Geometry and Spatial Intuition: A Genetic Approach

René Jagnow

Department of Philosophy
McGill University, Montréal

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Abstract

In this thesis, I investigate the nature of geometric knowledge and its relationship to spatial intuition. My goal is to rehabilitate the Kantian view that Euclid's geometry is a mathematical practice, which is grounded in spatial intuition, yet, nevertheless, yields a type of *a priori* knowledge about the structure of visual space. I argue for this by showing that Euclid's geometry allows us to derive knowledge from idealized visual objects, i.e., idealized diagrams by means of non-formal logical inferences. By developing such an account of Euclid's geometry, I complete the "standard view" that geometry is either a formal system (pure geometry) or an empirical science (applied geometry), which was developed mainly by the logical positivists and which is currently accepted by many mathematicians and philosophers. My thesis is divided into three parts. I use Hans Reichenbach's arguments against Kant and Edmund Husserl's genetic approach to the concept of space as a means of arguing that that the "standard view" has to be supplemented by a concept of a geometry whose propositions have genuine spatial content. I then develop a coherent interpretation of Euclid's method by investigating both the subject matter of Euclid's geometry and the nature of geometric inferences. In the final part of this thesis, I modify Husserl's phenomenological analysis of the constitution of visual space in order to define a concept of spatial intuition that allows me not only to explain how Euclid's practice is grounded in visual space, but also to account for the apriority of its results.

Résumé

Dans cette thèse, J'examine la nature de la connaissance géométrique et son rapport avec l'intuition spatiale. Mon but est de réhabiliter la vue de Kant que la géométrie de Euclid est une practique mathématique fondue dans l'intuition spatial, mais qui pourtant rapporte un type de connaissance *apriori* de la structure de l'espace visuel. Je me dispute pour ceci en montrant que cette géometrie de Euclid nous permit de dériver la conaissance d'objets visuels idéalisés ou diagrammes idéaliés avec d'inférences logiques non-formelles. En développant un tel compte de la géométrie eiclidienne je complète la vue standard que la géométrie est un system formel on une science empirique. Cette vue a été principalement développé par les positivistes logiques et est actuallement accepté par beaucoup de mathématiciens et philosophes. Ma thése est divisé en trois parties. J'utilise premièrement les arguments de Hans Reichenbach contre Kant et Husserl's approche génétique au concept d'espace, afin de se dispuiter que la vue standard droit être complétée par un concept d'une géométrie dont les propositiones ont un contenu spatial véritable. Je développe alors une interprétation cohérente de la méthode de Euclid en examinant les sujets de la géométrie euclidienne et la nature des inférences géométriques. Dans la partie finale de ma thèse je modifie l'analyse phénoménologique de Husserl de la constitution de l'espace visuel afin de définir un concept d'intuition spatiale qui me permet d'expliquer non seulement comment la practique de Euclid est fondue dans l'espace visuel, mais aussi l'apriorité de ses résultats.

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In writing this thesis, I have also profited from number of seminars and reading groups. I want to thank Emily Carson and Eric Lewis for their seminar on the philosophy of space, Stephen Menn for his course on Ancient Greek geometry, Phil Buckley and Ian McDonald for taking part in a reading group on Husserl's *Ding und Raum*, and Brian van den Broek and Lydia Patton for many discussions on the philosophy of mathematics. I am indebted to George DiGiovanni for his great support and to David Davies for his exceptional work as Director of Graduate Studies during my stay at McGill.

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1. General Introduction

Throughout history, philosophers and mathematicians have considered Euclidean geometry as a standard not only of mathematical precision and exactness, but also of knowledge. They have believed that Euclid's method enabled the derivation of absolutely certain knowledge about the structure of space through reasoning alone. Most contemporary mathematicians no longer agree with this assessment. They have shown that Euclid's axiomatic presentation is faulty, and have argued that his appeal to intuition degrades the knowledge derived by his method to empirical knowledge. I believe that their conclusions result from an inaccurate understanding of both Euclid's method and the nature of human spatial intuition. In this thesis, I defend the view that Euclid's geometry yields a type of knowledge about the structure of visual space that differs from knowledge derived in the empirical sciences.

The reassessment of Euclid's method and its results was largely a consequence of the emergence of logical positivist philosophy of science at the beginning of the twentieth century. Until then, Kant's view was dominant. Representatives of this view believed that the propositions of Euclidean geometry were both true *a priori* and synthetic and that the possibility of this kind of knowledge could be explained only by appeal to the construction of geometrical proofs in pure intuition. Kant argued, first, that logical means alone do not allow us to derive synthetic propositions, and that we therefore have to appeal to construction in intuition; and second, that empirical intuition allows us only to derive contingent truths, and that we must, therefore, presuppose a different kind of intuition, one that is free of experience -- pure intuition.[1] According to this Kantian view, intuition is intimately connected to geometry: Kantian pure spatial intuition does not only enable a geometer to construct geometrical propositions, but also to justify their truth in a non-empirical way. A series of discoveries in geometry and

[1] This is the core of Kant's argument in the Prolegomena, § 6-13, *Prolegomena zu einer jeden künftigen Metaphysik, die als Wissenschaft wird auftreten können*, *Kant's Werke* vol. 4, Akademie-Textausgabe (Berlin: Walter de Gruyter & Co., 1968), pp. 253-384.

physics shattered this view, however. The construction of non-Euclidean geometries by Bolyai and Lobatchevski and the application of non-Euclidean geometry in the general theory of relativity raised doubts as to the *a priori* validity of Euclidean geometry, while the construction of geometry as a purely formal logical system, first clearly exemplified in David Hilbert's work *Die Grundlagen der Geometrie*, showed that appeal to intuition was not necessary in order to derive geometrical propositions.[2] Drawing the consequences from these developments, logical positivists such as Hans Reichenbach, Moritz Schlick, and Rudolf Carnap formulated what became the standard view on geometry, which distinguished between pure and applied geometry.[3] Pure geometry is a branch of pure mathematics and as such a purely formal deductive system; the terms occurring in its sentences do not refer to either real or to imagined objects.[4] As a consequence, pure geometry does not appeal to intuition, its sentences are said to be true *a priori* because they follow from a consistent system of axioms. Intuition enters only when such a geometrical construct is applied to reality by interpreting its terms with real elements, like light-rays or pencil-lines. In this case, however, the sentences of geometry become empirical statements and there is no need to appeal to anything like pure intuition.[5] According to the logical positivists, the two types of geometry exhaust the

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[2] David Hilbert, *Grundlagen der Geometrie*, 1st. ed., published in *Festschrift zur Enthüllung des Gauss-Weber Denkmals* (Leipzig: Verlag von B. G. Teubner, 1899). An historical account of the genesis of the formal view on geometry can be found in Ernest Nagel, "The Formation of Modern Conceptions of Formal Logic in the Development of Geometry," *Osiris* 7 (1939): pp. 142-224.

[3] This standard view was first voiced by Albert Einstein in *Geometrie und Erfahrung* (Berlin: Springer Verlag, 1921). For other expressions, see Moritz Schlick, *Allgemeine Erkenntnislehre* (Berlin: Springer, 1925), pp. 320-329; Hans Reichenbach, *Die Philosophie der Raum-Zeit-Lehre* (Berlin: De Gruyter, 1928); Rudolf Carnap, *An Introduction to the Philosophy of Science*, Martin Gardner, ed., (New York: Dover, 1966), pp. 177-183; and Ernest Nagel, *The Structure of Science* (New York: Harcourt, Bruce & World, 1961), pp. 203-233.

[4] Einstein writes, for example: "Unter 'Punkt', 'Gerade' usw. sind in der axiomatischen Geometrie nur inhaltsleere Begriffsschemata zu verstehen. Was ihnen Inhalt gibt gehört nicht zur Mathematik." ["In axiomatic geometry the words 'point,' 'straight line,' etc., stand only for empty conceptual schemata. That which gives them substance is not relevant to mathematics."], Albert Einstein, *Geometrie und Erfahrung*, p. 5., translation from Albert Einstein, *Sidelights on Relativity*, translated by G.B. Jeffery and W. Perrett (London: Methuen & Co. LTD., 1922), p. 31.

[5] In a famous passage Einstein sums up his view as follows: "Insofern sich die Sätze der Mathematik auf die Wirklichkeit beziehen, sind sie nicht sicher, und insofern sie sicher sind beziehen sie sich nicht auf die Wirklichkeit." ["As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."], Albert Einstein, *Geometrie und Erfahrung*, p. 4., translation

possible alternatives. Thus, the standard view rejects the existence of a type of geometry whose propositions have genuine spatial content, but which is distinct from applied geometry. I will call a type of geometry that fulfills these two conditions a "material geometry."

My central purpose in this thesis is first to show that the standard view has to be supplemented by a coherent notion of a material geometry and then to develop such a notion. In order to do so, I will engage with the arguments of both early logical positivists and phenomenologists. In particular, I will consider the views of Hans Reichenbach, whose seminal work *Die Philosophie der Raum-Zeit-Lehre* contributed perhaps most to the emergence of the standard view, and Edmund Husserl, who attempted to defend a Kantian position while taking into account the new developments in mathematics and physical geometry. I choose this point of departure, because both of these authors were involved in intense debates about the nature of geometry which focused strongly on the role of spatial intuition or perception. Reichenbach and Husserl thought that an investigation into the nature of geometry could not be divorced from an analysis of the nature of perception itself. Their strong interest in perception was no doubt a result of the historical/philosophical situation which forced them to react to the views of Kant and of the neo-Kantians.[6] This interest was also fostered by the revolutionary developments in the psychology of perception at the end of the nineteenth and the beginning of the twentieth centuries, specifically by Hermann von Helmholtz's investigations into the role of motion in spatial perception and by the Gestalt psychologists' dismissal of inferentialist theories of perception.[7] Yet the standard view's popularity and success in the second half of the twentieth century shifted the focus of the philosophical discussions about the nature of geometry away from perception and

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from Albert Einstein, *Sidelights on Relativity*, p. 28.

[6] In particular Michael Friedman has emphasized the role of neo-Kantianism in the emergence of logical positivism. Cf. Michael Friedman, *Reconsidering Logical Positivism* (Cambridge: Cambridge University Press, 1999).

[7] Reichenbach explicitly acknowledges Helmholtz's influence on his concept of perception in *Die Philosophie der Raum-Zeit-Lehre*, p. 78. We will see later on in part three of this dissertation that Husserl's philosophy of perception was also influenced by Helmholtz.

towards other issues such as conventionalism and realism.[8] Since the proponents of the standard view believed that the distinction between pure and applied geometry exhausted the possible options, an inquiry into the intuitive basis of geometry was no longer required. I will take up the question of the relation between geometry and spatial intuition by reengaging with some aspects of the early twentieth century debate about the nature of geometry.

My thesis is divided into three parts. In part one, I show that the standard view does not exclude the existence of a material geometry and that the latter is a necessary presupposition for the former. In order to show that the standard view does not exclude a material geometry, I first criticize Hans Reichenbach's arguments against Kant's concept of geometry. An analysis of Reichenbach's early works shows that his philosophical method of the analysis of the sciences (*wissenschaftsanalytische Methode*) does not allow him to investigate the structure of intuitive space. To do so would have been necessary in order to dismiss Kant, however. Once I have criticized Reichenbach's reply to Kant, I will point out a conceptual problem that arises from the standard view for physical geometry. I then outline Husserl's genetic approach to the concept of space and show how it allows us to solve the conceptual problem with the standard view by appeal to the notion of a material geometry.

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[8] This move may also have had an historical reason. As Michael Friedman has pointed out, through the forced emigration of many of its main proponents, logical positivism was taken out of the philosophical context from which it emerged. I believe that this ended many discussions that might otherwise have continued. As an example, I want to mention here the discussion between Hans Reichenbach and the phenomenologist Oskar Becker, a pupil of Husserl's, about the role of intuition in geometry. The discussion was initiated by Becker's review of Reichenbach's *Die Philosophie der Raum-Zeit-Lehre*. Reichenbach responded in print. In his paper, he expressed a view that seemed to have been symptomatic for the nature of the philosophical situation at that time as expressed by Friedman. He writes: "Meiner Entgegnung möchte ich die Bemerkung vorausschicken, daß ich Beckers Kritik meiner Auffassung als eine auf philosophischem Gebiet sehr erfreuliche Erscheinung begrüße. Es wird hier von einem Gegner der neueren naturphilosophischen Schule der Versuch einer eingehenden sachlichen Auseinandersetzung mit unseren Argumenten gemacht, und es werden dabei die Ergebnisse der mathematischen und physikalischen Forschung in ihrem ganzen Umfange anerkannt und in die Erörterung einbezogen." ["I want to introduce my reply with the remark that I welcome Becker's critique of my view as a very desirable occurrence within the field of philosophy. In this critique, an opponent of the newer school of natural philosophy attempts a detailed, factual examination of our arguments, which acknowledges the results of mathematical and physical research in their full extend and includes them into the debate."], Hans Reichenbach, "Zum Anschaulichkeitsproblem der Geometrie. Erwiderung auf Oskar Becker," *Erkenntnis* 2 (1931): pp. 61-72, p. 61, (translation my own).

In the second part of my dissertation, I develop a coherent account of such a material geometry. Given the clear distinction between pure and applied geometry drawn by the proponents of the standard view, the main challenge is to specify the concept of a material geometry in such a way that it does not collapse into either of those two geometries. The main goal of this part is therefore to identify the subject matter of material geometry and to specify a notion of inference that can transport specifically spatial content from this subject matter into the geometric propositions. I first criticize the commonly held view that material geometry is an axiomatic theory about an idealization of perceptual space. In particular, I will consider the theories of Husserl and the early Carnap. Their views do not suffice to establish an essential difference with pure (formal) geometry. Subsequently, I will argue that any attempt to define material geometry as an axiomatic theory in the contemporary sense will lead necessarily to its collapse into either pure or applied geometry. In order to formulate an alternative concept of the subject matter of material geometry, I then analyze the method exhibited in Euclid's *Elements*. An investigation of the actual practice shows that it is not the precursor of an axiomatic theory in the contemporary sense, as many interpreters still assume. Rather, Euclid's geometry is a practice that allows a geometer to derive geometric knowledge from certain qualitative properties of idealized visual objects, i.e., geometric diagrams, by means of a certain type of non-formal logical inference. The fact that the diagrams serve as means for exploring the structure of intuitive space, and that they thus provide a source of specifically spatial knowledge, prevents the collapse of Euclid's geometry into formal geometry. Euclid's geometry will not collapse into applied geometry either, because it derives spatial knowledge through logical inferences. Finally, I will present Reviel Netz's definition of the general validity of Euclid's results and show how it differs from generality in Hilbert's geometry.

My main goal in the third part of this thesis is to formulate a concept of spatial intuition that will allow me to explain how the general validity of Euclid's results is established. In order to formulate an adequate notion of spatial intuition, I have to investigate how visual space is experienced phenomenally in everyday perception. This is the goal of many theories of perception. In the first section of the last part, I therefore consider a number of theories that claim to do so, namely Berkeley's, Helmholtz's, Irvin Rock's, and James Gibson's. We will see that these theories cannot describe the phenomenal characteristics of perceptual space, because they understand perception as an inferential process. I conclude that only a theory of direct perception can adequately describe the phenomenal structure of perceptual space. Subsequently, I consider Husserl's theory of perception and his phenomenological account of perceptual space. I argue that although Husserl himself understands his theory of perception as a direct theory, he falls victim to some of the problems of the inferentialists. Yet, the specific problems of his account allow us to formulate a number of principles that a theory of perception suited for my purpose has to fulfil. I will then modify Husserl's account of the phenomenal structure of spatial perception according to these principles and formulate a concept of spatial intuition. On the basis of this concept, I will then explain how the general validity of Euclid's results is established. I will also argue that if perceptual space is to allow the application of Euclid's method, the results derived by this method will necessarily be consistent with formal geometries describing a space with constant curvature. In my conclusion I draw the consequences of my arguments concerning the nature of geometric knowledge, suggesting that Kant was right in assuming that material geometry yields *a priori* knowledge about the structure of visual space.

Part I: The Positivists' Rejection of the Notion of a Material Geometry and Husserl's Genetic Approach

2. Hans Reichenbach's Philosophy of Geometry and his Critique of Kant's Conception of Geometry

My goal in this section is to examine and to criticize the validity of the standard view's arguments against Kant's position. I will do so by analyzing Hans Reichenbach's arguments against Kant's conception of a material geometry. I choose this point of departure because Reichenbach presented the most explicit expression and influential defence of the standard view. In several books -- *Relativitätstheorie und Erkenntnis Apriori*, *Axiomatik der relativistischen Raum-Zeit-Lehre*, and *Die Philosophie der Raum-Zeit-Lehre*[9] -- he developed a philosophical approach to the problem of space based on Hilbert's axiomatic method. He called his approach the "method of the analysis of the sciences" (*wissenschaftsanalytische Methode*) and used it to criticize and dismiss Kant's notion of a material geometry. Reichenbach understood that an adequate refutation of Kant must not only clarify the distinction between pure and applied geometry, but also show the impossibility of a third type of geometry whose propositions are grounded in a pure form of spatial intuition. In PRZL, Reichenbach, therefore, not only presented the standard view, but also argued that no *a priori* form of spatial intuition could exist. In my argument against Reichenbach, I will proceed in the following way. By appeal to a recent debate between various interpreters of Kant's conception of geometry, I will show that we can actually ascribe to Kant *two* reasons for believing that geometry was grounded in pure intuition, -- one deriving from his conception of logic and the other from his notion of what constitutes a meaningful

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[9] Hans Reichenbach, *Relativitätstheorie und Erkenntnis Apriori* (Berlin: Springer, 1920) (henceforth abbreviated as RA); Hans Reichenbach, *Axiomatik der relativistischen Raum-Zeit-Lehre* (Braunschweig: Vieweg, 1924) (henceforce abbreviated as ARZL); Hans Reichenbach, *Die Philosophie der Raum-Zeit-Lehre* (Berlin: De Gruyter, 1928) (henceforce abbreviated as PRZL).

concept of space.[10] I will then describe the evolution of Reichenbach's philosophical method. This will provide the basis for my discussion of Reichenbach's arguments against Kant's notion of pure spatial intuition, which will show that Reichenbach's method did not allow him to adequately refute Kant's second reason for believing in the existence of a material geometry. I will conclude my discussion of Reichenbach by outlining a problem arising for the standard view. This problem will motivate my subsequent discussion of Husserl's genetic approach to geometry, in which I will argue that the standard view presupposes a third type of geometry, i.e., a material geometry.

2.1 Kant's Reasons for Appealing to Intuition in Geometry

In *Kant and the Exact Sciences*, Michael Friedman suggests that we apply our modern conception of logic in order to understand Kant more fully and, in particular, his arguments in the Metaphysical Exposition (the first part of the Transcendental Aesthetic).[11] Friedman then shows that Kant was forced to appeal to pure intuition because his limited means of logic did not suffice in order to construct geometry as a purely formal system.

Friedman's argument is complex and I will sketch only its main points here. He first describes how Kant's understanding of the nature of geometrical proofs differs from the modern, Hilbertian, conception. According to the modern conception, geometrical proofs are purely formal or conceptual objects. In order to prove a given proposition, for example, Euclid's proposition I,1 (that an equilateral triangle can be constructed with any given line segment as its base), we have to appeal only to the axioms of geometry and the definitions of 'triangle,' 'equilateral,' and 'base.' The proposition that there exists an equilateral triangle on any base then follows purely deductively from the axioms in conjunction with these definitions. As Friedman says,

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[10] I am referring here to the debate between Michael Friedman and Emily Carson. Cf. Michael Friedman, *Kant and the Exact Sciences* (Cambridge, MA: Harvard University Press, 1992) and Emily Carson, "Kant on Intuition in Geometry," *Canadian Journal of Philosophy* 27 (1997): pp. 498-512.

[11] Michael Friedman, *Kant and the Exact Sciences*, p. 56.

such a proof is "ideally a string of expressions in a given formal language" and requires no reference to construction on paper or in intuition.[12] Kant, in contrast, believed that logical deduction did not suffice to produce correct proofs. Friedman quotes the following passage:

> Philosophy confines itself to general concepts; mathematics can achieve nothing by concepts alone but hastens at once to intuition, in which it considers the concept *in concreto*, although still not empirically, but only in an intuition which it presents *a priori*, that is, which it has constructed, and in which whatever follows from the general conditions of the construction must hold, in general for the object [*Objekte*] of the concept thus constructed. (A 715/B 743)[13]

In contrast to modern philosophers of geometry, Kant believed that a geometrical proof had to actually be constructed, either in thought or in reality, for example, on paper. This involved a spatial representation of its elements, that is, of its lines, angles, circles, etc., which in turn required the successive generation of these elements. A geometrical proof for Kant, then, was a spatio-temporal object.[14]

In order to give a reason for Kant's belief that a proof is a spatio-temporal construction, Friedman considers a modern standard objection to Euclid's proposition I,1. This proposition shows that an equilateral triangle can be constructed with any given line segment as base. Friedman presents its proof as follows (see, Figure 1):

> Given line segment AB, construct (by Postulate 3) the circles C_1 and C_2 with AB as radius. Let C be a point of intersection of C_1 and C_2, and draw lines AC and BC

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12 Ibid, p. 58.

13 ["Jene [philosophy] hält sich bloß an allgemeinen Begriffen, diese [mathematics] kann mit den bloßen Begriffen nichts ausrichten, sondern eilt sogleich zur Anschauung, in welcher sie den Begriff in concreto betrachtet, aber doch nicht empirisch, sondern bloß in einer solchen, die sie a priori darstellet, d.i. konstruieret hat, und welcher dasjenige, was aus den allgemeinen Bedingungen der Konstruktion folgt, auch von dem Objekte des konstruierten Begriffs allein gelten muß."], Immanuel Kant, *Kritik der reinen Vernunft*, (A 715/B 743).

14 The dichotomy described here by Friedman between the modern conception of proof as a purely formal or conceptual object and Kant's conception of it as a spatio-temporal object is somewhat problematic. A string of expressions in a purely formal language is also a spatio-temporal object in Kant's sense.

(by Postulate 1). Then since (by definition of a circle: Def. 15) AC = AB = BC, ABC is equilateral. Q.E.D.[15]

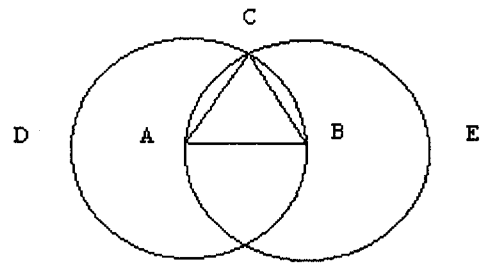

Figure 1: Euclid's Proof I,1

This proof constructs the point C as the intersection of two circles whose centres are the points A and B respectively and whose radii are equal to the line segment AB. Yet, Euclid's axioms and definitions do not specify the nature of circles in such a way as to exclude the following case. The circles might have gaps between their points and thus just "slip through" each other.[16] In this case, point C would not exist. From a contemporary point of view, it thus seems that Euclid's axiomatic system is defective. It does not allow a geometer to conclude the existence of point C, because it does not contain an axiom, for example, a continuity axiom, to secure this. Friedman concludes from this example that Kant must have thought that the existence of point C was guaranteed through the constructions by means of straightedge and compass.[17]

[15] Michael Friedman, *Kant and the Exact Sciences*, p. 59. For Euclid's original proof see: Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (New York: Dover Publications Inc., 1956), first published 1925, vol. 1, pp. 241-242.

[16] From a modern point of view, a model of Euclid's axioms in which point C does not exist can be constructed by covering the Euclidean plane with Cartesian coordinates and by assigning to the midpoint between AB coordinates (0, 0), to A (-1/2, 0), to B (1/2, 0). Given this, point C would have coordinates (0, $\sqrt{3}/2$). Euclid's axioms are satisfied, even if we eliminate all points with irrational coordinates from our model of the Euclidean plane. But in this case, point C does not exist. For this example, see Michael Friedman, *Kant and the Exact Sciences*, p. 60.

[17] Friedman writes: "Does this last 'counter-example' show that Euclid's axiomatization is hopelessly

According to this view, construction yields a kind of existence proof.

Friedman believes that our modern knowledge of logic allows us to interpret Kant's appeal to construction as a consequence of his limited means of logic. The proof of proposition I,1, like all Euclidean proofs involving intersections of geometrical figures, goes through only if we understand Euclid's lines as continuous in the sense required for Euclid's proofs.[18] But, according to our modern axiomatic point of view, this requires appeal to a potentially infinite number of elements. Yet an infinity can be represented conceptually only by means of polyadic logic. The reason for this is that an infinity of elements can be given only through an iterative process that produces its elements. So, for example, the concept of denseness present in modern axiom systems of Euclidean geometry, such as Hilbert's, involves the following series of quantifiers $\forall a \forall b \exists c \, (a > b \rightarrow (a > c > b))$. The conceptual representation of the iterative generation of geometric objects is accomplished through the dependence of the existential quantifier on the universal quantifiers. Kant, however, had at his disposal only monadic logic, which, of course, does not allow for quantifier iteration. We can show that for any finite consistent set of sentences of monadic logic, there is a model of this set with a domain containing at most 2^k objects, where k is the number of predicates in these sentences. Thus, Kant had to appeal to the indefinite iterability of constructive processes in order to express the idea of a potentially infinite number of elements. Two fundamental geometric properties involving an infinitude of elements, namely the idea of denseness (between any two points there is a third point) and the idea of an infinitely long straight line, thus require appeal to constructive procedures: the former to the

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'defective'? I think not. Rather, it underscores the fact that Euclid's system is not an axiomatic theory in our sense at all. Specifically, the existence of the necessary points is not logically deduced from appropriate existential axioms." Ibid., 61. In the second part of this dissertation, I will argue for precisely this position and explore more clearly the nature of the practice of Euclidean geometry.

18 Friedman points out that the type of continuity relevant to Euclidean geometry is not defined by Euclid's postulate 2, which allows the geometer "to produce a finite straight line continuously in a straight line." The main reason for this is that the notion of continuity appealed to in this definition is primitive and can thus be understood in various ways. It does not necessarily rule out cases like the one described in footnote 16. Friedman also emphasizes that Euclidean geometry does not require anything as strong as a continuity axiom. The Euclidean plane can be represented as "a Cartesian space (set of pairs) based on the so-called square-root (or 'Euclidean') extension Q* of the rationals, where Q* results from closing the rationals under the operation of taking the square-roots." The objects of the Euclidean plane thus represent a subset of the full Cartesian plane R^2. Michael Friedman, *Kant and the Exact Sciences*, p. 60 and 62/63.

recursive bisection of a given line segment and the latter to the indefinite recursive extension of a given line segment. As a result, Kant's understanding of a proof as a spatio-temporal object grounded in pure spatial intuition, according to Friedman, was a result of his limited means of logic.

Emily Carson grants Friedman's conviction that Kant's limited logical means represent one of the reasons for his appeal to construction in intuition. Yet, she believes that even if Kant had had polyadic logic at his disposal, he would still have thought it necessary to appeal to intuition in geometry. Carson substantiates her claims roughly in two steps. Firstly, she shows that Kant believed that pure intuition was a necessary condition for geometric constructions and thus for the derivation of geometric knowledge of space. She argues, secondly, that Kant's whole notion of a geometric concept demands an intimate connection to pure spatial intuition. Intuition confers objective reality on the geometric concepts, which, without it, would be mere inventions of the imagination. Thus, for Kant intuition in addition to its inferential role in the geometric derivations, has another role, namely to guarantee the correspondence of its concepts with objective reality, and thus to guarantee the truth of its propositions.

According to Carson, Kant's belief that the experience of space is a necessary condition for geometrical constructions, is a consequence of the fact that he ascribes priority to the intuitive experience of space and considers its conceptual construction as parasitic upon this experience.[19] This becomes clear from a distinction which Kant introduces in the context of a discussion of Kästner's treatises on space. There, Kant distinguishes between two different ways in which space can be treated. Whereas metaphysics is concerned with space 'as it is given before all determinations,' that is, with the original representation of space, geometry deals with space 'as it is generated.'[20] Carson further points out that Kant describes the relation between geometrical and metaphysical space as follows:

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[19] Kant believed that the distinction between intuition and concepts was exhaustive and that any representation was therefore either a concept or an intuition.

[20] Immanuel Kant, "Zur Rezension von Eberhards Magazin. II. Band," in *Kant's gesammelte Schriften* vol. 20, Georg Reimer, ed., p. 419-20, quoted in Carson, "Kant on Intuition in Geometry," p. 497.

> To say, however, that a straight line can be continued infinitely means that the space in which I describe the line is greater than any line which I might describe in it. Thus the geometer expressly grounds the possibility of his task of infinitely increasing a space (of which there are many) on the original representation of a single, infinite space.[21]

Kant thus holds that the space of geometry presupposes the space of metaphysics. He also describes the space of geometry as 'many spaces' which are 'derived' from metaphysical space as 'original and only one single space.' Accordingly, Kant considers the geometrical knowledge of space as secondary to metaphysical knowledge of space, i.e., of the space as it is given before its conceptual construction. In contrast to Friedman, who believes that Kant derives the properties of the original representation of space, i.e., its infinity and singularity, from geometric constructions, Carson concludes that Kant must have independent reasons for ascribing these properties to metaphysical space.

In order to explicate these reasons, Carson follows Parsons's suggestion that certain claims about the original representation of space in §15 of the *Inaugural Dissertation* and in the Transcendental Aesthetic can best be understood as assertions about the phenomenological character of the space of experience.[22] Carson believes that, according to Kant, space is originally given, that is, originally experienced, as a unique and boundless entity. Kant argues for the uniqueness of space as follows: "All spaces are only possible and thinkable only as parts of one single space," therefore, "the representation of parts already presupposes that of the whole."[23] In other words, one

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21 ["Sondern eine gerade Linie kann ins Unendliche fortgezogen werden, heißt so viel: der Raum, in welchem ich die gerade Linie beschreibe, ist größer, als jeder Raum, in welchen ich eine beschreiben mag, und so gründet der Geometer die Möglichkeit seiner Aufgabe, einen Raum (deren es viele giebt) ins Unendliche zu vergrößern, ausdrücklich auf die ursprüngliche Vorstellung eines einigen unendlichen Raums.], Immanuel Kant, "Zur Rezension von Eberhards Magazin. II. Band," *Kant's gesammelte Schriften* vol. 20, p. 419-21, quoted in Carson, "Kant on Intuition in Geometry," p. 498.

22 For Parsons suggestion, see his article "The Transcendental Aesthetic," in Paul Guyer, ed., *The Cambridge Companion to Kant* (Cambridge: Cambridge University Press, 1992), pp. 62-100, p. 72.

23 ["Alle Räume [sind] nur als Theile eines Einzigen möglich und denkbar . . . , und daher [setzt] die Vorstellung der Theile schon das Ganze voraus."], Immanuel Kant, "Zur Rezension von Eberhards Magazin. II. Band," *Kant's gesammelte Schriften* vol. 20, p. 419, quoted in Emily Carson, "Kant on Intuition in Geometry," p. 498.

would not be able to experience limited spaces, i.e. places and spatial objects, if they were not experienced as parts of a larger space surrounding them. The boundlessness, or infinity, of space, on the other hand, follows for Kant from the fact that every part of it, however large, must be given as surrounded by more of the same: "One can only view as infinite a magnitude in comparison to which any specified similar magnitude is equal only to a part."[24] In the *Inaugural Dissertation*, Kant sums up his view as follows:

> *The concept of space is a singular representation* comprehending all things *within itself*, not an abstract common notion containing them *under itself*. For what you speak of as *several places* are only parts of the same boundless space, related to one another by a fixed position, nor can you conceive to yourself a cubic foot unless it is bounded in all directions by the space that surround it.[25]

Accordingly, Kant believes that the properties of metaphysical space can be determined through phenomenological considerations. The original representation is infinite and singular, because it is experienced in this way *before* we derive conceptual knowledge of it by means of geometric constructions.

The specific phenomenological constitution of metaphysical space has consequences for the geometric concept of space. If objects and places can be perceived only as limits of the original representation of space, then geometry can determine the structure of space conceptually only by inscribing limits in its original representation. This task is accomplished through geometric constructions in pure intuition. The properties of the original representation of space, its uniqueness and boundlessness, do not only necessitate the geometric constructions, but also secure their possibility. In the *Critique of Pure Reason*, Kant writes:

> The property of magnitudes by which no part of them is the smallest possible, that is, by which no part is simple, is called their continuity. Space and time are *quanta*

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24 ["Nun kann man eine Größe, in Vergleichung mit welcher jede anzugebende gleichartige nur einem Theile von ihr gleich ist, nicht anders als unendlich benennen."], Immanuel Kant, *Kant's gesammelte Schriften* vol. 20, p. 419, quoted in Carson, "Kant on Intuition in Geometry," p. 498.

25 Immanuel Kant, "Inaugural Dissertation," in *Selected Pre-Critical Writings and Correspondence with Beck*, translated by G.B. Kerfert and D. E. Walford (Manchester, Manchester University Press, 1968), p. 68.

> *continua* because no part of them can be given save as enclosed between limits (points or instants), and therefore only in such fashion that this part is itself again a space or a time. Space consists therefore solely of space, time only of times. Points and instants are only limits, that is, mere positions which limit space and time. But positions always presuppose the intuitions which they limit or are intended to limit; and out of mere positions, viewed as constituents capable of being given prior to space or time, neither space nor time can be constructed. (A 169/B 211)[26]

The continuity required for geometry (including both infinite divisibility and infinite extendibility) is guaranteed by the uniqueness and boundlessness of metaphysical space. As unique representation, metaphysical space allows continuous limitation; as boundless representation, it allows infinite extension. In this way, the phenomenal features of our intuition of space secure the possibility of geometric constructions, thus grounding geometric knowledge.

Up to this point it is still open to Kant to consider geometry as a purely formal system, a purely conceptual construct, assuming he were to be supplied with polyadic logic. He could still say that once geometry is constructed, it could be treated as a purely formal system. Geometric space would then be a purely formal concept. Carson, therefore, also argues that Kant's specific understanding of the nature of mathematical or geometric concepts represents a second reason for maintaining that geometry requires reference to intuition. In the *Critique of Pure Reason*, Kant writes:

> It is, indeed, a necessary logical condition that a concept of the possible must not contain any contradiction; but this is not by any means sufficient to determine the objective reality of the concept, that is, the possibility of such an object as is thought through the concept. Thus there is no contradiction in the concept of a figure which is enclosed within two straight lines, since the concept of two straight lines and of their coming together contain no negation of a figure. The impossibility arises not from the concept in itself, but in connection with its

[26] ["Die Eigenschaft der Größen, nach welcher an ihnen kein Teil der kleinstmögliche (kein Teil einfach) ist, heißt die Kontinuität derselben. Raum und Zeit sind quanta continua, weil kein Teil derselben gegeben werden kann, ohne ihn zwischen Grenzen (Punkten und Augenblicken) einzuschließen, mithin nur so, daß dieser Teil selbst wiederum ein Raum oder eine Zeit ist. Der Raum besteht also nur aus Räumen, die Zeit aus Zeiten. Punkte und Augenblicke sind nur Grenzen, d.i. bloße Stellen ihrer Einschränkung; Stellen aber setzen jederzeit jene Anschauungen, die sie beschränken oder bestimmen sollen, voraus, und aus bloßen Stellen, als aus Bestandteilen, die noch vor dem Raume oder der Zeit gegeben werden könnten, kann weder Raum noch Zeit zusammengesetzt werden."], Immanuel Kant, *Kritik der reinen Vernunft*, (A 169/B 211), quoted in Carson, "Kant on Intuition in Geometry," p. 499.

> construction in space, that is from the conditions of space and of its determination. (A 220-1/B 268)[27]

Kant claims here that the logical possibility of a concept, that is, the possibility to form a concept of an object that is not logically contradictory, does not guarantee the existence of a corresponding object. In Kant's terminology, the logical possibility of a concept does not guarantee its objective reality. Only "the construction of an object in intuition exhibits the objective reality of geometrical concepts by showing the possibility of an object corresponding to the concept."[28] Thus, Kant's anti-formalism is based on the assumption that a legitimate mathematical concept like that of geometric space must be grounded in intuition. Whether this amounts to a coherent anti-formalist view about the nature of geometry depends on whether we can ascribe to Kant both the notion of mathematical concepts that are logically possible, but impossible in a narrower sense, and the idea of possibility corresponding to constructibility in pure intuition.[29] The former would show that Kant had the resources for representing conceptually the possibility of alternative geometries, or at least, of alternative geometrical concepts. The latter would show that Kant did not have to appeal to empirical intuition in order to single out one particular geometrical system as the one whose concepts have objective reality. I will clarify Kant's concepts of logical possibility and constructibility in pure intuition by considering Friedman's objections against them and Carson's reply.

Friedman argues that Kant could not entertain the notion of possibility required for this view. With respect to the idea of alternative geometries, Friedman claims that Kant is unable to accept mathematical concepts that are only logically possible. In the

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[27] ["Daß in einem solchen Begriffe kein Widerspruch enthalten sein müsse, ist zwar eine notwendige logische Bedingung; aber zur objektiven Realität des Begriffs, d.h. der Möglichkeit eines solchen Gegenstandes, als durch den Begriff gedacht wird, bei weitem nicht genug. So ist in dem Begriffe einer Figur, die in zwei geraden Linien eingeschlossen ist, kein Widerspruch, denn die Begriffe von zwei geraden Linien und deren Zusammestoßung enthalten keine Verneinung einer Figur; sondern die Unmöglichkeit beruht nicht auf dem Begriffe an sich selbst, sondern der Konstruktion desselben im Raume, d.i. den Bedingungen des Raumes und der Bestimmung desselben."], Immanuel Kant, *Kritik der reinen Vernunft*, (A 220-21/B 268), quoted in Carson, "Kant on Intuition in Geometry," p. 502.

[28] Ibid.

[29] Ibid.

Critique, Kant writes: 'I cannot think a line except by drawing it in thought.' (B154)[30] This seems to suggest that one cannot use the concept of a line in mathematics, unless one is able to form a corresponding intuition. Yet, since, according to Kant, humans do not have non-Euclidean spatial intuitions, they are unable to entertain the idea of alternative geometries.[31] Friedman concludes that Kant's philosophy of space has no place for mathematical concepts that are merely logically possible. Carson agues that Friedman's criteria for the notion of mathematical possibility are too strong. Kant's anti-formalist view requires only the means of representing the general idea of other possible geometries, or other geometrical concepts. The specific structure of these geometries is irrelevant. That Kant had at his disposal both the general idea and the means for conceptually representing it, follows from two facts. First, he 'recognized the possibility of other creatures with different modes of intuition' (e.g., A27/B43, B148-150, or the Inaugural *Dissertation*, §1). Second, he thought that although we can form no intuition of these different spaces, we can represent them by listing the properties which they do not have.[32] Such a space then is represented 'through all the predicates which are implied in the presupposition that it has none of the characteristics proper to sensible intuition.' (B149) So, although Kant does not have at his disposal axiomatic representations of non-Euclidean geometries, his theory nevertheless allows for other forms of sensibility which can be represented by distinguishing them from human sensibility. Accordingly, Kant can maintain that Euclidean geometry is true because humans can only form intuitions of Euclidean figures. This, however, suffices to formulate the anti-formalist thesis that human intuition sets the concepts of Euclidean geometry above all other thinkable geometric concepts.

Friedman also argues that Kant does not have available to him the notion of constructibility in pure intuition. Kant writes that a pure intuition 'can acquire its objects, and therefore objective validity, only through the empirical intuition of which it

[30] ["Wir können uns keine Linie denken, ohne sie in Gedanken zu ziehen."], Immanuel Kant, *Kritik der reinen Vernunft*, (B 154),

[31] See Michael Friedman, *Kant and the Exact Sciences*, p. 82.

[32] Emily Carson, "Kant on Intuition in Geometry," p. 503.

is the mere form.' (A239/B298-9). So, only construction in empirical intuition can confer objective reality onto geometrical concepts. Construction in pure intuition gives only knowledge of the form of an object and does not establish its possibility, however. Friedman concludes from this that, according to Kant, only transcendental philosophy, not geometry, could prove the possibility of objects corresponding to geometric concepts, thus giving them objective validity. Carson argues against this objection to Kant's anti-formalist view by showing that his appeal to transcendental philosophy is harmless. By establishing once and for all that intuition is a form of sensibility to which, therefore, all empirical objects must conform, transcendental philosophy proves that construction in intuition guarantees the possibility of its objects. Accordingly, geometrical concepts "earn their objectivity derivatively: in establishing that whatever geometry asserts of pure intuition is valid of empirical intuition, the objective reality of geometrical concepts which are constructible in pure intuition is thereby also established."[33] Carson concludes from this that Kant's anti-formalist view finds its expression in the fact that geometry is constrained by pure intuition, because only the latter can ensure that it is not a mere play of imagination, i.e., that its concepts and theorems have non-formal content.

The debate between Friedman and Carson shows that we can ascribe to Kant two reasons for believing that geometry was grounded in pure spatial intuition. The first reason is that his logic does not allow him to conceive of geometry as a formal axiomatic theory whose derivations are purely formal logical inferences. The second reason is his belief that a purely conceptual construct with no connection to intuition might be a mere fiction -- a meaningless conceptual game. This threat seems to be particularly pertinent in the case of Euclidean geometry, where, as Kant thought, we do not have to consult empirical intuition in order to derive propositions. He thought he could guarantee the meaningfulness of this concept of space by proving its *a priori* validity for the space of experience. The faculty of pure intuition enables the geometer

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[33] Ibid., p. 508.

to derive the geometric propositions and simultaneously guarantees that they are true of empirical reality.

I think that the most important point brought to light by Carson is that for Kant any *meaningful* mathematical concept must apply to empirical reality. Even when confronted with a Euclidean geometry constructed as an axiomatic system and based on formal logical inferences, Kant would still say that pure intuition was necessary if the propositions of this geometry were to be meaningful. According to him, the propositions of Euclidean geometry have genuine spatial content. This will remain true even after the construction of formal geometries or non-Euclidean axiomatic systems. Thus, in order to refute Kant's view, one has to show that Euclidean geometry, the type of geometry Kant was concerned with, does not have the properties which he ascribed to it, i.e., that it is not based in pure spatial intuition. In his *Die Philosophie der Raum-Zeit-Lehre*, Reichenbach tried to do just this by showing that there is no *a priori* form of spatial intuition. If his argument proves to be successful, it will show that the propositions of Euclidean geometry do not have objective reality as Kant believed. And so it is Reichenbach's argument to which I now turn.

2.2 Reichenbach's Method of the Analysis of the Sciences (*wissenschaftsanalytische Methode*)

In *Die Philosophie der Raum-Zeit-Lehre* from 1928, Hans Reichenbach refuted Kant's notion of a material geometry by applying his method of the analysis of the sciences (*wissenschaftsanalytische Methode*), which he had developed in two previous works, *Die Relativitätstheorie und das Apriori* from 1920 and *Axiomatik der Raum-Zeit-Lehre* from 1924.[34] In this section, I will reconstruct his method of the analysis of the sciences

[34] In *Die Philosophie der Raum-Zeit-Lehre*, Reichenbach explicitly presupposes the results of *Axiomatik der Raum-Zeit-Lehre*, thus indicating that his argument in the former is based on the application of the method of the analysis of the sciences. Reichenbach writes: "Wenn ausführliche mathematische Rechnungen vermieden werden konnten, so liegt dies daran, daß ein großer Teil der erforderlichen mathematischen Arbeit schon in des Verfassers Axiomatik der relativistischen Raum-Zeit-Lehre niedergelegt wurde; diese Schrift, auf welche die im Folgenden gegebene philosophische Deutung der Raum-Zeit-Lehre aufbaut, muß deshalb für eine strenge Begründung vieler Stellen des vorliegenden Buches herangezogen werden. " ["A considerable part of the necessary mathematical work was completed in the author's *Axiomatik der relativistischen Raum-Zeit-Lehre* and detailed mathematical computation could therefore be omitted from this book. The philosophical interpretation of space and time presupposes the earlier work to which I have to refer the reader for rigorous proofs of many

by reference to these two works. We will see that Reichenbach's definition of this method in ARZL represents a major shift in focus vis-à-vis RA. Whereas in RA the method was designed to disclose the transcendental presuppositions for the constitution of the objects of experience in general, in ARZL the method was restricted to the experience with respect to a particular scientific theory. This argument will lay the foundation for my critique of Reichenbach's argument against a notion of pure intuition in the next section.

In RA, Reichenbach expressed his dissatisfaction with contemporary philosophy, which he thought was alienated from the natural sciences, in contrast to the philosophy of the seventeenth and eighteenth centuries. Looking back, Reichenbach praised Kant for his prudence in orientating his analysis of reason on a contemporaneous concept of scientific knowledge. Reichenbach claimed that most of the value of Kant's theory derived from the fact that his philosophy reflected Galileian and Newtonian physics to such a high degree. Yet, the shortcomings of contemporary philosophy were already prefigured in Kant. Reichenbach writes:

> It seems to have been Kant's mistake that he who had discovered the essence of epistemology in his critical question confused two aims in his answers to this question. If he searched for the conditions of knowledge, he should have analyzed knowledge, but what he analyzed was reason. . . . It is correct that the nature of knowledge is determined by reason, but how this influence of reason manifests itself can be expressed only by knowledge, not by reason.[35]

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statements in the present book."], Hans Reichenbach, *Philosophie der Raum-Zeit-Lehre,* p. 6, translation in Hans Reichenbach, *Philosophy of Space and Time*, translated by M. Reichenbach and J. Freund (New York: Dover, 1958), p. xv.

[35] ["Es scheint uns der Fehler Kants zu sein, daß er, der mit der kritischen Frage den tiefsten Sinn aller Erkenntnistheorie aufgezeigt hatte, in ihrer Beantwortung zwei Absichten miteinander verwechlselte. Wenn er die Bedingungen der Erkenntnis suchte, so mußte er die Erkenntnis analysieren; aber was er analysierte, war die Vernunft. . . . Es ist ja richtig, daß die Art der Erkenntnis durch die Vernunft bestimmt ist; aber worin der Einfluß der Vernunft besteht, kann sich immer nur wieder in der Erkenntnis ausdrücken, nicht in der Vernunft."], Hans Reichenbach, *Relativitätstheorie und Erkenntnis Apriori*, p. 68, translation in Hans Reichenbach, *The Theory of Relativity and A Priori Knowledge*, translated and ed. by Maria Reichenbach (Berkeley and Los Angeles: University of California Press, 1965), p. 72; see also, "Logistic Empiricism in Germany and the Present State of its Problems," *The Journal of Philosophy* 33 (1936): p. 142.

Reichenbach recognized here that an analysis of the rational presuppositions of experience could not access reason independently of and prior to its results. Kant's philosophy was saved from the consequences of ignoring this principle only through his rigorous scientific instinct. According to Reichenbach, contemporary philosophers, however, did not share the same concern for the concept of scientific knowledge and thus produced faulty analyses of reason.

Reichenbach believed that these shortcomings could only be avoided by strictly following the method of the analysis of the sciences. He gives his first characterization of this method when he describes his way of proceeding in RA. He writes:

> We shall, therefore, choose the following procedure. First, we shall establish the contradictions existing between the theory of relativity and critical philosophy and indicate the assumptions and empirical data that the theory of relativity adduces for its assertions. Subsequently, starting with an analysis of the concept of knowledge, we shall investigate what assumptions are inherent in Kant's theory of knowledge. By confronting these assumptions with the results of our analysis of the theory of relativity, we shall decide in what sense Kant's theory has been refuted by experience. Finally, we shall modify the concept of *a priori* in such a way that it will no longer contradict the theory of relativity, but will, on the contrary, be confirmed by it on the basis of the theory's own concept of knowledge. The method of this investigation is called the method of logical analysis [*wissenschaftsanalytische Methode*].[36]

In RA, Reichenbach's proceeds from the results of the special and general theories of relativity in order to show how they contradict Kant's epistemological assumptions. Through a critique of the concept of knowledge in the physical sciences, he then defines

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[36] ["Wir wählen deshalb folgendes Arbeitsverfahren. Es muß zunächst festgestellt werden, welches die Widersprüche sind, die zwischen der Relativitätstheorie und der kritischen Philosophie bestehen, und welches die Voraussetzungen und Erfahrungsresultate sind, die die Relativitätstheorie für ihre Bedingungen anführt. Danach untersuchen wir, von einer Analyse des Erkenntnisbegriffs ausgehend, welche Voraussetzungen die Erkenntnistheorie Kants einschließt, und indem wir diese den Resultaten unserer Analyse der Relativitätstheorie gegenüberstellen, entscheiden wir, in welchem Sinne die Thoerie Kants durch die Erfahrung widerlegt worden ist. Wir werden sodann eine solche Änderung des Begriffs 'apriori' durchführen, daß dieser Begriff mit der Relativitätstheorie nicht mehr in Widerspruch tritt, daß vielmehr die Relativitätstheorie durch die Gestaltung ihres Erkenntnisbegriffs als eine Bestätigung seiner Bedeuting angesehen werden muß. Die Methode dieser Untersuchung nennen wir die wissenschaftsanalytische Methode."], Hans Reichenbach, *Relativitätstheorie und Erkenntnis Apriori*, p. 4, translation in Hans Reichenbach, *The Theory of Relativity and Apriori Knowledge*, p. 5

a new concept of the *a priori* that will be fully compatible with the theory of relativity. Accordingly, the central goal of the method of the analysis of the sciences in RA is an analysis of the *a priori*, that is, of the rational principles that underlie our physical knowledge of the world. Reichenbach believes that such an analysis has to start from a given scientific theory and to exhibit the rational principles on which it is based.

In order to gain deeper insight into Reichenbach's method, we have to understand what these principles of rationality are and why their identification constitutes an important philosophical task. Reichenbach's account of the principles of rationality is a consequence of his understanding of the concept of knowledge in physics, on the one hand, and his Kantian belief that we cannot access reality as it is in itself, on the other.

In RA and in ARZL, Reichenbach defines his concept of physical knowledge in opposition to that of pure mathematics. Mathematical knowledge is characterized by the fact that its objects are defined exclusively by the definitions and axioms of a given mathematical theory. In order to determine whether a certain theorem of the theory is true or false, the mathematician considers only its logical relationship to the definitions and axioms. As an example of a theory which allows us to see the nature of mathematical knowledge very clearly, Reichenbach mentions Hilbert's axiomatic presentation of geometry in his *Grundlagen der Geometrie.* The objects of Hilbert's geometric theory are exhaustively characterized by his axioms, which function as implicit definitions. More precisely, all the axioms together define the properties of all the objects, i.e., the points, lines, and planes, about which the theory speaks. Reichenbach concludes:

> Under these circumstance it is not surprising that the mathematical sentence possesses absolute validity. Because the mathematical sentence means nothing other that a new way of connecting known concepts according to known rules.[37]

[37] ["Es ist unter diesen Umständen nicht weiter verwunderlich, daß der mathematische Satz absolute Geltung besitzt. Denn er bedeutet nichts als eine neue Art der Verflechtung der bekannten Begriffe nach bekannten Regeln."], Hans Reichenbach, *Relativitätstheorie und Erkenntnis Apriori*, p. 34., (translation my own). In *Die Philosophie der Raum-Zeit-Lehre*, Reichenbach expresses the same idea in the following way: "Wir müssen deshalb an der Behauptung festhalten, daß die mathematische Geometrie überhaupt keine Wissenschaft vom Raume ist, sofern man darunter jenes anschauliche Gebilde versteht, daß sich mit Dingen erfüllen läßt — sie ist eine reine Mannigfaltigkeitslehre. In ihr spielt die Anschauung keine andere

Thus, knowledge in pure mathematics is synonymous with logical derivability from a consistent set of axioms. In contrast to mathematics, physics is concerned not only with logical relations between its propositions, but also with the truth of these. Since modern physical theories are expressed in mathematical language, this means that a system of physical equations of a given theory has to be true of reality. Reichenbach defines the concept of physical knowledge therefore as follows:

> Physics has developed the method of defining one magnitude in terms of others by relating them to more and more basic magnitudes and by ultimately basing them on a system of axioms, the fundamental equations of physics. Yet what is obtained in this way is always only a system of interrelated mathematical sentences. What is lacking in such a system is a statement expressing the significance of physics, the statement that the system of equations is true for reality. This relation is also totally different from the immanent truth relations of mathematics. *The physical relation can be conceived as a coordination: actual things are coordinated with equations.* (Emphasis mine.)[38]

Thus, physical knowledge is a coordination (*Zuordnung*) between the equations of a

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Rolle als in der Arithmetik oder Analysis; sie ist wie diese eine auf logische Grundbegriffe reduzierbare Disziplin; die logischen Grundbegriffe des Zuordnens, der Klasse usw. konstituieren den eigentlichen Inhalt der geometrischen Aussagen. Die geometrischen Axiome sind, als mathematische Sätze durch Formeln . . . völlig erschöpfend formuliert; alles anschaulich Räumliche ist überflüssige Zutat. Darum gibt es in der mathematischen Geometrie kein Geltungsproblem der Axiome; diese sind willkürlich gesetzte materiale Beziehungen, deren Inhalt sich völlig als gewisse Verknüpfung logischer Grundbegriffe darstellen läßt, und die mit gleichem Recht durch jede andere widerspruchsfreie Verknüpfung ersetzt werden können." ["We must therefore maintain that mathematical geometry is not a science of space insofar as we understand by space a visual structure that can be filled with objects - it is a pure theory of manifolds. In it, visualization plays the same role it does in arithmetic or in analysis; and, like the latter, it is reducible to basic logical concepts, namely the concepts of coordination, classes, etc., which constitute the actual content of geometrical assertions. The geometrical axioms are completely formulated as mathematical laws by formulae . . . . The visual elements of space are an unnecessary addition.], Hans Reichenbach, *Die Philosophie der Raum-Zeit-Lehre*, p. 121, translation in Hans Reichenbach, *The Philosophy of Space and Time*, p. 100.

[38] ["Es ist Methode der Physik geworden, eine Größe durch andere zu definieren, indem man sie zu immer weiter zurückliegenden Größen in Beziehung setzt und schließlich ein System von Axiomen, Grundgleichungen der Physik, and die Spitze stellt. Aber was wir auf diese Weise erreichen, ist immer nur ein System von verflochtenen mathematischen Sätzen, und es fehlt innerhalb dieses Systems gerade diejenige Behauptung, die den Sinn der Physik ausmacht, die Behauptung, daß dieses Systen von Gleichungen Geltung für die Wirklichkeit hat. Das ist eine ganz andere Beziehung als die immanente Wahrheitsrelation der Mathematik. *Wir können sie als eine Zuordnung auffassen: Die wirklichen Dinge werden Gleichungen zugeordnet.* (emphasis mine), Hans Reichenbach, *Relativitätstheorie und Erkenntnis Apriori*, p. 34, translation from Hans Reichenbach, *The Theory of Relativity and Apriori Knowledge*, p. 36, modified.

theory and physical reality.

We can clarify Reichenbach's notion of physical knowledge, by saying more explicitly what he takes to be the relata of this correspondence relation. On the side of theory, he seems to think that it is both the individual equations and the system of equations as a whole.[39] On the side of reality, he actually speaks of objects given in perception. Reichenbach gives as an example Boyle's gas law. As part of a physical theory, the formula $pV = RT$ must be coordinated with real perceptual objects, that is, with gases and their properties. But Reichenbach believes that an object of perception is never given directly in the senses. Rather, it is constituted through the application of concepts to that which is immediately given in the senses.[40] This also applies to the objects of a physical theory. The gas with its properties (pressure, volume, temperature) in Reichenbach's example is a conceptual extension of certain perceptual facts. For example, a physicist ascribes a certain pressure to a given volume of gas on the basis of his/her observation of a manometer, i.e., on the basis of certain measurements.[41] Yet, the extension itself depends on the physical theory.[42] Physical knowledge is therefore not

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39 Reichenbach writes: "Nicht nur die Gesamtheit der wirklichen Dinge ist der Gesamtheit des Gleichungssystems zugeordnet, sondern auch die einzelnen Dinge den einzelnen Gleichungen. ["Not only the totality of real things is coordinated to the total system of equations, but individual things are coordinated to individual equations."], Hans Reichenbach, *Relativitätstheorie und Erkenntnis Apriori*, p. 34, translation from Hans Reichenbach, *The Theory of Relativity and Apriori Knowledge*, p. 37.

40 Summarizing Kant's theory of the constitution of an object of experience, Reichenbach writes: "Der Gegenstand der Erkenntnis, das Ding der Erscheinung, ist nach Kant nicht unmittelbar gegeben. Die Wahrnehmung gibt nicht den Gegenstand, sondern nur den Stoff, aus dem er geformt wird; diese Formung wird durch den Urteilsakt vollzogen. Das Urteil ist die Synthesis, die das Mannigfaltige der Wahrnemung zum Objekt zusammenfaßt." ["According to Kant, the object of knowledge, the thing of appearance, is not immediately given. Perceptions do not give the object, only the material of which it is constructed. The judgement is the synthesis constructing the object from the manifold of perception."] He later continues "Unsere vorangegangenen Überlegungen können den Grundgedanken dieser Theorie nur bestätigen." ["Our previous analysis confirms the fundamental principle of this theory."], Hans Reichenbach, *Relativitätstheorie und Erkenntnis Apriori*, p. 46, translation from Hans Reichenbach, *The Theory of Relativity and Apriori Knowledge*, p. 48.

41 Hans Reichenbach, *Relativitätstheorie und Erkenntnis Apriori*, p. 35.

42 Reichenbach writes: "Es war die große Entdeckung Kants, daß der Gegenstand der Erkenntnis nicht schlechthin gegeben, sondern konstruiert ist, daß er begriffliche Elemente enthält, die in der reinen Wahrnehmung nicht enthalten sind." ["It was Kant's great discovery that the object of knowledge is not immediately given but constructed, and that it contains conceptual elements not contained in pure perception."], Hans Reichenbach, *Relativitätstheorie und Erkenntnis Apriori*, p. 47, translation from Hans Reichenbach, *The Theory of Relativity and Apriori Knowledge*, p. 49.

the coordination between a theory and an objectivity that is accessible independently of it. Rather, the coordination itself first "creates" one side of the relation, including both the order relations between its elements and these elements themselves. Reichenbach writes:

> Thus we are faced with the strange fact that in the realm of cognition two sets are coordinated, one of which not only attains its order through this coordination, but whose elements are defined by means of this coordination.[43]

Thus, physical cognition is a process that coordinates a theory with reality by constituting the latter.[44]

Reichenbach argues that coordination in this sense requires the introduction of certain general principles which he calls 'principles of coordination.' The objects of physical theories are determined by extensions of certain observable facts, such as measurements. Yet, such an extension cannot be achieved by the formalism of the theory on its own. Reichenbach presents a number of examples, in order to illustrate this point. For example, the observation of certain events in the Wilson chamber must be interpretable as observations of the same particle, say, the same electron. In order to do so, the physicist has to have the notion of genidentiy of subatomic particles. Only on the basis of this notion can he/she interpret the various observations as observations of the same object and thus coordinate it with the formalism. Reichenbach therefore calls the principle of genidentiy a 'principle of coordination.' Another example of a coordinating principle is the principle of probability, which allows the physicist to define a physical constant on the basis of individual measurements. Other principles of coordination are

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[43] ["Und wir stehen vor der merkwürdigen Tatsache, daß wir in der Erkenntnis eine Zuordnung zweier Mengen vollziehen, deren eine durch die Zuordnung nicht bloß ihre Ordnung erhält, sondern in ihren Elementen erst durch die Zuordnung definiert wird."], Hans Reichenbach, *Relativitätstheorie und Erkenntnis Apriori*, p. 38, translation from Hans Reichenbach, *The Theory of Relativity and Apriori Knowledge*, 40.

[44] Reichenbach writes: "Wir sahen, daß die Wahrnehmung das Wirkliche nicht definiert, daß erst die Zuordnung zu mathematischen Begriffen das Element der Wirklichkeit, den wirklichen Gegenstand, bestimmt." ["We saw that perception does not define reality, but that a coordination to mathematical concepts determines the element of reality, the real object."], Hans Reichenbach, *Relativitätstheorie und Erkenntnis Apriori*, p. 46, translation from Hans Reichenbach, *The Theory of Relativity and Apriori Knowledge*, 48.

the axioms of mathematics, which refer to vector operations that constitute the concept of physical force, and the principles of space and time.[45] These coordinating principles are not physical equations connecting state variables with others (Reichenbach calls these 'principles of connection'). Rather, coordinating principles are general rules according to which connections take place. With this notion of principles of coordination in place, we can formulate the concept of knowledge in physics more precisely. Physical knowledge consists in the coordination of a certain mathematically formulated theory with reality *via* general coordinating principles.

In order to answer our initial question about the nature of the rational principles underlying a given theory, we have to understand the conditions under which we accept a theory as true. His concept of physical knowledge allows Reichenbach to define the truth of a physical theory as the correctness of the coordination between mathematical theory and physical data. Correctness means that the correlation is unambiguous (*eindeutig*) in the sense that a given state variable will always have the same value, no matter how it is being determined. Reichenbach writes: "For cognitive coordination, unambiguity means that the physical state variable, as it is determined from different empirical data, will be represented by the same value."[46] This unambiguity does not require that a given mathematical theory be coordinated with reality in only one way, or that only one theory be successfully coordinated with reality. Rather, the totality of the mathematical equations and the principles of coordination must lead to a consistent assignment of values to the various variables of state.

Reichenbach further argues that an unambiguous assignment of values to the state variables of a given formalism can be achieved by using different sets of coordinating principles. We have seen that two such principles of coordination were space and time. If we consider the theory of relativity, for example, we can see that certain formulas allow us to transform one assignment of coordinates into another while

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[45] Hans Reichenbach, *Relativitätstheorie und Erkenntnis Apriori*, p. 51.

[46] ["Eindeutigkeit heißt für die Erkenntniszuordnung, daß eine physikalische Zustandsgröße bei ihrer Bestimmung aus verschiedenen Erfahrungsdaten durch dieselbe Messungszahl wiedergegeben wird."], Ibid., p. 43, translation my own. The English edition translates "Eindeutigkeit" as "uniqueness". This is highly misleading. Cf. Hans Reichenbach, *The Theory of Relativity and A Priori Knowledge*, p. 40.

maintaining an unambiguous coordination. This shows that, within certain limits, space-time coordinates can be chosen arbitrarily. Once one particular assignment of coordinates has been given, the metric functions $g_{\mu\nu}$ have determinate values. Thus, a given physical theory contains an arbitrary and a non-arbitrary aspect. According to Reichenbach, the former is expressed by the independent variables and the latter by the dependent variables. By showing how to transform one system of coordinates into another, the formulas of transformation express the objective content of a physical theory, namely the content that remains invariant vis-à-vis various arbitrary assignments of coordinates. Reichenbach concludes from this that as long as the coordination remains unambiguous, the physicist can choose arbitrarily between different sets of principles of coordination.[47]

We can now answer our initial question about the nature of the principles of rationality. Reichenbach claims that the arbitrary aspect of a given theory represents its rational content.[48] In other words, certain principles of a given theory are rational, because they are not uniquely determined by the empirical data. Reichenbach concludes from this with respect to the principles of coordination:

> Just as the invariance with respect to the transformations characterizes the objective nature of reality, the structure of reason expresses itself in the arbitrariness of admissible systems [of coordinating principles].[49]

Accordingly, the principles of coordination are the principles of rationality that underlie

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[47] The idea that the objective content of a physical theory is expressed through the invariants vis-à-vis various possible transformations of coordinates was also expressed by Hermann Weyl. Reichenbach explicitly mentions Weyl's *Raum-Zeit-Materie* (Berlin: Springer, 1918).

[48] Reichenbach writes: "Nicht darin drückt sich der Anteil der Vernunft aus, daß es unveränderte Elemente des Zuordnungssystems gibt, sondern darin, daß willkürliche Elemente im System auftreten." ["The contribution of reason is not expressed by the fact that the system of coordination contains unchanging elements, but in the fact that arbitrary elements occur in the system."], Hans Reichenbach, *Relativitätstheorie und Erkenntnis Apriori*, p. 85, translation from Hans Reichenbach, *The Theory of Relativity and Apriori Knowledge*, p. 88.

[49] ["Aber wie die Invarianz gegenüber den Transformationen den objektiven Gehalt der Wirklichkeit charakterisiert, drückt sich in der Beliebigkeit der zulässigen System die Struktur der Vernunft aus."], Hans Reichenbach, *Relativitätstheorie und Erkenntnis Apriori*, p. 86; translation in Hans Reichenbach, *The Theory of Relativity and Apriori Knowledge*, p. 90.

a physical theory and constitute its objects. They are principles of reason because their acceptance depends to a certain degree on an arbitrary choice.

We have seen at the beginning of this section that Reichenbach defined the method of the analysis of the sciences as a means for distinguishing the empirical content of a theory from its rational content. His analysis has shown that this is accomplished by ordering "its assumptions and classify[ing] them as special and general principles, and as principles of connection and coordination."[50] This categorization of the content of a given theory is based on a specific conception of axiomatic representation which makes the logical structure of the theory explicit, namely Hilbert's.

By applying his method to the theory of relativity, Reichenbach draws an important conclusion for Kant's notion of the *a priori*. Reichenbach believes that Kant's notion of the *a priori* has two meanings. On the one hand, it expresses the fact that certain judgements are valid for all times. On the other hand, it designates the conditions constituting the concept of an object.[51] According to Reichenbach, the discovery of the theory of relativity showed that certain experiences can force the physicist to change his/her theory, including the principles of coordination. Consequently, the principles of coordination are not valid for all times. Nevertheless, Reichenbach strongly confirms Kant's second claim. He writes:

> The concept of the *a priori* is fundamentally changed by our investigations. Because of the rejection of Kant's analysis of reason, one of its meanings, namely, that the *a priori* statement is to be eternally true, independently of experience, can no longer be maintained. The more important does its second meaning become: that only the *a priori* principles constitute the world of experience.[52]

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[50] [". . . der Philosoph aber will diese Annahmen ordnen und gliedern in spezielle und allgemeine, in Verknüpfungs- und Zurdnungsprinzipien."], Hans Reichenbach, *Relativitätstheorie und Erkenntnis Apriori*, p. 72; translation in Hans Reichenbach, *The Theory of Relativity and Apriori Knowledge*, p. 75.

[51] Ibid., p. 46.

[52] ["Der Begriff des Apriori erfährt durch unsere Überlegungen eine tiefgreifende Wandlung. Seine eine Bedeutung, daß der apriorische Satz unabhängig von der Erfahrung ewig gelten soll, können wir nach der Ablehnung der Kantischen Verunftsanalyse nicht mehr aufrecht erhalten. Um so wichtiger wird seine andere Bedeutung: daß die aprioren Prinzipien die Erfahrungswelt erst konstituieren."], Hans Reichenbach, *Relativitätstheorie und Erkenntnis Apriori*, p. 74; translation in Hans Reichenbach, *The Theory of Relativity and Apriori Knowledge*, p. 77, slightly modified.

Accordingly, in RA, Reichenbach strongly believed that the *a priori* principles were constitutive of the very object of experience and that it was the task of the method of the analysis of the sciences to identify these principles.

ARZL presents a significant clarification of the method of the analysis of the sciences which shows more precisely how we have to understand the notion of constitution, but simultaneously changes its focus. In this work, Reichenbach first reaffirms that the axiomatic method is the appropriate means for representing the logical structure of a given theory. He also repeats that the philosopher does not merely have to represent the sentences of the theory in a deductive form, but rather has to divide the propositions of a given theory into those with experiential and those with rational content. Yet, in ARZL, Reichenbach, no longer construes this distinction in terms of principles of coordination and principles of connection, but rather in terms of axioms and definitions. Accordingly, he develops an axiomatic representation of the theory of relativity in a procedure which he describes as follows: "It is possible to start with the observable facts and to end with the abstract conceptualization."[53] In other words, Reichenbach accepts certain fundamental empirical facts as axioms of the theory and then, by means of certain definitions, coordinates the theory with real objects. Thus, the axioms express the empirical and the definitions the rational content of the physical theory. Reichenbach calls this way of proceeding a 'constructive axiomatic,' distinguishing it from a 'deductive axiomatic' in which the axioms are implicit definitions.

This modification of the axiomatic method has consequences with respect to its possible results. In order to see this, we have to describe more precisely how Reichenbach understands axioms and definitions. In outlining the axiomatic method in ARZL, he does not naively assume that the axioms, i.e., assertions about observable facts, can be stated independently of theory. Reichenbach admits that statements of fact presuppose epistemological principles such as the principle of causality. Nevertheless,

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[53] ["Man kann es so einrichten, daß am Anfang gerade die beobachtbaren Tatsachen und am Ende die abstrakten Begriffsbildungen stehen."], Hans Reichenbach, *Axiomatik der Raum-Zeit-Lehre*, p. 2, translation from Hans Reichenbach, *Axiomatization of the Theory of Relativity*, translated by M. Reichenbach (Berkeley and Los Angeles: University of California Press, 1969), p. 4.

he believes that there exist certain elementary facts that represent relative invariants with respect to theoretical interpretation, because of their imprecision. For example, the fundamental facts of the theory of relativity can be stated in the vocabulary of classical physics and in that of everyday perception. Their interpretation, thus, does not depend on the theory of relativity itself.[54] According to Reichenbach, these considerations allow us to select suitable 'statements of experience' as axioms for our theory.

Reichenbach describes definitions in physics by distinguishing them from definitions in mathematics. Whereas the latter define concepts through concepts, the former assume that a concept has already been defined and coordinate it with a 'piece of reality.' Reichenbach calls these definitions 'coordinating definitions.' As in RA, he does not assume that real objects (pieces of reality) are just given to us. Rather, they have to be constituted. Yet, in analogy to the case of the axioms in constructive axiomatics, Reichenbach believes it is possible to choose objects in such a way that their existence is relatively independent of the theory under consideration.[55] So with respect to the theory of relativity, the concept of definition in the modified axiomatic method requires that the objects of classical physics and of everyday perception are already available to us as relative invariants with respect to the theory. Reichenbach illustrates his notion of a coordinating definition as follows:

> Physical definitions, therefore, consist in the coordination of a mathematical definition to a "piece of reality"; one might call them *real definitions*. Thus the concept of the unit of length is a mathematical one; it asserts that a certain particular interval is to serve as a [standard of] comparison for all other intervals. From this nothing can be inferred, however, as to which physical interval is to serve as a unit of length.[56]

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[54] Ibid., p. 4.

[55] Ibid., p. 5.

[56] ["Das physikalische Definieren besteht also in der Zuordnung einer mathematischen Definition zu einem 'Stück Realität'; man kann auch von Realdefinitionen sprechen. So ist der Begriff der Längeneinheit ein mathematischer; er besagt, daß eine gewisse Strecke als Vergleich für alle anderen Strecken dienen soll. Daraus folgt aber nichts darüber, welche reale Strecke Längeneinheit sein soll; dies vollzieht erst die Zuordnungsdefinition, welche den Pariser Urmeter zur Längeneinheit macht."], Hans Reichenbach, *Axiomatik der Raum-Zeit-Lehre*, p. 5, translation from Hans Reichenbach, *Axiomatization of the Theory of Relativity*, p. 8.

The *Urmeter* in this example is an already constituted perceptual object.

We can now see that Reichenbach's switch from 'principles of coordination' to 'coordinating definitions' and from 'the principles of connection' to 'axioms' represents a shift in focus between RA and ARZL with consequences for the question of constitution. In RA, Reichenbach identified Kant's main contribution to philosophy as recognizing that the object of experience is not simply given to us in sense-perception, but is constructed. The principles of coordination were Reichenbach's own attempt to formulate general conditions under which the object of experience could be constituted. In ARZL, in contrast, Reichenbach does not even address this problem, but rather simply presupposes the objects of experience as already constituted, namely in everyday perception and in classical physics. In contrast to the principles of coordination, coordinating definitions relate a physical theory to an already constituted reality. In this way, Reichenbach shifts his focus from a concern about the Kantian question of the constitution of the objects of experience to his own question about the constitution of the objects of a given scientific theory.[57] As a result, Reichenbach's axiomatic method restricts the scope of the transcendental question significantly.

This shift in focus between RA and ARZL, however, is not just Reichenbach's thematic reorientation from questions concerning the constitution of objects in general to those concerning the constitution of the objects of a given scientific theory. Rather, by defining the method of the analysis of the particular science as a method for partitioning the sentences of a given scientific theory into definitions and axioms, Reichenbach dismisses the sphere of pre-scientific experience from the field of philosophical inquiry. The method of the analysis of the sciences therefore contains a blind-spot with respect to this kind of experience. Reichenbach himself did not recognize this consequence. He believed that the method of the analysis of the sciences allows him to draw conclusions with respect to questions whose scope exceeds that of a given scientific theory and its

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[57] Hartmut Hecht argues for the more radical claim that Reichenbach gives up the constitutive question in ARZL. I do not agree with Hecht's conclusion. Reichenbach is still searching for conditions for the possibility of objects of the theory of relativity. Hartmut Hecht, "Hans Reichenbach zwischen transzendentaler und wissenschaftsanalytischer Methode," in Lutz Danneberg, Andreas Kamlah, Lothar Schäfer, eds., *Hans Reichenbach und die Berliner Gruppe* (Braunschweig/Wiesbaden: Friedrich Vieweg & Sohn Verlagsgesellschaft, 1994), pp. 219-228.

ontological and epistemological implications. In ARZL, he concluded, for example:

> The most important result that could be derived from this construction was an explication of the concept of time. Modern epistemology has clearly shown that time is not a form of pure intuition, as Kant believed; the intuitive experience of time is merely the psychological source from which stems a construction of the conceptual scheme of time holding form the physical world.[58]

In this passage, Reichenbach claims that his clarification of the theory of relativity by means of the method of the analysis of the sciences has shown that time is not a pure intuition. But, since for Kant, a pure intuition was the representation of time as it was originally given to the mind, Reichenbach's result is an assertion about the pre-scientific experience of time and thus not restricted to the theory of relativity.[59]

2.3 Reichenbach's Arguments Against Pure Intuition

Reichenbach emphasizes that the discovery of formal axiomatic geometries forced proponents of Kant's notion of a material geometry to reformulate it in the modern axiomatic context. As we have seen, Kant believed that pure intuition was necessary in order to construct geometrical proofs and to guarantee the existence of its objects. We have also seen that his notion of alternative geometries was still quite restricted, since he did not have at his disposal fully developed axiomatic systems of non-Euclidean geometries. As Reichenbach points out, modern geometry has constructed a host of different axiomatic geometries (such as Euclidean, hyperbolic, and elliptic geometry, non-Archimedian geometries, etc). All of these geometries share two features: they are true from a mathematical point of view (their sentences followed from a consistent set of axioms) and their proofs proceed by purely logical means. According to Reichenbach, neo-Kantians therefore formulated Kant's notion of a material geometry in the following

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58 ["Das wichtigste Resultat, welches sich mit der Durchführung dieser Konstruktion ergeben hat, besteht in der Aufklärung des Begriffes der Zeit. Daß die Zeit nicht eine reine Anschauung ist, wie Kant glaubte, sondern das anschauliche Erlebnis 'Zeit' nur die psychologische Quelle ist, aus der die Konstruktion des begrifflichen Ordnungsschemas 'Zeit' für die wirkliche Welt entspringt, ist nun in der erkenntnistheoretischen Entwicklung klar zutage getreten."], Hans Reichenbach, *Axiomatik der Raum-Zeit-Lehre*, p. 11, translation from Hans Reichenbach, *Axiomatization of the Theory of Relativity*, p. 14.

59 Kant refers to the original representation of time in the *Critique of Pure Reason* (A32/B48).

way:

> It has been argued that mathematics is not only a science of implications, but that it has to establish a preference for one particular axiomatic system. Whereas physics bases this choice on observation and experimentation, i.e., on applicability to reality, mathematics bases it on visualization (*Anschauung*), the analogue to perception in a theoretical science. Accordingly, mathematics may work with the non-Euclidean geometries, but in contrast to Euclidean geometry, which is said to be "intuitively understood," these systems consist of nothing but "logical relations" or "artificial manifolds." They belong to the field of analytical geometry, the study of manifolds and equations between variables, but not to geometry in the real sense which has visual significance.[60]

So, the question derived from Kant is whether human beings have an *a priori* spatial intuition which is Euclidean and represents a model of only one of the formal systems. If such an intuition existed, then Euclidean formal geometry would be more than just a formal system, it would also represent the structure of intuitive space. We could then say that there is a special kind of mathematical intuition. In the next part of this thesis, I will argue that the characterization of a material geometry as a formal axiomatic system, which is chosen by intuition is problematic on conceptual grounds. It simply collapses material either into formal or applied geometry. But at this point, I continue to follow Reichenbach's argumentative strategy.

In order to refute the view that Euclidean spatial intuition confers priority to one formal system, Reichenbach shows that humans can have non-Euclidean spatial intuitions. The central idea for his argument comes from Helmholtz. Reichenbach writes:

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[60] ["Es wird behauptet, daß die Mathematik eben doch nicht nur die Wissenschaft der Implikationen sei, daß sie auch die Bevorzugung eines Axiomensystems zu leisten hätte; beruhe für die Physik diese Bevorzugung auf Beobachtung und Experiment, also auf Wirklichkeitsgeltung, so beruhe sie für die Mathematik auf dem wissenschaftstheoretischen Äquivalent der Wahrnehmung, auf der Anschauung. Danach könne die Mathematik wohl mit den nichteuklidischen Geometrien arbeiten, aber sie seien im Gegensatz zu der 'rein wesenhaft geschauten' euklidischen Geometrie bloße 'Beziehungsgefüge', 'fingierte Mannigfaltigkeiten'; sie gehörten in die analytische Geometrie, die Lehre von den Mannigfaltigkeiten und Gleichungen zwischen mehreren Variablen, nicht aber in die Geometrie im eigentlichen Sinne, in der etwas anschaulich vorgestellt werde."], Hans Reichenbach, *Die Philosophie der Raum-Zeit-Lehre*, p. 100; translation in Hans Reichenbach, *The Philosophy of Space and Time*, p. 80. Reichenbach refers specifically to the views of Hans Driesch and Johann von Kries. Reichenbach uses the term 'analytic geometry' here in an unusual sense. He does not mean a synthetic geometry in Descartes sense, but rather an axiomatic system of geometry. This is indicated in particular by the first sentence of the quote where Reichenbach speaks of axiomatic systems.

> Helmholtz has already coined the following definition of 'visualizing': ' . . . that one imagine the series of sense-impressions that one would have if something like that would occur in an individual case.'[61]

In order to substantiate his claim that human beings can have non-Euclidean intuitions, Reichenbach therefore shows how we can visualize a non-Euclidean space. His argument is based on both an analysis of geometric intuition and a result from the logical analysis of physical geometry.

Reichenbach begins his analysis of geometric intuition with the following remark:

> When we attempt to visualize (*anschaulich vorstellen*) an object, for instance a triangle, blurred images emerge in our mind that are obviously connected with previous perceptions.[62]

Geometrical intuition thus presents its objects in the form of an image. In this way it is like a perception, except that the image does not represent an actually perceived external object, but rather an object conceived in imagination. This image can then be represented graphically, by pencil-lines on paper, for example. Reichenbach concludes from this that intuition includes a pictorial function (*bildhafte Funktion*). But, geometrical intuition does not represent its objects exclusively by means of the pictorial function. Reichenbach writes:

> I have a triangle and a straight line intersecting one of the sides of the triangle; if sufficiently prolonged, will the straight line also intersect another line of the triangle? Visualization (*Anschauung*) says "yes." It simply demands this answer

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[61] ["Helmholtz hat bereits für 'veranschaulichen' die Definition geprägt: '. . . daß man sich die Reihe der sinnlichen Eindrücke ausmalen könne, die man haben würde, wenn soetwas im einzelnen Falle vor sich ginge'"], Hans Reichenbach, *Die Philosophie der Raum-Zeit-Lehre*, p. 86. For the quote from Helmholtz see Hermann von Helmholtz, *Schriften zur Erkenntnislehre*, Moritz Schlick and Heinrich Hertz, eds., (Berlin: Springer, 1921).

[62] ["Versuchen wir uns einen geometrischen Gegenstand, etwa ein Dreieck, anschaulich vorzustellen, so tauchen zunächst verschwommene Vorstellungsbilder auf, die ersichtlich mit früheren Wahrnehmungsbildern zusammenhängen."], Hans Reichenbach, *Die Philosophie der Raum-Zeit-Lehre*, p. 52; translation in Hans Reichenbach, *The Philosophy of Space and Time*, p. 38.

> and I can do nothing about it. [63]

Geometrical intuition therefore also contains a normative element which forces the person who draws a specific geometrical figure to interpret the image in a particular way. Reichenbach calls this element the normative function of intuition (*normative Funktion*). Geometrical intuition visualizes geometrical concepts by interpreting them with imagined or real objects according to certain rules prescribed by the normative function.

Reichenbach further clarifies the notion of a normative function. He believes that for Kant the normative function was derived from the pictorial function and is therefore not conceptual. It is precisely this idea, says Reichenbach, that led Kant to the recognition of a synthetic *a priori*. But, according to Reichenbach, a closer examination of the actual act of visualizing a given geometrical problem shows this not to be the case. First, the geometer can draw a representation of a particular geometric problem only if he/she can appeal to something that is external to the picture itself, that is, some norm that tells him/her how to draw the lines on paper. This norm must be there *before* the picture is constructed. Further, if the geometer has drawn a particular representation of a geometrical figure on paper, he/she can decide that it is not accurate enough to solve a given problem. The geometer will then correct the picture. This also requires some function external to the pictorial function. Second, Reichenbach points out that human intuition has limits. Human beings cannot represent intuitively geometrical objects that are either very small or very big or make reference to the infinity of space. Yet, despite the impossibility of such intuition, we are able to conduct proofs about these objects with utmost accuracy. Reichenbach concludes:

> The normative function of visualization (*Anschauung*) is revealed as a correlate of the logical compulsion and achieves the same results by means of the elements furnished by the image-producing function as the logical inference does by means

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[63] ["Ich habe ein Dreieck und eine gerade Linie, welche eine Dreiecksseite schneidet; wird diese Linie bei genügender Verlängerung auch eine zweite Dreiecksseite schneiden? Ja, befiehlt die Anschauung. Sie befiehlt es einfach, und ich kann nichts dagegen machen."], Hans Reichenbach, *Die Philosophie der Raum-Zeit-Lehre*, p. 52; translation in Hans Reichenbach, *The Philosophy of Space and Time*, p. 39.

of the conceptual elements of thought.[64]

The normative function is a logical norm constraining the pictorial function. Given the distinction between the pictorial and normative functions, Reichenbach shows that human beings can visualize non-Euclidean spaces by specifying the normative function in such a case.

Reichenbach specifies the normative function required for the perception of non-Euclidean spaces in reference to a result derived from a philosophical consideration of physical geometry. In particular, he argues that the perception of a non-Euclidean space requires the human visual system to change its congruence-definition. He arrives at this result by means of a thought experiment which shows that the curvature of a given space cannot be determined merely on the basis of empirical data, but rather presupposes a coordinating definition that determines the congruence behaviour of non-contiguous physical bodies. He first considers a two-dimensional case and then extrapolates his results to three-dimensional space.

There are two different ways in which we can determine the curvature of a given two-dimensional surface. Consider Figure 2, which shows two surfaces G and E. From our (external) point of view, E is a normal plane and G is a plane with a bump in the middle, a half-sphere. We can measure the curvature of the half-sphere at a given point in one direction by finding the circle that best approximates it at this point.[65] This method requires us to be able to step outside of the plane, however, and would not be open to two-dimensional beings living on the surface. Accordingly, this method does not allow us to determine the curvature of three-dimensional space, since as three-dimensional beings, we cannot step outside of it. There is another method, however, that does not require an external point of view. The curvature of two-dimensional surfaces and thus also of three-dimensional space can be determined through internal measurements. Consider again the example in Figure 2. We can measure the diameter

[64] ["Die normative Funktion der Anschauung enthüllt sich als ein Korrelat des logischen Zwanges, welches mit den von der bildhaften Funktion gelieferten Elementen dasselbe leistet, was der logische Schluß mit den begrifflichen Elementen des Denkens vollzieht."], Hans Reichenbach, *Die Philosophie der Raum-Zeit-Lehre*, p. 56; translation in Hans Reichenbach, *The Philosophy of Space and Time*, p. 42.

[65] The curvature k of a curve is the inverse of the radius of this circle $k = 1/R$.

and the circumference of a given circle with its centre at A', the middle-point of the half-sphere, and then determine the ratio between the two. Since the measurement takes place on a half-sphere, the result will be smaller than π. If, in contrast, we measure a given circle on E in the same way, we will find that the ratio between its circumference and its diameter equals π. Thus, a curved surface differs from one that is not curved with respect to its internal metrical relations.

If we restrict ourselves to this second method, our results will depend on an assumption about the properties of our measuring instruments that requires us to introduce a coordinating definition. To see this, let us now project the points A', B', C' on G, which have equal distance from each other, onto the plane E.

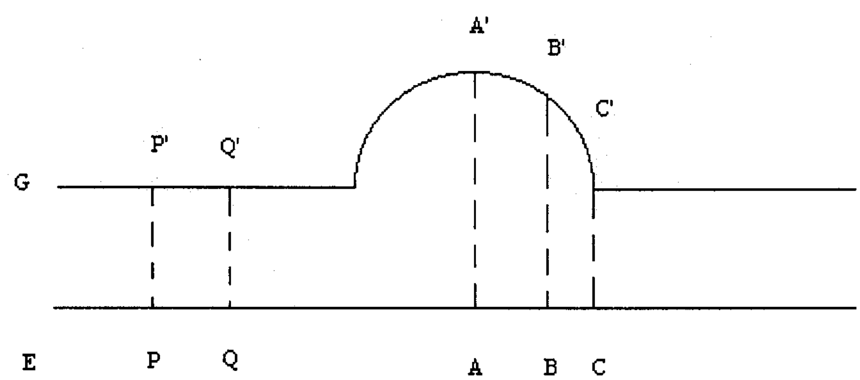

Figure 2: Projection of a Half-Sphere onto a Plane

A person on E measuring the distance between the projected points AB and BC will find that the interval BC is smaller than AB. Let us now postulate a force field in E in such a way that our measuring rod expands in the same ratio in which the distance between the projected points increases. Let us further say that this force field has the same effect on any possible object and cannot be shielded in such a way that the effect will not occur. If we now measure the distances AB and BC, we will get results that coincide with the measurements on the half-sphere. The plane E seems to be curved in the middle. Thus, if we postulate a force field, we can maintain the Euclidean metric and simply say that our measuring rod shrinks as we approach the centre. As a result, the internal geometry of a two-dimensional surface is determined only through both the measuring results and the force field. In other words, since the measurements allow us to compare only contiguous physical bodies, we have to introduce an assumption about the force field, if we want to know the congruence behaviour of non-contiguous bodies. This assumption may consist in simply stipulating that there is no force field.

We can appreciate the definitional character of congruence if we remind ourselves that the force field was defined in such a way that its existence could not be verified by its effects. A force can be measured only on the basis of its differential effects, i.e., its different effects on physical bodies with different physical properties. Yet, according to Reichenbach, the force field is such that it cannot be shielded and has the same effect on every physical body; it is, as he says, a "universal force." Due to the imperceptible nature of such universal forces, the decision whether to introduce a force field is arbitrary, equivalent to introducing a coordinating definition regulating the congruence of non-contiguous physical objects.[66]

Reichenbach extrapolates his example to three-dimensional space in the following way. He takes the projection of G to E as a cross-section of such a space (See Figure 3).

[66] Adolf Grünbaum has argued that Reichenbach's invocation of universal forces in this context is misleading. What Reichenbach's thought experiment with universal forces actually shows is that a continuous manifold has no intrinsic metric and can thus be metricized in different ways. This has nothing to do with physical forces, but concerns only the metrical properties of space. Grünbaum also points out that Reichenbach himself was not mislead by his own formulation. For Grünbaum's view, see his *Philosophical Problems of Space and Time* (Dordrecht: Reidel, 1973), pp. 80-105. See also, section 2.4 of this dissertation.

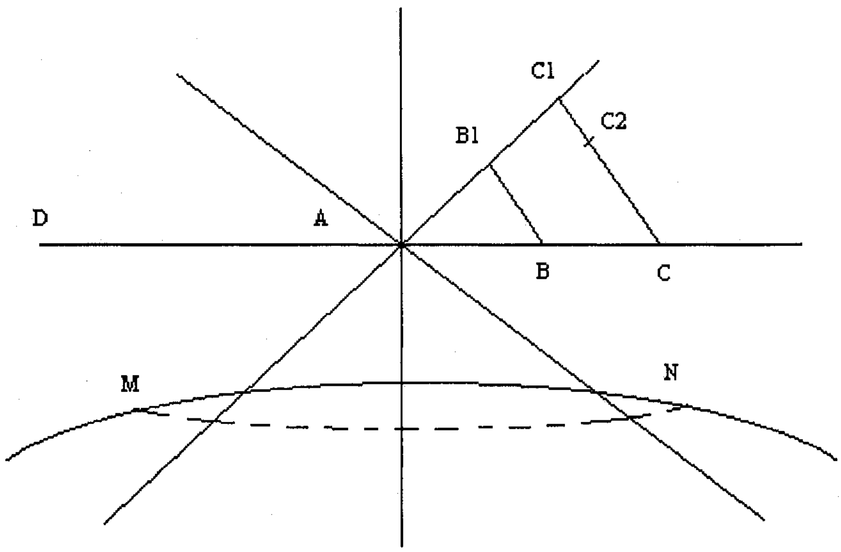

Figure 3: Cross-section through a Space

In this case, A represents some point from which rays (that is the meridians of G projected onto E) emanate in every direction. If a person in this space takes two of these rays which lie in the same plane and measures them, he/she will find that against his/her Euclidean expectations CC_1 is not $2BB_1$ although AB=BC. Again, two explanations are possible here. The person can interpret the measurements as indicating the presence of a force field or as showing the space to be non-Euclidean. In the second case, the measuring rod is taken to maintain its length, when transported to different places. At the same time, the deviations of the measurements from Euclidean geometry are said to be consequences of the metrical properties of space. In this case, it is no longer true that two objects that have the same length at any given place, continue to have it if they are transported to different places. As a result, whether we ascribe to a given space Euclidean or non-Euclidean geometric properties depends on the congruence-definition. In our example, objects are taken to expand when transported outwards along the rays emanating from A.

Reichenbach believes that this consideration about the logical status of the notion of congruence in physical geometry allows him to say how the normative function in perception has to be changed in order to enable human beings to visualize non-Euclidean space. He writes:

> The adjustment necessary for a visualization (*veranschaulichen*) of a curved space consists in projecting congruence differently into three-dimensional space.[67]

All one has to do, in order to perceive space as non-Euclidean is to change the congruence-definition of the visual system. Thus, whether we visualize a given picture as Euclidean or non-Euclidean depends only on the normative function, and, more specifically, on the congruence-definition.

Reichenbach concludes his argument by giving examples from everyday experience in which, as he believes, such changes of the congruence-definition take place.[68] When a myopic person first receives a pair of glasses, things on the periphery of the visual field will appear to move when the person moves. With time, however, the person will adjust and interpret the changes as perspectival changes, rather than as movements of the object. A similar adaptation takes place with respect to convex mirrors in cars, in which objects seem at first distorted. Later, when the driver gets used to the mirror, the objects appear to have normal proportions. In both cases, the person's visual system compensates for non-Euclidean distortions caused by the lenses of the glasses or the non-planar mirror by interpreting the images according to a non-Euclidean congruence-definition. Reichenbach concludes that our visual system has no natural preference for a Euclidean congruence-definition.

His consideration of the experience of non-Euclidean space now allows Reichenbach to draw the following consequence for geometrical intuition. In order to visualize a non-Euclidean space, one has to construct a space, in imagination or on paper, with a non-Euclidean congruence-definition. Reichenbach believes that his

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[67] ["Die ganze Umstellung, die wir nötig haben, um den gekrümmten Raum zu veranschaulichen, besteht also darin, daß wir die Kongruenz anders in den dreidimensionalen Raum hineinsehen."], Hans Reichenbach, *Die Philosophie der Raum-Zeit-Lehre*, p. 70; translation in Hans Reichenbach, *The Philosophy of Space and Time*, p. 54.

[68] Cf., Hans Reichenbach, *Die Philosophie der Raum-Zeit-Lehre*, pp. 70-71.

previous considerations show how this is possible. Since the visual system has no preference with respect to Euclidean geometry, one can perceive a given picture as one representing a non-Euclidean metric. Reichenbach explains this by considering again Figure 3. He asks how one can see the representation as a spherical space in which there are no parallels. In this case, line MN shall represent a line with constant distance to the straightest line (geodesic) DC. According to our Euclidean perception, the line MN is closer to DC in the middle than at its ends. Reichenbach now demands from us that we change the congruence-definition of our visual system. He believes that this allows us to see the lines MN and DC as having the same distance everywhere. He writes:

> If we adjust our eyes to the other congruence we can very well see the distance of the two lines being the same everywhere. We have to realize only that Euclidean congruence, in spite of its obtrusiveness, is likewise merely a definition which we see into the plane of drawing.[69]

Reichenbach then points out that the broken line MN is a shortest line between points M and N. Again, if we change our congruence-definition, we will see that it is shorter than the continuous line MN. Thus, we see that in this space, the shortest line between two points is not identical with the line that has equal distance from a given shortest line. Since, according to Reichenbach, this is synonymous with visualizing that there are no parallels, this example refutes the Kantian notion of a specifically Euclidean intuition. If human beings can adapt to a different congruence-definition, intuition simply cannot select one geometry over another.

Reichenbach concludes his rejection of a specifically mathematical intuition of Euclidean space by arguing that his example, derived from physical geometry, is also relevant to the mere imagination of geometric objects. More precisely, even if we imagine geometric objects in an imaginary space, we have to appeal to a congruence-

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[69] ["Stellen wir uns jetzt auf die andere Kongruenz ein, so können wir sehr wohl den Abstand beider Linien überall gleich 'sehen'. Wir müssen uns dabei nur ganz klar machen, daß die euklidische Kongruenz, die sich dagegen immer weiter vordrängen will, auch eine Definition ist und von uns in die Zeichenebene hineigesehen wird."], Hans Reichenbach, *Die Philosophie der Raum-Zeit-Lehre*, p. 71; translation in Hans Reichenbach, *The Philosophy of Space and Time*, p. 56.

definition. The reason for this is that imagination is also pictorial. The only difference with the physical case is that we do not see physical objects like measuring rods or pencil lines on paper, but rather sense-qualities.

To what extend does Reichenbach's argument answer Kant's challenge? As we have seen, we can ascribe to Kant two reasons for appealing to pure intuition in geometry, one derived from his limited means of logic and the other from his notion of a mathematical concept. Since Kant's first reason was made redundant by modern logic, which allows us to construct geometry as a purely formal science, the question is whether Reichenbach can adequately refute Kant's second reason. When reconstructing his method of the analysis of the sciences, I concluded that Reichenbach's constructive axiomatics in ARZL actually changes the focus of the method. Whereas in RA the goal of this method was to exhibit *a priori* principles necessary for the constitution of the objects of both scientific and pre-scientific experience, in ARZL, this goal is restricted only to the constitution of objects of the physical sciences. Accordingly, Reichenbach's method suffices at best to substantiate his result, i.e., the definitional character of congruence, for the objects of a physical theory. Consequently, whether the definitional character of congruence also applies to the space of pre-scientific experience requires further argument. Reichenbach thought that he did not have to provide such an argument, because he explicitly identified the space of the sciences with the space of pre-scientific experience. He wrote:

> It is incorrect to call the space of the physicists, or their substance, or their laws, something that exists independently, something fictitious, which differs in principle from that which we designate in everyday life with the same words.[70]

However, this identification of the two types of space is merely a conjecture, which Reichenbach does not substantiate.[71] As a result, he does not succeed in refuting Kant's

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[70] ["Es geht nicht an, den Raum der Physiker, oder ihre Substanz, oder ihre Gesetzlichkeit, etwas für sich Bestehendes, etwas Fiktives zu nennen, das von dem grundsätzlich verschieden ist, was der Mensch des täglichen Lebens mit denselben Wörtern bezeichnet."] Hans Reichenbach, "Die philosophische Bedeutung der modernen Physik," *Erkenntnis* 1 (1930-1931): pp. 49-71, p. 51 (translation my own).

[71] The identification of the object (*Gegenstand*) of experience with the object of physics is also pointed out by Renate Wahsner in "Hans Reichenbach's wissenschaftsanalytische Bestimmung des Raum-Zeit-

second reason. Reichenbach is not in a position to exclude a third type of geometry whose truths are grounded in pure intuition.

We can modify Reichenbach's argument in such a way that it does not appeal to the results of his analysis of the theory of relativity. However, even then it is unable to refute Kant's phenomenological reason for pure intuition. In order to see why, let me formulate Reichenbach's argument without reference to the definitional character of congruence. We can say that Reichenbach has shown two things. First, by reference to ordinary experience, i.e. the experience of drawing lines, circles, squares, etc. on paper, he proves that intuition contains a pictorial and a normative function. Second, by reference to ordinary experiences, i.e. experiences involving lenses and convex-mirrors, he showed that the normative function could be changed. These changes can compensate for what we would describe in geometrical terms as deviations from Euclidean metrical properties. Consequently, the visual system does not give preference to Euclidean geometry. This argument deals specifically with the space of pre-scientific experience and makes no use of the definitional character of congruence. From this point of view, we could understand Reichenbach's overall argument against pure intuition as combining phenomenological method and analysis of the sciences. The analysis of the sciences shows that physics and mathematics do not place Euclidean over non-Euclidean space and the phenomenological argument shows that ordinary experience does not do so either.

Yet, a closer look at Reichenbach's 'phenomenological argument' shows that his first result, the distinction between the pictorial and normative functions, was not derived from *pre-scientific* experience. According to his argument, the experience of lines, circles, squares, etc. drawn on paper presupposes knowledge of geometric axioms. This is why Reichenbach says that the normative function has logical force (*logischer Zwang*). He thus derived the distinction between the pictorial and normative functions not from ordinary, but rather from mathematical experience. Whether this distinction could legitimately be extrapolated to our ordinary experience of spatial objects and to

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Begriffs und ihe Kritik," in *Hans Reichenbach und die Berliner Gruppe*, pp. 203-211, p. 208.

the three-dimensional space surrounding them is, therefore, a different question, which Reichenbach does not address.

However, even if we grant Reichenbach the distinction between normative and pictorial functions, his second result, that the human visual system can change the normative function, thus compensating for non-Euclidean distortions of the space perceived, does not follow. His examples show only that certain visual clues are interpreted in a way that maintains the Euclidean character of experiential space. This concerns only the relation between the structure of the visual field and the space as it is experienced and is thus not equivalent to seeing in it a different definition of congruence. Consider Reichenbach's example of the myopic person who receives a new pair of glasses. At the beginning, certain objects on the periphery appear to be moving when the person moves. Later on, when the person gets used to the glasses, these objects appear steady. However, this fact does not indicate that the person now perceives the space of experience as non-Euclidean or that the person perceives a non-Euclidean space. The reason for this is that we can consider the glasses as part of the person's visual system. Accordingly, the process of getting used to the glasses can be described as one in which the visual system compensates for certain distortions *within itself.* In other words, in Reichenbach's example, the person does not see first a space that is non-Euclidean and then compensates for this by ascribing to it a different congruence-definition. Rather, the visual system compensates for certain abnormal experiences.

In sum, in the previous three sections, I argued that Reichenbach did not succeed in refuting Kant's grounds for assuming the existence of a material geometry. Reichenbach refutes Kant by showing that there is no such thing as a pure intuition. He believes that his method of the logical analysis of the sciences allows him to draw consequences as to the *a priori* structure of perceptual and imaginative space and that this suffices to refute Kant's notion of an *a priori* form of intuition. Yet, as we have seen, without further argument, the results gained by the method of the analysis of the science cannot be extrapolated to perceptual or imaginative space, which would have been necessary to refute Kant. As Emily Carson has argued, Kant considered the *a priori* features of the original representation of space as phenomenal properties of spatial experience. I will return to the phenomenological constitution of spatial experience in the third part of this thesis.

2.4 A Conceptual Problem for the Standard View

The standard view in itself poses a conceptual difficulty for applied (physical) geometry. In order to bring out this problem, I will consider two possible ways of understanding physical geometry, both of which do not allow us to explain the constitution of a concept of physical space. These two possible views, reductionism and conventionalism, result from Reichenbach's thought experiment about universal forces. Because the structure of physical space is necessarily underdetermined by the possible observations, physical geometry as an interpreted formal axiomatic system includes three types of statements: statements reporting physical observations about the coincidence of contiguous physical bodies, coordinating definitions, and statements expressing the behaviour of non-contiguous physical bodies.[72] "Reductionism" refers to a thesis about the spatial content or meaning of a physical geometry that reduces the content of such a theory to that given by observation statements. According to this view, the specifically spatial content of geometry is contained exclusively in those statements expressing empirical observations about the behaviour of contiguous physical bodies, and will be the same no matter what coordinating definitions we accept; Euclidean and non-Euclidean physical geometries express the same concept of physical space. The conventionalism view, in contrast, argues that the specifically spatial content of physical geometry is contained in both types of statements together, that is, in the empirical observations and in the statements expressing the behaviour of non-contiguous physical bodies. The term "conventionalism" expresses the fact that the content of the theory depends on the coordinating definition and is thus partly a conventional matter. According to this view, a multiplicity of physical geometries with different *spatial*

[72] I want to emphasize here that I am using the notion of an observation statement in the same sense as Reichenbach used it in his early writings, according to which they are not necessarily free of theory. Rather, observation statements are simply statements expressing results of measurements that enter into the theory in question. Such statements may presuppose other theories like Newtonian physics.

contents exist; Euclidean and non-Euclidean physical geometries express different concepts of space.[73]

By reducing the spatial content of a physical geometry to relations of incidence observable without the introduction of a definition of congruence, reductionism leads to an unacceptably narrow concept of physical space. According to the assumptions of this view, the formal axiomatic system from which physical geometry is constructed does not have any spatial content. The genuine spatial content of the sentences of physical geometry that express facts about the metrical properties of non-contiguous physical bodies is thus restricted to the content of the observational statements. Accordingly, if the reductionist view is correct, the concept of physical space expressed by a physical geometry is restricted to the metrical features of contiguous objects. We can show that this concept of physical space is unacceptably narrow, if we consider the question of what happens when we use it as the basis of a physical theory, say classical mechanics. Consider the case of a physical object moving uniformly in a straight line. Classical mechanics describes such an object as moving equal distances in equal times. If we fix the definition of congruence in such a way that Euclidean geometry holds, the reductionist physical concept of space seems to imply the very same description of the physical object's motion. Yet, since the spatial content of such a geometry is restricted to statements about contiguous physical bodies, the statement that an object moves equal distances has no *spatial* meaning. But what other meaning could it possibly have? Thus, reductionism restricts the concept of physical space in such a way that it can no longer function as a basis for a physical theory.

According to conventionalism, in contrast, the statements about the metrical properties of non-contiguous physical objects have genuine spatial meaning. Conventionalism thus defines the spatial content in sufficiently broad terms as to serve

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[73] Reichenbach himself defended such a reductionism. He believed that there was no difference between the two positions "Euclidean geometry plus universal forces" or "Non-Euclidean geometry with no universal forces" with respect to the content of a physical theory. Henri Poincaré, on the other hand, defended conventionalism, believing that the decision to chose one or the other of these two positions would change the content of the physical theory. For a more explicit description of the two positions see Lawrence Sklar, *Space, Time, and Spacetime* (Berkeley/Los Angeles/ London: University of California Press, 1977), pp. 80-146.

as a basis for a physical theory. Yet, a problem arises when we ask on what grounds we ascribe spatial content to the statements of the theory. According to the assumptions above, they can derive their spatial content neither from observation statements, nor from the formal axiomatic system, because the observation statements do not contain information about non-contiguous objects and the formal theory is simply a formal structure. Consequently, the conventionalist view on the concept of physical space is incoherent. It demands that the statements about non-contiguous physical bodies have spatial content, but is unable to explain how such content is being ascribed to them. As a result, we have to conclude that the standard view raises conceptual problems for applied geometry on both the reductionist and the conventionalist readings.

We can describe the problem for the standard view in an alternative way by considering a specific example of a coordinating definition. In order to apply a formal axiomatic system to physical reality, one can coordinate the concept of a straight line with the path of a light-ray.[74] In this case, it seems that one is giving spatial content to the formal axiomatic system, because the path of a light-ray is a genuine spatial object. Yet, the path of a light-ray is not an empirical spatial object, but rather a geometric interpretation of certain observable facts. To be sure, we can see light, and it is a spatial phenomenon. Yet, we cannot say that a light-ray is a spatial object independent of our physical geometry. Thus, some interpretations are unable to confer spatial content to a formal axiomatic system. With respect to these interpretations, it is not possible to determine whether they are spatial objects independently of the overall theory.

The problem of the spatial content of physical geometry is manifested in the differences between Reichenbach's and Grünbaum's views on the nature of congruence. We have seen that Reichenbach argument for the definitional character of congruence was based on the possibility of introducing universal forces. In a footnote, I pointed out that Grünbaum thinks that Reichenbach's appeal to universal forces is at the very least misleading. The actual reason for the definitional character of congruence, according to Grünbaum, is that in a spatial continuum, as Riemann has shown, "there are no intrinsic

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[74] For example, in the theory of relativity light-rays are taken to travel on geodesics, that is, straightest lines.

metric attributes of intervals which could be invoked to single out one of these congruence classes as unique."[75] Thus, "only the choice of a particular congruence standard which is *extrinsic* to the continuum itself can determine a unique congruence class, the rigidity or self-congruence of that standard under transport being *decreed by convention*."[76] Consequently, whereas Reichenbach argues from the testability, Grünbaum argues from the continuity of the manifold of physical space. If Grünbaum is right, then Reichenbach was mistakenly looking for a physical cause for the different congruence behaviour of physical bodies. Accordingly, the notion of universal forces is merely a metaphorical way of saying that congruence has definitional character.

In order to see whether Grünbaum is right, we have to ask on what grounds he bases his claim that Reichenbach uses the notion of a universal force in a metaphorical sense. Grünbaum simply states that physical space is a continuous manifold and shows that such a manifold does not have an intrinsic metric. If we want to determine the metrical properties of this space, we therefore first have to define congruence. Since we don't know whether this congruence definition is a result of physical properties, we cannot conclude that the metrical properties of physical space are spatial, rather than physical. Yet Grünbaum simply stipulates that metrical properties are purely spatial properties, even though the standard view does not yield any justification for doing so. If Grünbaum could not prove this thesis, than Reichenbach might be right in saying that the metrical properties are physical properties of space. This possibility is supported by the fact that we know of a force that is like a universal force, i.e., that cannot be shielded and has the same effect on every physical material, namely gravitation. Thus the differences between Reichenbach and Grünbaum were made possible because the concept of physical space is underdetermined with respect to the congruence of non-contiguous physical objects.

We can conclude that the non-empirical character of congruence, the very fact that led to the emergence of the standard view, also poses a conceptual problem for it. A conceptual gap opens up between empirical observations of objects in physical space and

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[75] Adolf Grünbaum, *Philosophical Problems of Space and Time*, p. 11.

[76] Ibid.

the latter's complete metrical structure. This gap deprives us of any reason for saying that the total structure expressed in applied geometry is a specifically *spatial* structure, and that physical geometry expresses a physical concept of space. I believe that this gap can be filled only by introducing a material geometry as the link between the space of perception and our formal concepts of space. In order to explain how this is possible, I will now consider Husserl's genetic account of geometry.

3. Edmund Husserl's Philosophy of Geometry

In December 1892, Husserl wrote to Brentano: "Lately, I occupied myself again and again with philosophico-geometrical problems."[77] In the same letter, Husserl pointed out the value of Riemann's and Helmholtz's analyses of the problem of space but at the same time deplored their lack of philosophical reflection. He demanded that the philosophical content of their scientific analyses be clarified. In this way, he recognized the necessity and possibility of the *philosophical* analysis of a given scientific theory. Although Husserl did not have a sophisticated idea of how such analysis would have to proceed, he thereby acknowledged the validity of an investigation in Reichenbach's sense. Husserl continued his letter in the following way, however:

> In my investigations, in contrast, I am pursuing other paths, which, if I am not wrong, lead much deeper and, simultaneously, are incomparably more straight forward and easier than the ones pursued by the mathematicians.[78]

In the following two years, 1893/94, he wrote a number of short studies which contain the basic ideas of his approach to space.[79] He intended these investigations to become part of a *Raumbuch*, an intention he soon abandoned, however, in order to pursue the inquiries into the foundation of logic which culminated in the *Logische Untersuchungen*.[80] On the basis of Husserl's notes to the *Raumbuch*, I will first reconstruct his understanding of a genetic approach to the philosophy of geometry and

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[77] ["In der letzten Zeit habe ich mich immer wieder mit den philosophisch-geometrischen Problemen beschäftigt."], Edmund Husserl, Letter to Franz Brentano, 29. December 1892, in Edmund Husserl, *Die Brentanoschule*: *Briefwechsel,* vol. 1, Karl Schuhmann, ed., (Dordrecht/ Boston/ London: Kluwer Academic Publishers, 1994), p. 10, (translation my own).

[78] ["In meinen Untersuchungen habe ich allerdings ganz andere Wege eingeschlagen, die, wenn ich nicht irre, viel tiefer führen und dabei unvergleichlich ebener und leichter sind, als die von den Mathematikern eingeschlagenen."], ibid., p. 11.

[79] These studies are published as *Philosophische Versuche über den Raum* in Edmund Husserl, *Studien zur Arithmetik und Geometrie. Texte aus dem Nachlass (1886-1901),* Husserliana XXI, ed. by Elisabeth Strohmeyer (The Hague/Boston/Lancaster: Martinus Nijhoff, 1983), pp. 261-310.

[80] Edmund Husserl, *Logische Untersuchungen. Erster Band,* in Husserliana XVIII, ed. by E. Holenstein (The Hague/Boston/Lancaster: Martinus Nijhoff, 1975); Edmund Husserl, *Logische Untersuchungen. Zweiter Band,* in Husserliana XIX,1, XIX,2, ed. by Ursula Panzer (The Hague/Boston/Lancaster: Martinus Nijhoff, 1984).

then outline his genetic account of the different concepts of space. This will allow me to show that the standard view presupposes a third type of geometry next to formal and applied geometry. In my presentation of Husserl's genetic view, I will focus on his early writings between 1890 and 1901. Although Husserl later changed his philosophical method in ways that I will indicate in parts II and III of this thesis, he held on to his early genetic approach to space.

3.1 Genetic Phenomenology

David Bell has argued that Husserl's method in the 1890's, exemplified above all in *Philosophie der Arithmetik* (1891), is identical with Brentano's method of descriptive psychology or descriptive phenomenology.[81] This thesis is defensible until one considers Husserl's own methodological considerations in the manuscripts from 1893/94, which contain an important deviation from Brentano's views. Husserl's philosophical method can thus best be described by contrasting it with Brentano's method of descriptive psychology.

In his *Psychologie vom empirischen Standpunkte*, Brentano defines psychology as the science of mental (*psychische*) phenomena or appearances, distinguishing it in this way from the natural sciences, which deal with physical phenomena.[82] These two types of science together exhaust the totality of all phenomena. Brentano gives three main criteria for the distinction between physical and mental phenomena: (i) Mental phenomena contain intentional objects (*intentionale Gegenstände*) as proper parts within themselves. (ii) Mental phenomena are either presentations (*Vorstellungen*) or founded on presentations. And (iii) they are accessible through inner perception, or introspection. Physical phenomena, in contrast, are not intentional in Brentano's sense; they do not involve presentations, and they are accessible only through outer perception.

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[81] Edmund Husserl, *Philosophie der Arithmetik. Mit Ergänzenden Texten* (1890-1901, in Husserliana XII, ed. by Lothar Eley (The Hague/Boston/Lancaster: Martinus Nijhoff, 1970); David Bell, *Husserl* (London/ New York: Routledge, 1990); David Bell, "A Brentanian Philosophy of Arithmetic," in *Brentano Studien* 2 (1989): pp. 139-144.

[82] Franz Brentano, *Psychologie vom empirischen Standpunkte* vol. 1 and 2, (Leipzig: Duncker und Humblot, 1874).

Sensations, acts of perception, emotions, feelings, acts of willing, acts of thinking, acts of remembering, acts of fantazising, etc. are all examples of mental phenomena, whereas colour, figure, landscape, warmth, coldness, etc. are examples of physical phenomena.[83] Since mental phenomena are the objects of inner perception, Brentano calls them the objects of secondary consciousness. Their contents are physical phenomena, which are the objects of a primary consciousness. His choice of terms here is based on the fact that our attention is normally directed towards external objects and not towards the experiences themselves, that is, the sensations and acts.

Brentano distinguishes between two areas of psychology -- descriptive and genetic. Descriptive psychology "aims at exhaustively determining (if possible) the elements of human consciousness and the ways in which they are connected."[84] The descriptive approach is very similar to that of the natural sciences, and includes: (i) a description of the unique characteristics of all mental phenomena; (ii) the categorization of mental phenomena; and (iii) an analysis of the primary elements which constitute complex phenomena. On the basis of these fundamental investigations, Brentano intends to search for general mental laws. This involves the induction to general laws, the deduction of specific laws, and the verification of specific laws through experience.[85] Brentano's descriptive method thus involves more than just description, namely also analysis, induction, deduction, and experimental verification. Why, then, did Brentano want to call this method descriptive in the first place?

We can answer this question by considering Brentano's understanding of the natural sciences, which he defined as follows:

> Natural science is that science which seeks to explain the succession of physical phenomena connected with normal and pure sensations (that is, sensations which are not influenced by special mental conditions and processes) on the basis of the assumption of the effect on our sense organs of a world which has three

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[83] Ibid., pp. 103-104.

[84] ["Sie sucht die Elemente des menschlichen Bewußtseins und ihre Verbindungsweisen (nach Möglichkeit) erschöpfend zu bestimmen."], Franz Brentano, *Deskriptive Psychologie* (Hamburg: Felix Meiner Verlag, 1982), p. 1, (translation my own).

[85] Franz Brentano, *Psychologie vom empirischen Standpunkte*, p. 93.

dimensional extension in space and flows in one direction in time.[86]

Natural science attempts to explain the primary contents of our sensations and their behaviour by considering them as causal consequences of a spatial and temporal external world. Since, according to Brentano, human beings do not have an immediate access to the external world, it is merely the stipulated cause of physical phenomena, or, in other words, an inferred object. For this reason, Brentano also calls the objects in the spatio-temporal world "non-real objects." Physical phenomena, in contrast, are real objects. For example, one's perception of a tree is real. The perceived tree, on the other hand, is not real, but merely stipulated or intended.[87] Given this understanding of the goal of the natural sciences and the epistemological thesis, that the spatio-temporal word is only inferred, we can say that Brentano's term 'descriptive' expresses the fact that this kind of psychology avoids the stipulation of non-real causes.

Genetic psychology, in contrast, "seeks to exhibit the conditions to which the particular phenomena [that is, appearances as mental phenomena] are causally connected."[88] The task of genetic psychology is to exhibit the physical causes of mental phenomena. The genetic psychologist seeks to determine what chemicals produce certain smell-sensations or what light frequencies produce certain colour sensations, for example. Brentano believes that these generalizations are in principle less exact than those of descriptive psychology. For example, light of a certain frequency usually produces a sensation of blue; yet, not always. A person might be colour blind, his/her nerves might be severed, or he/she might suffer hallucinations. Further, the blue might be obliterated by other sensations. For Brentano, the generalizations of genetic psychology differ from those of descriptive psychology in that the former, but not the

[86] ["Die Naturwissenschaft sei jene Wissenschaft, welche die Aufeinanderfolge der physischen Phänomene normaler und reiner (durch keine besonderen psychischen Zustände und Vorgänge mitbeeinflußten) Sensationen auf Grund der Annahme der Einwirkung einer räumlich in drei Dimensionen ausgebreiteten und zeitlich in einer Richtung verlaufenden Welt auf unsere Sinnesorgane zu erklären suche."], Ibid., p. 128, (translation my own).

[87] Ibid.

[88] ["[Genetic psychology] "sucht . . . die Bedingungen anzugeben, mit welchen die einzelnen Erscheinungen [that is, appearances as mental phenomena] ursächlich verknüpft sind."], Franz Brentano, *Deskriptive Psychologie*, p. 1, (translation my own).

latter, contain *ceteris paribus* clauses, which render them less reliable.[89] In that genetic psychology searches for physical or physiological causes of mental phenomena, it is a hybrid science which belongs partly to the natural sciences and partly to psychology. Nevertheless, since it is based on the stipulation of causes, it can achieve at best only the certainty of the natural sciences. For this reason, Brentano excludes genetic investigation from the realm of pure psychology.

For Brentano, there is a complex interrelation between genetic and descriptive psychology. In order for genetic psychology to get off the ground, descriptive psychology must first describe the phenomena, that is, the complex representations. Only then can genetic psychology commence its investigation into the physical causes of these phenomena. Yet, genetic psychology can also assist descriptive psychology. By exhibiting the regularities between physical causes and mental phenomena, genetic psychology teaches the psychologist how to obtain data for the generalizations of descriptive phenomenology. In this relation of dependency, descriptive psychology is primary, however, because it is still able to do its work without the help of genetic psychology. Genetic psychology, in contrast, requires the presentation of mental phenomena.

In the context of his investigations into the philosophy of space, Husserl also distinguishes between descriptive and genetic psychology, ascribing priority to the former. In a short fragment, called "Psychologische Analyse der Raumvorstellung," he formulates his view with respect to the representation of space:

> [Psychological analysis of the representation of space] may be understood in two ways:
> (i) as task of descriptive psychology; we can speak of a descriptive analysis. This requires exhibiting the elements of which the representation of space consists . . . . Here, we have to pay attention to two things:
> (a) the actual content of the representation, or, rather, the primary content (substance), which we find through description;
> (b) the judgements, which start from this content and which create the appearance

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89 Ibid., p. 5.

> that that which is a matter of idealization and other types of judgement is the result of a simple intuition, or a simple observation.
> (ii) The task of genetic psychology: genetic analysis. How and from what elements, by means of what psychological function of association, according to what laws did the representation of space come into existence. According to its nature, the descriptive analysis is the fundament of the genetic analysis.[90]

Accordingly, Husserl agrees with Brentano on three points: (i) that descriptive psychology "descriptively analyzes"[91] the elements which constitute the representation of space; (ii) that genetic psychology explains the process which leads to the formation of a given representation; and (iii) the priority of descriptive to genetic psychology.

Husserl's understanding of the task of descriptive psychology can be explained in greater detail by reference to his distinction between proper (*eigentlich*) and symbolic (*symbolisch*) representations.[92] Authentic representations for Husserl are primary contents of consciousness (*Urinhalte*), which present their objects so to speak 'in person,' with the whole content of the representation being intuitively given in it. Symbolic representations, in contrast, contain either intuitive or conceptual elements, which signify other elements which are not given intuitively, and yet also belong to the content of the representation. For example, a perception of the Golden Gate Bridge presents intuitively only one side of it. Yet, this intuitive representation indicates other possible perceptions as part of a network of representations of the bridge, which one

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90 ["[Die psychologische Analyse der Raumvorstellung] kann im doppelten Sinn verstanden werden:
1) als Aufgabe der deskriptiven Psychologie, wir können von einer deskriptiven Analyse sprechen. Es handelt sich hier um die Nachweisung der Elemente, aus denen die Raumvorstellung besteht . . . . Dabei muß wieder auf ein doppeltes geachtet werden:
a) auf den wirklichen Vorstellungsgehalt oder besser den primären Inhalt (Gehalt), den wir deskriptiv vorfinden,
b) auf die Beurteilungen, die sich an diesen knüpfen und den Schein erwecken, als wäre es Sache der einfachen Auffassung, des einfachen Bemerkens, was Sache einer Idealisierung und sonstigen Beurteilung ist.
2) Die Aufgabe der genetischen Psychologie: genetische Analyse. Wie, aus welchen Elementen, durch welche psychischen Funktionen der Verknüpfung, nach welchen Gesetzen ist die Raumvorstellung entstanden. Naturgemäß ist die deskriptive Analyse das Fundament für die genetische. Edmund Husserl, "Psychologische Analyse der Raumvorstellung," in Husserliana XXI, p. 267.

91 Ibid., p. 267.

92 Edmund Husserl, "Mehrfache Bedeutung des Terminus Raum," in Husserliana XXI, pp. 270-274, p. 271.

would have from different possible points of view. So, the perceived side signifies other sides with the result that the content of the representation of the Golden Gate Bridge is the entire bridge, the three-dimensional object. From this example, we see that many representations are not merely intuitions. Rather, they involve a synthetic connection between their different partial representations. In Husserl's early writings, this connection is the result of a psychological process of association.[93] The ultimate goal of descriptive analysis thus is not only to reveal the intuitive elements and the conceptual elements of a given representation, but also to identify and describe the psychological functions connecting them.[94] On the basis of the results of the descriptive analysis, genetic psychology then clarifies the genesis of the various representations of space from primary contents. In other words, the goal of genetic psychology is to explain how complex representations are constructed from less complex, and ultimately from intuitive, representations.

Husserl's definition of genetic psychology contains a striking difference vis-à-vis Brentano. Husserl's genetic psychologist is not searching for physical causes of mental phenomena, but rather for primary contents of consciousness (*Urinhalte*). The *Urinhalte* are immediately present to the mind and thus proper objects of pure psychology. By defining genetic psychology in this way, Husserl freed it from its physiological basis, from its basis in the natural sciences.[95]

However, Husserl's notion of genetic psychology also contains another element that goes beyond Brentano. In his notes to the *Raumbuch*, Husserl indicates that a philosophy of space has to begin with a clarification of the different meanings of the word *space*. In these sketches, he distinguishes between the space of intuition (which he also called "the space of everyday life"), the space of pure geometry, the space of

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93 Husserl later dismissed the idea of a psychological association, replacing it with that of a rational synthesis.

94 In part III of this thesis, we will see how Husserl modifies the distinction between authentic and symbolic representations in his later writings, and, in particular, in *Ding und Raum* (1907) and *Ideen I* (1913).

95 I will criticize Husserl's later phenomenological notion of a descriptive analysis in part III of this thesis.

applied geometry (which Husserl also calls "the space of natural science"), and the space of metaphysics.[96] Around 1900, Husserl deals with formal axiomatic systems and in this context refers to formal geometry.[97] We can therefore add formal space to his typology. With respect to these different concepts of space, Husserl demands:

> If it is true that the term 'space' is grounded in a multiplicity of concepts that are interconnected through genetic relations, rather than accidentally, then we have to show how the later, inauthentic, concepts are formed from original concepts and ultimately from primitive elements of intuition, through what dispositional or conscious, logical or extra-logical relations.[98]

Husserl's conviction that the concepts of space depend genetically on each other opens a new field of genetic inquiry. Psychology must not only explain the formation of a particular representation from its primary elements, but also describe the process in which it was derived from representations that correspond to more basic concepts.[99]

We can state Husserl's innovation in a different way. Brentano understands the genesis of mental representations in terms of physical causation. Husserl, in contrast, defines the notion of genesis as constitution, which has a synchronic and a diachronic

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[96] Edmund Husserl, "Mehrfache Bedeutung des Terminus Raum," in Husserliana XXI, p. 270-274, p. 270.

[97] Cf., Edmund Husserl, "Das Imaginäre in der Mathematik" in Husserlana XII, p. 430-447. He writes, for example: "If we conceive of the elements and their relations as formally defined by their laws and their points of relation as not further specified, then we have a formal geometry, i.e. the form of a geometry." ["Denken wir uns die Elemente und ihre Relationen formell durch ihre Gesetze definiert und die Beziehungspunkte im übrigen unbestimmt, so haben wir eine formale Geometrie, d.i. die Form einer Geometrie."], "Das Gebiet eines Axiomensystems/Axiomensystem-Operationssystem," in Husserliana XII, p. 470-488, p. 486, (translation my own). See also Edmund Husserl, *Logische Untersuchungen. Erster Band. Prolegomena zur reinen Logik*, §§ 69-70, Husserliana XVIII, ed. by E. Holenstein (The Hague: Martinus Nijhoff, 1975), pp. 247-252.

[98] ["Sollte dem Terminus Raum eine nicht durch bloßen Zufall, sondern genetische Beziehungen verknüpfte Mehrheit von Begriffen zugrunde liegen, dann ist zu zeigen, wie die späteren, uneigentlichen aus den ursprünglichen und zuletzt aus den primitiven Anschauungselementen entstanden sind, durch welche dispositionelle oder bewußte, außerlogische und logische Beziehungen."], Edmund Husserl, "Mehrfache Bedeutung des Terminus Raum," in Husserliana XXI, p. 263, (translation my own).

[99] Husserl's idea of genetic psychology not only deviated from Brentano's but also from Carl Stumpf's. The latter had formulated his view in *Über den psychologischen Ursprung der Raumvorstellung*. His main goal was to identify the simple representations, which made up the complex representation of space. Stumpf's investigation was well known to Husserl. Carl Stumpf, *Über den psychologischen Ursprung der Raumvorstellung* (Leipzig: Verlag von S. Hirzel, 1873).

aspect. The synchronic investigation exhibits the conscious parts that make up a particular representation. The diachronic investigation seeks to explain the constitution of higher-level from lower-level concepts, whereby the latter precede the former.

3.2 Husserl's Genetic Account of the Concepts of Space

From Husserl's notes to the *Raumbuch* and related writings, we can reconstruct the main features of his genetic analysis of the various representations of space.[100] Husserl does not distinguish explicitly between descriptive and genetic analyses. Yet, his main emphasis lies on the diachronic aspect of the genesis of the different concepts of space. I will follow his analysis and deal with intuitive, geometric, and formal space. Husserl provides us with the following characterization of intuitive space:

> By representation of space can first be meant the space of intuition, that is, the space of extra-scientific consciousness, the space which everyone, children or adults, scholars or laymen, experience in lived perception and fantasy.[101]

Husserl distinguishes everyday consciousness from other kinds, notably from scientific and ethic, by the interest or attitude a person takes towards the world. In contrast to the attitude of scientific consciousness, that of everyday experience takes the life-world unreflectively as an unanalysed whole. In order to establish a concept of intuitive space, one must extract intuitive spatial relations from the complex whole of experiences that constitute everyday consciousness.

In his notes to the *Raumbuch*, Husserl determined that the elements of intuitive

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[100] See Edmund Husserl, "Philosophische Versuche zum Raum" in *Studien zur Arithmetik und Geometrie. Texte aus dem Nachlass (1886-1901),* Husserliana XXI, ed. by Elisabeth Strohmeyer (The Hague/Boston/Lancaster: Martinus Nijhoff, 1983), pp. 261-310; in particular, pp. 275-293, and Husserl's investigations into formal axiomatic systems in Edmund Husserl, *Philosophie der Arithmetik. Mit Ergänzenden Texten* (1890-1901, Husserliana XII, ed. by Lothar Eley (The Hague/Boston/Lancaster: Martinus Nijhoff, 1970); in particular, pp. 340-503.

[101] ["Unter Raumvorstellung kann fürs erste gemeint sein der Raum der Anschauung, ich meine den Raum des außerwissenschaftlichen Bewußtseins, den Raum, wie ihn alle, ob Kinder oder Erwachsene, ob Gelehrte oder Laien, in lebendiger Wahrnehmung und Phantasie vorfinden."], Edmund Husserl, "Mehrfache Bedeutung des Terminus Raum," Husserliana XXI, pp. 270-274, p. 271.

space, spatial objects like houses, trees, and landscapes, etc., were not entirely intuitively accessible. As we have already seen in the example of the Golden Gate Bridge, they can be given to consciousness only as a continuous series of partial intuitions, with the objects themselves as the ideal limits. The elements of intuitive space are, therefore, unities of conceptual and intuitive elements - they are ideals.[102] As a result, the life-world is not an exclusively intuitive world. Rather, it is the world as given to us in our most concrete and most intuitive experiences. Consequently, we can define intuitive or perceptual space as an ideal itself containing ideal objects and relations as they are experienced in everyday consciousness.

According to Husserl, the space of intuition is distinguished from geometric space, which he describes as "a conceptual construct produced through logical treatment of the representation of space present in pre-scientific consciousness." Geometric space is therefore no longer "represented in intuition or intuitable, but rather only thinkable."[103] The concept of geometric space is expressed by an axiomatic system that results from two idealizing processes:

> The origin of a geometric concept of space already presupposes the origin of basic geometric concepts. Because only on the basis of these idealizations of original concepts of objects, as we find them in intuition, can we make such quasi-inductions, which we can also call idealizations. The ideal concept that we call 'geometric space,' however, is defined though ideal concepts and ideal propositions.[104]

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[102] Husserl writes: "We called landscapes, trees, houses, and so on spatial unities and showed that they were not contents of momentary intuitions, but rather ideal objects." ["Wir nannten Landschaften, Bäume, Häuser usw. räumliche Einheiten und wiesen nach, daß sie nicht Inhalte von Momentanschauungen, sondern ideelle Objekte sind."], Edmund Husserl, "Der anschauliche Raum," Husserliana XXI, pp. 275-284, p. 281.

[103] "From this space of intuition we distinguish the space of scientific thought, that is, geometric space, a conceptual construct produced through logical treatment of the representation of space present in pre-scientific consciousness. We can no longer speak of geometric space as represented in intuition or as intuitable, but rather only thinkable." ["Von diesem Raum der Anschauung ist zu scheiden der Raum des wissenschaftlichen Denkens, der geometrische Raum, ein begriffliches Gebilde logischer Bearbeitung der Raumvorstellung des außerwissenschaftlichen Bewußtseins, von dem nicht mehr gesagt werden kann, daß es anschaulich vorgestellt oder vorstellbar sei, sondern nur denkbar."], Edmund Husserl, "Mehrfache Bedeutung des Terminus Raum" Husserliana XXI, p. 271, (translation my own).

[104] ["Der Ursprung der geometrischen Vorstellung vom Raume setzt bereits den Ursprung der geometrischen Grundbegriffe voraus. Denn erst durch diese Idealisierungen der ursprünglichen Begriffe von Gebilden, wie wir sie in der Anschauung finden, sind jene Quasi-Induktionen, die wir auch

The first type of idealization constitutes the basic geometric concepts, whereas the second constitutes the axioms of a geometric theory such as Euclid's.

The process of idealization departs from everyday perception, which enables human beings to form pre-geometrical concepts such as points, lines, and planes, etc., through a process of partitioning:

> We can divide a physical body in fantasy multiple times, without destroying its unity. . . . Since each body is a physical part of the total space, it has to have a border to the remaining space. This border is its "surface." Similarly, we can divide planes physically. We can divide each plane into two planes, which do not have any parts in common, without destroying the unity of the original plane. The spatial object, which is common to both of them, and "by means of which" they border on each other, is called a "line." In the same way, the physical partition of lines leads to borders, which we call "points." Points cannot further be divided spatially.[105]

The concepts resulting from the partitions in experience are inexact. Husserl later calls them "morphological" in order to distinguish them from geometric concepts, which he understands as *ideal* boundaries.[106] The geometer must idealize correlating morphological concepts in order to form the basic concepts of Euclidean geometry. This can be accomplished only by ascribing continuity to the space of everyday experience.[107]

Idealisierungen nennen können, möglich. Der Idealbegriff, den wir auch geometrischen Raum nennen, ist aber definiert durch die Idealbegriffe und Idealsätze."], Ibid.

105 ["Ein Körper läßt sich, ohne seine Einheit zu verlieren, in Wirklichkeit oder Phantasie mannigfach teilen. . . Da jeder Körper physischer Teil des Gesamtraumes ist, so muß er gegenüber dem übrigen Raum eine Grenze besitzen. Dies ist seine 'Oberfläche.' In ähnlicher Weise wie Körper lassen auch Flächen physische Teilungen zu. Wir können jede Fläche, ohne ihre Einheit zu stören, in zwei Flächen zerstückt denken, welche also keinen Flächenteil gemein haben. Das Räumliche, das ihnen gemeinsam ist, und 'wodurch' sie aneinandergrenzen, heißt 'Linie.' Ebenso führt die physische Teilung von Linien auf Grenzen, die man 'Punkte' nennt. Punkte sind räumlich unteilbar. Edmund Husserl, "Der anschauliche Raum," Husserliana XXI, p. 278.

106 Cf., Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie. Erstes Buch*, Husserliana III, pp. 138-140.

107 Husserl writes: "Wir schreiben dem Raum, so wie den räumlichen Gebilden, Linien, Flächen, ferner Richtungsänderungen, Abständen, 'innere Unendlichkeit', d.h. 'Stetigkeit' zu." ["We ascribe 'inner infinity', i.e. 'continuity' to space, as well as to spatial forms, lines, planes, changes in direction, and distances."], Edmund Husserl, "Zur Entestehung der idealen räumlichen Vorstellung," Husserliana XXI, pp. 286-290, p. 286.

Since we cannot actually see the continuity of a given extension, say, of a given line segment, as it is relevant to geometry, Husserl concludes that it is an ideal concept constructed on the basis of certain perceptual processes familiar to everybody. He gives as his example the situation of two visible points bordering on each other. One can focus more sharply on these points by using instruments or by diminishing the distance between oneself and the points. By improving viewing conditions in this way, one can discover a visible distance between the points that can then be filled in with a further point. Since this process can be repeated over and over again, we can arrive at the general rule that it is always possible to place a further point between any two points under ideal conditions of observation. Although one cannot actually complete an infinite process of adding further points, one can stipulate ideal limits as the products of such an infinite division and add them to the line. As a result, the points can be understood as extensionless, and the line which they are a part of as continuous. According to Husserl, this example shows us how the process of idealization departs from the objects and relations of intuitive, i.e. experiential, space. It ends up constituting a completely new kind of object, however: geometric objects like points, lines, and planes that cannot be perceived, but rather only thought.[108]

Once the basic concepts of geometry have been constituted in this manner, the

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108 Husserl writes: "Freilich wird man es bezweifeln dürfen, ob von den geometrischen Objekten in Wahrheit Anschauung möglich ist, oder vielmehr ist es sicher, daß dies nicht der Fall ist. Die Anschauung und die empirisch-räumliche Auffassung enthält die Ausgangspunkte und die leitenden Motive für die geometrische Begriffsbildung, aber die den Begriffen zugehörigen abstrakten Gegenstände und deren Attribute sind nicht einfach durch 'Abstraktion' (in dem jetzt üblichen Sinn aufmerksamer Pointierung von Einzelzügen) aus den Anschauungen zu gewinnen, sie liegen in diesen nicht eingebettet wie die gesehene Gestalt in der gesehenen 'Fläche.' Das Dreieck als angeschautes Abstraktum ist keine geometrische Figur, es dient dem Geometer als bloßes Symbol, dessen charakteristischer Typus in seinem Geist dispositionelle Verknüpfung besitzt mit dem zugehörigen reinen Begriff und seinem idealen und bloß 'gedachten' Gegenstand." ["Admittedly, one can doubt whether one can have actual intuitions of geometric objects; or, rather, it is certain that this is not the case. Intuition and the empirico-spatial attitude contain points of departure and leading motives for the geometric formation of concepts. Yet, the abstract objects belonging to the concepts and the attributes of these concepts are not to be obtained simply through 'abstraction' (in the presently common sense of attentively emphasizing singular features) from intuitions. The concepts are not embedded in intuition like the seen shape in the seen 'plane.' The triangle as an intuited abstract object is not a geometric figure. The triangle serves the geometer as mere symbol whose characteristic type possesses dispositional connection in the geometer's mind with the correlating pure concept and its ideal, merely 'thought,' object."], "Intentionale Gegenstände," (1894) in Edmund Husserl, *Aufsätze und Rezensionen* (1890-1910), Husserliana XXI, Bernhard Lang (ed.), The Hague/Boston/London: Martinus Nijhoff, 1979, pp. 269-348, p. 327.

geometer can construct an axiom system, thus constituting the geometric concept of space. This construction is guided by experience and based on the observation of relations between empirical objects. These facts do not justify the axioms, however:

> Pure deduction is a matter of pure theory. Deduction never asks where the basic assumptions come from, it assumes them. One may disagree about the cognitive value of the basic concepts of geometry and the basic assumptions; the geometric sentences are beyond suspicion. Naturally so, since their validity has no meaning other than the correctness of the consequences drawn from the basic geometric assumptions.[109]

The geometer accepts the axioms, which for the early Husserl include axioms of existence and axioms defining properties of the objects postulated by them. The axioms themselves must be independent of each other and the resulting theory must be consistent.[110] We can conclude that the concept of geometric space is expressed by a

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109 ["Sache der reinen Theorie ist die reine Deduktion. Sie fragt überall nicht, woher die Grundsätze kommen, sie assumiert sie. Über den Erkenntniswert der geometrischen Grundbegriffe und Grundsätze mag Streit bestehen, die geometrischen Sätze sind über jeden Streit erhaben. Ganz natürlich, da ihre Gültigkeit keine andere Bedeutung hat als die Triftigkeit der Konsequenz aus den geometrischen Grundannahmen."], Edmund Husserl, "Assumption der Axiome in Geometrie, Mannigfaltigkeitslehre und reiner Mechanik," (1894) in Edmund Husserl, *Aufsätze und Rezensionen* (1890-1910), Husserliana XXI, pp. 430-431, p. 431.

110 Husserl expresses this in the following passage: "[Der] Geometer . . . nimmt die Grundsätze einfach hin und zieht seine Konsequenz. Höchstens, daß er die Zahl der Grundbegriffe und Grundsätze beschränkt und eine Minimalzahl zu fixieren sucht, welche, voneinander deduktiv unabhängig das ganze System der Deduktionen zu tragen vermögen."["[The] geometer simply accepts the axioms and draws his consequences. At most, he restricts the number of basic concepts and axioms and attempts to fix a minimal number of them that are independent of each other and able to carry the entire system of deductions."], Ibid., p. 431. It is interesting to contrast Husserl's view here with that of Moritz Pasch in his *Vorlesungen über neuere Geometrie* from 1882. Pasch believed that the geometric axioms were rational extensions of statements about observable facts. In his construction of geometry he therefore first formulated certain fundamental observational statements about visible objects, which he called "core propositions" (*Kernsätze*). Nuclear sentences are about physical bodies. For example, the term 'point' refers to a physical body whose subdivision into parts is incompatible with the limits set by perception. On the basis of these core propositions, Pasch then constructs an axiomatic system whose justifications, logical derivations, are entirely independent of the original meanings of these terms. By doing so, he changes the meanings of the original terms. They become relational concepts that are exhaustively defined by the axioms. No appeal to perception is required, in order to justify the theorems. In other words, the extension of the core propositions to the axioms of a geometric theory renders the latter a formal theory similar to Hilbert's. Such a formal theory can then be applied to physical reality by interpreting its terms with the original objects. Husserl also thought that the construction of a geometric axiomatic system begins with observations of perceptual objects. Yet, in contrast to Pasch, he did not believe that the idealization, and thus the subject matter of geometry, is a result of the axiomatization. Rather, according to Husserl, the subject matter of geometry (in the sense of material geometry) has to be constituted, at least partly, before

deductive discipline which is constituted through two types of idealization.

Husserl believes that in order to form the concept of formal space, the geometer has to formalize the axiomatic system of geometry. Husserl describes the process of formalization in the following passage:

> According to its highest and most comprehensive ideal, mathematics is the science of theoretical systems as such, and in abstraction from that which is being theorized in the given theories of the various sciences. If in any given theory, in any given deductive system, we abstract away from the matter, the specific objects that the theory seeks to bring under control, and substitute the materially determined object representations with mere formulas, that is, with representations of objects as such, which are governed by a theory of this form, then we have accomplished a generalization, which considers the given theories as a mere special case of a theory class, or, rather, of a theory-form (*Theorienform*).[111]

To formalize a deductive theory, the geometer has to substitute representations of concrete objects with representations of objects as such. But what does Husserl mean by representations of objects as such? He often calls these representations "object-forms" which he defines as follows:

> They are precisely determined neither directly as individual or specific particulars, nor indirectly through their material kinds or species, but rather exclusively through the form of the relations ascribed to them.[112]

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one can construct an axiomatic theory which captures its properties. That this was Husserl's view will also become clearer in the next paragraphs. For Pasch's view see his *Vorlesungen über neuere Geometrie* (Leipzig: B.G. Teubner, 1882). A short summary of Pasch's view can be found in Ernest Nagel, "The Formation of Modern Conceptions of Formal Logic in the Development of Geometry," pp. 193-201.

111 ["Mathematik im höchsten und umfassendsten Sinn ist die Wissenschaft von den theoretischen Systemen überhaupt und in Abstraktion von dem, was in den gegebenen Theorien der verschiedenen Wissenschaften theoretisiert wird; abstrahieren wir bei irgendeiner gegebenen Theorie, bei irgendeinem gegebenen deduktiven System von seiner Materie, von den besonderen Gattungen von Objekten, auf deren theoretische Beherrschung sie es abgesehen hat, und substituieren wir den materiell bestimmten Objektvorstellungen die bloßen Formeln, also die Vorstellung von Objekten überhaupt, die durch solch eine Theorie, durch eine Theorie dieser Form beherrscht wird, so haben wir eine Verallgemeinerung vollzogen, welche die gegebenen Theorien als einen bloßen Einzelfall einer Theorie-Klasse auffaßt oder vielmehr einer Theorienform."], Edmund Husserl, "Das Imaginäre in der Mathematik," Husserliana XII, ed. by Lothar Eley (Den Haag: Martinus Nijhoff, 1970), pp. 430-440, p. 430.

112 ["Sie sind eben weder direkt als individuelle oder spezifische Einzelheiten, noch indirekt durch ihre materiellen Arten oder Gattungen bestimmt, sondern ausschließlich durch die Form ihnen zugeschriebener Verknüpfungen."], Edmund Husserl, *Logische Untersuchungen. Erster Band. Prolegomena zur reinen Logik*, Husserliana XVIII, p. 250, (translation my own).

Accordingly, the object-forms are defined exclusively through the relations that hold between them. According to Husserl, these relations, and thus the properties of these object-forms, are defined implicitly by the axioms of the formal axiomatic system. He writes:

> The domain of objects is defined through axioms in the sense that it is delimited as any sphere of objects as such, whether real or ideal, for which basic assumptions [axioms] of such and such forms are valid.[113]

Further, the relations themselves are also defined exclusively through the axioms, and not through their geometric meanings. We can conclude that formal space is a relational structure whose properties are defined in such a formal axiomatic system. Husserl also calls such a structure a "formal manifold."[114] A relational structure of this kind can be common to an unlimited number of material manifolds, which are given in material axiomatic systems.

In order to make Husserl's notion of formalization more concrete, I will consider an example. Assume that we have at our disposal a given material axiomatic theory, say, Euclidean geometry. According to Husserl, the axioms and theorems of this material axiomatic system contain terms referring to ideal geometric concepts and relations. One of the axioms of Euclidean geometry states, for example: "For any two points a, c, there exists at least one point b on the line ac such that b lies between a and c." We can express this in the language of first-order symbolic logic as $\forall a \forall c$ (a $\neq$ c

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113 ["Das Objektgebiet aber ist durch Axiome in dem Sinn definiert, daß es umgrenzt ist als irgendeine Sphäre von Objekten überhaupt, gleichgültig ob realen oder idealen, für welche Grundsätze [Axiome] solcher und solcher Formen gelten."], Edmund Husserl, "Das Imaginäre in der Mathematik," p. 431, (translation my own). The idea that the axioms of formal geometry are implicit definitions is also well expressed in the following passage: "In der Geometrie werden die Elemente (Punkte, Geraden) nicht bestimmt; sie lassen sich nicht logisch abheben; sie sind definiert als Beziehungspunkte der in den Axiomen charakterisierten Relationen." ["The elements of geometry (points, straight lines) are indeterminate; they can not be logically characterized; they are defined as points connecting the relations defined by the axioms."], Edmund Husserl, "Das Gebiet eines Axiomensystems," Husserliana XII, ed. by Lothar Eley (Den Haag: Martinus Nijhoff, 1970), pp. 470-488, p. 486.

114 Cf., Edmund Husserl, *Logische Untersuchungen. Erster Band. Prolegomena zur reinen Logik*, p. 250.

$\rightarrow \exists b\, (Lbac \wedge Babc)$). The variables a, b, c stand for geometric points, Bxyz is ternary relation 'y is between x and z' and Lxyz is the ternary relation 'x lies on the line yz'. In order to formalize the theory in Husserl's sense, we have to abstract away from the meanings of the non-logical terms. In the above axiom, we can do this by simply abstracting away from their intended meanings. We can indicate this abstraction by replacing the symbols that refer to specifically geometric objects, namely a, b, c, Lxyz, and Bxyz, by other symbols that have no particular intended referent, say: $\mathcal{a}$, $\mathcal{b}$, $\mathcal{c}$, $\mathcal{L}$xyz, $\mathcal{B}$xyz. We get:

$\forall \mathcal{a} \forall \mathcal{c}\, (\mathcal{a} \neq \mathcal{c} \rightarrow \exists \mathcal{b}\, (\mathcal{L}\mathcal{b}\mathcal{a}\mathcal{c} \wedge \mathcal{B}\mathcal{a}\mathcal{b}\mathcal{c}))$. If we do this for the entire theory, we construct a formal axiomatic system in Husserl's sense. I want to emphasize that formalization in Husserl's sense does not require us to formulate the theory in symbolic language. In my example, I used the symbolization only as a visual aid. The difference between the material and formal axiomatic theory in Husserl's sense is that, in the former, we take the terms as referring to specifically geometric objects, and, in the latter, we think of them as devoid of any meaning.

Like the logical positivists, Husserl thought that Hilbert's development of Euclidean geometry in his *Grundlagen der Geometrie* was a paradigmatic representation of a formal theory. This is not an accident, since Husserl was familiar with Hilbert's work and understood the intentions behind it. In an excerpt from the correspondence between Hilbert and Frege, Husserl writes, for example:

> I remark to this point. Frege does not understand the meaning of Hilbert's "axiomatic" foundation of geometry, namely, that it is a purely formal system of conventions that is identical in its theoretical form with Euclid's.[115]

Husserl also quotes Hilbert's famous formulation (see, for example, his correspondence with Frege) that a geometric theory is a "*Fachwerk* (*System*) of concepts including

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[115] ["Ich merke dazu an. Frege versteht nicht den Sinn der Hilbertschen 'axiomatischen' Begründung der Geometrie, nämlich daß es sich um ein rein formales System von Konventionen handelt, das sich der Theorieform nach mit dem Euklidischen deckt."], Edmund Husserl, "Auszüge Husserl's aus einem Briefwechsel zwischen Hilbert und Frege," Husserliana XII, ed. by Lothar Eley (Den Haag: Martinus Nijhoff, 1970), pp. 447-451, p. 448.

necessary connections" where the basic elements have no specific referents.[116]

Husserl believed that the formalization of material axiomatic theories had an important epistemic value. The fact that a formal theory in his sense exhibits the form of a material discipline allows the geometer to treat the former as a meta-theory of the latter. By considering the axiomatic form of a theory independently of any contents, the geometer can improve its deductive structure and create a system of axioms which fulfills a number of conditions. All the facts of material geometry must follow from the axioms, and the axioms themselves have to be consistent and independent of each other.[117] But the epistemic value of formalization goes further. Once mathematical inquiry has reached the formal level, it is free from its genetic roots. At this level, the mathematician is not only delivered from the responsibility of paying attention to any content, but also from the constraints of the process in which the formal theory was generated. This freedom from the material domain finds its expression in the fact that it is legitimate to change the formal theory by adding or eliminating axioms. In Husserl's words:

> Once one ascends to the pure system of operations, leaving behind the original real domain of objects, be it line-segments or numbers, and one considers in utmost generality a domain which is determined solely by these forms of operation, one is able to modify the idea of such a system in various ways, either through an extended or restricted system of operations, or of axioms.[118]

Husserl cites the development of Riemann's conception of a continuous manifold as a case where the formalization and the modification of a material axiomatic system,

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[116] Ibid., p. 450.

[117] "Mathematik ist also ihrer höchsten Idee nach Theorienlehre, die allgemeinste Wissenschaft von den deduktiven Systemen überhaupt." ["According to its highest ideal, mathematics is a theory of theories, the most general science of deductive systems as such."], Edmund Husserl, "Das Imaginäre in der Mathematik," Husserliana XII, p. 431, (translation my own).

[118] "Erhebt man sich zum reinen Operationssystem, verläßt man das urprüngliche reale Objekt-Gebiet, ob es Strecken oder Anzahlen sind, und faßt in allgemeinster Allgemeinheit ein Gebiet überhaupt ins Auge, das durch solche Operationsformen definiert ist, dann kann man die Idee eines solchen Gebietes verschiedentlich modifizieren, bald im Sinn eines weiteren, bald in dem eines engeren Operationssystems bzw. Axiomensystems." "Das Imaginäre in der Mathematik" Husserliana XII, p. 437. See also Husserl's concept of "Erweiterung eines Axiomensystems," Husserliana XII, p. 439.

namely Euclidean geometry, led to the construction of more general formal manifolds.[119] According to Husserl, the fact that Euclidean space is just a special case of a more general manifold teaches us much about the nature of the former.

3. 3 The Genetic Point of View and the Concept of Physical Space

I now want to return to the conceptual problem I earlier discovered in the standard view, namely that of how applied or physical geometry can express a concept of physical space. Husserl's genetic approach to the concept of space yields a solution to this problem through its account of what constitutes a meaningful concept of space. Husserl agrees with Kant that in order to express a geometric concept of space, the propositions of material geometry must be true of empirical reality. However, Husserl also believes that formal geometry expresses a meaningful concept of space:

> If we use the term 'space' for the familiar type of order of the world of phenomena, then to talk of 'spaces' for which (for example) the axiom of parallels does not hold, is of course a contradiction. It is just as much of a contradiction to talk of different geometries, if 'geometry' designates the science of the space of the world of phenomena. But if we mean by 'space' the categorical form of world-space, and, correlatively, by 'geometry' the categorical theoretic form of geometry in the ordinary sense, then space falls under a genus (to be circumscribed by laws) of pure, categorically determined manifolds, in regard to which we will speak of 'space' in a yet more extended sense. In the same way, geometric theory falls under a corresponding genus of theoretically interrelated theory-forms determined in purely categorical fashion, which in a correspondingly extended sense can be called 'geometries' of these 'spatial' manifolds. At any rate, the theory of *n*-dimensional spaces forms a theoretically closed piece of the theory of theory in the sense defined above. The theory of a Euclidean manifold of three dimensions is an ultimate ideal particular in this series of *a priori*, purely categorical theoretic forms (formal deductive systems) which are interconnected according to a law.[120]

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[119] Husserl writes: "Die Euklidische Geometrie ist eine konkrete Theorie, welche formalisiert die Theorieform ergibt, die wir als dreifache Euklidische Mannigfaltigkeitslehre bezeichnen, und diese wieder ist nur ein Einzelfall aus der systematisch zusammenhängenden Klasse der Mannigfaltigkeit von variablem Krümmungsmaß." ["Euclidean geometry is a concrete theory, which, if formalized, represents the theory-form that we call theory of the three-dimensional Euclidean manifold, and this again is only a special case of the systematically connected class of manifolds with variable measure of curvature."], "Das Imaginäre in der Mathematik," Husserliana XII, p. 431, (translation my own).

[120] ["Nennen wir den Raum die bekannte Ordnungsform der Erscheinugswelt, so ist natürlich die Rede von 'Räumen', für welche z.B. das Parallelenaxiom nicht gilt, ein Widersinn. Ebenso, die Rede von verschiedenen Geometrien, wofern Geometrie eben die Wissenschaft vom Raume der Erscheinungswelt

Accordingly, a formal geometry expresses a categorical form of space, i.e., a formal concept of space, because it stands in a specific relation to formal Euclidean geometry. Such a formal concept of space is either a generalization of Euclidean geometry, as in projective geometry and topology, or a non-Euclidean specialization of such a generalization, as in Lobatchevskian geometry.[121] Consequently, certain formal axiomatic systems express formal concepts of space, because they occupy a certain position in the hierarchy of geometries of ever greater generality. In this way, these concepts of space are depend upon the logical form of a material geometry, and thereby upon material geometry itself.

We can now show how Husserl's genetic approach removes the standard view's problem with the physical concept of space. For Husserl, the propositions of material geometry have genuine spatial content. Likewise, the formal axiomatic theories, which express formal concepts of space, retain some connection to the logical form of this

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genannt wird. Verstehen wir aber unter Raum die kategoriale Form des Weltraums und korrelativ unter Geometrie die kategoriale Theorienform der Geometrie im gemeinen Sinn, dann ordnet sich der Raum unter eine gesetzlich zu umgrenzende Gattung von rein kategorial bestimmten Mannigfaltigkeiten, mit Beziehung auf welche man dann naturgemäß vom Raum in einem noch umfassenderen Sinne sprechen wird. Ebenso ordnet sich die geometrische Theorie einer entsprechenden Gattung von theoretisch zusammenhängenden und rein kategorial bestimmten Theorienformen ein, die man dann in entsprechend erweitertem Sinne 'Geometrien' dieser 'räumlichen' Mannigfaltigkeiten nennen mag. Jedenfalls realisiert die Lehre von den '*n*-dimensionalen Räumen' ein theoretisch geschlossenes Stück der Theorienlehre in dem oben definierten Sinn. Die Theorie der Euklidischen Mannigfaltigkeit von drei Dimensionen ist eine letzte ideale Einzelheit in dieser gesetzlich zusammenhängenden Reihe apriorischer und rein kategorialer Theorienformen (formaler deduktiver Systeme)."] Edmund Husserl, *Logische Untersuchungen. Erster Band. Prolegomena zur reinen Logik*, p. 251, (translation my own).

[121] At around 1900 Husserl was keenly interested in the relationship between formal and material mathematical concepts. In a lecture given at the Göttingen Mathematical Society in the winter 1900/01, he was concerned with the concept of a definite manifold. He believed that the definiteness of a formal theory would guarantee that the results derived within it would also be true for the material domain. Husserl's notion of definiteness was identical to the idea of decidability (*Entscheidbarkeit*) in the sense that for any proposition either it is provable or its negation is provable. Gödel's incompleteness results restrict the applicability of Husserl's notion of definiteness to relatively simple axiomatic systems and exclude plane geometry. I discussed Husserl's notion of a definite manifold in this context in my paper "Edmund Husserl on the Applicability of Formal Geometry" (forthcoming). Husserl never published the text of this lecture and it could not be found in the Husserl archive. However, we have a sketch of this lecture that is now published under the title "Das Imaginäre in der Mathematik" and a number of Husserl's studies on a definite manifold. Edmund Husserl, "Das Imaginäre in der Mathematik," Husserliana XII, pp. 431-451 and Edmund Husserl, "Drei Studien zur Definitheit und Erweiterung eines Axiomensystems," Husserliana XII, pp. 452-461. That Husserl ascribed a great significance to the ideas developed in this lecture is indicated by the fact that in both *Ideen I* and *Formale und transzendentale Logik* he mentions this unpublished text. See, *Ideen I*, Husserliana III, p. 137, Footnote 1, and *Formale und transzendentale Logik*, Husserliana XVII, p. 85.

original subject matter. In other words, certain logical formal systems are particularly closely related to the logical form of Euclidean geometry and are therefore considered as spatial forms. If such a formal axiomatic system is applied to reality, i.e., interpreted, it expresses a physical concept of space in virtue of this form. Although the formal geometry may be devoid of material content and thus unable to confer materiality to the physical concept of space, it nevertheless will provide a form in which we can meaningfully speak of *spatial* properties characterizing the relation between non-contiguous physical objects.

For clarity, I want to rehearse the problem with the standard view and Husserl's solution to it in a slightly different way. The standard view of the two geometries has no grounds for distinguishing formal geometries from other formal axiomatic systems. Thus, applied geometry must derive its entire spatial content from empirical observation. Yet, as Reichenbach has shown, empirical observation is restricted to contiguous physical bodies and thus cannot account for the spatial content of physical geometry in a manner that would be sufficient for grounding a physical theory. The standard view requires that spatial content flow exclusively from experience into physical geometry, that is bottom-up. But this is problematic, as we have seen. Husserl, in contrast, allows for a certain *spatial* form to flow from formal to physical geometry, that is, top-down. In this way, he removes the difficulties of the standard view and guarantees the possibility of a physical concept of space that will be adequate for founding a physical theory. We can conclude that a physical concept of space presupposes a formal concept of space, rather than a purely formal axiomatic system.

Husserl's genetic account presupposes a coherent concept of a material geometry. I will criticize Husserl's idea of a material geometry in the next part of this dissertation. Nevertheless, my previous considerations allow me to state three very general conditions that a material geometry would have to fulfill in order to function as the conceptual link between perceptual space and formal geometry. First, a material geometry would have to be distinct from a physical or applied geometry. Second, the propositions of material geometry would have to express the global spatial structure of perceptual space. And third, the propositions of a material geometry would have to be axiomatizable, thus exhibiting a particular formal logical structure. We will see that the concept of a material geometry developed in the second part of this thesis will fulfill all three conditions.

Let me summarize my argument in this part. In the first three sections, I showed that Reichenbach does not succeed in refuting the notion of a material geometry. In order to do so, I first argued that Kant had two reasons for assuming the existence of a material geometry: his limited means of logic and his anti-formalism with respect to mathematical concepts. The first reason has been dismissed historically through the construction of formal axiomatic systems. Reichenbach, therefore, attempted to dismiss Kant's second reason for the existence of a material geometry by arguing that there is no *a priori* form of spatial intuition. But, his argument fails, because his method does not allow him to investigate the structure of perceptual space. Analyzing this structure would have been necessary, however, to show that it has no *a priori* form. I then pointed out a problem for the concept of physical space arising from within the standard view. Subsequently, I turned to Husserl's genetic point of view. I first showed that Husserl distinguished between a synchronic and a diachronic investigation into the constitution of the concept of space. A diachronic investigation shows how our various concepts of space at different levels of abstraction are related to each other. I then outlined Husserl's early account of the genesis and the nature of the these concepts and argued that it allows us to solve the problem with the concept of physical space by introducing a material geometry, that is, a specifically geometric concept of space.

Part II: The Subject Matter of Material Geometry

4. Phenomenological Attempts: Material Geometry as the Science of an Idealized Perceptual Space

In the first part of this thesis, I defined a material geometry as a science that is distinct from applied geometry (i.e., from an interpreted formal axiomatic system) and whose propositions have genuine spatial content. In this part, I will argue that Euclid's method as it is exhibited in his famous *Elements* is such a material geometry. In order to show this, I will present an interpretation of Euclid's practice and investigate its relation to spatial perception. We will see that Euclid's method yields a type of logical knowledge about certain idealized visual objects, namely geometric diagrams, and, through them, about the structure of visual space. The propositions of Euclidean geometry have genuine spatial content, because they are established not through empirical generalization, but through a special type of logical inferences, departing from certain qualitative features of visual objects.

Before I proceed with this argument, I want to discard with an alternative view on the subject matter of Euclidean geometry. Many philosophers and mathematicians define the subject matter of material geometry as a point-manifold, which they understand as the product of an idealization of intuitive or perceptual space. According to this suggestion, material geometry is simply about an idealized intuitive or perceptual space. Edmund Husserl and the early Rudolf Carnap presented such accounts from a phenomenological point of view.[122] They believe that this way of proceeding allows them to maintain a connection to spatial experience, thereby securing the materiality of this type of geometry.

In sections 4.1 and 4.2 of this part, I will consider Husserl's and Carnap's views and show that they are insufficient to prevent the collapse of material into formal

122 Husserl's theory of the subject matter of Euclidean geometry is contained in various of his writings. These are specified in the next section. For Rudolf Carnap's suggestion about the subject matter of Euclidean geometry, see Rudolf Carnap, *Der Raum: Ein Beitrag zur Wissenschaftslehre* (Berlin: Verlag Reuther, 1922).

geometry. My critique seems at first to support strongly the positivists' dismissal of material geometry. In contrast to the positivists, however, I will also show in section 4.3 that the problems of the phenomenological accounts result from internal inconsistencies: it is simply incoherent to claim that the properties of geometric space are captured in an axiomatic theory and simultaneously ascribe a material character to it that differs from that of applied geometry. Thus, we cannot simply dismiss the possibility of a material geometry, but rather must search for a coherent account of its subject matter.

4.1 Husserl

We have seen in the first chapter that Husserl had already developed the basis of his account of Euclidean geometry by the early 1890's. He believed that its subject matter is constituted in a process that involves both idealizations of perceptions and logical ordering. This subject matter thus has a double character. On the one hand it is a three-dimensional Euclidean point-manifold whose properties are captured in a deductive axiomatic theory. Husserl expressed this by saying that the subject matter of Euclidean geometry is the correlate of an axiomatic system. On the other hand, the subject matter of Euclidean geometry is an idealization of intuitive, that is, pre-geometric space.[123] Thus, Husserl claims that the subject matter of Euclidean geometry is distinct from that of formal and applied geometry because it is the correlate of an axiomatic system and at the same time has an intimate connection to spatial experience (Oskar Becker calls this the "*anschaulich-kategorialer Doppelcharakter der Geometrie*"[124]). However, as it stands this claim names only two independent sources of the Euclidean manifold. Whether Husserl's suggestion is acceptable depends on whether it is able to explain the non-formal content of this manifold coherently and this depends, at least partly, on the

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[123] Evidence for this view can be found in many of Husserl's writings. Already in his outline to his *Raumbuch*, he planed a section on "geometrical space as Euclidean Manifold of three dimensions." Edmund Husserl, *Studien zur Arithmetik und Geometrie. Texte aus dem Nachlaß (1886-1901)*, Husserliana, XXI, p. 402. The same idea is expressed, for example, in Edmund Husserl, *Philosophie der Arithmetik. Mit Ergänzenden Texten (1809-1901)*, Husserliana, XII, p. 431 and pp. 78-92; *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie*, Husserliana, III, pp. 133-137.

[124] Oskar Becker, *Beiträge zur phänomenologischen Begründung der Geometrie und ihrer physikalischen Anwendungen,* in Edmund Husserl, ed., *Jahrbuch für Philosophie und phänomenologische Forschung* 6 (1923): pp. 385-560, in particular pp. 416-419.

underlying concept of idealization.

We can state Husserl's problem more precisely if we clarify the logical status of the material manifold of Euclidean geometry vis-à-vis different types of formal manifolds.[125] As we have seen, for Husserl a formal manifold is defined in a Hilbert-style axiomatic system, that is, a system of axioms whose non-logical terms function as mere *Platzhalter* for possible objects. Thus, a manifold is a systematically connected unity of elements whose properties are exhaustively defined by a formal axiomatic system.[126] As we have also seen, Husserl further believes that different formal axiomatic systems that define the same concept (such as the concept of space) stand in a hierarchical relation to each other. A manifold of a higher degree of abstraction relates to a manifold of the next lesser degree of abstraction like a species to a sub-species. There is no upper limit to the degree of abstraction of a given manifold, there is no most abstract formal manifold. Yet, there is a limit to the lowest degree of abstraction. Husserl believes that more complete axiomatic systems define their respective manifolds more perfectly, reaching a limit at which the addition of further axioms no longer leads to further specifications of the relations between the elements. This idea of Husserl's captures the notion of the categoricity of an axiomatic system. An axiomatic system is

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125 Husserl first mentions the concept of a manifold in the *Prolegomena* to his *Logische Untersuchungen*, elaborating it shortly afterwards (1900/01) in a number of studies now published as "Drei Studien zur Definitheit und Erweiterung eines Axiomensystems," "Das Imaginäre in der Mathematik", and "Das Gebiet eines Axiomensystems/Axiomensystem-Operationssystem." The notion of a manifold remained important in Husserl's later philosophy of science, however, as its many occurrences in his works show. Other descriptions of the concept of a manifold can be found in *Ideas* I, Husserliana, III, pp. 261-311; *Formale und transzendentale Logik*, Husserliana, XVII, pp. 93-102; *Die Krisis der Europäischen Wissenschaften und die tranzendentale Phänomenologie. Eine Einleitung in die phänomenologische Philosophie*, Husserliana VI, ed. by W. Biemel (The Hague: Martinus Nijhoff, 1954), pp. 42-45. Husserl uses the concept of a manifold in a sense that is similar, but not identical, to Riemann's concept of a continuous manifold. For Riemann, a manifold is a general concept of a magnitude (*allgemeiner Größenbegriff*) which is determined either by points or elements. If the general concept is determined through points, it is a continuous manifold, and if it is determined through elements, it is a discrete manifold. Yet, in contrast to Husserl, Riemann treated the continuous manifolds analytically. Bernard Riemann, "Über die Hypothesen, welche der Geometrie zugrunde liegen," *Göttinger Abhandlungen* 13 (1867): pp. 133-152, reprinted in *Riemanns gesammelte mathematische Werke*, ed., Raghavaran Narasimhan, (Berlin/New York: Springer Verlag, 1990), pp. 272-287, p. 273f. It should also be pointed out that Husserl sometimes uses the concept of a manifold in order to designate the set of sentences that form an axiomatic theory, rather than the system of objects picked out by it.

126 Oskar Becker attempted to clarify Husserl's notion of a manifold. Oskar Becker, *Beiträge zur phänomenologischen Begründung der Geometrie und ihrer Anwendungen*, pp, 402-414.

categorical, if all its interpretations are isomorphic. In other words, the addition of axioms does not demand further structural additions of an interpretation. Such a manifold is a formal individual that can be specified further only by reference to material properties, yielding a concrete individual.[127] If we apply this schema to geometry, we can formulate the challenge Husserl is faced with in the following way: he has to give a criterion that will distinguish the formal individual, i.e., the formally defined three-dimensional Euclidean manifold, from the concrete individual as the correlate of Euclidean geometry. In short, Husserl has to account for the difference between geometric and formal space under the assumption that both are constituted in axiomatic systems.

Husserl changed his views on the difference between the space of material geometry and formal space over time. In his early writings on geometry, he offered two very different suggestions; and later, in his mature philosophy of science beginning with the *Logische Untersuchungen* and *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie*, he put these earlier ideas into a new context, thus generating a third answer.[128] In my reconstruction, I will, therefore, follow this development by first considering his views in the early writings and then explicating this matter in his mature theory of science.

Husserl's first suggestion for a criterion that can distinguish the abstract individual of formal geometry from the *concretum* of Euclidean geometry is a consequence of his early account of what constitutes a philosophically adequate construction of an axiomatic geometric theory. I will first describe his understanding of correct axiomatization and then draw the respective consequences from it. In a short manuscript from 1892 published under the title "Funktionen-Mannigfaltigkeit im

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[127] That Husserl understood the different concepts of formal and material geometry as being related was shown in the first chapter.

[128] Husserl's different responses to this problem reflect his philosophical development. Initially, he was concerned only with mathematics, logic, and geometry. This changed with *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie*, in which he focused on the philosophy of science in general. This shift found its most distinct expression in the fact that from 1900 on, Husserl no longer wrote about geometry, but rather used it as his most prominent example in explaining his mature theory of science.

weitesten Sinn gegenüber dem engeren Mannigfaltigkeitsbegriff," Husserl criticizes analytic approaches to the axiomatization of geometry such as Lie's.[129] His main contention is that because these approaches define a geometric manifold by means of an external representational system, namely the system of real numbers, there is no guarantee that they capture all and only genuine geometric relations. Thus, there is no way of telling whether the formal systems constructed by the analytic method are actual geometric systems or mere formal inventions.[130] For Husserl, the only way to ensure that an axiomatization of geometry is correct is by first giving a criterion for identifying genuine geometric relations which is independent of the axiomatization itself. In order to find such a criterion, he turns to the subject matter of material geometry and writes: "My own approach is to consider manifolds of contents, which possess a continuous connection in virtue of their inner relations."[131] He believes that a genuine geometric manifold is a continuous point manifold whose continuity follows from internal relations between its elements. For Husserl, internal, as opposed to external, relations are grounded in the properties of the particular elements of the structure. Accordingly, a correct axiomatization contains only the formalized versions of internal relations of the manifold of material geometry.

To illustrate Husserl's view on genuine geometric relations, I want to consider

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129 Edmund Husserl, "Funktionen-Mannigfaltigkeit im weitesten Sinn gegenüber dem engeren Mannigfaltigkeitsbegriff," *Studien zur Arithemetik und Geometrie (1886-1901)*, Husserliana, XXI, pp. 408-411. In a later manuscript from around 1900, Husserl writes with respect to Lie's axiomatic system, for example: "Eine Unmenge von Dingen wird in den (anderen) Axiomen vorausgesetzt. . . . Alles ungeometrische Voraussetzungen, die man nicht übersieht, Voraussetzungen, die nichts mit der Geometrie zu tun haben, sondern Voraussetzungen, die an der Methode haften. Lie will eben seine Methode anwenden." ["The (other) axioms presuppose a host of things . . . All of them are non-geometric presuppositions, which one can not comprehend, presuppositions that have nothing to do with geometry, but depend on the method. Lie simply wants to apply his method."], Edmund Husserl, "Verschiedene Richtungen der Geometrie," Ibid., p. 412, (translation my own). As Ulrich Majer pointed out to me, this later criticism may have been influenced by Hilbert's views on geometry.

130 I have shown in the first part that Husserl drew a sharp distinction between formal systems that were mere inventions, i.e., calculi, and formal geometry.

131 ["Meine Betrachtungsweise geht dahin, Mannigfaltigkeiten von Inhalten zu betrachten, die durch ihre inneren Relationen stetigen Zusammenhang besitzen."], Edmund Husserl, "Funktionen-Mannigfaltigkeit und Mannigfaltigkeit im weitesten Sinn gegenüber dem engeren Mannigfaltigkeitsbegriff," Husserliana, XXI, p. 409, (translation my own).

his criticism of Riemann's distance-function.[132] The central feature of Riemann's geometry is that in it different spaces -- Euclidean as well as non-Euclidean spaces -- are all Euclidean from a differential point of view. Riemann expressed this by defining the distance between any two infinitely close points as a quadratic form: thus, for points $P_1(x_1, y_1, z_1)$ and P_2 (x_2, y_2, z_2) the infinitesimal distance is $ds^2= dx^2+dy^2+dz^2$ (s stands for the distance) According to Husserl, this distance-function illuminates the metrical nature of physical space by showing that it is a special case of a more general manifold, but he now enters a question:

> The theory sheds light on the analytic connection between the metric relations as they occur in our space by interpreting this connection as a special case of something more general. But does this suffice in order to speak of other "spatial forms?"[133]

The scepticism expressed in the last sentence of this quote is based on the fact that Riemann extends the validity of his distance-function to non-Euclidean spaces in a merely stipulative manner; it is not at all clear that this captures an *essential* feature of these geometric manifolds. In order to make clear why not, Husserl presents an analogy with the case of a continuous tone-manifold.[134] The different tones in a tone-manifold can be distinguished by their frequencies: higher tones have higher frequencies and lower tones have lower frequencies. Thus, there is a distance between two tones, i.e. the difference between their frequencies. However, one can also introduce a different

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[132] This criticism is given in Edmund Husserl, "Funktionen-Mannigfaltigkeit und Mannigfaltigkeit im weitesten Sinn gegenüber dem engeren Mannigfaltigkeitsbegriff," *Studien zur Arithmetik und Geometrie (1886-1901)*, Husserliana, XXI, pp. 408-411. Riemann formulated his view on the definition of a geometric manifold in his famous Inauguraldissertation "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen," *Abhandlungen der Königlichen Gesellschaft der Wissenschaft zu Göttingen* 13 (1867): pp. 271-287.

[133] ["Auf den analytischen Zusammenhang der Maßbestimmungen, wie sie in unserem Raum vorkommen, wirft die Theorie ein helles Licht, indem sie diesen analytischen Zusammenhang als Besonderheit eines allgemeineren faßt. Aber genügt dies auch, um von anderen 'Raumformen' zu sprechen?"], Edmund Husserl, "Riemann-Helmholtzsche Behandlungsweise," *Studien zur Arithemetik und Geometrie (1886-1901)*, Husserliana, XXI, pp. 406-407, p. 407.

[134] Edmund Husserl, "Funktionen-Mannigfaltigkeit und Mannigfaltigkeit im weitesten Sinn gegenüber dem engeren Mannigfaltigkeitsbegriff," *Studien zur Arithemetik und Geometrie (1886-1901)*, Husserliana, XXI, pp. 408-411, p. 409.

distance-function, as, for example, a distance between two tones as characterized by pitch and volume. This can be done by representing the different volume values on the x-axis and the pitch values on the y-axis of a Cartesian coordinate-system (Fig. 4) and by defining the distance between the two tones in Riemann's manner as $\Delta s^2=\Delta x^2+\Delta y^2$.

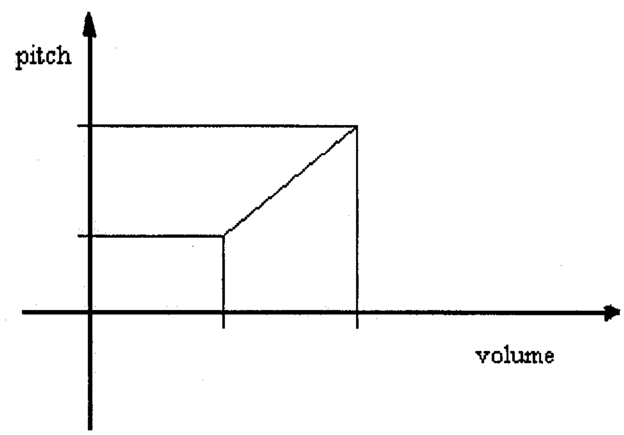

Figure 4: Volume-Pitch Diagram

In the case of the Euclidean manifold, the distance between two points defined by Riemann's formula corresponds to the distance that can actually be measured. For example, if we construct the coordinate system in such a way that it represents actual distances, we can simply measure the distance between two points. In Husserl's example of the tone-manifold we can also measure the distances displayed in the coordinate system. We would see that the value measured is the same as the value calculated by Riemann's formula. Yet the two cases are essentially different. In the first case, the properties represented by the coordinate system all belong to the same category -- they are all distances. In the second case, in contrast, the properties of pitch and volume represented by the coordinate system belong to different categories and no real property of the tone-manifold corresponds to the distance defined by Husserl. In other words, although we can define a property as Riemann does, no intuitively determinable property of the elements of the tone-manifold corresponds to it. Husserl concludes his example by stating that Riemann's distance-function, applied to non-Euclidean spaces, is like the definition of a distance between two tones characterized by pitch value and volume value, that is, a merely formal stipulation.

Husserl's view of correct axiomatization now allows us to state his first solution to the problem of the materiality of Euclidean geometry in the following way. The material manifold is characterized by the fact that its elements (the points) determine the relations holding between them. In a formal manifold the relation of dependence is inverted, in that the relations defined by the axioms determine its elements. The difference between the abstract individual of formal geometry and the *concretum* of material geometry is thus that in the latter there is an essential link between the relations and the non-formal properties of its elements -- a link that is necessarily missing in the former. We can summarize Husserl's first solution by saying that the subject matter of Euclidean geometry (an idealized intuitive space) does not collapse into that of formal geometry, because it has an additional property, namely an internal link between the non-formal properties of its elements and its relations. I will call this in the following the RED-property ('relation-element-dependence-property').

In order to evaluate Husserl's suggestion, we have to ask whether he can consistently ascribe the RED-property to the subject matter of Euclidean geometry under the assumption that the latter is a product of a process that involves idealization. This question has to be answered in the negative. As we have seen in the first chapter, in his early writings, Husserl characterized geometric idealizations as resulting in pure thought-objects that are devoid of any material content. So, for example, he characterized the notion of continuity as a merely formal relation on points. Idealization construed as a process that retains only formal properties is unable to do justice to the RED-property, for when one abstracts from the intuitive properties of the elements of Euclidean geometry, one simultaneously abstracts from the RED-property. As a result, Husserl's first suggestion fails, because there is no guarantee at all that the subject matter of Euclidean geometry as constituted in a process of idealization will retain the RED-property.

The impossibility of distinguishing the abstract individual of formal geometry from the *concretum* that forms the subject matter of Euclidean geometry by ascribing a particular status to the relations of the latter became clear to Husserl in the late 1890s. In a letter to Natorp in 1897, he put forward an implicit self-criticism and wrote:

> The Euclidean manifold [is] one of many species of manifold, just as the Two [is] one of many species of numbers in the number series. These species are pure, formal species, because they lack any material content, in contrast to the species of colour. As soon as one arrives at the lowest differences between colours, one can only indicate their distinguishing features, but not determine them conceptually.[135]

In this passage, Husserl claims that the Euclidean manifold can be characterized conceptually. Defined in this way, it does not differ from other formal manifolds with the same structure. Husserl then continues:

> Since the point as such is a primitive principle concretizing manifolds, and since geometry is not concerned with physical properties of space, then it is understandable that its systematic content can differ from the theory of the Euclidean manifold only in that instead of speaking about elements as such, it speaks about spatial elements.[136]

This view represents a major shift vis-à-vis his view from 1893. Husserl no longer attempts to characterize the material content of the subject matter of Euclidean geometry by reference to the RED-property. Rather, he now believes that such a manifold can only be distinguished from a purely formal manifold by reference to its very elements. What distinguishes the subject matter of Euclidean geometry from that characterized through

[135] ["Die Euklidische Mannigfaltigkeit [ist] eine neben anderen Mannigfaltigkeitsspezies, so wie die Zwei eine neben anderen Zahlspezies in der Reihe der Zahlen [ist]. Diese Spezies sind nun reine, formale Spezies, weil sie alles Stofflichen bar sind, ungleich der Farbenspezies, wo man bei den niedersten Differenzen angelangt, auf das Unterscheidende auch nur hinweisen, aber es nicht rein begrifflich bestimmen kann."], Edmund Husserl, Letter to Paul Natorp from 29.03.1897, *Studien zur Arithemetik und Geometrie (1886-1901)*, Husserliana, XXI, pp. 390-395, p. 390 (translation my own).

[136] ["Da der Punkt als solcher ein primitives Prinzip der Konkretion von Mannigfaltigkeiten ist und da die Geometrie es nicht mit physikalischen Raumbestimmungen zu tun hat, so ist es begreiflich, daß sie sich in ihrem systematischen Gehalt von der Theorie der Euklidischen Mannigfaltigkeit in nichts unterscheiden kann, als darin, daß sie statt von Elementen überhaupt von Raumelementen spricht."], Ibid, (translation my own).

merely formal relations is that its elements are *spatial* points. This view represents Husserl's second attempt to distinguish the subject matter of Euclidean geometry from the formal individual that is the correlate of formal geometry.[137]

This second suggestion fails for similar reasons as the first. We can see this, if we consider Husserl's explanation of how spatial points differ from the elements of the manifold of formal geometry. With respect to the difference between temporal and spatial points, he writes: "The difference between spatial and temporal points cannot be determined; one can only say: look!"[138] Thus, the difference between the two types of points is a merely intuitive feature of these objects. The property of 'being a spatial point' can only be pointed out. However, as we have seen repeatedly, Husserl construed the process of idealization as resulting in pure thought-objects. These objects are not intuitive objects: one cannot point to the points of an idealized intuitive space.[139] Again, Husserl's attempt to distinguish between Euclidean and formal geometry is incoherent. It is not possible to understand the subject matter of Euclidean geometry as a pure thought-object and at the same time ascribe material content to it.

Husserl's mature philosophy of science allows us to derive a third suggestion for the distinction between formal and material manifold. This suggestion can be stated in a relatively short way. Beginning with his *Logische Untersuchungen*, but most explicitly in his *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie*, Husserl distinguishes between two different types of essence, material and formal. Both these types of essence can define what he calls a definite manifold, that is, a manifold that is defined by a deductively complete (decidable in Husserl's sense) axiomatic

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[137] Gottlob Frege tried to defend a related view in his "Über die Grundlagen der Geometrie" in A. Gutzmer, ed., *Jahresbericht der deutschen Mathematiker-Vereinigung. Fünfzehnter Band* (Leipzig: Verlag von B.G. Teubner, 1906), pp. 293-309, 377-403, 423-430. He believes that a geometric theory is not a purely formal construct because the axioms speak about specific ideal (geometric) objects like points and lines.

[138] ["Worin sich aber Raum- und Zeitpunkt unterscheiden, das läßt sich nicht bestimmen, man kann nur sagen: siehe!"], Edmund Husserl, Letter to Paul Natorp from 29.03.1897, *Studien zur Arithemetik und Geometrie (1886-1901)*, Husserliana XXI, p. 390.

[139] Frege's view (see footnote 137) is not addressed by this argument, simply because he does not give an account of idealization. In section 4.3., I will present a general argument against Husserl's and related approaches to the idea of a material geometry, which also applies to Frege.

system.[140] In other words, in Husserl's mature philosophy of science, axiomatic systems express essences. Accordingly, the difference between material and formal geometry is that the manifold of the former is defined by a material essence and the manifold of the latter by a formal essence. In order to assess Husserl's third suggestion, we thus have to understand the difference between formal and material essences.

In Husserl's mature philosophy, essences are constituted in a certain type of intentional act, so-called acts of *Wesensschau*. These acts are like perceptual acts, in that they allow a subject direct access to the essences themselves. Husserl says that acts of *Wesensschau* give their objects in person (*leibhaftig*). In *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie*, he describes the constitution of a material essence in the following way:

> If we produce in free fantasy spatial forms, melodies, social practices, and the like, or if we fantasize acts of experiencing, of liking or disliking, of willing, etc., then on that basis, we can grasp various pure essences through "ideation" originarily, and perhaps even adequately: be it the essences of spatial form, melody, social practice as such, or be it the essences of form, melody, etc., of the particular type exemplified.[141]

According to this passage, acts of *Wesensschau* that give material essences contain three moments: (i) a concrete object must be perceived or imagined, as, for example, a certain spatial form; (ii) this object must be modified in fantasy, producing a manifold of

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[140] Husserl defines a definite manifold in the following way: "[A definite manifold] ist dadurch charakterisiert, daß eine endliche Anzahl gegebenenfalls aus dem Wesen des jeweiligen Gebietes zu schöpfender Begriffe und Sätze die Gesamtheit aller möglichen Gestaltungen des Gebietes in der Weise rein analytischer Notwendigkeit vollständig und eindeutig bestimmt, so daß also in ihm prinzipiell nichts mehr offen bleibt." ["[A definite manifold] is characterized by the fact that, if necessary, a finite number of concepts and propositions derivable from the essence of the domain in question determines the totality of all the possible forms belonging to this domain completely and unambiguously, in the manner characteristic of purely analytic necessity. Accordingly, nothing in the domain is left open."], Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie*, Husserliana, III, p. 135, (translation my own).

[141] ["Erzeugen wir in der freien Phantasie irgendwelche Raumgestaltungen, Melodien, soziale Vorgänge u. dgl., oder fingieren wir Akte des Erfahrens, des Gefallens oder Mißfallens, des Wollens u. dgl., so können wir daran durch 'Ideation' mannigfach reine Wesen originär erschauen und evtl. sogar adäquat: sei es die Wesen von räumlicher Gestalt, Melodie, sozialem Vorgang usw. überhaupt, sei es von Gestalt, Melodie usw. des betreffenden besonderen Typus."], Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie*, Husserliana, III, p. 13, (translation my own).

variations; and (iii) the modifications are compared and their invariant extracted. This invariant is the essence.[142] According to this description, a material essence is one that is constituted in an act of *Wesensschau* that proceeds from the representation of a concrete object. The essence is material, because it is given only *within* the imagined variations of this concrete object. A formal essence, in contrast, is constituted in second-order acts of *Wesensschau*, which proceed not from concrete objects, but from essences themselves. Although their genesis presupposes material essences and thus also the perception or imagination of concrete objects, these no longer play a role in the formal essence, which represents only the categorial form of a given material essence.[143]

Husserl's final attempt to secure the distinction between the manifolds of formal and material geometry also fails. His account can succeed only if the material essence retains a specifically spatial character. Otherwise, Husserl would simply have constituted two different kinds of formal essences. Yet it is precisely with respect to the specifically spatial character of the material essence where Husserl's account runs into problems. The fact that a material essence can only be grasped by reference to concrete sensible objects, does not create a special connection between them. Since a material essence is an invariant, it could be grasped by appeal to any other intuitable reality with an isomorphic structure. Since such a structure does not have to be spatial, there is only a genetic connection between geometric space and actual spatial experience. Once the axiomatic system has been established, it is independent of the specific sensible reality from which it was derived; and spatial experience no longer plays any justificatory role. Husserl's mature proposal thus collapses material into formal geometry.

One could argue that Husserl's distinction between formal and material essences suffices to distinguish between formal and material geometry in the following sense:

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[142] These three moments of an act of *Wesensschau* are not to be understood as successive steps, because doing so renders Husserl's account of these acts circular in that the variation already presupposes a knowledge of essences. Rather, the three moments represent the phenomenological structure of one single unified act.

[143] Husserl believes that acts of *Wesensschau* that constitute formal essences are based on a special faculty of categorial intuition. Cf., Edmund Husserl, *Logische Untersuchungen: Zweiter Band. Zweiter Teil, VI. Logische Untersuchung*, 2. Abschnitt, Husserliana XVIII., E. Holenstein, ed., (The Hague: Martinus Nijhoff, 1975), pp. 657-733.

Material essences are invariants of certain kinds of objects. As such, they are constrained by the nature of these objects. The same holds for the axiomatic systems that capture these essences. In contrast, as we have seen in the first chapter, Husserl believes that a formal axiomatic system is not constrained in this way, it can be extended, if only conservatively. So, there seems to be a clear sense in which the axiomatic system of material geometry is different from the formal axiomatic system. This is correct, but the constraints concern only the shape of the different axiomatic systems and thus do not constitute a qualitative difference between them. As I have tried to show, maintaining the distinction between formal and material geometry requires that the former, in contrast to the latter, captures some non-formal property of its manifold.

4.2 Carnap

Before drawing more general conclusions, I will examine Rudolf Carnap's account in his doctoral dissertation which was closely related to Husserl's phenomenological approach to the philosophy of space around 1913. Given Carnap's later views, this may sound surprising. Yet, in his dissertation, he shares two main presuppositions with Husserl's philosophy of geometry. He not only accepts Husserl's account of *Wesensschau,* but also distinguishes between different concepts of space in a manner similar to Husserl's.[144] Carnap believes that scientists belonging to different professions are concerned with different concepts of space -- philosophers with intuitive space, mathematicians with formal space, and physicists with physical space.[145] Carnap's aim in *Der Raum* is to show that philosophers', mathematicians', and physicists' disagreements about the *a priori* status of geometric propositions results from the fact that they are all dealing with fundamentally different concepts of space. By clearly distinguishing between them, Carnap hopes to resolve the debate. His study concludes that formal space is analytically *a priori*, intuitive space is synthetically *a priori*, and physical space is the product of experience.[146] These three concepts of space do not contradict one another, but represent

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[144] Rudolf Carnap, *Der Raum*, p. 6 and pp. 22-23.

[145] Ibid., p. 61.

[146] Ibid., p. 63.

a hierarchy. The relationship between the different types of formal space (topological, projective, metric) and their corresponding intuitive spaces is the same as that between a general and a particular object. But the particular object, the respective intuitive space, here is still general vis-à-vis a spatio-temporal individual. The relation between the different types of intuitive spaces and physical spaces therefore is that between a particular object and an individual.[147] Since I am interested in Carnap's concept of intuitive space, I will address these issues in the following only in so far as they facilitate our understanding of this notion.[148]

Carnap defines intuitive space in the following way:

> We understand intuitive space as the structure of the relations between 'spatial' forms understood in the usual sense, that is, the line-, surface-, and space-elements whose determinate peculiarities we apprehend on the occasion of perception or also in mere imagination. These peculiarities do not yet concern spatial facts present in empirical reality. Rather, they concern only the 'essence' of these objects, which can be recognized in any of their representatives.[149]

According to this passage, intuitive space is the subject matter of geometry. This seems to indicate a grave departure from Husserl's account of geometric space as a pure thought-object. Upon closer consideration, however, this difference turns out to be merely terminological: Carnap's intuitive space is identical with Husserl's geometric space. As we have seen in the previous section, in his mature philosophy of geometry, Husserl had defined geometric space as a manifold that was represented in an axiomatic

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[147] Ibid., p. 60-61.

[148] An explicit exposition and discussion of Carnap's notion of mathematical, physical, and intuitive space can be found in Alan W. Richardson, *Carnap's Construction of the World: The* Aufbau *and the Emergence of Logical Positivism* (Cambridge: Cambridge University Press, 1998), ch. 6, pp. 139-158. See also Michael Friedman, *Reconsidering Logical Positivism* (Cambridge: Cambridge University Press, 1999), ch. 2, pp. 44-58.

[149] ["Unter Anschauungsraum . . . wird das Gefüge der Beziehungen zwischen den im üblichen Sinne 'räumlichen' Gebilden verstanden, also den Linien-, Flächen- und Raumstücken, deren bestimmte Eigenheit wir bei Gelegenheit sinnlicher Wahrnehmung oder auch bloßer Vorstellung erfassen. Dabei handelt es sich aber noch nicht um die in der Erfahrungswirklichkeit vorliegenden räumlichen Tatsachen, sondern nur um das 'Wesen' jener Gebilde selbst, das an irgendwelchen Artvertretern erkannt werden kann."], Ibid., p. 6.

system constructed in a process of *Wesensschau* that allowed one to extract the essences of certain objects of empirical perception, for example, in the case of geometry the essences of points, lines, etc. He seems to believe that the essences of these geometric objects are exclusively relational properties. Carnap holds essentially the same view. In fact, the above definition of intuitive space not only shows that it is a product of a *Wesensschau*, but also that the spatial essence that results from this process delimits an *Ordnungs-* or *Relationsgefüge*, which can be represented in an axiomatic system. Like Husserl's notion of a definite manifold, Carnap's concept of *Ordnungsgefüge* indicates that the axiomatic system defines a closed domain, or in Carnap's terms, a *Gesamtgefüge*. Moreover, both Husserl and Carnap maintain that geometric space (in Carnap's terminology: intuitive space) retains an intimate connection to spatial intuition.

Although Carnap and Husserl agree with respect to the fundamental features of intuitive space (in Carnap's sense of the term), they disagree with respect to its constitution. This difference is grounded in divergent views about the point of departure of the *Wesensschau*. Husserl believes that the *Wesensschau* proceeds from intuitive space (in his sense of the term), i.e., the space of everyday spatial perception, which he sees as the correlate of objectifying intentional acts and thus as already comprising a total system of spatial relations.[150] Carnap, in contrast, holds that only a limited region of space can be immediately accessible to perception and that the *Wesensschau* thus departs from a system of spatial relations that is restricted with respect to its extension.[151] This difference leads to diverging views about the function of *Wesensschau*: whereas in Husserl's mature philosophy of science, it is as an instrument for idealization, in Carnap's dissertation, it is not only a tool for idealization, but also allows the geometer to extend the features of a limited spatial region to a total system of relations. In other

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[150] This is particularly clearly expressed in Husserl's explicit analysis of the constitution of perceptual space in *Ding und Raum: Vorlesungen 1907*, Husserliana XVI, Ulrich Claesges, ed., (The Hague: Martinus Nijhoff, 1973). See also Part III of this dissertation for an analysis of the properties of perceptual space.

[151] Carnap writes: "Die Anschauung bezieht sich immer auf ein beschränktes Raumgebiet. Daher lassen sich in ihr auch nur Erkenntnisse über räumliche Gebilde von beschränkter Größe entnehmen." ["Empirical intuition always remains restricted to a limited spatial region. Accordingly, from it we can derive only knowledge about spatial objects of a limited size."], Rudolf Carnap, *Der Raum*, p. 23, (translation my own).

words, for Carnap geometric space is not only an idealization of intuitive space (the term here taken in Husserl's sense as the space of pre-scientific spatial experience), but also an extension of it. His notion of geometric space thus depends on the particulars of this process of extension. In the account given in his dissertation, Carnap focuses exclusively on this feature of the *Wesensschau*.

The question now is whether Carnap's concept of intuitive space (I will continue using Carnap's term) as an extension of an intuitable region of space is able to secure a distinct subject matter for Euclidean geometry. This is possible only if Carnap's concept retains an intimate connection to spatial perception. In order to see whether Carnap is able to secure a distinct subject matter for Euclidean geometry, we first have to consider how he constitutes intuitive space, that is, how he extends the spatial relations of a limited region of space to a *Gesamtsystem*.

Carnap constructs geometric space in two steps. First, he departs from Hilbert's axiomatic system as presented in the *Grundlagen der Geometrie* and shows which of its axioms are valid within an intuitable spatial region. Afterwards, he establishes a number of principles that allow him to complete the axioms that are valid in a limited region to an axiomatic system that defines a totality of spatial relations, a *Gesamtsystem*. Since nothing in my argument depends on the particulars of his account, I will outline only its main features. Carnap finds that Hilbert's Axioms I, 1-8, II, 1-4, and III, 1-4 can be validated immediately by intuition in a *limited* region of space.[152] These are the following axioms:

Axioms of Incidence:

1. For any two points there exists (at least) one straight line which contains both of them.
2. For any two points there exists only one straight line.
3. On any line there exist at least two points; in any plane there exist at least three points that do not lie on a straight line.
4. For any three points, which do not lie on a straight line, there exists (at least) one plane that

[152] Carnap's presentation is based on the 4th edition of Hilbert's *Grundlagen der Geometrie*. Cf., David Hilbert, *Grundlagen der Geometrie*, 4th ed., (Leipzig/Berlin: Verlag von B.G. Teubner, 1913).

contains each of them.

5. For any three points, which do not lie on a straight line, there exists only one plane that contains each of them.
6. If two points of a straight line lie in a plane, then the same is true of all other points of the line.
7. If two planes have one point in common, then they have at least one more point in common.
8. There are at least four points which do not lie in a plane.

Axioms of Order:

9. If a point lies on a straight line between A and B, then it also lies between B and A.
10. If A and C are two points on a straight line, there exists at lest one point between A and C, and at least one point D, such that C lies between A and D.
11. Of any three point on a straight line there exists one and only one that lies between the other two.
12. If in a plane there exists a straight line and three points which do not lie on this line, and if the straight line cuts one of the three lines determined by the points, then it will also cut one other line determined by the points.

Axioms of Congruence:

13. For any given line segment there exists on any given straight line from any given point in any given direction exactly one congruent line segment. Every line segment is congruent with itself.
14. If two line segments are congruent to a third line segment, then they are congruent to each other.
15. Two line segments are congruent, if they consist of two congruent parts.
16. For any given angle there exists in any given plane on any given half ray in any direction always one and only one congruent angle. Every angle is congruent with itself.[153]

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[153] ["**Grundsätze der Verknüpfung:**

1. Durch zwei Punkte geht stets (mindestens) eine Gerade.
2. Durch zwei Punkte geht nur eine Gerade.
3. Auf jeder Geraden liegen mindestens zwei Punkte, in jeder Ebene mindestens drei nicht auf einer Geraden gelegene Punkte.
4. Durch drei Punkte, die nicht auf einer Geraden liegen, geht stets (mindestens) eine Ebene.
5. Durch drei Punkte, die nicht auf einer Geraden liegen, geht nur eine Ebene.
6. Liegen in einer Ebene zwei Punkte einer Geraden, so auch alle übrigen.
7. Haben zwei Ebenen einen Punkt gemeinsam, so auch noch mindestens einen andern.
8. Es gibt mindestens vier nicht in einer Ebene gelegene Punkte.

Grundsätze der Anordung:

9. Liegt ein Punkt auf einer Geraden zwischen A und B, so auch zwischen B und A.
10. Wenn A und C zwei Punkte einer Geraden sind, so gibt es stets wenigstens einen Punkt der zwischen A und C liegt, und wenigstens einen Punkt D, so daß C zwischen A und D liegt.

Carnap further modifies Hilbert's axioms III, 5 (congruence of triangles) and IV (Parallel Postulate) in such a way that they can also be validated by intuition. In order to accomplish this, he restricts their validity to *neighbouring* triangles and straight lines:

17. If two neighbouring triangles are have two sides and the angle enclosed by them in common, then also the two other angles.
18. If two neighbouring straight lines lying in one plane, do not cut one another, then two equally located angles under which some other straight line cuts the two lines are equal.

Carnap does not define the term 'neighbouring' here. But it seems to mean simply that two geometric objects, say two triangles, have to be close enough to each other for intuition to allow us to compare them. Carnap does not think that his axioms 13-16 have to be restricted to neighbouring objects, because they express *formal* properties of the concept of equality or congruence. In contrast, the axiom of triangle congruence and the parallel postulate express qualitative properties, and thus require qualitative comparison. Having determined which axioms of Hilbert's can be validated by intuition in a limited region of space, Carnap then extends his system according to the following principles:

1. Axioms 1-18 shall be true in any part of the *Gesamtsystem*.

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11. Unter irgend drei Punkten einer Geraden gibt es stets einen und nur einen, der zwischen den beiden andern liegt.
12. Liegen in einer Ebene eine Gerade und drei nicht auf ihr gelegene Punkte, und schneidet die Gerade einen der drei durch die Punkte bestimmten Strecken, so auch eine der beiden andern Strecken.

Grundsätze der Kongruenz:

13. Zu einer jeden gegebenen Strecke gibt es auf irgend einer Geraden von irgend einem Punkte aus nach jeder Seite stets eine und nur eine kongruente Strecke. Jede Strecke ist sich selbst kongruent.
14. Sind zwei Strecken einer dritten kongruent, so auch untereinander.
15. Zwei Strecken sind kongruent, wenn sie aus je zwei kongruenten Teilstrecken bestehen.
16. Zu einem jeden gegebenen Winkel gibt es in irgend einer Ebene an irgend einem Halbstrahl nach jeder Seite stets einen und nur einen kongruenten Winkel. Jeder Winkel ist sich selbst kongruent.
17. Stimmen zwei benachbarte Dreiecke in je zwei Seiten und dem von ihnen eingeschlossenen Winkel überein, so auch in den beiden andern Winkeln.
18. Schneiden zwei benachbarte Geraden einer Ebene einander nicht, so sind zwei gleichliegende Winkel, unter denen irgend eine andere Gerade sie schneidet, gleich."], Ibid., pp. 25-26, (translation my own).

2 Axioms 1-4 shall be true for the *Gesamtsystem.*

3. The process of marking off a line segment on a line from a given point can be repeated arbitrarily many times.

4. By marking off line segments in this way one can always reach a line segment on which any arbitrarily given point of the line is situated.

5. The formal properties of relations of identity between line segments and between angles shall be valid in the extended system.

6. The relations of identity which are defined in axioms 17 and 18 with respect to neighbouring places shall be extended in such a way that the relation of identity is replaced by a relation according to which two geometrical objects which approach each other approach a limit of identity.

The space thus constituted is characterized by the fact that Hilbert's axioms 1, 3-4, 7-10, 13-16 are valid in the entire domain and axioms 2, 5, 6, 11, 12 in every limited part of it. Carnap concludes that Hilbert's axioms hold in every small part of this space. He interprets 'small' here in the sense of infinitesimally small and says with Riemann that intuitive space is Euclidean in such areas.[154] Riemann showed that this type of space leaves open the particular curvature that is being ascribed to each point of it and thus its global metric properties.[155] Consequently, Carnap's concept of intuitive space is more general than Husserl's concept of geometric space, since the former contains Euclidean space as a special case.

Carnap broadens his concept of intuitive space even further. He believes that the concept of three-dimensional metric space can be generalized in various ways and that

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[154] Carnap's idea that *Wesensschau* allows a geometer to intuit the structure of space in infinitesimally small areas is problematic. If the results of *Wesensschau* are restricted because empirical intuition is unable to reach beyond a certain distance, why should this not also be the case with very small areas? Husserl's pupil Oskar Becker gives an extended argument showing how the rational modification (*Bearbeitung*) of an intuitively given continuum leads to a geometric continuum via transition to the limit (*Grenzübergang*). Becker's study was published in 1923, however, and Carnap could not have known it when he wrote his dissertation. Carnap probably just took over this idea from Husserl. Cf. Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie*, p. 138. See Oskar Becker, "Beiträge zur phänomenologischen Begründung der Geometrie und ihrer physikalischen Anwendung," in particular, pp. 398-476.

[155] Riemann calls this property "Ebenheit in kleinsten Teilen." Cf., Bernard Riemann, "Über die Hypothesen, welche der Geometrie zugrunde liegen," *Göttinger Abhandlungen* 13 (1867): pp. 133-152, reprinted in *Riemanns gesammelte mathematische Werke*, Raghavaran Narasimhan, ed., (Berlin/New York: Springer Verlag, 1990), pp. 272-287, p. 280.

the resulting spaces also deserve to be called intuitive. Further dimensions can be added to three-dimensional space, thus constituting n-dimensional metrical spaces. One can alternatively abstract from the congruence of lines and angles and consider only the system consisting of the basic concepts 'point,' 'line,' and 'plane,' thus constituting a three-dimensional projective space. One can further abstract from the geometric concepts 'straight line' and 'Euclidean plane' and consider only the relations between lines and surfaces in general, thus constituting a three-dimensional topological space. Finally, one can lift the restriction of three dimensions in projective and topological spaces, thus constituting n-dimensional projective and topological spaces.[156] An n-dimensional topological space is the most general structure common to this infinite multiplicity of spaces. Carnap believes that this structure represents the general condition for the perception of the matters of fact and, therefore, calls it the *a priori* form of spatial intuition.[157]

Having clarified Carnap's notion of geometric space, we can now turn to the question of how he wanted to secure the connection of this kind of space to intuition, thus distinguishing it from formal space. In order to answer this question, we have to consider separately the different types of geometric space, i.e., three-dimensional metric space, n-dimensional metric space, projective and topological space. Firstly, according Carnap, three-dimensional metric space is intuitive, because it contains intuitively accessible areas as its parts. Secondly, Carnap explains the connection of n-dimensional metric space to intuition in the following way:

> Even this structure should still be called an intuitive space, in spite of the impossibility of grasping in intuition its objects, in so far as they have more than three dimensions. It should be called this because, first, all intuitive objects, which we know in R'_{3m} [three-dimensional metric intuitive space] also exist in R'_{nm} [n-dimensional metric intuitive space], and, second, those higher-level objects are also built up of intuitively accessible parts.[158]

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156 Ibid., p. 30.

157 Ibid., p. 65.

158 ["Anschauungsraum soll auch dieses Gebilde noch heißen, trotz der Unmöglichkeit, seine Gebilde, soweit sie selbst mehr als drei Abmessungen haben, in der Anschauung zu erfassen, weil erstens auch alle Anschauungsgebilde, die wir im R'_{3m} [three-dimensional metric intuitive space] kennen, im R'_{nm} [n-

Thus the generalization of three-dimensional space is intuitive because either it 'contains' intuitive objects or its objects are 'built up' from intuitive objects likes points, lines, etc. Finally, Carnap states that projective and topological spaces deserve to be called intuitive spaces, because they relate to other intuitive spaces as a general to a particular.[159] As a result, we have four different relations between intuitively accessible regions and geometric space: 'part-whole', 'being-contained-in', 'being-built-up-of', and 'general-particular'.

Whether geometric space as defined by Carnap retains an intimate connection to intuition thus depends (a) on whether the relation 'part-whole' guarantees the intuitive character of three-dimensional metrical space and (b) on whether the remaining three types of relation ('being-built-up-of', 'being-contained-in', and 'general-particular') are able to transport this intuitive content to the higher concepts of geometric space. The decisive point is (a); if the 'part-whole' relation is unable to secure the intuitive character of three-dimensional metric space, then the other relations will have nothing to transport into the more abstract concepts of geometric space. Before I consider that 'part-whole' relation, however, I will critique Carnap's claim that the other relations can fulfill their respective functions.

Carnap uses the word 'contain' in the relation 'being-contained-in' in the sense of 'embedded'. He claims that we can say that an n-1-dimensional space is embedded in a n-dimensional space just as we speak of a plane being embedded in a three-dimensional space. The term 'contain' thus derives its meaning from an intuitable case. Similarly, Carnap believes that we can say that an n-dimensional object is built up of n-1-dimensional objects, just as we say that a cube is built up of planes. Thus, the term 'built up' as used in the relation 'built-up-of' also derives its meaning from an intuitive case. The fact that these terms derive their meaning from intuition, however, renders the corresponding relations unable to account for the materiality of n-dimensional metric spaces. In order to apply these terms in a non-metaphorical sense to a three-dimensional

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dimensional metric intuitive space] vorkommen, und zweitens auch jene höherstufigen Gebilde aus anschauungsgegebenen Gliedern zusammengefügt sind."], Ibid., p. 30.

[159] Ibid., p. 31.

Euclidean space, one already has to know that this space is intuitive.[160] The relation 'general-particular' has a clear meaning and expresses the relation between three-dimensional metric space and projective and topological spaces. However, the fact that three-dimensional metrical space is a special case of projective and topological space does not guarantee that the latter is intuitive if the former is.

Let me now turn to the question of whether the 'part-whole' relation ensures the intuitive character of three-dimensional metrical space. That this is not the case is a result of the fact that Carnap actually does not extend the intuitively accessible regions; but rather he extends a purely conceptual construct, namely an axiomatic system. In order to constitute intuitive space, he proceeds from Hilbert's axioms, isolates a subset that could be verified by *Wesensschau*, and extends it to a total structure. The intention that led Hilbert to construct his axiomatic system was to represent the structure of Euclidean space by purely conceptual means and to divorce geometry from intuition. Thus, not only the subset of axioms isolated by Carnap, but also its extension to a total structure is a purely conceptual construct -- a fact of which Carnap was well aware, as the following quote shows:

> Intuitive space is an order structure whose formal type can certainly be circumscribed conceptually, but, like everything intuitable, we cannot circumscribe its particular properties. Here we can only point to contents of experience, namely to intuitive-spatial objects and relations: points, line-segments, plane-elements, volume-elements, the lying of a point on a line or in a volume, the intersection of two lines, etc.[161]

Given this, his assertion about the whole-part relation is to be understood as the claim

[160] Michael Hallett provided me with the following example: Assume that straight line-segments are intuitable objects and that we know how two build lines, triangles, squares, etc. out of them. Suppose that we do not have any intuitive knowledge of the third dimension; but rather only know that there is one and that we can build up objects of line-segments which extend in the third dimension, as, for example, by constructing a cube. Clearly, from this it does not follow that we have intuitive knowledge of the cube. Rather, intuitive knowledge of the cube requires that we can intuit three-dimensional space.

[161] ["Der Anschauungsraum ist ein Ordnungsgefüge von dem wir wohl die formale Art begrifflich umgrenzen können, aber wie bei allem Anschauungsmäßigen nicht sein besonderes Sosein. Hier läßt sich nur auf Erlebnisinhalte hinweisen, nämlich auf die anschaulich-räumlichen Gebilde und Beziehungen: Punkte, Linienstücke, Flächenstücke, Raumstücke; das Liegen eines Punktes auf einer Linie, in einem Raumstück, das Sich-Schneiden zweier Linien usw."] Ibid., p. 22, (translation my own).

that idealizations of intuitively accessible regions are parts of the structure described by the axiomatic system. This, however, is not possible because the spatial regions and the newly constructed total space are entities belonging to different levels of abstraction. The spatial regions that are intuitively given are idealizations effected by acts of *Wesensschau*. Although the regions are idealizations, they are concrete in the sense that they are apprehended in intuitive acts and concern a domain of specifically spatial objects. (For the sake of argument, I grant Carnap the notion of a *Wesensschau*.) The total space, in contrast, is nothing other than the structure picked out by the formal axiomatic system. Thus, Carnap is faced with the following alternative: Either he restricts his concept of intuitive space to a limited region of space, in which case the material geometry that represents its structure must remain a torso that does not represent a total spatial structure. Or, he considers intuitive space as a total space whose structure is captured in the axioms of material geometry. By choosing this option, he commits a category-mistake, however. Intuitive space, in his account, is a conceptual construct and cannot serve as a source of synthetic *a priori* knowledge. Thus, Carnap's notion of intuitive space collapses material into formal geometry.

We can illustrate the fact that the objects of the intuitively accessible region belong to a different category than those in the space constructed through Carnap's extension by means of the following example. Carnap believed that Hilbert's axiom I,1 was verifiable through *Wesensschau*. The axiom states that "for any two points there exists (at least) one line that contains each of them."[162] *Wesensschau* allows the geometer to verify axiom I,1 because for any two intuited points he/she can also intuit a line on which they both lie. One of the principles by which Carnap extends his axioms to a total system demands that axiom I,1 should also hold in any part of the *Gesamtsystem*. Since intuition is always restricted to a limited region of space, there exist points in the total space that are further apart than the limits of any such region. For any two points of this kind, the geometer can no longer intuit a line, which contains both of them. He/she

[162] Rudolf Carnap, *Der Raum*, p. 24.

therefore cannot verify axiom I,1. Carnap's demand simply stipulates that such a line exist and that the axiom be true. As a result, the points and lines are characterized only by the relational properties defined through the axioms. Whereas points and lines were intuitable objects before the extension, after it they are purely formal objects.

4. 3 Material Geometry as an Axiomatic Theory

I will now argue that Husserl's and Carnap's problems in establishing a coherent account of material geometry result from an inconsistency in their approach. Both attempted to reconcile three different claims: (i) the subject matter of material geometry is an idealized perceptual space; (ii) material geometry is an axiomatic theory in the contemporary sense; (iii) material geometry thus defined differs from formal and applied geometry. But these claims cannot be reconciled. In particular, claim (iii) cannot be true, if claims (i) and (ii) are true, or, in other words, an axiomatic theory about an idealized perceptual space necessarily collapses into applied geometry. I will argue for this by showing that our contemporary notion of an axiomatic system does not provide any basis for a conceptual distinction between material and formal geometry. My argument will proceed in two steps: I will first define a material geometry as an axiomatic theory in the contemporary sense and then show how it collapses either into formal or applied geometry.

According to Husserl and Carnap, a material geometry is a certain type of axiomatic system, namely one whose propositions express genuine spatial content. Richard Trudeau calls such a system a material axiomatic system and characterizes it as follows:

> (1) The basic technical terms of the discourse are introduced and their meaning explained. These basic terms are called *primitive terms*.
>
> (2) A list of primary statements about the primitive terms is given. In order for the system to be significant to the reader, he or she must find these statements acceptable as true based on the explanations given in (1). These primary statements are called *axioms*.
>
> (3) All other technical terms are defined by means of previously introduced terms.

> Technical terms which are not primitive terms are accordingly called *defined terms*.
>
> (4) All other statements of the discourse are logically deduced from previously accepted or established statements. These derived statements are called *theorems*.[163]

The propositions of such an axiomatic system speak about a particular subject matter, namely about the objects given through the explanations. If the axioms are true of these objects, the other propositions derived from them by purely logical deductions will be true too.[164] Accordingly, a material geometry is an axiomatic system whose terms speak about spatial objects such as points, lines, and planes.

We can best see that this view collapses the concept of material geometry into either formal or applied geometry by considering the structure of such a theory more closely. The sentences of such a theory are held together exclusively by logical deductions, which according to our modern concept are content-neutral and thus will affect the particular content of the axioms in no way. But what do we mean by 'content' here? In the broadest sense, the term 'content' refers to anything captured by the axioms, except for their logically relevant structure. The latter is represented by a specific logical vocabulary, which includes the quantificational structure of the sentences and the truth-functional operators. In the language of first order logic, these are the connectives 'and', 'or', 'if-then', 'if-and-only if', 'not' and the existential and universal quantifiers. Given this, we can say more precisely that a logical deduction is a derivation of a given sentence from a previously accepted set of sentences, -- one that leaves the specific content of the non-logical terms occurring in these sentences unchanged. Since a sentence is included in a theory only on the basis of its logical vocabulary and its quantificational structure, we can replace all the non-logical terms by mere symbols in

[163] Richard L. Trudeau, *The Non-Euclidean Revolution* (Boston: Birkhäuser, 1987), p. 6.

[164] This notion of a material axiomatic system not only characterizes the views of Husserl and Carnap, as I presented them in the previous section, but also of Frege. Yet Frege did not accept Hilbert's point of view and wanted to resist the formalization of such a material axiomatic system. He believed that the formalization, or *disinterpretation*, of the propositions of the material axiomatic system would lead to a meaningless symbolism.

the way suggested by Hilbert and Husserl (for example). This, of course, has to be done consistently for the entire axiomatic theory. In this way, we construct a formal axiomatic system, that is, a logical structure. That this *disinterpretation* is always possible is a consequence of our understanding of the nature of logical deduction.

Given the possibility of disinterpretation, a builder of an axiomatic theory has two options. On the one hand, he/she can understand the non-logical terms occurring in its sentences as referring to the objects, properties, and relations of a particular subject matter.[165] On the other hand, he/she can understand them as mere place-holders (even though, they may have conventional referents). In his *Grundlagen der Geometrie*, Hilbert took the axioms of Euclidean geometry in the latter sense. He expressed this by replacing the terms designating geometric objects with symbols. He wrote:

> Consider three distinct systems of objects. Let the objects of the first system be called *points* and be denoted by A, B, C, . . .; let the objects of the second system be called *lines* and be denoted by a, b, c, . . . ; let the objects of the third system be called *planes* and be denoted by α, β, γ, ...[166]

Thus, depending on the intention of the builder, the axiomatic theory is either a formal axiomatic theory, as in the second case, or an interpreted axiomatic theory, as in the first.[167] But, due to the character of the logical deductions, there is no conceptual difference between the two views. They differ only with respect to the builder's intention to 'see' one as about a formal structure and the other as about a particular subject matter. Accordingly, an axiomatic theory is either a formal axiomatic system or an interpretation of such a system, depending on the builder's intentions.

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165 This was Frege's suggestion. He understands an axiomatic system as a theory containing interpreted sentences.

166 ["Wir denken drei verschiedene Systeme von Dingen: die Dinge des ersten Systems nennen wir *Punkte* und bezeichnen sie mit A, B, C, . . .; die Dinge des zweiten Systems nennen wir *Geraden* und bezeichnen sie mit a, b, c, . . .; die Dinge des dritten Systems nennen wir *Ebenen* und bezeichnen sie mit α, β, γ, ... "], David Hilbert, *Grundlagen der Geometrie*, 1st. ed., published in *Festschrift zur Enthüllung des Gauss-Weber Denkmals*, p. 2, (translation my own).

167 The fact that any axiomatic theory is abstract also means that it is formal, i.e., that it captures only formal features of its subject matter. But this does not mean that any theory is a formalized axiomatic theory. Rather, a formalized axiomatic theory is a purely syntactical object, an uninterpreted calculus. Cf., Roberto Torretti, *Philosophy of Geometry from Riemann to Poincaré*, p. 192.

In order to bring home the argument, we have to consider the second option more closely. If the builder of an axiomatic theory chooses to understand it as an interpreted system, he/she has two possibilities: he/she can consider the theory either as being about a domain of empirical objects or as being about a domain of ideal objects. The first possibility would render the theory an empirical science. Yet, the second possibility would not change the status of the theory at all. Rather, the theory would remain a formal axiomatic theory. The reason for this is that the objects in the ideal domain are constituted in a process of idealization that necessarily abstracts away from any material particularities of the original subject matter, retaining only formal properties captured in the system of axioms. If the process of idealization did not do so, the properties of the objects in the domain would differ from the properties captured in the axioms, and the theory would become an empirical science. As a result, the type of idealization relevant to an axiomatic theory constitutes formal objects; and, thus, the theory is about a formal structure. But this is precisely the nature of a formal axiomatic theory.

We can now apply the previous argument to geometry. A geometric axiomatic theory is either a formal axiomatic theory in Hilbert's sense (or a formalization of such a theory) or applied geometry, that is, an empirical science. Any attempt to understand geometry as an axiomatic theory about a domain of ideal objects necessarily collapses it into formal geometry. This is why Husserl and Carnap did not succeed in distinguishing material from formal geometry.

5. Euclid: Geometry as Science about Diagrams

5.1 The Received Interpretation of Euclid's Method

In section 4, I have demonstrated the manner in which Husserl and Carnap attempted to secure materiality or non-formal content for a third type of geometry other than formal or applied geometry: they believed that the subject matter of Euclidean, or material, geometry is captured by an axiomatic theory that is nevertheless more than just a formal structure. I argued that these attempts fail, partly, because they are based on the idea that a material geometry is an axiomatic system based on formal logical deductions. In my following characterization of material geometry, I will, therefore, suspend the contemporary concept of logical inference as well as the related account of the axiomatic method and turn my attention to the actual practice of geometry as it is exhibited in the text of Euclid's *Elements*.[168] By investigating the specific nature of the devices that are employed in Euclid's practice to generate geometric knowledge, that is, in its proofs, I will show that it is based on a non-formal type of logical reasoning for which diagrammatic representations are essential. This implies that the immediate subject matter of Euclidean geometry is not an idealized intuitive space or the structure captured by an axiomatic theory, but rather idealized diagrams, i.e., idealized visual, and thus intuitable objects. This subject matter establishes an intimate connection between spatial intuition (perception) and the statements of geometry, preventing its collapse into formal or applied geometry.

My analysis of Euclid's text is not an historical analysis in the narrow sense of the word. I am not interested so much in Euclid's own understanding of geometry. Rather, I am interested in the type of mathematical *practice* exhibited in the *Elements*, or, more specifically, in the question of how this practice allows those who grasp it properly to prove geometric propositions. Even though, I am not primarily focusing on

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[168] I do not intend to criticize the idea of a formal logic. My suggestion is rather, as we will see later, that Euclidean geometry involves another type of logically correct inference, namely an inference based on visual information.

an historical analysis of the *Elements*, I will support my view by reference to specific features of the text that facilitate a reader's grasping of the practice.

In order to prevent misunderstandings, I want to add that my analysis of the ontological presuppositions of geometric reasoning does not exclude material geometry from being also a science about space. Since this type of geometry can be applied in land surveying, for example, it is about the space of experience. Nevertheless, my argument will show that this type of content of geometric propositions is rather independent of the ontological presuppositions of geometric proofs. In other words, in order to establish a connection between geometry and intuition that is able to account for the materiality of geometry, I draw a distinction between two notions of its content, the first one designating those entities that are required for its proofs -- the *primary subject matter*; and the other referring to the structure of the space of experience -- the *secondary subject matter*.

In order to identify the type of geometric reasoning exemplified by the *Elements*, I will first criticize an interpretation of Euclidean geometry that is contained in many contemporary textbooks. Although it has been criticized by some interpreters of Euclid, it is so ubiquitous that I will call it the "received view."

Many textbooks interpret Euclid's geometry through the contemporary concept of the axiomatic method. In order to explicate this view, I want to take Greenberg's text *Euclidean and Non-Euclidean Geometries. Development and History* as an example.[169] Greenberg writes: "Ancient geometry was actually a collection of rule-of-thumb procedures arrived at through experimentation, observation of analogies, guessing, and occasional flashes of intuition."[170] This changed with "Thales of Miletus [who] insisted that geometric statements be established by deductive reasoning rather than by trial and error."[171] The next major step in the development of Ancient Greek geometry was Euclid

169 Marvin Jay Greenberg, *Euclidean and Non-Euclidean Geometries. Development and History* (New York: W. H. Freeman and Company, 1993). Many other introductions to geometry contain analogous views, usually stated in a much briefer form. See, for example, Earl Perry, *Geometry. Axiomatic Developments with Problem Solving* (New York: Marcel Dekker, Inc., 1992).

170 Op. cit., p. 7.

171 Ibid.

whose "monumental achievement was to single out a few simple postulates, statements that were acceptable without further justification, and then to deduce 465 propositions."[172] Accordingly, Euclid's axiomatic method is characterized by the fact that its proofs are based on "certain statements called 'axioms,' or 'postulates,' which are accepted without further justification" and that other statements follow logically from these axioms.[173] In a later section, Greenberg defines the concept of logical inference in the contemporary sense as based on formal relations between sentences. He thus concludes that "the axiomatic method used by Euclid," i.e., the method exemplified in the text of his *Elements*, "is the prototype of what we now call 'pure mathematics'." In short, according to Greenberg, Euclid's geometric method is a precursor of the modern axiomatic method in which a correct geometric proof is simply a correct formal deduction from a set of axioms.[174]

This interpretation of Euclid implies a specific view of his use of diagrams. Greenberg writes: "Geometry, for human beings (perhaps not for computers), is a visual subject. Correct diagrams are extremely helpful in understanding proofs and in discovering new results."[175] Accordingly, the diagrams facilitate the readers' understanding of geometric proofs and help them to discover geometric propositions, but are unfit to justify geometric knowledge.[176] The purpose of the diagrams is psychological, rather than logical. The same view has been expressed more clearly by Neil Tennant:

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172 Ibid.

173 Ibid., p. 11.

174 Zeuthen expresses the same point of view as Greenberg. He writes for example: "Die moderne Lehre von den geometrischen Axiomen (wie man jetzt oft sagt) verfolgt ja denselben Zweck wie die antike Aufstellung von Postulaten und stimmt in vielen Beziehungen mit dieser überein." ["The modern theory of geometric axioms (as is often said today) pursues the same goal as the Ancient stipulation of postulates and the former coincides in many respects with the former."], H. G. Zeuthen, *Die Mathematik im Altertum und im Mittelalter* (Stuttgart: B.G. Teubner Verlagsgesellschaft, 1966), first published 1912, p. 45, (translation my own).

175 Marvin Jay Greenberg, *Euclidean and Non-Euclidean Geometries. Development and History*, p. 25.

176 This view was also held by Pasch and Hilbert.

> It is now common place to observe that the [geometrical] diagram . . . is only a heuristic to prompt certain trains of inference; that it is dispensable as a proof-theoretic device; indeed, that it has no proper place in the proof as such. For the proof is a syntactic object consisting only of sentences arranged in a finite and inspectable array. . . . Thus, the "general triangle" drawn on the page has no genuine role to play in the reasoning.[177]

Given this understanding of geometric reasoning, the authors referred to in this passage have to conclude that Euclid himself was not always completely conscious of the status of his diagrams so that in some of his proofs he relied on intuitable features. We have already seen in the first chapter the description of such an appeal to non-logical features of the diagram in Michael Friedman's analysis of the proof of proposition I,1. Friedman claimed that Euclid was forced to show the existence of point C by constructing it, because of his limited logic. There are also cases in which Euclid appeals to intuitive features of the diagram, even though he could have proved the respective propositions by purely deductive means. One such example that recurs frequently in the *Elements* is the comparison of size. If one object is part of another and this is clearly shown by the diagram, then Euclid concludes that it is smaller than the other. Euclid uses this type of comparison first in proposition I,6; the same tool plays a significant role in some of the propositions of Book XII, as well where he applies the so-called method of exhaustion. Although such a comparison is justified by Common Notion 5 ('The whole is greater than the part.'), the above interpretation of Euclid's geometry as a prototype of pure mathematics would condemn it as illegitimate.

The standard interpretation of Euclid's geometry as put forward by Greenberg, for example, is incorrect, since it does not do justice to many features of the actual text. In the following, I will address three of these features: (i) the seeming imprecise formulation of the definitions; (ii) the problem-theorem and construction-proof distinctions; and (iii) the fact that the postulates are formulated as rules for construction.

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[177] Neil Tennant, "The Withering Away of Formal Semantics," *Mind and Language* 1 (1986): pp. 302-318, pp. 304-5.

According to contemporary standards of mathematical rigor, most of Euclid's definitions are problematic. In the geometric books of the *Elements*, that is, Books I-IV, VI, and XI-XIII, we can distinguish between two types of definitions. The first type, mainly the definitions 1-7 of Book I, defines the basic concepts of geometry by reference to idealized features of intuition. For example, the first definition defines a point as that which has no parts; the second definition states that a line is that which has no breadth. With respect to these definitions, Greenberg writes:

> We cannot define every term that we use. In order to define one term we must use other terms, and to define these terms, we must use still other terms, and so on. If we were not allowed to leave some terms undefined, we would get involved in infinite regress. Euclid did attempt to define all geometric terms. He defined a "straight line" to be "that which lies evenly with the points on itself." This definition is not very useful; to understand it, you must already have the image of a line. So it is better to take "line" as an undefined term.[178]

According to our contemporary understanding of Euclidean geometry, the first seven definitions of Book I are not actual definitions but rather merely what one might call elucidations of primitive terms. As such, they do not play any role in the geometric deductions and are better left undefined.[179] The second type of definition corresponds to what contemporary mathematicians call definitions of defined terms. These play a role in the proofs and are, therefore, indispensable. Yet, Euclid's definitions are often problematic, because they contain non-primitive undefined terms. For example, definition 8 of Book I states that a plane angle is the inclination to one another of two lines, yet Euclid neither defined the notion of an inclination, nor introduced it as a primitive term. Whereas the problems arising from the second type of definition can be eliminated by either introducing certain terms as primitives or by giving definitions for

[178] Greenberg, *Euclidean and Non-Euclidean Geometries. Development and History*, p. 11.

[179] This does not mean that they have no purpose whatsoever, even if one takes this negative view. The first seven definitions may serve to make clearer the kind of thing which is being analyzed in geometry. This view was held by Frege and Pasch, for example. Frege called the definitions therefore "elucidations" (*Erläuterungen*). Gottlob Frege, "Über die Grundlagen der Geometrie" I, p. 301. For Moritz Pasch's view see his *Vorlesungen über neuere Geometrie* (Leipzig: Verlag von B.G. Teubner, 1882).

them,[180] the first type of definition raises an explanatory problem for any interpreter of Euclid's method. Why did Euclid define the primitive concepts, and why did he include them among the other definitions without making their different function explicit? This is the more puzzling, since Euclid seems to have been aware of the difference between these two types of definitions -- he never used the first seven definitions in his proofs. Lucio Russo has argued that the first seven definitions were not present in Euclid's actual text and were inserted in an attempt to render Euclid's text more Platonist only in the first century A.D.[181] This may be correct and would explain the presence of the first seven definitions in the *Elements*. Yet, there still remains the question of whether these definitions also fulfill a practical function with respect to the actual derivations.

Greenberg's interpretation also fails to account for two closely linked structural features of Euclid's text. Euclid draws a distinction between two types of propositions, so-called theorems and problems.[182] Problems solve construction tasks, for example, as in proposition I,1, which, as we have seen, shows how to construct an equilateral triangle on a given straight line. Theorems, in contrast, make geometric assertions, for example, as in proposition I,6, which states: "If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another." This distinction is a consequence of the structure of the individual proofs of the various propositions. Let us consider as an example the proof of proposition I,20:

> *protasis* (enunciation)] *In any triangle two sides taken together in any manner are greater than the remaining one.*
>
> [*ekthesis* (setting-out)] For let ABC be a triangle;
>
> [*diorismos* (definition-of-goal)] I say that in the triangle ABC two sides taken together in any manner are greater than the remaining one, namely

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[180] For this view see also Richard J. Trudeau, *The Non-Euclidean Revolution*, pp. 32-39.

[181] Lucio Russo, "The Definitions of Fundamental Geometric Entities Contained in Book I of Euclid's *Elements*," *Archive for the History of Exact Sciences* 3 (1998): pp. 195-219.

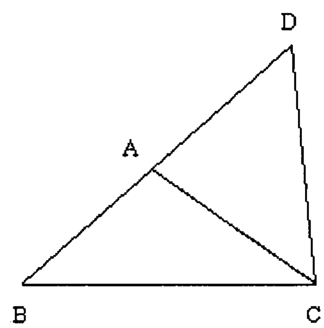
[182] The terms 'theorem' and 'problem' may have been introduced only after Euclid. Cf., Ian Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements* (Cambridge, Mass.: The MIT Press, 1981), p. 11.

BA, AC greater than BC,

AB, BC greater than AC,

BC, CA greater than AB.

[*kataskeue* (construction)] For let BA be drawn though the point D, let DA be made equal to CA, and let DC be joined.



[*apodeixis* (proof)] Then, since DA is equal to AC,
the angle ADC is also equal to the angle ACD;
therefore the angle BCD is greater that the angle ADC.
And since DCB is a triangle having the angle BCD greater than the angle BDC,
and the greater angle is subtended by the greater side, therefore DB is greater than BC.
But DA is equal to AC;
therefore BA, AC, are greater than BC.
Similarly we can prove that AB, BC are also greater than CA, and BC, CA that AB.
[*sumperasma* (conclusion)] Therefore, in any triangle two sides taken together in any manner are greater than the remaining one. Q.E.D.[183]

In the presentation of this proof, I have made explicit its structure by labelling its various parts according to ancient typology.[184] Like most of the proofs in the *Elements*, the proof

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[183] Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements*, vol. 1, pp. 286-287.

[184] Not all of Euclid's proofs contain all the structural elements. Some problems do not require a setting-out and some theorems do not require a construction. Proclus writes, for example:

of proposition I,20 contains six structural elements: *protasis* (enunciation), *ekthesis* (setting-out), *diorismos* (definition-of-goal), *kataskeue* (construction), *apodeixis* (proof), and *sumperasma* (conclusion).[185] For the present purpose, the most important feature of this structure is the clear distinction between the construction (*kataskeue*) and the actual proof (the *apodeixis*), the former always preceding the latter. In order to prove a given geometric proposition, Euclid first constructs an adequate lettered diagram and then proceeds with an *apodeixis*. Greenberg's interpretation of Euclid's method is unable to account for the presence of either distinction in the *Elements*. Firstly, if the use of the diagram is psychological rather than logical, as Greenberg claimed, then there is no reason to differentiate between construction and *apodeixis* in every proof, i.e., to ascribe to this distinction the status of a necessary structural feature of a geometric proof. Secondly, since from Greenberg's point of view, the constructions effected by the problems function merely in the production of psychological objects, the problems themselves are not mathematically relevant. Thus, they should not be included in the actual mathematical text. Moreover, for Greenberg, a geometric proof is a formal derivation from a set of axioms. This also excludes the presence of problems in the *Elements*.

> These are all parts of problems and theorems, but the most essential and those which are found in all are *enunciation, proof, conclusion*. For it is equally necessary to know beforehand what is sought, to prove this by means of the intermediate steps, and to state the proved fact as a conclusion; it is impossible to dispense with any of these three things. The remaining parts are often brought in, but are often left out as serving no purpose. Thus there is neither *setting-out* nor *definition* in the problem of constructing an isosceles triangle having each of the angles at the base double of the remaining angle, and in most theorems there is no *construction* because the *setting-out* suffices without any edition for proving the required property from the data. (Proclus, *A Commentary on Euclid's Elements I*, trans. by G. Morrow, with intro. by Ian Mueller (Princeton: Princeton University Press, 1992))

Proclus states here that the construction can be dispensed with provided the setting-out suffices for the proof, that is, if the setting-out contains sufficient data for proving the proposition. The construction is required only in order to refine the given data, that is, to add further elements to the diagram. The difference between the setting-out and the construction can thus also be put in terms of immediate and mediated access to the truth of the propositions. In those proofs in which a construction is not required, the proof proceeds immediately from the data given in the diagram presented in the setting-out. In all other proofs, the argument can proceed only through an emendation of the original figure. Further, Proclus states that if a proposition requires a geometer to construct the figure from the beginning, no setting out will be present.

[185] For a description of the various parts of an Euclidean proof see Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements*, pp. 129-131.

Finally, Greenberg's view is unable to explain the presence and specific formulation of Euclid's first three postulates. First, according to the modern concept of the axiomatic method, as we have seen, a proof of a theorem is a formal derivation from an independent and consistent set of axioms. These axioms are formulated as statements of existence: they guarantee the existence of the elements about which the axiomatic system speaks and specify all the properties that are relevant for the derivation. In contrast, Euclid states his first three postulates as rules for construction. For example, the first postulate legitimizes drawing a line between any two points. As such, the postulates guarantee the existence of their objects and can be seen as fulfilling one of the functions of axioms in modern axiomatics.[186] However, the postulates are unable to fulfill another important function. In formal logic, a deduction is valid if the truth of the conclusion always follows from the truth of the assumptions. This presupposes that the assumptions are formulated as assertions, that is, as statements that can be true or false. As rules for construction, Euclid's first three postulates are imperative statements that have different conditions of fulfillment and cannot be true or false. Accordingly, a proof in the *Elements* cannot be a derivation in the modern sense, that is, an inference from the actual or assumed truth of assumptions to the truth of their consequences. Second, in his *Grundlagen der Geometrie*, Hilbert constructed a system of axioms defining (at least implicitly) the properties of three systems of objects (points, lines, and planes) exhaustively.[187] An axiomatic system in his sense, therefore, characterizes a system of objects as the subject matter of the theory constructed from the axioms. A proof is a procedure that makes implicit structural properties of a system of objects (the subject matter) explicit. Euclid, in contrast, presents a diagram for every proof and indicates the individual geometric objects about which the *apodeixis* speaks. In the

[186] The thesis that Euclid's postulates fulfill the function of axioms in the modern sense, that is, guarantee the existence of their objects was first put forward by Zeuthen. More recent interpreters of Euclid disagree with this interpretation of the axioms, as for example, Wilbur Knorr. For Zeuthen's view, see his article "Construction als 'Existenzbeweis' in der antiken Mathematik," *Mathematische Annalen* 47 (1896): pp. 222-228; For Wilbur Knorr, see "Construction as Existence Proof in Ancient Geometry," *Ancient Philosophy* 3 (1983): pp. 125-148.

[187] Strictly speaking this is true only of the second edition of Hilbert's *Grundlagen* from 1903.

proof above, these are the straight lines AB, AC, BC, and DB and the angles DBC and ACB. A proof in the *Elements* can thus not be understood as a means for explicating implicit structural features of a given system of objects.[188] Euclid's geometry is constructive in the modern sense of the word, that is, its subject matter is constructed as the work proceeds. At no point does Euclid present a 'complete,' present and pre-existing domain whose properties are then explored in the axiomatic system. Greenberg could explain away the differences between the modern concept of proof and Euclid's actual practice by saying that Euclid was the first to use the axiomatic method and was thus not entirely clear about its actual nature. The differences between Euclid's text and Greenberg's interpretation are grave, however, and thus demand that one consider alternative interpretations of Euclid's axiomatic method and of the notion of proof relevant to it.

We can formulate an alternative notion of a proof in material geometry by reconsidering the role of diagrams in it. As we have already seen in Friedman's analysis of Kant, the development of formal logic has led many mathematicians to the view that a rigorous mathematical proof is ideally a formal logical deduction.[189] Hilbert, Husserl, Pasch, and Frege, for example, hold this view with respect to geometric proofs. They all draw a sharp distinction between the context of justification and the context of discovery. Whereas the context of justification is said to be purely logical, the context of discovery is said to be psychological or sociological. In contrast to the former, the latter involves appeal to empirical intuition, that is, to the observation of geometric diagrams. In the final section of the first part of this thesis, I argued that the view that the proofs of material geometry

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[188] Ian Mueller writes: "For Hilbert geometric axioms are characterized by an existent system of points, straight lines, etc. At no time in the *Grundlagen* is an object brought into existence, constructed. Rather its existence is inferred from the axioms. In general Euclid produces, or imagines produced, the objects he needs for a proof . . . . It seems fair to say then that in the geometry of the *Elements* there is no underlying system of points, straight lines, etc. which Euclid attempts to characterize. Rather, geometric objects are treated as isolated entities about which one reasons by bringing other entities into existence and into relation with the original objects and one another." Ian Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*, p. 14. See also Ian Mueller, "Euclid's *Elements* and the Axiomatic Method," *British Journal for the Philosophy of Science* 20 (1969): pp. 289-309.

[189] See, for example, Friedman's definition of the modern concept of a proof as outlined in section 2.1.

are purely formal logical deductions collapses material geometry into either formal or applied geometry. I therefore suggest that we change our understanding of the role of the diagrams. They not only allow the geometer to discover geometric propositions, but also convince him/her of their truth. Accordingly, Euclid's proofs are partial reconstructions of the context of discovery, which capture the justificatory function of the diagrams. In other words, the diagrams ground geometric inferences. A proof in the mathematical practice exhibited in the *Elements* thus is not a formal deduction from a set of axioms, and the diagrams are not mere psychological aids that serve to illustrate geometric propositions. Rather, a proof in the *Elements* is a procedure which makes explicit a certain type of information that is implicitly contained in a given physical object, that is, a correctly drawn diagram.[190] A direct proof is successful if it shows that a given diagram encodes simultaneously the assumptions from which the conclusion is to be drawn and the conclusion itself. An indirect proof is successful, if it shows that a given diagram encodes the assumptions, the negation of the conclusion, and a contradiction. I am using the term 'encode' here in a broad sense: everything that can be made explicit by reference to the diagram according to certain licensed rules of interpretation and logical argument is said to be encoded in the diagram. Geometric reasoning thus understood is intimately linked to visual features of physical objects and is not a purely formal process.

The obvious objection to this suggestion is that it degrades the knowledge of geometric propositions to empirical knowledge. On the one hand, the objects from which the inferences start, i.e., the diagrams, are physical objects. It seems therefore that we can

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190 Ian Mueller first suggested that a proof in the *Elements* is not a formal deduction in the contemporary sense. He writes: "It might be thought that Euclidean derivations of theorems are at least not significantly different from modern formal derivations. But, in fact, this is not true, for in almost every one of Euclid's derivations the carrying out of certain operations, previously shown possible, precedes argumentation in the usual sense. The characterization of a Euclidean derivation as a 'thought experiment' involving an idealized physical object which can be represented in a diagram seems clearly justified." We will see later on that the geometric object that is the source of geometric knowledge is indeed an idealized diagram. Yet, as will become clear later on, I do not believe that a Euclidean proof is a thought experiment. Rather, it is based on non-formal logical deductions. Ian Mueller, "Euclid's *Elements* and the Axiomatic Method," p. 291.

gain knowledge of their geometric properties only by measuring them. Since knowledge derived in this way is empirical, we need to explain how the diagrams can serve as a source of mathematical knowledge. On the other hand, it seems that inferences starting from empirical object are empirical generalizations. Thus, we also need to show that the inferences involved in the proofs of Euclid's geometry can indeed be defined as mathematical inferences -- ones that are based on the extraction of information from visual objects. In order to show that the type of reasoning exhibited by the *Elements* differs from empirical reasoning, I will argue, first, that there are logical inferences which depart from visual objects. I will then give an account of the constitution of the geometric object and show that it allows the geometer to derive a type of logical knowledge which is grounded in logical inferences from visual objects.

5.2 The Nature of Geometric Inference

Most contemporary logicians characterize logical inferences as purely formal derivations leading from sentences (or propositions) to other sentences (or propositions). As I pointed out in the final section of Part I, a derivation is formal if it is neutral to the content of the non-logical terms. Since formal deductions understood in this way cannot start from visual objects, but only from sentences (or propositions), we would have to accept that Euclid's proofs can generate only empirical knowledge. I therefore want to adopt a suggestion made by John Etchemendy and John Barwise and simply accept the existence of non-formal logical inferences. Etchemendy and Barwise argue that:

> [V]alid deductive inference is often described as the extraction or making explicit of information that is only implicit in information already obtained. Modern logic builds on this intuition by modeling inference as a relation between sentences of a formal language like the first-order predicate calculus. In particular, it views deductive proofs as structures built out of such sentences by means of certain predetermined formal rules. But of course language is just one of many forms in which information can be couched. Visual images, whether in the form of geometrical diagrams, maps, graphs, or visual scenes of real-world situations, are other forms.[191]

[191] Jon Barwise and John Etchemendy, "Visual Information and Valid Reasoning," in Gerard Allwein and Jon Barwise, eds., *Logical Reasoning with Diagrams* (New York: Oxford University Press, 1996), pp. 160-182, p. 161. See also Jon Barwise and John Etchemendy, "Heterogeneous Logic," in J. F. Glasgow,

According to this view, a geometric diagram, or more precisely, the system of geometric objects constructible by means of the postulates, can serve as a starting point for non-formal logical inferences.

The existence of non-formal logical inferences cannot be proved: whether we want to accept them depends on our intuitions about the concept of logical consequence. Thus, the only way to convince the reader of the existence of non-formal logical inferences which depart from visual objects is by considering particular examples, such as the following: The task is to determine how to get by subway from the Alexanderplatz to the Rathaus Neukölln in Berlin. In order to do so, we look at a plan of the subway system. (Fig. 5) We notice that although both places are accessible by subway, there is no direct connection between them. Yet, the plan shows many different ways of getting from one to the other. For example, we can first go to Charlottenburg, change trains, and then go to the Rathaus Neukölln; or we could go to the Friedrichstraße, change trains, go to the Mehringdamm station, change trains and go to the Rathaus Neukölln. Or, we can go to the Hermannplatz and then change trains to the Rathaus Neukölln. If we have not made a mistake in reading the map, all three connections are possible ways of getting to the desired station. Yet, stated in this way, the conclusion goes beyond what we can infer from the plan alone. If the plan is wrong, these three connections might not actually represent ways of getting to the Rathaus Neukölln. Since there may be a discrepancy between the representational system (the plan) and reality (the actual subway system), the result is empirical. But this is different if we restrict the conclusion to the representational system itself by saying that it *displays* the three respective connections. Restricted in this way, the conclusion is not reached by means of empirical generalization and is thus perfectly reliable. Further, we assume that any other rational being equipped with a visual sense similar to ours would accept the conclusion based on the same map.

N. H. Narayanan and B. Chandrasekaran, eds., *Diagrammatic Reasoning: Cognitive and Computational Perspective* (Cambridge: MIT Press, 1995), pp. 209-232.

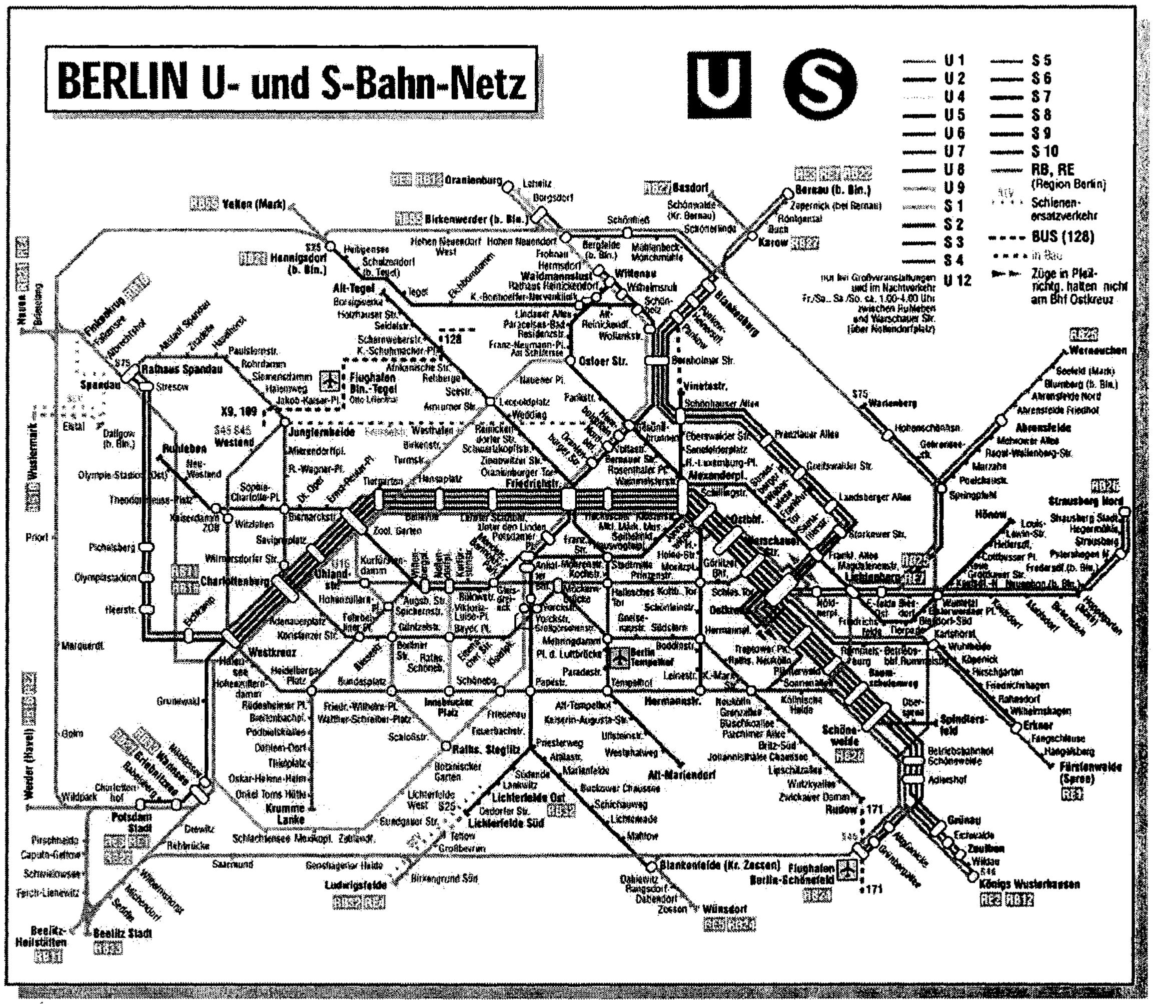


Figure 5: Map of the Subway System in Berlin

My suggestion that we distinguish empirical reasoning from the type of reasoning that uses information implicitly contained in representational structures like city maps seems intuitively plausible. Since both types of knowledge are ultimately justified by reference to observable facts, it is not entirely clear what the difference between them actually amounts to. I, therefore, want to make a suggestion that allows us to distinguish between logical and empirical knowledge.

There is a large philosophical tradition that attempts to distinguish these two types of knowledge by the different methods that lead to them. Logical knowledge is said to be derived by deduction and empirical knowledge by induction. It has been argued that this is problematic, however. Popper, for example, emphasized the role of deduction in the empirical sciences. I do not want to decide here the question of whether we can actually draw the distinction between logical and empirical knowledge in this way. But, I believe that the notion of logical knowledge defined by Etchemendy and Barwise gives us a way in which we can distinguish empirical from logical knowledge by reference to the different epistemic interests of logicians/mathematicians and scientists. The logician in general, and the geometer in particular, are concerned exclusively with the features of their respective representational systems. For example, the geometer working within the Euclidean framework is interested in the features of geometric diagrams. Knowledge of the actual structure of space, apart from the space occupied by the diagrams, does not enter into his/her justifications. We can see this by considering the types of error that can occur in geometric reasoning in the *Elements*. A proof is rejected: (a) on the basis of perceptual errors that result in misreadings of the diagrams; (b) errors in the interpretation of diagrams; and (c) errors in the chain of reasoning. Accordingly, no comparison to the space of experience is required. The mathematical sciences are also often, perhaps always, directly concerned with representational systems, or so-called models. The scientist has to avoid different kinds of errors: (a) perceptual errors, or more broadly, errors with respect to the data that enter as input into the theory; (b) errors concerning the inferences; (c) false predictions. (c) is the most important point with respect to the different epistemic conditions of logicians and scientists. In contrast to the logician, the scientist can accept the results of his/her reasoning only if (some of the time) they lead to adequate predictions of empirically observable facts. There is no such requirement with respect to logic or Euclidean geometry. This does not mean that Euclidean geometry cannot function as a physical theory; that is, that it cannot be applied to physical space. After all, this intention gave rise to its existence in the first place. Yet, as far as its proofs are concerned, no justification by reference to physical or experiential space is ever required, nor would such a reference be legitimate. Accordingly, we can say that the difference between logical/geometrical and empirical propositions results from the differences between their intended justifications. Whereas logical/geometrical propositions are justified by reference to features of certain representational systems, scientific propositions, or theories as webs of propositions, require further comparison between the representational system and empirical reality.

The above suggestion allows us to explain why the different types of justification lead to different types of certainty. Empirical knowledge has approximate character, because the ideal representational system has to be compared to empirical reality. Since we can determine the quantitative features of empirical reality only with approximate accuracy, this comparison can also only be approximate. Knowledge of the representational system, in contrast, is not approximative, because it does not involve measurements of quantities. This is clear with respect to formal logic. The formal deductions depend only on the correct application of rules to correctly formed strings of symbols. The logician does not have to measure these symbols in order to be able to properly recognize them. In other words, in addition to the correct application of the rules of deductive logic, correct formal logical reasoning depends only on qualitative rather than quantitative features of the representational system. This point will become important, when we consider the way in which the diagrams serve as visual sources of geometric inferences.

Before moving on, I want to prevent a misunderstanding concerning my use of the notion of a non-formal logical inference by contrasting it with Sellars's concept of a material inference. Sellars argues that in addition to formal inferences there exists so-called material inferences that "have an original authority not derived from formal rules of inference, and play an indispensable role in our thinking of matters of fact."[192] Consider the following example: Premise: "Pittsburgh is to the west of Princeton." Conclusion: "Princeton is to the East of Pittsburgh." If we were to understand this inference as a formal inference, we would have to say that it contains a conditional as an implicit premise, namely: "If A is to the West of B, then B is to the East of A." Yet, Sellars suggests that we should understand it as a material inference which we accept on the basis of our implicit knowledge of the inferential implications of the concepts 'east' and 'west'. Such an inference does not require appeal to a conditional. According to this suggestion, a material inference in Sellars's sense is a type of non-formal logical inference -- it does not only take into account the inferential function of logical terms

[192] Wilfrid Sellars, "Inference and Meaning," *Mind* (1953): pp. 313-338, p. 317.

like 'and', 'not', and 'if-then', but also of other concepts contained in the premises. Yet, Sellars's concept of a material inference is intimately tied to knowledge derivable from a linguistic representational system. My notion of a non-formal logical inference is broader, since it also takes into account other types of representational systems as sources of logical knowledge. I will give concrete examples of geometric inferences in later on when I analyze the proof structure of proof I,32. But this is possible only after giving an account of the geometric object.

In order to make plausible the thesis that Euclid's method yields logical knowledge as defined above and does not collapse into applied geometry, we have to show that the diagrams function as the source of geometric knowledge in a way similar to the linguistic symbols in formal deductions. In particular, we have to show that the properties relevant to the logical deductions are uniquely determined and do not have to be approximated through measurements. Since we are concerned here with a human practice, it is in principle impossible to specify conditions that are necessary and sufficient for a visual object to be uniquely determined. Yet the long tradition of Euclidean geometry confirms that questions about the particular features of the diagrams rarely arise, and if they do, they are answered without having to measure the diagrams. Thus, instead of attempting to specify necessary and sufficient conditions guaranteeing that the visual source of geometric inferences is uniquely determined, I will explain how it is possible that questions about the geometric object can be resolved without measuring the diagrams. In order to do so, I will show how the geometric object is constituted in the *Elements*.

5.3 The Constitution of the Geometric Object

The first step in the constitution of the geometric object is the construction of the diagram, the physical object. In order to understand how the physical object is produced, I will consider as an example Euclid's proof of propositions I,6. This proof is a so-called *apagogic* proof, that is, a proof by *reductio ad absurdum*.

[*protasis* (enunciation)]

If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.

[*ekthesis* (setting-out)]

Let ABC be a triangle having the angle ABC equal to the angle ACB;

[*diorismos* (definition-of-goal)

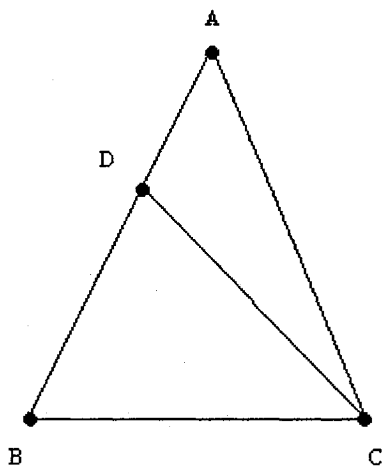
I say that the side AB is equal to the side AC.

[*kataskeue* (construction)]

For, if AB is unequal to AC, one of them is greater.

(a) Let AB be greater; and from AB the greater let DB be cut off equal to AC the less;

(b) Let DC be joined.



[*apodeixis* (proof)]

(a) Then, since DB is equal to AC and BC is common, the two sides DB, BC are equal to the two sides AC, CB respectively;

(b) and the angle DBC is equal to the angle ACB;

(c) therefore the base DC is equal to the base AB, and the triangle DBC will be equal to the triangle ACB, the less to the greater: which is absurd.

(d) Therefore AB is not unequal to AC.

[*sumperasma* (conclusion)]

Therefore, if in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another. Q.E.D.

We can distinguish two stages in the construction of the geometric diagram, the first being contained in the *ekthesis* and the second in the *kataskeue*. The *ekthesis* gives a simple geometric object, namely the triangle ABC and the *kataskeue* tells the geometer how this diagram has to be amended in order to allow him/her to derive the conclusion. The *ekthesis* does not actually require the geometer to construct such a triangle, as he/she can simply assume it as given. The construction that leads to the more complex geometric diagram, however, must be executed by means of certain legitimate tools of construction, namely a collapsing compass and unmarked straightedge.[193] These means of construction are legitimized through Euclid's first three postulates:

(1) To draw a straight line from any point to any point.
(2) To produce a finite straight line continuously in a straight line.
(3) To describe a circle with any centre and distance.[194]

Interpreted as describing a constructive procedure, the first postulate licenses the application of a straightedge in order to draw a line-segment between any two given points. The second postulate permits continuing such a line-segment indefinitely. This is also done by using a straightedge. The third postulate licenses the application of a compass. The construction (*kataskeue*) of Euclid's proof of proposition I,6 requires only two constructive procedures. The first consists of cutting a smaller line-segment off from a longer one, thus constructing point D. The second consists of joining two points by a straight line. The latter procedure is licensed immediately by the first postulate, which states that [it is possible] to draw a straight line between any two points. The former procedure is licensed by proposition I,3. This proposition is a so-called problem and requires the geometer to do the following: 'Given two unequal straight lines, [to] cut off

[193] Ian Mueller calls the postulates "licences to perform certain geometric operations". Ian Mueller, "Euclid's *Elements* and the Axiomatic Method," p. 290. Note also the difference of this view to Zeuthen's suggestion that the postulates are existence assertions.

[194] Sir Thomas L. Heath, *Euclid. The Thirteen Books of Euclid's Elements*, p. 154.

from the greater a straight line equal to the less.' The construction in proposition I,3 itself is licensed by proposition I,2, which shows the possibility of constructing "at a given point (as an extremity) a straight line equal to a given straight line," and postulate 3, which allows one "to describe a circle with any centre and distance." And finally, the construction in propositions I,1 is permitted by postulates 2 (which allows one "to produce a finite straight line continuously in a straight line") and 3. Thus, the construction in proposition I,6 is ultimately reducible to the first three postulates. Similarly, a correct construction in the *Elements* is one that is ultimately licensed by the postulates.[195] Thus, the representational system from which Euclid's proofs start consists of simple geometric objects as they are defined in his definitions, such as straight line, triangle, square, cube, sphere, and certain constructions executed by means of straightedge and compass.

The fact that Euclid limits the constructive procedures allowed in the *kataskeue* to straightedge and compass ensures a certain type of visual simplicity of the resulting perceptual object. Of course, using these procedures, we can produce highly complex diagrams by applying straightedge and compass repeatedly. Some of Euclid's diagrams are very complex in this sense. Nevertheless, the perceptual situation is relatively simple in a different sense, since it contains only two types of lines -- straight line-segments and circles, or arcs of circles -- and combinations thereof.[196] Thus, even if a given diagram has been drawn rather imperfectly, these two types of lines can often be properly identified. In this context, it is particularly interesting to note that Euclid does not expand his constructive procedures when he proceeds to solid geometry. This is possible because in this case the diagrams are two-dimensional representations of three-dimensional objects.

This interpretation of Euclid's postulates also illuminates the function of Euclid's fifth Postulate and its specific formulation. The so-called Parallel Postulate states:

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195 For further elaboration on the constructive structure of in *Book I* of the *Elements* see Ian Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's* Elements, ch. 1, pp. 1-57.

196 The Ancients also classified straight line and circle as the "two simplest and most fundamental species of line," Proclus, *A Commentary on Euclid's* Elements *I*, p. 84.

> (5) That, if a straight line falling on two straight lines make the interior angles on the same side less then two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.[197]

The historical investigation of this postulate, which culminated in the construction of non-Euclidean geometries, was based mainly on the assumption that Euclid had intended it as a self-evident principle or truth about the structure of space. Accordingly, many mathematicians and philosophers criticized it, because of its appeal to infinity.[198] I think that this postulate plays a very different role in the context of Euclid's practice, namely that of guaranteeing the possibility of constructing arbitrarily large triangles. In this sense, it functions in a similar way to Euclid's first three postulates.[199] I believe that this function finds its expression in Euclid's formulation, which emphasizes the coming into existence of a closed figure, namely a triangle. This stands in sharp contrast to other formulations of the Parallel Postulate, such as the following given by Hilbert:

> Let a be any line and A a point not on it. Then there is at most one line in the plane, determined by a and A, that passes through A and does not intersect a.

Hilbert here uses a negative formulation, saying, in effect, that there is at most one parallel line that does *not* intersect the given line a. The focus here is on the existence of a line with such and such properties, and not on the constructibility of a geometric figure.[200]

The geometric object that serves as the basis for the geometric inferences cannot

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[197] Sir Thomas L. Heath, *Euclid. The Thirteen Books of Euclid's Elements*, p. 155.

[198] One such criticism was raised by Proclus, for example. Proclus, *A Commentary on Euclid's Elements I*, p. 191, 21 sqq.

[199] David Reed arrives at a similar conclusion. He believes that Euclid's fifth Postulate affords a certain self-determination within the material given by the definitions. In particular, this Postulate guarantees that a certain figure consisting of straight lines and angles is completely determined. Without this postulate, additional information would be required to achieve such determination. Reed also agrees that the fifth Postulate does not determine the type of geometry. David Reed, *Figures of Thought. Mathematics and Mathematical Texts* (London/New York: Routledge, 1995), p. 18

[200] So far, I have left out of consideration Euclid's fourth Postulate. I did so, because I believe that it actually functions as an Axiom. I will therefore deal with it later on.

be identical to the diagram as it is seen, however. A visual object has many properties that diverge from those of Euclidean geometry. For example, we can always draw more than one line through a visible point in the same direction. The geometric object must therefore, at least partly, be an idealization of the visual object. Thus, to derive geometric propositions from a diagram, the geometer requires certain rules of idealization that allow him/her to remove their non-geometrical features.

One view that is widely accepted is that such rules can be stated by appeal to some type of recursive process. Yet appealing to processes is problematic. In order to show this, I will consider first an example of a rule that is specified by means of an infinite process and then one that requires only a finite process. An example of a rule specified by means of an infinite process is at work in Husserl's definition of a geometric point. He considers a geometric point as a limit concept that is formed by imagining completed an infinite process of dividing a visual point. The result can no loner be perceived and is a pure thought-object. Consider a diagram consisting of a straight line and a point on it. If we want to apply the procedure suggested by Husserl, we have to divide the point. But, in order to do so, we have to make a decision as to how we want to divide it first, say, in the middle, more to the top, more to the left, etc. Having divided it once, we again have to make the same decision, and so on. A point as a limit-object is the result of a decision process that could have been different, that is, the point could have a different location. Husserl's suggestion, therefore, leaves open an infinite number of possible positions of the ideal point on the line. In general, the problem with this type of idealization is that the process itself does not determine the specific geometric properties of the geometric object uniquely.

Kenneth Manders developed a suggestion based on a recursive process that involves an arbitrary, but finite, number of steps.[201] He assumes the principle that similarity transformations preserve qualitative features of a given figure (scaling principle). According to this principle, we can magnify a given diagram, and the resulting figure will retain all the qualitative features of the original. Manders then

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[201] Kenneth Manders, "On Geometric Intentionality," in Thomas M. Seebohm, Dagfin Føllesdal, and J.N. Mohanty, eds., *Phenomenology and Formal Science* (Dordrecht: Kluwer Academic Publishers, 1991), pp. 215-224.

suggests that we can redraw the magnified figure with lines and points that have the width of the original. He concludes:

> By the scaling principle, this new figure represents the original figure redrawn to a correspondingly smaller line width, one can thus see how non-Euclidean intersection points (and other non-Euclidean aspects) would disappear from the original figure if it was redrawn to the smaller line width.[202]

The hope then is that after an arbitrary, but finite, number of repetitions, only the relation between the two objects -- line and point -- are determined in agreement with the propositions of Euclid's geometry. Yet, this suggestion runs into problem similar to those encountered in Husserl. Manders himself points out that the magnified diagram can be redrawn in different ways. For example, a thinner line can be situated in different ways within the thicker line, that is, tilted in different angles. Thus, again arbitrary decisions will enter into the process of idealization such that the result is no longer uniquely determined. Manders therefore believes that further rules guiding the process of idealization have to be introduced. But this is not possible without appeal to the geometric properties of a point as defined by Euclid's theory. Idealization in this sense is circular, since it presupposes its own results.

Given these problems, I suggest that the rules of idealization relevant to material geometry do not appeal to recursive processes. The geometer constitutes an ideal particular simply by abstracting or thinking away from certain visual properties of the diagram. By doing so, he/she does not constitute an object that differs ontologically from the diagram; the visual object remains the referent. Rather, the ideal object is the diagram *considered* in a certain way. In order to support the suggestion that the ideal geometric object is the visual object considered in a certain way, I will look again at Euclid's *Elements*. I will show that the text conveys this concept of idealization and its specific rules to its reader mainly through the 23 definitions.

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202 Ibid., p. 219.

I already emphasized the special status of definitions 1-7, which are commonly understood as Euclid's attempts to give meaning to the primitive terms of his axiomatic system. Yet, within the framework of my suggestion, they should be understood as rules for the correct idealization of the visual features of a given diagram. In order to do so, they first restrict the vocabulary that can be used in speaking about the diagrams, thus picking out certain of its visual features.[203] The diagrams, which are part of our everyday perceptual world, are to be seen as objects consisting of points, lines, etc. Yet, the terms 'point,' 'line,' etc. are everyday concepts and as such ambiguous. The same holds for the visual objects designated by these terms. The definitions, therefore, fulfill a second function by eliminating these ambiguities. This procedure does not require guidance by the axioms of Euclidean geometry. In order to eliminate the possibility of multiple interpretations of visual objects of everyday experience, it suffices to say what such an object is not. According to the above suggestion, they have to do this by telling the reader of the *Elements* to simply abstract away from certain visual features of the diagram. I want to emphasize again that we cannot prove that the definitions necessarily succeed in doing so. Rather, I take this for granted and seek an explanation of how this is possible.

Euclid's first definition states that a "point is that which has no parts."[204] This definition is often interpreted as defining a point as a limit-object, i.e., an ideal object resulting from a process of continually diminishing some visual object or property. We have already seen this view in Husserl. My suggestion that the definitions specify criteria for the correct idealization of a diagram allows us to view this definition in a different way. A point is part of an already drawn diagram -- it is given through an appropriate mark, either by determining, more or less precisely, a certain place on a visual line, or by designating an intersection of different lines. Given as a feature of a diagram, a point is

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[203] Netz shows that the selection of a specific vocabulary for geometry was an important achievement of the entire text of the *Elements*, not only of the definitions. Reviel Netz, *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History* (Cambridge: Cambridge University Press, 1999), ch. 3, pp. 89-126.

[204] Sir Thomas L. Heath, *The Thirteen Books of Euclid's* Elements, p. 153.

ambiguous with respect to its size, however. Without further information, the geometer would not know how many lines could be drawn through the same point in the same direction, for example. Thus, in order to be able to generate unique results, the geometer requires additional information that disambiguates the relation between points and lines.

We can best see how Euclid's first definition achieves this disambiguation, if we consider an alternative definition that would have been available to him. Aristotle defined a point as an indivisible magnitude. This definition does not actually require the reader to repeatedly divide a visible object until this is no longer possible, it does not define a point as a limit-object. Nevertheless, Aristotle conveys the concept of a point by stating what kind of operation cannot be performed on such an object. A similar appeal to an operation is present in Heron's definition of a point, according to which "a point is that which has no parts or an extremity without extension, or the extremity of a line."[205] The first part of this definition is identical to Euclid's. Heron seems to have considered it as equivalent to the second and third definitions. Accordingly, he disambiguated the notion of a point by appeal to the operation of measurement. In particular, a point is that which has zero extension. Euclid's definition is often identified with Aristotle's, since one can say that something has no parts if it cannot be divided into parts. Yet, Euclid's definition displays a clear shift in emphasis away from processes and operations and toward static, immediately observable, figurative features of the diagram.[206] If a point is

[205] Heron wrote a commentary of the *Elements* and lived between 100 B.C. and A.D. 100. Thus, his definition was not available to Euclid. Heron, *Opera*, J. L. Heiberg, ed., (Leipzig: 1895-1914).

[206] Árpád Szabó also argued that Euclid's definitions attempted to characterize the subject matter of geometry without appeal to processes, operations, or movement. Szabó writes: "Ich glaube, daß wohl eben diese Schwierigkeiten die ersten griechischen Theoretiker veranlaßt haben mögen, eine sog. axiomatische Grundlegung - zunächst nur für die Geometrie - zu schaffen. Es mußten nämlich jene bloß empirischen Tatsachen zusammengestellt werden, ohne die man gar keine Wissenschaft vom 'Raum', keine Geometrie hätte aufbauen können, die aber dennoch die Ansprüche der Eleaten auf reine, nur intellektuelle Erkenntnisweise keineswegs zu befriedigen vermochten. Nachdem man betonte, daß die Gebilde der Geometrie – 'Linien', 'Strecken', 'Schnittpunkte', 'Winkel', 'Figuren' usw. - mitnichten dieselben sind, die man auch sinnlich wahrnimmt (z.B. sieht), sondern daß diese ebensolche nur gedachten Dinge wären, wie die 'Zahlen', vesuchte man in den geometrischen Definitionen jedes Element nur sinnlichen Ursprungs - z. B. das Anschauliche - zu vermeiden." ["I believe that precisely these difficulties may have prompted the first Greek theoreticians to develop a so-called axiomatic foundation - originally only for geometry. One had to collect those merely empirical matters of fact without which one could not construct a science of 'space,' a geometry, which, nevertheless, could satisfy the Eleats' claims to a pure, exclusively intellectual, type of knowledge. After one had emphasized that the objects of geometry - 'lines,' 'line segments,' 'points of intersection,' 'angles,' 'figures,' etc. - are by no means the same as the ones which one perceives through the senses (e.g. sees), but that the geometric objects were objects of thought just like

a visual object such as a mark drawn on paper, on a wax tableau, or in the sand, it will necessarily contain distinguishable parts. This can best be seen in analogy to another visual object such as a tree, which allows us to naturally distinguish stem from crown and leaves from branches without actually having first to divide the tree into these parts. If this is indeed the way in which Euclid thought of a part, his definition is best understood as requiring the reader to think away from the fact that he/she can distinguish parts in a visual point. The geometer is required to *consider* the visual point as an object which has no parts.

This interpretation is supported first by Euclid's formulation of the definition. He not only avoids appeal to processes and operations, but also uses a term for point that seems to emphasize the fact that a point is something drawn on a surface, which, as any other visual object, naturally falls into parts. Euclid's term was 'σημειον', which means a 'conventional mark'.[207] Earlier authors, such as Aristotle, used the word 'στιγμη', which means 'a puncture'. Second, Euclid seems to have thought of the part-whole relation as one that is readily recognizable in a given diagram. This can be seen by appeal to Common Notion 5, "The whole is greater than the part".[208] This notion allows the geometer to conclude that one geometric configuration is greater than another, if the latter is part of the former. Thus, in order to apply Common Notion 5 in an actual proof, the part-whole relation must be an immediately observable fact. It is likely then that Euclid thought of the relation between a visible point and its parts in the same way.

My interpretation of Euclid's definition of a point is further supported by the fact that he formulated it in a purely negative way. This was already pointed out in antiquity by Proclus, who believed that this demonstrated that Euclid did not want to define ideal

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'numbers,' one attempted to avoid in the definitions any element exclusively originating in the senses - e.g. intuition."] "Anfänge des euklidischen Axiomensystems," in *Archive for the History of the Sciences I* (1960): pp. 37-106, p. 91. But, as this quote shows, Szabó does not inquire into the actual practice of geometry. Rather, his argument is based on the hypothesis that Platonic and Eleatic metaphysical ideas contributed significantly to the rise of mathematics as an axiomatic discipline. Whether this interpretation is correct or not, it does not necessarily contradict my own suggestion. I am interested mainly in the particular effect of the definitions, rather than in the ideology that led to their formulation.

207 Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements*, p. 156.

208 Ibid., p. 155.

objects and was not a Platonist. If Euclid, or whoever introduced the definitions into his text, wanted to specify a point as an ideal object, he or she would have had to specify the property that belongs essentially to this ideal object or a process by which one could form the idea of such an object. If the goal of the definitions is to present criteria that disambiguate a given visual object, however, they would have to have been formulated negatively.

I believe that the same intention that led Euclid in the first definition also underlies the formulation of definitions 2-7. The second definition states that "a line is a breadthless length."[209] We can integrate this definition into my interpretation of Euclid's method by understanding the terms 'breadth' and 'length' not in the sense of results of measurements, but rather as immediately recognizable features of a visual line. If this is correct, then this definition requires a geometer to interpret a given visual line as having the property of length, but not that of breadth. This understanding of Euclid's second definition can again be supported indirectly by considering alternative definitions that would have been available at the time when the text of the *Elements* was composed. Aristotle defined a line as "a magnitude divisible only in one way."[210] Proclus gives an alternative definition that was also already present in Aristotle: "A line is a flux of a point," i.e. the product of moving a point.[211] These two definitions appeal to results of the process of division and of the movement of a point and are thus not immediately given by a diagram. Clearly, Euclid's definition avoided this.

The third definition states that "the extremities of lines are points."[212] In order to underline the particularities of this definition, I want to consider a criticism that was brought forward against Plato, who had defined points in the same way. Aristotle objected to this definition, which he found incomplete. He thought that we need to know

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209 Ibid. p. 153.

210 Aristotle, *Metaphysics* 1016b 25-27.

211 G. Morrow, ed., *Proclus: A Commentary on the First Book of Euclid's* Elements, p. 97 8-13; alluded to in Aristotle, *De Anima* I.4, 409 a 4.

212 Sir Thomas L. Heath, *The Thirteen Books of Euclid's* Elements, p. 153.

not only that a point is the end or the beginning of a line, but also the result of its division. Aristotle also noted that a point in a line is the intersection of two lines. According to my suggestion, however, Euclid's specific formulation of the third definition does not need to be extended in this way. As a feature of a diagram, a point is given simply through a mark. Wherever the geometer puts the mark, there is a point. Thus, there is no need to say that the division of one line or the intersection of two lines is a point. However, having defined (in the sense of 'disambiguation') a point in the first definition and a line in the second, Euclid now needs to specify their relation to each other. This, rather than giving a second definition of a point, is the task of definition 3. By stating that the limits of lines are points, Euclid disambiguates a given visual object, that is a line, by delimiting its two ends.

Definition 4 defines the notion of a straight line in a rather obscure manner by stating that "a straight line is a line which lies evenly (uniformly) with its points on itself."[213] I suggest that we understand this definition as idealizing the properties of a straight line given in the diagram in the same way as definitions 1 and 2. The idea here is that for this disambiguation, Euclid exploits the everyday concepts 'even' and 'uneven.' A given straight line in a diagram is never really straight. According to definition 4, this means that the path leading from one point given on it to another is visibly uneven. The definition now asks the geometer to abstract away from this fact. As in the previous cases, this interpretation finds indirect support by comparing it to other definitions that would have been available. Plato defines a straight line as "that of which the middle covers the ends."[214] Heath conjectures that Euclid (according to him a Platonist) reformulated this definition with the goal of eliminating the obvious appeal to the sense of sight implied in Plato's formulation. If it is indeed the case that Euclid's definition is a modification of Plato's, however, then it seems that it is not the appeal to the sense of sight that was eliminated, but rather Plato's implicit appeal to the operation of picking up a line from the two-dimensional plane in order to look at it from its side. The same point is expressed more clearly by the fact that Euclid did not use definitions that are similar to

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[213] Ibid.

[214] Plato, *Parmenides* 137 E.

that of Archimedes. Archimedes defines "that of all the lines which have the same extremities the straight line is the least."[215] This definition appeals to a process which allows us to determine that a certain line is shorter than others, thus, requiring measurements. Similarly, Euclid's definition has often been understood as defining a straight line as one whose points cover the minimal distance between two points. Again this definition would appeal to measurements.

Definitions 5-7 of surfaces and plane-surfaces mimic definitions 2-4 even in the way in which they are formulated. They apply the definitional procedures of definitions 2-4 to two-dimensional geometrical objects and also avoid reference to processes and operations. Because of the similarities between the formulations of the two groups of definitions, I will not deal with definitions 5-7 in detail. Definitions 5-7 differ from definitions 1-4 in one important respect, however. According to my interpretation, Euclidean geometry is concerned with diagrams drawn in the sand, on wax tableaux, or on paper. From this fact a disanalogy between straight lines and plane-surfaces arises. The drawn line, including its inexactness, is visually accessible as part of the diagram. The plane, in contrast, is the ground on which the diagrams are drawn. Although it can be seen, it is not part of the diagram and thus not a visual source of geometric knowledge. It seems therefore that Euclid did not have to define the properties of the plane, and, thus, that he did not need definitions 5-7. Yet, my interpretation gives a reason for their presence. If Euclidean geometry is about drawn diagrams, then the structure of the surface on which these diagrams are drawn becomes important. Since natural surfaces like a wax tableau or paper are uneven, Euclid has to require from the reader to abstract from this feature.

An observation of David Reed's allows us to describe another way in which the definitions specify idealizations of diagrams, namely by defining the area of specifically geometric interest.[216] Reed agrees with many interpreters of Euclid's text that the first seven definitions define primitive terms. Yet, in contrast to them, he observes a significant difference between the terms occurring in definitions 1-4 and those in

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[215] Archimedes, "On the Sphere and the Cylinder," in Sir Thomas L. Heath, ed., *The Works of Archimedes with the Method of Archimedes* (Dover Publications, New York, 1953).

definitions 5-7. As we have already seen, the former group defines (the word 'defined' is used here according to my interpretation, that is, in the sense of disambiguating visual features) points, lines, straight lines, and the relation between them. Reed points out that Euclid determines that lines are delimited by their end-points and by their location on a given line. There is no further question of how the two types of objects relate to each other. The case is different with respect to the second subgroup of definitions. Definition 6 says that the extremities of surfaces are lines. This does not determine completely how these two kinds of objects are related to each other. The lines of a triangle delimit a different surface than the lines of a square, for example. Reed concludes from this that definition 6 focuses the geometer's interest on the relation between lines-as-boundaries and surfaces-as-bounded-by-lines, or, more generally, on the relation between figures and their boundaries, thus opening up an area of geometric interest -- an area to be investigated. By delineating the area of geometric interest, the definitions specify further ways in which the perceptual object has to be idealized. For example, the geometer has to abstract away from the colour of the lines, or, more generally, from any property of the visual object that does not concern the relation between figures and their boundaries.

Reed further supports his interpretation by showing how Euclid's definitions of boundary and figure, i.e., definitions 13 and 14, fit into his picture: they generalize the area of investigation. This can be seen in the way in which the 23 definitions are arranged. After opening up an area of investigation in definitions 5-7, Euclid defines the notion of an angle. As an inclination of one line to another, an angle allows the geometer to focus only on a very few relations between lines and surfaces, thus providing a very limited context for geometric investigation. Reed believes that Euclid, therefore, introduces the notion of a figure as something that is limited by boundaries. This enables him to broaden the context of inquiry by specifying as relevant geometric figures circles (definitions 15-18), trilateral rectilineal figures (definitions 19-21), and quadrilateral rectilineal figures (definition 22). This context suffices for Euclid's investigations into plane-geometry, and is consequently broadened only one more time, namely at the beginning of his Books on solid geometry. Definitions XI, 1-2 define a solid as bounded

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[216] David Reed, *Figures of Thought. Mathematics and Mathematical Texts*, pp. 3-9.

by surfaces.

These considerations provide circumstantial evidence for my suggestion that idealization relevant to the practice of Euclidean geometry consists in abstracting away from certain readily recognizable qualitative features of the diagrams. This implies that the application of the rules of idealization laid down in the 23 definitions does not result in a geometric object which has to be measured. But this does not yet explain how the diagrams can serve as a source of logical knowledge in a way that is similar to the symbols of a formal logical deduction. In other words, the fact that the idealizations themselves do not require approximation does not guarantee that the ideal geometric object is uniquely determined. The reason for this is that most of Euclid's propositions are metrical, that is, they say something about the quantitative relations between various magnitudes such as line-segments, areas, and volumes. Accordingly, Euclid's propositions seem to require a metrical geometric object from which they are derived. To avoid this latter consequence, I suggest that the ideal geometric object is best understood as an idealized qualitatively determined visual object to which metrical properties are ascribed by means of logical argument. Such an object would not have to be approximated: if the visual properties relevant to a Euclidean proof are uniquely determined, then the geometric object constituted through both idealization and ascription of metrical properties is also uniquely determined. In order for this suggestion to work, we have to clarify two points. First, we have to say which visual properties are relevant to Euclid's proofs. Second, we have to show how the metrical properties are ascribed to the geometric object.

In order to clarify which visual properties of the diagram are relevant to a Euclidean proof, I will consider the case of a badly drawn diagram, which, nevertheless, suffices for a Euclidean proof. Consider Figure 6 which represents a correctly constructed diagram for Euclid's proof of proposition I,1.

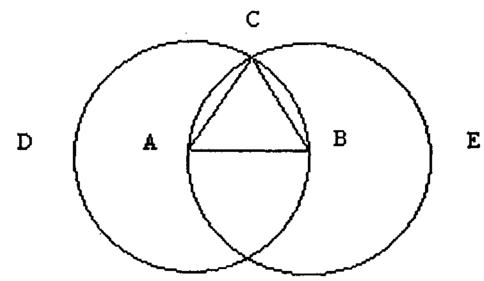


Figure 6: Euclid's Proof I,1

As we have seen, the proof of the proposition I,1 relies on two different types of information. First, by constructing the diagram, the geometer sees that the two circles intersect at point C and that drawing the lines AC and BC produces the triangle ABC. Second, by appeal to the definition of a circle, the geometer shows that the triangle ABC is equilateral. The first type of information is knowledge about the visual features of the diagram, and consists of observing an incidence between circles and recognizing a figure, the triangle ABC. Now consider the badly drawn diagram for proposition I,1 in Figure 7.

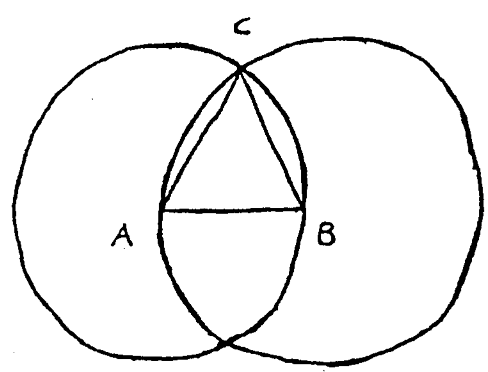


Figure 7: Euclid's Proof I,1

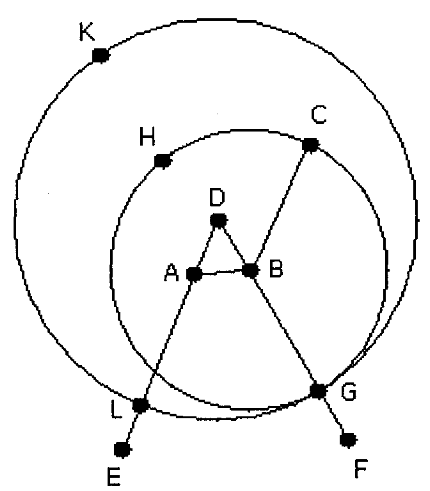
The first thing to point out is that the circles and the lines are not drawn with a compass and straightedge, and that they are crooked. Nevertheless, since Euclidean geometry contains only two types of lines, we can clearly identify straight lines and circles. Moreover, this diagram preserves the incidence of the two circles at point C and shows that the lines AB, BC, CA form a triangle. Thus, it suffices for the proof. Yet, in contrast to the correctly drawn diagram, the badly drawn diagram distorts the metrical relations. Whereas in the former the lines AB, BC, and CB are equal, at least within the limits permitted by our drawing instruments, in the latter this is not the case. I think that this example shows that the visual features of the diagram relevant for Euclid's proofs are not metrical properties, but rather incidence relations and configurative features. Moreover, as we can see by considering other examples, a feasible diagram must preserve order relations of points on lines and part-whole relations. I want to call these relations and features together "topological features of a diagram," since they are all associated with the situation of certain visual objects. I do not use the term 'topological' here in the usual sense of designating properties that are preserved under continuous transformations.[217] Accordingly, the fact that we can conduct a proof on a badly drawn diagram shows that the visual properties of the diagram relevant to a Euclidean proof are topological features in my sense. Due to the fact that our drawing instruments are necessarily inexact, all diagrams are actually badly drawn in this way. Nevertheless, if the geometer can identify the topological features of the diagram unambiguously, the visual properties of the geometric object will be uniquely determined. A diagram that

[217] Mark Greaves argues that diagrams are relevant to Euclid's proofs because they track certain properties of the actual ideal geometric subject matter veridically. He therefore distinguishes between two different types of geometric properties: (a) properties that are correctly represented by the diagram and (b) properties that are not correctly represented by the diagram. He concludes: "From a modern point of view, diagrammatic representations of properties of the first category tend to be linked to the ways that the graphical properties of the diagram reflect the overall topology of the represented domain, and importantly, are relatively stable with respect to many kinds of minor perturbations and variations in the drawn figures. On the other hand, representations of second-category properties tend to be more closely associated with the ways in which geometric diagrams or drawings require graphical exactness in the construction of the objects to which they referred." Mark T. Greaves, *The Philosophical Status of Diagrams* (Doctoral Dissertation: Stanford University, 1997). Although it should be clear that I disagree with the notion of an ideal geometric reality that is different from the diagrams, I believe that the distinction between two essentially different types of geometric properties is correct. Yet the basis for this distinction is the fact that only the topological features can serve as a visual source of geometric knowledge.

does not allow the geometer to identify the topological features unambiguously is not unacceptable. I will call the idealized, qualitatively determined visual geometric object the "topological grid."

I will now show, by means of an example, how the metrical properties are ascribed to the topological grid, and thus, how the metrical geometric object is generated from the idealized visual topological object. I will consider the proof of proposition I,2, which is not a theorem, but a problem:

[*protasis*] To place at a given point (as an extremity) a straight line equal to a given straight line.
[*ekthesis*] Let A be the given point, and BC the given straight line.
[*diorismos*] Thus it is required to place at the point A (as an extremity) a straight line equal to the given straight line BC.
[*kataskeue*] From the point A to the point B let the straight line be joined;
and on it let the equilateral triangle DAB be constructed.
Let the straight lines AE, BF be produced in a straight line with DA, DB;
with centre B at distance BC let the circle CGH be described;
and again, with centre D and distance DG let the circle GKL be described.



[*apodeixis*] Then, since the point B is the centre of the circle CGH, BC is equal to BG.
Again since the point D is the centre of the circle GKL, DL is equal to DG.
And in these DA is equal to DB;

therefore the remainder AL is equal to the remainder BG.

But BC was also proved equal to BG;

therefore each of the straight lines AL, BC is equal to BG.

And things which are equal to the same thing are also equal to one another;

therefore AL is equal to BC.

[*sumperasma*] Therefore at a given point A the straight line AL is placed equal to the given straight line BC. (Being) what it was required to do.

The properties that enter into the *apodeixis* are the equalities of line segments BC and BG, DG and DL, and DB and AD, which are ascribed as follows. Proposition I,1 allows the geometer to construct the equilateral triangle DAB. If we look back at the proof of proposition I,1, we can see that it is ultimately the definition of a circle that guarantees the equality of line segments AB, AD, and DB. The same holds for the equalities of line segments BC and BG, DG and DL, and DB and AD. Given that the geometric object has these metrical properties, Euclid then establishes the equality of line segments BC and AL as follows. Proposition I,1 shows that DA is equal to DB. Appeal to Common Notion 1 ("Things which are equal to the same thing are also equal to one another.") and Common Notion 3 ("If equals be subtracted from equals, the remainders are equal."), then allows him to draw his conclusion. Thus, we can generalize by saying that the metrical properties are ascribed to the topological grid by appeal to four different non-visual sources of geometric knowledge: (a) the definitions and, in particular, that of a circle; (b) previously established metrical results; (c) the common notions; (d) the constructive procedures. This ascription is a stipulative procedure that does not require measurements. We can conclude that the fully qualified geometric object is determined through disambiguated qualitative visual properties (topological properties of the diagram) and metrical properties (ascribed to it on the basis of argument). I will call this the "hybrid conception of the geometric object."

On the basis of the hybrid conception of the geometric object, we can now explain how it is possible that the latter is uniquely determined and can serve as source of logical knowledge. First, only visual topological properties of a diagram are relevant to a proof. These are readily recognizable (i.e., the diagrams do not have to be measured) and are uniquely determined (in all those diagrams that are adequate to a proof). The idealizations effected by the definitions do not introduce approximative procedures. Thus, they leave the topological properties unaltered. The metrical properties relevant to the proof, on the other hand, are neither determined by the visual properties of the diagram nor through measurement. Rather, they are ascribed to the diagram by the geometer through argument. Consequently, for the fully qualified geometric object to be uniquely determined it suffices that the topological properties are unambiguously given.

We can now say precisely how the concept of a non-formal logical inference applies to Euclid's method. In order to do so, I will consider Euclid's proof of proposition I,32. I will first explicate the topological features from which the inferences start and then say something about the rules of inference themselves. Consider this proof:

[*protasis*] In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.

[*ekthesis*]

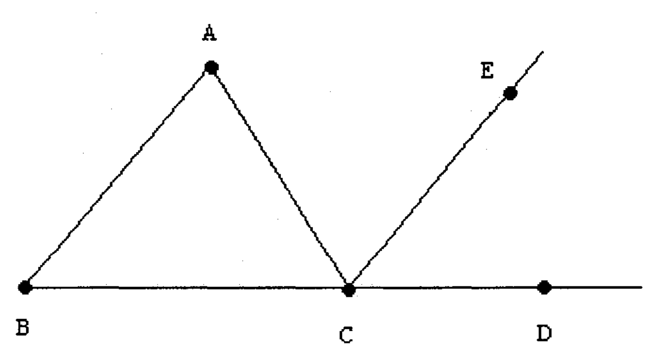
Let ABC be a triangle, and let one side of it BC be produced to D.

[*diorismos*]

I say that the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC, and the three interior angles of the triangle ABC, BCA, CAB are equal to two right angles.

[*kataskeue*]

For let CE be drawn through the point C parallel to the straight line AB.



[*apodeixis*]

(i) Then, since AB is parallel to CE (1), and AC has fallen upon them (2), the alternate angles BAC, ACE are equal to one another (3).

(ii) Again, since AB is parallel to CE (1), and the straight line BD has fallen upon them (2), the exterior angle ECD is equal to the interior and opposite angle ABC (3).

(iii) But the angle ACE was proved equal to the angle BAC (1); therefore the whole angle ACD is equal to the two interior and opposite angles BAC, ABC (2).

(iv) Let the angle ACB be added to each (1); therefore the angles ACD, ACB are equal to the three angles ABC, BCA, CAB (2).

(v) But the angles ACD, ACB are equal to two right angles (1); therefore the angles ABC, BCA, CAB are also equal to two right angles (2).

[*Sumperasma*]

Therefore etc.

Q.E.D.

We can explicate the topological, or configurative, properties that function as sources for Euclid's inferences by considering the steps (i) to (v) of the *apodeixis* separately.

Step (i):

(1) We know that AB is parallel to CE, simply because CE was constructed in this way. The possibility of such a construction was shown in proposition I, 31.[218]

(2) Inspection of the diagram shows that AC cuts AB and CE. A line cutting two other lines is a readily recognizable topological feature of the diagram.

(3) On the basis of this information, we can apply proposition I, 29 and conclude that angle BAC equals angle ACE.[219]

Step (ii):

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[218] Proposition I, 31 shows how "to draw a straight line parallel to a given straight line" through a given point. Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements*, p. 315.

[219] Proposition I, 29 states: "A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles." Ibid., p. 311.

(1) We know that AB is parallel to CE, simply because CE was constructed in this way. The possibility of such a construction was shown in proposition I, 31.

(2) The inspection of the diagram shows that BD cuts AB and CE. A line cutting two other lines is a simple topological feature.

(3) On the basis of this information, we can apply proposition I, 29 and conclude that angle ECD equals angle ABC.

Step (iii):

(1) Simply repeats the result of step (i).

(2) Given this and the result of step (ii), we can conclude that angle ACD is equal to the two angles BAC, ABC. Yet, this conclusion also includes knowledge derived through the inspection of the diagram, namely that angle ACD is identical with the two angles ACE and ECD. Again, this is a simple observation of topological features of the diagram.

Step (iv):

(1) We add the same angle ACB to the two angles ABC, BAC and also to angle ACD.

(2) According to Common Notion 2 and the result of step (iii), we can conclude that the angles ABC, BAC, ACB are equal to angles ACB, ACD.

Step (v):

(1) Inspecting the diagram shows that CA is a straight line set up on a straight line BD. Thus, we can apply proposition I,13 to angles ACD and ACB and conclude that they are equal to two right angles.[220]

(2) Given this and the result of step (iv)(2), we can conclude that angles ABC, BAC, ACB are equal to two right angles. Inspection of the diagram shows that these are the three interior angles of the triangle ABC. Thus, we reach the final result.

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[220] Proposition I, 13 states: "If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles." Ibid., p. 275.

The visual, topological, or configurative, features of the diagrams relevant for the inferences are given in steps (i)(2), (ii)(2), (iii)(2), (v)(1), and (v)(2). An example of such a feature is the condition for applying proposition I,29 in step (i)(2). The geometer has to recognize the configuration given in Figure 8 below and the position of angles BAC and ACE in it. Only if the geometer recognizes this configuration, can he/she apply proposition I,29 and conclude the equality of angles BAC and ACE. Accordingly, step (i)(3) contains an inference from the observation of certain *qualitative* properties of the diagram (a straight line cutting two parallel lines and the position of certain angles) to a *quantitative* property of the diagram (equality of angles BAC and ACE).

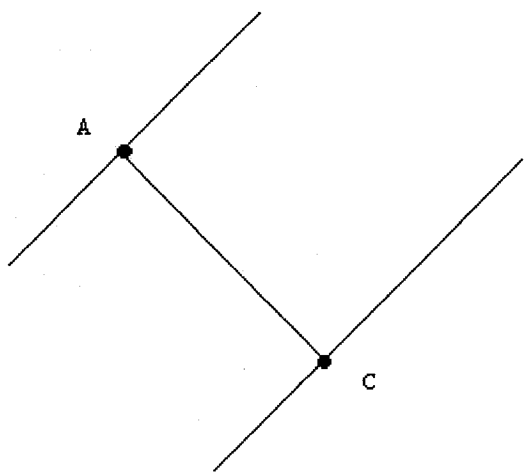

Figure 8: Configuration Involving Parallel Lines

A second example is the condition for applying proposition I,13 in step (v)(1). Here the geometer has to recognize the configuration in Figure 9 and the position of angles BCA and ACD.

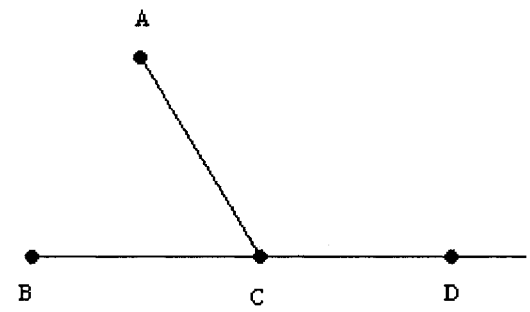

Figure 9: Configuration: Angle

Proposition I,13, then allows him/her to conclude that BCA and ACD together are equal to two right angles. Again, the geometer goes from the observation of a qualitative feature of the diagram to a quantitative result. I believe that we can say in general that Euclid's geometric practice differs from axiomatic geometry understood in the contemporary sense, because his proofs contain logical inferences leading from qualitative to quantitative properties of a given diagram. In the proof of proposition I,32, these inferences are based on previously derived results. We can say that in general the latter represent the rules for the former in Euclid's geometry. This raises the question of how this process got started in the first place. Are there any rules that legitimize Euclid's first inferences from qualitative to quantitative properties? I believe that Euclid's Common Notions or Axioms constitute such rules.

Before discussing the role of Euclid's axioms in my interpretation, however, I want to emphasize that I am not making a claim about Euclid's *complete* proofs, but rather only about certain partial inferences contained in them. We have already seen that many of his proofs are so-called problems that fulfill a constructive task and are not to be understood as inferences at all. Yet, even the proofs that are problems may contain an *apodeixis*, which, in turn, may contain logical inferences leading from qualitative to quantitative properties. Further, many of Euclid's propositions lead from *quantitative* assumptions to quantitative results, and yet their proofs contain *apodeixis* with inferences from qualitative to quantitative features.

I believe that Euclid stated the most basic inference rules that would enable a geometer to infer quantitative from qualitative properties in his Common Notions or Axioms. The *Elements* contain the following five axioms:

(1) Things which are equal to the same thing are also equal to one another.
(2) If equals be added to equals, the wholes are equal.
(3) If equals be subtracted from equals, the remainders are equal.
(4) Things which coincide with one another are equal to one another.
(5) The whole is greater than the part.

The fourth and fifth axioms lead directly from observable qualitative features to quantitative properties of the geometric object. The fourth axiom departs from the coincidence of two visual objects in order to allow us to conclude that they are equal. The fifth axiom leads from the observation that something is a part of something else to the conclusion that the former is smaller than the latter. Here, the whole-part-relation must be understood in a spatial sense; otherwise the conclusion would simply be false. For example, the number one is part of the whole number series; yet it does not make sense to say that the former is smaller than the latter. Axioms (4) and (5) thus state certain facts about the structure of visual space. The first three axioms differ from the fourth and the fifth, since they do not lead directly from qualitative to quantitative properties. Rather, they do so only in combination with the fourth axiom, which establishes the link between the notions of coincidence and equality. If we apply the fourth axiom to the first, we can state the latter's spatial content as follows: "Things (i.e., extensions, areas, or volumes) which can be made to coincide with the same thing can be made to coincide with each other." The notion of making two figures coincide here refers to the possibility of imagining the coincidence of the spatial forms of the two objects on the basis of our knowledge of the structure of visual space. This notion does not refer to the practical possibility of moving one figure to the place of the other.

Euclid's Axioms have been interpreted as general, self-evidently true principles that are common to every deductive science, in agreement with Aristotle's use of this term.[221] As emphasized before, I am not interested in the interpretative tradition, but rather in the actual function of the various textual elements within the proofs. I believe the Axioms are indeed self-evident principles, and that if Euclid conceived of them this way, then he was right. And yet, they are not universal principles common to every deductive science, because they express genuine truths about the structure of perceptual space. It is a truth of our perception of space that spatial wholes are greater than their parts, for example. This, as I indicated before, is not the case with non-spatial part-whole relations.

This interpretation also allows us to understand the very important role of Euclid's fourth Postulate, which states: "That all right angles are equal to one another."[222] Heath points out that this postulate not only determines a standard by means of which other angles may be measured, but also functions "as the equivalent to the principle of *invariability of figures* or its equivalent, the *homogeneity of space*."[223] In other words, the possibility of comparing non-contiguous right angles guarantees the homogeneity of space. Accordingly, this postulate clearly plays the role of an Axiom, expressing an obvious truth about the structure of visual space. We can best appreciate the importance of Euclid's fourth Postulate for a geometric concept of space by considering it in the context of my criticism of the standard view's problem with the concept of physical space. I have argued that a geometric concept of space was necessary to fill the gap between the measurements of contiguous physical objects and the global structure of space. I suggested that this gap be filled via a geometric concept of space. But this can only happen if the latter does not itself contain such a gap. Postulate four implies a global structure of space and thus constitutes a geometric concept of space that fill this gap in Euclid's geometry.

On the basis of this analysis, we can now say that Euclid's logical proofs are

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221 Heath writes for example: "On the whole I think it is from Aristotle that we get the best idea of what Euclid understood by a postulate and an axiom or common notion. Thus Aristotle's account of an axiom as a principle common to all sciences, which is self-evident, but incapable of proof, agrees sufficiently with the contents of Euclid's *common notions* as reduced to five in the most recent text." Ibid., p. 124.

222 Ibid., p. 154.

223 Ibid., p. 200.

grounded in visual inferences from uniquely determined visual objects. But does this analysis suffice to distinguish the knowledge derived by means of Euclid's axioms from empirical knowledge? In other words, are Euclid's axioms not just empirical generalizations of observable facts? This depends, first, on the process by which we come to know the structure of visual space, and second, on whether this knowledge expresses a contingent structure of visual space. It seems obvious that we do not find it necessary to make repeated observations or experiments about the behaviour of spatial objects. Rather, we simply seem to know the structural properties of visual space that are relevant to the axioms. In Part III, I will suggest that this knowledge is given through spatial intuition, i.e., through a system of expectations that constitutes visual space. I will also argue that Euclid's axioms express a necessary structure of visual space, which must be homogeneous, if it is to allow for the construction of a geometric concept of space.

The structure of the text of the *Elements* strongly supports my interpretation of a proof as a process making explicit information implicitly contained in idealized visual objects. First, the suggestion avoids the contradictions that arise from Greenberg's understanding of a proof as a formal logical derivation from a set of axioms. The presence of Euclid's first seven definitions is explained, if we understand them as rules for idealization. Second, the distinctions between problems and theorems and between construction and *apodeixis* is justified as reflecting the necessity to first produce a correct diagram. Third, according to my interpretation, the first three postulates have to be formulated as giving criteria for construction. Finally, the hybrid conception of the geometric object explains how we can construct proofs on the basis of badly or falsely drawn diagrams.

I have argued that the definitions play an important role in the constitution of the geometric object. Geometers like Greenberg who interpret Euclid's definitions according to our contemporary understanding of these concepts find them lacking mathematical rigour and condemn them as bad definitions. Reviel Netz also recognizes this lack of mathematical rigour, but draws a different conclusion from this. He believes that Euclid's definitions do not belong to the actual mathematical text, but rather form a second-order discourse. He takes this to be supported by the fact that Euclid's definitions were not numbered as in Heiberg's edition, but preceded the actual text as a kind of preamble. As such they seem to belong into a tradition which is most prominent in Archimedes' texts, that is, the tradition to preface the text with a non-mathematical introduction.[224] My interpretation disputes both these views and takes a third route. I believe that Euclid's specific approach to geometry, that is, the close connection to the visual features of a given diagram does not allow us to draw sharply the distinction between first-order and second-order mathematical discourse. Reading the diagram correctly is equally as important to the geometrical inference as correct logical reasoning and correct application of the common notions.[225] Thus, even if Euclid had not included definitions in his text, the reader would have to be able to grasp the rules for the correct idealization of a given diagram somehow, maybe through the oral teachings of other geometers, in order to follow and accept the proofs. Explanation of primitive terms and their role in the proofs are not to be divorced from each other.

Given my suggestion that a proof in the *Elements* is a logical derivation from an idealized visual object, it is obvious that the latter are necessary conditions for this practice. I will now argue that they are also sufficient in the sense that the geometer does not have to appeal to an independent geometric reality in order to conduct the *apodeixis*. I will refute two types of argument. The first requires the postulation of an ideal geometric objectivity that is independent of the diagrams, and the second demands independent concrete objects.

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[224] Reviel Netz, *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History*, pp. 91-101.

[225] Ian Mueller also argues that the definitions do not belong to the second-order mathematical discourse, but play an actual role in the deductions by making clear the nature of the idealized objects to be reasoned about. He writes: "Euclid's definitions have frequently been belittled by modern commentators on the *Elements* who look at them from the perspective of the modern axiomatic method. Of course, the definitions could never figure in a formal derivation, but that is just one more reason for denying that Euclid's proofs are formal derivations. The definitions should be looked at as attempts to make clear the meaning of the terms to be used before argumentation begins, that is, to make clear the nature of the objects to be studied. That the most fundamental definitions (e.g. of point, line, straight line) succeed only with persons who already have some idea of what the objects in question are does not really matter if these definitions are taken to represent preliminary agreements among people of presumably normal intelligence. But to say this is not to say that the terms are really taken as primitive because the understanding of the nature of the objects plays a role in Euclid's proofs." Ian Mueller, "Euclid's *Elements* and the Axiomatic Method," p. 294.

The postulation of an ideal reality seems to be necessary for two reasons. First, no matter how precise diagrams are, they are empirical objects. This seems to imply that the knowledge derived from the diagrams can only be true with a greater or smaller degree of probability. In contrast, the statements of geometry seem to be known with certainty. I have already given an explanation of how the geometric objects are determined uniquely and have thus challenged this argument. But, we can further show that the argument itself is incoherent. Second, according to the above argument, the criteria for correct construction and idealization of diagrams are conveyed to the reader of the *Elements* by the postulates and definitions. It seems that in order to be able to apply the notion of correctness to these criteria, one has to presuppose an independent geometric reality according to which the criteria can be evaluated. Thus, whether the diagrams form the sole subject matter of Euclidean geometry depends on the validity of these two arguments.

Both arguments are incoherent, however. The first argument fails because it is unable to explain how reference to an ideal objectivity is supposed to account for the certainty with which geometric statements are known. The main premise of such an argument is that geometric knowledge is certain *because* it is true of an ideal objectivity. The diagrams serve only to access this ideal reality. Yet, since the diagrams are empirical objects, they cannot establish the certainty of claims about a non-empirical reality. Thus, if the diagrams themselves do not allow the geometer to establish certainty, then the appeal to an ideal objectivity cannot do so either. The second argument for the necessity of an ideal geometric reality fails because in order to derive criteria that assure the correctness of the idealization, the geometer would have to have independent access to the ideal realm of geometry. Only the postulation of a mysterious cognitive faculty would thus allow a geometer to evaluate the criteria for correct idealization.

The appeal of the two arguments given above for the postulation of an ideal geometric reality could also be weakened by giving an alternative explanation of the genesis of the concept of idealization. But giving a full-fledged explanation is difficult, and I do not claim to be able to give such. Yet, I believe that the key to it is the theoretical interest. If geometry is to be a science that yields intersubjectively communicable and univocal results, then the criteria must successfully disambiguate the process of interpretation. This, of course, does not explain why the disambiguation led to the criteria stated in Euclid's definitions, that is, to the criteria for Euclidean geometry; it could theoretically have led to other definitions and thus to different geometries altogether. Presumably, the criteria stated in Euclid's definitions are the natural choice for humans, given the nature of their perceptual apparatus and their pre-scientific spatial experiences. This explanation of the genesis of the criteria for correct interpretation is admittedly vague. Yet, in contrast to the postulation of an ideal geometric reality, it allows us to define a field of research, namely an inquiry into the cognitive history of the process of the disambiguation of the interpretation of geometric diagrams.

The simplicity of this view gives it an explanatory advantage vis-à-vis the suggestion that the diagrams represent certain relevant features of ideal geometric objects. This advantage can best be appreciated if we consider how the latter view deals with a simple geometric object, let us say a triangle. According to this view, the triangle as a physical is an inexact copy of the ideal geometric reality. According to my suggestion, presented in this chapter, the triangle is an instantiation of the concept 'triangle'. Whereas the former view takes the triangle to derive the properties that are relevant for the proof from the idealization, I hold that they derive from the concept of a triangle plus the specific geometric interest specified in the definitions and postulates. In contrast to the former, the latter process is much less mysterious. In order to explain the ability to do Euclidean geometry, we do not have to appeal to a faculty that allows direct access to the laws governing an ideal geometric reality, such as Husserl's *Wesensschau.* All that is required is that the geometer has the concept of a triangle and is familiar with the definitions and postulates. The former is used in everyday discourse and can thus be acquired before he/she starts doing geometry and the latter is conveyed by the text of the *Elements* itself.

The fact that the practice of Euclidean geometry does not require the postulation of an ideal geometric reality does not prove that the diagrams are the actual objects of Euclidean geometry. Even though no separate ideal geometric reality is required it may be possible that geometry entails a distinction between the visible diagrams and an intended non-ideal geometric reality. Such a distinction seems to be necessary for *apagogic* proofs like the proof of proposition I,6. In this proof, the triangle ABC, which is assumed to have two equal base angles (ABC and ACB), yet unequal sides (AB and AC), cannot exist. Nevertheless, the proof itself refers to it and thus seems to presuppose an intended object that differs from the diagram itself and its features: the visible triangle ABC seems to symbolize, but not to represent, this intended object. This conclusion goes too far, however. All that is necessary for the reader to follow the proof is that he/she *thinks* of the geometric object as having side AB unequal to AC. This is possible because only the topological properties (in the broad sense defined above) of the diagram serve as a visual source of geometric knowledge. The metrical properties can thus be superimposed on the topological grid, even though the latter actually exhibits a different visual metric. Thus the representation of the counterfactual situation, i.e., the assumption that AB is unequal to AC, in this *apagogic* proofs does not require the reader of the *Elements* to intend an object that is different from the actual diagram.

5. 4 The Generality of the Results of Euclid's Method

The previous arguments show that, as far as the *apodeixis* is concerned, Euclid's proofs do not require appeal to a reality that differs ontologically from the individual visual object. In other words, the diagrams are ontologically sufficient conditions for the *apodeixis*. For this reason, I called the diagrams the primary subject matter of material geometry. Yet, Euclid's propositions claim to be true of geometric objects in general. Thus, we have to explain how the generality of his proofs is established. A geometric axiomatic system such as Hilbert's allows a geometer to conduct a proof with perfect generality. In order for such a proof to be general, it has to fulfill two conditions. First, the assumptions of the proof have to be formulated in a general way. For example, the proof of proposition I,32 has to begin with the assumption: Let ABC be *any* triangle. Second, the proof itself must appeal only to information specified in the axioms. Since the axioms characterize the system of objects about which the axiomatic theory speaks, the inferences will be true of all the objects mentioned in the proposition. We can make this clearer if we consider the formalization of Hilbert's axiomatic system. In such a formalized theory, we would formulate the above assumption as $\exists x\,(Tx)$ ['Tx' - 'x is a triangle']. Then, using the information given in the axioms we derive $\forall\alpha\beta\gamma\,(Ix\alpha \,\&\, Ix\beta \,\&\, Ix\gamma \rightarrow \alpha + \beta + \gamma = 180°)$ ['Ixy' - 'y is an interior angle of x']. Thus, by conditionalization, we get the conditional $\exists x\,(Tx) \rightarrow \forall\alpha\beta\gamma\,(Ix\alpha \,\&\, Ix\beta \,\&\, Ix\gamma \rightarrow \alpha + \beta + \gamma = 180°)$. Finally, since we used only information given in the axioms we can generalize: $\forall x\,(Tx \rightarrow \forall\alpha\beta\gamma\,(Ix\alpha \,\&\, Ix\beta \,\&\, Ix\gamma \rightarrow \alpha + \beta + \gamma = 180°))$.[226] Now, as we have seen, Euclid's proofs start from individual visual objects. For example, proof I, 32 starts from the particular triangle ABC. Euclid can thus not establish the generality through conditionalization and universal generalization. Thus, we have to ask how a Euclidean proof establishes the generality of its results.

Mueller conjectures that the Greeks themselves never solved the problem of generality satisfactorily. He writes:

> It is natural to ask about the legitimacy of such a proof. How can one move from an argument based upon a particular example to a general conclusion, from an argument about the straight line AB to a conclusion about any straight line? I do not believe that the Greeks ever answered this question satisfactorily, but I suspect that the threefold repetition of what is to be proved reflects a sense of the complexity of the question.[227]

This absolves my interpretation of Euclidean geometry from the task of explaining the universal validity of its propositions. Trying to explain the general validity of geometric propositions would be an attempt to explain a property of material geometry which it does not possess. Nevertheless, Netz has argued that Euclidean geometry is based on a different notion of generality, a notion which does not require appeal to a geometric reality that is given prior to the constructions. In order to see how Euclid's proofs secure the generality of his propositions, I want to outline the main points of Netz's argument.

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[226] For an account of the generality of Hilbert's proofs see Ian Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*, pp. 12-13.

[227] Ibid., p. 13.

Netz points out that the first interesting lesson about generality can be drawn from proofs like that of proposition III,1, which constructs the centre of a circle. In order to show how this can be done, Euclid gives the appropriate construction and then proves that the centre cannot be any other point than the one given by this procedure. After showing this for a given point F, Euclid concludes: "Similarly we can prove that neither is any other point except F." Accordingly, Euclid secures the validity of this proposition by taking some other point within the circle and showing that it cannot be the centre. He does this by saying that the same argument can be given for any other point within the circle. This is somewhat curious, because Euclid appeals here to a potentially infinite number of proofs. Netz concludes from this that the generality relevant to the *Elements* is likely to be generality with respect to provability, rather than with respect to the geometric results: A proof is universally valid, if it can be given for any adequate diagram. Generality thus is not truth about a domain of objects, but repeatability of proofs.

Netz supports this hypothesis by considering the structure and formulation of the proofs in the *Elements*. They consist of an enunciation or general statement that makes a conditional claim whose antecedent describes a constructed geometric situation and whose consequent states something that follows from this situation. Netz represents this general conditional as $C(x) \rightarrow P(x)$. This is followed by the setting-out, which states a particular situation, symbolized as $C(a)$. Given this particular situation, Euclid states the definition-of-goal (*diorismos*): $P(a)$. The construction then extends the given situation $C(a)$ in such a way that the *apodeixis* can derive the particular conclusion $P(a)$. Finally, the conclusion repeats the enunciation $C(x) \rightarrow P(x)$. Given this representation of the structural elements of a Euclidean proof, Netz states his theory of generality as follows: Construction and *apodeixis* prove a particular case, namely $P(a)$. As such they prove the definition-of-goal (*diorismos*), rather than the conclusion (*sumperasma*).[228] Now,

[228] Netz also believes the word 'for,' which connects construction and *apodeixis* to the definition of goal, indicates the fact that the former two are intended as proving the latter. Moreover, the same connector often introduces the setting-out, thus indicating that the conclusion is not supported by the *apodeixis*, but rather by the entire sequence from setting out to *apodeixis*. Finally, the 'I say that' preceding the definition of goal can be understood as an affirmation of the provability of the enunciation. Reviel Netz, *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History*, p. 255.

setting-out (*ekthesis*) and definition of goal (*diorismos*) together are able to support the general claim $C(x) \rightarrow P(x)$. This follows from the fact that construction and *apodeixis* do not only show that $P(a)$ follows from $C(a)$, but also the provability of $P(a)$ from $C(a)$.[229] In other words, by giving a proof for a particular geometric object, it is shown that "the same proof must be repeatable for any other object as long as the same *ekthesis* applies to that object."[230] This repeatability then shows the generality of the claim $C(x) \rightarrow P(x)$.

We can conclude from this argument that the generality relevant to Euclidean proofs is not generality with respect to the results of geometric reasoning, but rather generality with respect to repeatability of a proof for an alternative diagram. This type of generality does not require a pre-existing geometric reality which renders the results universally valid. It suffices that one can show that an adequate proof for each object given by the same setting-out *could* be produced. I will devote much of the next part of this dissertation to the question of how a geometer can establish the generality of his/her results. In order to do so, I will investigate how space is perceived and on the basis of this formulate a notion of spatial intuition.

My interpretation of Euclidean geometry and its ontological presuppositions now allows me to distinguish it from formal and applied geometry and to establish it as a third type of geometry that is based on spatial intuition. The criticism of Husserl's and Carnap's

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[229] Robert Tragesser has pointed out that in many cases showing the provability of a claim already amounts to a proof. Thus, even in the case of infinitely many proofs as in proposition III,1, Euclid is not faced with the task of actually providing the proofs. Rather he must give a sample that exhibits the general procedure that can be applied to alternative diagrams. See Robert S. Tragesser, "Three Insufficiently Attended to Aspects of most Mathematical Proofs: Phenomenological Studies," in Michael Detlefsen, ed., *Proof, Logic, and Formalization* (New York: Routledge, 1992), pp. 162-198. Michael Hallett pointed out to me that Euclid's approach has an analogue in modern formal logic. The rule of Universal Generalization on a Constant allows one to recast a deduction of a formula containing an individual constant to a deduction in which all occurrences of the individual constant are replaced by a suitable variable x, given that a is not mentioned in the premises. Machover formulates this rule as follows: "If $\mathbf{\Phi} \vdash \boldsymbol{\alpha}\ (\mathbf{x/c})$, where c is a constant that occurs neither in $\mathbf{\Phi}$ nor in α, then also $\mathbf{\Phi} \vdash \forall \mathbf{x}\ \boldsymbol{\alpha}$." See, Moshé Machover, *Set Theory, Logic and their Limitations* (Cambridge: Cambridge University Press, 1996), p, 182. To show that Euclid proofs have general validity, we need an equivalent to the Universal Generalization rule. I will show in the third part of this thesis that this equivalent is not a formal principle of Euclid's proofs, but rather spatial intuition itself.

[230] Reviel Netz, *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History*, p. 256.

attempts to secure a material geometry showed that they failed because they considered the axioms as the only legitimate source of geometric knowledge. The fact that the primary subject matter of material geometry, which they understood as an idealized intuitive space, had no influence on the truth of geometric propositions led to its collapse into formal or applied geometry. In contrast, according to my analysis, a proof in the *Elements* is partly based on visual information provided by the diagrams. As a result, subject matter of Euclidean geometry and source of geometric knowledge partly coincide, thus preventing the collapse of Euclidean into formal geometry. Further, Euclid's practice is grounded in a specific type of non-formal logical inference leading from qualitative visual features to quantitative features of the geometric objects. This prevents the collapse of material into applied geometry. Material geometry is not a deficient form of axiomatic geometry according to our contemporary understanding. Rather, Euclid's geometry is an essentially different activity based on the extraction of spatial information from visual objects.

The view that Euclidean geometry is an altogether different activity than formal and applied geometry whose subject matter are idealized diagrams brings us one step closer to clarifying the relationship between geometry and spatial perception. The diagrams that form the subject matter of geometry are spatial objects. Due to this property, they allow the geometer to explicate one part or aspect of the complex structure of his/her everyday spatial experience. The diagrams are the means for *tapping* the space perceived by human beings. In this process they function in a particular way. They represent the structure of the spatial intuition not in a symbolic way, but rather embody it. The cognitive value of these diagrams is a result of the fact that they represent this structure in a situation that is simplified vis-à-vis the complex everyday perception. I will, therefore, say that perceptual space is the secondary subject matter of Euclidean geometry

Let me summarize. In this Part of my thesis, I argued that the phenomenologists' attempts to define the subject matter of Euclidean geometry as an idealized intuitive space are inconsistent. In contrast to their views, I argued that the actual practice of geometry as exemplified by Euclid's *Elements* is based on the extraction of visual information from diagrams as idealized visual objects *via* a specific type of non-formal logical inference. This prevents its collapse into both formal and applied geometry. I further argued that, as far as the practice of geometry is concerned, the diagrams should be considered as the actual subject matter of Euclidean geometry. The phenomenologists' attempts failed, because they share the positivists' prejudice about the character of Euclidean geometry. Both believed that Euclidean geometry was to be understood in terms of an axiomatic system in the contemporary sense. Further, the phenomenologists share the positivists' view about legitimate deductive inferences. For both logic is a purely formal enterprise.

Part III: Material Geometry and Spatial Perception

6. Inferential Theories of Space Perception: Berkeley, Helmholtz, Rock, and Gibson

In Part II, I argued that the primary subject matter of material geometry is not an idealized intuitive space, but rather idealized diagrams, that is, physical and visual objects considered in a certain way. In particular, I have argued that certain qualitative features of these objects represent the immediate visual source of geometric knowledge. I also argued that the generality of Euclid's method is the repeatability of a given proof for all other diagrams falling under its *ekthesis*. In order to complete my account of a material geometry, that is, of Euclid's geometric practice, I will therefore now formulate an adequate concept of spatial intuition that explains how the generality of Euclid's proofs is established.

In order to formulate a concept of spatial intuition, I will investigate how visual space is experienced *phenomenally* in everyday perception.[231] In this section, I will first consider a number of inferential theories that *explicitly* aim at a phenomenal analysis of visual space, namely the theories of George Berkeley, Hermann von Helmholtz, Irvin Rock, and James Gibson.[232] I will argue that these theories cannot describe the phenomenal characteristics of perceptual space, because they all understand perception as an inferential process. This discussion will allow me to draw the more general conclusion that no inferential theory of perception can provide an analysis of the phenomenal qualities of visual space. The reason for this is that any such theory is faced with the impossible task of reconstructing the phenomenal structure of space, that is, a

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[231] I use the term 'phenomenal' here, and in the remainder of this thesis, as referring to the specific quality of a given experience. In other words, 'phenomenal' describes what a given experience is like for the experiencing subject. I am not implying any similarity between my account and Phenomenalism.

[232] The fact that I am counting Gibson's theory of space-perception among the inferential theories requires justification, because he believes that his theory is a direct theory of perception. I will return to this issue when I consider his theory in detail.

structure described from a first-person perspective, from a stimulus that is characterized by means of a particular special science, that is, from a third-person perspective. I conclude that only a direct theory of perception can adequately describe the phenomenal structure of visual space. In order to develop a direct approach to perceptual experience, I then turn to Husserl's phenomenological theory and his account of visual space. After outlining Husserl's theory, I will critique it and modify it. This will provide us with a concept of spatial intuition, which explains how the generality of Euclid's results is established.

Currently, inferential theories of the perception of visual space are widely accepted. They share three fundamental assumptions: (i) the structure of objective space is, at least for all practical purposes, three-dimensional and Euclidean; (ii) the information entering the visual system is impoverished and does not represent the complete structure of objective space; and (iii) we perceive space (roughly) as a Euclidean structure.[233] Given these assumptions, perception has to be understood as a process that reconstructs the structure of objective space from impoverished input-information. A theory of space perception would consequently have to give an account of how the visual system accomplishes this task. Some inferentialists claim that their theories explain how space is actually experienced. In other words, they believe that their theories provide a phenomenal analysis. Most prominent among these inferentialists were Berkeley, Hermann von Helmholtz, Irvin Rock, and James Gibson. I will therefore first consider these four inferentialists and ask whether their phenomenal analyses actually succeed. Since these author's accounts differ with respect to the descriptions of both the nature of the information that enters into the visual system and the actual process that recovers the structure of objective space, I will consider them separately. We will see that each theory has its own problems that prevent it from giving an adequate analysis of perceptual space.

In his *An Essay Towards a New Theory of Vision*, Berkeley specifies the central

[233] Behaviourists, for example, deny the third point. See Irvin Rock, *An Introduction to Perception* (New York: McMillan Publishing Co., Inc., 1975), p. 10.

question of a theory of space-perception as follows: "My design is to show the manner wherein we perceive by sight, the distance, magnitude, and situation of objects."[234] This goal needs to be motivated, however, since, at the outset, it is not clear why the 'manner wherein we perceive by sight, the distance, magnitude, and situation of objects' requires any explanation whatsoever. Do we not simply *see*, more or less accurately, how far away an object is, how large it is, and how it is oriented? Berkeley, therefore, provides a reason for each of the three explanatory tasks. His famous one-point-argument motivates his inquiry into the perception of distance. He writes:

> It is, I think, agreed by all, that distance of itself, and immediately, cannot be seen. For distance being a line directed end-wise to the eye, it projects only one point in the fund of the eye. Which point remains invariably the same, whether the distance be longer or shorter.[235]

Berkeley's one-point-argument is formulated in a vocabulary that derives directly from optical writers such as Kepler whom he subsequently criticizes. A line is understood here as the line of sight that projects a given point of the observed object on to the observer's retina (the 'fund of the eye'). Since the point projected by the light-ray that travels on this line of sight does not change when the object moves towards or away from the observer (on this line), the length of this line cannot be determined by vision. In more contemporary terms, we can say that Berkeley believes the one-point-argument to show that the proximal stimulus (the retinal image) does not contain information about the distance between observer and object -- it contains impoverished spatial information. He concludes from this that the idea of distance does not derive from the sense of sight. His theory of vision, therefore, has to explain how it is possible that, despite this optical fact, human beings cannot only orient themselves in three-dimensional space, but also do so by seemingly forming a visual idea of distance.[236]

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[234] George Berkeley, *An Essay Towards a New Theory of Vision* [1709] (London: J. M. Dent & Sons, Ltd, 1910), p. 13.

[235] Ibid.

[236] I agree with D. M. Armstrong who argued that it is a consequence of Berkeley's one-point argument that a two-dimensional field is originally given in vision. Thus, a theory of vision has to explain how this

In order to motivate his task of explaining the perception of an object's magnitude, Berkeley presents another argument from optics. In this context, he distinguishes between two types of object -- one perceived by sight and the other by touch. Both objects have their own type of magnitude or extension, depending on how many minimal visual or tactile points the perceived object occupies.[237] He then writes:

> The magnitude of the object which exists without the mind [i.e., the tactile object], and is at a distance, continues always invariably the same: but the visible object still changing as you approach to, or recede from the tangible object, it hath no fixed and determinate greatness.[238]

The claim that the visual object has no fixed size sounds like a phenomenal report on how we see rigid objects at different distances. However, due to the size-constancy, this cannot be what Berkeley has in mind here. As a matter of fact, under normal circumstances, we see a rigid object at different distances as being of the same size.[239] Thus, Berkeley must be appealing here to the fact that the retinal image becomes smaller when the object moves further away and larger when it approaches, a fact that he mentioned only a few paragraphs before. His argument then has to be formulated in the following way: We cannot see what we generally believe to see, namely, the constant size of objects at a greater or smaller distance. Rather, since, as geometry shows, the proximal stimulus changes its size, our visual idea of the extension of an object must also change. Given this, Berkeley's theory of perception has to explain how humans form the idea of the fixed extension of rigid objects and how they can assess extension correctly.

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field is transformed into an experience of three-dimensional space. D. M. Armstrong, *Berkeley's Theory of Vision: A Critical Examination of Bishop Berkeley's* "Essay towards a New Theory of Vision" (Melbourne: Melbourne University Press, 1960), pp. 2-9. However, this interpretation of Berkeley has also been challenged. See, for example, Robert Schwartz, *Vision: Variations on Some Berkleian Themes* (Oxford UK: Blackwell, 1994), p. 28.

[237] Ibid., p. 36.

[238] Ibid., p. 36f.

[239] For example, when I put my hand close to my eyes and then stretch my arm slowly, my hand does not appear to be shrinking, although its retinal image does. Analogous constancies apply to the perception of colours and shapes. For a description of the various perceptual constancies, see, for example, Maurice

Berkeley motivates the third task of his theory of space-perception, that of explaining how the 'situation' of an object, i.e., its orientation, is perceived, as follows:

> There is, at this day, no one ignorant, that the pictures of external objects are painted on the retina, or fund of the eye. That we can see nothing that is not so painted: and that, according as the picture is more distinct or confused, so also is the perception we have of the object: but then in this explication of vision, there occurs one mighty difficulty. The objects are painted in an inverted order on the bottom of the eye: the upper part of any object being painted on the lower part of the eye, and the lower part of the object on the upper part of the eye. Since therefore the pictures are thus inverted, it is demanded how it comes to pass, that we see the objects erect and in their natural position.[240]

Berkeley claims that human beings can see only those features of objects that are projected onto the retina. The second sentence in the above quote even seems to imply a one-to-one correspondence between the properties of the retinal image and the visual ideas that enter the mind. The problem resulting from this view is to explain how humans come to see objects the right-side up, despite the fact that the retinal image is inverted.

Berkeley answers these three questions by stipulating a unity between the visual and the tactile sense. He claims that the ideas of distance, extension, and orientation relevant to spatial-perception are not ideas of sight, but rather ideas of touch. As we have seen, according to Berkeley, we can *assess* distance, extension, and orientation correctly by sight, but not actually *see* them. He suggests that we must make such an assessment on the basis of a coordination between the ideas of the tactile and the visual senses. Berkeley argues extensively that there is no necessary connection between the ideas of the two senses and, therefore, concludes that our ability to assess distance, extension, and orientation is the result of a process of association. Repeated experience of the

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Hershenson, *Visual Space Perception: A Primer* (Cambridge, Mass.: The MIT Press, 1999), pp. 114 -118.
[240] Ibid., p. 54.

correlation between certain ideas of one sense with certain ideas of the other sense creates a union between them, which, in the case of distance, Berkeley describes as follows:

> In these and the like instances, the truth of the matter stands thus: having a long time experienced certain ideas, perceivable by touch, as distance, tangible figure, and solidity, to have been connected with certain ideas of sight, I do upon perceiving these ideas of sight, forthwith conclude what tangible ideas are, by wonted ordinary course of nature, like to follow. Looking at an object, I perceive a certain visible figure and colour, with some degree of faintness and other circumstances, which from what I have formerly observed, determine me to think, that if I advance forward so many paces or miles, I shall be affected with such and such ideas of touch.[241]

The union between the different types of ideas consists of an anticipation of certain tactile qualities prompted by certain visual qualities. Once presented with certain ideas of sight, we cannot but form the correlative ideas of touch.

As a result of his investigation of distance, Berkeley concludes that the union between visual and tactile ideas forged by experience is so strong that one is easily confused about the phenomenal content of one's own visual experience -- it *seems* that one can indeed *see* spatial qualities directly:

> The secondary objects, or those which are only suggested by sight [i.e., the tactile ideas], do often more strongly affect us, and are more regarded than the proper objects of that sense, along with which they enter into the mind, and with which they have a far more strict connection, than ideas have with words. Hence it is, we find it so difficult to discriminate between the immediate and mediate objects of sight, and are so prone to attribute to the former, what belongs only to the latter.[242]

The appearance that distance, extension, and orientation are ideas of the visual sense is a 'prejudice,' however, resulting from the fact that the union between the two types of ideas possesses a new phenomenal quality. The ideas are so closely linked that we are prone to misidentify their sources. Nevertheless, Berkeley believes that the original ideas

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[241] Ibid., p. 32.

[242] Ibid., p. 35.

do not change their phenomenal qualities. These qualities are specific to the respective senses and remain implicitly contained in the unified experience. As such they are recoverable if one keeps in mind that the term 'seeing' in its literal meaning designates only ideas given immediately through the sense of sight.[243] Thus, when we speak of 'seeing distance,' we use the phrase only in a metaphorical sense, applying it to the new phenomenal unity. In effect, Berkeley's argument purports to show that there is no such thing as a visual space, i.e., a space that can be experienced visually. We do not really *see* space. Rather, it only appears to us as if we were doing so. In fact, we only see signs for spatial qualities, and vision does not grant us a direct access to the phenomenal features of space.[244] According to Berkeley, it would therefore be a vain undertaking to describe the structure of visual space.

Denying the possibility of describing the phenomenal features of visual space is fatal for the material geometry I tried to establish in the previous chapter, because it prevents the diagrams as visual objects from serving as a sources of geometric knowledge. According to Berkeley, spatial ideas belong to the tactile sense. He attempts to do justice to this fact by defining geometric objects not as "abstract" or "visible extension[s]," but rather as "tangible figures" suggested by visual figures.[245] Following Berkeley, the diagrams would be only visible signs for actual, i.e., tactile, geometric

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[243] Cf., George Berkeley, *An Essay Towards a New Theory of Vision*, p. 34.; I thus disagree with Gary Hatfield's description of the phenomenal union of visual and tactile experience: "The immediate object of vision gives to the complex idea its visual character -- the phenomenal qualities of light and color; the tactile ideas, which do not preserve their phenomenally tactile character, serve to give the visual array its phenomenal three-dimensionality. Just as two substances may combine chemically to form some third substance that differs in quality from the other two, our ideas may become conjoined through association in such a way that the product of the association does not preserve the phenomenal character of the ingredients." Gary Hatfield, *The Natural and the Normative: Theories of Spatial Perception from Kant to Helmholtz* (Cambridge Mass.: The MIT Press, 1990), p. 43.

[244] Lorne Falkenstein sums up Berkeley's view in the following way:
"Localization on the Depth Axis
- we learn by experience to associate certain qualities of visual sensations with the distance over which we must reach to touch an object
- these qualities become signs that suggest the tangible distance and we come to infer the signification so rapidly from the sign that we seem to see depth."
Lorne Falkenstein "Reid's Account of Localization," *Philosophy and Phenomenological Research* 61 (2000): 305-328, p. 327

[245] George Berkeley, *An Essay Towards a New Theory of Vision*, p. 86 and p. 83.

objects. As I have argued in the previous chapter, the proofs of material geometry proceed from qualitative visual properties of the diagrams. Accordingly, on Berkeley's theory, the seen diagrams would have to have a structure that is isomorphic to that of the tactile figures. We can also say that the diagrams would have to function as iconic signs of tactile figures that veridically represent the latter's spatial properties. Yet, a large part of Berkeley's argument is dedicated to denying just this point. He argues, for example, that the visual extension of an object changes as it approaches an observer or moves away from him/her. A material geometry as a science grounded in the structure of visual space is thus not possible, if we accept Berkeley's conclusion.

Berkeley's rejection of a visual space is a result of his specific optico-psychological analysis of the visual system. His psychology treats the visual and tactile sensual systems as sources of information that work independently. This puts certain constraints on the possible information that can be conveyed from them to the mind. In particular, Berkeley argues that the visual sense is constrained by the laws of optics and can thus receive only information that is already contained in the retinal image. The same holds for the phenomenal qualities of the ideas produced by this sense. Berkeley thus implicitly defines the literal sense of the term 'to see' by appeal to the visual sense's optico-psychological properties. In other words, Berkeley makes the non-metaphorical sense of the term 'to see' dependent on a description of the input-information, i.e., the information contained in the proximal stimulus, in a vocabulary belonging to a particular special science. But if the literal meaning of the term 'to see' is defined by the optico-psychological properties of the visual sense, and if these properties do not account for the perception of three-dimensional space, then we cannot *see* space. This argumentative strategy will recur in different ways in the theories of Helmholtz, Rock, and Gibson. At the end of this section. I will show that this strategy is based on an illegitimate inference from a description from a third-person perspective to a description from a first-person perspective and *vice versa*.

In the second half of the nineteenth century, Hermann von Helmholtz radicalized Berkeley's empiricist theory of space-perception. In contrast to Berkeley, who thought that the proximal stimulus contained spatially impoverished information, Helmholtz argued that what was originally received by the mind had *no* spatial qualities whatsoever. We can best understand Helmholtz's argument by first considering how he defined the different aspects of a theory of vision. His main treatise on this subject, the *Handbuch der physiologischen Optik*, is divided into three parts, each dealing with a different aspect of vision.[246] The first part is concerned with the optical properties of the eye; the second with the properties of the nervous system conveying the sensations from the eye to the brain; and the final part with the way in which the brain combines the sensations into a representation of the world. In other words, the different parts take different approaches to the phenomenon of vision -- optical, physiological, and psychological --, thus illuminating different aspects of the phenomenon of vision. In agreement with Berkeley, Helmholtz believes that, due to its optical properties, the eye can receive only a two-dimensional retinal image, and can thus convey no information about the third dimension. Further, according to Helmholtz, new discoveries in physiology show that the signals transmitted by the nervous system to the brain eliminate any spatial properties from the information given by the eye. In particular, physiological results show that each light-sensitive cell on the retina produces an electro-motorical impulse that is transmitted to the brain by means of a separate nervous fibre. Each of these signals has the same quality and does not carry any spatial information. But, if the signals transmitted to the brain do not contain information as to their origin on the retina, they do not represent the spatial properties of the proximal stimulus. In separate essay on the nature of sensations, Helmholtz concludes:

> Sensations of light and colour are only symbols for real relations; the former are similar, or, related, to the latter no less and no more than a person's name, or its written representation, to the person him or herself.[247]

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[246] Hermann von Helmholtz, *Handbuch der physiologischen Optik*, vol. 1-3 [1867] (Hamburg und Leipzig: Leopold Voss, 1910). I am quoting from the English translation Hermann von Helmholtz, *Helmholtz's Treatise on Physiological Optics*, translated from the third edition by James P. C. Southhall (Rochester, N.Y.: The Optical Society of America, 1924-1925).

[247] ["Licht- und Farbenmpfindungen sind nur Symbole für Verhältnisse der Wirklichkeit; sie haben mit den letzteren ebensowenig und ebensoviel Aehnlichkeit oder Beziehung als der Name eines Menschen, oder der Schriftzug für den Namen mit dem Menschen selbst."], "Hermann von Helmholtz, "Über die Natur der menschlichen Sinnesempfindungen," Hermann von Helmholtz, *Wissenschaftliche Abhandlungen* vol. 2 (Leipzig: Barth, 1882), pp. 591-609, p. 608, (translation my own).

Nevertheless, Helmholtz believes that the visual system represents spatial relations in the actual world veridically, in spite of the fact that the information transmitted to the brain is non-spatial. He writes:

> Only the relations of time, of space, of equality, and those which are derived from them, of number, size, regularity, of coexistence and of sequence - 'mathematical relations' in short - are common to our outer and inner world, and here we may indeed look for a complete correspondence between our conceptions and the objects which excite them.[248]

Thus, in accordance with his more radical point of departure, the problem of perception for Helmholtz is to explain how an accurate representation of space is formed on the basis of non-spatial sensations.

By allowing genuine spatial sensations of touch, Berkeley could account for the perception of the third dimension by associating visual experiences with genuinely spatial experiences of the tactile sense. This way was not open to Helmholtz, since his physiological argument against the spatial character of visual sensations also applied to tactile sensations. Helmholtz therefore suggested that the representation of space was formed through the experience of a regular or law-like coordination between certain changes in the sensations and the position of the eyes of an observer. According to his physiological argument, the nervous impulses emanating from the light-receptors do not carry any information as to the location of their source on the retina. Nevertheless, Helmholtz believed that colour-sensations produced through light falling on one part of the retina were distinguishable from those produced by light falling on another part. Using a term introduced by Hermann Lotze, Helmholtz called the sensations produced by retinal cells 'local signs.'[249] In order to represent spatial relations, these sensations

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[248] Hermann von Helmholtz, "The Recent Progress of the Theory of Vision," in *Scientific Subjects*, trans. by E. Atkinson (London: Longmanns, Green, and Co., 1873), pp. 197-316, p. 316.

[249] Helmholtz remarks: "Lotze has named this difference between the sensations which the same colour excites when it affects different parts of the retina, the local sign of the sensation. We are for the present ignorant of the nature of this difference, but I adopt the name given by Lotze as a convenient expression. While it would be premature to form any further hypothesis as to the nature of these 'local signs,' there can

must be assigned to a particular position in the visual field -- a task that Helmholtz described as follows:

> Hence, when we find that a plane optical image of the objects in the field of vision is produced on the retina, and that the different parts of this image excite different fibres of the optical nerve, this is not sufficient ground for our referring the sensations thus produced to locally distinct regions in our field of vision. Something else must clearly be added to produce the notion of separation in space.[250]

He then suggested that the local signs are assigned to different positions in the visual field through experience, through actively experimenting with rigid objects:

> A movement of the eye which causes the retinal image to shift its place upon the retina always produces the same series of changes as often as it is repeated, whatever objects the visual field may contain. The effect is that the impressions which had before the local signs a_0, a_1, a_2, a_3, receive the new local signs b_0, b_1, b_2, b_3; and this may always occur in the same way, whatever be the quality of the impressions. By this means we learn to recognize the changes as belonging to the special phenomena which we call changes in space.[251]

Accordingly, experience teaches an observer how to interpret the local sign as representations of spatial relations.[252] More specifically, experience allows an observer to discover a law-like relation between changes of the positions of the eyes and the local signs, and thus to assign the latter to a particular position in the visual field.[253] Thus, for

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be no doubt of their existence, for it follows from the fact that we are able to distinguish local differences in the field of vision." Ibid., p. 275. Since the local sign has to be phenomenally accessible in some way, one must ask here what type of experience this is. Helmholtz's remarks so far indicate that it can not be an experience of space or colour. This leaves only the experience of the intensity of the local signs. However, it seems wrong to assume that differences in intensity of a sensation could correspond to changes in an observer's position in the way required by Helmholtz's theory.

[250] Ibid., p. 273.

[251] Ibid., p. 305f.

[252] Helmholtz writes: "[I]t is merely the qualities of the sensation that are to be considered as real, pure sensation; the great majority of space-apperceptions, however, being the product of experience and training." Hermann von Helmholtz, *Helmholtz's Treatise on Physiological Optics*, vol. 3, p. 13.

[253] Helmholtz believes that the observer becomes aware of the changes in the position of the eyes through certain acts or efforts of volition (*Willensanstrengungen*), namely the efforts of volition that it takes to bring about specific changes. With this thesis, he distances himself from the idea that the differences in the

Helmholtz, spatial perception is a result of an association between two types of change. He also claims that the association is the result of an unconscious inference.[254]

Like Berkeley, Helmholtz believes that his theory not only explains the ability of human beings to orient themselves in their Euclidean environments, but also why our representation of space is phenomenally spatial, i.e., why it *shows* objects arranged in three-dimensional space. Yet, in fact, this cannot be explained by his theory. If the original sensations that enter the mind are phenomenally non-spatial, they cannot be assigned a spatial position by means of experience. To do so would presuppose a phenomenal representation of space into which they could be projected. Helmholtz is concerned with the original constitution of spatiality, however, and thus does not have an already spatial world available to him. One way out of this would be to say that the local signs exhibit spatial phenomenal qualities. But this would contradict Helmholtz's own characterization of them and of his problem of space-perception in general. Another way out would be that suggested by Berkeley, who, as we have seen, claimed that sensations of touch are spatial in a phenomenal sense and could somehow transfer their phenomenal qualities to the visual sense. Again this would contradict Helmholtz's characterization of sensations. In effect, Helmholtz is unable to explain the phenomenal features of our representation of space. Yet, even more, he precludes the representation of space from exhibiting spatial phenomenal qualities. Consequently, like Berkeley's theory of perception, Helmholtz's cannot explain how material geometry is possible. The reasons for this failure are similar to Berkeley's. Helmholtz believes that what is given immediately to the mind through the senses must be described in terms of the vocabulary

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positional changes of the eyes are perceived through muscle sensations. We will encounter the latter view later on in the analysis of Husserl's theory of perception. Cf. Hermann von Helmholtz, "The Recent Progress of the Theory of Vision," p. 305.

254 Helmholtz writes: "Now we have exactly the same case in our sense-perceptions. When those nervous mechanisms whose terminals lie on the right-hand portions of the retinas of the two eyes have been stimulated, our usual experience, repeated a million times all through life, has been that a luminous object was over there in front of us on our left. We had to lift the hand toward the left to hide the light or to grasp the luminous object; or we had to move towards the left to get closer to it. Thus while in these cases no particular conscious conclusion may be present, yet the essential and original office of such a conclusion has been performed, and the result of it has been attained; simply, of course, by the unconscious processes of association of ideas going on in the dark background of our memory. Thus too its results are urged on our consciousness, so to speak, as if an external power had constrained us, over which our will has no control." Hermann von Helmholtz, *Helmholtz's Treatise on Physiological Optics*, vol. 3, p. 26f.

of a particular special science, which, in his case, is ultimately physiology.[255]

Irvin Rock's more recent theory of perception follows Berkeley and Helmholtz in both the assumption that perception is an inferential process and the claim that such a theory has to explain the phenomenal aspect of perception. Rock expresses his allegiance to the inferential approach in the following passage:

> My view follows Helmholtz's (1867) that perceptual processing is guided by the effort or search to interpret the proximal stimulus, i.e., the stimulus impinging on the sense organ, in terms of what object or event in the world it represents, what others have referred to as the "effort after Meaning."[256]

In his earlier book *An Introduction to Perception*, Rock states his interest in the phenomenal aspects of perception very clearly when he writes: "We wish to explain why things look as they do. This means that the starting point, the facts to be accounted for, are the various aspects of the phenomenal world."[257] Like Berkeley and Helmholtz, Rock believes that we can explain our ability to orient ourselves in the space of our environment by analyzing how the phenomenal features of representation of space arise. For Rock, such a analysis is a psychological question based on, but not reducible to, optical and physiological facts about the perceptual system.

Although Helmholtz and Rock agree that perception is a result of inferential processes, they disagree about both the nature of the inferences and the type of information that is originally given to the senses. Helmholtz, as we have seen, admitted

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[255] In this context, it is interesting to point out a contradiction in Helmholtz's writings. He describes visual sensations in two different ways. First, as we have seen, Helmholtz states that visual sensations are non-spatial local signs. Second, Helmholtz also claims that visual sensations are what is contained in the retinal image. He asserts, for example, that appropriate training and certain purified conditions of observation sometimes allow us to become aware of the simple sensations which experience has 'hidden' in a complex spatial perception. So, for example, the training of a draftsman allows him/her to see how objects are actually first given to us. Instead of seeing the opening of a glass as round, even when looked at from an angle, the draftsman has to focus his/her attention on the sensation and will then see it as an ellipse. But notice that Helmholtz here describes the sensation phenomenally as spatial; one can simply not conceive of an ellipse as a non-spatial object.

[256] Irvin Rock, *The Logic of Perception* (Cambridge Mass.: The MIT Press, 1983), p. 17.

[257] Irvin Rock, *An Introduction to Perception*, p. 21.

only one kind of inference, namely, unconscious association of ideas of the tactile and visual senses. Rock, in contrast, believes that perception is ultimately a result of a number of different mechanical and intelligent processes. These include *ad hoc* organizational decisions as well as inferences in a more robust sense such as solutions to certain perceptual problems and deductions from hypotheses.[258] Rock's reception of Gestalt theory and his own experimental investigations into perceptual processes lead him to establish this more sophisticated view.

Rock's understanding of what is originally given to the mind as the point of departure for inferences differs from Helmholtz's in two respects. First, we have seen that Helmholtz's physiological argument lead him to the assumption that the immediate sensations transmitted to the brain have no spatial content. As becomes clear from the above quote, Rock, in contrast, does not appeal to such arguments and is closer to Berkeley in assuming that what is immediately given to the mind closely resembles the proximal stimulus, i.e., the retinal image.[259] Rock implicitly justifies his decision to disregard the neurological level by saying that he is concerned with the cognitive aspects of perception. The neurological level, however, belongs to a different explanatory scheme and should not be confused with the cognitive level.[260] Second, Helmholtz believed that inferences departed from perceptual ideas. Taking into account certain experimental results, Rock thinks that the inferences are not based on sensations alone, but, as we will see, also involve a separate class of mental events, so-called "descriptions."

Given these differences, Rock describes the constitution of three-dimensional visual space as the result of a process containing two different types of cognitive constructive processes, both of which can be understood as inferences in a broad sense of this term. The first type of constructive process is more mechanical and plays a large role in the perception of oriented shapes in a two-dimensional visual field. Its inferences are

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[258] Irvin Rock, *The Logic of Perception*, p. 19.

[259] Cf., Ibid., p. 93.

[260] Ibid., p. 40f.

not deductions from rules or hypotheses. The second type is not mechanical and requires inferences from rules as well as the confirmation of hypotheses and plays a large role in interpreting these two-dimensional shapes as aspects of three-dimensional objects. These two types of processes are not to be understood as separate procedures that lead to two different results. Rather, they cooperate in constituting the final representation of three-dimensional space. Nevertheless, for the sake of his analysis, Rock distinguishes between these two types of processes and treats them separately.

In accordance with his distinction between the two types of constructive procedures, Rock first explains how we grasp shapes phenomenally.[261] He claims that we grasp shapes phenomenally through a certain type of mental event, namely, a non-conscious, non-verbal description, which is given by an executive agency outside the sensory domain. Rock believes that if we did not have these descriptions, we would have to assume, *pace* Berkeley, that perceptual inferences departs from a percept, i.e., a non-conceptual representation, that is structurally identical to the proximal stimulus. This assumption would contradict a number of perceptual phenomena, however. By considering three examples, I will illustrate both the necessity of introducing descriptions and the way in which they explain the phenomenal properties of perceived shapes. The first example is the fact that human beings can identify forms even if no extended, persistent retinal image is present. Take the case in which an observer follows with his/her eyes a moving point that describes a shape. The retinal image will be a point, continuously remaining at the same location. Nevertheless, as experiments show, the observer is able to describe the trajectory of the point, thus describing a shape to which no percept corresponds. Second, a phenomenon that cannot be explained on the assumption that a perceptual inference departs from a percept that is structurally identical to the proximal stimulus is the fact that the recognition of shapes depends on the assignment of directions to the image. For example, certain geographical shapes (like the outline of the African continent, which are easily recognized if the pointy part is viewed as bottom and the flatter part as top), become virtually non-identifiable if these directions

[261] Rock writes: "The problem of form perception is to explain why figures look the way they do and consequently appear to be similar to or different from other figures." Ibid., p. 43.

are changed. Differently orientated representations of the same shape *look* very different. If the point of departure of the perceptual inference was the percept corresponding to the proximal stimulus, the assigned directions should not play any role in the phenomenal appearance of the shape. A further example is given by ambiguous figures such as Jastrow's duck-rabbit. Again, if the phenomenal features of the representation was determined merely by the proximal stimulus, there would be no difference between the two perceived figures. By introducing descriptions, Rock is able to explain these three phenomena. First, we perceive shapes when no extended spatial proximal stimulus is present, because our eye-movements lead to the formation of a description. Such a description represents the object as continuous. Second, the changing appearance of the shape under different directional assignments is explained by saying that the description includes a specification of the relationship between the internal geometry of the figures and the system of spatial reference imposed on it. Finally, the two different appearances of the duck-rabbit is explained by saying that the figure appears differently under different aspectual descriptions.

In order to characterize the nature of these descriptions more explicitly, Rock distinguishes them from picture-like percepts. In contrast to the latter, descriptions are abstract propositions with predicative structure. They ascribe perceptually relevant properties to shapes by means of a pre-linguistic language, a language of thought, using concepts like 'parallelism,' 'convergence,' and 'straightness.'[262] Descriptions account for the phenomenal features of shapes by representing certain properties of the proximal stimulus while simultaneously disregarding others. Accordingly, we can describe the perceptual process leading to the constitution of the representation of shape as beginning with automated processes of selecting which objects are to be included into the field of vision and how they are to be organized. At this level, the visual system makes decisions as to the internal geometry of the given shapes. Rock repeatedly claims that descriptions at this level resemble very much the proximal stimulus, but are not identical with it.

Rock believes that these lower-level descriptions, which he also calls

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[262] Ibid., p. 51.

'descriptions in the proximal mode,' do no yet represent objective properties of the world and thus have to be complemented by higher-level descriptions. He claims that lower-level descriptions are implicitly present in perception and can be made explicit by shifting one's attention. For example, an observer who perceives as round the opening of a cup seen from an angle is conscious of the fact that this perception also includes seeing it as an oval. But this shows that these descriptions are subjective and do not yet represent objective features of our environment. In order to explain how spatial objects and objective space are constituted, Rock therefore introduces descriptions producing the various constancies of shape, size, and orientation, i.e., descriptions in the 'world mode.' This, according to Rock, is accomplished by higher-order inferences that interpret the descriptions in the proximal mode as subjective aspects of an objective world. Visual space is thus given to us in a representation that includes the presence of the proximal stimulus, a set of descriptions that describe the stimulus almost as it is, and a set of descriptions that interpret the properties described by the lower-level descriptions as subjective aspects of spatial objects.

Rock starts from the same assumption as Berkeley, namely that the original information entering the visual system is spatially impoverished. The descriptions at the different levels have to reconstruct a three-dimensional objective world from this impoverished information. Since the descriptions are abstract propositions, the visual process is simply an inference from propositions of one type to propositions of another type. This does not explain how the qualities of the stimulus are enriched in such a way as to represent a *phenomenally* fully qualified visual space. Thus, although Rock does not deny that we can *see* space in the literal sense, he nevertheless, does not succeed in explaining how this is possible. This is because, like Berkeley, Rock describes that which is originally given in perception in optical terms.

According to Rock, visual space is represented through higher-order properties that are present neither in the proximal stimulus nor in the first level description. Rather, it is given only in a propositional form expressed in the language of thought. As Rock states, the propositions are abstract, not concrete. According to this view, a material geometry in Euclid's sense is a method that makes explicit through diagrams spatial knowledge couched in the language of thought. Yet, Rock does not explain how this works, or how space is experienced phenomenally. He thus cannot explain how perceiving the diagrams allows the geometer to explicate the given knowledge in the language of thought.

James Gibson proposes what he calls a direct theory of perception, which challenges the views of Berkeley, Helmholtz, and Rock. Gibson agrees with these authors that a theory of perception has to explain "why things look as they do"[263] But Gibson criticizes Berkeley, Helmholtz, and Rock for attempting to reconstruct the phenomenal constitution of space from a meaningless stimulus. He argues that the concept of a stimulus, which is central to these approaches, is actually misapplied. A stimulus, according to Gibson, designates a form of energy acting upon some receptor. As such it cannot carry any psychological meaning. More precisely, the term 'stimulus' is physiological, rather than psychological. As a result, any theory that uses this concept in order to designate the information that is originally given to the visual system will contain an infinite regress. Depending on whether the theory is closer to Berkeley's and Rock's or to Helmholtz's, this regress can occur in two forms. First, any theory that starts from the assumption that the original information is given in pictorial form through the retinal image (Berkeley and Rock), requires a *homunculus* that looks at this picture and extracts spatial meaning from it. But this just pushes the problem one step back, leading to an infinite regress. Any theory that assumes that the original information is coded in the electro-chemical nerve impulses transmitted to the brain (Helmholtz on my interpretation), requires a *homunculus* who decodes this signal. Again, an infinite regress threatens.[264] Gibson, therefore, suggests that a theory of perception dismiss the notion of a stimulus as a mediator between the world and its representation and consider vision as a direct process. This implies that an organism first perceives not a meaningless stimulus, an energy pattern, but rather a meaningful world.[265] The task of a theory of

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[263] James, J. Gibson, *The Ecological Approach to Visual Perception* (Boston Mass.: Houghton Mifflin Company, 1979), p. 1.

[264] Ibid., p. 62.

[265] I want to point out here the similarity to Heidegger's understanding of the process of seeing. Heidegger describes the seeing of a lectern in the following way: "In den Hörsaal tretend, sehe ich das

perception would then be to explain how a meaningful environment can be perceived. I will outline the basic ideas of Gibson's approach to the perception of space. But I will also bring out the inferentialist nature of Gibson's approach. We will see that the problems of his approach arise from the same source as Berkeley's, Helmholtz's, and Rock's.

In order to specify a conceptual framework within which the task of a theory of perception in Gibson's sense can be solved, he develops what he calls an "ecological approach to vision" based on the assumption that perception is always the perception of an organism living in an environment. For the organism, this environment is not a physical structure, but rather a structure that is functionally significant. In other words, the perceived environment is structured according to what it *affords* for an organism. For example, for creatures living on land, the ground affords orientation, a path affords walking, a cave affords shelter, water affords drinking, etc. The first goal of a theory of space perception is, therefore, to describe this functional structure of the spatial environment. According to Gibson, such a description is always relative to the particular physiological and psychological organization of a given organism. Subsequently, the theory of space perception has to explain how the organism can perceive these features. As we have seen, inferential theories understand this as the double-task of first specifying the spatial information available to a given perceptual system and then explaining how objective properties of space are reconstructed from this information.

Katheder . . . Was sehe ich? Braune Flächen, die sich rechtwinklig schneiden? Nein, ich sehe etwas anderes: eine Kiste, und zwar eine größere, mit einer kleineren daraufgebaut. Keineswegs, ich sehe das Katheder, an dem ich sprechen soll. Sie sehen das Katheder, von dem aus zu ihnen gesprochen wird, an dem ich schon gesprochen habe . . . . In dem Erlebnis des Kathedersehens gibt sich mir etwas aus einer unmittelbaren Umwelt . . . das Bedeutsame ist das Primäre, gibt sich mir unmittelbar . . . . In einer Umwelt lebend bedeutet es mir überall und immer, es ist alles welthaft, es weltet." ["Entering the lecture hall, I see the lectern . . . What do I see? Brown surfaces, which cut one another perpendicularly? No, I see something different: a box, namely a larger box with a smaller one built on top of it. Not at all, I see the lectern, at which I am supposed to speak. You see the lectern, from which someone is lecturing to you, at which I have already lectured . . . In the experience of seeing the lectern, something gives itself to me from an immediate environment . . . the meaningful is the primary, it gives itself to me immediately . . . Existing in an environment, meaning is given to me everywhere and always, everything is worldly, everything is being the world."] Martin Heidegger, "Zur Bestimmung der Philosophie" (1919) in *Gesamtausgabe: Ausgabe letzter Hand* 56/57 (Frankfurt am Main: Vittorio Klostermann-Verlag), pp. 117-131, (translation my own).

Gibson, in contrast, wants to explain how information can be 'picked up' from the environment directly. In order to do this, he specifies the information not in terms of traditional optics, but rather in terms of what he calls "ecological optics." Whereas traditional optics assumes that the visual information available to a given organism is specified by the light-rays that are reflected from the environment and cross in the eye, ecological optics departs from the features of the ambient optic array, i.e., the structured light surrounding a possible point of observation. According to Gibson, traditional optics describe visual information in terms of angular size projected onto the retina. Ecological optics, in contrast, describes visual information contained in the ambient optic array, that is, information in the form of solid angles that represent the surfaces of actual objects. (See, Figure 10) The structure of the array is not static, but changes if the objects change or if the observer changes the point of view. Gibson then hypothesizes that information about the environment is picked up by the visual system as *invariants* of reversible changes in the optic array. As a result, a direct theory of perception in Gibson's sense has to specify the invariants representing certain functionally significant features of the environment. Or more specifically, a theory of visual space perception has to specify those invariants that represent spatial features of the environment.

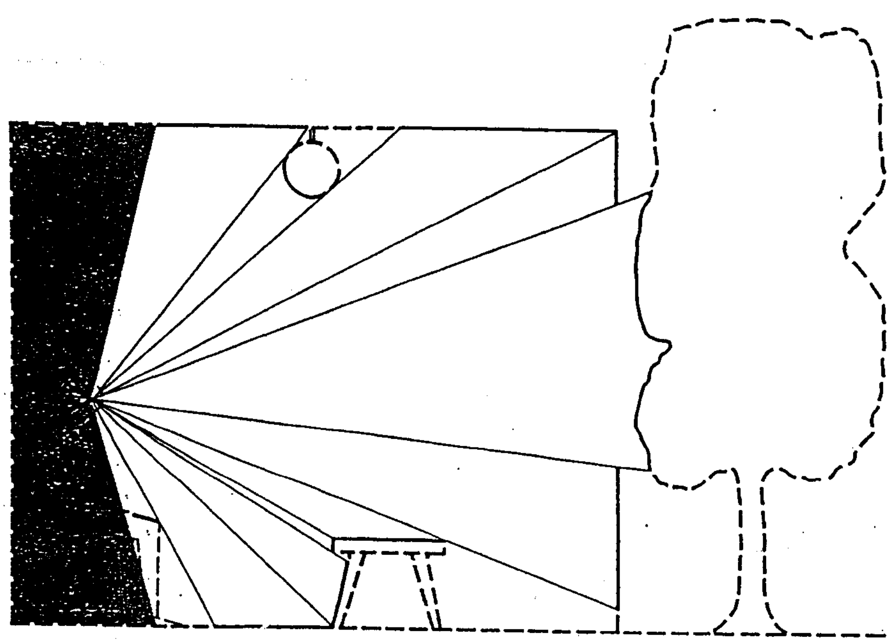

Figure 10: Ambient Optic Array

Gibson's approach to space perception leads him to the conclusion that space itself has no functional significance for humans. He writes: "Space has nothing to do with perception. Geometrical space is a pure abstraction. Outer space can be visualized but cannot be seen."[266] Yet, according to Gibson, specific features like distance, size, form, and orientation do have a functional significance. His theory must therefore explain how these features are picked up from the ambient optic array. Gibson makes the following proposal with respect to the perception of distance and size. Distance is seen in terms of the ground that stretches between a possible place of observation and an object. How is this possible? For an organism that lives on land, the ground has the significance of a surface that affords movement from one place to another. So, in this sense, the distance to an object is experienced as the path that leads to it. The length of this path can be evaluated because the ground has a texture and contains more or less evenly scattered structural elements. The size of these elements is given either through experience or through the organism's own body which is observed as part of this environment. The distance is given as texture-gradient, i.e., as the degree to which the structural units shrink with greater distance. Thus, according to Gibson, the texture-gradient is the invariant structure of the ambient optic array that allows perception of distances. (See, Figure 11)

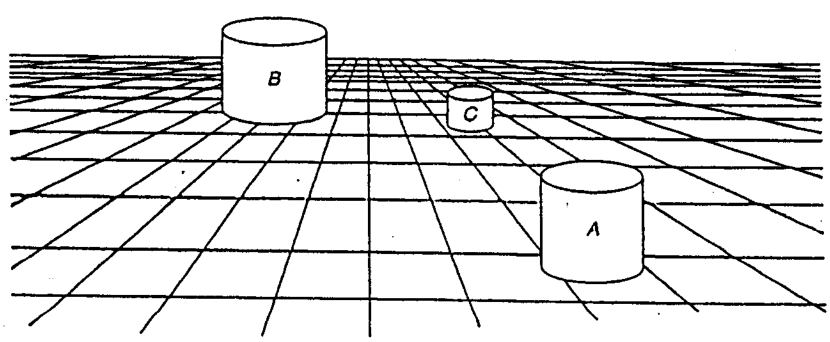

Figure 11: Texture Gradient Indicating Distance

[266] Ibid., p. 3.

Similarly, the perception of the sizes of objects is possible because they are usually arranged on the ground. Depending on their distance, they will occlude more or less structural units of the ground, thus allowing evaluation of their size. For Gibson, space is therefore experienced mainly as texture invariants of the ground or other surfaces as they are specified in the ambient optic array.

Gibson's approach has often been interpreted as fitting into the framework of inferential theories. Hershenson, for example, seems to believe that the notion of an optic ambient array simply designates an enriched proximal stimulus.[267] I believe that this is essentially correct. Although Gibson criticized the inferentialist notion of a stimulus, the ambient optic array seems to play the same role in his theory.[268] An organism sees the distribution of light in the array and simultaneously perceives certain invariants. Yet, the invariants are representations of features of the environment and not these features themselves. Thus, instead of explicating the claim that organisms perceive their environment directly, Gibson's theory actually implies that organisms perceive optical invariants which they subsequently interpret in terms of their functional significance.[269] The theory requires some sort of inference. Gibson, thus, falls victim to the same problems that he identifies in inferential theories of perception.[270] In effect, he defines

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267 Describing Gibson's theory of distance-perception, Hershenson writes: "When stimulating an eye above the ground, the respective distal points are represented by different proximal points and the spaces between them are represented by gradually decreasing proximal distances. The proximal array carries distance information." Maurice Hershenson, *Visual Space Perception: A Primer*, p. 133. This terminology seems to imply that the ambient optic array is just an enriched proximal stimulus.

268 Gunnar Johansson also points out the similarities between Gibson's approach and his own attempts to enrich the proximal stimulus by his vector analysis. Gunnar Johansson, "On Theories for Visual Space Perception. A Letter to Gibson" in *Scandinavian Journal of Psychology* 11 (1970): pp. 67-74. See also Jerry Fodor and Zenon Pylyshyn, "How Direct is Visual Perception?: Some Reflections on Gibson's 'Ecological Approach'" *Cognition* 9 (1981): pp. 139-196.

269 Some interpreters of Gibson have criticized his theory on the grounds that that the texture-gradient alone does not suffice to specify uniquely a given distance or size. But this type of criticism is not quite adequate to Gibson. Gibson's theory admits any kind of information into the theory that can be picked up by the visual system. He simply refutes the notion of an informationally impoverished stimulus. See, for example, the criticism of Robert Schwartz in *Vision: Variations on Some Berkeleian Themes*, p. 131.

270 By contrasting Gibson's notion of "affordance" with Heidegger's notion of "Bewandtnis," we can see a problem with the former that is also related to inferentialism. Gibson's term and his actual analysis of the significance of the environment for an organism betray an ecological prejudice. They should not be

the verb 'to see' in terms of perceived intensities and invariants, i.e., in terms of properties of the ambient optic array. Like the inferentialists, he describes the input to the visual system by appeal to a special science, namely ecological optics.

Gibson's appeal to the notion of the ambient optic array renders his theory incapable of accounting for the phenomenal structure of space. Spatial relations like distance and size are experienced phenomenally as certain invariants, namely as texture-gradients. Yet, like the two-dimensional retinal image or the nerve-impulses transmitted from the eyes to the brain, a texture-gradient is a non-spatial, abstract entity. Gibson's research shows at best that if the human visual system is presented with the image of a textured surface and if that surface is properly oriented within environmental space, the texture-gradient will enable the visual system to evaluate distance and size. Yet, if the surface cannot be properly oriented, i.e., if its place relative to the ground cannot be determined, the texture-gradient becomes ambiguous. It can be seen as both a surface receding into the distance and as a surface whose structural elements are of decreasing size. Thus, the visual system of an organism can interpret a texture-gradient only on the basis of a previous experience of spatial relations.

We can draw the general conclusion that no inferential theory of perception can provide an analysis of the phenomenal structure of perceptual space, if we consider the general structure of the inferentialist argument. Inferentialists believe that what is originally given to the mind, the stimulus, is informationally impoverished with respect to the structure of the three-dimensional spatial world. They argue for this by analyzing the stimulus by means of a particular special science, such as psychology, optics, or neurophysiology. Subsequently, they stipulate an inferential process that recovers the missing information. Simultaneously, inferentialists believe that the *informational* impoverishment of the stimulus leads to its *phenomenal* impoverishment. Consequently, they also assume that the same inferential process that recovers the missing information

restricted to elementary biological needs. Heidegger attempted to do justice to this insight by analyzing the human environment in terms of its significance for human existence in general. His term "Bewandtnis" is designed to capture this broad sense of meaning. See Martin Heidegger, *Sein und Zeit* [1927] (Tübingen: Max Niemeyer Verlag, 1986), in particular, § 18, pp. 83-89. Thus, Gibson formulates what is originally given to an organism in terms of a special science, here ecology.

also enriches the phenomenal properties of the stimulus to a fully qualified spatial experience. Yet, an inference from properties described in the vocabulary of a special science, i.e., from a third-person perspective, to properties described in phenomenal terms i.e., from a first-person perspective, is impossible. Otherwise we could have non-analogical knowledge of other beings' phenomenal experience. As a result, the inferentialists can simply not draw any conclusions as to the phenomenal features of the representation of space. Their approach allows them to say neither that the original stimulus is phenomenally impoverished, nor that the stipulated inferential process will enrich the phenomenal experience of space.

Let me explain with an example why an inference from properties described from a third-person perspective to properties described from a first-person perspective is impossible. As we have seen, it is an optical fact about our visual system that the image on the retina is two-dimensional. The two-dimensionality of the retinal image is a property described from a third-person perspective. Considered in itself, the image does not contain any information about the third direction. As we have also seen, the inferentialist concludes from this that the experience resulting from such an image is phenomenally impoverished, or, at least, would become impoverished without any intervening inferential processes. The experience itself would be two-dimensional. Accordingly, the inferentialist basis his/her theory on an inference from 'x is two-dimensional' to 'x causes/produces, or would cause/produce, an experience of something two-dimensional.' But this is neither a logical inference, nor an empirical inference. First, the fact that something has a certain property does not imply logically that it causes a certain experience. Second, such an inference cannot be justified empirically. There are, for example, two-dimensional objects that cause three-dimensional sensations such as stereograms. Thus, I believe that the basic assumption underlying the inferentialists' theories is mistaken.

In this section, I have argued that inferential approaches explain neither the phenomenal features of visual space, nor how material geometry is grounded upon it. I have pointed out a problem these theories have in common. They all claim to explain why we see in the way we do, i.e., why our representation of space exhibits the phenomenal features it does. But, this goal is illusory and based on the identification of two incompatible first-person and third-person perspectives. This, however, is not a reason to dismiss these theories and their results altogether. Rather, they should be understood as attempts to explain our ability to orient ourselves in our three-dimensional Euclidean environment on the basis of a representation of space. But, my argument also shows that we can analyze the phenomenal constitution of spatial experience only by means of non-inferential approach, or, in other words, by means of a direct theory of perception. I will develop such a theory in the following by criticizing Husserl's phenomenological analysis of perception.

7. Husserl's Theory of Perception and the Phenomenological Analysis of Space

In a number of works, most importantly the *Logische Untersuchungen*, *Ding und Raum. Vorlesungen 1907*, and *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie*, Husserl develops a complex theory of perception which also includes an analysis of visual space. Some interpreters have argued that this is a direct theory of perception.[271] If they were correct, Husserl would enable us to explain the possibility of geometric proofs according to Euclid's method. In this section, I will show that although Husserl's mature theory of perception fails as a direct theory, the specific problems arising from it nevertheless allow me to formulate three methodological principles fundamental to my phenomenological analysis of visual space. In section 7.1, I will consider how Husserl's theory explains two phenomenal features of perceptual experiences, namely that they present their objects in person and as identical. I will argue that both explanations involve inferences. Husserl's explanation of the fact that perceptions present their objects in person is based on an inference from subjective sense data to the intentional object. In order to avoid such an inference, I will then formulate a first methodological principle and demand that a phenomenological analysis describe a phenomenon only in object terms, i.e., that a phenomenological analysis of perception be a noematic analysis, as Husserl would say. Subsequently, I will show that Husserl's explanation of the identity of the perceptual object involves an inference from the intentional to the real object. I will then formulate a second methodological principle forbidding such an inference. In section 7.2, I will turn to Husserl's explanation of the constitution of the spatial object. After outlining his account, I will argue that it involves a further inference from a two-dimensional visual field to a three-dimensional spatial object. In order to avoid such an inference, I will formulate a third methodological

[271] See, for example, Izchak Miller, *Husserl, Perception, and Temporal Awareness* (Cambridge, Mass.: The MIT Press, 1984), John Drummond, *Husserlian Intentionality and Non-Foundational Realism* (Dordrecht/Boston/London: Kluwer Academic Publishers, 1990), and Kevin Mulligan, "Perception" in Barry Smith and David Woodruff Smith, eds., *The Cambridge Companion to Husserl* (Cambridge, Mass.: Cambridge University Press, 1995): pp. 168-238.

principle demanding that an analysis of perceptual space start from a fully qualified spatial experience. These three principles will then allow me to modify Husserl's account of spatial experience in section 8.1.

7.1 Husserl's Phenomenological Approach to Perception

Husserl introduces his analysis of perceptual experience in the context of his theory of knowledge. In order to outline Husserl's account of perceptual experience, I will therefore first consider his general approach to epistemology and his analysis of knowledge. In *Die Idee der Phänomenologie*, he sets the stage for his theory of knowledge by criticizing traditional epistemological theories, which he categorizes as Cartesian approaches to epistemology. Husserl believes that these traditional theories approach the problem of knowledge in a way that renders it unsolvable.[272] He thinks that the Cartesians are right in defining knowledge as an adequate correlation between experience and reality; yet they err in construing this correlation as one between immanent subjective experience and a transcendent objective world, and in understanding the terms 'immanent' and 'transcendent' as referring to two ontologically distinct realms. Consequently, the Cartesian epistemologists understand knowledge itself as the result of an inference that leads from one ontological realm, i.e., the realm of subjective experience, to another ontological realm, i.e., objective reality. This, however, renders knowledge impossible, because the correctness of the inference itself cannot be determined. Epistemological Cartesianism thus leads to scepticism. Husserl wants to escape from these consequences by means of a double move: he redefines the goal of a theory of knowledge and suggests a philosophical method that avoids construing the distinction between immanent subjective experience and transcendent objective reality as one between two different ontological realms.

Husserl believes that the theory of knowledge requires a phenomenological analysis, which he describes as follows:

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[272] Edmund Husserl, *Die Idee der Phänomenologie*: *Fünf Vorlesungen*, Husserliana II, ed. by Walter Biemel (The Hague: Martinus Nijhoff, 1950), given in 1905, pp. 29-39.

> What resides in the essence of experience, in its originary "sense"? . . . We do not ask how experience arises (i.e., as a totality of psychological experiences, interwoven in the real nexus of experiences and experiential dispositions of empirical persons), but what "resides" in it, what there is to draw out of it, in virtue of its essence, as absolute givenness, i.e., what it shows, purely phenomenologically, as its own proper sense and content.[273]

Phenomenology's goal is an internal analysis of the phenomenal features of experience, an analysis of the meaning of the various types of experience. The phrase 'residing in experience' here is used to indicate the fact that such an analysis is being conducted from a first-person perspective in which the observer describes his/her own experiences. A phenomenological analysis is a first-person account of phenomenal consciousness. In particular, a phenomenological analysis of knowledge has to describe the phenomenal structure of knowing-experiences, or what Husserl calls "acts of knowing."

One qualification must be made here. Husserl makes clear that consciousness in the sense relevant to phenomenology is intentional consciousness, that is, consciousness *of* something. He famously writes:

> In perception something is perceived, in imagination something imagined, in a statement something stated, in love something loved, in hate hated, in desire desired, etc.[274]

Phenomenology then has to clarify how the different kinds of objects are given in their

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273 ["Was liegt im Wesen der Erfahrung, in ihrem originären 'Sinn'?" . . . Wir fragen nicht, wie Erfahrung entsteht (nämlich als Inbegriff psychologischer Erlebnisse, eingeflochten in den realen Zusammenhang der Erlebnisse und Erlebnisdispositionen empirischer Personen), sondern was in ihr "liegt", was aus ihr als absolute Gegebenheit vermöge ihres Wesens zu entnehmen ist, was sie als ihren eigenen Gehalt und Sinn rein phänomenologisch ausweist."], Edmund Husserl, *Ding und Raum. Vorlesungen 1907. Vorlesungen 1907*, p. 141, translation from Edmund Husserl, *Thing and Space. Lectures 1907. Lectures of 1907*, translated by R. Rojcewicz (The Hague: Kluwer, 1997), p. 118, slightly modified.

274 ["In der Wahrnehmung wird etwas wahrgenommen, in der Bildvorstellung etwas bildlich vorgestellt, in der Aussage etwas ausgesagt, in der Liebe etwas geliebt, im Hasse etwas gehaßt, im Begehren etwas begehrt usw."], Edmund Husserl, *Logische Untersuchungen: Zweiter Band.* [1901] Husserliana XIX/1, ed. by Ursula Panzer (The Hague: Martinus Nijhoff, 1984), p. 366, translation from Edmund Husserl, *Logical Investigations: Second Volume*, translated by J. N. Findlay (London: Routledge & Kegan Paul, 1970), p. 554.

respective intentional acts. Husserl himself also describes this task of phenomenology as an analysis of the constitution of the object of experience. The term 'constitution' here lends itself to misinterpretation, since it seems to indicate that Husserl is thinking of something like a construction or creation of the intentional object. But constitutional analysis is really nothing other than a description of the way in which the intentional object is experienced. Husserl himself also says that it is an analysis of the way in which the object shows itself (he uses the term '*Sich-Beurkunden*').[275] Applied to acts of knowing, the constitutional analysis has to show how an object of cognition is experienced.

Given this general task, Husserl then designs a phenomenological method, the so-called "phenomenological reduction," whose primary goal is to avoid an ontological gap between the realm of subjective experience and that of objective reality. Husserl's account of the reduction takes its point of departure from what he calls the "natural attitude," that is, the attitude of everyday consciousness. As a human being, I am most of the time in this attitude. While being in it, I take the material world with its spatio-temporal order as existing. This world not only includes material objects, but also other sentient beings, some of whom are like me. I constantly experience this world with the help of my senses, although always from a given point of view.[276] The most important characteristic of the natural attitude is its naivety with respect to the ontological claims involved in it. The existence of the external material world is not questioned. Husserl calls this the *Generalthesis der natürlichen Einstellung* and formulates it as follows:

> As what confronts me, I continually find the one spatio-temporal actuality to which I belong like all other human beings who are to be found in it and who are related to it as I am. I find "actuality", the word already says it, as a factually existent

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[275] Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, p. 3. An extensive analysis of Husserl's concept of constitution along these lines can be found in Robert Sokolowski, *The Formation of Husserl's Concept of Constitution* (The Hague: Martinus Nijhoff, 1964).

[276] The world of the natural attitude is not to be understood as one world among many, but rather as the world of common sense beliefs. Barry Smith reconstructs in detail Husserl's account of common sense. See, Barry Smith, "Common Sense," in Barry Smith and David Woodruff Smith, eds., *The Cambridge Companion to Husserl*, pp. 394-437.

actuality and also accept it as it presents itself to me as factually existing.[277]

The natural attitude not only characterizes our everyday lives, but also the natural sciences, which explore the laws of the material world. The phenomenological reduction requires the philosopher to give up the naivety of the natural attitude by simply abstaining from the claims to existence contained within it. The assumptions concerning the existence of the natural world and its objects are not negated, but simply neutralized.[278] The result of this change in attitude, of this epoché, as Husserl calls it, is that the natural world is no longer considered as a region of external existence, a region that is ontologically transcendent to the immanent subjective experience of it. Rather, the external world is now considered only as given, as experienced, or, in other words, as a pure phenomenon. For Husserl, a phenomenological analysis is guided by a special kind of intuition, so-called "eidetic intuition" (*Wesensanschauung*), which we have already encountered at the beginning of Part II of this thesis.[279] He believes that eidetic intuition enables the phenomenologist to reveal the essences of acts, that is, the structures governing *a priori* the constitution of an object of experience.[280]

By applying his phenomenological method, Husserl analyzes knowledge as a correlation between two types of acts, namely between what he calls meaning acts (*signitive Akte/ Bedeutungsakte*) and intuitive acts (*Anschauungsakte*). The phenomenal difference between these two kinds of acts lies in the ways in which they present their

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[277] ["Ich finde beständig vorhanden als mein Gegenüber die eine räumlich-zeitliche Wirklichkeit, der ich selbst zugehöre, wie alle anderen in ihr vorfindlichen und auf sie in gleicher Weise bezogenen Menschen. Die 'Wirklichkeit', das sagt schon das Wort, finde ich als daseiende vor und nehme sie, wie sie sich mir gibt, auch als daseiende hin."], Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, p. 53, translation from Edmund Husserl, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. First Book*, translated by F. Kersten (The Hague/Boston/London: Martinus Nijhoff Publishers, 1982), p. 57.

[278] A very lucid description of the phenomenological reduction along these lines can be found in David Bell, *Husserl*, pp. 161-168.

[279] One of Husserl's early statements of the eidetic reduction can be found in *Die Idee der Phänomenologie*, pp. 55-63. See also *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, § 3, pp. 10-12.

[280] I will not argue here against the possibility of an eidetic reduction, but only express my reservations towards it. In my own analysis of visual space, I will not appeal to anything like eidetic intuition or essences in Husserl's sense.

respective objects. An intuitive act presents its object in person (*leibhaftig*). An act of meaning, in contrast, merely signifies its object which is thus not presented, but rather represented.[281] Thus, an act of knowing is a type of experience that presents a meaning act as being fulfilled by an adequate act of intuition.[282] In other words, knowledge is simply an experience in which a signified object or state of affairs is intuitively presented.

We can now characterize perceptual acts as a specific type of intuitive act. In order to do so, we have to consider two consequences of Husserl's analysis of knowledge. First, Husserl's analysis does not restrict knowledge to propositional knowledge. Acts of naming, for example, are meaning acts which can be fulfilled by intuitive acts that present the named individual adequately. For example, I can refer to Husserl simply by using the name 'Husserl.' Given the right circumstances, I can also use the name when Husserl stands right in front of me and I see him very well, i.e., when I have an adequate perceptual experience of him. In this case, the act of meaning that refers to Husserl simply by naming him is fulfilled, and there is a knowledge-relation between the two experiences. Thus, simple perceptual acts that present spatio-temporal individuals represent a fundamental class of intuitive acts. Second, Husserl's theory of knowledge requires a type of intuition that differs from mere perception of spatio-temporal individuals. A propositional act, for example, cannot be fulfilled by an experience of a spatio-temporal individual. Consider the meaning act expressed by the proposition "This roof is red." This act cannot be fulfilled by a simple perception of a red roof. Rather, the perceptual act itself must be adequately structured -- it has to afford the

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281 It is essential to note here that the description of the difference between intuitive and significative acts is only a preliminary phenomenological characterization which serves the sole purpose of making plausable that there is such a fundamental distinction between two types of experiences. Husserl gives an extended characterization of the phenomenological differences between the two types of act in the §§ 14-15, VI. *Logische Untersuchung*. Edmund Husserl, *Logische Untersuchungen: Zweiter Band*, pp. 586-595.

282 Husserl writes: "Wir erleben es [in acts of knowing], wie in der Anschauung dasselbe Gegenständliche intuitiv vergegenwärtigt ist, welches im symbolischen [signitiv] Akte 'bloß' gedacht war, und daß es gerade als das so und so Bestimmte anschaulich wird, als was es zunächst bloß gedacht (bloß bedeutet) war." ["We experience [in acts of knowing] how the same objective item which was 'merely thought of' in symbol is now presented in intuition, and that it is intuited as being precisely the determinate so-and-so that it was at first merely thought or meant to be."], Edmund Husserl, *Logische Untersuchungen: Zweiter Band*. p. 504, translation from Edmund Husserl, *Logical Investigations: Second Volume*, p. 694.

predication of the redness to the roof. In other words, the perceptual act has to be structured in such a way that the redness is seen *as* a property of the roof in question. In order to account for the fulfillment of these types of acts, so-called categorial acts, Husserl introduces a categorial intuition as opposed to a simple perception.[283] According to Husserl, a categorial act is also a perceptual experience, if the intuition of the categorial object, e.g. a given property or state of affairs, involves the perception of a spatio-temporal individual. Consequently, perceptual acts are either simple perceptions or categorial acts founded in simple perceptions. We can conclude that, for Husserl, simple and categorial perceptual acts are ultimate acts of fulfillment. In order to reconstruct and critique Husserl's theory of perception, I will restrict my analysis in the following to simple perceptual acts, i.e., acts in which a spatio-temporal individual is perceived.

Husserl believes that perceptual acts function as the most basic elements of his theory of perception because they present their object in a way that is different from all other acts. He writes: "In perception the object is given in person (*leibhaftig*), or (expressed more precisely) as actually presented, given as itself in the actual now."[284] In contrast to a memory, for example, the object of a perception is given *leibhaftig* or in person. As I already mentioned, some interpreters of Husserl's phenomenology have taken this statement to mean that he intended to develop a direct theory of perception.[285] I think, however, that the emphasis of this statement is not so much the direct character of perception, but the specific phenomenal nature of perceptual experience. In *Ideen I*, Husserl writes, for example:

> One mode of consciousness pertaining to the sense is the "intuitive" mode, which

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[283] For Husserl's theory of categorial intuition see: *Logische Untersuchungen: Zweiter Band*, VI. LU, pp. 533-750, in particular ch. 7, pp. 694-709.

[284] ["Der Gegenstand steht in der Wahrnehmung als leibhafter da, er steht, genauer noch gesprochen, als aktuell gegenwärtiger, als selbstgegebener im aktuellen Jetzt da."], Edmund Husserl, *Ding und Raum. Vorlesungen 1907. Vorlesungen 1907*, p. 14, (translation my own).

[285] See, for example, Kevin Mulligan, "Perception" in Barry Smith and David Woodruff Smith, eds., *The Cambridge Companion to Husserl*, p. 169, Izchak Miller, *Husserl, Perception, and Temporal Awareness*, and John Drummond, *Husserlian Intentionality and Non-Foundational Realism*.

> is such that the "meant object as meant" is intentively intuited; and an especially pre-eminent case here is the one in which the mode of intuition is precisely the originarily presentive mode. In the perception of the landscape the sense is fulfilled perceptually; in the mode of "itself in person" there is consciousness of the perceived object with its colors, forms, and other determinations (in so far as they "are included in the perception").[286]

Husserl's phenomenological analysis then aims primarily at explicating the specific phenomenal features of the experience that presents its object in person (*leibhaftig*). Given his characterization of simple perceptual acts as acts that present spatio-temporal individuals in person, we can distinguish three different aspects of Husserl's phenomenal analysis of perceptual acts: it shows (a) how simple perceptual acts characterize their object as given *leibhaftig*; (b) how they present their objects as identical objects; and (c) how they present their objects as spatio-temporal particulars. Although these three aspects are not independent of each other, for the sake of my presentation, I will deal with them separately.

The role played by simple perceptual acts in Husserl's theory of knowledge imposes the following general condition on their analysis. Since they are the ultimate acts of fulfillment, their description must not reintroduce the ontological dichotomy between immanent subjective experience and transcendent objective world. Such a dichotomy would simply force the perception theorist to introduce some kind of perceptual inference, thus opening up the possibility of scepticism. Husserl clearly states this in his recapitulation of his analysis of perception in *Ding und Raum. Vorlesungen 1907*:

> We will not ask, as Kant did in the year 1772: what ground supports the relation of

[286] ["Eine Erlebnisweise des Sinnes ist die 'intuitive', wobei der 'vermeinte Gegenstand als solcher' anschaulich bewußter ist, und ein besonders ausgezeichneter Fall ist dabei der, daß die Anschauungsweise eben die originär gebende ist. Der Sinn in der Wahrnehmnung der Landschaft ist perzeptiv erfüllt, der wahrgenommene Gegenstand mit seinen Farben, Formen usw. (soweit sie 'in die Wahrnehmung fallen') ist in der Weise des 'leibhaft' bewußt."], Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, p. 283, translation from Edmund Husserl, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. First Book*, p. 327. See also Ibid., p. 77, p. 99, p. 127. Husserl also writes: "Jedes wahrnehmende Bewußtsein hat das Eigene, daß es Bewußtsein der leibhaften Selbstgegenwart eines individuellen Objektes ist." ["Any perceiving consciousness has the peculiarity of being a consciousness of the own presence in person."], Ibid., p. 70.

> the which we call representation in us to an object existing in itself? We do not say that the things outside stimulate our sense organs and that to these excitations are linked certain psychophysical sensations and, subsequently, representations and other movements of the soul. How can we conclude back from these effect, present to us in consciousness, to their causes? Not will we say that all allegations and assumptions about thinks trace back to experiences, ultimately to perceptions. These subjective lived experiences are all that is given to us. Since they themselves are not the things (which are, on the contrary, supposed to exist outside the subject), there must be inferences which induce and justify our assuming the things outside.[287]

Husserl here explicitly distances himself from an inferentialist approach to perceptual analysis.[288] A phenomenological description of perceptual experience must avoid any type of inference. In the next section, I will outline Husserl's views with respect to the first two points, the constitution of *Leibhaftigkeit* and identity, and show that each of them introduces some kind of inference. In section 7.3, I will then deal with the constitution of the spatiality of the perceived object and the constitution of visual space in the next section.

7.2 The Constitution of *Leibhaftigkeit* and Identity

Husserl's analysis of the phenomenal features of perceptual acts is based on the fundamental distinction between two different types of phenomenological analysis, namely *reell* and intentional analysis. Thus, before we can outline his account of the phenomenal character of *Leibhaftigkeit*, we first have to understand these two types of

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[287] ["Wir sagen nicht: Draußen sind Dinge; wie können wir von ihnnen wissen? Wir sagen nicht wie Kant im Jahr 1772: Auf welchem Grund beruht die Beziehung desjenigen, was wir in uns Vorstellung nennen, auf einen an sich seienden Gegenstand? Wir sagen nicht, die Dinge draußen üben auf unsere Sinnesorgane Reize, an die sich psychophysische Empfindungen und in weiterer Folge Vorstellungen und sonstige Seelenregungen knüpfen. Wie können wir aus diesen uns im Bewußtsein vorliegenden Wirkungen auf ihre Ursachen zurückschließen? Und wir sagen wieder nicht: Alle Behauptungen und Annahmen über Dinge gehen zurück auf Erfahrungen, zuletzt auf Wahrnehmungen. Diese subjektiven Erlebnisse sind das uns allein Gegebene. Da sie nicht selbst die Dinge sind, die vielmehr außerhalb des Subjektes sein sollen, so müssen es Schlüsse sein, die uns veranlassen und die uns berechtigen, draußen Dinge anzunehmen."], Edmund Husserl, *Ding und Raum. Vorlesungen 1907. Vorlesungen 1907*, p. 139, translation from Edmund Husserl, *Thing and Space. Lectures 1907*, p. 117.

[288] Husserl also rejects explicitly the so-called picture-theory of perception. See *Logische Untersuchungen: Zweiter Band*, pp. 436-440.

analysis and some of the results derived through them.

Husserl bases his assumption of two different types of intentional analysis on a phenomenological distinction between two different components of intentional acts. Consider the experience of seeing a particular tree. On the one hand, the experience presents the tree, the object in the world as characterized by objective properties, its height, its colour, its shape, etc. On the other hand, the object and its properties are given in the perception only as they appear under subjective circumstances of observation. For example, the green of the tree's leaves is seen in a certain light and its shape is always presented only from one side, in profile so to speak. Husserl concludes that we have to distinguish in any perception, or more generally, in any intentional act, that which is actually contained in it, as for example, sensations, from that which is only intended. In a perceptual experience, the appearance is actually given, it is actually contained in the act. Husserl, therefore, calls it a "*reell*" part of the act.[289] *Reell* parts are not objectified, but rather experienced (*erlebt*). The intentional object, in contrast, is only given through the *reell* parts.[290] Husserl also characterizes the distinction between *reell* and intentional

[289] In the remainder of this thesis, I will simply continue using the German word *reell* when referring to this aspect of intentional experience.

[290] Husserl uses a similar example, in order to defend the distinction between intentional and *reell* components of an act in the fifth *Logische Untersuchung*. He writes: "Nicht selten mengt man beides, Farbempfindung und objektive Farbigkeit des Gegenstandes, zusammen. Gerade in unseren Tagen ist eine Darstellung sehr beliebt, die so spricht also wäre das eine und andere dasselbe, nur unter verschiedenen 'Gesichtspunkten und Interessen' betrachtet; psychologisch oder subjektiv betrachtet, heiße es Empfindung; physich oder objektiv betrachtet, Beschaffenheit des äußeren Dinges. Es genügt hier aber der Hinweis auf den leicht faßlichen Unterschied zwischen dem objektiv als gleichmäßig gesehenen Rot dieser Kugel und der gerade dann in der Wahrnehmung selbst unzweifelhaften und sogar notwendigen Abschattung der subjektive Farbabschattungen." ["These two, the colour-sensation and the object's objective colouring, are often confounded. In our time people have favoured a form of words according to which both are the same thing, only seen from a different standpoint, or with a different interest: psychologically or subjectively speaking, one has a sensation, physically or objectively speaking, one has a property of an external thing. Here it is enough to point to the readily grasped difference between the red of this ball, objectively seen as uniform, and the indubitable, unavoidable projective differences among subjective colour-sensation in our percept, a difference repeated in all sorts of objective properties and the sense-complexes which correspond to them."], Edmund Husserl, *Logische Untersuchungen: Zweiter Band*, p. 349, translation in Edmund Husserl, *Logical Investigations*, p. 538. In *Ding und Raum. Vorlesungen 1907*, Husserl describes the difference as follows: "Vergleichen wir den Inhalt der äußeren Wahrnehmung mit dem Inhalt ihres Gegenstandes, dann treten auseinander empfundene Farbe und wahrgenommene Farbe (d.i. Farbe des wahrgenommenen Hauses), empfundene Rauhigkeit und gegenständliche Rauhigkeit, empfundene Ausbreitung, empfundenes Gestalt- und Farbenmoment und wahrgenommene räumliche Ausdehnung, räumliche Größe und Gestalt, die von den 'sinnlichen Qualitäten' des Gegenstandes so und so ausgefüllt, überdeckt wird, bzw. so und so geschieden und geteilt." ["If we compare the content of outer perception with the content of its object, then the following separate themselves: sensed colour versus

parts as two different ways in which something is being experienced. The *reell* parts are given precisely *as* they are, there is no difference between their appearance and their being. Husserl compares this particular type of experience to a pain-sensation which simply *is* the pain. The intentional part, in contrast, is always only given in appearances. Since it is necessarily adumbrated, there is always a gap between its being and its appearing.

From the distinction between *reell* and intentional components of an experience, Husserl concludes that one can analyze an experience in two different ways. He writes:

> The evidence that perception is perception of this or that object tells us already that perception and object are not one and the same. And in fact it is manifest that two series of evident assertions are possible at any time, assertions about perception and assertions about the object in the sense it has in perception, and that in these the perception and the object presenting itself in person therein are not interchangeable.[291]

The phenomenological analysis can thus either focus on the intentional object of the experience, that is, on its intentional content or on its *reell* aspects. The former consists of a description of that which is actually experienced *as* it is experienced. For example, an analysis of perception has to describe the perceived as perceived or the perceived as such.[292] Husserl calls this the noematic analysis. The *reell* analysis, in contrast, requires

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perceived colour (i.e., the colour of the perceived house), sensed roughness versus the objects roughness, sensed extension, sensed structural moment, sensed moment of form versus perceived spatial expanse, perceived spatial size and structure. The latter are filled up and covered in such and such a way by the 'sensuous qualities' of the object, i.e., they are divided and distributed in this or that way."], Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, p. 42, translation from Edmund Husserl, *Thing and Space. Lectures 1907*, p. 42. The difference between intentional and *reell* aspects of an intentional act are also described in *Logische Untersuchungen: Zweiter Band*, pp. 356-361 and in *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie*, § 41, pp. 73-76.

291 ["Die Evidenz, daß Wahrnehmung Wahrnehmung von dem oder jenem Gegenstand sei, sagt uns schon, daß Wahrnehmung und Gegenstand nicht einerlei seien. Und in der Tat ist es evident, daß zwei Reihen evidenter Aussage jeweils möglich sind, Aussagen über die Wahrnehmung und Aussagen über den Gegenstand im Sinne der Wahrnehmung, und daß in diesen die Wahrnehmung und der sich in ihr leibhaft darstellende Gegenstand nicht vertauscht werden können."], Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, p. 17, translation from Edmund Husserl, *Thing and Space. Lectures 1907*, p. 14, slightly modified.

292 Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, p. 182.

the phenomenologist to re-focus his/her attention to the *reell* parts and to describe what is contained in the experience in this way. Since, as we have already seen, the intentional parts of an experience are given only through its *reell* parts, there is a very close correlation between *reell* and noematic analysis. As a matter of fact, every aspect of the latter has a counterpart in the former and *vice versa*.

We can now explicate some of the results of Husserl's *reell* and noematic analyses, i.e., the most basic *reell* and intentional aspects of a perceptual experience, by considering a number of examples. Let me begin with the *reell* aspects by considering again the perception of a particular tree. In order to analyze it in terms of the *reell* analysis, we have to direct our attention to its appearance. Take, for example, the green of a particular leaf as it appears in this perception. We could imagine that the same shade of green could fill a different shape than that of the particular leaf, say, for example, a square. According to Husserl this shows that the leaf is experienced as a shape filled with a particular quality. Husserl calls the qualitative aspects, the act's "hyletic parts" (*hyletische Bestandteile*). He also uses the terms "hyletic data" and "sensation," thus indicating that the qualitative aspects are the primary material of a perception.[293] Husserl distinguishes his notion of sensation from sensations as psychological entities as they appear in Berkeley's and Helmholtz's theories. Husserl claims that according to his view there is only one entity, namely the object of the perception. There is no ontological difference between this object and its appearance -- "Wir haben nicht zweierlei Ding und Dingseite oder Erscheinung."[294] Both can be distinguished from each other by shifting one's direction of interest and are thus abstractive aspects (Husserl calls them "moments"). The same holds for the sensations themselves.

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[293] Husserl writes: "Zum Ersteren [to the *reell* aspects of the act] gehören gewisse, der obersten Gattung nach einheitliche 'sensuelle' Erlebnisse, 'Empfindungsinhalte' wie Farbdaten, Tastdaten, Tondaten u. dgl., die wir nicht mehr mit erscheinenden dinglichen Momenten, Farbigkeit, Rauhigkeit usw. verwechseln werden." ["Among the former [among the *reell* aspects of the act] belong certain 'sensuous' mental processes which are unitary with respect to their highest genus, 'sensation-contents' such as colour-Data, touch-Data and tone-Data, and the like, which we shall no longer confuse with appearing moments of physical things - colourdness, roughness, etc."], Ibid., p. 172, translation from Edmund Husserl, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. First Book*, p. 172. Husserl uses the Greek term υλη to designate the sensuous contents.

[294] Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, p. 149.

As abstract moments of a given appearance, the sensations, or hyletic data, are, for Husserl, a 'formless stuff.'[295] In order to represent objects or objective properties, the sensations thus have to be interpreted as appearances. Husserl, therefore, concludes that perceptual experiences contain a second abstract moment, which he calls "interpretative characters" (*Auffassungscharaktere*).[296] He often says that the sensations, i.e., the hyletic data, are representative contents (*darstellende Inhalte*) that are animated (*beseelt*) by the interpretative characters.[297] The unity of hyletic data and interpretative characters is the "appearance" (*Erscheinung*) of the object.

According to Husserl, the *reell* analysis also reveals another important structural element, namely the so-called "thetic characters." Compare the perception of the tree to an act in which the existence of the tree is doubted. In the perception, the tree is given as existing; the act has a quality that ascribes existence to the object. The second act, in contrast, has a different quality, one that questions the existence of the tree. Husserl calls the qualitative aspect responsible for this difference the "thetic character" of an act (*thetischer Charakter*).[298] He introduces the term "noesis" and, accordingly, speaks of "noetic characters" in order to refer to both interpretative and thetic characters of intentional acts.[299]

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[295] Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, p. 173.

[296] Ibid., p. 213.

[297] Husserl writes: "Wir finden dergleichen konkrete Empfindungsdaten als Komponenten in umfassenderen konkreten Erlebnissen, die als Ganze intentionale sind, und zwar so, daß über jenen sensuellen Momenten eine gleichsam 'beseelende', sinngebende (bzw. Sinngebung wesentlich implizierende) Schicht liegt, eine Schicht, durch die aus dem Sensuellen, das in sich nichts von Intentionalität hat, eben das konkrete intentionale Erlebnis wird." ["We find such concrete sensuous Data as components in more inclusive concrete mental processes which are intentional as wholes; and, more particularly, we find those sensuous moments overlaid by a stratum which, as it were, 'animates,' which bestows sense (or essentially involves bestowing of sense) - a stratum by which precisely the concrete mental process arises from the sensuous, which has in itself nothing pertaining to intentionality."], Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, p. 172, translation from Edmund Husserl, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. First Book*, p. 203, slightly modified.

[298] Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, pp. 214-215.

[299] In order to emphasize the specific function of the noetic characters as opposed to the hyle, Husserl also speaks of morphé (μορφη). Ibid., p. 172.

If we now turn from the *reell* to the intentional analysis, we can see how the different aspects of the noema, the intended as such, correlate to the noetic aspects. First, the thetic characters correspond to characters of existence (*Seinscharaktere*) in the noema.[300] For example, the object of the tree-perception mentioned in the previous paragraph exists as an actual spatio-temporal object. The object of the doubting is not given as an existing object, but rather as one whose existence is doubted. Since both experiences present the same object, Husserl identifies within the noema a core (*noematischer Kern*) as that aspect in virtue of which both experiences are of the same object.[301] Second, the interpretative characters in combination with the hyletic data correspond to the objective properties of the tree. For example, a particular shade of green is interpreted as the appearance of the actual objective colour of a particular leaf.

Having introduced the fundamental results of Husserl's *reell* and intentional analyses, and, in particular, the notion of hyletic data, we can partially answer the question of what it means for an object of a perceptual act to be given in person (*leibhaftig*). In contrast to other types of acts, perceptual acts contain hyletic data (sensations), which, if interpreted in certain ways, represent qualitative aspects of the object. That an object is given in person in a particular intentional experience then means in one sense that the latter's *reell* aspects include hyletic data that have a representing function. This is only part of the explanation, however. The complete answer can only be given by considering the identity of the intentional object. I will therefore now turn to Husserl's analysis of the phenomenal character of identity and then return to *Leibhaftigkeit*.

In order to reconstruct Husserl's account of the constitution of the identical object, we have to depart from a merely static analysis of perception and consider it

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[300] Ibid., p. 214.

[301] Husserl writes: "Jedem noetischen Moment, speziell jedem thetisch-noetischen, entspricht ein Moment im Noema, und in diesem scheidet sich gegenüber dem Komplex thetischer Charaktere der durch sie charakterisierte noematische Kern." ["To each noetic moment, especially to each positing noetic one, there corresponds a moment in the noema and, in the latter, there is set apart from the complex posited characteristics the noematic core characterized by them."], Ibid., p. 268, translation in Edmund Husserl, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. First Book*, p. 310.

instead as an essentially dynamic process. If we do so, two problems arise. In the above analysis, we saw that hyletic data can be interpreted as appearances of objective properties of the act's object. By changing the circumstances of observation, say by going around the tree while fixing our eyes on the very same spot, we generate a continuously changing series of hyletic data in which the same objective colour appears. Yet even though the particular hyletic datum given in the experience at any moment is similar to the colour of the object, it is surely not identical with it. How then can the identical objective colour be given in a series of continuously changing hyletic representations? The same question arises with respect to the identity of the perceptual object itself. The object is always only given in appearances. We always only see one side or profile. By moving around the object, we generate a continuously changing series of profiles. Yet, at any moment the appearance presents the object only from a given point of view, a given perspective -- it never appears as it is in itself. How then is the identical object given in this continuous series of changing appearances, or as Husserl says, in this series of adumbrations (*Abschattungen*)? More generally, Husserl's has to answer the question of how an identical object, a unity, can be given in a manifold of adumbrations.[302]

Because of his distinction between *reell* and intentional analyses, Husserl presents two answers to the two questions of the previous paragraph. I will begin with his answer in terms of the *reell* analysis. Husserl offers a mathematical analogy. He believes that identical objective properties and the identical object itself are both given as some kind of ideal limit of a continuous, infinite series of adumbrations. With respect to qualitative adumbrations (hyletic data), a continuously changing series points towards an identical objective property; and in terms of perspectival adumbrations, the continuous series of profiles points towards the identical object. I think that this analogy is slightly misleading, however. In contrast, to a mathematical series with a limit, the continuous series of adumbrations does not converge on a particular value, one that can be said to be identical to the actual property or object. Rather, in terms of a *reell* analysis,

[302] Husserl describes the phenomenal occurrences associated with dynamic changes of perception in *Ding und Raum. Vorlesungen 1907*, ch. 5, pp. 85-105.

objective property and object are experienced as a unity of a series of changing adumbrations. With respect to a given colour, Husserl describes how this unity is experienced as follows:

> Let us reflect on sensations, on adumbrations: we then seize upon them as evident data and, in perfect evidence, changing the focus and direction of attention, we can also relate them and the corresponding objective moments, cognize them as corresponding and, in so doing, see at once that, e.g., the adumbrative colours pertaining to any fixed physical-thing colour are related to it as a continuous "multiplicity" is related to a "unity."[303]

Objective unity, in general, is experienced as the result of a law-like correlation between a manifold of adumbrational changes and the changes in the circumstances of observation.[304] If such a correlation is present, the subject will interpret the adumbrations as appearances *of* a given property or *of* an identical object. The laws of this correlation

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303 ["Vollziehen wir die Empfindungsreflexion, die auf die Abschattungen: so erfassen wir sie als evidente Gegebenheiten, und in vollkommener Evidenz können wir, in der Einstellung und Aufmerksamkeitsrichtung abwechselnd, sie und die entsprechenden gegenständlichen Momente auch in Beziehung setzen, sie als entsprechende erkennen und dabei auch ohne weiteres sehen, daß z.B. die zu irgendeiner fixierten Dingfarbe gehörigen Abschattungsfarben sich zu ihr verhalten wie kontinuierliche 'Mannigfaltigkeit' zu 'Einheit.'"], Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, p. 203, translation in Edmund Husserl, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. First Book*, 204.

304 In *Ideen*, Husserl writes: "Wir gewinnen sogar, im Vollzuge der phänomenologischen Reduktion, die generelle Wesenseinsicht, daß der Gegenstand Baum in einer Wahrnehmung überhaupt als objektiv so bestimmter, wie er in ihr erscheint, nur erscheinen kann, wenn die hyletischen Momente (oder falls es eine kontinuierliche Wahrnehmungsreihe ist - wenn die kontinuierlichen hyletischen Wandlungen) gerade die sind und keine anderen." ["Effecting the phenomenological reduction, we even acquire the generical eidetic insight that the object, tree, can only appear at all in a perception as objectively determined in the mode in which it does appear in the perception of hyletic moments (or, in the case of a continuous series of perceptions, if the continuous hyletic changes) are just those and no others."], Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, p. 203, translation in Edmund Husserl, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. First Book*, 204. Husserl also writes: "Die ungesehenen Bestimmtheiten eines Dinges sind, das wissen wir in apodiktischer Evidenz, wie Dingbestimmtheiten überhaupt, notwendig räumliche: das gibt eine gesetzmäßige Regel für mögliche räumliche Ergänzungsweisen; eine Regel, die voll entfaltet, reine Geometrie heißt. Weitere dingliche Bestimmtheiten sind zeitliche, sind materielle: zu ihnen gehören neue Regeln für mögliche (also nicht frei-beliebige) Sinnesergänzungen." ["We know in the apodictic evidence that the unseen determination of a physical thing are, like any physical thing-determinations whatever, necessarily spatial: this yields a law-conforming rule for the possible modes of spatial completion of the unseen sides of the appearing physical thing; a rule which, fully developed, is called pure geometry."], Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, p. 297, translation in Edmund Husserl, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. First Book*, 341.

are determined by the essence of the object at hand. We can summarize this by saying that an experience of an objective unity is an act of a specific type, namely a synthesis that unifies a given series by interpreting it as an appearance of a given property or object.[305] A manifold of *reell* aspects is unified in an intentional object.[306]

I will now turn to Husserl's answer to the question of the constitution of the objective properties and the identical object in terms of the intentional, or noematic, analysis. Husserl's point of departure for the noematic analysis exactly parallels that of the *reell* analysis but is formulated in object terms.[307] In respect to the perception of a

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305 Some authors interpret Husserl's synthesis as a process predicating sensations, i.e., hyletic data, to an object. For example, David Bell writes: "Roughly speaking, in the context of the example we have been examining, they [the notions of synthesis, unity, identity, permanence, intentionality, object, and property] are related to each other as follows: to synthesize a plurality of different colour-sensations is to unite them by predicating them, as properties, of one and the same intentional object which is taken to be capable of persisting through changes in those properties." David Bell, *Husserl*, p, 176. This interpretation does not do justice to the fact that sensations belong to the *reell* analysis. They are subjective aspects of the act and cannot be predicated of an object. Rather, as we will see in the next paragraph, the hyletic data first constitute the objective properties, which are indeed predicated to the identical object. Moreover, the unity cannot be a unity of predication because this would presuppose an already constituted intentional object that could function as the carrier of the predicates.

306 Husserl writes: "Mit alledem ist auch absolut zweifellos, daß hier 'Einheit' und 'Mannigfaltigkeit' total verschiedenen Dimensionen angehören, und zwar gehört alles Hyletische in das konkrete Erlebnis als reelles Bestandstück, dagegen das sich in ihm als Mannigfaltigem 'Darstellende', 'Abschattende' ins Noema." ["It is absolutely indubitable, then, that here 'unity' and 'multiplicity' belong to wholly different dimensions and, more particularly, that everything hyletic belongs in the concrete mental process as a really inherent component, whereas, in contrast, what is 'presented,' 'adumbrated,' in it as multiplicity belongs to noema."], Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, p. 203, translation in Edmund Husserl, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. First Book*, 338.

307 The following passage nicely expresses the specific character of the noematic analysis as opposed to the *reell* analysis: "Mit anderen Worten zu seinem Noema gehört eine 'Gegenständlichkeit' - in Anführungszeichen - mit einem gewissen noematischen Bestand, der sich in einer Beschreibung bestimmter Umgrenzung entfaltet, nämlich in einer solchen, die als Beschreibung des vermeinten 'Gegenständlichen, so wie es vermeint ist' alle 'subjektiven' Ausdrücke vermeidet. Es werden da formal-ontologische Ausdrücke verwendet, wie 'Gegenstand', 'Beschaffenheit', 'Sachverhalt'; material-ontologische Ausdrücke wie 'Ding', 'Figur', 'Ursache'; sachhaltige Bestimmungen wie 'rauh', 'hart', 'farbig' - alle haben ihre Anführungszeichen, als den noematisch modifizierten Sinn." ["In other words, there belongs to its noema 'something objective' - in inverted commas - with a certain noematic composition which becomes explicated in a description of determinate delimitation, that is to say, in such a description which, as a description of the 'meant objective something, as it is meant,' avoids all 'subjective' expressions. There formal-ontological expressions are applied, such as 'object,' 'determination,' [and] 'predicatively formed affair-complex;' material-ontological expressions, such as 'physical thing,' 'bodily figure,' [and] 'cause;' determinations with a material content, such as 'rough,' 'hard,' [and] 'coloured' - all have their inverted commas, accordingly the noematic-modified sense."], Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, p. 269, translation in Edmund Husserl, *Ideas Pertaining to a Pure Phenomenology and to a*

colour, he writes:

> Thus it is certain, for instance, that the appearing colour is a unity in contradistinction to noetic multiplicities and, specifically, multiplicities of noetic construing-characteristics. But more precise investigation reveals that changes in these characteristics correspond to noematic parallels - if not the "colour itself," which continues to appear there, then at least in their changing "modes of givenness," e.g., in their appearing "orientation with respect to me." In this way, then noetic "characterizations" are mirrored in the noematic ones.[308]

This example shows, according to Husserl, that the distinction between appearance and object (here between colour appearance and appearing colour) recurs on the noematic side as difference between the *object as it appears* and the *object which appears*. The term 'appearance' thus has two meanings, one corresponding to the *reell* and one corresponding to the intentional aspects of an act. In analogy to the *reell* analysis, the noematic analysis thus has to show how the identical object and its objective properties are constituted within a multiplicity of appearances.

In order to account for the identical object within these appearances in the noematic sense, Husserl includes a further structural element in the noematic core which he describes as follows:

> It becomes separated as a central noematic moment: the "objects" [Gegenstand], the "Object" [Objekt], the "Identical," the "determinable subject of its possible predicates" - the pure X in abstraction from all predicates - it becomes separated from these predicates or, more precisely, from the predicate-noemas.[309]

Phenomenological Philosophy. First Book, 312.

[308] ["Also gewiß ist z.B. die erscheinende Farbe eine Einheit gegnüber noetischen Mannigfaltigkeiten und speziell von solchen noetischer Auffassungscharaktere. Nähere Unterscheidung zeigt aber, daß allen Wandlungen dieser Charaktere, wenn auch nicht in der 'Farbe selbst', die da immer fort erscheint, so doch in ihrer wechselnden 'Gegebenheitsweise', z.B. in ihrer erscheinenden 'Orientierung zu mir' noematische Parallelen entsprechen. So spiegeln sich denn überhaupt in noematischen 'Charakterisierungen' noetische."], Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, p. 208, translation in Edmund Husserl, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. First Book*, 243.

[309] ["Es scheidet sich als zentrales noematisches Moment aus: der 'Gegenstand' das 'Objekt', das 'Identische', das 'bestimmbare Subjekt seiner möglichen Prädikate' - das pure X in Abstraktion von allen Prädikaten - und scheidet sich ab von diesen Prädikaten, oder genauer, von den Prädikatnoemen."], Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes*

As we have seen, the noematic core comprises all those structural aspects of an act that remain identical under different thetic characters. For example, the same tree can be the object of a perception, of an act of remembering, and of a hallucination, etc. Accordingly, the noematic core is a unity vis-à-vis different act-qualities. Yet, characterizing the unity in these terms does not guarantee the identity of the intentional object. The noematic cores of two different perceptions might have the very same phenomenal properties, but present different objects. This would be the case if two perceptions presented two numerically different, but visually indistinguishable, objects. According to the above quote, Husserl excludes cases like this one by introducing a further element into the noematic core that does not itself appear. This element has no properties, but is the bearer of objective properties. Husserl also characterizes this point of unity as the object as such (*der Gegenstand schlechthin*), thus contrasting it with the object in the how of its determinations (*der Gegenstand im Wie seiner Bestimmtheiten*) which is the noematic core. The object as such is the actual identical object of an intentional experience. Thus, in terms of a noematic analysis objective properties and identical object are given as the unity of appearances in the noematic sense. As the example of the colour mentioned in the previous paragraph shows, an analogous element has to be present in each of the particular properties of a given object.

Summarizing Husserl's analysis, we can say that problems of the constitution of objective properties and of the identical object have two aspects and thus two different answers. In terms of the *reell* analysis, objective properties and identical object are given in experiences of the synthetic unity of manifolds of adumbrations or appearances in the *reell* sense of the word. In terms of the noematic analysis, objective properties and identical object are unities of appearances in the noematic sense of the word. These unities are indicated through a special element within the noematic core.

Given this analysis of the identity of the intentional object, we are now in a

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Buch, p. 271, translation in Edmund Husserl, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. First Book*, p. 313.

situation to complete Husserl's explanation of the phenomenal character of *Leibhaftigkeit* of a perceptual experience. So far, we have seen that an object is given in person if the *reell* parts of the act contain hyletic data that are noetically interpreted as objective properties of the object. Yet, as we have just seen, the identical object is (speaking in terms of *reell* analysis) given only in synthetic experiences, in synthetic acts that unify a manifold of appearances. Since at any single moment, the object is given only from one side, this synthetic experience does not unify a set of actually given appearances; this it can not do. Rather, the synthesis is a synthesis of fulfillment. At any given moment the *reell* aspects of a perceptual act contain the appearance of one side of the object and in addition to it expectations of other appearances that would occur if the observer moved in certain ways. Husserl also says that the expectations are significative intentions that require intuitive fulfillment. The perceptual object can then be said to be given in person, only if these meaning intentions are continuously fulfilled.[310] If some observer movement leads to sensations that can no longer be interpreted as appearances of the same object, the latter will turn out to be an illusion. The dialectic between empty meaning acts and intuitive acts thus reappears at the level of perceptual acts themselves, as one between meaning intentions and their fulfilling appearances.

Husserl's analyses of the phenomenal character of *Leibhaftigkeit* and identity betray two fundamental problems in his theory of perception, which I will now explicate. The first problem arises within his explanation of *Leibhaftigkeit*. Husserl accounted for the fact that a perceptual object is given in person by introducing hyletic data as *reell* parts of experience. In order to constitute a spatial object, such data had to be interpreted

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[310] In this context, Husserl speaks of the teleological aspect of perception. Although, as we have seen, we can not say that a multiplicity of appearances converges against a limit, we can say that a perceptual act converges against an ideal of fulfillment. Namely when all the adumbrational expectations are completely fulfilled. The *telos* of such a fulfilling synthesis is the actually existing identical object. Husserl writes: "Vermeinen überhaupt, Bewußtsein überhaupt jeder Art und Sondergestalt untersteht einer möglichen teleologischen Beurteilung. Es hat entweder von vornherein oder kann in sich aufnehmen ein Sich-richten des Ich auf ein *Telos*, auf das Objekt selbst in seinem wahren Sein und Sosein." ["Intending as such, consciousness as such, of any kind and specific form, is subject to being teleologically judged. Such a consciousness either has from the very start, or can take on, a directing of the 'I' toward a *telos*, toward the object itself in its true being and so being."], Edmund Husserl, *Erste Philosophie (1923-1924). Erster Teil: Kritische Ideengeschichte*, Husserliana VII, ed. by Rudolf Boehm (The Hague: Martinus Nijhoff, 1956), p. 80, (translation my own). See also Edmund Husserl, *Ideen Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, p. 302; *Formale und transzendentale Logik*, p. 168, p. 269.

as appearances (in the *reell* sense) of a spatial object. But hyletic data are non-spatial abstractive entities. The only way to interpret them as appearance, therefore, is to project them onto some spatial object or into objective space. Thus, in effect, Husserl is advancing a projective theory of perception.[311] This creates a problem for his approach to perception, because projection is simply a kind of inference, leading from subjective data to their objective interpretation. Such an interpretation contradicts Husserl's own intention to avoid any such inferences.[312]

By adopting a projective theory of perception, Husserl also contradicts his intention of avoiding the pitfalls of traditional philosophical positions, and, in particular, of traditional transcendental idealism. As Hermann Philipse has pointed out, Husserl's projective theory of perception forces him ultimately into a form of transcendental idealism.[313] Philipse argues for this as follows. Given the projective theory, Husserl can no longer say that the intentional object of a perceptual act is not identical to the real object that exists in space and time; to do so would reinstate the ontological distinction he wants to avoid. This would introduce the possibility of scepticism. In this case, whether a perceptual act presents its object veridically would depend on whether the intentional object represents the real object adequately. This could not possibly be

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[311] Hermann Philipse also argues that Husserl held a projective theory of perception. See his "Transcendental Idealism," Barry Smith and David Woodruff Smith, eds., *The Cambridge Companion to Husserl*, pp. 239-322, p. 265. In particular, he points out that in a review from 1903, Husserl described perception as *deutende Hinausverlegung* (interpretative externalization). See Edmund Husserl, Review of Th. Elsenhans, "Das Verhältnis der Logik zur Psychologie (1896), in "Bericht über deutsche Schriften zur Logik in den Jahren 1895-99" III, *Archiv für systematische Philosophie* IX (1903): pp. 398-399, *Aufsätze und Rezensionen* (1890-1910), Husserliana XXII, ed. by B. Rang (The Hague: Martinus Nijhoff, 1979), p. 206

[312] Husserl's theory of perception has received various criticisms from other phenomenologists, most importantly, Aron Gurwitsch and Maurice Merleau-Ponty. Gurwitsch derives his arguments from Gestalt theoretic considerations. He argues that Husserl holds on to a version of the constancy-hypothesis according to which identical stimuli produce identical sensations (Husserl's hyletic data). But as Gestalt theoretic considerations have shown, perception contains no such originally given material. Rather, every aspect of the perceptual experience receives its meaning only within the context of the whole experience. Aron Gurwitsch, *The Field of Consciousness* (Duquesne University Press: Pittsburgh, 1964). Merleau-Ponty argues that Husserl's introduction of sensations forces him to explain how the mind can turn something meaningless into something meaningful. This is an impossible task, however. Maurice Merleau-Ponty, *The Phenomenology of Perception*, trans. by C. Smith (Routledge and Kegan Paul: London 1962).

[313] Philipse develops this argument in detail his article "Transcendental Idealism," Barry Smith and David Woodruff Smith, eds., *The Cambridge Companion to Husserl*, pp. 239-322.

determined from within the experience itself. In order to avoid scepticism, Husserl therefore equates intentional and real object:

> It need only be said to be acknowledged that the intentional object of a presentation is *the same* as its actual object, and on occasions its external object, and that it is *absurd* to distinguish between them. The transcendent object would not be the object of this presentation, if it was not *its* intentional object. (Husserl's emphasis)[314]

The fact that Husserl identifies intentional and real object and simultaneously accepts a projective theory of perception means that he subscribes to an epistemological idealism. According to this type of idealism, the phenomenal world depends ontologically on consciousness. But Husserl reinterprets Kant's *Ding-an-sich* as the identical X, the intentional object as such. This, so Philipse, turns his epistemological idealism into an ontological idealism according to which the world as it is in itself depends on consciousness. We can say that this constitutes Husserl's specific version of transcendental idealism, as expressed in *Ideen I.* Such a metaphysical thesis contradicts Husserl's own idea of phenomenology as an analysis of the meaning of experience. It also does not do justice to our realist intuitions which have to be explained as phenomenal features of the perceptual experience and not as the results of metaphysical assumptions.

Ultimately, Husserl adopted a projective theory of perception, because he accepted the possibility of a *reell* analysis as opposed to a noematic analysis. Examples like perceiving a tree under changing circumstances of observation, demonstrated to Husserl that there is a difference between the tree and its appearance, and that analysis of the latter, understood in *reell* terms, provided arguments for the existence of hyletic data or sensations. Accordingly, whether these data do indeed exist depends on the possibility

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[314] ["Man braucht es nur auszusprechen, und jedermann muß es anerkennen: daß der intentionale Gegenstand der Vorstellung *derselbe* ist wie ihr wirklicher und gegebenenfalls ihr äußerer Gegenstand und daß es *widersinnig* ist, zwischen beiden zu unterscheiden. Der transzendente Gegenstand wäre gar nicht der Gegenstand der Vorstellung, wenn er nicht *ihr* Gegenstand wäre. (Husserl's emphasis) Edmund Husserl, *Logische Untersuchungen: Zweiter Band*, p. 424, translation in Edmund Husserl, *Logical Investigations: Second Volume*, p. 595.

of the *reell* analysis. Here I am skeptical. Although Husserl takes it to be self-evident that we can focus our attention on the perceiving itself and explicate its phenomenal features, I think this is impossible. In order to bring this out, let me consider the tree-example once more. When engaged in such an experience, I see the tree from various points of view, and under varying conditions of perception. I see the particular green of a certain leaf, and I see it changing when I move and when the light changes, for example. What else can there be to this experience? How else could I say what is it like to perceive the tree than by describing what I see -- the tree as it is given to me? The fact that an object is always only given from one perspective and under certain circumstances of observation, i.e., always adumbrated, does not force me to assume the existence of immanent hyletic data or sensations.[315] Husserl's notion of a *reell* analysis must thus be understood as a method that turns the description of the object as it is given into a description of subjective processes. But this is legitimate only on the assumption that there are such processes, and this is a metaphysical thesis that cannot be verified phenomenologically. As a consequence, I must conclude that a *reell* analysis of experience in Husserl's sense is not possible and contradicts his own phenomenological premises. A phenomenological analysis of perception in my interpretation would have to be restricted to a purely noematic description that analyzes the object as perceived. I will observe this first methodological principle in my own analysis of perceptual space.

The second problem with Husserl's theory concerns his explanation of the constitution of the identical perceptual object. As we have seen, Husserl argues that the identical object is constituted as the bearer of its predicates or determinations. The object is the identical X, i.e., that which remains identical if the determinations change in certain predetermined ways. Since these determinations, or more precisely, the chain of appearances in which these determinations are given, are necessarily incomplete, as Husserl asserts repeatedly, one object could always replace another one that has the same determinations. In other words, any series of appearances necessarily underdetermines

[315] Arguments against the inference from the fact that perception is always perspectival to the existence of sense-data were already put forward by Austin in his critique of the so-called 'argument from illusion' given by Price and Ayer. John L. Austin, *Sense and Sensibilia*, reconstructed from the manuscript by G. J. Warnock (New York: Oxford University Press, 1964).

the object as such, the identical X. Nevertheless, as we have also seen, Husserl believes that an act is directed to a unique object. This is clearly implied in Husserl's following example. Imagine that someone who enters a panopticum first sees a woman. As he gets closer, the woman turns out to be a wax figure. Husserl comments: "It is the same lady who appears on both occasions, and who appears endowed with the same set of phenomenal properties."[316] Some commentators believe that Husserl, therefore, also introduced into the noema an indexical element that fixes the referent uniquely, namely the noematic sense.[317] However, introducing an indexical element into the noematic core is problematic for Husserl, because doing so entails a distinction between the intentional object, the noema, and the real object, the referent. Such a distinction contradicts Husserl's own statement that the noema itself *is* the object of the intentional act in the how of its determinations. It also reintroduces an ontological gap between intentional and real object. I think that the noematic sense, a term that Husserl uses in *Ideen I*, does not refer to this indexical element, but rather to the intentional object.[318] Yet Husserl's

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316 ["Es ist dieselbe Dame, die beiderseits erscheint, und sie tut dies hier und dort mit identisch denselben phänomenalen Bestimmtheiten."], Edmund Husserl, *Logische Untersuchungen: Zweiter Band*, p. 444, translation in Edmund Husserl, *Logical Investigations: Second Volume*, p. 610.

317 This is an aspect primarily of theories that understand the noema as a 'sense' in Frege's use of the word. Izchak Miller writes, for example: "Given these various considerations, it is reasonable to conclude that there is a feature in the noematic *Sinn* of our perceptual experience in virtues of which our experience maintains a "fix" on its object, if it exists . . . . I will refer to that feature as an 'indexical' feature, or a 'demonstrative' feature, in the perceptual act. I will, indeed, maintain that - what Husserl means by - determinable-X in the perceptual noematic Sinn corresponds to that 'indexical' element." Izchak Miller, *Husserl, Perception, and Temporal Awareness*, p. 69. For a critical account of the indexical moment of the noema see Ullrich Melle's discussion of Husserl's theory of perception. In *Das Wahrnehmungsproblem und seine Verwandlung in phänomenologischer Einstellung* (The Hague/Boston/Lancaster: Martinus Nijhoff Publishers, 1983), pp. 67-82.

318 Husserl's writings contain many passages that seem to indicate that he drew a distinction between the noema and the real object and that he understood the noema as a kind of Fregean sense through which the act is directed to its object. The noematic sense must then contain an indexical component. Husserl writes, for example: "Jedes Noema hat einen 'Inhalt', nämlich seinen 'Sinn', und bezieht sich durch ihn auf 'seinen' Gegenstand." ["Each noema has a 'content,' that is to say, its 'sense,' and is related through it to 'its' object."], Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, p. 267, translation in Edmund Husserl, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. First Book*, p. 309. This interpretation of the noema was first proposed by Dagfinn Føllesdal in his seminal article, "Husserl's Notion of Noema," in Hubert L. Dreyfuss with Harrison Hall, eds., *Husserl, Intentionality, and Cognitive Science* (The MIT Press: Cambridge, Mass., 1982): pp. 73-81. This "Fregean" interpretation has been criticized by, for example, John Drummond in his book *Husserlian Intentionality and Non-Foundational Realism: Noema and Object* (Kluwer Academic Publishers: Dordrecht/Boston/London, 1990), in particular chapter 5, pp. 104-141 and by Rudolf Bernet in his article "Husserls Begriff des Noema" in Samuel Ijsseling, ed.,

description of cases like the panopticum illusion seems to confirm his belief that a phenomenological analysis can clarify questions of numerical identity, and this would force him into accepting an indexical component within the noema. In order to avoid the problems with this analysis, I will make the demand that a phenomenological analysis abstain from asking questions that introduce an ontological dualism between real object and noema. This is my second methodological principle.

7.3 The Constitution of the Spatial Object

In this section, I will turn to the third phenomenal character of simple perceptual acts and outline Husserl's explanation of how the perceptual object is constituted as a spatial object. We will see that Husserl's account of the spatiality of the perceptual object poses a problem similar to that which we encountered in his explanation of *Leibhaftigkeit*: the fully constituted three-dimensional spatial object is a projection, or inference, from a theoretical construction, the two-dimensional visual field, into a three-dimensional space.

In analyzing the constitution of the spatial object, Husserl pursues the following argumentative strategy. He departs from the fully constituted physical object in everyday experience, introducing a series of abstractions that lead him to the most basic constituents of spatial experience. He then explains how these elements are synthesized or interpreted as appearances of three-dimensional spatial objects. His analysis will do justice to the phenomenon of spatiality only if his abstractions can be justified phenomenologically, i.e., if the most basic constituents of spatial experience can be shown to be phenomenal aspects of it. Otherwise, these constituents will become postulated entities, just like the proximal stimulus in the inferential theories. Husserl believes that the phenomenologist can appeal to eidetic intuition as a means of isolating the constitutive elements of spatial experience. Yet, eidetic intuition is not only a highly dubious faculty, but also presupposes the availability of a series of phenomenal occurrences as its point of departure. In the following, I will argue that no such

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Husserl-Ausgabe und Husserl-Forschung (Kluwer Academic Publishers: Dordrecht/Boston/London, 1990), pp. 61-80. Both interpret the noema as the phenomenologically reduced object of the intentional act. This is the interpretation that am adopting in this thesis.

phenomenal occurrences exist and that Husserl's abstractions, at least at a very basic level, are driven by metaphysical stipulations, similar to those of Berkeley, Helmholtz, and Rock.

For Husserl, the fully constituted physical (*physisch*) object of everyday perception is given as an object in space and time. Such an object is part of the universal causal connection that structures the world of everyday experience and as such has causal properties. The same physical object can be seen and touched -- it is visually and tactually qualified. Husserl believes that eidetic intuition allows us to distinguish three typologically different layers of properties within the fully constituted physical object. He writes:

> In its ideal essence, the physical thing is given as *res temporalis*, in the necessary "form" of time. . . . The physical thing is furthermore, according to its idea, *res extensa*; it is capable, e.g., with respect to space, of infinitely multiple changes in form and, in the case where the shape and alterations in shape are retained as identical, of infinitely multiple alterations in place; it is "moveable" in infinitum. . . . Finally, the physical thing is a *res materialis*; it is a substantial unity and as such a unity it is a unity of causalities and, with respect to possibility, of infinitely complex causalities.[319]

Thus, any individual perceptual object has temporal, spatial, and physical properties. Husserl also claims that spatial properties are founded in temporal properties, and that physical properties are founded in both. Husserl uses the term "foundation" here in the sense of a necessary presupposition. The *res temporalis* is necessary for the *res extensa* and both together are necessary for the *res materialis*. This implies that the *res temporalis* is the basis for the experience of spatial properties and that the *res extensa* is the basis for the experience of causal and physical interaction.[320] Husserl's terms can be

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319 ["Das Ding gibt sich in seinem idealen Wesen als *res temporalis*, in der notwendigen 'Form' der Zeit . . . . Das Ding ist seiner Idee gemäß ferner *res extensa*, es ist z.B. in räumlicher Hinsicht unendlich mannigfaltiger Formverwandlungen und, bei identisch festgehaltener Gestalt und Gestaltveränderung, unendlich mannigfacher Veränderungen der Lage fähig, es ist in infinitum 'beweglich' . . . . Das Ding ist endlich *res materialis*, es ist substanzielle Einheit von Kausalitäten und der Möglichkeit nach von unendlich vielgestaltigen."], Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, p. 312, translation in Edmund Husserl, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. First Book*, p. 359.

320 Husserl believes that each layer, *res temporalis*, *res extensa*, and *res materialis*, represents a unity that

misunderstood as referring to three ontologically distinct substances. He guards himself against this interpretation, however, by pointing out that these terms simply designate three layers of qualitatively distinguishable properties within the noema of a perceptual act.[321] This noematic structure allows Husserl to introduce his first abstraction. He writes:

> Within that which is 'actually' given nothing would change if we deleted the entire layer of materiality [*res materialis*] from apperception. This is indeed imaginable.[322]

Accordingly, the phenomenological analysis can focus exclusively on the structure of the *res extensa*. The analysis of the constitution of a perceptual object's spatiality is an analysis of the way in which the *res extensa*, the carrier of physical properties, is given.

By introducing a distinction between different types of quality, we can say more clearly how Husserl's analyses spatiality. According to him, the *res extensa* can be qualified in many different ways. A physical object has a certain colour; it appears as bright or dark; its surface can also be experienced as smooth or rough; it can have a certain temperature, emit a smell, and sometimes be heard. These different qualities fall into two distinct classes: necessary and accidental properties. The visual and tactile qualities necessarily cover, or "fill," as Husserl says, the *res extensa*. The latter can never

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is governed by its respective essence and therefore forms an ontological region that is relatively closed. This allows him to claim that each of these layers provides a transcendental guiding principle (*transzendentaler Leitfaden*) for a separate area of investigation, a regional ontology (*regionale Ontologie*). I have already expressed my doubts as to the concept of an essence and will argue later that my own approach allows me to replace Husserl's transcendental guiding principle with what one could call a "hermeneutical guiding principle." Husserl describes his concept of a transcendental guiding principle in *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, pp. 344-352. Ulrich Claesges gives a systematic reconstruction of Husserl's theory of the ontological layers of an object with respect to the constitution of space. Cf., Ulrich Claesges, *Husserls Theorie der Raumkonstitution* (The Haag: Martinus Nijhoff, 1964), ch. 2, pp. 35-54.

321 Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, p. 348.

322 ["Im 'eigentlich' Gegebenen würde sich nichts ändern, wenn die ganze Schicht der Materialität [*res materialis*] aus der Apperzeption weggestrichen würde. Das ist in der Tat denkbar."], Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie. Zweites Buch*, ed. by Marly Biemel (The Hague: Martinus Nijhoff, 1952), p. 37, (translation my own). This text was written by Husserl in 1912.

be experienced apart from these qualifications. Tone, smell, and temperature, in contrast, are only accidental properties, they might or might not qualify a given *res extensa*. This distinction allows us to introduce Husserl's second abstraction. He restricts the analysis of the constitution of the spatial object to the visually qualified *res extensa*, which he calls the "phantom," "pure shape filled with colours."[323] Thus, the phenomenological analysis of the perceptual object's spatiality is a description of the way in which the phantom is given.

In order to make the possibility of such an analysis plausible, Husserl gives an example that purports to show that phantoms can sometimes be experienced. If this was true, the abstraction of the visual phantom would be phenomenologically justified. He writes:

> A mere phantom is present, for example, if we learn to bring to a bodily unity correlating formations in the stereoscope. Then we see a body in space and we can ask meaningful questions about its shape, colour, smoothness or roughness. And these can be truthfully answered, as, for example, in the words: this is a red, rough pyramid. Yet, on the other hand, that which appears can be given in such a way that the question of whether it is heavy or light, elastic, magnetic, etc., makes no sense, better: has no grounding in the perceptual sense.[324]

It seems to me correct that objects are sometimes experienced in this way. Objects seen

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[323] ["pure farbig erfüllte Gestalt"], Ibid., p. 22. In a manuscript from 1910, Husserl describes the phantom as follows: "Wollen wir von allem Kausalen abstrahieren, so behalten wir allerdings etwas übrig: ein Phantom, ein geometrischer Körper, mit Qualitäten erfüllt, aber nicht mit Materie. Und die Qualitäten haben nun gar nichts reales mehr. Vom Ding als solchem ist nur ein Schatten da, ein Nichts." ["If we abstract away from everything causal, we do indeed retain something: a phantom, a geometric body, filled with qualities, but not with matter. And now, the qualities have nothing real anymore. Of the thing as such only a shadow, a nothing, exists."], "Phantom und Ding," Ms. D13 III / 24a (1910), Husserl-Archives Louvain (translation my own). Husserl also calls the phantom a "schema" (*Schema*) and distinguishes between an empty schema (*Leerschema*) which is the unqualified extension and the qualified schema. The latter with all its qualities is called the "complete schema" (*Vollschema*). See *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie. Zweites Buch*, p. 38.

[324] ["Ein bloßes Phantom liegt z.B. vor, wenn wir im Stereoskop lernen, passende Gruppierungen zu körperlicher Verschmelzung zu bringen. Wir sehen dann einen Raumkörper, für den hinsichtlich seiner Gestalt, hinsichtlich seiner Farbe, seiner Glätte oder Rauhigkeit und ähnlich geordnete Bestimmungen sinnvolle Fragen zu stellen sind, die also der Wahrheit gemäß Beantwortung finden können, wie etwa in den Worten: dies ist eine rote, rauhe Pyramide. Andererseits kann das Erscheinende so gegeben sein, daß die Frage ob es schwer ist oder leicht, ob elastisch, magnetisch usw. gar keinen Sinn, besser: keinen Anhalt am Wahrnehmungssinn hat."], Ibid., p. 36, (translation my own).

in the stereoscope will evoke assessments with respect to their spatial dimensions and it is imaginable that such an object will not *look* light or heavy. This thought-experiment seems to justify Husserl's two abstractions phenomenologically. However, as will become clear later on, phantoms of this kind cannot support Husserl's own analysis of spatiality, since the latter is intimately bound to the experience of perspectival changes, yet objects in the stereoscope appear only from one side.

I want to emphasize that Husserl does not equate an analysis of the phantom with an analysis of the spatial object or objective space itself. He writes:

> We can therefore separately ask how visual space is constituted and how tactile space is, insofar as in general they are constituted independently of one another. (In any case, we must be able to establish their shares.)[325]

Thus, he leaves open the possibility that tactile and maybe even causal properties could be relevant to the constitution of the spatial object and objective space. Indeed, in *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Zweites Buch* and in his later manuscripts on the constitution of space, Husserl investigates the contributions of tactile qualities and even causal experiences involving forces like pulling and pushing.[326] Yet, he is convinced that one can analyze the constitution of the

[325] ["Wir können also scheiden: Wie konstituiert sich der visuelle und der taktuelle Raum, wofern sie überhaupt voneinander unabhängig sich konstituieren? - Jedenfalls ihren Anteil müssen wir feststellen."], Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, p. 156, translation in Edmund Husserl, *Thing and Space. Lectures 1907*, p. 132.

[326] Edmund Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie. Zweites Buch*, in particular §§ 12-18, pp. 27-89. A reconstruction of Husserl's account of the tactile and causal aspects of spatial constitution can be found in Ulrich Claesges, *Husserls Theorie der Raumkonstitution*, pp. 90-115. In a manuscript from 1910, Husserl motivates further analyses of the constitution of the tactile and causal spatial object as follows: "Um eine klare Vorstellung von der Konstitution der materiellen Dinglichkeit zu gewinnen, genügt nicht eine erste Vorstellung von der Konstitution der Räumlichkeit und der Raumphantome, mit den sie ursprünglich qualifizierenden Momenten [. .]. Es bedarf vielmehr, und das fehlt, wenn ich mich recht entsinne, eines Studiums der Konstitution der anhängenden Qualitäten, und das befaßt gerade die spezifisch mechanischen und sonstigen Qualitäten als 'Sinnesqualitäten'. Es genügt durchaus nicht, vom ursprünglichen Raumphantom überzugehen zu geregelten funktionellen Abhängigkeiten, die in dem Gang der Entwicklung konstituierender Erfahrung ihre Rolle spielen. In den Vorlesungen von 1907 sind wohl erste und zweite Qualitäten unterschieden worden, aber dem wurde nicht entsprechend nachgegangen." ["In order to gain a clear idea of the constitution of material objectivity, a first idea of the constitution of spatiality and of spatial phantoms, with the moments that originally qualified them, is not sufficient. . . . Rather, what is required, and what is missing, is, if I recollect correctly, is a study of the constitution of the accompanying

visual phantom on its own.

Husserl introduces his third abstraction on the basis of the observation that the phantom and the entire visual scene of which it is a part appear always only from one side. Husserl says, only one side falls within the *actual* appearance. Thus, he demands that we focus exclusively on the actual appearance of a given object in terms of his *reell* analysis. Husserls calls this abstraction the "visual field" and characterizes it generally as a visually qualified pre-empirical extension.[327] The term "pre-empirical" here means that the visual field as a notion of *reell* analysis does not designate the objective, or empirical, aspects of the perceptual act; it is not an object of experience, but a constituent whose interpretation makes possible the perception of such an object.

On the basis of his various abstractions from the fully qualified experience of a physical object, Husserl now defines the constitutive analysis of spatiality as an analysis of the way in which representational means (*Darstellungsmittel*) in the visual field manage to present spatial, individual objects. The analysis of the constitution of the spatial object is thus a two-fold problem: Husserl has to describe the representational means of the field, or, more generally, its nature, and he has to show how these features constitute spatiality. I will deal with these two points in turn.

In order to analyze the means of representation in the visual field, Husserl has to show how they can be phenomenologically accessed and described. Although, he does not give an explicit account, we can reconstruct it from his general approach in *Ding und Raum. Vorlesungen 1907*. The key to this reconstruction is his notion of a kinaesthetic

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qualities, including precisely the specifically mechanical and other qualities as 'sensuous qualities.' It is absolutely not enough, to proceed from the original spatial phantom to the law-like functional dependencies which play a role in the development of the constituting experience. In the lectures from1907, primary and secondary qualities have been distinguished; but this was not pursued sufficiently."], "Raumkonstitution," Ms. D 13 II / 17a (1918, partly 1910), Husserl-Archives Louvain, (translation my own).

[327] Husserl writes: "Die darstellenden Inhalte der visuellen Gesamtwahrnehmnung bilden einen kontinuierlichen Zusammenhang: Wir nennen ihn das visuelle Feld. Das visualle Feld ist eine präempirische Ausdehnung, und hat eine so und so bestimmte visuelle Fülle." ["The presentational contents of the total visual appearance form a continuous nexus: we call it the visual field.], Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, p. 82, translation in Edmund Husserl, *Thing and Space. Lectures 1907*, p. 68.

system. I will therefore first say why Husserl introduces this notion and then show how it is supposed to ground his phenomenological description of the visual field.

Husserl introduces the kinaesthetic system in order to explain the constitution of the distinction between physical change and purely spatial change. Without further information purely visual experiences do not allow an observer to draw a distinction between physical changes of a given object and changes in its appearance due to movements of the object or the observer. The reason for this is that certain types of change in the appearing visual scene, namely the perspectival changes, can be interpreted both as a result of the movement of the object or the observer and alternatively as a change of the object's physical state. Moreover, a non-changing visual scene may indicate both rest and movement of an object, because object and observer can move in a way that cancels out possible changes.[328] Husserl therefore demands that we take into account the movements initiated by the observer. Since Husserl is concerned with a *reell* analysis, he cannot appeal to objective movements directly, but rather has to rely on the *reell* elements that indicate the latter. Husserl believes that the so-called "kinetic sensations" (*Bewegungsempfindungen*) considered not as psychological entities, but as phenomenological data serve this function. He marks his non-psychological understanding of these sensations by calling them "kinaesthetic sensations."[329] They are simply those experiences that indicate the movement of an observer's own body to him or herself. One example is given by the sensations that indicate different tona of the eye muscles when the observer moves his/her eyes.[330] According to Husserl, these kinaesthetic sensations reflect the structure of the body's various kinetic systems, moving eyes, moving head, moving upper body, and walking, and therefore represent systems that are relatively closed and can be considered independently of each other.[331]

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[328] Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, pp. 157f.

[329] Ibid., pp. 160f.

[330] The idea that spatial perception requires kinetic sensations was known to Husserl through both his teacher Carl Stumpf and through Alexander Bain. Husserl studied Bain's work *The Senses and the Intellect* (London: Parker, 1855) in German translation. Stumpf's study *Über den psychologischen Ursprung der Raumvorstellung* (Leipzig: Verlag von S. Hirzel, 1873) was well-known to Husserl.

[331] Husserl writes: "den Leib [. . .] haben [. . .] wir als Gesamtorgan anzusehen, in dem vielerlei Teilorgane zu unterscheiden sind. Die Teilung in Organe, wie die Rede vom Gesamtorgan, hat Beziehung

An observer can move his/her eyes alone while the rest of the body remains fixed. He/she can then combine eye movements with movements of the head, the upper body, and, ultimately, the entire body (through walking). The different kinaesthetic systems (*Sonderkinesen*) together form a total kinaesthetic system (*Gesamtsystem, Totalkinästhese*), i.e., a total system of possible observer movements.[332] Husserl then concludes that purely spatial changes are changes that can be reversed through changes in the total kinaesthetic system.

The fact that the total kinaesthetic system is a compound of specific kinaesthetic systems (*Sonderkinästhesen*) now allows Husserl to describe the possible visual occurrences corresponding to each system, that is, the representational means of the visual field.[333] In order to do so, he simply arrests the other kinaesthetic systems.[334] He

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auf Kinästhesen; jedes Organ hat sein 'kinästhetisches System', jedes mit anderen zusammen fungierend konstituiert sozusagen ein kinästhetisches Organ höherer Stufe und so das All der Organe ein Organ gemäß einer Totalkinästhese als Synthesis aller, nämlich aller miteinander fungierender oder <möglicherweise> mit einander fungierender, Organe." ["we have to consider the body (*Leib*) . . . as a total-organ within which we can distinguish many partial organs. The division into organs, as well as our talk of the total-organ, are related to kinaestheses; every organ has its own 'kinaesthetic system;' every kinaesthetic system, functioning together with other kinaesthetic systems, constitutes so to speak a kinaesthetic organ of higher order and, in this way, the totality of organs, an organ resulting from the total kinaesthesis as the synthesis of all organs, namely all organs functioning together, or possibly functioning together."], "Assoziative Passivität des Ich und Ichaktivität in der untersten Stufe; Kinästhese in der praktischen und nicht-praktischen Funktion," Ms. D 12 / 15b (05/10/1931), Husserl-Archives Louvain, (translation my own).

332 Carl Stumpf considered the different types of kinetic sensations and their function in his theory in § 12 of *Über den psychologischen Ursprung der Raumvorstellung*, pp. 217-244. He distinguishes between sensations concerning the accommodation, convergence, and sensations of muscles responsible for moving different limbs. Husserl's analysis of the latter is far more refined due to his notion of the partial kinaestheses. Husserl describes the relation between total kinaesthetic system and partial kinaestheses as follows: "Ich sprach immer von Totalkinästhese und Sonderkinästhese. Inwiefern sind diese Kinästhesen eine Totalität? [. . .] Wir unterschieden die Kinästhesen der Augen, des Kopfes, der Hand usw. Sie sind in der Tat in sich artmäßig unterschieden. Allerdings die Kinästhesen des einen und anderen Auges sind doch in der funktionellen Verschmelzung auch inhaltlich so verschmolzen, dass wir sie kaum auseinanderhalten können - obschon doch auch können." ["I always spoke of total kinaesthesis and partial kinaesthesis. In how far are the kinaestheses a totality? . . . We distinguish between the kinaestheses of the eyes, the head, the hands, etc. They indeed differ with respect to their kinds. But, in their functional unity, the kinaesthesis of the one eye and the other also form a unity of content in such a way that we can hardly differentiate between them - although, we can, nevertheless, do so."], "Schwierigkeiten der Kinästhese," Ms. D 10 / 58a (June 1932), Husserl-Archives Louvain, (translation my own).

333 Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, p. 201.

334 Husserl justifies the possibility of considering the various systems separately by appealing to their possible genesis: "Ich gebrauche alle Kinästhesen und kann Stillhaltungen verschiedener Sonderkinästhesen vollziehen. Die Sonderkinästhesen , das sind Teilsysteme, die in sich geschlossen sind, in Sonderheit eingeübt, was wohl darauf zurückweist, daß hier ursprünglich Gründe vorliegen, die es

first considers what he believes to be the most basic kinaesthetic system, i.e., the system restricted to movements of the eyes, the so-called "oculomotoric system." He then extends the kinaestheses and considers the occurrences corresponding to the total system.

In order to describe the representational means of the field corresponding to the oculomotoric system, i.e., the oculomotoric field, Husserl considers first the field corresponding to one eye without any movement. He describes the phenomenal features of this field as follows. The field is completely qualified, that is, each part of it has its colour. If the field is not entirely homogenous, the different colours will delineate various shapes. If we then consider a particular shape in the field, a pre-empirical object, say a red oval, we can imagine it to be qualified differently, say as blue or green. This, according to Husserl, allows us to distinguish between the shape of objects in the field and their matter (their qualitative filling). Further, we can imagine the same patch to be at a different location in the field, up or down, more to the left or more to the right. Thus, the field is a system of places (*Lagensystem*) ordered in two directions. Since any object in it can continuously change its place, the field itself is continuous.[335] Finally, the field is finite and bounded. Husserl believes that these features allow us to draw a conclusion as to the dimensionality of the visual field. Since the field is continuous, we can cut it into arbitrary regions by changing the colour in the areas. The regions will be delineated by visual lines that are formed by the qualitative differences between them. A border will be a contrast between red and green, for example. These lines can also be cut into pieces arbitrarily by introducing visual points. Points cannot be further divided, however. If we reason backwards, it follows that the field is two-dimensional: the points have no

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machen, dass bestimmte kinästhetische Gruppen zunächst außer Spiel bleiben." ["I use all kinaestheses and can arrest various partial kinaestheses. The partial kinaestheses are partial systems, which are closed in themselves and separately acquired, indicating that probably original reasons are responsible for the fact that certain kinaestheses remain initially left aside."], "Schwierigkeiten der Kinästhese," Ms. D 10 / 67b (June 1932), Husserl-Archives Louvain, (translation my own).

[335] For Husserl, continuity of the oculomotoric field means only that we can transform any figure of the visual field into any other continuously. He does not consider this a decision as to whether the division of the field will lead to *minima visibilia* or whether it can be continued *ad infinitum*. But, since the points are the result of the division of the field itself, this is problematic. Husserl can maintain the continuity of the field only if he rejects the idea of points as *minima visibilia*. Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, p. 166.

dimension, the lines have one dimension, and the field, therefore, two. The structure of the oculomotoric field becomes qualitatively enriched when we consider both eyes together. Husserl points out that two eyes present two "pictures" of the same visual scene that are almost identical.[336] If the eyes are optimally accommodated, the two "pictures" melt with each other. This experience confers a new quality to the field, namely depth or relief.[337] As a pre-empirical qualification of the field, the depth-values do not constitute a third dimension, however. Husserl summarizes his discussion as follows: "The visual field [is] a two-dimensional manifold, self-congruent, continuous, simple connected, finite and bounded; it has a border which has no beyond."[338] In this quote, Husserl seems to identify visual and oculomotoric fields. But since the oculomotoric field is an abstract layer within the visual field, it is likely that Husserl simply used the term "visual field" to refer to the abstraction. The oculomotoric field, thus, presents the primary elements of experience from which the three-dimensional world is constituted. Given this, we can now explain how the representational means of the oculomotoric field, the primary elements of spatial experience, constitute spatial objects.

According to Husserl, a first type of objectification takes place already in the oculomotoric field, which contains pre-empirical spatial objects. The description of the oculomotoric field in the previous paragraph was based on a certain type of object,

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336 The term 'picture' here refers to the appearance of an object or a visual scene.

337 Ibid., p. 172. This constitutes a significant deviation from the views of Stumpf. In contrast to Husserl, Stumpf argued that we have an immediate representation of three-dimensional space. This holds even for a one-eyed perception. Stumpf writes, for example: "Eine reine Flächenvorstellung ist demnach so wenig möglich wie eine reine Linien- oder Punctvorstellung; und so wenig wie eine raumlose Qualitätsvorstellung. Jeder Gesichtsinhalt schließt notwendig die dritte Dimension bereits ein. Und dies liegt ebensosehr in seiner Natur, wie daß er in einer Farbqualität vorgestellt wird." ["A pure flat representation is as impossible as any pure representation of a line or a point, and as impossible as a non-spatial representation of a quality. Any visual content necessarily includes a third dimension. And this is as much part of its nature as the fact that it can be represented only as a colour quality."], Carl Stumpf, *Über den psychologischen Ursprung der Raumvorstellung*, p. 182, (translation my own). Stumpf further believes that this representation is phenomenally enriched (*Ausbildung*) through association with further experiences like accomodation, conversion, and kinesis, etc. Ibid., pp. 222-225. He writes: "Die ursprüngliche Tiefe wird durch Association in der mannigfachsten und ausgiebigsten Weise verändert." ["The original depth is modified in the most varied and substantial way through association."], Ibid., p. 276, (translation my own).

338 ["Das Sehfeld [ist] eine zweidimensionale Mannigfaltigkeit, in sich kongruent, stetig, einfach zusammenhängend, endlich und zwar begrenzt; es hat einen Rand, der kein Jenseits hat."], Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, p. 165, (translation my own).

namely a qualitatively filled shape, as, for example, a red patch. That this is an object of some kind can be seen by means of the following argument: Consider the red patch and imagine that it has the shape of a square. I can fix my eyes on one corner and then move them slowly around the square until I arrive at the point of departure. In doing so the patch will change its appearance. The corner from which I departed will move out of focus and the other corners will subsequently come into focus, until the original situation is reconstituted. Each of the stages in between will give a different appearance of the patch. In other words, the patch is an identical object appearing in different ways.[339] Husserl explains this type of objectification in the following way: The objectified patch is experienced as a unity of certain visual expectations. (Husserl speaks here of "quasi-intentions.") When I direct my eyes from one corner of the patch to another, I will experience a change in both the kinaesthetic sensations and the appearance of the patch (its "picture"). But these two types of changes are not unrelated. Although a specific kinaesthetic sensation is not correlated (associated) with a particular picture, it will nevertheless prompt an expectation of a *typical* change in the picture. This expectation will then be fulfilled or disappointed by actual changes in the picture. The object, in this case the red patch, is thus experienced as an intentional unity given through a law-like correlation between typical changes in the kinaesthetic sensations and typical changes in the appearance. The same holds for all the qualitatively different shapes in the oculomotoric field and even for the field as a whole.[340] The oculomotoric field remains two-dimensional and contains only objects that are flat, i.e., objects that have only one side. Husserl writes: "The oculomotoric field is not a field of things. The oculomotoric

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[339] This example is adapted from Husserl's considerations in § 51 and 52 of *Ding und Raum. Vorlesungen 1907*, pp. 176-186. Husserl uses 'appearance' in this example as a concept designating the difference between something that is optimally given, that is, for him, given as it actually is, like the particular aspect of the square that is seen at a given point in time, and something that is not given in this way, like the square as whole. Moreover, at this stage of objectification, it is principally impossible to see the square optimally. To do so would require seeing it as a drawing on a two-dimensional surface in a three-dimensional space. Husserl generally uses the term 'appearance' in this way. Thus, each further level of objectification, initiated through further kinaesthetic systems (see next paragraphs), reduces the previous level to an appearance of an object of the higher level.

[340] The oculomotoric field can also change in other ways. For example, when I move my eyes, new objects will enter into the field and other objects will disappear. But these changes are not important for the process of objectification and I will leave them aside here.

unities, although unities in manifolds, are still 'pictures.'"[341] Actual spatial objects (in the sense of phantoms), in contrast, are not given in a manifold of appearances, but rather in a manifold of appearing sides. Consequently, in order to explain the constitution of the spatial object, Husserl shows how the occurrences in the two-dimensional oculomotoric field are interpreted as appearances of spatial objects.[342]

In order to show how the changes in the two-dimensional oculomotoric field are objectified as spatial objects, Husserl takes into account changes in all kinaesthetic systems. The observer can now freely move around, turn his/her head, upper body, and walk. By considering the total kinaesthetic system in this way, Husserl describes two further types of phenomenal changes in the visual field: first, a particular object in the field can stretch (*dehnen*) or shrink (*zusammenziehen*) in such a way that all its internal geometric relations change proportionally.[343] A stretching object will occupy an increasingly larger part of the visual field and a shrinking one will uncover an increasing part of the field. Second, a particular object can turn (*drehen*) and present itself from

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[341] ["Das okulomotorische Feld ist kein Dingfeld. Die okulomotorischen Einheiten, obschon Einheiten in Mannigfaltigkeiten, sind immer noch 'Bilder.'"], Edmund Husserl, *Ding und Raum. Vorlesungen 1907,* p. 234, (translation my own).

[342] This understanding of Husserl's theory of the constitution of spatiality has also been proposed by Oskar Becker and, more recently, by Smail Rapic. Becker writes, for example: "Der Sehraum konstituiert sich aus dem okulomotorischen Feld durch die Umdeutung einer gewissen Qualität seiner Elemente, der sog. "Sehtiefe", in eine dritte Raumdimension, die mit beiden im Felde ausgebreiteten Dimensionen eine im Wesentlichen homogene dreidimensionale Mannigfaltigkeit bildet." ["Visual space is constituted from the oculomotoric field through the reinterpretation of a certain quality of its elements, the so-called 'visual depth,' as a third spatial dimension, which, together with the two dimensions extending in the field, forms a homogeneous three-dimensional manifold."], Oskar Becker, "Beiträge zur phänomenologischen Begründung der Geometrie und ihrer physikalischen Anwendungen,'"p. 455, (translation my own). For Rapic see "Einführung," in Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, ed. by Karl-Heinz Hahnengress and Smail Rapic (Hamburg: Felix Meiner Verlag, 1991), p. LXXV. Ulrich Claesges criticized this view. He thinks that interpreting Husserl's notion of the visual field as two-dimensional manifold would turn it into a phenomenologically illegitimate abstraction. (*Husserls Theorie der Raumkonstitution*, pp. 84-89) As we will see, I agree with Cleasges conclusion, but I also think that Becker and Rapic are correct in their interpretation of Husserl. Husserl writes, for example: "Das zweidimensionale hyletische Feld, immerfort ausgefüllt, ist also der Kern, der alle Darstellungen zweidimensional koexistieren macht. Die Dreidimensionalität baut sich aus intentionalen Abwandlungen der Zweidimensionalität auf." ["The two-dimensional hyletic field, continuously qualified, is thus the core through which all representations coexist two-dimensionally. Three-dimensionality is constituted though intentional modifications of two-dimensionality."], "Konstition als Perspektivierung . . . ," Ms. D10 / 32b (June 1910), Husserl-Archives Louvain. Thus, the problem is inherent in Husserl's theory and not a consequence of tbe incorrectness of Becker's or Rapic's interpretations.

[343] Ibid., pp. 225-243.

different sides. Such a turn can be completed when the object returns to its original position. Objectification is achieved when these changes in the visual field are interpreted as changes of an object's appearing side, that is, when an object itself is seen as that which remains identical within these changes. In order for this to happen, an association between typical changes in the field and certain types of kinaesthetic sensation has to be established. For example, certain kinaesthetic changes that indicate walking will lead to a stretching or shrinking of the object and certain other sensations will lead to a turning of the object. If this correlation is present, the phenomenon of turning will represent the closed surface of a spatial object and the phenomena of stretching and shrinking will be interpreted as indicating changes in distance concerning the third dimension. As in the oculomotoric field, objectification in the total kinaesthetic field is thus synonymous with a law-like relation between typical motivating kinaesthetic sensations and typical changes in the appearance. The three-dimensional spatial object is an intentional unity given in the law-like correlation between certain types of kinaesthetic changes and certain types of changes in the visual field.[344] Husserl also says that the three-dimensional object is the correlate of the total kinaesthetic system.[345] We can alternatively say that the spatial object is experienced as an invariant of certain typical transformations within the *reell* aspects of the visual field, namely those transformations that are motivated through changes in the total kinaesthetic system.

Husserl extends this explanation to visual space in general. An oculomotoric field, as we have seen, not only presents the appearance of one object. Rather, it is a total system of places, and can thus represent the appearances of a multiplicity of objects. The phenomena of stretching, shrinking, and turning affect the oculomotoric field as a whole; all the objects in it stretch, shrink, or turn in related ways. Using a mathematical analogy, Husserl says that the entire field is mapped into itself through these phenomena.[346] In

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344 Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, pp. 227-240.

345 Husserl's account of the constitution of the spatial object as the correlate of the total kinaesthetic system is outlined in Ulrich Claesges, *Husserls Theorie der Raumkonstitution*, pp. 79-84.

346 Husserl says: "Das [. . .] Raumfeld verschiebt sich gleichsam [. . .] in sich selbst" ["The field of space is so to speak mapped . . . into itself."], "Zur Konstitution des Raumes . . .," Ms. D13 I / 12b (5.-7./10/1921), Husserl-Archives Louvain.

this way, the phenomenal changes introduced when we considered the total kinaesthetic system transform the entire oculomotoric field into an appearance of three-dimensional visual space. Husserl concludes:

> Thereby [through the phenomena of stretching, shrinking, and turning] the two-dimensional oculomotoric field is transformed into the three-dimensional field of space as a conjunction of the one-dimensional linear manifold of receding with the two-dimensional cyclical manifold of turning.[347]

Thus, phenomena of stretching and turning affect the entire two-dimensional oculomotoric field and transform it into a three-dimensional visual field, thus constituting visual space. Husserl makes clear that this transformation is not a geometric operation of embedding a two-dimensional plane in a three-dimensional space. Rather, the phenomena of stretching and turning qualify the oculomotoric field phenomenally.[348]

Husserl's account of the constitution of the spatial object and of visual space represents an interesting development from the traditional inferential view. He agrees with the inferentialists that three-dimensional visual space is constituted from two-dimensional sensations. This becomes particularly clear from the last quote above. But Husserl also realizes that a phenomenological account of the constitution of visual space can not succeed by simply interpreting the occurrences in the two-dimensional field as appearances of three-dimensional objects. In particular, the process of constitution cannot be understood as an interpretation of sensations as signs of spatial reality. This was the problem with the traditional inferential theories of perception. In contrast, Husserl suggests that constitution is a process that transforms the two-dimensional field in such a way that the third dimension becomes a *phenomenal* feature of it. The total kinaesthetic system has a completely new phenomenal quality. The viability of Husserl's approach thus depends on two presuppositions. First, the abstraction that leads to the

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[347] ["Dadurch [through the phenomena of stretching, shrinking, and turning] wird das zweidimensionale okulomotorische Feld in das dreidimensionale Raumfeld verwandelt als Verbindung der eindimensional linearen Entfernungsmannigfaltigkeit mit der zweidimensional zyklischen Wendungsmannigfaltigkeit."], Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, p. 255, translation in Edmund Husserl, *Thing and Space. Lectures 1907*, p. 216.

[348] Ibid., p. 204f.

two-dimensional oculomotoric field as that which contains the most basic elements of visual spatial experience must be phenomenologically legitimate. This means that we must be able to exhibit certain phenomenal occurrences under restricted conditions of observation that show the two-dimensionality of the oculomotoric field. According to Husserl, this can be done by restricting the kinaesthetic movements to mere eye movements. Second, the process by which the oculomotoric field is transformed must be such that it can actually explain the phenomenal change.

Husserl's account fails with respect to both presuppositions. First, the abstraction of the two-dimensional oculomotoric field is phenomenologically illegitimate. A particular visual scene maintains its three-dimensionality, even when the conditions of observation are restricted to mere eye-movements. This is even the case for one eye. Certain lines, for example, are seen as receding and not as perspectival projections onto a two-dimensional plane. Thus, the actual oculomotoric field is phenomenally much richer than admitted by Husserl. His notion of an oculomotoric field is, therefore, not the correlate of the oculomotoric kinaesthetic system, but rather an abstraction that is presumably driven by remnants of physiological and optical arguments like those encountered in Berkeley, Helmholtz, and Rock. In effect, we have to say that Husserl's theory of space perception is an inferential theory. Second, Husserl defines the phenomenal transformation of the field as the result of a law-like coordination between two different types of sensation, namely kinaesthetic sensations and hyletic data. But, as in the case of Berkeley and Helmholtz, appeal to an inferential mechanism of this kind, namely, to an association between sensations of different types, does not legitimize Husserl's conclusions about the phenomenal features of the result. As we have seen, such inferences are illegitimate. In order to avoid these problems, a phenomenological analysis of space has to start from the fully qualified spatial experience, rather than from an abstraction. This then is my third principle for a phenomenological analysis of perceptual space.[349]

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[349] I want to point out here that Stumpf also believes that the visual field, even of the one eye, is phenomenally much richer than admitted by Husserl and displays depth. Stumpf writes: "Die Tiefe des Gesichtssinnes ist ein besonderer Inhalt; derselbe ist nicht zum Teil aus anderen Sinnen dazugefügt; er ist auch nicht durch spontane Production des Vorstellungsvermögens entstanden. Es bleibt nur übrig, dass er direct empfunden wird." ["The depth of the visual sense is a special content; it is not partly added through

We can describe the problematic aspects of Husserl's analysis in a more general way. He attempts to bring together two types of phenomenological description. On the one hand, he wants to explain the genesis of our spatial experience and concludes that it is a process in which a two-dimensional field is transformed into an appearance of a three-dimensional objective space. On the other hand, he is concerned with the synchronic phenomenological description of spatial experience, i.e., the law-like coordination between different types of changes. Yet, as we have seen in the first part, these types of analysis cannot be mixed.

Husserl's later analyses contain an alternative account of the constitution of the third dimension. In 1921, he wrote:

> Of course, all of this is preceded by the question of whether "Depth" does not mean something that is foreign to the specifically visual domain; that is, something that can only be co-represented within the visual sphere and is actually represented within the tactile sphere.[350]

Moreover, in the context of his account of the constitution of the subject of kinaesthetic and visual sensations, namely the living body (*Leib*), he argues that distance is originally experienced as distance from one's own body.[351] In this way, he seems to introduce an original idea of distance that depends on the tactile, rather than the visual sense. We can describe this idea in the following way: I can be in immediate contact with an object, as,

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other senses; it is also not the result of a spontaneous production of the faculty of imagination. The only alternative is that depth is directly experienced."], Stumpf does not think that the visual field is phenomenally fully qualified. But the idea that it contains depth points in the right direction. Husserl seems to have been unaware of Stumpf's idea, thus falling back into a more traditional view. Carl Stumpf, *Über den psychologischen Ursprung der Raumvorstellung*, p. 176.

[350] ["Selbstverständlich geht all dem aber die Streitfrage voraus, ob 'Tiefe' nicht überhaupt etwas dem speziell visuellen Gebiete Fremdes besagt, also innerhalb der Sehsphäre ausschließlich mitrepräsentiert ist und eigentlich präsentiert innerhalb der taktuellen Sphäre."], Edmund Husserl, "Zu Hoffmanns Arbeit, besonders über Empfindungsbegriff, Sehding, Sehgröße, Raumwahrnehmung", Ms. D 13 III / 218b (1921), Husserl-Archives Louvain, (translation my own).

[351] Husserl's analyses of the constitution of the body are contained in the following manuscripts: D10 (1932) and D 12 (1931), Husserl-Archives Lovain. See also *Ideen II*, ch. 3, pp. 55-89. Smail Rapic argues that it is ultimately the constitution of the body as a presupposition for the kinaesthetic system that leads Husserl to believe that distance is not an original product of the visual sense. Smail Rapic, "Einleitung," in Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, ed. by Elisabeth Ströker (Hamburg: Felix Meiner Verlag, 1991), p. LXXI.

for example, when my feet touch the ground or my hands rest on a table. In such cases, the distance between the objects and the body part that touches them is zero. Yet, most objects are not immediately experienced in this way. Some will require very little effort to touch, something as minute as a very small movement, like when I reach for the cup right in front of me. Other objects require me to walk towards them before I can touch them. Given this, Husserl defines distance as follows: "The distance of any body is determined in relation to my initial position and the measure of the movement necessary to approach the body."[352] Thus, distance between my body and an object is the effort that is required to reach the latter. Since I have a general ability to reach any thing, at least in principle, I can determine not only the distances between my own body and these objects, but also between these objects themselves. Husserl writes:

> I can reach the place of any object, and in this way any object acquires a distance from any other object related to its distance from my own body . . . . The realization of the perception of distance lies therein that I can move to the position of the one and to the position of the other and directly move from one to the other. That is, I test how far the second object is away from me when I stand or would stand in the position of the first. The distance of a body from myself - its distance - is the original distance, and it is extrapolated to positions (which initially are positions in relation to me) of foreign bodies in relation to each other.[353]

If my interpretation of Husserl's later writings on space is correct, then the constitution of tactile distance becomes a necessary condition for the constitution of visual distance. Or more generally, tactile space becomes a presupposition for visual space. But, just as

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[352] ["Die Entfernung jedes Körpers ist bestimmt in Bezug auf meine Ausgangsstellung und das Maß meiner Annäherungsbewegung."], Edmund Husserl, "Zur Konstitution des Raumes . . .," Ms. D 13 I 19b (5.-7./10/1921), Husserl-Archives Louvain, (translation my own).

[353] ["An jeden Dinges Stelle kann ich heran, und so bekommt jedes Ding Abstand von jedem andern als Abstand meines Leibes von jedem . . . . Die Realisierung der Abstandswahrnehmung liegt also darin, daß ich mich an den Ort des einen und an den des anderen versetze und geradehin von einem zum anderen mich bewege. Das ist, ich erprobe, wie weit der zweite von mir ist, wenn ich an Stelle des ersten stehe oder stünde. Der Abstand eines Körpers von mir - seine Entfernung -, das ist der Urabstand, und das wird übertragen auf Orte (die auch zunächst Orte in Beziehung zu mir sind) fremder Körper in Beziehung aufeinander."], Edmund Husserl, *Zur Phänomenologie der Intersubjektivität. Texte aus dem Nachlaß: Zweiter Teil: 1921-1928*, Husserliana XIV, ed. by Iso Kern (The Hague: Martinus Nijhoff, 1973), pp. 541-544, (translation my own).

in the case of Berkeley, this suggestion has devastating consequences for Husserl's phenomenological analysis of visual space.

In order to bring out these consequences, I will state more clearly what the relation between tactile and visual space is. Husserl says very little about this; but we can reconstruct his argument in the following way. As we have seen, the visual spatial object and visual space in general were given in terms of a specific law-like correlation between typical changes in the visual field and changes in the kinaesthetic system. The same is true of tactile space -- it is a correlation between certain kinaesthetic sensations and tactile experiences. Husserl believes that this correlation is experienced as the ability to bring a given object into the reach of the tactile sense. He also calls the correlation a "system of *Vermöglichkeiten*."[354] Tactile space is constituted as the correlate of a system of tactile *Vermöglichkeiten*.[355] Thus, we have two systems of law-like correlations, one concerning visual and the other concerning tactile changes. Given this, we can say that the visual field is transformed through a process of association. Certain occurrences in the visual field are associated with certain tactile experiences or expectations of tactile experiences because both are motivated by the same kinaesthetic sensations. As a result, the stretching of a certain object in the visual field is accompanied by a certain kinaesthetic sensation that can be interpreted as a reduction of the distance between observer and object. In this alternative explanation of depth, Husserl's similarity to Berkeley and Helmholtz becomes even more obvious. Spatial perception becomes the result of an association between sensations of different kinds. Again, this does not

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[354] See, for example, manuscript D10 (1932), Husserl-Archives Louvain.

[355] For Husserl, the tactile kinaesthetic system is a result of experience, acquired during the ontogenesis. He writes about the kinaesthetic system: "Es ist ein System möglicher 'subjektiver Bewegungen', das durch vielfältiges, sei es auch regelloses Durchlaufen verschmolzen ist zu einem vertrauten habituellen Bewegungssystem (jede mögliche Bewegung also eine bekannte und praktisch zu intendierende). Durch Übung ist Herrschaft über dieses System erwachsen, jede intendierte Bewegung 'kann ich' also, und darin liegt, sie ist jederzeit für mich ausführbar und als das in eins mit ihrer Vorstellung bewußt." ["It is a system of possible 'subjective movements' that is unified into a familiar habitual system of movement (every possible movement is thus familiar and can practically be intended) through divers repetition, be it without following a rule."], "Typologie des visuellen Feldes und die zugehörige Kinästhese. . . ," Ms. D13 IV (1921), Husserl-Archives Louvain, (translation my own).

explain how the visual field is *phenomenally* transformed in such a way as to appear three-dimensional.

By criticizing Husserl's theory of perception, and, in particular, his accounts of the phenomenal constitution of *Leibhaftigkeit*, identity, and spatiality, I have argued that a phenomenological analysis of visual space has to obey three methodological principles. First, such an analysis has to be formulated in object terms, i.e., as a noematic, rather than a *reell* analysis. Second, this analysis has to take as its point of departure the full experience of visual space. And finally, this analysis must not introduce a distinction between the intentional and the real object. In the next section, I will modify Husserl's analysis of the constitution of visual space according to these principles. We will see that visual space is experienced as a perspectival system. On the basis of this result, I will then formulate a concept of spatial intuition that allows me to explain the generality of the results of Euclid's method.

8. The Phenomenological Constitution of Visual Space and the Apriority and Generality of the Propositions of Material Geometry

8.1 The Phenomenological Constitution of Visual Space and the Concept of Spatial Intuition

In *Ding und Raum*, Husserl describes how visual space is experienced in terms of his noematic analysis:

> One bodily objectivity is seen, but it leaves open infinitely many possibilities for further objectivities in the "between." The "between" however is constituted because discrete extensions, no matter what they are, can be mediated by continuous extensions in different ways and finally continuously. Although we cannot say that empty space is seen, we have the between as an empty space, that can be filled continuously, as a mere possibility of actual mediations that are characterized by laws. We can see only bodies and with them we see the between. Space is rather implied in actual perception.[356]

According to this passage, visual space is experienced as a system of places, some of which are occupied with visual spatial objects. I believe that this description of visual space is correct. Thus, in order to analyze its phenomenal properties, we have to show how a system of places is constituted. This, in turn, requires an analysis of the experience of individual spatial objects.

According to Husserl's description of the constitution of spatial objects in terms of the *reell* analysis, it is an invariant of certain kinaesthetically motivated transformations of pre-empirical objects in the oculomotoric field. In other words, a

[356] ["Eine Körperlichkeit ist gesehen, aber unendlich viele Möglichkeiten für weitere Körperlichkeiten läßt sie offen, nämlich in dem 'Zwischen'; das Zwischen aber konstituiert sich dadurch, daß Dehnungsdiskretionen, wie immer sie bestehen, durch Dehnungskontinua in verschiedener Weise vermittelt werden können und schließlich in kontinuierlicher Weise. Das Zwischen als leeren, aber kontinuierlich erfüllbaren Raum, als bloße Möglichkeit gesetzlich bestimmt charakterisierter realer Vermittlungen hätten wir hier also, obschon wir nicht sagen können, daß der leere Raum gesehen sei. Gesehen sind die Körper, und mit dem Gesehenen erfaßt ist das Zwischen, das die Phantasie dann körperlich so oder so ausfüllen kann. [Der Raum] ist also eher mitgesehen."], Edmund Husserl, *Ding und Raum. Vorlesungen 1907*: *Vorlesungen 1907*, p. 261, (translation my own).

spatial object is the correlate of the total kinaesthetic system. As we have seen, Husserl arrives at this result through a number of abstractions. He departs from the fully constituted physical object, the *res materialis*, and first isolates certain ideal essences from it, namely the *res extensa* and the *res temporalis*. Husserl takes it that the description of the constitution of the spatial object is nothing other than the analysis of the *res extensa*. In this context, he calls the *res extensa* the 'transcendental guiding principle' (*transzendentaler Leitfaden*).[357] Husserl further introduces the abstraction of the oculomotoric field, in order to describe the constitution of the *res extensa*, the visual phantom. In the previous section, I showed that this approach was misguided. Husserl's abstraction of the oculomotoric field is phenomenologically illegitimate. By extension, this also means that the *res extensa* cannot function as an adequate transcendental guiding principle. This is no surprise, since Husserl's method of eidetic intuition (*Wesensschau*) itself is highly problematic. Nevertheless, in order to modify Husserl's phenomenological analysis and to formulate it in object terms, we require a 'guiding principle.' Thus, before entering into the actual analysis of the constitution of spatial objects, I will formulate such a principle. In contrast to Husserl, I will do so by reference to the explanatory purpose of the phenomenological analysis, namely the possibility of a material geometry, thus avoiding alleged insights into *a priori* essences. In contrast to Husserl's, my guiding principle will be a hermeneutical, rather than a transcendental, principle.

In order to formulate a principle that can guide my phenomenological analysis of visual space, I will focus on two presuppositions of the application of Euclid's method.

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[357] Husserl writes: "Die Stufenfolge der formalen und materialen Wesenslehren zeichnet in gewisser Weise die Stufenfolge der konstitutiven Phänomenologien vor, bestimmt ihre Allgemeinheitsstufen und gibt ihnen in den ontologischen und material eidetischen Grundbegriffen und Grundsätzen die 'Leitfäden'." ["The sequence of levels of formal and material theories of essence prescribes in a certain way the sequence of levels of the constitutive phenomenologies, determines their levels of universality and provides them with 'guiding threads' in the ontological and materially fundamental concepts and principles."], *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie: Erstes Buch*, p. 322, translation from Edmund Husserl, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. First Book*, p. 369, slightly modified. The term 'transzendentaler Leitfaden' appears in Edmund Husserl, *Formale und transzendentale Logik*, p. 252. See also Ulrich Claesges, *Edmund Husserl's Theorie der Raumkonstitution*, pp. 27-32.

The first presupposition is that the geometer has to understand geometric diagrams, not as perspectival representations of the surfaces of spatial objects, but rather as figures drawn without perspective. I want to illustrate this presupposition by means of an example. Consider the diagram in Figure 12, which we will presume is serving in a Euclidean proof in plane geometry.

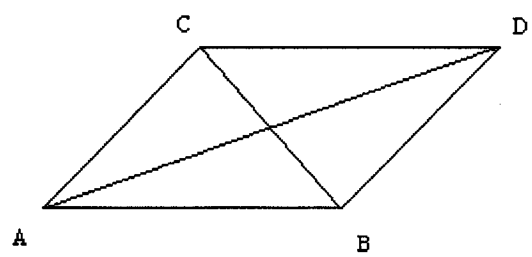

Figure 12: Rhombus

The rhombus in this diagram consists of two isosceles triangles. We can produce a rhombus like this one by constructing two right-angled triangles ABD and ACD which have AD in common and angles ACD and ABD are congruent and larger than 90°. For the rhombus ABDC, we can show, for example, that its diagonals AD and BC have different lengths, more precisely, we can show that AD > BC. Yet, if we were simply given the figure ABDC, we could see it as representing the side of a cube as in Figure 13.

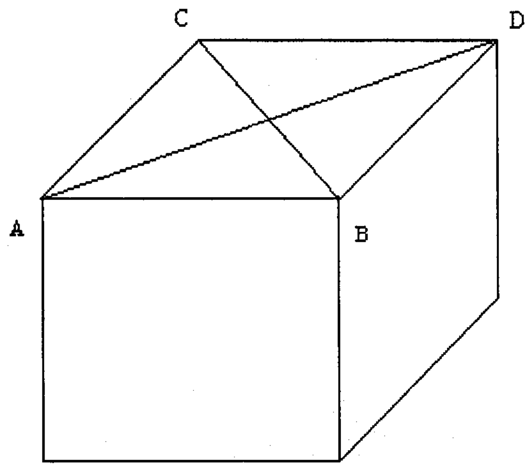

Figure 13: Cube

The shape of rhombus ABDC is distorted by the perspective from which it is seen. If seen in this way, the rhombus's diagonals would have the same length. Further geometric inferences would lead to results different from those of Euclidean geometry. This example shows that the correct reading of Euclid's diagrams requires the geometer to understand them not as figures drawn as perspectival representations, but rather as figure drawn without perspective. The case for solid-geometry is slightly different, in that the diagrams used as sources of knowledge are drawn according to certain perspectival conventions. For example, the fact that a line recedes into the third dimension is indicated by the angle it forms with a horizontal line. In Figure 13, line BD is drawn with an angle of about 45°, indicating that it recedes into the third dimension. The geometer who examines these figures has to understand that they are drawn as conventional representations of actual shapes, rather than as conventional representations of perspectival appearances.

The second presupposition for the practice of Euclidean geometry is that the geometer has to neutralize the perspectival distortions that occur while inspecting the diagram. Any figure drawn on paper or some other background can be seen from different angles and distances. In accordance with the angle of observation, the figure will change its appearance. A rectangle, for example, may look like a rhombus, or right angles might appear to have less or more than 90°. The geometer has to neutralize these distortions and consider the figure not in perspective. This is not a complex cognitive task, but something that we do all the time in everyday perception. If we want to see the "actual" shape of a given object, we change angle and distance of observation until the non-perspective shape is seen. This usually requires looking at the shape from an angle of approximately 90° and a suitable distance.[358]

It is interesting to observe that these two presuppositions are implicitly addressed in Euclid's definitions. In Part II, I cited Reed's conclusion that Euclid's definitions allow a geometer to appeal only to the relation between surfaces (solids) and their boundaries. Euclid even adds the definition of a figure as that "which is contained by any boundary or boundaries."[359] But specifying the geometric interest in this way implicitly assumes that geometric objects (except for points, lines, and angles) are nothing other than bounded areas. In particular, they are *not* perspectival representations of actual geometric shapes. Euclid was not necessarily consciously aware of these geometric presuppositions.[360] In fact, they seem almost too obvious to be noticed. Nevertheless, Euclid's definitions clearly express them.

The two presuppositions allow me to formulate a hermeneutical guiding principle for my analysis of the constitution of the spatial object. The presuppositions show that the practice of Euclidean geometry requires both an implicit understanding of the nature of perspectival representation and the adoption of certain perspectival conventions. I believe that we can therefore take the pre-geometric concept of spatial perspective as the hermeneutical principle that will guide an analysis of the constitution of the spatial object sufficient for explaining the generality of Euclid's results. Thus, the pre-geometric analysis of perceptual space has to describe the latter under the aspect of perspective, i.e.,

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358 Husserl's account of the constitution of the actual, i.e., non-perspective, shape of a figure in the perceptual process is along these lines. He believes that the actual shape of an object is nothing other than the cognitive equivalent of a phenomenal optimum. The optimum is simply the object as seen under optimal conditions of observation, including optimal distance, angle, and illumination, etc.

359 Euclid's definition 14. Cf., Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements*, p. 153.

360 Nevertheless, Euclid was aware of the phenomenon of perspective distortion. This constitutes the subject-matter of his Optics.

describe the way in which objects of spatial perception and perceptual space itself are given in perspectival appearances.

Given the guiding principle of perspective, I will now modify Husserl's account of the constitution of spatial objects. For Husserl, as we have seen, a spatial object is experienced as an intentional unity, or invariant, given in a law-like correlation between certain typical changes in the oculomotoric field and in the kinaesthetic system. We can reinterpret this account in terms of perspective by analyzing an example. Assume an observer perceives a given, visual, spatially extended object from a fixed point in space. In this case, the object is seen only from one aspect; it is given only by means of a certain fixed, adumbrated representation. Since the object is experienced as a spatial object, however, this perception of one side entails a system of other possible sides which the observer could see, if he/she moved in certain ways. So, for example, if the observer perceives a car from the front, he/she can expect that successively different points of observation will give successively different aspects, like the car's doors, and then its back, etc. Being only in one single position at a time, the observer will not have a fully specified idea of how these sides actually look. Nevertheless, he/she will have a general idea of what to expect. This means that a spatial object is given through a law-like connection, not between concrete changes in the observer's position and concrete changes of the adumbrated representation, but rather between certain types of movements and certain types of perspectival changes. We can conclude that a spatial object is experienced as the correlate of a perspectival system, that is, as a result of a law-like correlation between typical changes in an object's appearance (i.e., in the way in which the object is seen from a given perspective) and typical changes in an observer's position. At any specific instant, only one of these infinitely many possible appearances is actually given. The others are implied by it.

Husserl points out that a spatial object is characterized by the fact that it has a closed surface. This means that among the series of perspectival changes there are some that are cyclical. For example, when an observer completes a 360° circle around the object, he/she sees the very same side of it from which he/she departed. The same circle, with the same positional and perspectival changes can be repeated indefinitely. There are also infinitely many such cyclical perspectival systems for each spatial object. Husserl writes:

> The closure of corporeal form thus manifests itself in the cyclical nexus of appearance, and, in different cyclical lines of this sort, different cyclical nexuses of the sides of the same body appear. We could say that in this way cyclically closed sides of the body appear and that the completely closed body is what is identical in all such cyclical sides.[361]

As a result, the entire object in all its concreteness is actually given only as the correlate of the infinite number of cyclical perspectival systems.

This description of the constitution of the spatial object fulfills the three methodological principles formulated in the previous two sections. The elementary experiences are described in object terms, i.e., in terms of Husserl's noematic analysis. This can best be seen by comparing the main concepts used in the two accounts. Husserl described the constitution of a spatial object by reference to the concepts "oculomotoric field" and "kinaesthetic system." These are two terms specifically designed for a *reell* analysis. My analysis, in contrast, appeals to the object as it appears from a given perspective and the objective position of the observer. These are both ideas using only object terms. Second, both the position of the observer and the object appearing are given in a fully constituted three-dimensional space. Thus, although the experience of a particular series of positional and perspectival changes is described from a first-person perspective, it does not appeal to private data that may not be accessed by anyone else. The data are objective in the sense that any observer can occupy the same position and move in the same direction, thereby seeing the very same series of changes in the appearing object. Finally, my description does not imply a distinction between intentional and real object.

[361] ["Es kommt die Geschlossenheit der Körpergestalt im zyklischen Erscheinungszusammenhang zur Erscheinung, und in verschiedenen solchen zyklischen Linien verschiedene zykische Seitenzusammenhänge desselben Körpers. Wir können sagen, es erscheinen so zyklisch geschlossene Körperseiten, und der voll geschlossene Körper ist das Identische all solcher zyklisch geschlossenen Seiten."], Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, p. 206, translation from Edmund Husserl, *Thing and Space. Lectures 1907*, p. 174.

I want to illustrate the phenomenological structure of the experience of a spatial object by means of a diagram. Figure 14 represents the structure of a spatial act in which an observer gradually changes his/her position with respect to a given object.

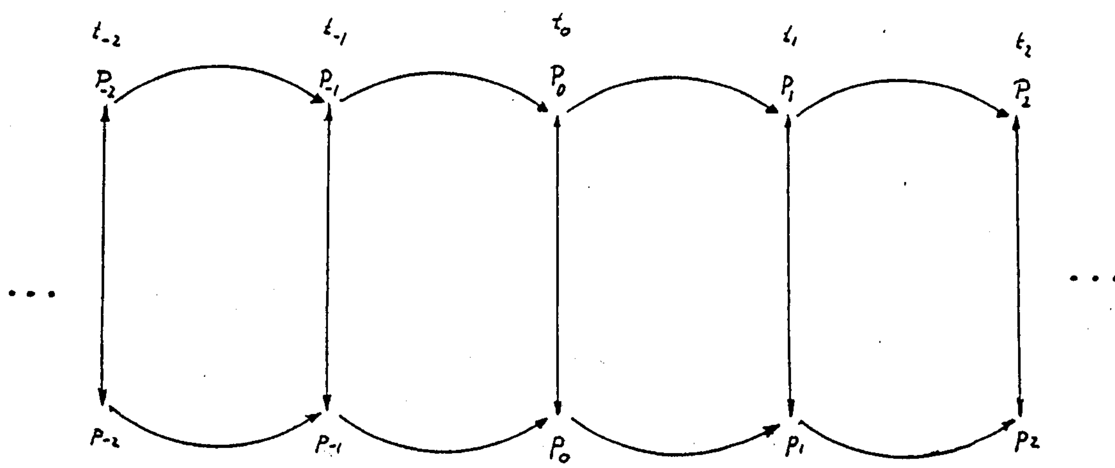

Figure 14: Structure of a Spatial Act

t_0 indicates the actual moment of observation; P_0 is the side of the object actually seen ; p_0 is the objective position of the observer; t_{-1} to t_{-n} are previously experienced moments picked out from the actual continuum of moments; p_{-1} to p_{-n} and P_{-1} to P_{-1} are the corresponding positions and appearances; t_1 to t_n are future moments at which certain positional changes p_1 to p_n will motivate certain expectations as to the appearances P_1 to P_n. These expectations are governed by the type of object under observation. The structure of a perceptual act is more complex than shown in Figure 14, because the future expectations should not only correlate with one particular series of perspectival changes, but rather with an infinite number.

Having explained the constitution of visual spatial objects, we can now say how visual space is experienced. Following Husserl, I defined visual space as a system of places for visual objects. Since visual objects are given in perspectival systems, it follows that perceptual space is a system of infinitely many possible perspectival systems. An individual perspectival system is actually experienced only if a particular object occupies the place under observation.

My analysis of the constitution of visual space contains two restrictions that I must lift if I am to describe further perspectival occurrences. On the one hand, I restricted myself to observer movements and ignored the movements of the objects. On the other hand, I did not take into account that an observer usually sees not only one object, but many, located in different places. If objects move in space, the observer will experience certain perspectival changes that will be specified according to the object-types at hand. Thus, the experience of perceptual space must include perspectival changes not only motivated by the observer's but also by the object's movements. Moreover, the experience of seeing moving objects also includes the occlusion of some aspects of the moving object by others.[362] If we take into account the fact that an observer usually sees a multiplicity of perceptual objects, then we must conclude that he/she will experience changes in their relative position to each other.[363] Thus, phenomena like occlusion and parallactic displacement will occur.

On the basis of this description of the constitution of visual space I will now formulate a phenomenological notion of spatial intuition. I argued that a spatial object is given as a law-like coordination between changes in the object as it appears and changes in the observer's position. Space itself was given as a generalized perspectival system. The law-like coordination between the two types of change is expressed in an observer's perspectival expectations. In other words, the fact that there is a law-like correlation between the two types of change means that an observer has certain perspectival expectations. For example, an observer who fixes his/her gaze on a house will expect its appearance to change in certain, predetermined ways, when he/she moves. Thus, from a phenomenological point of view, the perception of spatial objects and thus of space includes expectations which express a perceiver's knowledge of possible perspectival changes. I will call the ability to anticipate certain perspectival changes based on this knowledge "spatial intuition."

In section 8.3, I will exploit this concept of spatial intuition by explaining how the generality of the results of Euclid's method is established. But before I turn to this

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[362] Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, p. 249.

[363] Ibid., p. 234.

discussion, I will draw conclusions as to the necessary structure of visual space.

8.2 The Necessary Structure of Spatial Intuition

Husserl believed that the fact that space was given as a transformation of two-dimensional "pictures" into appearances of three-dimensional objects dictated the general form of the laws correlating kinaesthetic sensations with changes in the appearances of a given spatial object. Accordingly, he felt that spatial intuition had a necessary structure:

> On the other hand, it is to be understood, and to be founded mathematically in every case, that not ever arbitrary lawful variation of images that are continuously expanding, and whose orientation is constantly changing, needs to be apprehensible through the projection of a three-dimensional body in three-dimensional space. Thus the constitution of precisely a three-dimensional space, on the basis of the conversion of the oculomotoric field into a manifold of expansion or of displacement, will then certainly not be deducible a priori. We can only say this much: . . . The forms of this lawfulness are then circumscribed by the further requirement that this Objectivity has to be thing-like, i.e., one in which things in manifolds are posited in fixed relations such that possibilities of movement and change remain open. of movement and change. But identity in motion requires a continuum of places, one that is congruent in itself.[364]

According to this passage, the fact that human beings experience rigid, or, as Husserl says, "thing-like" objects presupposes that these can change their place without deformation. Husserl then appeals to a result of Helmholtz's and Lie's mathematical analysis that shows that this is possible only if the objects are in a space that is a "self-congruent continuum," i.e., that has a constant curvature, be it negative, positive, or

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[364] ["Andererseits ist es einzusehen und jedenfalls mathematisch zu begründen, daß nicht jede beliebige gesetzmäßige Variation von sich kontinuierlich dehnenden und ihre Orientierung ändernden Bildern durch Projektion eines dreidimensionalen Körpers auffaßbar sein muß, und so wird die Konstituition eines dreidimensionalen Körpers aufgrund der Überführung des okulomotorischen Feldes in eine Dehnungs- und Wendungsmannigfaltigkeit sicherlich nicht *a priori* deduzierbar sein. Aber soviel können wir sagen: . . . Die Formen dieser Gesetzmäßigkeit sind dann umschränkt durch die weitere Forderung, daß diese Objektität eine dingliche sein soll, nämlich eine solche, in der Mannigfaltigkeitsdinge sich in feste Verhältnisse setzen derart, daß Möglichkeiten der Bewegung und Veränderung offen bleiben. Identität in der Bewegung setzt aber voraus ein stetiges Ortkontinuum, und zwar ein in sich kongruentes."], Edmund Husserl, *Ding und Raum. Vorlesungen 1907*, p. 243, translation in Edmund Husserl, *Thing and Space. Lectures 1907*, p. 205, slightly modified.

zero.[365] Thus, for Husserl a rigid spatial object can be experienced (i.e., perceived or imagined) only as part of an intuitive space whose global structure has a constant curvature.

Husserl's conclusion cannot be reformulated in terms of my perspectival analysis because it appeals to Helmholtz's scientific results concerning an idealized concept of space. Helmholtz defined a continuous point manifold and then explicated the conditions under which this construct allowed for free mobility. Yet, the space under consideration, as well as his notion of free mobility, are ideal mathematical concepts that cannot describe pre-geometric spatial experience. Even if we understand Helmholtz's analysis as being of a physical space, we still cannot escape the problem of idealization.[366] In this case, his results can show that a physical object can be moved without being deformed only if a space has constant curvature. As I demonstrated in Part I of this thesis, however, Husserl believed that physical space was constituted in a physical theory and that it was thus an ideal construct. Again, the results reached by analyzing such an idealization do not necessarily apply to the space of pre-scientific experience. In this case, too, the results of Helmholtz's analysis would not apply to visual space. As a consequence, since Husserl's analysis of the necessary structure of visual space was based on Helmholtz's results, it fails.

Although Husserl's conclusions are based on unacceptable premises, he is, nevertheless, correct in assuming that the experience of spatial objects and of space has a necessary structure. In order to see why this is the case, we have to do two things, namely supply a further premise and reformulate Husserl's argument in terms of the perspectical analysis of visual space. Material geometry establishes a specifically geometric concept of space. In other words, material geometry deals with purely

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[365] Husserl refers to Helmholtz's results concerning manifolds that allow for free mobility in his "Über die Tatsachen, welche der Geometrie zugrunde liegen," *Göttinger gelehrte Nachrichten*, 1868, p. 193-222. Husserl was also familiar with the mathematically more sophisticated account of Helmholz's view by Sophus Lie. See his criticism of Lie in "*Verschiedene Richtungen in der Geometrie*," Husserliana XXI, p. 412.

[366] For the distinction between the two ways of reading Helmholtz's program, see Roberto Torretti, *Philosophy of Geometry from Riemann to Poincaré* (Dordrecht/Boston/Lancaster: D. Reidel Publishing Company, 1978), p. 314.

geometric, as opposed to physical, properties of space. In the previous part of this thesis, I argued that Euclid's method departs from diagrams, that is, from objects of pre-geometric visual experience. Accordingly, for material geometry to be possible, pre-geometric spatial experience must allow the geometer to isolate purely spatial properties. Given my phenomenological description of pre-geometric spatial experience, this is accomplished by the perspectival system. Yet, in order to do so, the perspectival system has to follow the same laws everywhere. In other words, an observer can experience purely spatial properties only if the space constituted by the perspectival system is homogeneous. I am using the term 'homogeneous' here in a pre-geometric sense, designating a universal perspectival law. In order to show that the perspectival system has to follow the same laws everywhere, I want to consider a world in which this is not the case, i.e., a world in which the laws governing the perspectival system change from place to place. Assume that in such a world an observer holds a pencil in his/her hand and slowly turns it. Spatial intuition would allow him/her to predict the typical perspectival changes according to a given law. Now assume that the observer turns 30° without changing his/her position relative to the pencil and then repeats the very same movements. Since, according to our assumption, space is not homogeneous, the perspectival changes of the pencil would generally not exhibit the same law-like behaviour. Simply by moving the pencil, the observer would have changed its shape. If occurrences like this one were possible in pre-geometric experience, a geometer would not be able to draw a distinction between purely spatial changes and other types of change. Thus, homogeneity is a necessary feature of perceptual space, if the latter is to allow a geometer to construct a material geometry, i.e., a specifically geometric concept of space.

The homogeneity of perceptual space allows me to draw conclusions as to the necessary form of material geometry. In the previous section, I argued that Euclid's practice requires the geometer to neutralize perspectival deformations of the diagrams and to see them as figures defined by their boundaries. The fact that perceptual space is homogeneous (in the pre-geometric sense of this term) means that the results of the neutralization are the same for a given geometric figure no matter where it is located.

Thus, the application of the same constructive procedures will constitute the same geometric figure, independent of location. Consequently, no matter where the geometer constructs a given proof, the results will be the same. This means that after geometric idealization, the propositions of Euclidean geometry describe a space that is *geometrically* homogeneous. But, as we know from Helmholtz and Lie, this is possible only if the propositions constructed by Euclid's procedure describe a space with constant curvature, be it negative, positive, or zero. As a result, the fact that only a homogeneous (in the pre-geometric sense of the term) visual space can allow us to construct a geometric concept of space necessitates the latter's form: the space constructed by the propositions of a material geometry will have to have a constant curvature.

My conclusion that geometric space must have constant curvature concurs well with a point made by Michael Friedman, who showed that elliptic and hyperbolic geometries with constant curvature can also be constructed by collapsing circle and unmarked straightedge.[367] A geometer applying Euclid's method in a space with constant negative or positive curvature would be able to derive propositions that are consistent with the two types of non-Euclidean geometry. Further, a geometer not living in a space with constant curvature, would not be able to explicate the structure of this space by means of Euclid's method. We can also put the point in terms of the notion of spatial intuition, which I defined above as a system of perspectival expectations. If a system of specifically spatial expectations is to be possible at all, it must exhibit a necessary form such that it generates a material geometry describing a space with constant curvature.

In Part II, I argued that Euclid's Axioms (including the fourth Postulate) express fundamental truths about the structure of visual space. We can now see that they are implied by the homogeneity of space. I think it is obvious that Axioms four and five are true in any metrical space, and thus, in particular, in a homogenous metrical space. The other Axioms, in contrast, are specific to homogenous space. Postulate four ("That all right angles are equal to one another") expresses the fact that we can compare right angles. As we have already seen, This is possible, only in a space in which the

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[367] Michael Friedman, "Geometry, Construction, and Intuition in Kant and His Successors," in G. Sher and R. Tieszen, eds. *Between Logic and Intuition: Essays in Honor of Charles Parsons* (Cambridge: Cambridge University Press, 2000), pp. 206-209.

movement of a geometric object does not change its shape, that is, in a homogenous space.[368] The same is required for Axioms one to three. These axioms deal with the comparison, addition, and subtraction of extension. In order to be compared, added, or subtracted, geometric objects must not change their shapes, when transported through space. Again this is possible only in a homogenous space. My account of spatial intuition also explains why a geometer has immediate access to these axioms and does not have to derive them by means of empirical generalization. The axioms simply make explicit some fundamental features of the system of perspectival expectations.

At the end of the nineteenth and the beginning of the twentieth centuries, a number of authors attempted to rescue a Kantian theory geometry and argued that perceptual space had a necessary structure. Some of the most prominent among them were Russell (1897), Husserl (1907), and Becker (1923). With the exception of Becker, they differed from Kant in that they believed that the necessary structure of perceptual space was more general than Euclidean geometry. In order to show the specific character of my own argument more clearly, I will briefly consider Russell's and Becker's arguments.

In his *An Essay on the Foundations of Geometry*, Russell argues that sense-perception has a necessary form that is given by the axioms of projective geometry.[369] He presents a two-step argument. He first shows that the axioms of projective geometry can be analytically deduced from the concept of a form of externality. He then argues that sense-perception, if it is to ground knowledge, has to have a form of externality. Both arguments together show that if knowledge is to be possible at all, the axioms of projective geometry must be true of perceptual space. The particular curvature of this space, in contrast, is a matter for empirical investigation.

In order to establish his first point, Russell first states the axioms of projective geometry and then shows how they can be derived from the concept of a form of

[368] See, Sir Thomas L. Heath, *Euclid. The Thirteen Books of Euclid's* <u>Elements</u>, p. 200.

[369] Bertrand Russell, *The Foundations of Geometry* [1897] (London and New York: Routledge, 1996).

externality. The axioms, according to him, are the following:

> (1) We can distinguish different parts of space, but all parts are qualitatively similar, and are distinguished only by the immediate fact that they lie outside one another.
>
> (2) Space is continuous and infinitely divisible; the result of infinite division, the zero of extension, is called a *point*.
>
> (3) Any two points determine a unique figure, called a straight line, any three in general determine a unique figure, the plane. Any four determine a corresponding figure of three dimensions, and for aught that appears to the contrary, the same may be true of any number of points. But this process comes to an end, sooner or later, with some number of points which determine the whole of space. For if this were not the case, no number of relations of a point to a collection of points could ever determine its relation to fresh points, and Geometry would become impossible.[370]

Russell's derivation of the first two axioms is much shorter than that of the third. Nevertheless, they suffice for the purpose of my criticism, and so I will restrict my argument to them.

In order to derive the axioms of projective geometry, Russell first defines a form of externality as follows:

> In any world in which perception presents us with various things, with discriminated and differentiated contents, there must be, in perception, at least one 'principle of differentiation,' an element, that is, by which the things presented are distinguished as various. This element, taken in isolation, and abstracted from the content which it differentiates, we may call a form of externality.[371]

According to this definition, a form of externality is the 'principle of bare diversity.' Russell then explicates three properties entailed by this notion. First, since the form of externality is the result of a process of abstraction from material content, it does not contain any intrinsic qualities. Thus, the form of externality differentiates between the different positions of perceptual particulars only on the basis of their external relation to

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[370] Ibid., p. 133.

[371] Ibid., p. 136f.

each other. In other words, position becomes relative. Second, since the form of externality is a result of abstracting away from material contents, the principle of differentiation itself must be uniform and undifferentiated, i.e., homogeneous. And finally, since the form is homogenous, the relation between any two perceptual particulars must be capable of continuous alteration, i.e., infinite division. The products of this division will be points. Russell's first axiom of projective geometry thus becomes identical with the relativity of position in a homogenous space, while his second axiom becomes identical with the continuity of space.

Russell then argues that the concept of a form of externality is a condition for the possibility of knowledge. If knowledge is to be possible, knowledge understood here as an inference from sense-perception, then individual sensations must necessarily be in relation to other sensations. Otherwise, we would be presented with an undifferentiated perceptual content from which nothing could be inferred. Thus, the objects of perception must contain a diversity, and this is possible only if perception itself contains as an element some form of externality. This form must have more than one dimension, because different things have to be presented at the same time for inferences to be possible. We need a diversity of simultaneously existing things. Russell concludes that the possibility of knowledge presupposes a form of externality that, by virtue of the previous argument, is governed by the axioms of projective geometry.[372]

Russell's argument was criticized early on by Poincaré, who pointed out that Russell's three axioms did not suffice to characterize projective geometry.[373] In his reply, Russell admitted the justice of this criticism and reformulated his axioms. But at that time he had already changed his view and no longer attempted to derive the axioms from the notion of externality. Independently of this technical criticism, Russell's argumentative strategy contains two fundamental problems. The first problem is that his point of departure in the first step of his argument contains a vicious circle. He begins by abstracting away from the material content of perception in an attempt to exhibit its pure

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[372] Ibid., p. 181.

[373] Henri Poincaré, "Sur le hypothèses fondamentales de la géométrie, in *Bulletin de la Société Mathématique de France*, 15 (1899): pp. 203-216.

form. Yet, the possibility of such an abstraction presupposes the distinction between form and matter. For Russell, this distinction is equivalent to that between qualitative and quantitative (extensional) features of sense-perception. Such a distinction between quality and quantity might be drawn in sense-perception. But in this case, quantity would be an imprecise, rather that an ideal, geometric concept. Thus, it becomes clear that Russell is presupposing an ideal geometric concept of extension -- the same one he referred to in his analysis of projective geometry. If this were not the case, he would not be able to derive the notion of infinite divisibility from the concept of externality, since that would require a concept of continuity (denseness) that is not available in our perceptual world. His argument is, therefore, circular. In my own argument for the homogeneity of perceptual space, I avoid such a circularity by choosing a descriptive vocabulary that is pre-geometrical. The notion of homogeneity I used refers only to the fact that there is only one perspectival system.

The second problem arises from Russell's assumption that the possibility of knowledge requires a difference in simultaneity and thus a form of externality. This premise for the possibility of knowledge is too general. We can see this by considering an argument from Strawson, who believed that in spite of the fact that *our* actual conceptual scheme requires a spatio-temporal framework in order to differentiate and reidentify particulars, we can nevertheless imagine a different conceptual scheme in which such a framework is tonal rather than spatial.[374] In order to make this possibility intelligible, Strawson described a "world of sound," i.e., a world in which the most basic particulars are not material bodies, but individual sounds. Since this scenario will also allow me to criticize Becker's point of view later, I will briefly outline it here.

Strawson's central question is the following: "could a being whose experience was wholly non-spatial have a conceptual scheme which provided for objective particulars?"[375] In order to answer this question, we have to make intelligible to ourselves the idea of a being whose sensory input is essentially non-spatial. Strawson

[374] P.F. Strawson, *Individuals. An Essay in Descriptive Metaphysics* (New York and London: Routledge, 1959), pp. 59-76.

[375] Ibid., p. 66.

believes that such a being could be understood as one whose senses present it only with auditory data, which, when considered on their own, i.e., without any kinaesthetic sensations, do not convey any spatial content. In order to answer Strawson's question, we have to abstract from all other sense-modalities and to see whether it is possible for a being whose world is purely auditory to experience objective sound particulars.[376] Strawson takes the notion of objectivity here in a relatively weak sense, as "distinct from myself and my own states." But he points out that this notion is connected with a much stronger sense that is required not only for Russell's argument, but also for any type of objective experience. The being whose conceptual scheme we are trying to imagine must be able not only to distinguish between its own mental states and objects existing independently of them, but in addition to reidentify objective particulars. The question, then, is whether a being whose world is purely audible can experience reidentifiable sound particulars.

In our familiar world, the reidentification of particulars is achieved by a "spatial system of objects, through which oneself, another object, moves, but which extends beyond the limits of one's observation at any moment, or, more generally, is never fully revealed to observation at any moment."[377] This system is able to "house" objects even when they are not being observed. Thus, according to Strawson, we have to find a tonal analogue in the world of sound of a location where sounds can be heard, even when no one is there. He suggests that a continuously heard master-sound can fulfill this function. Such a sound has constant volume, but varies its pitch. Any other sound or continuous series of sounds, like a melody, would be accompanied by this master-sound. The auditory world would then "sound" as follows. We would hear a certain sequence of tones, say a particular melody, while the master-sound was not changing its pitch. When the master-sound changes its pitch, the sequence is interrupted and another sequence of tones would be heard. We would be able to continue changing the sequences by changing the pitch of the master-sound. But we would also be able to return to any of the

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[376] Ibid., pp. 64-66.

[377] Ibid., p. 74.

sequences by turning the mater-sound back to the pitch that it was originally perceived at. Thus, every sequence would be associated with a particular pitch of the master-sound, which, therefore, would function as a tonal framework, in which objects, i.e., sound-sequences, can be reidentified. Strawson compares the process of reidentification in such a tonal world with the turning of the station knob on the radio. Many positions of the knob would be associated with particular stations, or better, with the series of sounds transmitted by them. Thus, the position of the knob would allow us to identify and reidentify a given station. In the auditory world, the master-sound plays the role of the knob.

Strawson's thought experiment shows that Russell's point of departure was too general. He wants to derive the necessary existence of a form of externality from the mere possibility of knowledge. In particular, he argued that empirical knowledge presupposes the simultaneous existence of sensual objects and that that this was possible only in a spatial world. Yet, in Strawson's auditory world, an object is individuated by qualitative, rather than quantitative features. No spatial form of externality is required for the possibility of knowledge in such a scheme. In contrast to Russell's, my own argument departed from the fact that we can construct a material geometry grounded in perception. In contrast to Russell, I merely claimed that the possibility of this particular practice required that perceptual space allows us to draw a distinction between purely spatial and physical change. Thus, in my view both the existence and necessary structure of perceptual space are conditional, based on the existence of this particular type of knowledge.

Oskar Becker also argues that perceptual space has a necessary form, and gives a transcendental and a phenomenological argument to prove it. The latter is based on the faulty description of perceptual space that I have already criticized in Husserl.[378] I will therefore focus here only on Becker's transcendental argument, which is based on the same premise as Russell's, namely on the fact that space allows us to differentiate and reidentify particulars; that it is, in Becker's terms, a principle of individuation. In

[378] For Becker's analysis of the constitution of perceptual space see his "Beiträge zur phänomenologischen Begründung der Geometrie und ihrer physikalischen Anwendungen," pp. 484-489.

contrast to Russell, however, Becker wants to conclude from this visual space is necessarily Euclidean.

Becker believes that as a principle of individuation, space is characterized by the property of repeatability (*Wiederholbarkeit*) which he defines as follows:

> It lies now obviously in the sense of a *principium individuationis* as a medium of repeatability everywhere, that, in accordance with its "inner" properties (i.e., with the exclusion of the "position") identical bodies can exist in it.[379]

By appeal to the results of Riemann and Helmholtz/Lie, he then concludes:

> Already Riemann and Helmholtz, as is known, have expressed the idea that the requirement of the "existence of bodies independently of their position" (Riemann) or "of free mobility" (Helmholtz) is mathematically equivalent to the constancy of the measure of curvature everywhere in space . . . The simple reflection on the essential meaning of a *principium individuationis* suffices, however, in order to justify the existence of congruent movement and thereby the constancy of the measure of curvature ontologically and transcendentally (as the condition of the possibility, i.e., to function as *principium individuationis*).[380]

So far, this conclusion coincides with Husserl's, namely that perceptual space must have a constant curvature. But Becker further argues as follows:

> Repeatability in the strictest sense means that none of the repeated objects has to be distinct in any sense from any other. The geometric group of transformation that expresses the "repetition" (the *individuatio* of the full *concretum* spatial figure) must not contain any specially characterized points. From this it follows, however, that this group of transformation is a translation . . . . This means, however,

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379 ["Es liegt nun offenbar im Sinne des principium individuationis als Mediums der Wiederholbarkeit, daß allenthalben in ihm ihren 'inneren' Eigenschaften nach (d.h. bis auf die 'Lage') identische Körper existieren können."], Oskar Becker, "Beiträge zur phänomenologischen Begründung der Geometrie und ihrer physikalischen Anwendungen," p. 483, (translation my own).

380 ["Schon von Riemann und Helmholtz ist bekanntlich der Gedanke ausgesprochen worden, daß die Forderung der 'Existenz von Körpern unabhängig vom Ort' (Riemann) oder 'der freien Beweglichkeit' (Helmholtz) mathematisch äquivalent sei mit der Konstanz des Krümmungsmaßes überall im Raum . . . . Die einfache Besinnung auf die wesensmäßige Bedeutung eines principium individuationis genügt aber, um die Existenz der kongruenten Verpflanzung und damit die Konstanz des Krümmungsmaßes ontologisch und transzendental (als Bedingung der Möglichkeit, d.h. als principium individuationis zu fungieren) zu begründen."], Ibid, (translation my own).

> according to known mathematical theorems, that the metric of space is Euclidean, and that the Riemannian measure of curvature disappears everywhere.[381]

Accordingly, from a particular aspect of a principle of individuation, i.e., from the repeatability of the objects individuated by it, it follows not only that perceptual space has a constant curvature, but also that this curvature is zero.

It should be obvious that the same arguments that I put forward against Russell also apply to Becker's account. First, he too applies specifically geometric results to his analysis of perceptual space, thus giving a circular account of its necessary features. Second, as Strawson's scenario showed, a principle of individuation does not have to be spatial and does not have to have geometric properties at all.

8.3 Spatial Intuition and the Generality of the Results of Material Geometry

I will now turn to the question of the generality of the results of material geometry. As we have seen, the result of a proof is general if it can be repeated for various possible diagrams constructed under the same *ekthesis*. We have also seen that the visual properties relevant for a proof are topological properties as defined in the broad sense above, which includes incidence, order and whole-part relations, as well as configurative properties. Accordingly, the geometer can conclude that the result of a given proof is generally valid only if its *apodeixis* does not appeal to topological properties that change under the *ekthesis*. I suggest that this knowledge is provided by spatial intuition, which allows the geometer to grasp which topological features of a given diagram remain invariant under the same *ekthesis*. Yet, the structure of spatial intuition as a system of perspectival expectations does not suffice for the geometer to grasp all the topological invariants necessary for Euclid's proofs. To prove this, one need only consider the fact that in a non-Euclidean space with constant curvature it is not possible to construct a triangle that is similar, yet not congruent, to a given one. The angle sum of a given

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[381] ["Wiederholbarkeit im strengsten Sinn besagt, daß keines der wiederholten Objekte vor dem anderen in irgendeiner Hinsicht ausgezeichnet zu sein braucht. Die die 'Wiederholung' (die individuatio des vollen Konkretums Raumfigur) ausdrückende geometrische Transformationsgruppe darf also keine ausgezeichneten Punkte besitzen. Daraus folgt aber, daß jene Transformationsgruppe eine Translation ist. . . . Das besagt aber nach bekannten mathematischen Sätzen, daß die Metrik des Raumes euklidisch ist, daß das Riemann'sche Krümmungsmaß überall verschwindet."], Ibid., (translation my own).

triangle magnified in hyperbolic space, for example, will become smaller, and its individual angles will look smaller.[382] In geometric terms, we can say that the larger triangle will have a larger defect. In this situation, Euclid's proof that the interior angle sum of a given triangle equals two right angles will not be able to be generalized because it will not apply to the different sized triangles. This example demonstrates that in order to allow the geometer to establish the general validity of his results, spatial intuition will necessarily have to include specific metrical expectations, for example, concerning the congruence of angles in triangles of different sizes.

The fact that spatial intuition includes metric expectations can be seen by considering pre-scientific experience. We often measure distances for everyday purposes with relatively imprecise instruments without appealing to particular geometrical or physical idealizations. Thus, even this experience of space is metrical in that everyday space is covered as a whole by a metric. Further, Berkeley, Helmholtz, and others have argued, that we are also able to estimate sizes and distances visually.[383] For example, we can represent the size of a small or middle-sized object seen in the distance by moving our hands closer together or further away from each other. We can also estimate, within reasonable limits, whether a certain object is, let us say, twice as far away from us than another object. Thus, we can conclude that spatial intuition is not restricted to perspectival expectations, but rather includes a pre-geometrical and pre-physical metrical structure.

I want to express the previous point by saying that pre-geometric visual space has a certain type of imprecise metric, which I will call a quasi-Euclidean structure. This conclusion raises a problem for my account of the generality of the results of Euclid's method. As we have seen, a geometer must be able to predict how the diagrams will look

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[382] For a mathematically precise derivation of these results see, for example, Richard J. Trudeau, *The Non-Euclidean Revolution*, pp. 173-231, in particular, p. 222f.

[383] Helmholtz writes, for example: "When we perform measurements, we do but employ the best and most reliable means we know of in order to determine what we habitually determine by forming an estimate by sight, touch, or steps. In these habitual measurements it is our own body with its organs which is the measuring instrument we carry around with us in space. Now it is our hands, then our legs, which serve as a compass, or our eyes, turning in all directions, are our theodolite for measuring arcs and angles in the visual field." Hermann von Helmholtz, "On the Origin and Meaning of Geometrical Axioms" in *Popular Lectures*, vol. II, p. 56.

if he or she is to know that a given proposition can be proven for all diagrams falling under a given *ekthesis*. This seems to imply that the geometer must have an intuition of a Euclidean, and not what I am calling a quasi-Euclidean structure. It seems that a proof conducted on one single diagram can carry generality only if visual space has a precise metrical structure. But I would still argue that this is not necessary. The only demands we need to make of this visual space are that it be homogenous and that it allow the geometer to create a material geometry whose propositions will describe a Euclidean space, or space without constant positive or negative curvature. The quasi-Euclidean visual space I have presented fulfills these criteria, in that it exhibits no phenomena that would occur in a curved space.

Let me summarize. My primary goal in this part of my thesis was to demonstrate how the geometer can establish the generality of the results of Euclid's method. I first investigated the phenomenal structure of visual space in order to formulate a concept of spatial intuition. By criticizing a number of inferential theories of perception, I argued that only a direct approach to visual space allows us to explicate its phenomenal structure. I then considered Husserl's phenomenological approach to space perception. Although he claimed to develop a direct theory of perception, Husserl fell victim to some of the problems of the inferentialists. Nevertheless, these problems allowed me to formulate three methodological principles on the basis of which I modified Husserl's own account of visual spatial experience. I concluded that visual space is experienced as a generalized perspectival system and that spatial intuition is a system of perspectival expectations. This concept of spatial intuition then allowed me to show that the propositions of material geometry are necessarily consistent with the propositions of a formal geometry that describes a space with constant curvature. Finally, I exploited the notion of spatial intuition in order to explain how the generality of Euclid's results is established.

9. Conclusion: The Nature of Geometric Knowledge

In this thesis, I have investigated the nature of geometric knowledge and its relationship to spatial intuition. In part one, I argued that the standard view presupposes a material geometry. I began by considering Reichenbach's arguments against a material geometry and argued that they did not succeed in refuting Kant's reasons for believing in the existence of a material geometry, and his idea that there is a specifically geometric concept of space based on pure spatial intuition. Reichenbach failed because his method of analyzing the sciences did not allow him to investigate the constitution of the space of experience. I then pointed out that the standard view raises a conceptual problem for applied or physical geometry, which I solved by turning to Husserl's genetic approach to the concept of space and his appeal to a geometric concept of space constituted in a material geometry.

In the second part of my thesis, I formulated a coherent notion of a material geometry. I first rejected the idea that such a geometry is in essence an axiomatic system in the contemporary sense. I then considered the mathematical practice exhibited in Euclid's *Elements* and argued that the propositions of material geometry are derived through non-formal logical inferences from idealized visual objects, namely diagrams. As a visual source of geometric knowledge, the diagrams prevent the collapse of material into formal or applied geometry. I explained how diagrams could serve as a source of non-empirical geometric knowledge by giving a theory of the constitution of the geometric object, which I define as the result of a process of idealization that simply consists in thinking away certain features of the visual object, thereby disambiguating it. In my interpretation, the idealized diagram functions as the source of geometric knowledge, and only certain topological features are relevant. Metrical features are ascribed to topological object by means of rational argument. As a result, geometric inferences do not require approximation of metrical properties and are thus non-empirical. This, explanation applied to geometric proofs only in so far as they say something about an individual diagram, i.e., up to the *apodeixis*. In the last section of this part, I argued that the generality of Euclid's propositions is repeatability of the same proof for alternative diagrams.

In the third and final part of my thesis, I developed a concept of spatial intuition that showed how the geometer establishes the generality of his/her results. I first investigated how visual space is given phenomenally in everyday experience. By criticizing a number of inferential theories of perception, I showed that this can be done only by means of a direct theory of perception. I then turned to Husserl's account of spatial perception. Although he himself believed that he developed a direct theory of perception, he fell victim to some of the problems of the inferentialists. I therefore modified his theory and concluded that space is experienced as a generalized perspectival system and that spatial intuition is a system of perspectival expectations. In the final section of this part, I then extended the notion of spatial intuition to comprise not only persepectival but also metric expectations, which allow the geometer to grasp which topological features of a diagram will remain invariant under the same *ekthesis*, and thus to establish the general validity of the proposition in question. In this conclusion, I will draw from my dissertation consequences concerning the nature of geometric knowledge and the notion of axiomatization. I will argue that the knowledge about the structure of visual space established by means of Euclid's method is *a priori*, because the process in which it is derived guarantees its validity.

In the course of this dissertation, I have described two different interpretations of the nature of geometric knowledge in Euclid's method. Apriorists like Kant believed that Euclid's method yielded *a priori* knowledge about the structure of the space of experience, whereas empiricists like Moritz Pasch and Hilbert thought it was a natural science yielding only empirical knowledge of space. Despite their differences concerning the epistemic status of Euclid's propositions, they agreed that his results were true *of* space. Many interpreters in both camps also agreed that Euclidean geometry was an axiomatic system whose axioms (Euclid's Postulates) expressed facts about the space of experience. Consequently, Euclidean geometry was a science about space. In my dissertation, I argued, however, that Euclid's Postulates do not express truths about the structure of space, and that the primary subject matter of material geometry was not space, but rather idealized diagrams. My conclusions thus raise two questions related to the nature of geometric knowledge in Euclid's geometry: 1) in what sense is it a science about space?; and 2) what is the epistemic status of Euclid's propositions? Let me take up these questions in turn.

At the end of Part II, I suggested that diagrams allow the geometer to explore the structure of visual space. In my interpretation, Euclid's geometry was a science about space not because its axioms (in the sense of Postulates) captured essential facts about space, but rather because it exhibited the structure of space *through* the behaviour of visual objects in it. Given Husserl's characterization of visual space as something that cannot be seen directly but is only implied (*mitgesehen*) in the perception of visual objects, this is necessarily so. Visual space reveals its structure only through objects in it. To use a metaphor, the structure of visual space emerges as the mirror image of the spatial properties of visual objects. Thus, material geometry becomes a science about space only because it is a science about spatial objects.

A consequence of this view is that a material geometry reflects to a large degree the qualitative structure of space, which consists of incidence, order, and whole-part relations between certain defined configurations. This qualitative structure of perceptual space is principally inexhaustible. A geometer who wants to construct a material geometry can begin by first choosing the objects by means of which he/she will explore visual space. Euclid selected these objects (points, lines, angles, certain types of quadrilaterals, spheres, cubes, octahedrons, etc.) through his definitions. In so doing, he relied on a traditional specification of the subject matter of geometry. Yet, it is clear that his choice was somewhat arbitrary. He could have specified different spatial configurations in his definitions, thereby constructing an altogether different material geometry. Or, he could have extended this subject matter, thus deriving further propositions. As a result, one can say that a material geometry has no natural sense of completeness. We can indefinitely continue to generate new propositions, each of which will express new qualitative features of visual space.

Let me now turn to the second question about the epistemic status of Euclid's propositions. I believe that Euclid's method yields *a priori* knowledge about the structure of visual space. I will first describe some characteristic features of the type of knowledge derived by Euclid's method and then give my reasons for calling it *a priori*.

According to the hybrid-conception of the geometric object, only readily recognizable topological or configurative features of the diagrams serve as sources of geometric inferences. The metrical properties are ascribed to such an object by means of Euclid's Axioms (including his fourth Postulate), which are rules for deriving quantitative properties from qualitative properties (whole-part relation and coincidence). Given this, the epistemic status of Euclid's propositions depends on the nature of the visual source, on the one hand, and on the nature of the inferences (Axioms), on the other. We can show, however, that both these features of Euclid's proofs are such as to render the results established by means of them necessarily true of visual space.

Let me first restrict my consideration to the qualitative features of the diagrams and of visual space. Since the only condition for the possibility of a material geometry is the homogeneity of visual space, the latter may not have a quasi-Euclidean structure. If it does not, as in a homogeneous, non-Euclidean visual space with an imprecise metric, Euclid's geometric objects and constructions will look different. The qualitative properties of the diagrams and qualitative properties of visual space in this case would also necessarily coincide, because the diagrams are objects in visual space and, thus, *must* represent its qualitative structure, whatever it may be. The qualitative properties would then be ascribed to all geometric objects of the same kind by appealing to the very same system of expectations that constituted the visual space in the first place. Thus, the general results would coincide *necessarily* with the overall structure of visual space. Consequently, the qualitative properties that form the point of departure for Euclid's inferences can be said to represent the qualitative structure of visual space.

But what about the axioms, that is, the rules of inference leading from these qualitative to quantitative results? It seems that if the axioms are empirical generalizations, the results of Euclid's method will not necessarily represent the metrical structure of visual space. But in Part II, I argued that the homogeneity of visual space necessitates Euclid's axioms. In other words, the rules of inference express features of visual space that must apply to it if the construction of a geometric concept of space is to be possible at all. Given the nature of the visual source of the inferences, the knowledge derived by means of Euclid's method is necessarily true of visual space, no matter whether this space is quasi-Euclidean or not. Yet visual space is inexact and has no precise metrical structure. Thus, the results derived by Euclid's method will always only be approximately true.

I think that we are justified in calling the results derived by means of Euclid's method a type of *a priori* knowledge (defined not in the sense of being independent of experience). Although the geometer derives his/her propositions partly through the observation of individual spatial objects, this process does not require empirical generalizations. The inferences depart only from qualitative features of the diagrams and thus do not require repeated experiments to establish the properties of the geometric objects. More importantly, the results reached are necessarily true of visual space. They constitute a type of *a priori* knowledge because the process through which they are derived guarantees their truth. In this sense, then, I side with Kant. Yet, I believe that Kant was mistaken in restricting this knowledge to Euclidean geometry. In non-Euclidean spaces, Euclid's method would yield non-Euclidean geometries. Moreover, nothing in my argument excludes the possibility of a non-homogenous visual space.

My account of Euclid's geometry requires that we reconsider exactly what is happening when it is axiomatized. The standard history of geometry considers subsequent attempts to develop a consistent axiom system of Euclidean geometry, culminating in Hilbert's *Grundlagen der Geometrie,* as successive improvements on Euclid's axioms. Yet if Euclid's method represents a mathematical practice that differs significantly from Hilbert's geometry, as I have argued, then the standard account of geometry's history cannot be accepted without qualification. I think that the axiomatization of Euclid's geometry did not patch up Euclid's alleged axiomatic system, but rather transformed one mathematical practice into another. This is evident by the manner in which modern geometers reinterpreted Euclid's Postulates. I have described these postulates as having the function of legitimizing certain constructive procedures for extending the diagrams. Given that they can fulfill this function only if their application leads to the same results everywhere, I concluded that they express the *possibility* of constructing all objects, as long as the structure of visual space is homogenous. In axiomatizing Euclid's geometry, it was an easy step for geometers like Hilbert to just assume the existence of all these objects and to reformulate the Postulates as existence assumptions. The standard history of geometry hides the conceptual shift occurring here, and thus distorts the nature of Euclid's original geometry.

On the basis of these conclusions, I want to distinguish my view of material geometry from those of Pasch, Hilbert, Husserl, and Carnap. First, they all wrongly equated material geometry and axiomatic system in the contemporary sense. According to my view, the essential property of a material geometry is that its propositions are derived from its primary subject matter through non-formal logical inferences. Pasch and Hilbert were further wrong in assuming that the propositions of material geometry were true, because the axioms captured certain empirical facts about space. They were right about the fact that material geometry approximates the metrical structure of visual space. Yet, they drew a wrong conclusion from this by saying that that Euclid's geometry is a natural science. Husserl and Carnap, in contrast, correctly claimed that Euclidean geometry is grounded in spatial intuition. Yet, spatial intuition does not contain the categorial form of space as it is expressed in a Hilbert-stile axiomatic system, as Husserl believed, or of a limited region of space, as Carnap thought. Rather, spatial intuition is a system of expectations about the perspectival changes of certain types of visual objects.

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