mu}(\beta + 1) \setminus {}^{\mu}\beta \rvert$
(b) $\delta_0 < \delta_{\aleph_0}$	$\aleph_0$	$\lvert \beta + \aleph_0 \rvert$
(iii) $\mu = \aleph_0$		
(a) $\delta_0 = \delta_{\aleph_0}$	1	$\lvert \beta + 1 \rvert^{\aleph_0}$
(b) $\delta_0(p) < \aleph_0$ for at least one but only finitely many $p \in P$	$\aleph_0$	
(c) $\delta_0(p) < \aleph_0$ for infinitely many $p \in P$	$2^{\aleph_0}$	

PROOF: By the Basis Theorem (v), every model is determined by its dimension; by (ix) every dimension vector $\delta \geqq \delta_0$, the dimension of the prime model, occurs as the dimension vector of some model; and so by (vii) the number of models of T of cardinality $\kappa = \aleph_\beta$ is just $|D_\beta|$ where D_β is the set of all possible dimension vectors of models of T of cardinality $\aleph_\beta = \kappa$, $D_\beta = \{\delta\colon\ P \to \text{Cardinals} \mid \delta_0 \leqq \delta \leqq \delta_\kappa$, and if $\kappa = \aleph_\beta > \aleph_\alpha, \bigcup\delta[P] = \kappa\}$. Note that if $\mu < cf(\kappa)$ (in particular when

μ is finite) the final condition can be written "$\kappa \in \delta[P]$" .

Now it is a fairly straightforward combinatorial problem to check the cardinalities given in (3.2).

Let $\mu_0 = \{p \in P | \delta_0(p) < \aleph_0\}$ and let $\mu_1 = \{p \in P | \delta_0(p) = \aleph_0\}$. Then $|\mu_0| + |\mu_1| = \mu \leq |T|$, so if μ is infinite at least one of μ_0, μ_1 has cardinality μ .

Notice that for any $\aleph_\beta$ there are $|\beta + 1|$ infinite cardinals $\leq \aleph_\beta$ and $|\beta + \aleph_0|$ cardinals $\leq \aleph_\beta$. I will use repeatedly some simple cardinal arithmetic: for any ordinal β and finite λ , $|\beta + \aleph_0|^\lambda =$ $= |\beta + \aleph_0| \geq |\beta + 1|^\lambda$; and for any ordinal $\beta > 0$ and infinite cardinal λ , $|\beta + 1|^\lambda = |\beta + \aleph_0|^\lambda$.

If $\mu = 1$, then δ_0 is a single cardinal $\leq \aleph_0$ and so $D_\alpha = \{\delta | \delta \text{ a cardinal}, \delta_0 \leq \delta \leq \aleph_\alpha\}$ and $D_\beta = \{\aleph_\beta\}$ for $\beta > \alpha$. Thus $|D_\alpha|$ is $|\alpha + 1|$ or $|\alpha + \aleph_0|$ as δ_0 is infinite or finite respectively, and $|D_\beta| = 1$ for $\beta > \alpha$.

For cases (ii) and (iii) ($\mu < cf(\aleph_\beta)$) first notice that D_α and D_β ($\beta > \alpha$) can be regarded as being the following sets:

$$D_\alpha = {}^{\mu_0}(\omega + \alpha + 1) \times {}^{\mu_1}(\alpha + 1)$$

$$D_\beta = {}^{\mu_0}(\omega + \beta + 1) \times {}^{\mu_1}(\beta + 1) \setminus {}^{\mu_0}(\omega + \beta) \times {}^{\mu_1}\beta$$

The expression for D_β is not correct when $\mu \geq cf(\aleph_\beta)$ in case (iii), but nonetheless it is clear that when μ is infinite, the number of μ-sequences $< \delta_\kappa$ whose limit is $\kappa = \aleph_\beta$ is the same as the number of μ-sequences containing κ , so the cardinality is correct.

Thus for all $\beta \geq \alpha$, $|D_\beta| \leq |\beta + \aleph_0|^\mu$ so if μ is finite, $|D_\beta| \leq |\beta + \aleph_0|$, and if μ is infinite, $\beta > 0$, $|D_\beta| \leq |\beta + 1|^\mu$.

In part (a) of (ii) and (iii), $\mu_0 = \emptyset$, $|\mu_1| = \mu$, so $|D_\alpha| = |\alpha + 1|^\mu$ and $|D_\beta| = |{}^\mu(\beta + 1) \setminus {}^\mu\beta|$ for $\beta > \alpha$, giving (iia) immediately as well as the first part of (iiia). For the other part of (iiia), it suffices to show that $|D_\beta| \geqq |\beta + 1|^\mu$, which is easy: Given $f \in {}^\mu(\beta + 1)$, define $\hat{f} \in D_\beta$ by $\hat{f}(\gamma) = \beta$ when $\gamma = \cup\gamma$ is an ordinal $< \mu$, and $\hat{f}(\varepsilon + 1) = f(\varepsilon)$ for all $\varepsilon < \mu$. Then $f \neq g \in {}^\mu(\beta + 1) \Rightarrow \hat{f} \neq \hat{g} \in D_\beta$.

The second part of (iiib, c) now also follows immediately, since I already have for $\beta > \alpha$ that $|D_\beta| \leqq |\beta + 1|^\mu$. But (iiib, c) allow more dimension vectors than (iiia) and for (iiia) I have $|D_\beta| = |\beta + 1|^\mu$. Therefore the same holds for (iiib, c) as well.

In (iib) both μ_0 and μ_1 are finite and $\mu_0 \neq \emptyset$. Thus $|D_\beta| \geqq |\beta + \aleph_0|$ for all $\beta \geqq \alpha$. Fix $p \in \mu_0$. Since $\mu > 1$ there is at least one $q \in P$, $q \neq p$. For each cardinal κ , $\delta_0(p) \leqq \kappa \leqq \aleph_\beta$ (there are $|\beta + \aleph_0|$ such since $p \in \mu_0$) , define δ^*_κ by:

$$\delta^*_\kappa(q) = \begin{cases} \kappa & q = p \\ \aleph_\beta & \text{otherwise} \end{cases} \qquad (q \in P)$$

Then $\delta^*_\kappa \in D_\beta$, so $|D_\beta| \geqq |\beta + \aleph_0|$.

The only cases remaining are (iiib, c) for $\beta = \alpha$. $|D_\alpha| = |\omega + \alpha + 1|^{|\mu_0|} |\alpha + 1|^{|\mu_1|}$. When μ_0 is finite this reduces to $|\alpha + \aleph_0||\alpha + 1|^\mu$, so $|D_\alpha| = \aleph_0$ when $\alpha = 0$ and $|D_\alpha| = |\alpha + 1|^\mu$ when $\alpha > 0$. Finally, when μ_0 is infinite, $|D_\alpha| = |\alpha + \aleph_0|^\mu$.

<u>3.3 REMARKS</u>: (x) of the Basis Theorem tells us how to define D^λ_β , the set of all possible dimensions of λ-saturated models of T of power $\aleph_\beta$ $(\aleph_0 \leqq \lambda \leqq \aleph_\beta = \kappa)$:

$D^{\lambda}_{\beta} = \{\delta\colon\ P \to \text{Cardinals} \mid \delta_{\lambda} \leqq \delta \leqq \delta_{\kappa}$, and if $\kappa = \aleph_{\beta} > \aleph_{\alpha}$, $\bigcup \delta[P] = \kappa\}$.

The calculation of $|D^{\lambda}_{\beta}|$ proceeds in a manner similar to that for $|D_{\beta}|$.

I 4. APPLICATION TO GENERAL nmd tt THEORIES

4.0 INTRODUCTION

By Theorem 1.2(iii) every non-multidimensional tt theory T is close to being basic: in fact, $T(M_0)$ is a basic theory, as noted in (1.4). So a great deal is already known about the structure of the models of such a T by the Basis Theorem and its corollaries. However, not enough is known to calculate any of the spectrum functions of T: clearly it may be possible to expand a given model M of T to a model of $T(M_0)$ in many different ways.

This problem and its solution are the main points of one of the culminating theorems of Shelah's book [Sh IX, 2.4]. Several points about this theorem and its proof should be made. Firstly, it is obvious that Corollary 3.2 is just a special case. What is not at all obvious is that the main points of the Basis Theorem are contained in the proof of [Sh IX, 2.4]. However, there is something very important here: the proof of the Basis Theorem is quite elementary, whereas that of [Sh IX, 2.4] is not. Furthermore, I feel that it is very important to separate the structural details (Basis Theorem) from the combinatorial details (Cor. 3.2) to gain a clear understanding of the models of a tt basic theory.

To what extent can the Basis Theorem be used to clarify the statement and proof of [Sh IX, 2.4]? As it turns out, a great deal can be said. A. Pillay [Pi], building on a nice characterization of nmd tt theories discovered by Bouscaren and Lascar [Bo-L 3.4] (but see also [Sh V, Theorem 3.4 and V, Definition 5.2]), has recently given a very nice explication of [Sh IX, 2.4], covering the spectrum calculations for $I(\aleph_\beta, T)$, T countable. However, Pillay's treatment, as does Shelah's more complicated version, does not separate the structural from the combinatorial aspects of the theorem.

This section is intended as an informal sketch of how to adapt Pillay's paper to the present context. The reader should be familiar with [Pi] or at the very least with [Sh IX, 2.4]; this discussion is not entirely self-contained. It should be emphasized that ultimately the general idea of the proof is due to Shelah.

The main idea behind the solution of the problem mentioned at the end of the first paragraph is that $T(M_0)$ is too large an expansion of T. We want to choose $A \subset M_0$ "in as nice a way as possible" so that $T' = T(A)$ is basic and so that the relation "$M_1, M_2 \models T'$ are isomorphic as models of T" can be characterized in some simple way. Unfortunately, the choice of A cannot be made canonically, but we can come close enough to gain a fair understanding of the structure of models of T and to calculate the various spectrum functions.

I would like to acknowledge the helpful comments of Anand Pillay who encouraged me to include this section.

<u>4.1 DEFINITIONS</u>: (i) (Refer to the finite equivalence relation theorem, 0, 1.8ii) $\vec{a} \equiv^s_C \vec{b}$ means that for every finite equivalence relation $E(\vec{x}, \vec{y})$ definable over C, $\models E(\vec{a}, \vec{b})$. (Clearly $\vec{a} \equiv^s_C \vec{b}$ implies $\vec{a} \equiv_C \vec{b}$). $\mathrm{stp}(\vec{b}, C) = \{E(\vec{x}, \vec{b}) | E$ as above$\}$.
(ii) If $\vec{A} \equiv \vec{B}$ then there is an automorphism α of $\mathfrak{C}$ sending A to B. If $p \in S(A)$ then p_B denotes the image of p under α.

4.2 THEOREM: Let T be tt.

(i) [Bo L 3.4] See also [Pi, III.2]. (Extends 1.2)

T is nmd iff for every $\vec{a}$, every sr $p(\vec{x}) \in S(\vec{a})$, and every $\vec{b} \equiv^s \vec{a}$, $p \not\perp p_{\vec{b}}$.

(ii) [Bo L 4.6] See also [Pi III.5, IV.1] (T nmd)

$p \in S(a)$ sr, $M \supset \vec{a} \cup \vec{b}$, $M \models T$ and: $b \equiv^s \vec{a}$ or ($\vec{b} \equiv \vec{a}$, $t(\vec{a})$ isolated, $p \not\perp p_{\vec{b}}$). Then $\dim(p, M) = \dim(p_b, M)$.

4.3 REMARKS: In the remainder of this section T is a fixed nmd tt theory, $\mu = \mu(T)$.

4.4 LEMMA: Let $\Xi = \langle \xi_i \rangle_{i<\mu}$ be the equivalence classes under $\not\perp$ of the weight one types of T. Let α be an automorphism of $\mathfrak{C}$. Then α induces a permutation $\pi ::= \pi_\alpha$ on Ξ by the relation "$\alpha(p) \in \pi(\xi)$ for every $\xi \in \Xi$ and $p \in \xi$".

Alternatively, once an enumeration of Ξ in order type μ has been fixed, π can be regarded as a permutation of μ by $\pi(\xi_i) = \xi_{\pi(i)}$. As well, if P is a set of representatives of Ξ, (say, by strongly regular 1-types) then π can be regarded as a permutation of P by the relation "$\pi(p)$ is the representative in P of π(class of p)".

All three viewpoints are used freely.

PROOF: Obvious.

4.5 DEFINITION: [Sh, Theorem IX, 2.4]. $G(T) = \langle \{\pi_\alpha \mid \alpha \in \mathrm{Aut}(\mathfrak{C})\}, \circ \rangle$.

4.6 REMARKS: (i) The group G(T) is an important invariant of T. Clearly, if μ is finite then $|G(T)| \leq \mu!$, and if μ is infinite then

$|G(T)| \leq 2^{\mu}$. In fact, we will see that for infinite μ , $|G(T)| \leq \mu$.

(ii) T basic $\Rightarrow G(T) = \{id\}$ (id the identity permutation).

(iii) T uni-dimensional $\Rightarrow G(T) = \{id\}$.

(iv) Let T_n be the theory of Example 1.5(ii). Then $G(T_n)$ is isomorphic to the full permutation group on n .

4.7 DEFINITION: Let $x = \{\xi_0, \ldots, \xi_{n-1}\}$, $z = \langle\zeta_0, \ldots, \zeta_{n-1}\rangle$ be n-tuples in Ξ (or alternatively n-tuples in P) . Then x and z are associates if for some $\pi \in G(T)$, $\zeta_i = \pi(\xi_i)$ for all $i < n$.

4.8 PROPOSITION: [Pi, page 18, immediately preceding Lemma IV.6] (for part (ii)).

(i) Association is an equivalence relation on n-typles from Ξ .

(ii) Any n-typle $\xi \in \Xi$ has only finitely many associates.

PROOF: (ii) Clearly it suffices to prove this for $n = 1$. Suppose ξ has infinitely many associates. Let $p \in \xi$ be sr, $p \in S_1(\vec{a})$ for some $\vec{a}$ (possible since T is tt). There are $\alpha_i \in \mathrm{Aut}(\mathfrak{C})$ $(i < \omega)$ with $i \neq j \Rightarrow \alpha_i(p) \perp \alpha_j(p)$. Let $\vec{a}_j = \alpha_j(\vec{a})$. Then $t(\vec{a}_i) = t(\vec{a}_j)$ for all $i, j < \omega$. But T is tt, so all types have finite multiplicity (0, 1.10(i)) and thus by the finite equivalence relation theorem (0, 1.8ii), for some $i \neq j$ $\vec{a}_i \equiv^s \vec{a}_j$. But then by (4.2(i)), $\alpha_i(p) \not\perp \alpha_j(p)$, contradiction.

4.9 REMARKS: In choosing a base set A so that T(A) is basic, we attempt to code up the relation of association by suitable types. Unfortunately we can only do this for elements Ξ , not for arbitrary n-tuples. This contributes, in part, to our failure to get a precise result later on (4.14). It follows from 4.8(ii) that $|G(T)| \leq \mu + \aleph_0$.

4.10 PROPOSITION: [Pi IV.6, IV.7]

Fix M_0 a copy of the prime model. Let μ' be the number of associate-classes of elements of Ξ. (Clearly $1 \leq \mu' \leq \mu$; $\mu = 1 \Rightarrow \mu' = 1$; μ' is infinite iff $\mu' = \mu$ is infinite by 4.8).

There are types $q_i(x, \vec{y}_i)$ and $r_i(\vec{y}_i) \in S(\emptyset)$, r_i isolated, $n_i < \omega$, $(i < \mu')$; and sequences $\vec{a}_i^j$ in M_0 , $t(\vec{a}_i^j) = r_i$, $(i < \mu'$, $j < n_i)$ such that, letting $p_i^j = q(x, \vec{a}_i^j)$ and $P = \{p_i^j \mid i < \mu', j < n_i\}$

(i) P is a complete set of sr representatives of Ξ , $(p_i^j)_{j<n_i}$ is a complete set of representatives of an associate-class for each $i < \mu'$.

(ii) $\dim(p_i^j, M_0) \in \{0, \aleph_0\}$.

(iii) For any $i_0 \neq i_1 < \mu'$, for every $\vec{b}_0, \vec{b}_1 \in \mathfrak{C}$, $\models r_{i_j}(\vec{b}_j)$ $(j \in 2) \Rightarrow q_{i_0}(x, \vec{b}_0) \perp q_{i_1}(x, \vec{b}_1)$.

REMARKS: This is somewhat modified from Pillay's version. In particular I do not restrict $|T|$. For completeness I give the proof of this important result, but the reader should note that except for some small details it is copied from [Pi].

Note that even with the q's and r's fixed, the type of the sequence of all the $\vec{a}_i^j$'s is not determined.

PROOF: Recall that each class has only finitely many associates, so the comments on μ' apply and we can enumerate Ξ as $(\xi_i^j)_{i<\mu'}^{j<n_i}$, so that for each $i < \mu'$, $n_i < \aleph_0$, $(\xi_i^j)^{j<n_i}$ is an enumeration of an associate class.

For each $i < \mu'$ pick $p_i \in \xi_i^0$, an sr 1-type over M_0 , based on $\vec{a}_i^0 \in M_0$ such that if $\dim(p_i \restriction \vec{a}_i^0, M_0) < \aleph_0$, it is in fact 0 . Such a representative p_i exists by (1.2); such an $\vec{a}_i^0$ exists, since p_i is based on a finite set $\vec{a}_i$ (0, 1.10(v)), so we let $\vec{a}_i^0$ be $\vec{a}_i$ together with any finite basis for $p_i \upharpoonright \vec{a}_i$ in M_0. Furthermore, if $\dim(p_i \upharpoonright \vec{a}_i^0, M_0)$ is infinite, it equals $\aleph_0$ by the characterization of prime models (0, 1.10(ii)). Let the type $q_i(x, \vec{y}^i)$ be determined by $p_i \upharpoonright \vec{a}_i^0 = q_i(x, \vec{a}_i^0)$ and let $r_i(\vec{y}^0) = t(\vec{a}_i^0)$, isolated since $\vec{a}_i^0 \in M_0$.

For each j, $0 < j < n_i$, there is $\alpha \in \mathrm{Aut}(\mathfrak{C})$ such that $\alpha(q_i(x, a_i^0)) \in \xi_i^j$, an associate of ξ_i^0. Since r_i is isolated and T is tt, $\mathrm{stp}(\alpha(a_i^0))$ is also isolated. Let $\vec{a}_i^j$ realize this $\mathrm{stp}(\alpha(\vec{a}_i^0))$ in M_0 and let $p_i^j = q_i(x, \vec{a}_i^j)$. Then, since T is nmd, $p_i^j \in \xi_i^j$. Since $\vec{a}_i^0 \equiv \vec{a}_i^j$ and M_0 is prime, hence homogeneous, there is $\beta \in \mathrm{Aut}(M_0)$, $\beta(a_i^0) = a_j^0$. Hence $\dim(p_i^0, M_0) = \dim(p_i^j, M_0)$.

Thus (i) and (ii) are ensured. It only remains to check (iii).

Note that for any b with $\models r_i(\vec{b})$, $q_i(x, \vec{b})$ is sr (there is an automorphism of $\mathfrak{C}$ taking $\vec{a}_i^0$ to $\vec{b}$), and hence $q_i(x, \vec{b})$ is an associate of $q_i(x, \vec{a}_i^0)$, i.e. $q_i(x, \vec{b}) \not\perp q_i(x, \vec{a}_i^j)$ for some $j < n_i$. (iii) follows immediately.

4.11 DEFINITION: Fix once and for all a "standard copy" of M_0 in $\mathfrak{C}$ together with all the things yielded by the preceding proposition. Let $A = \{a_i^j \mid i < \mu', j < n_i\} \subset M_0$.

A dimension schema s for $N \models T$ is an elementary map $s\colon M_0 \to N$; and for each such, N has a dimension relative to s,

$$\dim_s(N) = \langle \dim(s(p_i^j), N \rangle_{i<\mu', j<n_i}$$

4.12 PROPOSITION: $T(A)$ is a basic theory.

4.13 REMARKS AND NOTATION: As before δ denotes a dimension vector, $\delta = \langle \delta_i^j \rangle_{i<\mu'}^{j<n_i}$ where each δ_i^j is a cardinal. I denote by $A(\delta)$ the

reduct to T of the (unique) model of $T(A)$ with dimension δ .

What we would like to be true is the following:

$$A(\delta) \cong A(\delta') \Leftrightarrow (\exists \pi \in G(T)) [\delta' = \delta \circ \pi]$$

(interpreting π as a permutation on the appropriate index set). For $\mu(T)$ finite, this fact is "Stage H" of Shelah's proof of [Sh, IX, 2.4]. Shelah points out that the result is not known for $\mu(T)$ infinite.

4.14 THEOREM:

$A(\delta) \cong A(\delta') \Rightarrow (\exists \pi \in G(T)) [\delta' = \delta \circ \pi]$ and if $\mu < \aleph_0$ the converse holds.

PROOF: ($\Rightarrow$) Let $f\colon A(\delta) \cong A(\delta')$. WLOG $A(\delta) = A(\delta') = N$; f is an automorphism of $\mathfrak{C}$ with $f\colon N \cong N$. Let $\pi := \pi_f \in G(T)$, $A^* := f[A]$.

Take any $p_i^j = q_i(x, \vec{a}_i^j)$. Then $q_i(x, f(\vec{a}_i^j)) = f(p_i^j) \not\perp p_i^{j'}$ for some $j' < n_i$ by (4.10), and $\langle i, j'\rangle = \pi_f(\langle i, j\rangle)$ by the definition of π_f . But $t(f(\vec{a}_i^j)) = t(\vec{a}_i^{j'}) = r_i$, isolated, so by (4.2(ii)), $\dim(p_i^{j'}, N) = \dim(f(p_i^j), N)$. That is,

$$\delta'(\langle i, j\rangle) = \delta(\langle i, j'\rangle) = \delta \circ \pi_f(\langle i, j\rangle)$$

($\Leftarrow$) Now assume that $\mu' < \aleph_0$.

Since $\mu' < \aleph_0$, $|A| < \aleph_0$ and since $A \subset M_0$, $t(A)$ is isolated. Since T is tt, each of the finitely many strong types extending $t(A)$ is isolated. Let $N = A(\delta)$. Let $\pi = \pi_\alpha \in G(T)$. I construct $s\colon A \to N$ so that $\dim_s(N) = \delta \circ \pi$ which does it.

Consider $\alpha[A]$. $stp(\alpha[A])$ is isolated, so realized in N . Define s by the fact that $s[A]$ realizes $stp(\alpha[A])$. For each $i < \mu'$, $j < n_i$, $q_i(x, s(\vec{a}_i^j)) \not\perp q_i(x, \alpha(\vec{a}_i^j))$ since $s(\vec{a}_i^j) \equiv^s \alpha(\vec{a}_i^j)$

(4.2(i)); and $q_i(x, \alpha(\vec{a}_i^j)) \not\vdash q_i(x, \vec{a}(\pi(i, j)))$ by the definition of $\pi = \pi_\alpha$. Now, by (4.2(iii)) $\dim(q_i(x, s(\vec{a}_i^j)), N) = \dim(q_i(x, \vec{a}(\pi(i, j)), N)$. Hence $\dim_s(N) = \delta \circ \pi$.

<u>4.15 PROPOSITION</u>: $(\lambda \geq \aleph_0)$ (i) $N \models T$ is λ-saturated iff $\dim_s(N) \geq \delta_\lambda$ for some (any) dimension schema s.

(ii) $N \models T$ is the saturated model of power $\lambda(|N| = \lambda)$ iff $\dim_s(N) = \delta_\lambda$ for some (any) s.

<u>PROOF</u>: (ii) is immediate from (i). For (i) suppose N is λ-saturated, and $s: A \to N$ a dimension schema. Since $s(p_i^j)$ is a type over a finite set, by λ-saturation there is an independent family of cardinality at least λ realizing $s(p_i^j)$ in N, so $\dim_s(N) \geq \delta_\lambda$.

Conversely, suppose for some s, $\dim_s(N) \geq \delta_\lambda$. Then by the Basis Theorem (x), $\langle N, s(A)\rangle$ is λ-saturated. Hence N is λ-saturated.

<u>4.16 THEOREM</u>: Let $\aleph_\alpha$ be the cardinality of the prime model, δ_0 its dimension. (Note that δ_0 is well-defined by (4.10(ii)) and the homogeneity of the prime model, and $\delta_0 \leq \delta_{\aleph_0}$). Let $\beta \geq \alpha$ and for fixed β, let γ range over $0 \leq \gamma \leq \beta$.

Let $D_\beta = \{\delta | \delta_0 \leq \delta \leq \delta_{\aleph_\beta}$, and if $\aleph_\beta > \aleph_\alpha$ then $\bigcup\{\delta_i^j | i < \mu', j < n_i\} = \aleph_\beta\}$.

Let $D_\beta^\gamma = \{\delta \in D_\beta | \delta_{\aleph_\gamma} \leq \delta\}$.

(i) $\delta \in D_\beta$ iff for some $M \models T$, $|M| = \aleph_\beta$, and some dimension schema s, $\dim_s(M) = \delta$.

(ii) $\delta \in D_\beta^\gamma$ iff for some $\aleph_\gamma$-saturated $M \models T$, $|M| = \aleph_\beta$, and some dimension schema s, $\dim_s(M) = \delta$.

(iii) $I(\aleph_\beta, T) \geq |D_\beta/G(T)|$

(iv) $I^S(\aleph_\beta, \aleph_\gamma, T) \geq |D^\gamma_\beta/G(T)|$

(v) If $\mu(T)$ is finite, equality holds in (iii), (iv).

PROOF: (i) is immediate by the Basis Theorem, (vii) and (ix); (ii) follows by (4.15).

By (4.14) the equivalence relation induced by $G(T)$ on D_β or D^γ_β is coarser than that induced by the isomorphism of models, and the two equivalence relations are the same if μ is finite.

4.17 REMARKS: (4.10), (4.12), (4.14), (4.15) constitute a structural description of the (saturated) models of T, and (4.16) nearly reduces the combinatorial problem of calculating the various spectrum functions to a direct appeal to the Basis Theorem, where I already calculated $|D_\beta|$. Below I will indicate the possible values of $I(\aleph_\beta, T)$ $(\aleph_\beta \geq \aleph_\alpha)$ and follow it with a brief sketch of how to compute the values from Corollary (3.2) and Theorem (4.16). As before, I will leave the similar computations required for $I^S(\aleph_\beta, \aleph_\alpha, T)$ to the reader.

4.18 THEOREM: [Sh, Theorem IX, 2.4] See also [Pi IV.10] Let $\aleph_\alpha$, δ_0 be as in 4.16, $\mu = \mu(T)$, $G = G(T)$.

	$I(\aleph_\alpha, T)$	$I(\aleph_\beta, T)$
(i) $\mu = 1$		
(a) $\delta_0 = \aleph_0$	$\lvert\alpha + 1\rvert$	1
(b) $\delta_0 < \aleph_0$	$\lvert\alpha + \aleph_0\rvert$	1
(ii) $1 < \mu < \aleph_0$		
(a) $\delta_0 = \aleph_0$	$\lvert{}^{\mu}(\alpha + 1)/G(T)\rvert$	$\lvert({}^{\mu}(\beta + 1) \setminus {}^{\mu}\beta)/G(T)\rvert$
(b) $\delta_0 < \aleph_0$	$\lvert\alpha + \aleph_0\rvert$	$\lvert\beta + \aleph_0\rvert$
(iii) $\aleph_0 \leqq \mu \leqq \lvert T\rvert$		
(a) $\delta_0 = \aleph_0$	$\lvert\alpha + 1\rvert^{\mu}$	$\lvert\beta + 1\rvert^{\mu}$
(b) $\delta_0(p_i^j) < \aleph_0$ for at least one but only finitely many $i < \mu'$	$\begin{cases} \aleph_0 & \text{if } \alpha = 0 \\ \lvert\alpha + 1\rvert^{\mu} & \text{if } \alpha > 0 \end{cases}$	$\lvert\beta + 1\rvert^{\mu}$
(c) $\delta_0(p_i^j) < \aleph_0$ for infinitely many $i < \mu'$	$\lvert\alpha + \aleph_0\rvert^{\mu}$	$\lvert\beta + 1\rvert^{\mu}$

PROOF: As noted in the proof of (3.2), D_α and D_β can be regarded as ${}^{\mu}(\alpha + 1)$, ${}^{\mu}(\beta + 1) \setminus {}^{\mu}\beta$ respectively, so the entries in (ii)(a) make sense.

By (4.16) in (i), (ii) $I(\aleph_\beta, T) = \lvert D_\beta / G(T)\rvert$ for $\beta \geqq \alpha$. In case (i), $G(T) = \{id\}$, and so all of (i) follows immediately by (3.2). In case (ii)(b), D_β $(\beta \geqq \alpha)$ is infinite, and $G(T)$ is finite, so $\lvert D_\beta / G(T)\rvert = \lvert D_\beta\rvert$. Thus all of (ii) follows immediately from (3.2).

The only really new part of the proof is to see why the entries in (iii) are the same as in (3.2(iii)).

We must examine more closely the calculation of $|D_\beta|$. Let $\chi_\beta = |D_\beta|$ $(\beta \geqq \alpha)$. By checking how each dimension can vary, clearly $\chi_\beta = \Pi_{i<\mu'}^{j<n_i} \kappa_i^j$ where

$$\kappa_i^j = \begin{cases} |\beta| + \aleph_0 & \text{if } \delta_0(i, j) = 0 \\ |\beta + 1| & \text{if } \delta_0(i, j) = \aleph_0 . \end{cases}$$

As noted in the proof of (4.10), $\delta_0(i, j)$ does not depend on j . Let $\kappa_i = \kappa_i^0$. Then $\chi_\beta = \Pi_{i<\mu'} \kappa_i^{n_i}$. But in case (iii), μ is infinite, hence $\mu' = \mu$, and, with $n_i < \aleph_0$, $\chi_\beta = \Pi_{i<\mu} \kappa_i$.

Call δ <u>uniform</u> if $\delta(i, j)$ does not depend on j . Clearly two distinct uniform δ are inequivalent mod $G(T)$, and there are $\Pi_{i<\mu'} \kappa_i$ different uniform δ which are dimensions of models of power $\aleph_\beta$. Thus $|D_\beta / G(T)| \geq \chi_\beta$. Also, by (4.16), $|D_\beta| \geqq I(\aleph_\beta, T) \geqq |D_\beta / G(T)|$. Thus $I(\aleph_\beta, T) = \chi_\beta = |D_\beta|$, and $|D_\beta|$ was calculated in (3.2(iii)).

<u>4.19 SUMMARY</u>: The structure of the models of a tt nmd theory T can be understood as follows: there is a nice set A such that $T(A)$ is basic, and the structure of models of $T(A)$ is given by the Basis Theorem. The group $G(T)$ contains information about the different ways models of T can be expanded to models of $T(A)$. The problem of counting the number of models of T is handled by means of the basic theory $T(A)$ and the equivalence relation induced by $G(T)$. This theorem is <u>the</u> generalization of the famous theorem of Baldwin and Lachlan on $\aleph_1$-categorical theories (which is, in fact (i) for T countable).

II AN EXTENDED EXAMPLE: EXISTENTIALLY CLOSED MODULES OVER A COHERENT RING

II 0. INTRODUCTION

The intent of this chapter is to present an extended example, and to show how, in the context of this example, the concepts of stability theory relate to concepts and theorems of classical mathematics. I study in detail the stability theory of the theories T_Λ^* introduced by Eklof and Sabbagh [ES]. Here Λ is a coherent ring and T_Λ^* is the model completion of the theory of unitary left Λ-modules: T_Λ^* is the theory of existentially closed modules.

These theories are especially interesting because their model theory is very algebraic in nature, a consequence of the fact that T_Λ^* has elimination of quantifiers. Thus I will be able to show that many standard concepts of stability theory as they appear in T_Λ^* correspond to standard concepts of ring theory or module theory. With some trivial qualifications to be made explicit in the text, I establish the following correspondences (N: Λ Noetherian only; CN: Λ commutative Noetherian only):

1-types over $\emptyset$	$\longleftrightarrow$	left ideals
Lascar-Poizat fundamental order	$\longleftrightarrow$	ideal lattice
independence	$\longleftrightarrow$	linear independence
product of types	$\longleftrightarrow$	intersection of ideals
orthogonality	$\longleftrightarrow$	no common prime factor (CN)
weight 1	$\longleftrightarrow$	indecomposable injective
strongly regular type	$\longleftrightarrow$	prime ideal (CN)

regular decomposition of types $\longleftrightarrow$ Lesieur-Croisot tertiary decomposition of ideals (N)

In fact I will show that two classical theorems of Noetherian rings — the Matlis theorem that every injective module over a Noetherian ring can be written uniquely as a direct sum of indecomposable injectives, and the Lesieur-Croisot theorem extending the Lasker-Noether theorem on the primary decomposition of an ideal — follow directly from the stability theory of T^*_Λ.

In section 1 I gather together the basic facts about T^*_Λ from [ES] and the basic facts about injective modules needed for this chapter. In section 2, I develop the "easy" stability theory of T^*_Λ, including the first four correspondences from the list above. In section 3 I characterize orthogonality, weight, and regular types. I show that if Λ is Noetherian then T^*_Λ is basic, and I show how to find sr types. In section 4 I use these results to prove decomposition theorems, including the two mentioned above.

These results form the basis of my abstract [K2] and represent work done mostly in 1978-1980. The abstract contains a misstatement which will be pointed out where appropriate (3.9). In addition, some of these results, essentially those of section 2, were discovered independently by E. Bouscaren and reported in her thesis [Bo 1]. Some of the terminology of section 2 is borrowed from B. Poizat's description of theories of linearly closed vector spaces with operator in his thesis [Po, Ch 10, Ex 8].

Subsequent and for the most part independent work by other researchers has greatly extended the scope of these results. See especially the work of Ziegler [Z] and Prest [Pr 1, Pr 2], both of whom saw an early version of these results. My own work extending the results of this chapter will be reported in Chapter III. In addition, where appropriate in these two chapters, I will discuss the subsequent work of other researchers which relates to my own results, and I will summarize all such references at the end of Chapter III.

II 1. THE THEORIES T_Λ^* AND INJECTIVE MODULES

1.0 REMARKS: I assume that the elementary definitions and theorems in the theory of rings, ideals and modules, particularly injective modules, are known, for which I refer the reader to the books of Northcott [N] and Sharpe and Vamos [SV] cited in the bibliography. In what follows, I emphasize those definitions and theorems which are the most significant for this chapter. For the most part I do not give proofs.

1.1 DEFINITION/THEOREM:

(i) Λ is a ring with unit. λ, μ etc. denote elements of Λ . I, J, P, Q etc. denote left ideals of Λ . All ideals are assumed to be one sided unless explicitly stated otherwise. $I(\Lambda)$ is the lattice of left ideals of Λ . M, N, etc. denote unitary left Λ-modules (for simplicity, just "Λ-modules"). If $A \subset M$, $((A))$ denotes the submodule generated by A , so in particular if $A \subset \Lambda$, $((A))$ is the left ideal generated by A .

(ii) [ES] The first order language $L(\Lambda)$ for the theory of Λ-modules has the symbols for the language of abelian groups (constant 0, unary -, binary +) and unary operation symbols $\lambda(\cdot)$ for each $\lambda \in \Lambda$. The theory of unitary left Λ modules T_Λ has the obvious axiomatization in this language.

(iii) (a) A set of linear equations $\Sigma(\vec{x})$ in arbitrarily many variables $\vec{x}$ and parameters from the Λ-module M is consistent if it has a solution in some extension of M .

(b) Consistency is preserved by homomorphisms. That is, if $f: M \to N$ is a Λ-module homomorphism, $f(\sigma(\vec{x}, m_0, \ldots, m_{n-1}))$ is defined as $\sigma(\vec{x}, f(m_0), \ldots, f(m_{n-1}))$ for each linear equation σ over M , and

$\Sigma(\vec{x})$ is a consistent set of linear equations over M, then $f[\Sigma]$ is a consistent set of equations over N.

Algebraically, this is expressed as follows:

$$\begin{tikzcd}
N \arrow[r, hook, dashed, "g'"] & L \\
M \arrow[u, "f"] \arrow[r, hook, "g"'] \arrow[ur, phantom, "\circlearrowright"] & K \arrow[u, dashed, "f'"']
\end{tikzcd}$$

If g is an embedding of $M \to K$ then the diagram can be completed as shown, with g' an embedding. L is constructed as $N \oplus K / M_0$ where M_0 is the submodule $\{\langle f(m), -g(m)\rangle | m \in M\}$. If $\Sigma(\vec{x})$ has a solution $\vec{k}$ in K, then $f[\Sigma]$ has the solution $f'(\vec{k})$ in L.

(c) A set of linear equations $\Sigma(x) = \{\lambda_i x + a_i | i \in I\}$ in the single variable x is consistent iff for every finite $J \subset I$, $\{\mu_i | i \in J\} \subset \Lambda$, if $\Sigma_{i \in J} \mu_i \lambda_i = 0$ then $\Sigma_{i \in J} \mu_i a_i = 0$. [ES, Lemma 3.2].

(iv) A Λ-module M is <u>injective</u> if every consistent set $\Sigma(\vec{x})$ of linear equations over M has a solution in M. E, F, etc. are used to denote injective modules. The following are equivalent:

(a) E is injective.

(a') Every consistent set $\Sigma(x)$ of linear equations in one variable has a solution in E.

(b) If $f: M \to E$ is a homomorphism and $g: M \to N$ is an embedding, then there is $f': N \to E$ such that $f = f' \circ g$.

(c) E is a direct summand of every extension.

(v) If E is injective then every homomorphism between submodules of E can be lifted to an endomorphism of E.

(vi) A finite direct sum of injective modules is injective.

REMARKS: I choose the "logical" or "model-theoretical" form of the definition of injectivity rather than the categorical form (b). The equivalences of (iv) are all easy. I remind the reader of how to prove that (a) $\Rightarrow$ (b): Enumerate N as $\vec{n}$, let $\vec{x}$ be a sequence of distinct variables indexed by $\vec{n}$, let $\Sigma(\vec{x})$ be the set of all linear equations $\sigma(\vec{x})$ over M such that $N \models \sigma[\vec{n}/\vec{x}]$. Then Σ is consistent and so by (iiib) so is $f[\Sigma]$. Thus $f[\Sigma]$ has a solution $\vec{e}$ in E since E is injective. Define $f'\colon N \to E\colon \vec{n} \mapsto \vec{e}$. A similar proof of (b) from the weaker condition (a') can be given by induction, extending f to the elements of $N \setminus M$ one at a time.

(v) is an easy consequence of (ivb): Suppose M, N are submodules of E and $f\colon M \to N$. Consider the following diagram:

$$\begin{tikzcd}
E \arrow[ddrr, dashed, "f'"] & & \\
\cup & & \\
M \arrow[r, "f"'] & N \arrow[r, phantom, "\subset"] & E
\end{tikzcd}$$

Since $M \subset E$ is an embedding, f' exists by (ivb) and is the required endomorphism of E extending f.

1.2 DEFINITION/LEMMA:

(i) $I{:}\lambda ::= \{\mu \mid \mu\lambda \in I\}$

$I{:}J ::= \{\mu \mid \mu J \subset I\}$

$\mathrm{ann}(m) ::= \{\mu \mid \mu m = 0\}$ is the annihilator of m.

Remark: If Λ is not commutative, it is not necessarily true that $I{:}\lambda = I{:}((\lambda))$

(ii) $I:\lambda$, $I:J$, and $\operatorname{ann}(m)$ are all left ideals.

(iii) $I:\lambda = \Lambda$ iff $\lambda \in I$.

(iv) $I \subset J \Rightarrow I:\lambda \subset J:\lambda$

(v) $(I:\lambda)\lambda \subseteq I$; if I is two-sided then $I \subseteq I:\lambda$

(vi) $(I:\lambda):\mu = I:(\mu\lambda)$

$\operatorname{ann}(\lambda m) = \operatorname{ann}(m):\lambda$

REMARKS: All of these are obvious. The notation may be extended to a submodule M and subset A of some N $(M:A = \{\mu \mid \mu A \subseteq M\})$ so that $\operatorname{ann}(m) = (0):m$. The fact mentioned in (vi) is particularly useful, so I give the proof: $\mu \in \operatorname{ann}(\lambda m)$ iff $\mu(\lambda m) = 0$ iff $(\mu\lambda)m = 0$ iff $\mu\lambda \in \operatorname{ann}(m)$ iff $\mu \in \operatorname{ann}(m):\lambda$.

1.3 DEFINITION/THEOREM: See [SV Chapter 2.4]

(i) A module E is an injective envelope of a module M , and we write $E = E(M)$, if E is injective, M is embedded in E , and every embedding of M into an injective module F may be lifted to an embedding of E into F .

(ii) Every module M has an injective envelope, unique up to isomorphism over M .

(iii) N is an essential extension of M if $N \supset M$ and every $n \in N \setminus M$ satisfies a non-trivial equation over M (a trivial equation being $\lambda x = 0$ for some $\lambda \in \Lambda$) . In algebraic terms, this says that if $n \in N \setminus M$ then $M \cap ((n)) \neq \emptyset$.

E(M) is characterized as being a minimal injective extension, as a maximal essential extension, or as an injective essential extension of M . The fact that every $n \in E(M)$, $n \neq 0$ satisfies a non-trivial equation

over M will have a very important model-theoretic consequence (2.12).
(iv) $E(\cdot)$ commutes with finite direct sums, and if Λ is Noetherian, with arbitrary direct sums.

1.4 DEFINITION/THEOREM: See [SV, Chapter 2.5] and [Ma].
(i) E is indecomposable if $E \neq \{0\}$ and the only direct summands of E are $\{0\}$ and E.

$M \subset N$ is irreducible if $M \neq N$ and $M_1, M_2 \subseteq N$, $M_1 \cap M_2 = M$ implies that $M_1 = M$ or $M_2 = M$.
(ii) An injective module E is indecomposable iff $E \neq \{0\}$ and $\{0\} \neq M \subseteq E$ implies $E = E(M)$ iff $\{0\} \subseteq E$ is irreducible.

1.5 COROLLARY: (i) $E(M)$ is indecomposable iff $\{0\} \subseteq M$ is irreducible.
(ii) If $0 \neq e \in E$ an indecomposable injective Λ-module, then $E \cong E(\Lambda/\mathrm{ann}(e))$, and $\mathrm{ann}(e)$ is irreducible.
(iii) E is an indecomposable injective Λ-module iff there is an irreducible left ideal I of Λ such that $E \cong E(\Lambda/I)$.
(iv) If E is an indecomposable injective Λ-module, $e_0 \neq 0 \neq e_1$ are elements of E, then for some $\lambda_0, \lambda_1 \in \Lambda$, $0 \neq \lambda_0 e_0 = \lambda_1 e_1$, that is, any two non-zero elements of E satisfy a non-trivial equation.

1.6 REMARKS: (1.5(iv)) is used repeatedly without comment. Except for the existence of injective envelopes, the proofs of (1.3-1.5) are all straightforward. All proofs may be found in [SV].

The following material (1.7-1.13) is all taken from the paper of Eklof and Sabbagh [ES] where the proofs may be found. Most of these results are quite straightforward, among the first ones only (1.10(i)) requiring any great sophistication. (The proof of [ES 3.12] uses an ultraproduct construction.) The final theorem, however (1.13(i) or [ES 4.1, 4.8]) is much more complicated, the proof building on all the preceding model-theoretic and algebraic development. (1.10-1.12) are included primarily for information; (1.13) is used in a substantial way.

1.7 DEFINITION/THEOREM: [ES 3.13, 3.14]

The ring Λ is coherent iff it satisfies the following two equivalent conditions:

(i) The kernel of every homomorphism of $\Lambda^n \to \Lambda$ is finitely generated, for all $n < \omega$.

(ii) Every finitely generated left ideal is finitely presented.

REMARKS: Suppose Λ is coherent and $\{\lambda_i | i < n\} \subset \Lambda$. $f(\langle\mu_i\rangle_{i<n}) = \Sigma_{i<n}\mu_i\lambda_i$ defines a homomorphism $f: \Lambda^n \to \Lambda$ (and in fact all homomorphisms $\Lambda^n \to \Lambda$ have this form). Let $\{b_j | j < m\}$ be a generating set of $\ker(f)$, $b_j = \langle\beta_{ji}\rangle_{i<n}$, $\beta_{ji} \in \Lambda$. By (1.1 iiic) for any left Λ-module M and $\{a_i | i < n\} \subset M$, the system of equations $\{\lambda_i x + a_i = 0 | i < n\}$ is consistent iff $M \models \bigwedge_{j<m}(\Sigma_{i<n}\beta_{ji}a_i = 0)$. Thus if Λ is coherent, the consistency of a finite system of equations in x can be expressed by a single formula. (These remarks are the central part of the proof of the case (iii) $\Rightarrow$ (i) of [ES, Theorem 3.12] and show some of the "model-theoretic" content of coherency.)

1.8 DEFINITION/THEOREM: [ES 3.4, 3.5, 3.5½]

(i) Let α be a finite or infinite cardinal. M is α-injective if M satisfies the following equivalent conditions:

(a) Every consistent system of fewer than α equations "$\lambda x = m$" with $\lambda \in \Lambda$ and $m \in M$ has a solution in M .

(b) For every ideal I generated by fewer than α elements and every homomorphism $f\colon I \to M$, there is a homomorphism $f\colon \Lambda \to M$ extending f .

(ii) $\gamma(\Lambda)$ is the least (finite or infinite) cardinal γ such that every ideal of Λ is generated by fewer than γ elements. Clearly $\gamma \leqq |\Lambda|^+$ and Λ is Noetherian iff $\gamma(\Lambda) \leqq \aleph_0$.

(iii) M is α-injective for all cardinals $\alpha \geqq 2$ iff M is $\gamma(\Lambda)$-injective iff M is injective.

<u>1.9 PROPOSITION</u>: [ES 3.10] If $\alpha \leqq \aleph_0$, then the direct sum of α-injective modules is α-injective. Thus arbitrary direct sums of injective modules are $\aleph_0$-injective.

<u>1.10 THEOREM</u>: (i) [ES 3.12] The $\aleph_0$-injective modules form an elementary class (in the wider sense) iff Λ is coherent.

(ii) [ES 3.17] If Λ is coherent, every $\aleph_0$-injective module is an elementary substructure of an injective module.

<u>1.10 PROPOSITION</u>: [ES 3.18] The following are equivalent:

(i) Λ is Noetherian.

(ii) Every $\aleph_0$-injective module is injective

(iii) The injective modules form an elementary class (in the wider sense).

<u>1.12 REMARKS</u>: A module M is <u>absolutely pure</u> if it is a pure submodule of every extension. Recall that $M \subset N$ is <u>pure</u> if every finite system of equations $\Sigma(\vec{x})$ with parameters in M that is solvable in N is solvable in M . Thus M is absolutely pure if every finite consistent

set of equations $\Sigma(\vec{x})$ over M is solvable in M. Comparing this to (1.8(i)), we see that M is $\aleph_0$-injective iff every finite consistent set of equations $\Sigma(x)$ over M in one free variable x is solvable in M, so every absolutely pure module is $\aleph_0$-injective. Eklof and Sabbagh prove [ES, 3.23] that if Λ is coherent, then the converse holds. It is not known for which rings Λ that $\aleph_0$-injective implies absolutely pure.

When the definition of a compact (pure-injective) module is given in Chapter III, it will be obvious that M is injective iff M is compact and absolutely pure. In fact, from (1.1(iva')) it will follow that M is injective iff M is compact and $\aleph_0$-injective.

1.13 THEOREM:

(i) [ES 4.1, 4.8] T_Λ has a model completion T_Λ^* iff Λ is coherent; in fact T_Λ^* is obtained as $Th(M_0)$ where

$$M_0 = \oplus(E(\Lambda/I)^{(\aleph_0)} \mid I \in \mathcal{I}(\Lambda))$$

(ii) [ES 4.7(i)] Let Λ be coherent. Let M, N be $\aleph_0$-injective left Λ-modules which contain submodules elementarily equivalent to M_0. Let $f\colon A \to B$ be an isomorphism of finitely generated submodules of M, N respectively. Then $\langle M, a\rangle_{a\in A} \equiv \langle N, f(a)\rangle_{a\in A}$.

(iii) [ES 4.7(ii)] Let Λ be coherent. $E(M_0) \equiv M_0$. In fact, by (1.10(ii)) if M is any model of T_Λ^* then $M \prec E(M)$.

(iv) Let Λ be coherent. Every model of T_Λ^* is $\aleph_0$-injective. If $M \models T_\Lambda^*$ and N is any $\aleph_0$-injective left Λ-module, then $M \prec M \oplus N$.

(v) Let Λ be coherent. T_Λ^* has elimination of quantifiers.

REMARKS: (iv) follows from (1.10(i)) and (1.13(ii)). (v) follows since T_Λ^* is the model completion of the universal theory T_Λ .

1.14 REMARKS: For the remaining sections of this chapter, Λ is a coherent ring.

II 2. TYPES, STABILITY AND INDEPENDENCE IN T^*_Λ

2.0 DEFINITION: Let $M \subset N$ be Λ-modules, $a \in N$. An equation of a over M is a formula $\lambda x + m = 0$, $\lambda \in \Lambda$, $m \in M$, such that $N \models \lambda a + m = 0$. $f = Eq(a, M)$ is the set of all pairs $\langle\lambda, m\rangle$ such that $\lambda x + m = 0$ is an equation of a over M. $I ::= I(f) ::= \{\lambda | \langle\lambda, m\rangle \in f\}$. I write $Eq(a, M) = (I, f)$.

2.1 LEMMA: (i) The definition of $Eq(a, M)$ does not depend on N (insofar as $M \subset N$, $a \in N$).

(ii) A set $f \subset \Lambda \oplus M$ is equal to $Eq(a, M)$ for some a iff f is a submodule satisfying $f \cap (\{0\} \oplus M) = \{\langle 0, 0\rangle\}$ iff $I(f)$ is a left ideal of Λ and f is a homomorphism of I into M.

(iii) There is a 1-1 correspondence between sets of the form $Eq(a, M)$ and $S_1(M)$. That is, the sets $Eq(a, M)$ are exactly the 1-types over M.

PROOF: (i) Obvious.

(ii) That the first clause implies the second and that the second and third clauses are equivalent is obvious. Suppose $f \subset \Lambda \oplus M$ satisfies the conditions of the second clause. Let $N = (\Lambda \oplus M)/f$. Clearly M is embedded in N and $f = Eq(\langle 1, 0\rangle/f, M)$.

(iii) Suppose $Eq(a, M) = Eq(a', M)$. Then $((M \cup \{a\})) \cong_M ((M \cup \{a'\}))$ by the map $\alpha: m + \mu a \to m + \mu a'$. But T^*_Λ is the model completion of T_Λ, so α is elementary.

2.2 DEFINITION: $Eq(a, M)$ is called the module of equations of a over M.

(This definition extends the definition of Poizat [Po, Example 10.8].) (The notation (I, f) for a module of equations is the same as Bouscaren's [Bo, Bo1]).

<u>2.3 REMARKS AND NOTATION</u>: (2.0)-(2.2) clearly extend to types in many (possibly infinitely many) variables.

Since 0, $+$, and scalar multiplication are part of our language, for any set A $t(a, A) \vdash t(a, ((A)))$. Thus it suffices to consider types over modules, which I do from now on.

In light of the above, the following notational conventions are adopted for the balance of this chapter:

A type over a module M in α variables (α a finite or infinite cardinal) is determined by a pair (I, f) , I a submodule of $\Lambda^{(\alpha)}$, f a homomorphism of I into M . If M must be made explicit I write the type as (I, f, M) . A type over $\emptyset$ (that is, a type over the zero module $\{0\}$) is determined by I and says "$\mathrm{ann}(\vec{x}) = I$" where $\mathrm{ann}(\vec{a}) = \{\vec{\lambda} \in \Lambda^{(\alpha)} | \Sigma_{i<\alpha}\lambda_i a_i = 0\}$. Thus I freely confuse α-types over $\emptyset$ and submodules of $\Lambda^{(\alpha)}$. Clearly then if $p = (I, f)$ is a type, $p \restriction \emptyset$ is the kernel of f , $\ker(f) = \{\vec{\lambda} \in \Lambda^{(\alpha)} | f(\vec{\lambda}) = 0\}$. If p is a type, then p^+ is the set of equations in p . So if $p = (I, f)$, then $p^+ = \{\Sigma_{i<\alpha}\lambda_i x_i + f(\vec{\lambda}) = 0 | \vec{\lambda} \in I\}$.

(The very free use of the equality sign in expressions like "$p = (I, f)$" should cause no confusion.)

<u>2.4 LEMMA</u>: Let $M \subset N$. A type $q = (J, g, N)$ is an extension of the type $p = (I, f, M)$ iff $I \subset J$, $f = g \restriction I$, $I = g^{-1}[M]$.

PROOF: (I prove it for 1-types). Suppose $p \subset q$. Then "$\lambda x + m = 0$" $\epsilon\ p$ implies that "$\lambda x + m = 0$" $\epsilon\ q$, so $I \subset J$ and $f = g \upharpoonright I$. Also, "$\lambda x + m \neq 0$" $\epsilon\ p$ implies "$\lambda x + m \neq 0$" $\epsilon\ q$ so $I = g^{-1}[M]$. Conversely suppose that p, q satisfy the conditions of the lemma. Then "$\lambda x + m = 0$" $\epsilon\ q$, $m \in M$, imply $\lambda \in g^{-1}[M] = I$, hence $m = f(\lambda)$ so "$\lambda x + m = 0$" $\epsilon\ p$. If "$\lambda x + m = 0$" $\epsilon\ p$, then $\lambda \in I$ and $m = f(\lambda) = g(\lambda)$, so "$\lambda x + m = 0$" $\epsilon\ q$. By (2.1(iii)), $p \subset q$.

2.5 THEOREM:

(i) T^*_Λ is stable, in fact for any M

$$|S_1(M)| \leq |\mathcal{I}(\Lambda)|\,|M|^{<\gamma(\Lambda)}$$

so T^*_Λ is stable in every $\kappa \geq |\mathcal{I}(\Lambda)|$ such that $\kappa^{<\gamma(\Lambda)} = \kappa$.

(ii) T^*_Λ is superstable iff T^*_Λ is totally transcendental iff Λ is Noetherian.

REMARKS: In fact, every complete theory of modules is stable. See III 1.11.

PROOF: (i) By (2.1) $|S_1(M)| = \Sigma_{I \in \mathcal{I}(\Lambda)} |\mathrm{Hom}_\Lambda(I, M)|$. Let γ_I be the least cardinality of a generating set of $I \in \mathcal{I}(\Lambda)$, so $\gamma_I < \gamma(\Lambda)$. Thus $|\mathrm{Hom}_\Lambda(I, M)| \leq |M|^{\gamma_I} \leq |M|^{<\gamma(\Lambda)}$. So $|S_1(M)| \leq |\mathcal{I}(\Lambda)|\,|M|^{<\gamma(\Lambda)}$.

(ii) First I show that if Λ is not Noetherian, then T^*_Λ is not superstable. So let I be an ideal of Λ such that $I = ((\lambda_i | i < \omega))$ and with $I_n ::= ((\lambda_i | i < n))$, $I_n \subsetneq I_{n+1}$ for all $n < \omega$. Let η_n be the natural homomorphism $I \to I/I_n$. Pick $\kappa \geq |\mathcal{I}(\Lambda)| + |\Lambda|$ such that $\kappa^\omega > \kappa$. Let $M = \oplus_{n<\omega} (I/I_n)^{(\kappa)}$. Clearly $|M| = \kappa$. For $\alpha < \kappa$, let $\eta_{n,\alpha}$ be the projection of I onto the α-th copy of I/I_n. For

$f \in {}^{\omega}\kappa$ define $\phi_f\colon I \to M$ by $\phi_f(\lambda) = \Sigma_{n<\omega}\eta_{n,f(n)}(\lambda)$. ϕ_f is well defined since for any $\lambda \in I$ there is $N < \omega$ such that for every $m \geq N$, $\eta_m(\lambda) = 0$. (N is the least n such that $\lambda \in I_n$) . Clearly ϕ_f is a homomorphism. Now suppose $f \neq g \in {}^{\omega}\kappa$. Pick any $n < \omega$ such that $f(n) \neq g(n)$. Now $\lambda_n \notin I_n$, so $\eta_n(\lambda_n) \neq 0$. Thus the $\langle n, f(n)\rangle$ coordinate of $\phi_f(\lambda_n)$ is nonzero, whereas the $\langle n, f(n)\rangle$ coordinate of $\phi_g(\lambda)$ must be zero since $g(n) \neq f(n)$. Therefore there are $\kappa^{\omega} > \kappa$ homomorphisms of I into M , hence more than κ 1-types over M , so T^*_Λ is not superstable.

If Λ is Noetherian, then $\gamma(\Lambda) \leq \aleph_0$ and $|\mathcal{I}(\Lambda)| \leq |\Lambda| + \aleph_0$ so by part (i) T^*_Λ is stable in every cardinal $\kappa \geq |\Lambda| + \aleph_0$, i.e. T^*_Λ is superstable. Notice that by results of Shelah, this implies that T^*_Λ is tt if $|\Lambda| < 2^{\aleph_0}$. However, much more work is required to show that T^*_Λ is always tt when Λ is Noetherian.

First I show that if Λ is Noetherian then there are no proper descending sequences under $\to$ in T^*_Λ of conjunctions of equations. Let $(\phi_i(x, \vec{a}_i))_{i<\omega}$ be conjunctions of equations such that $\mathfrak{C} \models (\forall x)[\phi_j(x, \vec{a}_j) \to \phi_i(x, \vec{a}_i)]$ for $i \leq j < \omega$. Let I be the left ideal of Λ generated by all the coefficients of x appearing in the ϕ_j's . Since Λ is Noetherian, I is finitely generated, say by the coefficients of x appearing in $(\phi_i)_{i<n}$. Enumerate the equations appearing as conjuncts of $\Lambda_{i<n}\phi_i (\equiv \phi_{n-1})$ as $\langle\lambda_j x + b_j = 0\rangle_{j<N}$. I claim that for $k \geq n$, ϕ_j is a consequence of ϕ_{n-1} . Let $\nu x + c = 0$ be a conjunct of ϕ_k . Then $\nu = \Sigma_{j<N}\mu_j\lambda_j$ for some $(\mu_j)_{j<N} \subset \Lambda$. Thus $-c = \nu x = (\Sigma_{j<N}\mu_j\lambda_j)x = \Sigma_{j<N}\mu_j b_j$, since ϕ_k and ϕ_{n-1} are mutually consistent.

Therefore for any subring Λ_0 of Λ, $T^*_\Lambda \upharpoonright L(\Lambda_0)$ satisfies the same condition on conjunctions of equations. Now it is clear that if $\Lambda_0 \subset \Lambda$ is a countable subring, then there is a countable subring $\Lambda_1 \supset \Lambda_0$ such that $T_1 = T^* \upharpoonright L(\Lambda_1)$ has elimination of quantifiers (just close down under coefficients required to eliminate quantifiers in T^*_Λ). T_1 is a complete theory of modules and, just as in T^*_Λ, types over a Λ_1-module M are determined by the equations they contain. The condition in fact implies that every type is determined by finitely many equations, hence $|S_1(M)| \leq |M| + \aleph_0$ for all Λ_1-modules M. Therefore T_1 is ω-stable, hence T^*_Λ is totally transcendental.

2.6 THEOREM: (In what follows, $[\![a, b]\!]$ denotes a closed interval in a partially ordered set; * denotes the converse of a partial ordering; $\amalg$ denotes the disjoint union of partially ordered sets.)

The fundamental order (0, 1.3) of 1-types of T^*_Λ is isomorphic to $\amalg\{[\![I, \Lambda]\!]^* \mid I \in \mathcal{I}(\Lambda)\}$.

PROOF: Recall the following elementary facts from [LP; immediately following 2.1]: The minimal classes of the fundamental order are in 1-1 correspondence with the types over $\emptyset$, and distinct types over $\emptyset$ have no common extension. Since the 1-types over $\emptyset$ of T^*_Λ correspond to the left ideals of Λ, it will be enough to show that the fundamental order among the extensions of "ann x = I" is isomorphic to $[\![I, \Lambda]\!]^*$. By (2.4), the extensions of "ann x = I" have the form (J, g) where $I \subset J$ and $\ker(g) = I$, and all such occur by (2.1(iii)).

Suppose now that $M \models T^*_\Lambda$ and $p = (J, g)$ is such a type over M. A formula $\phi(x) :=: \lambda x + \Sigma_{j<n}\mu_j v_j = 0$ is represented in p iff $\lambda \in J$ and

$\delta\mu_j = 0$ for all $j < n$ implies that $\delta\lambda \in I = \ker g$. For clearly if ϕ is represented in p by $\langle m_j | j<n \rangle$, then $\Sigma_{j<n}\mu_j m_j \in M$ and so by the definition of $p = (J,g)$, $\lambda \in J$ and $\Sigma_{j<n}\mu_j m_j = g(\lambda)$. Now $\Sigma_{j<n}\mu_j v_j = g(\lambda)$ has a solution in M, a model T^*_Λ, iff $0 = \delta\mu_j$ for all $j < n$ implies that $0 = \delta g(\lambda) = g(\delta\lambda)$, that is, $\delta\lambda \in \ker(g) = I$. Thus the fact that ϕ be represented or not depends only on I and J, not on g.

A formula of the form $\neg\phi$ is always represented in p if $\Sigma_{j<n}\mu_j v_j$ can be made different from $g(\lambda)$ (if the latter is defined at all). If $\lambda \notin I$, then the choice of $v_j = 0$ for all $j < n$ will do, and so such a $\neg\phi$ is always represented. If $\lambda \in I$, $\neg\phi$ is represented in p iff $\Sigma_{j<n}\mu_j m_j \neq 0$ for some $\langle m_j | j<n \rangle$ in M, a condition independent of I, J and g.

Any formula $\psi(x,\vec{v})$ is equivalent to a positive Boolean combination of formulas of the form of ϕ and $\neg\phi$. Hence by the above arguments, the class of p in the fundamental order depends only on the pair $(J, I = \ker(g))$ and not on g itself. In particular, for the fundamental order between extensions of "ann $x = I$" it suffices to check only the formulas of the form of ϕ, since the representation of formulas of the form of $\neg\phi$ depends only on I, which is fixed. Clearly if $q = (J',g')$ is another such type over M, by the above characterization q represents more formulas like ϕ than p iff $J' \supset J$; so $q \leq p$ in the fundamental order iff $J' \supset J$ and $q \approx p$ iff $J' = J$.

<u>2.7 COROLLARY</u>: (i) Let $p = (J, g)$ be a type, $I = \ker g$. Then $U(p)$, the Lascar rank of p, is the ordinal $\upsilon(J)$ defined by: $\upsilon(J)$ is the least ordinal greater than $\upsilon(J')$ for all $J' \supsetneqq J$, if such exists, or ∞.

(ii) [Bo, Proposition 3] (J, g) is ranked by U iff the Λ-module Λ/J is Noetherian.

<u>2.8 REMARKS</u>: (i) By an argument identical to that of Theorem 2.6, it is easy to see that the fundamental order on the n-types of T_Λ^* is isomorphic to $\coprod\{[\![I, \Lambda^n]\!]^* | I \text{ a submodule of } \Lambda^n\}$.

(ii) In regards to (2.7) it is interesting to note the following result from Bouscaren's thesis [Bo, Proposition 4]: Let R denote Morley rank. $p = (J, g)$ is ranked by R iff J is finitely generated and Λ/J is Noetherian, in which case $R(p) = U(p)$.

<u>2.9 COROLLARY</u>: Let $M \subset N$. Then (J, g, N) is a non-forking extension of (I, f, M) iff $J = I$ and $g = f$; hence every type is stationary, and furthermore, $p = (I, f, M)$ is based on $f[I]$, so every type is based on a set of cardinality $< \gamma(\Lambda)$.

PROOF: (J, g, N) is a non-forking extension of (I, f, M) iff the two types have the same class and the former extends the latter. So by (2.6), $J = I$ and by (2.4), $g = f$. Thus (I, f, N) is the unique non-forking extension of (I, f, M) to N, so (I, f, M) is stationary. Clearly p is the non-forking extension of $(I, f, f[I])$ and $f[I]$ is generated by a set B of cardinality $< \gamma(\Lambda)$ since I is. Thus p is the unique non-forking extension of $p \restriction B$, i.e. p is based on B.

2.10 COROLLARY: The concepts of heir, representability, etc. (0, 1.3) are well defined over submodules M of $\mathfrak{C}$, not just over elementary submodels.

2.11 THEOREM: "Non-forking independence = linear independence"

(i) If A, B, C are Λ-modules then $B \underset{A}{\downarrow} C$ iff $(A + B) \cap (A + C) = A$. In particular, if $A \subseteq C$ then $B \underset{A}{\downarrow} C$ iff $B \cap C \subseteq A$.

(ii) Let $(A_i)_{i \in I}$ be Λ-modules. Then $\{A_i | i \in I\}$ is independent (over $\emptyset$) iff $\Sigma_{i \in I} A_i$ is direct. Hence if $\vec{a}_i \in A_i$ and $\Sigma_{i \in I} A_i$ is direct, then $\{\vec{a}_i | i \in I\}$ is independent.

(iii) If M is a Λ-module, $(\vec{m}_i)_{i \in I}$ are tuples in $\mathfrak{C}$, then $\{m_i | i \in I\}$ is independent over M iff for every family $\{\vec{\lambda}_i | i \in I\} \subseteq \Lambda$, $\vec{\lambda}_i$ of same length as $\vec{m}_i$, $\vec{\lambda}_i = \vec{0}$ for all but finitely many $i \in I$, $\Sigma_{i \in I} \vec{\lambda}_i \cdot \vec{m}_i \in M \Rightarrow \vec{\lambda}_i \cdot \vec{m}_i \in M$ for all $i \in I$.

Note: For finite sequences $\vec{\lambda} \in \Lambda$, $\vec{m} \in \mathfrak{C}$ of the same length, $\vec{\lambda} \cdot \vec{m} = \Sigma_j \lambda_j m_j$.

PROOF: (i) Without loss of generality $A \subseteq C$.

($\Leftarrow$) I must show that for every finite $\vec{b} \in B$, $t(\vec{b}, C)$ is the non-forking extension of $t(\vec{b}, A)$. Let $t(\vec{b}, C) = (I, f)$. By (2.9) I must show that $f[I] \subset A$. Let $n = \text{length}(\vec{b})$, so $I \subset \Lambda^n$. Let $\vec{\lambda} \in I$. Then $\vec{\lambda} \cdot \vec{b} + f(\vec{\lambda}) = 0$. Now $f(\vec{\lambda}) \in C$, and $f(\vec{\lambda}) = -\vec{\lambda} \cdot \vec{b} \in B$, so $f(\vec{\lambda}) \in B \cap C \subset A$.

($\Rightarrow$) Suppose $m \in B \cap C$ and $B \underset{A}{\downarrow} C$. Thus $t(m, C)$ is the non-forking extension of $t(m, A)$, and $t(m, C) = (\Lambda, f)$ with $f(1) = m$. By (2.9), $t(m, A) = (\Lambda, f)$, so $m \in A$.

(ii) Note that $\Sigma_{i \in I} A_i$ is generated by $\bigcup_{i \in I} A_i$. $\Sigma_{i \in I} A_i$ is direct iff for every $j \in I$, $(\Sigma_{i \in I \setminus \{j\}} A_i) \cap A_j = \{0\}$; iff for every $j \in I$, $(\Sigma_{i \in I \setminus \{j\}} A_i) \downarrow A_j$ iff for every $j \in I$, $\bigcup_{i \in I \setminus \{j\}} A_i \downarrow A_j$ iff $\{A_i | i \in I\}$ is independent. (iii) is similar to (ii). Let A_i be the module generated by $\vec{m}_i \cup M$ (so for any family of sequences $(\vec{\lambda}_i)_{i \in I}$ as described, $\vec{\lambda}_i \cdot \vec{m}_i \in A_i$). Now $(\vec{m}_i)_{i \in I}$ is independent over M iff $(A_i)_{i \in I}$ is independent over M iff $(\Sigma_{i \in I \setminus \{j\}} A_i) \cap A_j \subset M$ for all $j \in I$ (by (i)) iff for all families $(\vec{\lambda}_i)_{i \in I}$ as described, all $j \in I$, $n = \Sigma_{i \in I \setminus \{j\}} \vec{\lambda}_i \cdot \vec{m}_i \in A_j \Rightarrow n \in M$ and the latter is easily seen to be equivalent to the condition of (iii).

<u>2.12 COROLLARY</u>: Let A be any set. Then A dominates $E(A)$ over $\emptyset$, that is, for every B, $A \downarrow B \Rightarrow E(A) \downarrow B$.

(ii) Let $0 \neq b \in E(A)$. Then $b \not\downarrow A$.

<u>PROOF</u>: (i) WLOG A and B are modules. Suppose $E(A) \not\downarrow B$. Then by (2.11) $E(A) \cap B \neq \{0\}$. Since $E(A)$ is an essential extension of A, $A \cap B \neq \{0\}$ that is, $A \not\downarrow B$.

(ii) Immediate since $E(A)$ is an essential extension of A (1.3(iii)).

<u>2.13 DEFINITION</u>: Let $p_i = (I_i, f_i, M_i)$ $(i \in 2)$ be types. Then $p_0 \oplus p_1 ::= (I_0 \oplus I_1, f_0 \oplus f_1, M_0 + M_1)$.

<u>2.14 LEMMA</u>: $p_0 \oplus p_1$ is a type over $M_0 + M_1$ and $p_0 \oplus p_1 = (p_0|M_0 + M_1)) \otimes (p_1|M_0 + M_1))$ where $\otimes$ denotes the usual Lascar product of types (0, 1.5(iii)).

<u>PROOF</u>: Immediate by (2.9).

<u>2.15 COROLLARY</u>: Let p_i be 1-types over $\emptyset$ determined by the ideals I_i $(i \in 2)$. Let $q(x_0, x_1) := p_0(x_0) \otimes p_1(x_1)$ and let $\langle a_0, a_1 \rangle$ realize q . Then $t(a_0 + a_1, \emptyset) = I_0 \cap I_1$.

<u>REMARKS</u>: This apparently trivial result ties the product theory of types (regular decomposition, etc.) to the intersection theory of ideals, and so has very important consequences throughout the rest of this chapter.

<u>PROOF</u>: $\lambda_0 a_0 + \lambda_1 a_1 = 0$ iff $\lambda_0 \in I_0$ and $\lambda_1 \in I_1$ by (2.11(iii)). Hence $\lambda(a_0 + a_1) = 0$ iff $\lambda \in I_0 \cap I_1$.

<u>2.16 DEFINITION</u>: Let $p = I$ be an n-type over $\emptyset$. Let $A \subset \Lambda^n \setminus I$ p is <u>constrained by A</u> if for any type $J \supsetneq I$ over $\emptyset$, $J \cap A \neq \emptyset$.

p is finitely constrained if it is constrained by some finite set A .

Similarly, if $p = (I, f, M)$ is an n-type over M , $A \subset \Lambda^n \oplus M \setminus f$, then p is <u>constrained by A</u> if for any n-type $q = (J, g, M)$, $J \supsetneq I$ and $g \restriction I = f \Rightarrow g(\vec{\lambda}) = m$ for some $\langle \vec{\lambda}, m \rangle \in A$.

2.17 PROPOSITION: $p = (I, f, M)$ is isolated iff I is finitely generated and p is finitely constrained.

REMARK: The principal interest is for 1-types over $\emptyset$: "ann $x = I$" is isolated iff I is finitely generated and finitely constrained.

PROOF: By the elimination of quantifiers for T^*_Λ, p is isolated iff p is axiomatized by a finite set of equations and inequations over M, and this is exactly what the conditions says.

2.18 PROPOSITION: The only algebraic n-types over M are of the form $\{x_i = m_i | i < n\}$ for some $m_i \in M$.

PROOF: $p = (J, g, M)$ is algebraic iff $U(p) = 0$ iff $J = \Lambda^n$ (2.7), and Λ is a ring with unit.

2.19 PROPOSITION: Let Λ be Noetherian.

(i) Let $M \models T^*_\Lambda$, $q \in S_*(M)$, $\vec{b} \models q$, $\vec{b} \in M(q)$. Then $M(q) \cong_M E(M \cup \{\vec{b}\})$. In particular if $\vec{b} \downarrow M$ then $M(q) \cong_M M \oplus E(\vec{b})$.

(ii) As a consequence it makes sense to define $M(q)$ as above for an arbitrary injective module M, and with this definition (0, 1.24) holds for such M: Let $p \in S(M)$ be sr. Then for every q, p is realized in $M(q)$ iff $p \not\perp q$.

PROOF: (i) $E(M \cup \{\vec{b}\})$ is a model of T^*_Λ by (1.13). If q is realized in $N \succ M$, say by $\vec{b}'$, there is an embedding of $((M \cup \{\vec{b}\}))$ into N over M with $\vec{b} \mapsto \vec{b}'$. Since N is a model, it is injective (1.11), and by the definition of injective envelope there is an embedding of $E(M \cup \{\vec{b}\})$ into N. This embedding is elementary since T^*_Λ is model complete. The prime model over $M \cup \{\vec{b}\}$ is unique up to isomorphism, so $M(q) \cong_M E(M \cup \{b\})$.

The comment follows by the characterization of independence (2.11) and the fact that M is injective.

(ii) So for any injective M , $q \in S_*(M)$, I define $M(q) = E(M \cup \{\vec{b}\})$ for some $\vec{b}$ realizing q . Let $p \in S(M)$ be sr.

Let N be any model of T^*_Λ . Then $N \prec N \oplus M$. Let $p' = p|N \oplus M$, $q' = q|N \oplus M$. By (0, 1.24) p' is realized in $N \oplus M(q') \cong N \oplus E(M \cup \vec{b}')$ (some $\vec{b}' \models q'$) iff $p' \not\perp q'$ iff $p \not\perp q$. Since $N \mathop{\smile\!\!\!\!\!\!|}\, E(M \cup \vec{b}')$, if $\langle n, e\rangle \in N \oplus E(M \cup \vec{b}')$ realizes p' then so does $\langle 0, e\rangle$. To see this, let $p = (I, f, M)$. Since $p' = p|N \oplus M$, $p' = (I, f, N \oplus M)$ by (2.9). Let $t(\langle 0, e\rangle, N \oplus M) = (J, g, N \oplus M)$. It is enough to show that $J = I$, $g = f$. Take $\lambda x + a' \in p'$. Then $\lambda \in I$ and $a = \langle 0, m\rangle$ where $m = f(\lambda)$. Thus $\models \lambda\langle n, e\rangle + \langle 0, m\rangle = 0$ and so $\models \lambda e + m = 0$. Thus $J \supset I$ and $g \restriction I = f$. Let $\mu \in J$ and $g(\mu) = \langle n', m'\rangle$. Thus $\models \mu\langle 0, e\rangle + \langle n', m'\rangle = 0$ and so $\models \mu e + m' = 0$. Let $n'' = -\mu n$. Then $\models \mu\langle n, e\rangle + \langle n'', m'\rangle = 0$ and so $\mu \in I$. Thus $J = I$ and $e \in E(M \cup \vec{b}')$ realizes p .

II 3. ORTHOGONALITY, WEIGHT AND REGULARITY IN T^*_Λ

3.0 REMARKS: In (3.1) and (3.2) I make some trivial observations and introduce some notation which allows me to express the concept of orthogonality (0, 1.11) algebraically, and thus tie it in with the algebraic treatment of independence in section 2. This eventually results in some strong analogies between parts of stability theory and the ideal theory of Λ .

3.1 DEFINITION/LEMMA:

(i) Let $p_i \in S_{\vec{x}_i}(M_i)$ $(i \in 2)$ be types, $M \supset M_0 + M_1$, $q \in S_{\vec{x}_0 \hat{} \vec{x}_1}(M)$. q is compatible with (p_0, p_1) if for $i \in 2$, $q \restriction \vec{x}_i = p_i | M$.

(ii) Let (I_i, f_i) be m_i-types $(i \in 2)$, (J, g) an $m_0 + m_1$ type (so that $J \subseteq \Lambda^{m_0} \oplus \Lambda^{m_1}$) . (J, g) is compatible with $((I_0, f_0)$, $(I_1, f_1))$ if for $i \in 2$, $(J \cap \Lambda^{m_i}, g \restriction \Lambda^{m_i}) = (I_i, f_i)$.
(Note that this definition (ii) is independent of the co-domains of f_0, f_1, g).

(iii) Suppose $p_i = (I_i, f_i)$, $q = (J, g)$ as in (i), (ii). Then q is compatible with (p_0, p_1) iff (J, g) is compatible with $((I_0, f_0)$, $(I_1, f_1))$.

(iv) For types over $\emptyset$, I drop "f_0" and "f_1" in the notation, as they are both the zero homomorphism.

PROOF: (iii) is immediate from the definitions and (2.3).

3.2 PROPOSITION: (Notation as in 3.1)

$p_0 \perp p_1$ iff (J, g) compatible with $((I_0, f_0), (I_1, f_1))$ implies that $J = I_0 \oplus I_1$.

PROOF: Let M be any module $\supset M_0 + M_1$. The unique non-forking extension of $p_i = (I_i, f_i, M_i)$ to M is (I_i, f_i, M) by (2.9) $(i \in 2)$ and $(p_0|M) \otimes (p_1|M) = (I_0 \oplus I_1, f_0 \oplus f_1, M)$ by (2.14) and (2.9). Thus the condition of the proposition is seen to be a direct restatement of the definition of $\perp$ (0, 1.11).

REMARKS: (Same notation) Suppose in addition that $M_0 = M_1$. We say that p_0 and p_1 are weakly orthogonal $(p_0 \perp^w p_1)$ if $p_0(\vec{x}_0) \cup p_1(\vec{x}_1)$ is a complete type. If this is the case, then necessarily $(I_0, f_0) \oplus (I_1, f_1)|M$ is the only type over M compatible with $((I_0, f_0), (I_1, f_1))$, so $p_0 \perp p_1$. This contrasts strongly with the general stable case, where weak orthogonality implies orthogonality only over a-models, and even with the general tt case, where weak orthogonality implies orthogonality over models. In all the theories T^*_Λ this holds over arbitrary Λ-modules.

3.3 THEOREM: Let $p_0, p_1 \in S_1(\emptyset)$, $p_i = I_i$ $(i \in 2)$. Then $p_0 \perp p_1$ iff $(\forall\lambda_0)(\forall\lambda_1)\,[(I_0:\lambda_0 = I_1:\lambda_1) \to (\lambda_0 \in I_0 \vee \lambda_1 \in I_1)]$.

REMARKS: (i) Note that if $I_0:\lambda_0 = I_1:\lambda_1$ and $\lambda_0 \in I_0$ then necessarily $\lambda_1 \in I_1$ (1.3(iii)) so I could just as well write "$\wedge$" in the consequent of the right hand side of the equivalence.

(ii) This theorem generalizes easily to a many variable version but I will not need it: Let $p_i \in S_{m_i}(\emptyset)$, $p_i = I_i \subset \Lambda^{m_i}$ $(i \in 2)$. Then $p_0 \perp p_1$ iff $(\forall\vec{\lambda}_0\vec{\lambda}_1)\,[(I_0:\vec{\lambda}_0 = I_1:\vec{\lambda}_1) \to (\vec{\lambda}_0 \in I_0 \vee \vec{\lambda}_1 \in I_1)]$, where for $I \subset \Lambda^m$, $\vec{\lambda} \in \Lambda^m$, $I:\vec{\lambda} = \{\alpha \in \Lambda | \alpha\vec{\lambda} \in I\}$.

(iii) Continuing the policy of confusing 1-types over $\emptyset$ and ideals of Λ, I write $I_0 \perp I_1$ to mean that the types "ann $x = I_0$" and "ann $x = I_1$" are orthogonal.

(iv) Recall that in a commutative ring Λ we say that a prime ideal P belongs to an ideal I, or is an associated prime ideal of I, if for some λ, $P = I:\lambda$. (See, e.g., [SV, Theorem 4.16]). Thus, for a prime ideal P, P belongs to I iff $P \not\perp I$. See corollary (3.14) for a further application, as well as the material from (4.3) onwards.

PROOF: ($\Rightarrow$) Assume that $p_0 \perp p_1$ and that $I_0:\lambda_0 = I_1:\lambda_1$. Let $J = ((I_0 \oplus I_1 \cup \{<\lambda_0, \lambda_1>\}))$, let $g\colon J \to \{0\}$. (J, g) is compatible with (I_0, I_1), for a typical element of J is $<\mu_0 + \alpha\lambda_0, \mu_1 + \alpha\lambda_1>$ where $\mu_i \in I_i$ and $\alpha \in \Lambda$. If, say, $\mu_1 + \alpha\lambda_1 = 0$, then $\alpha\lambda_1 \in I_1$, hence $\alpha \in I_1:\lambda_1 = I_0:\lambda_0$ so $\mu_0 + \alpha\lambda_0 \in I_0$. Thus, since $p_0 \perp p_1$, by (3.2) $J = I_0 \oplus I_1$, that is, $\lambda_0 \in I_0$ and $\lambda_1 \in I_1$.

($\Leftarrow$) Suppose $p_0 \not\perp p_1$, so there is (J, g) compatible with (I_0, I_1) but $J \neq I_0 \oplus I_1$. Thus there is $<\lambda_0, \lambda_1> \in J \setminus I_0 \oplus I_1$. By compatibility $\lambda_0 \notin I_0$ and $\lambda_1 \notin I_1$. Now $\alpha \in I_0:\lambda_0$ iff $\alpha\lambda_0 \in I_0$ iff $<\alpha\lambda_0, 0> \in J$; and $\alpha<\lambda_0, \lambda_1> = <\alpha\lambda_0, \alpha\lambda_1> \in J$; so $\alpha \in I_0:\lambda_0$ iff $<0, \alpha\lambda_1> \in J$ iff $\alpha \in I_1:\lambda_1$.

3.4 COROLLARY: $I_0 \perp I_1$ iff $E(\Lambda/I_0)$ and $E(\Lambda/I_1)$ have no non-zero direct summand in common.

PROOF: ($\Rightarrow$) Suppose $I_0 \perp I_1$. Then by orthogonality, for any $a_i \in \mathfrak{C}$ with $\mathrm{ann}(a_i) = I_i$ $(i \in 2)$, $a_0 \downarrow a_1$, and so by (2.12) $E(a_0) \downarrow E(a_1)$. But $E(\Lambda/I_i) \cong E(a_i)$, $(i \in 2)$, so if the $E(\Lambda/I_i)$ have a non-zero direct summand E in common, then (by a suitable automorphism of $\mathfrak{C}$)

there is an isomorphic copy $E(a_1')$ of $E(\Lambda/I_1)$ with $E(a_0) \cap E(a_1') = E$. Thus by (2.11) $E(a_0) \not\downarrow E(a_1')$, contradiction.

($\Leftarrow$) Suppose $I_0 \neq I_1$. Then by (3.3), there are $\lambda_i \notin I_i$ $(i \in 2)$ such that $I_0:\lambda_0 = I_1:\lambda_1 = J$, say $(J \neq \Lambda)$. Therefore there is $e_i \in \Lambda/I_i$ with $\mathrm{ann}(e_i) = J$ $(i \in 2)$. Hence, each $E(\Lambda/I_i)$ has $E(\Lambda/J) \neq 0$ as a direct summand by the definition of injective envelope.

<u>3.5 THEOREM</u>: Let A be a Λ-module. $w(A)$ denotes the weight of A (0, 1.16) over $\emptyset$.

(i) $w(A) = 0$ iff $A = \{0\}$.

(ii) $w(A) = w(E(A))$.

(iii) If $0 \neq A$ is an indecomposable injective Λ-module then $w(A) = 1$.

(iv) If $A \supset \oplus_{i \in I} A_i$ with $A_i \neq 0$ for all $i \in I$, then $w(A) \geq |I|$.

(v) If $A \subset \oplus_{i \in I} A_i$, where $0 \neq E(A_i)$ are indecomposable, then $w(A) \leq |I|$.

<u>PROOF</u>: First I remind the reader that $A \subset A'$ implies that $w(A) \leq w(A')$ (0, 1.17(i)).

(i) $w(A) = 0$ iff A is algebraic $/\emptyset$ (0, 1.17(iii)) iff $A = \{0\}$ (2.18).

(ii) Since $A \subset E(A)$, $w(A) \leq w(E(A))$. Since A dominates $E(A)$ over $\emptyset$ (2.12), $w(E(A)) \leq w(A)$ (0, 1.17(v)).

(iii) By (i) $w(A) \geq 1$. By the definition of weight (0, 1.16) it suffices to show that $pw(A/B) \leq 1$ for all B such that $A \downarrow B$. So suppose that $A \downarrow B$ and $pw(A/B) \geq 2$. Then there are $C_0 \underset{B}{\downarrow} C_1$, with $A \underset{B}{\not\downarrow} C_i$ for $i \in 2$, and wlog $B \subset C_0$, C_1 are all modules. By (2.11)

$A \cap C_i \not\subseteq B$ so in particular $A \cap C_i \neq \{0\}$ $(i \in 2)$. Since A is an indecomposable injective, $A \cap C_0 \cap C_1 \neq \{0\}$ also (1.4(ii)). Again by (2.11), $A \cap B = \{0\}$ and $C_0 \cap C_1 \subset B$. Hence $A \cap C_0 \cap C_1 = \{0\}$, contradiction.

(iv) $w(A) \geq w(\oplus_{i\in I} A_i)$ and $\{A_i | i \in I\}$ is independent (2.11), so $w(\oplus_{i\in I} A_i) = \Sigma_{i\in I}\, w(A_i)$ (0, 1.18i). But $A_i \neq \{0\}$ so by (i) $w(A_i) \geq 1$ for all $i \in I$. Therefore $w(A) \geq |I|$.

(v) By (i), (ii), $w(A_i) = 1$ for each $i \in I$. $w(A) \leq w(\oplus_{i\in I} A_i)$ (0, 1.17i) , and $w(\oplus_{i\in I} A_i) = \Sigma_{i\in I}\, w(A_i)$ just as in (iv) so $w(A) \leq \Sigma_{i\in I}\, 1 = |I|$.

3.6 THEOREM: Let $p \in S_1(\emptyset)$, $p = I$.

(i) p is regular iff I is maximal among the annihilators of non-zero elements of $E(\Lambda/I)$ (I say that I is critical), in which case I is irreducible and $E(\Lambda/I)$ is indecomposable.

(ii) If I is finitely generated, then p is regular iff p is strongly regular iff p is strongly regular via the formula "ann $x \supset I$" .

(iii) If Λ is commutative then p is regular iff I is prime.

PROOF: Notice that if Λ is commutative then I is prime iff I is critical. (If Λ is commutative then I is prime iff $I{:}\lambda = \Lambda$ or $I{:}\lambda = I$ just as $\lambda \in I$ or $\lambda \notin I$. Now in Λ/I , $\mathrm{ann}(\lambda/I) = I{:}\lambda \supset I$. Since $E(\Lambda/I)$ is an essential extension of Λ/I , I is a maximal annihilator of non-zero elements of $E(\Lambda/I)$ iff I is a maximal annihilator of non-zero elements of Λ/I iff I is prime, by the first remarks). Thus (iii) follows immediately from (i).

The proof of (i) and (ii) contains several subtleties. Note that the description in (ii) of sr types holds in arbitrary T_Λ^*, not just the tt theories. The first thing I prove is that if I is critical and $\phi(x)$ is the set of formulas "ann $x \supset I$" then p and ϕ satisfy the characterization of sr types in tt theories (0, 1.22iv). I then imitate the usual proof that every sr type is regular (see, e.g. [L, proposition 6.3]) to show that p is regular. In fact, the proof shows more: it shows that p and ϕ satisfy the definition of sr type (0, 1.2(ii)). When I is finitely generated, $\phi(x)$ is equivalent to a single formula, so one direction of both (i) and (ii) is proved. For the converses, it suffices to show that if p is regular, then I is irreducible and critical.

Now assume that I is critical. Let $M \models T_\Lambda^*$ and let $N := M \oplus E(\Lambda/I)$; $M \prec N$ by (1.13(iv)). Clearly $p|M$ is realized in N by $\langle 0, 1/I\rangle$. I show that for any $\langle m, b\rangle \in N \setminus M$ (so $b \neq 0$), $\mathrm{ann}(\langle m, b\rangle) \supset I$ implies that $\langle m, b\rangle$ satisfies $p|M$. But $\mathrm{ann}(\langle m, b\rangle) = \mathrm{ann}(m) \cap \mathrm{ann}(b) \supset I$, so $\mathrm{ann}(b) \supset I$, and thus since I is critical and $b \neq 0$, $\mathrm{ann}(b) = I$. Hence $\mathrm{ann}(\langle m, b\rangle) = I$, that is, $\langle m, b\rangle$ satisfies p. I only have to check that $\langle m, b\rangle \downarrow M$. Suppose that $\lambda\langle m, b\rangle \in M$, that is, for some $\langle n, 0\rangle \in M$, $\lambda\langle m, b\rangle + \langle n, 0\rangle = \langle 0, 0\rangle$. Then $\lambda b = 0$, hence $\lambda \in I$, so $\lambda\langle m, b\rangle = \langle 0, 0\rangle$ and by (2.11), $\langle m, b\rangle \downarrow M$. Thus p and $\phi(x)$ satisfy the characterization (1.22(iv)) of sr types in tt theories.

Now I show that if $\phi(x) \subset q$, then $p \perp q$ or $p || q$. Since p is a stationary type over $\emptyset$, $p || q$ iff q is a non-forking extension of p. Assume that $\phi(x) \subset q$ and that q is either a forking extension of p or not an extension of p. I will show that $p \perp q$; comparing this situation with the definition (0, 1.21(i) and (ii)) this is enough to show that p is regular and, if I is finitely generated, that p is sr via $\phi(x)$.

Let $M \supset \mathrm{dom}(q)$; wlog M is an injective model of T^*_Λ (1.13(iii)). Let p_1, q_1 be the non-forking extensions of p, q respectively to M and let $N = M \oplus E(\Lambda/I)$. N is also an injective model of T^*_Λ (1.13(iv)) . p_1 is realized in $N \setminus M$ by $a ::= \langle 0, 1/I \rangle$, but not even q_1^+ is realized in $N \setminus M$. For $q_1^+ \supset$ "ann $x \supset I$" and by the first part of this proof, the only elements of $N \setminus M$ satisfying "ann $x \supset I$" are those satisfying p_1 . Let b realize q_1 . It suffices to show that $a \underset{M}{\downarrow} b$. By (2.11(iii)) to establish this I must show that if $\lambda a + \mu b + m = 0$ $(\lambda, \mu \in \Lambda, m \in M)$ then $\lambda a, \mu b \in M$. Let $\Sigma(y) ::= \{\lambda a + \mu y + m = 0\} \cup q_1^+(y)$. $\Sigma(y)$ is a consistent set of equations (satisfied by b) with parameters from the injective module N , so $\Sigma(y)$ is satisfied in N , say by c . But c satisfies $q_1^+(y) \supset$ "ann$(y) \supset I$" , so $c \notin N \setminus M$, that is, $c \in M$. Thus $\lambda a + (\mu c + m) = 0$ establishes that $\lambda a \in M$, and therefore also $\mu b \in M$. Thus $a \underset{M}{\downarrow} b$.

For the converses, suppose that p is regular. I is irreducible iff $E = E(\Lambda/I)$ is indecomposable (1.5(iii)). If $E \cong E_0 \oplus E_1$, $E_i \neq \{0\}$ $(i \in 2)$ then by (3.5(iv)), $w(E) \geq 2$. Now p dominates $t(E, \emptyset)$ (2.12), so by (1.17(v)), $w(p) \geq 2$ and p is not regular (0, 1.22(i)). Thus E is indecomposable and I is irreducible.

Now suppose that $J \supsetneqq I$, $0 \neq b \in E$ with ann $b = J$. Since E is essential over Λ/I , b and $1/I$ satisfy some non-trivial equation:

$0 \neq \lambda_0/I = \mu_0 b$ for some $\lambda_0, \mu_0 \in \Lambda$. In particular, $\mu_0 \notin J$. Consider $A = E(\Lambda/I) \oplus \{0\} \subseteq B = E(\Lambda/I) \oplus E(\Lambda/I)$. Let $a_0 = \langle 0, 1/I \rangle$ and $a_1 = \langle 1/I, b \rangle$. Let $p_i = t(a_i, A)$ $(i \in 2)$. Clearly p_0 is the non-forking extension of p to A but p_1 is a forking extension of p to A. ($\lambda a_1 = 0$ iff $\lambda \in I \cap J = I$ so p_1 extends p, but if $\lambda \in J \setminus I$ then $0 \neq \lambda a_1 \in A$). Therefore by the regularity of p, $p_0 \perp p_1$, hence $a_0 \underset{A}{\downarrow} a_1$. But $\mu_0 a_1 - \lambda_0 a_0 = \langle \mu_0/I, 0 \rangle \in A$, and $\lambda_0 a_0 = \langle 0, \lambda_0/I \rangle \notin A$, contradicting $a_0 \underset{A}{\downarrow} a_1$. Thus I is critical.

<u>3.7 REMARKS</u>: (i) Much more general characterizations of a similar nature (and with similar proofs) have been found independently and somewhat later by M. Prest [Pr2]. In addition I have characterized strongly regular types for certain tt theories of modules. This will be discussed in Chapter III where I will make some more comments about the connections with Prest's work.

(ii) The word "critical" to describe the ideals with the property described in the theorem is taken from a paper of Lambek and Michler [LM, proposition 2.7]. Many of the properties they discuss are seen to be manifestations of aspects of the theory of regular types in T^*_Λ. For instance, the concept of relatedness introduced just before proposition 2.2 of [LM] is just non-orthogonality (theorem 3.3), and proposition 2.2 itself will be proved below (Lemma 3.8(i)). Corollary 2.4 of [LM] is a direct consequence of the fact that $\not\perp$ is an equivalence relation on weight one types. Lemma (3.8(ii)) below gives proposition 2.8 of [LM]. All of these correspondences are immediate and direct. More remarks will be made later, as appropriate.

(iii) As a consequence of Theorem (3.6), I will refer to critical left ideals as regular ideals.

3.8 LEMMA: (i) Let I_0, I_1 be regular left ideals. Then $I_0 \not\perp I_1$ iff $E(\Lambda/I_0) \cong E(\Lambda/I_1)$.

(ii) Let I_0 be regular, $\lambda \notin I_0$. Then $I_0:\lambda$ is regular and $I_0:\lambda \not\perp I_0$.

PROOF: (i) ($\Rightarrow$) If $I_0 \not\perp I_1$ then by (3.3) there are λ_0, λ_1 such that $\Lambda \neq I_0:\lambda_0 = I_1:\lambda_1 = J$, say. Hence there are $e_i \in E(\Lambda/I_i)$ such that $\mathrm{ann}(e_i) = J$ $(i \in 2)$. But $E(\Lambda/I_i)$ is indecomposable since each I_i is regular, so $E(\Lambda/I_i) \cong E(\Lambda/J)$ $(i \in 2)$.

($\Leftarrow$) If $E(\Lambda/I_0) \cong E(\Lambda/I_1)$ then $I_0 \not\perp I_1$ by (3.4).

(ii) Let $I_1 = I_0:\lambda$. Then $I_1:1 = I_0:\lambda$ so $I_0 \not\perp I_1$ by (3.3). It is enough to see that I_1 is critical. Suppose $0 \neq y \in E(\Lambda/I_0)$ and $I_1 \subset \mathrm{ann}(y)$. Consider the map $f:((\lambda/I_0)) \to ((y)):\mu\lambda/I_0 \mapsto \mu y$. f is a well defined homomorphism since $\mathrm{ann}(\lambda/I_0) = I_0:\lambda = I_1 \subset \mathrm{ann}(y)$. By (1.1(v)) there is an endomorphism α of $E(\Lambda/I_0)$ extending f , and $y = f(\lambda/I_0) = \alpha(\lambda/I_0) = \lambda\alpha(1/I_0)$. Thus $\mathrm{ann}(y) = \mathrm{ann}(\alpha(1/I_0)):\lambda$. Since α is a homomorphism, $\mathrm{ann}(\alpha(1/I_0)) \supset \mathrm{ann}(1/I_0) = I_0$. But I_0 is critical. Therefore $\mathrm{ann}(\alpha(1/I_0)) = I_0$, so $\mathrm{ann}(y) = I_0:\lambda = I_1$. Thus I_1 is critical.

3.9 REMARKS: In my abstract [K2] reporting these results, the sentence "Any two such types [sr types] which are non-orthogonal are in fact equal" occurs. This is incorrect, and the sentence should begin "If Λ is commutative, then ...". The following sentence should begin "A representative set P of primes ..." instead of "The set P of primes ...". This does not affect the validity of the main result announced there, for which see section 4 of the current chapter.

<u>3.10 COROLLARY</u>: (See [LM, Theorem 2.13]) Let Λ be left Noetherian.
(i) If E is an indecomposable injective Λ-module, $0 \neq e \in E$, then for some λ, $\mathrm{ann}(e):\lambda$ is regular, that is, $0 \neq \lambda e$ has a (strongly) regular type.
(ii) If M is a direct sum of indecomposable injective Λ modules and $0 \neq b \in M$, then for some λ, $0 \neq \lambda b \in M$ has a strongly regular type.

<u>PROOF</u>: (i) Since Λ is Noetherian there is I, maximal among the annihilators of non-zero elements of E, and since E is indecomposable, $E = E(\Lambda/I)$. Therefore I is critical. But E is essential over Λ/I, so for some λ, μ, $0 \neq \lambda e = \mu/I$, and by (3.8(ii)) μ/I has regular type.
(ii) WLOG M is a finite direct sum of indecomposable injectives, $M = \bigoplus_{i<n} E_i$, and $b = \langle b_i \rangle_{i<n}$. <u>Claim</u>: There is μ such that $0 \neq \mu b$ and for every $i < n$, $\mathrm{ann}(\mu b_i) = \mathrm{ann}(\mu b)$ or $\mathrm{ann}(\mu b_i) = \Lambda$. Proof of claim: If $\Lambda \neq \mathrm{ann}(b_i)$, $\Lambda \neq \mathrm{ann}(b_j)$, and $\mathrm{ann}(b_i) \setminus \mathrm{ann}(b_j) \neq \emptyset$, pick $\mu_0 \in \mathrm{ann}(b_i) \setminus \mathrm{ann}(b_j)$. Then $\mu_0 b \neq 0$ (since $\mu_0 b_j \neq 0$) and $\mu_0 b$ has (at least) one more component $(\mu_0 b_i)$ equal to zero than b. So, by recursion, I find $\mu = \mu_0 \cdot \ldots \cdot \mu_m$ (some $m < n$) with the desired property. Suppose $\mu b_i \neq 0$. By part (i) there is λ such that $\lambda\mu b_i$ has regular type. By the choice of μ, $t(\lambda\mu b_i, \emptyset) = t(\lambda\mu b, \emptyset)$, that is, $\lambda\mu b$ has regular type.

<u>3.11 THEOREM</u>: Every non-algebraic type p is non-orthogonal to some 1-type p' over $\emptyset$. Furthermore, if p is regular (sr) then p' may be taken to be so, too.

<u>PROOF</u>: WLOG $p \in S_1(M)$ where M is an injective model of T_Λ^*, and p is realized in an injective model $M \oplus N$, say by $\langle m, n\rangle$ with $n \neq 0$. Let $p' = t(\langle 0, n\rangle, \emptyset)$. Clearly $\langle 0, n\rangle$ realizes $p'|M$ by (2.11), so it is enough to check that $\langle m, n\rangle \not\downarrow_M \langle 0, n\rangle$ which is again obvious by (2.11).

Now assume that p is regular and that p' is not. Let $I = \mathrm{ann}(n)$, so wlog $n = 1/I \in E(\Lambda/I)$. Since p' is not regular, I is not critical, and there is J, $\Lambda \neq J \supsetneq I$, and $b_0 \in E(\Lambda/I)$, $\mathrm{ann}(b_0) = J$. Consider $M \oplus E(\Lambda/I) \oplus E(\Lambda/I)$, a model of T_Λ^*, and elements $a = \langle m, 0, n\rangle$, $b = \langle m, n, b_0\rangle$. Clearly both satisfy p over M, and the former satisfies $p|M'$, where $M' = M \oplus E(\Lambda/I)$, while the latter satisfies a forking extension of p to M'. Thus by the regularity of p, $t(a, M') \perp t(b, M')$. But $E(\Lambda/I) = E(n)$ is an essential extension of Λ/I, so there is a non-trivial relation $\lambda n + \mu b_0 = 0$. Hence $\lambda a + \mu b \in M'$, $\mu b \notin M'$, i.e. $a \not\downarrow_{M'} b$, a contradiction. Clearly if p is sr via $\phi(x, \vec{c})$, $\vec{c} \in M$, then p' is sr via $\phi(x, \vec{0})$.

<u>3.12 THEOREM</u>: Let Λ be left Noetherian.

(i) If $M \subseteq N$ are injective modules, then the non-forking extension to M of some sr 1-type over $\emptyset$ is realized in $N \setminus M$.

(ii) Every type is non-orthogonal to some sr 1-type over $\emptyset$.

(iii) T_Λ^* is basic.

PROOF: Let $M \subseteq N$ be injective. Then $N = M \oplus M'$ for some (injective) M'. Let M_0 be any model of T^*_Λ. Then $M_0 \oplus M \prec M_0 \oplus M \oplus M'$ are models (1.13(iv)) and so there is an sr type realized between them (0, 1.22v). By (the proof of) 3.11, wlog it is the non-forking extension to $M_0 \oplus M$ of an sr 1-type over $\emptyset$. Applying the method of the proof of (3.11) once again, we find an sr 1-type over $\emptyset$ realized in $N \setminus M$.

Parts (ii) and (iii) follow immediately.

3.13 COROLLARY: Let Λ be Noetherian, let N be an injective Λ-module. Then N has an indecomposable direct summand.

PROOF: In (3.12(i)) take $M = \{0\}$, then apply (3.6).

3.14 COROLLARY: Let Λ be commutative Noetherian. Then $I_0 \perp I_1$ iff no prime ideal belongs to both I_0, I_1.

PROOF: Recall remark (iv) to Theorem 3.3. This corollary restates a basic property of $\perp$ and sr types in tt theories: see (0, 1.23(iii) and (iv)). (3.12) establishes that the sr 1-types over $\emptyset$ (i.e. the prime ideals of Λ) are enough.

3.15 THEOREM: Let Λ be Noetherian, M an injective Λ-module, $q \in S_1(M)$ a non algebraic type, $b \models q$. Then there are $(a_i)_{i<n}$, independent over $\emptyset$ and from M, each a_i definable from b over M such that $\langle t(a_i, \emptyset)\rangle_{i<n}$ is a sr decomposition of q (see 0, 1.23).

PROOF: Since M is injective, $E(M \cup \{b\}) = M \oplus N$ for some injective N. Let $b = \langle m, b'\rangle$ in $M \oplus N$. Since b' (i.e. $\langle 0, b'\rangle$) is definable from b over M, it will be sufficient to find $(a_i)_{i<n}$ as described, definable from b' (over $\emptyset$).

Let M_0 be any model of T_Λ^*, $M_0 \perp M \cup \{b\}$, so M_0 is injective (1.11). Let $M_1 = M_0 \oplus M$. Now $M_1 < M_1 \oplus N$ are models of T_Λ^* (1.13(iv)) and clearly b' realizes $q' = q|M$ by the same argument as in the proof of (2.19(ii)), and $M_1(q') = M_1 \oplus N$ by (2.19). By (0, 1.25) $M_1 \oplus N$ has an sr resolution over M_1, and by (3.12) I may take the types involved to be the appropriate non-forking extensions of sr 1-types $(p_i)_{i<n}$ over $\emptyset$, where $(p_i)_{i<n}$ is a sr decomposition of q. Thus, by a slight abuse of notation, $M_1 \oplus N = M_1(p_0)(p_1) \ldots (p_{n-1})$. If E_i is the indecomposable injective associated with p_i $(i < n)$ (3.6(i)), then by (2.19) $M_1 \oplus N \cong_{M_1} M_1 \oplus \bigoplus_{i<n} E_i$. Thus b' can be written as an element of $\bigoplus_{i<n} E_i$, that is, $b' = \langle b_i \rangle_{i<n}$.

Now I claim that $\mathrm{ann}(b') = \bigcap_{i<n} \mathrm{ann}(b_i)$ is an irredundant decomposition. For suppose, say, that $\mathrm{ann}(b_0) \supset \bigcap_{0<i<n} \mathrm{ann}(b_i)$. Let $b'' = \langle 0, b_i \rangle_{0<i<n}$. From $\mathrm{ann}(b') = \mathrm{ann}(b'')$ and $b' \perp M_1$, $b'' \perp M_1$ it follows that $t(b'', M_1) = t(b', M_1) = q'$. But I do not need the component E_0 to write b'', thus I have $M_1(q') = M_1(p_1) \ldots (p_{n-1})$, contradicting that $(p_i)_{i<n}$ is a sr decomposition of q $(q' = q|M_1)$.

Thus, for each $i < n$, there is $\mu_i \in (\bigcap_{j<n,\, j \neq i} \mathrm{ann}\ b_j) \setminus (\mathrm{ann}\ b_i)$. (In case $n = 1$ note that $\bigcap \emptyset = \Lambda$). Thus $\mu_i b_j \neq 0$ iff $i = j$. Now $0 \neq \mu_i b_i \in E_i$, an indecomposable injective, so by (3.10) there is λ_i such that $\lambda_i \mu_i b_i$ has strongly regular type. Let $a_i = \lambda_i \mu_i b'$. Therefore $t(a_i)$ is sr and $t(a_i) \not\perp p_i$, that is, $(t(a_i, \emptyset))_{i<n}$ is a regular decomposition of q definable from b over M.

<u>3.16 REMARKS:</u> This theorem picks out a "best possible" sr decomposition of the type q: all the types are over $\emptyset$, and the decomposition is

"definable". It is not claimed that every decomposition is definable in this way.

This is much stronger than what is true in the general case [Sh, Theorem V4.11]. Shelah's result requires that we work in $\mathfrak{C}^{eq.}$, and we only get "semi-regular" types.

II 4. INVARIANTS AND UNIQUE DECOMPOSITIONS

4.0 THEOREM: Let E be an indecomposable injective Λ-module, and M any Λ-module. Let $\{A_i | i \in I\}$ be a maximal independent family of submodules of M such that $E(A_i) \cong E$ for all $i \in I$. Then $|I|$ is an invariant of M.

In particular, if M can be written as a direct sum of indecomposable injective modules, its expression as such is unique up to order.

PROOF: By (3.5iii) $w(A_i) = 1$ for each $i \in I$, and by (2.12), $A_i \not\sim E$. Hence by (0, 1.18) $|I|$ is determined by M and E.

Suppose $M \cong \oplus_{i \in I} E_i \cong \oplus_{j \in J} F_j$ are two expressions of M as a direct sum of indecomposable injectives. From (3.4) it is easy to see that for indecomposable injectives E and F, $t(E, \emptyset) \not\sim t(F, \emptyset)$ iff $E \cong F$. It is also clear that $(\forall i \in I)(\exists j \in J)[E_i \not\perp F_j]$ hence $t(E_i, \emptyset) \not\sim t(F_j, \emptyset)$. Thus for any indecomposable injective E, by the first part of this theorem $|\{i \in I | E_i = E\}| = |\{j \in J | F_j = E\}|$ and there is thus a bijection $\alpha: I \to J$ such that $E_i \cong F_{\alpha(i)}$ for each $i \in I$.

4.1 COROLLARY: [Ma]. Let M be an injective module over a Noetherian ring. Then M may be written uniquely as a direct sum of indecomposable injective modules.

Furthermore, if P is a representative set of the $\not\sim$ classes of critical ideals (sr 1-types over $\emptyset$) the decomposition is given by

$$M \cong \oplus_{P \in P} E(\Lambda/P)^{(\alpha_P)}$$

where $\alpha_P = \dim(P, M)$.

PROOF: Let $\{E_i | i \in I\}$ be a maximal independent family of indecomposable injective submodules of M. $I \neq \emptyset$ by (3.13). Again, since Λ is Noetherian, $\Sigma_{i \in I} E_i$ is injective and thus a direct summand of M; thus by (3.13) and the maximality of the family, $M \cong \bigoplus_{i \in I} E_i$. By (4.0) this expression is unique.

Note that each E_i is $E(\Lambda/P)$ for some critical P, (3.6) and wlog I may choose $P \in \mathcal{P}$ (3.4), and finally $\dim(P, M) = \dim(E(\Lambda/P), M) = \alpha_p$ (2.12).

<u>4.2 REMARK</u>: Note that this corollary extends to all injective Λ-modules what the Basis Theorem says about models of T^*_Λ, namely that they are determined by the vector $\langle \dim(p, M) \rangle_{p \in \mathcal{P}}$.

Eklof and Sabbagh [ES 5.2] give conditions on $\dim(M)$ that determines whether or not the injective module M is a model of T^*_Λ.

The classical proof of this theorem (and others of a similar nature) appeals to the Krull-Remak-Schmidt-Azumaya Theorem by showing that if E is an indecomposable injective, then $\mathrm{End}_\Lambda(E)$ is a local ring. Here "weight one" serves the same purpose. I will have more to say about this in Chapter III.

<u>4.3 REMARKS</u>: In the remainder of this section I give a sketch of the relation between the regular decomposition of types (0, 1.23), (3.15) and the classical decomposition theorems for ideals in Noetherian rings (Lasker-Noether in the commutative case, Lesieur-Croisot in the non-commutative case. See [N] and [LC]). I give a sketch only, because I can prove the theorems in a more general context, which I do in Chapter III. However the results are presented here because the correspondences between the type theory and the classical theory of ideals in Noetherian rings is so striking.

I remind the reader of the simple result of corollary (2.15) which may be rephrased as follows: $a \downarrow b \Rightarrow \mathrm{ann}(a+b) = \mathrm{ann}(a) \cap \mathrm{ann}(b)$.

In what follows Λ is a Noetherian ring, I, J, P, Q etc. are left ideals, and in notation I seldom make any distinction between the ideal I and the 1-type over $\emptyset$ "ann $x = I$" .

Recall that by (3.6) P is regular iff P is strongly regular iff P is critical, and if Λ is commutative, P is regular iff P is prime. In this discussion, then, the regular types are made to play the role of prime ideals, P, Q are always assumed to be regular, and for the purposes of emphasizing the algebraic analogy, I refer to them as <u>pseudo-prime</u> ideals here.

<u>4.4 LEMMA</u>: (i) $\not\perp$ is an equivalence relation on the pseudo-prime ideals.
(ii) $P \not\perp I \cap J \Rightarrow P \not\perp I \otimes J \Leftrightarrow P \not\perp I$ or $P \not\perp J$.

<u>PROOF</u>: (i) The basic property (0, 1.20(iii)).
(ii) (2.15) and (0, 1.23).

<u>4.5 DEFINITION</u>: (Extending the algebraic analogy, and recalling remark (iv) to Theorem 3.3):

<u>I is P-primary</u> (and I say that P belongs to I) iff for every pseudo-prime Q , $(I \not\perp Q$ iff $P \not\perp Q)$. I is primary if it is P-primary for some P .

In general, P belongs to J iff $P \not\perp J$.

<u>4.6 LEMMA</u>: (i) $P \not\perp P' \Rightarrow$ (I is P-primary iff I is P' primary).
(ii) I is P-primary iff I is realized in $E(\Lambda/P)^n$ for some $n < \omega$.
(iii) I is P-primary iff $(\forall J)(I \not\perp J \Rightarrow P \not\perp J)$.

(iv) I is irreducible $\Rightarrow$ I is primary.

(v) A finite intersection of P-primary ideals is again P-primary; and conversely any P-primary ideal is a finite intersection of irreducible P-primary ideals.

PROOF: I just make the following comments: I is irreducible iff $E(\Lambda/I)$ is indecomposable (1.5(iii)). (i) is immediate, (ii) is the important one and uses the regular decomposition for ($\Rightarrow$). (v) follows immediately from (ii) and (2.15).

4.7 REMARKS: By (4.6) it is immediate that if Λ is commutative, then "I is P-primary" has its usual meaning. See [N].

4.8 LEMMA: Let I be P primary. Let $P^{ⓝ}$ be a regular decomposition of I. Then n is uniquely determined by I, $n = w(I)$. n is also of course the least possible n appearing in (4.6(ii)).

4.9 THEOREM: Let I be a left ideal. Then:

(i) $I = \bigcap_{i<m} I_i$ with I_i irreducible for $i < m$ and the decomposition is irredundant.

(ii) Let $\mathcal{P} = \{P_j | j < n\}$ represent the distinct $\not\sim$-classes of primes belonging to the I_i $(i < m)$. For each $j < n$, let $n_j = |\{i | I_i \not\sim P_j\}|$.

Then $\mathcal{P}$, $\langle n_j | j < n\rangle$ are uniquely determined by I.

PROOF: This is just the regular decomposition of I using an argument similar to that of (3.15). In brief, $E(\Lambda/I) \cong \oplus_{i<m} E(\Lambda/P_i)$, and $1/I = \langle b_i\rangle_{i<m}$. Then $I = \bigcap_{i<m} \operatorname{ann}(b_i)$ is the desired decomposition. The uniqueness results of (ii) follow from the uniqueness of regular decompositions.

4.10 COROLLARY: Let I be a left ideal. Then I has a normal decomposition, that is, $I = \bigcap_{j<n} J_j$ where for $j < n$, J_j is P_j-primary, for $j < j' < n$, $P_j \perp P_{j'}$, and the decomposition is irredundant.

Such a decomposition is uniquely determined up to the $\not\perp$-classes of prime ideals occurring in it and the weights of the corresponding primary ideals.

4.11 REMARKS: Theorem (4.9) is of course the classical Lasker-Noether (commutative) or Lesieur-Croisot (non-commutative) decomposition theorem for the ideals of a Noetherian ring, with some slight differences in the non-commutative case, as pointed out below. In addition, I have not seen the parts of (4.9) and (4.10) which refer to the weight stated in the usual treatments. (What this says is that even though the P-primary ideals occurring in a decomposition are not uniquely determined, the number of irreducible P-primary ideals is.)

I've had to abuse the classical terminology in the non-commutative case somewhat. The usual algebraic definition of "primary" can be generalized directly to the non-commutative case; this is not particularly useful. Lesieur and Croisot are forced to look for a stronger property and introduce "tertiary ideals". My uniform definition (4.5) gives the primary ideals in the commutative case and the tertiary ideals in the non-commutative case.

In the non-commutative case, regular ideals are not prime, even in the weak sense of prime left ideals. As noted before (3.7(ii)) they are exactly the critical ideals of Lambek and Michler [LM]. This paper gives several important properties of prime and critical ideals, which may be rephrased in stability theoretic terms. (In reading [LM] the reader is reminded to keep in mind that they talk about right ideals in a right Noetherian ring, whereas I work on the left.) The following numbers all refer to results from [LM]. Corollary 2.9 shows that a critical ideal P satisfies an algebraic property like primeness: $I \supsetneqq P$, $J \supsetneqq P \Rightarrow IJ \not\subset P$. By 2.13, every regular left ideal is non-orthogonal to some critical prime left ideal. Theorem 3.9 says that every two-sided prime ideal P is primary in my sense, and $w(P)$ is the "left Goldie dimension" of the ring Λ/P.

III. <u>TOTALLY TRANSCENDENTAL THEORIES OF MODULES: TYPES AND DECOMPOSITION THEOREMS</u>

III 0. <u>INTRODUCTION</u>

In this chapter I extend some of the results of Chapter II and in some cases provide proofs on a more elementary level. $S_1(\emptyset)$ is developed as an analogue of the ideal lattice of a ring.

In the first section I begin by putting things into the most general context possible, the <u>abelian structures</u> of Fisher [F]. These are many-sorted structures such that each sort is an abelian group and such that every atomic formula defines a (many-sorted) subgroup. Since "abelian structure" generalizes the idea of module, I refer to such structures as <u>abstract modules</u> or even just <u>modules</u>. When I want to refer to the usual concept, I say <u>ordinary module</u>. The associated many-sorted first order language L is called the <u>language of modules</u>. A particular example of a family of such structures is of course the class of all modules-over-a-fixed-ring-Λ as considered in Chapter II. Chapter IV will introduce a new example.

The elementary model-theoretic facts about modules over a fixed ring Λ actually depend only on the fact that positive primitive formulas (1.3) define subgroups, not on any explicit ring-theoretic properties of the ring Λ. Thus all of the standard theorems on pp-elimination of quantifiers, stability, and so on, as well as the characterizations of stability-theoretic concepts like independence, go through without modification. All of these basic results from the literature are gathered together in section 1, with appropriate explanations where necessary.

The two substantial results quoted are the pp-elimination of quantifiers (1.6) and Garavaglia's characterization of forking (1.19). The rest of the results from the literature are straightforward. Corollary 1.21 is one of the first facts I noticed about forking theory when I began this research. (1.24) and (1.25) are fairly straightforward consequences of Garavaglia's result which I find necessary for my particular approach to the problems of sections 3 and 4. My goal is always to use as much of "model-theoretic techniques" and as little of "algebraic techniques" as is possible. I will continually re-emphasize this point in the remarks I make. Because of the pp-elimination of quantifiers, the division between the two techniques is sometimes obscured.

In particular the treatment of independence is elementary throughout. By this I mean that I make only weak appeals to algebraic concepts. The main fact about tt modules is that they are equationally compact (= pure injective). Beyond this, I use very little: I do not need to appeal to the rather deep theory of compact hulls as developed by Prest, Ziegler and others in recent years. Thus the proofs presented here are at a more elementary level than the ones in Chapter II, where I do not hesitate to appeal to the well-established theory of injective hulls. In fact, I am able to recover the beginnings of the theory of compact hulls as consequences of abstract stability theory: the existence of unique prime models over any set in a tt theory is the central point. As to independence, the starting point is Garavaglia's 1980 characterization [G5] already mentioned. The more recent work of Pillay and Prest [PP] and Ziegler [Z] would apparently allow simpler proofs in some cases, but this is illusory, because the underlying theory of compact hulls is itself quite complicated.

The context of abelian structures naturally leads to questions about the effect of expanding a language for modules by a new relation defining a subgroup. In section 2 I briefly consider questions of this nature and show that adding a pure congruence relation does not change the stability classification.

In sections 3 and 4 I study totally transcendental theories of modules T, mostly under the assumption that T is closed under products. As will be seen, such an assumption is innocuous for the applications that I have in mind.

In section 3 I study regularity, weight, and indecomposable modules. I show that such a theory T is basic and use this to prove the existence, uniqueness, and minimality of the compact hull $H(q)$ of $q \in S_*(\emptyset)$ (3.5). I characterize those types $p \in S_1(\emptyset)$ which are strongly regular (3.7) and when $q \in S_*(\emptyset)$ has weight 1 ($\Leftrightarrow$ $H(q)$ is indecomposable) (3.10).

In section 4 I deal with decomposition theorems. First I prove Garavaglia's theorem [G4] that every tt module can be written uniquely as a direct sum of indecomposable modules (4.2, 4.3). My proof is entirely model-theoretic in nature, primarily appealing to weight and referring back to the Basis Theorem. In particular, I do not need to use the usual algebraic argument involving the Krull-Remak-Schmidt-Azumaya theorem. The existence of such a decomposition is essentially the fact that for any tt theory T, if $M \prec N \models T$ then there is an sr type over M realized in N; the uniqueness is the fact that dimension is well defined for weight 1 types. Pillay and Prest were aware that such a proof might exist; I thank them for encouraging me to complete and publish these results. Secondly, I characterize the decomposition of $H(q)$ in terms of the sr decomposition of $\underline{q}$ (4.4). Finally I prove a Lasker-Noether style "primary decomposition" for $S_1(\emptyset)$ (4.9, 4.10). This is again a consequence of the sr decomposition of types. By the remarks at the end of Chapter II, these results properly extend the classical Lasker-Noether primary decomposition and Lesieur-Croisot tertiary decomposition of ideals in a Noetherian ring.

At the end of section 4 I make some historical remarks and summarize the references made in Chapters II and III to the work of other people which overlaps or extends these results.

III 1. ABSTRACT MODULES, STABILITY AND INDEPENDENCE

1.0 DEFINITION: An abstract module (in the future I will mostly say just module) is an abelian structure in the sense of E. Fisher [F]. So a module is a many sorted structure $M = \langle\langle M_s\rangle_{s\in S}; \langle R_i\rangle_{i\in I}, \langle f_j\rangle_{j\in J}; Ad; Val\rangle$, where

(i) S is the set of sorts, $I \cap J = \emptyset$, $s \neq s' \in S \Rightarrow M_s \cap M_{s'} = \emptyset$.

(ii) Each R_i (f_j) is a (finitary) relation (operation) on $\bigcup_{s\in S} M_s$.

(iii) Ad is a function on $I \cup J$, with $Ad(k) \subset S^n$ for some $n < \omega$, for $k \in I \cup J$.

(iv) Val is a function on J with $Val(j){:}Ad(j) \to S$. ($Ad(k)$ is the set of admissible sort assignments for the n-ary relation R_k (the n-ary function f_k) (as the case may be). $Val(j)$ is the sort valuation map for the function f_j . For convenience I will write $Ad(R_i)$, $Ad(f_j)$, $Val(f_j)$ instead of $Ad(i)$, $Ad(j)$, $Val(j)$ respectively).

(v) Among the $(R_i)_{i\in I}$ is the relation "=" with $Ad(=) = \{\langle s, s\rangle | s \in S\}$. "=" is interpreted as equality.

(vi) Among the $(f_j)_{j\in J}$ are the functions "+", "-", and "0_s" ($s \in S$) with $Ad(+) = \{\langle s, s\rangle | s \in S\}$ and $Val(+)(\langle s, s\rangle) = s$; $Ad(-) = \{s | s \in S\}$ and $Val(-)(s) = s$; and $Ad(0_s) = \emptyset$ and $Val(0_s)() = s$ for each $s \in S$.

(vii) For each $s \in S$, $\langle M_s, +\restriction s, -\restriction s, 0_s\rangle$ is an abelian group.

(viii) Each R_i ($i \in I$) is additive, that is, assuming that $\vec{x}, \vec{y}$ are chosen from $\bigcup_{s\in S} M_s$ compatible with $Ad(R_i)$ and pointwise of the same sort, then $\vec{x}, \vec{y} \in R_i \Rightarrow \vec{x} + (-\vec{y}) \in R_i$, where the latter expression is to be evaluated co-ordinatewise.

(ix) Each f_j ($j \in J$) is a group homomorphism, that is, assuming $\vec{x}, \vec{y}$ are chosen from $\bigcup_{s\in S} M_s$ compatible with $Ad(f_j)$ and pointwise of the same sort, then $f_j(\vec{x} + (-\vec{y})) = f(\vec{x}) + (-f(\vec{y}))$.

1.1 DEFINITION: Associated with any abelian structure are the corresponding many-sorted language (with, in particular, sorted variables) and its many-sorted model theory. The rules for well-formed formulas of this language are quite standard and insist the Ad and Val be respected. Thus the language consists of predicate symbols $(R_i)_{i\in I}$; function symbols $(f_j)_{j\in J}$, Ad, Val and S and is provided with infinitely many variables of each sort $s \in S$.

1.2 REMARKS: For a full and very formal treatment of the many-sorted logic associated with abelian structures, the reader is referred to the first two sections of Fisher's article. The main point to keep in mind is that virtually all the differences with ordinary (1-sorted) model theory are notational rather than fundamental. The one significant difference is in fact one of the strongest motivations for a many-sorted approach: every element of a model has a unique sort, even if there are infinitely many sorts. Of course an interpretation of many-sorted logic into ordinary logic loses this property.

Natural examples of abelian structures abound. Among them are modules over a ring Λ , chain complexes of modules, additive group valued functors on an abelian category (for all of which, see [F]), and the examples arising from my study of topological modules in Chapter IV, which are new.

Abelian structures are the natural starting point for the study of the model theory of (ordinary) modules, for reasons which I will try to make clear now.

The fundamental fact about the first order logic of (ordinary) modules is the family of theorems due to Baur, Monk, and Garavaglia (1.6) which describes first order properties in terms of positive primitive formulas (1.3). Since parameter-free positive primitive formulas define subgroups (1.4) the proof of this fundamental fact lies entirely within the theory of abelian groups. Conditions (viii) and (ix) are all that are needed for (1.4) - and after that is established, the relations and functions $(R_i)_{i\in I}$, $(f_j)_{j\in J}$ disappear from the discussion and are seldom if ever explicitly mentioned again. Thus with a few exceptions, the reader will be able to treat all of the following material as if it applied only to ordinary modules. The only extra work involved in taking the many sorted approach are the definitions (1.0) and (1.1).

1.3 DEFINITION: A positive primitive formula (afterwards: ppf) is a formula of the form $(\exists\vec{y})\bigwedge_{k<n}\alpha_k(\vec{x}, \vec{y})$ for some n , where for each $k < n, \alpha_k$ is an atomic formula.

1.4 LEMMA: (i) If ϕ is a ppf then so is $(\exists x)\phi$. If ϕ and ψ are ppf's then $\phi \wedge \psi$ is logically equivalent to a ppf.
(ii) If $\phi(x)$ is a ppf, then ϕ is additive, that is for every abstract module M ,

$$M \models (\forall\vec{x}\vec{y})[\phi(\vec{x}) \wedge \phi(\vec{y}) \rightarrow \phi(\vec{x} - \vec{y})]$$

In particular if $\vec{x}$ is of sort $<s, \ldots, s> \in S^n$, then $\phi[\vec{M}_s] ::= \{\vec{m} \in M_s \mid \models \phi[\vec{m}]\}$ is a subgroup of M_s^n . In general, $\phi[\vec{M}]$ is a subgroup of some product $M_{s_1} \times \ldots \times M_{s_n}$.

(iii) For every $\vec{a}$, $M \models (\forall xy)[\phi(x, \vec{a}) \rightarrow (\phi(y, \vec{a}) \leftrightarrow \phi(x - y, \vec{0})]$ for ppf ϕ and $\vec{a}$ of the appropriate sorts. Thus, if x is of sort s , for every $\vec{a}$, $\phi[M_s, \vec{a})$ is either $\emptyset$ or a coset of $\phi[M_s, \vec{0})$ in M_s .

Hence $\phi(x, \vec{a})$, $\phi(x, \vec{b})$ are contradictory or equivalent.

(iv) ppf's factor across direct sums, that is, for every ppf $\phi(\vec{x})$, every pair of modules M, N, and every $\vec{m} \in M$, $\vec{n} \in N$ of the right sorts, $M \oplus N \models \phi[\overrightarrow{<m, n>}]$ iff $M \models \phi[\vec{m}]$ and $N \models \phi[\vec{n}]$.

(v) For ppf's $\phi(\vec{x})$, $\psi(\vec{x})$, let $\phi \cap \psi$ ::='$\phi \wedge \psi$', $\phi + \psi$ ::='$(\exists \vec{y}\vec{z})[\phi(\vec{y}) \wedge \psi(\vec{z}) \wedge \vec{x} = \vec{y} + \vec{z}]$', $\phi \subset \psi$ ::='$M \models (\forall \vec{x})[\phi(\vec{x}) \rightarrow \psi(\vec{x})]$'. Let $L_{\vec{x}}$ be the set of equivalence classes of ppf's in $\vec{x}$ (by the relation '$\phi \subset \psi \wedge \psi \subset \phi$') . Let $\underline{0}$ ::= "$\vec{x} = \vec{0}$" , $\underline{1}$ ::= "$\vec{x} = \vec{x}$" .

Then $\langle L_{\vec{x}}; \cap , +, \underline{0}, \underline{1} \rangle$ is a bounded modular lattice with induced order $\subset$.

PROOF: All of these are quite elementary and are left as an exercise.

1.5 DEFINITION: Let $\phi(x)$, $\psi(x)$ be ppf's over $\emptyset$. Then Ind(M, ϕ, ψ) is $(\phi[M]: \phi \wedge \psi[M])$ (the group theoretic index) if finite, "∞" otherwise. Note: if x is of sort s , then by (1.4(ii)), these formulas define subgroups of the abelian group M_s , so the definition makes sense: we are not trying to calculate the cardinality of a many-sorted object.

1.6 THEOREM: Monk [Mo], Baur [Ba].

(i) Let M, N be modules. Then $M \equiv N$ iff for all ppf's $\phi(x)$, $\psi(x)$, Ind(M, ϕ, ψ) = Ind(N, ϕ, ψ) .

(ii) If M is a module, $\phi(\vec{x})$ a formula in the language of M , then there is a Boolean combination of positive primitive formulas $\psi(\vec{x})$ such that $M \models (\forall \vec{x})[\phi(\vec{x}) \leftrightarrow \psi(\vec{x})]$.

1.7 REMARKS: This result was first proved by Monk in his thesis, for abelian groups. It was also proved, in whole or in part, independently by Baur, Garavaglia, and Martyanov. For a nice proof, the reader is referred to Ziegler [Z, Theorem 1.1]. (1.6(ii)) is the single most important fact about the model-theory of abstract modules. It is referred to as "the pp-elimination of quantifiers".

Since the standard proof eliminates quantifiers one at a time, and consists entirely of a group-theoretic/combinatorial argument, the very general context taken here has no effect whatsoever.

1.8 DEFINITION: Let $p = t(\vec{a}, M)$ be a complete type. (M some (abstract) module).

(i) $p^+ ::= t^+(\vec{a}, M) ::= \{\phi \mid \phi \text{ ppf}, \phi \in p\}$

(ii) $p^- ::= t^-(\vec{a}, M) ::= \{\neg\phi \mid \phi \text{ ppf}, \neg\phi \in p\}$

(iii) $p^\pm ::= t^\pm(\vec{a}, M) ::= p^+ \cup p^-$

1.9 LEMMA: (i) p^+ uniquely determines p^- and conversely.

(ii) $p^\pm \vdash p$

PROOF: (i) p is complete, so for every ϕ, $\phi \in p$ or $\neg\phi \in p$.

(ii) By (1.6(ii)) every formula in p is equivalent to a disjunction of conjunctions of formulas in $p^\pm$ (since p is consistent and complete).

1.10 REMARKS: The intuition behind many of my results in sections 3 and 4 is that the pp 1-types over $\emptyset$ play the role of the (left) ideals of a ring. Certainly this was clear for the theories considered in Chapter II, where the correspondence was exact and explicit: a 1-type over $\emptyset$ says exactly "ann $x = I$" for some left ideal I.

It is interesting to note in passing that (in the case of ordinary modules over a ring Λ) it is possible to characterize syntactically those pp 1-types over $\emptyset$ which correspond to two-sided ideals:

<u>DEFINITION</u>: p^+ is <u>two-sided</u> iff $\phi(x) \in p^+ \Rightarrow \phi(\lambda x) \in p^+$ for all $\lambda \in \Lambda$. Clearly this is exactly the right condition for the theories considered in Chapter II, and is a reasonable extension to arbitrary theories of ordinary modules. In such a case, clearly $p^+[M]$ is a submodule of M. In the general case of a complete theory T of abstract modules, I offer the following:

<u>DEFINITION</u>: Let s be the sort of x. $p^+(x)$ is <u>two-sided</u> if for every $\phi \in p^+(x)$ and term t of L such that $\langle s, \ldots, s\rangle \in Ad(t)$, $Val(t)(\vec{s}) = s$, then $\phi(t(x, \ldots, x)) \in p^+(x)$. Similarly, T is <u>commutative</u> if for every sort s, every ppf $\phi(x)$, $x, y_0, \ldots, y_{n-1}$ of sort s, and every term $t(\vec{y})$ of sort s, $T \vdash (\forall\vec{y})[\bigwedge_{i<n}\phi(y_i) \rightarrow \phi(t(\vec{y}))]$. As a consequence each ppf $\phi(x)$, x of sort s, defines a substructure of M_s for each model M of T. An even stronger condition would be to require that every ppf $\phi(\vec{x})$ should define a substructure of each $M \models T$.

<u>1.11 THEOREM</u>: (i) Baur [Ba]; (ii), (iii) MacIntyre and Garavaglia [G3]. For a proof see [Z].

(i) All modules are stable.

Let M be a module.

(ii) M is superstable iff there is no infinite descending sequence of pp-definable subgroups of M, each of infinite index in its predecessor.

(iii) M is totally transcendental iff there is no infinite descending sequence of pp-definable subgroups of M.

<u>REMARKS</u>: These are all fairly straightforward from the definitions and

(1.6). It should be noted in (ii) and (iii) that we only need consider formulas in one variable. (iii) is the result of main interest in this chapter. Two interesting and useful ways of rephrasing it are the following (for T a complete theory of modules):

(iii') T is tt iff for each sort s, each variable x of sort s, there is no sequence $\langle\phi_i(x)\rangle_{i<\omega}$ of ppf's such that $T \vdash (\forall x)[(\phi_{i+1} \to \phi_i) \wedge \neg(\phi_i \to \phi_{i+1})]$ for each $i < \omega$.

(iii") T is tt iff for each sort s, each variable x of sort s, there is no sequence $\langle p_i(x)\rangle_{i<\omega}$ of complete types over $\emptyset$ in x such that $p_i^+ \subsetneqq p_{i+1}^+$ for each $i < \omega$.

Thus "T tt" is seen to be exactly a Noetherian condition on the pp 1-types over $\emptyset$. I exploit this to develop analogies between the ideal theory in a Noetherian ring and the type structure of T.

Also note that the conditions of (ii) and (iii) can easily be rephrased in terms of the elementary invariants (1.5):

The module M is not tt (ss) iff there is a sequence $(\phi_i(x))_{i<\omega}$ of ppf's, for all $i < \omega$ $M \models (\forall x)[\phi_{i+1} \to \phi_i]$, and for all $i < \omega$ $\mathrm{Ind}(M, \phi_i, \phi_{i+1}) \geq 2$ $(= \infty)$.

1.12 DEFINITION: [F]. Let M, N be modules, $f\colon M \to N$.

(i) f is a homomorphism if for every atomic formula $\alpha(\vec{x})$ and $\vec{m} \in M$, $M \models \alpha[\vec{m}] \Rightarrow N \models \alpha[f(\vec{m})]$.

(ii) f is an embedding if for every atomic formula $\alpha(\vec{x})$ and $\vec{m} \in M$, $M \models \alpha[\vec{m}] \Leftrightarrow N \models \alpha[f(\vec{m})]$

(iii) f is a pure embedding iff for every positive primitive formula $\phi(\vec{x})$ and $\vec{m} \in M$, $M \models \phi[\vec{m}]$ iff $N \models \phi[f(\vec{m})]$.

NOTATION: For (ii) I write $f: M \hookrightarrow N$ or if f is the inclusion map, just $M \subseteq N$. For (iii) with f the inclusion map I write $M \prec_{ppf} N$.

REMARKS: It follows from the definitions that in each case f is a sorted map, $f[M_s] \subseteq N_s$ for each $s \in S$.

If there are relation symbols in the language, a 1-1 homomorphism need not be an embedding.

Because of the pp-elimination of quantifiers, the concept of pure embedding will be of central importance. Notice that for any M, N, $M \prec_{ppf} M \oplus N$. Notice also that $M \prec N$ implies $M \prec_{ppf} N$.

1.13 PROPOSITION: (Ziegler [Z, Corollary 2.2]). Suppose $R \prec_{ppf} M$. Then M is ss (tt) iff both R and M/R are ss (tt).

REMARKS: Ziegler proves this by establishing the following two formulas in the case $R \prec_{ppf} M$:

(1) For each ppf ϕ, $\phi[M/R] = \phi[M]/R$.

(2) For every pair of ppf's ϕ, ψ, $\mathrm{Ind}(M, \phi, \psi) = \mathrm{Ind}(R, \phi, \psi) \cdot \mathrm{Ind}(M/R, \phi, \psi)$.

The result then follows easily by considering the characterization of ss (tt) in terms of the invariants.

1.14 DEFINITION: Let M be a module.

(i) M is compact (also: algebraically compact, equationally compact) if for every set $\Sigma(\vec{x})$ of atomic formulas with parameters in M ($\vec{x}$ may be infinite), $M \models (\exists\vec{x}) \bigwedge \Sigma(\vec{x})$ iff for every finite $\Sigma' \subseteq \Sigma$, $M \models (\exists\vec{x}) \bigwedge \Sigma'(\vec{x})$.

(ii) M is <u>pure-injective</u> iff for every N, N', homomorphism $f: N \to M$ and pure embedding $g: N \to N'$, there is a homomorphism $f': N' \to M$ such that $f = f' \circ g$.

$$\begin{tikzcd} M & \\ N \arrow[u, "f"] \arrow[r, "g"', "\prec \text{ppf}"'] & N' \arrow[ul, dashed, "\exists f'"'] \end{tikzcd}$$

<u>1.15 LEMMA</u>: [F, 3.1, 3.7] [Z, 3.1].

Let M be a module. The following are equivalent:

(i) M is compact.

(ii) M is pure-injective.

(iii) (1.14(i)) holds for $\Sigma(\vec{x})$ any set of ppf's with parameters in M.

(iv) M is a direct summand of every pure extension.

<u>REMARKS</u>: The proofs are all elementary and similar to those discussed after (III 1.1). In [F], Fisher gives a unified treatment of these different kinds of injectivity. For another discussion and many more references, see [W].

<u>1.16 THEOREM</u>: (Garavaglia [G3]). Let M be a module. The following are equivalent:

(i) M is totally transcendental.

(ii) $M^{(\kappa)}$ is compact for all cardinals κ.

(iii) $M^{(\aleph_0)}$ is compact.

<u>REMARKS AND PROOF</u>: I have stated this theorem in its full strength, but

all that I actually need is that every tt module is compact. This is easy, and I give the proof, modelled on [G3, Lemma 6].

Let $\Sigma(\vec{x})$ be a set of ppf's over M, finitely satisfiable in M. Enumerate $\vec{x}$ as $\langle x_\alpha \rangle_{\alpha<\kappa}$ for some cardinal κ. It is enough to find $m_0 \in M$ of the appropriate sort such that $\Sigma(m_0/x_0, \langle x_\alpha \rangle_{0<\alpha<\kappa})$ is finitely satisfiable in M, because then the desired solution may be constructed by recursion on $\alpha < \kappa$.

For each finite subset $\Sigma'(\vec{x})$ of Σ, the formula $\phi_{\Sigma'}(x_0, \vec{n}) ::= (\exists x_\alpha)_{0<\alpha<\kappa} \bigwedge \Sigma'(\vec{x})$ is actually (equivalent to) an ordinary finitary ppf over M, where $\vec{n}$ is the finite list of parameters from M occurring in Σ'. $\phi_{\Sigma'}[M, \vec{n})$ is a coset of $\phi_{\Sigma'}[M, \vec{o})$ by (1.4(iii)) (it is non-empty by the finite satisfiability of Σ in M). Since T is tt, there is no infinite descending sequence of subgroups $\phi_{\Sigma'}[M, \vec{o})$ (Σ' a finite subset of Σ), hence no infinite descending sequence of cosets $\phi_{\Sigma'}[M, \vec{n})$. Thus, since the family $\{\phi_{\Sigma'}[M, \vec{n}) \mid \Sigma' \text{ finite} \subset \Sigma\}$ is downwards directed by $\subset$, there is a smallest member ϕ_{Σ_0}: $\phi_{\Sigma_0}[M, \vec{n}_0) \subset \phi_{\Sigma'}[M, \vec{n})$ for all finite $\Sigma' \subset \Sigma$. Let $m_0 \in \phi_{\Sigma_0}[M, \vec{n}_0)$.

<u>1.17 COROLLARY</u>: Let N be tt, $M \prec_{ppf} N$. Then M is a direct summand of N. In particular, if $M \prec N$ then for some $A \subset N$, $N = M + A(\text{dir})$.

<u>PROOF</u>: $M \prec_{ppf} N$ implies that M is tt (1.13) and hence M is compact (1.16), so M is a direct summand of N (1.15).

REMARK: Thus if T is a tt theory of modules "A is a pure submodule of $\mathfrak{C}$" and "A is a direct summand of $\mathfrak{C}$" are equivalent.

1.18 REMARKS: In all of sections 3 and 4, the following assumption about a complete theory T of modules will be of central importance:

$(T = T^{\aleph_0})$: $\mathrm{Mod}(T)$ is closed under finite direct products.

By considering the invariants (1.6(i)) associated with T, it is easy to see that the following are all equivalent:

(i) $T = T^{\aleph_0}$

(ii) for any ppf's $\phi(x)$, $\psi(x)$ and $M \models T$, $\mathrm{Ind}(M, \phi, \psi)$ is 1 or ∞.

(iii) For any $M \models T$, $M \times M \models T$.

(iv) $\mathrm{Mod}(T)$ is closed under arbitrary direct sums.

(v) If $M \models T$, N is a direct summand of a model of T, then $M \oplus N \models T$.

In addition, if $T = T^{\aleph_0}$, $a \in M \models T$, then a is algebraic over $\emptyset$ iff $a = 0$ (for if $a \neq 0$ then $t^+(a, \emptyset)$ is realized infinitely often in $M^{(\aleph_0)} \models T$.

For many algebraic applications, the assumption $T = T^{\aleph_0}$ is innocuous, for if M is any module, then M is a direct summand of $M^{(\aleph_0)}$ and $T = \mathrm{Th}(M^{(\aleph_0)})$ satisfies $T = T^{\aleph_0}$. Since pp formulas factor across direct sums (1.4(iv)), most facts about M expressible in terms of pp formulas (and these are quite a few by (1.6)) can be dealt with in T just as easily as in $\mathrm{Th}(M)$. Again, by (1.4(iv)) and the characterization of tt modules (1.11(iii)), if M is tt then so is $M^{(\aleph_0)}$.

The reason for preferring theories that satisfy $T = T^{\aleph_0}$ is that the forking relation has an especially simple characterization. Along with

(v) above, this characterization will allow me to prove theorems of an algebraic nature about totally transcendental modules.

The characterization is due to Garavaglia [G5, Lemma 1] who only states a simple version of it, but his proof actually contains the stronger version as stated by Pillay and Prest [PP, Theorem 3.3]. The proof and very general statement of the result given here is due to Makkai, and is similar to the treatment of the topic by Ziegler [Z, 11.1, 11.2]. Note that the characterization applies in a broader context than theories of modules alone, and that the proof is of a much more elementary nature than that of Garavaglia or Pillay and Prest.

<u>1.19 THEOREM</u>: Assume that:

(i) T is a complete stable theory.

(ii) Every formula is equivalent (modulo T) to a Boolean combination of ppf's.

(iii) $M, N \models T \Rightarrow M \times N \models T$.

Then if $p \subset q$ are complete types, q is a non-forking extension of p iff $p^+ \vdash q^+$.

<u>PROOF</u>: It suffices to prove that for every A , every $p \in S(A)$, there is a unique $\overline{p} \in S(\mathfrak{C})$ such that $p \subset \overline{p}$ and $p^+ \vdash \overline{p}^+$. For if this is true, every A-automorphism of $\mathfrak{C}$ leaves $\overline{p}$ fixed ($\overline{p}^+$ determines $\overline{p}$ by (ii)), hence $\{\overline{p}\}$ is the unique small orbit of ideal extensions of p under A-automorphisms of $\mathfrak{C}$, that is, $\overline{p}$ is the unique non-forking extension of p to $\mathfrak{C}$. The result then follows immediately.

If the above statement is true, then $\overline{p}$ must be defined by $\overline{p} \supset \Sigma := p^+ \cup \{\neg\phi(\vec{x}, \vec{b}) \mid \vec{b} \in \mathfrak{C}, \; p^+ \nvdash \phi(x, \vec{b}), \; \phi \text{ ppf}\}$. I claim that Σ is consistent. If it is not consistent, then for some $\phi_i(\vec{x}, \vec{b}_i)$ such that $p^+ \not\vdash \phi_i(\vec{x}, \vec{b}_i)$, ϕ_i ppf $(i < n)$, $p^+ \vdash \bigvee_{i<n} \phi_i(\vec{x}, \vec{b}_i)$ (*). But $p^+ \not\vdash \phi_i(\vec{x}, \vec{b}_i)$ so there are $\vec{c}_i \in \mathfrak{C}$, $\vec{c}_i \models p^+$, $\models \neg\phi_i(\vec{c}_i, \vec{b}_i)$ for each $i < n$. Now since a ppf can be checked component by component across a direct product by (ii) and (iii) the diagonal embedding $\Delta: \mathfrak{C} \longrightarrow \mathfrak{C}^n$ is elementary. Let $\vec{c}$ be the sequence in $\mathfrak{C}^n$ whose i-th projection is $\vec{c}_i$. Now $\mathfrak{C}^n \models p^+[\vec{c}]$ and $\mathfrak{C}^n \models \neg\phi_i[\vec{c}, \Delta(\vec{b}_i)]$, the latter since it holds in the i-th projection. Therefore $\mathfrak{C}^n \models \neg\bigvee_{i<n} \phi_i[\vec{c}, \Delta(\vec{b}_i)]$, contradicting (*) since Δ is elementary.

<u>REMARK</u>: When we work over $\emptyset$ as in [G5] it is convenient to reformulate (1.19) as follows:

<u>1.20 COROLLARY</u>: [G5] $(T = T^{\aleph_0})$ $\vec{a} \not\perp \vec{b}$ iff for some ppf $\phi(\vec{x}, \vec{y})$ over $\emptyset$, $\models \phi[\vec{a}, \vec{b}] \wedge \neg\phi[\vec{a}, \vec{0}]$ (equivalently, $\models \phi[\vec{a}, \vec{b}] \wedge \neg\phi[\vec{0}, \vec{b}]$).

<u>1.21 COROLLARY</u>: $(T = T^{\aleph_0})$ Let a, b have the same sort. $a \perp b \Rightarrow t^+(a + b) = t^+(a) \cap t^+(b)$.

<u>PROOF</u>: Suppose $a \perp b$. $t^+(a + b) \supset t^+(a) \cap t^+(b)$ is automatic since ppf's are additive (1.4(ii)). On the other hand, suppose $\phi(x) \in t^+(a + b)$ Let $\psi(x, y) := \phi(x + y)$. Thus $\models \psi[a, b]$. By (1.20), $\models \psi[a, 0]$ and $\models \psi[0, b]$. That is, $\models \phi[a]$ and $\models \phi[b]$.

<u>REMARKS</u>: Just as in Chapter II, this corollary will allow me to establish a relationship between the stability-theoretic regular decomposition of types and an $\cap$-decomposition paralleling the Lasker-Noether primary decomposition of ideals. (1.21) clearly generalizes to arbitrary (possibly infinite) sequences $\vec{a}, \vec{b}$ as long as the sorts match.

<u>1.22 COROLLARY</u>: [G5] $(T = T^{\aleph_0})$. All types are stationary.

<u>PROOF</u>: This is immediate by the first paragraph of the proof of (1.19).

<u>1.23 PROPOSITION</u>: $(T = T^{\aleph_0})$. Suppose $N \prec_{ppf} \mathfrak{C}$, $A, N_0, N_1 \subset N$ and $N = N_0 + A + N_1$ (dir). Then $N_0 \underset{A}{\perp} N_1$.

<u>REMARKS AND PROOF</u>: Actually a much stronger version of this proposition is true, for which see [PP, Theorem 5.3], or, for $A = \emptyset$, [G5, Theorem 1]. However, the converse requires the theory of compact hulls which I avoid. The proposition that I give is quite an easy consequence of (1.19), as follows:

Let $b_i \in N_i$ $(i \in 2)$. I must show that $t(\vec{b}_0, A \cup \vec{b}_1)$ is a non-forking extension of $t(\vec{b}_0, A)$. By (1.19) it is enough to show that for every ppf $\phi(\vec{x}_0, \vec{x}_1, \vec{a})$ over A, if $\models \phi[\vec{b}_0, \vec{b}_1, \vec{a}]$ then there is a ppf $\psi(\vec{x}_0, \vec{a}')$ over A such that $\models \psi[\vec{b}_0, \vec{a}']$ and $\models (\forall x_0)[\psi(\vec{x}_0, \vec{a}') \rightarrow \phi(\vec{x}_0, \vec{b}_1, \vec{a})]$. For ψ I take the formula $\phi(\vec{x}_0, \vec{0}, \vec{a})$. Now since $N \prec_{ppf} \mathfrak{C}$, $N \models \phi[\vec{b}_0, \vec{b}_1, \vec{a}]$, and since ppf's factor across direct sums, $N \models \phi[\vec{b}_0, \vec{0}, \vec{a}]$, and so $\mathfrak{C} \models \psi[\vec{b}_0, \vec{a}]$. In particular, $\phi[\vec{\mathfrak{C}}, \vec{b}_1, \vec{a})$ and $\phi[\vec{\mathfrak{C}}, \vec{0}, \vec{a})$ are the same coset of $\phi[\vec{\mathfrak{C}}, \vec{0}, \vec{0})$, so $\mathfrak{C} \models (\forall \vec{x}_0)[\psi \rightarrow \phi]$ as required.

<u>1.24 PROPOSITION</u>: $(T = T^{\aleph_0}$, T tt). Let $p \in S_*(\emptyset)$, $M \models T$, $M(p|M) \cong_M M \oplus A$. Then p is realized in A.

<u>PROOF</u>: In fact I prove a slightly stronger statement: if $\overrightarrow{\langle m, a\rangle} \in M \oplus A$ realizes $p|M$, then so does $\overrightarrow{\langle 0, a\rangle}$.

First note that $M(p|M)$ has a representation as stated by (1.17).

Next note that if p is trivial (i.e. the type of $\vec{0}$) then $A = \{0\}$ and I am done. So assume p is non-trivial, let $\overrightarrow{<m, a>}$ realize $p|M$ in $M \oplus A$ with $\vec{a} \neq \vec{0}$. By (1.23) $M \downarrow \overrightarrow{<0, a>}$. Thus it suffices to prove that $t^+(\overrightarrow{<0, a>}) = p^+$. Fix some assignment $\vec{x}$ of variables of the appropriate sorts to $\overrightarrow{<m, a>}$. In each ppf $\phi(\vec{x})$ only finitely many of the variables in $\vec{x}$ actually occur of course.

Since ppf's factor across direct sums, for any ppf ϕ , $\models \phi[\overrightarrow{<m, a>}] \Rightarrow \models \phi[\overrightarrow{<0, a>}]$, so $t^+(\overrightarrow{<0, a>}) \supset p^+$. On the other hand, since $\overrightarrow{<m, a>}$ realizes $p|M$, $\overrightarrow{<m, a>} \downarrow M$ and so $\overrightarrow{<m, a>} \downarrow -\overrightarrow{<m, 0>}$. By (1.21), $t^+(\overrightarrow{<m, a>}) \cap t^+(-\overrightarrow{<m, 0>}) = t^+(\overrightarrow{<0, a>})$, and so $t^+(\overrightarrow{<0, a>}) \subset p^+$.

REMARKS: This proposition is in some sense new. It is almost vacuous if one accepts the theory of compact hulls as a prerequisite for studying independence, and both Prest [Pr 1, Pr 2] and Ziegler [Z] use something of this sort freely. This proposition is the first step in illustrating that the compact hull theory is not a prerequisite for the study of tt modules. This will be completed in section 3 where I show that A is uniquely determined by p and minimal over a realization of p (3.5). I then define the hull of p as $H(p) = A$.

1.25 LEMMA: $(T = T^{\aleph_0})$. (i) Suppose $(A_i)_{i \in I}$ are submodules of $\mathfrak{C}$, independent over $\emptyset$. Then $\Sigma_{i \in I} A_i$ is direct.
(ii) In addition suppose that T is tt and for each $i \in I$, $A_i \prec_{ppf} \mathfrak{C}$. Then $\Sigma_{i \in I} A_i \prec_{ppf} \mathfrak{C}$.

PROOF: (i) Assume $\Sigma_{i \in I} A_i$ is not direct. Then there are $(a_j)_{j \leq n}$ taken from distinct A_i's $(i \in I)$, each non-zero, but $\Sigma_{j \leq n} a_j = 0$.

Consider the ppf $\phi(x, \vec{y}) ::= "x + y_1 + \ldots + y_n = 0"$. I have $\models \phi[a_0, a_1, \ldots, a_n] \wedge \neg\phi[a_0, 0, \ldots, 0]$. Hence, by (1.20) $a_0 \not\downarrow \{a_1, \ldots, a_n\}$, contradicting the independence of $(A_i)_{i\in I}$.

(ii) Let $M \models T$, $\{M\} \cup \{A_i | i\in I\}$ independent. Then $N = M + \Sigma_{i\in I} A_i$ is direct by part (i), so it suffices to show that $N \prec \mathfrak{C}$.

Well order $\{A_i | i \in I\}$ as $\{A_\alpha | \alpha < \kappa\}$ and let $M_\beta \subseteq N$ be defined as $M_\beta = M + \Sigma_{\alpha<\beta} A_\alpha$ for each $\beta \leq \kappa$ (so $M_0 = M$, $M_\kappa = N$ and $(M_\beta)_{\beta<\kappa}$ is an increasing continuous chain). It is easy to see that this is in fact an elementary chain, by an induction using (1.18(v)) (taking note of the remark to (1.17)) at successor stages. Thus in particular, $N = M_\kappa \prec \mathfrak{C}$.

<u>REMARKS</u>: In regards to (ii) if $T \neq T^{\aleph_0}$ it is not always true that a direct sum of summands of $\mathfrak{C}$ is a summand of $\mathfrak{C}$. If T is not tt, it is not always true that a direct sum of compact pure submodules of $\mathfrak{C}$ is compact.

<u>1.26 COROLLARY</u>: ($T = T^{\aleph_0}$, tt) Suppose $A_i \prec_{ppf} \mathfrak{C}$ for each $i \in I$. Then $\langle A_i \rangle_{i\in I}$ is independent over $\emptyset$ iff $\Sigma_{i\in I} A_i$ is direct and pure in $\mathfrak{C}$.

<u>Proof</u>: Immediate by (1.23) and (1.25).

III 2. EXPANSIONS OF ABELIAN STRUCTURES

2.0 REMARKS: In this section I briefly consider some simple cases of the following question:

If M is tt(ss) and $(R_i)_{i \in I}$ and $(f_j)_{j \in J}$ are new additive relations and functions on the sorted structure M, what is the stability classification of the expansion $\langle M, (R_i)_{i \in I}, (f_j)_{j \in J} \rangle$ to the new language containing the appropriate relation and operation symbols?

Of course, if all of the R_i, f_j are definable in $\mathrm{Th}(M)$, the stability classification does not change. Clearly the expansion is no more stable that M itself because $\mathrm{Th}(M)$ is interpretable in the expanded theory. In fact, if we regard each relation R_i and each relation "$x = f_j(\vec{y})$" as abelian structures, we see that the theory of each such relation is interpretable in the theory of the expanded structure, so the new structure is no more stable than any of the new relations. Note that since the relation "$x = f_j(\vec{y})$" and the homomorphism f_j are interdefinable, we might as well assume $J = \emptyset$. All of these remarks are essentially trivial.

I am able to give two results on the preservation of the stability classification and a limiting counter-example. The two main tools are the characterization of stability for modules (1.11) and the corollary of Ziegler (1.13).

In (2.1) and (2.2) a pure substructure $R \prec_{ppf} M$ is used to induce a new relation on M. As a many sorted structure $R = \langle R_s | s \in S\rangle$, so as a relation $Ad(R) = \langle s | s \in S\rangle$, so as a relation $Ad(R) = \{s | s \in S\}$. $Ad(R)$ can be considerably restricted without changing anything by setting $Ad(R) = \{s | s \in S, R_s \neq \{0_s\}\}$. It is less confusing on first reading to assume that for some $s \in S$, $R \prec_{ppf} M_s$ and $Ad(R) = \{s\}$.

In (2.3) and afterwards for the sake of simplicity (and readability) I restrict myself to one sort at a time.

2.1 LEMMA: $R \prec_{ppf} M \Rightarrow \langle R, R\rangle \prec_{ppf} \langle M, R\rangle$.

PROOF: WLOG a ppf $\phi(\vec{x})$ in $L \cup \{R\}$ has the form $(\exists\vec{y}) \bigwedge_j \alpha_j(\vec{x}, \vec{y}) \wedge \beta(\vec{x}, \vec{y})]$ where each α_j is an atomic formula of L and β is a conjunction of expressions $R(x_i)$, $R(y_j)$. In the structure $\langle R, R\rangle$, $R(x)$ is always true. Suppose ϕ is a ppf with parameters in R satisfied in $\langle M, R\rangle$. Then $(\exists\vec{y}) \bigwedge_j \alpha_j(\vec{x}, \vec{y})$ is a ppf of L with parameters in R satisfied in M . Since $R \prec_{ppf} M$ it is witnessed in R , hence $\phi(\vec{x})$ is witnessed in $\langle R, R\rangle$.

2.2 THEOREM: $R \prec_{ppf} M \Rightarrow \langle M, R\rangle$ has the same stability as M .

PROOF: By (2.1) $\langle R, R\rangle \prec_{ppf} \langle M, R\rangle$. By (1.13) $\langle M, R\rangle$ is tt (ss) iff both $\langle R, R\rangle$ and $\langle M, R\rangle/\langle R, R\rangle$ are tt (ss) . Both of the latter structures allow elimination of "$R(x)$" :

$$\langle R, R\rangle \models (\forall x)[R(x) \leftrightarrow x = x]$$

$$\langle M, R\rangle/\langle R, R\rangle \models (\forall x)[R(x) \leftrightarrow x = 0] .$$

(Note that these actually refer to families of formulas, one for each sort admissible for R).

Thus the structures $\langle R, R\rangle$ and R are essentially the same, as are $\langle M, R\rangle/\langle R, R\rangle$ and M/R . But R and M/R are tt (ss) since $R \prec_{ppf} M$ and M is tt (ss) .

Hence $\langle M, R\rangle$ is tt (ss) .

2.3 DEFINITION: (i) $\theta(x_0, \ldots, x_{n-1})$ is a generalized pure congruence relation $(n \geq 2)$ if

(a) $\theta \prec_{ppf} M_s^n$ for some sort s .

(b) $(\forall x)\,\theta\,(x, x, \vec{0})$.

(c) For each permutation π on n ,

$$(\forall x_0, \ldots, x_{n-1})[\theta(x_0, \ldots, x_{n-1}) \rightarrow \theta(x_{\pi(0)}, \ldots, x_{\pi(n-1)})]$$

(For $n = 2$, θ really is a pure congruence relation. Furthermore when $n = 2$ (c) of the definition is unnecessary since symmetry and transitivity of $\theta(x_0, x_1)$ both follow easily from (a) and (b).)

(ii) For any generalized pure congruence relation $\theta \subset M_s^n$, $(n \geq 2)$, $\theta' \subset M_s^{n-1}$ is defined as $\{\langle x_1, \ldots, x_{n-1}\rangle \in M_s^{n-1} \mid \langle x_1, \ldots, x_{n-1}, 0\rangle \in \theta\}$. In general for $m \in \omega$, $0 \leq m < n$, $\theta^{(m)}$ is defined by $\theta^{(0)} = \theta$, $\theta^{(m+1)} = (\theta^{(m)})'$. Thus in particular $\theta^{(n-1)} \subset M$.

<u>2.4 LEMMA</u>: $(n \geq 2)$ Let θ be a generalized pure congruence relation, $\theta \subset M_s^n$. Then $\theta' \prec_{ppf} M_s^{n-1}$ and if $n \geq 3$, θ' is a generalized pure congruence relation.

<u>PROOF</u>: For simplicity of notation I check ppf's $\phi(x, y)$ in two variables only. Suppose $M^{n-1} \models (\exists x)\phi(x, a)$ where ϕ is a ppf and $a \in \theta'$, $a = \langle a_1, \ldots, a_{n-1}\rangle$. (I must show that $\theta' \models (\exists x)\phi(x, a)$.) Let $c = \langle a_1, \ldots, a_{n-1}, 0\rangle$. Since $a \in \theta'$, $c \in \theta$. I claim that $M^n \models (\exists x)\phi(x, c)$. But this is easy: if $M^{n-1} \models \phi[b, a]$ then $M^n \models \phi[b\frown 0, c]$. Since $\theta \prec_{ppf} M^n$, $\theta \models (\exists x)\phi(x, c)$. Let $b \in \theta$ witness this, $b = \langle b_1, \ldots, b_n\rangle$. Now by (b) and (c) of the definition, $\langle b_n, \vec{0}, b_n\rangle \in \theta$, hence, since θ is a subgroup of M^n , also $\langle b_1 - b_n, b_2, \ldots, b_{n-1}, 0\rangle \in \theta$. Now $M^n \models \phi[\langle b_1, \ldots, b_n\rangle, \langle a_1, \ldots, a_{n-1}, 0\rangle]$, hence, since ppf's factor across direct sums, $M \models \phi[b_i, a_i]$ for $1 \leq i < n$ and $M \models \phi[b_n, 0]$. Since ppf's are additive,

$M \models \phi[b_1 - b_n, a_1]$ and so $M^{n-1} \models \phi[\langle b_1 - b_n, b_2, \ldots, b_{n-1}\rangle, \langle a_1, \ldots, a_{n-1}\rangle]$. But $\langle b_1 - b_n, b_2, \ldots, b_{n-1}\rangle \in \theta'$.

Hence $\theta' \prec_{ppf} M^{n-1}$.

If $n \geqq 3$, conditions (b) and (c) of the definition are immediate for θ' , so θ' is a generalized pure congruence relation.

<u>2.5 LEMMA</u>: Let $\theta \subseteq M^n$ be a generalized pure congruence relation. $\langle M, \theta\rangle \models \theta[a_1, \ldots, a_n]$ iff $\langle M, \theta^{(n-1)}\rangle \models \theta^{(n-1)}[a_1 - a_2 - \ldots - a_n]$.

<u>PROOF</u>: The result follows by an easy induction once I establish that $\langle M, \theta\rangle \models \theta[a_1, \ldots, a_n]$ iff $\langle M, \theta'\rangle \models \theta'[a_1 - a_n, a_2, \ldots, a_{n-1}]$. But $\langle a_n, \vec{0}, a_n\rangle \in \theta$ by (b) and (c) of the definition, so if $\langle a_1, \ldots, a_n\rangle \in \theta$, then by (a), $\langle a_1 - a_n, a_2, \ldots, a_{n-1}, 0\rangle \in \theta$. Thus $\langle a_1 - a_n, a_2, \ldots, a_{n-1}\rangle \in \theta'$. These steps are reversible, so the claim is proved.

<u>2.6 COROLLARY</u>: Let $\theta \subset M^n$ be a generalized pure congruence relation. Then $\langle M, \theta\rangle$ and $\langle M, \theta^{(n-1)}\rangle$ have the same stability classification.

<u>PROOF</u>: By (2.5) the two structures are mutually interdefinable.

<u>2.7 THEOREM</u>: Let $\theta \subseteq M^n$ be a generalized pure congruence relation. Then $\langle M, \theta\rangle$ and M have the same stability classification.

<u>PROOF</u>: By (2.6) $\langle M, \theta\rangle$ and $\langle M, \theta^{(n-1)}\rangle$ have the same stability. But $\theta^{(n-1)} \prec_{ppf} M$ by (2.4) and so $\langle M, \theta^{(n-1)}\rangle$ and M have the same stability by (2.2).

<u>2.8 LEMMA</u>: Let M be a module, $f: M \to M$ a homomorphism, and θ the graph of f $(\theta = \{\langle m, f(m)\rangle | m \in M\})$. Then $\theta \prec_{ppf} M^2$.

PROOF: Suppose $\phi(\vec{x}, \vec{a})$ is a ppf with $\vec{a} = \overrightarrow{\langle a^0, f(a^0)\rangle}$ satisfied in M^2 by $\vec{b} = \overrightarrow{\langle b^0, b^1\rangle}$. Then since ppf's factor across direct sums, $M \models \phi[\vec{b}^0, \vec{a}^0]$. Since f is a homomorphism, $M \models \phi[\overrightarrow{f(b^0)}, \overrightarrow{f(a^0)}]$. Thus $M^2 \models \phi[\overrightarrow{\langle b^0, f(b^0)\rangle}, \overrightarrow{\langle a^0, f(a^0)\rangle}]$ and $\overrightarrow{\langle b^0, f(b^0)\rangle}$ is a sequence of elements of θ.

Thus $\theta \prec_{ppf} M^2$.

2.9 THEOREM, EXAMPLE: Let M, f, θ be as in (2.8) and in addition suppose $f: M \cong f[M]$.

(i) If f is not onto, then $\langle M, \theta\rangle$ is not tt.

(ii) If $M/f[M]$ is infinite then $\langle M, \theta\rangle$ is not ss.

(iii) Thus, in particular, taking $M = \mathbb{Q}^{(\omega)}$ and $f: M \to M$, the "right shift operator" defined as follows: for $m = \langle m_i\rangle_{i<\omega} \in \mathbb{Q}^{(\omega)}$, $(f(m))_0 = 0$, $(f(m))_i = m_{i-1}$ $(i > 0)$, I obtain an example of a tt module M and relation $\theta \prec_{ppf} M^2$ such that $\langle M, \theta\rangle$ is not even superstable.

PROOF: Let ϕ_n be the ppf $(\exists y_1, \ldots, y_n)\,[\theta(y_1, x) \wedge \bigwedge_{i=1}^{n-1} \theta(y_{i+1}, y_i)]$. $\phi_n(x)$ holds iff x is in the image of f^n, thus (since $f[M] \cong M$) $\mathrm{Ind}(M, \phi_n, \phi_{n+1}) = |M/f[M]|$. So (i) and (ii) follow immediately by the characterization of stability (1.11). For (iii), $\mathbb{Q}^{(\omega)}$ is clearly a tt abelian group (in fact it is ω_1-categorical) and in this case $M/f[M] \cong \mathbb{Q}$, so $\langle M, \theta\rangle$ is not superstable.

III 3. tt MODULES: REGULARITY, WEIGHT, INDECOMPOSABLES

3.0 REMARKS: Throughout this section T is a complete totally transcendental theory of (abstract) modules satisfying $T = T^{\aleph_0}$. L is the language of T and if I must refer to sorts, notation is as in (1.0, 1.1). Recall that all types are stationary (1.22), that if A is a summand of $\mathfrak{C}$ and M a model, then $M \oplus A$ is a model (1.18v) and that if $M \prec N$ are models then $N \cong_M M \oplus A$ for some A (1.17).

3.1 DEFINITION: (Compare II, 3.6(i)) Let $A \neq 0$ be a direct summand of $\mathfrak{C}$. $p \in S_1(\emptyset)$ is <u>critical in A</u> if p^+ is maximal (under $\subset$) in $\{t^+(a, \emptyset) | 0 \neq a \in A\}$. $p \in S_1(\emptyset)$ is <u>critical</u> if for some $A \prec_{ppf} \mathfrak{C}$, p is critical in A . Prest uses the same terminology [Pr 1, Pr 2].

3.2 THEOREM: (i) Every non-zero direct summand A of $\mathfrak{C}$ contains an element realizing a critical type.

(ii) Let $p \in S_1(\emptyset)$ be critical. Let $\phi \in p^+$ be minimal under $\rightarrow$, i.e. $\models \phi \rightarrow \psi$ for all $\psi \in p^+$. Then p is sr via ϕ .

PROOF: (i) is immediate by the characterization of tt theories of modules (1.11(iii")).

(ii) Such a ϕ exists because T is tt. Suppose p^+ is maximal in $\{t^+(a, \emptyset) | 0 \neq a \in A\}$, $p = t(a_0, \emptyset)$, $a_0 \in A$. Let $M \models T$. By (1.18v) $M \oplus A \models T$, in fact $M \prec M \oplus A$. By (0, 1.22(iv)) it is enough to show that any $a \in M \oplus A \setminus M$ satisfying ϕ satisfies $p_0 | M$.

So suppose $a \in M \oplus A \setminus M$ satisfies ϕ . Then $a = \langle m, b \rangle$ for some $m \in M$, $b \in A$, $b \neq 0$. Since ppf's factor across direct sums, $A \models \phi[b]$. Now ϕ is minimal in p^+ , thus $t^+(b, \emptyset) \supset p^+$. But

$0 \neq b \in A$, and p is critical, so $t^+(b, \emptyset) = p^+$, i.e. $t(b, \emptyset) = p$. Also, since ϕ is minimal in p^+ , and $\models \phi[a]$, $p^+ \subset t^+(a, \emptyset)$. But $a = \langle m, b \rangle$ so $t^+(a, \emptyset) \subset t^+(b, \emptyset) = p^+$. Therefore $t(a, \emptyset) = p$.

To check that $a \downarrow M$, I use (1.20). Let $\psi(x, \vec{y})$ be a ppf over $\emptyset$, $\overrightarrow{\langle n, 0 \rangle}$ elements of M , and suppose $\models \psi[a, \overrightarrow{\langle n, 0 \rangle}]$. Taking the second projection, $\models \psi[b, \vec{0}]$. But $t^+(b, \emptyset) = p^+ = t^+(a, \emptyset)$, hence $\models \psi[a, \vec{0}]$. Thus $a \downarrow M$.

Hence p is sr via ϕ .

<u>3.3 COROLLARY</u>: T is basic.

<u>PROOF</u>: Let $p \in S(M)$, $M \models T$. Let $N = M(p)$. Then $M \prec N$, and since T is tt, M is compact so for some A , $N \cong M \oplus A$. By (3.2) A realizes a sr 1-type q over $\emptyset$, say by a , and by (1.23) $a \downarrow M$. So $q|M$ is realized in $M(p)$. By (0, 1.24) $q|M \not\perp p$. Hence $q \not\perp p$, that is, every type is non-orthogonal to a sr 1-type over $\emptyset$.

<u>3.4 REMARKS</u>: The trick used in Corollary 3.3 to construct a type q over $\emptyset$ not orthogonal to p is actually much more general than the current situation would indicate. In his work (independent of mine) M. Prest uses this construction to great advantage (he calls it p_*) and as a consequence of Prest's work it is immediate that every tt theory of modules is basic, in fact in any complete theory of modules whatsoever, every type is non-orthogonal to a type over $\emptyset$, regular if the given type was regular.

The construction of (3.2) can also be made much more general as Prest's work shows. In the general case, a direct summand of $\mathfrak{C}$ need not realize a critical type, but if it does this type is regular, and every regular 1-type over $\emptyset$ is critical. The outline of the proof in the general case is the same, however it does seem to require some of the theory of compact hulls. See [Pr 2] for details.

3.5 THEOREM: (i) For any set A there is a unique (up to isomorphism) $B \prec_{ppf} \mathfrak{C}$, $A \subseteq B$, called the compact hull of A in $\mathfrak{C}$, $B = H(A)$, such that if $A \subseteq N \prec_{ppf} \mathfrak{C}$ there is a pure embedding $f\colon B \to N$ fixing A.
(ii) Let $p = t(\vec{A}, \emptyset)$. Then for any $M \models T$, $M(p|M) \cong_M M \oplus H(A)$. So in particular $H(A)$ can be recovered as $M(p|M)/M$ for some (any) $M \models T$. I write $H(p) = H(A)$.
(ii') Suppose $M \prec \mathfrak{C}$, $M \downarrow A$. Then there is a copy of $H(A) \supset A$ such that $M(A) = M + H(A)(\mathrm{dir})$.
(iii) $H(A)$ is minimal over A in the following sense:
$A \subset N \prec_{ppf} H(A) \Rightarrow N = H(A)$.
(iv) A dominates $H(A)$ over $\emptyset$.

PROOF: Fix $M \models T$. By (1.24) for some B, $M(p|M) \cong M \oplus B$ and wlog $A \subseteq B$. Since T is basic, $M(p|M)$ is minimal over $M \cup A$ (I, 2.3). Thus $A \subseteq B' \prec_{ppf} B$ implies that $B' = B$ (B' is a summand of $\mathfrak{C}$ by (1.17), hence $M \oplus B'$ is a model (1.18v) containing A). Thus (iii) is proved. Now let $N \models T$, $N(p|N) \cong N \oplus B'$, and again wlog $A \subseteq B'$. But $p|N$ is realized in $N \oplus B$, thus for some C, $N \oplus B' \oplus C \cong_{N \cup A} N \oplus B$. Since the isomorphism is over N, $B' \oplus C \cong_A B$, and so $A \subset B' \prec_{ppf} B$, so $B' = B$.

Let $H(p) ::= H(A) ::= B$. (ii) is proved.

For (i), suppose $A \subset N \prec_{ppf} \mathfrak{C}$. Let $M \models T$, $M \downarrow A$. By (ii) $M(A) \cong M \oplus H(A)$. Now $M \oplus N \models T$ and $A \subset N$, so $M(A) \prec M \oplus N$. Thus for some C, $M \oplus H(A) \oplus C \cong_{M \cup A} M \oplus N$ and so $H(A) \prec_{ppf} N$.

For (iv) I must show that $X \downarrow A \Rightarrow X \downarrow H(A)$. Pick some $X \downarrow A$, pick $M \models T$, $M \downarrow X \cup H(A)$. Then $X \underset{M}{\downarrow} A$. Now $M(A) = M \oplus H(A)$ and A dominates $M(A)$ over M (0, 1.14). Thus $X \underset{M}{\downarrow} H(A)$, and by the choice of M , $X \downarrow H(A)$.

(ii') is a simple rephrasing of (ii) in light of the characterizations of independence.

3.6 REMARKS: The hulls produced by (3.5) are the T-injective hulls of Prest [Pr 1]. They are also treated in a somewhat different fashion by Ziegler [Z] where all the elements of the theory of compact hulls are thoroughly explored. The idea was first developed fully by Fisher [F], and has been explored from a purely algebraic standpoint by several authors. For further references consult the three papers cited as well as the survey [W].

I do not need to use very much about compact hulls in what follows, but it would certainly be interesting to go through the papers of Prest and Ziegler and see how much can be rephrased in suitable terms and proved by entirely model-theoretic means.

One should note, however, that my model-theoretic approach depends strongly on the assumption that $T = T^{\aleph_0}$, T tt and so theorems like (3.4) are only very special cases of the results cited above.

3.7 THEOREM: Let $p \in S_1(\emptyset)$. Let $\phi \in p^+$ be minimal (under $\rightarrow$). The following are equivalent:

(i) p is sr via some ψ

(ii) p is sr via ϕ

(iii) p is critical in $H(p)$

(iv) p is critical.

<u>PROOF</u>: (iv) $\Rightarrow$ (ii) is exactly (3.2) and (ii) $\Rightarrow$ (i) is immediate, as is (iii) $\Rightarrow$ (iv).

[(ii) $\Rightarrow$ (iii)] Suppose $0 \neq c \in H(p)$, $t^+(c, \emptyset) \supset p^+$. Then $\phi \in t^+(c, \emptyset)$. Let $M \models T$, $M(p|M) = M \oplus H(p)$. Now $\models \phi[<0, c>]$, $<0, c> \notin M$, hence since p is sr via ϕ, $<0, c>$ satisfies $p|M$. Hence $t^+(c, \emptyset) = p^+$, that is, p is critical in $H(p)$.

[(i) $\Rightarrow$ (ii)] Suppose p is sr via ψ. Clearly if $\psi' \to \psi$ and $\psi' \in p$ then p is sr via ψ'. Since ϕ is minimal in p^+, wlog by the pp-elimination of quantifiers, ψ has the form $\phi \wedge \bigwedge_{j<n} \neg\alpha_j$ where the α_j's are all ppf's.

<u>Claim</u>: Let $b \in H(p)$, $b \neq 0$, $\models \phi[b]$. Then $t^+(b) = p^+$. The claim suffices to prove the theorem for the following reasons: Suppose $M \models T$, so $M(p|M) = M \oplus H(p)$, and $<m, c> \in M(p|M)$ satisfies ϕ. I must show that $<m, c>$ satisfies $p|M$. Since $<m, c>$ satisfies ϕ which is minimal in p^+, then $p^+ \subset t^+(<m, c>, \emptyset)$. But c also satisfies ϕ since ppf's factor through direct sums, so by the claim, $t^+(c, \emptyset) = p^+$. Thus $t^+(<m, c>, \emptyset) = p^+$. Suppose β is a ppf, $\overrightarrow{<n, 0>} \in M$, $\models \beta[<m, c>, \overrightarrow{<n, 0>}]$. Taking the projection on to $H(p)$, $\models \beta[c, \vec{0}]$, so $\beta \in t^+(c, \emptyset) = p^+ = t^+(<m, c>, \emptyset)$. Thus $\models \beta[<m, c>, \overrightarrow{<0, 0>}]$ and so by (1.20), $<m, c> \perp M$. Thus $<m, c>$ satisfies $p|M$.

Now I prove the claim. Let $M \models T$, $M' = M(p|M) = M \oplus H(p)$, and $N = M' \oplus H(p)$. Let $a \in M'$ realize $p|M$. Consider $<a, b> \in N$. Clearly $\models \phi[<a, b>]$. Since $\models \neg\alpha_j[a]$ for each $j < n$, $\models \neg\alpha_j[<a, b>]$ for each $j < n$. Therefore $\models \psi[<a, b>]$. But $N = M'(p|M')$ and p is sr via ψ, $b \neq 0$ (hence $<a, b> \notin M'$), so $<a, b>$ realizes $p|M'$.

Consider, for each $j < n$ the formula $\alpha_j(x - \langle a, 0\rangle)$, a ppf over M' . If $\models \alpha_j[\langle a, b\rangle - \langle a, 0\rangle]$, then, since $\langle a, b\rangle \downarrow M'$, $\models \alpha_j[\langle a, b\rangle - \langle 0, 0\rangle]$, a contradiction. Thus for each $j < n$, $\models \neg\alpha_j[\langle a, b\rangle - \langle a, 0\rangle]$, i.e. $\models \neg\alpha_j[\langle 0, b\rangle]$. But α_j is a ppf, so $\models \neg\alpha_j[b]$. Thus, since $\models \phi[b]$, I have $\models \psi[b]$.

Now, in $M' = M(p|M) = M \oplus H(p)$, $b \in H(p)$, $b \neq 0$, and $\models \psi[b]$. Since p is sr via ψ , b satisfies $p|M$. In particular, $t^+(b) = p^+$.

<u>3.8 REMARKS</u>: (i) $\Rightarrow$ (ii) is the subtle part of the theorem. Similar theorems hold in a more general context, as discovered independently by Prest [Pr 2, Theorems 20-23] and Ziegler [Z, 11.4]. In light of the above theorem and in analogy with Chapter II, I make the following definitions:

<u>3.9 DEFINITIONS</u>: (i) p is a <u>pseudo-prime</u> iff p is a sr 1-type over $\emptyset$, that is, iff p is critical.

(ii) $q \in S_{\bar{x}}(\emptyset)$ is <u>irreducible</u> iff for no $q_0, q_1 \in S_{\bar{x}}(\emptyset)$, $q_0 \neq q \neq q_1$, is $q^+ = q_0^+ \cap q_1^+$. (Note that by (1.21) for any $q_0, q_1 \in S_{\bar{x}}(\emptyset)$ there is $r \in S_{\bar{x}}(\emptyset)$ such that $r^+ = q_0^+ \cap q_1^+$) .

(iii) Let A be a submodule of $\mathfrak{C}$ (usually $A \prec_{ppf} \mathfrak{C}$) A is <u>indecomposable</u> if $A_0, A_1 \subseteq A$, $A = A_0 + A_1(\text{dir})$ implies that $A_0 = \{0\}$ or $A_1 = \{0\}$.

<u>3.10 THEOREM</u>: Let $q \in S_*(\emptyset)$. The following are equivalent:

(i) $w(q) = 1$

(ii) $w(H(q)) = 1$

(iii) $H(q)$ is indecomposable

(iv) $H(q) = H(p)$ for some pseudoprime p

(v) q is irreducible.

PROOF: $[(i) \Rightarrow (ii)]$ Let $\vec{b} \in H(q)$ realize q, so that $H(q) = H(\vec{b})$. By (3.5(iv)), $\vec{b}$ dominates $H(q)$ over $\emptyset$, so $w(H(q)) \leq w(q) = 1$ by (0,1.17v). Clearly $w(H(q)) \neq 0$.

$[(ii) \Rightarrow (iii)]$ Suppose $H(q) \cong A_0 \oplus A_1$, $A_i \neq \{0\}$, $i \in 2$. Since $\{0\} \neq A_0, A_1 \subseteq H(q)$, clearly $A_i \not\downarrow H(q)$ $(i \in 2)$. But by (1.23) $A_0 \downarrow A_1$. Therefore by the definition of weight, $w(H(q)) \geq 2$.

$[(iii) \Rightarrow (iv)]$ By (3.2) $H(q)$ realizes a pseudo-prime p. By (3.5(i)) $H(p)$ is a summand of $H(q)$. Since $H(q)$ is indecomposable, $H(p) = H(q)$.

$[(iv) \Rightarrow (i)]$ Since $H(q) = H(p)$, by (3.5(iv)) p dominates q. p is a pseudo-prime, so $w(p) = 1$, hence $w(q) = 1$.

$[(i) \Rightarrow (v)]$ Suppose $q_0, q_1 \in S_{\vec{x}}(\emptyset)$ and $q^+ = q_0^+ \cap q_1^+$. Consider $H(q_0) \oplus H(q_1)$, let $\vec{a}_i \in H(q_i)$ realize q_i $(i \in 2)$, so by (1.21) $\vec{a} = \overrightarrow{\langle a_0, a_1 \rangle}$ realizes q. Now $\vec{a}_0 \downarrow \vec{a}_1$ by (1.23) and $w(\vec{a}) = 1$, so either $\vec{a} \downarrow \vec{a}_0$ or $\vec{a} \downarrow \vec{a}_1$. Suppose $\vec{a} \downarrow \vec{a}_0$. I claim that $q_1^+ \subseteq q^+$, which completes the proof. Since $\vec{a} \downarrow \vec{a}_0$, for every ppf $\phi(\vec{z}, \vec{y})$, $\models \phi[\vec{a}, \vec{a}_0] \Rightarrow \models \phi[\vec{a}, \vec{0}]$ by (1.20). Let $\psi(\vec{z}) \in q_1^+$ ($\vec{z}$ some finite subset of $\vec{x}$), let $\phi(\vec{z}, \vec{y}) ::= \psi(\vec{z} - \vec{y})$. Since $\psi \in q_1^+$, $\models \psi[\vec{a}, \vec{a}_0]$, hence $\models \phi[\vec{a}, \vec{0}]$ so $\models \psi[\vec{a}]$. That is, $\psi \in q^+$.

$[(v) \Rightarrow (iii)]$ Suppose $H(q) = A_0 \oplus A_1$, $A_0 \neq \{0\} \neq A_1$. q is realized in $H(q)$ by some $\overrightarrow{\langle a_0, a_1 \rangle}$, and by (3.5(iii)), $\vec{a}_0 \neq \vec{0} \neq \vec{a}_1$, $\vec{a}_0, \vec{a}_1$ do not realize q. Therefore $q^+ = t^+(\vec{a}_0, \emptyset) \cap t^+(\vec{a}_1, \emptyset)$ is a proper reduction of q.

3.11 COROLLARY: $0 \neq A \prec_{ppf} \mathfrak{C}$, A indecomposable implies that $A = H(a)$ for every a, $0 \neq a \in A$. In particular, $A \cong H(p)$ for some pseudo-prime p.

PROOF: $A = H(a)$ by (3.5(iii)). By (3.2) for some $a \in A$, $0 \neq a$, $t(a, \emptyset)$ is a pseudo-prime.

3.12 COROLLARY: Let $0 \neq A \prec_{ppf} \mathfrak{C}$. Then A has a non-zero indecomposable direct summand.

PROOF: For some $a \in A$, $0 \neq a$, $t(a, \emptyset)$ is a pseudo-prime p. Hence $H(p)$ is a summand of A.

3.13 REMARKS: Prest [Pr 2] and Ziegler [Z, 11.5] both establish that if p is regular then $H(p)$ is indecomposable. As (3.10) reveals, the central connection is between weight 1 and indecomposability, not between regularity and indecomposability. The arguments in terms of weight go through in the more general contexts of arbitrary complete theories of modules studied by Prest. It is interesting to note that both Prest and Ziegler define q to be irreducible (indecomposable) iff $H(q)$ is indecomposable, whereas (3.10) shows that "irreducible" has its natural meaning.

3.14 PROPOSITION: Let $q_0, q_1 \in S_1(\emptyset)$ have weight 1. Then $H(q_0) \cong H(q_1) \Leftrightarrow q_0 \not\perp q_1$.

PROOF: First note that wlog q_0 and q_1 are strongly regular, by (3.10) on the left and by (0, 1.23(iii)) on the right. Let $M \models T$, $N = M \oplus H(q_0) \cong M(q_0|M)$. If $H(q_0) \cong H(q_1)$ then $q_1|M$ is realized in $M(q_0|M)$, hence $q_0 \not\perp q_1$ (0, 1.24). Conversely, if $q_0 \not\perp q_1$, then $q_1|M$ is realized in N, so $M \prec M \oplus H(q_1) \prec M \oplus H(q_0)$, and by the standard argument $H(q_1) \cong H(q_0)$ since $H(q_0)$ is indecomposable.

III 4. DECOMPOSITION THEOREMS

4.0 REMARKS: As in section 3, throughout this section $T = T^{\aleph_0}$ is a totally transcendental theory of modules.

First I give an entirely model-theoretic proof of Garavaglia's thorem on the unique direct sum decomposition of tt modules. Then I prove a primary decomposition theorem for $S_1(\emptyset)$ analogous to the Lasker-Noether decomposition. I establish more general versions of those results (II4.3-4.11) left unproved or only sketched in Chapter II.

4.1 PROPOSITION: Let A be a direct summand of $\mathfrak{C}$. Then A can be written as a direct sum of indecomposable modules.

PROOF: By (3.12) A has a non-zero indecomposable direct summand.

I call a collection F of non-zero indecomposable summands of A nice if F is independent over $\emptyset$ and ΣF is pure in A. I claim that the union of an increasing chain of nice families is nice. Certainly it is independent. To check the other part of the definition it is enough to see that if $(A_i)_{i<\kappa}$ is an increasing chain of submodules pure in A, then $\bigcup_{i<\kappa} A_i$ is pure in A. But this too is obvious. By Zorn's lemma I choose a maximal nice family F. By (1.25), ΣF is direct. Since ΣF is pure in A, it is a summand of A by (1.17). If $A = \Sigma F + A'(\mathrm{dir})$ with $A' \neq \{0\}$, then A' again contains an indecomposable direct summand A'' and $F \cup \{A''\}$ contradicts the maximality of F. Hence $A \cong \oplus F$.

4.2 THEOREM: Let A be a direct summand of $\mathfrak{C}$. Then A may be written uniquely (up to order) as a direct sum of indecomposable modules.

Furthermore, if P is a representative set of pseudo-primes for T

(I, 1.3) then $A \cong \oplus_{p \in P} H(p)^{(\alpha_p)}$ where $\alpha_p = \dim(p, A)$.

PROOF: By (4.1) A may be written as a direct sum of indecomposable modules, by (3.11) and (3.14) these may be taken to be $H(p)$ for some $p \in P$, so $A \cong \oplus_{p \in P} H(p)^{(\alpha_p)}$ for some cardinals α_p . By (1.23) the family of indecomposables in this representation is independent, and by (3.10) the weight of each $H(p)$ is 1. Thus the cardinals α_p are uniquely determined as $\dim(p, A)$ for each $p \in P$ (0, 1.18, 1.19).

4.3 COROLLARY: (Garavaglia's Theorem [G4, Theorem 4]). Let M be any tt module. Then M may be written uniquely as a direct sum of indecomposable modules.

PROOF: $\mathrm{Th}(M^{(\aleph_0)})$ satisfies the conditions set out in (4.0), hence (4.2) applies to M , a direct summand of $M^{(\aleph_0)}$.

4.4 THEOREM: Let $q \in S_*(\emptyset)$. $H(q)$ has a decomposition as in (4.2). The following are equivalent:

(i) $H(q) \cong \oplus_{i \in I} H(p_i)$ (p_i pseudo-primes)

(ii) $H(q) \cong H(\otimes_{i \in I} p_i)$

(iii) $q \supsetneq \otimes_{i \in I} p_i$.

In addition, (i) - (iii) imply that $w(q) = |I|$.

PROOF: The final comment is immediate from (iii), the fact that each p_i has weight 1, and the additivity of weight (0, 1.18).

[(i) $\Rightarrow$ (ii)] Let $A = H(\otimes_{i \in I} p_i)$, $B = \oplus_{i \in I} H(p_i)$. Let $\{a_i | i \in I\} \subseteq A$ realize $\otimes_{i \in I} p_i$, that is, a_i realizes p_i for each $i \in I$ and $\{a_i | i \in I\}$ is independent over $\emptyset$. By (3.5) for each $i \in I$ there is

$A_i <_{ppf} A$, $a_i \in A_i \cong H(p_i)$ and a_i dominates A_i over $\emptyset$. Thus $\{A_i | i \in I\}$ is independent and by (1.25), $\Sigma_{i\in I} A_i$ is direct and pure in $\mathfrak{C}$, hence in A. But again by (3.5), A is minimal over $\{a_i | i \in I\}$, so $A = \Sigma_{i\in I} A_i(\mathrm{dir})$, that is, $A \cong B$.

[(ii) $\Rightarrow$ (iii)] Let A, $\{a_i | i \in I\}$ be as in the preceding paragraph and let $\vec{b} \in A$ realize q. By (3.5), since $A = H(q)$, $\vec{b}$ dominates $\{a_i | i \in I\}$ over $\emptyset$, and since $A = H(\otimes_{i\in I} p_i)$, $\{a_i | i \in I\}$ dominates $\vec{b}$ over $\emptyset$. Thus by the definition (0, 1.13(ii)) $q \trianglelefteq\!\!\!\approx \otimes_{i\in I} p_i$.

[(iii) $\Rightarrow$ (i)] Suppose $q \trianglelefteq\!\!\!\approx \otimes_{i\in I} p_i$, and also for some pseudo-primes p_j' $(j \in J)$, $H(q) \cong \oplus_{j\in J} H(p_j')$. Then by (i) $\Rightarrow$ (iii) already established, $q \trianglelefteq\!\!\!\approx \otimes_{j\in J} p_j'$. By the uniqueness of sr decompositions there is a bijection $f: I \to J$ with $p_i \not\perp p_{f(i)}'$ for all i. By (3.14) $H(p_i) \cong H(p_{f(i)}')$ for all i, so $H(q) \cong \oplus_{i\in I} H(p_i)$.

<u>REMARK</u>: The regular decomposition theorem (0, 1.23) is usually stated only for $q \in S(M)$ some M, so $w(q)$ is finite. Thus the uniqueness result used above is only stated for finite products of regular types. In actual fact the finiteness assumption is not necessary.

<u>4.5 LEMMA</u>: (i) $\not\perp$ is an equivalence relation on the pseudo-primes.
(ii) Let p be a pseudo-prime, $r, q_0, q_1 \in S_1(\emptyset)$, $r^+ = q_0^+ \cap q_1^+$. Then $p \not\perp r \Rightarrow p \not\perp q_0 \otimes q_1$ and $p \not\perp q_0 \otimes q_1$ iff $p \not\perp q_0$ or $p \not\perp q_1$.

<u>PROOF</u>: (i) (0, 1.20(iii)) (since pseudo-primes have weight 1.)
(ii) The first part is immediate from (1.21). The second part is (0, 1.23(iii)).

<u>4.6 DEFINITION</u>: (Extending the analogies drawn in (II, 4.5)).

(i) Let p be a pseudo-prime, $q \in S_1(\emptyset)$. q is *p-primary* if for every pseudo-prime p' , $q \not< p'$ iff $p \not< p'$. q *is primary* if it is p-primary for some p .

(ii) Let p be a pseudo-prime, $q \in S_1(\emptyset)$. Then *p belongs to q* iff $p \not< q$.

4.7 PROPOSITION: Let $p, q \in S_1(\emptyset)$, p a pseudo-prime. The following are equivalent:

(i) q is p-primary

(ii) q is realized in $H(p)^{(n)}$ for some $n < \omega$

(iii) $q \underset{\approx}{\trianglelefteq} p^{n}$ for some $n < \omega$

(iv) $H(q) \cong H(p)^{(n)}$ for some $n < \omega$.

PROOF: (iii) and (iv) are equivalent by (4.4) and (iv) $\Rightarrow$ (ii) is immediate.

[(i) $\Rightarrow$ (iv)] By (0, 1.23) $w(q)$ is finite so by (4.4), $H(q) \cong \oplus_{i<n} H(p_i)$ for some pseudo-primes p_i . Since q is p-primary, by (3.14) $H(p_i) = H(p)$ for all $i < n$, so $H(q) \cong H(p)^{(n)}$.

[(ii) $\Rightarrow$ (i)] Since q is realized in $H(p)^{(n)}$, $H(q) \prec_{ppf} H(p)^{(n)}$. If p' is a pseudoprime, $p' \not< q$, then p' is realized in $H(q)$, hence in $H(p)^{(n)} \cong H(p^{n})$. Therefore $p' \not< p^{n}$, so $p' \not< p$.

4.8 PROPOSITION: (All types are in $S_1(\emptyset)$ and p, p' are pseudo-primes.)

(i) $p \not< p' \Rightarrow$ (q is p-primary iff q is p'-primary)

(ii) q is p-primary iff $(\forall r)[q \not< r \Rightarrow p \not< r]$

(iii) q is irreducible $\Rightarrow q$ is primary

(iv) A finite intersection of p-primary types is p-primary, and conversely any p-primary type is a finite intersection of irreducible p-primary types. (Here $r = q_0 \cap \dots \cap q_{n-1}$ means that r is the unique type such that $r^+ = q_0^+ \cap \dots \cap q_{n-1}^+$ (1.21).)

PROOF: (i) and (ii) are both immediate by (0, 1.23(iii)) and the definition of q being primary. (iii) follows immediately from (3.10) and (4.7). (iv) Let $(q_i)_{i<n}$ be p-primary types, q_i realized in $H(p)^{(n_i)}$. Then $\otimes_{i<n} q_i$ is realized in $H(p)^{(m)}$ (where $m = \Sigma_{i<n} n_i$), hence, by (1.21), so is $\bigcap_{i<n} q_i$.

For the converse, suppose q is p-primary, q realized in $H(p)^{(n)}$ by $\langle a_i \rangle_{i<n}$, and wlog, $a_i \neq 0$ for all $i < n$. Then each $t(a_i, \emptyset)$ is irreducible p-primary by (3.10) and (iii), and $q^+ = \bigcap_{i<n} t^+(a_i, \emptyset)$ by (1.21).

4.9 THEOREM: Let $q \in S_1(\emptyset)$, $w(q) = m$. Then:

(i) $q^+ = \bigcap_{i<m} r_i^+$ with r_i irreducible for $i < m$ and the decomposition is irredundant.

(ii) Let r_i be p_i-primary. Then $q \precapprox \otimes_{i<m} p_i$.

(iii) In (ii) group together equivalent pseudo-primes and write $q \precapprox \otimes_{j<n} p_j^{(n_j)}$ with $j < j' < n \Rightarrow p_j \perp p_{j'}$. Then $\{p_j | j < n\}$ and the map $p_j \to n_j$ are uniquely determined by q (up to $\not\perp$).

PROOF: $w(q)$ is finite so by (4.4), $H(q) \cong \oplus_{i<m} H(p_i)$ for some pseudo-primes $(p_i)_{i<m}$ where $m = w(q)$ and $q \precapprox \otimes_{i<m} p_i$. Now q is realized by some $\langle a_i \rangle_{i<m}$ so $q^+ = \bigcap_{i<m} t^+(a_i, \emptyset)$ where $t(a_i, \emptyset) = r_i$ is irreducible p_i-primary. The decomposition is irredundant (else a factor could be omitted from the essentially unique decomposition $H(q) \cong \oplus_{i<m} H(p_i)$). The regular decomposition of q is unique up to $\not\perp$ (0, 1.23(ii)) so the results of (iii) follow.

4.10 COROLLARY: Let P be a set of representatives of the $\not\perp$-classes of pseudo-primes. Let $q \in S_1(\emptyset)$. Then q has a normal decomposition, that is, $q^+ = \bigcap_{j<n} r_j^+$ where for $j < n$, r_j is p_j-primary, some $p_j \in P$, for $j < j' < n$, $p_j \neq p_{j'}$, and the decomposition is irredundant.

$\{p_j | j < n\}$ is the set of pseudo-primes (in P) belonging to q and is uniquely determined by q . As well, for each $j < n$ the number of irreducible p_j-primary types necessary to represent r_j is uniquely determined by $w(r_j)$.

PROOF: Immediate from (4.9(iii)).

4.11 REMARKS: As noted in Chapter II, these results have as corollaries the usual Lasker-Noether and Lesieur-Croisot decomposition theorems for Noetherian rings (with some minor qualifications on the latter). My decomposition theorem for types in $S_1(\emptyset)$ appears to be an entirely new generalization of these algebraic results, as is of course the model-theoretic proof. The model-theoretic approach allows me to emphasize a point usually not mentioned explicitly in the usual algebraic statement of the normal decomposition (4.10), namely that the number of irreducibles contributing to each p_j-primary type r_j is uniquely determined by q and p_j , and in fact this number is an important model-theoretic measure, the weight of r_j .

4.12 HISTORICAL REMARKS: At various points throughout Chapters II and III I have remarked on research done independently of my own which overlaps or extends my work. I will try to summarize those remarks here, re-emphasize which parts of these chapters should be regarded as original, and indicate the influence of this other work on the final version of my results as they appear here.

One of these influences was somewhat negative: I have felt that an important goal of the model-theoretic approach to algebra is to show the generality of certain results by showing that they are essentially model-theoretic in nature. Thus in each of the results presented in these two chapters I have tried to use as much model theory and as little algebra as possible, most particularly in Chapter III. This is quite different from the approach of other researchers.

The paper of Prest [Pr 2] develops the theory of orthogonality and regular types in arbitrary theories of modules. It is interesting to note that although the context is often quite different, the theorems and their proofs are very similar to the ones that I have presented in Chapters II and III. In particular, Prest's theorems 18-23 are natural generalizations of my results in II 3 and III 3. My results and Prest's were obtained independently. However I have modified somewhat the presentation of my results, especially in Chapter III, in the light of kind suggestions made by Mike Prest after he saw my original manuscript [private communication, 1981].

Another substantial change from the early (1980-1981) version of these results is the emphasis on weight one sets, rather than on strongly regular types alone. This has resulted in a substantial clarification of proofs and a more concise statement of results.

Both Prest [Pr 1] and Ziegler [Z] discuss compact hulls and indecomposable compact modules in great generality and in a generally model-theoretic setting. There are two important points to what I have done that seem at first somewhat contradictory. Firstly, in the context $T = T^{\aleph_0}$, tt, the results are very reminiscent of well-known algebraic results, especially as has been seen in this section. Secondly, in the same context, the proofs can be accomplished with very little algebra. Indeed, the compact hulls which are so important in the very general treatment by Prest and Ziegler arise in my work as a natural consequence of the existence of prime models. [Pr 2] makes explicit the connection between indecomposable compact modules and regular types, as does, to a lesser extent, the final chapter of [Z]. As remarked in (3.13) both Prest and Ziegler seem to have missed the fact that $H(q)$ is indecomposable iff q^+ is irreducible in the usual sense (at least in my special context). However [Z, Theorem 4.4] appears to be related to this.

Ziegler also proves the most general theorem about the decomposition of compact modules, due, apparently to Fisher [See Z, Theorem 5.1]. Again the proof of uniqueness is by means of the Krull-Remak-Schmidt-Azumaya theorem, and one might ask if the general stability theory of modules developed in [PP], [Pr 1] and [Pr 2] is enough to give a proof of Fisher's theorem in the same spirit as my proof of Garavaglia's theorem: namely, that uniqueness is a consequence of the fact that indecomposables have weight 1. This is a matter for further research.

The proper context for such research is probably nothing more than $T = T^{\aleph_0}$. Thus I propose the following question related to the KRSA theorem and Fisher's theorem:

PROBLEM: Suppose $(A_i)_{i \in I}$ are modules with $\mathrm{End}(A_i)$ a local ring for all $i \in I$. Let $T = \mathrm{Th}(\oplus_{i \in I} A_i^{(\aleph_0)})$ (so that $T = T^{\aleph_0}$ and each A_i

is a summand of a model of T). Is it the case that $w(A_i) = 1$ for all $i \in I$?

As to the elementary theory of forking, I only need the theorem of Garavaglia quoted as (1.19) here, in fact for the most part I use only its corollary (1.20). This has the important consequences (1.21) (this simple idea dates back to my earliest work on the subject) and (1.23). Because of the approach I take, avoiding the introduction of a lot of algebra to establish the theory of compact hulls, I need the two additional lemmas (1.24) and (1.25). I do not need any of the more general (and more complicated) results of Pillay and Prest [PP].

The results of III 2 appear not to have been considered at all by other authors, which is perhaps natural. Although everyone seemed to be aware that the model theory of modules can be done in a much more general context such as Fisher's abelian structures, they do not appear to have realized that this context opens up some new and interesting problems. At first it seems as if there is no model-theoretic difference at all between ordinary modules and abelian structures.

IV . TOPOLOGICAL LOGIC: STABILITY THEORY AND MODULES

IV 0. INTRODUCTION

In this chapter I study the topological logic first introduced by McKee [Mc] and studied in detail by Garavaglia [G1] and Flum and Ziegler [FZ]. This is a logic suitable for the study of topological structures.

After a summary of basic results on topological model theory (1.0-1.14), mostly taken from [FZ], I develop a translation of the topological language L_t into an ordinary first order language L^*. This translation is faithful to the model-theoretic content of L_t (1.18). This allows me to consider individual stability theory for L_t : the stability-theoretic study of those types of L_t in which only individual variables occur freely and in which only individuals occur as parameters. That is, I do not allow an open set to be represented by a free variable or a constant symbol. I originally developed this stability theory entirely within L_t ; the fact that the theorems and their proofs were identical to those in ordinary first order logic suggested the reduction from L_t to L^*.

This reduction allows me to prove the following basic fact about stable L_t-theories (1.22): Let $\langle M, \sigma\rangle$ be a saturated (more generally, special) model of a stable L_t theory, p a type over a small subset A of M. Two extensions p_1, p_2 to M of p are equivalent if for some topological automorphism α of $\langle M, \sigma\rangle$ fixing A, $\alpha(p_1) = p_2$. Then there is exactly one equivalence class of extensions of p with cardinality $< |M|$, in fact $\leq 2^{|T|}$.

In the second section I study topological modules. First I modify a result of Garavaglia [G1] to prove an analogue of the pp-elimination of quantifiers for ordinary modules (III 1.6(ii)). Although the proof of my theorem parallels Garavaglia's proof almost exactly, my theorem is not a corollary of Garavaglia's. This theorem allows me to state some elementary facts about the stability theory of topological modules analagous to the results of Chapter III.

Finally I prove a strong quantifier-elimination result for compact Hausdorff modules. The complete L_t theory of such admits elimination of set quantifiers from formulas with only individual variables free.

IV 1. TOPOLOGICAL LOGIC AND STABILITY THEORY

1.0 DEFINITION AND REMARKS: A topological structure is a pair $\langle M, \tau\rangle$ where M is a structure for ordinary first order logic and τ is a topology on M .

Generally, ordinary first order model theory is not suitable for the study of such structures, since topology involves genuinely second order concepts. The topological language L_t extending a given first order language L and the associated topological model theory are introduced as a best possible approximation to the model-theoretic study of full topology. Here "best possible" means that L_t satisfies a Lindström-type theorem [FZ, p. 48].

For completeness and to fix terminology, I include the following articles (1.1-1.14) which follow sections I1 to I3 of [FZ] closely. (1.8(ii), (iii)) and (1.9), however, are new.

1.1 DEFINITIONS AND DISCUSSION: Let L be a fixed language for finitary first order logic. (In general L may be many-sorted, but for simplicity I assume here that L has a unique sort.) L_2 denotes the two sorted language with variables of the first sort (x, y, z, etc.) corresponding to L , variables of the second sort (X, Y, Z, etc.), and a single new relation symbol ϵ with sorting determined by "$x \epsilon X$" . The intended (standard) interpretation of variables of the second sort is as sets of elements of the first sort. Variables of the first sort are called individual variables, those of the second sort set variables. If L was initially many-sorted, introduce one new sort $\bar{s}$ for each of the original sorts s , and new relation symbols ϵ_s with sort $s \, \epsilon_s \, \bar{s}$.)

A <u>weak structure</u> for the second order logic of L is a pair (A, σ) where A is an ordinary first order structure for L and $\emptyset \neq \sigma \subset \mathfrak{P}(A)$. Such weak structures interpret L_2 by giving $\in$ its standard meaning and requiring the variables of the second sort to range over σ. Conversely, any model (A, S, ε) of the L_2 sentence

ext: $(\forall X)(\forall Y)[X = Y \leftrightarrow (\forall z)(z \in X \leftrightarrow z \in Y)]$ (extensionality)

can be construed as a weak structure for the second order logic of L by replacing S by σ, where $\sigma ::= \{\{a \in A | \langle A, S, \varepsilon\rangle \models a \in s\} | s \in S\}$ and ε by the real $\in$. In the sequel, all structures for L_2 are assumed to be weak structures, in particular in the use of the symbol $\models$.

It is clear that it is impossible to write down a set of axioms Σ in L_2 such that $\langle A, \sigma\rangle \models \Sigma$ iff Σ is a topology on A, since this concept is second order over σ. However, the concept of σ being a basis for a topology on A is axiomatizable (1.3).

Since L_2 is a standard 2-sorted first order logic, and by the remarks on <u>ext</u> above, all the usual elementary theorems hold. In particular:

<u>1.2 THEOREM</u>: (Compactness) A set Σ of L_2 sentences has a weak model iff every finite subset of Σ does.

<u>1.3 DEFINITION AND DISCUSSION</u>: Let <u>bas</u> be the following sentence of L_2:

$$(\forall x)(\exists X)[x \in X] \wedge (\forall x)(\forall X)(\forall Y)[(x \in X \wedge y \in Y) \rightarrow$$
$$\rightarrow (\exists Z)[x \in Z \wedge (\forall z)[z \in Z \rightarrow (z \in X \wedge z \in Y)]]].$$

Clearly $\langle A, \sigma\rangle$ is a weak model of <u>bas</u> iff σ is a basis for a topology on A. In such a case σ^* denotes the topology on A generated by σ. Many topological properties can be described in terms of the basis σ rather than the topology σ^*. One of the goals of topological logic is to make the relevant logical concepts invariant for choice of basis as well. In particular, two weak models $\langle A_0, \sigma_0\rangle$ and $\langle A_1, \sigma_1\rangle$ of <u>bas</u> are regarded as being the same if $A_0 = A_1$ and $\sigma_0^* = \sigma_1^*$. Clearly every topological structure is a weak model of <u>bas</u>.

<u>1.4 DEFINITION</u>: Let ϕ be a sentence of L_2. ϕ is <u>invariant</u> for topologies if for all weak models of <u>bas</u> $\langle A, \sigma\rangle$, $\langle A, \sigma\rangle \models \phi$ iff $\langle A, \sigma^*\rangle \models \phi$. Similarly, a formula $\phi(\vec{x}, \vec{X})$ of L_2 is invariant for topologies if for all weak models of <u>bas</u> $\langle A, \sigma\rangle$, $\vec{a} \in A$, $\vec{U} \in \sigma$, $\langle A, \sigma\rangle \models \phi[\vec{a}, \vec{U}]$ iff $\langle A, \sigma^*\rangle \models \phi[\vec{a}, \vec{U}]$.

<u>1.5 DEFINITION AND REMARKS</u>: Let $\phi(X)$ be a formula of L_2, wlog ϕ is in negation normal form. X occurs positively (negatively) in ϕ if all the occurrences of X as a free variable of ϕ are of the form "$t \in X$" ("$\neg t \in X$") for some term t of L.

Note that if X is not a free variable of ϕ, then X occurs both positively and negatively in ϕ, and that if X has occurrences of both forms "$t \in X$" and "$\neg t \in X$" then X occurs neither positively nor negatively in ϕ. The notation $\phi :=: \phi(x_1, \ldots, x_n, X_1^+, \ldots, X_m^+, Y_1^-, \ldots, Y_p^-)$ indicates that the free variables of ϕ are among $x_1, \ldots, x_n$, $X_1, \ldots, X_m, Y_1, \ldots, Y_p$, with each X_i occurring positively and each Y_i occurring negatively.

<u>1.6 DEFINITION</u>: (McKee [Mc], Garavaglia [G1], Ziegler [Z1]). L_t is the least set of L_2 formulas containing the atomic formulas other than equalities of the second sort (i.e. of the form "$X = Y$"), closed under the formation rules of $L_{\omega\omega}$ allowing only individual quantifiers, and closed under

(i) If t is a term of L, X occurs positively in $\phi \in L_t$, then "$(\forall X)[t \in X \to \phi]$" $\in L_t$.

(ii) If t is a term of L, X occurs negatively in $\phi \in L_t$, then "$(\exists X)[t \in X \wedge \phi]$ $\in L_t$.

Clearly up to logical equivalence (ii) is an immediate consequence of (i) and closure under $\neg$. The formula in (i) is abbreviated as "$(\forall X)_t\phi$", that in (ii) as "$(\exists X)_t\phi$". The intended interpretation of these formulas is as "for every open neighbourhood U of t, $\phi[U]$ holds" and "there is an open neighbourhood U of t such that $\phi[U]$ holds".

<u>1.7 THEOREM</u>: [Mc] [G1] A sentence ϕ of L_2 is invariant for topologies iff <u>bas</u> $\models \phi \leftrightarrow \psi$ for some $\psi \in L_t$. A formula $\phi(\vec{x}, \vec{X})$ of L_2 is invariant for topologies iff <u>bas</u> $\models (\forall\vec{x})(\forall\vec{X})[\phi \leftrightarrow \psi]$ for some $\psi(\vec{x}, \vec{X}) \in L_t$.

<u>REMARKS</u>: The forward direction is quite deep. Garavaglia first derives a topological form of Keisler's ultrapower theorem [G1, Theorem 1] and then (1.7) follows immediately by a standard argument. McKee proves it by a consistency property argument.

<u>1.8 DEFINITIONS</u>: (i) The model-theoretic relations $\models$, $\equiv$, $\cong$ are relativized to L_t as $\models_t$, $\equiv_t$, $\cong_t$ as follows: they hold only between weak models of <u>bas</u> and refer only to sentences of L_t.

f: $\langle M, \sigma\rangle \cong_t \langle N, \tau\rangle$ iff f is an isomorphism of the first order structures and a homeomorphism of the topological spaces. f is called a <u>topological isomorphism</u>.

(ii) $\langle M, \sigma\rangle \prec^*_t \langle N, \tau\rangle$ iff M is a substructure of N in the usual sense, and there is $f: \sigma \to \tau$ such that for every formula $\phi(\vec{x}, \vec{X})$ of L_t, all

$\vec{m} \in M$, all $\vec{U} \in \sigma$, $\langle M, \sigma\rangle \models \phi[\vec{m}, \vec{U}]$ iff $\langle N, \tau\rangle \models \phi[\vec{m}, f(\vec{U})]$.

(iii) $\langle M, \sigma\rangle \prec_t \langle N, \tau\rangle$ iff $M \subset N$ and for all $\phi(\vec{x}) \in L_t$ with only individual variables free, all $\vec{m} \in M$, $\langle M, \sigma\rangle \models \phi[\vec{m}]$ iff $\langle N, \tau\rangle \models \phi[\vec{m}]$.

<u>REMARKS</u>: $\prec_t^*$ and $\prec_t$ have not been considered previously by other authors. Both satisfy some nice properties (1.9) but $\prec_t^*$ is too strong for my present purposes. Part (iv) of the important theorem (1.18) only holds for the weaker relation $\prec_t$. Of the two, $\prec_t$ is clearly basis invariant, but it is not at all clear whether or not $\langle M, \sigma\rangle \prec_t^* \langle N, \tau\rangle$ implies that $\langle M, \sigma^*\rangle \prec_t^* \langle N, \tau^*\rangle$. The function f is introduced because I insist that all structures be weak structures. The reader should note that neither relation implies that the first structure is a topological subspace of the second. Also note that if f is a map defined on $M \cup \sigma$, then $f\colon \langle M, \sigma\rangle \prec_2 \langle N, \tau\rangle$ implies that $f\colon \langle M, \sigma\rangle \prec_t^* \langle N, \tau\rangle$, and the latter implies that $f\colon \langle M, \sigma\rangle \prec_t \langle N, \tau\rangle$.

<u>1.9 PROPOSITION</u>: (i) $\langle M, \sigma\rangle \prec_t \langle N, \tau\rangle \Rightarrow \langle M, \sigma\rangle \equiv_t \langle N, \tau\rangle$.

(ii) $\langle M, \sigma\rangle \prec_t \langle M, \sigma\rangle$.

(iii) $\langle M_0, \sigma_0\rangle \prec_t \langle M_1, \sigma_1\rangle \prec_t \langle M_2, \sigma_2\rangle \Rightarrow \langle M_0, \sigma_0\rangle \prec_t \langle M_2, \sigma_2\rangle$.

(iv) $\langle M_0, \sigma_0\rangle \prec_t \langle M_2, \sigma_2\rangle$, $\langle M_1, \sigma_1\rangle \prec_t \langle M_2, \sigma_2\rangle$ and $M_0 \subset M_1$ imply that $\langle M_0, \sigma_0\rangle \prec_t \langle M_1, \sigma_1\rangle$.

(i)*-(iii)*: replace $\prec_t$ by $\prec_t^*$ in (i)-(iii).

(iv)* $f\colon \langle M_0, \sigma_0\rangle \prec_t^* \langle M_2, \sigma_2\rangle$, $g\colon \langle M_1, \sigma_1\rangle \prec_t^* \langle M_2, \sigma_2\rangle$, $f[M_0] \subset g[M_1]$, and $f[\sigma_0] \subset g[\sigma_1]$ imply that $h\colon \langle M_0, \sigma_0\rangle \prec_t^* \langle M_1, \sigma_1\rangle$ where for $x \in M_0 \cup \sigma_0$, $h(x)$ is the unique $y \in M_1 \cup \sigma_1$ such that $f(x) = g(y)$.

<u>PROOF</u>: These are all trivial. For (iv)* note that by the definition of $\prec_t^*$, g must be 1-1 (consider the formula "$x \in X \wedge x \notin Y$").

1.10 EXAMPLES: [FZ, pp. 6-7]. The following sentences axiomatize topological concepts in L_t :

T_0: $(\forall x)(\forall y)[x = y \vee (\exists X)_x[y \notin X] \vee (\exists Y)_y[x \notin Y]]$

T_1: $(\forall x)(\forall y)[x = y \vee (\exists X)_x(\exists Y)_y[y \notin X \wedge x \notin Y]]$

haus: $(\forall x)(\forall y)[x = y \vee (\exists X)_x(\exists Y)_y(\forall z) \neg [z \in X \wedge z \in Y]]$

reg: $(\forall x)(\forall X)_x(\exists Y)_x(\forall y)[y \in X \vee (\exists W)_y(\forall z)[z \notin W \vee z \notin Y]]$

disc: $(\forall x)(\exists X)_x(\forall y)[y \in X \rightarrow y = x]$

triv: $(\forall x)(\forall X)_x(\forall y)[y \in X]$

perf: $(\forall x)(\forall X)_x(\exists y)[x \neq y \wedge y \in X]$

Suppose t is an n-ary term of L . Continuity of t is expressed by:

$$(\forall x_1, \ldots, x_n)(\forall Y)_{t(x_1,\ldots x_n)}(\exists X_1)_{x_1} \ldots (\exists X_n)_{x_n}$$
$$(\forall y_1, \ldots, y_n)[(y_1 \in X_1 \wedge \ldots \wedge y_n \in X_n) \rightarrow t(y_1, \ldots, y_n) \in Y]$$

If $\phi(x_1, \ldots, x_n, \vec{y}) \in L_t$ then

$$(\forall \vec{y})(\forall \vec{x})[\neg \phi(\vec{x}, \vec{y}) \rightarrow (\exists X_1)_{x_1} \ldots (\exists X_n)_{x_n} (\forall \vec{z})$$
$$[(z_1 \in X_1 \wedge \ldots \wedge z_n \in X_n) \rightarrow \neg \phi(\vec{z}, \vec{y})]$$

expresses "for every $\vec{a} \in M$, $\{\vec{x} | \langle M, \sigma\rangle \models_t \phi[\vec{x}, \vec{a}]\}$ is closed". As a consequence, one can express "P is an open relation", "P is a closed relation" and "graph of f is closed". "f is an open map" is also expressible in L_t .

1.11 EXAMPLES: [FZ, Section 3], [Ban].

There is no set Σ of L_t sentences axiomatizing connected spaces, compact spaces, spaces with topology induced by a uniformity, normal spaces, separable, first and second countable spaces, and "f is a closed map". Refer to [Ban] for many more interesting examples.

1.12 FUNDAMENTAL THEOREMS:

(i) Compactness Theorem: A set Σ of L_t sentences has a topological model iff every finite subset of Σ does.

(ii) Completeness Theorem: If L is recursive, the set of L_t sentences which hold in all topological structures for L is r.e.

(iii) Lowenheim-Skolem Theorem: A countable set of L_t sentences which has a topological model has a countable topological model ($\langle M, \tau\rangle$ is countable if M is countable and τ has a countable basis.)

(iv) Ultrapower Theorem: (Garavaglia [G7]). $\langle M, \sigma\rangle \equiv_t \langle N, \tau\rangle$ if $\langle M, \sigma\rangle$ and $\langle N, \tau\rangle$ have topologically isomorphic ultrapowers.

REMARKS: (i) is the main result used in the rest of this chapter. (ii)-(iv) are quoted to show the full strength of the logic for L_t. (i)-(iii) are immediate consequences of the corresponding results for L_2 and Theorem 1.7. In (iv), the ultrapowers are of the two-sorted structures, so by the ordinary Łoś theorem, the ultrapowers satisfy ext and bas and thus wlog are weak structures. The proof of (iv) in [G1] is quite substantial and is used by Garavaglia to prove (1.7). Although Bankston does not directly treat the language L_t in [Ban], his paper does give facts about topological ultraproducts and topological ultrapowers which are directly relevant to the model theory of L_t.

1.13 DEFINITION: $\langle M, \sigma\rangle$ is saturated if it is saturated as a two-sorted structure for L_2. Note that saturation is not a basis-invariant concept.

1.14 THEOREM: [FZ, I, Lemma 4.7]. Let $\langle M, \sigma\rangle \equiv_t \langle N, \tau\rangle$ be saturated structures of the same cardinality (i.e. $|M| = |\sigma| = |N| = |\tau|$). Then $\langle M, \sigma\rangle \cong_t \langle N, \tau\rangle$.

REMARKS: Flum and Ziegler state this theorem only for countable saturated structures. I do not include the rather lengthy proof of the more general statement given here because it is virtually identical to the proof given by Flum and Ziegler for the countable case with the following addition: it is necessary to prove (and to use) that the set I of partial homeomorphisms constructed in the proof is closed under the union of increasing chains of length less than $|M|$.

As in ordinary first order logic I introduce a universe $\mathfrak{C}_t = \langle \mathfrak{C}, \sigma_{\mathfrak{C}} \rangle$ which is a saturated model of large cardinality. Since I will want to be able to embed any model in a small saturated elementary submodel of $\mathfrak{C}_t$, I insist that $|\mathfrak{C}| = \Theta$ be hyperinaccessible. (Recall that a cardinal κ is hyperinaccessible if it is regular and for every $\lambda < \kappa$ there is μ, $\lambda < \mu < \kappa$ and μ inaccessible. Thus if $\langle M, \sigma \rangle$ is small, $|M \cup \sigma| = \lambda < \kappa$, there is a saturated submodel of $\mathfrak{C}_t$ of cardinality μ containing $\langle M, \sigma \rangle$). Note also that the assumption on Θ is inessential and serves only to simplify the expression of proofs. For the results that follow the assumption can be eliminated by an absoluteness argument or better yet, by an appeal to special models. See (1.23) for further remarks.

Let T be the complete L_t theory of $\mathfrak{C}_t$. For each small $A \subset \mathfrak{C}$, $\mathrm{Aut}_A^{L_t}(\mathfrak{C}_t)$ denotes the set of all topological automorphisms of $\mathfrak{C}_t$ fixing A. The following facts are immediately apparent by (1.13), (1.14) and elementary facts about saturation in L_2.

1.16 LEMMA: (Notation as in (1.15)).

(i) If $\langle M, \sigma \rangle \models_t T$, $|M| < \Theta$ then there is $f: M \cup \sigma \to \mathfrak{C} \cup \sigma_{\mathfrak{C}}$, respecting the sorts, such that for every formula $\phi(\vec{x}, \vec{Y})$ of L_t, $\vec{m} \in M$, $\vec{U} \in \sigma$,

$\langle M, \sigma\rangle \models \phi[\vec{m}, \vec{U}]$ iff $\mathfrak{C}_t \models \phi[f(\vec{m}), f(\vec{U})]$, that is, $f\colon \langle M, \sigma\rangle \prec_t^* \langle\mathfrak{C}, \sigma_{\mathfrak{C}}\rangle$.

(ii) If $A \subset \mathfrak{C}$, $\vec{B}, \vec{C}$ sequences of individuals of $\mathfrak{C}$, $|A \cup \vec{B} \cup C| < \Theta$, and $B \equiv_t C(A)$ (that is, they satisfy the same formulas of L_t with parameters from A) then there is a topological automorphism α of $\mathfrak{C}$ fixing A and such that $\alpha[\vec{B}] = \vec{C}$.

(iii) If $A \subset \mathfrak{C}$, $|A| < \Theta$ then there is $\langle M, \sigma\rangle \prec_t^* \langle\mathfrak{C}, \sigma_{\mathfrak{C}}\rangle$, two-sorted saturated with $A \subset M$, $|M| < \Theta$.

PROOF: As noted, (i) and (ii) are straightforward. For (iii), there is certainly a two-sorted elementary substructure $\langle M, \sigma'\rangle \prec_2 \langle\mathfrak{C}, \sigma_{\mathfrak{C}}\rangle$ which is saturated, $A \subset M$, $|M| < \Theta$. Since $\langle M, \sigma'\rangle \models \underline{ext} \wedge \underline{bas}$, σ' induces a basis σ for a topology on M. $\langle M, \sigma'\rangle \cong \langle M, \sigma\rangle$ as two-sorted structures and clearly $\langle M, \sigma\rangle \prec_t^* \langle\mathfrak{C}, \sigma_{\mathfrak{C}}\rangle$.

1.17 DEFINITION: (i) Given a language L define a language $L^* \supset L$ as follows: for each formula $(\forall X)\phi(X, \vec{x})$, $(\exists X)\phi(X, \vec{x})$ of L_t with only individual variables free, introduce a new relation symbol $R_{(\forall X)\phi}(\vec{x})$, $R_{(\exists X)\phi}(\vec{x})$.

(ii) If $\langle M, \sigma\rangle$ is a weak structure for L, define M^* a structure for L^* so that $M^* \upharpoonright L = M$ and $M^* \models R_{(\forall X)\phi}[\vec{m}]$ iff $\langle M, \sigma\rangle \models_t (\forall X)\phi(X, \vec{m}]$ and similarly for $R_{(\exists X)\phi}$.

(iii) Given a universe $(\mathfrak{C}, \sigma_{\mathfrak{C}})$, let $T^* = Th(\mathfrak{C}^*)$.

1.18 THEOREM: (i) there is a bijection * between the formulas of L_t with only individual variables free and the formulas of L^*.

(ii) for $\vec{c} \in \mathfrak{C}$, $\phi \in L_t$, $(\mathfrak{C}, \sigma_{\mathfrak{C}}) \models_t \phi[\vec{c}]$ iff $\mathfrak{C}^* \models \phi^*[\vec{c}]$.

(iii) T^* is uniquely determined by $T = Th(\mathfrak{C}_t)$, and conversely.

(iv) $\langle M, \sigma\rangle \prec_t \langle \mathfrak{C}, \sigma_{\mathfrak{C}}\rangle$ iff $M^* \prec \mathfrak{C}^*$.

(v) If p is a complete L_t-type over $A \subset \mathfrak{C}$ with only individual variables free, then p^* is a complete L^*-type over A in the same free variables.

(vi) $p \to p^*$ is a bijection between the set of L_t-types over A and the set of L^* types over A .

(vii) $\alpha \in \mathrm{Aut}_A^{L_t}(\mathfrak{C}_t)$ implies that $\alpha \in \mathrm{Aut}_A^{L^*}(\mathfrak{C}^*)$.

(viii) If $\alpha \in \mathrm{Aut}_A^{L^*}(\mathfrak{C}^*)$, $A \subset B \subset \mathfrak{C}$, $|B| < \Theta$, then there is $\alpha^t \in \mathrm{Aut}_A^{L_t}(\mathfrak{C}_t)$ such that $\alpha^t \restriction B = \alpha \restriction B$. (I say that $\mathrm{Aut}_A^{L_t}(\mathfrak{C}_t)$ is <u>dense</u> in $\mathrm{Aut}_A^{L^*}(\mathfrak{C}^*)$.)

(ix) $\mathfrak{C}^*$ is saturated.

<u>PROOF</u>: (i) For $\phi(\vec{x}) \in L_t$ define $\phi^* \in L^*$ by recursion: $\phi^* ::= \phi$ if ϕ is atomic, * commutes with the first-order connectives and quantifiers, and $((\forall X)\phi(X, \vec{x}))^* ::= R_{(\forall X)\phi}(\vec{x})$, $((\exists X)\phi(X, \vec{x}))^* ::= R_{(\exists X)\phi}(\vec{x})$. Clearly * satisfies (i). The inverse of * is denoted by ^{t} . Note that, e.g., $(R_{(\forall X)\phi}(\vec{x}))^t = (\forall X)\phi(X, \vec{x})$ is completely well defined since the result of ^{t} is encoded in the subscript of R .

(ii)-(vii) are then all obvious. Note that in (v) $p^* = \{\phi^* | \phi \in p\}$. Note also that (iv) fails for $\prec_t^*$. (viii) follows immediately by the saturation of $\mathfrak{C}_t$, (1.16(ii)) and (ii). (ix) is immediate by (vi) and the saturation of $\mathfrak{C}_t$.

<u>1.19 REMARKS</u>: The essential content of (1.16(ii)) is that the L_t-elementary type of $\vec{b}$ over A , $t(\vec{b}, A) = \{\phi(\vec{x}, \vec{a}) | \phi(\vec{x}, \vec{y}) \in L_t , \vec{a} \in A\}$, and the $\mathfrak{C}_t$-automorphism type of $\vec{b}$ over A , $\{\alpha(\vec{a}) | \alpha \in \mathrm{Aut}_A^{L_t}(\mathfrak{C}_t)\}$ contain the same information. Reviewing the definition of forking adopted here (0, 1.5) it is seen that Theorem 1.18, in particular the density property (vii) and (viii), allows me to reduce the whole problem of stability theory in L_t to the ordinary stability of L^* .

<u>1.20 DEFINITION</u>: Let T be a complete L_t theory. Then T is stable, superstable or totally transcendental just as T^* is.

It is clear by (1.18(vi)) that I could equally well have defined stability for L_t in the traditional fashion by counting complete L_t-types.

<u>1.21 REMARKS</u>: Recall the following things from Chapter 0. When I talk about $A \subset \mathfrak{C}$, etc., it is always assumed that $|A| < \Theta$. An <u>ideal type</u> is a type $\bar{p}$ over $\mathfrak{C}$. $\mathrm{Aut}_A^{L_t}(\mathfrak{C}_t)$ acts on the ideal types, in particular $\mathcal{O}(\bar{p})$, the <u>orbit</u> of $\bar{p}$, is $\{\alpha(\bar{p}) \mid \alpha \in \mathrm{Aut}_A^{L_t}(\mathfrak{C}_t)\}$. If $p \in S(A)$ the orbits of ideal extensions of p under topological A-automorphisms are important: for brevity I call such an <u>orbit of p-extensions</u>.

<u>1.22 THEOREM</u>: Let T be a stable L_t theory, $p(\vec{x})$ a complete L_t type over $A \subset \mathfrak{C}$. Among the orbits of p-extensions there is a unique orbit of cardinality $< \Theta$, in fact this orbit has cardinality $\leq 2^{|T|}$.

<u>PROOF</u>: T^* is stable and p^* is a complete type over $A \subset \mathfrak{C}^*$ so by (0, 1.4) there is exactly one small orbit of ideal extensions of p^* to $\mathfrak{C}^*$ under A-automorphisms of $\mathfrak{C}^*$. By (1.18(vii)) every orbit of p-extensions is contained in an orbit of p^*-extensions, so there is at least one small orbit of p-extensions.

I claim that in fact the two kinds of orbit are the same. It is enough to show that if $\bar{p}^*$, $\bar{q}^* = \alpha(\bar{p}^*)$ are ideal types in the same orbit of p^*-extensions, $\alpha \in \mathrm{Aut}_A^{L^*}(\mathfrak{C}^*)$, then for some $\alpha^t \in \mathrm{Aut}_A^{L_t}(\mathfrak{C}^t)$, $\alpha^t(\bar{p}) = \bar{q}$. But for some $B \supset A$, $\bar{p}^* \restriction B$ is stationary, as is $\bar{q}^* \restriction \alpha[B]$. By (1.18(viii)) there is $\alpha^t \in \mathrm{Aut}_A^{L_t}(\mathfrak{C}^t)$ such that $\alpha \restriction B = \alpha^t \restriction B$. By (1.18(vii)),

$\alpha^t \in \mathrm{Aut}_A^{L^*}(\mathfrak{C}^*)$ so by stationarity in L^*, $\alpha^t(\vec{p}^*) = \vec{q}^*$, i.e. $\alpha^t(\bar{p}) = \bar{q}$. So p, q are in the same orbit of p-extensions.

1.23 REMARKS: This theorem is a version for stable L_t-theories of the fundamental property (0, 1.4) of stable theories. It is not just a simple restatement of that basic fact, but is a new result depending on the density property, (1.18(vii), (viii)), and says something about stable L_t theories independently of the translation into ordinary first order logic L^*.

In fact of course the proof does not depend on any assumptions on Θ and it suffices that $(\mathfrak{C}, \sigma_{\mathfrak{C}})$ be some saturated model (see the proof in [LP]). Somewhat more is true: with a slight restatement of the theorem, it suffices that $(\mathfrak{C}, \sigma_{\mathfrak{C}})$ be a special model (in the two-sorted sense). Recall that M is special if for some elementary chain $(M_\lambda)_{\lambda<\kappa}$ indexed by cardinals $\lambda < \kappa = |M|$, $M = \bigcup_{\lambda<\kappa} M_\lambda$ and each M_λ is λ^+-saturated. Every theory T has special models in all powers κ, $2^{<\kappa} = \kappa > |T|$. $(M_\lambda)_{\lambda<\kappa}$ is called a specializing chain for M. $B \subset M$ is called bounded if for some specializing chain and some $\lambda < \kappa$, $B \subset M_\lambda$. Bounded sets replace sets of cardinality $< \kappa$ in saturated models, in particular, if M is special and B is bounded, then:

(i) Let $p \in S(B)$. Then p is realized in M.

(ii) Let $\vec{c}, \vec{d}$ be finite sequences in M (or more generally, sequences such that $B \cup \vec{c} \cup \vec{d}$ is bounded). If $\vec{c} \equiv_B \vec{d}$ then there is an automorphism of M fixing B and sending $\vec{c}$ to $\vec{d}$.

It is straightforward but tedious exercise to verify that (0, 1.4) holds for special models M and types p over a bounded subset of M. Thus I have the following:

<u>1.22' THEOREM</u>: Let T be a stable L_t theory, $(M, \sigma) \models_t T$ and special as a two sorted structure. Let $A \subset M$ be bounded, $p \in S(A)$. Among the orbits of extensions of p to M under topological automorphisms of (M, σ) fixing A, there is a unique orbit of cardinality $< |M|$, in fact, of cardinality $\leq 2^{|T|}$.

<u>1.24 DEFINITION</u>: Let T be a stable L_t theory.

(i) an ideal type $\bar{p}$ <u>does not fork over</u> A (dnf over A) or is a <u>nonforking extension</u> of $\bar{p}\restriction A$ (nf ext. of A) if $\bar{p}$ lies in the unique small orbit of ideal extensions of $\bar{p}\restriction A$ under topological automorphisms of $\mathfrak{C}_t$ fixing A.

(ii) Let $p \supset q \in S(A)$. p dnf over A, or p is a nf ext. of q iff each ideal type $\bar{p}$ which is a nf ext. of p is also a nf ext. of q.

(iii) The <u>multiplicity</u> of p is the cardinality of the unique small orbit of p-extensions, i.e. the maximum number of nf extensions of p to any set (so the multiplicity of p is $\leq 2^{|T|}$).

<u>1.25 COROLLARY</u>: Let p, q be types of T, a stable L_t theory. Then p is a nf ext. of q iff p^* is a nf ext. of q^*.

<u>PROOF</u>: By the proof of (1.22), the unique small orbit of p-extensions to $\mathfrak{C}_t$ is contained in the unique small orbit of p^* extensions to $\mathfrak{C}^*$. Therefore for an ideal type $\bar{p}$ over $\mathfrak{C}_t$, $\bar{p}$ is a nf ext. of p iff $\bar{p}^*$ is a nf ext. of p^* to $\mathfrak{C}^*$.

<u>1.26 THEOREM</u>: Let T be a stable L_t-theory, $\bar{p}$ an ideal type. Then there is $A \subset \mathfrak{C}$, $|A| \leq |T|$, such that $\bar{p}$ is the unique nf ext. of $\bar{p}\restriction A$ to $\mathfrak{C}_t$.

PROOF: By (0, 1.6) there is $A \subseteq \mathfrak{C}^*$, $|A| \leq |T|$ such that $\bar{p}^*$ is the unique nf ext. of $\bar{p}^* \upharpoonright A$ to $\mathfrak{C}^*$. By (1.18(vi)) and (1.25), $\bar{p}$ is the unique nf ext. of $\bar{p} \upharpoonright A$ to $\mathfrak{C}_t$.

1.27 REMARKS: (1.22) and (1.26) are the two basic facts underlying stability theory. Together with (1.18) and (1.25) they justify developing stability theory for L_t in L^* by means of the map *. Thus all results of ordinary stability theory have their analogues for the stability theory of topological structures. In particular, the results of Chapter I are applicable to L_t.

IV 2. TOPOLOGICAL MODULES

2.0 DEFINITION: A topological module is a structure (M, τ) such that M is a module (III 1.0) and τ is a topology on M. (For simplicity I assume that M has a single sort, but the results are equally applicable in the general case when $\boldsymbol{\tau}$ is assumed to be a family of topologies, one for each sort of M.) Each function of M is continuous in the topology τ, and each relation of M is closed. In all of this section L is some fixed language for modules. It is clear that there is a set of L_t sentences T such that for each weak structure $\langle M, \sigma\rangle$ for L, $\langle M, \sigma\rangle \models_t T$ iff $\langle M, \sigma^*\rangle$ is a topological module.

2.1 DEFINITION: (Garavaglia [G1]). A topological positive primitive formula (tppf) is a formula of L_t of the form $Q_1Q_2 \ldots Q_n\phi$ where ϕ is a conjunction of atomic formulas of L_t and each Q_i is of the form $(\exists x)$ or $(\forall X)_0$ for some variables x, X. (Recall (1.6) that "$X = Y$" is not allowed.)

REMARKS: Because of the abelian group structure of M, we may restrict to set quantifiers of the type $(\forall X)_0$, $(\exists X)_0$. The formula $(\forall X)_t\phi$ is equivalent to $(\forall X)_0\psi$ where ψ is obtained from ϕ by replacing all occurrences "$s \in X$" of X, s some term, by "$s - t \in X$". Note that any free set variables of a tppf occur only positively.

2.2 REMARKS: I am now going to present a theorem analagous to (III1.6(ii)) for topological modules: for every topological module $\langle M, \sigma\rangle$ every L_t formula with only individual variables free is equivalent in $\langle M, \sigma\rangle$ to a Boolean combination of tppf's.

A few comments on the history and proof of this result are in order. Garavaglia proves in [G1] that if $\langle M, \sigma\rangle$ and $\langle N, \tau\rangle$ satisfy exactly the same L_t-sentences which are $\forall\exists$ with respect to individual quantifiers, then $\langle M, \sigma\rangle \equiv_t \langle N, \tau\rangle$. This is a direct generalization of an early result of G. Sabbagh [Sa] for ordinary modules. Baur [Ba] first proved the pp-elimination of quantifiers for modules inspired by this same result of Sabbagh's. All three authors use a saturated model method similar to what I present here. My theorem and proof has somewhat the same relation to Garavaglia's as Baur's does to Sabbagh's original result.

In his excellent article [Z], Ziegler gives an entirely different proof of the pp-elimination of quantifiers which is "local" in the sense that it works in any module, not just a saturated model, and "constructive" in the sense that it is shown how the Boolean combination of ppf's equivalent to a given formula is determined by the definable-submodule-structure of the module. Monk's invariants (III 1.6(i)) follow directly. Unfortunately, there appear to be several obstacles to giving a similar proof for my theorem. Although the main steps of Ziegler's proof can be adapted to the topological context, set parameters must be carried through the induction steps and it is not at all clear how it is possible to preserve positive/negative occurrences of set variables in this method. Thus the proof breaks down because some intermediate formulas cannot be shown to be in L_t. Also, the Ziegler-style proof seems to be intimately tied up with questions of recursiveness and decidability (Monk's elementary invariants). As is pointed out by Flum and Ziegler [FZ] and in the articles by Cherlin and Schmidt [CS1, CS2] there are many undecidable theories of topological modules. Thus I prove the theorem in the style of Baur's original proof. A somewhat peculiar situation arises:

although my result is not a corollary of Garavaglia's theorem, the proof itself is virtually identical, except of course for the details of the initial set up and final interpretation. Thus the work I present here should not be regarded as a new proof, but instead as a significant interpretation of an already existing result.

I begin by quoting a series of technical results from [G1] that establish that a sufficiently saturated topological module has a very special neighbourhood basis at 0.

2.3 LEMMA: Let $\phi(\vec{x}, \vec{X})$, $\psi(\vec{x}, \vec{X})$ be tppf's. Then $(\exists x)\phi$, $(\forall X)_0\phi$ are tppf's and $\phi \wedge \psi$ is equivalent (in weak models of bas) to a tppf.

PROOF: Obvious.

2.4 LEMMA: [G1, 4, Lemma 1]. Let $\langle M, \tau\rangle$ be a topological module and suppose that τ has a basis at 0 consisting of subgroups of M . Let $A_1, \ldots, A_m \in \tau$ be subgroups of M , and let $\phi(\vec{x}, \vec{X})$ be a tppf. Then $\phi[\vec{M}, A_1, \ldots, A_m)$ is a subgroup of $\vec{M}$.

PROOF: Let $\{H_i | i \in I\}$ be a basis at 0 consisting of subgroups. Then $(\forall Y)_0[\vec{M}, A_1, \ldots, A_m, Y) = \bigcap_{i\in I} \phi[\vec{M}, A_1, \ldots, A_m, H_i)$. The result follows immediately by a simple induction.

REMARKS: Note that if $\{H_i | i \in I\}$ is a basis at 0 consisting of subgroups, then $\{a + H_i | i \in I\}$ is a basis at a consisting of cosets. Thus under the assumptions of (2.2), if $a_{i+1}, \ldots, a_n \in M$ then $\phi[\vec{M}, a_{i+1}, \ldots, a_n, A_1, \ldots, A_m)$ is either empty or a coset of $\phi[\vec{M}, 0, \ldots, 0, A_1, \ldots, A_m)$.

2.5 DEFINITION/LEMMA: [G1]. Let $\langle M, \sigma\rangle$ be a weak model of the axioms for topological modules and $\aleph_1$-saturated in the two-sorted sense.

$$\sigma_0 ::= \{U \in \sigma \mid 0 \in U\}$$

$$\sigma_0^* ::= \{U \in \sigma^* \mid 0 \in U\}$$

$$\sigma_0' ::= \{\textstyle\bigcap_{n<\omega} U_n \mid (U_n)_{n\in\omega} \text{ is a decreasing sequence in } \sigma_0\}$$

(i) $\sigma_0 \subset \sigma_0' \subset \sigma_0^*$

(ii) $\langle M, \sigma\rangle$ satisfies the conditions of (2.4), that is, σ^* has a basis at 0 consisting of subgroups of M. In fact, such a basis can be found in σ_0'.

PROOF: (i) Let $V \in \sigma_0'$, $V = \bigcap_{n<\omega} U_n$. I must show that V is open. Let $a \in V$; I will find $U \in \sigma_0$, $a \in U \subset V$. Consider the set of L_2 formulas $\Sigma(X) ::= \{a \in X \wedge 0 \in X\} \cup \{(\forall x)[x \in X \to x \in U_n] \mid n < \omega\}$ over countably many parameters in $\langle M, \sigma\rangle$. Σ is finitely consistent because σ is a basis for a topology on M, so is realized in $\langle M, \sigma\rangle$ by some U, $0 \in U$, $a \in U \subset V$.

(ii) Since "-" is continuous, given any $U \in \sigma_0$, there is $V \in \sigma_0$, $V - V \subset U$ (where $V - V = \{a - b \mid a, b \in V\}$). Thus I define by recursion on $n < \omega$ $(U_n)_{n<\omega}$ a decreasing sequence in σ_0 such that $U_0 = U$, $U_{n+1} - U_{n+1} \subset U_n$. Then $V = \bigcap_{n<\omega} U_n \in \sigma_0'$, $0 \in V \subset U$ is clearly a subgroup of M.

2.6 DEFINITION: [G1]. Let $\phi_1, \ldots, \phi_m$, $\psi_1, \ldots, \psi_k$ be tppf's with free variables among $x_1, \ldots, x_p$, $U_1, \ldots, U_n$. For each s, $0 \leq s \leq n$, let $\sigma(\phi_1, \ldots, \phi_m; \psi_1, \ldots, \psi_k; s)$ denote the following formula of L_t:

$$(\forall U_{s+1})_0(\exists V_{s+1})_0 \ldots (\forall U_n)_0(\exists V_n)_0(\forall x_1,\ldots,x_p)$$
$$\bigwedge_{i=1}^{k} \psi_i(x_1, \ldots, x_p, V_1, \ldots, V_n) \to$$
$$\to \bigvee_{i=1}^{m} \phi_i(x_1, \ldots, x_p, U_1, \ldots, U_n)] .$$

Note that each V_i $(1 \leq i \leq s)$ occurs only negatively as a free variable, each U_i occurs only positvely as a free variable, and no other variables are free. I abbreviate as follows: Let $\Phi ::= \{\phi_1, \ldots, \phi_m\}$, $\Psi ::= \{\psi_1, \ldots, \psi_k\}$; then $\sigma(\phi_1, \ldots, \phi_m; \psi_1, \ldots, \psi_k; s) :=:$ $\sigma(\Phi; \psi; s)(U_1^+, \ldots, U_s^+; V_1^-, \ldots, V_s^-)$.

<u>2.7 LEMMA</u>: (a) [G1, 4, Lemma 4] Let $<M, \tau>$ be a weak model of the axioms for topological groups, κ-saturated in the two-sorted sense, $|L|^+ + |M| + |\tau| \leq \kappa$. Then there is a sequence $(A_\beta)_{\beta<\kappa}$ of elements of τ_0' such that

(i) $A_\beta \supset A_\gamma$ for $\beta < \gamma < \kappa$.

(ii) Each A_β is a subgroup of M .

(iii) If $A \in \tau_0$ then there is $\beta < \kappa$ such that $A_{\beta'} \subset A$.

(iv) For all tppf $\phi_1, \ldots, \phi_n$, $\psi_1, \ldots, \psi_k$ if $<M, \tau> \models \sigma(\Phi; \Psi; 0)$ then for all s , all $\beta_1 < \ldots < \beta_s < \kappa$, $<M, \tau^*> \models \sigma(\Phi; \Psi; s)[A_{\beta_1}, \ldots, A_{\beta_s}; A_{\beta_1}, \ldots, A_{\beta_s}]$.

(v) For all tppf $\phi_1, \ldots, \phi_n$, $\psi_1, \ldots, \psi_k$ if $<M, \tau> \models \neg\sigma(\Phi; \Psi; 0)$ then for all s , all $\beta_1 < \ldots < \beta_s < \kappa$, $<M, \tau> \models \neg\sigma(\Phi; \Psi; s)[A_{\beta_1}, \ldots, A_{\beta_s}; A_{\beta_1}, \ldots, A_{\beta_s}]$.

(b) (From the proof of [G4, 4, Theorem 1]) Suppose $<M, \tau>$ is as described in (a) and in fact $\tau_0 = \{A_\beta | \beta < \kappa\}$. Suppose $\phi(x, U_1, \ldots, U_n)$, $\psi(x, U_1, \ldots, U_n)$ are tppf's, $\gamma_1 < \ldots < \gamma_n < \kappa$, $\beta_1 < \ldots < \beta_n < \kappa$. Then:

$$|\phi[M, A_{\beta_1}, \ldots, A_{\beta_n})/(\phi \wedge \psi)[M, A_{\beta_1}, \ldots, A_{\beta_n})| =$$

$$= |\phi[M, A_{\gamma_1}, \ldots, A_{\gamma_n})/(\phi \wedge \psi)[M, A_{\gamma_1}, \ldots, A_{\gamma_n})| \text{ if either is finite.}$$

REMARKS AND PROOF: Part (a) is a generalization and strengthening of (2.5(ii)). I refer the reader to Garavaglia's paper for the proof. Note that (ii) and (iii) say that $\{A_\beta | \beta < \kappa\}$ is a basis of open subgroups at 0, so it is possible to assume in (b) that $\tau_0 = \{A_\beta | \beta < \kappa\}$.

For (b), note that the left-hand side is $\leq m$ iff $M \models \sigma(\Phi; \Psi; n)[A_{\beta_1}, \ldots, A_{\beta_n}; A_{\beta_1}, \ldots, A_{\beta_n}]$ where $\Phi ::= \{\phi(x_i, U_1, \ldots, U_n) | 1 \leq i \leq m + 1\}$ and $\Psi ::= \{\psi(x_i - x_j, U_1, \ldots, U_n) | 1 \leq i < j \leq m + 1\}$ (and similarly for the right-hand side), so (b) follows immediately by (a(iv)) and (a(v)).

2.8 LEMMA: [G1, 4, lemma 6]. If G, G' are abelian groups, $S, S_1, \ldots, S_n, S', S'_1, \ldots, S'_n$ are subgroups of G, G' respectively, $g, g_1, \ldots, g_n \in G$, $g', g'_1, \ldots, g'_n \in G'$, for every $C \subset \{1, \ldots, n\}$, $(g + S) \cap \bigcap_{i \in C}(g_i + S_i) = \emptyset$ iff $(g' + s') \cap \bigcap_{i \in C}(g'_i + S'_i) = \emptyset$ and for every $C \subset \{1, \ldots, n\}$ $|\bigcap_{i \in C}(S \cap S_i) / \bigcap_{i=1}^n (S \cap S_i)| =$

$$= |\bigcap_{i \in C}(S' \cap S'_i) / \bigcap_{i=1}^n (S' \cap S'_i)| \quad \text{then}$$

$$g + S = \bigcup_{i=1}^n ((g + S) \cap (g_i + S_i)) \text{ iff } g' + S' = \bigcup_{i=1}^n ((g' + S') \cap (g'_i + S'_i))$$

REMARK: This is essentially the same as the main combinatorial/group-theoretical step in Ziegler's proof of Baur's theorem [Z, Theorem 1.1] and indeed, in that case is practically the whole proof. Here much more is needed. It is pointed out in Garavaglia's proof that without loss of generality, all the cardinalities mentioned are finite.

2.9 THEOREM. Let $\langle M, \tau \rangle$ be a topological module. Then for every $\vec{c}::\vec{d} \in M$, $\langle M, \tau, \vec{c} \rangle \equiv_{tppf} \langle M, \tau, \vec{d} \rangle$ implies $\langle M, \tau, \vec{c} \rangle \equiv_t \langle M, \tau, \vec{d} \rangle$.

PROOF: (See [G1, 4, Theorem 1]) Without loss of generality I can assume that $\langle M, \tau\rangle$ is a κ-saturated weak structure, $|L|^+ + |M| + |\tau| \leqq \kappa$ and $\tau_0 = (A_\beta)_{\beta<\kappa}$ is a described in Lemma 2.7 .

Now I construct sequences $\{a_\beta | \beta < \kappa\}$ and $\{b_\beta | \beta < \kappa\}$ exhausting M, and increasing functions $p, q: \kappa \to \kappa$ such that for all tppf $\phi(x_1, \ldots, x_m, \vec{y}, U_1, \ldots, U_m)$, $\vec{y}::\vec{c}$, and all $\beta_1 < \ldots < \beta_m < \kappa$, $\langle M, \tau\rangle = \psi[a_{\vec{\beta}}, \vec{c}, A_{p(\vec{\beta})}]$ iff $\langle M, \tau\rangle = \phi[b_{\vec{\beta}}, \vec{d}, A_{q(\vec{\beta})}]$. (Here and elsewhere I abbreviate as follows: $a_{\vec{\beta}} :=: a_{\beta_1}, \ldots, a_{\beta_m}$; $A_{p(\vec{\beta})} :=: A_{p(\beta_1)}, \ldots, A_{p(\beta_m)})$. Note that this construction will establish the theorem since then the map $\alpha: a_\beta \to b_\beta$ is a topological automorphism of $\langle M, \tau\rangle$ taking $\vec{c} \mapsto \vec{d}$, because the necessary conditions can be expressed by tppf's.

Well order M in order type κ . I procede by recursion on $\gamma < \kappa$. Notice that the initial condition $(\gamma = 0)$ says simply "for all tppf $\phi(\vec{y})$, $\langle M, \tau\rangle \models \phi[\vec{c}]$ iff $\langle M, \tau\rangle \models \phi[\vec{d}]$" which is precisely the hypothesis of the theorem.

Suppose that the construction has been carried out for all $\beta < \gamma$ and $\gamma < \kappa$ is even. (When γ is odd carry out similar steps with the roles of a_γ, p and b_γ, q interchanged.) Let a_γ be the first element of $M \setminus \{a_\beta | \beta < \gamma\}$ if non-empty, 0 otherwise. Now I choose b so that for every tppf $\phi(x_1, \ldots, x_{m+1}, \vec{y}, U_1, \ldots, U_m)$ and all $\beta_1 < \ldots < \beta_m < \gamma$, $\langle M, \gamma\rangle \models \phi[a_{\vec{\beta}}, a_\gamma, \vec{c}, A_{p(\vec{\beta})}]$ iff $\langle M, \tau\rangle \models \phi[b_{\vec{\beta}}, b_\gamma, \vec{d}, A_{q(\vec{\beta})}]$. Since $\langle M, \tau\rangle$ is saturated, it is sufficient to show that a certain set of formulas (determined by this condition) is finitely consistent. In terms of the cosets defined by the tppf's involved (see the remarks to (2.4)), it is enough to show that for all tppf $\phi(x_1, \ldots, x_{n+1}, \vec{y}, U_1, \ldots, U_n)$

and $\psi_i(\vec{x}, \vec{y}, \vec{U})$ $(1 \leq i \leq m)$ and all $\beta_1 < \ldots < \beta_n < \gamma$, if $\phi(a_{\vec{\beta}}, [M], \vec{c}, A_{p(\vec{\beta})}) \neq \bigcup_{i=1}^{m}(\phi \wedge \psi_i)(\vec{a}_\beta, [M], \vec{c}, A_{p(\vec{\beta})})$ then $\phi(b_{\vec{\beta}}, [M], \vec{d}, A_{q(\vec{\beta})}) \neq \bigcup_{i=1}^{m}(\phi \wedge \psi_i)(b_{\vec{\beta}}, [M], \vec{d}, A_{q(\vec{\beta})})$.

For this I use (2.8). (For the subgroups referred to in (2.8) take those defined by the above formulas with the individual parameters replaced by 0's.) The cardinality conditions in (2.8) are satisfied by (2.7(b)). For the other condition of (2.8), let $C \subset \{1, \ldots, m\}$. Then $\bigcap_{i\in C}(\phi \wedge \psi_i)(b_{\vec{\beta}}, [M], \vec{d}, A_{q(\vec{\beta})}) = \emptyset$ iff $\langle M, \tau\rangle \models \neg(\exists x_{n+1}) \bigwedge_{i\in C}(\phi \wedge \psi_i)$ $[b_{\vec{\beta}}, x_{n+1}, \vec{d}, A_{q(\vec{\beta})}]$ iff (induction hypothesis) $\langle M, \tau\rangle \models \neg(\exists x_{n+1}) \bigwedge_{i\in C}(\phi \wedge \psi_i)[a_{\vec{\beta}}, x_{n+1}, \vec{c}, A_{p(\vec{\beta})}]$ iff $\bigcap_{i\in C}(\phi \wedge \psi_i)(a_{\vec{\beta}}, [M], \vec{c}, A_{p(\vec{\beta})}) = \emptyset$.

Hence b_γ can be chosen as specified. Now I choose $p(\gamma)$ and $q(\gamma)$ so that the induction hypothesis is verified for all $\beta \leq \gamma$.

To choose $p(\gamma)$, $q(\gamma)$ first note that set variables appear only positively in tppf's so that if $\langle M, \tau\rangle \models \phi[A_\alpha]$ and $\beta < \alpha$ then $\langle M, \tau\rangle \models \phi[A_\beta]$. There are $\leq \aleph_0 \cdot |L| \cdot |\gamma| < \kappa$ formulas $\phi(a_{\vec{\beta}}, a_\gamma, \vec{c}, A_{p(\vec{\beta})}, V)$ with $\beta_1 < \ldots < \beta_n < \gamma$, so by choosing $p(\gamma)$ and $q(\gamma)$ large enough (that is, $A_{p(\gamma)}$ and $A_{q(\gamma)}$ small enough) I can ensure that for all tppf $\phi(x_1, \ldots, x_{n+1}, \vec{y}, U_1, \ldots, U_{n+1})$ and all $\beta_1 < \ldots < \beta_n < \gamma$,

$$\langle M, \tau\rangle \models \phi[a_{\vec{\beta}}, a_\gamma, \vec{c}, A_{p(\vec{\beta})}, A_{p(\gamma)}]$$

iff $\langle M, \tau\rangle \models (\forall U_{n+1})_0\ \phi[a_{\vec{\beta}}, a_\gamma, \vec{c}, A_{p(\vec{\beta})}, U_{n+1}]$

iff (by choice of b_γ)

$$\langle M, \tau\rangle \models (\forall U_{n+1})_0\ \phi[b_{\vec{\beta}}, b_\gamma, \vec{d}, A_{q(\vec{\beta})}, U_{n+1}]$$

iff $\langle M, \tau\rangle \models \phi[b_{\vec{\beta}}, b_\gamma, \vec{d}, A_{q(\vec{\beta})}, A_{q(\gamma)}]$

as was required.

<u>2.10 COROLLARY</u>: Every formula $\phi(\vec{x})$ of L_t with only individual variables free is equivalent to a Boolean combination of tppf's.

<u>PROOF</u>: This follows immediately by a standard compactness argument. See for instance the proofs of the various preservation theorems in [CK], section 5.2 and especially Lemma 3.2.1 of [CK].

<u>2.11 DEFINITION</u>: (See III 1.8). Let $\langle M, \tau \rangle$ be a topological module, $A \subset M$, $\vec{b} \in M$. Let $p := t(\vec{b}, A)$ where $t(\vec{b}, A) ::= \{\phi(\vec{x}, \vec{a}) | \phi(\vec{x}, \vec{y}) \in L_t$, $\vec{a} \in A$, $\langle M, \tau \rangle \models \phi[\vec{b}, \vec{a}]\}$.

Then $p^+ ::= t^+(\vec{b}, A) ::= \{\phi(\vec{x}, \vec{a}) \in p | \phi(\vec{x}, \vec{y}) \text{ tppf}\}$

$p^- ::= t^-(\vec{b}, A) ::= \{\neg\, \phi(\vec{x}, \vec{a}) \in p | \phi(\vec{x}, \vec{y}) \text{ tppf}\}$

$p^\pm ::= t^\pm(\vec{b}, A) ::= p^+ \cup p^-$

<u>2.12 COROLLARY</u>: (Notation as in 2.11). (i) p^+ uniquely determines p^-, and conversely.

(ii) $p^\pm \vdash p$

<u>2.13 DEFINITION</u>: (Recall (1.17), (1.18) for L^*, etc.) Let $L' = L \cup \{R_\phi \in L^* | \phi \text{ a tppf of } L\} \subset L^*$. Define ϕ', Σ', T', $\mathfrak{C}'$ by analogy with ϕ^*, Σ^*, T^*, $\mathfrak{C}^*$.

<u>2.14 COROLLARY</u>: (i) If $\phi \in L_t$ is a tppf, then $\phi^* \in L'$ and $\phi^* = \phi'$ is a ppf of L'.

(ii) L' is a language for modules (III 1.1).

(iii) Every formula of L^* is equivalent to a formula of L', in particular to a Boolean combination of ppf's of L'.

(iv) Up to equivalence, $'$ defines a bijection from the formulas of L_t to the formulas of L'.

(v) Theorem 1.18(ii)-(ix) holds for ' in place of * .

PROOF: (i) is obvious. (ii) follows from (2.4) and (2.5) taking a sufficiently saturated extension (thus in any topological module, a tppf with only individual variables free defines a subgroup). (iii) follows from (2.10) and (1.18(ii)). (iv) and (v) are then immediate.

2.15 COROLLARY: (See III 1.11)

(i) Every topological module is stable.

(ii) $\langle M, \tau \rangle$ is tt iff there is no infinite descending sequence of subgroups of M definable by tppf's.

(iii) $\langle M, \tau \rangle$ is ss iff there is no infinite descending sequence of subgroups of M, each of infinite index in its predecessor, definable by ttpf's.

2.16 REMARKS: Thus the results of Chapter III apply to topological modules via the correspondence between L_t and L'. As well, the results (1.22) and (1.22') apply to topological modules. Furthermore, this correspondence provides an opportunity for investigating by model-theoretic means the interaction between the algebraic structure of a topological module and its topology.

For example, consider one of the standard topological groups: the circle groups $C = \mathbb{R}/\mathbb{Z}$. The first order structure is tt and, following the results of Chapter III, is decomposed uniquely as $\oplus_{p \text{ prime}} \mathbb{Z}(p^\infty) \oplus \mathbb{Q}^{(c)}$. The unique copies of each $\mathbb{Z}(p^\infty)$ are rigidly fixed in C and dense in the topology of C. Thus if the topological group C is written as a direct sum, $C = A \oplus B$ with A, B closed, either A or B must contain the copy of $\mathbb{Z}(2^\infty)$, say, and by density is then all of C. Thus the ordinary decomposition theory shows that the circle group is topogically indecomposable.

By the L_t-analogue of (III, 4.2), C has a unique decomposition as a direct sum of indecomposable modules, all of these concepts being taken in the sense of L_t structures (weak models) rather than topologically. What is this decomposition and how does it relate to the first-order decomposition and the topology? Since C is compact, the following results answer this question.

2.17 REMARKS: The following facts about convergence in topological spaces can be found in Chapters 2 and 3 of Kelley's text [Ke]. Kelley presents these facts in terms of convergence via nets; I find it more convenient here to treat convergence via filters, for which see [Ke, 2, Problem L].

Let F be a filter of subsets of a topological space X . F converges to $c \in X$, $\lim(F) = c$, iff every open neighbourhood of c is in F . If $C \subset X$, F is in C iff for all $A \in F$, $A \cap C \neq \emptyset$. The following hold:

(i) $f: X \to Y$ is continuous iff for all filters F , $\lim(F) = c$ in X implies that $\lim(f[F]) = f(c)$ in Y .

(ii) $C \subset X$ is closed iff for all F in C , $\lim(F) = c$ implies $c \in C$.

(iii) X is Hausdorff iff for all F , $\lim(F) = c$ and $\lim(F) = c'$ implies $c = c'$.

(iv) X is compact iff every ultrafilter converges to some point of X .

2.18 DEFINITION: Let $\phi(\vec{x})$ be a tppf of L_t with only individual variables free. Let $\phi_0(\vec{x}, \vec{y}, \vec{Y})$ be the open matrix of ϕ . Let $\underline{\phi}(\vec{x})$ be $(\exists\vec{y})(\forall\vec{Y})_0\phi_0$, let $\overline{\phi}(\vec{x})$ be $(\forall Y)_0(\exists\vec{y})\phi_0$, and let $\hat{\phi}(\vec{x})$ be $(\exists\vec{y})\hat{\phi}_0$, where $\hat{\phi}_0$ is obtained from ϕ_0 by replacing every conjunct "$t \in Y$" , t some term of L , by "$t = 0$" .

2.19 LEMMA: (Notation as in 2.18)

(i) "$\psi_0 \leq \psi_1$" iff $\models_t (\forall\vec{x})(\psi_0 \to \psi_1)$ is a preorder on the set of tppf's $\phi(\vec{x})$ of L_t with the same free variables as ϕ and open matrix ϕ_0 . $\underline{\phi}$ is the least element (strongest element) of this order and $\overline{\phi}$ is the greatest element (weakest element). In particular $\models_t (\forall\vec{x})[\underline{\phi} \to \psi \wedge \psi \to \overline{\phi}]$.

(ii) $\models_t (\forall\vec{x})[\hat{\phi} \to \underline{\phi}]$.

PROOF: Obvious.

2.20 THEOREM: Let T be a complete L_t theory of Hausdorff topological modules with a (topologically) compact model. Then for every tppf $\phi(\vec{x})$ of L_t

$$T \models_t (\forall\vec{x})[\overline{\phi} \to \hat{\phi}] .$$

Thus T has complete elimination of set quantifiers from formulas with only individual variables free.

PROOF: The final comment follows by (2.19) and (2.10).

Let $\langle M, \tau\rangle$ be a compact model of T , $\tau = \tau^*$. $\overline{\phi}$ is a formula of the form $(\forall\vec{Y})_0(\exists\vec{y}) \bigwedge_j \phi_j(\vec{x}, \vec{y}, \vec{Y})$ where each ϕ_j is an atomic formula of L_t , hence of one of the following forms:

(a) $t_j(\vec{x}, \vec{y}) = 0$

(b) $R(t_j^1(\vec{x}, \vec{y}), \ldots, t_j^k(\vec{x}, \vec{y}))$

(c) $t_j(\vec{x}, \vec{y}) \in Y$, some Y in $\vec{Y}$

where the t's are terms of L and R is a relation symbol of L .

Let $\tau_0 = \{U \in \tau | 0 \in U\}$, a neighbourhood basis around 0. Fix $\vec{a} \in M$, $\vec{a}::\vec{x}$, and assume that $\langle M, \tau\rangle \models (\forall\vec{Y})_0(\exists\vec{y}) \bigwedge_j \phi_j(\vec{a}, \vec{y}, \vec{Y})$. Suppose $\vec{y}::n$

and consider the compact space M^n. For each $\vec{U} \in \tau_0$, $\vec{U}::\vec{Y}$, let $B_{\vec{U}} ::= \{\vec{b} \in M^n | \langle M, \tau\rangle \models \bigwedge_j \phi_j(\vec{a}, \vec{b}, \vec{U})\}$. By assumption each $B_{\vec{U}}$ is non-empty. Since the Y's appear only positively in each tppf ϕ_j, $\vec{U} \subset \vec{U}'$ (componentwise) implies that $B_{\vec{U}} \subset B_{\vec{U}'}$. By the definition of τ_0, if $\vec{U}^0, \vec{U}^1 \in \tau_0$ and $\vec{U} = \vec{U}^0 \cap \vec{U}^1$ (componentwise) then $\vec{U} \in \tau_0$ and by the preceding remark, $B_{\vec{U}} \subset B_{\vec{U}^0} \cap B_{\vec{U}^1}$. Thus $\mathcal{B} = \{B_{\vec{U}} | \vec{U} \in \tau_0\}$ is a filter base in M^n.

Therefore $\mathcal{B}$ may be extended to an ultrafilter $\mathcal{U}$ over the compact space M^n, and $\mathcal{U}$ converges to a unique limit $\vec{b}$ since M is Hausdorff. For each term $t(\vec{x}, \vec{y})$ occurring in the ϕ_j's, $\{\{t(\vec{a}, \vec{c}) | \vec{c} \in C\} | C \in \mathcal{U}\}$ is a filter converging to $t(\vec{a}, \vec{b})$ by the continuity of terms. If ϕ_j is of form (a), then $t_j(\vec{a}, \vec{b}) = 0$ by continuity; if ϕ_j is of form (b), then $R(t_j^1(a, \vec{b}), \ldots, t_j^k(\vec{a}, \vec{b}))$ holds since R is closed; and if ϕ_j is of form (c) then $t_j(\vec{a}, \vec{b}) = 0$ by convergence, since for each $\vec{U} \in \tau_0$, $\{t_j(\vec{a}, \vec{c}) | \vec{c} \in B_{\vec{U}}\} \subset \bigcup \vec{U}$. Thus b witnesses that $\langle M, \tau\rangle \models \hat{\phi}[\vec{a}]$.

<u>2.21 REMARKS</u>: Let T be a complete L_t-theory of Hausdorff topological modules with a compact model $\langle M, \tau\rangle$ and let $T°$ be the ordinary theory of M in L. Then the stability theory of T is essentially the same as the stability theory of $T°$.

Theorem 2.20 thus can be regarded as a strong reduction theorem in the style of (1.18) and (2.14). In particular it says that every topologically definable subset of a compact Hausdorff topological module is already first order definable (in the sense of L_t and L respectively).

Theorem 2.20 can be generalized somewhat to arbitrary topological structures $\langle M, \tau\rangle$ with continuous functions, closed relations, and at least one constant symbol. A tppf is defined as before, but with the universal set quantifiers restricted to a constant term of L_t. If $\langle M, \tau\rangle$ is compact Hausdorff, T is the L_t theory of $\langle M, \tau\rangle$ and $\phi(x)$ is a tppf as before, $T \models_t (\forall\vec{x})[\phi \leftrightarrow \hat{\phi}]$. Of course the second part of (2.20) depends on the tpp-elimination of quantifiers (2.10). A stronger statement along these lines seems unlikely by the present methods since the proof depends heavily on the form of tppf's.

A generalization in a different direction might be possible. It follows from [FZ, II, 2.9, 2.10] that the L_t theory of the topological group of the reals has complete elimination of quantifiers (Flum and Ziegler suggest a proof by a saturated model method). $\mathbb{R}$ is locally compact and complete. Here I mean completeness in the sense of complete uniform spaces, see [Ke, Ch. 6]. Thus $\mathbb{R}$ is very close to being compact.

PROBLEM: Does theorem 2.20 hold when T has a locally compact complete model? The problem with applying the current proof occurs at the crucial point of showing that $\mathcal{B}$ may be extended to a convergent filter.

I conjecture that the answer to the problem is in the affirmative.

V. EXTENSIONS OF DEISSLER'S RANK WITH APPLICATIONS TO MODULES

V 0. INTRODUCTION

In [D], R. Deissler introduced a "minimality rank" which I denote by "rk" here. This rank provides an ordinal measure on the difficulty of defining a given element b in a structure M: if $A \subseteq M$, $rk(b, A, M) = 0$ iff b is definable in M by a formula with parameters from A. Roughly speaking, in the general case $rk(b, A, M)$ measures how hard we have to work at adding new parameters to A from definable sets in order to be able to define b. "rk" is called a "minimality rank" because of the following: M is a minimal model of the complete theory $T = Th(M)$ iff $rk(b, \emptyset, M) < \infty$ for every $b \in M$. Deissler's rank was studied further by R. Woodrow and J. Knight [WK].

The central concept underlying Deissler's rank is that of a definable set. In section 1 I introduce the idea of a <u>context for definability</u> $\Phi(x)$. A set B is Φ-definable over A if for some $\phi(x, \vec{v}) \in \Phi(x)$ and $\vec{a} \in A$, $B = \phi[M, \vec{a})$. For Deissler's rank, Φ is the set of all formulas; for rk^+ used in the study of modules, Φ^+ is the set of all ppf's. Associated with each Φ is a relation $\prec_\Phi$ between structures which says that Φ-definitions are preserved. In section 1 I develop the basic properties of these two concepts and give a list of examples.

Associated with each context for definability $\Phi(x)$ is a Deissler-type rank rk^Φ, although for rk^Φ to have all the nice properties of Deissler's rk I must impose additional restrictions on $\Phi(x)$. With these restrictions I obtain: M is minimal among all structures for the same language, ordered by $\prec_\Phi$, iff $rk^\Phi(b, \emptyset, M) < \infty$ for every $b \in M$. In particular, if E is an injective module over a Noetherian ring and $A \subset E$ then E is the injective envelope of A iff $rk^+(b, A, E) < \infty$ for all $b \in E$. Section 2 develops these ideas.

In section 3 I introduce the concept of an analysis of rk^Φ ; this is a tree structure which records the steps taken in the recursive computation of $rk^\Phi(b, A, M)$. In attempting to provide estimates of $rk^\Phi(b, A, M)$, such tree structures are often easier to deal with than the original definition, although all the arguments using analyses could in practice be converted to inductive arguments which appeal directly to the definition of rk^Φ. I prove several elementary lemmas on rk^Φ and on analyses, and prove a simple rank inequality (3.9) which turns out to be quite useful in estimating upper bounds.

In section 4 I begin to concentrate more on modules and I establish some relationships between rk and rk^+ in modules. I also prove a theorem relating the Lascar rank U to rk: If M is the minimal model of a tt theory T, then for any $b \in M$, $rk(b, A, M) \leq \omega \cdot (U(b, A) + 1)$, and as a consequence I give a substantial generalization of a result of Deissler's stated but not proved in [D], and answer a question posed about this result, namely the relation between $rk(M)$ and α_T.

Section 5 is devoted to the study of rk^+ in injective modules over a commutative Noetherian ring Λ. If E is an indecomposable injective, $E = E(\Lambda/P)$ for P a prime ideal of Λ, then the rk^+ of E over Λ/P is 1 or 2, and rk^+ of $E^{(\kappa)}$ over $(\Lambda/P)^{(\kappa)}$ is bounded by ω for any cardinal κ. Finally, if P is a set of prime ideals of Λ, I define the Krull dimension ρ of P, and show that for A some direct sum $\oplus_{P \in P}(\Lambda/P)^{(\alpha_P)}$ and E the injective envelope of A, then $rk^+(b, A, E) < \omega^{\rho(P)}$ for all $b \in E$.

I would like to thank Robert Woodrow for listening to me talk through the first draft of this chapter and for offering some constructive criticism on the work presented here.

V 1. A GENERAL CONTEXT FOR THE STUDY OF DEFINABLE SETS

1.0 DEFINITION: Let L be a fixed first order language. A context for definability in L is a set $\Phi ::= \Phi(x)$ of formulas $\phi(x, \vec{v})$ of L with the one distinguished variable "x" and "$\vec{v}$" an arbitrary string of parameter variables (it is understood that "x" is distinct from any parameter variables or bound variables appearing in ϕ) and satisfying:

(i) Φ is closed under the arbitrary renaming of parameter variables;

(ii) Φ contains all formulas "$x = t(\vec{v})$" where t is some term of L ;

(iii) If $\phi(x, u, \vec{v}) \in \Phi$ and $\psi(x, \vec{w}) \in \Phi$ then $(\exists y)[\phi(x, y, \vec{v}) \wedge \psi(y, \vec{w})] \in \Phi$ (where y is some variable not occurring in ϕ, ψ). The formula resulting from the application of (iii) is denoted by $\phi \circ \psi(x, \vec{v}, \vec{w})$ for reasons which will be made clear below.

1.1 DEFINITION: Let $L, \Phi(x)$ be as in (1.0), let M be an L-structure, let $A, C \subset M$, $c \in M$.

(i) C is Φ-definable over A in M iff for some $\phi(x, \vec{v}) \in \Phi$, $\vec{a} \in A$, $C = \phi[M, \vec{a})$.

(ii) c is Φ-algebraic of degree $n < \omega$ over A in M iff for some C , $|C| = n$, $c \in C$, C is Φ-definable over A in M and n is the least cardinal for which such a C exists.

(iii) c is Φ-definable over A in M iff c is Φ-algebraic of degree 1 over A in M , i.e. iff $\{c\}$ is Φ-definable over A in M .

(iv) $A^{\Phi} ::= \{c \in M | c$ is Φ-definable over A in $M\}$.

(v) $\Phi(A) ::= \{\phi(x, \vec{a}) | \phi(x, \vec{v}) \in \Phi , \vec{v} :: \vec{a} \in A\}$.

1.2 REMARKS: (Continue the notation of 1.0, 1.1).

(i) In the phrase "Φ-definable over A in M" , I suppress "M" when the context makes clear what $\mathcal{M}$ is, or when a specific $\mathcal{M}$ is irrelevant; and I suppress "A" when $A = \emptyset$.

(ii) Note that neither M nor $\emptyset$ are necessarily Φ-definable over A .

(iii) Rule (ii) of definition 1.0 implies that any element of M computable from A by a term of L is Φ-definable over A . This rule provides a bare minimum of definable sets: I do not even require that sets definable by a predicate of L be Φ-definable (that is, sets $\{m \in M | \mathcal{M} \models R[m]\}$).

(iv) Rule (iii) may be extended by a simple induction to the following rule:

1.0(iii'): If $\phi(x, u_0, \ldots, u_{n-1}, \vec{v}) \in \Phi$ and for $i < n$, $\psi_i(x, \vec{w}_i) \in \Phi$, then $\phi\circ\langle\psi_i\rangle_{i<n}(x, \vec{v}, \vec{w}_0, \ldots, \vec{w}_{n-1}) \in \Phi$ where $\phi\circ\langle\psi\rangle ::= (\exists y_0, \ldots, y_{n-1})$ $[\phi(x, y_0, \ldots, y_{n-1}, \vec{v}) \wedge \bigwedge_{i<n}\psi_i(y_i, \vec{w}_i)]$.

(v) The rule (iii') essentially says that the composition of Φ-definitions over A is again a Φ-definition over A . For instance, suppose that ϕ is "$x = f(u_0, \ldots, u_{n-1}, \vec{v})$" and for $i < n$, ψ_i is "$x = g_i(\vec{w}_i)$" where f, g are function symbols of L , then $\phi\circ\langle\psi_i\rangle$ is equivalent to "$x = f(g_0(\vec{w}_0), \ldots, g_{n-1}(\vec{w}_{n-1}), \vec{v})$" . When the ψ_i's do not have unique solutions, we must take a more liberal definition of the word "composition" in this remark.

(vii) Another way of expressing a consequence of rule (iii') is the following: suppose $c_0, \ldots, c_{n-1}$ are Φ-definable over A and C is Φ-definable over $A \cup \{c_0, \ldots, c_{n-1}\}$. Then C is Φ-definable over A .

(viii) It is often useful to consider additional conditions on Φ beyond those of definition 1.0. In particular, closure under conjunctions and closure under existential quantification of parameter variables are natural conditions.

1.3 DEFINITION: Let M, N be L-structures, M a substructure of N, and $\Phi(x)$ a context for definability in L . $M \prec_\Phi N$ iff for all $\phi(x, \vec{v}) \in \Phi$ and $m, \vec{n} \in M$,

$$M \models \phi[m, \vec{n}] \Leftrightarrow N \models \phi[m, \vec{n}] .$$

I say that M is a Φ-elementary substructure of N.

1.4 PROPOSITION:

(i) $M \prec_\Phi M$

(ii) $M_0 \prec_\Phi M_1$ and $M_1 \prec_\Phi M_2$ implies $M_0 \prec_\Phi M_2$

(iii) $M_0 \prec_\Phi M_2$, $M_1 \prec_\Phi M_2$ and $M_0 \subset M_1$ imply that $M_0 \prec_\Phi M_1$

(iv) $M \prec_\Phi N$, $A \subset M$, $\phi(x) \in \Phi(A)$ imply that $\phi[M] = \phi[N] \cap M$.

PROOF: All obvious.

1.5 DEFINITION: Let $\Phi(x)$ be a context for definablity in L . Φ is T-V if $\prec_\Phi$ satisfies a Tarski-Vaught type criterion: for an L-structure N and $M \subset N$, M is a Φ-elementary substructure of N under the operations and relations induced from N iff for all $\phi(x) \in \Phi(M)$, $N \models (\exists x)\phi(x) \Rightarrow$ for some $m \in M$, $N \models \phi[m]$.

1.6 REMARKS: If L has no relation symbols, it suffices in definition 1.3 to assume that $M \subset N$. The same applies if Φ contains all the atomic formulas of L . If M, N are as in (1.5) and satisfy the condition of the Tarski-Vaught criterion, then definition 1.0(ii) ensures that M is closed under the operations of L .

1.7 DEFINITION: (Examples of Φ's)

(i) Φ^0 is the least set of formulas satisfying definition 1.0.

(ii) Φ^1 is the least set of formulas containing all the atomic formulas $\alpha(x, \vec{v})$ of L and satisfying the conditions of definition 1.0.

(iii) Φ^+ is the set of all formulas of L logically equivalent to a positive primitive formula of L .

(iv) Φ^p is the set of all formulas of L logically equivalent to a primitive formula of L (<u>primitive</u> formulas are defined like positive primitive formulas but allowing negated atomic formulas to appear).

(v) $\Phi^\exists$ is the set of all formulas of L logically equivalent to an existential formula of L .

(vi) Φ^L is the set of all formulas of L .

<u>1.8 LEMMA</u>:

(i) $\Phi^0 \subset \Phi^1 \subset \Phi^+ \subset \Phi^p \subset \Phi^\exists \subset \Phi^L$ and there is a language L in which all the inclusions are strict.

(ii) $\Phi^0, \Phi^1, \Phi^+, \Phi^p, \Phi^\exists, \Phi^L$ are all contexts for definability in L .

<u>PROOF</u>: Obvious. Note that it is possible in (i) to have some equalities.

<u>1.9 REMARKS</u>: (Meanings of some of the Φ's and $\prec_\Phi$'s)

(i) c is Φ^0-definable over A in M iff c is computable from A in M by some term of L . This follows from (1.2(iii), (iv)), which establish from the definition of Φ^0 that Φ^0 contains only formulas logically equivalent to "$x = t(\vec{v})$" for some term t of L . There are no other Φ^0 definable subsets over A in M . $M \prec_{\Phi^0} N$ iff M is a substructure of N .

(ii) Φ^1 contains, in addition to the atomic formulas $x = t(\vec{v})$ of L , atomic formulas $s(x, \vec{v}) = t(x, \vec{v})$ and $R(x, \vec{v})$ for s, t terms of L and R a predicate symbol of L . Thus in Φ^1 , sets of solutions to equations and relations are definable sets, and there is no general simple characterization, independent of L , of when an element is Φ^1-definable. Repeated application of rule (1.0(iv) may yield very complicated definable sets.

(iii) Φ^+ is naturally applicable to theories of modules by the Baur-Monk elimination of quantifiers (III 1.6). Φ^+ will be the major example studied in this chapter, with some quite specific calculations in section 5. As observed in Chapter III, the Φ^+-definable subsets of a module M are cosets of a Φ^+-definable subgroup or empty. In this context, $M \prec_{\Phi^+} N$ iff $M \prec_{\mathrm{ppf}} N$ iff M is a pure submodule of N .

In a more general context, recall the following definition (e.g. [ES, 7.1]): M is algebraically closed in N iff M is a substructure of N and every finite set of equations over M which has a solution in M has a solution in M . Thus in general, $M \prec_{\Phi^+} N$ iff M is algebraically closed in N .

(iv) Recall the following definition (e.g. [ES, 7.2]): M is existentially closed in N iff M is a substructure of N and every finite system of equations and inequations over M which has a solution N has a solution in M . It is easy to see that M is existentially closed in its extension N iff any existential sentence over M true in N is true in M (whence the name). Thus $M \prec_{\Phi^\exists} N$ iff $M \prec_{\Phi^p} N$ iff M is existentially closed in N .

(v) An element or set is Φ^L definable iff it is first order definable in the usual sense. $M \prec_{\Phi^L} N$ iff $M \prec N$.

1.10 PROPOSITION: If L has only function and constant symbols then Φ^0 is T-V. Φ^+, Φ^p, $\Phi^\exists$, Φ^L are all T-V.

REMARK: The situation for Φ^1 is problematic and Φ^1 may or may not be T-V depending on L .

PROOF: For Φ^0 by the remarks (1.9(i)) if $M \subset N$, $\vec{a} \in M$, then $\phi(x, \vec{a})$ is either inconsistent or has a unique solution $t(\vec{a})$ for some term t of L . M is the underlying set of a substructure of N iff all such solutions lie in M (since L has no relation symbols). Thus $\prec_{\Phi^0}$ satisfies a T-V condition.

For Φ^+, Φ^p, and $\Phi^\exists$ a proof like the usual proof for $\prec_{\Phi^L}$ ($= \prec$) can be given (see [CK 3.1.2]). The important point is that each of these is obtained (up to logical equivalence) by closing a certain set of open formulas under existential quantification.

V 2. GENERALIZED DEISSLER RANKS

2.0 DEFINITION: (After Deissler [D, 1.1]). Let M be an L-structure, $b \in M$, $A \subset M$, Φ a context for definability in L.

(i) $rk^{\Phi}(b, A, M) = 0$ iff b is Φ-definable over A in M.

(ii) For $\xi > 0$, $rk^{\Phi}(b, A, M) = \xi$ iff ξ is the least ordinal α such that for some $\phi(x) \in \Phi(A)$, $\phi[M] \neq \emptyset$ and for every $c \in \phi[M]$, $rk^{\Phi}(b, A \cup \{c\}, M) < \alpha$.

(iii) If $rk^{\Phi}(b, A, M) \neq \xi$ for all ordinals ξ, then $rk^{\Phi}(b, A, M) = \infty$.

2.1 LEMMA: Let M, L, b, A, Φ be as in (2.0).

(i) $A \subset B \subset M \Rightarrow rk^{\Phi}(b, A, M) \geq rk^{\Phi}(b, B, M)$

(ii) $M \prec_{\Phi} N \Rightarrow rk^{\Phi}(b, A, M) \leq rk^{\Phi}(b, A, N)$

(iii) $rk^{\Phi}(b, A, M) = rk^{\Phi}(b, A^{\Phi}, M)$

(iv) Let Ψ be a context for definability in L, $\Phi \subset \Psi$

Then $rk^{\Phi}(b, A, M) \geq rk^{\Psi}(b, A, M)$

PROOF: (i), (ii), (iv) are similar and easy, so I prove only (i) in detail:

By induction on α I show that for all b, $A \subset B \subset M$,

$(*) rk^{\Phi}(b, A, M) = \alpha \Rightarrow rk^{\Phi}(b, B, M) \leq \alpha$. The main point is that $\Phi(A) \subset \Phi(B)$. Thus (*) holds for $\alpha = 0$. Now assume that (*) holds for all b, all A, B, $A \subset B \subset M$ for all $\beta < \alpha$. Suppose $rk^{\Phi}(b, A, M) = \alpha$. Then there is $\phi(x) \in \Phi(A) \subset \Phi(B)$, $\phi[M] \neq \emptyset$ and for every $c \in \phi[M]$, $rk^{\Phi}(b, A \cup \{c\}, M) = \beta < \alpha$ for some β (depending on c). By the induction hypothesis, $rk^{\Phi}(b, B \cup \{c\}, M) \leq \beta$. Hence, since $\phi \in \Phi(B)$, $rk^{\Phi}(b, B, M) \leq \alpha$.

(ii) By induction on α show that for all b, all $A \subset M$, $rk^{\Phi}(b, A, N) = \alpha \Rightarrow rk^{\Phi}(b, A, M) \leq \alpha$ and use (1.4(iv)).

(iii) Immediate by rule (iii) of definition 1.0 (see remark 1.2(vii)).

(iv) By induction on α show that for all b, all $A \subset M$, $rk^{\Phi}(b, A, M) = \alpha \Rightarrow rk^{\Psi}(b, A, M) \leq \alpha$.

2.2 NOTATION: Let $*$ be one of $0, +, p, \exists$. Then $rk^{*}(b, A, M) ::= rk^{\Phi^{*}}(b, A, M)$. $rk(b, A, M) ::= rk^{\Phi^{L}}(b, A, M)$ is Deissler's rank. I call rk^{+} positive Deissler rank, rk^{p} primitive Deissler rank, and $rk^{\exists}$ existential Deissler rank.

2.3 LEMMA: If $rk^{\Phi}(b, A, M) > 0$ then

$$rk^{\Phi}(b, A, M) = \inf_{\substack{\phi\in\Phi(A) \\ \phi[M]\neq\emptyset}} \sup_{c\in\phi[M]} (rk^{\Phi}(b, A \cup \{c\}, M) + 1) .$$

PROOF: Recall that for X a set of ordinals, $\inf X$ is the least element of X and $\sup X$ is the least ordinal α such that for all $x \in X$, $x \leq \alpha$. In particular, $\sup \emptyset = 0$.

The lemma is immediate from the definition.

2.4 DEFINITION:

(i) $rk^{\Phi}(b, M) ::= rk^{\Phi}(b, \emptyset, M)$

(ii) $rk^{\Phi}(M/A) ::= \sup\{rk^{\Phi}(b, A, M) + 1 | b \in M\}$

(iii) $rk^{\Phi}(M) ::= rk^{\Phi}(M/\emptyset)$.

2.5 PROPOSITION: (See [D, 1.3]). $rk^{\Phi}(b, A, M) = \infty \Rightarrow$

$\Rightarrow (\forall\phi(x) \in \Phi(A))(\exists c \in M)\{[\phi[M] \neq \emptyset \Rightarrow \models \phi[c]]$ & $rk^{\Phi}(b, A \cup \{c\}, M) = \infty\}$.

PROOF: Supose not. Then there is $\phi(x) \in \Phi(A)$, $\phi[M] \neq \emptyset$, such that for each $c \in \phi[M]$, $rk^{\Phi}(b, A \cup \{c\}, M) < \infty$. Then by (2.3), $rk^{\Phi}(b, A, M) \leq \sup\{rk^{\Phi}(b, A \cup \{c\}, M) + 1 | c \in \phi[M]\} < \infty$.

2.6 THEOREM: (See [D, 1.4]). Suppose Φ is T-V. Then $rk^{\Phi}(b, A, M) < \infty$ iff for all M', $A \subset M' \prec_{\Phi} M \Rightarrow b \in M'$.

PROOF: The proof is essentially the same as that given by Deissler.

($\Rightarrow$) Suppose $rk^{\Phi}(b, A, M) < \infty$. By induction on $\alpha < \infty$ I show that

(*) for all M', $A \subset M' \prec_{\Phi} M$, for all $C \subset M'$,

$$rk^{\Phi}(b, A \cup C, M) = \alpha \Rightarrow b \in M'.$$

(Then ($\Rightarrow$) follows by taking $C = \emptyset$.)

($\alpha = 0$): Then b is defined in M by some $\phi(x) \in \Phi(A \cup C)$, $A \cup C \subset M'$. Since $M \models (\exists x)\phi(x)$ and $\prec_{\Phi}$ is T-V, there is $m \in M'$, $M \models \phi[m]$. But ϕ defines b, i.e. $b = m$, $b \in M'$.

($\alpha > 0$) Suppose (*) is verified for all $\beta < \alpha$ and we are given M', C as described and $rk^{\Phi}(b, A \cup C, M) = \alpha$. Then there is a formula $\phi(x) \in \Phi(A \cup C)$, $\phi[M] \neq \emptyset$ and for each $d \in \phi[M]$, $rk^{\Phi}(b, A \cup C \cup \{d\}, M) < \alpha$. But $M' \prec_{\Phi} M$ and $\prec_{\Phi}$ is T-V, so $\phi[M'] \neq \emptyset$, that is, there is $d \in M'$ with $rk^{\Phi}(b, A \cup C \cup \{d\}, M) < \alpha$. Therefore by the induction hypothesis (*), $b \in M'$.

($\Leftarrow$): Suppose $rk^{\Phi}(b, A, M) = \infty$. I construct M', $A \subset M' \prec_{\Phi} M$ so that $b \notin M'$.

Let $C = \{c_{\alpha} | \alpha < \kappa\}$ be new distinct constant symbols, $\kappa = |L| + |A|$. Let $L' := L \cup C$, and give expressions like $\Phi(C \cup A)$ their obvious interpretation. Let $\langle\phi_{\alpha}\rangle_{\alpha<\kappa}$ enumerate $\Phi(C \cup A)$ so that the new constants of ϕ_{α} are contained in $\{c_{\beta} | \beta < \alpha\}$. I find interpretations of the c_{β} in M (also denoted by c_{β}) such that $rk^{\Phi}(b, A \cup \{c_{\beta} | \beta < \alpha\}, M) = \infty$ and $\langle M, c_{\beta}\rangle_{\beta \leq \alpha} \models (\exists x)\phi_{\alpha}(x) \rightarrow \phi_{\alpha}(c_{\alpha})$. Once given $\{c_{\beta} | \beta < \alpha\}$ satisfying this property, the new constants of ϕ_{α} are among those already interpreted. Since $rk^{\Phi}(b, A \cup \{c_{\beta} | \beta < \alpha\}, M) = \infty$, by (2.5) there is $d \in M$ such that

$\phi_\alpha[M] \neq \emptyset$ implies that $M \models \phi_\alpha[d]$ and $rk^\Phi(b, A \cup \{c_\beta | \beta < \alpha\} \cup \{d\}, M) = \infty$. So I interpret c_α by d.

CLAIM: Let M' be the structure induced on $\{c_\alpha | \alpha < \kappa\}$ by their interpretations in M. Then $A \subset M' \prec_\Phi M$ and $b \notin M'$.

Since $\prec_\Phi$ satisfies a T-V criterion, the first part of the claim holds because the interpretations of the c_α were chosen to witness all the conditions of the criterion. Now $b \notin M'$ because if $b = c_\alpha$, then $rk^\Phi(b, A \cup \{c_\beta | \beta \leq \alpha\}, M) = 0$ since "$x = v$" $\in \Phi$. But by construction, $rk^\Phi(b, A \cup \{c_\beta | \beta \leq \alpha\}, M) = \infty$.

2.7 COROLLARY: Let $M_0 \subset M$.

(i) $rk^\Phi(M/M_0) < \infty$ iff M is a minimal member of the class of extensions of M_0 ordered by $\prec_\Phi$.

(ii) (Theories of modules) Let $\{0\}$ denote the trivial module (i.e. every sort is trivial). Then $rk^+(M/\{0\}) < \infty$ iff $M = \{0\}$.

(iii) (Theories of modules) Among the compact extensions M of a module M_0, $rk^+(M/M_0) < \infty$ implies that M is a compact hull of M_0.

(iv) (Theories of modules over a Noetherian ring) Among the injective extensions M of a module M_0, $rk^+(M/M_0) < \infty$ iff M is the injective envelope of M_0.

(v) (Theories of modules) Suppose M is tt, $M \supset M_0$. Then $rk^+(M/M_0) < \infty$ iff M is a compact hull of M_0.

(vi) Among the existentially closed extensions M of a structure M_0, $rk^\exists(M/M_0) < \infty$ iff M is an existential closure of M_0.

REMARK: For elementary facts about compact hulls beyond those given in Chapter III, consult [Z, Section 3]. In particular, if $M_0 \subset M$, M

compact, then the compact hull of M_0 in M exists and satisfies (III, 3.5(i)). The hull is denoted $H_M(M_0)$, $M_0 \subset H_M(M_0) \prec_{ppf} M$.

PROOF: (i) ($\Rightarrow$) Suppose M is not minimal. Then there is M', $M_0 \subset M' \prec_\Phi M$, $M' \neq M$. Let $b \in M \setminus M'$. By the theorem $rk^\Phi(b, M_0, M) = \infty$. Hence $rk^\Phi(M/M_0) = \infty$.

($\Leftarrow$) If M is minimal, $b \in M$, then the only M' satisfying $M_0 \subset M' \prec_\Phi M$ is M itself. Hence $b \in M'$ and by the theorem $rk^\Phi(b, M_0, M) < \infty$. Hence $rk^\Phi(M/M_0) < \infty$.

(ii) The trivial module is a pure submodule of every module.

(iii) Suppose M is a compact extension of M_0 . Then $M_0 \subset H_M(M_0) \prec_{ppf} M$. If M is not a compact hull of M_0 , then $M \neq H_M(M_0)$ and by (i), $rk^+(M/M_0) = \infty$.

(iv) Since the underlying ring is Noetherian, M injective and $M' \prec_{ppf} M$ implies that M' is injective. Thus if M is injective, $M_0 \subset M$, then $H_M(M_0)$ is the injective envelope of M_0 , i.e. a minimal injective extension of M_0 . Thus ($\Leftarrow$) follows immediately and ($\Rightarrow$) follows by (iii).

(v) and (vi) are similar to (iv). For (v) use that M tt and $M' \prec_{ppf} M$ implies that M' is tt. For (vi) use that (for arbitrary languages) M existentially closed and $M' \prec_{\Phi^\exists} M$ implies that M' is existentially closed [Es, 7.7].

<u>2.8 REMARKS</u>: (i) Although (2.7(iv), (v)) are statements abour rk^+ in modules parallel to (i), the converse of (iii) is false in general. For suppose M_0 is any module which is not compact. The <u>pure hull</u> of M_0 is that compact hull of M in which M is purely embedded. (The pure

hull of M is $H_N(M_0)$ where N is a $|L|^+$-saturated elementary extension of M_0 .) Since M_0 is not compact, $M \neq M_0$. Since M_0 is pure in M , $rk^+(M/M_0) = \infty$.

(ii) Note by an example of Eklof and Sabbagh [ES, Appendix], (vi) does not hold for algebraically closed structures, even with $rk^\exists$. ((i) taken together with (1.9(iii)) shows that the analogue of (vi) with algebraically closed structures and rk^+ fails).

(iii) Recall the theories T^*_Λ of Chapter II. Let Λ be any coherent but not Noetherian ring. Let $M_0 \models T^*_\Lambda$ as in (II 1.13(i)), $M_0 = \oplus(E(\Lambda/I)^{(\aleph_0)} | I \in \mathcal{I}(\Lambda))$. Then M is the existential closure of the module $A ::= \oplus((\Lambda/I)^{(\aleph_0)} | I \in \mathcal{I}(\Lambda))$ but M is not injective. Thus $rk^\exists(M_0/A) < \infty$ but since by (II 1.13(iii)) $M_0 \prec E(M_0) = E(A)$, even $rk(E(M_0)/A) = \infty$, not to mention $rk^+(E(M_0)/A)$.

(iv) Of course, Deissler's theorem also follows [D, 1.4(i)]: M is a minimal model of $Th(M)$ iff $rk(M) < \infty$.

<u>2.9 EXAMPLES</u>: Recall the examples of Baldwin and Lachlan (I,1.6). Rereading part (i) we see that it is established there that $rk(M) = 3$ (where M was the prime model of the theory discussed): Let $a \in M$. Up to equivalence the only consistent formula over $\emptyset$ is "x = x" . Thus $rk(a, \emptyset, M) \neq 0$. Let b be some solution of "x = x" in M different from a . The only new formulas over $\{b\}$ (up to equivalence) are "x = b" and "x ≠ b" . Thus $rk(a, \{b\}, M) \neq 0$ and so $rk(a, \emptyset, M) > 1$. Let c be some solution of "x ≠ b" . It was shown in (I,1.6(i)) that every element of M is definable from two distinct elements of M . Thus $rk(a, \{b, c\}, M) = 0$ and $rk(a, \emptyset, M) = 2$, and in fact $rk^p(a, \emptyset, M) = 2$.

It is also relatively easy to check that in part (ii), $\mathrm{rk}(N) = 4$. (The elements of the fibres of N have rk 3. One way of showing that $b \in C_a$ has rk 3 is to define a in two steps as above, pick out an arbitrary $c \in C_a$ by the formula $R(a, x)$ and then use the definable isomorphism $T(a, c, x)$ from C to C_a and the fact that, in defining a, we have accumulated enough parameters to make every element of C definable.)

V 3. RANK ANALYSES AND SOME SIMPLE RANK CALCULATIONS

3.0 DEFINITION: An analysis of $rk^{\Phi}(b, A, M)$ is a labelled rooted tree $\langle T, <, \lambda\rangle$ of the following sort:

(i) For each node $t \in T$, the label $\lambda(t)$ is a pair $\langle\phi, C\rangle$ where $C \subset M$ and $\phi \in \Phi(C)$ is such that $\phi[M] \neq \emptyset$.

(ii) If t is the root of T, $\lambda(t) = \langle\phi, A\rangle$ for some ϕ.

(iii) For each $t \in T$, if t is a leaf of T and $\lambda(t) = \langle\phi, C\rangle$, then b is defined by ϕ over C in M.

(iv) For each $t \in T$, if t is not a leaf of T and $\lambda(t) = \langle\phi, C\rangle$ then the successors of t are in 1-1 correspondence with $\phi[M]$, the second component of the labels ranging over the sets $C \cup \{m\}$, $m \in \phi[M]$.

(v) The tree is well founded, i.e. every branch is finite.

"rank (T)" denotes the usual foundation rank for well-founded trees. Sometimes I suppress the labelling function λ and refer to each node by its label, as if each label corresponded to a unique node. Although this is in fact false no harm is done as long as we keep in mind what we are doing.

3.1 LEMMA: (i) If $rk^{\Phi}(b, A, M) < \infty$, an analysis of it exists, say $\langle T, <, \lambda\rangle$, with rank $(T) = rk^{\Phi}(b, A, M)$.

(ii) If $\langle T, <, \lambda\rangle$ is an analysis of $rk^{\Phi}(b, A, M)$ then $rk^{\Phi}(b, A, M) < \infty$ and in fact $rk^{\Phi}(b, A, M) \leq \text{rank}(T)$.

PROOF: (i) I show by induction on $\alpha \in On$ that (*) for all C, $A \subset C \subset M$, if $rk^{\Phi}(b, C, M) = \alpha$ then an analysis $\langle T, <, \lambda\rangle$ with rank$(T) = \alpha$ exists. Then (i) follows by taking $C = A$. If $\alpha = 0$ then b is definable over C in M by some formula $\phi(x) \in \Phi(C)$. Let $T := \{0\}$, $< := \emptyset$,

$\lambda(0) := \langle\phi, C\rangle$. Then $\langle T, <, \lambda\rangle$ is the required tree.

Suppose $\alpha > 0$ and (*) holds for all $\beta < \alpha$. Suppose $A \subset C \subset M$ and $rk^{\Phi}(b, C, M) = \alpha$. Then for some $\phi(x) \in \Phi(C)$, $\phi[M] \neq \emptyset$ and for every $d \in \phi[M]$, $rk^{\Phi}(b, C \cup \{d\}, M) < \alpha$, and furthermore $\alpha = \sup_{d\in\phi[M]} rk^{\Phi}(b, C \cup \{d\}, M)$. By the induction hypothesis there are trees $\langle T_d, <_d, \lambda_d\rangle$ $(d \in \phi[M])$ such that rank $(T_d) = rk^{\Phi}(b, C \cup \{d\}, M\rangle$ for each $d \in \phi[M]$. Without loss of generality these trees are disjoint. Let $\langle T, <, \lambda\rangle$ be the tree with root $\emptyset$, $\lambda(\emptyset) = \langle\phi, C\rangle$, the subtrees succeeding the root exactly the trees T_d $(d \in \phi[M])$, and $<, \lambda$ extending the various $<_d, \lambda_d$'s .

(ii) This follows by an easy induction which I will not give here: By induction on $\alpha \in On$ show that (*) for all $C, A \subset C \subset M$, if $\langle T, <, \lambda\rangle$ is an analysis of $rk^{\Phi}(b, C, M)$ with rank $(T) = \alpha$ then $rk^{\Phi}(b, C, M) \leq \alpha$.

<u>3.2 COROLLARY</u>: $rk^{\Phi}(b, A, M) = \inf \{rank(T) \mid \langle T, <, \lambda\rangle$ is an analysis of $rk^{\Phi}(b, A, M)\}$.

<u>3.3 DEFINITION</u>: An analysis as in (3.1(i)) is called an <u>accurate</u> analysis of $rk^{\Phi}(b, A, M)$.

<u>3.4 PROPOSITION</u>: Let M be a structure for L , $A \subset M$. Suppose $\Phi(x)$ is a context for definability in L such that for each $\phi(x, \vec{v}) \in \Phi$, there is $\psi(x, u, \vec{v}) \in \Phi$ such that $M \models (\forall x, u, \vec{v})[\psi(x, u, \vec{v}) \leftrightarrow (\phi(x, \vec{v}) \wedge x \neq u)]$. Let $b \in M$. If b is Φ-algebraic of degree n over A in M , then $rk^{\Phi}(b, A, M) < n$.

<u>PROOF</u>: By induction on n , $1 \leq n < \omega$, I show that for all $C, A \subset C \subset M$, if b is Φ-algebraic of degree n over C in M , then $rk^{\Phi}(b, C, M) < n$. The proposition then follows by taking $A = C$.

If b is Φ-algebraic of degree 1 over C in M, then b is Φ-definable over C in M, hence $rk^{\Phi}(b, C, M) = 0$.

If the statement is true for all $m < n$, and b is Φ-algebraic of degree n, then there is $\phi(x) \in \Phi(C)$, $|\phi[M]| = n$, and $M \models \phi[b]$. For each $c \in \phi[M] \setminus \{b\}$ consider the formula $\psi(x, c)$ given by the hypothesis of the proposition, so $\psi[M, c) = \psi[M] \setminus \{c\}$. Thus b is Φ-algebraic of degree $\leq n - 1$ over $C \cup \{c\}$, hence by the induction hypothesis, $rk^{\Phi}(b, C \cup \{c\}, M) < n - 1$. But c ranges over all solutions of $\phi(x) \in \Phi(C)$, thus by (2.3) $rk^{\Phi}(b, C, M) < n$.

<u>3.5 LEMMA</u>: (Strengthening formulas) Suppose $<T, <, \lambda>$ is an analysis of $rk^{\Phi}(b, A, M)$, $t \in T$, $\lambda(t) = <\phi, C>$, and $\phi' \in \Phi(C)$, $\phi'[M] \neq \emptyset$, $M \models (\forall x)[\phi' \to \phi]$. Define a new tree $<T', <', \lambda'>$ as follows: Let T' be obtained from T by deleting all subtrees above t whose initial node t' satisfies $\lambda(t') = <\phi', C \cup \{c\}>$ where $c \in \phi[M] \setminus \phi'[M]$. Let $<' = < \restriction T'$, and let $\lambda' = \lambda \restriction (T' \setminus \{t\}) \cup <t, <\phi', C>>$.

Then $<T', <', \lambda'>$ is an analysis of $rk^{\Phi}(b, A, M)$ and $\mathrm{rank}(T') \leq \mathrm{rank}(T)$.

<u>PROOF</u>: Obvious.

<u>3.6 COROLLARY</u>: Suppose M is atomic over A, more precisely, for every C, $A \subset C \in M$ and every $b \in M$, $t(b, C)$ is isolated. Suppose $rk(b, A, M) < \infty$. Then without loss of generality every formula in an analysis of $rk(b, A, M)$ is complete in $Th(M)$, in particular there is an accurate analysis as described.

<u>3.7 DEFINITION</u>: Let ξ be a finite sequence of ordinals, $\xi \in {}^{n}On$.

Then $\bigvee \xi$ ("join ξ") is $\inf\{\Sigma_n \xi \circ \pi \mid \pi$ a permutation of $n\}$. Thus $\bigvee \xi$ is the least possible sum formed by adding the components of ξ in any order.

<u>3.8 REMARK*</u>: In the following starred (*) articles I suppress parts of the notation to improve readability: Φ is a context for definability in L and M is an L structure, and for $rk^{\Phi}(b, A, M)$ I write $rk(b, A)$.

<u>3.9 PROPOSITION*</u>:

(i) $rk(b, A) \leq rk(b, A \cup \{c_i \mid i < n\}) + \bigvee_{i<n} rk(c_i, A)$

(ii) $rk(b, A) \leq rk(b, A \cup \{c_i \mid i < n\}) + \Sigma_{i<n} rk(c_{n-1-i}, A \cup \{c_j \mid j < n - 1 - i\})$

<u>REMARKS</u>: Of course (ii) is the sharper statement but the main application will be where we know $rk(c_i, A)$ for each $i < n$.

<u>PROOF</u>: <u>Claim</u>: For any $c \in M$, $rk(b, A) \leq rk(b, A \cup \{c\}) + rk(c, A)$.

<u>Proof of Claim</u>: Let T_0 be an accurate analysis of $rk(b, A \cup \{c\})$ and let T_1 be an accurate analysis of $rk(c, A)$. Create a new tree T by attaching copies of T_0 to the leaves of T_1 as follows: Each leaf of T_1 is labelled by some $\langle\phi, D\rangle$ where $\phi(x) \in \Phi(D)$ and ϕ defines c . Replace this leaf by a copy of T_0 modified by replacing each label $\langle\psi(x, c), D'\rangle$ of T_0 by the label $\langle(\exists y)[\psi(x, y) \wedge \phi(y)], D \cup (D' \setminus \{c\})\rangle$. The formula of this label is in $\Phi(D \cup (D' \setminus \{c\}))$ by rule (iii) of (1.0) and has exactly the same solutions in M as $\psi(x, c)$. Thus it is easy to see that the resulting tree T is an analysis of $rk(b, A)$, and clearly $rank(T) = rank(T_0) + rank(T_1) = rk(b, A \cup \{c\}) + rk(c, A)$.

Now (ii) follows easily by induction on $n < \omega$. The case $n = 1$ is given by the claim. If (ii) holds for $n - 1$ and we are given $\{c_i \mid i < n\}$

then $(*) rk(b, A) \leq rk(b, A \cup \{c_i | i < n - 1\}) + \Sigma_{i<n-1} rk(c_{n-2-i}, A \cup \{c_j | j < n - 2 - i\})$, by the claim $rk(b, A \cup \{c_i | i < n - 1\}) \leq rk(b, A \cup \{c_i | i < n\}) + rk(c_{n-1}, A \cup \{c_i | i < n - 1\})$, and replacing this in the right hand side of (*) I am done.

(i) follows immediately since $rk(c_{n-1-i}, A \cup \{c_j | j < n - 1 - i\}) \leq rk(c_{n-1-i}, A)$ by (2.1(i)), and clearly I can arrange $\{c_i | i < n\}$ in order so that $\Sigma_{i<n} rk(c_i, A)$ is as small as possible.

<u>3.10 COROLLARY*</u>: Let $A \subset M$, $b, c_0, \ldots, c_{n-1} \in M$. Then $b \in (A \cup \{c_0, \ldots, c_{n-1}\})^\Phi$ implies that

(i) $rk(b, A) \leq \bigvee_{i<n} rk(c_i, A)$.

(ii) $rk(b, A) \leq \Sigma_{i<n} rk(c_{n-1-i}, A \cup \{c_j | j < n - 1 - i\})$

<u>3.11 COROLLARY*</u>: (Suppose also the hypotheses of proposition 3.4). If b is Φ-algebraic of degree m over $A \cup \{c_i | i < n\}$ then

(i) $rk(b, A) \leq m + \bigvee_{i<n} rk(c_i, A)$

(ii) $rk(b, A) \leq m + \Sigma_{i<n} rk(c_{n-1-i}, A \cup \{c_j | j < n - 1 - i\})$

<u>3.12 COROLLARY*</u>: Let $A \subset M$, $b, c, \vec{d} \in M$, and suppose $b \in \{c, \vec{d}\}^\Phi$, $c \in \{b, \vec{d}\}^\Phi$. Then $rk(b, A \cup \{\vec{d}\}) = rk(c, A \cup \{\vec{d}\})$.

<u>PROOF</u>: $rk(b, A \cup \{\vec{d}\}) \leq rk(b, A \cup \{c, \vec{d}\}) + rk(c, A \cup \{\vec{d}\})$ by the theorem. But since $b \in \{c, \vec{d}\}^\Phi$, the first term on the right is 0. Similarly $rk(c, A \cup \{\vec{d}\}) \leq rk(b, A \cup \{\vec{d}\})$.

<u>REMARK</u>: In the context of modules, a simple and useful form of 3.12 is:

<u>3.12' COROLLARY</u>: Let M be a module, $A \subset M$, $b, c \in M$. Then $rk^+(b, A \cup \{b + c\}, M) = rk^+(c, A \cup \{b + c\}, M)$.

V 4. RELATIONS BETWEEN U-RANK AND rk; BETWEEN rk AND rk^+ IN MODULES

4.0 LEMMA: Let T be a complete tt theory, $M \models T$, $A \subset B \subset M$, M minimal over A, $p \in S(B)$. Then there is $k \in \omega$ such that if $I \subset M$ is an independent set of realizations of p over B, then $|I| < k$.

PROOF: Note that M is also minimal over B, so wlog I assume that $B = A$. Since T is tt and M is minimal over A, there is no infinite set of indiscernibles over A in M and M is prime over A (0, 1.10(iii), (ii)). Hence M is homogeneous over A. Furthermore, since T is tt, p has finite multiplicity $d < \omega$. Thus there are d strong types over A extending p and any independent set of realizations of p over A can be partitioned into at most d sets of indiscernibles in these strong types. I claim that if p_1 and p_2 are two strong types over A extending p and I_1, I_2 are maximal independent sets of realizations of p_1, p_2 respectively in M then $|I_1| = |I_2| = \ell$, some $\ell < \omega$. It then follows that $k = d \cdot \ell + 1$ suffices. (For details on strong types see [Sh, III 2]).

That both I_1 and I_2 are finite follows from the second remark above. So suppose $|I_1| \geq |I_2| = \ell > 0$, let $I_2 = \{b_i^2 | i < \ell\}$, and let $\{b_i^1 | i < \ell\} \subset I_1$. By homogeneity there is $\alpha \in Aut_A(M)$ with $\alpha(b_0^1) = b_0^2$. Then $\alpha[I_1]$ is an independent set in the strong type p_2, and so $\langle\alpha(b_i^1)\rangle_{i<\ell} \equiv_A \langle\alpha(b_i^2)\rangle_{i<\ell}$. Again by homogeneity there is $\beta \in Aut_A(M)$ such that $\beta\circ\alpha\,(b_i^1) = b_i^2$ for $i < \ell$, and so $\beta\circ\alpha\,[I_1]$ is an independent set in the strong type p_2 extending I_2. But I_2 was maximal, so $|I_1| = |I_2| = \ell$.

4.1 THEOREM: (Recall Lascar's rank U, (0, 1.9)) Let T be a complete tt theory and assume the prime model M of T is minimal. Then:

$$rk(M) \leqq \sup\{\omega(U(b, \emptyset) + 1) | b \in M\} .$$

In fact, for any $b \in M$, $A \subset M$, $rk(b, A, M) < \omega(U(b, A) + 1)$.

PROOF: The first statement is an immediate consequence of the second. I prove by induction on $\alpha \in On$ that for all $A \subset M$, if $U(b, A) = \alpha$ then $rk(b, A, M) < \omega(\alpha + 1)$.

If $U(b, A) = 0$, then a is algebraic of degree n over A for some $n < \omega$, and by (3.4) $rk(b, A, M) < n < \omega = \omega(0 + 1)$.

Now assume that the claim is proved for all $B \subset M$ and all $\beta < \alpha$, and that $U(b, A) = \alpha$. Construct by recursion on its levels a tree $\langle T, <, \lambda\rangle$ satisfying (i), (ii) and (iv) of the definition (3.0) of an analysis of $rk(b, A, M)$ and such that for each $t \in T$, $\lambda(t) = \langle\phi, C\rangle$ where ϕ isolates $t(b, C)$ and t is a leaf of T iff $t(b, C)$ forks over A. This constitutes a complete set of instructions for building T, and the construction is possible since $b \in M$, and M is prime, hence atomic, over all $C \subset M$. Let $k < \omega$ be determined by lemma (4.0) so that if $I \subset M$ is an independent set of elements equivalent to b over A, then $|I| < k$. I claim that $\langle T, <, \lambda\rangle$ as constructed has at most k levels, i.e. every branch of T has length $\leqq k$. Clearly every branch of T consists of elements equivalent to b over A by the choice of the formulas ϕ, so it is enough to show independence. Suppose that $(t_i)_{i\leqq n} \subset T$, t_i on level i, t_{i+1} a successor of t_i. Then there are $\{c_i | 1 \leqq i \leqq n\}$ in M such that for $C_i = \{c_j | 1 \leqq j \leqq i\}$ $(i \leqq n)$ and $\lambda(t_i) = \langle\phi_i, A \cup C_i\rangle$ with ϕ_i isolating $t(b, A \cup C_i)$, then c_{i+1} satisfies ϕ_i and $b \underset{A}{\downarrow} C_i$ $(i < n)$. Note that by the construction $b \underset{A}{\not\downarrow} C_n$ if t_n is a leaf. Then by the choice of the c_i's, $c_{i+1} \underset{A}{\downarrow} C_i$

$(i < n)$ so C_{n-1} is independent. Thus by (4.0) $n - 1 < k$, and so each branch of T has length $\leq k$.

Thus each branch of T terminates with a node t, $\lambda(t) = \langle\phi, C\rangle$ and $t(b, C)$ forks over A. Thus $U(b, C) = \beta < \alpha$ for some β, and by the induction hypothesis $rk(b, C, M) < \omega(\beta + 1) \leq \omega\alpha$. For each such t, let $\langle T_t, <_t, \lambda_t\rangle$ be an accurate analysis (3.0(i)) of $rk(b, C, M)$, and replace the leaf t by T_t so that T_t becomes a subtree of T. Naming the root of T_t by t again, $rank(t) = rk(b, C, M) \leq \omega\alpha$. After replacing all leaves of T in this manner, and calling the resulting tree T', it is clear that T' is an analysis of $rk(b, A, M)$ and $rank(T') \leq \omega\alpha + k < \omega(\alpha + 1)$.

4.2 COROLLARY:

(i) Let T, M be as in the theorem. Then $rk(M) \leq \omega\cdot(\alpha_T + 1)$.

(ii) Suppose in addition that T is countable and $\aleph_1$-categorical not $\aleph_0$-categorical. Then $rk(M) < \omega^2$.

PROOF: Recall that α_T is the least ordinal greater than or equal to the Morley rank of p for all types p; hence in particular $U(b, \emptyset) \leq \alpha_T$ for all $b \in M$ since Lascar's rank U is the smallest rank. Thus by the theorem $rk(M) \leq \omega\cdot(\alpha_T + 1)$. A well-known theorem of Baldwin [B] states that for $\aleph_1$-categorical not $\aleph_0$-categorical T, $\alpha_T < \omega$. So in this case $rk(M) < \omega^2$.

4.3 REMARKS: Deissler states (4.2(ii)) in his article [D] without proof. He then asks if for $\aleph_1$-categorical not $\aleph_0$-categorical T there is any relation between $rk(M)$ and α_T. Part (i) of the corollary directly answers this question and considerably generalizes the result of Deissler. Of course, by the theorem I can replace α_T in (4.2(i)) by α_T^U defined in the same way as α_T but with Lascar's rank U. Note that in (4.1, 4.2(i)) I place no restriction on $|T|$.

In (4.4, 4.5) I give two examples relating to this estimate of rk . The first example is of a theory T, prime model M , and $b_n \in M$ such that $U(b_n, \emptyset) = n$ $(n \in \omega)$. In each case, $rk(b_n, \emptyset, M) \leq 1$. The second example gives $\aleph_1$-categorical not $\aleph_0$-categorical T_n with all types p of U-rank 0 or 1, but the prime model M_n has $rk(M_n) = n + 2$.

<u>4.4 EXAMPLE</u>: Let M_0 be the prime model of the theory of existentially closed abelian groups (i.e. take $\Lambda = \mathbb{Z}$ in Chapter II). Let $N := \langle M_0 \oplus \mathbb{Q}, \langle 0, 1\rangle, \langle m, 0\rangle\rangle_{m \in M_0}$ and $T = Th(N)$. T is tt and N is the minimal model of T . From Chapter II it is clear that N realizes types of arbitrarily large U-rank $< \omega$, for instance $U(\langle 0, 1/2^n\rangle, \emptyset) = n$. I claim that for any $b \in N$, $rk(b, \emptyset, N) \leq 1$.

Let $b \in N$, so $b = \langle m_0, c/d\rangle$ where $m_0 \in M_0$ and $c, d \in \mathbb{Z}$, $d \neq 0$, c, d relatively prime. Since all elements $\langle m_0, 0\rangle$ are constants of the language, wlog $m_0 = 0$. If $c = 0$, then b is the definable element $\langle 0, 0\rangle$. Otherwise $t(b, \emptyset)$ is isolated by the formula "$dx = c \cdot \langle 0, 1\rangle$" which has solutions $b' = \langle m, c/d\rangle$ where $m \in M_0$ is such that $dm = 0$. Given any such b' , b is defined by the formula $x = b' - \langle m, 0\rangle$. Thus $rk(b, \emptyset, N) \leq 1$.

Thus U-rank can be large while rk remains small.

<u>4.5 EXAMPLE</u>: For results about stable theories of lattices, refer to the work of K. W. Smith [Sm 1, Sm 2]. The example I present is quite simple and the facts stated about it are fairly obvious. For detailed proofs, the reader will have to refer to [Sm 2, Section 6]. (In Smith's terminology the examples I present are all "type 3 $\aleph_1$-categorical lattices in S" and

so they are not $\aleph_0$-categorical (Corollary 6.10)).

I define a family of lattices $(L_n)_{n<\omega}$. L_3 is pictured in figure (4.5a). An element like b which has a unique predecessor and a unique successor is called a <u>dead end</u> in Smith's terminology. The elements denoted by open circles form an <u>infinite fence</u>. L_n is the four level lattice with an infinite fence such that every element on level 2 is covered by exactly n dead ends on level 3. T_n is the complete theory of L_n .

The theories T_n are clearly $\aleph_1$-categorical not $\aleph_0$-categorical. In fact L_n is the minimal model of T_n and the elementary extensions of L_n are formed by adding new copies of the infinite fence with its associated dead ends between the 0 and 1 of L_n . The types of T_n have a very simple structure: the realized types are the types of 0 and 1, the other 1-types over $\emptyset$ are strongly regular and are the type of an element on level 2, the type of a non-dead-end on level 3, and the type of a dead-end on level 3. These types all have U-rank 1, and so every 1-type over $\emptyset$ of T_n has U-rank ≤ 1 .

It is easy to check that 0, 1 have rk 0 and that the elements on level 2 and the non-dead-ends of level 3 have rk 2 . I will only sketch a proof that the dead ends on level 3 have rk $n + 1$, leaving the details to the reader. In figure (4.5b) I list the general form of the formulas appearing in an analysis of $rk(b, \emptyset, L_n)$.

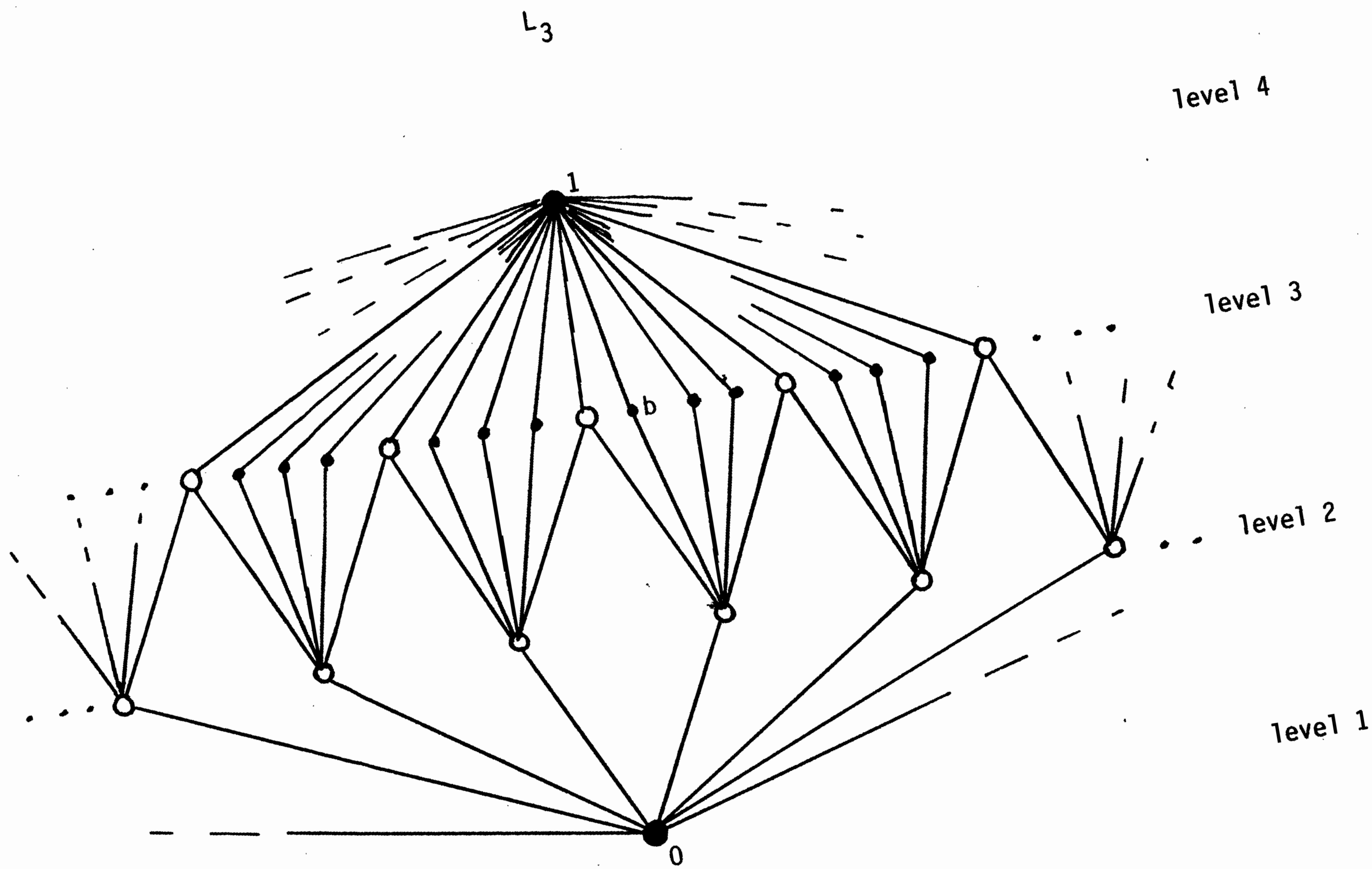


FIGURE 4.5a

4.5b FIGURE: (how to construct an analysis of rk(b, ∅, L_n)).

level of tree	formula, general solution
0	"x is a dead end on level 3" (solutions c, ...)
1	"x is a dead end on level 3 accessible by a fence of length k from c" (two groups of solutions, one to the "left" of c , the other to the "right" of c . Call one such d .)
n - 1 levels	(If b, d are in the <u>same</u> family of solutions) "x is a dead end on level 3 covering the same point as d"
OR	
n levels	(If b, d are in different families of solutions) "x is a dead end on level 3, accessible by a fence of length k from c , not covering the same point as d"

In the two possibilities for the last (n - 1) or n steps, we need this many steps because we must find b among the n different (and indistinguishable) dead ends covering the same point that b covers.

4.6 REMARKS: I wish to establish a relationship between rk and rk^+ for indecomposable tt modules. Recall that all tt modules are compact (III 1.16) and recall the concept of compact hull (III 3.5(i), V 2.7). I will need two additional results from Ziegler's in-depth study of the theory of compact hulls, namely [Z, lemma 3.7] which I take as the definition of small and [Z, corollary 3.10].

4.7 DEFINITION: [Z, 3.7]

Let M be a module, $A, B \subset M$. B is small over A in M iff $t_M^+(\vec{b}, A) \vdash t(\vec{b}, A)$ for every finite $\vec{b} \in B$.

4.8 LEMMA: [Z, 3.10] Let M be compact. B is a compact hull of A in M iff B is small over A in M, compact, and pure in M.

4.9 COROLLARY: Let M be a tt module, $A \subset M$, $b \in M$, M a compact hull of A. Then $t(b, A)$ is isolated by a ppf.

PROOF: Since M is tt, $t_M^+(b, A)$ is equivalent to a single formula by the characterization of tt modules (III 1.11).

REMARK: This is also proposition 5.5 of [Pr 1].

4.10 THEOREM: Let M be a tt module, $A \subset M$, M a compact hull of A, $b \in M$. Then

(i) $rk^+(b, A, M) = rk(b, A, M)$.

(ii) if in addition, M is indecomposable associated with the sr type (p, ψ) of $Th(M^{(\aleph_0)})$ (see III 3.10) and $A = \psi[M]$, then
$rk^+(b, A, M) \leq rk(b, \emptyset, M) \leq \sup\{rk^+(b, \{a\}, M) | 0 \neq a \in A\} + 1$.

(iii) if Λ is a commutative Noetherian ring, $A = \Lambda/P$ for some prime ideal P of Λ and $M = E(A)$, then

$$rk^+(M/A) \leq rk(M/\emptyset) \leq rk^+(M/A) + 1 .$$

REMARK: All the ranks mentioned are $< \infty$ by (2.7(v)).

PROOF: (i) By (2.1(iv)) $rk(b, A, M) \leq rk^+(b, A, M)$. Let $(T, <, \lambda)$ be an accurate analysis of $rk(b, A, M)$. Clearly $\langle M, a\rangle_{a\in A}$ is the prime model of its complete theory so by (3.6) wlog for every $t \in T$, $\lambda(t) = \langle\phi, C\rangle$ with ϕ a complete formula over C . Since $A \subseteq C \subseteq M$, M is also a compact hull of C so M is small over C and by (4.9) wlog ϕ is in fact a ppf. Thus $\langle T, <, \lambda\rangle$ is also an analysis of $rk^+(b, A, M)$ and so $rk^+(b, A, M) \leq rk(b, A, M)$.

(ii) The first inequality is immediate by (i) and (2.1(i)). If a is any non-zero solution of ψ in M then M is the hull of $\{a\}$ in M since M is indecomposable. Construct an analysis of $rk(b, \emptyset, M)$ as follows: the root of the tree is labelled by $\langle\psi(x) \wedge x \neq 0, \emptyset\rangle$; the subtree succeeding the root and labelled with a solution a at its root is an accurate analysis of $rk^+(b, \{a\}, M)$. The resulting tree has rank $\sup\{rk^+(b, \{a\}, M) \mid 0 \neq a \in A\} + 1$ as required.

(iii) In (ii) take $\psi(x)$ to be "Ann x = P" which can be done by (II, 3.7). Then any $a \neq 0$ realizing $\psi(x)$ generates A as a submodule of M , so the second inequality of (ii) becomes $rk(b, M) \leq rk^+(b, A, M) + 1$. Then (iii) is immediate.

4.11 LEMMA: Let Λ be a left Noetherian ring and L the language for left Λ-modules. Let $\phi(\vec{v})$ be a ppf of L . Then there is $\phi^*(\vec{v})$, a conjunction of equations of L , such that for all injective left Λ-modules E, $E \models (\forall\vec{v})[\phi \leftrightarrow \phi^*]$.

REMARK: This well-known but I have not found it stated explicitly anywhere. The following simple proof is based on part of the proof of a theorem of Eklof and Sabbagh [ES, Theorem 3.12].

PROOF: I define ϕ^* by recursion on the quantifier depth of ϕ. So suppose $\phi(\vec{v}) :=: (\exists y)\bigwedge_{j<n}(\lambda_j y + p_j(\vec{v}) = 0)$ where each p_j is some term of L where y does not appear. Since Λ is left Noetherian, the kernel of the homomorphism $\Lambda^n \to \Lambda$: $\langle\alpha_j\rangle_{j<n} \to \Sigma_{j<n}\alpha_j\lambda_j$ is finitely generated. Let the generators be $\langle\delta^i\rangle_{i<k}$, where $\delta^i = \langle\delta^i_j\rangle_{j<n} \in \Lambda^n$. Then by [ES, lemma 3.2] (See (II, 1.1(iii)c), the criterion for consistency in injective modules) $(\forall\vec{v})[(\exists y)\bigwedge_{j<n}(\lambda_j y + p_j(\vec{v}) = 0) \leftrightarrow \bigwedge_{i<k} \Sigma_{j<n}\delta^i_j p_j(v) = 0]$ holds in all injective left Λ-modules.

4.12 REMARKS: As a consequence, when studying rk^+ for injective modules over a Noetherian ring, I may restrict my attention to formulas which are conjunctions of equations.

Of course, by the results of Eklof and Sabbagh [ES] we know that every formula is equivalent to a Boolean combination of equations. (4.11) gives a more precise result for ppf's.

V 5. rk^+ IN INJECTIVE MODULES OVER A COMMUTATIVE NOETHERIAN RING

5.0 ASSUMPTIONS: Throughout this section, Λ is a fixed commutative Noetherian ring and all modules are unitary modules over Λ. I study rk^+ for injective Λ-modules. By the decomposition theorem of Matlis (II 4.2) any injective E may be written as $E(M)$ where M is a direct sum of cyclic modules Λ/P, P some prime ideal of Λ. Thus a reasonable understanding of the complexity of injective envelopes may be obtained through estimates of $rk^+(E/M)$ where E, M are as above. Recall that by (2.7(ii)) if $E \neq \{0\}$ then $rk^+(E/\{0\}) = \infty$. If $M = \oplus_{P \in P} (\Lambda/P)^{(\alpha_P)}$ then $E(M) = \oplus_{P \in P} E(\Lambda/P)^{(\alpha_P)}$, so this study progresses through several natural stages: indecomposables, powers of indecomposables, arbitrary injectives. By (4.12) every ppf I consider in analyzing rk^+ will be taken to be a conjunction of equations.

As well as the results about injective modules and injective envelopes quoted in Chapter II, I will need some more detailed results of Matlis [Ma] (5.1, 5.2, 5.7).

For elementary results on primary ideals, consult Northcott [N, Section 1.5, 1.9].

5.1 LEMMA: (i), (ii): [Ma, 3.2(2)]) Let P be a prime ideal of Λ, $E = E(\Lambda/P)$. Then:

(i) Q is an irreducible P-primary ideal of Λ iff for some $0 \neq x \in E$, $\mathrm{ann}(x) = Q$.

(ii) Let $\lambda \in \Lambda \setminus P$. Then there is unique division by λ in E.

(iii) If $\lambda \in P \setminus \{0\}$, division by λ is defined and not unique.

REMARKS: The proof is easy, for which see Matlis. For (iii), note that

$\lambda x = 0$ has many solutions when $0 \neq \lambda \in P$ (e.g., μ/P, any $\mu \notin P$). See also (III 4.5) and the related material for my model-theoretic approach to (i).

5.2 LEMMA: [Ma, 3.8(1)] Let P be a prime ideal of Λ, $E = E(\Lambda/P)$, $y, x_1, \ldots, x_n \in E$. Then $\bigcap_j \operatorname{ann}(x_j) \subset \operatorname{ann}(y)$ iff there are $\lambda \in \Lambda \setminus P$ and $\mu_1, \ldots, \mu_n \in \Lambda$ such that $\lambda y = \Sigma_{i=1}^n \mu_i x_i$.

5.3 COROLLARY: Let P, Λ, E be as in (5.2), $x, y \in E$. Then $\operatorname{ann}(x) = \operatorname{ann}(y)$ iff x and y are definable in terms of each other by an equation.

PROOF: ($\Leftarrow$) is immediate. For ($\Rightarrow$), by the lemma for some $\lambda \in \Lambda \setminus P$ and $\mu \in \Lambda$, $\lambda x = \mu y$ and by (5.1(ii)), x is the unique solution of $\lambda v = \mu y$ in E.

REMARKS: (5.2) arises as a corollary to some of the deepest results of Matlis' paper. The corollary (5.3) apparently states a much simpler fact; the question arises whether or not a more elementary proof can be found.

5.4 THEOREM: Let E be an indecomposable injective Λ-module, so $E = E(A)$, $A = \Lambda/P$, P some prime ideal of Λ. Then $rk^+(E/A) \leq 2$.

PROOF: Let $b \in E$. It suffices to prove that $rk^+(b, A, E) \leq 1$. By (4.9), $t(b, A)$ is isolated by a ppf $\phi(x) \in A$. Let $b' \in \phi[E]$. Then $\operatorname{ann}(b) = \operatorname{ann}(b')$, so by (5.3), b is definable from b'. Hence $rk^+(b, A, E) \leq 1$.

5.5 EXAMPLE: Suppose that Λ is Noetherian domain, so that $\{0\}$ is a prime ideal. Then $E(\Lambda) \cong K$, the quotient field of Λ [Ma]. It is easy to see that every element of K is definable over Λ, hence $rk^+(K/\Lambda) = 1$.

5.6 EXAMPLE: Let $\Lambda = \mathbb{Z}$, $P = ((p))$ any non-zero prime ideal. Then $E(\mathbb{Z}/P)$ is the Prufer group $\mathbb{Z}(p^\infty)$. Let $E = \mathbb{Z}(p^\infty)$ and $A = \mathbb{Z}/P$. Represent E in the usual way as the set of all fractions $0 \leq m/p^n < 1$ with addition modulo 1 (hence $A = \{0, 1/p, \ldots, (p-1)/p\}$). Consider any element $b = m/p^n$. b is pp-definable over A iff $b \in A$ by (4.12) and (5.1). Thus every b represented in lowest form as m/p^n with $n \geq 2$ has $rk^+ \geq 1$. So by the theorem, $rk^+(E/A) = 2$.

5.7 THEOREM: The Matlis Hierarchy [Ma, 3.4]. Let P be a prime ideal of Λ, $E ::= E(\Lambda/P)$. Define $A_i ::= \{x \in E \mid P^i x = 0\}$. Then:

(i) A_i is a submodule of E, $A_i \subset A_{i+1}$, $E = \bigcup_{i<\omega} A_i$

(ii) The set of non-zero elements of A_{i+1}/A_i is the set of elements of E/A_i with annihilator P. In other words, for $x \in E \setminus A_i$, $Px \subset A_i$ iff $(\lambda x \in A_i \Leftrightarrow \lambda \in P)$ iff $x \in A_{i+1}$.

(iii) Let K be the quotient field of Λ/P. Then A_{i+1}/A_i is a vector space over K and $A_1 \cong K$.

REMARKS: The proof is quite easy and as I need to refer to some of the details in the later work, I include it essentially as given by Matlis.

PROOF: (i) Since Λ is commutative, each A_i is a submodule of E and $A_i \subset A_{i+1}$. If $x \in E$, $x \neq 0$ then $\mathrm{ann}(x)$ is a P-primary ideal (5.1) and so since Λ is Noetherian, $P^i \subset \mathrm{ann}(x)$ for some i, thus $x \in A_i$.

(ii) Let $x \in E \setminus A_i$. If $Px \subset A_i$ then $P^i(Px) = 0$, that is,

$P^{i+1}x = 0$ or $x \in A_{i+1}$. If $x \in A_{i+1}$, then $P^{i+1}x = 0$ or $P^i(Px) = 0$, that is, $Px \subset A_i$. Furthermore, if $\lambda \notin P$, $\mathrm{ann}(\lambda x) = \mathrm{ann}(x)$ by (5.1) so $\lambda x \notin A_i$. Finally, if $\lambda \in P$, then $\lambda x \in A_i$.

(iii) Define the operation of K on A_{i+1}/A_i as follows: For $k \in K$, $k = (\alpha/P) \div (\lambda/P)$ where $\lambda \notin P$, $\alpha, \lambda \in \Lambda$. Let $k(x/A_i) ::= \alpha y/A_i$ where y is the unique solution of $\lambda y = x$. It is straightforward that this operation is well defined and makes A_{i+1}/A_i a vector space over K.

Since $A_0 = \{0\}$, A_1 is then a vector space over K. Choose $0 \neq x \in A_1$ and define $g\colon\ K \to A_1 : k \to kx$. g is an embedding. Let $z \in A_1$, $z \neq 0$. Since E is an essential extension of A_1, x and z satisfy a non-trivial equation $\alpha x = \beta z \neq 0$. Since $x, z \in A_1$, $\alpha x = \beta z \neq 0$ implies that $\alpha, \beta \in \Lambda \setminus P$ by part (ii). Thus $g((\alpha/P) \div (\beta/P)) = z$ by the definition of the action of K on A_1, and so g is an isomorphism.

<u>REMARK</u>: The careful reader may wonder why I don't simply quote (5.2) to prove the latter part of (iii). These results are taken out of their logical order in Matlis' paper, and in fact (5.7) is one of the prerequisites for (5.2).

<u>5.8 COROLLARY</u>: Let P, Λ, E, A_i be as in the theorem. Consider $E^{(\kappa)}$ for some finite or infinite cardinal κ. Define $A_{\kappa,i} ::= \{x \in E^{(\kappa)} \mid P^i x = 0\}$. Then:

(i) $A_{\kappa,i}$ is a submodule of $E^{(\kappa)}$, $A_{\kappa,i} \cong (A_i)^{(\kappa)}$, $A_{\kappa,i} \subset A_{\kappa,i+1}$, and $E^{(\kappa)} = \bigcup_{i<\omega} A_{\kappa,i}$.

(ii) For $x \in E^{(\kappa)} \setminus A_{\kappa,i}$, $Px \subset A_{\kappa,i}$ iff $(\lambda x \in A_{\kappa,i} \Leftrightarrow \lambda \in P)$ iff $x \in A_{\kappa,i+1}$.

(iii) Let Q be an ideal of Λ (which can be written as an intersection of $\leq \kappa$ irreducible ideals if κ is finite). Then Q is a P-primary ideal iff for some $0 \neq x \in E^{(\kappa)}$, $\mathrm{ann}(x) = Q$.

PROOF: (i) and (ii) are immediate consequences of the theorem and the definitions. Since Λ is Noetherian, every ideal is the intersection of finitely many irreducible ideals, in particular a P-primary ideal is the intersection of finitely many irreducible P-primary ideals, so (iii) follows from (5.1(i)) and the fact that $\mathrm{ann}(\langle x_j \rangle_{j<\kappa}) = \bigcap_{j<\kappa} \mathrm{ann}(x_j)$.

5.9 THEOREM: Let P, Λ, E, A_i be as in (5.7). Then:

(i) If $E = A_1$ then $rk^+(E/A) = 1$

(ii) If $E \neq A_1$ then $rk^+(E/A) = 2$.

By (5.4) these are all the possibilities. Examples (5.5) and (5.6) illustrate the two cases.

PROOF: (i) If $E = A_1$ then since $A_1 \cong K$, the quotient field of Λ/P, every element of A_1 is definable from Λ/P: if $x \in A_1$ then $\lambda x \in \Lambda/P$ for some $\lambda \in \Lambda \setminus P$.

(ii) By (5.1(ii), (iii)) no element of E not in A_1 is definable from Λ/P.

5.10 THEOREM: Let P be a prime ideal of Λ, κ a finite or infinite cardinal, $E := E(\Lambda/P)^{(\kappa)}$, $B := (\Lambda/P)^{(\kappa)}$ (so that $E = E(B)$), let B_i be $A_{\kappa,i}$ as defined in (5.8), and let k be the minimal number of generators of P, say $P = ((\lambda_j \mid j < k))$.

(i) If $a \in B_{n+1} \setminus B_n$, then

$$rk^+(a, B, E) \leq \begin{cases} n & (k = 1) \\ \dfrac{k^n - 1}{k - 1} & (k > 1) \end{cases}$$

(ii) $rk^+(E/B) \leqq \omega$.

PROOF: (ii) follows immediately from (i).

(i) By induction on $n \geq 0$ I show that $a \in B_{n+1}$ implies that (*)$rk^+(a, B, E) \leq \Sigma_{i=0}^{n-1} k^i$. Let $a = \langle a_i \rangle_{i<\kappa}$ with $a_i = 0$ almost everywhere.

If $n = 0$, then $a \in B_1$ and so for each non-zero component a_i of a , $ann(a_i) = P = ann(1/P)$; hence by (5.3) there is $\mu_i \in \Lambda \setminus P$ such that $0 \neq \mu_i a_i \in \Lambda/P$. Let $\mu := \Pi_{i<\kappa} \mu_i$. μ is well defined since $a_i = 0$ almost everywhere, and $\mu \notin P$ since P is prime. Thus $0 \neq \mu a \in B$ and a is the unique solution in E of $\mu x - (\mu a) = 0$. Thus $rk^+(a, B, E) = 0$. Notice that for $n = 0$, the right hand side of (*) is also 0.

Now since any $b \in B_1$ is definable over B by the previous paragraph, $rk^+(a, B, E) = rk^+(a, B_1, E)$ by (2.1(iii)) for any $a \in E$. So I proceed to prove (*) with B replaced by B_1 .

Assume now that the statement is proved for $n - 1$ and that $a \in B_{n+1} \setminus B_n$. By (5.8(ii)) $Pa \subset B_n$, so in particular for each $j < k$, $b_j ::= \lambda_j a \in B_n$. Let $\phi(x, \vec{b}) ::= \bigwedge_{j<k}(\lambda_j x + b_j = 0)$, a ppf over B_n . Suppose $c \in E$, $E \models \phi[c, \vec{b}]$. Then $E \models [a - c, \vec{0}]$, and since $((\lambda_j)) = P$, $P(a - c) = 0$, hence $a - c \in B_1$. Thus a is definable over $B_1 \cup \{c\}$ for every solution c of ϕ and so $rk^+(a, B_1 \cup \{b_j | j < k\}, E) \leq 1$. Hence by (3.9(i)), $rk^+(a, B_1, E) \leqq 1 + \bigvee_{j<k} rk^+(b_j, B_1, E)$.

But $b_j \in B_n$ for all $j < k$ and so by the induction hypothesis, $rk^+(b_j, B_1, E) \leq \Sigma_{i=0}^{n-2} k^i < \omega$, so the $\bigvee$ is just Σ and:
$rk^+(a, B_1, E) \leqq 1 + k(\Sigma_{i=0}^{n-2} k^i) = \Sigma_{i=0}^{n-1} k^i$.

(i) follows immediately.

5.11 COROLLARY: For any $a \in E$, there is an accurate analysis $\langle T, \leq, \lambda \rangle$ of $rk^+(a, B, E)$ with rank $(T) < \omega$ and for every $t \in T$, $\lambda(t) = \langle \phi, C \rangle$ with ϕ a conjunction of equations which isolates the type of some element over C, in particular ϕ decides the annihilator I of its solutions by conjuncts $\mu_i x = 0$ with $((\mu_i)) = I$.

PROOF: Apply (3.5), (4.9) and (4.11) to an accurate analysis of $rk^+(a, B, E)$. The resulting analysis is necessarily also accurate by (3.5), has rank $< \omega$ by the theorem, and satisfies the other conditions by (4.9) and (4.11).

5.12 COROLLARY: If $E = B_1$, $rk^+(E/B) = 1$. (Notation as in (5.10), compare (5.9).)

5.13 EXAMPLE: In certain special cases, different techniques can lead to more precise estimates of $rk^+(E/B)$. Consider again $\mathbb{Z}(p^\infty)$ as in (5.6) and look at $E = (\mathbb{Z}(p^\infty))^k$, $1 \leq k < \aleph_0$, and take other notation as in (5.10). Then $B_1 = B \cong (\mathbb{Z}/P)^k$, and in general B_n consists of those vectors whose components can be written with denominator p^n. I claim that $rk^+(E/B) \leq k + 2$.

Note that for any $d, C, \{d\} \cup C \subset B_{n+1} \setminus B_n$, $t(d, B \cup C)$ decides whether or not $\{d\} \cup C$ is linearly independent over B_n. ($\{d\} \cup C$ is linearly independent over B_n iff for all $j < \omega$, $(d_j)_{i<j} \subset \{d\} \cup C$ with $d_0 = d$, all $(\lambda_i)_{i<j} \subset \mathbb{Z}$, $\Sigma_{i<j} \lambda_i d_i \in B_n$ implies that $\lambda_i d_i \in B_n$ for all $i < j$. Now $x \in B_n$ iff $p^n x = 0$.) Also, $t(d, B \cup C)$ determines the level of d: $(p^{n+1}x = 0) \wedge (p^n x \neq 0) \in t(d, B \cup C)$ iff $d \in B_{n+1} \setminus B_n$. Therefore by (4.9) if $\{d\} \cup C \subset B_{n+1} \setminus B_n$ is linearly independent over B_n, there is a ppf $\phi(x)$ over $B \cup C$ which implies "$x \in B_{n+1} \setminus B_n$ and $\{x\} \cup C$ is linearly independent over B_n".

Now $\{x_i | i < j\} \subset B_{n+1} \setminus B_n$ is linearly independent over B_n iff $\{x_i/B_n | i < j\}$ is linearly independent in the vector space B_{n+1}/B_n . (Here B_{n+1}/B_n is clearly isomorphic to $(\mathbb{Z}/P)^k$, the k-dimensional vector space over the p-element field), and so any such set has cardinality $\leq k$, and if $\{x_i | i < k\} \subset B_{n+1}$ is linearly independent over B_n , then it generates B_{n+1} as a submodule of E .

Now it is clear how to construct by recursion on the levels an analysis $\langle T, <, \lambda \rangle$ of $rk^+(a, B, E)$ $(a \in B_{n+1} \setminus B_n)$ with $rank(T) = k + 1$. Let the label of the root be $\langle \phi_0(x), B \rangle$ where ϕ_0 is a ppf over B implying that $x \in B_{n+1} \setminus B_n$. The induction hypothesis will be that for each $t \in T$, $\lambda(t) = \langle \phi(x), B \cup C \rangle$ where $C \subset B_{n+1} \setminus B_n$ is linearly independent over B_n . If $|C| < k$ then the ppf ϕ is chosen so as to imply that $x \in B_{n+1} \setminus B_n$ and $\{x\} \cup C$ is linearly independent over B_n ; if $|C| = k$ then ϕ is chosen so as to define a . This all works by the remarks of the preceding two paragraphs.

<u>5.14 CONJECTURE</u>: (Notation of (5.10)). If $\kappa \geq \aleph_0$ then $rk^+(E/B) = 1$ or $rk^+(E/B) = \omega$.

This seems reasonable, since if $E \setminus B_1 \neq \emptyset$ and $\kappa \geq \aleph_0$, there are arbitrarily large finite linearly independent sets. Thus there is no reason to expect the calculation of rk^+ to be much easier than the method indicated in the proof of (5.10).

<u>5.15 NOTATION</u>: I introduce the following notation and conventions for the subsequent definition and theorem.

Λ as always is a commutative Noetherian ring. $\mathcal{P}$ is a set of prime ideals of Λ , $(\alpha_P)_{P \in \mathcal{P}}$ is a family of cardinals, $A_P ::= (\Lambda/P)^{(\alpha_P)}$,

$E_P ::= E(A_P) \cong E(\Lambda/P)^{(\alpha_P)}$; $A ::= \oplus_{P \in \mathcal{P}} A_P$, $E ::= E(A) \cong \oplus_{P \in \mathcal{P}} E(\Lambda/P)^{(\alpha_P)}$.
The module E together with the various submodules just defined are regarded as being fixed once and for all.

Let $b \in E$. Then $b = \langle b(P, i)\rangle_{P \in \mathcal{P}, i<\alpha_P}$, that is, $b(P, i)$ is the component of b in the i-th copy of E_P . $b \restriction P$ will mean one of two things, depending on the context:

(a) that element $b' :=: \langle b_i' \rangle_{i<\alpha_P}$ of E_P satisfying $b_i' = b(P, i)$ for all $i < \alpha_P$.

(b) the image of the element described in (a) under the inclusion of E_P in E .

<u>5.16 DEFINITION</u>: For each $P \in \mathcal{P}$ let $\rho(P)$ be the least ordinal $> \rho(Q)$ for all $Q \in \mathcal{P}$, $Q \supsetneqq P$. Let $\rho(\mathcal{P})$ be the least ordinal $> \rho(P)$ for all $P \in \mathcal{P}$.

<u>5.17 REMARKS</u>: For maximal members P of $\mathcal{P}$, $\rho(P) = 0$. For $\mathcal{P}$ a chain (necessarily finite since Λ is Noetherian), $\rho(\mathcal{P})$ is the length of the chain. If $\mathcal{P}$ is the set of all prime ideals of Λ , then for $P \in \mathcal{P}$, $\rho(P)$ is the <u>Krull dimension</u> of P and $\rho(\mathcal{P})$ is the Krull dimension of Λ . For many more facts about Krull dimension, I refer the reader to the monograph of Gordon and Robson [GR].

<u>5.18 THEOREM</u>: $rk^+(E/A) \leq \omega^{\rho(\mathcal{P})}$.

<u>PROOF</u>: First note the following facts about ordinal addition: if $0 < n, m < \omega$ and $\alpha < \omega^\delta n$, $\beta < \omega^\delta m$, then $\alpha + \beta < \omega^\delta(n + m - 1)$. In particular, for each ordinal δ , ω^δ is closed under addition.

I must show that for any $b \in E$, $rk^+(b, A, E) < \omega^{\rho(\mathcal{P})}$. Since b

may be written as a finite sum $b = \Sigma_{i<n}(b\restriction Q_i)$ for some $Q_i \in \mathcal{P}$, by the first paragraph and (3.11) it is enough to establish that $rk^+(b, A, E) < \omega^{\rho(\mathcal{P})}$ for each b satisfying $b = b\restriction P$ for some $P \in \mathcal{P}$.

I prove by induction on $\delta < \rho(\mathcal{P})$:

(A): For any $P \in \mathcal{P}$, $\rho(\mathcal{P}) = \delta$, for any $b \in E$, $b = b\restriction P$, $rk^+(b, A, E) < \omega^{\delta+1}$.

Let $<T_0, <_0, \lambda_0>$ be an (accurate) analysis of $rk^+(b\restriction P, A_P, E_P)$ as described in (5.11). Let T', $<'$ be copies of T_0, $<_0$ respectively and for $t \in T_0$ with copy $t' \in T'$, define $\lambda'(t') ::= <\phi', C'>$ where $\lambda_0(t) = <\phi, C>$ by renaming the parameters of ϕ and the elements of C according to the inclusion of E_P in E, then letting $C' = C \cup A$. (Remark: I preserve a formal distinction between the elements of E_P (of the form $<e_i>_{i<\alpha_P}$) and the elements of E (of the form $<e(Q, i)>_{Q\in\mathcal{P}, i<\alpha_Q}$) even though $E_P \subseteq E$ because I have to deal with rank analyses in E_P and in E at the same time.)

The result T' is not necessarily an analysis of rk^+ in E; in fact it is possible that a formula ϕ' appearing as a label in T' has solutions d in E with $d\restriction Q \neq 0$ for some Q different from P. Such a solution will not appear in the label of a successor of the node where ϕ' occurs. Thus additional subtrees must be added to T' to turn it into an analysis of $rk^+(b, A, E)$. I do this, keeping track of the ranks, by recursion on the rank $n < \omega$ of the nodes of T_0. I do not complicate notation any further by using an index to denote the successive modifed versions of T': they are all called T'.

So by recursion on $n < \omega$ I fix up T' so that:

(B): For every node t of T_0, rank $t < n$, the subtree $T'_{t'} = \{s \in T' | s \geq t'\}$ is an analysis of $rk^+(b, C, E)$ where $\lambda'(t') = <\psi, C>$ and furthermore rank $(t') < \omega^\delta$ (rank $(t) + 1)$.

Note that this is vacuous for $n = 0$.

Now I need:

(C): CLAIM: If $t \in T_0$, $\lambda'(t') = <\phi', C>$, and $c \in E$, $E \models \phi[c]$, $c \restriction Q \neq 0$, $Q \in \mathcal{P}$, then $P \subseteq Q$.

PROOF: Recall that $\mathrm{ann}(c \restriction Q)$ is Q-primary (5.8(iii)) and that if I is P-primary, J is Q-primary, $J \supset I$, then $Q \supset P$. (I again refer the reader to [N]). By the choices made in (5.11), ϕ' asserts that $\mathrm{ann}(c)$ contains a P-primary ideal. But $\mathrm{ann}(c \restriction Q) \supset \mathrm{ann}(c)$. Hence $Q \supset P$. (Note that ϕ is a complete formula in $Th(E_P)$ and decides $\mathrm{ann}(c \restriction Q)$ in that theory. But ϕ' is not necessarily a complete formula in $Th(E)$ and so I need the specific form of ϕ guaranteed by (5.11) to obtain the claim.)

If $\delta = 0$, then P is a maximal in $\mathcal{P}$. By the claim, for each $t \in T_0$, $\lambda(t) = <\phi, C>$, the solutions of ϕ' in E satisfy $c = c \restriction P$, that is, they are the solutions of ϕ in E_P, lifted to E by the inclusion of E_P in E. Thus T' is an analysis of $rk^+(b, A, E)$ and $\mathrm{rank}(T') < \omega = \omega^{0+1}$.

Now assume that (A) holds for all $Q \in \mathcal{P}$ with $\rho(Q) < \delta$, $\rho(P) = \delta$, $b = b \restriction P$, and that T_0, T' are as described above. Assume that (B) holds for n and that $t_0 \in T_0$, $\mathrm{rank}(t_0) = n$, with $\lambda(t_0) = <\phi, C>$, so $\lambda'(t_0') = <\phi', C'>$. Every solution d of ϕ' in E satisfies

(i) $d \restriction P$ is a solution of ϕ in E_P.

(ii) $d \upharpoonright Q \neq 0$, $Q \neq P \Rightarrow Q \supsetneqq P$ and hence $\rho(Q) < \rho(P) = \delta$ (by claim (C)).

(iii) $\delta \upharpoonright Q \neq 0$ for at most finitely many $Q \neq P$, say for $(Q_i)_{i<m}$.

When $m \geq 1$, d is a "new" solution of ϕ' , that is, one not obtained from a solution of ϕ . Thus I must add to T' a subtree which is an analysis of $rk^+(b, C' \cup\{d\}, E)$ as a successor of t'_0 . By (3.9):

$$\text{(D):}\quad rk^+(b, C' \cup\{d\}, E) \leq rk^+(b, C' \cup\{d\} \cup\{d \upharpoonright Q_i \mid i < m\} , E)$$
$$+\bigvee_{i<m} rk^+(d \upharpoonright Q_i, C' \cup\{d\} , E) .$$

Now, on the r.h.s. of (D),
$rk^+(b, C' \cup\{d\} \cup\{d \upharpoonright Q_i \mid i < m\}, E) \leq rk^+(b, C' \cup\{d \upharpoonright P\}, E)$ because $d \upharpoonright P$ is definable from $\{d\} \cup\{d \upharpoonright Q_i \mid i < m\}$ $(d \upharpoonright P = d - \Sigma_{i<m}\, d \upharpoonright Q_i)$, and $rk^+(d \upharpoonright Q_i, C' \cup\{d\}, E) \leq rk^+(d \upharpoonright Q_i, A, E)$ because $A \subset C' \cup\{d\}$. Thus:
$rk^+(b, C' \cup\{d\}, E) \leq rk^+(b, C' \cup\{d \upharpoonright P\}, E) +\bigvee_{i<m} rk^+(d \upharpoonright Q_i, A, E)$.
Now $d \upharpoonright P$ is a solution of ϕ , so $C' \cup\{d \upharpoonright P\}$ is the set label of some node t'_1 of T' , t'_1 being the copy of t_1 an immediate successor of t_0 in T_0 . Thus $\text{rank}(t_1) = n - 1$ and by the induction hypothesis on n , $\text{rank}(t'_1) < \omega^\delta(\text{rank}(t_1) + 1) \leq \omega^\delta n$. Furthermore, by (ii) above, $\rho(Q_i) < \rho(P)$ for each $i < m$ and so by the induction hypothesis on δ , for each $i < m$ $rk^+(d \upharpoonright Q_i, A, E) < \omega^\delta$. Thus by the comments on ordinal arithmetic and the inequality above, $rk^+(b, C' \cup\{d\}, E) < \omega^\delta n$.

Let T_d be an analysis of $rk^+(b, C' \cup\{d\}, E)$ with $\text{rank}(T_d) < \omega^\delta n$, and add this to T' as the successor of t'_0 corresponding to the solution d of ϕ' . Note that this does not change any subtrees of T' which are rooted on a node t' of T' with $\text{rank}(t) < n$ (in T_0). Thus the validity of (B) is preserved for such nodes t' .

Now the rank of t_0' is the least ordinal greater than the ranks of all the successors of t' ; thus, after all T_d , $d \in \phi'[E]$ have been added, $\mathrm{rank}(t_0') \leq \omega^\delta n < \omega^\delta(n + 1)$ as was to be proved for (B). Note that by the remarks in the last paragraph, $\mathrm{rank}(t_0')$ is not changed at any subsequent stage of the construction.

Now, applying (B) to the node at the root of T' , $\mathrm{rk}^+(b, A, E) < \omega^\delta(\mathrm{rk}^+(b{\restriction}P, A_p, E_p) + 1) < \omega^{\delta+1}$, establishing (A).

<u>5.19 THEOREM</u>: Let Λ be commutative Noetherian, $\mathcal{P}$, $\mathcal{Q}$ sets of prime ideals of Λ such that if $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ then $P \not\supseteq Q$. Fix sequences of cardinals $\langle\alpha_P | P \in \mathcal{P}\rangle$ and $\langle\beta_Q | Q \in \mathcal{Q}\rangle$. Let $M_0 ::= \bigoplus_{P\in\mathcal{P}}(\Lambda/P)^{(\alpha_P)}$ and $N_0 ::= \bigoplus_{Q\in\mathcal{Q}}(\Lambda/Q)^{(\beta_Q)}$. Let $\mathcal{M} ::= E(M_0)$, $\mathcal{N} ::= E(N_0)$. Let $A ::= M_0 \oplus N_0$, $E ::= E(A) = \mathcal{M} \oplus \mathcal{N}$, let $b \in N$. Then:

$$\mathrm{rk}^+(b, N_0, \mathcal{N}) = \mathrm{rk}^+(\langle 0, b\rangle, A, E)$$

<u>PROOF</u>: Let T be an accurate analysis of $\mathrm{rk}^+(\langle 0, b\rangle, A, E)$, $T = \langle T_0, <, \lambda\rangle$. Define an analysis $T' = \langle T_0, <, \lambda'\rangle$ by the following: if $t \in T_0$, $\lambda(t) = \langle\phi(x, \overrightarrow{\langle c, d\rangle}), B\rangle$, $B \supset A$, then $\lambda'(t) = \langle\phi(x, \vec{d}), (B \cap (0 \oplus N)){\restriction}\mathcal{N}\rangle$, that is, $\lambda'(t)$ is the restriction of $\lambda(t)$ to $\mathcal{N}$. T' is clearly an analysis, so $\mathrm{rk}^+(b, N_0, \mathcal{N}) \leq \mathrm{rank}(T') = \mathrm{rank}(T) = \mathrm{rk}^+(\langle 0, b\rangle, A, E)$.

Now let T be an accurate analysis of $\mathrm{rk}^+(b, N_0, \mathcal{N})$. By (3.5, 4.9) every formula occurring in a label of T can be taken to be complete; hence each formula decides the annihilator I of its solutions in $\mathcal{N}$. So without loss of generality each formula of T contains explicitly conjuncts $\{\mu_i x = 0 | i < n\}$ with $\{\mu_i | i < n\}$ generating I . Let $\phi(x, \vec{c})$ be such a formula. Then $\phi(x, \overrightarrow{\langle 0, c\rangle})$ is a ppf (over some B, $A \subset B \subset E$)

and any solution of it in E has the form $\langle d, a\rangle$ with a a solution of $\phi(x, \vec{c})$ in N and $\mathrm{ann}(d) \supset I$. However, since I is the annihilator of an element of N, I is the intersection of Q-primary ideals, $Q \in \mathcal{Q}$. As well, $\mathrm{ann}(d)$ is the intersection of P-primary ideals, $P \in \mathcal{P}$, or $\mathrm{ann}(d) = \Lambda$. But the first alternative is impossible, since then $P \supset Q$ for some $P \in \mathcal{P}$, $Q \in \mathcal{Q}$, contradicting the assumption on $\mathcal{P}$ and $\mathcal{Q}$. Thus $\mathrm{ann}(d) = \Lambda$, i.e. $d = 0$. As a result, the solutions of $\phi(x, \overrightarrow{\langle 0, c\rangle})$ in E are exactly of the form $\langle 0, a\rangle$ where a is a solution of $\phi(x, \vec{c})$ in N. Supose $T = \langle T_0, <, \lambda\rangle$ and define $T' = \langle T_0, <, \lambda'\rangle$ by the following: if $t \in T_0$, $\lambda(t) = \langle \phi(x, \vec{c}), B\rangle$ where $B \supset N_0$, then $\lambda'(t) = \langle \phi(x, \overrightarrow{\langle 0, c\rangle}), A \cup (\{0\} \oplus B)\rangle$. Clearly T' is an analysis of $rk^+(\langle 0, b\rangle, A, E)$ so $rk^+(\langle 0, b\rangle, A, E) \leq \mathrm{rank}(T') = \mathrm{rank}(T) = rk^+(b, N_0, N)$.

<u>5.20 REMARKS</u>: Corresponding to theorem (5.18) there should be examples showing that the upper bound may be attained. Theorem (5.19) is possibly a useful tool in studying examples, because it identfies cases when adding new direct summands to a module does not change the rk^+ of old elements.

However, the problem of calculating exact ranks (or even lower bounds for rk^+) is a much more difficult problem than that of finding upper bounds. The difference in difficulty is a very fundamental one: to obtain an upper bound on rk^+ it is enough to check an existential quantifier (there exists a formula $\phi(x)$ verifying that $rk^+(---) \leq \alpha$) whereas for lower bounds one must verify a universal quantifier (no formula $\phi(x)$ establishes that $rk^+(---) \leq \alpha$). What is needed then to provide estimates of lower bounds for rk^+ are some new general properties of rk^+ which would simplify this kind of checking.

It seems to me to be likely that the upper bound of theorem (5.18) can be attained. In particular the rings constructed by Gordon and Robson [GR, Section 9] should provide such examples. These rings are commutative Noetherian unique factorization domains with large Krull dimension; because they satisfy strong properties and are given explicitly, it should be easier to work with their injective modules than with arbitrary ones. Consequently, I make the following:

5.21 CONJECTURE: Let $\Lambda = F\{G\}$ be the commutative Noetherian ring constructed in [GR, Section 9] with Krull dimension α, P the set of prime ideals of Λ, $A = \bigoplus_{P \in P} (\Lambda/P)^{(\aleph_0)}$, $E = E(A)$. Then $rk^+(E/A) = \omega^\alpha$.

Given the difficulties of calculating rk^+ exactly, it might be more reasonable to first attempt to show

5.21 a: $rk^+(E/A) \geqq \omega\alpha$, or even $rk^+(E/A) \geqq \alpha$.

I have some very fragmentary results along these lines (Λ, prime ideals $Q \subset P$, $b \in E(\Lambda/P) \oplus E(\Lambda/Q) = E$ such that $rk^+(b, A, E) = 2$) but the result is too special and the proof too involved to include here. Essentially it is a proof by exhaustion checking every possible formula ϕ to show that ϕ does not establish that $rk^+(b, A, E) = 1$. The proof also requires some very technical results from the theory of injective modules beyond those considered here.

For a published example of how difficult it may be to calculate rk exactly, I refer the reader to the article of Woodrow and Knight [WK]. Their example is of quite a different sort than those considered here, and an affirmative resolution of conjecture (5.21) would augment the examples of Deissler, Woodrow and Knight by examples which are much more natural and familiar.

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