Bundle Extension Problems and Characteristic Classes

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2 Abstract

We present the construction of infinite flag manifolds as classifying spaces of filtrations of vector bundles and compute their cohomology groups and rings. With these, we prove results relating to characteristic classes and extension problems of vector bundles, in particular the study of Chern classes and their corresponding Schubert cells, as well as studying when sub-bundles of restricted vector bundles can be extended to the ambient bundle.

Abrégé

Nous présentons la construction des varietés de drapeaux en dimension infinie comme espaces de classifications pour des filtrations d'espaces fibrés vectoriels et nous calculons leurs groupes et anneaux de cohomologie. Avec ceux-ci, nous démontrons des résultats concernant les classes caractéristiques et des problèmes d'extension de fibrés vectoriels, en particulier l'étude des classes de Chern et des cellules de Schubert, ainsi que le problème de comment des sous-fibrés de fibrés restreints peuvent être étendus au fibré ambiant.

Contents

3 Introduction

The majority of this thesis is focused on the study of Grassmannians and flag manifolds, spaces whose objects are vector spaces and filtrations of vector spaces, respectively. These closed manifolds are ubiquitous in several areas of mathematics, particularly differential and algebraic geometry. For instance, Grassmannians serve as classifying spaces for vector bundles, another important class of object studied thoroughly in this thesis. We will see that, much like Grassmannians, flag manifolds also serve as classifying spaces for filtrations of vector bundles.

Vector bundles and fiber bundles appear here as important topological tools to study the structure and properties of flag manifolds. They have historically provided widespread applications in geometry and topology and have been the focus of extensive research in algebraic topology. Section 4 presents an overview of these objects as well as introducing the basics of characteristic classes and obstruction theory, further tools used in the theory of bundles.

Sections 5 and 6 present Grassmannians and flag manifolds, with a particular focus on their infinite dimensional counterparts. In section 5, besides the cohomology of Grassmannians, the main result presented is the correspondence result between isomorphism classes of vector bundles over a base space and homotopy classes of maps into an infinite Grassmannian. In section 6, the key result is the description of flag manifolds as iterated fiber bundles constructed from smaller Grassmannians.

Section 7 looks at the cell structure of flag manifolds in terms of Schubert cells and uses these to study the cohomology of these manifolds, as well as the Chern classes of a natural vector bundle, the tautological bundle of a flag manifold. Section 8 now turns to generalizing the result of section 5 to flag manifolds, where isomorphism classes of filtrations of vector bundles over a base space are in correspondence with homotopy classes of maps into an infinite flag manifold. A full description of the homotopy groups of infinite flag manifolds is then presented, based on their iterated fiber bundle structure.

Section 9 looks at problems relating to extending sub-bundles of vectors restricted to submanifolds to the whole ambient manifold, as applications to the previous results. Section 10 presents a couple miscellaneous and interesting results about vector bundles relevant to the work presented here, with the main results being summarized as finding important properties of closed manifolds which admit only trivial vector bundles.

A thorough treatment of the cohomology of principle fiber bundles and homogeneous spaces, which includes in particular the study of generalized flag varieties, is done by Borel in his seminal 1953 paper "Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts". A more specialized treatment of flag varieties is presented later by Bernstein, Gel'fand and Gel'fand in their 1973 paper "Schubert cells and Cohomology of the Spaces G/P", where these cells and the corresponding cohomology are detailed more explicitly.

Although the results are classical, the proofs presented in this thesis for some of the cohomology results are original to my supervisor Professor Jacques Hurtubise and I. Overall, these results and proofs can be obtained as special cases of those in Borel's paper. These include all the results in section 7.2 of this thesis, excluding lemma 7.4. We note also that original proofs are presented in section 6, 8 and 9 as well, in particular those of theorems 6.1, 8.3, 8.10 and 9.2, proposition 9.3 and corollary 9.4. We make no claims as to the originality of these later results, although a more thorough review of the literature is required to determine whether some of these appear in said literature.

4 Fiber bundles and vector bundles

4.1 Preliminary definitions and basic results

We shall assume every topological space we're dealing with is a smooth connected manifold.

Definition 4.1. *A fiber bundle consists of three spaces: the base space M, the fiber F and the total space E, usually written as $F \to E \to M$, such that the following holds:*

(1) *there is a continuous projection map $\pi : E \to M$,*

(2) *for each $p \in M$, $\pi^{-1}(p) \cong F$, and is denoted $F_p(E)$ or F_p, and*

(3) *the fiber bundle satisfies the local triviality condition:*

For every $p \in M$ there exists a neighborhood U and a homeomorphism

$$h : \pi^{-1}(U) \to U \times F$$

such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{h} & U \times F \\ & \searrow^{\pi} & \downarrow p_1 \\ & & U \end{array}$$

Ranging over every point in M, a covering set $\{(\pi^{-1}(U_i), h_i)\}$ is called a *local trivialization* of the bundle.

When there is no risk of confusion, we will denote a fiber bundle simply by its total space E.

In particular, we have that if F is a vector space, we call the fiber bundle E a vector bundle of rank n, where n is the dimension of F, and we further require that the homeomorphism h be an isomorphism of vector spaces on every fiber. A rank 1 vector bundle will be called a line bundle.

The canonical example of a smooth vector bundle is the tangent bundle of a manifold, TM, with fibers the tangent spaces T_pM.

A bundle is called trivial if $E \cong M \times F$. A manifold is called parallelizable if its tangent bundle is trivial. Here are a few early results about trivial bundles.

Proposition 4.2. *Every Lie group is parallelizable [4].*

This essentially follows from the fact that the tangent space at the identity element is isomorphic to the Lie algebra of the Lie group, and then one defines a bundle isomorphism (see definition 4.3) using the left translation map.

So in particular, since S^1 and S^3 are Lie groups, they are parallelizable. In fact, the only other sphere that is parallelizable is S^7 [11].

Definition 4.3. *Two vector bundles E_1 and E_2 over the same base space M are isomorphic if there exists a homeomorphism $\phi : E_1 \to E_2$ such that $F_p(E_1)$ is mapped isomorphically as a vector space to $F_p(E_2)$. In particular, this gives a commuting diagram*

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & M & \end{array}$$

This generalizes to arbitrary fiber bundles E_1, E_2 over distinct bases M, N as follows.

Definition 4.4. *A bundle map between fiber bundles E_1, E_2 over bases M, N respectively is a pair of maps $\phi : E_1 \to E_2$, $f : M \to N$ such that the following commutes:*

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow[f]{} & N \end{array}$$

Particularly interesting cases are when $M = N$ and f is the identity map, such as it was the case above for vector bundles.

We now define the pullback of a bundle, which is a way of constructing new vector bundles out of other ones.

Definition 4.5. *The pullback of a bundle $E \to N$ over M given a map $f : M \to N$ is the bundle $f^*E \to M$ defined as*

$$f^*E = \{(p, x) \in M \times E : f(p) = \pi(x)\}.$$

Generalizing the concept of a vector field in a tangent bundle, we have the following.

Definition 4.6. *A section of a fiber bundle is a continuous map $s : M \to E$ such that $\pi \circ s = id$.*

Although not every fiber bundle admits a section, every vector bundle admits the trivial zero-section, that which maps to the zero vector in each fiber.

A parallelizable manifold admits n sections that are linearly independent over every fiber, in particular they are nowhere zero. But not every manifold admits nowhere zero vector fields. The classic example is S^2 via the hairy ball theorem. This will be further elaborated on when discussing the Euler class and Euler characteristic.

Definition 4.7. *A principal bundle is a fiber bundle $G \to P \to M$ with fibers homeomorphic to a topological group G, together with a continuous right action of G on P such that if $p \in P_x$, then $pg \in P_x$ for all $g \in G$. Furthermore, the action is free and transitive on the fibers such that for fixed $p \in P_x$, the map $G \to P_x$ sending g to pg is a homeomorphism.*

The following result produces many examples of fiber bundles that do not admit any sections.

Proposition 4.8. *A principal bundle admits a section if and only if the bundle is trivial.*

We now define an orientable vector bundle.

Definition 4.9. *A vector bundle E is said to be orientable if every fiber can be given an orientation as a vector space such that, given suitable local trivializations of the bundle, the orientations all agree on intersections over the whole manifold.*

One must be careful to distinguish the orientability of E as a manifold and as a vector bundle. For example, the tangent bundle of any manifold is an orientable manifold, but is orientable as a vector bundle if and only if the base manifold is orientable as a manifold.

We now turn to sub-bundles of vector bundles, which are vector bundles of smaller rank embedded inside larger ones over a common base manifold.

Definition 4.10. *A sub-bundle E_1 of E_2 over M, denoted $E_1 \subset E_2$, is a subset of E_2 which is also a vector bundle over M where each fiber $F_p(E_1)$ is a linear subspace of the fiber $F_p(E_2)$.*

In terms of bundle maps, this is more concisely stated as having an injective bundle map $\phi : E_1 \to E_2$ over M.

Basic examples of sub-bundles arise when constructing higher rank vector bundles from smaller ones via direct summation.

Definition 4.11. *Given two vector bundles E_1 and E_2 over M, of ranks n_1, n_2 respectively,* *their direct sum $E_1 \oplus E_2$ is a rank $n_1 + n_2$ vector bundle over M whose fibers are the direct sum of the fibers of the smaller bundles; specifically,*

$$E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 : \pi_1(v_1) = \pi_2(v_2)\}$$

where $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ are the bundle projection maps.

An important example is the splitting of trivial vector bundles into a direct sum of line bundles, i.e. if E is trivial or rank n, then $E = \varepsilon^n \cong \varepsilon^1 \oplus \ldots \oplus \varepsilon^1$, where ε^1 is the trivial line bundle over a given space.

In particular, a tangent bundle admits a nowhere zero vector field if and only if it admits a trivial line sub-bundle.

Definition 4.12. *A vector bundle E is stably trivial if there exists some $k > 0$ such that $E \oplus \varepsilon^k$ is trivial.*

A classic example of a stably trivial vector bundle is the tangent bundle of S^n. Indeed, S^n embeds into $\mathbb{R}^{n+1}$ and one can see that its normal bundle is trivial. The sum of the tangent bundle and the normal bundle is clearly trivial, so TS^n is stably trivial.

4.2 Characteristic classes

Definition 4.13. *Given a vector bundle $E \to M$, a characteristic class c of E is a cohomology class $c(E) \in H^*(M)$ that is natural, in the sense that given a map $f : N \to M$, $c(f^*E) = f^*(c(E)) \in H^*(N)$.*

The primary characteristic classes are the Stiefel-Whitney classes in mod 2 cohomology for real bundles, and the Chern classes in $\mathbb{Z}$ cohomology for complex bundles. We will focus on Chern classes for now, which are largely analogous to Stiefel-Whitney classes. We will present two equivalent approaches, on which defines these classes axiomatically, and another which uses linear dependence of generic sections of bundles to define them.

Given a complex vector bundle $E \to M$, we define for each $k > 0$ the k^{th} Chern class of E to be the unique classes $c_k(E) \in H^{2k}(M, \mathbb{Z})$ that satisfy naturality and the following three axioms. We also denote the total Chern class to be $c(E) = 1 + c_1(E) + \ldots \in H^*(M, \mathbb{Z})$.

(1) $c_k(E) = 0$ if k is larger than the rank of E,

(2) For two bundles E, F over M, $c(E \oplus F) = c(E) \smile c(F) = c(E)c(F)$ for short, and hence

$$c_k(E \oplus F) = \sum_{i+j=k} c_i(E)c_j(F).$$

(3) If L is the tautological line bundle over $\mathbb{C}P^1$, $c_1(L)$ is non-zero, a generator of $H^2(\mathbb{C}P^1, \mathbb{Z})$.

As for Stiefel-Whitney classes, they are defined analogously: the k^{th} Stiefel-Whitney class of a real vector bundle $E \to M$ is the unique class $w_k(E) \in H^k(M, \mathbb{Z}_2)$ that satisfies the same axioms as the Chern classes, except for axiom 3, with $\mathbb{R}P^1$ replacing $\mathbb{C}P^1$.

The fact that these Chern classes exist and are unique is a theorem. Here are some consequences of the axiomatic approach.

For a trivial bundle $\varepsilon^n \to M$, we immediately get by naturality that $c(\varepsilon^n) = 1$, since ε^n is the pullback of the trivial bundle over a singleton, which has only trivial Chern classes.

Lemma 4.14. *For any vector bundle E, $c(E \oplus \varepsilon^1) = c(E)$.*

Proof. This follows from the formula $c(E \oplus \varepsilon^1) = c(E)c(\varepsilon^1) = c(E)$. □

As a corollary, we get that for a stably trivial bundle E, $c(E) = 1$, since there is some k such that $E \oplus \varepsilon^k$ is trivial, and hence $1 = c(E \oplus \varepsilon^k) = c(E)$. So already we get elementary obstruction results, namely to a bundle being trivial or stably trivial.

From an earlier result, we know that TS^n is stably trivial, thus its Stiefel-Whitney classes are trivial.

We also get, from naturality, that if $E_1 \cong E_2$, then $c_k(E_1) = c_k(E_2)$ for all k.

Now we turn to the second approach, which requires that our manifold be compact and orientable. First, we need Poincaré duality, a key result for later on.

Proposition 4.15. *Let M be a compact orientable manifold and let A be an abelian group. An orientation of M determines an isomorphism $PD : H^k(M, A) \to H_{n-k}(M, A)$ where $PD(\alpha) = [M] \frown \alpha$, $[M]$ being the fundamental class.*

In particular, if we choose $A = \mathbb{Z}_2$, then an orientation of M is not required.

Let $E \to M$ be a complex vector bundle of rank r over a compact orientable smooth manifold of dimension n. Let $s : M \to E$ be a generic section, i.e. a section which intersects the zero section transversally. Two sections are said to intersect transversally if at every point of intersection, the tangent spaces of the embeddings of M via the two sections generate the tangent space of E at that point.

Let $X(s_1, \ldots, s_k)$ denote the cycle of points $p \in M$ where the generic sections $s_1(p), \ldots, s_k(p)$ are linearly dependent. By transversality this is a smooth submanifold of M of real codimension k.

We then define the k^{th} Chern class to be the Poincaré dual of the fundamental class of $X(s_1, \ldots, s_{r-k+1})$, that is, $c_k(E) = PD([X(s_1, \ldots, s_{r-k+1})]) \in H^{2k}(M)$. It can be shown to be independent of the choice of sections. We further define $c_0(E) = 1$ as well.

This reveals that a Chern class is an obstruction to finding a set of everywhere linearly independent sections of a vector bundle.

We now turn to the construction of the Euler class of a vector bundle. This class is unique in the sense that it depends on the rank of the bundle.

Let E be a rank n complex oriented vector bundle over M. The Euler class $e(E)$ is the element of $H^n(M, \mathbb{Z})$ corresponding to the top Chern class of E. Given $E = TM$, and e its Euler class, evaluating it on its fundamental class gives $e[M] = \chi(M)$, its Euler characteristic.

Along with naturality and the Whitney-sum formula, the Euler class satisfies two further properties.

(1) If E possess a nowhere zero section, then $e(E) = 0$,

(2) If $\bar{E}$ is E with the opposite orientation, then $e(\bar{E}) = -e(E)$.

It is possible for a vector bundle to have zero Euler class but no nowhere-zero sections. Using the tools from section 5 on infinite Grassmannians, one can show that, over S^4, all rank 1 and rank 2 real vector bundles are trivial, but there are $\mathbb{Z}$ worth of real rank 3 vector bundles. Take a non-trivial vector bundle of rank 3 over S^4. Since $H^3(S^4) = 0$, its Euler class is zero. If E were to have a nowhere-zero section, it would split as a direct sum $L \oplus F$ of a line sub-bundle L and a rank 2 sub-bundle F. But both must be trivial, hence E would have to be trivial, a contradiction.

Finally, we construct the Pontryagin classes.

Definition 4.16. *The complexification of a real vector bundle $E \to M$, denoted $E \otimes \mathbb{C}$, is the complex vector bundle over M obtained from replacing each fiber E_p with $E_p \otimes \mathbb{C}$.*

Note that the underlying real vector bundle $(E \otimes \mathbb{C})_{\mathbb{R}}$ is canonically isomorphic to $E \oplus E$ [9].

Definition 4.17. *The k^{th} Pontryagin class*

$$p_k(E) \in H^{4k}(M, \mathbb{Z})$$

is defined to be equal to $(-1)^k c_{2k}(E \otimes \mathbb{C})$.

Similarly, the total Pontryagin class is defined to be

$$p(E) = 1 + p_1(E) + \ldots + p_{[n/2]}(E),$$

where $[n/2]$ is the largest integer less than or equal to $n/2$.

It can be shown that the Pontryagin classes satisfy naturality as before, but care must be taken when trying to invoke the previous axioms of the Chern and Stiefel-Whitney classes.

Proposition 4.18. *Given two vector bundles E_1, E_2 over M, the total Pontryagin class $p(E_1 \oplus E_2)$ is congruent to $p(E_1)p(E_2)$ modulo elements of order 2. [9]*

4.3 Examples of obstructions

Using the Euler class, we can find an obstruction to even dimensional spheres admitting non-trivial sub-bundles of their tangent bundles.

Proposition 4.19. *The tangent bundle of an even dimensional sphere S^{2n} does not admit a sub-bundle of rank $k, 0 < k < 2n$.*

We are grateful to Mark Grant for providing the following proof in his answer to a MathOverflow question.

Proof. Assume that $TS^{2n} \cong E_1 \oplus E_2$ for non-trivial vector bundles E_1 and E_2 over S^{2n} of dimensions k and l, respectively. Thus $0 < k, l < 2n$. Both bundles must be oriented since S^{2n} is simply connected, and so we can obtain Euler classes:

$$e(E_1) \in H^k(S^{2n}, \mathbb{Z}), \quad e(E_2) \in H^l(S^{2n}, \mathbb{Z}),$$

but for dimension reasons both are zero.

We also get that

$$e(TS^{2n}) = e(E_1)e(E_2) = 0,$$

however,

$$2 = \chi(S^{2n}) = \langle e(TS^{2n}), [S^{2n}] \rangle = 0,$$

a contradiction. □

Next, we turn to the question of whether every element of a homotopy group of a smooth manifold can be represented by an immersion. If $dim(M) = n$, then for $k \leq n/2$, every $\alpha \in \pi_k(M)$ has a representative $f : S^k \to M$ that is an immersion by transversality. But this inequality turns out to be strict.

Proposition 4.20. *There exists a simply-connected closed 6-manifold M with a homotopy class $\alpha \in \pi_4(M)$ which does not contain an immersion.*

The proof is also attributed to Mark Grant and his colleague Diarmuid Crowley as an answer to a question of mine on MathOverflow.[1] The obstruction this time arises from a Pontryagin class.

Proof. According to C. T. C. Wall in [13], there exists a simply connected 6-manifold M such that $H^*(M, \mathbb{Z}) \cong H^*(\mathbb{C}P^3, \mathbb{Z})$, where the cup product

$$H^2(M) \times H^2(M) \to H^4(M)$$

[1]https://mathoverflow.net/q/375919/143629

is trivial, and the first Pontryagin class $p_1(M) \in H^4(M)$ is non-zero.

We first need to show that the Hurewicz map $\pi_4(M) \to H_4(M)$ is surjective.

Due to the cohomology isomorphism, M has the homotopy type of a CW-complex with one cell in each even dimension between 0 and 6, since that is the case for $\mathbb{C}P^3$. From the trivial cup product described above, the cup square of the generator in $H^2(M)$ is trivial, and this implies that the attaching map $S^3 \to S^2$ of the 4-cell has trivial Hopf invariant, so is also trivial. This can be seen with a Mayer-Vietoris argument. This means $M^{(4)} \sim S^2 \vee S^4$, which proves that the Hurewicz map is surjective.

Next, let $\alpha \in \pi_4(M)$ be a homotopy class whose image via the Hurewicz map is a generator $x \in H_4(M)$. For contradiction, assume that α is represented by an immersion $f : S^4 \to M$. We get

$$f^*(TM) \cong \nu(f) \oplus TS^4,$$

where $\nu(f)$ is the normal bundle of the immersion. Since $\nu(f)$ has rank 2 over S^4 and $\pi_4 G(2) \cong \pi_3 O(2) = 0$ (see section 5 on infinite Grassmannians), we have that $\nu(f)$ is trivial. Furthermore, TS^4 is stably trivial. From the bundle isomorphism and the naturality of Pontrygin classes, we get that

$$f^*(p_1(M)) = p_1(f^*(TM)) = p_1(\nu(f)) + p_1(S^4) = 0 \in H^4(S^4).$$

Finally,

$$0 = \langle f^*(p_1(M)), [S^4] \rangle = \langle p_1(M), f_*[S^4] \rangle = \langle p_1(M), x \rangle,$$

but this means $p_1(M) = 0$, a contradiction. □

5 Finite and Infinite Grassmannians

The real (resp. complex) Grassmannian $Gr(k, n)$ is the closed manifold whose elements are all the dimension k subspaces in $\mathbb{R}^n$ (resp. $\mathbb{C}^n$). It is naturally diffeomorphic to $Gr(n-k, n)$ by taking subspace complements. The classic example where $k = 1$ gives the projective space $\mathbb{P}^{n-1}$.

One obtains in general that Grassmannians are homogeneous spaces. In the real case,

$$Gr(k,n) \cong O(n)/(O(k) \times O(n-k)),$$

and in the complex case,

$$Gr(k,n) \cong U(n)/(U(k) \times U(n-k)).$$

This gives it a dimension of $k(n-k)$.

From here on out, we will be looking at the complex Grassmannian exclusively.

The tautological bundle of $Gr(k,n)$ is the rank k vector bundle E with total space

$$\{(V,v) \in Gr(k,n) \times \mathbb{C}^n : v \in V\},$$

that is, the fibers correspond to the elements of the Grassmannian.

One also constructs a complementary bundle F or rank $n-k$, with total space also lying in $Gr(k,n) \times \mathbb{C}^n$, whose fibers are the orthogonal complements of the fibers of E. Then immediately one gets that $E \oplus F = Gr(k,n) \times \mathbb{C}^n$ is trivial.

Thus, we get the following relation between the total Chern classes of E and F, which will be useful in constructing the cohomology ring of Grassmannians:

$$c(E)c(F) = 1.$$

Taking the direct limit of $Gr(n,m)$ with respect to m gives the infinite Grassmannian $G(n,\mathbb{C}^\infty)$ of n-planes in $\mathbb{C}^\infty$. It yields many important properties that classify vector bundles. Thus it is called a classifying space.

The important results are the following, see [9].

Proposition 5.1. *There is a correspondence between isomorphism classes of rank n vector bundles over a manifold M and homotopy classes of maps from M to $G(n,\mathbb{C}^\infty)$. Specifically, two maps $f,g : M \to G(n,\mathbb{C}^\infty)$ are homotopic if and only if the vector bundles E and F corresponding to f and g are isomorphic.*

Let $c_1, \ldots, c_n$ be the Chern classes of the tautological bundle of $G(n, \mathbb{C}^\infty)$, where $c_i \in H^{2i}(G(n, \mathbb{C}^\infty), \mathbb{Z})$.

Proposition 5.2. *The integral cohomology ring of $G(n, \mathbb{C}^\infty)$ is isomorphic to $\mathbb{Z}[c_1, ..., c_n]$, the polynomial algebra freely generated by the Chern classes of the tautological bundle, where the c_i are of degree $2i$.*

For the case of finite Grassmannians, we have

Proposition 5.3.

$$H^*(Gr(k,n), \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_k]/\langle \bar{c}_{n-k+1}, \ldots, \bar{c}_n \rangle,$$

where $\bar{c}_i$ is the dual of c_i subject to the relation $c\bar{c} = 1$.

As for the cohomology groups, we first need the following. Let $p(a; b, c)$ be the number of partitions of the integer a into at most b parts of size at most c.

Proposition 5.4. *$H^{2r}(Gr(k,n), \mathbb{Z})$ is torsion free and its dimension is equal to $p(r; k, n-k)$, while the odd degree cohomology groups are all trivial.*

Letting n tend to infinity in the direct limit also gives the dimension of the cohomology groups of the infinite Grassmannian $G(k, \mathbb{C}^\infty)$ with the same formula.

6 Finite and Infinite Flag Manifolds

Similar to how Grassmannians are defined, the complex flag manifold $Fl(n_1, \ldots, n_k, \mathbb{C}^m)$ is the closed manifold whose elements are flags of subspaces in $\mathbb{C}^m$ of dimensions $n_1 < \ldots < n_k$, denoted by $V_1 \subset \ldots \subset V_k$.

Taking the direct limit of m to infinity, we get the infinite flag manifold $F(n_1, \ldots, n_k, \mathbb{C}^\infty)$ of flags in $\mathbb{C}^\infty$.

An important property of flag manifolds that allows us to prove many results is that they are the total spaces of special fiber bundles.

Theorem 6.1. *In the finite dimensional case, $Fl(n_1, \ldots, n_k, \mathbb{C}^m)$ is the total space of a fiber bundle with base space $Gr(n_k, m)$ and with fiber the "smaller" flag manifold $Fl(n_1, \ldots, n_{k-1}, \mathbb{C}^{n_k})$.*

In the infinite dimensional case, $F(n_1, \ldots, n_k, \mathbb{C}^\infty)$ has base space $G(n_k, \mathbb{C}^\infty)$ and fiber $Fl(n_1, \ldots, n_{k-1}, \mathbb{C}^{n_k})$.

Outline of Proof. Begin with the principal bundle

$$U(n_k) \to V(n_k, \mathbb{C}^m) \to Gr(n_k, m).$$

Take the quotient of the fiber and the total space by the subgroup $U(n_1) \times U(n_2 - n_1) \times \ldots \times U(n_k - n_{k-1})$. The fiber and the total space now become the required flag manifolds and we are left with the required fiber bundle. The infinite dimensional case is obtained by taking the direct limit with respect to m. □

There are k natural projection maps on the infinite flag manifold, given by

$$\rho_i : F(n_1, \ldots, n_k, \mathbb{C}^\infty) \to G(n_i, \mathbb{C}^\infty)$$

where a flag is projected to the n_i-dimensional subspace of $\mathbb{C}^\infty$ in it.

Lemma 6.2. *The existence of a section of $\rho_2 : F(n_1, n_2, \mathbb{C}^\infty) \to G(n_2, \mathbb{C}^\infty)$ is equivalent to the existence of a non-trivial sub-bundle of the tautological bundle of $G(n_2, \mathbb{C}^\infty)$.*

Proof. Let $s : G(n_2, \mathbb{C}^\infty) \to F(n_1, n_2, \mathbb{C}^\infty)$ be a section. This is equivalent to the existence of a continuous map $f : G(n_2, \mathbb{C}^\infty) \to G(n_1, \mathbb{C}^\infty)$ sending each $V \in G(n_2, \mathbb{C}^\infty)$ to a dimension n_1 subspace of V. This furthermore is equivalent to a rank n_1 sub-bundle of the tautological bundle over $G(n_2, \mathbb{C}^\infty)$. □

Proposition 6.3. *For all natural n, the tautological bundle over $G(n, \mathbb{C}^\infty)$ has no non-trivial rank k sub-bundles for all $1 \leq k < n$.*

Corollary 6.4. *Thus, the fiber bundle map ρ_2 above does not admit a section for all natural n_1, n_2.*

Proof of proposition 6.3. Assume that the tautological bundle T^n splits as a direct sum $E_1 \oplus E_2$ of sub-bundles of ranks k and $n - k$. Then we get the following factorization of the total Chern class of T^n:

$$(1 + c_1 + \ldots + c_n) = (1 + c'_1 + \ldots + c'_k)(1 + c''_1 + \ldots + c''_{n-k}).$$

Since $H^*(G(n,\mathbb{C}^\infty),\mathbb{Z}) = \mathbb{Z}[c_1,\dots,c_n]$, each c_i is a generator of degree $2i$, and we get $c_n = c'_k c''_{n-k}$ as the only term. But since c_n is a generator, it cannot be the product of lower degree terms, which gives the required contradiction. □

One can easily use this argument to also show the following.

Corollary 6.5. *None of the projection maps $\rho_k : F(n_1,\dots,n_k,\mathbb{C}^\infty) \to G(n_k,\mathbb{C}^\infty)$ admit sections.*

7 Cell Structure and Cohomology of Flag Manifolds

7.1 Schubert Cells and Cell Structure

Recall that

$$Fl(n_1,\dots,n_k,\mathbb{C}^m) \cong U(m)/U(n_1) \times U(n_2-n_1) \times \dots \times U(m-n_k).$$

This is equivalent to the description of flag manifolds as the quotient of $GL(m,\mathbb{C})$ by lower block-triangular matrices with invertible blocks in the diagonal.

Let $G = GL(m,\mathbb{C})$, P be the upper block triangular matrices with invertible blocks in the diagonal, and $X = Fl(n_1,\dots,n_k,\mathbb{C}^m)$, so that $X = G/P$.

Let B be the Borel subgroup contained in P, which in general is the maximal Zariski closed and connected solvable algebraic subgroup of an algebraic group. In this case, B is the invertible upper triangular matrices. Let $\bar{B}$ be the lower triangular version.

Let $\bar{N}$ be the unipotent radical of $\bar{B}$, that is, the largest normal subgroup of $\bar{B}$ consisting entirely of unipotent elements, elements x which satisfy $(x-I)^n = 0$ for some n. This equals the subgroup of $\bar{B}$ consisting of elements whose diagonal entries are all equal to 1.

Let W be the Weyl group of G and W_P be the corresponding Weyl group of P, a subgroup of W.

The Schubert cells of X are then defined as follows. Let $o \in X$ be the image of P in X. Let w represent a class of W/W_P. Then $X_w = \bar{N}wo \subset X$, as w varies, decomposes X into

$\bar{N}$-orbits [8]. These give us the Schubert cells.

We will now explicitly describe the cell structure with matrices, following section 50 of [5]. We will first introduce the general computation by means of specific examples.

Take $X = Fl(2,4,\mathbb{C}^6)$. Consider the following matrix, which represents a dimension 12 (codimension 0) cell, the orbit of I under $\bar{N}$.

$$\left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & 1 & 0 & 0 \\ * & * & * & * & 1 & 0 \\ * & * & * & * & 0 & 1 \end{array}\right)$$

The rules are as follows. Below all 1's, we put stars, which represent the free values, each adding one to the dimension count. Each star is a "copy" of $\mathbb{C}$. Above all 1's there are 0's. On the right of every 1, there should only be 0's. On the left of every 1 within two consecutive vertical lines, there should be 0's.

Now we want to permute the columns to obtain different cells with different number of stars at various positions following the above rules. Note that permutations interchanging columns within two consecutive vertical lines amounts to no significant change. By convention we will arrange every column in decreasing order of 1's between each such line.

Let's look at the cell obtained from permuting columns 2 and 3, i.e. about the first vertical line.

$$\left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & 1 & 0 & 0 \\ * & * & * & * & 1 & 0 \\ * & * & * & * & 0 & 1 \end{array}\right)$$

We see that it has 11 stars, so is of codimension 1 and so its closure will correspond to a first Chern class of a tautological bundle (see section 8).

The second and last cell of codimension 1 corresponds to the permutation of the 4th and 5th column, about the second line:

$$\left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 \\ * & * & * & 0 & 1 & 0 \\ * & * & 0 & 1 & 0 & 0 \\ * & * & * & * & 0 & 1 \end{array}\right)$$

Now for codimension 2 cells, we have 5 of them. This number can be verified by counting the dimension of the 4^{th} cohomology group, using corollary 7.6 later on.

$$\left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 & 0 \\ * & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & * & * & 1 & 0 \\ * & * & * & * & 0 & 1 \end{array}\right), \quad \left(\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & 1 & 0 & 0 \\ * & * & * & * & 1 & 0 \\ * & * & * & * & 0 & 1 \end{array}\right), \quad \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 1 & 0 \\ * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & 1 & 0 & 0 \\ * & * & * & * & 0 & 1 \end{array}\right),$$

$$
\left(\begin{array}{cc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
* & * & 1 & 0 & 0 & 0 \\
* & * & * & 0 & 1 & 0 \\
* & * & * & 0 & 0 & 1 \\
* & * & 0 & 1 & 0 & 0
\end{array}\right), \qquad
\left(\begin{array}{cc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 \\
* & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
* & * & * & 0 & 1 & 0 \\
* & * & 0 & 1 & 0 & 0 \\
* & * & * & * & 0 & 1
\end{array}\right).
$$

The reason these matrices represent the correct cells of the flag manifold is as follows. First start with the trivial permutation matrix (the identity matrix).

$$
\left(\begin{array}{cc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

Acting on this by $\bar{N}$ places stars underneath each 1 as arbitrary elements in the orbit.

$$
\left(\begin{array}{cc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 \\
* & 1 & 0 & 0 & 0 & 0 \\
* & * & 1 & 0 & 0 & 0 \\
* & * & * & 1 & 0 & 0 \\
* & * & * & * & 1 & 0 \\
* & * & * & * & * & 1
\end{array}\right)
$$

But taking into consideration the action by elements of the parabolic group P, we see that within columns separated by the vertical lines, we may do column reductions, and the stars on the left of the 1's get replaced by 0's after normalization, leaving us with

$$\left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & 1 & 0 & 0 \\ * & * & * & * & 1 & 0 \\ * & * & * & * & 0 & 1 \end{array}\right)$$

as desired when we begin with the trivial permutation. Choosing different starting elements of W/W_P, i.e. different permutation matrices, with 1's in descending order withing two vertical lines, gives us every cell in the flag manifold. Note that each element of W/W_P has a representative that allows the 1's to be in descending order between the appropriate vertical lines.

7.2 Cohomology of Flag Manifolds

Theorem 7.1. *The integral cohomology ring $H^*(F(n_1, \ldots, n_k, \mathbb{C}^\infty))$ is isomorphic to*

$$H^*(Fl(n_1, \ldots, n_{k-1}, \mathbb{C}^{n_k})) \otimes H^*(G(n_k, \mathbb{C}^\infty)).$$

Proof. From the results of section 5, we see that there is no torsion in the fiber bundle, since $H^*(G(n_k, \mathbb{C}^\infty))$ is free and finitely generated.

By applying the Serre spectral sequence, we see that at the E_2 page, it collapses for degree reasons, since all the cohomology groups of the fiber and base space lie in even degrees. So this immediately gives us the required ring structure as a tensor product. □

This gives the following two corollaries.

Corollary 7.2. *The integral cohomology ring $H^*(Fl(n_1, \ldots, n_k, \mathbb{C}^m)$ is isomorphic to*

$$H^*(Gr(n_1, n_2)) \otimes \ldots \otimes H^*(Gr(n_{k-1}, n_k)) \otimes H^*(Gr(n_k, m)).$$

Thus, the integral cohomology ring $H^*(F(n_1, \ldots, n_k, \mathbb{C}^\infty))$ is isomorphic to

$$H^*(Gr(n_1, n_2)) \otimes \ldots \otimes H^*(Gr(n_{k-1}, n_k)) \otimes H^*(G(n_k, \mathbb{C}^\infty)).$$

Corollary 7.3. *The integral cohomology groups satisfy*

$$H^m(F(n_1,\ldots,n_k,\mathbb{C}^\infty)) \cong$$
$$\bigoplus_{p_1+\ldots+p_k=m} H^{p_1}(Gr(n_1,n_2))\otimes\ldots\otimes H^{p_{k-1}}(Gr(n_{k-1},n_k))\otimes H^{p_k}(G(n_k,\mathbb{C}^\infty)).$$

In particular, its cohomology groups in odd dimensions are all trivial, and are all torsion free in even dimensions.

We can count the number of Schubert cells of Grassmannians and flag manifolds [9].

Lemma 7.4. *The total number of Schubert cells of $Gr(n,m)$ is $\binom{m}{n}$.*

Corollary 7.5. *The total number of Schubert cells of $Fl(n_1,\ldots,n_k,\mathbb{C}^m)$ is*

$$\binom{n_2}{n_1}\cdots\binom{n_k}{n_{k-1}}\binom{m}{n_k}$$
$$=\binom{m}{n_1,n_2-n_1,\ldots,m-n_k}$$

where the large term in the equality is a multinomial coefficient.

This formula is obtained from the fact that the number of cells of a product of CW-complexes is the product of their number of cells, generalized to the previously shown fact that flag manifolds are iterated fiber bundles.

Recall that $p(a;b,c)$ denotes the number of partitions of a into at most b parts of size at most c, and that the dimension of $H^{2r}(Gr(n,m),\mathbb{Z})$ equals $p(r;n,m-n)$.

Thus we can compute the dimensions of the cohomology groups of flag manifolds in terms of those of Grassmannians, which corresponds to the number of Schubert cells of a given codimension.

Corollary 7.6.

$$
\begin{aligned}
& dim\ H^{2r}(Fl(n_1,\ldots,n_k,\mathbb{C}^m)) \\
&= dim\left[\bigoplus_{2q_1+\ldots+2q_k=2r} H^{2q_1}(Gr(n_1,n_2))\otimes\ldots\otimes H^{2q_k}(Gr(n_k,m))\right] \\
&= \sum_{q_1+\ldots+q_k=r}\left[dim\ H^{2q_1}(Gr(n_1,n_2))\cdot\ldots\cdot dim\ H^{2q_k}(Gr(n_k,m))\right] \\
&= \sum_{q_1+\ldots+q_k=r}\left[p(q_1;n_1,n_2-n_1)\cdot\ldots\cdot p(q_k;n_k,m-n_k)\right].
\end{aligned}
$$

8 Classifying Spaces of Bundle Filtrations

8.1 Correspondence theorem

Let $F(n_1,\ldots,n_k,\mathbb{C}^\infty)$ be an infinite flag manifold. We will show that, similar to the infinite Grassmannians, infinite flag manifolds are classifying spaces of filtrations of vector bundles.

Definition 8.1. *A bundle filtration is a filtration of vector bundles over a common manifold, i.e. bundles $E_1,\ldots,E_k$ of increasing ranks $n_1<\ldots<n_k$ such that*

$$E_1\subset E_2\subset\ldots\subset E_k$$

as sub-bundles.

Definition 8.2. *Two bundle filtrations $E_1\subset E_2\subset\ldots\subset E_k$ and $F_1\subset F_2\subset\ldots\subset F_k$ over M are isomorphic if there exists bundle maps g_i,b_i,h_i such that the following diagram commutes, where the g_i and h_i are injective and the b_i are bundle isomorphisms.*

$$
\begin{array}{ccccccc}
E_1 & \xrightarrow{g_1} & E_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{k-1}} & E_k \\
{\scriptstyle b_1}\downarrow & & {\scriptstyle b_2}\downarrow & & & & {\scriptstyle b_k}\downarrow \\
F_1 & \xrightarrow[h_1]{} & F_2 & \xrightarrow[h_2]{} & \cdots & \xrightarrow[h_{k-1}]{} & F_k
\end{array}
$$

The main result is as follows.

Theorem 8.3. *There is a bijective correspondence between bundle filtrations over M and homotopy classes of maps $f:M\to F(n_1,\ldots,n_k,\mathbb{C}^\infty)$. In particular, two bundle filtrations are isomorphic if and only if the corresponding maps are homotopic.*

Definition 8.4. *The tautological bundle of $G(n, \mathbb{C}^\infty)$ is*

$$T^n = \{(V, v) \in G(n, \mathbb{C}^\infty) \times \mathbb{C}^\infty : v \in V\}.$$

By abuse of notation, we denote the rank n_i tautological bundle of $F(n_1, \ldots, n_k, \mathbb{C}^\infty)$ to be

$$\tilde{T}^{n_i} = \{(P, v) \in F(n_1, \ldots, n_k, \mathbb{C}^\infty) \times \mathbb{C}^\infty : v \in \rho_i(P)\},$$

where $\rho_i : F(n_1, \ldots, n_k, \mathbb{C}^\infty) \to G(n_i, \mathbb{C}^\infty)$ is the previously defined projection map.

So in particular,

$$f^*\tilde{T}^{n_1} = \{(p, x) \in M \times \tilde{T}^{n_1} : f(p) = \pi(x)\}$$

where $\pi : \tilde{T}^{n_1} \to F(n_1, \ldots, n_k, \mathbb{C}^\infty)$ is the projection map of the tautological bundle.

We can now generalize theorem 5.6 in [9].

Theorem 8.5. *Any filtration $E_1 \subset \ldots \subset E_k$ over a paracompact base space B admits k bundle maps $f_1, \ldots, f_k$ such that*

$$\begin{array}{ccccccc}
E_1 & \xrightarrow{i} & E_2 & \xrightarrow{i} & \ldots & \xrightarrow{i} & E_k \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & & \downarrow{\scriptstyle f_k} \\
\tilde{T}^{n_1} & \xrightarrow[j]{} & \tilde{T}^{n_2} & \xrightarrow[j]{} & \ldots & \xrightarrow[j]{} & \tilde{T}^{n_k}
\end{array}$$

commutes, where i, j are the inclusion maps.

Proof. The proof follows that of lemma 5.3 in [9].

Since B is paracompact, there exists a locally finite open cover $\{U_j\}_{j=1}^\infty$ such that each $E_k|_{U_j}$ is trivial. There also exists another such open cover $\{V_j\}_j$ with $\bar{V}_j \subset U_j$. Similarly, there exists $\{W_j\}_j$ such that $\bar{W}_j \subset V_j$.

Let $\lambda_j : B \to \mathbb{R}$ be a bump function that takes on the value of 1 on $\bar{W}_j$ and 0 outside of V_j. Since $E_k|_{U_j}$ is trivial, there exists maps

$$h_{ij} : E_i|_{U_j} \to \mathbb{C}^{n_i}$$

which are linear isomorphisms when restricted to each fiber in the domain, with $1 \le i \le k$.

Next, set

$$h'_{ij} : E_i \to \mathbb{C}^{n_i}$$

which sends e to $\lambda_j(\pi(e)) \cdot h_{ij}(e)$. This is linear on each fiber, since on each fiber, $\lambda_j(\pi(e))$ is constant.

Set

$$\hat{f}_i : E_i \to \mathbb{C}^\infty$$

which sends e to

$$(h'_{11}(e), h'_{12}(e), \ldots))$$

$$\in \mathbb{C}^{n_i} \times \mathbb{C}^{n_i} \times \ldots$$

$$\cong \mathbb{C}^\infty.$$

Finally, let $\pi_i : E_i \to B$ be the projection maps, and $V_j(e)$ be the fiber in E_j passing through $\pi_i(e)$ for $e \in E_i$, j arbitrary. Then set

$$f_i : E_i \to \tilde{T}^{n_i}$$

which sends e to

$$(\hat{f}_1(V_1(e)) \subset \ldots \subset \hat{f}_k(V_k(e)), \hat{f}_i(e)).$$

This completes the construction of the f_i and it is readily verified that the diagram commutes with these maps.

□

We now generalize theorem 5.7 in [9].

Definition 8.6. *Two bundle maps $f, g : E \to F$ are called bundle homotopic if there is a homotopy $H : E \times [0,1] \to F$ between them such that for each $t \in [0,1]$, $H(\cdot, t)$ is a bundle map.*

Theorem 8.7. *Let $E_1 \subset \ldots \subset E_k$ be a bundle filtration of ranks $n_1 < \ldots < n_k$ over a common paracompact base space B. Let $\{f_1, \ldots, f_k\}$ and $\{g_1, \ldots, g_k\}$ be bundle maps that make the following diagram commute when inserted into the vertical arrows:*

$$\begin{array}{ccccccc} E_1 & \xrightarrow{i} & E_2 & \xrightarrow{i} & \dots & \xrightarrow{i} & E_k \\ \downarrow & & \downarrow & & & & \downarrow \\ \tilde{T}^{n_1} & \xrightarrow[j]{} & \tilde{T}^{n_2} & \xrightarrow[j]{} & \dots & \xrightarrow[j]{} & \tilde{T}^{n_k} \end{array}$$

Then each f_i and g_i are bundle homotopic via k homotopies

$$H_i : E_i \times [0,1] \to \tilde{T}^{n_i}$$

such that for each t, the k maps $H_i(\cdot, t)$ make the diagram commute.

The proof is entirely similar to that of theorem 5.7 in [9] and we omit the details.

Corollary 8.8. *Let $E_1 \subset \ldots \subset E_k$ be a bundle filtration over a paracompact base space B of ranks $n_1 < \ldots < n_k$. Then this filtration determines a unique homotopy class of maps $f : M \to F(n_1, \ldots, n_k, \mathbb{C}^\infty)$.*

This follows immediately from theorems 8.5 and 8.7.

We can now prove the main theorem.

Proof of Theorem 8.3. It remains to show that given two homotopic maps

$$f, g : B \to F(n_1, \ldots, n_k, \mathbb{C}^\infty),$$

they correspond to isomorphic bundle filtrations over B.

We will start with the case $k = 2$. Let $G \to P \to F(n_1, n_2, \mathbb{C}^\infty)$ be the principal bundle with fiber G the parabolic subgroup of $GL(n_2, \mathbb{C})$ of block upper triangular matrices, with diagonal blocks of size n_1 and $n_2 - n_1$.

Recall the classical result, as found in [12] section 11, that given a principal bundle $P \to X$ and homotopic maps $f, g : B \to X$, that the pullback bundles f^*P and g^*P are isomorphic.

Setting $X = F(n_1, n_2, \mathbb{C}^\infty)$ and P the principal bundle above, we have that the principal bundle f^*P naturally induces two vector bundles E_1 and E_2 of ranks n_1, n_2, respectively, as follows.

Let G act on $\mathbb{C}^{n_1}$ by the natural action induced by the first $n_1 \times n_1$ diagonal block. Similarly G acts on $\mathbb{C}^{n_2}$ by the action of the whole matrix group. Then let E_1 and E_2 be the associated bundles obtained by taking the quotients of these two actions, respectively, on $f^*P \times \mathbb{C}^{n_1}$ and $f^*P \times \mathbb{C}^{n_2}$. This gives that $E_1 = f^*P \times_G \mathbb{C}^{n_1}$ is a sub-bundle of $E_2 = f^*P \times_G \mathbb{C}^{n_2}$. Similarly, we obtain bundles F_1 and F_2 from g^*P.

We can now clearly see that the following diagram commutes with the natural maps in each arrow:

$$\begin{array}{ccc} E_1 & \longrightarrow & E_2 \\ \downarrow & & \downarrow \\ F_1 & \longrightarrow & F_2 \end{array}$$

It thus follows that the filtrations obtained from f^*P and g^*P are isomorphic as in definition 8.2.

The proof for general k then follows similarly. □

We will describe two sub-bundles F_1, F_2 of the same vector bundle E as being *homotopic* if they are isomorphic as bundles and if their embeddings into E are bundle homotopic as bundle maps.

8.2 Computations of homotopy groups of infinite flag manifolds

Theorem 8.9. *In the real case, the higher homotopy groups $\pi_n F(1, 2, \mathbb{R}^\infty)$ are trivial for all $n \geq 2$.*

Proof. We begin with the case $n = 2$. Let $f : S^2 \to F(1, 2, \mathbb{R}^\infty)$ be any map. This corresponds to a line sub-bundle L of a plane bundle E over S^2. Since S^2 is simply connected, any line bundle must be trivial. Otherwise, there exists a circle where the restricted bundle is non-orientable, and contracting that circle to a point yields a contradiction.

So L must be trivial. Now, $E = L \oplus L^\perp$ and $L^\perp$ must also be trivial, thus E is trivial too. It remains to show that any two embeddings of L into E are bundle-homotopic. This follows from a winding number argument. If there is a non-trivial embedding of L into E, one can find a circle where the restricted bundle winds around a non-zero number of times. Contracting that circle yields again a contradiction.

Now for $n > 2$, we pass to the long exact sequence of homotopy groups from the fiber bundle structure of $F(1,2)$. We have

$$\cdots \longrightarrow \pi_n Gr(1,2) \longrightarrow \pi_n F(1,2,\mathbb{R}^\infty) \xrightarrow{\rho_{2*}} \pi_n G(2,\mathbb{R}^\infty) \longrightarrow \cdots$$

and $Gr(1,2) \cong \mathbb{R}P^1 \cong S^1$, so for $n > 2$, $\pi_n Gr(1,2) = 0$. So this gives $\pi_n F(1,2,\mathbb{R}^\infty) \cong \pi_n G(2,\mathbb{R}^\infty)$. But, $\pi_n G(2,\mathbb{R}^\infty) \cong \pi_{n-1} O(2) \cong \pi_{n-1} S^1 = 0$ for $n > 2$. Thus $\pi_n F(1,2,\mathbb{R}^\infty) = 0$ for $n > 2$ as required. □

In general, we have the following.

Theorem 8.10. *For the complex case,*

$$\pi_m F(n_1,\ldots,n_k,\mathbb{C}^\infty) \cong \pi_{m-1}(U(n_1) \times U(n_2 - n_1) \times \ldots \times U(n_k - n_{k-1})).$$

Proof. We will first start with the case $k = 2$. There is a principal $U(n_1) \times U(n_2 - n_1)$-bundle given by the projection

$$\pi : V(n_2,\mathbb{C}^\infty) \to F(n_1,n_2,\mathbb{C}^\infty)$$

sending

$$(u_1,\ldots,u_{n_1},u_{n_1+1},\ldots,u_{n_2})$$

to

$$(Span\{u_1,\ldots,u_{n_1}\}, Span\{u_1,\ldots,u_{n_2}\}).$$

The proof then follows immediately from the long exact sequence of homotopy groups for this principal bundle, since $V(n_2,\mathbb{C}^\infty)$ is contractible. The case of general k is identical.

□

In particular, we get that $\pi_3 F(1,3,\mathbb{R}^\infty) = 0$. This means that any two line sub-bundles of a trivial (the only possible) 3-bundle over S^3 are homotopic. It's interesting to note that there is another result in [6] that says that the homotopy class of nowhere vanishing vector fields over S^3 (without passing through the origin at each fiber) is in bijection with $\mathbb{Z}$. This shows that, even though one can obtain a line sub-bundle from a nowhere vanishing vector field, there is some subtlety that differentiates between the two.

9 Bundle Extension Problems

9.1 Sub-bundle extension problem

Let N, M be manifolds with N embedded in M. Consider the following diagram, where i is the embedding map.

$$\begin{array}{ccc} N & \xrightarrow{h} & F(k,n,\mathbb{R}^\infty) \\ {\scriptstyle i}\downarrow & \overset{g}{\nearrow} & \downarrow{\scriptstyle \rho_2} \\ M & \xrightarrow[f]{} & G(n,\mathbb{R}^\infty) \end{array}$$

If the surrounding square commutes, then the map g exists and makes the two triangles commute if and only if the following situation holds. Given the rank n bundle E_2 induced by f, which restricts to a bundle E_1 over N, and which has a rank k sub-bundle L given by h, then L extends to a sub-bundle over all of M.

One can find obstructions to this extension problem when passing to cohomology as in the following diagram.

$$\begin{array}{ccc} H^*(N) & \xleftarrow{h^*} & H^*(F(k,n,\mathbb{R}^\infty)) \\ {\scriptstyle i^*}\uparrow & \overset{g^*}{\swarrow} & \uparrow{\scriptstyle \rho_2^*} \\ H^*(M) & \xleftarrow[f^*]{} & H^*(G(n,\mathbb{R}^\infty)) \end{array}$$

As a first simple result, we have:

Lemma 9.1. *Let g exist so as to make everything commute, and let the embedding i be nullhomotopic. Then h^* must be the 0 map.*

Proof. Since i is nullhomotopic, i^* is the 0 map. Then $h^* = i^* \circ g^* = 0$. $\square$

So in this special case, when i is nullhomotopic, the obstruction is whenever

$$h^* : H^r(F(k, n, \mathbb{R}^\infty)) \to H^r(N)$$

is not the 0 map for some r.

As a hand-wavy example, let $N = S^1$, $M = T^2$ the 2-torus, $k = 1$, $n = 2$. Let i and f be nullhomotopic. Thus the bundle over T^2 is trivial. If $h : S^1 \to F(1, 2)$ is also nullhomotopic, the picture is that of a contractible circle on the torus, the ambient bundle of T^2 being the tangent bundle (since it is trivial), and parallel lines placed at each point of the circle representing the trivial sub-bundle. Since the winding number of these lines is 0, one easily sees that this sub-bundle extends over the entire torus in a simple way.

However, if h is not nullhomotopic, then the lines about the circle have non-zero winding number, which makes it clear that they cannot extend continuously to cover the whole torus, as they cannot extend to the entire interior disc of the circle.

One also notices that if the embedding of S^1 is non-trivial, e.g. one of the standard generators of homology, then by simply "sliding" the knot around the other generating circle, we cover the whole torus as in a foliation, and so any sub-bundle on the knot extends to the whole torus.

Next we turn to studying possible extensions of the tangent bundle of complex projective spaces.

Theorem 9.2. *The tangent bundle $T\mathbb{C}P^{n-1}$ does not extend to a rank $n-1$ complex sub-bundle of $T\mathbb{C}P^n$.*

Proof. By the Lefschetz hyperplane theorem for complex projective varieties, if $Y \subset X$, X and n-dimensional complex projective algebraic variety and Y a hyperplane section of X such that $X - Y$ is smooth, then, in particular, we have that $i^* : H^k(X, \mathbb{Z}) \to H^k(Y, \mathbb{Z})$ is injective for $k \leq n - 1$.

Setting $X = \mathbb{C}P^n$ and $Y = \mathbb{C}P^{n-1}$ with the natural inclusion, the above result holds. Now assume TY, a sub-bundle of $TX|_Y$, has a rank $n-1$ complex bundle extension E over X, i.e. $E|_Y = TY$. Then $i^*c(E) = c(TY)$, and by injectivity we get $c(E) = c(TY) = (1+a)^n$ for a a generator of $H^2(\mathbb{C}P^{n-1}, \mathbb{Z})$.

It turns out that $(1+a)^n = 1 + na + \binom{n}{2}a^2 + \ldots + na^{n-1}$, with the last expected term of the expansion killed due to dimension reasons. Now let N be a complementary sub-bundle to E inside TX, i.e. $c(E)c(N) = c(TX)$. N must be a line bundle with $c(N) = 1 + a$. So

$$\begin{aligned} c(E)c(N) &= (1+a)^n(1+a) \\ &= (1 + na + \binom{n}{2}a^2 + \ldots + na^{n-1})(1+a) \\ &= 1 + (n+1)a + \binom{n+1}{2}a^2 + \ldots + \binom{n+1}{n-1}a^{n-1} + na^n, \end{aligned}$$

yet

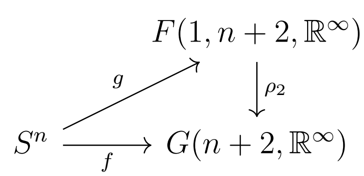
$$\begin{aligned} c(TX) &= (1+a)^{n+1} \\ &= 1 + (n+1)a + \binom{n+1}{2}a^2 + \ldots + \binom{n+1}{n-1}a^{n-1} + (n+1)a^n, \end{aligned}$$

a contraction.

□

9.2 Extension of sub-bundles over spheres

Now consider the following commutative diagram involving the n-sphere S^n.



I want to show that, given a homotopy class of f, there is at most one homotopy class of the lift g that makes the diagram commute up to homotopy.

This is equivalent to showing the following.

Proposition 9.3. *For all $n \geq 1$, the induced homomorphism*

$$\rho_{2*} : \pi_n F(1, n+2, \mathbb{R}^\infty) \to \pi_n G(n+2, \mathbb{R}^\infty)$$

is injective.

This implies that, if there is such a lift g, then any two line sub-bundles inside the rank $n+2$ bundle E induced by f over S^n are homotopic, in the previously defined sense that their bundle maps are bundle homotopic.

Proof. We have the following long exact sequence for fiber bundles:

$$\cdots \longrightarrow \pi_n Gr(1, n+2) \longrightarrow \pi_n F(1, n+2, \mathbb{R}^\infty) \xrightarrow{\rho_{2*}} \pi_n G(n+2, \mathbb{R}^\infty) \longrightarrow \cdots$$

Now since $Gr(1, n+2) \cong \mathbb{R}P^{n+1}$, and $\pi_n \mathbb{R}P^{n+1} \cong \pi_n S^{n+1} = 0$, This proves that π_* is injective since its kernel is 0. □

In particular, we get the following result. By two line bundles of the same orientability, we mean that either both are orientable or they both aren't.

Corollary 9.4. *If S^n is embedded in a manifold M, E is a rank $n+2$ bundle over M containing a line sub-bundle L, and L' is a line sub-bundle of $E|_{S^n}$ of the same orientability of L, then L' extends to another line sub-bundle over all of M.*

Proof. Let L'' be the restriction of L to S^n inside $E|_{S^n}$. Then by the previous proposition, L'' is homotopic to L'. So we can homotope L to a new line sub-bundle $\tilde{L}$ of E such that its restriction equals L'. □

10 Miscellaneous Problems and Results

We first state the following result, called the "real cancellation theorem", as a useful tool for later on.

Lemma 10.1. *If E is a rank r vector bundle over M, with $r > n = dim(M)$, then E has a rank $r - n$ trivial sub-bundle. Furthermore, for any other rank r vector bundle F and for any $k \in \mathbb{N}$, if $E \oplus \varepsilon_M^k \cong F \oplus \varepsilon_M^k$, then in fact $E \cong F$.*

The proofs of the following theorems are attributed to Michael Albanese and Jason DeVito on MathOverflow, to whom I extend my gratitude for their time and answers to my questions. [2] [3]

I began by asking for examples and properties of spaces that admit only trivial vector bundles, such as S^3. We can rule out closed manifolds of dimensions 1 and 2: if M is such a manifold, we would get $H^1(M, \mathbb{Z}_2) = 0$ since this group classifies line bundles. This implies that every bundle of any rank is orientable over M. Furthermore, orientable rank two bundles over M are classified by $H^2(M, \mathbb{Z})$, which must equal 0. These two points contradict the fact that in dimensions 1 and 2, either H^1 or H^2 must be non-zero.

We now look at the case where $dim(M) = 3$.

Theorem 10.2. *Let M be a closed 3-manifold. Every vector bundle over M is trivial if and only if M is an integral homology sphere.*

Proof. Let M have only trivial vector bundles. By Poincaré duality and the previous results, $H_1(M, \mathbb{Z}) \cong H^2(M, \mathbb{Z}) = 0$, and we have $H^3(M, \mathbb{Z}) \cong \mathbb{Z}$. Thus M is an integral homology sphere.

Now let M be an integral homology sphere. Then, since $H^1(M, \mathbb{Z}) \cong H^2(M, \mathbb{Z}) = 0$, all bundles of rank 1 and 2 are trivial. Let E be a rank 3 bundle. Since it is orientable, we can define its Euler class $e(E)$. As $dim(M) = rank(E)$, one result is that $e(E)$ is the only obstruction to a nowhere zero section. Another result is that since E has odd rank, $e(E)$ must be two-torsion, see section 9 in [9]. But $e(E) \in H^3(M, \mathbb{Z}) \cong \mathbb{Z}$ which is torsion-free, hence $e(E) = 0$. This implies that E has a trivial line sub-bundle, i.e. $E \cong E_0 \oplus \varepsilon^1$ with E_2 of rank 2. But E_0 must then be trivial, so E is trivial as well.

Lastly, assume E has rank greater than 3. Then by the real cancellation theorem above,

[2]https://mathoverflow.net/q/416977/143629

[3]https://math.stackexchange.com/q/4409482/354855

$E \cong E_0 \oplus \varepsilon^k$ for some rank 3 bundle E_0. Since E_0 has just been shown to be trivial, E is also trivial. Thus every bundle over M is trivial. □

Theorem 10.3. *Let M be a closed manifold with only trivial vector bundles. Then each of the following hold:*

1. *M must be orientable*
2. *M must have odd dimension*
3. *The first integral homology group is 0*
4. *M is a rational homology sphere.*

Proof. Property 1 has been shown in the previous theorem.

Property 2: If M were even dimensional and orientable, we can construct a non-trivial bundle as follows. One can define a map $f : M \to S^n$, where $n = dim(M)$, of degree 1. Since TS^n has non-trivial Euler class, so does f^*TS^n, hence is non-trivial.

Property 3: Assume that $H_1(M, \mathbb{Z}) \neq 0$, and that it contains a torsion element. By the universal coefficient theorem, $H^2(M, \mathbb{Z})$ must also contain a torsion element. Since H^2 classifies complex line bundles, there is a non-trivial map $f : M \to \mathbb{C}P^\infty$ corresponding to the torsion element. Pulling back the tautological bundle over $\mathbb{C}P^\infty$ gives a vector bundle over M whose Euler class is this same torsion element. Thus this vector bundle is non-trivial.

So wlog $H_1(M, \mathbb{Z})$ is torsion free. But being non-zero implies that $H^1(M, \mathbb{Z}_2)$ is non-trivial, which gives a non-trivial line bundle over M, a contradiction.

Property 4: Given that M is orientable and odd dimensional, assume that $H_k(M, \mathbb{Q}) \neq 0$ for some $0 < k < n$. Using both Poincaré duality and universal coefficients, we have that $H^k(M, \mathbb{Q})$ and $H^{n-k}(M, \mathbb{Q})$ must be non-trivial as well. Jason DeVito then cites a paper of Belegradek and Kapovitch [1] which shows there is a vector bundle over such an M with non-trivial Euler class in either degree d or $n - d$, whichever is even. But one of them must be even due to parity reasons, so we reach a contradiction since this produces a non-trivial vector bundle over M. □

The proofs of the next two results are beyond the scope of this thesis and will be omitted, but can be found in the two previously mentioned posts.

Theorem 10.4. *Let M be a closed simply connected manifold of dimension n which admits only trivial vector bundles. Then M cannot be a $\mathbb{Z}_2$-homology sphere, unless $n = 3$.*

A brief sketch of the proof involves working wlog with $n = dim(M) \geq 5$, and M being orientable and odd dimensional. For contradiction one assumes that all the torsion in $H^*(M, \mathbb{Z})$ is of odd order, and then constructing appropriate non-trivial bundles over M as pullbacks of non-trivial bundles over S^n that have even torsion.

Corollary 10.5. *Every closed simply connected 5-manifold admits a non-trivial vector bundle.*

Essentially the proof of the corollary would be to show that if a 5-manifold M with only trivial vector bundles exists, it must be a $\mathbb{Z}_2$-homology sphere, and thus the result would follow from the previous theorem.

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