CONVERGENCE RESULTS ON FOURIER SERIES IN ONE VARIABLE ON THE UNIT CIRCLE

Ryan Ferns

Master of Science

Department of Mathematics and Statistics

McGill University

Montreal,Quebec

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ABSTRACT

This thesis is an analysis of convergence results on Fourier series. Convergence of Fourier series is studied in two ways in this thesis. The first way is in the context of Banach spaces, where the set of functions is restricted to a certain Banach space. Then the problem is in determining whether the Fourier series of a function can be represented as an element of that Banach space. The second way is in the context of pointwise convergence. Here, the problem is in determining what conditions need to be placed on an arbitrary function for its Fourier series to converge at a point.

ABRÉGÉ

Cette thèse est une analyse de résultats de convergence sur les séries de Fourier. On a deux façons d'étudier les séries de Fourier. La première façon est dans le contexte des espaces de Banach, où l'ensemble de fonctions est limité à un certain espace de Banach. Alors le problème est en déterminant si la série de Fourier d'une fonction peut être représentée comme un élément de cet espace de Banach. La deuxième façon est dans le contexte de convergence simple. Ici, le problème est en déterminant quelle conditions doivent être placées sur une fonction arbitraire pour que sa série de Fourier converge à un point.

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CHAPTER 1
Introduction

This thesis is a survey of results concerning the summability and convergence of Fourier series. Specifically, the Fourier series of 2π-periodic complex-valued functions will be examined. Note that summability is studied in the technical sense of summability kernels with the focus only on one particular summability kernel known as the Fejér kernel. In this case, summability is also referred to as Cesàro summability and corresponds to the study of the Cesàro sums of Fourier series. This is as opposed to the usual idea of convergence which corresponds to the study of the partial sums of Fourier series. Furthermore, the concepts of convergence and summability are related by the fact that the Cesàro sums are just the averages of the partial sums.

Here, summability and convergence will be explored in two contexts. The first context is in norm and the second context is pointwise. These will be explained further down. In both contexts, the problem of summability is much simpler than the problem of convergence. In fact, many of the results, that we would like to hold for convergence but do not, actually do hold for summability.

Chapter 2 is a review of background information that will be needed later in the thesis. The topics include subjects from Functional Analysis, Riemann integration, and Measure Theory.

Chapter 3 is an analysis of summability and convergence in norm of Fourier series. In this chapter, a special class of Banach spaces of functions called homogeneous Banach spaces is introduced. These Banach spaces will be the primary spaces of functions that will be used throughout the text. Then the Fourier series of a function is defined as well as the n-th partial and Cesàro sums of Fourier series. The n-th partial sums are used for the convergence of Fourier series and the n-th Cesàro sums are used for the summability of Fourier series. Now, let B be a homogeneous Banach space and $f \in B$. It can subsequently be shown that the n-th partial and Cesàro sums of the Fourier series of f belong to B. Summability in norm is concerned with showing that the sequence of Cesàro sums converges in B and convergence in norm is concerned with showing that the sequence of partial sums converges in B. Then the rest of the chapter deals mainly with what conditions need to be placed on f and B for summability and convergence in norm to occur.

Chapter 4 is an analysis of pointwise summability and convergence of Fourier series. Unlike the previous chapter where everything was in the setting of Banach spaces, this chapter deals with the usual concepts of summability and convergence. Pointwise summability is concerned with showing that the sequence of Cesàro sums converges at a point and pointwise convergence is concerned with showing that the sequence of partial sums converges at a point. This chapter deals mainly with what conditions need to be placed on a function f for its Fourier series to be convergent or summable at a point.

Chapter 5 is about a test, created by Boris Korenblum, for pointwise convergence that generalizes the Dirichlet-Jordan and Dini-Lipschitz pointwise convergence tests. This test involves Korenblum's theory of κ-entropy, which will also be presented in this chapter.

Before preceding to the next chapter, two theorems about the pointwise convergence of Fourier series will be stated. These two theorems are the results of greatest importance in this entire subject area. Unfortunately, these theorems are beyond the scope of this thesis and will not be discussed in further detail. The following theorems are from (Katznelson, 2004, P. 80) and (Edwards, 1979, P. 170). All the notation will be introduced in Chapter 3.

Theorem 1.0.1 (Carleson). If $f \in L^2(\mathbb{T})$, then $\lim_{n\to\infty} S_n(f)(t) = f(t)$ m-a.e.

Theorem 1.0.2 (Carleson-Hunt). If $f \in L^p(\mathbb{T})$, where $1 < p < \infty$, then $\lim_{n\to\infty} S_n(f)(t) = f(t)$ m-a.e.

CHAPTER 2
Preliminaries

2.1 Banach Spaces and Functional Analysis

This section is a review of some definitions and results in Functional Analysis from (Folland, 1999). In this section, let $Z = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.1. Let X be a vector space over Z. A norm on X is a function $X \longrightarrow [0, \infty)$, $x \mapsto \|x\|$, $x \in X$ s.t.:

(i) $\|x\| = 0 \iff x = \vec{0}$, where $\vec{0}$ is the zero vector in X.

(ii) $\forall x \in X \ \forall \lambda \in Z$, $\|\lambda x\| = |\lambda| \|x\|$

(iii) $\forall x, y \in X$, $\|x + y\| \leq \|x\| + \|y\|$

$(X, \| \ \|)$ is called a normed vector space over Z.

Note. X is also a metric space, where $d(x, y) = \|x - y\|$ is called the metric induced by $\| \ \|$ or the norm metric. For any normed vector space $(X, \| \ \|)$, the norm is uniformly continuous because $\forall x, y \in X$, $|\|x\| - \|y\|| \leq \|x - y\| = d(x, y)$ and so if $\delta(\epsilon) = \epsilon$, then $d(x, y) < \delta(\epsilon) \Rightarrow |\|x\| - \|y\|| < \epsilon$. $\Longrightarrow$ If $\lim_{n \to \infty} x_n = x$ in X, then $\lim_{n \to \infty} \|x_n\| = \|x\|$.

Definition 2.1.2. If $(X, \| \ \|)$ is complete w.r.t. the norm metric, then it is called a Banach space.

Theorem 2.1.1. $(X, \| \ \|)$ is a Banach space $\iff$ every absolutely convergent series converges in X. ($\sum_{n=1}^{\infty} \|x_n\| < \infty \Rightarrow \exists x \in X$, $\lim_{n \to \infty} \| \sum_{k=1}^{n} x_k - x\| = 0$.)

Definition 2.1.3. Let $(X, \| \|_a)$ and $(Y, \| \|_b)$ be normed vector spaces over Z. $T\colon X \to Y$ is linear iff $\forall \lambda \in Z\ \forall x_1, x_2 \in X$, $T(\lambda x_1 + x_2) = \lambda T(x_1) + T(x_2)$. T is bounded if $\exists c \in [0, \infty)\ \forall x \in X$, $\|T(x)\|_b \leq c\|x\|_a$.

Definition 2.1.4. Let $T\colon (X, \| \|_a) \to (Y, \| \|_b)$ be a bounded linear operator.

$$\|T\|_{op} = \sup_{\substack{x \in X \\ x \neq \vec{0}}} \frac{\|T(x)\|_b}{\|x\|_a} = \sup_{\substack{x \in X \\ \|x\|_a = 1}} \|T(x)\|_b = \inf\{c \in [0, \infty) | \forall x \in X, \|T(x)\|_b \leq c\|x\|_a\}.$$

Note. $T\colon (X, \| \|_a) \to (Y, \| \|_b)$ is bounded iff $\|T\|_{op} < \infty$.

Theorem 2.1.2. Let $L(X, Y) = \{T\colon X \to Y |\ T \text{ is a bounded linear operator.}\}$. Then $L(X, Y)$ is a normed vector space over Z. If $(Y, \| \|_b)$ is a Banach space, then $L(X, Y)$ is a Banach space.

Notation. Let $X^* = L(X, Z)$ be the set of bounded linear functionals, which is a Banach space by the previous theorem since Z is complete. X^* is called the dual of X. If $(X, \| \|_a) = (Y, \| \|_b)$, then let $L(X) = L(X, X)$.

Theorem 2.1.3 (Principle of Uniform Boundedness). Let $(X, \| \|_a)$ be a Banach space and $(Y, \| \|_b)$ be a normed vector space over Z. Let $\mathcal{F}$ be a family of bounded linear transformations from X to Y. Suppose that for each $x \in X$, $\{\|T(x)\|_b : T \in \mathcal{F}\}$ is bounded. Then $\{\|T\|_{op} : T \in \mathcal{F}\}$ is bounded.

2.2 Riemann Integration of Vector-Valued Functions

In this section, let $Z = \mathbb{R}$ or $\mathbb{C}$ and note that $[a, b]$ is always a finite closed interval in $\mathbb{R}$.

2.2.1 Riemann-Stieltjes Integrals

Before the Riemann integration of vector-valued functions is discussed in the next subsection, a review of some of the theory of Riemann-Stieltjes integrals will be presented from (Labute, 2003) and (Bartle & Sherbert, 2000). This is because the main results here can be generalized in the next subsection. Note that in this subsection all functions are Z-valued.

Definition 2.2.1. A partition of the closed interval $[a, b]$ is a subset $P = \{x_j\}_{j=0}^n$ of $[a, b]$, where $n \in \mathbb{N}$, $x_0 = a$, $x_n = b$, and $\forall j$ s.t. $1 \leq j \leq n$, $x_{j-1} < x_j$. The norm of a partition P is $\|P\| = \max\limits_{1 \leq j \leq n} \Delta x_j$, where $\Delta x_j = x_j - x_{j-1}$. If P and Q are partitions of $[a, b]$, then P is finer than Q if $Q \subseteq P$. Note that if P is finer than Q, then $\|P\| \leq \|Q\|$. A tagged partition of $[a, b]$ is a pair (P, t), where $P = \{x_j\}_{j=0}^n$ is a partition of $[a, b]$ and $t = \{t_j\}_{j=1}^n$ with $x_{j-1} \leq t_j \leq x_j$ is called a tag. If $(P, t), (Q, s)$ are tagged of partitions of $[a, b]$, then (P, t) is finer than (Q, s), i.e. $(P, t) > (Q, s)$, if P is finer than Q.

Definition 2.2.2. Let f and α be functions defined on $[a, b]$. If (P, t) is a tagged partition, then the Riemann-Stieltjes sum of f w.r.t. α for the tagged partition (P, t) is $S(P, t, f, \alpha) = \sum_{j=1}^n f(t_j)\Delta\alpha_j = \sum_{j=1}^n f(t_j)(\alpha(x_j) - \alpha(x_{j-1}))$.

Now, Riemann-Stieltjes integrability will be defined in two ways.

Definition 2.2.3. f is Riemann-Stieltjes integrable w.r.t. α, denoted by $f \in R(\alpha, a, b)$, if $\exists L \; \forall \epsilon > 0 \; \exists (Q, s) \; \forall (P, t) > (Q, s)$, $|S(P, t, f, \alpha) - L| < \epsilon$.

Definition 2.2.4. f is strictly Riemann-Stieltjes integrable w.r.t. α, denoted by $f \in R^*(\alpha, a, b)$, if $\exists L \; \forall \epsilon > 0 \; \exists \delta > 0 \; \forall (P, t)$, $\|P\| < \delta \Rightarrow |S(P, t, f, \alpha) - L| < \epsilon$.

Remark. In both cases, L is unique and is called the Riemann-Stieltjes integral of f w.r.t. α. L is denoted by $\int_a^b f(x)\, d\alpha(x)$.

It is easy to show that the second definition implies the first. When $\alpha(x) = x$, the two definitions agree and then f is called Riemann integrable, which is denoted by $f \in R(a,b)$. The second definition is the one that will be generalized in the next subsection.

Theorem 2.2.1 (Cauchy Criterion). $f \in R^*(\alpha, a, b) \iff$
$\forall \epsilon > 0\ \exists \delta > 0\ \forall (P,t),(P',t'), \|P\|, \|P'\| < \delta \Rightarrow |S(P,t,f,\alpha) - S(P',t',f,\alpha)| < \epsilon.$

Proof. ($\Rightarrow$) Let $f \in R^*(\alpha, a, b)$, $\int_a^b f(x)\, d\alpha(x) = L$, and $\epsilon > 0$. Then,

$\exists \delta > 0\ \forall (P,t),\ \|P\| < \delta \Rightarrow |S(P,t,f,\alpha) - L| < \frac{\epsilon}{2}$ and so

$\forall (P,t),(P',t'),\ \|P\|, \|P'\| < \delta \implies |S(P,t,f,\alpha) - S(P',t',f,\alpha)| \leq$
$|S(P,t,f,\alpha) - L| + |L - S(P',t',f,\alpha)| < 2\left(\frac{\epsilon}{2}\right) < \epsilon.$

$\therefore \forall \epsilon > 0\ \exists \delta > 0\ \forall (P,t),(P',t'), \|P\|, \|P'\| < \delta \implies$
$|S(P,t,f,\alpha) - S(P',t',f,\alpha)| < \epsilon$

($\Leftarrow$) By induction, a sequence of tagged partitions $\{(P_n, t_n)\}_{n=1}^{\infty}$ can be defined as follows: $\forall n \in \mathbb{N}, (P_{n+1}, t_{n+1}) > (P_n, t_n)$,
$\exists \delta_n > 0\ \forall (P,t), \|P\| < \delta_n \Rightarrow |S(P,t,f,\alpha) - S(P_n,t_n,f,\alpha)| < \frac{1}{n}$, and
$\|P_n\| < \delta_n$.

Let $n \in \mathbb{N}$. Then $\forall j \geq n$, $(P_j, t_j) > (P_n, t_n) \Rightarrow \|P_j\| \leq \|P_n\| < \delta_n \Rightarrow$
$|S(P_j,t_j,f,\alpha) - S(P_n,t_n,f,\alpha)| < \frac{1}{n}$.

Let $\epsilon > 0$ and $n \in \mathbb{N}$ be s.t. $\frac{1}{n} < \frac{\epsilon}{2}$. Then $\forall i,j \geq n$,

$$(P_i,t_i),(P_j,t_j) > (P_n,t_n) \implies |S(P_j,t_j,f,\alpha) - S(P_i,t_i,f,\alpha)|$$
$$\leq |S(P_j,t_j,f,\alpha) - S(P_n,t_n,f,\alpha)| + |S(P_n,t_n,f,\alpha) - S(P_i,t_i,f,\alpha)|$$
$$< 2\left(\frac{1}{n}\right) < 2\left(\frac{\epsilon}{2}\right) = \epsilon.$$

$$\therefore \forall \epsilon > 0\ \exists n \in \mathbb{N}\ \forall i,j \geq n, |S(P_j,t_j,f,\alpha) - S(P_i,t_i,f,\alpha)| < \epsilon$$

$\implies \{S(P_n, t_n, f, \alpha)\}_{n=1}^{\infty}$ is a Cauchy sequence in Z. Since Z is complete, the sequence $\{S(P_n, t_n, f, \alpha)\}_{n=1}^{\infty}$ converges in Z to a limit L. Since $\forall n \in \mathbb{N}\ \forall j \geq n, |S(P_j, t_j, f, \alpha) - S(P_n, t_n, f, \alpha)| < \dfrac{1}{n}$, then by letting $j \to \infty$, $\forall n \in \mathbb{N}, |S(P_n, t_n, f, \alpha) - L| = |L - S(P_n, t_n, f, \alpha)| \leq \dfrac{1}{n}$.

Now let $\epsilon > 0$ and $n \in \mathbb{N}$ be s.t. $\dfrac{1}{n} < \dfrac{\epsilon}{2}$. Let (P, t) be s.t. $\|P\| < \delta = \delta_n$

$$\implies |S(P, t, f, \alpha) - S(P_n, t_n, f, \alpha)| < \frac{1}{n}. \implies |S(P, t, f, \alpha) - L|$$

$$\leq |S(P, t, f, \alpha) - S(P_n, t_n, f, \alpha)| + |S(P_n, t_n, f, \alpha) - L| < 2\left(\frac{1}{n}\right) < \epsilon.$$

$$\therefore \exists L\ \forall \epsilon > 0\ \exists \delta > 0\ \forall (P, t), \|P\| < \delta \Rightarrow |S(P, t, f, \alpha) - L| < \epsilon.$$

$$\implies f \in R^*(\alpha, a, b) \text{ and } \int_a^b f(x)\, d\alpha(x) = L.$$

□

Note. The second part of the proof depends only on the completeness of Z.

Theorem 2.2.2 (Integration by Parts). If $f \in R(g, a, b)$, then $g \in R(f, a, b)$ and $\int_a^b f(x)\, dg(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x)\, df(x)$.

Theorem 2.2.3. Let α be a function on $[a, b]$ with a continuous derivative α' and f be a bounded function on $[a, b]$. If $g(x) = f(x)\alpha'(x)$ and $g \in R(a, b)$, then $f \in R(\alpha, a, b)$ and $\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\alpha'(x)\, dx$.

2.2.2 Riemann Integration of Vector-Valued Functions

This section is a review of the main results from (Katznelson, 2004, P. 295-296). In this section, let $(B, \| \|_B)$ be a Banach space over Z and $F\colon [a, b] \to B$. The following is a description of how the Riemann integral can be defined for

B-valued functions. The same notation can be used from the previous subsection like (tagged) partitions, etc.

Definition 2.2.5. If (P, t) is a tagged partition given as above, then the Riemann sum of f for the tagged partition (P, t) is $S(P, t, F) = \sum_{j=1}^{n} \Delta x_j F(t_j) = \sum_{j=1}^{n} (x_{j+1} - x_j) F(t_j)$.

Note. Unlike the previous section where the Riemann-Stieltjes sum is a number, $S(P, t, F)$ is a vector, i.e. $S(P, t, F) \in B$.

Definition 2.2.6. F is Riemann integrable, denoted by $F \in R(a, b)$, if $\exists L \in B \ \forall \epsilon > 0 \ \exists \delta > 0 \ \forall (P, t), \|P\| < \delta \Rightarrow \|S(P, t, F) - L\|_B < \epsilon$. L is unique and is called the Riemann integral of F. L is denoted by $\int_a^b F(x)\, dx$.

Note. While the Riemann integral of a Z-valued function is an element of Z, the Riemann integral of a B-valued function is actually an element of B. Also when $B = Z$, the definition of the Riemann integral in the last subsection coincides with the definition in this subsection. Also, the Cauchy Criterion still holds with the exact same proof except that the absolute value signs are replaced by $\| \ \|_B$ and the completeness of $(B, \| \ \|_B)$ is used instead of the completeness of Z.

Theorem 2.2.4 (Cauchy Criterion). $F \in R(a, b) \iff \forall \epsilon > 0 \ \exists \delta > 0 \ \forall (P, t), (P', t'), \ \|P\|, \|P'\| < \delta \Rightarrow \|S(P, t, F) - S(P', t', F)\|_B < \epsilon$.

Remark. Let F be continuous, i.e. $\forall x \in [a, b] \ \forall \epsilon > 0 \ \exists \delta_x(\epsilon) > 0$ s.t. $|y - x| < \delta_x(\epsilon) \Rightarrow \|F(y) - F(x)\|_B < \epsilon$. Note that since $[a, b]$ is compact and $F\colon [a, b] \to B$ is continuous, then $F\colon [a, b] \to B$ is uniformly continuous, i.e. $\forall \epsilon > 0 \ \exists \delta(\epsilon) > 0$ s.t. $|y - x| < \delta(\epsilon) \implies \|F(y) - F(x)\|_B < \epsilon$.

Corollary 2.2.5. If F is continuous, then $F \in R(a, b)$.

Proof. The Cauchy Criterion will be used to show that $F \in R(a,b)$. Let $\epsilon > 0$. Since F is uniformly continuous, then $\exists \delta = \delta(\epsilon) > 0$ s.t. $|y-x| < \delta(\epsilon) \Rightarrow \|F(y)-F(x)\|_B < \dfrac{\epsilon}{2(b-a)}$. Let (P,t) and (P',t') satisfy $\|P\|, \|P'\| < \delta$. Let (Q,s) be a tagged partition of $[a,b]$, where $Q = P \cup P'$ and s is any tag for Q. Note that $(Q,s) > (P,t)$ and that $(Q,s) > (P',t')$.

Now consider $\|S(Q,s,F) - S(P,t,F)\|_B$. Let $Q = \{x_j\}_{j=0}^n$, $P = \{y_j\}_{j=0}^m$ and so $S(Q,s,F) = \sum_{j=1}^n \Delta x_j F(s_j)$. Since $(Q,s) > (P,t)$, then $S(P,t,F)$ can be written as $S(P,t,F) = \sum_{j=1}^n \Delta x_j F(u_j)$ for a tag u except that now it may not be true that $x_{j-1} \le u_j \le x_j$. However, it is true that $\forall\ 1 \le j \le n\ \exists k$ s.t. $1 \le k \le m$ and $y_{k-1} \le s_j, u_j \le y_k$.

$$\Longrightarrow\ |s_j - u_j| \le \Delta y_k \le \|P\| < \delta \Rightarrow \|F(s_j) - F(u_j)\|_B < \frac{\epsilon}{2(b-a)}$$

$$\Longrightarrow\ \|S(Q,s,F) - S(P,t,F)\|_B = \left\| \sum_{j=1}^n \Delta x_j F(s_j) - \sum_{j=1}^n \Delta x_j F(u_j) \right\|_B$$

$$= \left\| \sum_{j=1}^n \Delta x_j (F(s_j) - F(u_j)) \right\|_B \le \sum_{j=1}^n \|\Delta x_j (F(s_j) - F(u_j))\|_B$$

$$= \sum_{j=1}^n \Delta x_j \|F(s_j) - F(u_j)\|_B < \sum_{j=1}^n \Delta x_j \left(\frac{\epsilon}{2(b-a)}\right) = \left(\frac{\epsilon}{2(b-a)}\right)\left(\sum_{j=1}^n \Delta x_j\right)$$

$$= \left(\frac{\epsilon}{2(b-a)}\right)(x_n - x_0) = \left(\frac{\epsilon}{2(b-a)}\right)(b-a) = \frac{\epsilon}{2}.$$

Similarly, $\|S(Q,s,F) - S(P',t',F)\|_B < \dfrac{\epsilon}{2}$. Thus, $\|S(P,t,F) - S(P',t',F)\|_B$ $\le \|S(P,t,F) - S(Q,s,F)\|_B + \|S(Q,s,F) - S(P',t',F)\|_B < 2\left(\frac{\epsilon}{2}\right) = \epsilon$.

$\therefore \forall \epsilon > 0\ \exists \delta > 0\ \forall (P,t), (P',t'), \|P\|, \|P'\| < \delta \Rightarrow \|S(P,t,F) - S(P',t',F)\|_B < \epsilon$

By the Cauchy Criterion, $F \in R(a,b)$. $\square$

Note. Define $H\colon [a,b] \to [0,\infty)$ by $H(x) = \|F(x)\|_B$. H is continuous because it is a composition of the continuous functions $F\colon [a,b] \to B$ and $\|\ \|_B : B \to [0,\infty)$.

Here are the main properties of the Riemann integral.

Theorem 2.2.6. Let $F, G \in R(a,b)$ and $c_1, c_2 \in Z$. Then,

1) $\int_a^b (c_1 F(x) + c_2 G(x))\,dx = c_1 \int_a^b F(x)\,dx + c_2 \int_a^b G(x)\,dx$
2) $\forall c \in (a,b)$, $\int_a^b F(x)\,dx = \int_a^c F(x)\,dx + \int_c^b F(x)\,dx$
3) $\left\| \int_a^b F(x)\,dx \right\|_B \leq \int_a^b \|F(x)\|_B\,dx$

Remark. In 3), the integral on the RHS is the Riemann integral of the real-valued function $H(x) = \|F(x)\|_B$.

Proposition 2.2.7. Let F be continuous and $k\colon [a,b] \to Z$ be continuous. Define $G\colon [a,b] \to B$ by $G(x) = k(x)F(x)$. Then G is continuous and $G \in R(a,b)$.

Proof. $k(x)$ and $H(x)$ as described above are continuous on $[a,b]$. $\Rightarrow$ They are both bounded, i.e. $\exists M > 0$ s.t. $\forall x \in [a,b]$, $|k(x)|, H(x) \leq M$.

$$[G(y) - G(x)] = k(y)F(y) - k(x)F(x) = k(y)[F(y) - F(x)] + [k(y) - k(x)]F(x)$$

$$\implies \|G(y) - G(x)\|_B = \|k(y)[F(y) - F(x)] + [k(y) - k(x)]F(x)\|_B$$

$$\leq |k(y)|\ \|F(y) - F(x)\|_B + |k(y) - k(x)|\ \|F(x)\|_B$$

$$\leq |k(y)|\ \|F(y) - F(x)\|_B + H(x)\ |k(y) - k(x)|$$

$$\leq M[\|F(y) - F(x)\|_B + |k(y) - k(x)|]$$

Let $\epsilon > 0$. Since F and k are continuous, $\exists \delta_x(\epsilon) > 0$ s.t. $|y - x| < \delta_x(\epsilon) \Rightarrow$ $\|F(y) - F(x)\|_B, |k(y) - k(x)| < \dfrac{\epsilon}{2M}$. By the previous inequality, if $|y - x| < \delta_x(\epsilon)$, then $\|G(y) - G(x)\|_B < \epsilon$.

$\therefore \forall x \in [a, b]\ \forall \epsilon > 0\ \exists \delta_x(\epsilon) > 0$ s.t. $|y - x| < \delta_x(\epsilon) \Rightarrow \|G(y) - G(x)\|_B < \epsilon$

Hence, G is continuous and by Corollary 2.2.5, $G \in R(a, b)$. □

2.3 Measure Theory

This section is a short review of Measure Theory from (Folland, 1999). In this section, $(X, \mathfrak{M}, \mu)$ is a measure space, where X is a set, $\mathfrak{M}$ is a σ-algebra on X, and μ is a measure on $\mathfrak{M}$.

2.3.1 The Riemann Integral and The Lebesgue Integral

Here, $(X, \mathfrak{M}, \mu) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$, where m is Lebesgue measure.

Note. The following convention will be made for the remainder of the text. Unless otherwise stated, two functions f and g defined on $\mathbb{R}$ are said to be equal if they are equal m-a.e. .

In the following theorem, $(X, \mathfrak{M}, \mu) = (I, \mathcal{B}_{\mathbb{R}} \cap I, m)$ for any finite closed interval I in $\mathbb{R}$ and $W = \mathbb{R}$ or $\mathbb{C}$.

Theorem 2.3.1. 1) If $f \colon I \to W$ is a bounded Riemann integrable function, then $f \in L^1(I, \mathcal{B}_{\mathbb{R}} \cap I, m)$ and $\int_I f\, dm = \int_I f(x)\, dx$, where the LHS of the equality is the Lebesgue integral of f w.r.t. m over I and the RHS of the equality is the Riemann integral of f over I.

2) If $f \colon I \to W$ is a bounded function, then f is Riemann integrable iff the set $N = \{x \in I |\ f$ is not continuous at $x.\}$ is a m-null set, i.e. f is continuous m-a.e.

Remark. Due to this theorem, no distinction will be made between the Riemann and Lebesgue integrals of Riemann integrable functions. In particular, if f is a bounded continuous function on I, then f is Riemann integrable on I and so by the Theorem, the Riemann and Lebesgue integrals of f agree.

Notation. Let $f\colon \mathbb{R} \to W$ be measurable. For simplicity, $\int f(t)\, dm(t)$ will be denoted by $\int f(t)\, dt$. Let $A \in \mathcal{B}_{\mathbb{R}}$ and χ_A be the indicator function of A, which is measurable. Let $a, b \in \mathbb{R}$ and $a \le b$. Let $g_1, g_2 \in \{\chi_{(a,b)}, \chi_{[a,b)}, \chi_{(a,b]}, \chi_{[a,b]}\}$. Then $fg_1 = fg_2$ m-a.e. because $\{x \in \mathbb{R} : fg_1 \neq fg_2\} \subseteq \{a, b\}$ and $m(\{a,b\}) = 0$.

$$fg_1 = fg_2 \text{ } m\text{-a.e.} \implies \int f(t) g_1(t)\, dt = \int f(t) g_2(t)\, dt$$

Let $g \in \{\chi_{(a,b)}, \chi_{[a,b)}, \chi_{(a,b]}, \chi_{[a,b]}\}$ and define $\int_a^b f(x)\, dx = \int f(x) g(x)\, dx$ and $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$. Note that when $a = b$, $\int_a^b f(x)\, dx = \int_b^a f(x)\, dx = 0$.

Note. If I is any finite interval in $\mathbb{R}$, then

$f \in L^1(I, \mathcal{B}_{\mathbb{R}} \cap I, m) \iff f\chi_I \in L^1(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$.

Definition 2.3.1. $f\colon \mathbb{R} \to W$ is locally integrable if f is measurable and f is integrable over any compact set. Let $L^1_{\text{loc}}(\mathbb{R})$ be the set of locally integrable functions.

Note. Let $f \in L^1_{\text{loc}}(\mathbb{R})$ and $A \in \mathcal{B}_{\mathbb{R}}$ be s.t. A is bounded in $\mathbb{R}$. Then $\exists N \in \mathbb{N}$ s.t. $A \subseteq [-N, N]$. Since $[-N, N]$ is compact, then $\int_A |f(x)|\, dx \le \int_{-N}^N |f(x)|\, dx < \infty$. Thus, f is integrable over any bounded set in $\mathcal{B}_{\mathbb{R}}$. Also, by the argument used here, if $f\colon \mathbb{R} \to W$ is measurable, then to show $f \in L^1_{\text{loc}}(\mathbb{R})$, it is enough to show that f is integrable over any finite interval since any compact subset A of $\mathbb{R}$ is closed and bounded and so $\exists N \in \mathbb{N}$ s.t. $A \subseteq [-N, N]$.

The following proposition just states that certain properties of the Riemann integral still hold for the Lebesgue integral over finite intervals.

Proposition 2.3.2. Let $f \in L^1_{\text{loc}}(\mathbb{R})$ and $a, b \in \mathbb{R}$.

1) $\forall c \in \mathbb{R}, \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

2) $\forall c \in \mathbb{R}, \int_a^b f(x)\,dx = \int_{a-c}^{b-c} f(x+c)\,dx$

3) $\forall c \in \mathbb{R}, \int_a^b f(x)\,dx = \int_{c-b}^{c-a} f(c-x)\,dx$

Note. By the previous note, if $f \in L^1_{\text{loc}}(\mathbb{R})$, then $\forall a \in \{-1, 1\}$ $\forall c \in \mathbb{R}$, $g \in L^1_{\text{loc}}(\mathbb{R})$, where $g \colon \mathbb{R} \to W$ is given by $g(x) = f(ax + c)$. This is because if f is integrable over any finite interval, then g is integrable over any finite interval by Proposition 2.3.2.(2-3).

Remark. The only hypothesis that is needed for Proposition 2.3.2.(2-3) is $f\chi_{[a,b]} \in L^1(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$, which is equivalent to $f \in L^1([a,b], \mathcal{B}_{\mathbb{R}} \cap [a,b], m)$. Also, note that $[a,b]$ could be replaced by (a,b), $[a,b)$, or $(a,b]$.

Theorem 2.3.3. If $f \in L^1_{\text{loc}}(\mathbb{R})$, then $\displaystyle\lim_{h\to 0^+} \frac{1}{2h} \int_{x-h}^{x+h} |f(t) - f(x)|\,dt = 0$ m-a.e. .

Corollary 2.3.4. If $f \in L^1_{\text{loc}}(\mathbb{R})$, then $\displaystyle\lim_{h\to 0^+} \frac{1}{2h} \int_{x-h}^{x+h} f(t)\,dt = f(x)$ m-a.e. .

Definition 2.3.2. If $f \in L^1_{\text{loc}}(\mathbb{R})$, then the Lebesgue set of f is $L_f = \left\{ x \in \mathbb{R} \,\middle|\, \displaystyle\lim_{h\to 0^+} \frac{1}{2h} \int_{x-h}^{x+h} |f(t) - f(x)|\,dt = 0 \right\}$. By the theorem, $m(L_f^c) = 0$, where L_f^c is the complement of L_f.

Corollary 2.3.5. If $f \in L^1_{\text{loc}}(\mathbb{R})$, then

$$\lim_{h\to 0^+} \frac{1}{h} \int_0^h \left| \left(\frac{f(x+t) + f(x-t)}{2} \right) - f(x) \right| dt = 0 \ m\text{-a.e.}$$

Proof. Let $x \in L_f$.

$$\begin{aligned}
&\int_0^h \left|\left(\frac{f(x+t)+f(x-t)}{2}\right) - f(x)\right| dt \\
&= \int_0^h \left|\left(\frac{f(x+t)-f(x)}{2}\right) + \left(\frac{f(x-t)-f(x)}{2}\right)\right| dt \\
&\leq \int_0^h \left[\left|\frac{f(x+t)-f(x)}{2}\right| + \left|\frac{f(x-t)-f(x)}{2}\right|\right] dt \\
&= \int_0^h \left|\frac{f(x+t)-f(x)}{2}\right| dt + \int_0^h \left|\frac{f(x-t)-f(x)}{2}\right| dt \\
&= \frac{1}{2}\left[\int_0^h |f(x+t)-f(x)|\, dt + \int_0^h |f(x-t)-f(x)|\, dt\right] \\
&= \frac{1}{2}\left[\int_x^{x+h} |f(t)-f(x)|\, dt + \int_{x-h}^{x} |f(t)-f(x)|\, dt\right] \text{ [by Proposition 2.3.2]} \\
&= \frac{1}{2}\int_{x-h}^{x+h} |f(t)-f(x)|\, dt \\
&\implies 0 \leq \frac{1}{h}\int_0^h \left|\left(\frac{f(x+t)+f(x-t)}{2}\right) - f(x)\right| dt \leq \frac{1}{2h}\int_{x-h}^{x+h} |f(t)-f(x)|\, dt
\end{aligned}$$

Since $\lim_{h\to 0^+} \frac{1}{2h}\int_{x-h}^{x+h} |f(t)-f(x)|\, dt = 0$, then
$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \left|\left(\frac{f(x+t)+f(x-t)}{2}\right) - f(x)\right| dt = 0.$

Let $S = \left\{x \in \mathbb{R} \,\middle|\, \lim_{h\to 0^+} \frac{1}{h}\int_0^h \left|\left(\frac{f(x+t)+f(x-t)}{2}\right) - f(x)\right| = 0\right\}$.

Then, $\lim_{h\to 0^+} \frac{1}{h}\int_0^h \left|\left(\frac{f(x+t)+f(x-t)}{2}\right) - f(x)\right| dt = 0$ m-a.e.,
i.e. $m(S^c) = 0$, because $L_f \subseteq S \implies S^c \subseteq L_f^c$ and $m(L_f^c) = 0$. □

Remark. If f is continuous at $x \in \mathbb{R}$, then the limits in the previous theorem and corollary hold at x. In particular, if f is continuous, then the limits hold $\forall x \in \mathbb{R}$. If f is continuous on a finite closed interval I, then the uniform continuity of f on I implies that the limits hold uniformly for $x \in I$.

2.3.2 Functions of Bounded Variation

In the next two subsections, $Z = \mathbb{R}$ or $\mathbb{C}$ and $[a, b]$ is always a finite closed interval in $\mathbb{R}$. In this subsection, the notation from Section 2.2.1 will be used.

Definition 2.3.3. Let $\alpha\colon [a, b] \to Z$. Given a partition P of $[a, b]$, define $A(P) = \sum_{j=1}^{n} |\Delta\alpha_j|$. α is said to be of bounded variation on $[a, b]$, denoted $\alpha \in BV([a, b])$, if $\sup\limits_{P} A(P) < \infty$, where the supremum is taken over all partitions P of $[a, b]$. If $\alpha \in BV([a, b])$, then $V_\alpha([a, b]) = \sup\limits_{P} A(P)$ is called the total variation of α on $[a, b]$.

Remark. It is easy to show that if α is of bounded variation on $[a, b]$, then α is bounded on $[a, b]$.

Theorem 2.3.6. (i) $\alpha \in BV([a, b]) \iff \Re[\alpha], \Im[\alpha] \in BV([a, b])$

(ii) $\alpha \in BV([a, b]) \implies \forall x \in [a, b]$, $F(x-) = \lim\limits_{y \to x^-} F(y)$ and $F(x+) = \lim\limits_{y \to x^+} F(y)$ exist.

(iii) $\alpha \in BV([a, b]) \implies \{x \in [a, b] |\ f$ is not continuous at $x.\}$ is countable and so it is an m-null set.

(iv) $\alpha \in BV([a, b]) \implies \alpha$ is differentiable m-a.e. .

Proposition 2.3.7. If $\alpha \in BV([a, b])$, $f \in R(\alpha, a, b)$, and f is bounded by M on $[a, b]$, then $\left|\int_a^b f(x)\, d\alpha(x)\right| \leq M V_\alpha([a, b])$.

2.3.3 Absolutely Continuous Functions

Definition 2.3.4. A function $F\colon \mathbb{R} \to Z$ is absolutely continuous if $\forall \epsilon > 0$ $\exists \delta(\epsilon) > 0$ s.t. for any finite set of disjoint intervals $\{(a_j, b_j)\}_{j=1}^{n}$ satisfying $\sum_{j=1}^{n} m((a_j, b_j)) = \sum_{j=1}^{n} (b_j - a_j) < \delta(\epsilon)$, $\sum_{j=1}^{n} |F(b_j) - F(a_j)| < \epsilon$. A function

$F\colon [a,b] \to Z$ is absolutely continuous on $[a,b]$ if the above condition holds except that the disjoint intervals all lie in $[a,b]$.

Remark. It follows from the definition (let $n = 1$) that if F is absolutely continuous, then F is uniformly continuous. It also follows that if F is differentiable everywhere and F' is bounded, then F is absolutely continuous.

Theorem 2.3.8. If F is absolutely continuous on $[a,b]$, then $F \in BV([a,b])$.

Theorem 2.3.9 (The Fundamental Theorem of Calculus for Lebesgue Integrals). If $-\infty < a < b < \infty$ and $F\colon [a,b] \to Z$, then TFAE:

1) F is absolutely continuous on $[a,b]$.
2) $F(x) - F(a) = \int_a^x f(t)\,dt$ for some $f \in L^1([a,b], \mathcal{B}_{\mathbb{R}} \cap [a,b], m)$.
3) F is differentiable m-a.e. on $[a,b]$, $F' \in L^1([a,b], \mathcal{B}_{\mathbb{R}} \cap [a,b], m)$, and $F(x) - F(a) = \int_a^x F'(t)\,dt$

Note. It can be shown that $F' = f$ m-a.e. in the above theorem.

Theorem 2.3.10 (Integration by Parts). If F and G are absolutely continuous on $[a,b]$, then $\int_a^b F(x)G'(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b G(x)F'(x)\,dx$.

2.4 Miscellaneous Results in Real and Complex Analysis

Notation. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

Definition 2.4.1. The signum function $\operatorname{sgn}\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}.$$

2.4.1 Big O and Little o Notation

Definition 2.4.2. Let $g\colon \mathbb{N} \to \mathbb{C}$.

$$\mathrm{O}(g(n)) = \{f\colon \mathbb{N} \to \mathbb{C} | \exists c \geq 0 \text{ and } n_0 \in \mathbb{N} \text{ s.t. } \forall n \geq n_0, |f(n)| \leq c|g(n)|\}$$

$$\mathrm{o}(g(n)) = \left\{ f\colon \mathbb{N} \to \mathbb{C} \,\middle|\, \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \right\}$$

Note. 1) $f(n) = \mathrm{O}(g(n))$ means $f(n) \in \mathrm{O}(g(n))$.

2) Since the behaviour as $n \to \infty$ is considered, it is sometimes written $f(n) = \mathrm{O}(g(n))$ or $f(n) = \mathrm{o}(g(n))$ as $n \to \infty$.

3) Also, note that similar definitions can be made if f is a function defined on $\mathbb{R}$, where the behaviour as $x \to a$ is examined for some fixed $a \in \bar{\mathbb{R}}$ or f is a function defined on $\mathbb{Z}$, where the behaviour as $|n| \to \infty$ is examined.

4) It is easy to show that $\mathrm{o}(g(n)) \subseteq \mathrm{O}(g(n))$.

2.4.2 Trigonometric Functions

Let $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Then $\operatorname{sgn}(\sin(x)) = \begin{cases} (-1)^k, & \text{if } x \in (k\pi, (k+1)\pi) \\ 0, & \text{if } x = k\pi \end{cases}$.

This implies if $c > 0$, then $\operatorname{sgn}(\sin(cx)) = \begin{cases} (-1)^k, & \text{if } x \in \left(\frac{k\pi}{c}, \frac{(k+1)\pi}{c}\right) \\ 0, & \text{if } x = \frac{k\pi}{c} \end{cases}$.

$$\forall u \in \left[0, \frac{\pi}{2}\right],\ \frac{2}{\pi}u \leq \sin(u) \leq u \implies \forall u \in [0, \pi],\ \frac{u}{\pi} \leq \sin\left(\frac{u}{2}\right) \leq \frac{u}{2}$$
$$\implies \frac{2}{\pi} \leq \frac{\sin\left(\frac{u}{2}\right)}{\frac{u}{2}} \leq 1 \implies \frac{2}{\pi} \leq \left|\frac{\sin\left(\frac{u}{2}\right)}{\frac{u}{2}}\right| \leq 1$$

$$\forall u \in [0,\pi],\ 0 \leq \frac{u}{\pi} \leq \sin\left(\frac{u}{2}\right) \implies 0 \leq \csc\left(\frac{u}{2}\right) \leq \frac{\pi}{u}$$

Note that the above is okay at $u = 0$ because $\lim_{u\to 0} \dfrac{\sin\left(\frac{u}{2}\right)}{\frac{u}{2}} = 1$.

Also, $\forall u \in [0,\pi]$, $\dfrac{\sin\left(\frac{-u}{2}\right)}{\frac{-u}{2}} = \dfrac{-\sin\left(\frac{u}{2}\right)}{-\frac{u}{2}} = \dfrac{\sin\left(\frac{u}{2}\right)}{\frac{u}{2}}$.

This implies $\forall u \in [-\pi,\pi]$, $\dfrac{2}{\pi} \leq \left|\dfrac{\sin\left(\frac{u}{2}\right)}{\frac{u}{2}}\right| \leq 1$ and $1 \leq \left|\dfrac{\frac{u}{2}}{\sin\left(\frac{u}{2}\right)}\right| \leq \dfrac{\pi}{2}$.

$\forall u \in \mathbb{R}, |\sin u| \leq |u| \implies \left|\dfrac{\sin u}{u}\right| \leq 1$; Note that at $u = 0$, $\lim_{u\to 0} \dfrac{\sin u}{u} = 1$.

$\forall u \in \left[0, \frac{\pi}{2}\right], \tan(u) \geq u \implies \forall u \in [0,\pi], \tan\left(\frac{u}{2}\right) \geq \frac{u}{2} \implies 0 \leq \cot\left(\frac{u}{2}\right) \leq \frac{2}{u}$

$\implies \forall u \in [-\pi,\pi], \left|\cot\left(\frac{u}{2}\right)\right| \leq \dfrac{2}{|u|}$

CHAPTER 3
Summability and Convergence in Norm of Fourier Series on $\mathbb{T}$

This chapter is a review of the main results from Chapters 1 and 2 of (Katznelson, 2004).

3.1 The Spaces $L^p(\mathbb{T})$ and $C(\mathbb{T})$

Consider the following equivalence relation :

Example 3.1.1. Let $X = \mathbb{R}$ and define xRy if $x - y$ is an integer multiple of 2π, i.e. $x - y = 2k\pi$, where $k \in \mathbb{Z}$.

Let $\mathbb{T} = X/R = \{[x] : x \in \mathbb{R}\}$ be the set of equivalence classes.
$\forall x \in \mathbb{R}$, $[x] = \{y \in \mathbb{R} : yRx\} = \{x + 2k\pi : k \in \mathbb{Z}\}$. Let $c \in \mathbb{R}$ and $I = [c, c + 2\pi)$. Then $\mathbb{T} = \{[x] : x \in I\}$ and all of these elements are distinct because each distinct $x \in I$ corresponds to a unique element $[x]$ of $\mathbb{T}$. Since $[c] = [c + 2\pi]$, then by replacing $[c]$ with $[c + 2\pi]$, it follows that $\mathbb{T}$ is in bijective correspondence with any half-open interval of length 2π.

Notation. Let Z be a set. Then $f\colon \mathbb{T} \to Z$ means that $f\colon \mathbb{R} \to Z$ is 2π-periodic. In addition, if $Z = \mathbb{C}$, then it is also assumed that f is measurable.

There are two important measure spaces that will be considered throughout the rest of the text. The first is the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \lambda)$, where $\lambda = \dfrac{m}{2\pi}$ and m is Lebesgue measure. Using the same notation from Section 2.3.1, $\displaystyle\int f(t)\, d\lambda(t) = \frac{1}{2\pi}\int f(t)\, dm(t) = \frac{1}{2\pi}\int f(t)\, dt$. The second and maybe more important measure space is $(I, \mathcal{B}_{\mathbb{R}} \cap I, \lambda)$, where I is a fixed interval of length 2π. I is usually $[-\pi, \pi]$ or $[0, 2\pi]$ where at most one of the endpoints might be removed. Since $\dfrac{1}{2\pi}$ is a positive constant, then $\forall\ 1 \le p \le \infty$, $L^p(I, \mathcal{B}_{\mathbb{R}} \cap I, m) = L^p(I, \mathcal{B}_{\mathbb{R}} \cap I, \lambda)$ except that now $\forall\ 1 \le p < \infty$, $\|f\|_p = \left[\dfrac{1}{2\pi} \displaystyle\int_I |f(x)|^p\, dx\right]^{\frac{1}{p}}$. Note that when $p = \infty$, the ∞-norms corresponding to m and λ are the same.

The following proposition states that the Lebesgue integral of a 2π-periodic function over an interval of length 2π is independent of which interval of length 2π is chosen.

Proposition 3.1.1. Let $f\colon \mathbb{T} \to \mathbb{C}$ be s.t. $f \in L^1([-\pi, \pi], \mathcal{B}_{\mathbb{R}} \cap [-\pi, \pi], m)$. Then, $f \in L^1_{\text{loc}}(\mathbb{R})$ and $\forall a \in \{-1, 1\}$ $\forall c \in \mathbb{R}$, $\int_{-\pi}^{\pi} f(x)\, dx = \int_{c-\pi}^{c+\pi} f(x)\, dx = \int_{-\pi}^{\pi} f(ax - c)\, dx$.

Note. In the hypothesis of the proposition, $[-\pi, \pi]$ could have been replaced with any (not necessarily closed) interval I of length 2π and the proposition would still hold. Also, the only hypothesis that was needed for the second part of the proposition is that $f\colon \mathbb{T} \to \mathbb{C}$ and $f \in L^1_{\text{loc}}(\mathbb{R})$. Also, ($f\colon \mathbb{T} \to \mathbb{C}$ and $f \in L^1_{\text{loc}}(\mathbb{R})$) $\implies f \in L^1([-\pi, \pi], \mathcal{B}_{\mathbb{R}} \cap [-\pi, \pi], m)$.

Now the $L^p(\mathbb{T})$ spaces will be defined.

Definition 3.1.1. $L^1(\mathbb{T}) = \left\{ f\colon \mathbb{T} \to \mathbb{C} \,\middle|\, f \in L^1_{\text{loc}}(\mathbb{R}) \right\}$

$\forall\ 1 < p < \infty$, $L^p(\mathbb{T}) = \left\{ f\colon \mathbb{T} \to \mathbb{C} \,\middle|\, |f|^p \in L^1_{\text{loc}}(\mathbb{R}) \right\}$

$L^\infty(\mathbb{T}) = \{ f\colon \mathbb{T} \to \mathbb{C} |\ f \in L^\infty(I, \mathcal{B}_{\mathbb{R}} \cap I, \lambda)\}$, where I is any half-open interval of length 2π.

Proposition 3.1.2. $\forall\ 1 \le p \le \infty$, $L^p(\mathbb{T})$ can be identified with $L^p(I, \mathcal{B}_{\mathbb{R}} \cap I, \lambda)$, where I is any half-open interval of length 2π.

Proof. First, it will be shown that if $f \in L^p(\mathbb{T})$, then $f \in L^p(I, \mathcal{B}_{\mathbb{R}} \cap I, \lambda)$.

$f \in L^p(\mathbb{T}) \implies |f|^p \in L^1_{\text{loc}}(\mathbb{R}) \implies |f|^p \in L^1(I, \mathcal{B}_{\mathbb{R}} \cap I, \lambda) \implies f \in L^p(I, \mathcal{B}_{\mathbb{R}} \cap I, \lambda)$

Now, it will be shown that if $f \in L^p(I, \mathcal{B}_{\mathbb{R}} \cap I, \lambda)$, then $f \in L^p(\mathbb{T})$.

If $f \in L^p(I, \mathcal{B}_{\mathbb{R}} \cap I, \lambda)$, then extend f periodically to the rest of $\mathbb{R}$. Then, $f : \mathbb{T} \to \mathbb{C}$. If $p = \infty$, then it is clear that $f \in L^p(\mathbb{T})$. Let $p < \infty$. Since $f : \mathbb{T} \to \mathbb{C}$, then $|f|^p : \mathbb{T} \to \mathbb{C}$.

$f \in L^p(I, \mathcal{B}_{\mathbb{R}} \cap I, \lambda) \implies |f|^p \in L^1(I, \mathcal{B}_{\mathbb{R}} \cap I, \lambda) = L^1(I, \mathcal{B}_{\mathbb{R}} \cap I, m)$

By Proposition 3.1.1, $|f|^p \in L^1_{\text{loc}}(\mathbb{R})$ and so $f \in L^p(\mathbb{T})$. $\square$

Remark. This identification makes $\forall\ 1 \leq p \leq \infty$, $L^p(\mathbb{T})$ into a Banach space because $L^p(I, \mathcal{B}_{\mathbb{R}} \cap I, \lambda)$ is a Banach space, where the corresponding norm is given by if $p < \infty$, $\|f\|_p = \left[\frac{1}{2\pi}\int_I |f(x)|^p\,dx\right]^{\frac{1}{p}}$ and if $p = \infty$, the ∞-norm on $L^\infty(I, \mathcal{B}_{\mathbb{R}} \cap I, \lambda)$.

Note. The first part of the the proof of the previous proposition still holds if I is any interval of length 2π.

Corollary 3.1.3. $\forall\ 1 \leq p \leq \infty$, $L^p(\mathbb{T})$ is a Banach space.

By the previous proposition, the following alternative definition could have been made for $L^p(\mathbb{T})$.

Definition 3.1.2. $\forall\ 1 \leq p \leq \infty$, let $L^p(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C} |\ f \in L^p(I, \mathcal{B}_{\mathbb{R}} \cap I, \lambda)\}$

Notation. The notation is for the following theorem. Let $1 < p < \infty$ and $q = \dfrac{p}{p-1}$ so that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. When $p = 1$, let $q = \infty$ and when $p = \infty$, let $q = 1$. In both cases, it is agreed that $\dfrac{1}{\infty} = 0$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Using the identification between $L^p(\mathbb{T})$ and $L^p(I, \mathcal{B}_{\mathbb{R}} \cap I, \lambda)$, then by (Folland, 1999, P. 182, Theorem 6.2) and (Folland, 1999, P. 184, Theorem 6.8.(a)) with $(X, \mathfrak{M}, \mu) = (I, \mathcal{B}_{\mathbb{R}} \cap I, \lambda)$, the following holds.

Theorem 3.1.4 (Holder's Inequality). Let $1 \leq p, q, r \leq \infty$ be s.t. $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $f \in L^p(\mathbb{T})$ and $g \in L^q(\mathbb{T})$, then $fg \in L^r(\mathbb{T})$ and $\|fg\|_r \leq \|f\|_p \|g\|_q$.

Note. When $q = \infty$, then $r = p$ and by the corollary, $f \in L^p(\mathbb{T})$ and $g \in L^\infty(\mathbb{T})$ $\implies fg \in L^p(\mathbb{T})$ and $\|fg\|_p \leq \|f\|_p \|g\|_\infty$.

Corollary 3.1.5. Let $1 \leq r < p \leq \infty$. Then $\forall f \in L^p(\mathbb{T})$, $f \in L^r(\mathbb{T})$, i.e. $L^p(\mathbb{T}) \subseteq L^r(\mathbb{T})$ and $\|f\|_r \leq \|f\|_p$.

Proof. Let $g = \chi_{\mathbb{R}} = 1$. Then $\forall\ 1 \leq q \leq \infty$, $g \in L^q(\mathbb{T})$ and $\|g\|_q = 1$. By the previous corollary, the result holds. The only thing that needs to be checked is if $\forall\ 1 \leq r < p \leq \infty\ \exists q \geq 1$ s.t. $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Note that $1 \leq r < \infty$. By the comments before Definition 3.1.2, if $r = 1$, then q exists. If $r > 1$, then divide both sides of $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ by $\frac{1}{r}$ to get $\frac{1}{\left(\frac{p}{r}\right)} + \frac{1}{\left(\frac{q}{r}\right)} = 1$. Then $\left(\frac{q}{r}\right)$ exists and $\left(\frac{q}{r}\right) \geq 1 \Rightarrow q = \left(\frac{q}{r}\right) r$ exists and $q \geq r \geq 1$, i.e. $q \geq 1$.

Thus, $\forall\ 1 \leq r < p \leq \infty\ \exists q \geq 1$ s.t. $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. □

In particular, when $r = 1$ or $p = \infty$, the following hold:

Corollary 3.1.6. $\forall\ 1 < p \leq \infty$, $L^p(\mathbb{T}) \subseteq L^1(\mathbb{T})$ and $\forall f \in L^p(\mathbb{T})$, $\|f\|_1 \leq \|f\|_p$. $\forall\ 1 \leq r < \infty$, $L^\infty(\mathbb{T}) \subseteq L^r(\mathbb{T})$ and $\forall f \in L^\infty(\mathbb{T})$, $\|f\|_r \leq \|f\|_\infty$.

Definition 3.1.3. Let $C(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C} \mid f \text{ is continuous.}\}$.

The following proposition lists some important properties of $C(\mathbb{T})$.

Proposition 3.1.7. (i) Define $\| \ \|_\infty : C(\mathbb{T}) \to [0, \infty)$ by $\|f\|_\infty = \sup_{t \in \mathbb{R}} |f(t)|$. $(C(\mathbb{T}), \| \ \|_\infty)$ is a Banach space.

(ii) $\forall\ 1 \le p \le \infty$, $C(\mathbb{T}) \subseteq L^p(\mathbb{T})$

(iii) Furthermore, if $p < \infty$, then $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$, i.e. $\forall f \in L^p(\mathbb{T})\ \forall \epsilon > 0\ \exists g \in C(\mathbb{T})$ s.t. $\|f - g\|_p < \epsilon$.

Remark. It can be shown that on $C(\mathbb{T})$, the ∞-norm corresponding to $L^\infty(\mathbb{T})$ is equal to the ∞-norm in the first part of the proposition.

Note. (i) $\forall f \in C(\mathbb{T})$, $\|f\|_\infty = \sup_{t \in \mathbb{R}} |f(t)| = \sup_{t \in J} |f(t)|$, where J is any interval of length 2π, because f is 2π-periodic.

(ii) If $f\colon \mathbb{R} \to \mathbb{C}$ is a 2π-periodic function, then to show $f \in C(\mathbb{T})$, it is enough to show that f is continuous on any interval J of length 2π.

By the previous corollary and proposition, the following holds.

Corollary 3.1.8. $\forall\ 1 < p < \infty$, $C(\mathbb{T}) \subseteq L^\infty(\mathbb{T}) \subseteq L^p(\mathbb{T}) \subseteq L^1(\mathbb{T})$

Note. Due to this corollary, it will now be assumed that all functions belong to $L^1(\mathbb{T})$.

Notation. The following notation will be used for convenience.

$$\forall f \in L^1(\mathbb{T}), \text{ let } \int f(t)\, dt = \int_{\mathbb{T}} f(t)\, dt = \int_0^{2\pi} f(x)\, dx = \int_I f(x)\, dx,$$

where I is any interval of length 2π by Proposition 3.1.1.

Corollary 3.1.9. $\forall f \in L^1(\mathbb{T})\ \forall a \in \{-1, 1\}\ \forall c \in \mathbb{R}$, $g \in L^1(\mathbb{T})$, where $g\colon \mathbb{T} \to \mathbb{C}$ is given by $g(t) = f(at - c)$ and $\int f(t)\, dt = \int f(at - c)\, dt$.

Proof. Fix a and c. By the note after Proposition 2.3.2, $g \in L^1_{\text{loc}}(\mathbb{R})$ and also g is 2π-periodic because f is 2π-periodic. Therefore, $g\colon \mathbb{T} \to \mathbb{C}$ and $g \in L^1_{\text{loc}}(\mathbb{R})$, i.e.

$g \in L^1(\mathbb{T})$. By Proposition 3.1.1, $\int f(t)\,dt = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} f(ax-c)\,dx = \int f(at-c)\,dt$. $\square$

Remark. For $a = 1$, the corollary is called the translation invariance property of $\mathbb{T}$.

3.2 Fourier Series

Definition 3.2.1. A trigonometric polynomial is a function $P\colon \mathbb{T} \to \mathbb{C}$ of the form $P(t) = \sum_{n=-N}^{N} a_n e^{int}$, where $N \in \mathbb{N}_0$ and $a_n \in \mathbb{C}$. The degree of P, $\deg(P)$, is the largest $N \in \mathbb{N}_0$ s.t. $|a_{-N}| + |a_N| \neq 0$.

Remark. Since $e^{it} = \cos t + i \sin t$, then any trigonometric polynomial can be expressed as a finite sum of sine and cosine terms and vice-versa.

$$P(t) = \sum_{n=-N}^{N} a_n e^{int} = \frac{c_0}{2} + \sum_{n=1}^{N} (c_n \cos(nt) + b_n \sin(nt))$$

where $\forall\ 0 \leq n \leq N$, $c_n = (a_{-n} + a_n)$, $b_n = \dfrac{(a_{-n} - a_n)}{i}$, $a_n = \dfrac{(c_n - b_n i)}{2}$, and $a_{-n} = \dfrac{(c_n + b_n i)}{2}$.

Note. It is easy to show that if $\mathbf{T}$ is the set of trigonometric polynomials, then $\mathbf{T} \subseteq C(\mathbb{T})$.

The following proposition is easy to prove by elementary calculus.

Proposition 3.2.1. $\forall n \in \mathbb{Z}$, $\dfrac{1}{2\pi} \displaystyle\int e^{int}\,dt = \delta_{n,0}$, where $\delta_{n,0} = \begin{cases} 0, & \text{if } n \neq 0 \\ 1, & \text{if } n = 0 \end{cases}$.

Remark. It follows from the proposition, that if $P \in \mathbf{T}$, then the coefficients a_n of P can be computed by the formula $a_n = \dfrac{1}{2\pi} \displaystyle\int P(t) e^{-int}\,dt$. This motivates the definition of the Fourier coefficients, which will be defined shortly.

Definition 3.2.2. A trigonometric series is an expression of the form

$$S \sim \sum_{n=-\infty}^{\infty} a_n e^{int} = \frac{c_0}{2} + \sum_{n=1}^{\infty} (c_n \cos(nt) + b_n \sin(nt))$$

where $\forall n \in \mathbb{Z}$, $a_n \in \mathbb{C}$ and $\forall n \in \mathbb{N}_0$, b_n, c_n are defined as above. (Note that here it is not necessarily true that the series converges and that it may even diverge for all values of t.) The conjugate trigonometric series of S is $\tilde{S} \sim \sum_{n=-\infty}^{\infty} -i \operatorname{sgn}(n) a_n e^{int}$.

Definition 3.2.3. Let $f \in L^1(\mathbb{T})$. Then $\forall n \in \mathbb{Z}$, the n-th Fourier coefficient of f is $\hat{f}(n) = \dfrac{1}{2\pi} \displaystyle\int f(t) e^{-int}\, dt$ and the Fourier series of f is the trigonometric series $S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}$ and $S[f](t)$ will denote the Fourier series of f at the fixed value $t \in \mathbb{R}$. The conjugate Fourier series of f is the conjugate trigonometric series of $S[f]$ given by $\tilde{S}[f] \sim \sum_{n=-\infty}^{\infty} -i \operatorname{sgn}(n) \hat{f}(n) e^{int}$.

Note.

$S[f] \sim \dfrac{c_0(f)}{2} + \displaystyle\sum_{n=1}^{\infty} (c_n(f) \cos(nt) + b_n(f) \sin(nt))$ where $\forall n \in \mathbb{N}_0$ and $\forall m \in \mathbb{N}$,

$$c_n(f) = \hat{f}(-n) + \hat{f}(n) = \frac{1}{\pi} \int f(t) \left(\frac{e^{int} + e^{-int}}{2} \right) dt = \frac{1}{\pi} \int f(t) \cos(nt)\, dt$$

and $b_m(f) = \dfrac{\hat{f}(-m) - \hat{f}(m)}{i} = \dfrac{1}{\pi} \displaystyle\int f(t) \left(\frac{e^{imt} - e^{-imt}}{2i} \right) dt = \dfrac{1}{\pi} \displaystyle\int f(t) \sin(mt)\, dt.$

$\{c_n(f)\}_{n=0}^{\infty}$ and $\{b_n(f)\}_{n=1}^{\infty}$ are also referred to as the Fourier coefficients of f.

Remark. The conjugate Fourier series will not be referred to again until near the end of the chapter, where it has an important role in the section concerning convergence in norm of Fourier series.

The following theorem is a list of some of the important properties of the Fourier coefficients such as linearity.

Theorem 3.2.2. Let $f, g \in L^1(\mathbb{T})$ and $\alpha \in \mathbb{C}$. Then

1) $(\widehat{\alpha f + g})(n) = \alpha \hat{f}(n) + \hat{g}(n)$
2) If $\overline{f}$ is the complex conjugate of f, then $\widehat{\overline{f}}(n) = \overline{\hat{f}(-n)}$
3) $\forall \tau \in \mathbb{R}$, let $f_\tau(t) = f(t-\tau)$. Then $(\widehat{f_\tau})(n) = \hat{f}(n)e^{-in\tau}$.
4) $|\hat{f}(n)| \leq \|f\|_1$

Corollary 3.2.3. If $\{f_j\}_{j=0}^{\infty} \subseteq L^1(\mathbb{T})$ and $\lim_{j\to\infty} f_j = f_0$ in $L^1(\mathbb{T})$, i.e. $\lim_{j\to\infty} \|f_j - f_0\|_1 = 0$, then $\lim_{j\to\infty} \hat{f}_j(n) = \hat{f}_0(n)$ uniformly in n.

Proof. The proof follows from the last part of the previous theorem because $|\hat{f}_j(n) - \hat{f}_0(n)| = |(\widehat{f_j - f_0})(n)| \leq \|f_j - f_0\|_1$. □

Theorem 3.2.4. Let $f \in L^1(\mathbb{T})$ be s.t. $\int f(t)\,dt = 0$. Define $F\colon \mathbb{T} \to \mathbb{C}$ by $F(y) = \int_0^y f(x)\,dx$. Then $F \in C(\mathbb{T})$ and $\forall n \in \mathbb{Z}$ s.t. $n \neq 0$, $\hat{F}(n) = \dfrac{\hat{f}(n)}{in}$.

Proof. By Proposition 2.3.2.(1), $F(y+2\pi) = \int_0^{y+2\pi} f(x)\,dx = \int_0^y f(x)\,dx + \int_y^{y+2\pi} f(x)\,dx = F(y) + \int f(t)\,dt = F(y) + 0 = F(y)$ and so F is 2π-periodic. By the note after Proposition 3.1.2, $f \in L^1(\mathbb{T}) \implies L^1([0,2\pi], \mathcal{B}_{\mathbb{R}} \cap [0,2\pi], m)$. By Theorem 2.3.9, F is absolutely continuous on $[0,2\pi]$ which implies that F is continuous on $[0,2\pi]$. By the note after Proposition 3.1.7, $F \in C(\mathbb{T})$ because F is a 2π-periodic function which is continuous on $[0,2\pi]$. Also, by Theorem 2.3.9, $F' = f$ m-a.e on $[0,2\pi]$. Let $n \in \mathbb{Z}$ be s.t. $n \neq 0$ and $G(x) = \dfrac{e^{-inx}}{-in}$. G is absolutely continuous because G is differentiable everywhere and $G'(x) = e^{-inx}$ is bounded in absolute value by 1. Also, $G \in C(\mathbb{T})$. Now, by Integration by

Parts, $2\pi\hat{F}(n) = \int F(t)e^{-int}\,dt = \int_0^{2\pi} F(x)e^{-inx}\,dx = \int_0^{2\pi} F(x)G'(x)\,dx = F(2\pi)G(2\pi) - F(0)G(0) - \int_0^{2\pi} G(x)F'(x)\,dx$. Since F and G are 2π-periodic functions, then FG is also a 2π-periodic function and $F(2\pi)G(2\pi) - F(0)G(0) = 0$.

$$\begin{aligned}
\int_0^{2\pi} G(x)F'(x)\,dx &= \int_0^{2\pi} \frac{e^{-inx}}{-in} f(x)\,dx = \frac{1}{-in}\int_0^{2\pi} f(x)e^{-inx}\,dx \\
&= \frac{1}{-in}\int f(t)e^{-int}\,dt = \frac{1}{-in}(2\pi\hat{f}(n)) = -2\pi\frac{\hat{f}(n)}{in} \\
&\implies 2\pi\hat{F}(n) = 2\pi\frac{\hat{f}(n)}{in} \implies \hat{F}(n) = \frac{\hat{f}(n)}{in}
\end{aligned}$$

□

Theorem 3.2.5. Let $f, g \in L^1(\mathbb{T})$. Define $h\colon \mathbb{T} \to \mathbb{C}$ by

$$h(t) = \frac{1}{2\pi}\int f_\tau(t)g(\tau)\,d\tau = \frac{1}{2\pi}\int f(t-\tau)g(\tau)\,d\tau.$$

Then h is well-defined m-a.e., $h \in L^1(\mathbb{T})$, $\|h\|_1 \le \|f\|_1\|g\|_1$, and $\forall n \in \mathbb{Z}$, $\hat{h}(n) = \hat{f}(n)\hat{g}(n)$.

Definition 3.2.4. The function h from the previous theorem is called the convolution of f and g and is denoted by $f * g$. By the previous theorem, $\forall n \in \mathbb{Z}$, $\widehat{(f * g)}(n) = \hat{f}(n)\hat{g}(n)$.

The next theorem is a list of some of the properties of the convolution operator.

Theorem 3.2.6. Let $f, g, h \in L^1(\mathbb{T})$ and $\alpha \in \mathbb{C}$. Then

1) $f * (\alpha g + h) = \alpha(f * g) + f * h$ and $(\alpha f + g) * h = \alpha(f * h) + g * h$
2) $f * g = g * f$
3) $(f * g) * h = f * (g * h)$

4) If $f \in C(\mathbb{T})$ or $g \in C(\mathbb{T})$, then $(f * g)(t)$ is well-defined $\forall t \in \mathbb{R}$ and $(f * g) \in C(\mathbb{T})$.

Lemma 3.2.7. Let $f \in L^1(\mathbb{T})$ and $\phi(t) = e^{int}$, where $n \in \mathbb{Z}$.

Then $(\phi * f)(t) = \hat{f}(n)e^{int}$.

Corollary 3.2.8. If $f \in L^1(\mathbb{T})$ and $P \in \mathbf{T}$, where $P(t) = \sum\limits_{n=-N}^{N} a_n e^{int}$, then $(P * f)(t) = \sum\limits_{n=-N}^{N} a_n \hat{f}(n) e^{int}$.

3.3 Summability in Norm

Following (Katznelson, 2004, P. 14, Definition 1.2.10), a useful class of Banach spaces called homogeneous Banach spaces on $\mathbb{T}$ will now be introduced.

Definition 3.3.1. A Banach space $(B, \| \|_B)$ is called a homogeneous Banach space on $\mathbb{T}$ if B is a subspace of $L^1(\mathbb{T})$ satisfying $\forall f \in B$,

(H-1) $\|f\|_1 \leq \|f\|_B$

(H-2) $\forall \tau \in \mathbb{R}$, $f_\tau \in B$ and $\|f_\tau\|_B = \|f\|_B$.

(H-3) The function $\phi \colon \mathbb{T} \to B$ given by $\phi(\tau) = f_\tau$ is continuous, i.e.

$$\forall \tau_0 \in \mathbb{R}, \lim_{\tau \to \tau_0} \|f_\tau - f_{\tau_0}\|_B = 0.$$

Remark. (H-2) is referred to as translation invariance and (H-3) is referred to as the continuity of the translation.

Note. If (H-2) is true, then to show (H-3) is true, it is enough to show (H-3) is true when $\tau_0 = 0$ because by (H-2), $\|f_\tau - f_{\tau_0}\|_B = \|f_{(\tau-\tau_0)} - f\|_B$.

Lemma 3.3.1. $C(\mathbb{T})$ is a homogeneous Banach space on $\mathbb{T}$.

Proof. By Proposition 3.1.7, $C(\mathbb{T})$ is a Banach space and $C(\mathbb{T})$ is a subspace of $L^1(\mathbb{T})$. By Corollary 3.1.6, (H-1) holds. (H-2) holds because if $f \in C(\mathbb{T})$, then

$f_\tau \in C(\mathbb{T})$ and $\|f_\tau\|_\infty = \sup\limits_{t\in[0,2\pi]} |f_\tau(t)| = \sup\limits_{t\in[0,2\pi]} |f(t-\tau)| = \sup\limits_{t\in[-\tau,-\tau+2\pi]} |f(t)| = \|f\|_\infty$. Thus, only (H-3) needs to be proven. By the previous note, it is enough to show if $f \in C(\mathbb{T})$, then $\lim\limits_{\tau\to 0} \|f_\tau - f\|_\infty = 0$, i.e. $\forall\epsilon > 0\ \exists\delta(\epsilon) > 0$ s.t. $|\tau| < \delta(\epsilon) \implies \|f_\tau - f\|_\infty < \epsilon$. Let $t \in [0, 2\pi]$ and $\tau \in [-2\pi, 2\pi]$. Then $(t-\tau) \in [-2\pi, 4\pi]$. f is uniformly continuous on $[-2\pi, 4\pi]$ because f is continuous on $[-2\pi, 4\pi]$ and $[-2\pi, 4\pi]$ is compact. Let $\epsilon > 0$. Then $\exists\ 0 < \delta(\epsilon) < 2\pi$ s.t. $\forall x, y \in [-2\pi, 4\pi]$, $|y - x| < \delta(\epsilon) \implies |f(y) - f(x)| < \frac{\epsilon}{2}$. If $|\tau| < \delta(\epsilon)$, then $\forall t \in [0, 2\pi]$, $|(t-\tau)-t| = |-\tau| = |\tau| < \delta(\epsilon) \implies |f_\tau(t) - f(t)| = |f(t-\tau) - f(t)| < \frac{\epsilon}{2}$ and $\|f_\tau - f\|_\infty = \sup\limits_{t\in[0,2\pi]} |f_\tau(t) - f(t)| \le \frac{\epsilon}{2} < \epsilon$.

$\therefore \forall\epsilon > 0\ \exists\delta(\epsilon) > 0$ s.t. $|\tau| < \delta(\epsilon) \implies \|f_\tau - f\|_\infty < \epsilon$

Therefore, (H-3) holds and by all of the above, $C(\mathbb{T})$ is a homogeneous Banach space on $\mathbb{T}$. □

Theorem 3.3.2. $\forall\ 1 \le p < \infty$, $L^p(\mathbb{T})$ is a homogeneous Banach space on $\mathbb{T}$.

Proof. By Corollary 3.1.3, $L^p(\mathbb{T})$ is a Banach space. By Corollary 3.1.6, $L^p(\mathbb{T})$ is a subspace of $L^1(\mathbb{T})$ and (H-1) holds. By Corollary 3.1.9 and noting that $f \in L^p(\mathbb{T}) \iff |f|^p \in L^1(\mathbb{T})$, (H-2) holds. Thus, only (H-3) needs to be proven. By the previous note, it is enough to show if $f \in L^p(\mathbb{T})$, then $\lim\limits_{\tau\to 0} \|f_\tau - f\|_p = 0$, i.e. $\forall\epsilon > 0\ \exists\delta(\epsilon) > 0$ s.t. $|\tau| < \delta(\epsilon) \implies \|f_\tau - f\|_p < \epsilon$.

Let $\epsilon > 0$. By Proposition 3.1.7, $\exists g \in C(\mathbb{T})$ s.t. $\|f - g\|_p < \frac{\epsilon}{3}$. By the previous lemma, $\exists\delta(\epsilon) > 0$ s.t. $|\tau| < \delta(\epsilon) \implies \|g_\tau - g\|_\infty < \frac{\epsilon}{3}$. By Corollary 3.1.6, if

$|\tau| < \delta(\epsilon)$, then $\|g_\tau - g\|_p \leq \|g_\tau - g\|_\infty < \dfrac{\epsilon}{3}$.

$$(f_\tau - f) = (f_\tau - g_\tau) + (g_\tau - g) + (g - f)$$

$$\implies \|f_\tau - f\|_p \leq \|f_\tau - g_\tau\|_p + \|g_\tau - g\|_p + \|g - f\|_p$$

$$= \|f - g\|_p + \|g_\tau - g\|_p + \|f - g\|_p \text{ [by H-2]}$$

$$= \|g_\tau - g\|_p + 2\|f - g\|_p < \frac{\epsilon}{3} + 2\left(\frac{\epsilon}{3}\right) = \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon.$$

$\therefore \forall \epsilon > 0 \ \exists \delta(\epsilon) > 0$ s.t. $|\tau| < \delta(\epsilon) \implies \|f_\tau - f\|_p < \epsilon$

Therefore, (H-3) holds and by all of the above, $L^p(\mathbb{T})$ is a homogeneous Banach space on $\mathbb{T}$. □

Note. It can be shown that $L^\infty(\mathbb{T})$ is not a homogeneous Banach space on $\mathbb{T}$.

Remark. An important homogeneous Banach space on $\mathbb{T}$ is $L^1(\mathbb{T})$ because all functions are assumed to be in $L^1(\mathbb{T})$. Another important homogeneous Banach space on $\mathbb{T}$ is $C(\mathbb{T})$. This is because if a certain property needs to be proven for $L^p(\mathbb{T})$, where $p < \infty$, then sometimes, the property can first be proven for $C(\mathbb{T})$ and then the fact that $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$ can be used to show that the property holds for $L^p(\mathbb{T})$. This argument was used for the proof of (H-3) in the previous theorem.

Definition 3.3.2. A summability kernel is a sequence $\{k_n\}_{n=0}^\infty \subseteq C(\mathbb{T})$ satisfying:

(S-1) $\forall n \in \mathbb{N}_0, \ \dfrac{1}{2\pi} \displaystyle\int k_n(t)\, dt = 1$

(S-2) $\exists K > 0 \ \forall n \in \mathbb{N}_0, \ \|k_n\|_1 \leq K$

(S-3) $\forall\ 0 < \delta < \pi, \ \displaystyle\lim_{n \to \infty} \int_\delta^{2\pi - \delta} |k_n(t)|\, dt = 0$

Remark. A positive summability kernel is one s.t. $\forall n \in \mathbb{N}_0$, $k_n \geq 0$. Note that for a positive summability kernel, (S-1) $\Rightarrow$ (S-2). Also, $0 < \delta < \pi \iff 0 < \delta < 2\pi - \delta$.

The theory from Section 2.2.2 will be used now. The following theorem is from (Katznelson, 2004, P. 10-11, Lemma 1.2.2).

Theorem 3.3.3. Let B be a Banach space, $\phi\colon \mathbb{T} \to B$ be continuous, and $\{k_n\}_{n=0}^{\infty}$ be a summability kernel. Then $\lim_{n\to\infty} \frac{1}{2\pi}\int k_n(\tau)\phi(\tau)\,d\tau = \phi(0)$ in B.

Proof. It must be shown that $\lim_{n\to\infty} \left\| \frac{1}{2\pi}\int k_n(\tau)\phi(\tau)\,d\tau - \phi(0) \right\|_B = 0$.
Let $0 < \delta < \pi$. By (S-1),

$$\left[\frac{1}{2\pi}\int k_n(\tau)\phi(\tau)\,d\tau - \phi(0)\right] = \left[\frac{1}{2\pi}\int k_n(\tau)\phi(\tau)\,d\tau - \left(\frac{1}{2\pi}\int k_n(\tau)\,d\tau\right)\phi(0)\right]$$

$$= \left[\frac{1}{2\pi}\int k_n(\tau)(\phi(\tau) - \phi(0))\,d\tau\right] = \left[\frac{1}{2\pi}\int_{-\delta}^{2\pi-\delta} k_n(\tau)(\phi(\tau) - \phi(0))\,d\tau\right]$$

$$= \left[\frac{1}{2\pi}\int_{-\delta}^{\delta} k_n(\tau)(\phi(\tau) - \phi(0))\,d\tau\right] + \left[\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} k_n(\tau)(\phi(\tau) - \phi(0))\,d\tau\right]$$

$$\Longrightarrow \left\| \frac{1}{2\pi}\int k_n(\tau)\phi(\tau)\,d\tau - \phi(0) \right\|_B \leq$$

$$\left\| \frac{1}{2\pi}\int_{-\delta}^{\delta} k_n(\tau)(\phi(\tau) - \phi(0))\,d\tau \right\|_B + \left\| \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} k_n(\tau)(\phi(\tau) - \phi(0))\,d\tau \right\|_B$$

By Theorem 2.2.6.(3),

$$\left\|\frac{1}{2\pi}\int_{-\delta}^{\delta} k_n(\tau)(\phi(\tau)-\phi(0))\,d\tau\right\|_B \le \frac{1}{2\pi}\int_{-\delta}^{\delta} \|k_n(\tau)(\phi(\tau)-\phi(0))\|_B\,d\tau$$

$$\le \frac{1}{2\pi}\int_{-\delta}^{\delta} |k_n(\tau)|\ \|\phi(\tau)-\phi(0)\|_B\,d\tau \le \frac{1}{2\pi}\int_{-\delta}^{\delta} |k_n(\tau)| \left(\sup_{|t|\le\delta} \|\phi(t)-\phi(0)\|_B\right) d\tau$$

$$= \left(\frac{1}{2\pi}\int_{-\delta}^{\delta} |k_n(\tau)|\,d\tau\right)\left(\sup_{|t|\le\delta}\|\phi(t)-\phi(0)\|_B\right)$$

$$\le \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |k_n(\tau)|\,d\tau\right)\left(\sup_{|t|\le\delta}\|\phi(t)-\phi(0)\|_B\right) = \|k_n\|_1\left(\sup_{|t|\le\delta}\|\phi(t)-\phi(0)\|_B\right)$$

$$\le K\left(\sup_{|t|\le\delta}\|\phi(t)-\phi(0)\|_B\right).$$

Define $H\colon \mathbb{T} \to [0,\infty)$ by $H(\tau) = \|\phi(\tau)\|_B$. $H \in C(\mathbb{T})$ because it is a composition of the continuous functions $\phi\colon \mathbb{T} \to B$ and $\|\ \|_B\colon B \to [0,\infty)$.

$\Longrightarrow$ H is bounded, i.e. $\exists M > 0$ s.t. $\forall\tau \in \mathbb{R}$, $H(\tau) = \|\phi(\tau)\|_B \le M$.

$\Longrightarrow$ $\forall\tau \in \mathbb{R}$, $\|\phi(\tau)-\phi(0)\|_B \le \|\phi(\tau)\|_B + \|\phi(0)\|_B \le 2M$

$$\Longrightarrow \left\|\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} k_n(\tau)(\phi(\tau)-\phi(0))\,d\tau\right\|_B \le \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} \|k_n(\tau)(\phi(\tau)-\phi(0))\|_B\,d\tau$$

$$\le \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} |k_n(\tau)|\ \|\phi(\tau)-\phi(0)\|_B\,d\tau \le \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} 2M|k_n(\tau)|\,d\tau$$

$$= \frac{M}{\pi}\left(\int_{\delta}^{2\pi-\delta} |k_n(\tau)|\,d\tau\right).$$

$$\Longrightarrow \left\|\frac{1}{2\pi}\int k_n(\tau)\phi(\tau)\,d\tau - \phi(0)\right\|_B \le$$

$$K\left(\sup_{|t|\le\delta}\|\phi(t)-\phi(0)\|_B\right) + \frac{M}{\pi}\left(\int_{\delta}^{2\pi-\delta} |k_n(\tau)|\,d\tau\right)$$

Let $\epsilon > 0$. ϕ is continuous at $\tau = 0$.

$$\implies \exists\delta > 0 \text{ s.t. } |\tau| \le \delta \Rightarrow \|\phi(\tau) - \phi(0)\|_B < \frac{\epsilon}{2K} \Rightarrow \sup_{|t|\le\delta} \|\phi(t) - \phi(0)\|_B \le \frac{\epsilon}{2K}.$$

By (S-3), $\displaystyle\lim_{n\to\infty} \int_\delta^{2\pi-\delta} |k_n(t)|\,dt = 0 \implies$

$$\exists N \in \mathbb{N}\ \forall n \ge N,\ \left|\int_\delta^{2\pi-\delta} |k_n(t)|\,dt\right| = \int_\delta^{2\pi-\delta} |k_n(t)|\,dt < \frac{\pi\epsilon}{2M}.$$

All of the above $\implies \left\|\dfrac{1}{2\pi}\displaystyle\int k_n(\tau)\phi(\tau)\,d\tau - \phi(0)\right\|_B < \epsilon.$

$\therefore \forall\epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \ge N,\ \left\|\dfrac{1}{2\pi}\displaystyle\int k_n(\tau)\phi(\tau)\,d\tau - \phi(0)\right\|_B < \epsilon$

Hence, $\displaystyle\lim_{n\to\infty} \left\|\frac{1}{2\pi}\int k_n(\tau)\phi(\tau)\,d\tau - \phi(0)\right\|_B = 0.$ □

Note. (i) It can be shown that the integral in the theorem is independent of which interval of length 2π is chosen as the interval of integration so that the integral is well-defined.

(ii) Let $k \in C(\mathbb{T})$ and ϕ be as in the theorem. Define $F\colon \mathbb{T} \to B$ by $F(\tau) = \frac{1}{2\pi}k(\tau)\phi(\tau)$. By Proposition 2.2.7, F is continuous and Riemann integrable because $\phi\colon \mathbb{T} \to B$ is continuous and $\frac{1}{2\pi}k \in C(\mathbb{T})$. This shows that the Riemann integral in the theorem exists.

Remark. From now on, B is a homogeneous Banach space on $\mathbb{T}$.

Note. Let $F\colon \mathbb{T} \to B$ be Riemann integrable and $S(P,t,F)$ be the Riemann sum of F for the tagged partition (P,t), where P is a partition of an interval of length 2π. By Definition 2.2.6, $\int F(\tau)\,d\tau$ is the unique element of B which satisfies $\forall\epsilon > 0\ \exists\delta > 0\ \forall(P,t),\ \|P\| < \delta \Rightarrow \left\|S(P,t,F) - \int F(\tau)\,d\tau\right\|_B < \epsilon$. Since $\left\|S(P,t,F) - \int F(\tau)\,d\tau\right\|_1 \le \left\|S(P,t,F) - \int F(\tau)\,d\tau\right\|_B$, then

$\forall \epsilon > 0\ \exists \delta > 0\ \forall (P, t)$, $\|P\| < \delta \Rightarrow \left\|S(P, t, F) - \int F(\tau)\, d\tau\right\|_1 < \epsilon$. Thus, F is Riemann integrable in $L^1(\mathbb{T})$ and since the Riemann integral of F in $L^1(\mathbb{T})$ is unique, then the Riemann integral of F in B is the same as the Riemann integral of F in $L^1(\mathbb{T})$.

Remark. Here, the presentation of homogeneous Banach spaces on $\mathbb{T}$ differs from that in (Katznelson, 2004). In (Katznelson, 2004), the following corollary and theorems are proven only when $B = L^1(\mathbb{T})$. Then homogeneous Banach spaces on $\mathbb{T}$ are defined. Finally, Theorem 3.3.3, the first part of the proof of the following theorem, and the case when $B = L^1(\mathbb{T})$ are used to prove the general case. The following corollary corresponds to (Katznelson, 2004, P. 11, Theorem 1.2.3), the following lemma and theorem correspond to (Katznelson, 2004, P. 11-12, Lemma 1.2.4), and Theorem 3.3.7 corresponds to (Katznelson, 2004, P. 15-16, Theorem 1.2.11).

In the previous theorem, let ϕ be as in (H-3) of Definition 3.3.1 and note that $\phi(0) = f$. Then,

Corollary 3.3.4. If $f \in B$ and $\{k_n\}_{n=0}^{\infty}$ is a summability kernel, then $\lim\limits_{n\to\infty} \dfrac{1}{2\pi} \displaystyle\int k_n(\tau) f_\tau\, d\tau = f$ in B.

In the next lemma, let $B = C(\mathbb{T})$.

Lemma 3.3.5. If $k, f \in C(\mathbb{T})$, then $\dfrac{1}{2\pi} \displaystyle\int k(\tau) f_\tau\, d\tau = k * f$.

Proof. Define $F\colon \mathbb{T} \to C(\mathbb{T})$ by $F(\tau) = \dfrac{1}{2\pi} k(\tau)\phi(\tau) = \dfrac{1}{2\pi} k(\tau) f_\tau$. By the note (ii) after the previous theorem, F is continuous and Riemann integrable. Note that $\displaystyle\int F(\tau)\, d\tau = \dfrac{1}{2\pi} \int k(\tau) f_\tau\, d\tau$ and that by Theorem 3.2.6.(2),

$$\int [F(\tau)](x)\,d\tau = \frac{1}{2\pi}\int k(\tau)f_\tau(x)\,d\tau = (f*k)(x) = (k*f)(x).$$

Thus, to show that $\frac{1}{2\pi}\int k(\tau)f_\tau\,d\tau = k*f$, it must be shown that $\forall x \in \mathbb{R}$, $\left[\int F(\tau)\,d\tau\right](x) = \int [F(\tau)](x)\,d\tau$.

By Definition 2.2.6,

$\forall \epsilon > 0\ \exists \delta > 0\ \forall (P,t),\ \|P\| < \delta \Rightarrow \left\|S(P,t,F) - \int F(\tau)\,d\tau\right\|_\infty < \epsilon$. Fix $x \in \mathbb{R}$. Since $\left|[S(P,t,F)](x) - \left[\int F(\tau)\,d\tau\right](x)\right| \le \left\|S(P,t,F) - \int F(\tau)\,d\tau\right\|_\infty$, then $\forall \epsilon > 0\ \exists \delta > 0\ \forall (P,t), \|P\| < \delta \Rightarrow \left|[S(P,t,F)](x) - \left[\int F(\tau)\,d\tau\right](x)\right| < \epsilon$.

Let $f^x(\tau) = f(x-\tau)$. Since $f \in C(\mathbb{T})$, then $f^x \in C(\mathbb{T})$. Define $G\colon \mathbb{T} \to \mathbb{C}$ by $G(\tau) = \frac{1}{2\pi}k(\tau)f^x(\tau)$. $G \in C(\mathbb{T})$ because $k, f^x \in C(\mathbb{T})$. Also, $[F(\tau)](x) = \frac{1}{2\pi}k(\tau)f_\tau(x) = \frac{1}{2\pi}k(\tau)f^x(\tau) = G(\tau)$. Let $P = \{\tau_j\}_{j=0}^n$, where $n \in \mathbb{N}$. By Definition 2.2.5, $S(P,t,F) = \sum_{j=1}^n (\tau_{j+1} - \tau_j)F(t_j)$ and $[S(P,t,F)](x) = [\sum_{j=1}^n (\tau_{j+1} - \tau_j)F(t_j)](x) = \sum_{j=1}^n (\tau_{j+1} - \tau_j)[F(t_j)](x) = \sum_{j=1}^n (\tau_{j+1} - \tau_j)G(t_j) = S(P,t,G)$.

$$\Longrightarrow \forall \epsilon > 0\ \exists \delta > 0\ \forall (P,t), \|P\| < \delta \Rightarrow \left|S(P,t,G) - \left[\int F(\tau)\,d\tau\right](x)\right| < \epsilon$$

By Corollary 2.2.5, G is Riemann integrable because $G \in C(\mathbb{T})$ and by Definition 2.2.6, the uniqueness of $\int G(\tau)\,d\tau \implies \int G(\tau)\,d\tau = \left[\int F(\tau)\,d\tau\right](x)$. Since $\int G(\tau)\,d\tau = \int [F(\tau)](x)\,d\tau$, then $\left[\int F(\tau)\,d\tau\right](x) = \int [F(\tau)](x)\,d\tau$.

Therefore, $\forall x \in \mathbb{R}$, $\left[\int F(\tau)\,d\tau\right](x) = \int [F(\tau)](x)\,d\tau$.

Hence, $\frac{1}{2\pi}\int k(\tau)f_\tau\,d\tau = k*f$. □

Theorem 3.3.6. If $k \in C(\mathbb{T})$ and $f \in B$, then $\frac{1}{2\pi}\int k(\tau)f_\tau\,d\tau = k*f$ in B.

Proof. Let ϕ be as in (H-3) and F be as in the note (ii) after the previous theorem. Then, $\int F(\tau)\,d\tau = \frac{1}{2\pi}\int k(\tau)f_\tau\,d\tau$. By the note before Corollary 3.3.4, the Riemann integral of F in B is the same as the Riemann integral of F in $L^1(\mathbb{T})$ and this implies the following. If it is shown that $\frac{1}{2\pi}\int k(\tau)f_\tau\,d\tau = k * f$ in $L^1(\mathbb{T})$, then $\frac{1}{2\pi}\int k(\tau)f_\tau\,d\tau = k * f$ in B. Therefore, the theorem only needs to be proven when $B = L^1(\mathbb{T})$.

Let $\epsilon > 0$. By Proposition 3.1.7, $\exists g \in C(\mathbb{T})$ s.t. $\|f-g\|_1 < \frac{\epsilon}{2\|k\|_1+1}$. By the previous lemma, $\frac{1}{2\pi}\int k(\tau)g_\tau\,d\tau = k * g$.

$$\left[\frac{1}{2\pi}\int k(\tau)f_\tau\,d\tau - (k*f)\right] =$$
$$\left[\frac{1}{2\pi}\int k(\tau)f_\tau\,d\tau - \frac{1}{2\pi}\int k(\tau)g_\tau\,d\tau\right] + [(k*g)-(k*f)]$$
$$= \left[\frac{1}{2\pi}\int k(\tau)(f-g)_\tau\,d\tau\right] + [k*(g-f)]$$

$$\implies \left\|\frac{1}{2\pi}\int k(\tau)f_\tau\,d\tau - (k*f)\right\|_1 \le \left\|\frac{1}{2\pi}\int k(\tau)(f-g)_\tau\,d\tau\right\|_1 + \|k*(g-f)\|_1$$

By Theorem 2.2.6.(3) and (H-2),

$$\left\|\frac{1}{2\pi}\int k(\tau)(f-g)_\tau\,d\tau\right\|_1 \le \frac{1}{2\pi}\int \|k(\tau)(f-g)_\tau\|_1\,d\tau = \frac{1}{2\pi}\int |k(\tau)|\;\|(f-g)_\tau\|_1\,d\tau$$
$$= \frac{1}{2\pi}\int |k(\tau)|\;\|f-g\|_1\,d\tau = \left(\frac{1}{2\pi}\int |k(\tau)|\,d\tau\right)\|f-g\|_1 = \|k\|_1\|f-g\|_1.$$

By Theorem 3.2.5, $\|k * (g-f)\|_1 \leq \|k\|_1 \|g-f\|_1 = \|k\|_1 \|f-g\|_1$.

$$\implies \left\| \frac{1}{2\pi} \int k(\tau) f_\tau \, d\tau - (k * f) \right\|_1 \leq 2\|k\|_1 \|f-g\|_1 < 2\|k\|_1 \left(\frac{\epsilon}{2\|k\|_1 + 1} \right)$$

$$= \left(\frac{2\|k\|_1}{2\|k\|_1 + 1} \right) \epsilon < \epsilon$$

$$\therefore \forall \epsilon > 0, \ 0 \leq \left\| \frac{1}{2\pi} \int k(\tau) f_\tau \, d\tau - (k * f) \right\|_1 < \epsilon$$

Hence, $\left\| \frac{1}{2\pi} \int k(\tau) f_\tau \, d\tau - (k * f) \right\|_1 = 0$ and $\frac{1}{2\pi} \int k(\tau) f_\tau \, d\tau = k * f$ in $L^1(\mathbb{T})$.

□

Note. It follows from the theorem that if $k \in C(\mathbb{T})$ and $f \in B$, then $(k * f) \in B$. Also, by Theorem 2.2.6.(3) and (H-2),

$$\|k * f\|_B = \left\| \frac{1}{2\pi} \int k(\tau) f_\tau \, d\tau \right\|_B \leq \frac{1}{2\pi} \int \|k(\tau) f_\tau\|_B \, d\tau = \frac{1}{2\pi} \int |k(\tau)| \ \|f_\tau\|_B \, d\tau$$

$$= \frac{1}{2\pi} \int |k(\tau)| \ \|f\|_B \, d\tau = \left(\frac{1}{2\pi} \int |k(\tau)| \, d\tau \right) \|f\|_B = \|k\|_1 \|f\|_B.$$

By Corollary 3.3.4 and Theorem 3.3.6, the following holds.

Theorem 3.3.7. If $f \in B$ and $\{k_n\}_{n=0}^{\infty}$ is a summability kernel, then $\lim_{n \to \infty} (k_n * f) = f$ in B.

Remark. The only summability kernel that will be considered here is the Fejér kernel. For more examples of summability kernels, see (Katznelson, 2004, P. 16-17). Before the Fejér kernel is defined, the Dirichlet kernel will be defined because, even though the Dirichlet kernel is not a summability kernel, the Fejér kernel is derived from the Dirichlet kernel.

Definition 3.3.3. $\forall n \in \mathbb{N}_0$, the n-th partial sum of $S[f]$ is $S_n(f)$, where $S_n(f)(t) = \sum_{j=-n}^{n} \hat{f}(j) e^{ijt}$.

Note. By the remark after Definition 3.2.1 and the note after Definition 3.2.3,

$$S_n(f)(t) = \frac{c_0(f)}{2} + \sum_{j=1}^{n}(c_j(f)\cos(jt) + b_j(f)\sin(jt)).$$

Remark. Recall from the note after Definition 3.2.1 that $\mathbf{T}$ is the set of trigonometric polynomials. Let $P \in \mathbf{T}$ be as in Definition 3.2.1. By the remark after Proposition 3.2.1, $\forall n \in \mathbb{Z}$, $\hat{P}(n) = a_n$. This implies that $\forall n \in \mathbb{N}_0$ s.t. $n \geq \deg(P)$, $S_n(P) = P$ and so $S[P] = P$.

Definition 3.3.4. The Dirichlet kernel is the sequence $\{D_n\}_{n=0}^{\infty} \subseteq C(\mathbb{T})$ given by $D_n(t) = \sum_{j=-n}^{n} e^{ijt}$. By Corollary 3.2.8, $S_n(f) = (D_n * f)$.

The following proposition lists some properties of the Dirichlet kernel.

Proposition 3.3.8. (i) $D_n(t) = 1 + 2\sum_{j=1}^{n}\cos(jt)$

(ii) D_n is an even function.

(iii)

$$D_n(t) = \begin{cases} \dfrac{\sin\left(\dfrac{(2n+1)t}{2}\right)}{\sin\left(\dfrac{t}{2}\right)}, & \text{if } t \notin [0] \\ 2n+1, & \text{if } t \in [0] \end{cases}$$

(iv) $\forall k \in \mathbb{Z}$, $\operatorname{sgn}(D_n(t))$ is nonzero and constant on $\left(\dfrac{2k\pi}{2n+1}, \dfrac{2(k+1)\pi}{2n+1}\right)$.

Proof. (i) It follows from the remark after Definition 3.2.1.

(ii) This follows from (i) because the cosine function is an even function.

(iii) If $t \in [0]$, then $D_n(t) = D_n(0) = 2n + 1$. Now assume $t \notin [0]$, which implies that $(e^{it} - 1) \neq 0$.

$$D_n(t) = \sum_{j=-n}^{n} (e^{it})^j = \sum_{j=0}^{2n} (e^{it})^{j-n} = e^{-int} \sum_{j=0}^{2n} (e^{it})^j = e^{-int} \left(\frac{e^{i(2n+1)t} - 1}{e^{it} - 1} \right)$$

$$= \frac{e^{i(n+1)t} - e^{-int}}{e^{it} - 1} = \frac{e^{-\frac{it}{2}}(e^{i(n+1)t} - e^{-int})}{e^{-\frac{it}{2}}(e^{it} - 1)} = \frac{\dfrac{e^{i\left(\frac{2n+1}{2}\right)t} - e^{-i\left(\frac{2n+1}{2}\right)t}}{2i}}{\dfrac{e^{\frac{it}{2}} - e^{-\frac{it}{2}}}{2i}}$$

$$= \frac{\sin\left(\frac{(2n+1)t}{2}\right)}{\sin\left(\frac{t}{2}\right)}$$

(iv) By (iii), if $t \neq \left(\frac{2}{2n+1}\right) k\pi$, where $k \in \mathbb{Z}$, then $\operatorname{sgn}(D_n(t))$ is nonzero and $\operatorname{sgn}(D_n(t)) = \operatorname{sgn}\left(\sin\left(\frac{t}{2}\right)\right) \operatorname{sgn}\left(\sin\left(\frac{(2n+1)t}{2}\right)\right)$. By Section 2.4.2,

$$\operatorname{sgn}\left(\sin\left(\frac{t}{2}\right)\right) = \begin{cases} (-1)^k, & \text{if } x \in (2k\pi, 2(k+1)\pi) \\ 0, & \text{if } x = 2k\pi \end{cases}$$

and

$$\operatorname{sgn}\left(\sin\left(\frac{(2n+1)t}{2}\right)\right) = \begin{cases} (-1)^k, & \text{if } x \in \left(\frac{2k\pi}{2n+1}, \frac{2(k+1)\pi}{2n+1}\right) \\ 0, & \text{if } x = \frac{2k\pi}{2n+1} \end{cases}$$

, where $k \in \mathbb{Z}$.

This implies that the result holds because $\operatorname{sgn}\left(\sin\left(\frac{t}{2}\right)\right)$ and $\operatorname{sgn}\left(\sin\left(\frac{(2n+1)t}{2}\right)\right)$ are constant on $\left(\frac{2k\pi}{2n+1}, \frac{2(k+1)\pi}{2n+1}\right)$, where $k \in \mathbb{Z}$.

$\square$

Note. By the remark after Proposition 3.2.1, the Dirichlet kernel satisfies (S-1). However, the Dirichlet kernel is not a summability kernel because it can be shown that (S-2) and (S-3) are not satisfied. (In the next chapter, it will be shown that (S-2) is not satisfied and more specifically, $\lim_{n\to\infty} \|D_n\|_1 = \infty$.) This is the reason why the problem of convergence for Fourier series is very difficult compared to the problem of summability.

The motivation for summability comes from the following lemma which is proven in (Korner, 1988, P. 4).

Lemma 3.3.9. Let $\{s_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$. Define the sequence of arithmetic means $\{\sigma_n\}_{n=0}^{\infty}$ by $\sigma_n = \dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} s_n$.

(i) If $\lim_{n\to\infty} s_n = s$, then $\lim_{n\to\infty} \sigma_n = s$.

(ii) There exist sequences $\{s_n\}_{n=0}^{\infty}$ s.t. $\lim_{n\to\infty} s_n$ does not exist but $\lim_{n\to\infty} \sigma_n$ exists.

Definition 3.3.5. $\forall n \in \mathbb{N}_0$, the n-th Cesàro sum of $S[f]$ is $\sigma_n(f)$, where $\sigma_n(f) = \dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} S_n(f)$.

Remark. By the first part of the lemma, if $\lim_{n\to\infty} S_n(f)(t)$ exists, then $\lim_{n\to\infty} \sigma_n(f)(t) = \lim_{n\to\infty} S_n(f)(t)$.

Note. By Theorem 3.2.6.(1), $\sigma_n(f) = \dfrac{1}{n+1}\sum_{j=0}^{n} S_n(f) = \dfrac{1}{n+1}\sum_{j=0}^{n} (D_n * f) = \left(\dfrac{1}{n+1}\sum_{j=0}^{n} D_n\right) * f$.

Definition 3.3.6. The Fejér kernel is the sequence $\{K_n\}_{n=0}^{\infty} \subseteq C(\mathbb{T})$, where $K_n = \dfrac{1}{n+1}\sum_{j=0}^{n} D_n$. It is easy to see that $K_n(t) = \dfrac{1}{n+1}\sum_{j=0}^{n} D_j(t) =$

$\sum_{j=-n}^{n}\left(1-\frac{|j|}{n+1}\right)e^{ijt}$. Since $\sigma_n(f) = (K_n * f)$, then by Corollary 3.2.8, $\sigma_n(f)(t) = (K_n * f)(t) = \sum_{j=-n}^{n}\left(1-\frac{|j|}{n+1}\right)\hat{f}(j)e^{ijt}$.

The following lemma is used to show that the Fejér kernel is a summability kernel.

Lemma 3.3.10.

$$K_n(t) = \begin{cases} \dfrac{1}{n+1}\left[\dfrac{\sin\left(\dfrac{(n+1)t}{2}\right)}{\sin\left(\dfrac{t}{2}\right)}\right]^2, & \text{if } t \notin [0] \\ n+1, & \text{if } t \in [0] \end{cases}$$

Proof. The proof will not be shown because it is very long even though it is simple. For a proof, see (Katznelson, 2004, P. 12-13) or (Korner, 1988, P. 6-7). □

Proposition 3.3.11. The Fejér kernel is a positive summability kernel.

Proof. By the lemma, $\forall n \in \mathbb{N}_0$, $K_n \geq 0$. By the remark after Proposition 3.2.1, (S-1) is satisfied. By the remark after Definition 3.3.2, (S-2) is satisfied. Thus, it is only necessary to show (S-3). Let $0 < \delta < \pi$ and $t \in [\delta, 2\pi - \delta]$. Then, $\frac{t}{2} \in \left[\frac{\delta}{2}, \pi - \frac{\delta}{2}\right]$. First, let $t \in [\delta, \pi]$. Then $\frac{t}{2} \in \left[\frac{\delta}{2}, \frac{\pi}{2}\right]$. Since $\sin(x)$ is increasing on $\left[0, \frac{\pi}{2}\right]$ and $\forall u \in [0, \pi]$, $\sin\left(\frac{u}{2}\right) \geq \frac{u}{\pi}$, then $\sin\left(\frac{t}{2}\right) \geq \sin\left(\frac{\delta}{2}\right) \geq \frac{\delta}{\pi}$.

Now, let $t \in [\pi, 2\pi - \delta]$. Since $\sin(x) = \sin(\pi - x)$, then $\frac{t}{2} \in \left[\frac{\pi}{2}, \pi - \frac{\delta}{2}\right]$, $\left(\pi - \frac{t}{2}\right) \in \left[\frac{\delta}{2}, \frac{\pi}{2}\right]$ and $\sin\left(\frac{t}{2}\right) = \sin\left(\pi - \frac{t}{2}\right) \geq \frac{\delta}{\pi}$.

$\implies \forall t \in [\delta, 2\pi - \delta]$, $\sin\left(\dfrac{t}{2}\right) \geq \dfrac{\delta}{\pi} > 0$ and $|K_n(t)| = K_n(t) =$

$$\frac{1}{n+1}\left[\frac{\sin\left(\dfrac{(n+1)t}{2}\right)}{\sin\left(\dfrac{t}{2}\right)}\right]^2 \leq \frac{1}{n+1}\left[\frac{1}{\sin\left(\dfrac{t}{2}\right)}\right]^2 \leq \frac{1}{n+1}\left(\frac{\pi}{\delta}\right)^2.$$

$$\implies \int_\delta^{2\pi-\delta} |K_n(t)|\, dt \leq \int_\delta^{2\pi-\delta} \frac{1}{n+1}\left(\frac{\pi}{\delta}\right)^2 dt = \frac{2\pi - 2\delta}{n+1}\left(\frac{\pi}{\delta}\right)^2 = \frac{2(\pi-\delta)}{n+1}\left(\frac{\pi}{\delta}\right)^2$$

Since $0 \leq \displaystyle\int_\delta^{2\pi-\delta} |K_n(t)|\, dt \leq \frac{2(\pi-\delta)}{n+1}\left(\frac{\pi}{\delta}\right)^2$ and $\displaystyle\lim_{n\to\infty} \frac{2(\pi-\delta)}{n+1}\left(\frac{\pi}{\delta}\right)^2 = 0$, then $\displaystyle\lim_{n\to\infty}\int_\delta^{2\pi-\delta} |K_n(t)|\, dt = 0$.

$\therefore \forall\ 0 < \delta < \pi$, $\displaystyle\lim_{n\to\infty}\int_\delta^{2\pi-\delta} |K_n(t)|\, dt = 0$

Hence, (S-3) holds and so the Fejér kernel is a positive summability kernel.

□

By applying Theorem 3.3.7 with the Fejér kernel yields

Corollary 3.3.12. If $f \in B$, then $\displaystyle\lim_{n\to\infty} \sigma_n(f) = f$ in B.

Note. Since $\sigma_n(f) = \dfrac{1}{n+1}\sum_{j=0}^n S_n(f)$, it follows that if $S[f]$ converges in B, then the limit must be f.

Remark. Due to the corollary, it is said that B admits summability in norm and that $S[f]$ is summable in norm to f.

The following corollary corresponds to (Katznelson, 2004, P. 16, Theorem 1.2.12).

Corollary 3.3.13. $\mathbf{T} \cap B$ is dense in B.

Proof. By the note after Theorem 3.3.6, if $f \in B$, then $\sigma_n(f) = (K_n * f) \in B$ and by Definition 3.3.6, $\sigma_n(f) \in \mathbf{T}$, which implies that $\sigma_n(f) \in \mathbf{T} \cap B$. Then the result follows from the previous corollary. □

Remark. If $B = C(\mathbb{T})$ or $L^p(\mathbb{T})$, where $p < \infty$, then by the note after Definition 3.2.1 and Corollary 3.1.8, $\mathbf{T} \cap B = \mathbf{T}$. Thus, $\mathbf{T}$ is dense in B.

The following corollary and theorems correspond resp. to (Katznelson, 2004, P. 16, Corollary 1.2.12), (Katznelson, 2004, P. 13, Theorem 1.2.7), and (Katznelson, 2004, P. 13, Theorem 1.2.8).

The next corollary follows from Corollary 3.3.12 with $B = C(\mathbb{T})$.

Corollary 3.3.14 (Weierstrass Approximation Theorem). Every continuous 2π-periodic function can be approximated uniformly by trigonometric polynomials.

Theorem 3.3.15 (Uniqueness Theorem). If $f \in B$ and $\forall n \in \mathbb{Z}$, $\hat{f}(n) = 0$, then $f = 0$ in B.

Proof. Since $\forall n \in \mathbb{N}_0$, $\sigma_n(f) = 0$ and $\lim_{n\to\infty} \sigma_n(f) = f$ in B, then $f = 0$ in B. □

Remark. The previous theorem is equivalent to the following:
If $f, g \in B$ and $\forall n \in \mathbb{Z}$, $\hat{f}(n) = \hat{g}(n)$, then $f = g$ in B.

Theorem 3.3.16 (Riemann-Lebesgue lemma). If $f \in L^1(\mathbb{T})$, then $\lim_{|n|\to\infty} \hat{f}(n) = 0$.

Proof. Let $B = L^1(\mathbb{T})$ and $\epsilon > 0$. By the remark after Corollary 3.3.13, $\exists P \in \mathbf{T}$ s.t. $\|f - P\|_1 < \epsilon$. If $|n| > \deg(P)$, then $\hat{P}(n) = 0$ and by Theorem 3.2.2.(1), $\hat{f}(n) = \hat{f}(n) - \hat{P}(n) = (\widehat{f - P})(n)$. Then, $|\hat{f}(n)| = |(\widehat{f - P})(n)| \le \|f - P\|_1 < \epsilon$.

Let $N = (\deg(P) + 1)$. Hence, $\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall\ |n| \geq N, |\hat{f}(n)| < \epsilon$ and so $\lim\limits_{|n| \to \infty} \hat{f}(n) = 0$. □

The following remark is from (Katznelson, 2004, P. 14).

Remark. Let C be a compact set in $L^1(\mathbb{T})$ and $\epsilon > 0$. Then there exists a subset $\{P_j\}_{j=1}^N \subseteq \mathbf{T}$ s.t. $\forall f \in C\ \exists\ 1 \leq j \leq N$ satisfying $\|f - P_j\|_1 < \epsilon$. If $|n| > \max_{1 \leq j \leq N} \deg(P_j)$, then $\forall f \in C$, $|\hat{f}(n)| < \epsilon$. Thus, the Riemann-Lebesgue lemma holds uniformly on compact subsets of $L^1(\mathbb{T})$.

3.4 Boundedness of the Fourier Coefficients

Remark. By the note after Definition 3.2.3, if $\forall n \in \mathbb{N}$, $\hat{f}(n) = -\hat{f}(-n)$, then $S[f]$ is a sine series and if $\forall n \in \mathbb{N}$, $\hat{f}(n) = \hat{f}(-n)$, then $S[f]$ is a cosine series.

The following lemma is a simple application of the Monotone Convergence Theorem in Measure Theory.

Lemma 3.4.1. Given $\{a_{n,j}\}_{n,j=1}^{\infty}$ s.t. $\forall n, j \in \mathbb{N}$, $a_{n,j} \in [0, \infty]$ and $a_{n,j} \leq a_{n+1,j}$, then $\lim\limits_{n \to \infty} \sum\limits_{j=1}^{\infty} a_{n,j} = \sum\limits_{j=1}^{\infty} \lim\limits_{n \to \infty} a_{n,j}$.

Remark. The following theorem uses Fejér's Theorem, which will be stated and proven in Section 4.1.

Theorem 3.4.2. If $f \in L^1(\mathbb{T})$ and $\forall n \in \mathbb{N}$, $\hat{f}(n) = -\hat{f}(-n) \geq 0$, then $\sum\limits_{n=1}^{\infty} \dfrac{\hat{f}(n)}{n} < \infty$.

Proof. W.L.O.G. $\hat{f}(0) = 0$ [Otherwise, replace f by g, where $g(t) = f(t) - \hat{f}(0)$. By Theorem 3.2.2.(1) and the remark after Definition 3.3.3, $\hat{g}(0) = 0$ and $\forall n \in \mathbb{Z}\backslash\{0\}$, $\hat{g}(n) = \hat{f}(n)$.] Then $\int f(t)\, dt = 2\pi \hat{f}(0) = 0$. Define $F\colon \mathbb{T} \to \mathbb{C}$

by $F(y) = \int_0^y f(x)\,dx$. By Theorem 3.2.4, $F \in C(\mathbb{T})$ and $\forall n \in \mathbb{Z}$ s.t. $n \neq 0$, $\hat{F}(n) = \dfrac{\hat{f}(n)}{in}$.

$$\forall n \in \mathbb{N},\ \hat{F}(-n) = \frac{\hat{f}(-n)}{i(-n)} = \frac{-\hat{f}(n)}{-in} = \frac{\hat{f}(n)}{in} = \hat{F}(n)$$

By Fejér's Theorem, $\lim_{n\to\infty} \sigma_n(F)(0) = F(0) = 0$ because F is continuous at $y = 0$.

$$\sigma_n(F)(0) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) \hat{F}(j) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) \hat{F}(|j|)$$
$$= \hat{F}(0) + 2\sum_{j=1}^{n} \left(1 - \frac{j}{n+1}\right) \hat{F}(j) = \hat{F}(0) + 2\sum_{j=1}^{n} \left(1 - \frac{j}{n+1}\right) \left(\frac{\hat{f}(j)}{ij}\right)$$
$$= \hat{F}(0) + \frac{2}{i}\sum_{j=1}^{n} \left(1 - \frac{j}{n+1}\right) \left(\frac{\hat{f}(j)}{j}\right)$$
$$\Longrightarrow \lim_{n\to\infty} \left[\hat{F}(0) + \frac{2}{i}\sum_{j=1}^{n} \left(1 - \frac{j}{n+1}\right) \left(\frac{\hat{f}(j)}{j}\right)\right] = 0$$
$$\Longrightarrow \lim_{n\to\infty} \sum_{j=1}^{n} \left(1 - \frac{j}{n+1}\right) \left(\frac{\hat{f}(j)}{j}\right) = \frac{-\hat{F}(0)i}{2} = \frac{\hat{F}(0)}{2i}$$

Define $\{a_{n,j}\}_{n,j=1}^{\infty}$ as follows.

$$a_{n,j} = \begin{cases} 0, & \text{if } n < j \\ \left(1 - \dfrac{j}{n+1}\right)\left(\dfrac{\hat{f}(j)}{j}\right), & \text{if } n \geq j \end{cases}$$

It is easy to see that $\{a_{n,j}\}_{n,j=1}^{\infty}$ satisfies the conditions of the previous lemma. By the lemma, $\lim_{n\to\infty}\sum_{j=1}^{\infty} a_{n,j} = \sum_{j=1}^{\infty}\lim_{n\to\infty} a_{n,j}$.

$$\lim_{n\to\infty}\sum_{j=1}^{\infty} a_{n,j} = \lim_{n\to\infty}\sum_{j=1}^{n}\left(1-\frac{j}{n+1}\right)\left(\frac{\hat{f}(j)}{j}\right) \text{ and } \sum_{j=1}^{\infty}\lim_{n\to\infty} a_{n,j} = \sum_{j=1}^{\infty}\left(\frac{\hat{f}(j)}{j}\right)$$
$$\Longrightarrow \sum_{j=1}^{\infty}\left(\frac{\hat{f}(j)}{j}\right) = \lim_{n\to\infty}\sum_{j=1}^{n}\left(1-\frac{j}{n+1}\right)\left(\frac{\hat{f}(j)}{j}\right) = \frac{\hat{F}(0)}{2i} < \infty$$

□

Corollary 3.4.3. If $\forall n \in \mathbb{N}$, $a_n > 0$ and $\sum_{n=1}^{\infty}\left(\frac{a_n}{n}\right) = \infty$, then $\sum_{n=1}^{\infty} a_n \sin(nt)$ is not a Fourier series. Hence there exist trigonometric series with coefficients tending to 0 which are not Fourier series.

Remark. By the corollary, $\sum_{n=2}^{\infty}\frac{\sin(nt)}{\log n} = -i\sum_{|n|\geq 2}\frac{\operatorname{sgn}(n)}{2\log|n|}e^{int}$ is not a Fourier series.

Definition 3.4.1. $F\colon \mathbb{T} \to \mathbb{C}$ is absolutely continuous if $\exists f \in L^1(\mathbb{T})$ s.t. $\int f(t)\,dt = 0$ and $F(y) = F(0) + \int_0^y f(x)\,dx$. Let $AC(\mathbb{T})$ be the set of absolutely continuous functions.

Note. By the argument used in the proof of Theorem 3.2.4, $AC(\mathbb{T}) \subsetneq C(\mathbb{T})$. Also, it can be shown that $F \in AC(\mathbb{T})$ iff F is absolutely continuous on an interval of length 2π.

Theorem 3.4.4. If $F \in AC(\mathbb{T})$, then $\hat{F}(n) = o\left(\frac{1}{n}\right)$ as $|n| \to \infty$.

Proof. $F \in AC(\mathbb{T}) \Rightarrow \exists f \in L^1(\mathbb{T})$ s.t. $\int f(t)\,dt = 0$ and $F(y) = F(0) + \int_0^y f(x)\,dx$. By the remark after Definition 3.3.3, Theorem 3.2.2.(1), and Theorem 3.2.4, $\forall n \in \mathbb{Z}$ s.t. $n \neq 0$, $\hat{F}(n) = \frac{\hat{f}(n)}{in}$. By the Riemann-Lebesgue Lemma, $\lim_{|n|\to\infty} n\hat{F}(n) = \lim_{|n|\to\infty}\frac{\hat{f}(n)}{i} = \frac{1}{i}\lim_{|n|\to\infty}\hat{f}(n) = 0$ and the result holds. □

The following remark is from (Katznelson, 2004, P. 26).

Remark. If f is k-times differentiable and $f^{(k-1)} \in AC(\mathbb{T})$, then by repeated application of the previous theorem, $\hat{f}(n) = o(n^{-k})$ as $|n| \to \infty$. Moreover, if $0 \leq j \leq k$, then $\hat{f}(n) = (in)^j \left(\widehat{f^{(j)}}(n)\right)$ and so $|\hat{f}(n)| \leq |n|^{-j} \|f^{(j)}\|_1$.

Theorem 3.4.5. If f is k-times differentiable and $f^{(k-1)} \in AC(\mathbb{T})$, then $|\hat{f}(n)| \leq \min\limits_{0 \leq j \leq k} \dfrac{\|f^{(j)}\|_1}{|n|^j}$. If f is infinitely differentiable, then $|\hat{f}(n)| \leq \min\limits_{j \geq 0} \dfrac{\|f^{(j)}\|_1}{|n|^j}$.

The notation from Section 2.3.2 will be used for the following definitions. The following definition is from (Edwards, 1979, P. 33).

Definition 3.4.2. $f\colon \mathbb{T} \to \mathbb{C}$ is of bounded variation if $V(f) = \sup\limits_{P} A(P) < \infty$, where the supremum is taken over all partitions P of intervals of length 2π. Let $BV(\mathbb{T})$ be the set of functions of bounded variation.

Note. $f \in BV(\mathbb{T}) \Rightarrow f \in BV(I)$ for any interval I of length 2π. Also, according to (Edwards, 1979, P. 16, 33), it can be shown that if $f\colon \mathbb{T} \to \mathbb{C}$ is of bounded variation on some interval of length 2π, then f is of bounded variation on all intervals of length 2π, $f \in BV(\mathbb{T})$, and for any interval I of length 2π, $V_f(I) = V(f)$.

Thus, $BV(\mathbb{T})$ could be defined alternatively as follows:

Definition 3.4.3. $BV(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C} \mid f \in BV(I)\}$, where I is any interval of length 2π.

Proposition 3.4.6. $BV(\mathbb{T}) \subsetneq L^\infty(\mathbb{T})$ and so $BV(\mathbb{T}) \subsetneq L^1(\mathbb{T})$.

Proof. $f \in BV(\mathbb{T}) \implies f \in BV([0, 2\pi])$ By the remark after Definition 2.3.3, f is bounded on $[0, 2\pi]$. Since f is 2π-periodic, then f is bounded on $\mathbb{R}$. Then

$f \in L^\infty(\mathbb{T})$ because f is bounded. Hence, $BV(\mathbb{T}) \subseteq L^\infty(\mathbb{T})$ and by Corollary 3.1.6, the last part holds. □

Remark. By Theorem 2.3.8 and the notes after Definitions 3.4.1 and 3.4.2, $AC(\mathbb{T}) \subseteq BV(\mathbb{T})$.

Theorem 3.4.7. If $f \in BV(\mathbb{T})$, then $\forall n \in \mathbb{Z}\backslash\{0\}$, $|\hat{f}(n)| \leq \dfrac{V(f)}{2\pi|n|}$.

Proof.

$$\hat{f}(n) = \frac{1}{2\pi}\int f(t)e^{-int}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,dt$$

$f \in BV(\mathbb{T}) \Rightarrow f \in BV([0,2\pi])$; By Theorem 2.3.6.(iii), f is continuous m-a.e. on $[0,2\pi]$ because $f \in BV([0,2\pi])$. Since e^{-int} is continuous on $[0,2\pi]$, then $f(t)e^{-int}$ is continuous m-a.e. on $[0,2\pi]$. $f(t)e^{-int}$ is bounded on $[0,2\pi]$ because e^{-int} is bounded and f is bounded by the proof of the previous proposition. By Theorem 2.3.1, $f(t)e^{-int}$ is Riemann integrable and the Lebesgue integral in the above equality is a Riemann integral.

$\alpha(t) = \dfrac{e^{-int}}{-in}$ is a function with a continuous derivative $\alpha'(t) = e^{-int}$ and f is bounded on $[0,2\pi]$. By Theorem 2.2.3, $f \in R(\alpha, a, b)$ and

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(t)\alpha'(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,d\alpha(t).$$

By Integration by Parts, $\alpha \in R(f, a, b)$ and

$$\int_0^{2\pi} f(t)\,d\alpha(t) = f(2\pi)\alpha(2\pi) - f(0)\alpha(0) - \int_0^{2\pi} \alpha(t)\,df(t).$$

Since f and α are 2π-periodic functions , then $[f(2\pi)\alpha(2\pi) - f(0)\alpha(0)] = 0$ and so

$$\int_0^{2\pi} f(t)\,d\alpha(t) = -\int_0^{2\pi} \alpha(t)\,df(t) = \frac{1}{in}\int_0^{2\pi} e^{-int}\,df(t).$$

$$\implies \hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,d\alpha(t) = \frac{1}{2\pi in}\int_0^{2\pi} e^{-int}\,df(t)$$

By Proposition 2.3.7 and noting that e^{-int} is bounded by 1 & $V_f([0,2\pi]) = V(f)$,

$$|\hat{f}(n)| = \left|\frac{1}{2\pi in}\int_0^{2\pi} e^{-int}\,df(t)\right| = \frac{1}{2\pi|n|}\left|\int_0^{2\pi} e^{-int}\,df(t)\right| \le \frac{1}{2\pi|n|}(V(f)) = \frac{V(f)}{2\pi|n|}.$$

$$\therefore \forall n \in \mathbb{Z}\backslash\{0\},\ |\hat{f}(n)| \le \frac{V(f)}{2\pi|n|}$$

□

Definition 3.4.4. For $f \in C(\mathbb{T})$, the modulus of continuity of f is $\omega(f,h) = \sup\limits_{|\eta|\le h} \|f(t+\eta) - f(t)\|_\infty$ and for $f \in L^1(\mathbb{T})$, the integral modulus of continuity of f is $\Omega(f,h) = \sup\limits_{|\eta|\le h} \|f(t+\eta) - f(t)\|_1$. By (H-1) with $B = C(\mathbb{T})$, $\forall f \in C(\mathbb{T})$, $\Omega(f,h) \le \omega(f,h)$.

Proposition 3.4.8. $\forall n \in \mathbb{Z}\backslash\{0\}$, $|\hat{f}(n)| \le \frac{1}{2}\Omega\left(f, \frac{\pi}{|n|}\right)$

Proof. By Corollary 3.1.9,

$$\hat{f}(n) = \frac{1}{2\pi}\int f(t)e^{-int}\,dt = \frac{1}{2\pi}\int f\left(t+\frac{\pi}{n}\right)e^{-in\left(t+\frac{\pi}{n}\right)}\,dt$$

$$= -\frac{1}{2\pi}\int f\left(t+\frac{\pi}{n}\right)e^{-int}\,dt$$

$$\implies \hat{f}(n) = \frac{\hat{f}(n)+\hat{f}(n)}{2} = -\frac{1}{2}\left[\frac{1}{2\pi}\int\left(f\left(t+\frac{\pi}{n}\right) - f(t)\right)e^{-int}\,dt\right]$$

$$\implies |\hat{f}(n)| \le \frac{1}{2}\left\|f\left(t+\frac{\pi}{n}\right) - f(t)\right\|_1 \le \frac{1}{2}\Omega\left(f, \frac{\pi}{|n|}\right)$$

□

3.5 Fourier series of Functions in $L^2(\mathbb{T})$

The main results concerning Fourier series of $L^2(\mathbb{T})$ functions in this section are just corollaries of some theorems in the theory of Hilbert spaces. Thus, a review of Hilbert spaces will be presented before the main result.

3.5.1 Hilbert Spaces

All the results in this section are from (Katznelson, 2004, P. 29-31) and (Folland, 1999, P. 171-177).

Definition 3.5.1. A complex vector space V is called an inner product space if there is a complex-valued function $\langle \cdot, \cdot \rangle$ on $V \times V$ that satisfies the following four conditions for all $x, y, z \in V$ and $\alpha \in \mathbb{C}$:

(i) $\langle x, x \rangle \geq 0$ with equality $\iff x = \vec{0}$.

(ii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(iii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

(iv) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

Note. An inner product space V is also a normed vector space where the norm induced by the inner product is given by $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$.

Definition 3.5.2. An inner product space V is called a Hilbert space if $(V, \| \ \|)$ is a Banach space.

Remark. For the rest of this subsection, V is always a Hilbert space.

Definition 3.5.3. Let $x, y \in V$ and $X \subseteq V$. x is orthogonal to y if $\langle x, y \rangle = 0$. y is orthogonal to X if y is orthogonal to every element of X. X is called orthogonal if any two distinct vectors in X are orthogonal to each other. An orthogonal set X is called an orthonormal system if $\forall x \in X$, $\langle x, x \rangle = 1$.

Lemma 3.5.1. If $\{\phi_n\}_{n=1}^N$ is a finite orthonormal system and $\{a_n\}_{n=1}^N \subseteq \mathbb{C}$, then $\left\|\sum_{n=1}^N a_n\phi_n\right\|^2 = \sum_{n=1}^N |a_n|^2$.

Corollary 3.5.2. If $\{\phi_n\}_{n=1}^\infty$ is an orthonormal system and $\{a_n\}_{n=1}^\infty \subseteq \mathbb{C}$ is s.t. $\sum_{n=1}^\infty |a_n|^2 < \infty$, then the series $\sum_{n=1}^\infty a_n\phi_n$ converges in V.

Lemma 3.5.3. If $\{\phi_n\}_{n=1}^N$ is a finite orthonormal system, $x \in V$, and $\{a_n\}_{n=1}^N \subseteq \mathbb{C}$ is defined by $a_n = \langle x, \phi_n\rangle$, then $0 \leq \left\|x - \sum_{n=1}^N a_n\phi_n\right\|^2 = \|x\|^2 - \sum_{n=1}^N |a_n|^2$.

Corollary 3.5.4 (Bessel's Inequality). Let $\{\phi_\alpha\}_{\alpha\in A}$ be an orthonormal system, $x \in V$ and $\forall\alpha \in A$, $a_\alpha = \langle x, \phi_\alpha\rangle$. Then $\sum_{\alpha\in A} |a_\alpha|^2 \leq \|x\|^2$ and the set $\{\alpha \in A : a_\alpha \neq 0\}$ is countable.

Definition 3.5.4. A complete orthonormal system is an orthonormal system with the condition that the only vector orthogonal to it is $\vec{0}$.

Theorem 3.5.5. If $\{\phi_\alpha\}_{\alpha\in A}$ is an orthonormal system, then TFAE:

1) $\{\phi_\alpha\}_{\alpha\in A}$ is complete.
2) $\forall x \in V$, $\|x\|^2 = \sum_{\alpha\in A} |\langle x, \phi_\alpha\rangle|^2$
3) $\forall x \in V$, $x = \sum_{\alpha\in A} \langle x, \phi_\alpha\rangle\phi_\alpha$, where the sum on the right has only countably many nonzero terms and converges in the norm topology no matter how these terms are ordered.

Lemma 3.5.6 (Parseval). Let $\{\phi_\alpha\}_{\alpha\in A}$ be a complete orthonormal system and $x, y \in V$. Then $\langle x, y\rangle = \sum_{\alpha\in A} \langle x, \phi_\alpha\rangle\langle\phi_\alpha, y\rangle$.

Note. By Bessel's Inequality, the sum in the above lemma is a countable sum because the sets $\{\alpha \in A : \langle x, \phi_\alpha\rangle \neq 0\}, \{\alpha \in A : \langle\phi_\alpha, y\rangle \neq 0\}$ are countable.

3.5.2 The Hilbert Space $L^2(\mathbb{T})$

Now let $V = L^2(\mathbb{T})$. It is easy to show that $L^2(\mathbb{T})$ is a Hilbert space where the inner product is given by $\langle f, g\rangle = \dfrac{1}{2\pi}\displaystyle\int f(t)\overline{g(t)}\,dt$ and the norm induced by the inner product is the 2-norm.

Proposition 3.5.7. $\{e^{int}\}_{n=-\infty}^{\infty}$ is a complete orthonormal system.

Proof. $\{e^{int}\}_{n=-\infty}^{\infty}$ is an orthonormal system because by Proposition 3.2.1,

$$\forall m, n \in \mathbb{Z},\ \langle e^{imt}, e^{int}\rangle = \frac{1}{2\pi}\int e^{imt}\overline{e^{int}}\,dt = \frac{1}{2\pi}\int e^{i(m-n)t}\,dt = \delta_{(m-n),0} = \delta_{m,n}.$$

Let $f \in L^2(\mathbb{T})$ and assume that $\forall n \in \mathbb{Z}$, $\langle f, e^{int}\rangle = 0$.

$$\langle f, e^{int}\rangle = \frac{1}{2\pi}\int f(t)\overline{e^{int}}\,dt = \frac{1}{2\pi}\int f(t)e^{-int}\,dt = \hat{f}(n)$$

Thus, $\forall n \in \mathbb{Z}$, $\hat{f}(n) = 0$. By the Uniqueness Theorem with $B = L^2(\mathbb{T})$, $f = 0$ in $L^2(\mathbb{T})$. Therefore, the only vector orthogonal to the set $\{e^{int}\}_{n=-\infty}^{\infty}$ is the zero vector. Hence, $\{e^{int}\}_{n=-\infty}^{\infty}$ is a complete orthonormal system. □

Now the main results from the Hilbert space section can be applied to $L^2(\mathbb{T})$ with the complete orthonormal system $\{e^{int}\}_{n=-\infty}^{\infty}$.

Remark. Let $f \in L^2(\mathbb{T})$. By the proof of the proposition, $\forall n \in \mathbb{Z}$, $\langle f, e^{int}\rangle = \hat{f}(n)$.

$$\Longrightarrow \forall n \in \mathbb{N}_0,\ S_n(f) = \sum_{j=-n}^{n} \langle f, e^{int}\rangle e^{ijt} \text{ and } S[f] \sim \sum_{n=-\infty}^{\infty} \langle f, e^{int}\rangle e^{int}$$

The following theorem is from (Katznelson, 2004, P. 32, Theorem 1.5.5).

Theorem 3.5.8. Let $f \in L^2(\mathbb{T})$. Then,

1) $\displaystyle\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi}\int |f(t)|^2\,dt$

2) $\lim_{n\to\infty} S_n(f) = f$ in $L^2(\mathbb{T})$

3) For any square summable sequence $\{a_n\}_{n=-\infty}^{\infty}$ of complex numbers, that is, s.t. $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$, there exists a unique $f \in L^2(\mathbb{T})$ s.t. $a_n = \hat{f}(n)$.

4) Let $f, g \in L^2(\mathbb{T})$. Then $\frac{1}{2\pi} \int f(t)\overline{g(t)}\, dt = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$.

3.6 Convergence in Norm

Definition 3.6.1. Let $f \in B$. $S[f]$ converges in norm to f if $\lim_{n\to\infty} S_n(f) = f$ in B. B admits convergence in norm if $\forall f \in B$, $S[f]$ converges in norm to f.

By Theorem 3.2.6.(1) and the note after Theorem 3.3.6, the following proposition holds.

Proposition 3.6.1. Let $k \in C(\mathbb{T})$. Define $\mathbf{K}\colon B \to B$ by $\mathbf{K}(f) = k * f$. Then $\mathbf{K} \in L(B)$ and $\|\mathbf{K}\|_{op} \leq \|k\|_1$.

Notation. When $k = D_n$, $\mathbf{K}$ is denoted by $\mathbf{S}_n$ and $\mathbf{S}_n(f) = (D_n * f) = S_n(f)$.

The following theorem is from (Katznelson, 2004, P. 68, Theorem 2.1.1) and the proof is basically reproduced as in the book.

Theorem 3.6.2. B admits convergence in norm $\iff$ $\{\|\mathbf{S}_n\|_{op}\}_{n=0}^{\infty}$ is bounded, i.e. $\exists K > 0\ \forall n \in \mathbb{N}_0\ \forall f \in B$, $\|S_n(f)\|_B \leq K\|f\|_B$.

Proof. ($\Rightarrow$) $\forall f \in B$, $\lim_{n\to\infty} S_n(f) = f$ in B. $\implies$ $\lim_{n\to\infty} \|S_n(f)\|_B = \|f\|_B$ $\implies$ $\{\|S_n(f)\|_B\}_{n=0}^{\infty}$ is bounded in $\mathbb{R}$. By the Principle of Uniform Boundedness, $\{\mathbf{S}_n\}_{n=0}^{\infty} \subseteq L(B)$ and $\forall f \in B$, $\{\|S_n(f)\|_B\}_{n=0}^{\infty}$ is bounded in $\mathbb{R}$ $\implies$ $\{\|\mathbf{S}_n\|_{op}\}_{n=0}^{\infty}$ is bounded.

($\Leftarrow$) Let $f \in B$ and $\epsilon > 0$. By Corollary 3.3.13, $\exists P \in \mathbf{T} \cap B$ s.t.

$\|f - P\|_B < \dfrac{\epsilon}{K+1}$. Let $N = \deg(P)$ and $n \geq N$. Then $S_n(P) = P$ and

$$[S_n(f) - f] = [(S_n(f) - S_n(P)] + [P - f] = [S_n(f - P)] + [P - f]$$

$$\implies \|S_n(f) - f\|_B \leq \|S_n(f - P)\|_B + \|P - f\|_B \leq K\|f - P\|_B + \|P - f\|_B$$

$$= K\|f - P\|_B + \|f - P\|_B = (K+1)\|f - P\|_B < \epsilon.$$

$$\therefore \ \forall \epsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N, \|S_n(f) - f\|_B < \epsilon$$

$$\implies \forall f \in B, \lim_{n \to \infty} \|S_n(f) - f\|_B = 0, \text{ i.e. } S[f] \text{ converges in norm to } f.$$

Hence, B admits convergence in norm.

□

The following notation will be used from (Katznelson, 2004, P. 68).

Notation. $\forall n \in \mathbb{N}_0$, let $L_n = \|D_n\|_1$. $\{L_n\}_{n=0}^{\infty}$ are called the Lebesgue constants. In the next chapter, it will be shown that $\lim\limits_{n \to \infty} L_n = \infty$ like a constant multiple of $\log n$.

Note. By the previous proposition, $\forall n \in \mathbb{N}_0$, $\|\mathbf{S}_n\|_{op} \leq L_n$.

Suppose $\exists N \in \mathbb{N}_0 \ \forall n \geq N$, $\|\mathbf{S}_n\|_{op} \geq L_n$. Then $\forall n \geq N$, $\|\mathbf{S}_n\|_{op} = L_n$ and $\{\|\mathbf{S}_n\|_{op}\}_{n=0}^{\infty}$ is not bounded because $\lim\limits_{n \to \infty} L_n = \infty$. By the previous theorem, B does not admit convergence in norm. Hence, to show that B does not admit convergence in norm, it is enough to show that $\exists N \in \mathbb{N}_0 \ \forall n \geq N$, $\|\mathbf{S}_n\|_{op} \geq L_n$.

The following proposition and theorem are stated and proven informally on (Katznelson, 2004, P. 68-69). The proof of the proposition is basically reproduced as in the book.

Proposition 3.6.3. $L^1(\mathbb{T})$ does not admit convergence in norm.

Proof. By the note, it is enough to show that $\exists N \in \mathbb{N}_0\ \forall n \geq N$, $\|\mathbf{S}_n\|_{op} \geq L_n$. Fix $n \in \mathbb{N}_0$. Since the Fejér kernel is a positive summability kernel, then by (S-1), $\forall N \in \mathbb{N}_0$, $\|K_N\|_1 = 1$. By Theorem 3.2.6.(2), $S_n(K_N) = D_n * K_N = K_N * D_n = \sigma_N(D_n)$. By Definition 2.1.4, $\forall N \in \mathbb{N}_0$, $\|\mathbf{S}_n\|_{op} \geq \|S_n(K_N)\|_1 = \|\sigma_N(D_n)\|_1$. By Corollary 3.3.12 with $B = L^1(\mathbb{T})$, $\lim_{N\to\infty} \sigma_N(D_n) = D_n$ in $L^1(\mathbb{T})$. $\implies \lim_{N\to\infty} \|\sigma_N(D_n)\|_1 = \|D_n\|_1 = L_n \implies \|\mathbf{S}_n\|_{op} \geq L_n$.

Hence, $\forall n \in \mathbb{N}_0$, $\|\mathbf{S}_n\|_{op} \geq L_n$. □

Theorem 3.6.4. $C(\mathbb{T})$ does not admit convergence in norm.

As the proof is long, a quick summary of the proof will be presented first which is basically the proof presented on (Katznelson, 2004, P. 69).

Summary. A sequence $\{\psi_n\}_{n=1}^{\infty} \subseteq C(\mathbb{T})$ can be constructed s.t. $\|\psi_n\|_\infty = 1$ and $\psi_n(t) = \operatorname{sgn}(D_n(t))$ except in small intervals around the points of discontinuity of $\operatorname{sgn}(D_n(t))$. If the sum of the lengths of these intervals is less than $\dfrac{\epsilon}{2n}$, then

$$\|\mathbf{S}_n\|_{op} \geq \|S_n(\psi_n)\|_\infty \geq |S_n(\psi_n)(0)| = \left|\frac{1}{2\pi}\int D_n(t)\psi_n(t)\,dt\right| > L_n - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, then $\|\mathbf{S}_n\|_{op} \geq L_n$. Thus, $\forall n \in \mathbb{N}_0$, $\|\mathbf{S}_n\|_{op} \geq L_n$.

Remark. The proof here contains all the details that were omitted in (Katznelson, 2004) like the construction of the sequence $\{\psi_n\}_{n=1}^{\infty}$. The proof is based on a careful examination of $\operatorname{sgn}(D_n(t))$, which was suggested by Professor Klemes.

Proof. By the note, it is enough to show that $\forall n \in \mathbb{N}$, $\|\mathbf{S}_n\|_{op} \geq L_n$. Fix $n \in \mathbb{N}$. The proof will be split up into four parts.

(i) Define $\phi_n \colon \mathbb{R} \to \mathbb{C}$ by $\phi_n(t) = \operatorname{sgn}(D_n(t))$. ϕ_n is a simple function because $\phi_n(\mathbb{R}) = \{-1, 0, 1\}$. ϕ_n is measurable because, by Proposition 3.3.8,

$\phi_n^{-1}(\{-1\}), \phi_n^{-1}(\{1\}) \in \mathcal{B}_{\mathbb{R}}$ as they are each countable unions of intervals and $\phi_n^{-1}(\{0\}) \in \mathcal{B}_{\mathbb{R}}$ because it is a countable set. ϕ_n is an even 2π-periodic function because D_n is an even 2π-periodic function. (Therefore, $\phi_n \colon \mathbb{T} \to \mathbb{C}$ and ϕ_n is an even function.) $\implies$ ϕ_n only needs to be examined on $[0, \pi]$. By Proposition 3.3.8.(iii) and the proof of Proposition 3.3.8.(iv), $\phi_n(0) = 1$ and $\forall x \in (0, \pi]$, $\phi_n(t) = \operatorname{sgn}\left(\sin\left(\frac{(2n+1)t}{2}\right)\right)$ as $\operatorname{sgn}\left(\sin\left(\frac{t}{2}\right)\right) = 1$.

Let $\alpha = \dfrac{2\pi}{2n+1}$ and note that $\dfrac{(2n+1)\alpha}{2} = \pi$.

$$\phi_n(t) = \begin{cases} 1, & \text{if } t \in [0, \alpha) \\ 0, & \text{if } t = k\alpha, \text{ where } 1 \leq k \leq n \\ (-1)^k, & \text{if } t \in (k\alpha, (k+1)\alpha), \text{ where } 1 \leq k \leq n-1 \\ (-1)^n, & \text{if } t \in (n\alpha, \pi] \end{cases}$$

This implies that the points of discontinuity of ϕ_n on $[0, \pi]$ are $\{k\alpha\}_{k=1}^{n}$.

(ii) Let $\epsilon > 0$. Let $\epsilon_0 > 0$ be sufficiently small. ϵ_0 will be chosen later.

Let $A = \bigcup\limits_{k=1}^{n} [k\alpha - \epsilon_0, k\alpha + \epsilon_0]$ and

$B = [0, \alpha - \epsilon_0] \cup \bigcup\limits_{k=1}^{n-1} [k\alpha + \epsilon_0, (k+1)\alpha - \epsilon_0] \cup [n\alpha + \epsilon_0, \pi]$.

The union in each of A and B is assumed to be disjoint and $A \cup B = [0, \pi]$. Now $\psi_n \in C(\mathbb{T})$ will be defined as follows. ψ_n will be an even 2π-periodic function so that it only needs to be defined on $[0, \pi]$. $\psi_n(t) = \phi_n(t)$ on B and is extended linearly on A.

$$\psi_n(t) = \begin{cases} 1, & \text{if } t \in [0, \alpha - \epsilon_0] \\ \dfrac{(-1)^k}{\epsilon_0}(t - k\alpha), & \text{if } t \in [k\alpha - \epsilon_0, k\alpha + \epsilon_0], \text{ where } 1 \leq k \leq n. \\ (-1)^k, & \text{if } t \in [k\alpha + \epsilon_0, (k+1)\alpha - \epsilon_0], \\ & \text{where } 1 \leq k \leq n-1. \\ (-1)^n, & \text{if } t \in [n\alpha + \epsilon_0, \pi] \end{cases}$$

From the above, ψ_n is continuous on $[0, \pi]$. Since ψ_n is an even function, then ψ_n is continuous on $[-\pi, \pi]$. By the note after Proposition 3.1.7, $\psi_n \in C(\mathbb{T})$ because ψ_n is a 2π-periodic function which is continuous on $[-\pi, \pi]$.

(iii) Let $c_n = \psi_n - \phi_n$. $\phi_n \in L^\infty(\mathbb{T})$ because ϕ_n is bounded. By Corollary 3.1.8, $\psi_n \in C(\mathbb{T}) \implies \psi_n \in L^\infty(\mathbb{T})$. Then $c_n \in L^\infty(\mathbb{T})$ because $\psi_n, \phi_n \in L^\infty(\mathbb{T})$. c_n is an even function because ϕ_n and ψ_n are even functions. On B, $c_n = 0$ and on A, $|c_n| \leq 1$.

By Definition 2.1.4, $\|\psi_n\|_\infty = 1 \implies \|\mathbf{S}_n\|_{op} \geq \|S_n(\psi_n)\|_\infty \geq |S_n(\psi_n)(0)|$.

$$\begin{aligned} \psi_n = \phi_n + c_n &\implies S_n(\psi_n)(0) = S_n(\phi_n)(0) + S_n(c_n)(0) \\ &\implies |S_n(\psi_n)(0)| \geq |S_n(\phi_n)(0)| - |S_n(c_n)(0)| \end{aligned}$$

$$S_n(\phi_n)(0) = (D_n * \phi_n)(0) = \frac{1}{2\pi}\int D_n(-t)\phi_n(t)\,dt = \frac{1}{2\pi}\int D_n(t)\phi_n(t)\,dt$$

because D_n is an even function. Since $\phi_n(t) = \operatorname{sgn}(D_n(t))$, then $D_n\phi_n = |D_n|$ and $S_n(\phi_n)(0) = \dfrac{1}{2\pi}\displaystyle\int |D_n(t)|\,dt = L_n \implies |S_n(\phi_n)(0)| = S_n(\phi_n)(0) = L_n$.
By the above argument with ϕ_n replaced by c_n,
then $S_n(c_n)(0) = \dfrac{1}{2\pi}\displaystyle\int D_n(t)c_n(t)\,dt$.

$D_n c_n$ is an even function because D_n and c_n are even functions.

$$\Longrightarrow \; S_n(c_n)(0) = \frac{1}{2\pi}\int D_n(t)c_n(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)c_n(t)\,dt$$
$$= \frac{1}{2\pi}\left[2\int_0^{\pi} D_n(t)c_n(t)\,dt\right] = \frac{1}{\pi}\int_0^{\pi} D_n(t)c_n(t)\,dt$$

Note that $|D_n(t)| = \left|\sum_{j=-n}^{n} e^{ijt}\right| \leq \sum_{j=-n}^{n} |e^{ijt}| = \sum_{j=-n}^{n} 1 = 2n+1$.

$D_n c_n = 0$ on B, $|D_n c_n| = |D_n|\;|c_n| \leq |D_n| \leq 2n+1$ on A, and

$m(A) = \sum_{k=1}^{n} m([k\alpha - \epsilon_0, k\alpha + \epsilon_0]) = \sum_{k=1}^{n} 2\epsilon_0 = 2n\epsilon_0$.

$$|S_n(c_n)(0)| = \left|\frac{1}{\pi}\int_0^{\pi} D_n(t)c_n(t)\,dt\right| = \left|\frac{1}{\pi}\int_A D_n(t)c_n(t)\,dt\right|$$
$$\leq \frac{1}{\pi}\int_A |D_n(t)c_n(t)|\,dt \leq \frac{1}{\pi}\int_A (2n+1)\,dt = \left(\frac{2n+1}{\pi}\right) m(A)$$
$$= \left(\frac{2n+1}{\pi}\right)(2n\epsilon_0) = \left(\frac{2}{\alpha}\right)(2n\epsilon_0) = \frac{4n\epsilon_0}{\alpha}$$

(iv) Now the conditions on ϵ_0 will be chosen.

The first condition is that $0 < (\alpha - \epsilon_0)$, i.e. $\epsilon_0 < \alpha$. The second condition is that $[k\alpha + \epsilon_0] < [(k+1)\alpha - \epsilon_0]$, i.e. $\epsilon_0 < \frac{\alpha}{2}$. The third condition is that $(n\alpha + \epsilon_0) < \pi$, i.e. $\epsilon_0 < (\pi - n\alpha) = \frac{\alpha}{2}$ which is the same as the second condition. These conditions imply that the union in each of A and B is disjoint. The last condition is that $\frac{4n\epsilon_0}{\alpha} < \epsilon$, i.e. $\epsilon_0 < \frac{\alpha\epsilon}{4n}$.

Now, assume $0 < \epsilon_0 < \min\left(\frac{\alpha}{2}, \frac{\alpha\epsilon}{4n}\right)$. Then all the conditions hold.

$$\Longrightarrow \; |S_n(c_n)(0)| \leq \frac{4n\epsilon_0}{\alpha} < \epsilon \text{ and } |S_n(\psi_n)(0)| \geq L_n - |S_n(c_n)(0)| > L_n - \epsilon$$
$$\Longrightarrow \; \|\mathbf{S}_n\|_{op} \geq |S_n(\psi_n)(0)| > L_n - \epsilon$$

Since $\epsilon > 0$ is arbitrary, then $\|\mathbf{S}_n\|_{op} \geq L_n$.

Hence, $\forall n \in \mathbb{N}$, $\|\mathbf{S}_n\|_{op} \geq L_n$.

$\square$

The following lemma is a generalization of Corollary 3.2.3.

Lemma 3.6.5. If $\lim_{j\to\infty} f_j = f$ in B, then $\lim_{j\to\infty} \hat{f}_j(n) = \hat{f}(n)$ uniformly in n.

Proof. By (H-1), $\forall j \in \mathbb{N}_0$, $\|f_j - f\|_1 \leq \|f_j - f\|_B$. Then, $\lim_{j\to\infty} f_j = f$ in B $\implies \lim_{j\to\infty} f_j = f$ in $L^1(\mathbb{T})$. By Corollary 3.2.3, the result holds. $\square$

Although $C(\mathbb{T})$ does not admit convergence in norm, there is a simple case when $S[f]$ does converge in norm to f in $C(\mathbb{T})$. The following proposition corresponds to (Korner, 1988, P. 32-33, Theorems 1.9.1-1.9.2). Due to the assumption that two functions f and g are equal if they are equal m-a.e., the additional hypothesis in (Korner, 1988, P. 32, Theorem 1.9.1) that $f \in C(\mathbb{T})$ is removed. The proof is reproduced as in the book except that the Theory of Banach spaces is applied instead of the Weierstrass M test in (Korner, 1988).

Proposition 3.6.6. Let $f \in L^1(\mathbb{T})$ be s.t. $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. Then $f \in C(\mathbb{T})$ and $S[f]$ converges in norm to f in $C(\mathbb{T})$.

Proof. $\forall n \in \mathbb{N}_0$, let $s_n = \sum_{j=-n}^{n} |\hat{f}(j)|$. By Theorem 3.2.6.(4), $S_n(f) = (D_n * f) \in C(\mathbb{T})$ because $D_n \in C(\mathbb{T})$ and $f \in L^1(\mathbb{T})$.

Let $m, n \in \mathbb{N}_0$ be s.t. $m \geq n$. Then, $\forall t \in \mathbb{R}$,

$$|S_m(f)(t) - S_n(f)(t)| = \left| \sum_{n+1\leq|j|\leq m} \hat{f}(j)e^{ijt} \right| \leq \sum_{n+1\leq|j|\leq m} |\hat{f}(j)| = |s_m - s_n|.$$

$\implies \forall m, n \in \mathbb{N}_0$ s.t. $m \geq n$, $\|S_m(f) - S_n(f)\|_\infty \leq |s_m - s_n|$

Since $\{s_n\}_{n=0}^{\infty}$ converges, then $\{s_n\}_{n=0}^{\infty}$ is a Cauchy sequence. This implies that $\{S_n(f)\}_{n=0}^{\infty}$ is a Cauchy sequence in $C(\mathbb{T})$. Since $C(\mathbb{T})$ is a Banach space, then $g = \lim_{j\to\infty} S_j(f)$ exists in $C(\mathbb{T})$. By the previous lemma, $\hat{g}(n) = \lim_{j\to\infty} \widehat{S_j(f)}(n)$ uniformly in n. Since $\forall j \geq |n|$, $\widehat{S_j(f)}(n) = \hat{f}(n)$, then $\forall n \in \mathbb{Z}$, $\hat{g}(n) = \lim_{j\to\infty} \widehat{S_j(f)}(n) = \hat{f}(n)$. Thus, $\forall n \in \mathbb{Z}$, $\hat{g}(n) = \hat{f}(n)$. By the Uniqueness Theorem with $B = L^1(\mathbb{T})$, $g = f$ in $L^1(\mathbb{T})$. $\implies \|g - f\|_1 = 0 \implies g = f$ m-a.e. $\implies g = f$.

Hence, $f \in C(\mathbb{T})$ and $S[f]$ converges in norm to f in $C(\mathbb{T})$. □

The following remark corresponds to (Korner, 1988, P. 34, Theorem 1.9.6)

Remark. By the remark after Theorem 3.4.4, if f is twice differentiable and $f' \in AC(\mathbb{T})$, then $\hat{f}(n) = O(n^{-2})$ as $|n| \to \infty$ which implies $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. By the proposition, $S[f]$ converges in norm to f in $C(\mathbb{T})$. A special case is when f is twice continuously differentiable.

As on (Katznelson, 2004, P. 69, Definition 2.1.4), the following definitions are made.

Definition 3.6.2. Let $f \in L^1(\mathbb{T})$. If $\exists g \in L^1(\mathbb{T})$ s.t. $S[g] = \tilde{S}[f]$, i.e. $\forall n \in \mathbb{Z}$, $\hat{g}(n) = -i \operatorname{sgn}(n)\hat{f}(n)$, then g is called the conjugate function of f and is denoted by $\tilde{f}$. B admits conjugation if $\forall f \in B, \tilde{f}$ exists in B.

The following theorem is stated and proved on (Katznelson, 2004, P. 70, Theorem 2.1.4).

Theorem 3.6.7. Assume $\forall f \in B$ $\forall n \in \mathbb{Z}$, $e^{int}f \in B$ and $\|e^{int}f\|_B = \|f\|_B$. Then, B admits conjugation iff B admits convergence in norm.

The following theorem is from (Katznelson, 2004, P. 71, Theorem 2.1.5).

Theorem 3.6.8. If $1 < p < \infty$, then $L^p(\mathbb{T})$ admits convergence in norm.

Remark. Only an outline of the proof will be presented. The detailed proof involves a whole lot of theory in the subject of conjugation, which cannot be discussed here due to the constraints of this paper.

Proof. It is easy to see that $L^p(\mathbb{T})$ satisfies the assumption of the previous theorem. In (Katznelson, 2004, Chapter 3), it is shown that $L^p(\mathbb{T})$ admits conjugation. By the previous theorem, $L^p(\mathbb{T})$ admits convergence in norm. $\square$

Note. The case $p = 2$ is just Theorem 3.5.8.(2).

CHAPTER 4
Pointwise Summability and Convergence of Fourier Series on $\mathbb{T}$

4.1 Pointwise Summability of $S[f]$

Definition 4.1.1. $S[f](t)$ is summable to $s \in \mathbb{C}$ if $\lim_{n\to\infty} \sigma_n(f)(t) = s$.

Lemma 4.1.1. (i) K_n is an even function.

(ii) $\forall\, 0 < \delta < \pi$, $\lim_{n\to\infty} \left(\sup_{t\in[\delta, 2\pi-\delta]} K_n(t) \right) = 0$.

Proof. (i) By the remark after Definition 3.2.1,

$K_n(t) = 1 + 2\sum_{j=1}^{n} \left(1 - \frac{j}{n+1}\right) \cos(jt)$. Then the result follows because the cosine function is an even function.

(ii) By the proof of Proposition 3.3.11, $0 \le \left(\sup_{t\in[\delta,2\pi-\delta]} K_n(t) \right) \le \frac{1}{n+1} \left(\frac{\pi}{\delta}\right)^2$ and then the result follows because $\lim_{n\to\infty} \left[\frac{1}{n+1} \left(\frac{\pi}{\delta}\right)^2 \right] = 0$.

□

The following theorem is from (Katznelson, 2004, P. 19-20, Theorem 1.3.1). The proof is essentially the same as in the book.

Theorem 4.1.2 (Fejér). Let $f \in L^1(\mathbb{T})$.

1) Let $t_0 \in \mathbb{R}$ and assume that $\lim_{h\to 0^+} [f(t_0 + h) + f(t_0 - h)]$ exists. Then, $\lim_{n\to\infty} \sigma_n(f)(t_0) = \lim_{h\to 0^+} \left[\frac{f(t_0 + h) + f(t_0 - h)}{2} \right]$. In particular, if t_0 is a point of continuity of f, then $\lim_{n\to\infty} \sigma_n(f)(t_0) = f(t_0)$.

2) If every point of a closed interval I is a point of continuity for f, then $\sigma_n(f)(t)$ converges to $f(t)$ uniformly on I.

3) Let f be real-valued and $m_0, M \in \mathbb{R}$. If $f \geq m_0$ m-a.e., then $\sigma_n(f) \geq m_0$. Similarly, if $f \leq M$ m-a.e., then $\sigma_n(f) \leq M$.

Proof. 1) Let $0 < \delta < \pi$, $\epsilon > 0$, and $\check{f}(t_0) = \lim_{h \to 0^+} \left[\dfrac{f(t_0 + h) + f(t_0 - h)}{2} \right]$. By Theorem 3.2.6.(2),

$$\sigma_n(f)(t_0) = (K_n * f)(t_0) = (f * K_n)(t_0) = \frac{1}{2\pi} \int K_n(\tau) f(t_0 - \tau)\, d\tau.$$

By the argument used in Theorem 3.3.3 with $\phi(\tau)$ and $\phi(0)$ replaced by $f(t_0 - \tau)$ and $\check{f}(t_0)$ resp.,

$$[\sigma_n(f)(t_0) - \check{f}(t_0)] = \left[\frac{1}{2\pi} \int_{-\delta}^{\delta} K_n(\tau)(f(t_0 - \tau) - \check{f}(t_0))\, d\tau \right]$$
$$+ \left[\frac{1}{2\pi} \int_{\delta}^{2\pi - \delta} K_n(\tau)(f(t_0 - \tau) - \check{f}(t_0))\, d\tau \right].$$
$$\Longrightarrow |\sigma_n(f)(t_0) - \check{f}(t_0)| \leq \left| \frac{1}{2\pi} \int_{-\delta}^{\delta} K_n(\tau)(f(t_0 - \tau) - \check{f}(t_0))\, d\tau \right|$$
$$+ \left| \frac{1}{2\pi} \int_{\delta}^{2\pi - \delta} K_n(\tau)(f(t_0 - \tau) - \check{f}(t_0))\, d\tau \right|$$

$$
\begin{aligned}
&\left|\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} K_n(\tau)(f(t_0-\tau)-\check{f}(t_0))\,d\tau\right| \\
&\leq \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} |K_n(\tau)(f(t_0-\tau)-\check{f}(t_0))|\,d\tau \\
&= \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} K_n(\tau)|f(t_0-\tau)-\check{f}(t_0)|\,d\tau \\
&\leq \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} \left(\sup_{t\in[\delta,2\pi-\delta]} K_n(t)\right)|f(t_0-\tau)-\check{f}(t_0)|\,d\tau \\
&= \left(\sup_{t\in[\delta,2\pi-\delta]} K_n(t)\right)\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} |f(t_0-\tau)-\check{f}(t_0)|\,d\tau \\
&\leq \left(\sup_{t\in[\delta,2\pi-\delta]} K_n(t)\right)\frac{1}{2\pi}\int_{0}^{2\pi} |f(t_0-\tau)-\check{f}(t_0)|\,d\tau \\
&= \left(\sup_{t\in[\delta,2\pi-\delta]} K_n(t)\right)\frac{1}{2\pi}\int |f(t_0-\tau)-\check{f}(t_0)|\,d\tau \\
&= \left(\sup_{t\in[\delta,2\pi-\delta]} K_n(t)\right)\frac{1}{2\pi}\int |f(\tau)-\check{f}(t_0)|\,d\tau \text{ [by Corollary 3.1.9.]} \\
&= \|f-\check{f}(t_0)\|_1\left(\sup_{t\in[\delta,2\pi-\delta]} K_n(t)\right) \leq (\|f\|_1+\|\check{f}(t_0)\|_1)\left(\sup_{t\in[\delta,2\pi-\delta]} K_n(t)\right) \\
&= (\|f\|_1+|\check{f}(t_0)|)\left(\sup_{t\in[\delta,2\pi-\delta]} K_n(t)\right) \text{ [because } \check{f}(t_0) \text{ is a constant.]}
\end{aligned}
$$

By Proposition 2.3.2.(1), $\frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(\tau)(f(t_0-\tau)-\check{f}(t_0))\,d\tau =$
$\frac{1}{2\pi}\int_{-\delta}^{0} K_n(\tau)(f(t_0-\tau)-\check{f}(t_0))\,d\tau + \frac{1}{2\pi}\int_{0}^{\delta} K_n(\tau)(f(t_0-\tau)-\check{f}(t_0))\,d\tau.$

By Proposition 2.3.2.(3), $\frac{1}{2\pi}\int_{-\delta}^{0} K_n(\tau)(f(t_0-\tau)-\check{f}(t_0))\,d\tau$

$$= \frac{1}{2\pi}\int_0^{\delta} K_n(-\tau)(f(t_0+\tau)-\check{f}(t_0))\,d\tau = \frac{1}{2\pi}\int_0^{\delta} K_n(\tau)(f(t_0+\tau)-\check{f}(t_0))\,d\tau,$$

where the last equality follows from Lemma 4.1.1.(i).

Let $g_{t_0}(\tau) = \left[\left(\frac{f(t_0+\tau)+f(t_0-\tau)}{2}\right) - \check{f}(t_0)\right]$. Then

$$\frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(\tau)(f(t_0-\tau)-\check{f}(t_0))\,d\tau$$
$$= \frac{1}{2\pi}\int_0^{\delta} K_n(\tau)(f(t_0+\tau)-\check{f}(t_0))\,d\tau + \frac{1}{2\pi}\int_0^{\delta} K_n(\tau)(f(t_0-\tau)-\check{f}(t_0))\,d\tau$$
$$= \frac{1}{2\pi}\int_0^{\delta} K_n(\tau)[(f(t_0+\tau)+f(t_0-\tau))-2\check{f}(t_0)]\,d\tau = \frac{1}{\pi}\int_0^{\delta} K_n(\tau)g_{t_0}(\tau)\,d\tau$$

Since K_n is an even function, then $\frac{1}{\pi}\int_0^{\pi} K_n(\tau)\,d\tau = \frac{1}{2\pi}\left[2\int_0^{\pi} K_n(\tau)\,d\tau\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(\tau)\,d\tau = 1$, where the last equality is by (S-1).

By the argument used at the beginning of the proof,

$$\left|\frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(\tau)(f(t_0-\tau)-\check{f}(t_0))\,d\tau\right| = \left|\frac{1}{\pi}\int_0^{\delta} K_n(\tau)g_{t_0}(\tau)\,d\tau\right|$$
$$\leq \left(\frac{1}{\pi}\int_0^{\pi} K_n(\tau)\,d\tau\right)\left(\sup_{t\in(0,\delta]}|g_{t_0}(t)|\right) = \left(\sup_{t\in(0,\delta]}|g_{t_0}(t)|\right).$$
$$\therefore\ |\sigma_n(f)(t_0)-\check{f}(t_0)| \leq \left(\sup_{t\in(0,\delta]}|g_{t_0}(t)|\right) + (\|f\|_1 + |\check{f}(t_0)|)\left(\sup_{t\in[\delta,2\pi-\delta]} K_n(t)\right)$$

$$\check{f}(t_0) = \lim_{h\to 0^+}\left[\frac{f(t_0+h)+f(t_0-h)}{2}\right] \implies \lim_{\tau\to 0^+} g_{t_0}(\tau) = 0$$
$$\implies \exists\delta>0 \text{ s.t. } 0<\tau<\delta \Rightarrow |g_{t_0}(\tau)| < \frac{\epsilon}{3} \implies \left(\sup_{t\in(0,\delta]}|g_{t_0}(t)|\right) \leq \frac{\epsilon}{3} < \frac{\epsilon}{2}$$

By Lemma 4.1.1.(ii), $\lim_{n\to\infty}\left(\sup_{t\in[\delta,2\pi-\delta]} K_n(t)\right) = 0 \implies \exists N \in \mathbb{N}\ \forall n \geq N,$

$$\left|\sup_{t\in[\delta,2\pi-\delta]} K_n(t)\right| = \left(\sup_{t\in[\delta,2\pi-\delta]} K_n(t)\right) < \frac{\epsilon}{2[(\|f\|_1 + |\check{f}(t_0)|) + 1]}.$$

All of the above $\implies |\sigma_n(f)(t_0) - \check{f}(t_0)| < \epsilon.$

$\therefore\ \forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N, |\sigma_n(f)(t_0) - \check{f}(t_0)| < \epsilon$

Hence, $\lim_{n\to\infty} \sigma_n(f)(t_0) = \check{f}(t_0).$

2) Assume f is continuous on I. Then $\forall t_0 \in I$, $\check{f}(t_0) = f(t_0)$. f is uniformly continuous on I because f is continuous on I and I is compact. Let $\epsilon > 0$.

$$\implies \exists \delta > 0\ \forall x, y \in I, |x - y| < \delta' \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{2} \implies$$

$$\forall\ 0 < \tau < \delta\ \forall t_0 \in I, |(t_0 \pm \tau) - t_0| = |\tau| = \tau < \delta \implies$$

$$\begin{aligned}
|g_{t_0}(\tau)| &= \left|\left(\frac{f(t_0+\tau) + f(t_0-\tau)}{2}\right) - f(t_0)\right| \\
&= \left|\frac{(f(t_0+\tau) - f(t_0)) + (f(t_0-\tau) - f(t_0))}{2}\right| \\
&\leq \left(\frac{|f(t_0+\tau) - f(t_0)| + |f(t_0-\tau) - f(t_0)|}{2}\right) < \frac{2\epsilon}{2} = \epsilon
\end{aligned}$$

$$\therefore\ \forall \epsilon > 0\ \exists \delta > 0\ \forall t_0 \in I\ \forall\ 0 < \tau < \delta, |g_{t_0}(\tau)| < \epsilon$$

Hence, $\lim_{\tau\to 0^+} g_{t_0}(\tau) = 0$ uniformly in $t_0 \in I$ and this implies that δ in (1) can be chosen independently of $t_0 \in I$. Also, f is continuous on $I \Rightarrow f$ is bounded on I by some $M > 0$. Since $(\|f\|_1 + |\check{f}(t_0)|) = (\|f\|_1 + |f(t_0)|) \leq (\|f\|_1 + M)$, then $(\|f\|_1 + |\check{f}(t_0)|)$ can be replaced by $(\|f\|_1 + M)$ in (1). By all of the above, $\lim_{n\to\infty} \sigma_n(f)(t_0) = f(t_0)$ uniformly in $t_0 \in I$.

3)

$$\sigma_n(f)(t) = (K_n * f)(t) = \frac{1}{2\pi}\int K_n(\tau) f(t-\tau)\, d\tau$$
$$\geq \frac{1}{2\pi}\int m_0 K_n(\tau)\, d\tau = m_0 \left(\frac{1}{2\pi}\int K_n(\tau)\, d\tau\right) = m_0$$

, i.e. $\sigma_n(f)(t) \geq m_0$ because $f \geq m_0$ m-a.e. and $K_n \geq 0$. By a similar argument, $\sigma_n(f)(t) \leq M$.

□

Note. The proof of Theorem 4.1.2.(1) can be modified so that Theorem 4.1.2.(1) still holds even if $\lim_{h\to 0^+} [f(t_0+h) + f(t_0-h)] = \pm\infty$.

Lemma 4.1.3. $\forall t \in [0,\pi]$, $K_n(t) \leq \min\left(n+1, \frac{\pi^2}{(n+1)t^2}\right)$

Proof. By the proof of Lemma 4.1.1.(1) and Lemma 3.3.10,

$$K_n(t) = 1 + 2\sum_{j=1}^{n}\left(1 - \frac{j}{n+1}\right)\cos(jt) \leq 1 + 2\sum_{j=1}^{n}\left(1-\frac{j}{n+1}\right) = K_n(0) = n+1.$$

Since $\forall t \in [0,\pi]$, $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi} \geq 0$, then $K_n(t) = \frac{1}{n+1}\left[\frac{\sin\left(\frac{(2n+1)t}{2}\right)}{\sin\left(\frac{t}{2}\right)}\right]^2$

$$\leq \frac{1}{n+1}\left[\frac{1}{\sin\left(\frac{t}{2}\right)}\right]^2 \leq \frac{1}{n+1}\left(\frac{\pi}{t}\right)^2 = \frac{\pi^2}{(n+1)t^2}.$$

Hence, $\forall t \in [0,\pi]$, $K_n(t) \leq \min\left(n+1, \frac{\pi^2}{(n+1)t^2}\right)$.

□

It is easy to show that Fejér's condition, $\check{f}(t_0) = \lim_{h\to 0^+}\left[\frac{f(t_0+h)+f(t_0-h)}{2}\right]$, implies $\lim_{h\to 0^+}\frac{1}{h}\int_0^h |g_{t_0}(\tau)|\,d\tau = 0$. By Corollary 2.3.5, this new condition holds for m-a.a. $t_0 \in \mathbb{R}$ with $\check{f}(t_0) = f(t_0)$.

The following theorem is from (Katznelson, 2004, P. 21, Theorem 1.3.2). The proof is the same as in the book except that the proofs of $\lim_{n\to\infty}\left[\frac{1}{\pi}\int_{\delta_n}^{\pi} K_n(\tau)g(\tau)\,d\tau\right] = 0$ and $\lim_{n\to\infty}\left[\frac{1}{\pi}\int_{\frac{1}{n}}^{\delta_n} K_n(\tau)g(\tau)\,d\tau\right] = 0$ are given here in full detail. The elements of the proof of $\lim_{n\to\infty}\left[\frac{1}{\pi}\int_{\frac{1}{n}}^{\delta_n} K_n(\tau)g(\tau)\,d\tau\right] = 0$ arose in a discussion with Professor Klemes.

Theorem 4.1.4 (Lebesgue). Let $f \in L^1(\mathbb{T})$, $t_0 \in \mathbb{R}$, $\check{f}(t_0) \in \mathbb{C}$, and $g(\tau) = \left[\left(\frac{f(t_0+\tau)+f(t_0-\tau)}{2}\right) - \check{f}(t_0)\right]$. If $\lim_{h\to 0^+}\frac{1}{h}\int_0^h |g(\tau)|\,d\tau = 0$, then $\lim_{n\to\infty}\sigma_n(f)(t_0) = \check{f}(t_0)$. In particular, $\lim_{n\to\infty}\sigma_n(f)(t) = f(t)$ m-a.e.

Proof. By the argument used in Theorem 4.1.2.(1) with $\delta = \pi$,

$$[\sigma_n(f)(t_0) - \check{f}(t_0)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(\tau)(f(t_0-\tau) - \check{f}(t_0))\,d\tau = \frac{1}{\pi}\int_0^{\pi} K_n(\tau)g(\tau)\,d\tau.$$

Now, let $\delta_n = n^{-\frac{1}{4}}$. By Proposition 2.3.2.(1),

$$\begin{aligned}[\sigma_n(f)(t_0) - \check{f}(t_0)] &= \frac{1}{\pi}\int_0^{\pi} K_n(\tau)g(\tau)\,d\tau \\ &= \left[\frac{1}{\pi}\int_0^{\frac{1}{n}} K_n(\tau)g(\tau)\,d\tau\right] + \left[\frac{1}{\pi}\int_{\frac{1}{n}}^{\delta_n} K_n(\tau)g(\tau)\,d\tau\right] + \left[\frac{1}{\pi}\int_{\delta_n}^{\pi} K_n(\tau)g(\tau)\,d\tau\right]\end{aligned}$$

To show that $\lim_{n\to\infty}\sigma_n(f)(t_0) = \check{f}(t_0)$, it is enough to show that as $n \to \infty$ each of the three integrals on the RHS of the previous equation tend to 0.

First, it will be shown that $\lim_{n\to\infty}\left[\frac{1}{\pi}\int_{\delta_n}^{\pi} K_n(\tau)g(\tau)\,d\tau\right] = 0.$

$$\left|\frac{1}{\pi}\int_{\delta_n}^{\pi} K_n(\tau)g(\tau)\,d\tau\right| \le \frac{1}{\pi}\int_{\delta_n}^{\pi} |K_n(\tau)g(\tau)|\,d\tau = \frac{1}{\pi}\int_{\delta_n}^{\pi} K_n(\tau)|g(\tau)|\,d\tau$$
$$\le \frac{1}{\pi}\int_{\delta_n}^{\pi}\left(\frac{\pi^2}{(n+1)\tau^2}\right)|g(\tau)|\,d\tau \text{ [by Lemma 4.1.3.] } \le \frac{1}{\pi}\int_{\delta_n}^{\pi}\left(\frac{\pi^2}{(n+1)\delta_n{}^2}\right)|g(\tau)|\,d\tau$$
$$\le \left(\frac{\pi^2}{(n+1)\delta_n{}^2}\right)\left(\frac{1}{\pi}\int_{\delta_n}^{\pi}|g(\tau)|\,d\tau\right) \le \left(\frac{\pi^2}{(n+1)\delta_n{}^2}\right)\left(\frac{1}{\pi}\int_{-\pi}^{\pi}|g(\tau)|\,d\tau\right)$$
$$= \left(\frac{2\pi^2}{(n+1)\delta_n{}^2}\right)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(\tau)|\,d\tau\right) = \frac{2\pi^2\|g\|_1}{(n+1)\delta_n{}^2}$$

Note that $f \in L^1(\mathbb{T}) \implies g \in L^1(\mathbb{T}) \implies \|g\|_1 < \infty$.

$$\therefore\ 0 \le \left|\frac{1}{\pi}\int_{\delta_n}^{\pi} K_n(\tau)g(\tau)\,d\tau\right| \le \frac{2\pi^2\|g\|_1}{(n+1)\delta_n{}^2}$$

Since $\lim_{n\to\infty}\left[\frac{2\pi^2\|g\|_1}{(n+1)\delta_n{}^2}\right] = 0$, then $\lim_{n\to\infty}\left[\frac{1}{\pi}\int_{\delta_n}^{\pi} K_n(\tau)g(\tau)\,d\tau\right] = 0.$

Now, it will be shown that $\lim_{n\to\infty}\left[\frac{1}{\pi}\int_{0}^{\frac{1}{n}} K_n(\tau)g(\tau)\,d\tau\right] = 0.$

Let $\Phi(h) = \int_0^h |g(\tau)|\,d\tau$. By Theorem 2.3.9, Φ is absolutely continuous on $[0,\pi]$ because $g \in L^1(\mathbb{T}) \implies g \in L^1_{\text{loc}}(\mathbb{R}) \implies g \in L^1([0,\pi], \mathcal{B}_{\mathbb{R}} \cap [0,\pi], m)$.

$$\left|\frac{1}{\pi}\int_{0}^{\frac{1}{n}} K_n(\tau)g(\tau)\,d\tau\right| \le \frac{1}{\pi}\int_{0}^{\frac{1}{n}} |K_n(\tau)g(\tau)|\,d\tau = \frac{1}{\pi}\int_{0}^{\frac{1}{n}} K_n(\tau)|g(\tau)|\,d\tau$$
$$\le \frac{1}{\pi}\int_{0}^{\frac{1}{n}}(n+1)|g(\tau)|\,d\tau \text{ [by Lemma 4.1.3.]}$$
$$= \frac{(n+1)}{\pi}\left(\int_0^{\frac{1}{n}}|g(\tau)|\,d\tau\right) = \frac{(n+1)}{n\pi}\left[n\Phi\left(\frac{1}{n}\right)\right]$$

Since $\lim_{n\to\infty} \dfrac{(n+1)}{n\pi} = \dfrac{1}{\pi}$ and $\left(\lim_{h\to 0^+} \dfrac{\Phi(h)}{h} = 0 \implies \lim_{n\to\infty}\left[n\Phi\left(\dfrac{1}{n}\right)\right] = 0\right)$,

then $\lim_{n\to\infty}\left(\dfrac{(n+1)}{n\pi}\left[n\Phi\left(\dfrac{1}{n}\right)\right]\right) = 0$.

$\therefore\ 0 \leq \left|\dfrac{1}{\pi}\displaystyle\int_0^{\frac{1}{n}} K_n(\tau)g(\tau)\,d\tau\right| \leq \dfrac{(n+1)}{n\pi}\left[n\Phi\left(\dfrac{1}{n}\right)\right]$

Since $\lim_{n\to\infty}\left(\dfrac{(n+1)}{n\pi}\left[n\Phi\left(\dfrac{1}{n}\right)\right]\right) = 0$, then $\lim_{n\to\infty}\left[\dfrac{1}{\pi}\displaystyle\int_0^{\frac{1}{n}} K_n(\tau)g(\tau)\,d\tau\right] = 0$.

Now, it is only necessary to show that $\lim_{n\to\infty}\left[\dfrac{1}{\pi}\displaystyle\int_{\frac{1}{n}}^{\delta_n} K_n(\tau)g(\tau)\,d\tau\right] = 0$.

$$\left|\frac{1}{\pi}\int_{\frac{1}{n}}^{\delta_n} K_n(\tau)g(\tau)\,d\tau\right| \leq \frac{1}{\pi}\int_{\frac{1}{n}}^{\delta_n} |K_n(\tau)g(\tau)|\,d\tau = \frac{1}{\pi}\int_{\frac{1}{n}}^{\delta_n} K_n(\tau)|g(\tau)|\,d\tau$$
$$\leq \frac{1}{\pi}\int_{\frac{1}{n}}^{\delta_n}\left(\frac{\pi^2}{(n+1)\tau^2}\right)|g(\tau)|\,d\tau \text{ [by Lemma 4.1.3.] } = \frac{\pi}{n+1}\int_{\frac{1}{n}}^{\delta_n}\frac{|g(\tau)|}{\tau^2}\,d\tau$$

Let $h(\tau) = \dfrac{1}{\tau^2}$. Then $h'(\tau) = -\dfrac{2}{\tau^3}$. $\forall a \in (0,\pi)$, h is absolutely continuous on $[a,\pi]$ because on $[a,\pi]$, h is differentiable and h' is bounded. Also, by Theorem 2.3.9, $\Phi' = |g|$ m-a.e on $[a,\pi]$. By Integration by Parts,

$$\int_{\frac{1}{n}}^{\delta_n}\frac{|g(\tau)|}{\tau^2}\,d\tau = \int_{\frac{1}{n}}^{\delta_n} h(\tau)\Phi'(\tau)\,d\tau = [h(\tau)\Phi(\tau)]\Big|_{\tau=\frac{1}{n}}^{\delta_n} - \int_{\frac{1}{n}}^{\delta_n}\Phi(\tau)h'(\tau)\,d\tau$$
$$= \left[\frac{\Phi(\tau)}{\tau^2}\right]\Bigg|_{\tau=\frac{1}{n}}^{\delta_n} + 2\int_{\frac{1}{n}}^{\delta_n}\frac{\Phi(\tau)}{\tau^3}\,d\tau \leq \frac{\Phi(\delta_n)}{{\delta_n}^2} + 2\int_{\frac{1}{n}}^{\delta_n}\frac{\Phi(\tau)}{\tau^3}\,d\tau.$$
$$\implies \left|\frac{1}{\pi}\int_{\frac{1}{n}}^{\delta_n} K_n(\tau)g(\tau)\,d\tau\right| \leq \frac{\pi}{n+1}\int_{\frac{1}{n}}^{\delta_n}\frac{|g(\tau)|}{\tau^2}\,d\tau$$
$$\leq \frac{\pi}{n+1}\left[\frac{\Phi(\delta_n)}{{\delta_n}^2}\right] + \frac{2\pi}{n+1}\int_{\frac{1}{n}}^{\delta_n}\frac{\Phi(\tau)}{\tau^3}\,d\tau$$

Let $\epsilon > 0$. $\lim\limits_{h\to 0+} \dfrac{\Phi(h)}{h} = 0 \implies \exists \delta > 0 \; \forall \, 0 < h < \delta, \dfrac{\Phi(h)}{h} < \dfrac{\epsilon}{3\pi}$

$\lim\limits_{n\to\infty} \delta_n = 0 \Rightarrow \exists N \in \mathbb{N} \; \forall n \geq N, \; \delta_n < \delta \implies \forall \, 0 < \tau \leq \delta_n < \delta, \; \dfrac{\phi(\tau)}{\tau} < \dfrac{\epsilon}{3\pi}$

Also, $0 < \dfrac{1}{\delta_n} \leq n$.

$$\implies \frac{\pi}{n+1}\left[\frac{\Phi(\delta_n)}{{\delta_n}^2}\right] < \frac{\pi}{n+1}\left(\frac{\epsilon}{3\pi}\right)\left(\frac{1}{\delta_n}\right) \leq \left(\frac{n}{n+1}\right)\left(\frac{\epsilon}{3}\right) < \frac{\epsilon}{3} \text{ and}$$

$$\frac{2\pi}{n+1}\int_{\frac{1}{n}}^{\delta_n} \frac{\Phi(\tau)}{\tau^3}\,d\tau < \left(\frac{2\pi}{n+1}\right)\left(\frac{\epsilon}{3\pi}\right)\int_{\frac{1}{n}}^{\delta_n} \frac{1}{\tau^2}\,d\tau = \left(\frac{2\pi}{n+1}\right)\left(\frac{\epsilon}{3\pi}\right)\left[-\frac{1}{\tau}\right]\Bigg|_{\tau=\frac{1}{n}}^{\delta_n}$$

$$= \left(\frac{2\pi}{n+1}\right)\left(\frac{\epsilon}{3\pi}\right)(n - \delta_n^{-1}) < \left(\frac{2\pi}{n+1}\right)\left(\frac{\epsilon}{3\pi}\right)n = \left(\frac{n}{n+1}\right)\left(\frac{2\epsilon}{3}\right) < \frac{2\epsilon}{3}$$

$$\implies \left|\frac{1}{\pi}\int_{\frac{1}{n}}^{\delta_n} K_n(\tau)g(\tau)\,d\tau\right| < \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon$$

$\therefore \forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N, \left|\dfrac{1}{\pi}\displaystyle\int_{\frac{1}{n}}^{\delta_n} K_n(\tau)g(\tau)\,d\tau\right| < \epsilon$

Hence, $\lim\limits_{n\to\infty}\left[\dfrac{1}{\pi}\displaystyle\int_{\frac{1}{n}}^{\delta_n} K_n(\tau)g(\tau)\,d\tau\right] = 0.$

By all of the above, $\lim\limits_{n\to\infty} \sigma_n(f)(t_0) = \check{f}(t_0)$. □

The following corollary is from (Katznelson, 2004, P. 20, Corollary 1.3.1).

Corollary 4.1.5. If $f \in L^1(\mathbb{T})$ and $S[f]$ converges on a set E of positive measure, then $S[f] = f$ m-a.e. on E. In particular, if $S[f]$ converges to 0 m-a.e., then $f = 0$ and all the Fourier coefficients must vanish.

4.2 Pointwise Divergence of $S[f]$

The following theorem is from (Katznelson, 2004, P. 72-73, Theorem 2.2.1). Two proofs will be presented which are basically reproduced as in the book.

The first proof uses the Principle of Uniform Boundedness to show that f exists without giving an example of such a f. The second proof is the construction of a concrete example. Both proofs will use the proof of Theorem 3.6.4. Here, $B = C(\mathbb{T})$.

Theorem 4.2.1. $\exists f \in C(\mathbb{T})$ s.t. $S[f]$ diverges at a point.

Proof 1. Define $\forall n \in \mathbb{N}_0$, $E_n \colon C(\mathbb{T}) \to \mathbb{C}$ by $E_n(f) = [\mathbf{S}_n(f)](0) = S_n(f)(0)$. E_n is linear because $\mathbf{S}_n$ is linear. E_n is bounded because $\mathbf{S}_n$ is bounded since $\forall f \in C(\mathbb{T})$, $|E_n(f)| \leq \|S_n(f)\|_\infty \leq L_n \, \|f\|_\infty$. Thus, $E_n \in C(\mathbb{T})^*$ and $\|E_n\|_{op} \leq L_n$. Now, let $n \in \mathbb{N}$ and $\epsilon > 0$. By Definition 2.1.4, $\|E_n\|_{op} \geq |E_n(\psi_n)| = |S_n(\psi_n)(0)| > L_n - \epsilon$ because $\psi_n \in C(\mathbb{T})$ and $\|\psi_n\|_\infty = 1$. Since $\epsilon > 0$ is arbitrary, then $\|E_n\|_{op} \geq L_n$ and so $\|E_n\|_{op} = L_n$. Therefore, $C(\mathbb{T})$ is a Banach space, $\{E_n\}_{n=0}^\infty \subseteq C(\mathbb{T})^*$ and $\{\|E_n\|_{op}\}_{n=0}^\infty$ is not bounded. By the Principle of Uniform Boundedness, $\exists f \in C(\mathbb{T})$ s.t. $\{|E_n(f)|\}_{n=0}^\infty = \{|S_n(f)(0)|\}_{n=0}^\infty$ is not bounded. Since the sequence $\{|S_n(f)(0)|\}_{n=0}^\infty$ is not bounded, $\lim\limits_{n\to\infty} S_n(f)(0)$ does not exist. Hence, $S[f]$ diverges unboundedly when $t = 0$. $\square$

Proof 2. In the proof of Theorem 3.6.4, let $\epsilon = \dfrac{L_n}{2}$. Then $\exists \{\psi_n\}_{n=1}^\infty \subseteq C(\mathbb{T})$ s.t. $\|\psi_n\|_\infty = 1$ and $|S_n(\psi_n)(0)| > \dfrac{L_n}{2} > \dfrac{\log n}{10}$. Let $\rho_n = \sigma_{n^2}(\psi_n) = (K_{n^2} * \psi_n)$ which is a trigonometric polynomial of degree n^2. By the note after Theorem 3.3.6 and (S-1), $\|\rho_n\|_\infty \leq \|K_{n^2}\|_1 \|\psi_n\|_\infty = 1 \cdot 1 = 1$. By Theorem 3.2.2.(4) and (H-1), $\forall j \in \mathbb{Z}$,

$|\widehat{\psi_n}(j)| \leq \|\psi_n\|_1 \leq \|\psi_n\|_\infty = 1.$

$$\forall\ |j| \leq n,\ \left[\widehat{\psi_n}(j) - \widehat{\rho_n}(j)\right] = \widehat{\psi_n}(j) - \left(1 - \frac{|j|}{n^2+1}\right)\widehat{\psi_n}(j) = \left(\frac{|j|}{n^2+1}\right)\widehat{\psi_n}(j)$$

$$\implies S_n(\psi_n)(t) - S_n(\rho_n)(t) = \sum_{j=-n}^{n}\left[\widehat{\psi_n}(j) - \widehat{\rho_n}(j)\right]e^{ijt} = \sum_{j=-n}^{n}\left(\frac{|j|}{n^2+1}\right)\widehat{\psi_n}(j)e^{ijt}$$

$$\implies |S_n(\psi_n)(t) - S_n(\rho_n)(t)| \leq \left|\sum_{j=-n}^{n}\left(\frac{|j|}{n^2+1}\right)\widehat{\psi_n}(j)e^{ijt}\right|$$

$$\leq \sum_{j=-n}^{n}\left|\left(\frac{|j|}{n^2+1}\right)\widehat{\psi_n}(j)e^{ijt}\right| = \sum_{j=-n}^{n}\left(\frac{|j|}{n^2+1}\right)|\widehat{\psi_n}(j)|$$

$$\leq \sum_{j=-n}^{n}\frac{|j|}{n^2+1} = \frac{1}{n^2+1}\sum_{j=-n}^{n}|j| = \frac{2}{n^2+1}\sum_{j=1}^{n}j = \left(\frac{2}{n^2+1}\right)\left(\frac{n(n+1)}{2}\right)$$

$$= \frac{n(n+1)}{n^2+1} \leq \frac{n(n+1)}{n^2} = \left(1 + \frac{1}{n}\right) \leq 2$$

Since $S_n(\rho_n)(0) = S_n(\psi_n)(0) - [S_n(\psi_n)(0) - S_n(\rho_n)(0)]$,
then $|S_n(\rho_n)(0)| \geq |S_n(\psi_n)(0)| - |S_n(\psi_n)(0) - S_n(\rho_n)(0)| > \dfrac{\log n}{10} - 2.$

Let $\lambda_n = 2^{3^n}$ and $f(t) = \displaystyle\sum_{n=1}^{\infty}\frac{\rho_{\lambda_n}(\lambda_n t)}{n^2}$. Since $\lambda_n \in \mathbb{N}$, then $\rho_{\lambda_n}(\lambda_n t) \in C(\mathbb{T})$ and $\|\rho_{\lambda_n}(\lambda_n t)\|_\infty = \|\rho_{\lambda_n}\|_\infty \leq 1$. By Theorem 2.1.1, $f \in C(\mathbb{T})$ because

$$\sum_{n=1}^{\infty}\left\|\frac{\rho_{\lambda_n}(\lambda_n t)}{n^2}\right\|_\infty = \sum_{n=1}^{\infty}\frac{\|\rho_{\lambda_n}(\lambda_n t)\|_\infty}{n^2} \leq \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} < \infty.$$

Now it will be shown that $S[f]$ diverges at $t = 0$ by showing that it diverges unboundedly when $t = 0$. $\rho_{\lambda_j}(\lambda_j t)$ is a trigonometric polynomial of degree ${\lambda_j}^3$ because $\rho_{\lambda_j}(\lambda_j t) = \sum_{|m| \leq {\lambda_j}^2}\widehat{\rho_{\lambda_j}}(m)e^{i\lambda_j m t}$. This implies $S_{{\lambda_n}^2}(\rho_{\lambda_n}(\lambda_n t))(0) = S_{\lambda_n}(\rho_{\lambda_n})(0)$. Also, ${\lambda_j}^2 < {\lambda_j}^3 = \lambda_{j+1} < {\lambda_{j+1}}^2$.

$$\implies S_{\lambda_n{}^2}(f)(0) = S_{\lambda_n{}^2}\left(\sum_{j=1}^{n} \frac{\rho_{\lambda_j}(\lambda_j t)}{j^2}\right)(0) + \sum_{j=n+1}^{\infty} \frac{\widehat{\rho_{\lambda_j}}(0)}{j^2}$$
$$= \frac{S_{\lambda_n}(\rho_{\lambda_n})(0)}{n^2} + \sum_{j=1}^{n-1} \frac{\rho_{\lambda_j}(0)}{j^2} + \sum_{j=n+1}^{\infty} \frac{\widehat{\rho_{\lambda_j}}(0)}{j^2}$$

$$\implies |S_{\lambda_n{}^2}(f)(0)| \geq \left|\frac{S_{\lambda_n}(\rho_{\lambda_n})(0)}{n^2}\right| - \left|\sum_{j=1}^{n-1} \frac{\rho_{\lambda_j}(0)}{j^2} + \sum_{j=n+1}^{\infty} \frac{\widehat{\rho_{\lambda_j}}(0)}{j^2}\right|$$

Note that $|\rho_{\lambda_j}(0)| \leq \|\rho_{\lambda_j}\|_\infty \leq 1$ and $|\widehat{\rho_{\lambda_j}}(0)| \leq \|\rho_{\lambda_j}\|_1 \leq \|\rho_{\lambda_j}\|_\infty \leq 1$.

$$\left|\frac{S_{\lambda_n}(\rho_{\lambda_n})(0)}{n^2}\right| = \frac{|S_{\lambda_n}(\rho_{\lambda_n})(0)|}{n^2} > \frac{1}{n^2}\left(\frac{\log \lambda_n}{10} - 2\right) = \frac{1}{n^2}\left(\frac{3^n \log 2}{10} - 2\right)$$
$$= \left(\frac{3^n \log 2}{10n^2} - \frac{2}{n^2}\right) \text{ and } \left|\sum_{j=1}^{n-1} \frac{\rho_{\lambda_j}(0)}{j^2} + \sum_{j=n+1}^{\infty} \frac{\widehat{\rho_{\lambda_j}}(0)}{j^2}\right|$$
$$\leq \left|\sum_{j=1}^{n-1} \frac{\rho_{\lambda_j}(0)}{j^2}\right| + \left|\sum_{j=n+1}^{\infty} \frac{\widehat{\rho_{\lambda_j}}(0)}{j^2}\right| \leq \sum_{j=1}^{n-1} \frac{|\rho_{\lambda_j}(0)|}{j^2} + \sum_{j=n+1}^{\infty} \frac{|\widehat{\rho_{\lambda_j}}(0)|}{j^2}$$
$$\leq \sum_{j=1}^{n-1} \frac{1}{j^2} + \sum_{j=n+1}^{\infty} \frac{1}{j^2} < \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$$

$$\implies \left|S_{\lambda_n{}^2}(f)(0)\right| > \left(\frac{3^n \log 2}{10n^2} - \frac{2}{n^2}\right) - \frac{\pi^2}{6} \geq \left(\frac{3^n \log 2}{10n^2} - 3\right),$$

where $n \geq 2$ because $\forall n \geq 2$, $\left(\frac{2}{n^2} + \frac{\pi^2}{6}\right) \leq 3$. Since $\lim_{n\to\infty}\left(\frac{3^n \log 2}{10n^2} - 3\right) = \infty$, then $\lim_{n\to\infty} |S_{\lambda_n{}^2}(f)(0)| = \infty$. $\implies$ $\{|S_{\lambda_n{}^2}(f)(0)|\}_{n=1}^{\infty}$ is not bounded. $\implies$ $\{|S_n(f)(0)|\}_{n=0}^{\infty}$ is not bounded. Hence, $S[f]$ diverges unboundedly when $t = 0$. $\square$

The following remark is from (Katznelson, 2004, P. 73).

Remark. From the second proof, $\forall m \in \mathbb{N}$, $f(t) = \sum_{n=1}^{m} \frac{\rho_n(\lambda_n t)}{n^2} + \sum_{n=m+1}^{\infty} \frac{\rho_n(\lambda_n t)}{n^2}$. The first term on the RHS is a trigonometric polynomial and so does not affect the convergence of $S[f]$. The second term on the RHS is periodic with period $\dfrac{2\pi}{\lambda_m}$ since $\forall k \geq m$, λ_m divides λ_k. This implies $\{S_n(f)(t)\}_{n=0}^{\infty}$ is not bounded when $t = \dfrac{2\pi j}{\lambda_m}$, where $j, m \in \mathbb{N}$. To obtain divergence at every rational multiple of 2π, redefine $\{\lambda_n\}_{n=1}^{\infty}$ by $\lambda_n = n!2^{3^n}$.

4.3 The Modified Dirichlet Kernel

Lemma 4.3.1. Let $k \in C(\mathbb{T})$ be an even function that satisfies $\dfrac{1}{2\pi}\displaystyle\int k(\tau)\,d\tau = 1$, $f \in L^1(\mathbb{T})$, $\mathbf{K}(f) = (k * f)$, and $t \in \mathbb{R}$. Let $\beta_t(\tau) = \left[\dfrac{f(t+\tau)+f(t-\tau)}{2}\right]$ and $\phi_t(\tau) = [\beta_t(\tau) - f(t)] = \left[\left(\dfrac{f(t+\tau)+f(t-\tau)}{2}\right) - f(t)\right]$. Then,

(i) $\mathbf{K}(f)(t) = \dfrac{1}{2\pi}\displaystyle\int f(t\pm\tau)k(\tau)\,d\tau = \dfrac{1}{2\pi}\displaystyle\int \beta_t(\tau)k(\tau)\,d\tau.$

(ii) $[\mathbf{K}(f)(t) - f(t)] = \dfrac{1}{2\pi}\displaystyle\int [f(t\pm\tau) - f(t)]k(\tau)\,d\tau = \dfrac{1}{2\pi}\displaystyle\int \phi_t(\tau)k(\tau)\,d\tau = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \phi_t(\tau)k(\tau)\,d\tau.$

Proof. (i) By Theorem 3.2.6.(2), $\mathbf{K}(f)(t) = (f * k)(t) = \dfrac{1}{2\pi}\displaystyle\int f(t-\tau)k(\tau)\,d\tau$. Since k is even, then by Corollary 3.1.9, $\mathbf{K}(f)(t) = \dfrac{1}{2\pi}\displaystyle\int f(t+\tau)k(-\tau)\,d\tau = \dfrac{1}{2\pi}\displaystyle\int f(t+\tau)k(\tau)\,d\tau$. By taking the average of the previous two equalities, it follows that $\mathbf{K}(f)(t) = \dfrac{1}{2\pi}\displaystyle\int \beta_t(\tau)k(\tau)\,d\tau$.

(ii) Since $f(t) = f(t)\left(\dfrac{1}{2\pi}\displaystyle\int k(\tau)\,d\tau\right) = \dfrac{1}{2\pi}\displaystyle\int f(t)k(\tau)\,d\tau$, then by subtracting this equation from the equation in (i), the first two equalities in (ii) hold. Since ϕ_t and k are even functions, then $\phi_t k$ is an even function and $[\mathbf{K}(f)(t) - f(t)] = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \phi_t(\tau)k(\tau)\,d\tau = \dfrac{1}{2\pi}\left(2\displaystyle\int_0^{\pi} \phi_t(\tau)k(\tau)\,d\tau\right) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \phi_t(\tau)k(\tau)\,d\tau.$

□

Note. The Dirichlet (Fejér) kernel satisfies the hypotheses of the lemma and then $\mathbf{K}(f) = S_n(f)$ ($\sigma_n(f)$ resp.).

Following (Zygmund, 1977, P. 50), the modified Dirichlet kernel and the modified n-th partial sums of $S[f]$ will be presented.

Definition 4.3.1. The modified Dirichlet kernel is the sequence $\{D_n^*\}_{n=1}^{\infty} \subseteq C(\mathbb{T})$ given by $D_n^*(t) = D_n(t) - \cos(nt)$ and $\forall n \in \mathbb{N}$, the modified n-th partial sum of $S[f]$ is $S_n^*(f) = (D_n^* * f)$.

Proposition 4.3.2. (i) D_n^* is an even function.

(ii)

$$D_n^*(t) = \begin{cases} \cot\left(\frac{t}{2}\right)\sin(nt), & \text{if } t \notin [0] \\ 2n, & \text{if } t \in [0] \end{cases}$$

(iii) $\forall t \in \mathbb{R}$, $|D_n^*(t)| \leq 2n$ and $\forall t \in [-\pi, \pi]$, $|D_n^*(t)| \leq \dfrac{2}{|t|}$

(iv) $\dfrac{1}{2\pi} \displaystyle\int D_n^*(\tau)\, d\tau = 1$

Proof. (i) D_n^* is an even function because D_n and $\cos(nt)$ are even functions.

(ii) If $t \in [0]$, then $D_n^*(t) = D_n^*(0) = D_n(0) - \cos(0) = (2n+1) - 1 = 2n$. Now assume $t \notin [0]$.

By Proposition 3.3.8.(iii), $D_n(t) = \dfrac{\sin\left(\dfrac{(2n+1)t}{2}\right)}{\sin\left(\dfrac{t}{2}\right)}$.

Since $\sin\left(\dfrac{(2n+1)t}{2}\right) = \sin\left(nt + \dfrac{t}{2}\right) = \cos(nt)\sin\left(\dfrac{t}{2}\right) + \cos\left(\dfrac{t}{2}\right)\sin(nt)$,

then $D_n(t) = \cos(nt) + \left(\dfrac{\cos\left(\dfrac{t}{2}\right)}{\sin\left(\dfrac{t}{2}\right)}\right)\sin(nt) = \cos(nt) + \cot\left(\dfrac{t}{2}\right)\sin(nt)$.

$\implies D_n^*(t) = D_n(t) - \cos(nt) = \cot\left(\dfrac{t}{2}\right)\sin(nt)$

(iii) By Proposition 3.3.8.(i), $D_n(t) = 1 + 2\sum\limits_{j=1}^{n}\cos(jt)$ which implies $D_n^*(t) = 1 + 2\sum\limits_{j=1}^{n-1}\cos(jt) + \cos(nt)$. Then $|D_n^*(t)| \leq 1 + 2\sum\limits_{j=1}^{n-1}|\cos(jt)| + |\cos(nt)| \leq 1 + 2\sum\limits_{j=1}^{n-1} 1 + 1 = 2n$. Since $\forall t \in [-\pi, \pi]$, $\left|\cot\left(\dfrac{t}{2}\right)\right| \leq \dfrac{2}{|t|}$ and $|\sin(nt)| \leq 1$, then $|D_n^*(t)| = \left|\cot\left(\dfrac{t}{2}\right)\sin(nt)\right| = \left|\cot\left(\dfrac{t}{2}\right)\right| |\sin(nt)| \leq \dfrac{2}{|t|}$.

(iv) Let $P_n(t) = \cos(nt) = \left(\dfrac{e^{int} + e^{-int}}{2}\right)$. By Theorem 3.2.2.(1) and the remark after Definition 3.3.3, $D_n^* = [D_n - P_n] \implies \dfrac{1}{2\pi}\displaystyle\int D_n^*(\tau)\,d\tau = \widehat{D_n^*}(0) = \widehat{D_n}(0) - \widehat{P_n}(0) = 1 - 0 = 1$.

□

Note. By the proposition, the modified Dirichlet kernel satisfies the hypotheses of the previous lemma and then $\mathbf{K}(f) = S_n^*(f)$.

Theorem 4.3.3. $\forall f \in L^1(\mathbb{T})$, $\lim\limits_{n\to\infty} \|S_n(f) - S_n^*(f)\|_\infty = 0$

Proof.

By Theorem 3.2.6.(1), $D_n^* = [D_n - P_n] \Rightarrow S_n^*(f) = (D_n^* * f) = (D_n * f) - (P_n * f)$.

By Corollary 3.2.8, $(P_n * f)(t) = \dfrac{\hat{f}(-n)e^{-int} + \hat{f}(n)e^{int}}{2}$ and $(D_n * f) = S_n(f)$.

$$\Longrightarrow \; S_n^*(f)(t) = S_n(f)(t) - \left[\frac{\hat{f}(-n)e^{-int} + \hat{f}(n)e^{int}}{2}\right]$$

$$\Longrightarrow \; [S_n(f)(t) - S_n^*(f)(t)] = \left[\frac{\hat{f}(-n)e^{-int} + \hat{f}(n)e^{int}}{2}\right].$$

$$\Longrightarrow \; |S_n(f)(t) - S_n^*(f)(t)| \leq \left|\frac{\hat{f}(-n)e^{-int} + \hat{f}(n)e^{int}}{2}\right| \leq \frac{|\hat{f}(-n)| + |\hat{f}(n)|}{2}$$

$$\Longrightarrow \; 0 \leq \|S_n(f) - S_n^*(f)\|_\infty = \sup_{t\in\mathbb{R}} |S_n(f)(t) - S_n^*(f)(t)| \leq \frac{|\hat{f}(-n)| + |\hat{f}(n)|}{2}$$

By the Riemann-Lebesgue Lemma, $\lim\limits_{n\to\infty} \left[\dfrac{|\hat{f}(-n)| + |\hat{f}(n)|}{2}\right] = 0$.

$$\Longrightarrow \; \lim_{n\to\infty} \|S_n(f) - S_n^*(f)\|_\infty = 0$$ □

Remark. By the theorem, $[S_n(f) - S_n^*(f)]$ converges uniformly to 0. This implies that to show $\lim\limits_{n\to\infty} S_n(f)(t) = f(t)$, it is enough to show that $\lim\limits_{n\to\infty} S_n^*(f)(t) = f(t)$ and that if the second limit holds uniformly, then so does the first. This idea will be used in the rest of this chapter. As will be seen later, many results can be obtained from this minor substitution.

Except for the last section, the rest of this chapter is concerned with results on the pointwise convergence of Fourier series.

4.4 Dini's Test and the Principle of Localization

Two different versions of Dini's Test will be presented. Although they appear distinct, they are actually equivalent. The first version is (Zygmund, 1977, P. 52, Theorem 6.1). Unlike in (Zygmund, 1977), a formal proof will be given.

Lemma 4.4.1. If $f \in L^1(\mathbb{T})$, then $\lim_{n\to\infty} c_n(f) = \lim_{n\to\infty} b_n(f) = 0$.

Proof. By the note after Definition 3.2.3, $c_n(f) = \hat{f}(-n) + \hat{f}(n)$ and $b_n(f) = \dfrac{\hat{f}(-n) - \hat{f}(n)}{i}$. By the Riemann-Lebesgue Lemma, the the result follows. □

Note. By the remark after the Riemann-Lebesgue Lemma, the above lemma holds uniformly on compact subsets of $L^1(\mathbb{T})$.

Theorem 4.4.2 (Dini's Test : Version 1). Let $f \in L^1(\mathbb{T})$ and $t \in \mathbb{R}$. If $\displaystyle\int_0^\pi |\phi_t(\tau)| \cot\left(\frac{\tau}{2}\right) d\tau < \infty$, then $\lim_{n\to\infty} S_n(f)(t) = f(t)$.

Proof. By Lemma 4.3.1.(ii), $[S_n^*(f)(t) - f(t)] = \dfrac{1}{2\pi}\displaystyle\int \phi_t(\tau) D_n^*(\tau)\, d\tau$. By Proposition 4.3.2.(ii), $\forall \tau \notin [0]$, $D_n^*(\tau) = \cot\left(\frac{\tau}{2}\right)\sin(n\tau)$. Since $[0]$ is an m-null set, then $[S_n^*(f)(t) - f(t)] = \dfrac{1}{2\pi}\displaystyle\int \phi_t(\tau)\cot\left(\frac{\tau}{2}\right)\sin(n\tau)\, d\tau$.

Define $h\colon \mathbb{T} \to \mathbb{C}$ by $h(\tau) = \phi_t(\tau)\cot\left(\frac{\tau}{2}\right)$. [Note that on the m-null set $[0]$, $h(\tau) = 0$ because $\phi_t(\tau) = 0$ and that $h\colon \mathbb{T} \to \mathbb{C}$ because h is a product of 2π-periodic measurable functions.] h is an odd function because ϕ_t is an even function and $\cot\left(\frac{\tau}{2}\right)$ is an odd function. This implies that $|h|$ is an even function.

$$\implies \|h\|_1 = \frac{1}{2\pi}\int |h(\tau)|\, d\tau = \frac{1}{2\pi}\int_{-\pi}^{\pi} |h(\tau)|\, d\tau = \frac{1}{2\pi}\left(2\int_0^\pi |h(\tau)|\, d\tau\right)$$
$$= \frac{1}{\pi}\int_0^\pi |h(\tau)|\, d\tau = \frac{1}{\pi}\int_0^\pi |\phi_t(\tau)|\cot\left(\frac{\tau}{2}\right) d\tau < \infty \implies h \in L^1(\mathbb{T})$$

Then $[S_n^*(f)(t) - f(t)] = \dfrac{1}{2\pi}\displaystyle\int h(\tau)\sin(n\tau)\,d\tau = \dfrac{b_n(h)}{2}$. By the previous lemma, $\lim_{n\to\infty} b_n(h) = 0$ which implies $\lim_{n\to\infty} S_n^*(f)(t) = f(t)$.

Hence, $\lim_{n\to\infty} S_n(f)(t) = f(t)$. □

The second version is (Katznelson, 2004, P. 74-75, Lemma 2.3 and Theorem 2.5). Unlike the proof of the lemma in (Katznelson, 2004), it is shown here that $\left\|\dfrac{f(t)}{t}\right\|_1 < \infty$ and $h \in L^1(\mathbb{T})$, where $\left\|\dfrac{f(t)}{t}\right\|_1$ will be defined shortly and h is as in the following lemma. Also, unlike in (Katznelson, 2004), a formal proof will be given for the theorem.

Notation. In the following lemma and theorems, $\displaystyle\int_{-1}^{1}\left|\frac{f(t)}{t}\right| dt < \infty$ means that the integral of $\left|\dfrac{f(t)}{t}\right|$ over an interval around 0 is bounded. Although $\left(\dfrac{f(t)}{t}\right) \notin L^1(\mathbb{T})$ as $g(t) = t$ is not a 2π-periodic function, the following notation will be used for convenience. $\left\|\dfrac{f(t)}{t}\right\|_1 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\left|\frac{f(t)}{t}\right| dt$

Lemma 4.4.3. Let $f \in L^1(\mathbb{T})$. If $\displaystyle\int_{-1}^{1}\left|\frac{f(t)}{t}\right| dt < \infty$, then $\lim_{n\to\infty} S_n(f)(0) = 0$.

Proof. $\displaystyle\int_{-1}^{1}\left|\frac{f(t)}{t}\right| dt < \infty \implies \exists c \in (0,\pi]$ s.t. $\displaystyle\int_{-c}^{c}\left|\frac{f(t)}{t}\right| dt < \infty$

Let $c \le |t| \le \pi$, so that $\dfrac{1}{\pi} \le \dfrac{1}{|t|} \le \dfrac{1}{c}$.

$$\implies \frac{1}{2\pi}\int_{c\le|t|\le\pi}\left|\frac{f(t)}{t}\right| dt = \frac{1}{2\pi}\int_{c\le|t|\le\pi}\frac{|f(t)|}{|t|}\,dt \le \frac{1}{2\pi}\int_{c\le|t|\le\pi}\frac{|f(t)|}{c}\,dt$$
$$= \frac{1}{c}\left(\frac{1}{2\pi}\int_{c\le|t|\le\pi}|f(t)|\,dt\right) \le \frac{1}{c}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|\,dt\right) = \frac{1}{c}\left(\frac{1}{2\pi}\int|f(t)|\,dt\right)$$
$$= \frac{\|f\|_1}{c} < \infty$$
$$\implies \left\|\frac{f(t)}{t}\right\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{f(t)}{t}\right| dt = \frac{1}{2\pi}\int_{-c}^{c}\left|\frac{f(t)}{t}\right| dt + \frac{1}{2\pi}\int_{c\le|t|\le\pi}\left|\frac{f(t)}{t}\right| dt < \infty$$

Define $h\colon \mathbb{T} \to \mathbb{C}$ by $h(t) = f(t)\cot\left(\frac{t}{2}\right)$. [Note that on the m-null set $[0]$, h can be redefined so that it is well-defined on all of $\mathbb{R}$ and that $h\colon \mathbb{T} \to \mathbb{C}$ because h is a product of 2π-periodic measurable functions.]

$$\forall t \in [-\pi, \pi], \left|\cot\left(\frac{t}{2}\right)\right| \leq \frac{2}{|t|}$$
$$\implies \forall t \in [-\pi, \pi], |h(t)| = |f(t)| \left|\cot\left(\frac{t}{2}\right)\right| \leq |f(t)| \left(\frac{2}{|t|}\right) = 2\left|\frac{f(t)}{t}\right|$$
$$\implies \|h\|_1 = \frac{1}{2\pi}\int |h(t)|\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} |h(t)|\, dt \leq 2\left\|\frac{f(t)}{t}\right\|_1 < \infty \implies h \in L^1(\mathbb{T})$$

By the proof of Proposition 4.3.2.(ii), $\forall t \notin [0]$, $D_n(t) = \cos(nt) + \cot\left(\frac{t}{2}\right)\sin(nt)$ and $f(t)D_n(t) = f(t)\cos(nt) + h(t)\sin(nt)$. By Lemma 4.3.1.(i), $S_n(f)(0) =$ $\frac{1}{2\pi}\int f(t)D_n(t)\, dt = \frac{1}{2\pi}\int f(t)\cos(nt)\, dt + \frac{1}{2\pi}\int h(t)\sin(nt)\, dt = \frac{c_n(f) + b_n(h)}{2}$.

By the previous lemma, $\lim_{n\to\infty} c_n(f) = \lim_{n\to\infty} b_n(h) = 0 \implies \lim_{n\to\infty} S_n(f)(0) = 0$. □

Theorem 4.4.4 (Dini's Test : Version 2). Let $f \in L^1(\mathbb{T})$ and $t_0 \in \mathbb{R}$. If $\int_{-1}^{1}\left|\frac{f(t+t_0) - f(t_0)}{t}\right| dt < \infty$, then $\lim_{n\to\infty} S_n(f)(t_0) = f(t_0)$.

Proof. Define $g\colon \mathbb{T} \to \mathbb{C}$ by $g(t) = f(t+t_0) - f(t_0)$. Then, $g \in L^1(\mathbb{T})$ and $\int_{-1}^{1}\left|\frac{g(t)}{t}\right| dt < \infty$. By the previous lemma, $\lim_{n\to\infty} S_n(g)(0) = 0$. By Lemma 4.3.1, $\forall n \in \mathbb{N}_0$, $S_n(g)(0) = [S_n(f)(t_0) - f(t_0)]$. This implies $\lim_{n\to\infty} [S_n(f)(t_0) - f(t_0)] = 0$ which implies $\lim_{n\to\infty} S_n(f)(t_0) = f(t_0)$. □

The following remark from (Zygmund, 1977, P. 52) briefly explains why the two versions of Dini's Test are equivalent.

Remark. Since $\lim_{\tau \to 0} \left[\dfrac{\cot\left(\frac{\tau}{2}\right)}{\frac{2}{\tau}} \right] = 1$, then the integral in the first version can be replaced by $2\int_0^{\pi} \dfrac{|\phi_t(\tau)|}{\tau}\, d\tau$ and consequently it can be shown that the two versions are equivalent.

The following lemma is stated and proved on (Zygmund, 1977, P. 52-53, Lemma 6.3).

Lemma 4.4.5. Let $f \in L^1(\mathbb{T})$, $g \in L^\infty(\mathbb{T})$. Then the Fourier coefficients of the function $h(\tau) = f(t+\tau)g(\tau)$ tend to 0 as $|n| \to \infty$ uniformly in t.

Theorem 4.4.6 (Principle of Localization). Let $f \in L^1(\mathbb{T})$. If f vanishes in an open interval I, then $S_n(f)(t)$ converges to 0 for $t \in I$ and the convergence is uniform on closed subsets of I.

Remark. Two proofs of the uniform convergence will be presented. The first proof is from (Katznelson, 2004, P. 75, Theorem 2.4). This proof was worked out in detail in many discussions with Professor Klemes. The proof differs from the proof in (Katznelson, 2004) as follows. Here, it is shown that $\forall t_0 \in I$, $\lim_{n\to\infty} S_n(f)(t_0) = 0$. In (Katznelson, 2004), $f(t - t_0)$ should be replaced by $f(t + t_0)$, which is done here. The continuity of Φ and Ψ is shown here which is the justification for the compactness of $\Phi(I_0)$ and $\Psi(I_0)$.

Proof 1. Let $t_0 \in I$. Since I is an open interval, $\exists r > 0$ s.t. $(t_0 - r, t_0 + r) \subseteq I$. Then $\forall t \in (-r, r)$, $f(t + t_0) = 0$ and so $\int_{-r}^{r} \left| \dfrac{f(t+t_0)}{t} \right| dt = 0 < \infty$. By Dini's Test, $\lim_{n\to\infty} S_n(f)(t_0) = 0$. Therefore, $\forall t_0 \in I$, $\lim_{n\to\infty} S_n(f)(t_0) = 0$.

Now, let I_0 be a closed subinterval of I. By (H-3) with $B = L^1(\mathbb{T})$,

$\phi\colon \mathbb{T} \to L^1(\mathbb{T})$ is continuous. $\implies$ $\Phi\colon \mathbb{T} \to L^1(\mathbb{T})$ is continuous where $\Phi(t_0) = \Phi_{t_0} = \phi(-t_0)$, i.e. $\Phi_{t_0}(t) = f_{(-t_0)}(t) = f(t+t_0)$. $\Phi\colon I_0 \to L^1(\mathbb{T})$ is continuous because $I_0 \subseteq \mathbb{R}$. $\Phi(I_0) = \{\Phi_{t_0}\}_{t_0 \in I_0}$ is compact because I_0 is compact and $\Phi\colon I_0 \to L^1(\mathbb{T})$ is continuous.

$\forall t_0 \in I_0$, define $\Psi_{t_0}\colon \mathbb{T} \to \mathbb{C}$ by $\Psi_{t_0}(t) = \Phi_{t_0}(t)\cot\left(\frac{t}{2}\right)$. By the above, with $t_0 \in I_0$ and r as given, $\displaystyle\int_{-r}^{r}\left|\frac{\Phi_{t_0}(t)}{t}\right| dt = 0 < \infty$. By the proof of Lemma 4.4.3 with $f = \Phi_{t_0}$, $\Psi_{t_0} \in L^1(\mathbb{T})$. Define $\Psi\colon I_0 \to L^1(\mathbb{T})$ by $\Psi(t_0) = \Psi_{t_0}$.

Suppose it is shown that $\Psi\colon I_0 \to L^1(\mathbb{T})$ is continuous. $\Psi(I_0) = \{\Psi_{t_0}\}_{t_0 \in I_0}$ is compact because I_0 is compact and $\Psi\colon I_0 \to L^1(\mathbb{T})$ is continuous. By Lemma 4.3.1 and the proof of Lemma 4.4.3, $\forall t_0 \in I_0$, $S_n(f)(t_0) = S_n(\Phi_{t_0})(0) = \dfrac{c_n(\Phi_{t_0}) + b_n(\Psi_{t_0})}{2}$. By the note after Lemma 4.4.1, $\Phi(I_0)$ and $\Psi(I_0)$ are compact $\implies \lim\limits_{n\to\infty} c_n(\Phi_{t_0}) = \lim\limits_{n\to\infty} b_n(\Psi_{t_0}) = 0$ uniformly for $t_0 \in I_0$. $\implies \lim\limits_{n\to\infty} S_n(f)(t_0) = 0$ uniformly for $t_0 \in I_0$. Hence, $\lim\limits_{n\to\infty} S_n(f)(t) = 0$ uniformly on I_0.

Thus, it is enough to show $\Psi\colon I_0 \to L^1(\mathbb{T})$ is continuous. Let $I = (c,d)$ and $I_0 = [a,b]$, where $c < a < b < d$. Let $r = \dfrac{\min(d-b, a-c, \pi)}{2}$. Then, $0 < r \leq \dfrac{\pi}{2} < \pi$ and $r \leq \dfrac{\min(d-b, a-c)}{2} < \min(d-b, a-c)$. Let $t_0 \in I_0$. Then $c = a-(a-c) < a-r \leq t_0+r \leq b+r < b+(d-b) = d$, i.e. $c < t_0 - r < t_0 + r < d$ and so $(t_0 - r, t_0 + r) \subseteq (c,d) = I$. Therefore, $\forall t_0 \in I_0\ \forall t \in (-r,r)$, $\Phi_{t_0}(t) = 0$. Now, let $t_0, s_0 \in I_0$ and $g = [\Phi_{t_0} - \Phi_{s_0}]$. $\forall t \in (-r,r)$, $g(t) = [\Phi_{t_0}(t) - \Phi_{s_0}(t)] = 0$ and $[\Psi_{t_0}(t) - \Psi_{s_0}(t)] = g(t)\cot\left(\frac{t}{2}\right)$

By the proof of Lemma 4.4.3 with f and c replaced by g and r,

$$\left\|\frac{g(t)}{t}\right\|_1 = \frac{1}{2\pi}\int_{-r}^{r}\left|\frac{g(t)}{t}\right| dt + \frac{1}{2\pi}\int_{r\leq|t|\leq\pi}\left|\frac{g(t)}{t}\right| dt = 0 + \frac{1}{2\pi}\int_{r\leq|t|\leq\pi}\left|\frac{g(t)}{t}\right| dt$$
$$= \frac{1}{2\pi}\int_{r\leq|t|\leq\pi}\left|\frac{g(t)}{t}\right| dt \leq \frac{\|g\|_1}{r}.$$

By the proof of Lemma 4.4.3 with f, c, and h replaced by g, r, and $[\Psi_{t_0} - \Psi_{s_0}]$, $\|\Psi_{t_0} - \Psi_{s_0}\|_1 \leq 2\left\|\frac{g(t)}{t}\right\|_1 \leq \frac{2}{r}\|g\|_1 = \frac{2}{r}\|\Phi_{t_0} - \Phi_{s_0}\|_1$,

i.e. $\|\Psi(t_0) - \Psi(s_0)\|_1 = \|\Psi_{t_0} - \Psi_{s_0}\|_1 \leq \frac{2}{r}\|\Phi_{t_0} - \Phi_{s_0}\|_1 = \frac{2}{r}\|\Phi(t_0) - \Phi(s_0)\|_1$.

Therefore, $\forall t_0, s_0 \in I_0$, $\|\Psi(t_0) - \Psi(s_0)\|_1 \leq \frac{2}{r}\|\Phi(t_0) - \Phi(s_0)\|_1$, which implies that $\Psi\colon I_0 \to L^1(\mathbb{T})$ is continuous because $\Phi\colon I_0 \to L^1(\mathbb{T})$ is continuous.

Hence, $\Psi\colon I_0 \to L^1(\mathbb{T})$ is continuous and the theorem holds. □

The second proof is from (Zygmund, 1977, P. 52-53, Theorem 6.3). The proof is essentially the same as in (Zygmund, 1977).

Proof 2. Let I_0 be as above. By the first proof, $\exists r \in (0,\pi)\ \forall t_0 \in I_0\ \forall t \in (-r,r)$, $f(t_0+t) = 0$. By Lemma 4.3.1.(i) and the argument used in the proof of Dini's Test (Version 1), $S_n^*(f)(t_0) = \frac{1}{2\pi}\int f(t_0+t)D_n^*(t)\,dt = \frac{1}{2\pi}\int f(t_0+t)\cot\left(\frac{t}{2}\right)\sin(nt)\,dt$.

Define $g\colon [-\pi,\pi] \to \mathbb{C}$ by $g(t) = 0$ on $(-r,r)$ and $g(t) = \cot\left(\frac{t}{2}\right)$ if $r \leq |t| \leq \pi$. Extend g 2π-periodically so that $g\colon \mathbb{T} \to \mathbb{C}$. g is bounded because on $(-r,r)$, $|g| = 0$ and if $r \leq |t| \leq \pi$, then $|g(t)| \leq \frac{2}{|t|} \leq \frac{2}{r}$. This implies $g \in L^\infty(\mathbb{T})$.

Let $h(t) = f(t_0 + t)g(t)$. Then $S_n^*(f)(t_0) = \dfrac{1}{2\pi}\displaystyle\int f(t_0+t)\cot\left(\frac{t}{2}\right)\sin(nt)\,dt$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t_0+t)\cot\left(\frac{t}{2}\right)\sin(nt)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t_0+t)g(t)\sin(nt)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(t)\sin(nt)\,dt = \frac{1}{2\pi}\int h(t)\sin(nt)\,dt = \frac{b_n(h)}{2}.$$

By the previous lemma, $\lim_{n\to\infty} b_n(h) = 0$ uniformly for $t_0 \in I_0$. $\Longrightarrow$ $\lim_{n\to\infty} S_n^*(f)(t_0) = 0$ uniformly for $t_0 \in I_0$. Hence, $\lim_{n\to\infty} S_n(f)(t) = 0$ uniformly on I_0. □

The following remark is from (Katznelson, 2004, P. 75).

Remark. The Principle of Localization can be restated as follows: Let $f, g \in L^1(\mathbb{T})$ and assume that $f(t) = g(t)$ in some neighbourhood of a point t_0. Then $S[f](t_0)$ and $S[g](t_0)$ are either both convergent and to the same limit or both divergent and in the same manner.

The following definition is from (Zygmund, 1977, P. 42).

Definition 4.4.1. Let $f \in L^1(\mathbb{T})$ and I be a closed interval. Then the function $\omega\colon [0,\infty) \to [0,\infty]$ defined by $\omega(\delta) = \omega(f,\delta) = \sup_{\substack{x_1,x_2\in I\\ |x_2-x_1|\le\delta}} |f(x_2) - f(x_1)|$ is called the modulus of continuity of f on I. f is continuous on I iff $\lim_{\delta\to 0^+} \omega(\delta) = 0$.

Remark. If $f \in C(\mathbb{T})$ and I is an interval of length 2π, then the modulus of continuity of f on I is the same as the modulus of continuity of f as defined in Definition 3.4.4.

The following theorem is from (Zygmund, 1977, P. 54, Theorem 6.8). The proof is basically the same as in (Zygmund, 1977) except for the following differences. Unlike in (Zygmund, 1977), the proof that $|P| \leq \frac{2}{\pi}\int_0^\delta \frac{\xi(t)}{t}\,dt$ is presented here. The part of the proof where it is shown that $\int_{t_0}^\delta \frac{|f(x+t)-f(x)|}{t}\,dt \leq \int_0^\delta \frac{|f(b+t)-f(b)|}{t}\,dt$ was provided by Professor Klemes. Also, here it is shown in detail that $\lim_{n\to\infty} Q = 0$ uniformly for $x \in I$.

Theorem 4.4.7. Let $f \in L^1(\mathbb{T})$. Let f be continuous on a closed interval I and ω be the modulus of continuity of f on I.

Let $I = [a, b]$ and $\xi(t) = \omega(t)+|f(a)-f(a-t)|+|f(b+t)-f(b)|$. If $\int_0^\pi \frac{\xi(t)}{t}\,dt < \infty$, then $\lim_{n\to\infty} S_n(f)(x) = f(x)$ uniformly for $x \in I$.

Proof. Let $\epsilon > 0$. By Lemma 4.3.1.(ii),

$$[S_n^*(f)(x)-f(x)] = \frac{1}{2\pi}\int [f(x+t)-f(x)]D_n^*(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x+t)-f(x)]D_n^*(t)\,dt.$$

Let $\delta \in (0,\pi]$ be arbitrary. Then $[S_n^*(f)(x) - f(x)] = P + Q$, where $P = \frac{1}{2\pi}\int_{-\delta}^{\delta} [f(x+t)-f(x)]D_n^*(t)\,dt$ and $Q = \frac{1}{2\pi}\int_{\delta\leq|t|\leq\pi} [f(x+t)-f(x)]D_n^*(t)\,dt$.

By Proposition 4.3.2.(iii), $\forall t \in (-\delta,\delta)$, $|D_n^*(t)| \leq \frac{2}{|t|}$.

$$\implies |P| \leq \frac{1}{2\pi}\int_{-\delta}^{\delta} |[f(x+t)-f(x)]D_n^*(t)|\,dt = \frac{1}{2\pi}\int_{-\delta}^{\delta} |f(x+t)-f(x)|\,|D_n^*(t)|\,dt$$
$$\leq \frac{1}{\pi}\int_{-\delta}^{\delta} \frac{|f(x+t)-f(t)|}{|t|}\,dt.$$

First, consider $\int_0^\delta \frac{|f(x+t)-f(t)|}{|t|}\,dt = \int_0^\delta \frac{|f(x+t)-f(t)|}{t}\,dt.$

Fix $x \in I$ and let t_0 be s.t. $x + t_0 = b$. Then $t_0 = (b - x) \geq 0$ and $x = b - t_0$. If $t \in [0, t_0]$, then $|f(x+t) - f(x)| \leq \omega(t)$ and if $t \in [t_0, \delta]$, then $|f(x+t) - f(x)| \leq |f(x+t) - f(b)| + |f(b) - f(x)| \leq |f(x+t) - f(b)| + \omega(t) = |f(b + (t - t_0)) - f(b)| + \omega(t)$.

$$\implies \int_0^\delta \frac{|f(x+t) - f(x)|}{t}\,dt = \int_0^{t_0} \frac{|f(x+t) - f(x)|}{t}\,dt + \int_{t_0}^\delta \frac{|f(x+t) - f(x)|}{t}\,dt$$
$$\leq \int_0^{t_0} \frac{\omega(t)}{t}\,dt + \int_{t_0}^\delta \frac{|f(b + (t - t_0)) - f(b)| + \omega(t)}{t}\,dt$$
$$= \int_0^\delta \frac{\omega(t)}{t}\,dt + \int_{t_0}^\delta \frac{|f(b + (t - t_0)) - f(b)|}{t}\,dt$$

By Proposition 2.3.2.(2), $\displaystyle\int_{t_0}^\delta \frac{|f(b + (t - t_0)) - f(b)|}{t}\,dt$
$$= \int_0^{\delta - t_0} \frac{|f(b+t) - f(b)|}{t + t_0}\,dt \leq \int_0^{\delta - t_0} \frac{|f(b+t) - f(b)|}{t}\,dt \leq \int_0^\delta \frac{|f(b+t) - f(b)|}{t}\,dt.$$

$$\implies \int_0^\delta \frac{|f(x+t) - f(x)|}{t}\,dt \leq \int_0^\delta \frac{\omega(t)}{t}\,dt + \int_0^\delta \frac{|f(b+t) - f(b)|}{t}\,dt$$
$$= \int_0^\delta \frac{\omega(t) + |f(b+t) - f(b)|}{t}\,dt \leq \int_0^\delta \frac{\xi(t)}{t}\,dt$$

$$\therefore \int_0^\delta \frac{|f(x+t) - f(t)|}{|t|}\,dt = \int_0^\delta \frac{|f(x+t) - f(t)|}{t}\,dt \leq \int_0^\delta \frac{\xi(t)}{t}\,dt$$

By a similar argument, $\displaystyle\int_{-\delta}^0 \frac{|f(x+t) - f(t)|}{|t|}\,dt \leq \int_0^\delta \frac{\xi(t)}{t}\,dt.$

$$\implies \int_{-\delta}^\delta \frac{|f(x+t) - f(t)|}{|t|}\,dt = \int_{-\delta}^0 \frac{|f(x+t) - f(t)|}{|t|}\,dt + \int_0^\delta \frac{|f(x+t) - f(t)|}{|t|}\,dt$$
$$\leq 2\int_0^\delta \frac{\xi(t)}{t}\,dt \implies |P| \leq \frac{2}{\pi}\int_0^\delta \frac{\xi(t)}{t}\,dt$$

Since $\int_0^{\pi} \frac{\xi(t)}{t}\, dt < \infty$, then $\delta \in (0, \pi]$ can be chosen sufficiently small so that $|P| \leq \frac{2}{\pi} \int_0^{\delta} \frac{\xi(t)}{t}\, dt < \frac{\epsilon}{2}$ and this is independent of $x \in I$.

Define $g\colon [-\pi, \pi] \to \mathbb{C}$ by $g(t) = 0$ on $(-\delta, \delta)$ and $g(t) = \cot\left(\frac{t}{2}\right)$ on $[-\pi, \pi]\backslash(-\delta, \delta)$. Extend g 2π-periodically so that $g\colon \mathbb{T} \to \mathbb{C}$. g is bounded because on $(-\delta, \delta)$, $|g| = 0$ and on $[-\pi, \pi]\backslash(-\delta, \delta)$, $|g(t)| \leq \frac{2}{|t|} \leq \frac{2}{\delta}$. This implies $g \in L^{\infty}(\mathbb{T})$ and by Corollary 3.1.6, $g \in L^1(\mathbb{T})$.

$$
\begin{aligned}
Q &= \frac{1}{2\pi} \int_{\delta \leq |t| \leq \pi} [f(x+t) - f(x)] D_n^*(t)\, dt \\
&= \frac{1}{2\pi} \int_{\delta \leq |t| \leq \pi} [f(x+t) - f(x)] \cot\left(\frac{t}{2}\right) \sin(nt)\, dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x+t) - f(x)] g(t) \sin(nt)\, dt = \frac{1}{2\pi} \int [f(x+t) - f(x)] g(t) \sin(nt)\, dt \\
&= \frac{1}{2\pi} \int f(x+t) g(t) \sin(nt)\, dt - f(x) \left(\frac{1}{2\pi} \int g(t) \sin(nt)\, dt \right)
\end{aligned}
$$

Let $h(t) = f(x+t)g(t)$. Then $Q = \left(\frac{b_n(h) - f(x) b_n(g)}{2} \right)$. Since f is continuous on I, then f is bounded and so $\exists M > 0$ s.t. $|f(x)| \leq M$. This implies $0 \leq |Q| \leq \frac{|b_n(h)| + |f(x)|\, |b_n(g)|}{2} \leq \frac{|b_n(h)| + M |b_n(g)|}{2}$. By the previous lemma and Lemma 4.4.1, $\lim_{n \to \infty} \left[\frac{|b_n(h)| + M |b_n(g)|}{2} \right] = 0$ uniformly for $x \in I$. This implies $\lim_{n \to \infty} Q = 0$ uniformly for $x \in I$.

$$
\begin{aligned}
&\implies \exists N \in \mathbb{N}\ \forall x \in I\ \forall n \geq N, |Q| < \frac{\epsilon}{2} \\
&\implies |S_n^*(f)(x) - f(x)| = |P + Q| \leq |P| + |Q| \leq 2\left(\frac{\epsilon}{2}\right) = \epsilon \\
&\therefore\ \forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall x \in I\ \forall n \geq N, |S_n^*(f)(x) - f(x)| < \epsilon
\end{aligned}
$$

Therefore, $\lim_{n \to \infty} S_n^*(f)(x) = f(x)$ uniformly for $x \in I$.

Hence, $\lim_{n\to\infty} S_n(f)(x) = f(x)$ uniformly for $x \in I$. $\square$

4.5 Dirichlet-Jordan Test

The following theorem is from (Katznelson, 2004, P. 73-74, Theorem 2.2.2). The proof here differs from the one in (Katznelson, 2004) as follows. The following lemmas are used implicitly in the proof of the theorem and are not stated or proved in (Katznelson, 2004). Here, it is shown in detail that for $c > 1$ sufficiently close to 1,
$\left|\frac{[cn]+1}{[cn]-n} \sum_{n<|j|\leq[cn]} \left(1 - \frac{|j|}{[cn]+1}\right) \hat{f}(j)e^{ijt}\right|$ can be made as small as possible and
$\lim_{n\to\infty} \left[\frac{[cn]+1}{[cn]-n}\sigma_{[cn]}(f)(t) - \frac{n+1}{[cn]-n}\sigma_n(f)(t)\right] = \sigma(f)(t)$. Also, a full explanation is given on why $S_n(f)(t)$ converges uniformly on some set if $\sigma_n(f)(t)$ converges uniformly on that same set.

Notation. For the next lemmas and theorem only, let $\forall x \in \mathbb{R}$, $[x]$ denote the greatest integer less than or equal to x. Note that $x - 1 \leq [x] \leq x$.

Lemma 4.5.1. Let $c \in \mathbb{R}$ and $c > 0$. Then $\lim_{n\to\infty} \frac{[cn]}{n} = c$.

Proof.

$$\forall n \in \mathbb{N},\ cn - 1 \leq [cn] \leq cn \Rightarrow c - \frac{1}{n} \leq \frac{[cn]}{n} \leq c \Rightarrow -\frac{1}{n} \leq \frac{[cn]}{n} - c \leq 0$$
$$\Rightarrow 0 \leq \left|\frac{[cn]}{n} - c\right| \leq \frac{1}{n}$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, then $\lim_{n\to\infty} \frac{[cn]}{n} = c$. $\square$

Remark. In the lemma, let $c > 1$. Then $\exists N \in \mathbb{N}\ \forall n \geq N$, $[cn] \geq cn - 1 > n$.

Lemma 4.5.2. Let $c \in \mathbb{R}$ and $c > 1$. Then $\exists N \in \mathbb{N}\ \forall n \geq N$,

$$S_n(f)(t) + \frac{[cn]+1}{[cn]-n} \sum_{n<|j|\leq[cn]} \left(1 - \frac{|j|}{[cn]+1}\right) \hat{f}(j)e^{ijt}$$
$$= \frac{[cn]+1}{[cn]-n}\sigma_{[cn]}(f)(t) - \frac{n+1}{[cn]-n}\sigma_n(f)(t).$$

Proof. Let N be as in the remark and $n \geq N$. Then $([cn]-n) > 0$.

$$\begin{aligned}
\text{RHS} &= \frac{[cn]+1}{[cn]-n}\sigma_{[cn]}(f)(t) - \frac{n+1}{[cn]-n}\sigma_n(f)(t) \\
&= \frac{[cn]+1}{[cn]-n} \sum_{j=-[cn]}^{[cn]} \left(1 - \frac{|j|}{[cn]+1}\right) \hat{f}(j)e^{ijt} - \frac{n+1}{[cn]-n} \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) \hat{f}(j)e^{ijt} \\
&= \sum_{j=-[cn]}^{[cn]} \left(\frac{[cn]+1-|j|}{[cn]-n}\right) \hat{f}(j)e^{ijt} - \sum_{j=-n}^{n} \left(\frac{n+1-|j|}{[cn]-n}\right) \hat{f}(j)e^{ijt} \\
&= \sum_{j=-n}^{n} \left(\frac{[cn]+1-|j|}{[cn]-n}\right) \hat{f}(j)e^{ijt} + \sum_{n<|j|\leq[cn]} \left(\frac{[cn]+1-|j|}{[cn]-n}\right) \hat{f}(j)e^{ijt} \\
&\quad - \sum_{j=-n}^{n} \left(\frac{n+1-|j|}{[cn]-n}\right) \hat{f}(j)e^{ijt} \\
&= \sum_{j=-n}^{n} \left(\frac{[cn]-n}{[cn]-n}\right) \hat{f}(j)e^{ijt} + \frac{1}{[cn]-n} \sum_{n<|j|\leq[cn]} ([cn]+1-|j|)\hat{f}(j)e^{ijt} \\
&= \sum_{j=-n}^{n} \hat{f}(j)e^{ijt} + \frac{[cn]+1}{[cn]-n} \sum_{n<|j|\leq[cn]} \left(1 - \frac{|j|}{[cn]+1}\right) \hat{f}(j)e^{ijt} \\
&= S_n(f)(t) + \frac{[cn]+1}{[cn]-n} \sum_{n<|j|\leq[cn]} \left(1 - \frac{|j|}{[cn]+1}\right) \hat{f}(j)e^{ijt} = \text{LHS}.
\end{aligned}$$

□

Theorem 4.5.3. Let $f \in L^1(\mathbb{T})$ and assume that $\hat{f}(n) = \mathrm{O}\left(\frac{1}{n}\right)$ as $|n| \to \infty$. Then $S_n(f)(t)$ and $\sigma_n(f)(t)$ converge for the same values of t and to the same limit. Also, if $\sigma_n(f)(t)$ converges uniformly on some set, then so does $S_n(f)(t)$.

Proof. By the remark after Definition 3.3.5, if $\lim_{n\to\infty} S_n(f)(t)$ exists, then $\lim_{n\to\infty} \sigma_n(f)(t) = \lim_{n\to\infty} S_n(f)(t)$. Now assume $\lim_{n\to\infty} \sigma_n(f)(t)$ exists. Let $\sigma = \sigma(f)(t) = \lim_{n\to\infty} \sigma_n(f)(t)$.

$$\hat{f}(n) = \mathrm{O}\left(\frac{1}{n}\right) \text{ as } |n| \to \infty \Rightarrow \exists M > 0\ \exists n_0 \in \mathbb{N}\ \forall |n| \geq n_0,\ |\hat{f}(n)| \leq \frac{M}{|n|}$$

Let $n \in \mathbb{N}$ and $c \in \mathbb{R}$, where $c > 1$. By the previous lemma, $\exists N \in \mathbb{N}\ \forall n \geq N$,

$$\begin{aligned} S_n(f)(t) = {} & \frac{[cn]+1}{[cn]-n}\sigma_{[cn]}(f)(t) - \frac{n+1}{[cn]-n}\sigma_n(f)(t) \\ & - \frac{[cn]+1}{[cn]-n} \sum_{n<|j|\leq[cn]} \left(1 - \frac{|j|}{[cn]+1}\right) \hat{f}(j)e^{ijt}. \end{aligned}$$

Let $N_0 = \max(n_0, N)$ and $n \geq N_0$.

$$\begin{aligned} & \left| \sum_{n<|j|\leq[cn]} \left(1 - \frac{|j|}{[cn]+1}\right) \hat{f}(j)e^{ijt} \right| \leq \sum_{n<|j|\leq[cn]} \left| \left(1 - \frac{|j|}{[cn]+1}\right) \hat{f}(j)e^{ijt} \right| \\ & = \sum_{n<|j|\leq[cn]} \left(1 - \frac{|j|}{[cn]+1}\right) |\hat{f}(j)| \leq \sum_{n<|j|\leq[cn]} \left(1 - \frac{|j|}{[cn]+1}\right) \frac{M}{|j|} \\ & = M \sum_{n<|j|\leq[cn]} \left(1 - \frac{|j|}{[cn]+1}\right) \frac{1}{|j|} = 2M \sum_{n<j\leq[cn]} \left(1 - \frac{j}{[cn]+1}\right) \frac{1}{j} \\ & = 2M \sum_{j=n+1}^{[cn]} \left(1 - \frac{j}{[cn]+1}\right) \frac{1}{j} \leq 2M \sum_{j=n+1}^{[cn]} \left(1 - \frac{n+1}{[cn]+1}\right) \frac{1}{n} \\ & = \left(\frac{[cn]-n}{[cn]+1}\right) 2M \sum_{j=n+1}^{[cn]} \frac{1}{n} = \left(\frac{[cn]-n}{[cn]+1}\right) \frac{2M([cn]-n)}{n} \\ & \leq \left(\frac{[cn]-n}{[cn]+1}\right) \frac{2M(cn-n)}{n} = \left(\frac{[cn]-n}{[cn]+1}\right) \frac{2Mn(c-1)}{n} = \left(\frac{[cn]-n}{[cn]+1}\right) 2M(c-1) \end{aligned}$$

$$\implies \left| \frac{[cn]+1}{[cn]-n} \sum_{n<|j|\leq [cn]} \left(1 - \frac{|j|}{[cn]+1}\right) \hat{f}(j) e^{ijt} \right|$$
$$= \left(\frac{[cn]+1}{[cn]-n}\right) \left| \sum_{n<|j|\leq [cn]} \left(1 - \frac{|j|}{[cn]+1}\right) \hat{f}(j) e^{ijt} \right|$$
$$\leq \left(\frac{[cn]+1}{[cn]-n}\right) \left(\frac{[cn]-n}{[cn]+1}\right) 2M(c-1) = 2M(c-1)$$

Let $\epsilon > 0$. Choose $c > 1$ s.t $2M(c-1) < \frac{\epsilon}{2}$. Then, $\forall n \geq N_0$,

$$\left| \frac{[cn]+1}{[cn]-n} \sum_{n<|j|\leq [cn]} \left(1 - \frac{|j|}{[cn]+1}\right) \hat{f}(j) e^{ijt} \right| \leq 2M(c-1) < \frac{\epsilon}{2}.$$

$$\lim_{n\to\infty} \frac{[cn]+1}{[cn]-n} = \lim_{n\to\infty} \frac{\frac{[cn]}{n} + \frac{1}{n}}{\frac{[cn]}{n} - 1} = \frac{c}{c-1} \text{ and } \lim_{n\to\infty} \frac{n+1}{[cn]-n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{\frac{[cn]}{n}-1} = \frac{1}{c-1}$$

$$\lim_{n\to\infty} \sigma_n(f)(t) = \sigma \implies \lim_{n\to\infty} \sigma_{[cn]}(f)(t) = \sigma$$
$$\implies \lim_{n\to\infty} \left[\frac{[cn]+1}{[cn]-n} \sigma_{[cn]}(f)(t) - \frac{n+1}{[cn]-n} \sigma_n(f)(t) \right]$$
$$= \left(\frac{c}{c-1}\right) \sigma - \left(\frac{1}{c-1}\right) \sigma = \left(\frac{c-1}{c-1}\right) \sigma = \sigma$$
$$\implies \exists m \in \mathbb{N}\ \forall n \geq m, \left| \left[\frac{[cn]+1}{[cn]-n} \sigma_{[cn]}(f)(t) - \frac{n+1}{[cn]-n} \sigma_n(f)(t) \right] - \sigma \right| < \frac{\epsilon}{2}$$

Let $M_0 = \max\{N_0, m\}$ and $n \geq M_0$.

$$[S_n(f)(t) - \sigma] = \left(\left[\frac{[cn]+1}{[cn]-n} \sigma_{[cn]}(f)(t) - \frac{n+1}{[cn]-n} \sigma_n(f)(t) \right] - \sigma \right)$$
$$- \frac{[cn]+1}{[cn]-n} \sum_{n<|j|\leq [cn]} \left(1 - \frac{|j|}{[cn]+1}\right) \hat{f}(j) e^{ijt}$$

$$\implies |S_n(f)(t) - \sigma| < \left| \left[\frac{[cn]+1}{[cn]-n} \sigma_{[cn]}(f)(t) - \frac{n+1}{[cn]-n} \sigma_n(f)(t) \right] - \sigma \right|$$

$$+ \left| \frac{[cn]+1}{[cn]-n} \sum_{n<|j|\leq[cn]} \left(1 - \frac{|j|}{[cn]+1}\right) \hat{f}(j) e^{ijt} \right| < 2\left(\frac{\epsilon}{2}\right) = \epsilon$$

$$\therefore \ \forall \epsilon > 0 \ \exists M_0 \in \mathbb{N} \ \forall n \geq M_0, \ |S_n(f)(t) - \sigma| < \epsilon$$

Thus, $\lim_{n\to\infty} S_n(f)(t) = \sigma = \lim_{n\to\infty} \sigma_n(f)(t)$. Therefore, if $\lim_{n\to\infty} \sigma_n(f)(t)$ exists, then $\lim_{n\to\infty} S_n(f)(t) = \lim_{n\to\infty} \sigma_n(f)(t)$. Hence, $S_n(f)(t)$ and $\sigma_n(f)(t)$ converge for the same values of t and to the same limit.

Now, assume that $\sigma_n(f)(t)$ converges to $\sigma(f)(t)$ uniformly on some set A. In the above, n_0 is independent of t and N depends on c which depends only on ϵ. This implies that N_0 depends only on ϵ. m depends on the rate of convergence of $\lim_{n\to\infty} \left[\frac{[cn]+1}{[cn]-n} \sigma_{[cn]}(f)(t) - \frac{n+1}{[cn]-n} \sigma_n(f)(t) \right] = \sigma(f)(t)$, which only depends on the rates of convergence of $\lim_{n\to\infty} \sigma_n(f)(t) = \sigma(f)(t)$, $\lim_{n\to\infty} \frac{[cn]+1}{[cn]-n} = \frac{c}{c-1}$, and $\lim_{n\to\infty} \frac{n+1}{[cn]-n} = \frac{1}{c-1}$ because $\{\sigma_{[cn]}(f)(t)\}_{n=1}^{\infty}$ is a subsequence of $\{\sigma_n(f)(t)\}_{n=1}^{\infty}$. The last two rates of convergence depend only on c which depends only on ϵ. The rate of convergence of $\lim_{n\to\infty} \sigma_n(f)(t) = \sigma(f)(t)$ depends only on t and ϵ. This implies that m depends only on t and ϵ and that m can be made independent of t if the rate of convergence of $\lim_{n\to\infty} \sigma_n(f)(t) = \sigma(f)(t)$ is made independent of t. Since M_0 depends on N_0 and m, then M_0 can be made independent of t if m is made independent of t. Since $\sigma_n(f)(t)$ converges to $\sigma(f)(t)$ uniformly on A, then m can be chosen independently of t and it follows that $S_n(f)(t)$ converges to $\sigma(f)(t)$ uniformly on A. Hence, if $\sigma_n(f)(t)$ converges uniformly on A, then $S_n(f)(t)$ converges uniformly on A. $\square$

The following corollary is from (Katznelson, 2004, P. 74, Corollary 2.2.2). The proof is essentially the same as in the book.

Corollary 4.5.4 (Dirichlet-Jordan Test). Let $f \in BV(\mathbb{T})$. Then $\forall t \in \mathbb{R}$, $\lim_{n\to\infty} S_n(f)(t) = \dfrac{f(t+)+f(t-)}{2}$. In particular, $\lim_{n\to\infty} S_n(f)(t) = f(t)$ at every point of continuity t. The convergence is uniform on closed intervals of continuity of f.

Proof. By Proposition 3.4.6 and Theorem 3.4.7, $f \in BV(\mathbb{T}) \Rightarrow f \in L^1(\mathbb{T})$ and $\hat{f}(n) = O\left(\dfrac{1}{n}\right)$ as $|n| \to \infty$. Thus, f satisfies the hypotheses of the previous theorem. Let $t \in \mathbb{R}$ and $I = [t-\pi, t+\pi]$ which is an interval of length 2π that contains t. By the note after Definition 3.4.2, $f \in BV(I)$. By Theorem 2.3.6.(ii), $f(t+), f(t-)$ exist and so $\lim_{h\to 0^+}[f(t+h)+f(t-h)] = [f(t+)+f(t-)]$. Then the results follow from Fejér's Theorem and the previous theorem. $\square$

Remark. An alternate proof of the Dirichlet-Jordan Test can be found in (Zygmund, 1977, P. 57-58, Theorem 8.1).

The following theorem is from (Zygmund, 1977, P. 60, Theorem 8.14). The proof is basically the same as in the book.

Theorem 4.5.5. Let $f \in L^1(\mathbb{T})$ and I be an open interval. If $f \in BV(I)$, then $\forall t \in I$, $\lim_{n\to\infty} S_n(f)(t) = \dfrac{f(t+)+f(t-)}{2}$. Moreover if f is continuous on I, then the convergence is uniform on closed subsets of I.

Proof. First note that if $m(I) \geq 2\pi$, then $f \in BV(\mathbb{T})$ and the Dirichlet-Jordan Test can be applied to get the result. Now assume $m(I) < 2\pi$. Let J be a closed interval of length 2π s.t. $I \subsetneq J$. By the Principle of Localization, f can be replaced by a function $g \in L^1(\mathbb{T})$ s.t. $g = f$ on I and $g = 0$ on $J \backslash I$. Then $f \in BV(I) \Rightarrow g \in BV(\mathbb{T})$ and the result follows by the Dirichlet-Jordan Test. $\square$

4.6 Dini-Lipschitz Test

The following lemma is stated and proved on (Folland, 1999, P. 89, Corollary 3.6).

Lemma 4.6.1. Let $(X, \mathfrak{M}, \mu)$ be a measure space and $f \in L^1(X, \mathfrak{M}, \mu)$. Then $\forall \epsilon > 0\ \exists \delta > 0\ \forall E \in \mathfrak{M},\ \mu(E) < \delta \implies \left|\int_E f\, d\mu\right| < \epsilon$.

Remark. The lemma will be applied when $(X, \mathfrak{M}, \mu) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \lambda)$. The hypothesis that $f \in L^1(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \lambda)$ can be replaced by the hypothesis $f \in L^1_{\text{loc}}(\mathbb{R})$ because only sets of finite measure are considered. Also, f will be replaced by $|f|$ in the result.

The following theorem is from (Zygmund, 1977, P. 62, Theorem 10.1). The proof is essentially the same as in the book except for the following differences. Unlike in (Zygmund, 1977), complete details are given in showing the bounds for $|I_1|$ and $|I_2|$. The application of the previous lemma in the proof of the bound for $|I_2|$ was suggested by Professor Klemes. Also in the proof of the bound for $|I_2|$, it is explained why the bound is o(1) uniformly in every interval where f is bounded.

Theorem 4.6.2. Let $f \in L^1(\mathbb{T})$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, $\eta = \dfrac{\pi}{n}$, and $\phi = \phi_x$ be as in Lemma 4.3.1. Then, $|S_n^*(f)(x) - f(x)|$ is majorized by

$$\frac{1}{\pi}\int_\eta^\pi \frac{|\phi(t) - \phi(t+\eta)|}{t}\, dt + \eta \int_\eta^\pi \frac{|\phi(t)|}{t^2}\, dt + \frac{2}{\eta}\int_0^{2\eta} |\phi(t)|\, dt + \mathrm{o}(1),$$

where the o(1) term is uniform in every interval where f is bounded.

Proof. By Lemma 4.3.1.(ii), $H(x) = [S_n^*(f)(x) - f(x)] = \frac{1}{\pi}\int_0^\pi \phi(t)D_n^*(t)\,dt$. By Proposition 2.3.2.(2),

$$H(x) = \frac{1}{\pi}\int_0^\pi \phi(t)D_n^*(t)\,dt = \frac{1}{\pi}\int_{-\eta}^{\pi-\eta} \phi(t+\eta)D_n^*(t+\eta)\,dt.$$

$$\implies H(x) = \frac{H(x)+H(x)}{2} = \frac{1}{2\pi}\int_0^\pi \phi(t)D_n^*(t)\,dt + \frac{1}{2\pi}\int_{-\eta}^{\pi-\eta} \phi(t+\eta)D_n^*(t+\eta)\,dt$$

$$= \frac{1}{2\pi}\left[\int_0^\eta \phi(t)D_n^*(t)\,dt + \int_\eta^{\pi-\eta} \phi(t)D_n^*(t)\,dt + \int_{\pi-\eta}^{\pi} \phi(t)D_n^*(t)\,dt\right]$$

$$+ \frac{1}{2\pi}\left[\int_{-\eta}^{\eta} \phi(t+\eta)D_n^*(t+\eta)\,dt + \int_\eta^{\pi-\eta} \phi(t+\eta)D_n^*(t+\eta)\,dt\right] = I_1 + I_2 + I_3 + I_4,$$

where $I_1 = \frac{1}{2\pi}\int_\eta^{\pi-\eta} [\phi(t)D_n^*(t) + \phi(t+\eta)D_n^*(t+\eta)]\,dt$, $I_2 = \frac{1}{2\pi}\int_{\pi-\eta}^{\pi} \phi(t)D_n^*(t)\,dt$,

$I_3 = \frac{1}{2\pi}\int_0^\eta \phi(t)D_n^*(t)\,dt$, and $I_4 = \frac{1}{2\pi}\int_{-\eta}^{\eta} \phi(t+\eta)D_n^*(t+\eta)\,dt$.

$$\implies |H(x)| \le |I_1| + |I_2| + |I_3| + |I_4|$$

By Proposition 4.3.2.(iii), $\forall t \in \mathbb{R}$, $|D_n^*(t)| \le 2n$.

$$\implies |I_3| \le \frac{1}{2\pi}\int_0^\eta |\phi(t)D_n^*(t)|\,dt \le \frac{1}{2\pi}\int_0^\eta |\phi(t)|\,|D_n^*(t)|\,dt \le \frac{2n}{2\pi}\int_0^\eta |\phi(t)|\,dt$$

$$= \frac{1}{\eta}\int_0^\eta |\phi(t)|\,dt \le \frac{1}{\eta}\int_0^{2\eta} |\phi(t)|\,dt$$

By Proposition 2.3.2.(2), $I_4 = \frac{1}{2\pi}\int_{-\eta}^{\eta} \phi(t+\eta)D_n^*(t+\eta)\,dt = \frac{1}{2\pi}\int_0^{2\eta} \phi(t)D_n^*(t)\,dt$.

By the argument used for I_3, $|I_4| \le \frac{1}{\eta}\int_0^{2\eta} |\phi(t)|\,dt$.

$$\implies |I_3| + |I_4| \le \frac{2}{\eta}\int_0^{2\eta} |\phi(t)|\,dt$$

Consider I_2. Let $n \geq 2$.

By Proposition 4.3.2.(iii), $\forall t \in [0, \pi],\ |D_n^*(t)| \leq \dfrac{2}{t}$.

$$\implies \forall t \in [\pi - \eta, \pi]\,,\ t \geq (\pi - \eta) = \left(\pi - \frac{\pi}{n}\right) = \pi\left(1 - \frac{1}{n}\right) \geq \frac{\pi}{2} \geq 1$$

and $|\phi(t) D_n^*(t)| = |\phi(t)|\ |D_n^*(t)| \leq |\phi(t)| \left(\dfrac{2}{t}\right) \leq 2|\phi(t)|$

$$= 2\left|\left(\frac{f(x+t) + f(x-t)}{2}\right) - f(x)\right| = 2\left|\frac{f(x+t) + f(x-t) - 2f(x)}{2}\right|$$

$$= |f(x+t) + f(x-t) - 2f(x)| \leq |f(x+t)| + |f(x-t)| + 2|f(x)|.$$

$$\implies |I_2| \leq \frac{1}{2\pi}\int_{\pi-\eta}^{\pi} |\phi(t) D_n^*(t)|\, dt \leq \frac{1}{2\pi}\int_{\pi-\eta}^{\pi} [|f(x+t)| + |f(x-t)| + 2|f(x)|]\, dt$$

$$= \frac{1}{2\pi}\int_{\pi-\eta}^{\pi} |f(x+t)|\, dt + \frac{1}{2\pi}\int_{\pi-\eta}^{\pi} |f(x-t)|\, dt + \left(\frac{\eta}{\pi}\right)|f(x)|$$

$$= \frac{1}{2\pi}\int_{\pi-\eta}^{\pi} |f(x+t)|\, dt + \frac{1}{2\pi}\int_{\pi-\eta}^{\pi} |f(x-t)|\, dt + \frac{|f(x)|}{n}$$

By Proposition 2.3.2.(2)-(3), $\dfrac{1}{2\pi}\displaystyle\int_{\pi-\eta}^{\pi} |f(x+t)|\, dt = \dfrac{1}{2\pi}\int_{x+\pi-\eta}^{x+\pi} |f(t)|\, dt$

and $\dfrac{1}{2\pi}\displaystyle\int_{\pi-\eta}^{\pi} |f(x-t)|\, dt = \dfrac{1}{2\pi}\int_{x-\pi}^{x-\pi+\eta} |f(t)|\, dt.$

$$\implies |I_2| \leq \frac{1}{2\pi}\int_{x+\pi-\eta}^{x+\pi} |f(t)|\, dt + \frac{1}{2\pi}\int_{x-\pi}^{x-\pi+\eta} |f(t)|\, dt + \frac{|f(x)|}{n}$$

By the previous lemma,

$$f \in L^1(\mathbb{T}) \implies \forall \epsilon > 0\ \exists \delta > 0\ \forall E \in \mathcal{B}_{\mathbb{R}},\ \lambda(E) < \delta \implies \left|\frac{1}{2\pi}\int_E |f(t)|\, dt\right| < \epsilon.$$

Since $\lambda((x + \pi - \eta, x + \pi)) = \lambda((x - \pi, x - \pi + \eta)) = \dfrac{\eta}{2\pi} = \dfrac{1}{2n}$ and $\displaystyle\lim_{n\to\infty} \frac{1}{2n} = 0$,

then $\displaystyle\lim_{n\to\infty}\left[\frac{1}{2\pi}\int_{x+\pi-\eta}^{x+\pi} |f(t)|\, dt\right] = \lim_{n\to\infty}\left[\frac{1}{2\pi}\int_{x-\pi}^{x-\pi+\eta} |f(t)|\, dt\right] = 0$

and the convergence is uniform in x.

Also, $\lim_{n\to\infty} \frac{|f(x)|}{n} = 0$. If f is bounded by a positive constant M in an interval I, then $\forall x \in I$, $\frac{|f(x)|}{n} \le \frac{M}{n}$ and $\lim_{n\to\infty} \frac{M}{n} = 0$ independently of x.

Let $a_n(x) = \left[\frac{1}{2\pi}\int_{\pi-\eta}^{\pi} |f(x+t)|\,dt + \frac{1}{2\pi}\int_{\pi-\eta}^{\pi} |f(x-t)|\,dt + \frac{|f(x)|}{n}\right]$

and $b_n(x) = \left[\frac{1}{2\pi}\int_{\pi-\eta}^{\pi} |f(x+t)|\,dt + \frac{1}{2\pi}\int_{\pi-\eta}^{\pi} |f(x-t)|\,dt + \frac{M}{n}\right]$.

Thus, $|I_2| \le a_n(x)$ and $\lim_{n\to\infty} a_n(x) = 0$, i.e. $a_n(x) = o(1)$. Moreover, if f is bounded by a positive constant M in an interval I, then $|I_2| \le b_n(x)$ and $b_n(x) = o(1)$ uniformly for $x \in I$. Therefore, $|I_2| \le o(1)$, where the $o(1)$ term is uniform in every interval where f is bounded.

Now consider I_1. Let $n \ge 2$. By Proposition 4.3.2.(ii), $\forall t \in (0,\pi]$, $D_n^*(t) = \cot\left(\frac{t}{2}\right)\sin(nt)$. Note that $\forall t \in [\eta, \pi-\eta]$, $(t+\eta) \in [2\eta, \pi] \subseteq (0,\pi]$.

$$\Longrightarrow\ D_n^*(t+\eta) = \cot\left(\frac{t+\eta}{2}\right)\sin(n(t+\eta)) = \cot\left(\frac{t+\eta}{2}\right)\sin(nt+\pi)$$

$$= -\cot\left(\frac{t+\eta}{2}\right)\sin(nt) \text{ and } [\phi(t)D_n^*(t) + \phi(t+\eta)D_n^*(t+\eta)]$$

$$= \left[\phi(t)\cot\left(\frac{t}{2}\right)\sin(nt) - \phi(t+\eta)\cot\left(\frac{t+\eta}{2}\right)\sin(nt)\right]$$

$$= \left[\phi(t)\cot\left(\frac{t}{2}\right) - \phi(t+\eta)\cot\left(\frac{t+\eta}{2}\right)\right]\sin(nt)$$

$$= [\phi(t) - \phi(t+\eta)]\cot\left(\frac{t+\eta}{2}\right)\sin(nt) + \phi(t)\left[\cot\left(\frac{t}{2}\right) - \cot\left(\frac{t+\eta}{2}\right)\right]\sin(nt)$$

$$\Longrightarrow\ |\phi(t)D_n^*(t)+\phi(t+\eta)D_n^*(t+\eta)| \leq |\phi(t)-\phi(t+\eta)|\left|\cot\left(\frac{t+\eta}{2}\right)\right| |\sin(nt)|$$

$$+\ |\phi(t)|\left|\cot\left(\frac{t}{2}\right)-\cot\left(\frac{t+\eta}{2}\right)\right| |\sin(nt)|$$

$$\leq |\phi(t)-\phi(t+\eta)|\left|\cot\left(\frac{t+\eta}{2}\right)\right| + |\phi(t)|\left|\cot\left(\frac{t}{2}\right)-\cot\left(\frac{t+\eta}{2}\right)\right|$$

As $(t+\eta)\in(0,\pi]$, $\left|\cot\left(\frac{t+\eta}{2}\right)\right| \leq \frac{2}{(t+\eta)} \leq \frac{2}{t}$. As $\cot\left(\frac{t}{2}\right)$ is decreasing and nonnegative on $(0,\pi]$, $\left|\cot\left(\frac{t}{2}\right)-\cot\left(\frac{t+\eta}{2}\right)\right| = \cot\left(\frac{t}{2}\right)-\cot\left(\frac{t+\eta}{2}\right)$

$$= \frac{\cos\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)} - \frac{\cos\left(\frac{t+\eta}{2}\right)}{\sin\left(\frac{t+\eta}{2}\right)} = \frac{\sin\left(\frac{t+\eta}{2}\right)\cos\left(\frac{t}{2}\right)-\sin\left(\frac{t}{2}\right)\cos\left(\frac{t+\eta}{2}\right)}{\sin\left(\frac{t}{2}\right)\sin\left(\frac{t+\eta}{2}\right)}$$

$$= \sin\left(\frac{\eta}{2}\right)\csc\left(\frac{t}{2}\right)\csc\left(\frac{t+\eta}{2}\right) \leq \left(\frac{\eta}{2}\right)\left(\frac{\pi}{t}\right)\left(\frac{\pi}{t+\eta}\right) = \frac{\pi^2\eta}{2t(t+\eta)} \leq \frac{\pi^2\eta}{2t^2}.$$

$$\Longrightarrow\ |\phi(t)D_n^*(t)+\phi(t+\eta)D_n^*(t+\eta)| \leq \frac{2|\phi(t)-\phi(t+\eta)|}{t} + \left(\frac{\pi^2\eta}{2}\right)\frac{|\phi(t)|}{t^2}$$

$$\Longrightarrow\ |I_1| \leq \frac{1}{2\pi}\int_\eta^{\pi-\eta} |\phi(t)D_n^*(t)+\phi(t+\eta)D_n^*(t+\eta)|\,dt$$

$$\leq \frac{1}{2\pi}\int_\eta^{\pi-\eta}\left[\frac{2|\phi(t)-\phi(t+\eta)|}{t} + \left(\frac{\pi^2\eta}{2}\right)\frac{|\phi(t)|}{t^2}\right]dt$$

$$\leq \frac{1}{\pi}\int_\eta^{\pi-\eta}\frac{|\phi(t)-\phi(t+\eta)|}{t}\,dt + \frac{\pi\eta}{4}\int_\eta^{\pi-\eta}\frac{|\phi(t)|}{t^2}\,dt$$

$$\leq \frac{1}{\pi}\int_\eta^{\pi}\frac{|\phi(t)-\phi(t+\eta)|}{t}\,dt + \eta\int_\eta^{\pi}\frac{|\phi(t)|}{t^2}\,dt$$

$$\Longrightarrow |S_n^*(f)(x) - f(x)| \leq |I_1| + |I_2| + |I_3| + |I_4|$$
$$\leq \frac{1}{\pi}\int_\eta^\pi \frac{|\phi(t)-\phi(t+\eta)|}{t}\,dt + \eta\int_\eta^\pi \frac{|\phi(t)|}{t^2}\,dt + \frac{2}{\eta}\int_0^{2\eta}|\phi(t)|\,dt + \mathrm{o}(1),$$

where the o(1) term is uniform in every interval where f is bounded. □

The following lemma corresponds to (Zygmund, 1977, P. 14, Theorem 8.1).

Lemma 4.6.3. Let $(a, b]$ be a half-open interval in $\mathbb{R}$, $h, g\colon (a, b] \to \mathbb{C}$ be s.t. $\forall c \in (a, b)$, $h, g \in L^1((c, b), \mathcal{B}_{\mathbb{R}} \cap (c, b), m)$, and $g \geq 0$. Define $H, G\colon (a, b] \to \mathbb{C}$ by $H(x) = \int_x^b f(t)\,dt$ and $G(x) = \int_x^b g(t)\,dt$. If $\lim\limits_{x\to a^+} G(x) = \infty$ and $h(x) = \mathrm{o}(g(x))$ as $x \to a^+$, then $H(x) = \mathrm{o}(G(x))$ as $x \to a^+$.

The following theorem is from (Zygmund, 1977, P. 63, Theorem 10.3). Unlike in (Zygmund, 1977), a full proof is given here and in particular, it is explained why each term in the sum which majorizes $|S_n^*(f)(x) - f(x)|$ from the previous theorem converges uniformly to 0.

Theorem 4.6.4 (Dini-Lipschitz Test). Let $f \in C(\mathbb{T})$ and ω be the modulus of continuity of f. If $\omega(\delta) = \mathrm{o}([\log\delta]^{-1})$, i.e. $\lim\limits_{\delta\to 0^+} \omega(\delta)\log\delta = 0$, then $S[f]$ converges in norm to f in $C(\mathbb{T})$.

Proof. By the note after Theorem 3.3.6, $S_n^*(f) = (D_n^* * f) \in C(\mathbb{T})$ because $D_n^*, f \in C(\mathbb{T})$. By the remark after Theorem 4.3.3, it is enough to show $\lim\limits_{n\to\infty} S_n^*(f) = f$ in $C(\mathbb{T})$, i.e. $\lim\limits_{n\to\infty} \|S_n^*(f) - f\|_\infty = 0$. Since $\|S_n^*(f) - f\|_\infty = \sup\limits_{x\in J} |S_n^*(f)(x) - f(x)|$, where J is a closed interval of length 2π, then x can be restricted to J. This implies that it is enough to show that $\lim\limits_{n\to\infty} S_n^*(f)(x) = f(x)$ uniformly for $x \in J$ because convergence in norm in $C(\mathbb{T})$ is the same as uniform convergence. By the previous theorem, it is enough to show that each term in the sum, which majorizes $|S_n^*(f)(x) - f(x)|$, converges uniformly to 0 for $x \in J$.

$f \in C(\mathbb{T}) \implies f$ is bounded $\implies$ The o(1) term is uniform.

Consider $\displaystyle \frac{1}{\pi}\int_\eta^\pi \frac{|\phi(t)-\phi(t+\eta)|}{t}\,dt.$

$$|\phi(t)-\phi(t+\eta)| = \left|\left(\frac{f(x+t)+f(x-t)}{2}\right) - \left(\frac{f(x+t+\eta)+f(x-t-\eta)}{2}\right)\right|$$

$$= \left|\frac{(f(x+t)-f(x+t+\eta))+(f(x-t)-f(x-t-\eta))}{2}\right|$$

$$\leq \left(\frac{|f(x+t)-f(x+t+\eta)|+|f(x-t)-f(x-t-\eta)|}{2}\right) \leq \frac{2\omega(\eta)}{2} = \omega(\eta)$$

$$\implies \frac{1}{\pi}\int_\eta^\pi \frac{|\phi(t)-\phi(t+\eta)|}{t}\,dt \leq \frac{\omega(\eta)}{\pi}\int_\eta^\pi \frac{1}{t}\,dt = \frac{\omega(\eta)}{\pi}[\log\pi - \log\eta]$$

$$= \frac{\omega(\eta)\log\pi}{\pi} - \frac{\omega(\eta)\log\eta}{\pi}$$

$$\therefore\ 0 \leq \frac{1}{\pi}\int_\eta^\pi \frac{|\phi(t)-\phi(t+\eta)|}{t}\,dt \leq \frac{\omega(\eta)\log\pi}{\pi} - \frac{\omega(\eta)\log\eta}{\pi}$$

Since $\omega(\delta) = o([\log\delta]^{-1})$, $f \in C(\mathbb{T}) \implies \lim\limits_{\delta\to 0^+}\omega(\delta) = 0$, and $\lim\limits_{n\to\infty}\eta = 0$, then

$\displaystyle \lim_{n\to\infty}\left[\frac{\omega(\eta)\log\pi}{\pi} - \frac{\omega(\eta)\log\eta}{\pi}\right] = 0$ which implies

$\displaystyle \lim_{n\to\infty}\frac{1}{\pi}\int_\eta^\pi \frac{|\phi(t)-\phi(t+\eta)|}{t}\,dt = 0$ uniformly for $x \in J$.

Consider $\frac{2}{\eta}\int_0^{2\eta} |\phi(t)|\,dt$. By the remark after Corollary 2.3.5, $f \in C(\mathbb{T})$

$\implies \lim_{h\to 0^+} \frac{1}{h}\int_0^h |\phi(t)|\,dt = 0$ uniformly for $x \in J$. This implies, as $\lim_{n\to\infty} \eta = 0$,

$$\lim_{n\to\infty} \frac{2}{\eta}\int_0^{2\eta} |\phi(t)|\,dt = \lim_{n\to\infty} 4\left(\frac{1}{2\eta}\int_0^{2\eta} |\phi(t)|\,dt\right) = 0 \text{ uniformly for } x \in J.$$

Finally, consider $\eta\int_\eta^\pi \frac{|\phi(t)|}{t^2}\,dt$. Now the previous lemma will be applied with

$(a, b] = (0, \pi]$, $h(t) = \frac{|\phi(t)|}{t^2}$, and $g(t) = \frac{1}{t^2}$. Note that $g \geq 0$.

Since f is bounded, then ϕ is bounded, say by a positive constant M.

$$\forall u \in (0, \pi],\ G(u) = \int_u^\pi \frac{1}{t^2}\,dt = \left[-\frac{1}{t}\right]\Bigg|_{t=u}^{\pi} = \left(\frac{1}{u} - \frac{1}{\pi}\right) = \frac{\pi - u}{u\pi} < \infty$$

$$\text{and } H(u) = \int_u^\pi \frac{|\phi(t)|}{t^2}\,dt \leq M\int_u^\pi \frac{1}{t^2}\,dt = MG(u) < \infty$$

$$\implies \forall u \in (0, \pi),\ h, g \in L^1((u, \pi), \mathcal{B}_{\mathbb{R}} \cap (u, \pi), m)$$

It is easy to see that $\lim_{u\to 0^+} G(u) = \infty$.

$f \in C(\mathbb{T}) \implies \lim_{t\to 0^+} |\phi(t)| = 0$ and the uniform continuity of f on finite closed intervals implies that $\lim_{t\to 0^+} |\phi(t)| = 0$ uniformly for $x \in J$. $\implies h(t) = \mathrm{o}(g(t))$ uniformly for $x \in J$ as $t \to 0^+$.

Since the hypotheses of the previous lemma are satisfied, then $H(t) = \mathrm{o}(G(t))$ as $t \to 0^+$ and by a careful examination of the proof of the previous lemma, $H(t) = \mathrm{o}(G(t))$ as $t \to 0^+$ uniformly for $x \in J$. $\implies \lim_{u\to 0^+} \frac{H(u)}{G(u)} = 0$ uniformly for $x \in J$.

Since $\dfrac{H(u)}{G(u)} = \left(\dfrac{u\pi}{\pi - u}\right) H(u)$ and $\lim\limits_{u \to 0^+} \left(\dfrac{\pi - u}{\pi}\right) = 1$ independently of x,

then $\lim\limits_{u \to 0^+} \left[\left(\dfrac{\pi - u}{\pi}\right) \dfrac{H(u)}{G(u)}\right] = \lim\limits_{u \to 0^+} uH(u) = 0$ uniformly for $x \in J$.

Since $\lim\limits_{n \to \infty} \eta = 0$, then $\lim\limits_{n \to \infty} \eta H(\eta) = \lim\limits_{n \to \infty} \eta \displaystyle\int_{\eta}^{\pi} \dfrac{|\phi(t)|}{t^2}\, dt = 0$ uniformly for $x \in J$.

$\square$

The following theorem is from (Zygmund, 1977, P. 63, Theorem 10.5). The proof is basically reproduced as in the book.

Theorem 4.6.5. Let $f \in L^1(\mathbb{T})$, I be a closed interval, and ω be the modulus of continuity of f on I. If $\omega(\delta) = \mathrm{o}([\log \delta]^{-1})$, then $\lim\limits_{n \to \infty} S_n(f)(x) = f(x)$ uniformly on I.

Proof. Since $\omega(\delta) = \mathrm{o}([\log \delta]^{-1})$ and $\lim\limits_{\delta \to 0^+} [\log \delta]^{-1} = 0$, then $\lim\limits_{\delta \to 0^+} \omega(\delta) = \lim\limits_{\delta \to 0^+} (\omega(\delta) \log \delta)[\log \delta]^{-1} = 0$. This implies that f is continuous on I.

First note that if $m(I) \geq 2\pi$, then the Dini-Lipschitz Test can be applied to get the result because its hypotheses are satisfied. Now assume $m(I) < 2\pi$. Let J be a closed interval of length 2π s.t. $I \subseteq J$. By the Principle of Localization, f can be replaced by a function $g \in C(\mathbb{T})$ s.t. $g = f$ on I and g is extended linearly on $J \backslash I$. By the Dini-Lipschitz Test, the result follows because g satisfies its hypotheses. $\square$

4.7 Lebesgue's Test

The following notation will be used in this section.

Notation. Let $f \in L^1(\mathbb{T})$, $x \in \mathbb{R}$, $\phi = \phi_x$ be as in Lemma 4.3.1. $\forall h \in [0, \infty)$, let $\Phi(h) = \Phi_x(h) = \int_0^h |\phi(t)|\,dt$. By Corollary 2.3.5, $\lim_{h \to 0^+} \dfrac{\Phi_x(h)}{h} = 0$ for m-a.a. $x \in \mathbb{R}$. By the remark after Corollary 2.3.5, if f is continuous on a finite closed interval I, then $\lim_{h \to 0^+} \dfrac{\Phi_x(h)}{h} = 0$ uniformly for $x \in I$.

The following theorem is from (Zygmund, 1977, P. 65, Theorem 11.5). The proof is basically reproduced as in the book.

Theorem 4.7.1 (Lebesgue's Test). Let $f \in L^1(\mathbb{T})$ and $x \in \mathbb{R}$. If $\lim_{h \to 0^+} \dfrac{\Phi(h)}{h} = 0$ and $\lim_{n \to \infty} \displaystyle\int_\eta^\pi \frac{|\phi(t) - \phi(t + \eta)|}{t}\,dt = 0$, then $\lim_{n \to \infty} S_n(f)(x) = f(x)$. The convergence is uniform on any closed interval of continuity of f where the second condition holds uniformly.

Proof. Let I be a closed interval of continuity of f where the second condition holds uniformly, i.e. $\lim_{n \to \infty} \displaystyle\int_\eta^\pi \frac{|\phi(t) - \phi(t + \eta)|}{t}\,dt = 0$ uniformly for $x \in I$. By the remark after Theorem 4.3.3, it is enough to show that $\lim_{n \to \infty} S_n^*(f)(x) = f(x)$ and that the convergence is uniform on I. By Theorem 4.6.2, it is enough to show that each term in the sum, which majorizes $|S_n^*(f)(x) - f(x)|$, converges to 0 and that the convergence is uniform on I.

First, note that f is bounded on I because f is continuous on I. This implies that the o(1) term is uniform on I.

By hypothesis, $\lim_{n \to \infty} \dfrac{1}{\pi} \displaystyle\int_\eta^\pi \frac{|\phi(t) - \phi(t + \eta)|}{t}\,dt = 0$ and the convergence is uniform on I.

Consider $\dfrac{2}{\eta} \displaystyle\int_0^{2\eta} |\phi(t)|\,dt = 4 \left(\frac{\Phi(2\eta)}{2\eta} \right)$. Since $\lim_{h \to 0^+} \dfrac{\Phi(h)}{h} = 0$ and $\lim_{n \to \infty} 2\eta = 0$, then $\lim_{n \to \infty} \dfrac{2}{\eta} \displaystyle\int_0^{2\eta} |\phi(t)|\,dt = 0$ and, by the statement before the theorem, the convergence is uniform on I.

Finally, consider $\eta \int_{\eta}^{\pi} \frac{|\phi(t)|}{t^2}\, dt$. By Integration by Parts and the exact same argument used in Theorem 4.1.4 with $\frac{1}{n}$, δ_n, and g replaced resp. by η, π, and ϕ,

$$\eta \int_{\eta}^{\pi} \frac{|\phi(t)|}{t^2}\, dt = \eta\left[\left[\frac{\Phi(t)}{t^2}\right]\Bigg|_{t=\eta}^{\pi} + 2\int_{\eta}^{\pi} \frac{\Phi(t)}{t^3}\, dt\right] = \eta\left[\frac{\Phi(\pi)}{\pi^2} - \frac{\Phi(\eta)}{\eta^2} + 2\int_{\eta}^{\pi} \frac{\Phi(t)}{t^3}\, dt\right]$$

$$= \left(\frac{\Phi(\pi)}{\pi^2}\right)\eta - \frac{\Phi(\eta)}{\eta} + 2\eta \int_{\eta}^{\pi} \frac{\Phi(t)}{t^3}\, dt$$

Since $\lim_{n\to\infty} \eta = 0$, then $\lim_{n\to\infty} \left(\frac{\Phi(\pi)}{\pi^2}\right)\eta = 0$ and the convergence is uniform on I because $\left(\frac{\Phi(\pi)}{\pi^2}\right)$ is uniformly bounded for $x \in I$ as f is bounded on I. Also, $\lim_{n\to\infty} \frac{\Phi(\eta)}{\eta} = 0$ and, by the statement before the theorem, the convergence is uniform on I. Since $\lim_{h\to 0^+} \frac{\Phi(h)}{h} = 0$ and the convergence is uniform on I, then by the argument used in the proof of the Dini-Lipschitz Test to show that $\lim_{n\to\infty} \eta \int_{\eta}^{\pi} \frac{|\phi(t)|}{t^2}\, dt = 0$ with $|\phi(t)|$ replaced by $\frac{\Phi(t)}{t}$, $\lim_{n\to\infty}\left(2\eta \int_{\eta}^{\pi} \frac{\Phi(t)}{t^3}\, dt\right) = 0$ and the convergence is uniform on I.

All of this implies $\lim_{n\to\infty} \eta \int_{\eta}^{\pi} \frac{|\phi(t)|}{t^2}\, dt = 0$ and that the convergence is uniform on I. $\square$

The following remark is from (Zygmund, 1977, P. 66).

Remark. Although the most important tests for the pointwise convergence of Fourier series are Dini's Test, the Dirichlet-Jordan Test, and the Dini-Lipschitz Test, it can be shown that they are all included in Lebesgue's Test. The main difficulty of applying Lebesgue's Test is that the second condition in Lebesgue's Test does not correspond to any simple condition on f. Also, while both conditions of Lebesgue's Test are necessary for $S[f](x)$ to be convergent to $f(x)$, only the first condition of Lebesgue's Test is necessary for $S[f](x)$ to be summable to $f(x)$ by Lebesgue's Theorem (Theorem 4.1.4).

The following definition is from (Zygmund, 1977, P. 45).

Definition 4.7.1. For $f \in L^p(\mathbb{T})$, where $1 < p < \infty$, the integral modulus of continuity of f in $L^p(\mathbb{T})$ is $\omega_p(\delta) = \omega_p(f, \delta) = \sup_{|h| \leq \delta} \|f(t+h) - f(t)\|_p$.

The following theorem is from (Zygmund, 1977, P. 66, Theorem 11.10). The proof is basically reproduced as in the book.

Theorem 4.7.2. Let $f \in L^p(\mathbb{T})$, where $1 < p < \infty$, and $x \in \mathbb{R}$. If $\omega_p(\delta) = o(\delta^{\frac{1}{p}})$ and $\lim_{h \to 0^+} \dfrac{\Phi(h)}{h} = 0$, then $\lim_{n \to \infty} S_n(f)(x) = f(x)$. The convergence is uniform on any closed interval of continuity of f.

Proof. By Lebesgue's Test, it is enough to show that the second condition in Lebesgue's Test holds uniformly on $\mathbb{R}$

From the proof of the Dini-Lipschitz Test,

$$[\phi(t) - \phi(t+\eta)] = \frac{1}{2}[f(x+t) - f(x+t+\eta)] + \frac{1}{2}[f(x-t) - f(x-t-\eta)]. \implies$$

$$\|\phi(t) - \phi(t+\eta)\|_p \leq \frac{1}{2}\|f(x+t) - f(x+t+\eta)\|_p + \frac{1}{2}\|f(x-t) - f(x-t-\eta)\|_p$$

By Corollary 3.1.9, $\|f(x+t) - f(x+t+\eta)\|_p = \|f(x-t) - f(x-t-\eta)\|_p$

$$= \|f(t+\eta) - f(t)\|_p \leq \omega_p(\eta). \implies \|\phi(t) - \phi(t+\eta)\|_p \leq 2\left(\frac{\omega_p(\eta)}{2}\right) = \omega_p(\eta).$$

By Holder's Inequality ((Folland, 1999, P. 182, Theorem 6.2))

with $(X, \mathfrak{M}, \mu) = ([\eta, \pi], \mathcal{B}_{\mathbb{R}} \cap [\eta, \pi], m)$,

$$\int_{\eta}^{\pi} \frac{|\phi(t) - \phi(t+\eta)|}{t}\, dt \leq \left[\int_{\eta}^{\pi} |\phi(t) - \phi(t+\eta)|^p \, dt\right]^{\frac{1}{p}} \left[\int_{\eta}^{\pi} \frac{1}{t^q}\, dt\right]^{\frac{1}{q}},$$

where $1 < q = \dfrac{p}{p-1} < \infty$.

$$\left[\int_{\eta}^{\pi} |\phi(t) - \phi(t+\eta)|^p \, dt\right]^{\frac{1}{p}} = (2\pi)^{\frac{1}{p}} \left[\frac{1}{2\pi}\int_{\eta}^{\pi} |\phi(t) - \phi(t+\eta)|^p \, dt\right]^{\frac{1}{p}}$$

$$\leq (2\pi)^{\frac{1}{p}} \|\phi(t) - \phi(t+\eta)\|_p \leq (2\pi)^{\frac{1}{p}} \omega_p(\eta)$$

$$\left[\int_{\eta}^{\pi} \frac{1}{t^q}\, dt\right]^{\frac{1}{q}} = \left[\left[-\frac{1}{(q-1)t^{q-1}}\right]\Bigg|_{t=\eta}^{\pi}\right]^{\frac{1}{q}} = \left[\frac{1}{(q-1)}\left(\frac{1}{\eta^{q-1}} - \frac{1}{\pi^{q-1}}\right)\right]^{\frac{1}{q}}$$

$$\leq \left[\frac{1}{(q-1)\eta^{q-1}}\right]^{\frac{1}{q}} = \frac{1}{(q-1)^{\frac{1}{q}}\eta^{\frac{q-1}{q}}} = \frac{1}{(q-1)^{\frac{1}{q}}\eta^{\frac{1}{p}}}, \text{ as } p = \frac{q}{q-1}.$$

$$\Longrightarrow\ 0 \leq \int_{\eta}^{\pi} \frac{|\phi(t) - \phi(t+\eta)|}{t}\, dt \leq C\left(\frac{\omega_p(\eta)}{\eta^{\frac{1}{p}}}\right), \text{ where } C = \frac{(2\pi)^{\frac{1}{p}}}{(q-1)^{\frac{1}{q}}}.$$

Since $\omega_p(\delta) = o(\delta^{\frac{1}{p}})$ and $\lim_{n\to\infty} \eta = 0$, then $\lim_{n\to\infty} C\left(\dfrac{\omega_p(\eta)}{\eta^{\frac{1}{p}}}\right) = 0$, which implies $\lim_{n\to\infty} \displaystyle\int_{\eta}^{\pi} \frac{|\phi(t) - \phi(t+\eta)|}{t}\, dt = 0$ uniformly for $x \in \mathbb{R}$.

Hence, the second condition in Lebesgue's Test holds uniformly on $\mathbb{R}$. □

4.8 Lebesgue Constants

Let $\{L_n\}_{n=0}^{\infty}$ be the sequence of Lebesgue constants as defined in Section 3.6.

The following theorem is proved on (Zygmund, 1977, P. 67). The proof is essentially reproduced as in the book.

Theorem 4.8.1. $\forall n \in \mathbb{N}$, $L_n = \dfrac{4}{\pi^2} \log n + \mathrm{O}(1)$; In particular, $\lim\limits_{n\to\infty} L_n = \infty$.

Proof. Let $n \in \mathbb{N}$. Since D_n is an even function, then $|D_n|$ is an even function and

$$L_n = \|D_n\|_1 = \frac{1}{2\pi} \int |D_n(t)|\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt = \frac{1}{2\pi} \left(2 \int_0^{\pi} |D_n(t)|\, dt \right)$$
$$= \frac{1}{\pi} \int_0^{\pi} |D_n(t)|\, dt.$$

$$\forall t \in [0,\pi],\ \big||D_n(t)| - |D_n^*(t)|\big| \le |D_n(t) - D_n^*(t)| = |\cos(nt)| \le 1$$
$$\implies \left| L_n - \frac{1}{\pi} \int_0^{\pi} |D_n^*(t)|\, dt \right| = \left| \frac{1}{\pi} \int_0^{\pi} [|D_n(t)| - |D_n^*(t)|]\, dt \right|$$
$$\le \frac{1}{\pi} \int_0^{\pi} \big||D_n(t)| - |D_n^*(t)|\big|\, dt \le \frac{1}{\pi} \int_0^{\pi} dt = 1$$
$$\implies \left[L_n - \frac{1}{\pi} \int_0^{\pi} |D_n^*(t)|\, dt \right] = \mathrm{O}(1) \implies L_n = \frac{1}{\pi} \int_0^{\pi} |D_n^*(t)|\, dt + \mathrm{O}(1)$$

By Proposition 4.3.2.(ii), $\forall t \in (0,\pi]$, $|D_n^*(t)| = \left| \sin(nt) \cot\left(\dfrac{t}{2}\right) \right|$
$= |\sin(nt)| \cot\left(\dfrac{t}{2}\right)$. Since $[0]$ is an m-null set, then $\dfrac{1}{\pi} \displaystyle\int_0^{\pi} |D_n^*(t)|\, dt$
$= \dfrac{1}{\pi} \displaystyle\int_0^{\pi} |\sin(nt)| \cot\left(\dfrac{t}{2}\right) dt. \implies L_n = \dfrac{1}{\pi} \displaystyle\int_0^{\pi} |\sin(nt)| \cot\left(\dfrac{t}{2}\right) dt + \mathrm{O}(1)$

On $[0,\pi]$, $\left[\cot\left(\frac{t}{2}\right) - \frac{2}{t}\right]$ is bounded, i.e. $\exists M > 0\ \forall t \in [0,\pi], \left|\cot\left(\frac{t}{2}\right) - \frac{2}{t}\right| \le M$.

$$\implies \left| |\sin(nt)| \left[\cot\left(\frac{t}{2}\right) - \frac{2}{t}\right]\right| = |\sin(nt)| \left|\cot\left(\frac{t}{2}\right) - \frac{2}{t}\right| \le 1 \cdot M = M$$

$$\implies \left|\frac{1}{\pi}\int_0^\pi |\sin(nt)| \cot\left(\frac{t}{2}\right) dt - \frac{2}{\pi}\int_0^\pi \frac{|\sin(nt)|}{t} dt\right|$$

$$= \left|\frac{1}{\pi}\int_0^\pi |\sin(nt)| \left[\cot\left(\frac{t}{2}\right) - \frac{2}{t}\right] dt\right| \le \frac{1}{\pi}\int_0^\pi \left||\sin(nt)| \left[\cot\left(\frac{t}{2}\right) - \frac{2}{t}\right]\right| dt$$

$$\le \frac{1}{\pi}\int_0^\pi M\, dt = M$$

$$\implies \left[\frac{1}{\pi}\int_0^\pi |\sin(nt)| \cot\left(\frac{t}{2}\right) dt - \frac{2}{\pi}\int_0^\pi \frac{|\sin(nt)|}{t} dt\right] = \mathrm{O}(1)$$

$$\implies \frac{1}{\pi}\int_0^\pi |\sin(nt)| \cot\left(\frac{t}{2}\right) dt = \frac{2}{\pi}\int_0^\pi \frac{|\sin(nt)|}{t} dt + \mathrm{O}(1)$$

$$\implies L_n = \left(\frac{2}{\pi}\int_0^\pi \frac{|\sin(nt)|}{t} dt + \mathrm{O}(1)\right) + \mathrm{O}(1) = \frac{2}{\pi}\int_0^\pi \frac{|\sin(nt)|}{t} dt + \mathrm{O}(1)$$

By Section 2.4.2, $\forall t \in \left[0, \dfrac{\pi}{n}\right]$, $|\sin(nt)| = \sin(nt)$.

$$\int_0^{\pi} \frac{|\sin(nt)|}{t}\, dt = \sum_{k=0}^{n-1} \int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} \frac{|\sin(nt)|}{t}\, dt$$

Fix $0 \le k \le n-1$ and consider $\displaystyle\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} \frac{|\sin(nt)|}{t}\, dt$.

By Proposition 2.3.2.(2), $\displaystyle\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} \frac{|\sin(nt)|}{t}\, dt = \int_0^{\frac{\pi}{n}} \frac{\left|\sin\left(n\left(t + \frac{k\pi}{n}\right)\right)\right|}{\left(t + \frac{k\pi}{n}\right)}\, dt.$

$$\forall t \in \left[0, \frac{\pi}{n}\right], \ \left|\sin\left(n\left(t + \frac{k\pi}{n}\right)\right)\right| = |\sin(nt + k\pi)| = |(-1)^k \sin(nt)|$$

$$= |\sin(nt)| = \sin(nt) \implies \int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} \frac{|\sin(nt)|}{t}\, dt = \int_0^{\frac{\pi}{n}} \frac{\sin(nt)}{\left(t + \frac{k\pi}{n}\right)}\, dt$$

$$\implies \int_0^{\pi} \frac{|\sin(nt)|}{t}\, dt = \sum_{k=0}^{n-1} \int_0^{\frac{\pi}{n}} \frac{\sin(nt)}{\left(t + \frac{k\pi}{n}\right)}\, dt = \int_0^{\frac{\pi}{n}} \frac{\sin(nt)}{t}\, dt + \sum_{k=1}^{n-1} \int_0^{\frac{\pi}{n}} \frac{\sin(nt)}{\left(t + \frac{k\pi}{n}\right)}\, dt$$

$$= \int_0^{\frac{\pi}{n}} \frac{\sin(nt)}{t}\, dt + \int_0^{\frac{\pi}{n}} \sin(nt) \left[\sum_{k=1}^{n-1} \left(t + \frac{k\pi}{n}\right)^{-1}\right] dt$$

$$\forall t \in \left[0, \frac{\pi}{n}\right], \ \left|\frac{\sin(nt)}{nt}\right| \le 1 \implies \left|\frac{\sin(nt)}{t}\right| \le n$$

$$\implies \left|\int_0^{\frac{\pi}{n}} \frac{\sin(nt)}{t}\, dt\right| \le \int_0^{\frac{\pi}{n}} \left|\frac{\sin(nt)}{t}\right| dt \le \int_0^{\frac{\pi}{n}} n\, dt = n\left(\frac{\pi}{n}\right) = \pi$$

$$\implies \int_0^{\frac{\pi}{n}} \frac{\sin(nt)}{t}\, dt = \mathrm{O}(1)$$

Let $Y_n(t) = \sum_{k=1}^{n-1} \left(t + \frac{k\pi}{n} \right)^{-1}$.

$$\forall t \in \left[0, \frac{\pi}{n}\right],\ Y_n(t) \le \sum_{k=1}^{n-1} \left(\frac{k\pi}{n} \right)^{-1} = \sum_{k=1}^{n-1} \frac{n}{k\pi} = \frac{n}{\pi} \sum_{k=1}^{n-1} \frac{1}{k} \text{ and}$$

$$Y_n(t) \ge \sum_{k=1}^{n-1} \left(\frac{\pi}{n} + \frac{k\pi}{n} \right)^{-1} = \sum_{k=1}^{n-1} \left[\frac{(k+1)\pi}{n} \right]^{-1} = \sum_{k=2}^{n} \left(\frac{k\pi}{n} \right)^{-1} = \sum_{k=2}^{n} \frac{n}{k\pi} = \frac{n}{\pi} \sum_{k=2}^{n} \frac{1}{k}$$

$$\implies \forall t \in \left[0, \frac{\pi}{n}\right],\ \frac{n}{\pi} \sum_{k=2}^{n} \frac{1}{k} \le Y_n(t) \le \frac{n}{\pi} \sum_{k=1}^{n-1} \frac{1}{k}$$

By (Zygmund, 1977, P. 15), $\lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \log n \right]$ exists.

$$\implies \lim_{n\to\infty} \left[\sum_{k=1}^{n-1} \frac{1}{k} - \log n \right] \text{ and } \lim_{n\to\infty} \left[\sum_{k=2}^{n} \frac{1}{k} - \log n \right] \text{ exist.}$$

$$\implies \left[\sum_{k=1}^{n-1} \frac{1}{k} - \log n \right] = \mathrm{O}(1) \text{ and } \left[\sum_{k=2}^{n} \frac{1}{k} - \log n \right] = \mathrm{O}(1)$$

$$\implies \sum_{k=1}^{n-1} \frac{1}{k} = \log n + \mathrm{O}(1) \text{ and } \sum_{k=2}^{n} \frac{1}{k} = \log n + \mathrm{O}(1)$$

$$\implies \forall t \in \left[0, \frac{\pi}{n}\right],\ \frac{n}{\pi} [\log n + \mathrm{O}(1)] \le Y_n(t) \le \frac{n}{\pi} [\log n + \mathrm{O}(1)]$$

$$\implies \forall t \in \left[0, \frac{\pi}{n}\right],\ Y_n(t) = \frac{n}{\pi} [\log n + \mathrm{O}(1)]$$

$$\int_0^{\frac{\pi}{n}} \sin(nt)\, dt = -\frac{1}{n} \left[\cos(nt)\right]\Big|_{t=0}^{\frac{\pi}{n}} = -\frac{1}{n}[\cos\pi - \cos 0] = -\frac{1}{n}(-2) = \frac{2}{n}$$

$$\implies \int_0^{\frac{\pi}{n}} \sin(nt) \left[\sum_{k=1}^{n-1} \left(t + \frac{k\pi}{n} \right)^{-1} \right] dt = \int_0^{\frac{\pi}{n}} \sin(nt) Y_n(t)\, dt$$

$$= \int_0^{\frac{\pi}{n}} \sin(nt) \left(\frac{n}{\pi}[\log n + \mathrm{O}(1)] \right) dt = \left(\frac{n}{\pi}[\log n + \mathrm{O}(1)] \right) \int_0^{\frac{\pi}{n}} \sin(nt)\, dt$$

$$= \left(\frac{n}{\pi}[\log n + \mathrm{O}(1)] \right) \left(\frac{2}{n} \right) = \frac{2}{\pi} \log n + \frac{2}{\pi} \mathrm{O}(1) = \frac{2}{\pi} \log n + \mathrm{O}(1)$$

$$\implies \int_0^{\pi} \frac{|\sin(nt)|}{t}\,dt = \mathrm{O}(1) + \left(\frac{2}{\pi}\log n + \mathrm{O}(1)\right) = \frac{2}{\pi}\log n + \mathrm{O}(1)$$

$$\implies L_n = \frac{2}{\pi}\left(\frac{2}{\pi}\log n + \mathrm{O}(1)\right) + \mathrm{O}(1) = \frac{4}{\pi^2}\log n + \frac{2}{\pi}\mathrm{O}(1) + \mathrm{O}(1)$$

$$= \frac{4}{\pi^2}\log n + \mathrm{O}(1) + \mathrm{O}(1) = \frac{4}{\pi^2}\log n + \mathrm{O}(1)$$

$$\therefore\ L_n = \frac{4}{\pi^2}\log n + \mathrm{O}(1)$$

$\square$

CHAPTER 5
κ-Entropy

This chapter is a brief summary of the main results from (Korenblum, 1983) and (Korenblum, 1985). The purpose of this chapter is to generalize the Dirichlet-Jordan Test so that it includes the Dini-Lipschitz Test by using the notion of κ-entropy.

5.1 κ-Entropy

Note. Let κ be a positive nondecreasing concave function on $[0,1]$ s.t. $\kappa(0) = 0$ and $\kappa(1) = 1$. It can be shown that κ is continuous on $(0,1]$ and $\forall s \in [0,1]$, $s \leq \kappa(s) \leq 1$.

Definition 5.1.1. Let $E = \{x_j\}_{j=1}^{n}$ be a finite subset of an interval of length 2π, which means E is a partition of an interval of length at most 2π, where $n \in \mathbb{N}, \forall\, 1 \leq j \leq n-1$, $x_j < x_{j+1}$ and $x_n \leq x_1 + 2\pi$. Let $\{I_j\}_{j=1}^{n}$ be the complementary intervals of E where $\forall\, 1 \leq j \leq n-1$, $I_j = (x_j, x_{j+1})$ and $I_n = (x_n, x_1 + 2\pi)$. Then the κ-entropy of E is $\kappa(E) = \sum_{j=1}^{n} \kappa(\lambda(I_j))$, where λ is defined as in Section 3.1. If $E = \emptyset$, then $\kappa(E)$ is defined to be 0. If E is an infinite closed subset of an interval of length 2π, i.e. $\sup\limits_{x,y\in E} |x-y| \leq 2\pi$, then $\kappa(E) = \sup\{\kappa(E_1) : E_1 \text{ is a finite subset of } E\}$.

The following theorem is a list of some of the important properties of κ-entropy.

Theorem 5.1.1. (i) $\kappa(E) = 1$ if $|E| = 1$, where $|E|$ is the cardinality of E.

(ii) If $\kappa(s) \neq s$ and $|E| > 1$, then $\kappa(E) > 1$.

(iii) If $\kappa(s) = s$ and $E \neq \emptyset$, then $\kappa(E) = 1$.

(iv) If $\kappa(s) = 1$, then $\kappa(E) = |E|$.

(v) If $E \neq \emptyset$, then $1 \leq \kappa(E) \leq |E|$.

(vi) $F_1 \subseteq F_2 \implies \kappa(F_1) \leq \kappa(F_2)$

(vii) $\kappa(E_1 \cup E_2) \leq \kappa(E_1) + \kappa(E_2)$

(viii) $\forall t \in \mathbb{R}$, $\kappa(E + t) = \kappa(E)$

Here are three special examples of κ-entropy.

Example 5.1.1. (i) $\kappa(s) = s(1 - \log s)$; The corresponding κ-entropy is called the Shannon entropy.

(ii) $\kappa(s) = s^\alpha$, where $\alpha \in (0, 1)$; Here, κ-entropy is called the Lipschitz entropy.

(iii) $\kappa(s) = \left(1 - \frac{1}{2}\log s\right)^{-1}$; Here, κ-entropy is called the Dini entropy. This is the most important example.

5.2 C_κ

Definition 5.2.1. Let $\Re C(\mathbb{T})$ be the subset of $C(\mathbb{T})$ consisting of real-valued functions. If $f \in \Re C(\mathbb{T})$, then the κ-norm of f is $\|f\|_\kappa = \int_{-\infty}^{\infty} \kappa(E_y[f] \cap J)\, dy$, where $E_y[f] = \{\tau \in \mathbb{R} : f(\tau) = y\}$ and J is an interval of length 2π.

Note. (i) By the fact that f is 2π-periodic and by Theorem 5.1.1.(viii), $\kappa(E_y[f] \cap J)$ is independent of the choice of J.

(ii) If $\kappa(s) = s$, then the κ-norm of f is denoted by $\|f\|_C$ and $\|f\|_C = \max_{\tau \in \mathbb{T}} f(\tau) - \min_{\tau \in \mathbb{T}} f(\tau)$. If $\kappa(s) = 1$, then the κ-norm of f is denoted by $\|f\|_V$ and $\|f\|_V = V(f)$, where $V(f)$ is defined as in Section 3.4. By Theorem 5.1.1.(v), $\|f\|_C \leq \|f\|_\kappa \leq \|f\|_V$.

Remark. It can be shown that if $\kappa(0+) > 0$, then the κ-norm is equivalent to the V-norm and if $\lim_{s \to 0^+} \frac{\kappa(s)}{s} < \infty$, then the κ-norm is equivalent to the C-norm. It will now be assumed that $\kappa(0+) = 0$ and $\lim_{s \to 0^+} \frac{\kappa(s)}{s} = \infty$.

Definition 5.2.2. Let $\Re C_\kappa = \{f \in \Re C(\mathbb{T}) : \|f\|_\kappa < \infty\}$.

Note. The following assumption will now be used for the rest of this chapter. Two functions $f, g \in \Re C_\kappa$ are considered to be equal if they differ by a constant, i.e. $\exists c \in \mathbb{R}$ s.t. $g = f + c$.

The following theorem is from (Korenblum, 1985, P. 538, Corollary 3.6).

Theorem 5.2.1. With the above assumption, $(\Re C_\kappa, \| \|_\kappa)$ is a separable Banach space over $\mathbb{R}$ and $\mathbf{T} \cap \Re C_\kappa$ is dense in $\Re C_\kappa$.

Remark. The assumption is needed because $\forall f \in \Re C(\mathbb{T})$, $\|f\|_\kappa = 0$ iff f is a constant function. In (Korenblum, 1983, P. 216), instead of using this assumption, the κ-norm of f is redefined as $\|f\|'_\kappa = \|f\|_\infty + \|f\|_\kappa$. Then the above theorem is true with $\| \|_\kappa$ replaced by $\| \|'_\kappa$. The reason why this convention is not being used is because the argument from (Korenblum, 1985) is being used.

Definition 5.2.3. Let $C_\kappa = \{f \in C(\mathbb{T}) : f = x + iy$, where $x, y \in \Re C_\kappa.\}$. With f as given, then the κ-norm of f is $\|f\|_\kappa = \sup_{\vec{\lambda} \in \partial B(\vec{0},1)} [\|\lambda_1 x - \lambda_2 y\|_\kappa^2 + \|\lambda_2 x + \lambda_1 y\|_\kappa^2]^{\frac{1}{2}}$.

Remark. It can be shown that $\forall f \in C_\kappa$, $[\|x\|_\kappa^2 + \|y\|_\kappa^2]^{\frac{1}{2}} \leq \|f\|_\kappa \leq \|x\|_\kappa + \|y\|_\kappa$.

The following theorem follows from the previous theorem.

Theorem 5.2.2. $(C_\kappa, \| \|_\kappa)$ is a separable Banach space over $\mathbb{C}$ and $\mathbf{T} \cap C_\kappa$ is dense in C_κ.

5.3 Premeasures and the κ-Integral

Definition 5.3.1. Let $\mathfrak{I}$ be the collection of all intervals in $\mathbb{R}$ of length at most 2π with $\emptyset$ and all one-point sets. All elements of $\mathfrak{I}$ will be called intervals. A premeasure is a function $\mu\colon \mathfrak{I} \to \mathbb{C}$ satisfying :

(i) $\mu(\emptyset) = \mu(I) = 0$ where I is any interval of length 2π

(ii) $\forall t \in \mathbb{R}\ \forall I \in \mathfrak{I}$, $\mu(I + t) = \mu(I)$

(iii) $\forall I_1, I_2 \in \mathfrak{I}$ s.t. $I_1 \cap I_2 = \emptyset$, $\mu(I_1 \cup I_2) = \mu(I_1) + \mu(I_2)$

(iv) If $\{I_n\}_{n=1}^{\infty} \subseteq \mathfrak{I}$ is a decreasing sequence of intervals and $\bigcap\limits_{n=1}^{\infty} I_n = \emptyset$, then $\lim\limits_{n\to\infty} \mu(I_n) = 0$.

Note. A premeasure as defined here is not the same as the usual premeasure in Measure Theory, as in (Folland, 1999), for example.

Remark. Let μ be a premeasure. Define $\tilde{\mu}\colon \mathbb{T} \to \mathbb{C}$ by $\tilde{\mu}(t) = \mu((0, t])$ on $(0, 2\pi]$ and extend $\tilde{\mu}$ periodically. Then $\tilde{\mu}$ is right-continuous, $\forall t \in \mathbb{R}$, $\tilde{\mu}(t-)$ exists, and $\forall t \in [0]$, $\tilde{\mu}(t) = 0$, where $[0]$ is the equivalence class of 0 as defined in Section 3.1.

Definition 5.3.2. The κ-variation of a premeasure μ is
$\mathrm{Var}_\kappa \mu = \sup\limits_P \left(\dfrac{\sum_{j=1}^n |\mu(I_j)|}{\kappa(P)} \right)$, where the supremum is taken over all partitions P of intervals of length 2π and $\{I_j\}_{j=1}^n$ are the complementary intervals of P.
If $\mathrm{Var}_\kappa \mu < \infty$, then μ is called a premeasure of bounded κ-variation and $\tilde{\mu}$ is called a function of bounded κ-variation. Let V_κ be the set of all premeasures of bounded κ-variation and $\Re V_\kappa$ be the set of all real-valued premeasures of bounded κ-variation.

Note. If $\mu \in V_\kappa$, then μ can be extended to all open and closed sets F which are subsets of an interval of length 2π and satisfy $\kappa(\partial F) < \infty$.

Theorem 5.3.1. $(V_\kappa, \|\ \|)$ is a Banach space over $\mathbb{C}$ and $(\Re V_\kappa, \|\ \|)$ is a Banach space over $\mathbb{R}$, where $\forall \mu \in V_\kappa$, $\|\mu\| = \mathrm{Var}_\kappa \mu$.

Definition 5.3.3. Let $\mu \in V_\kappa$. If $f \in \Re C_\kappa$, then the κ-integral of f w.r.t. μ is $\int f\, d\mu = \int_{-\infty}^{\infty} \mu(F_y[f] \cap J)\, dy$, where $F_y[f] = \{t \in \mathbb{R} : f(t) \geq y\}$ and J is an interval of length 2π. If $f \in C_\kappa$, then the κ-integral of f w.r.t. μ is $\int f\, d\mu = \int x\, d\mu + i \int y\, d\mu$, where x and y are as in Definition 5.2.3.

Note. By the fact that f is 2π-periodic and by Definition 5.3.1.(ii), $\mu(F_y[f] \cap J)$ is independent of the choice of J.

Remark. If $f \in \Re C_\kappa$, then $\{F_y[f]\}_{y \in \mathbb{R}}$ are called the Lebesgue sets of f. The Lebesgue sets of f have no relation to the Lebesgue set of f which is defined only for $f \in L^1_{\mathrm{loc}}(\mathbb{R})$.

The following theorem is from (Korenblum, 1985, P. 540, Proposition 4.2).

Theorem 5.3.2. The κ-integral is bilinear in $f \in C_\kappa$ and $\mu \in V_\kappa$. If $f \in \Re C_\kappa$, then $\left|\int f\, d\mu\right| \leq \left(\frac{\mathrm{Var}_\kappa \mu}{2}\right) \|f\|_\kappa$ and if $f \in C_\kappa$, then $\left|\int f\, d\mu\right| \leq \left[\left(\frac{\sqrt{2}}{2}\right) \mathrm{Var}_\kappa \mu\right] \|f\|_\kappa$.

The following theorem is from (Korenblum, 1985, P. 540, Theorem 5.1).

Theorem 5.3.3. 1) $\Re V_\kappa = (\Re C_\kappa)^*$;

In more detail, $\forall F \in (\Re C_\kappa)^*\ \exists \mu \in \Re V_\kappa$ s.t. $F(f) = \int f\, d\mu$ and conversely every $\mu \in \Re V_\kappa$ defines an $F \in (\Re C_\kappa)^*$ by the same formula. Moreover, $\|F\|_{op} = \dfrac{\mathrm{Var}_\kappa \mu}{2}$.

2) $V_\kappa = (C_\kappa)^*$;

In more detail, $\forall F \in (C_\kappa)^*\ \exists \mu \in V_\kappa$ s.t. $F(f) = \int f\, d\mu$ and conversely every $\mu \in V_\kappa$ defines an $F \in (C_\kappa)^*$ by the same formula. Moreover, $\left(\frac{1}{2\sqrt{2}}\right) \mathrm{Var}_\kappa \mu \leq \|F\|_{op} \leq \left(\frac{\sqrt{2}}{2}\right) \mathrm{Var}_\kappa \mu$.

5.4 Generalization of the Dirichlet-Jordan and Dini-Lipschitz Tests

The motivation for the generalization comes from the following remark from (Korenblum, 1985, P. 527).

Remark. The hypothesis of the Dirichlet-Jordan Test, $f \in BV(\mathbb{T})$, is a restrictive condition to impose on f to ensure the pointwise convergence of $S[f]$. This is shown by the Dini-Lipschitz Test because the hypothesis that $\omega(\delta) = o([\log \delta]^{-1})$, where ω is the modulus of continuity of f, is a weaker condition. This led to the question of whether the Dirichlet-Jordan Test could be generalized so that it includes the Dini-Lipschitz Test. This question was answered affirmatively in (Korenblum, 1983; Korenblum, 1985).

Some of the notation from Section 2.3.2 will be used here.

The only κ-entropy that will be considered here is the Dini entropy where $C_d = C_\kappa$ and $V_d = V_\kappa$. Also, instead of working with premeasures of bounded Dini-variation directly, functions of bounded Dini-variation will be focused on. This means that if f is a function of bounded Dini-variation, then it will be written $f \in V_d$ and that the Dini-variation of f is $\mathrm{Var}_d f = \sup_P \left(\frac{A(P)}{\kappa(P)}\right)$, where the supremum is taken over all partitions P of intervals of length 2π.

Note. If $f \in V_d$, then $\mathrm{Var}_d f = \mathrm{Var}_d \mu$, where μ is a premeasure of bounded Dini-variation s.t. $\tilde{\mu} = f$.

Proposition 5.4.1. (i) $BV(\mathbb{T}) \subseteq V_d$ and $\forall f \in BV(\mathbb{T}), \mathrm{Var}_d f \leq V(f)$.

(ii) $V_d \subseteq L^\infty(\mathbb{T})$ and so $V_d \subseteq L^1(\mathbb{T})$.

Proof. (i) Let $f \in BV(\mathbb{T})$ and P be a partition of an interval of length 2π. By Theorem 5.1.1.(v), $P \neq \emptyset \implies \kappa(P) \geq 1$.

$$\implies \frac{A(P)}{\kappa(P)} \leq A(P) \leq V(f) \Rightarrow \mathrm{Var}_d f = \sup_P \left(\frac{A(P)}{\kappa(P)}\right) \leq V(f) < \infty \Rightarrow f \in V_d$$

Hence, $BV(\mathbb{T}) \subseteq V_d$ and $\forall f \in BV(\mathbb{T})$, $\mathrm{Var}_d f \leq V(f)$.

(ii) By Corollary 3.1.6, it is enough to show $V_d \subseteq L^\infty(\mathbb{T})$. Let $f \in V_d$ and $t \in [0, 2\pi)$. Let $P = \{0, t, 2\pi\}$ which is a partition of $[0, 2\pi]$ satisfying $2 \leq |P| \leq 3$. By Theorem 5.1.1.(v), $\kappa(P) \leq |P| \leq 3$. Then $|f(t) - f(0)| \leq A(P) = \kappa(P)\left(\frac{A(P)}{\kappa(P)}\right) \leq 3\mathrm{Var}_d f$ and $|f(t)| \leq |f(0)| + |f(t) - f(0)| \leq |f(0)| + 3\mathrm{Var}_d f$.

Thus, $\forall t \in [0, 2\pi)$, $|f(t)| \leq |f(0)| + 3\mathrm{Var}_d f$. By the identification between $L^\infty(\mathbb{T})$ and $L^\infty([0, 2\pi), \mathcal{B}_{\mathbb{R}} \cap [0, 2\pi), \lambda)$, $\|f\|_\infty \leq |f(0)| + 3\mathrm{Var}_d f < \infty$. Hence, $f \in L^\infty(\mathbb{T})$ and $V_d \subseteq L^\infty(\mathbb{T})$.

□

Note. It shall also be assumed that every $f \in V_d$ is normalized so that $\forall t \in \mathbb{R}$, $f(t) = \dfrac{f(t+) + f(t-)}{2}$.

The following remark is from (Korenblum, 1983, P. 217) and (Korenblum, 1985, P. 549).

Remark. The proof of the Dirichlet-Jordan Test, like the one in (Zygmund, 1977, P. 57-58, Theorem 8.1), is based on the classical $C(\mathbb{T}) - BV(\mathbb{T})$ duality. The generalized test is obtained by using the (Dini-entropy-norm)-(Dini-variation) duality, i.e. $C_d - V_d$ duality.

Before the generalized test is presented, the following theorem from (Korenblum, 1985, P. 551, Theorem 9.2) will be stated because a uniform bound on the partial sums of $S[f]$ is given with the only assumption being $f \in V_d$.

Theorem 5.4.2. If $f \in V_d$, then $\forall n \in \mathbb{N}_0$, $\|S_n(f)\|_\infty \leq \|f\|_\infty + \frac{7}{2}\mathrm{Var}_d f$.

Definition 5.4.1. Let $f \in V_d$, $t \in \mathbb{R}$, $\delta > 0$ and $g_t(\tau) = [(f(\tau) - f(t))\chi_{(t-\delta,t+\delta)}(\tau)]$. f is of vanishing d-variation at t if $\lim\limits_{\delta \to 0^+} \mathrm{Var}_d g_t = 0$. If $\forall t \in J$, where J is a closed interval of length 2π, f is of vanishing d-variation at t, then f is of vanishing d-variation on $\mathbb{T}$.

The following remarks are from (Korenblum, 1985, P. 551, Remarks 1 and 3).

Remark. (i) Let $f \in BV(\mathbb{T})$ and $t \in \mathbb{R}$. If f is continuous at t, then f is of vanishing d-variation at t. Moreover, if $f \in C(\mathbb{T})$ as well, then f is of vanishing d-variation on $\mathbb{T}$.

(ii) Let $f \in C(\mathbb{T})$ and ω be the modulus of continuity of f.
If $\omega(\delta) = \mathrm{O}([\log \delta]^{-1})$, then $f \in V_d$ and if $\omega(\delta) = \mathrm{o}([\log \delta]^{-1})$, then f is of vanishing d-variation on $\mathbb{T}$.

The following theorem is from (Korenblum, 1983, P. 218, Theorem 3) and (Korenblum, 1985, P. 552, Theorem 9.3). This is the generalization of the Dirichlet-Jordan Test and the Dini-Lipschitz Test.

Theorem 5.4.3. 1) If f is of vanishing d-variation at t, then $\lim\limits_{n\to\infty} S_n(f)(t) = f(t)$.

2) If f is of vanishing d-variation on $\mathbb{T}$, then $\lim\limits_{n\to\infty} S_n(f)(t) = f(t)$ uniformly on $\mathbb{R}$.

Note. By the previous remarks, it follows that the generalized test includes the Dirichlet-Jordan Test and the Dini-Lipschitz Test.

CHAPTER 6
Conclusion

In conclusion, this thesis was a summary of some important results in the convergence of Fourier series. Specifically, the Cesàro summability and convergence of Fourier series was examined. Here, summability was viewed in connection with summability kernels. The only summability kernel that was used in this text was the Fejér kernel and in this event, summability was also called Cesàro summability. In addition, while convergence of Fourier series was an investigation of the partial sums of Fourier series, summability of Fourier series was an investigation of the Cesàro sums of Fourier series.

Before the main subject of Fourier series was discussed, a brief review in Chapter 2 was given of topics that would be needed later on in the text. First, the Riemann integral was defined for complex-valued functions and then the definition was generalized for vector-valued functions. Next, a short comparison was given of the Riemann and Lebesgue integrals. Finally, three significant classes of functions were introduced. These classes were locally integrable functions, functions of bounded variation, and absolutely continuous functions.

In Chapter 3, the summability and convergence in norm of Fourier series was studied. To begin with, the Banach spaces of functions $L^p(\mathbb{T})$, where $1 \leq p \leq \infty$, and $C(\mathbb{T})$ were presented. These Banach spaces of functions were the most important classes of functions in this text. One reason is that all functions considered here were assumed to be in at least one of these classes. Next, the Fourier coefficients and Fourier series were defined. Then homogeneous Banach spaces on $\mathbb{T}$ were introduced. It was soon proven that $L^p(\mathbb{T})$, where $1 \leq p < \infty$, and $C(\mathbb{T})$ belong to this collection of Banach spaces. Following this, the n-th partial and Cesàro sums of Fourier series were defined. Subsequently, it was shown that all homogeneous Banach spaces on $\mathbb{T}$ admit summability in norm. This result was probably the most substantial result of this chapter. This result also gave rise to two consequential theorems which were the Weierstrass Approximation Theorem and the Riemann-Lebesgue lemma. Afterwards, the Banach space $L^2(\mathbb{T})$ was studied in more detail because of the special properties it inherits as a Hilbert space. Lastly, convergence in norm was defined and discussed.

In Chapter 4, the pointwise summability and convergence of Fourier series was studied. First, pointwise summability was discussed and in particular, Lebesgue's Theorem which states that $\forall f \in L^1(\mathbb{T})$, the Fourier series of f is pointwise summable to f m-a.e. . Later, an example was given of a continuous function whose Fourier series diverges at a point. Finally, the three most important tests for pointwise convergence, Dini's Test, the Dirichlet-Jordan Test, and the Dini-Lipschitz Test, were presented.

In Chapter 5, a generalization, due to Boris Korenblum, of the Dirichlet-Jordan Test and the Dini-Lipschitz Test was studied. All the results in this chapter were presented from the papers of Korenblum, (Korenblum, 1983) and (Korenblum, 1985). The Dirichlet-Jordan Test was generalized to give a new pointwise convergence test that included the Dini-Lipschitz Test. To obtain the generalized test, first the notion of κ-entropy was introduced. Then the κ-norm and an associated Banach space of functions denoted by C_κ were defined. Thereupon, premeasures and the κ-integral were introduced. At the end, the generalized test was stated.

If I continue my current research, I would investigate the following. Initially, I would study Theorem 3.6.8 and its proof in more detail. I would also see if there are alternate proofs that do not involve the theory of conjugation. After, I would explore the theory of κ-entropy more carefully so that I could understand the proof of the generalized test. Ultimately, I would analyze the proofs of the Carleson and Carleson-Hunt theorems and see if they could be simplified or generalized.

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