Conformally Induced Mean Curvature Flow

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April 2, 2024

A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of a Master of Science

Contents

Abstract

This thesis aims to loosely cover the subject of geometric flows, and more specifically a variant of the mean curvature flow. The subject of geometric flows uses results in PDE theory theory, more specifically parabolic PDEs, to gain new insight about Riemannian geometry.

One of the biggest problems often tackled by mean curvature flows is the Isoperimetric problem. The Isoperimetric problem asks us to classify the spaces that minimize perimeter for a given volume (hence the name), the first use of mean curvature flows to attack this problem was due to Huisken in 1984 [6] who proved it in case of boundaries of convex domains in Euclidean space, and since then there have been attempts to push his methods further. More recently, in 2013, Guan and Li [4] constructed a new normalized flow which allows them to prove the inequality in the case of boundaries of star-shaped domains in Space forms. Shortly after, in 2018, Guan, Li and Wang pushed this flow even further which allowed them to prove the result in a certain class of warped product spaces [5].

Then, concurrently with the writing of this thesis, Li and Pan reframed the technique in terms of conformal vector fields, allowing them to weaken the assumptions on the ambient space [11].

This thesis continues the effort to use such flows, together with my collaborator Joshua Flynn we pushed the flow even further, allowing us to weaken the assumptions on the ambient space even further and even weaken the star-shapedness assumption, which was key to all previous results.

Abrégé

Cette thèse vise à couvrir de manière générale le sujet des flots géométriques, et plus spécifiquement une variante du flot de courbure moyenne. Le domaine des flots géométriques utilise des résultats de la théorie des équations aux dérivées partielles (EDP), plus précisément des EDP paraboliques, pour obtenir de nouvelles perspectives sur la géométrie riemannienne.

L'un des plus grands défis souvent abordés par les flots de courbures moyennes est le problème isopérimétrique. Ce problème demande de classer les espaces qui minimisent le périmètre pour un volume donné (d'où le nom). La première utilisation des flots de courbures moyennes pour attaquer ce problème remonte à Huisken en 1984 [6], qui l'a prouvé dans le cas des frontières de domaines convexes dans l'espace euclidien. Depuis lors, des tentatives ont été faites pour pousser ses méthodes plus loin. Plus récemment, en 2013, Guan et Li [4] ont construit un nouveau flot normalisé qui leur a permis de prouver l'inégalité dans le cas de frontières de domaines en forme d'étoile dans les formes spatiales. Peu de temps après, en 2018, Guan, Li et Wang ont poussé ce flot encore plus loin, ce qui leur a permis de prouver le résultat dans une certaine classe d'espaces de produits déformés [5].

Puis, simultanément à la rédaction de cette thèse, Li et Pan ont reformulé la technique en termes de champs vectoriels conformes, ce qui leur a permis de relâcher les hypothèses sur l'espace ambiant [11].

Cette thèse poursuit l'effort d'utiliser de tels écoulements, avec mon collaborateur Joshua Flynn, nous avons poussé le flot encore plus loin, nous permettant de relâcher encore davantage les hypothèses sur l'espace ambiant et même affaiblir l'hypothèse de domaines en forme d'étoile, qui était cruciale pour tous les résultats précédents.

Acknowledgements

First I would like to thank my supervisor, Professor Pengfei Guan, for his regular assistance both before and after the start of my Masters program. He was always there to help me understand concepts I would otherwise spend weeks grappling with.

Secondly, I would like to thank my collaborator, Doctor Joshua Flynn, without whom this project would have never happened. Bouncing ideas off one another was one of the best parts of my Masters.

I would also like to thank my good friends Sammir Rahman, William Holman-Bissegger, Jenny Xu, Dao-Chen Yuan, Viet Nguyen, Margaret Zhou, Tiffany Yong, Sam Zeitler, Bart Syroka, Jessie Meanwell, Aaron Shalev, Michael Cimino as well as the whole of the graduate community at McGill, for making my last year at McGill the best year of my life.

Finally I would like to thank my family, and especially my mom, for always being there to help and support me whenever I would feel down and lost.

Contribution

This thesis and each chapter within was written solely by myself, with occasional assistance from colleagues regarding phrasing. The body of the thesis is largely taken from the paper co-authored by myself and Joshua Flynn.

Chapter 1

Introduction

1.1 Background on the Isoperimetric Problem

The Isoperimetric Problem asks a seemingly simple question:
Among all regions occupying a given volume which has the least perimeter?

In the simple case of 2 and 3 dimensions we all know that the answer is a circle and a sphere respectively, and when we extend it to n dimensions in Euclidean space, the answer remains the $n-1$ dimensional hypersphere, for a large variety of various proofs in this simple case see [1]. However, if we try to generalize this question to a more general ambient domain, the problem very quickly becomes intractable.

One of the ways to answer such a question is with the method of geometric flows, we will start with a shape which will likely not be optimal, and then we will morph it over time to make it optimal.

The thesis will be split into 3 Chapters,
This first chapter will establish all standard theory of Geometric Analysis used in the results of this thesis.
The second chapter will discuss the method of geometric flows to solve the Isoperimetric problem, as well as the geometric setting introduced by Li and Pan in [11].
The third chapter introduces a more general setting and extends the results to that setting too, first by computing the evolution equations for various geometric quantities and then by proving existence of the flow.

This will all lead to the proof of the following theorem.

Theorem 1.1.1: Let N be an ambient manifold admitting a conformal vector field X and a foliation $\mathcal{F}$, such that both are compatible (see Section 3.2) and satisfy Assumptions 3.2.1. Then the leaves S_α of the foliation $\mathcal{F}$ are the Isoperimetric profile of the class of all hypersurfaces satisfying $\langle X, \nu \rangle > 0$.

1.2 Concepts and Notation

This document assumes general knowledge of differential and Riemannian geometry, see [9] and [10] for great introductions, respectively.

For the rest of this document we will use the following notation, N is an $n+1$ dimensional Riemannian manifold with metric $\overline{g}$ within which we have a compact domain Ω with boundary $\partial\Omega = M$ such that $F : M \to N$ is an embedding making M a Riemannian hypersurface. We then set $g := F^*\overline{g}$ to be the induced metric on M. We will in general identify M with its image $F(M)$ and use the two interchangeably. We will write $\mathfrak{X}(N)$ to the set of surfaces that can be defined as above and call any such surface any such surface an *admissible* hypersurface.

In general, tensorial constructions defined on N will be written with an overline and their versions on M will be written normally. We will write the covariant derivatives on M and N as ∇ and $\overline{\nabla}$ respectively. We will write the Laplacian on N and M as Δ and $\overline{\Delta}$ respectively. We will use Einstein summation notation for all tensor equations.

Often for a matrix M_{ij} we will use the notation $M_{ij} \geq (>)\ 0$ to denote the fact that M_{ij} is positive semi-definite (definite), and similarly for $M_{ij} \leq (<)\ 0$.

We can use the Riemannian metric $\overline{g}$ to take inner products of tangent vectors in the same tangent space T_pN, for tangent vectors $X, Y \in T_pN$ we will write this as $\langle X, Y\rangle$. Since the metric g is just the restriction of $\overline{g}$ onto T_pM when we think of it as a subspace of T_pN, we will use the same notation $\langle X, Y\rangle$ for $X, Y \in T_pM$.

The Riemannian metric $\overline{g}$ defines with it a Riemannian volume form which we will call $\mathrm{d}V$, this form can be restricted to Ω to allow us to define

$$\mathrm{Volume}(\Omega) = \int_\Omega \mathrm{d}V,$$

we will often write $V(M)$ as our Ω can be determined uniquely an orientation by on M. Similarly the metric g defines a volume form on M which we will call $\mathrm{d}S$, using which we define

$$\mathrm{Area}(M) = \int_M \mathrm{d}S.$$

We will often write $A(M)$ for brevity. We can now define the Isoperimetric profile of N to be the function

$$I(v) := \inf\{A(M) : M \in \mathfrak{X}(N) \text{ and } V(M) = v\}.$$

If for some family of surfaces S_α we have $I(V(S_\alpha)) = A(S_\alpha)$ then we will also refer to $\{S_\alpha\}$ as the Isoperimetric profile, the meaning will be clear from context.

The Isoperimetric Problem now asks us to

1. Show there exists a family of hypersurfaces $S_\alpha \in \mathfrak{X}(N)$ which is the Isoperimetric profile.
2. Characterize this family.

In practice the above problem is extremely difficult so we often restrict ourself to a subclass of surfaces. For a subclass $\mathcal{Z} \subseteq \mathfrak{X}(N)$ the Isoperimetric profile of $\mathcal{Z}$ is the function

$$I(v) := \inf\{A(M) : M \in \mathcal{Z} \text{ and } V(M) = v\}$$

and the same questions can be asked of this case.

We will now start to build up the concepts that allow us to solve this problem.

1.3 Riemannian Geometry

Recall that since M is the boundary of a manifold it must be orientable, it thus has a canonical 'outward' pointing unit normal vector field, which we will call ν.

Working with Riemannian geometry is almost always easier when done with coordinates. In extrinsic geometry, there are two coordinate systems that we will be using repeatedly so we will list some of their properties.

Proposition 1.3.1: Let M be a Riemannian manifold, at any point p there exists a chart (U, φ) with the property that the frame

$$e_i = \frac{\partial}{\partial x^i}$$

forms an orthonormal basis *at the point p*. These are called *orthonormal coordinates.*

Then we define the second fundamental form h to be the bilinear form given by

$$h(X, Y) = \left\langle X, \overline{\nabla}_Y \nu \right\rangle.$$

This second fundamental form encodes within itself how the manifold M lies inside N, it also carries with it a number of useful properties, the most important of which is that it is symmetric, see [10, p. 227] for details. Also of much importance is the trace of this form, taken with respect to the metric, which we write as $H = h_{ii}$, which is called the *mean curvature*. Also its eigenvalues κ_i taken with respect to the metric are called *principle curvatures*.

Let us write up some properties of the second fundamental form.

Proposition 1.3.2: Let e_i be an orthonormal frame at p, the following are true:

1. h can be written in coordinates as $h_{ij} = \left\langle e_i, \overline{\nabla}_{e_j} \nu \right\rangle$.
2. $\overline{\nabla}_i \nu = h_{ij} e_j$.
3. $\overline{\nabla}_i e_j = -h_{ij} \nu$.
4. If f is a function $N \to \mathbb{R}$, then $\overline{\Delta} f = \Delta f + \overline{\text{Hess}}_f(\nu, \nu) + H\nu(f)$.

Proof: (a) is directly from definition, to see (b) note that $\{e_1, ..., e_n\} \cup \{\nu\}$ form a basis for the tangent space $T_p M$ and thus we have

$$\nabla_i \nu = a^j e_j + b\nu$$

for some coefficients $a^j, b \in \mathbb{R}$. But now consider,

$$0 = \nabla_j \langle \nu, \nu \rangle = 2\left\langle \nabla_j \nu, \nu \right\rangle = 2b$$

and so we have $b = 0$. We then get,

$$a^j = \left\langle \nabla_i \nu, e_j \right\rangle = h_{ij}$$

proving the claim.

Now for (c) we note first that $\nabla_X Y = \pi\left(\overline{\nabla}_Y X\right)$ for $X, Y \in T_p M$ where π is the orthogonal projection to $T_p M$, see, for instance, [7, p. 223]. This will mean that since e_i are orthonormal then

$$\pi\left(\overline{\nabla}_i e_j\right) = \nabla_i e_j = 0$$

for all i, j and so $\overline{\nabla}_i e_j = b_{ij}\nu$ for some matrix b of coefficients. Now we have

$$0 = \overline{\nabla}_i \langle e_j, \nu\rangle = \langle \overline{\nabla}_i e_j, \nu\rangle + \langle e_j, \overline{\nabla}_i \nu\rangle = b_{ij} + h_{ij}$$

which proves the claim.

Finally for (d), we have

$$\begin{aligned}\overline{\Delta} f &= \langle \overline{\nabla}_i \overline{\nabla} f, e_i\rangle + \langle \overline{\nabla}_\nu \overline{\nabla} f, \nu\rangle = \overline{\nabla}_i \langle \overline{\nabla} f, e_i\rangle - \langle \overline{\nabla} f, \overline{\nabla}_i e_i\rangle + \langle \overline{\nabla}_\nu \overline{\nabla}, \nu\rangle \\ &= \nabla_i \langle \nabla f, e_i\rangle - \langle \overline{\nabla} f, -H\nu\rangle + \langle \overline{\nabla}_\nu \overline{\nabla}, \nu\rangle \\ &= \Delta f + H\nu(f) + \langle \overline{\nabla}_\nu \overline{\nabla}, \nu\rangle\end{aligned}$$

□

Remark: We will also use the notation $A(X)$ to mean the endomorphism satisfying

$$h(X, Y) = \langle A(X), Y\rangle$$

which is given in coordinates by $A^i_j = g^{ik} h_{kj}$ and we will also use the notation $|A|^2$ to denote the squared norm of h or A.

We will also need another well known geometric identity,

Lemma 1.3.3 (Codazzi Equation): We have for any $X, Y, Z \in T_p M$

$$\overline{\mathrm{Rm}}(X, Y, Z, \nu) = -(\nabla_X h)(Y, Z) + (\nabla_Y h)(X, Z)$$

in particular in coordinates we have

$$\overline{\mathrm{Rm}}_{ijk\nu} = -\nabla_i h_{jk} + \nabla_j h_{ik}$$

Proof: See [13, p. 93], note that some references have a similar equation of opposite sign due to a difference in defining the second fundamental form. □

1.4 Conformal Vector Fields

A conformal vector field is a vector field X with the property that $\mathcal{L}_X \overline{g} = 2\varphi g$, where φ is any smooth function called the *conformal factor* of X. The Lie derivative is a little hard to work with for our purposes so we will follow the calculations of [2] to obtain better formulations for the properties of conformal vector fields

Proposition 1.4.1: Let X be a vector field on N, then for any $Y, Z \in T_pN$ we have

$$2\left\langle \overline{\nabla}_Y X, Z \right\rangle = (\mathcal{L}_X g)(Y, Z) + \mathrm{d}\eta(Y, Z)$$

where η is the dual one form to X defined by $\eta(Z) = \langle X, Z \rangle$

Proof: We have by Koszul's formula ([10, p. 123])

$$\begin{aligned}
2\left\langle \overline{\nabla}_Y X, Z \right\rangle &= Y(\langle X, Z \rangle) + X(\langle Y, Z \rangle) - Z(\langle X, Y \rangle) \\
&\quad -\langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle - \langle [X, Z], Y \rangle \\
&= \mathcal{L}_X(\langle Y, Z \rangle) + \mathcal{L}_Y(\eta(Z)) - \mathcal{L}_Z(\eta(Y)) \\
&\quad -\langle Z, \mathcal{L}_X Y \rangle - \eta(\mathcal{L}_Y Z) - \langle Y, \mathcal{L}_X Z \rangle \\
&= (\mathcal{L}_X \overline{g})(Y, Z) + \mathcal{L}_Y(\eta(Z)) - \mathcal{L}_Z(\eta(Y)) - \eta(\mathcal{L}_Y Z) \\
&= (\mathcal{L}_X \overline{g})(Y, Z) + (\mathcal{L}_Y \eta)(Z) - \mathcal{L}_Z(\eta(Y))
\end{aligned}$$

Now we can use Cartan's magic formula ([9, p. 372]) applied to the second term to get

$$\begin{aligned}
2\left\langle \overline{\nabla}_Y X, Z \right\rangle &= (\mathcal{L}_X \overline{g})(Y, Z) + \mathrm{d}\eta(Y, Z) + \mathrm{d}(\eta(Y))(Z) - \mathcal{L}_Z(\eta(Y)) \\
&= (\mathcal{L}_X \overline{g})(Y, Z) + \mathrm{d}\eta(Y, Z)
\end{aligned}$$

□

From this we see that an equivalent characterization of a Conformal vector field is $2\left\langle \overline{\nabla}_Y X, Z \right\rangle = 2\varphi\langle Y, Z \rangle + \mathrm{d}\eta(Y, Z)$

We now define the skew-symmetric endomorphism ψ by

$$\mathrm{d}\eta(Y, Z) = 2\langle \psi Y, Z \rangle$$

This endomorphism is then called the *associated tensor field* of X, and with it we can rewrite the above equation as

$$\langle \nabla_Y X, Z \rangle = \varphi\langle Y, Z \rangle + \langle \psi Y, Z \rangle.$$

Note that this is also the decomposition of the ∇X into its symmetric and anti-symmetric parts, that is

$$\mathrm{Sym}(\nabla X) = \varphi g \quad \text{and} \quad \mathrm{ASym}(\nabla X) = \langle \psi(\cdot), \cdot \rangle$$

In the special case that $\varphi = 0$ we call X a Killing vector field.

Definition 1.4.1: Given a fixed vector field X on N we define the support function u_X on M by

$$u_X := \langle X, \nu \rangle$$

where ν is the normal vector to M.

To see why conformal vector fields are so useful in the study of the Isoperimetric inequality, we will now derive a key result that was the basis of the results by Guan, Li and Wang and will also be the basis of the results in this thesis.

Lemma 1.4.2 (Conformal Integral Identities): Let $M \subseteq N$ be an admissible hypersurface as defined above, and let X be a conformal vector field on N with conformal factor φ, the following identities hold

$$\int_M (n\varphi - Hu)\,\mathrm{d}S = 0$$

and

$$\int_M H(n\varphi - Hu)\,\mathrm{d}S = \frac{n}{n-1}\int_M \overline{\mathrm{Ric}}(\nu, X - u\nu)\,\mathrm{d}S - \frac{1}{n}\int_M u\sum_{i<j}\left(\kappa_i - \kappa_j\right)^2\mathrm{d}S$$

where $u = \langle X, \nu \rangle$ is called the support function.

Proof: First we will define the vector field $Y = X - u\nu$, which is the projection of X onto the tangent space of M. Now consider the divergence of Y on M, for an orthonormal frame e_i of M we have

$$\mathrm{div}_M(Y) = \mathrm{tr}\big(\nabla_j\langle Y, e_i\rangle\big) = \mathrm{tr}\Big(\overline{\nabla}_j\langle X, e_i\rangle\Big) = \mathrm{tr}\Big(\langle\overline{\nabla}_j X, e_i\rangle + \langle X, \overline{\nabla}_j e_i\rangle\Big).$$

Now we combine the fact that X is a conformal vector field and the fact that the trace of an endomorphism is the same as the trace of its symmetrization, giving us that

$$\mathrm{tr}\Big(\langle\overline{\nabla}_j X, e_i\rangle\Big) = \mathrm{tr}\bigg(\Big(\overline{\nabla}X\Big)_{ji}\bigg) = \mathrm{tr}\bigg(\mathrm{Sym}\Big(\overline{\nabla}X\Big)_{ij}\bigg) = \mathrm{tr}\Big(\varphi\overline{g}_{ij}\Big)$$

Now knowing that in our coordinates $\overline{g}_{ij} = \delta_{ij}$ and that the trace of δ_{ij} is $\dim M$ we get

$$\mathrm{div}_M(Y) = \mathrm{tr}\big(\varphi\overline{g} + \langle X, -h_{ij}\nu\rangle\big) = n\varphi - Hu$$

then since M is closed we have by divergence theorem

$$\int_M \operatorname{div}_M(Y)\,\mathrm{d}S = \int_M (n\varphi - Hu)\,\mathrm{d}S = 0$$

Secondly we will consider the vector field $Y' = (HI - A)Y$, its divergence gives us

$$\begin{aligned}
\operatorname{div}_M Y' &= \operatorname{tr}\Big(\nabla_j (HI - A)^i_k Y^k\Big) = \operatorname{tr}\Big(\nabla_j \Big((HI - A)^i_k \langle X, e_k\rangle\Big)\Big) \\
&= \operatorname{tr}\Big(\langle X, e_k\rangle \nabla_j (HI - A)^i_k + (HI - A)^i_k \overline{\nabla}_j \langle X, e_k\rangle\Big) \\
&= \operatorname{tr}\Big(\langle X, e_k\rangle \big(\nabla_j h_{\ell\ell}\delta^i_k - \nabla_j h_{ik}\big) + (HI - A)^i_k \big(\varphi\delta_{jk} - uh_{jk}\big)\Big) \\
&= \langle X, e_k\rangle \big(\nabla_k h_{\ell\ell} - \nabla_i h_{jk}\big) + H(n\varphi - Hu) - H\varphi + u|A|^2 \\
&= \langle X, e_k\rangle (\nabla_k h_{\ell\ell} - \nabla_\ell h_{\ell k}) + H((n-1)\varphi - Hu) + u|A|^2
\end{aligned}$$

We now use the Codazzi equation to get

$$\begin{aligned}
\operatorname{div}_M Y' &= \langle X, e_k\rangle\, \overline{\mathrm{Rm}}_{\ell k \ell \nu} + H((n-1)\varphi - Hu) + u|A|^2 \\
&= \big\langle \overline{\mathrm{Rm}}(e_\ell, Y)e_\ell, \nu\big\rangle + H((n-1)\varphi - Hu) + u|A|^2
\end{aligned}$$

now swapping the last two indices on the Riemann tensor flips its sign gives us the Ricci tensor, then applying divergence theorem once more gives us

$$\begin{aligned}
\int_M H((n-1)\varphi - Hu)\,\mathrm{d}S &= \int_M \overline{\mathrm{Ric}}(\nu, Y)\,\mathrm{d}S - \int_M u|A|^2\,\mathrm{d}S \\
\frac{n-1}{n}\int_M H\Big(n\varphi - \frac{n}{n-1}Hu\Big)\,\mathrm{d}S &= \int_M \overline{\mathrm{Ric}}(\nu, Y)\,\mathrm{d}S - \int_M u|A|^2\,\mathrm{d}S \\
\int_M H\Big(n\varphi - \frac{n}{n-1}Hu\Big)\,\mathrm{d}S &= \frac{n}{n-1}\int_M \overline{\mathrm{Ric}}(\nu, Y)\,\mathrm{d}S - \int_M \frac{n}{n-1}u|A|^2\,\mathrm{d}S \\
\int_M H(n\varphi - Hu)\,\mathrm{d}S &= \frac{n}{n-1}\int_M \overline{\mathrm{Ric}}(\nu, Y)\,\mathrm{d}S - \int_M \frac{u(n|A|^2 - H^2)}{n-1}\,\mathrm{d}S \\
\int_M H(n\varphi - Hu)\,\mathrm{d}S &= \frac{1}{n-1}\left(\int_M n\,\overline{\mathrm{Ric}}(\nu, Y)\,\mathrm{d}S - \int_M u\sum_{i<j}\big(\kappa_i - \kappa_j\big)^2\,\mathrm{d}S\right)
\end{aligned}$$

□

1.5 Partial Differential Equations

The Partial Differential Equations (PDEs) we will be dealing with most in this thesis are parabolic PDEs, so we will dedicate this section to going over their properties.

Let $T \in (0, \infty]$ and $U \subseteq M$ be a smooth open domain, a function $u : [0, T] \times U$ is said to solve a **quasi-linear parabolic PDE** if it satisfies a differential equation of the form

$$\partial_t u(x,t) = a^{ij}(x,t,u,\nabla u)\nabla_i\nabla_j u + G(x,t,u,\nabla u) \tag{1.1}$$

where a is symmetric positive definite matrix depending smoothly on its inputs and G is a function depending smoothly on its inputs.

The equation is said to be in *divergence form* if it can be written instead as

$$\partial_t u(x,t) = \nabla_i\big(b^{ij}(x,t,u,\nabla u)\nabla_j u\big) + G(x,t,u,\nabla u) \tag{1.2}$$

We say that the PDE in (1.1) is *uniformly parabolic* if there exist constants A, B such that

$$A|v|^2 \le a^{ij} v_i v_j \le B|v|^2$$

for all $v \in T_p M$ everywhere.

The most important tool in the analysis of parabolic PDEs is the maximum principle, a form of which we will now prove.

Proposition 1.5.1: Assume u solves (1.1) and that at a spacial maximum of u the inequality $G(x,t,u,\nabla u) < f(t)$ holds, then we have for all $t \in [0,T]$

$$\sup_{x\in U} u(x,t) \le \sup_{x\in U} u(x,0) + \int_0^t f(s)\,\mathrm{d}s \tag{1.3}$$

if instead we have $G(x,t,u,\nabla u) < Bu(x,t)$ for some constant $B \in \mathbb{R}$ then we have

$$\sup_{x\in U} u(x,t) \le \left(\sup_{x\in U} u(x,0)\right) e^{Bt} \tag{1.4}$$

Proof: First for (1.3) consider the auxiliary function

$$v(x,t) = u(x,t) - \int_0^t f(s)\,\mathrm{d}s - \sup_{x\in U} u(x,0)$$

which then solves

$$\partial_t v(x,t) = a^{ij}(x,t,v,\nabla v)\nabla_i\nabla_j v + G(x,t,v,\nabla v) - f(t)$$

and also $v(x,0) \le 0$ on U. Now assume that (1.3) fails to hold, that is, at some point (y,t), we have

$$u(y,t) > \sup_{x\in U} u(x,0) + \int_0^t f(s)\,\mathrm{d}s$$

then we also have

$$v(y,t) > 0$$

and so the maximum of v is positive. But now let (z,t') be said maximum, we have that the maximum is either on the interior of $[0,T] \times U$ or on the boundary $\{T\} \times U$, it cannot be on $\{0\} \times U$ since there we have $v(x,0) \le 0$. Thus we have that

$$\nabla v(z,t') = 0, \nabla_i\nabla_j v(z,t') \le 0 \text{ as well as } \partial_t v(z,t') \ge 0$$

and so

$$0 \le \partial_t v(x,t) = a^{ij}(x,t,v,\nabla v)\nabla_i \nabla_j v + G(x,t,v,\nabla v) - f(t) < 0$$

this is a contradiction.

For (1.4) we use an identical argument except that we instead use

$$v(x,t) = e^{-Bt}u(x,t) - \sup_{x \in U} u(x,0)$$

□

The second most important tool is short-time existence, it will be extremely important as we want to use the derivatives of geometric quantities to characterize them, so we need the flow to exists for some non-zero amount of time.

Theorem 1.5.2: If $u(0,\cdot)$ is a smooth initial condition and (1.1) is uniformly parabolic then (1.1) has a solution u for some time $T > 0$ which is smooth on $[0,T)$. Furthermore, if there is an apriori uniform bound

$$\|u(t,\cdot)\|_{C^{1+r}} \le K \text{ for all } t \in [0,s)$$

for some constants $r > 0$, $K > 0$, then the solution exists on $[0,s)$ and satisfies a bound

$$\|u(t,\cdot)\|_{C^{2+r}} \le B(K)$$

where B is some constant depending on K.

Proof: Proposition 8.2 in [15, p. 411] for the first statement, and Theorem 4.28 in [12, p. 77] for the second statement. □

The last PDE results which we will need are the famous Nash-Moser estimates, for full details see [15, 8].

Theorem 1.5.3 (Nash-Moser estimates): Let u be a solution to uniformly parabolic (1.1) on $[0,T)$ with smooth initial condition, if we know that

$$\|u(t,\cdot)\|_{C^0(U)} < c_1 \text{ and } \|\nabla u(t,\cdot)\|_{C^0(U)} \le c_2 \text{ on } [0,T)$$

then on any subdomain U' with $\overline{U'} \subseteq U$ we have for some $r > 0$ depending only c_1, c_2, A, B that

$$\|u(t,\cdot)\|_{C^{1+r}(U')} \le C(c_1, c_2, A, B, d)$$

where d is the distance between $\partial U'$ and ∂U.

Proof: We will use Theorem 1.1 in [8, p. 517], it is enough to show that the functions

$$\partial_{x^k} a^{ij}(x,t,v,p), \partial_v a^{ij}(x,t,v,p), \partial_{p^k} a^{ij}(x,t,v,p), G(x,t,v,p)$$

are uniformly bounded on the set

$$\{(x,t,v,p) : x \in \overline{U}, |v| \le c_1 \text{ and } \|p\| \le c_2\}.$$

But this is immediate since these functions depend smoothly on their inputs and thus are continuous and so since the set above is compact they must attain their maximum inside that set and thus they are bounded by that maximum. □

1.6 Evolving Hypersurfaces

Now that we are familiar with geometry and PDEs we can start to use them together. This is done by use of **geometric flows**.

Definition 1.6.1: Let $F : M \to N$ be an admissible hypersurface. Let F_t be a function $F : I \times M \to N$, where $I = [0, T)$ for some fixed T and $F_0 = F$ on M. F_t is called a *normal flow* with *normal velocity* f if

$$\partial_t F_t(x) = f(t,x)\nu(x)$$

where $\nu(x)$ is the normal vector to $F_t(M)$ at $F_t(x)$.

Remark: We will often refer to $F_t(M)$ as M_t for brevity. Additionally many constructions on M_t will be denoted without explicit reference to t, i.e g instead of $g(t)$, even though the metric of M_t will depend on t. Keep in mind that any construction of the metric will also depend on t.

As a manifold flows it's various properties, both local and global, will change, the equations governing these changes are called *evolution equations*. For ambient objects, i.e. those objects that are simply restricted to the hypersurface, this evolution is simple.

Proposition 1.6.1: Let T be any tensor on N, then we write $T|_{M_t}$ to denote the orthogonal projection of T onto $T_p M_t$. We then have along the flow M_t

$$\partial_t\left(T|_{M_t}\right) = \left(f\overline{\nabla}_\nu T\right)|_{M_t}$$

For objects that depend on the induced metric on M_t, these objects depend on the embedding of a whole neighborhood of a point, so their evolution equations are more complicated, but we can still compute them. We will first start with the most important evolving tensor, the metric.

Remark: We will also adapt two important coordinate systems, we will be working in normal coordinates around a point $p \in M$ which will call these coordinates x^i, we will denote their partial derivatives ∂_i or e_i and the covariant derivatives with respect to the induced metric ∇_i. Secondly we will also have normal coordinates at $F(p) \in N$, we will call these coordinates y^i, their partials ∂_{y_i} or $\overline{e}_i$ and the covariant derivatives $\overline{\nabla}_i$. Note that we can rotate the normal coordinates y^i so that they align with x^i, in the sense that *at the point* p

$$\partial_i F = \overline{e}_i, \forall i \leq n \quad \text{and} \quad \nu = \overline{e}_{n+1}$$

Since we are working in normal coordinates, note that the Christoffel symbols Γ and $\overline{\Gamma}$ both vanish at p, but their derivatives might not, so we have to be very careful when working with these expressions.

Proposition 1.6.2: The evolution equation for the metric is

$$\partial_t g = 2fh$$

Proof: We prove by using Fermi coordinates, recall that we define the metric as the restriction of the ambient metric like so

$$g_{ij} = \langle \partial_i F, \partial_j F \rangle,$$

and thus we can differentiate in the ambient space to get an expression for the time derivative of the restriction

$$\begin{aligned}
\partial_t g_{ij} &= \partial_t \langle \partial_i F, \partial_j F \rangle = \langle \partial_t \partial_i F, \partial_j F \rangle + \langle \partial_i F, \partial_t \partial_j F \rangle \\
&= \langle \partial_i (f\nu), e_j \rangle + \langle e_i, \partial_j (f\nu) \rangle \\
&= \langle \nabla_i (f\nu), e_j \rangle + \langle e_i, \nabla_j (f\nu) \rangle \quad \text{because Christoffel symbols vanish} \\
&= \langle f\nabla_i \nu + \nu \nabla_i f, e_j \rangle + \langle e_i, f\nabla_j \nu + \nu \nabla_j f \rangle \\
&= f\langle \nabla_i \nu, e_j \rangle + f\langle e_i, \nabla_j \nu \rangle \qquad \text{by orthogonality} \\
&= f\langle h_{ki} e_k, e_j \rangle + f\langle e_i, h_{kj} e_k \rangle = fh_{ji} + fh_{ij} = 2fh_{ij}
\end{aligned}$$

where we used Proposition 1.3.2 in the final step. □

Now that we know how the metric evolves there are some immediate consequences that we can show.

Proposition 1.6.3: The evolution equations for $\nu, \mathrm{d}S$ are

$$\partial_t \nu = -\nabla f \quad \text{and} \quad \partial_t \,\mathrm{d}S = fH \,\mathrm{d}S$$

respectively.

Proof: First note that $\partial_t \langle \nu, \nu \rangle = 0$ and so we have that in Fermi coordinates

$$\partial_t \nu = \langle \partial_t \nu, e_j \rangle e_j$$

then we also have that for any j

$$0 = \partial_t \langle \nu, e_j \rangle = \langle \partial_t \nu, e_j \rangle + \langle \nu, \partial_t e_j \rangle$$

and so

$$\begin{aligned}\partial_t \nu &= -\langle \nu, \partial_t e_j \rangle e_j = -\langle \nu, \partial_t \partial_j F \rangle e_j = -\langle \nu, \partial_j (f\nu) \rangle e_j = -\langle \nu, f\partial_j \nu + \nu \partial_j f \rangle e_j \\ &= -\langle \nu, f h_{ij} e_i + \nu \nabla_j f \rangle e_j \qquad \text{apply orthogonality of } \nu \text{ and } e_i \\ &= -\langle \nu, \nu \nabla_j f \rangle e_j = -\nabla_j f e_j = -\nabla f.\end{aligned}$$

Note that on line 2 we also implicitly used the fact that the Christoffel symbols vanish in normal coordinates around the point p.

For the volume form, we know that $\mathrm{d}S = \sqrt{\det(g)}\,\mathrm{d}x_1 ... \,\mathrm{d}x_n$ and so we can compute

$$\partial_t (\mathrm{d}S) = \partial_t \left(\sqrt{\det(g)} \right) \mathrm{d}x_1 ... \,\mathrm{d}x_n.$$

Now recall that for a parametrized matrix $A(t)$ we have

$$\partial_t \det(A(t)) = \det(A(t)) \operatorname{tr}(\partial_t A(t))$$

and so

$$\begin{aligned}\partial_t \left(\sqrt{\det(g)} \right) \mathrm{d}x ... \,\mathrm{d}x &= \frac{1}{2\sqrt{\det(g)}} \partial_t (\det(g)) \,\mathrm{d}x_1 ... \,\mathrm{d}x_n \\ &= \sqrt{\det(g)} \operatorname{tr}\left(\partial_t \left(g_{ij} \right) \right) \mathrm{d}x_1 ... \,\mathrm{d}x_n \\ &= \sqrt{\det(g)} \operatorname{tr}\left(f h_{ij} \right) \mathrm{d}x_1 ... \,\mathrm{d}x_n \\ &= fH \,\mathrm{d}S\end{aligned}$$

□

Now that we have evolution equation for some local properties, we can extend those to evolution equation of global quantities.

Proposition 1.6.4: We have the following evolution equations for $V(M_t)$ and $A(M_t)$,

$$\partial_t V(M_t) = \int_{M_t} f \, \mathrm{d}S, \quad \partial_t A(M_t) = \int_{M_t} H f \, \mathrm{d}S$$

Proof: First for the volume, extend the vector field $f\nu$ to a global vector Y field on N. Now by classic geometry theorems [9, p. 425] we get that the change in volume for a domain evolving under a global vector field is

$$\partial_t V(M_t) = \int_{\Omega} \operatorname{div} Y \, \mathrm{d}V.$$

Now by the divergence theorem we get that

$$\int_{\Omega} \operatorname{div} Y \, \mathrm{d}V = \int_{M_t} \langle Y, \nu \rangle \, \mathrm{d}S,$$

but we know that $Y = f\nu$ along M_t so

$$\partial_t V(M_t) = \int_{M_t} f \, \mathrm{d}S.$$

For the area, we get

$$\partial_t A(M_t) = \partial_t \int_{M_t} \mathrm{d}S = \int_{M_t} \partial_t \, \mathrm{d}S = \int_{M_t} f H \, \mathrm{d}S.$$

□

We have one final evolution equation to find, and that is the one for the second fundamental form h_{ij}.

Proposition 1.6.5: We have the following evolution equations for h_{ij}

$$\partial_t h_{ij} = -\nabla_i \nabla_j f + f\left(h_{i\ell} g^{\ell k} h_{kj} - \overline{R}^{\nu}_{\nu ij}\right)$$

Proof: Recall that $h_{ij} = \langle e_i, \overline{\nabla}_j \nu \rangle = \langle \partial_i F, \overline{\nabla}_j \nu \rangle$. Then since ν is orthogonal to all $\partial_i F$'s we get

$$0 = \overline{\nabla}_j \langle \partial_i F, \nu \rangle = \langle \partial_i F, \overline{\nabla}_j \nu \rangle + \langle \overline{\nabla}_j \partial_i F, \nu \rangle$$

and thus

$$\langle \overline{\nabla}_j \partial_i F, \nu \rangle = -h_{ij}.$$

With this in mind we can compute

$$\begin{aligned}
-\partial_t h_{ij} &= \partial_t \langle \nabla_j \partial_i F, \nu \rangle = \langle \partial_t \nabla_j \partial_i F, \nu \rangle + \langle \partial_j \partial_i F, \partial_t \nu \rangle \\
&= \left\langle \partial_t \left(\partial_j \partial_i F + \overline{\Gamma}^k_{\rho\sigma} \partial_i F^\rho \partial_j F^\sigma \overline{e}_k \right), \nu \right\rangle + \langle \partial_j \partial_i F, \partial_t \nu \rangle \\
&= \left\langle \partial_j \partial_i (f\nu) + \left(\partial_t \overline{\Gamma}^k_{\rho\sigma} \right) (\partial_i F^\rho \partial_j F^\sigma) \overline{e}_k + \overline{\Gamma}^k_{\rho\sigma} \partial_t (\partial_i F^\rho \partial_j F^\sigma \overline{e}_k), \nu \right\rangle \\
&\quad - \langle \partial_j \partial_i F, \nabla f \rangle
\end{aligned}$$

now the Christoffel symbols vanish at p, so wherever they appear without a time derivative they vanish there, we hence get

$$\begin{aligned}
-\partial_t h_{ij} &= \left\langle \partial_j (\nu \partial_i f + f \partial_i \nu) + \left(f \partial_\nu \overline{\Gamma}^k_{\rho\sigma} \right) (\partial_i F^\rho \partial_j F^\sigma) \overline{e}_k, \nu \right\rangle - \langle \partial_j \partial_i F, \nabla f \rangle \\
&= \left\langle (\partial_j \nu) \partial_i f + (\partial_j f) \partial_i \nu + \nu (\partial_j \partial_i f) + f \partial_j (\partial_i \nu) + \left(f \partial_\nu \overline{\Gamma}^\nu_{\rho\sigma} \right) (\partial_i F^\rho \partial_j F^\sigma) \nu, \nu \right\rangle \\
&\quad - \langle -h_{ij} \nu, \nabla f \rangle
\end{aligned}$$

Now using the fact that ν is orthogonal to any derivation of ν (since it is a unit vector), the expression above simplifies to

$$\begin{aligned}
-\partial_t h_{ij} &= \partial_i \partial_j f + f \langle \partial_j \partial_i \nu, \nu \rangle + f \left(\partial_\nu \overline{\Gamma}^\nu_{\rho\sigma} \right) \partial_i F^\rho \partial_j F^\sigma \\
&= \partial_i \partial_j f + f \left\langle \partial_j \left(\nabla_i \nu - \overline{\Gamma}^k_{\rho\sigma} \partial_i F^\rho \nu^\sigma \overline{e}_k \right), \nu \right\rangle + f \left(\partial_\nu \overline{\Gamma}^\nu_{\rho\sigma} \right) \partial_i F^\rho \partial_j F^\sigma \\
&= \partial_i \partial_j f + f \left\langle \partial_j \left(h_{ik} \partial_k F - \overline{\Gamma}^k_{\rho\sigma} \partial_i F^\rho \nu^\sigma \overline{e}_k \right), \nu \right\rangle + f \left(\partial_\nu \overline{\Gamma}^\nu_{\rho\sigma} \right) \partial_i F^\rho \partial_j F^\sigma \\
&= \partial_i \partial_j f + f \left(\partial_\nu \overline{\Gamma}^\nu_{\rho\sigma} \right) \partial_i F^\rho \partial_j F^\sigma \\
&\quad + f \left\langle h_{ik} \partial_j \partial_k F + (\partial_j h_{ik})(\partial_k F) - \left(\partial_j \overline{\Gamma}^k_{\rho\sigma} \right) \partial_i F^\rho \nu^\sigma \overline{e}_k - \overline{\Gamma}^k_{\rho\sigma} \partial_j (\partial_i F^\rho v^\sigma \overline{e}_k), \nu \right\rangle
\end{aligned}$$

but now again the Christoffel symbols vanish and since ν is orthogonal to all tangent vectors, the second and fourth term in the inner product vanish and so we are left with

$$f \left\langle h_{ik} \partial_j \partial_k F - \left(\partial_j \overline{\Gamma}^k_{\rho\sigma} \right) \partial_i F^\rho \nu^\sigma \overline{e}_k, \nu \right\rangle = -f h_{ik} h_{jk} - f \left(\partial_j \overline{\Gamma}^\nu_{\rho\sigma} \right) \partial_i F^\rho \nu^\sigma.$$

Now recall that in orthonormal coordinates the Riemann tensor is given by

$$\overline{R}^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^\ell_{ik}$$

and so we get

$$\begin{aligned}
-\partial_t h_{ij} &= \partial_i \partial_j f + f \left(\partial_\nu \overline{\Gamma}^\nu_{\rho\sigma} \right) \partial_i F^\rho \partial_j F^\sigma - f \left(\partial_j \overline{\Gamma}^\nu_{\rho\sigma} \right) \partial_i F^\rho \nu^\sigma - f h_{ik} h_{jk} \\
&= \partial_i \partial_j f - f h_{ik} h_{jk} + f \overline{R}^\nu_{\nu ij}
\end{aligned}$$

which then since we are in orthonormal coordinates we know that $\partial_i \partial_j f = \nabla_i \nabla_j f$, and since the middle term is not tensorial we make it tensorial by contracting with the metric and so we get the desired result. □

Corollary 1.6.5.1: Immediately from Proposition 1.6.5 we get the following evolution equation for H

$$\partial_t H = -\Delta f - f\Big(|A|^2 + \overline{\mathrm{Ric}}(\nu, \nu)\Big)$$

Proof: We have $H = g^{ij} h_{ij}$ in coordinates so

$$\begin{aligned}
\partial_t H &= \partial_t\big(g^{ij} h_{ij}\big) = h_{ij}\partial_t\big(g^{ij}\big) + g^{ij}\partial_t\big(h_{ij}\big) \\
&= h_{ij}\big(-2fh^{ij}\big) + g^{ij}\Big(-\nabla_i\nabla_j f + f\Big(h_{ik}h_{kj} - \overline{R}^{\nu}_{\nu ij}\Big)\Big) \\
&= -2f|A|^2 - \Delta f + f\Big(h_{ik}h_{ik} - \overline{R}^{\nu}_{\nu ii}\Big) \\
&= -\Delta f - f|A|^2 - f\,\overline{\mathrm{Ric}}(\nu, \nu)
\end{aligned}$$

□

Chapter 2

Constrained Geometric Flow Method

With the preliminaries out of the way we can begin to discuss how we can attempt to attack the Isoperimetric problem. This is formalized in the **flow method**.

This method was first used by Gage and Hamilton in their curve shortening flow in $\mathbb{R}^2$ [3]. Then in 1984 Huisken extended these methods to the case of convex hypersurfaces in $\mathbb{R}^n$ [6].

Most recently Guan-Li used instead the **constrained flow method** to relax the convexity requirement to star-shapedness [4].

This constrained method highly depends on the Minkowski Identities which allows us to get a handle on the change in area and in volume along a special class of flows.

Theorem 2.1 (Constrained Flow Method): Consider two classes of admissible hypersurfaces $\mathcal{Z}, \mathcal{P}$ such that the following conditions hold.

1. For each hypersurface $M \in \mathcal{Z}$ we can define a flow M_t which exists for all time.
2. $V(M_t)$ is constant and $A(M_t)$ is non-decreasing.
3. The flow converges to a hypersurface $M_t \to M_\infty$ with $M_\infty \in \mathcal{P}$.

Then $\mathcal{P}$ is the Isoperimetric profile of $\mathcal{Z}$, in the sense that for each $M \in \mathcal{Z}$ there is a hypersurface $S \in \mathcal{P}$ with

$$V(M) = V(S) \text{ and } A(M) \geq A(S)$$

Proof: Let M_t be the flow of M. Then $M_t \to S$ for some hypersurface $S \in \mathcal{P}$ and so

$$V(M) = V(M_t) = \lim_{t\to\infty} V(M_t) = V(S)$$

and

$$A(M) \geq \lim_{t\to\infty} A(M_t) = A(S)$$

which proves the theorem. □

We will now consider two previous uses of this method that will motivate our use of it in the third chapter.

2.1 Warped Product Spaces

Warped products are in essence a generalization of the Polar coordinates in $\mathbb{R}^2$ so let us first look at those. The Polar coordinates (r, θ) in $\mathbb{R}^2$ are defined implicitly in terms of standard Euclidean coordinates, through $(x, y) = (r\cos(\theta), r\sin(\theta))$, where we have $r > 0$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Now the Euclidean metric is given by $g = \mathrm{d}x^2 + \mathrm{d}y^2$ and so we can compute its form in polar coordinates as

$$
\begin{aligned}
g &= \mathrm{d}x^2 + \mathrm{d}y^2 = (\mathrm{d}(r\cos\theta))^2 + (\mathrm{d}(r\sin\theta))^2 \\
&= (\cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta)^2 + (\sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta)^2 \\
&= \cos^2\theta\,\mathrm{d}r^2 - 2r\cos\theta\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta + r^2\sin^2\theta\,\mathrm{d}\theta^2 \\
&+ \sin^2\theta\,\mathrm{d}r^2 + 2r\sin\theta\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta + r^2\cos^2\theta\,\mathrm{d}\theta^2 \\
&= \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2
\end{aligned}
$$

Note that if we instead consider the function $(r, \theta) \to (r\cos(\theta), r\sin(\theta))$ as a function from $\mathbb{R}_+ \times S^1 \to \mathbb{R}^2$, where S^1 is the unit circle, then this almost gives us a decomposition

$$g_{\mathbb{R}^2} = g_{\mathbb{R}} + r^2 g_S^1$$

Note that this is not exactly the case because we first need to project a given vector down to its components in $\mathbb{R}$ and S^1 respectively and then apply the appropriate metrics. That is we actually have

$$g_{\mathbb{R}^2} = g_{\mathbb{R}} \circ \pi_1 + g_S^1 \circ \pi_2$$

where π_1, π_2 are projections onto the tangent spaces of $\mathbb{R}$ and S^1 respectively.

A similar constructions works in higher dimensions, where we have $\mathbb{R}^n = \mathbb{R}_+ \times S^{n-1}$.

It is this decomposition that we aim to generalize with the warped product space.

Definition 2.1.1: Let (M, g_M) and (N, g_n) be Riemannian manifolds, we can define a metric on $M \times N$ by

$$g(x, y) = g_M(x) \circ \pi_1 + f^2(x) g_N(y) \circ \pi_2,$$

where x, y are points of M and N respectively. This is called the *warped product space* with the *warping factor* f being a function $f : M \to \mathbb{R}_{>0}$, it is often denoted $M \times_f N$

Note that in practice we will always suppress the projections π_1 and π_2 for clarity.

The most common warped product spaces we see in practice are those of the form $\mathbb{R}_+ \times_f N$ for some N, for example the 3 space forms, $S^n, \mathbb{R}^n, \mathbb{H}^n$, are of the form

$$\mathbb{R}_+ \times_{\sin r} S^n, \quad \mathbb{R}_+ \times_r S^n, \quad \mathbb{R}_+ \times_{\sinh r} S^n$$

respectively.

These spaces carry a lot of nice properties, but the most important one for us is that they carry a natural conformal vector field.

Proposition 2.1.1: Let $\mathbb{R}_+ \times_f N$ be a warped space and let r be a coordinate on $\mathbb{R}_+$, then the vector field $X = f(r)\partial_r$ is a conformal vector field with conformal factor $f'(r)$. Furthermore its associated tensor ψ vanishes.

Proof: Let us compute the Lie derivative $\mathcal{L}_X g$,

$$\begin{aligned}\mathcal{L}_X g &= \mathcal{L}_X(dr^2 + f^2(r)g_N^2) = 2dr\mathcal{L}_X(dr) + g_N^2\mathcal{L}_X(f^2(r)) \\ &= 2dr(d(\mathcal{L}_X r)) + 2g_N^2 f^2(r)f'(r) = 2dr(df(r)) + 2f'(r)f^2(r)g_N^2 \\ &= 2f'(r)dr^2 + 2f'(r)f^2(r)g_N^2 = 2f'(r)(dr^2 + f^2(r)g_N^2).\end{aligned}$$

Note that if we set $\Phi(r) = \int_0^r f(s)\,\mathrm{d}s$ then $\nabla\Phi(r) = f(r)\partial_r$ and so $\langle\nabla_Y X, Z\rangle$ is just $\mathrm{Hess}_f(Y, Z)$ and so it is symmetric and so its anti-symmetric component vanishes. □

Note that since its a gradient, X in the above proposition is closed.

Recall that in Euclidean space spheres are the optimal shapes for the Isoperimetric inequality, in polar coordinates spheres take the simple form of sets where $r = r_0$ for some fixed r_0. Using the geometric properties of warped product spaces and the vector field X, Guan, Li and Wang were able to use the flow method to prove that this is fact generalizes, the Isoperimetric profile for a large class of warped product spaces are exactly the level sets of the projection onto $\mathbb{R}$.

Theorem 2.1.2 (Guan, Li, Wang): Let $N = \mathbb{R}_+ \times_f B$ with B closed and f satisfying some technical conditions. Then among the admissible hypersurfaces such that $\langle f(r)\partial_r, \nu\rangle$ is everywhere positive, the Isoperimetric profile consists of level sets of r.

The proof is detailed in [5], we will not consider these details too much as they will quickly get generalized by the next work.

2.2 Manifolds Admitting Compatible Conformal Vector Fields

Now it turns out that these closed conformal vector fields characterize warped products of the form we saw in the previous section, namely, if a manifold admits a closed conformal vector field then it can locally be written in the form $\mathbb{R}_+ \times_f N$ for some manifolds N and function f, see [14, Lemma 1.2]. This suggests that if we want to weaken the conditions of Guan, Li and Wang we should perhaps consider the case of non-closed conformal vector fields.

This idea was first explored by Li and Pan [11], where they formalized the necessary conditions on the ambient manifold in terms of a conformal vector field, and used this to weaken the assumptions on the vector field. They also derive a number of key properties for a conformal vector field satisfying their conditions.

They start with with a conformal vector field X on N with conformal factor φ which is non-zero on an open dense set U. They then assume X satisfies the following conditions.

Conditions 2.2.1:

1. The distribution $\mathcal{D}(X) \subseteq TU$ defined by $\mathcal{D}(X)|_p = \{v \in T_pN \mid \langle v, X\rangle = 0\}$ is integrable on the set U.

2. $\varphi > 0$ everywhere on U.
3. The integral surfaces of $\mathcal{D}(X)$ are compact level sets of $\frac{\|X\|}{\varphi}$ on U.
4. $\varphi^2 - X(\varphi) > 0$ everywhere on U.
5. The normal vector $\mathcal{N}$ to the integral surfaces of $\mathcal{D}(X)$ is the only direction of minimal Ricci curvature, that is for every unit vector v which is not colinear with $\mathcal{N}$ we have

$$\overline{\mathrm{Ric}}(\mathcal{N},\mathcal{N}) < \overline{\mathrm{Ric}}(v,v).$$

They consider the class $\mathfrak{Y}$ of admissible hypersurfaces which is star-shaped with respect to X, that is admissible hypersurfaces M for which $u = \langle X, \nu \rangle$ is positive along M. They define the normal flow with velocity $f = n\varphi - Hu$ and prove the following results about this flow.

Lemma 2.2.1:
1. For a hypersurface $M \in \mathfrak{Y}$ the normal flow with velocity $f = n\varphi - Hu$ exists for all time $t \in [0, \infty)$.
2. Along this flow volume is fixed and area is non-increasing.
3. The flow converges in the limit to an integral hypersurface of $\mathcal{D}(X)$.

Applying this to Theorem 2.1 they then prove the following

Theorem 2.2.2 (Li and Pan): Let N be an ambient manifold admitting a conformal vector field X satisfying Conditions 2.2.1, then for any star-shaped hypersurface M there exists an integral surface S of $\mathcal{D}(X)$ with

$$V(S) = V(M) \text{ and } A(M) \geq A(S)$$

2.3 Quasi-Closed Conformal Vector Fields

In this section we will rewrite some of the conditions of Li and Pan and the results that follow from those conditions in a form that is easier to use.

Let us recall the setting, we let N be the complete ambient manifold which admits a conformal vector field X on some open subset U which does not vanish on that subset, with conformal factor φ.

We will start with the first condition.

Proposition 2.3.1: Let X be a conformal vector field X such that condition 1 holds, then the associated tensor field ψ satisfies

$$\langle \psi(v), w \rangle = 0 \text{ for } v, w \in \mathcal{D}(X) \text{ eveywhere on } U$$

Proof: By definition the one form defined by $\omega(v) = \langle X, v\rangle$ annihilates $\mathcal{D}(X)$, then by the one form condition for integrability [9, p. 495] we get that $\mathrm{d}\omega$ also annihilates $\mathcal{D}(X)$, that is $\mathrm{d}\omega$ restricts to zero on $\mathcal{D}(X)$, hence by definition of the associated tensor field, we get the result above. $\square$

We see then that such a conformal vector field is 'almost' closed since its associated tensor field 'almost' vanishes.

Definition 2.3.1: We will call a conformal vector field X satisfying condition 1, a *quasi-closed* conformal vector field.

The rest of this section will be devoted to properties of quasi-closed conformal vector fields. We will fix a point $p \in U$ and an arbitrary vector $Y \in T_pU$. We will denote $\mathcal{N} = \frac{X}{\|X\|}$ the normal vector to the integral surfaces of $\mathcal{D}(X)$. We will also refer to $\mathcal{D}(X)$ as $\mathcal{D}$ for brevity. We will start with a key property regarding the integral surfaces of $\mathcal{D}$.

Proposition 2.3.2: Let S be an integral surface of $\mathcal{D}$, then S is totally umbilical, that is at every point $p \in S$ we have

$$h_{ij} = f(p)g_{ij},$$

for some function f on S. Furthermore we have $f = \frac{\varphi}{\|X\|}$.

Proof: We have in coordinates on S,

$$h_{ij} = \langle \overline{\nabla}_i \mathcal{N}, e_j\rangle = \left\langle \overline{\nabla}_i \frac{X}{\|X\|}, e_j \right\rangle = \left\langle \frac{\overline{\nabla}_i X}{\|X\|} - \frac{X}{\|X\|^2}\left(\overline{\nabla}_i \|X\|\right), e_j \right\rangle$$

then since X is orthogonal to the tangent vector e_j we get

$$h_{ij} = \left\langle \frac{\overline{\nabla}_i X}{\|X\|}, e_j \right\rangle = \frac{1}{\|X\|}\langle \overline{\nabla}_i X, e_j\rangle = \frac{1}{\|X\|}\left(\varphi g_{ij} + \langle \psi(e_i), e_j\rangle\right)$$

then by Proposition 2.3.1 we get

$$h_{ij} = \frac{\varphi}{\|X\|} g_{ij}$$

$\square$

Now we want the integral surfaces S of $\mathcal{D}$ to be our Isoperimetric profile, hence they should be critical points of the surface area functional with respect to fixed volume. Hence by Proposition 1.6.4 we need to have $H = n\frac{\varphi}{\|X\|}$ be constant, which motivates condition 2.

Corollary 2.3.2.1: If X satisfies condition 2, then the integral surfaces of $\mathcal{D}$ are totally umbilical with constant mean curvature $H = n\frac{\varphi}{\|X\|}$.

Definition 2.3.2: We will call quasi-closed conformal vector field X *symmetric* if it satisfies condition 2.

Now consider, for a moment, the spheres in $\mathbb{R}^n$ of radius r. They are the integral surfaces for the orthogonal distribution to $X = x^i\partial_i$ which is a closed conformal vector field with factor $\varphi = 2$. We see that their mean curvature is $H = \frac{2n}{r}$, we thus see that the mean curvature is inversely proportional to a certain 'scale' function, in this case r. This scale function will turn out to be extremely useful in many of our future calculations, so we give it a name here.

Definition 2.3.3: Wherever $\varphi \neq 0$ on U, we will call the following function the *scale* function for X

$$\lambda = \frac{\|X\|^2}{\varphi^2}$$

Since we know λ is constant on integral surfaces, its gradient must be colinear with X, that is $\overline{\nabla}\lambda = 2\Lambda X$ for some function Λ. We note a convenient expression for Λ.

Proposition 2.3.3: Where λ is defined, we have

$$\overline{\nabla}\lambda = 2\Lambda X = 2\frac{\varphi^2 - X(\varphi)}{\varphi^3}X$$

Proof: We see that

$$\begin{aligned}\overline{\nabla}\lambda &= \langle\overline{\nabla}\lambda, \mathcal{N}\rangle\mathcal{N} = X(\lambda)\frac{X}{\|X\|^2} = X\left(\frac{\|X\|^2}{\varphi^2}\right)\frac{X}{\|X\|^2} \\ &= \frac{2\langle\overline{\nabla}_X X, X\rangle}{\varphi^2}\frac{X}{\|X\|^2} + \left(-\frac{2}{\varphi^3}\right)X(\varphi)X = 2\frac{\varphi\langle X, X\rangle}{\varphi^2}\frac{X}{\|X\|^2} - 2\left(\frac{X(\varphi)}{\varphi^3}\right)X \\ &= 2\frac{\varphi^2 - X(\varphi)}{\varphi^3}X\end{aligned}$$

□

Now as we saw, for a quasi-closed conformal vector field we have that $\langle \psi(v), w \rangle$ vanishes on $\mathcal{D}$ and this turns out to be enough to get a precise equation for ψ even when v, w are not in $\mathcal{D}$.

Proposition 2.3.4: We have

$$\psi(Y) = \frac{\langle Y, \overline{\nabla}\|X\| \rangle X - \langle Y, X \rangle \overline{\nabla}\|X\|}{\|X\|}.$$

Furthermore if X symmetric, then wherever $\varphi \neq 0$ we have

$$\psi(Y) = \frac{\langle Y, \overline{\nabla}\varphi \rangle X - \langle Y, X \rangle \overline{\nabla}\varphi}{\varphi}.$$

Proof: Recall that $\langle \psi(Y), Z \rangle$ is anti-symmetric in Y, Z so we may assume WLOG one of the two is in $\mathcal{D}$ and hence also WLOG assume that it is Y. Then by Proposition 2.3.1 we may assume that Z is colinear with X. We thus have $\langle Y, Z \rangle = 0$ so

$$\begin{aligned} \langle \overline{\nabla}_Y X, Z \rangle &= \frac{\langle Z, X \rangle}{\|X\|^2} \langle \overline{\nabla}_Y X, X \rangle = \frac{1}{2} \frac{\langle Z, X \rangle}{\|X\|^2} \overline{\nabla}_Y \langle X, X \rangle \\ &= \frac{1}{2} \frac{\langle Z, X \rangle}{\|X\|^2} \langle Y, \overline{\nabla}\|X\|^2 \rangle = \frac{\langle Z, X \rangle}{\|X\|} \langle Y, \overline{\nabla}\|X\| \rangle \end{aligned}$$

We can then anti-symmetrize this to get that for arbitrary Y, Z

$$\langle \psi(Y), Z \rangle = \frac{\langle Z, X \rangle \langle Y, \overline{\nabla}\|X\| \rangle - \langle Y, X \rangle \langle Z, \overline{\nabla}\|X\| \rangle}{\|X\|}$$

which gives us the first result.

For the second result, we compute

$$\begin{aligned} \psi(Y) &= \frac{\left\langle Y, \overline{\nabla}\left(\varphi \cdot \frac{\|X\|}{\varphi}\right) \right\rangle X - \langle Y, X \rangle \overline{\nabla}\left(\varphi \cdot \frac{\|X\|}{\varphi}\right)}{\|X\|} \\ &= \frac{\left\langle Y, \frac{\|X\|}{\varphi} \overline{\nabla}\varphi + \varphi \overline{\nabla}\left(\frac{\|X\|}{\varphi}\right) \right\rangle X - \langle Y, X \rangle \left(\frac{\|X\|}{\varphi} \overline{\nabla}\varphi + \varphi \overline{\nabla}\left(\frac{\|X\|}{\varphi}\right)\right)}{\|X\|} \\ &= \frac{\langle Y, \overline{\nabla}\varphi \rangle X - \langle Y, X \rangle \overline{\nabla}\varphi}{\varphi} + \varphi \frac{\left\langle Y, \overline{\nabla}\left(\frac{\|X\|}{\varphi}\right) \right\rangle X - \langle Y, X \rangle \overline{\nabla}\left(\frac{\|X\|}{\varphi}\right)}{\|X\|}. \end{aligned}$$

Now we recall that $\frac{\|X\|}{\varphi}$ is constant along integral surfaces and thus its gradient is colinear with X. At the point p we then write $\overline{\nabla}\left(\frac{\|X\|}{\varphi}\right) = aX$ and get

$$\psi(Y) = \frac{\langle Y, \overline{\nabla}\varphi\rangle X - \langle Y, X\rangle\overline{\nabla}\varphi}{\varphi} + \varphi\frac{\langle Y, aX\rangle X - \langle Y, X\rangle aX}{\|X\|}$$
$$= \frac{\langle Y, \overline{\nabla}\varphi\rangle X - \langle Y, X\rangle\overline{\nabla}\varphi}{\varphi} + 0$$

which gets us the second result. □

We note here two other useful ways to write the covariant derivative of X

Corollary 2.3.4.1: Where $\varphi = 0$ we have

$$\overline{\nabla}_Y \frac{X}{\|X\|} = -\frac{\langle X, Y\rangle}{\|X\|}\overline{\nabla}\|X\|$$

if X is symmetric then where $\varphi \neq 0$ we have

$$\overline{\nabla}_Y \frac{X}{\varphi} = X - \frac{\langle X, Y\rangle}{\varphi^2}\overline{\nabla}\varphi$$

Proof: We prove directly, for the first case

$$\overline{\nabla}_Y \frac{X}{\|X\|} = \frac{1}{\|X\|}\overline{\nabla}_Y X - \frac{X}{\|X\|^2}\overline{\nabla}_Y\|X\|$$
$$= \frac{1}{\|X\|}\left(\frac{\langle Y, \overline{\nabla}\|X\|\rangle X - \langle Y, X\rangle\overline{\nabla}\|X\|}{\|X\|}\right) - \frac{X}{\|X\|^2}\overline{\nabla}_Y\|X\|$$
$$= -\frac{\langle Y, X\rangle\overline{\nabla}\|X\|}{\|X\|^2}.$$

And in the second case

$$\overline{\nabla}_Y \frac{X}{\varphi} = \frac{1}{\varphi}\overline{\nabla}_Y X - \frac{X}{\varphi^2}\overline{\nabla}_Y\varphi$$
$$= \frac{1}{\varphi}\left(\varphi X + \frac{\langle Y, \overline{\nabla}\varphi\rangle X - \langle Y, X\rangle\overline{\nabla}\varphi}{\varphi}\right) - \frac{X}{\varphi^2}\overline{\nabla}_Y\varphi$$
$$= X - \frac{\langle Y, X\rangle\overline{\nabla}\varphi}{\varphi^2}.$$

□

Corollary 2.3.4.2:

$$(\psi(Y), Z) = 2\frac{\mathrm{ASym}\left(\overline{\nabla}\|X\|^\flat \otimes X^\flat\right)(Y, Z)}{\|X\|}.$$

Furthermore if X is symmetric, wherever $\varphi \neq 0$ we have

$$(\psi(Y), Z) = 2\frac{\mathrm{ASym}\left(\overline{\nabla}\varphi^\flat \otimes X^\flat\right)(Y, Z)}{\varphi}.$$

Here $\flat$ represents raising an index and ASym represents the anti-symmetrization.

Proof: Immediate from Proposition 2.3.4. □

We can also rewrite some of the Riemann and Ricci curvatures of the ambient manifold in terms of X.

Proposition 2.3.5: Wherever $\varphi = 0$ we have

$$\overline{R}(Y,X,Y,X) = -\|X\|\big\langle\overline{\nabla}_Y\overline{\nabla}\|X\|,Y\big\rangle + \frac{\langle X,Y\rangle^2}{\|X\|}\big\langle\overline{\nabla}_{\mathcal{N}}\overline{\nabla}\|X\|,\mathcal{N}\big\rangle$$
$$\overline{\mathrm{Ric}}(X,Y) = -\frac{\langle X,Y\rangle}{\|X\|}\big(\overline{\Delta}\|X\| - \big\langle\overline{\nabla}_{\mathcal{N}}\overline{\nabla}\varphi,\mathcal{N}\big\rangle\big)$$

and in addition, if $Y \in \mathcal{D}$

$$\|X\|\,\overline{\mathrm{Ric}}(X,Y) = \big\langle\overline{\nabla}_Y\overline{\nabla}\|X\|,X\big\rangle = \big\langle\overline{\nabla}_X\overline{\nabla}\|X\|,Y\big\rangle = 0 \tag{2.5}$$

If X is symmetric, then wherever $\varphi \neq 0$ we have

$$\begin{aligned}\overline{R}(Y,X,Y,X) &= -\varphi\big\langle\overline{\nabla}_Y\overline{\nabla}\varphi,Y\big\rangle + \frac{\langle X,Y\rangle^2}{\varphi}\big\langle\overline{\nabla}_{\mathcal{N}}\overline{\nabla}\varphi,\mathcal{N}\big\rangle \\ \overline{\mathrm{Ric}}(X,Y) &= -\frac{\langle X,Y\rangle}{\varphi}\big(\overline{\Delta}\varphi - \big\langle\overline{\nabla}_{\mathcal{N}}\overline{\nabla}\varphi,\mathcal{N}\big\rangle\big)\end{aligned} \tag{2.6}$$

and in addition, if $Y \in \mathcal{D}$

$$\varphi\,\overline{\mathrm{Ric}}(X,Y) = \big\langle\overline{\nabla}_Y\overline{\nabla}\varphi,X\big\rangle = \big\langle\overline{\nabla}_X\overline{\nabla}\varphi,Y\big\rangle = 0 \tag{2.7}$$

Proof: For the first case, we will consider $\overline{R}\Big(\overline{e}_i,\overline{e}_j,\frac{X}{\|X\|},\overline{e}_k\Big)$, then we will use linearity of the Ricci tensor to remove the denominator, we start with a use of Corollary 2.3.4.1

$$\begin{aligned}\overline{R}\left(\overline{e}_i,\overline{e}_j,\frac{X}{\|X\|},\overline{e}_k\right) &= \left\langle\overline{\nabla}_i\overline{\nabla}_j\frac{X}{\|X\|} - \overline{\nabla}_j\overline{\nabla}_i\frac{X}{\|X\|},\overline{e}_k\right\rangle \\ &= \left\langle\overline{\nabla}_i\left(-\frac{\langle\overline{e}_j,X\rangle\overline{\nabla}\|X\|}{\|X\|^2}\right) - \overline{\nabla}_j\left(-\frac{\langle\overline{e}_i,X\rangle\overline{\nabla}\|X\|}{\|X\|^2}\right),\overline{e}_k\right\rangle \\ &= -\overline{\nabla}_i\left(\frac{\langle\overline{e}_j,X\rangle}{\|X\|^2}\right)\overline{e}_k(\|X\|) + \overline{\nabla}_j\left(\frac{\langle\overline{e}_i,X\rangle}{\|X\|^2}\right)\overline{e}_k(\|X\|) \\ &\quad -\frac{\langle\overline{e}_j,X\rangle}{\|X\|^2}\big\langle\overline{\nabla}_i\overline{\nabla}\|X\|,\overline{e}_k\big\rangle + \frac{\langle\overline{e}_i,X\rangle}{\|X\|^2}\big\langle\overline{\nabla}_j\overline{\nabla}\|X\|,\overline{e}_k\big\rangle.\end{aligned}$$

Now let us deal with the first two terms, expanding gives us

$$\overline{e}_k(\|X\|)\left(\frac{\big\langle\overline{\nabla}_jX,\overline{e}_i\big\rangle - \big\langle\overline{\nabla}_iX,\overline{e}_j\big\rangle}{\|X\|^2} + 2\frac{\langle X,\overline{e}_j\rangle\overline{e}_i(\|X\|) - \langle X,\overline{e}_i\rangle\overline{e}_j(\|X\|)}{\|X\|^3}\right),$$

then by definition the left denominator here is $2\langle\psi(\overline{e}_j),\overline{e}_i\rangle$ which we can expand by Proposition 2.3.4, this quickly shows that these terms exactly cancel the other terms in this above expression.

Now we are left with

$$\overline{R}\left(\overline{e}_i,\overline{e}_j,\frac{X}{\|X\|},\overline{e}_k\right)=\frac{\langle\overline{e}_i,X\rangle}{\|X\|^2}\langle\overline{\nabla}_j\overline{\nabla}\|X\|,\overline{e}_k\rangle-\frac{\langle\overline{e}_j,X\rangle}{\|X\|^2}\langle\overline{\nabla}_i\overline{\nabla}\|X\|,\overline{e}_k\rangle, \quad (2.8)$$

by linearity we can substitute $\overline{e}_i=\overline{e}_k=Y$ and $\overline{e}_j=X$, this gives us

$$\overline{R}\left(Y,X,\frac{X}{\|X\|},Y\right)=\frac{\langle Y,X\rangle}{\|X\|^2}\langle\overline{\nabla}_X\overline{\nabla}\|X\|,Y\rangle-\frac{\langle X,X\rangle}{\|X\|^2}\langle\overline{\nabla}_Y\overline{\nabla}\|X\|,Y\rangle$$

we now multiply by $\|X\|$ to get

$$\overline{R}(Y,X,X,Y)=-\|X\|\langle\overline{\nabla}_Y\overline{\nabla}\|X\|,Y\rangle+\frac{\langle Y,X\rangle}{\|X\|}\langle\overline{\nabla}_X\overline{\nabla}\|X\|,Y\rangle \quad (2.9)$$

Next consider the integral hypersurface S of $\mathcal{D}$ that passes through p, we know that h_{ij} is identically zero everywhere on this hypersurface. Hence, by Lemma 1.3.3, we have for any $e_i,e_j,e_k\in T_pS$

$$\overline{R}(e_i,e_j,e_k,\mathcal{N})=-\nabla_i h_{jk}+\nabla_j h_{ik}=0$$

and so in particular, by taking trace over j,k and using linearity to substitute $e_i=Y$ we get

$$\overline{\mathrm{Ric}}(Y,X)=0$$

for any $Y\in\mathcal{D}$.

Now by using (2.8) but tracing over e_i and e_k and plugging in $\overline{e}_j=Y$ gives us

$$\overline{\mathrm{Ric}}(Y,X)=-\frac{\langle Y,X\rangle}{\|X\|}\overline{\Delta}\|X\|+\frac{1}{\|X\|}\langle\overline{\nabla}_Y\overline{\nabla}\|X\|,X\rangle \quad (2.10)$$

then for any $Y\in\mathcal{D}$ we get

$$0=-\frac{\langle Y,X\rangle}{\|X\|}\overline{\Delta}\|X\|+\frac{1}{\|X\|}\langle\overline{\nabla}_Y\overline{\nabla}\|X\|,X\rangle=\frac{1}{\|X\|}\langle\overline{\nabla}_Y\overline{\nabla}\|X\|,X\rangle$$

which gives us (2.5).

Now plugging (2.5) into (2.9) gives us

$$\begin{aligned}\overline{R}(Y,X,X,Y)&=-\|X\|\langle\overline{\nabla}_Y\overline{\nabla}\|X\|,Y\rangle+\frac{\langle Y,X\rangle}{\|X\|}\left\langle\overline{\nabla}_{\|X\|\mathcal{N}}\overline{\nabla}\|X\|,\mathcal{N}\left\langle\frac{X}{\|X\|},Y\right\rangle\right\rangle\\&=-\|X\|\langle\overline{\nabla}_Y\overline{\nabla}\|X\|,Y\rangle+\frac{\langle Y,X\rangle^2}{\|X\|}\langle\overline{\nabla}_{\mathcal{N}}\overline{\nabla}\|X\|,\mathcal{N}\rangle\end{aligned}$$

which is the first result.

For the second result we do the same thing with (2.10), we get

$$\begin{aligned}\overline{\mathrm{Ric}}(Y,X) &= -\frac{\langle Y,X\rangle}{\|X\|}\overline{\Delta}\|X\| + \frac{1}{\|X\|}\Big\langle \overline{\nabla}_{\mathcal{N}\langle\mathcal{N},Y\rangle}\overline{\nabla}\|X\|, \mathcal{N}\|X\|\Big\rangle \\ &= -\frac{\langle Y,X\rangle}{\|X\|}\big(\overline{\Delta}\|X\| - \langle\overline{\nabla}_{\mathcal{N}}\overline{\nabla}\|X\|,\mathcal{N}\rangle\big)\end{aligned}$$

which is our second result.

Finally for the case where $\varphi \neq 0$, we start off similarly

$$\begin{aligned}\overline{R}\left(\overline{e}_i,\overline{e}_j,\frac{X}{\|X\|},\overline{e}_k\right) &= \left\langle \overline{\nabla}_i\overline{\nabla}_j\frac{X}{\varphi} - \overline{\nabla}_j\overline{\nabla}_i\frac{X}{\varphi},\overline{e}_k\right\rangle \\ &= \left\langle \overline{\nabla}_i\left(\overline{e}_j - \frac{\langle\overline{e}_j,X\rangle\overline{\nabla}\varphi}{\varphi^2}\right) - \overline{\nabla}_j\left(\overline{e}_i - \frac{\langle\overline{e}_i,X\rangle\overline{\nabla}\varphi}{\varphi^2}\right),\overline{e}_k\right\rangle\end{aligned}$$

We now note that $\overline{\nabla}_i\overline{e}_j$ and $\overline{\nabla}_j\overline{e}_i$ are both zero because we are working in normal coordinates. After that the calculation is identical to the first case. □

Chapter 3

Main Results

3.1 Motivation

As we saw in the previous section, already with quasi-closed conformal vector fields we can prove a strong result regarding Isoperimetric inequalities. However, there are still cases which we would expect these techniques to be applicable to which cannot be reached with their approach.

Example: Consider $N = \mathbb{R}^{n+1}$ with the following conformal vector field

$$X(x^1, ..., x^n) = x^i \partial_i - x^2 \partial_1 + x^1 \partial_2$$

this is indeed a quasi-closed conformal vector field with conformal factor φ, but its integral surfaces are not compact and they do not have fixed mean curvature. Thus we have no hope of attaining a useful Isoperimetric inequality for star-shaped surfaces with respect to this flow.

We could instead work with $Y = x^i \partial_i$ but then there are hypersurfaces that will never be star-shaped with respect to Y but are star-shaped with respect to X, we really need star-shapedness since it is needed to guarantee convergence and to guarantee that area decreases. An example of such a surface can be seen in Figure 1.

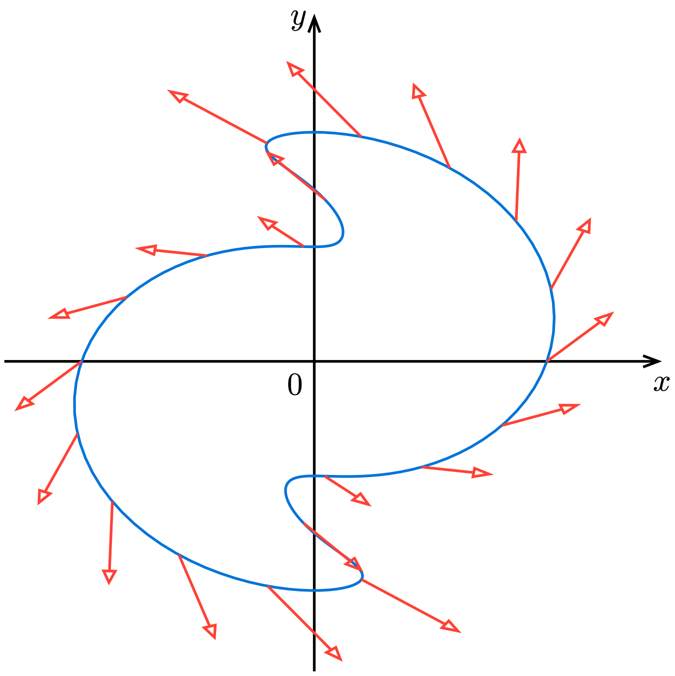

Figure 1: Hypersurface in $\mathbb{R}^2$ which is star shaped with respect to X but not Y

We will shortly introduce the tools we will need to deal with this issue, but before we do that we want to motivate these tools a little. We can think of the quasi-closedness condition on X as a compatibility condition between X and a foliation S_α, namely that X is everywhere orthogonal to S_α.

We can then try to consider foliations which are in some sense 'compatible' with X given in the example above.

Example: Consider the foliation $\mathcal{F}$ of $\mathbb{R}^{n+1} \setminus 0$ given by spheres S_α centered at the origin, this foliation is not everywhere orthogonal to X, however, the foliation is fixed under the normal flow $n\varphi - Hu$ and the foliation induces a decomposition $X = X^\perp + X^\top$ where $X^\perp$ is orthogonal to S_r and $X^\top$ is tangent to S_r.

Now importantly $X^\perp$ is just Y and thus is also a conformal vector field, and thus since $X^\top = X - X^\perp$ then $X^\top$ is also conformal. Now $X^\perp$ is a quasi-closed conformal vector field which we can manage with the preexisting techniques, so our goal is to find a way to use this decomposition to reduce to the case of just $X^\perp$.

3.2 Setting

We will consider a complete $n+1$ dimensional Riemannian manifold N, with $n \geq 2$. On this manifold we consider a complete conformal vector field X which is non-zero on an open set U, along with a foliation $\mathcal{F}$. We assume the two are are compatible, in the sense that the foliation $\mathcal{F}$ induces a decomposition $X = X^\perp + X^\top$ where $X^\perp$ is a quasi-closed symmetric conformal vector field with integral surfaces $S_\alpha \in \mathcal{F}$ and $X^\top$ is a quasi-closed Killing vector field, that is its conformal factor is zero.

We will associate with $X^\perp$ its conformal factor φ which is the same as that of X, we will also associate the scale function (Definition 2.3.3) λ and its derivative Λ (Proposition 2.3.3). We will denote by $\psi^\perp$ and $\psi^\top$ the associated tensor fields (Proposition 2.3.4) of $X^\perp$ and $X^\top$ respectively. We will also define

$$\mathcal{N}^\perp = \frac{X^\perp}{\|X^\perp\|} \text{ and } \mathcal{N}^\top = \frac{X^\top}{\|X^\top\|}$$

We will also make the following assumptions

Assumptions 3.2.1:
1. The conformal factor φ of $X^\perp$ is positive on U.
2. The function Λ (Proposition 2.3.3) is positive on U.
3. The function $\Lambda\varphi^3 + X^\top(\varphi)$ is positive on U.
4. The integral hypersurfaces S_α are compact.
5. The directions $X^\perp$ and $X^\top$ are both of least Ricci curvature, that is for any unit vector $Y \in T_pU$ we have

$$\overline{\mathrm{Ric}}(Y, Y) \geq \overline{\mathrm{Ric}}(\mathcal{N}^\perp, \mathcal{N}^\perp) = \overline{\mathrm{Ric}}(\mathcal{N}^\top, \mathcal{N}^\top)$$

The first condition informally means that $X^\perp$ is a dilation-like vector field, because under its first order vector field flow volumes increase.
The second condition informally means that our scale function λ is increasing in the direction of $X^\perp$, so just like in Euclidean space as balls increase in radius their mean curvature decreases.
The third and fourth conditions are technical conditions needed for convergence. The last condition is necessary for area to decrease along the normal flow we will construct.

Our flow will consist of two steps,

1. First we will use a time dependent conformal vector field $X(t) = X^{\perp} + X^{\top}\left(1 - \frac{t}{T_0}\right)$ for some constant T_0, we will consider the flow with velocity
$$f = n\varphi - H\langle X(t), \nu\rangle, \tag{3.11}$$
we will run this flow until $t = T_0$.
2. If the flow survives after $t = T_0$ we will stop the flow, and then set $X(t) = X^{\perp}$, we then continue with the flow
$$f = n\varphi - H\langle X^{\perp}, \nu\rangle$$
for however long the flow lasts.

Note that in this setting, u's definition depends on time but we will drop this dependence in our notation and only explicitly mention it when it comes up. Note that, if $u > 0$ when $t = T_0$, then at that point in time the surface is star-shaped with respect to just $X^{\perp}$ so we can apply the methods of Li and Pan.

For now we will assume that the flow exists on some interval $[0, T)$, we will show this must be the case later, in Proposition 3.7.6. We will also assume that u remains positive on $[0, T)$, this will be proven in Section 3.5. We will now start computing the evolution of various geometric quantities along our flow. For convenience we will define the factor

$$\Xi(t) = \left(1 - \frac{t}{T_0}\right).$$

We will also define the parabolic operator

$$L = \partial_t - u\Delta$$

as well as the functions

$$u^{\perp} = \langle X^{\perp}, \nu\rangle, \quad u^{\top} = \Xi(t)\langle X^{\top}, \nu\rangle.$$

Notice $u = u^{\perp} + u^{\top}$.

Finally we will also use the notation

$$\pi(X^{\perp}), \pi(X^{\top})$$

to denote the orthogonal projection of these vector fields onto T_pM_t, notice that

$$\pi(X^{\perp}) = X^{\perp} - u^{\perp}\nu = \langle X^{\perp}, e_i\rangle e_i \quad \text{and}$$
$$\pi(X^{\top}) = X^{\top} - u^{\top}\nu = \langle X^{\top}, e_i\rangle e_i.$$

For some calculations we will assume $t \leq T_0$, for $t > T_0$ we can simply set $\Xi(t) = \Xi'(t) = 0$ and the calculations still follow.

3.3 Variation of Area and Volume

Information about the variation of area and volume along our flow is crucial to the success of the fow method, so we will start with that.

Proposition 3.3.1: Let M_t be a solution to the flow for $t \in [0, T)$, we have

$$\partial_t V(M_t) = 0 \quad \text{and} \quad \partial_t A(M_t) \leq 0.$$

It thus follows that volume is fixed and area is non-increasing.

Proof: We Proposition 1.6.4 to calculate the variation of volume and area. For volume we have due to Lemma 1.4.2

$$\partial_t V(M_t) = \int_{M_t} f \, \mathrm{d}S = \int_{M_t} n\varphi - Hu \, \mathrm{d}S = 0.$$

For area we get

$$\begin{aligned}\partial_t A(M_t) &= \int_{M_t} Hf \, \mathrm{d}S = \int_{M_t} H(n\varphi - Hu) \, \mathrm{d}S \\ &= \frac{n}{n-1} \int_{M_t} \overline{\mathrm{Ric}}(\nu, X(t) - u\nu) \, \mathrm{d}S - \frac{1}{n} \int_{M_t} u \sum_{i<j} \left(\kappa_i - \kappa_j\right)^2 \mathrm{d}S\end{aligned}$$

now we write $X(t) = X^{\perp} + \Xi(t) X^{\top}$ and $u = u^{\perp} + u^{\top}$ to get

$$\begin{aligned}\overline{\mathrm{Ric}}(\nu, X(t) - u\nu) &= \overline{\mathrm{Ric}}(\nu, X^{\perp} - u^{\perp}\nu) + \overline{\mathrm{Ric}}(\nu, \Xi(t) X^{\top} - u^{\top}\nu) \\ &= \overline{\mathrm{Ric}}(\nu, X^{\perp}) + \Xi(t) \, \overline{\mathrm{Ric}}(\nu, X^{\top}) - (u^{\perp} + u^{\top}) \, \overline{\mathrm{Ric}}(\nu, \nu)\end{aligned}$$

then by (2.5) we get

$$\begin{aligned}\overline{\mathrm{Ric}}(\nu, X(t) - u\nu) &= \langle \nu, X^{\perp} \rangle \, \overline{\mathrm{Ric}}(\mathcal{N}^{\perp}, \mathcal{N}^{\perp}) + \langle \nu, \Xi(t) X^{\top} \rangle \, \overline{\mathrm{Ric}}(\mathcal{N}^{\top}, \mathcal{N}^{\top}) \\ &\quad - (u^{\perp} + u^{\top}) \, \overline{\mathrm{Ric}}(\nu, \nu) \\ &= u^{\perp} \, \overline{\mathrm{Ric}}(\mathcal{N}^{\perp}, \mathcal{N}^{\perp}) + u^{\top} \, \overline{\mathrm{Ric}}(\mathcal{N}^{\top}, \mathcal{N}^{\top}) - (u^{\perp} + u^{\top}) \, \overline{\mathrm{Ric}}(\nu, \nu).\end{aligned}$$

Now due to assumption 5 we get that $\overline{\mathrm{Ric}}(\mathcal{N}^{\perp}, \mathcal{N}^{\perp}) = \overline{\mathrm{Ric}}(\mathcal{N}^{\top}, \mathcal{N}^{\top})$ and so

$$\begin{aligned}\overline{\mathrm{Ric}}(\nu, X(t) - u\nu) &= (u^{\perp} + u^{\top}) \, \overline{\mathrm{Ric}}(\mathcal{N}^{\perp}, \mathcal{N}^{\perp}) - (u^{\perp} + u^{\top}) \, \overline{\mathrm{Ric}}(\nu, \nu). \\ &= u\Big(\overline{\mathrm{Ric}}(\mathcal{N}^{\perp}, \mathcal{N}^{\perp}) - \overline{\mathrm{Ric}}(\nu, \nu)\Big).\end{aligned}$$

Plugging this into the variation of area we get

$$\begin{aligned}\partial_t A(M_t) &= \frac{n}{n-1} \int_{M_t} \overline{\mathrm{Ric}}(\nu, X(t) - u\nu) \, \mathrm{d}S - \frac{1}{n} \int_{M_t} u \sum_{i<j} \left(\kappa_i - \kappa_j\right)^2 \mathrm{d}S \\ &= \frac{n}{n-1} \int_{M_t} u\Big(\overline{\mathrm{Ric}}(\mathcal{N}^{\perp}, \mathcal{N}^{\perp}) - \overline{\mathrm{Ric}}(\nu, \nu)\Big) \, \mathrm{d}S - \frac{1}{n} \int_{M_t} u \sum_{i<j} \left(\kappa_i - \kappa_j\right)^2 \mathrm{d}S.\end{aligned}$$

But now again by assumption 5 we get that the term $\overline{\mathrm{Ric}}(\mathcal{N}^{\perp}, \mathcal{N}^{\perp}) - \overline{\mathrm{Ric}}(\nu, \nu)$ is always non-positive and the term $\left(\kappa_i - \kappa_j\right)$ is clearly always non-positive so both of these integrals are non-positive and thus $\partial_t A(M_t) \leq 0$. □

By this theorem we get the second condition of Theorem 2.1.

3.4 Evolution Equation for λ

The first result we will prove is arguably the most important result, as it will guarantee our hypersurface remains within a compact subset.

Proposition 3.4.1: The evolution equation for λ under the flow is

$$L\lambda = -2\Lambda n\varphi u^{\top} - 2u\langle \nabla\Lambda, X^{\perp}\rangle.$$

Proof: First we compute the time derivative of λ, since it is an ambient quantity this is easy by Proposition 1.6.1. We get

$$\begin{aligned}\partial_t\lambda &= (n\varphi - Hu)\overline{\nabla}_{\nu}\lambda = (n\varphi - Hu)\langle \nu, \overline{\nabla}\lambda\rangle = (n\varphi - Hu)2\Lambda\langle \nu, X^{\perp}\rangle \\ &= 2(n\varphi - Hu)\Lambda u^{\perp}.\end{aligned}$$

For the induced Laplacian we get

$$\begin{aligned}\Delta\lambda &= \nabla_i\nabla_i\lambda = \nabla_i(2\Lambda\langle X^{\perp}, e_i\rangle) = 2\Lambda(\nabla_i\langle X^{\perp}, e_i\rangle) + 2\langle X^{\perp}, e_i\rangle\nabla_i\Lambda \\ &= 2\Lambda\left(\langle \overline{\nabla}_i X^{\perp}, e_i\rangle + \langle X^{\perp}, \overline{\nabla}_i e_i\rangle\right) + 2\langle \nabla\Lambda, \pi(X^{\perp})\rangle.\end{aligned}$$

Now since the trace of a tensor is the same as the trace of its symmetrization so

$$\langle \overline{\nabla}_i X^{\perp}, e_i\rangle = \operatorname{tr}\left(\overline{\nabla}X^{\perp}\right) = \operatorname{tr}\left(\operatorname{Sym}\left(\overline{\nabla}X^{\perp}\right)\right) = \operatorname{tr}(\varphi g_{ij}) = n\varphi.$$

Next by Proposition 1.3.2 we know that $\overline{\nabla}_i e_i = -h_{ii}\nu = -H\nu$. Combining these we get that

$$\Delta\lambda = 2\Lambda(n\varphi - H\langle X^{\perp}, \nu\rangle) + 2\langle \nabla\Lambda, X^{\perp}\rangle = 2\Lambda(n\varphi - Hu^{\perp}) + 2\langle \nabla\Lambda, X^{\perp}\rangle.$$

Finally we compute

$$\begin{aligned}(\partial_t - u\Delta)\lambda &= 2\Lambda((n\varphi - Hu)u^{\perp} - (n\varphi - Hu^{\perp})u) - 2\langle \nabla\Lambda, X^{\perp}\rangle \\ &= 2\Lambda(-n\varphi u^{\top}) - 2\langle \nabla\Lambda, X^{\perp}\rangle = -2\Lambda n\varphi u^{\top} - 2\langle \nabla\Lambda, X^{\perp}\rangle\end{aligned}$$

□

Corollary 3.4.1.1: For all $t \in [0, T)$ and all $p \in M_t$ we have

$$\min_{p\in M_0}\lambda(p,0) \le \lambda(p,t) \le \max_{p\in M_0}\lambda(p,0)$$

Proof: At a maximal or minimal point of λ we have by Lagrange multipliers that $\overline{\nabla}$ is colinear with ν, so we must have that $X^{\perp}$ is colinear with ν and thus $\pi(X^{\perp}) = 0$. Also since $X^{\top}$ is orthogonal to $X^{\perp}$ we have that at a maximal or minimal point $X^\top$ is orthogonal to ν and thus $u^\top = 0$. Thus we get that at a maximal or minimal point $L\lambda = 0$ and so by Proposition 1.5.1 applied to λ and $-\lambda$ we get

$$\min_{x \in M_0} \lambda(x, 0) \leq \lambda(x, t) \leq \max_{x \in M_0} \lambda(x, 0)$$

□

Corollary 3.4.1.2: Given a hypersurface M, there is a compact region Γ such that M_t is contained in Γ for as long as it exists.

Proof: We use Corollary 3.4.1.1 along with Proposition 3.7.2. □

3.5 Evolution Equation for u

This next evolution is nearly as important, our parabolic operator has a u factor and so it ceases to be uniformly parabolic if we do not have a uniform lower bound on u.

Proposition 3.5.1: The evolution equation for u under the flow is

$$\begin{aligned} Lu = n\big(\Lambda\varphi^3 - \Xi(t)X^\top(\varphi)\big) + \Xi'(t)u^\top - 2\varphi Hu + |A|^2\ u^2 + 2nu\nu(\varphi) \\ + u^2\,\overline{\mathrm{Ric}}(\nu, \nu) + H\langle X, \nabla u\rangle \end{aligned}$$

Proof: This is quite the long calculation so we will split it into multiple steps, first for the time derivative

$$\begin{aligned} \partial_t u &= \partial_t \big\langle X^\perp + \Xi(t)X^\top, \nu\big\rangle = \big\langle \partial_t \big(X^\perp + \Xi(t)X^\top\big), \nu\big\rangle + \big\langle X^\perp + \Xi(t)X^\top, \partial_t \nu\big\rangle \\ &= (n\varphi - Hu)\big\langle \overline{\nabla}_\nu \big(X^\perp + \Xi(t)X^\top\big), \nu\big\rangle + \Xi'(t)\big\langle X^\top, \nu\big\rangle \\ &+ \big\langle X^\perp + \Xi(t)X^\top, -\nabla(n\varphi - Hu)\big\rangle. \end{aligned}$$

Using the fact that $X^\perp + \Xi(t)X^\top$ is conformal with factor φ we can simplify the first term and continue calculating

$$\begin{aligned} \partial_t u &= \varphi(n\varphi - Hu) + \Xi'(t)u^\perp + \big\langle X^\perp + \Xi(t)X^\top, -\nabla(n\varphi - Hu)\big\rangle \\ &= \varphi(n\varphi - Hu) + \Xi'(t)u^\perp - n\langle X, \nabla\varphi\rangle + H\langle X, \nabla u\rangle + u\langle X, \nabla H\rangle \qquad (3.12) \\ &= \varphi(n\varphi - Hu) + \Xi'(t)u^\perp - n\big\langle X, \overline{\nabla}\varphi\big\rangle + nu\nu(\varphi) + H\langle X, \nabla u\rangle + u\langle X, \nabla H\rangle. \end{aligned}$$

Now we switch to the Laplacian, it will be helpful to decompose $u = u^\perp + u^\top$. First we deal with $u^\perp$.

Claim 3.5.1: We have

$$\Delta u^{\perp} = -n\nu(\varphi) - u^{\perp}\,\overline{\mathrm{Ric}}(\nu,\nu) + \left\langle \nabla H, X^{\perp}\right\rangle + \varphi H - |A|^2\, u^{\perp}$$

We start with computing from definitions

$$\Delta u^{\perp} = \nabla_i \nabla_i \langle X^{\perp}, \nu\rangle = \nabla_i\Big(\big\langle \overline{\nabla}_i X^{\perp}, \nu\big\rangle + \big\langle X^{\perp}, \overline{\nabla}_i \nu\big\rangle\Big).$$

Now for the first term e_i is orthogonal to ν and thus it simplifies to $\langle \psi^{\perp}(e_i), \nu\rangle$, we continue computing,

$$\Delta u^{\perp} = \nabla_i\big(\langle \psi^{\perp}(e_i), \nu\rangle + \big\langle X^{\perp}, h_{ij} e_j\big\rangle\big). \tag{3.13}$$

Let us now deal with the first term, we use Corollary 2.3.4.2

$$\begin{aligned}
\nabla_i(\langle \psi^{\perp}(e_i), \nu\rangle) &= \nabla_i\left(\frac{2\,\mathrm{ASym}\big(\overline{\nabla}\varphi^{\flat} \otimes X^{\perp\flat}\big)(e_i,\nu)}{\varphi}\right) \\
&= (\frac{\overline{\nabla}_i 2\,\mathrm{ASym}\big(\overline{\nabla}\varphi^{\flat} \otimes X^{\perp\flat}\big)(e_i,\nu)}{\varphi} \\
&+ \frac{2\,\mathrm{ASym}\big(\overline{\nabla}\varphi^{\flat} \otimes X^{\perp\flat}\big)\big(\overline{\nabla}_i e_i,\nu\big)}{\varphi} \\
&+ \frac{2\,\mathrm{ASym}\big(\overline{\nabla}\varphi^{\flat} \otimes X^{\perp\flat}\big)\big(e_i,\overline{\nabla}_i\nu\big)}{\varphi} \\
&- \frac{2\,\mathrm{ASym}\big(\overline{\nabla}\varphi^{\flat} \otimes X^{\perp\flat}\big)(e_i,\nu)}{\varphi^2}\overline{\nabla}_i\varphi.
\end{aligned}$$

We notice that since $\overline{\nabla}_i e_i = -H\nu$ the second term will have two ν inputs into an anti-symmetrization, making it vanish. Similarly, since $\overline{\nabla}_i \nu = h_{ij} e_j$ the third term will have the inputs $\big(e_i, e_j\big)$ symmetrized by h_{ij} and thus will also vanish. We are thus left with

$$\nabla_i(\langle \psi^{\perp}(e_i), \nu\rangle) = \frac{\Big(\overline{\nabla}_i 2\,\mathrm{ASym}\big(\overline{\nabla}\varphi^{\flat} \otimes X^{\perp\flat}\big)\Big)(e_i,\nu)}{\varphi} - \langle \psi^{\perp}(e_i), \nu\rangle \frac{\overline{\nabla}_i \varphi}{\varphi}. \tag{3.14}$$

Now we can compute the covariant derivative of the anti-symmetrization

$$2\,\mathrm{ASym}\Big(\overline{\nabla}\varphi^{\flat} \otimes \big(\varphi e_i + \psi^{\perp}(e_i)\big)^{\flat}\Big) + 2\,\mathrm{ASym}\Big(\overline{\nabla}_i \overline{\nabla}\varphi^{\flat} \otimes X^{\perp\flat}\Big),$$

now when we plug this back into (3.14) we get

$$\frac{\left(\overline{\nabla}_i \varphi\right)\langle \psi^\perp(e_i), \nu\rangle}{\varphi} - \langle e_i, e_i\rangle \nu(\varphi) + \overline{\text{Hess}}_\varphi(e_i, e_i)\frac{u^\perp}{\varphi} - \overline{\text{Hess}}_\varphi(e_i, \nu)\frac{\langle X^\perp, e_i\rangle}{\varphi}$$
$$-\langle \psi^\perp(e_i), \nu\rangle \frac{\overline{\nabla}_i \varphi}{\varphi},$$

which simplifies into

$$-n\nu(\varphi) + \overline{\text{Hess}}_\varphi(e_i, e_i)\frac{u^\perp}{\varphi} - \frac{\overline{\text{Hess}}_\varphi(\pi(X^\perp), \nu)}{\varphi}.$$

Now this Hessian term is almost the ambient Laplacian of φ, so we can rewrite this as

$$-n\nu(\varphi) + \overline{\Delta}\varphi \frac{u^\perp}{\varphi} - u^\perp \frac{\overline{\text{Hess}}_\varphi(\nu, \nu)}{\varphi} - \frac{\overline{\text{Hess}}_\varphi(\pi(X^\perp), \nu)}{\varphi},$$

but now since $X^\perp = \pi(X^\perp) + u^\perp \nu$ we further simplify this into

$$-n\nu(\varphi) + \overline{\Delta}\varphi \frac{u^\perp}{\varphi} - \frac{\overline{\text{Hess}}_\varphi(X^\perp, \nu)}{\varphi}.$$

and then we use (2.7) to get

$$\begin{aligned} &-n\nu(\varphi) + \overline{\Delta}\varphi \frac{u^\perp}{\varphi} - \langle X^\perp, \nu\rangle \frac{\overline{\text{Hess}}_\varphi(\mathcal{N}^\perp, \mathcal{N}^\perp)}{\varphi} \\ &= -n\nu(\varphi) + \frac{u^\perp}{\varphi}\left(\overline{\Delta}\varphi - \left(\overline{\text{Hess}}_\varphi(\mathcal{N}^\perp, \mathcal{N}^\perp)\right)\right). \end{aligned}$$

This form allows us to use (2.6) to get

$$-n\nu(\varphi) - \overline{\text{Ric}}(X^\perp, \nu). \tag{3.15}$$

Next for the second term of (3.13) we get

$$\begin{aligned} \overline{\nabla}_i\left(h_{ij}\langle X^\perp, e_j\rangle\right) &= \left(\overline{\nabla}_i h_{ij}\right)\langle X^\perp, e_j\rangle + h_{ij}\langle \overline{\nabla}_i X^\perp, e_j\rangle + h_{ij}\langle X^\perp, \overline{\nabla}_i e_j\rangle \\ &= \left(\overline{\nabla}_i h_{ij}\right)\langle X^\perp, e_j\rangle + h_{ij}\left(\varphi\langle e_i, e_j\rangle + \langle \psi^\perp(e_i), e_j\rangle\right) \\ &\quad -h_{ij}h_{ij}\langle X^\perp, \nu\rangle. \end{aligned}$$

Since h_{ij} is symmetric the third term here vanishes and so we are left with

$$\overline{\nabla}_i\left(h_{ij}\langle X^\perp, e_j\rangle\right) = \left(\overline{\nabla}_i h_{ij}\right)\langle X^\perp, e_j\rangle + \varphi H - |A|^2\, u^\perp. \tag{3.16}$$

Now plugging (3.15) and (3.16) into (3.13) gives us

$$\Delta u^\perp = -n\nu(\varphi) - \overline{\text{Ric}}(X^\perp, \nu) + \left(\overline{\nabla}_i h_{ij}\right)\langle X^\perp, e_j\rangle + \varphi H - |A|^2\, u^\perp \tag{3.17}$$

now we can use Lemma 1.3.3 to get

$$\left(\overline{\nabla}_i h_{ij}\right)\langle X^\perp, e_j\rangle = \left(\overline{\mathrm{Rm}}_{jii\nu} + \nabla_j h_{ii}\right)\langle X^\perp, e_j\rangle = \left(\overline{\mathrm{Ric}}(e_j,\nu) + \nabla_j H\right)\langle X^\perp, e_j\rangle$$
$$= \overline{\mathrm{Ric}}(\pi(X^\perp),\nu) + \langle \nabla H, X^\perp\rangle$$
$$= \overline{\mathrm{Ric}}(X^\perp,\nu) - u^\perp\,\overline{\mathrm{Ric}}(\nu,\nu) + \langle \nabla H, X^\perp\rangle$$

which we can plug back into (3.17) to get

$$\Delta u^\perp = -n\nu(\varphi) - u^\perp\,\overline{\mathrm{Ric}}(\nu,\nu) + \langle \nabla H, X^\perp\rangle + \varphi H - |A|^2\, u^\perp$$

Now we deal with $u^\top$

Claim 3.5.2: We have

$$\Delta u^\top = -u^\top\,\overline{\mathrm{Ric}}(\nu,\nu) + \langle \nabla H, X^\top\rangle - |A|^2\, u^\top$$

Again we compute from definitions, we will use $X^\top$ instead of $\Xi(t)X^\top$ since that does not change any of the calculations and both are Killing vector fields,

$$\Delta u^\top = \nabla_i\nabla_i\langle X^\top,\nu\rangle = \nabla_i\left(\langle \overline{\nabla}_i X^\top,\nu\rangle + \langle X^\top, \overline{\nabla}_i\nu\rangle\right).$$

Now for the first term e_i is orthogonal to ν and thus it simplifies to $\langle \psi^\top(e_i),\nu\rangle$, we continue computing,

$$\Delta u^\top = \nabla_i\left(\langle \psi^\top(e_i),\nu\rangle + \langle X^\top, h_{ij}e_j\rangle\right). \tag{3.18}$$

Let us now deal with the first term, we use Corollary 2.3.4.2

$$\nabla_i(\langle \psi^\top(e_i),\nu\rangle) = \nabla_i\left(\frac{2\,\mathrm{ASym}\left(\overline{\nabla}\|X^\top\|^\flat \otimes X^{\top\flat}\right)(e_i,\nu)}{\|X^\top\|}\right)$$
$$= \frac{\overline{\nabla}_i 2\,\mathrm{ASym}\left(\overline{\nabla}\|X^\top\|^\flat \otimes X^{\top\flat}\right)(e_i,\nu)}{\|X^\top\|}$$
$$+ \frac{2\,\mathrm{ASym}\left(\overline{\nabla}\|X^\top\|^\flat \otimes X^{\top\flat}\right)\left(\overline{\nabla}_i e_i,\nu\right)}{\|X^\top\|}$$
$$+ \frac{2\,\mathrm{ASym}\left(\overline{\nabla}\|X^\top\|^\flat \otimes X^{\top\flat}\right)\left(e_i,\overline{\nabla}_i\nu\right)}{\|X^\top\|}$$
$$- \frac{2\,\mathrm{ASym}\left(\overline{\nabla}\|X^\top\|^\flat \otimes X^{\top\flat}\right)(e_i,\nu)}{\|X^\top\|^2}\overline{\nabla}_i\|X^\top\|.$$

We notice that since $\overline{\nabla}_i e_i = -H\nu$ the second term will have two ν inputs into an anti-symmetrization, making it vanish. Similarly, since $\overline{\nabla}_i\nu = h_{ij}e_j$ the third term will have the inputs (e_i, e_j) symmetrized by h_{ij} and thus will also vanish. We are thus left with

$$\nabla_i(\langle\psi^\top(e_i),\nu\rangle)$$
$$= 2\frac{\left(\overline{\nabla}_i \operatorname{ASym}\left(\overline{\nabla}\|X^\top\|^\flat \otimes X^{\top\flat}\right)\right)(e_i,\nu)}{\|X^\top\|} - \langle\psi^\top(e_i),\nu\rangle\frac{\overline{\nabla}_i\|X^\top\|}{\|X^\top\|}. \tag{3.19}$$

Now we can compute the covariant derivative of the anti-symmetrization

$$2\operatorname{ASym}\left(\overline{\nabla}\|X^\top\|^\flat \otimes \left(\psi^\top(e_i)\right)^\flat\right) + 2\operatorname{ASym}\left(\overline{\nabla}_i\overline{\nabla}\|X^\top\|^\flat \otimes X^{\top\flat}\right),$$

now when we plug this back into (3.19) we get

$$\frac{\left(\overline{\nabla}_i\|X^\top\|\right)\langle\psi^\top(e_i),\nu\rangle}{\|X^\top\|} + \overline{\operatorname{Hess}}_{\|X^\top\|}(e_i,e_i)\frac{u^\top}{\|X^\top\|} - \overline{\operatorname{Hess}}_{\|X^\top\|}(e_i,\nu)\frac{\langle X^\top,e_i\rangle}{\|X^\top\|}$$
$$-\langle\psi^\top(e_i),\nu\rangle\frac{\overline{\nabla}_i(\|X^\top\|)}{\|X^\top\|},$$

which simplifies into

$$\overline{\operatorname{Hess}}_{\|X^\top\|}(e_i,e_i)\frac{u^\top}{\|X^\top\|} - \frac{\overline{\operatorname{Hess}}_{\|X^\top\|}(\pi(X^\top),\nu)}{\|X^\top\|}.$$

Now this Hessian term is almost the ambient Laplacian of $\|X^\top\|$, so we can rewrite this as

$$\overline{\Delta}\|X^\top\|\frac{u^\top}{\|X^\top\|} - u^\top\frac{\overline{\operatorname{Hess}}_{\|X^\top\|}(\nu,\nu)}{\|X^\top\|} - \frac{\overline{\operatorname{Hess}}_{\|X^\top\|}(\pi(X^\top),\nu)}{\|X^\top\|},$$

but now since $X^\top = \pi(X^\top) + u^\top\nu$ we further simplify this into

$$\overline{\Delta}\|X^\top\|\frac{u^\top}{\|X^\top\|} - \frac{\overline{\operatorname{Hess}}_{\|X^\top\|}(X^\top,\nu)}{\|X^\top\|}.$$

and then we use (2.7) to get

$$\overline{\Delta}\|X^\top\|\frac{u^\top}{\|X^\top\|} - \langle X^\top,\nu\rangle\frac{\overline{\operatorname{Hess}}_{\|X^\top\|}(\mathcal{N}^\top,\mathcal{N}^\top)}{\|X^\top\|}$$
$$= -\frac{u^\top}{\|X^\top\|}\left(\overline{\Delta}\|X^\top\| - \overline{\operatorname{Hess}}_{\|X^\top\|}(\mathcal{N}^\top,\mathcal{N}^\top)\right).$$

Now we use (2.6) to get

$$-\overline{\operatorname{Ric}}(X^\top,\nu). \tag{3.20}$$

Next for the second term of (3.18) we get

$$\overline{\nabla}_i\left(h_{ij}\langle X^\top,e_j\rangle\right) = \left(\overline{\nabla}_i h_{ij}\right)\langle X^\top,e_j\rangle + h_{ij}\langle\overline{\nabla}_i X^\top,e_j\rangle + h_{ij}\langle X^\top,\overline{\nabla}_i e_j\rangle$$
$$= \left(\overline{\nabla}_i h_{ij}\right)\langle X^\top,e_j\rangle + h_{ij}\left(\langle\psi^\top(e_i),e_j\rangle\right) - h_{ij}h_{ij}\langle X^\top,\nu\rangle.$$

Since h_{ij} is symmetric the third term here vanishes and so we are left with

$$\overline{\nabla}_i\big(h_{ij}\langle X^\top, e_j\rangle\big) = \Big(\overline{\nabla}_i h_{ij}\Big)\langle X^\top, e_j\rangle - |A|^2\ u^\top. \tag{3.21}$$

Now plugging (3.20) and (3.21) into (3.18) gives us

$$\Delta u^\top = -\overline{\mathrm{Ric}}(X^\top, \nu) + \Big(\overline{\nabla}_i h_{ij}\Big)\langle X^\top, e_j\rangle - |A|^2\ u^\top \tag{3.22}$$

now we can use Lemma 1.3.3 to get

$$\begin{aligned}\Big(\overline{\nabla}_i h_{ij}\Big)\langle X^\top, e_j\rangle &= \Big(\overline{\mathrm{Rm}}_{jii\nu} + \nabla_j h_{ii}\Big)\langle X^\top, e_j\rangle = \Big(\overline{\mathrm{Ric}}(e_j, \nu) + \nabla_j H\Big)\langle X^\top, e_j\rangle \\ &= \overline{\mathrm{Ric}}(\pi(X^\top), \nu) + \langle \nabla H, X^\top\rangle \\ &= \overline{\mathrm{Ric}}(X^\top, \nu) - u^\top\,\overline{\mathrm{Ric}}(\nu, \nu) + \langle \nabla H, X^\top\rangle\end{aligned}$$

which we can plug back into (3.22) to get

$$\Delta u^\top = -u^\top\,\overline{\mathrm{Ric}}(\nu, \nu) + \langle \nabla H, X^\top\rangle - |A|^2\ u^\top$$

which proves the claim.

Combined with the previous claim we get that

$$\Delta u = -n\nu(\varphi) - u\,\overline{\mathrm{Ric}}(\nu, \nu) + \langle \nabla H, X\rangle + \varphi H - |A|^2\ u$$

and then combining with (3.12) we get

$$\begin{aligned}Lu &= n\varphi^2 + \Xi'(t)u^\top - nX(\varphi) + 2nu\nu(\varphi) + H\langle X, \nabla u\rangle + u^2\,\overline{\mathrm{Ric}}(\nu, \nu) \\ &\quad -2\varphi u H + |A|^2\ u^2 \\ &= n(\varphi^2 - X^\perp(\varphi) - \Xi(t)X^\top(\varphi)) + \Xi'(t)u^\top + 2nu\nu(\varphi) + H\langle X, \nabla u\rangle + u^2\,\overline{\mathrm{Ric}}(\nu, \nu) \\ &\quad -2\varphi u H + |A|^2\ u^2 \\ &= n\Lambda\varphi^3 - n\Xi(t)X^\top(\varphi) + \Xi'(t)u^\top + 2nu\nu(\varphi) + H\langle X, \nabla u\rangle + u^2\,\overline{\mathrm{Ric}}(\nu, \nu) \\ &\quad -2\varphi u H + |A|^2\ u^2.\end{aligned}$$

This finishes the proof. □

Now we can start to analyse this evolution equation to get results about u.

Corollary 3.5.1.1: There is a constant $\varepsilon > 0$ such that for any $t \in [0, T)$

$$\min_{p\in M_t} u(p, t) \geq \frac{\varepsilon}{1 + \max_{p\in M_t}|H(p, t)|}$$

Proof: At a minimum point of u we have that ∇u vanishes and so we get

$$Lu = n(\Lambda\varphi^3 - \Xi(t)X^\top(\varphi)) + \Xi'(t)u^\top + 2nu\nu(\varphi) + u^2\Big(\overline{\mathrm{Ric}}(\nu, \nu)\Big) - 2\varphi u H + |A|^2\ u^2,$$

now recall that all ambient objects are uniformly bounded for all time, so there exists a constant M such that

$$Lu \geq n\big(\Lambda\varphi^3 - \Xi(t)X^\top(\varphi)\big) + \Xi'(t)u^\top - uM - 2\varphi uH + |A|^2\ u^2.$$

By our assumptions both $\Lambda\varphi^3$ and $\Lambda\varphi^3 - \Xi(t)X^\top(\varphi)$ are positive, we thus have that any convex combinations of them is positive so since these are ambient quantities they must be uniformly bounded and so

$$n\big(\Lambda\varphi^3 - \Xi(t)X^\top(\varphi)\big) \geq \varepsilon_1 > 0$$

for some ε_1. We thus have

$$Lu \geq \varepsilon_1 + \Xi'(t)u^\top - uM - 2\varphi uH + |A|^2\ u^2.$$

Now we are free to pick T_0 such that $\frac{\|X^\top\|}{T_0} \leq \frac{\varepsilon_1}{2}$. Then we can use the Newton-Maclaurin inequality to get

$$Lu \geq \varepsilon_2 + \Xi'(t)u^\top - uM - 2\varphi uH + H^2\frac{u^2}{n}.$$

Now assume that $u < \frac{\varepsilon}{1+\max|H(p,t)|}$, then

$$Lu \geq \varepsilon_2 + \Xi'(t)u^\top - \varepsilon M - 2\varphi\varepsilon$$

so by setting $\varepsilon < \frac{\varepsilon_2}{2(M+2\varphi)}$ then

$$Lu \geq \varepsilon_3 + \Xi'(t)u^\top$$

now we can pick T_0 so that $|\Xi'(t)| < \frac{1}{\|X^\top\|}$ and we get

$$Lu \geq \varepsilon_4.$$

Thus by using Proposition 1.5.1 we get that

$$u \leq \frac{\varepsilon}{1 + \max_{p\in M_t} H(p,t)} \quad \text{implies} \quad \min_{p\in M_t} u(p,t) \geq \min_{p\in M_0} u(p,0)$$

and so by choosing ε appropriately we get the desired result. □

Now that we can bound u using H, we just need to show that H grows sufficiently slowly, to guarantee the flow exists until $t = T_0$.

3.6 Evolution Equation for H

Proposition 3.6.1: The evolution equation for H is

$$\begin{aligned} LH &= 2\langle \nabla H, \nabla u\rangle + H\langle X, \nabla H\rangle - \varphi(H^2 - n\ |A|^2) \\ &+n\Big(\overline{\text{Hess}}_\varphi(\nu,\nu) - \overline{\text{Hess}}_\varphi(\mathcal{N}^\perp,\mathcal{N}^\perp)\Big) + n\varphi\Big(\overline{\text{Ric}}(\mathcal{N}^\perp,\mathcal{N}^\perp) - \overline{\text{Ric}}(\nu,\nu)\Big) \end{aligned}$$

Proof: We use Corollary 1.6.5.1 to get

$$\partial_t H = -\Delta(n\varphi - Hu) - (n\varphi - Hu)\Big(|A|^2 + \overline{\text{Ric}}(\nu,\nu)\Big)$$

then simplifying this we get

$$LH = -n\Delta\varphi + 2\langle \nabla H, \nabla u\rangle + H\Delta u - (n\varphi - Hu)\Big(|A|^2 + \overline{\text{Ric}}(\nu,\nu)\Big)$$

then using the results of Proposition 3.5.1 we get

$$\begin{aligned} LH &= -n\Delta\varphi + 2\langle \nabla H, \nabla u\rangle - Hn\nu(\varphi) - Hu\,\overline{\text{Ric}}(\nu,\nu) + H\langle \nabla H, X\rangle + \varphi H^2 \\ &- H|A|^2\ u - (n\varphi - Hu)\Big(|A|^2 + \overline{\text{Ric}}(\nu,\nu)\Big) \\ &= -n\Delta\varphi + 2\langle \nabla H, \nabla u\rangle - Hn\nu(\varphi) + H\langle \nabla H, X\rangle + \varphi H^2 - n\varphi\Big(|A|^2 + \overline{\text{Ric}}(\nu,\nu)\Big) \\ &= -n\Delta\varphi + 2\langle \nabla H, \nabla u\rangle - Hn\nu(\varphi) + H\langle \nabla H, X\rangle + \varphi(H^2 - n|A|^2) - n\varphi\,\overline{\text{Ric}}(\nu,\nu). \end{aligned}$$

Now we use Proposition 1.3.2 to get

$$\begin{aligned} LH &= -n\overline{\Delta}\varphi + n\,\overline{\text{Hess}}_\varphi(\nu,\nu) + 2\langle \nabla H, \nabla u\rangle + H\langle \nabla H, X\rangle + \varphi(H^2 - n|A|^2) \\ &- n\varphi\,\overline{\text{Ric}}(\nu,\nu) \end{aligned}$$

but now we use (2.5) to get

$$\begin{aligned} LH &= n\Big(\varphi\,\overline{\text{Ric}}(\mathcal{N}^\perp,\mathcal{N}^\perp) - \overline{\text{Hess}}_\varphi(\mathcal{N}^\perp,\mathcal{N}^\perp)\Big) + n\,\overline{\text{Hess}}_\varphi(\nu,\nu) + 2\langle \nabla H, \nabla u\rangle \\ &+H\langle \nabla H, X\rangle + \varphi(H^2 - n|A|^2) - n\varphi\,\overline{\text{Ric}}(\nu,\nu) \\ &= 2\langle \nabla H, \nabla u\rangle + H\langle \nabla H, X\rangle + \varphi(H^2 - n|A|^2) + n\varphi\Big(\overline{\text{Ric}}(\mathcal{N}^\perp,\mathcal{N}^\perp) - \overline{\text{Ric}}(\nu,\nu)\Big) \\ &+ n\Big(\overline{\text{Hess}}_\varphi(\nu,\nu) - \overline{\text{Hess}}_\varphi(\mathcal{N}^\perp,\mathcal{N}^\perp)\Big) \end{aligned}$$

□

Corollary 3.6.1.1: There are constants $a, b > 0$ such that for any $t \in [0, T)$

$$\max_{p\in M_t} H(p,t) \leq a + bt$$

Proof: At a maximum point of H we have $\nabla H = 0$, hence the evolution equation simplifies to

$$\begin{aligned} LH &= \varphi(H^2 - n|A|^2) + n\varphi\Big(\overline{\mathrm{Ric}}(\mathcal{N}^\perp, \mathcal{N}^\perp) - \overline{\mathrm{Ric}}(\nu, \nu)\Big) \\ &\quad + n\Big(\overline{\mathrm{Hess}}_\varphi(\nu, \nu) - \overline{\mathrm{Hess}}_\varphi(\mathcal{N}^\perp, \mathcal{N}^\perp)\Big), \end{aligned}$$

but then again using the Newton-Maclaurin inequality we get

$$LH \le n\varphi\Big(\overline{\mathrm{Ric}}(\mathcal{N}^\perp, \mathcal{N}^\perp) - \overline{\mathrm{Ric}}(\nu, \nu)\Big) + n\Big(\overline{\mathrm{Hess}}_\varphi(\nu, \nu) - \overline{\mathrm{Hess}}_\varphi(\mathcal{N}^\perp, \mathcal{N}^\perp)\Big).$$

Now on the right hand side these are all ambient objects and thus are uniformly bounded by some constant M, hence we have

$$LH \le M.$$

Hence by Proposition 1.5.1 we get that $H \le \max_{p \in M_0} H(p, 0) + Mt$, proving the desired result. □

With this linear bound we get an inverse linear lower bound on u.

Corollary 3.6.1.2: There exists $\varepsilon > 0$ such that

$$u(p, t) \ge \frac{\varepsilon}{1 + T_0}$$

for all $t \in [0, T)$ and $p \in M_t$.

Proof: Combining Corollary 3.6.1.1 with Corollary 3.5.1.1 we immediately get the desired result. □

3.7 Existence and Convergence

We now have everything we need to prove the flow exists until $t = T_0$.

Proposition 3.7.1: If a surface M which is star-shaped with respect to $X^\perp + X^\top$ admits a flow M_t, then M_t remains star-shaped with respect to $X(t)$ for all $t \in [0, T_0)$, furthermore if the flow exists at $t = T_0$ then there the surface is star-shaped with respect to $X^\perp$.

Proof: We have showed that u is uniformly bounded for $t \in [0, T_0)$, hence it is also uniformly bounded in the limit $t = T_0$. □

We now shift out focus to rewriting this flow as a flow of functions instead of hypersurfaces, which will allow us to apply the results of Section 1.5 to it. We want to write our hypersurface as a graph over an integral hypersurface of S_λ where

$$S_\lambda = \{p \in N : \lambda(p) = \lambda\},$$

since this is not a warped product space we need to be careful with this construction. We will fix a starting hypersurface M, and set

$$\lambda_0 = \min_{p \in M} \lambda(p) \quad \text{and} \quad \lambda_1 = \max_{p \in M} \lambda(p),$$

and we want to construct nice coordinates on

$$D := \{p \in N : \lambda_0 \leq \lambda(p) \leq \lambda_1\} \tag{3.23}$$

We will start with proving that D is compact, allowing us to lower bound important quantities uniformly.

Proposition 3.7.2: For any $\lambda_1 > \lambda_0 > 0$ in the image of λ, the subset D defined by (3.23) is compact.

Proof: First we will show that D is a fiber bundle over $[\lambda_0, \lambda_1]$, to see this fix $\lambda \in [\lambda_0, \lambda_1]$, then set $S_\lambda = \{p \in N : \lambda(p) = \lambda\}$, it is an integrable hypersurface of $X^\perp$ and is compact. Then consider the flow of $X^\perp$, since S_λ is compact we can pick $\varepsilon > 0$ such that the flow of $X^\perp$ exists for $t \in [-\varepsilon, \varepsilon]$ for all points $p \in S_\lambda$. Now the image of S_λ under this flow is another integrable hypersurface, this is because $X^\perp$ is a conformal vector field and so under its flow, itself and orthogonality are preserved. Hence the flow of $X^\perp$ fixes its orthogonal distribution $\mathcal{D}(X^\perp)$, and thus also its foliation. Hence for some $\lambda' < \lambda < \lambda''$ we can reparametrize the flow of $X^\perp$ to get the homeomorphism

$$\mathcal{F} : S_\lambda \times [\lambda', \lambda''] \to \lambda^{-1}([\lambda', \lambda'']),$$

and thus D is a fiber bundle.

We now prove a lemma regarding compactness of fiber bundles.

Lemma 3.7.3: If (E, B, F) is a fiber bundle with E, B, F metric spaces and B, F compact, then E is also compact.

Proof: Let x_n be a sequence of points in E, since $\pi(x_n)$ is a sequence of points in B it has a convergence subsequence $\pi\left(x_{n_k}\right)$. Assume that it converges to x, then let U be a neighborhood of x which trivializes the bundle E, in the sense that $\pi^{-1}(U) = U \times F$. Consider a precompact subset $V \subseteq U$, then since $\pi\left(x_{n_k}\right) \to x$ we know that $\pi\left(x_{n_k}\right)$ is eventually always contained in V, hence x_{n_k} is eventually always contained in $\pi^{-1}\left(\overline{V}\right) = \overline{V} \times F$. Now there is a subsequence of x_{n_k} which is entirely contained in $\overline{V} \times F$ which is a product of compact sets hence compact, so x_{n_k} has a convergent subsequence in E, which is also a convergent subsequence of E. Hence E is compact. □

Now in our situation we have D as a fiber bundle over $[\lambda_0, \lambda_1]$ with fiber S_λ which are both compact, hence D is compact which proves the proposition. □

We can now use these lower bounds to construct a nice coordinate system for D, we will do this by flowing the surface S_{λ_0} to cover the entirety of D. For brevity we will shorten S_{λ_0} to S.

Proposition 3.7.4: For any $\lambda \in [\lambda_0, \lambda_1]$, and any point $p \in S_\lambda$ there exists an integral curve of $X(t)$ going through p which intersects S at exactly one point.

Proof: First we prove existence, consider the flow of $-X(t)$ acting on p, lets call this flow $\mathcal{F}$. Notice that

$$\partial_t \lambda(\mathcal{F}(p,t)) = -2\Lambda < 0$$

and so this function is decreasing. Then at some point $\lambda(\mathcal{F}(p,t)) = \lambda_0$ since otherwise $\mathcal{F}(p,t)$ remains forever in D where $-2\Lambda < -\varepsilon < 0$ for some positive ε which is a contradiction.

To show uniqueness assume that the flow $\mathcal{F}(p,t)$ intersects S at more than one point. Then we have $\lambda(\mathcal{F}(p,t_1)) = \lambda(\mathcal{F}(p,t_2))$ and so by Rolle's theorem we have that $\partial_t \lambda(\mathcal{F}(p,t_3)) = 0$ which contradicts the fact that $-2\Lambda < 0$. □

Using the unique intersection point we found above as a 'projection map' onto S we get a diffeomorphism $F_t : D \to S \times [\lambda_0, \lambda_1]$. Note that this diffeomorphism depends on t because $X(t)$ depends on t.

We can now convert our hypersurface flow into a flow of functions.

Proposition 3.7.5: A hypersurface M contained in D is star-shaped with respect to $X(t)$ if and only if it can be identified using F_t with a graph of smooth function $f : S \to [\lambda_0, \lambda_1]$.

Proof: First assume that M can be identified with the graph of f, then set $F : S \to S \times [\lambda_0, \lambda_1]$ be the embedding of the graph

$$F : y \mapsto (y, f(y)).$$

One can easily compute that for $v \in T_p S$

$$F_* v = \hat{v} + v(f)\partial_\lambda$$

where $\hat{v}$ is the extension of v over the integral curve containing p through the flow of $X(t)$. Then let v be the unit vector which maximizes the length of the orthogonal projection of $X(t)$ onto $F_* v$, this length is

$$\frac{\langle v + v(f)\partial_\lambda, X(t)\rangle}{\|v + v(f)\partial_\lambda\|} = \frac{\langle v + v(f)\partial_\lambda, X(t)\rangle}{\|v + v(f)\partial_\lambda\|}$$

now orthogonally decompose v as $v = a\partial_\lambda + bz$ where z is a unit vector orthogonal to ∂_λ and by extension also $X(t)$. We then have

$$\begin{aligned}\frac{\langle a\partial_\lambda + bz + v(f)\partial_\lambda, X(t)\rangle}{\|a\partial_\lambda + bz + v(f)\partial_\lambda\|} &= \frac{\langle bz + (a + v(f))\partial_\lambda, X(t)\rangle}{\|bz + (a+v(f))\partial_\lambda\|} \\ &= \frac{(a+v(f))\langle\partial_\lambda, X(t)\rangle}{\sqrt{b^2 + (a+v(f))^2\|\partial_\lambda\|^2}} \\ &= \frac{(a+v(f))}{\sqrt{\frac{b^2}{\|\partial_\lambda\|^2} + (a+v(f))^2}}\|X(t)\|.\end{aligned}$$

Now since we are on a compact surface we have uniform bounds on $v(f)$ and b is non-zero (since v is not colinear with $X(t)$) we have

$$\frac{(a+v(f))}{\sqrt{\frac{b^2}{\|\partial_\lambda\|^2} + (a+v(f))^2}} < 1 - \varepsilon$$

for some $\varepsilon > 0$. Now from this we get that the projection π of $X(t)$ onto M satisfies

$$\|\pi(X(t))\|^2 < \|X(t)\|^2\ (1 - \varepsilon')$$

and so we have

$$u^2 = \|X(t)\|^2 - \|\pi(X(t))\|^2 > \|X(t)\|^2\varepsilon'$$

and so up to a change of orientation our surface is star-shaped.

On the other hand assume that a surface is star-shaped, then first we prove that it intersects every integral curve of $X(t)$ at most once, to see this note that if it were to intersect it twice, then we would have $\langle\nu, X(t_1)\rangle$ be positive and $\langle\nu, X(t_2)\rangle$ be negative or vice versa, where t_1, t_2 are the intersection times. But this directly contradicts the fact that it is star-shaped. We thus have an injective map $\pi : M \to S$ since the all the integral curves intersect S at exactly one point.

It will be enough to show that π is also a diffeomorphism, as then its inverse will exactly be the embedding of the graph of a function. It in fact suffices just to show it is a local diffeomorphism, since then it is a bijection onto its image and thus a diffeomorphism onto its image. Now we check that this is indeed the case, fix a point $p \in M$ and take an orthonormal frame $e_1, ..., e_n$ of S centered at $f(p)$ and extend it to n vector fields $\hat{e}_1, ..., \hat{e}_n, \hat{\nu}$ of D through the flow of $X(t)$. Now at the point p consider the projection $P : T_pN \to \mathrm{Span}(\hat{e}_1, ..., \hat{e}_n)$ induced by the frame $\hat{e}_1, ..., \hat{e}_n, X(t)$ (i.e. the projection with kernel $\mathrm{Span}(X(t))$). If we restrict this map to T_pN the projection kernel of this linear map is zero since anything in the kernel must be colinear with $X(t)$ and thus cannot be in T_pM since that would mean $\langle X(t), \nu\rangle = 0$ which contradicts star-shapedness.

Now we see that the projection is precisely the differential of π and it is a linear isomorphism and thus π is a local diffeomorphism. □

In fact, by the first argument in the proof above we get that lower bounds on u are equivalent to upper bounds on $\|\tilde{\nabla} f\|$ where $\tilde{\nabla}$ is the connection on S>

Corollary 3.7.5.1: There are functions $M(\varepsilon)$ and $\varepsilon(M)$ such that

$$\sup_S \|\tilde{\nabla} f\| < M \Rightarrow \inf_S u > \varepsilon(M) \quad \text{and}$$

$$\inf_S u > \varepsilon \Rightarrow \sup_S \|\tilde{\nabla} f\| < M(\varepsilon)$$

We will now construct the flow in the following way, first we use Proposition 3.7.5 to identify our initial surface M_0 with a graph of the function λ over S. Then M_t solving (3.11) is equivalent to it being the graph of a function f solving

$$\begin{cases} Lf = -2\Lambda n\varphi u^{\top} - 2u\langle \nabla\Lambda, X^{\perp}\rangle, \\ f(x,0) = \lambda(x), \end{cases} \tag{3.24}$$

up to an appropriate diffeomorphism that handles points changing which integral curve they are on (this is alright as all normal flows are diffeomorphism invariant). This identification is through $G : (x,t,\lambda) \mapsto F_t(x,\lambda)$.

Next we solve this (3.23) purely in function space. We then apply appropriate diffeomorphisms to convert this solution to a solution of our normal flow.

Proposition 3.7.6 (Short Time Existence): For any star-shaped hypersurface M_0, the normal flow with velocity $n\varphi - Hu$ with initial condition M_0 exists for some time interval $[0,T)$.

Proof: By the processed described above it is enough to show that (3.23) has a solution for some time interval $[0,T)$.

To see this we need to rewrite all geometric objects of the PDE in terms of w and its derivatives. We will work with in normal coordinates $x^1, ..., x^n$ on S, in which the induced metric is given by

$$g_{ij} = \langle F_* e_i, F_* e_j\rangle = \langle \hat{e}_i, \hat{e}_j\rangle + \partial_j f\langle \hat{e}_i, \partial_\lambda\rangle + \partial_i f\langle \hat{e}_j, \partial_\lambda\rangle + (\partial_i f)(\partial_j f)\langle \partial_\lambda, \partial_\lambda\rangle.$$

Notice that all 4 inner products in the expression are smooth functions of (x,λ) and so the entries are smooth functions of x, f, Df. We then immediately get that

$$g^{ij}(x,f,Df), \quad \det(g)(x,f,Df)$$

are both also smooth functions of their inputs. Now by the Gram-Schmidt method we get that the normal vector to the graph can be given by

$$\nu = X(t) - \sum_i \frac{\langle \hat{e}_i + \partial_i f\partial_\lambda, X(t)\rangle}{\langle \hat{e}_i + \partial_i f\partial_\lambda, \hat{e}_i + \partial_i f\partial_\lambda\rangle}(\hat{e}_i + \partial_i f\partial_\lambda)$$

appropriately normalized, which is once again a smooth function of x, f, Df, hence $u^\top$ and u are also smooth functions of x, f, Df. Finally all ambient objects like $X^\perp, \Lambda, \varphi$ are all smooth functions of x, λ and hence x, f. We can thus rewrite (3.24) as

$$\partial_t f = -u(x, f, Df)\Delta f + B(x, f, Df). \tag{3.25}$$

At first glance this seems like a standard parabolic PDE but in fact our Laplacian is with respect to the induced metric which depends on the gradient of f in a complicated manner. To counteract this we will use a technique called DeTurcks trick, which will allow us to exploit the diffeomorphism invariance of our geometric flow to sidestep this complexity.

We recall that in coordinates the Laplacian takes the form

$$\Delta f = g^{ij}\left(\partial_i \partial_j f - \Gamma^k_{ij}\partial_k f\right)$$

where both the inverse metric and the Christoffel Symbols depend on the induced metric. Now let us fix some other metric, for example the metric on M_0, we will call this metric $\tilde{g}$ and its Christoffel symbols $\tilde{\Gamma}$. We recall that object

$$\Gamma^k_{ij} - \tilde{\Gamma}^k_{ij}$$

is actually coordinate independent and is a tensor. Hence we can define the time dependent vector field

$$V^k(t) = g^{ij}(t)\left(\Gamma^k_{ij}(t) - \tilde{\Gamma}^k_{ij}\right).$$

We will now apply a time-dependent diffeomorphism Φ_t to our flow to see how it changes. Assume that f solves (3.25), then we get that $h := f \circ \Phi_t^{-1}$ solves

$$\partial_t(h \circ \Phi_t) = -u(\Phi_t(x), h \circ \Phi_t, D(h \circ \Phi_t))\Delta(h \circ \Phi_t) + B(\Phi_t(x), h \circ \Phi_t, D(h \circ \Phi_t))$$

now recall that the right hand side here is actually $(n\varphi - Hu)\langle \overline{\nabla}\lambda, \nu\rangle$ and is actually a geometric quantity, hence it does not depend on parametrization and so we can rewrite the right hand side as

$$(-u(x, h, Dh)\Delta h + B(x, h, Dh)) \circ \Phi_t$$

and so h solves

$$\partial_t(h \circ \Phi_t) = (-u(x, h, Dh)\Delta h + B(x, h, Dh)) \circ \Phi_t.$$

But now we can simplify the left hand side by chain rule to get

$$(\partial_t h) \circ \Phi_t + (dh \circ \Phi_t)(\partial_t \Phi_t) = (-u(x, h, Dh)\Delta h + B(x, h, Dh)) \circ \Phi_t.$$

Next we assume that $\partial_t \Phi_t = (u(x, h, Dh)V(t)) \circ \Phi_t$, this gives us

$$\begin{gathered}(\partial_t h) \circ \Phi_t + (u(x, h, Dh)dh(V(t))) \circ \Phi_t = (-u(x, h, Dh)\Delta h + B(x, h, Dh)) \circ \Phi_t \\ \partial_t h + u(x, h, Dh)dh(V(t)) = -u(x, h, Dh)\Delta h + B(x, h, Dh),\end{gathered}$$

and from the definition of $V^k(t)$ we get

$$\partial_t h = -(ug^{ij})(x, h, Dh)\Big(\partial_i\partial_j h - \tilde{\Gamma}^k_{ij}\partial_k h\Big) + B(x, h, Dh).$$

Now importantly in this PDE $\tilde{\Gamma}$ is fixed and $(ug^{ij})(x, h, Dh)$ is uniformly positive semi-definite since they are uniformly positive semi-definite at $t = 0$ and are also positive-semi-definite for some non-zero time interval. Hence by Theorem 1.5.2 we get that this PDE does have a solution for some time interval $[0, T)$.

Once we have this solution h, we can use it to construct the diffeomorphism Φ_t by simply considering the flow of the time dependent vector field $u(x, h, Dh)V(t)$ which exists for all time by standard ODE theory. Then $h \circ \Phi_t$ is a solution to (3.25) which proves short time existence. □

Now that we showed short-time existence we can use Corollary 3.6.1.2 along with Corollary 3.7.5.1 to get that as long as $T \leq T_0$

$$\|\nabla f\| < \varepsilon$$

for all $t \in [0, T)$. This along with Theorem 1.5.3 this gives us estimates on $\|\nabla f\|_{C^{1+r}}$ which then together with Theorem 1.5.2 gives us a stronger existence statement.

Corollary 3.7.6.1: For any star-shaped hypersurface M_0, the normal flow with velocity $n\varphi - Hu$ with initial condition M_0 exists on $[0, T_0]$.

Proof: We will again first pass to the function space and consider the evolution of the graph of the function f. Existence on $[0, T_0)$ is immediate by Theorem 1.5.2 along with Theorem 1.5.3. Then to get existence at $t = T_0$ we will take a sequence $t_n \to T_0$ with $f(t, \cdot) \to g$ and use Arzelà–Ascoli Theorem to prove g is smooth. Bounds on f_t then imply that f cannot infinitely oscillate in t and so g is independent of choice of sequence t_n.

To use Arzelà–Ascoli like this, we will need uniform bounds on all derivatives of f, here we will use Theorem 1.5.2 once again. Note that by the second part of Theorem 1.5.2 we get that

$$\|f(t, \cdot)\|_{C^{2+r}} \leq B(\|f(t, \cdot)\|_{C^{1+r}})$$

and so by setting $r' = 1 + r$ we can repeat this process to get bounds on higher and higher derivatives of f, which completes the proof. □

We now know that the flow survives until $t = T_0$. When the flow reaches this point we stop it and change the flow by removing the tangential component $X^\top$ entirely, only leaving $X^\perp$. With this simpler flow Li and Pan [11] showed that u is uniformly bounded from below for *all* time and thus our results prove that the flow exists for all $t \in [0, \infty)$.

Proposition 3.7.7: The flow described in Section 3.2 exists for all time $t \in [0, \infty)$.

This proves the first condition of Theorem 2.1.

We now want to show that the limit of this flow is precisely an integral hypersurface S_λ, to do this we will use a trick where we take a limit of an entire interval of our flow. To be more precise assume that $F : M \times [0, \infty) \to N$ solves the flow, then consider the functions $F_n : M \times [0, 1] \to N$ defined by

$$F_n(t, p) = F(n + t, p).$$

These are all solutions to the flow and by Arzelà–Ascoli we can, after passing to a subsequence, get that

$$F_n \to F_\infty$$

for some function $F_\infty : M \times [0, 1] \to N$ which is also a solution to the flow.

Proposition 3.7.8: If Q is a positive continuous geometric property of an embedding $F : M \to N$ such that $Q(F(t, \cdot))$ is non-increasing, then $Q(F_\infty(t, \cdot))$ is constant on $[0, 1]$.

Proof: Since for any $t \in [0, 1]$ the sequence $Q(F_n(t, \cdot))$ is positive non increasing and thus its limit exists. Hence we have

$$Q(F_\infty(t, \cdot)) = \lim_{n\to\infty} Q(F_n(t, \cdot))$$

which after plugging in $t = 0, 1$ gives us

$$Q(F_\infty(0, \cdot)) = \lim_{n\to\infty} Q(F_n(0, \cdot)) = \lim_{n\to\infty} Q(F(n, \cdot)),$$
$$Q(F_\infty(1, \cdot)) = \lim_{n\to\infty} Q(F_n(1, \cdot)) = \lim_{n\to\infty} Q(F(n + 1, \cdot)),$$

which immediately gives us that $Q(F_\infty(0, \cdot)) = Q(F_\infty(1, \cdot))$. But now Q is non-increasing along the flow and thus $Q(F_\infty(t, \cdot))$ is constant. □

We now first apply this to the surface area $A(M_t)$. Since it is clearly continuous and non-increasing by Proposition 3.3.1, the above proposition implies that $F_\infty(t, \cdot)$ has constant surface area. Due to the variation formula for surface area we get that

$$0 = \int_M = \frac{n}{n-1} \int_M u\Big(\overline{\mathrm{Ric}}(\mathcal{N}^\perp, \mathcal{N}^\perp) - \overline{\mathrm{Ric}}(\nu, \nu)\Big)\,\mathrm{d}S - \frac{1}{n} \int_M u \sum_{i<j} \left(\kappa_i - \kappa_j\right)^2 \mathrm{d}S$$

and so we get that $\kappa_i = \kappa_j$ for all i, j and so $F_\infty(t, \cdot)$ is totally umbilical. Secondly we get that $\overline{\mathrm{Ric}}(\mathcal{N}^\perp, \mathcal{N}^\perp) = \overline{\mathrm{Ric}}(\nu, \nu)$, this is important due to the following lemma.

Lemma 3.7.9: Let S be a symmetric bilinear form and $\langle \cdot, \cdot \rangle$ an inner product on a vector space V. If X is a unit vector with respect to $\langle \cdot, \cdot \rangle$ such that $S(X, X)$ is minimal/maximal among all other such unit vectors, then X is an eigenvalue of

$$T(S(X, \cdot))$$

where T is the isomorphism $V^* \to V$ induced by $\langle \cdot, \cdot \rangle$.

Proof: Since S is symmetric bilinear it has a basis of eigenvectors e_i that is orthonormal with respect to $\langle \cdot, \cdot \rangle$. Let λ_i be their eigenvalues which we can assume are ordered in increasing order. For all unit vectors $v = a_1 e_1 + ... + a_n e_n$ we have

$$\langle v, v \rangle = a_1^2 + a_2^2 + ... + a_n^2 = 1$$

and we also have

$$S(v, v) = \lambda_1 a_1^2 + \lambda_2 a_2^2 + ... + \lambda_n a_n^2 \leq \lambda_n a_1^2 + \lambda_n a_2^2 + ... + \lambda_n a_n^2 = \lambda_n$$

and so since $S(e_n, e_n) = \lambda_n$ we have that e_n is a unit vector with $S(e_n, e_n)$ maximal. Now assume that v is another unit vector with $S(v, v)$ also maximal, then we must have $S(v, v) = \lambda_n$ and so since

$$\lambda_n = S(v, v) = \lambda_1 a_1^2 + \lambda_2 a_2^2 + ... + \lambda_n a_n^2 = \lambda_n + (\lambda_1 - \lambda_n) a_1^2 + ... + (\lambda_{n-1} - \lambda_n) a_{n-1}^2$$

and since we know that $\lambda_i - \lambda_n$ is always negative and a_i^2 always positive, we must have that for each $i < n$, either $\lambda_i = \lambda_n$ or $a_i = 0$. We thus have that v can be written as a sum of eigenvectors all with eigenvalue λ_n and thus v is itself also an eigenvector. □

Applying this lemma to our situation we get that ν is an eigenvector of the Ricci tensor since it is symmetric bilinear. We can use this along with Lemma 1.3.3 to get that

$$0 = \overline{\mathrm{Ric}}(\nu, e_i) = \overline{\mathrm{Rm}}(e_i, e_j, e_j, \nu) = -(\nabla_i h)(e_j, e_j) + (\nabla_j h)(e_i, e_j)$$

t Now since $F_\infty(t, \cdot)$ is totally umbilical we know that $h = \frac{H}{n} g_{ij}$ and so we have

$$\nabla_k h = \frac{\nabla_k H}{n} g_{ij}$$

which when plugged into the equation above gives us

$$0 = -(\nabla_i h)(e_j, e_j) + (\nabla_j h)(e_i, e_j) = -\frac{\nabla_i H}{n} \langle e_j, e_j \rangle + \frac{\nabla_j H}{n} \langle e_i, e_j \rangle = -\nabla_i H + \frac{\nabla_i H}{n}$$
$$= -\nabla_i H \frac{n-1}{n}$$

and so $\nabla H = 0$ so we get that H is constant along $F_\infty(t, \cdot)$ (though not necessarily constant in time).

Next we get apply Proposition 3.7.8 to $\max_M \lambda$, this is also a non-increasing quantity due to Proposition 3.4.1 and is continuous so this is valid, and so we get that $\max_M \lambda(F_\infty(t,\cdot))$ is constant on $[0,1]$.

Proposition 3.7.10: At $t = \frac{1}{2}$ there is at least one maximal point which is stationary, that is we have a point p with $\partial_t \lambda(t,p) = 0$ and $\lambda\left(\frac{1}{2}, p\right) = \max_M \lambda\left(F_\infty\left(\frac{1}{2}, p\right)\right)$.

Proof: For brevity we will write $\lambda_{\max(t)} := \max_M \lambda(F_\infty(t,p))$. We prove by contrapositive, assume that there are no stationary maximal points, if any maximal point p satisfies

$$\partial_t \lambda\left(\frac{1}{2}, p\right) > 0$$

then we have

$$\lambda_{\max}\left(\frac{1}{2} + h\right) \geq \lambda\left(\frac{1}{2} + h, p\right) = \lambda\left(\frac{1}{2}, p\right) + h\left(\partial_t \lambda\left(\frac{1}{2}, p\right)\right) + O(h^2)$$

and so for small enough $h > 0$ we get that $\lambda_{\max}(t)$ is not constant.

Otherwise we have that $\partial_t \lambda\left(\frac{1}{2}, p\right) < 0$ for all maximal points p let S denote the set of all maximum points, since S is the preimage of $\lambda_{\max}\left(\frac{1}{2}\right)$ under a continuous function it is closed and thus compact in M. Hence there is positive ε such that $\partial_t \lambda\left(\frac{1}{2}, \cdot\right) < -\varepsilon$ on S. We can now define the open set

$$U := \left(\lambda\left(\frac{1}{2}, \cdot\right)\right)^{-1}\left(\left(\lambda_{\max}\left(\frac{1}{2}\right) - \varepsilon, \lambda_{\max}\left(\frac{1}{2}\right) + \varepsilon\right)\right)$$

which is clearly a neighborhood of S. Now the set $M \setminus U$ is closed, hence compact, hence on it $\lambda\left(\frac{1}{2}, \cdot\right)$ achieves a maximum. But this maximum cannot be $\lambda_{\max}\left(\frac{1}{2}\right)$ since this set does not contain S and so on $M \setminus U$ we have that $\lambda\left(\frac{1}{2}, \cdot\right) < \lambda_{\max}\left(\frac{1}{2}\right) - \varepsilon'$ for some positive $\varepsilon' > 0$ and also on $M \setminus U$ we have $\partial_t \lambda\left(\frac{1}{2}, \cdot\right) < B$ for some large positive B. But then we get that for $t = \frac{1}{2} + h$ we have

$$\lambda\left(\frac{1}{2} + h, p\right) \leq \begin{cases} \lambda_{\max}\left(\frac{1}{2}\right) - h\varepsilon + O(h^2) : p \in U \\ \lambda_{\max}\left(\frac{1}{2}\right) - \varepsilon' + Bh : p \in M \setminus U \end{cases}$$

and so by picking small enough positive h we get that

$$\lambda\left(\frac{1}{2} + h, p\right) < \lambda_{\max}\left(\frac{1}{2}\right)$$

everywhere and so we again get that $\lambda_{\max}(t)$ is not constant.

By contrapositive we get that since it is constant at least one stationary maximal point exists. □

Now at a stationary maximal point we know that

$$0 = \partial_t \lambda(t,p) = 2(n\varphi - Hu)\Lambda u$$

and so since Λ and u are positive we must have that $n\varphi - Hu = 0$. Now again, at a maximum point, we know that ν is colinear with the gradient of λ and so $\nu = \mathcal{N}^\perp$ and so $u = \|X^\perp\|$. This then gives us that

$$0 = n\varphi - Hu = n\varphi - H\|X^\perp\| = \varphi\left(n - \frac{H(\|X^\perp\|)}{\varphi}\right) = \varphi\left(n - H\lambda^{\frac{1}{2}}\right)$$

and so since φ is positive we get that $n - H\lambda^{\frac{1}{2}} = 0$ and thus $H = n\lambda^{-\frac{1}{2}}$ at a stationary maximal point. But now we recall that H is constant along M for any fixed time and so at $t = \frac{1}{2}$ we get that

$$H = n(\lambda_{\max})^{-\frac{1}{2}}.$$

We can now calculate that at any point p of M at $t = \frac{1}{2}$ we have

$$n\varphi - Hu \geq n\varphi - n(\lambda_{\max})^{-\frac{1}{2}}\|X^\perp\|\|\nu\| = n\varphi\left(1 - \left(\frac{\lambda}{\lambda_{\max}}\right)^{\frac{1}{2}}\right).$$

and so the speed function is nowhere negative. But now by Lemma 1.4.2 we get that

$$\int_M (n\varphi - Hu)\,\mathrm{d}S = 0$$

and so since it is nowhere negative the speed function must be everywhere zero and so we must have $\lambda = \lambda_{\max}$ on all of M. Since $\lambda_{\min}$ is also constant on $[0,1]$ we get that λ is constant on all of M for all time $[0,1]$ and thus the limit of the flow is an integral hypersurface S_λ.

Proposition 3.7.11: The limit as $t \to \infty$ of M_t is an integral hypersurface S_λ

This proves the last condition of Theorem 2.1, and thus proves Theorem 1.1.1.

3.8 Conclusion

The result of Theorem 1.1.1 provides is the best known result for the Isoperimetric inequality using the flow method. And we suspect that it is unlikely to be improved without a major change of approach. The reason is that already the evolution equations are quite difficult to handle and require a 2 step flow which is extremely rare in the literature. Because of this the author does not believe that there are many fruitful research directions stemming from this specific result

However, many of the tricks and methodologies used to prove this result are novel and not specific to this setup. For example, the proof of Proposition 3.7.11 used a novel from linear algebra applied along with the Codazzi equation to greatly reduce the necessary conditions for convergence, this trick is likely to be useful in other extrinsic flows like Inverse Mean Curvature Flow. The aforementioned 2 step flow could also likely be used in other geometric flows to ‘smooth out’ the target manifold before applying a known canonical flows. An interesting case could be the normalized Ricci flow which shares many of the properties of Mean Curvature Flow.

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