On the Geometry of Twistor Spaces of Hypercomplex and Hyperkähler Manifolds

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ABSTRACT

We start by generalizing the result of Kaledin and Verbitsky that twistor spaces of hyperkähler manifolds admit balanced metrics. It is shown that in fact the twistor space of any compact hypercomplex manifold is balanced. We then study holomorphic vector bundles on the twistor space $\mathrm{Tw}(M)$ of a simple hyperkähler manifold M and the stability of their restrictions to the fibres of the twistor projection $\mathrm{Tw}(M) \to \mathbb{CP}^1$. Extending an argument of Teleman, we show that fibrewise stability and semi-stability of a bundle on $\mathrm{Tw}(M)$ are Zariski open conditions on the base $\mathbb{CP}^1$. We prove a partial converse to another result of Kaledin and Verbitsky, namely that a generically fibrewise stable bundle E on the twistor space $\mathrm{Tw}(M)$ is irreducible, in the sense of having no proper subsheaves of lower rank. The converse is established for the case when the rank of E is 2 or 3, as well as for bundles E of general rank that are generically fibrewise simple. Finally, we construct an example of a stable vector bundle E on $\mathrm{Tw}(M)$ for M a K3 surface which is nowhere fibrewise stable.

ABRÉGÉ

On commence avec la généralisation du résultat de Kaledin et Verbitsky que les espaces twistoriels des variétés hyperkählériennes admettent des métriques semi-kählériennes. On démontre qu'en fait l'espace twistoriel d'une variété hypercomplexe compacte quelconque est semi-kählérien. On étudie ensuite des fibrés vectoriels holomorphes sur l'espace twistoriel $\mathrm{Tw}(M)$ d'une variété hyperkählérienne simple M et la stabilité de leurs restrictions aux fibres de la projection twistorielle $\mathrm{Tw}(M) \to \mathbb{CP}^1$. On démontre, en étendant un argument de Teleman, que la stabilité et la semi-stabilité des restrictions aux fibres d'un fibré sur $\mathrm{Tw}(M)$ sont des conditions ouvertes de Zariski sur la base $\mathbb{CP}^1$. On prouve une réciproque partielle d'un autre résultat de Kaledin et Verbitsky, soit qu'un fibré E sur l'espace twistoriel $\mathrm{Tw}(M)$ dont la restriction à la fibre générique est stable est irréductible, c'est-à-dire, ne possède pas de sous-faisceaux propres de rang inférieur. La réciproque est établie pour le cas où le rang de E est 2 ou 3, ainsi que pour des fibrés E de rang quelconque dont la restriction à la fibre générique est simple. Finalement, on construit un exemple d'un fibré vectoriel stable E sur $\mathrm{Tw}(M)$ pour M une surface K3 dont la restriction à chaque fibre est non stable.

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CHAPTER 1
Introduction

Quaternions were introduced by Hamilton in the XIX$^{\text{th}}$ century in his quest to generalize the operations of multiplication and division of complex numbers, viewed as points in $\mathbb{R}^2$, to higher dimensions. Although Hamilton's ambitious plan to completely recast the geometry and physics of his time in terms of quaternions ultimately fell short, it is fitting that in the XX$^{\text{th}}$ century the great advances in geometry naturally led mathematicians to search for suitable quaternionic analogues of the concept of complex manifold, and it turned out that these in turn lead to a rich theory with important applications to modern physics. It should be noted that the naive definition, namely that of a manifold with an atlas of charts taking values in $\mathbb{H}^n$, is too restrictive: the requirement that differentials of the transition maps $\mathbb{H}^n \to \mathbb{H}^n$ be $\mathbb{H}$-linear force the transition maps themselves to be $\mathbb{H}$-affine, giving us only affine manifolds. We follow a different approach that leads us to the definition of hypercomplex manifolds and their special case hyperkähler manifolds, which are the main objects studied in this thesis.

Recall that an alternative definition of a complex manifold is that of a smooth manifold M equipped with an almost complex structure, that is, an endomorphism of the tangent bundle $I : TM \to TM$ satisfying $I^2 = -1$, which is integrable in the sense of Newlander-Nirenberg [38]. An immediate generalization to the quaternionic setting is that of a smooth manifold M with a triple of integrable almost complex structures $I, J, K : TM \to TM$ satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = -1, \; IJ = -JI = K.$$

We call such a manifold hypercomplex. These were first studied by Boyer [6] in the 1980s, who gave a complete classification of compact hypercomplex manifolds in quaternionic dimension 1, showing that the only such are complex tori, K3 surfaces and Hopf surfaces. No such general classification exists for higher dimensions, but a wealth of examples were subsequently found; in particular, Joyce [26] constructed a large family of left-invariant hypercomplex structures on compact homogeneous spaces, rediscovering and generalizing an earlier work by string theorists [45].

Given a hypercomplex manifold (M, I, J, K), suppose we have a Riemannian metric g on M which is Hermitian with respect to the three complex structures I, J, K (such g is called hyperhermitian). Letting $\omega_I, \omega_J, \omega_K$ denote the corresponding Hermitian forms, we say that the manifold M is hyperkähler if each one of these is closed. Equivalently, letting

$$\Omega_I = \omega_J + \sqrt{-1}\omega_K,$$

it can be shown that the above condition is equivalent to the form Ω_I being closed. A compact hyperkähler manifold M is called simple if it is simply connected and satisfies $H^{2,0}(M) = \mathbb{C}$. The study of hyperkähler structures predates that of general hypercomplex structures: the above definition was first given by Calabi [11] in the 1970s, but hyperkähler manifolds also appeared in the much earlier work of Berger [4] in the 1950s on the classification of irreducible holonomy groups on Riemannian manifolds, where they correspond to the holonomy group $Sp(n)$. It's not hard to see that the form Ω_I given above is a non-degenerate $(2,0)$-form with respect to the complex structure I on M. The condition $d\Omega_I = 0$ forces it to be a holomorphic symplectic form. On the other hand, it follows from Yau's proof of the Calabi conjecture [53] that a compact Kähler holomorphic symplectic manifold is hyperkähler, showing the equivalence of these concepts. Hyperkähler geometry plays an important role in quantum gravity through its ties with supersymmetry [24] and has a rich and developed theory, but in stark contrast to the general hypercomplex case, there is a dearth of examples of hyperkähler manifolds. Two families of simple hyperkähler manifolds in each quaternionic dimension, namely Hilbert schemes of points on a K3 surface and generalized Kummer manifolds, were constructed by Beauville [3], while two sporadic examples in complex dimensions 6 and 10 (quaternionic dimensions 3 and 5, respectively) were given by O'Grady [39, 40]. It is currently unknown whether there exist examples that are not deformationally equivalent to these.

An important tool for studying both hypercomplex and hyperkähler manifolds is the twistor formalism. Twistor theory was introduced by Penrose [43] in the context of theoretical physics in the 1960s and has since played an important role in both physics and mathematics. In the hypercomplex setting, it takes the following form. Given a hypercomplex manifold (M, I, J, K), it's not hard to see that there is a whole 2-sphere of integrable almost complex structures on M:

$$S^2 = \left\{ x_1 I + x_2 J + x_3 K : x_1^2 + x_2^2 + x_3^2 = 1 \right\},$$

called induced complex structures. The product $M \times S^2$ parametrizes the induced complex structures at points of M, and identifying S^2 with $\mathbb{CP}^1$ in the usual way, we call $\mathrm{Tw}(M) = M \times S^2 \cong M \times \mathbb{CP}^1$ the twistor space of the hypercomplex manifold M. It turns out that there is a natural complex structure on $\mathrm{Tw}(M)$ making the projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$ holomorphic; in case there is also a hyperkähler metric g on M, it gives rise to a natural Hermitian metric on $\mathrm{Tw}(M)$. The usefulness of the twistor space $\mathrm{Tw}(M)$ comes from the fact that its complex structure completely encodes the quaternionic structure of the hypercomplex manifold M. For example, M can be recovered from $\mathrm{Tw}(M)$. More generally and more importantly, there is a notion of twistor correspondence, which takes many forms and associates to an object on M somehow compatible with the quaternionic structure a corresponding holomorphic object on $\mathrm{Tw}(M)$, in a one-to-one fashion. For the case of hyperkähler M, this correspondence can be roughly described as associating to a vector bundle on M which is simultaneously holomorphic with respect to all the induced complex structures a holomorphic bundle on $\mathrm{Tw}(M)$ satisfying certain restriction conditions (Theorem 2.2.7, originally from [28]). This leads to an identification of the corresponding moduli spaces, and in this way the geometry of the twistor space $\mathrm{Tw}(M)$ helps in our understanding of the geometry of the original manifold M.

Recall that to have a meaningful structure on the moduli space of vector bundles we need to look at stable vector bundles. The notion of stability was first introduced by Mumford in a purely algebro-geometric setting in [36] for projective varieties and then generalized to Kähler manifolds and then to general Hermitian manifolds. In the early 1980s, it was independently conjectured by Kobayashi and Hitchin that moduli spaces of stable vector bundles are essentially one and the same as moduli spaces of Hermitian-Einstein vector bundles, a purely differential-geometric notion introduced by Kobayashi in [29]. While this was known to be true for curves from the earlier work of Narasimhan and Seshadri [37], generalizing this result took a considerable amount of time and effort from many mathematicians. Known today as the Kobayashi-Hitchin correspondence, it was gradually proved in increasing generality: Donaldson gave a new differential-geometric proof for curves [13], then settled the case of algebraic surfaces [14] and manifolds [15], Uhlenbeck and Yau proved it for Kähler manifolds [48, 49], Buchdahl gave the proof for general surfaces [10], and finally Li and Yau settled the case of general Hermitian manifolds [31]. The Kobayashi-Hitchin correspondence, among its many other applications, leads to a better understanding of the geometry of moduli spaces of stable bundles. While there is a rich theory of stability of vector bundles in the algebraic context (see for example [25]), the non-Kähler case is more difficult. Historically, the first explicit determination of a moduli space of stable bundles on a non-Kähler manifold is due to Braam and Hurtubise [7]. They described the moduli space of stable $SL(2,\mathbb{C})$-bundles on primary elliptic Hopf surfaces, which by the Kobayashi-Hitchin correspondence can be identified with $SU(2)$-instantons. Since then, there has been a lot of interest in the subject of moduli spaces of stable bundles over non-Kähler manifolds, but while there were some results in this direction (see [9, 8]), much remains unknown.

One of the reasons why the non-Kähler case remains elusive is that, although the Kobayashi-Hitchin correspondence holds for arbitrary Hermitian manifolds as noted above, the absence of a Kähler metric in general indicates a loss of structure on the moduli spaces since the concept of stability becomes harder to work with. Without going into too much detail at this point, we note that the crucial ingredient in the definition of stability is the notion of degree, which for a holomorphic vector bundle E on a compact Kähler manifold of dimension n takes the form

$$\deg(E) := \int c_1(E) \wedge \omega^{n-1}.$$

Here, $c_1(E)$ is the first Chern class viewed as an element of the second de Rham cohomology group, while ω is the Kähler form. To see that this definition does not depend on the choice of representative of $c_1(E)$, note that for any 1-form η, we have

$$\int d\eta \wedge \omega^{n-1} = \int d\left(\eta \wedge \omega^{n-1}\right) + \int \eta \wedge d\left(\omega^{n-1}\right) = 0. \tag{1.1}$$

The first term on the right is zero by Stokes' theorem, while the second term is zero by the Kähler condition $d\omega = 0$. It follows from this that $\deg(E)$ is a topological invariant of E, since $c_1(E)$ is. In case ω is a general Hermitian form that is not closed, the degree as given above is not well-defined. In this case, one needs to adjust the above definition of $\deg(E)$, as will be explained in Section 2.3; it should be noted that for general Hermitian manifolds $\deg(E)$ is no longer a topological invariant of E but only a holomorphic one, which makes the theory more complicated and the corresponding moduli spaces less tractable. However, if the Hermitian form ω satisfies the condition $d\left(\omega^{n-1}\right) = 0$, one can see easily that the equality (1.1) still holds, and the definition of $\deg(E)$ as above goes through. A metric whose Hermitian form ω satisfies $d\left(\omega^{n-1}\right) = 0$ is called balanced; balanced metrics were first studied by Michelsohn [34]. They form a strictly larger class than Kähler metrics for manifolds of dimension ≥ 3, and in view of what was said above, one can argue that the class of balanced metrics is the largest one for which one can generally hope to have a nice structure on the moduli space of stable vector bundles.

1.1 Overview of results

The article [28] of Kaledin and Verbitsky studies autodual connections on complex vector bundles E over a hyperkähler manifold M. These are connections ∇ with curvature invariant under the natural action of $SU(2)$ on the bundle $\Lambda^* M$ of differential forms of the hyperkähler M. They prove that the moduli space of autodual connections on E is locally a complexification of the moduli space of Hermitian-Einstein structures on E. They then work with the twistor space $\mathrm{Tw}(M) = M \times \mathbb{CP}^1$ and its natural projections

$$\begin{array}{ccc} & \mathrm{Tw}(M) & \\ {}^{\sigma}\swarrow & & \searrow^{\pi} \\ M & & \mathbb{CP}^1 \end{array}$$

and establish a twistor correspondence for autodual connections, which takes (E, ∇) to the holomorphic vector bundle $(\sigma^* E, (\sigma^* \nabla)^{0,1})$ on the twistor space $\mathrm{Tw}(M)$. They explicitly describe which holomorphic bundles on $\mathrm{Tw}(M)$ can be obtained in this way and show that the correspondence is bijective (Theorem 2.2.7). Furthermore, they show that the image of an autodual connection via this map is semi-stable on $\mathrm{Tw}(M)$, thus exhibiting the twistor transform as an injective map of the corresponding moduli spaces. Finally, given a complex vector bundle E on M with first two Chern classes $c_1(E)$, $c_2(E)$ $SU(2)$-invariant, they study holomorphic structures on the complex bundle $\sigma^* E$ on $\mathrm{Tw}(M)$ whose restrictions to all the fibres of the holomorphic twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$ are stable; these are called fibrewise stable bundles.

They prove that the moduli space of fibrewise stable bundles on $\mathrm{Tw}(M)$ is isomorphic to the space of sections of the twistor projection $\widehat{\pi} : \mathrm{Tw}(\widehat{M}) \to \mathbb{CP}^1$ (called twistor lines), where $\widehat{M}$ is the space of deformations of E, which carries a natural hyperkähler metric and thus also has a twistor space $\mathrm{Tw}(\widehat{M})$; $\widehat{M}$ is called the Mukai dual of M, after the work of Mukai [35] on duality on K3 surfaces. In view of the twistor correspondence described above, one obtains an identification of the moduli space of autodual connections on E inducing stable holomorphic structures for every $I \in \mathbb{CP}^1$ with a space of lines in the twistor space $\mathrm{Tw}(\widehat{M})$ of the Mukai dual $\widehat{M}$.

This thesis is concerned with extending two results from [28]. One of these is generalized, while for the other a partial converse is proven. We presently give a brief description of these results.

To work with the moduli space of holomorphic bundles on the twistor space, and in particular to construct the twistor transform as an inclusion of moduli spaces, one needs a good notion of stability on $\mathrm{Tw}(M)$. As mentioned before, the twistor space $\mathrm{Tw}(M)$ of a hyperkähler manifold M comes equipped with a natural Hermitian metric. However, this metric need not be Kähler, and in fact when M is compact, it never is (Corollary 3.2.4). In Section 4.4 of [28], the authors show that the natural Hermitian metric on the twistor space $\mathrm{Tw}(M)$ of a hyperkähler M is balanced (Theorem 3.1.1), which, considering the discussion above, is the next best case scenario; this balancedness result is crucial in their discussion of maps of moduli spaces arising from the twistor transform and the Mukai dual correspondence. In this thesis we generalize this result and show that the twistor space $\mathrm{Tw}(M)$ of a general compact hypercomplex manifold M also admits balanced metrics (Theorem 3.2.3); in other words, the balancedness condition holds on $\mathrm{Tw}(M)$ without any metric assumptions on the base manifold M whatsoever. In the absence of a hyperkähler metric on M, the balanced metric on $\mathrm{Tw}(M)$ is constructed implicitly. The key result in the proof is that on a general complex manifold of complex dimension n, for every strictly positive $(n-1,n-1)$-form η there always exists a strictly positive $(1,1)$-form ω such that $\omega^{n-1} = \eta$; thus, the existence of a closed strictly positive $(n-1,n-1)$-form η guarantees that the corresponding $(1,1)$-form ω is the Hermitian form of a balanced metric (Lemma 3.2.1). One then proceeds to construct such a form η on $\mathrm{Tw}(M)$ as a certain linear combination of forms obtained from an arbitrary hyperhermitian metric on M. This work was previously published in the article [47] by the author of the present thesis.

When working with fibrewise stable bundles on the twistor space $\mathrm{Tw}(M)$ of a hyperkähler M, Kaledin and Verbitsky prove a short technical lemma (Lemma 7.3 in [28]) that shows that such bundles are stable as holomorphic bundles on $\mathrm{Tw}(M)$; this is then used to establish the Mukai dual correspondence as an identification of moduli spaces. In fact, their result is stronger: they show that a holomorphic vector bundle E on $\mathrm{Tw}(M)$ that stably restricts to a generic fibre of the twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$ (we call such E generically fibrewise stable) is irreducible, in the sense that it has no proper subsheaves of lower rank. Here, genericity is understood in the sense of Zariski topology on $\mathbb{CP}^1$. In this thesis, we prove a partial converse to this result (Theorem 4.2.1): we show that an irreducible holomorphic bundle E on the twistor space $\mathrm{Tw}(M)$ of a simple hyperkähler manifold M is generically fibrewise stable with respect to the twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$ in case the rank of

E is 2 or 3, as well as in the general case, provided that E is generically fibrewise simple, i.e. the restriction E_I of E to the fibre $\pi^{-1}(I)$ for generic $I \in \mathbb{CP}^1$ is simple, in the sense that $\mathrm{Hom}(E_I, E_I) = \mathbb{C}$. The first ingredient in the proof is the fact that fibrewise stability of E with respect to $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$ is a Zariski open condition on $\mathbb{CP}^1$ (Theorem 4.1.1). This is proved using a generalization of an argument of Teleman from [46], where the Zariski openness of stability is shown for families of type $X \times Y \to Y$, where X, Y are complex manifolds satisfying certain conditions. Using this fact, we argue by contradiction: if E is not generically fibrewise stable, then there are destabilizing subsheaves of some rank s for E_I for every $I \in \mathbb{CP}^1$. Using the one-to-one correspondence between subsheaves of E_I of rank s and line subsheaves of $\Lambda^s(E_I)$, one can show that there exists a line bundle L on $\mathrm{Tw}(M)$ such that for every $I \in \mathbb{CP}^1$ there are nontrivial morphisms

$$L_I = L|_{\pi^{-1}(I)} \longrightarrow C^s(E_I) \subseteq \Lambda^s(E_I)$$

where $C^s(E_I)$ denotes the cone of exterior monomials, a closed analytic subset of $\Lambda^s(E_I)$. The fact that these can be "glued" into a proper subsheaf $\mathcal{F} \subseteq E$ of rank s (which contradicts the irreducibility of E) is equivalent to the purely algebraic condition that there exists a section of

$$\begin{array}{ccc} Y & \hookrightarrow & \mathbb{P}(\pi_*(L^* \otimes \Lambda^s E)) \\ \downarrow & \swarrow & \\ \mathbb{CP}^1 & & \end{array}$$

where $Y \subseteq \mathbb{P}(\pi_*(L^* \otimes \Lambda^s E))$ is the closed analytic subset parametrizing morphisms $L_I \to \Lambda^s(E_I)$ with image in $C^s(E_I)$. If the rank of E is 2 or 3, such sections always exist, since in this case $C^s(E_I) = \Lambda^s(E_I)$. Otherwise, one can construct a multisection, i.e. a section of Y over a branched covering $f : X \to \mathbb{CP}^1$ (Lemma 4.2.2). The strategy then consists of taking the fibred product

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & \mathrm{Tw}(M) \\ \rho\downarrow & & \downarrow\pi \\ X & \xrightarrow[f]{} & \mathbb{CP}^1 \end{array}$$

and using the multisection to construct a proper subsheaf $\mathcal{F} \subsetneq \varphi^*(E)$ on Z. One can then show that if E is generically fibrewise simple, this condition together with the irreducibility E imply that the pullback bundle $\varphi^*(E)$ on the fibred product Z is irreducible as well, giving a contradiction, and thus proving our result. We also construct an example of a stable vector bundle on $\mathrm{Tw}(M)$ for M = K3 surface which is nowhere fibrewise stable, showing that mere stability doesn't imply fibrewise stability on the twistor space of a hyperkähler manifold.

These results are both interesting in themselves and have potential to be generalized and built upon. As concerns balanced metrics, these present a well-behaved generalization of the Kähler condition with many interesting properties, and thus examples of balanced manifolds are of interest in their own right. While it was known from the work of Kaledin and Verbitsky that twistor spaces $\mathrm{Tw}(M)$ of hyperkähler manifolds M are balanced, there are not many explicit examples of hyperkähler manifolds themselves, as mentioned above. On the other hand, the corresponding result for hypercomplex manifolds gives us a lot of new balanced manifolds $\mathrm{Tw}(M)$ since hypercomplex M are plentiful. Furthermore, the existence of a balanced metric on the twistor space $\mathrm{Tw}(M)$ of a hypercomplex manifold M naturally leads one to a discussion of stable bundles on $\mathrm{Tw}(M)$, and it would be interesting to see if any of the results of Kaledin and Verbitsky in [28] relating various moduli spaces would generalize to the case that M is hypercomplex. As concerns fibrewise stability of bundles on $\mathrm{Tw}(M)$ for hyperkähler M, a natural direction of research would be to try to prove the full converse of Theorem 4.2.1, i.e. that an arbitrary irreducible bundle on $\mathrm{Tw}(M)$ is generically fibrewise stable. It should be noted that irreducible bundles only appear in the non-algebraic setting. For algebraic manifolds, irreducible bundles don't exist and one always has recourse to various filtrations, but the lack of such techniques for irreducible bundles on non-algebraic manifolds make their study challenging. Thus, in case the full converse to Theorem 4.2.1 is true, it would give a very nice description of irreducible bundles on $\mathrm{Tw}(M)$, especially in view of the Mukai dual correspondence in [28].

We now give a concise description of the contents of each chapter of this thesis.

- Chapter 1 is the present introduction in which the results of the thesis are briefly outlined.
- Chapter 2 gives an overview of balanced manifolds, hypercomplex and hyperkähler geometry and stability of vector bundles. It consists mostly of definitions and statements of results that are needed in the next two chapters. None of the results are original, and the corresponding references are given at the beginning of the chapter and throughout the text. Most of the content in Section 2.1 appeared in the article [47] of the author and is reproduced here with only minor additions.

- Chapter 3 starts with the proof in Section 3.1 of the fact that for a hyperkähler manifold M, the twistor space $\mathrm{Tw}(M)$ is balanced. The argument is due to Kaledin and Verbitsky and appeared in [28]; we reproduce it here for convenience and because ideas from the proof are needed in the next section. In Section 3.2, it is shown that the twistor space $\mathrm{Tw}(M)$ of a compact hyperhermitian manifold M is balanced, which also establishes the result for arbitrary compact hypercomplex M, since these always admit hyperhermitian metrics. The material in this chapter is reproduced from the article [47] with only minor adjustments.
- Chapter 4 is concerned with irreducibility and fibrewise stability of bundles on the twistor space $\mathrm{Tw}(M)$ of a simple hyperkähler manifold M. In Section 4.1 it is proven that fibrewise stability and semi-stability are Zariski open conditions on the base of the twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$. The proof is basically a verification that the argument of Teleman from [46] works in the case of the twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$. In Section 4.2 it is proved that a generically fibrewise stable bundle E on $\mathrm{Tw}(M)$ is irreducible; the proof is due to Kaledin and Verbitsky [28]. A partial converse to this statement is proved, which covers the cases $\mathrm{rk}\, E = 2, 3$ and the general case for E generically fibrewise simple. In Section 4.3 an example of a stable but nowhere fibrewise stable bundle on $\mathrm{Tw}(M)$ for M = K3 surface is costructed.
- Chapter 5 is the conclusion which presents a number of possible generalizations of the results above and potential avenues of research to build upon these results.

CHAPTER 2
Preliminaries

In this chapter, we introduce the main notions with which we will be working, giving the definitions and stating important results that will be useful to us in the subsequent chapters, mostly without proof. The main references are the article [34] and the book [30] for Section 2.1, the papers [50, 51, 27, 28, 52] for Section 2.2, and the books [30, 32, 41] for Section 2.3.

2.1 Balanced manifolds

Our first goal is to give the definition of balanced metrics on manifolds. We start with some preliminaries from differential geometry. In what follows, let M denote a (real) C^∞ manifold and $E \to M$ a (real) C^∞ vector bundle over M. In what follows, we will denote by $\Gamma(E)$ the space of C^∞ sections of E.

Definition 2.1.1. A *connection* on E is an $\mathbb{R}$-linear operator $\nabla : \Gamma(E) \to \Gamma(\Lambda^1 M \otimes E)$ satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s \quad \forall f \in C^\infty M, s \in \Gamma(E).$$

Given a vector field $X \in \Gamma(TM)$, we denote by $\nabla_X s \in \Gamma(E)$ the usual pairing of X with $\nabla s \in \Gamma\left(\Lambda^1 M \otimes E\right)$. Associated to a connection ∇ is its *curvature* R^∇ : $\Lambda^2(TM) \to \mathrm{End}(E)$ defined by

$$R^\nabla(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \forall X, Y \in \Gamma\left(TM\right).$$

In the special case of $E = TM$ being the tangent bundle, we can also define the *torsion* $T^\nabla : \Lambda^2(TM) \to TM$ of the connection by

$$T^\nabla(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \quad \forall X,Y \in \Gamma(TM).$$

In fact, it's easy to verify that both R^∇ and T^∇ are C^∞ linear operators, hence we can think of them as tensors: $R^\nabla \in \Gamma(\Lambda^2 M \otimes \mathrm{End}(E))$, $T^\nabla \in \Gamma(\Lambda^2 M \otimes TM)$. If $R^\nabla = 0$, the connection is said to be *flat*, while if $T^\nabla = 0$, it is called *torsion-free*.

Observe that a connection $\nabla : \Gamma(E) \to \Gamma(\Lambda^1 M \otimes E)$ on E induces a canonical connection on the dual bundle $E^* = \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R})$, also denoted by ∇, and defined by

$$\langle \nabla\eta, s\rangle + \langle \eta, \nabla s\rangle = d\left(\langle \eta, s\rangle\right) \quad \forall \eta \in \Gamma(E^*),\ s \in \Gamma(E),$$

where we denote by $\langle\ ,\ \rangle$ the pairing of E^* with E. Given connections ∇^E, ∇^F on vector bundles E, F, we can consider the induced connections $\nabla^{E\oplus F}$, $\nabla^{E\otimes F}$ on $E \oplus F$, $E \otimes F$ defined by

$$\nabla^{E\oplus F}(s \oplus t) := \left(\nabla^E s\right) \oplus \left(\nabla^F t\right) \quad \forall s \in \Gamma(E),\ t \in \Gamma(F),$$

$$\nabla^{E\otimes F}(s \otimes t) := \left(\nabla^E s\right) \otimes t + s \otimes \left(\nabla^F t\right) \quad \forall s \in \Gamma(E),\ t \in \Gamma(F).$$

Thus, starting with a single connection ∇ on E, we can form induced connections on all tensor products $(E^*)^{\otimes r} \otimes E^{\otimes q}$. Moreover, it's not hard to see that the subspaces of symmetric and antisymmetric tensors are invariant under these connections. In what follows, all these induced connections on tensor powers of E will be denoted by the same symbol ∇, and when $\nabla s = 0$ for some tensor s, we will say that the connection *preserves* s, or that s is *parallel* with respect to ∇.

We now specialize to the case when M is a complex manifold and $E \to M$ is a complex vector bundle. Since E is in particular a real vector bundle, we can have connections on E defined as above, but this time we can single out those that are $\mathbb{C}$-linear as operators $\Gamma(E) \to \Gamma(\Lambda^1 M \otimes E)$; these are precisely the connections which preserve the operator $I : E \to E$, $I^2 = -1$, of multiplication by the imaginary unit in E viewed as a complex vector bundle. In addition to the induced connections described in the previous paragraph, a $\mathbb{C}$-linear connection ∇ on E induces $\mathbb{C}$-linear connections on the complex dual $E^* = \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})$ and the conjugate bundle $\bar{E}$. For the special case that $E = TM$ is the tangent bundle, the operator $I : TM \to TM$ above is called the *almost complex structure* of M. It is a well-known result that the condition of (M, I) being a complex manifold is equivalent to the *integrability* of I, i.e. the existence of a torsion-free connection ∇ that preserves I [38].

There is a canonical eigenvalue decomposition of the operator I on the complexified tangent bundle $T_{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$, where

$$T^{1,0}M = \left\{ v \in T_{\mathbb{C}}M : Iv = \sqrt{-1}v \right\} = \left\{ X - \sqrt{-1}IX : X \in TM \right\},$$

$$T^{0,1}M = \left\{ v \in T_{\mathbb{C}}M : Iv = -\sqrt{-1}v \right\} = \left\{ X + \sqrt{-1}IX : X \in TM \right\}.$$

Observe that $TM \cong T^{1,0}M$ as complex bundles, while $T^{0,1}M$ is the dual of $T^{1,0}M$. We can also define the induced operator $I : T^*M \to T^*M$ on the cotangent bundle by putting $I\Omega(X) := -\Omega(IX)$, and more generally on $\Lambda^n M$ by $I(\Omega_1 \wedge \ldots \wedge \Omega_n) = (I\Omega_1) \wedge \ldots \wedge (I\Omega_n)$. There is a similar decomposition $T^*_{\mathbb{C}}M = T^*M \otimes_{\mathbb{R}} \mathbb{C} = (T^*)^{1,0} M \oplus$

$(T^*)^{0,1} M$, where

$$(T^*)^{1,0} M = \left\{\omega \in T^*_{\mathbb{C}}M : \omega(v) = 0 \ \forall v \in T^{0,1}M\right\} = \left\{\Omega + \sqrt{-1}I\Omega : \Omega \in T^*M\right\},$$

$$(T^*)^{0,1} M = \left\{\omega \in T^*_{\mathbb{C}}M : \omega(v) = 0 \ \forall v \in T^{1,0}M\right\} = \left\{\Omega - \sqrt{-1}I\Omega : \Omega \in T^*M\right\}.$$

The higher differential forms on M can then be decomposed as

$$\Lambda^k_{\mathbb{C}}M = \Lambda^k M \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^{k,0}M \oplus \Lambda^{k-1,1}M \oplus \ldots \Lambda^{1,k-1}M \oplus \Lambda^{0,k}M,$$

where

$$\Lambda^{p,q}M \cong \Lambda^p\left((T^*)^{1,0} M\right) \otimes \Lambda^q\left((T^*)^{0,1} M\right).$$

The (real) exterior derivative operator $d : \Gamma(\Lambda^k M) \to \Gamma(\Lambda^{k+1}M)$ can be extended by $\mathbb{C}$-linearity to $\Lambda^k_{\mathbb{C}}M$, and on the spaces $\Lambda^{p,q}M$ as above, it decomposes as $d = \partial + \bar{\partial}$, where

$$\partial : \Gamma(\Lambda^{p,q}M) \longrightarrow \Gamma(\Lambda^{p+1,q}M), \quad \bar{\partial} : \Gamma(\Lambda^{p,q}M) \longrightarrow \Gamma(\Lambda^{p,q+1}M).$$

We also introduce the differential operator $d^c = \sqrt{-1}\left(\bar{\partial} - \partial\right)$ for convenience. Observe that d^c is a real operator like d, i.e. it takes real forms to real forms.

We now turn our attention to Hermitian structures on complex vector bundles $E \to M$ over a complex manifold M.

Definition 2.1.2. A *Hermitian metric* h on E is a section of $E^* \otimes \bar{E}^*$ which is an inner product over every point in M. In other words, for any $x \in M$, viewing h as a sesquilinear form on E_x, we have:

$$h(u,v) = \overline{h(v,u)} \ \ \forall u, v \in E_x, \quad h(u,u) > 0 \ \ \forall u \neq 0 \in E_x.$$

Given a holomorphic vector bundle E with a Hermitian metric h, it can be shown (see [30], Proposition I.4.9) that there exists a unique $\mathbb{C}$-linear connection ∇^h on E that preserves h and such that for every local holomorphic section s of E, $\nabla^h s$ is of degree $(1,0)$, in the sense that $\nabla^h s$ lies in $\Lambda^{1,0}M \otimes E \subseteq \Lambda^1_{\mathbb{C}}M \otimes E$. This connection is called the *Chern connection* of the Hermitian vector bundle (E,h). It can be shown that the curvature R^h of the Chern connection is of degree $(1,1)$, i.e. taking values in $\Lambda^{1,1}M \otimes \mathrm{End}(E) \subseteq \Lambda^2_{\mathbb{C}}M \otimes \mathrm{End}(E)$.

For the special case that $E = TM$ is the tangent bundle, a Hermitian metric on the complex manifold M is a Riemannian metric g on TM satisfying

$$g(IX, IY) = g(X,Y) \quad \forall X, Y \in \Gamma(TM),$$

where I is the almost complex structure. To see that this is equivalent to the above, observe that given such g, $h(X,Y) \coloneqq g(X,Y) - \sqrt{-1}g(IX,Y)$ is a Hermitian metric on TM in the sense of Definition 2.1.2. Conversely, given a Hermitian metric h on TM viewed as a complex vector bundle, $g \coloneqq \mathrm{Re}(h)$ clearly satisfies the above condition, showing the equivalence of the two definitions. Hermitian metrics always exist, in fact, starting with an arbitrary Riemannian metric g_0 on TM, we can define

$$g(X,Y) \coloneqq g_0(X,Y) + g_0(IX,IY) \quad \forall X, Y \in \Gamma(TM),$$

and this is clearly Hermitian. Observe that, as a Riemannian metric, g induces a (real) bundle isomorphism $TM \cong \Lambda^1 M$. On the other hand, identifying $TM \cong T^{1,0}M$ as complex bundles, h as constructed above induces an isomorphism of complex vector bundles $T^{1,0}M \cong \Lambda^{0,1}M$.

Given an arbitrary Hermitian manifold (M, I, g), there are two canonical connections on its tangent bundle TM:

1. The Levi-Civita connection ∇^{LC} is the unique $\mathbb{R}$-linear connection which preserves the metric tensor g and whose torsion is zero.
2. The Chern connection ∇^{Ch} is the unique $\mathbb{C}$-linear connection which preserves g and whose torsion tensor lies in $\Lambda^{2,0}M \otimes_{\mathbb{C}} T^{1,0}M \subseteq \left(\Lambda^2 M \otimes_{\mathbb{R}} \mathbb{C}\right) \otimes_{\mathbb{C}} T^{1,0}M \cong \Lambda^2 M \otimes_{\mathbb{R}} TM$, where $T^{1,0}M$ and (TM, I) are identified as complex vector bundles. It can be shown (see Proposition I.7.6 in [30]) that this coincides with our previous definition of Chern connection for the Hermitian metric h on TM constructed from g as above.

Associated to each Hermitian metric g on M is its *Hermitian form* $\omega \in \Lambda^2 M$ given by

$$\omega(X, Y) \coloneqq g(IX, Y) \quad \forall X, Y \in \Gamma(TM).$$

It's easy to verify that ω is a non-degenerate real $(1,1)$-form which satisfies the *strict positivity* property:

$$\omega(X, IX) > 0 \quad \forall X \neq 0 \in TM.$$

We are now ready to go ahead with the definition of Kähler and balanced metrics.

Definition 2.1.3. Let (M, I, g) be a Hermitian manifold of complex dimension n. It is called a *Kähler manifold* if its Hermitian form ω is closed; it is called a *balanced manifold* if the weaker condition $d(\omega^{n-1}) = 0$ is satisfied.

It can be shown that the Kähler condition is equivalent to $\nabla^{LC} I = 0$, and also to the vanishing of the torsion tensor T^{Ch} of the Chern connection ∇^{Ch}. On the other hand, observe that T^{Ch} lies in

$$\Lambda^{2,0}M \otimes_{\mathbb{C}} T^{1,0}M \subseteq \Lambda^{1,0}M \otimes_{\mathbb{C}} \left(\Lambda^{1,0}M \otimes_{\mathbb{C}} T^{1,0}M\right) \cong \Lambda^{1,0}M \otimes_{\mathbb{C}} \operatorname{End}_{\mathbb{C}}(T^{1,0}M),$$

and it can be shown (see Theorem 1.6 in [34]) that the vanishing of the $(1,0)$-form obtained by taking the complex trace pairing on $\operatorname{End}_{\mathbb{C}}(T^{1,0}M)$ of the tensor T^{Ch} is equivalent to the balancedness condition on the metric.

Observe that in dimension $\dim_{\mathbb{C}} M = 2$, the balancedness condition is equivalent to the Kähler condition, since in this special case $\omega^{n-1} = \omega$. In general dimension, however, the condition of being Kähler is stronger than that of being balanced. Examples of balanced non-Kähler manifolds are twistor spaces $\mathrm{Tw}(M)$ of certain self-dual Riemannian 4-manifolds M. These are 3-dimensional complex manifolds which encode the conformal structure of M. They are always balanced (see [34], Section 6), but, as shown by Hitchin in [23], the twistor space $\mathrm{Tw}(M)$ is Kähler only if $M = S^4$ or $\mathbb{CP}^2$. In Chapter 3, we will extend the balancedness result to twistor spaces of hyperkähler manifolds (following [28]) and general compact hypercomplex manifolds.

2.2 Hypercomplex and hyperkähler geometry

We now introduce hypercomplex and hyperkähler manifolds and their twistor spaces and state some results in hyperkähler geometry that we will need later on.

Definition 2.2.1. A C^∞ manifold M is called *hypercomplex* if it admits a triple of almost complex structures I, J, K that are integrable and satisfy the quaternionic relations

$$I^2 = J^2 = K^2 = -1, \ IJ = -JI = K.$$

A *hyperhermitian metric* on a hypercomplex manifold M is a Riemannian metric g on TM which is Hermitian with respect to the three complex structures I, J, K. In case the Levi-Civita connection ∇^{LC} of g satisfies $\nabla^{LC} I = \nabla^{LC} J = \nabla^{LC} K = 0$, we call g a *hyperkähler metric*.

Similarly to Hermitian metrics, hyperhermitian metrics always exist: for an arbitrary Riemannian metric g_0 on a hypercomplex manifold M, define $\forall X, Y \in \Gamma(TM)$,

$$g(X,Y) := g_0(X,Y) + g_0(IX,IY) + g_0(JX,JY) + g_0(KX,KY).$$

It's straightforward to verify that this is hyperhermitian. Hyperkähler metrics, on the other hand, are rare, and their existence puts rigid restrictions on the geometry of M. For the condition $\nabla^{LC} I = \nabla^{LC} J = \nabla^{LC} K = 0$ defining a hyperkähler metric, it's not hard to see that in fact one needs to check that only two of I, J, K are parallel: for example, if $\nabla^{LC} J = \nabla^{LC} K = 0$, then $\nabla^{LC} I = \nabla^{LC} JK = (\nabla^{LC} J) K + J (\nabla^{LC} K) = 0$. Letting $\omega_I, \omega_J, \omega_K$ denote the corresponding Kähler forms, since

$$d\omega_J = 0 \iff \nabla^{LC} J = 0, \quad d\omega_K = 0 \iff \nabla^{LC} K = 0,$$

we see that the hyperkähler condition on g is equivalent to the condition that the form

$$\Omega_I = \omega_J + \sqrt{-1}\omega_K.$$

is closed on M. It is straightforward to check that this 2-form is nondegenerate and of type $(2,0)$ with respect to the complex structure I. This shows that a hyperkähler M is a *holomorphic symplectic manifold*. Conversely, the Calabi-Yau theorem [53] implies that every compact Kähler holomorphic symplectic manifold admits a hyperkähler structure. We say that a compact hyperkähler manifold M is *simple* if it is simply connected and satisfies $H^{2,0}(M) = \mathbb{C}$.

For a hypercomplex manifold (M, I, J, K), note that the complex structures I, J, K induce an action of the quaternion algebra $\mathbb{H}$ on the tangent bundle TM, making each tangent space $T_m M$ into a quaternionic vector space; in case M is hyperkähler, this action is moreover parallel with respect to the Levi-Civita connection ∇^{LC}, since I, J and K are. A straightforward verification shows that any combination

$$A = x_1 I + x_2 J + x_3 K,\ \ x_1, x_2, x_3 \in \mathbb{R},\ \ x_1^2 + x_2^2 + x_3^2 = 1,$$

satisfies the relation $A^2 = -1$, and is thus an almost complex structure on M. In fact, it is integrable, since I, J and K are. We call such A the *induced complex structures* of the hypercomplex manifold M. A hyperhermitian metric g on M is easily seen to be Hermitian with respect to each A as above; we let

$$\omega_A(X, Y) := g(AX, Y) \quad \forall X, Y \in \Gamma(TM)$$

denote the corresponding Hermitian form. The hyperkähler condition on g is easily seen to be equivalent to the condition that all such ω_A are closed. Topologically, the set of induced complex structures on a hypercomplex manifold M forms a 2-sphere:

$$S^2 = \left\{A = x_1 I + x_2 J + x_3 K : x_1^2 + x_2^2 + x_3^2 = 1\right\} = \left\{A \in \mathbb{H} : A^2 = -1\right\} \subseteq \operatorname{Im}\mathbb{H}.$$

We would now like to assemble the induced complex structures at each point of M into a single geometrical object.

Definition 2.2.2. Let (M, I, J, K) be a hypercomplex manifold. The product manifold $\operatorname{Tw}(M) = M \times S^2$ is called the *twistor space* of M.

In this definition, we think of S^2 as the set of induced complex structures of M, as above. Identifying the 2-sphere S^2 with the complex projective line $\mathbb{CP}^1$, we can give $\operatorname{Tw}(M) \cong M \times \mathbb{CP}^1$ a natural complex structure. If $I_{\mathbb{CP}^1} : T\mathbb{CP}^1 \to T\mathbb{CP}^1$ denotes the complex structure on $\mathbb{CP}^1$, then for any point $(m, A) \in M \times \mathbb{CP}^1$ we define $\mathcal{I} : T_{(m,A)} \operatorname{Tw}(M) \to T_{(m,A)} \operatorname{Tw}(M)$ as follows:

$$\begin{array}{ccccc} \mathcal{I} & : & T_mM \oplus T_A\mathbb{CP}^1 & \longrightarrow & T_mM \oplus T_A\mathbb{CP}^1. \\ & & (X, V) & \longmapsto & (AX, I_{\mathbb{CP}^1} V) \end{array}$$

It's clear that this defines an almost complex structure on $\operatorname{Tw}(M)$, which is in fact integrable [27], thus making $\operatorname{Tw}(M)$ into a complex manifold of complex dimension $n + 1$, where $\dim_{\mathbb{C}} M = n$. There are canonical projections

$$\begin{array}{ccccc} & & \operatorname{Tw}(M) & & \\ & {\scriptstyle \sigma}\swarrow & & \searrow{\scriptstyle \pi} & \\ M & & & & \mathbb{CP}^1, \end{array}$$

the second of which is a holomorphic map. The fibres of π are copies of M with the corresponding induced complex structures, and it will be useful to think of $\operatorname{Tw}(M)$ as the collection of complex manifolds (M, A) lying above the points $A \in \mathbb{CP}^1$ via the map π. Following this analogy, in the canonical decomposition of the tangent space $T_{(m,A)} \operatorname{Tw}(M) \cong T_mM \oplus T_A\mathbb{CP}^1$, we will call vectors in T_mM *vertical* and vectors in

$T_A\mathbb{CP}^1$ *horizontal*, and similarly for 1-forms. Sections of the map π will be called *twistor lines*, while constant sections

$$\begin{array}{rcccl} s_m & : & \mathbb{CP}^1 & \longrightarrow & \mathrm{Tw}(M) \\ & & A & \longmapsto & (m, A) \end{array}$$

for $m \in M$ will be called horizontal twistor lines. There is a canonical antiholomorphic involution on the twistor space $\mathrm{Tw}(M) = M \times \mathbb{CP}^1$, given by

$$\iota' : \mathrm{id} \times \iota : M \times \mathbb{CP}^1 \longrightarrow M \times \mathbb{CP}^1,$$

where $\iota : \mathbb{CP}^1 \to \mathbb{CP}^1$ is the antipodal map on $\mathbb{CP}^1 \cong S^2$; clearly, we have $\iota \circ \pi = \pi \circ \iota'$. The hypercomplex structure on M can in fact be recovered from the horizontal twistor lines in $\mathrm{Tw}(M)$, which can be completely characterized as sections of the holomorphic projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$ that commute with the antiholomorphic involutions ι, ι' and whose normal bundle is isomorphic to $\mathcal{O}_{\mathbb{CP}^1}(1)^{\oplus n}$ (see [42] for details).

In case M has a hyperhermitian metric (in particular, if it has a hyperkähler metric), there is a natural Hermitian metric defined on the twistor space $\mathrm{Tw}(M)$. Letting g_M denote the hyperhermitian metric on M and $g_{\mathbb{CP}^1}$ the usual Fubini-Study metric on $\mathbb{CP}^1$,

$$g := \sigma^*(g_M) + \pi^*(g_{\mathbb{CP}^1})$$

is easily verified to be a Hermitian metric on $\mathrm{Tw}(M)$; simplifying notation, we will write $g = g_M + g_{\mathbb{CP}^1}$. At a point $(m, A) \in \mathrm{Tw}(M)$, the corresponding Hermitian form

ω decomposes as

$$\omega\left((X,V),(X',V')\right) = \omega_M(X,X') + \omega_{\mathbb{CP}^1}(V,V') = g_M(AX,X') + g_{\mathbb{CP}^1}(I_{\mathbb{CP}^1}V,V'),$$

where $(X,V),(X',V') \in T_mM \oplus T_A\mathbb{CP}^1 = T_{(m,A)}\,\mathrm{Tw}(M)$.

When considering the totality of the induced complex structures on a hypercomplex manifold M, sometimes the particular structures I,J,K no longer play any vital role, in which case we will denote an arbitrary induced complex structure in $\mathbb{CP}^1$ by I. On the other hand, in case the original complex structures I,J,K are important in our discussion, we will use $A \in \mathbb{CP}^1$ to denote an arbitrary induced complex structure. It will be clear from the context which is the case; generally speaking, in Chapter 3 the complex structures I,J,K will appear throughout our calculations, so we'll use the latter notation, while in Chapter 4 we'll use the former. For the rest of this section, we will assume that M is a compact hyperkähler manifold and we will denote an arbitrary induced complex structure on M by I, while the corresponding Kähler manifold (M,I) will be denoted by M_I.

Recall that a hyperkähler manifold M is equipped with a parallel action of the quaternion algebra $\mathbb{H}$ on its tangent bundle. Restricting to the group of unitary quaternions in $\mathbb{H}$, we get an action of $SU(2)$ on TM, hence on all of its tensor bundles, and in particular on the bundle of differential forms Λ^*M. Since the action is parallel, it commutes with the Laplace operator, and thus preserves harmonic forms. Applying Hodge theory, we get a natural action of $SU(2)$ on the cohomology $H^*(M,\mathbb{C})$.

Lemma 2.2.3. *A differential form η over a hyperkähler manifold M is $SU(2)$-invariant if and only if it is of Hodge type (p, p) with respect to all induced complex structures M_I.*

Proof. Proposition 1.2 in [51]. □

Definition 2.2.4. Let M be hyperkähler and I an induced complex structure. We say that I is *generic* with respect to the hyperkähler structure on M if all elements in

$$\bigoplus_p H^{p,p}(M_I) \cap H^{2p}(M, \mathbb{Z}) \subseteq H^*(M, \mathbb{C})$$

are $SU(2)$-invariant.

This terminology is justified: most induced complex structures are generic, in a sense made precise in the following proposition.

Proposition 2.2.5. *Let M be a hyperkähler manifold. The set $S_0 \subseteq S^2$ of generic induced complex structures is dense in S^2 and its complement is countable.*

Proof. Proposition 2.2 in [50]. □

As we will see, the genericity of the complex structure I puts rigid conditions on the geometric structure of the manifold M_I. For instance, all line bundles on M_I have only zero or nowhere vanishing sections (see the proof of Corollary 2.3.14), hence M_I can never be algebraic since it has no effective divisors.

We would now like to give a notion of a vector bundle on M which is simultaneously holomorphic with respect to all the complex structures induced by the hyperkähler structure on M.

Definition 2.2.6. Let M be hyperkähler and let E be a (smooth) complex vector bundle on M. If E admits a connection ∇ whose curvature $R^{\nabla} \in \Gamma(\Lambda^2 M \otimes \operatorname{End}(E))$ is $SU(2)$-invariant, the bundle with connection (E, ∇) is called *autodual*. In case ∇ can be chosen so as to also preserve a Hermitian metric h on E, the bundle (E, h) is called *hyperholomorphic*.

By Lemma 2.2.3, the $SU(2)$-invariance is equivalent to R^{∇} being a section of $\Lambda^{1,1} M_I \otimes E$, for any induced complex structure I. By a version of the Newlander-Nirenberg theorem (see [30], Proposition I.3.7), the $(0,1)$-part $\nabla_I^{0,1}$ of such a connection with respect to I induces a holomorphic structure on E over M_I. In this way, an autodual connection ∇ gives a family of holomorphic vector bundles E_I over the Kähler manifolds M_I, and in case of a hyperholomorphic structure, ∇ is simultaneously the Chern connection for all Hermitian bundles (E_I, h). To assemble these bundles into one object, we use the twistor formalism.

Recall that the twistor space $\operatorname{Tw}(M)$ comes equipped with a (non-holomorphic) projection $\sigma : \operatorname{Tw}(M) \to M$. Given an autodual bundle (E, ∇) on M, we take the pullback bundle and connection $(\sigma^* E, \sigma^* \nabla)$ on $\operatorname{Tw}(M)$. By the considerations in the previous paragraph and the structure of $\operatorname{Tw}(M)$, the curvature of the connection $\sigma^* \nabla$ is of type $(1,1)$, hence the $(0,1)$-part $(\sigma^* \nabla)^{0,1}$ of the connection defines a holomorphic structure on the bundle $\sigma^* E$ over $\operatorname{Tw}(M)$, which we denote by $\operatorname{Tw}(E)$. The correspondence

$$\sigma^* : (E, \nabla) \longmapsto \operatorname{Tw}(E) := (\sigma^* E, (\sigma^* \nabla)^{0,1})$$

defines a functor from the category of autodual bundles over M to the category of holomorphic bundles over $\mathrm{Tw}(M)$, called the *twistor transform*. It turns out that this functor is invertible and its image can be described explicitly.

Theorem 2.2.7. *The twistor transform* $(E, \nabla) \mapsto \mathrm{Tw}(E)$ *defines an equivalence between the categories of autodual bundles on* M *and holomorphic vector bundles on* $\mathrm{Tw}(M)$*, whose restrictions to all horizontal twistor lines are trivial.*

Proof. Theorem 5.12 in [28]. □

We end this section by stating one further technical result, which computes the higher direct images of the twistor transform $\mathrm{Tw}(E)$ of a hyperholomorphic bundle with respect to the holomorphic twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$.

Proposition 2.2.8. *Let* M *be a hyperkähler manifold with holomorphic twistor projection* $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$*,* I *an induced complex structure and* E *a hyperholomorphic bundle on* M *with the corresponding holomorphic bundle* $\mathrm{Tw}(E)$ *on* $\mathrm{Tw}(M)$*. Then for any* $i \geq 0$*,*

$$R^i \pi_* \mathrm{Tw}(E) \cong \mathcal{O}_{\mathbb{CP}^1}(i) \otimes_{\mathbb{C}} H^i(M_I, E_I),$$

where $H^i(M_I, E_I)$ *is the* i*-th holomorphic sheaf cohomology of* E_I *on* M_I*.*

Proof. Proposition 6.3 in [52]. □

Note that for a hyperholomorphic E, the cohomology groups $H^i(M_I, E_I)$ are isomorphic for different $I \in \mathbb{CP}^1$, albeit non-canonically (see Corollary 8.1 in [51]).

2.3 Stability

Let (M, I, g) be a compact Hermitian manifold of complex dimension n and ω its Hermitian form. We will denote by $\mathcal{O}$ the sheaf of holomorphic functions on

M. Our goal in this section is to define the notion of stability for (holomorphic) vector bundles E over M, Hermitian-Einstein metrics and the Kobayashi-Hitchin correspondence.

For a coherent sheaf $\mathcal{F}$ over M, we have the *dual sheaf* $\mathcal{F}^* = \operatorname{Hom}(\mathcal{F}, \mathcal{O})$, as well as a natural morphism into the double dual $\sigma : \mathcal{F} \longrightarrow \mathcal{F}^{**}$. It can be shown that the kernel of σ is the torsion subsheaf of $\mathcal{F}$:

$$\ker \sigma_x = \{a \in \mathcal{F}_x : fa = 0 \text{ for some } f \neq 0 \text{ in } \mathcal{O}_x\} \quad \forall x \in M.$$

Definition 2.3.1. A coherent sheaf $\mathcal{F}$ over M is called *torsion-free* if $\forall x \in M$, the stalk $\mathcal{F}_x$ is a torsion-free $\mathcal{O}_x$-module, or equivalently, if the natural morphism of sheaves

$$\sigma : \mathcal{F} \longrightarrow \mathcal{F}^{**}$$

is injective. If it is an isomorphism, we say that $\mathcal{F}$ is *reflexive*. We call the sheaf $\mathcal{F}$ *normal* if for every open set $U \subseteq M$ and every analytic subset $A \subseteq U$ of codimension at least 2, the restriction map

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(U \setminus A)$$

is an isomorphism.

Clearly, a vector bundle E, viewed as a locally free sheaf, is reflexive (and hence torsion-free). On the other hand, for an arbitrary coherent sheaf $\mathcal{F}$, let

$$S(\mathcal{F}) = \{x \in M : \mathcal{F}_x \text{ is not free over } \mathcal{O}_x\}$$

denote the *singularity set* of $\mathcal{F}$. It can be shown (see Section §1 of Chapter 2 in [41]) that for an arbitrary coherent sheaf this is a closed analytic subset of M of codimension ≥ 1 (≥ 2 for a torsion-free sheaf, ≥ 3 for a reflexive sheaf), so that $\mathcal{F}$ restricted to $M \smallsetminus S(\mathcal{F})$ is locally free. This justifies the following definition.

Definition 2.3.2. The *rank* $\operatorname{rk}\mathcal{F}$ of a coherent sheaf $\mathcal{F}$ over M is the rank of the locally free sheaf

$$\mathcal{F}|_{M\smallsetminus S(\mathcal{F})} \text{ over } M \smallsetminus S(\mathcal{F}).$$

For an arbitrary coherent sheaf $\mathcal{F}$ and any integer $s \geq 0$, we can define the exterior power sheaf $\Lambda^s\mathcal{F}$. If s is the rank of $\mathcal{F}$, then $\Lambda^s\mathcal{F}$ has rank 1, and the *determinant* of $\mathcal{F}$

$$\det\mathcal{F} := (\Lambda^s\mathcal{F})^{**}$$

is actually a line bundle on M, as the following two results show.

Lemma 2.3.3. *The dual of an arbitrary coherent sheaf is reflexive.*

Proof. Proposition V.5.18 in [30]. □

Lemma 2.3.4. *A reflexive sheaf of rank 1 is a line bundle.*

Proof. Lemma 1.1.15 in Chapter 2 of [41]. □

To proceed with the definition of degree of a coherent sheaf on M, we need to impose a certain differential condition on the metic g.

Definition 2.3.5. The metric g is called *Gauduchon* if it satisfies the condition

$$\partial\bar{\partial}\left(\omega^{n-1}\right) = 0.$$

Definition 2.3.6. Let g be Gauduchon. The *degree* of a coherent sheaf $\mathcal{F}$ on M with respect to g is given by

$$\deg_g(\mathcal{F}) = \int_M c_1(\det \mathcal{F}, h) \wedge \omega^{n-1},$$

where h is an arbitrary Hermitian metric on the line bundle $\det \mathcal{F}$, and

$$c_1(\det \mathcal{F}, h) := \frac{\sqrt{-1}}{2\pi} R^h,$$

where $R^h \in \Gamma(\Lambda^{1,1} M)$ is the curvature form of the Chern connection on $(\det \mathcal{F}, h)$. We will write $\deg(\mathcal{F})$ when the metric will be clear from the context.

The Gauduchon condition on g ensures that $\deg_g(\mathcal{F})$ is well-defined and does not depend on the metric h (see Lemma 1.1.18 in [32]). If the metric g satisfies the balancedness condition $d(\omega^{n-1}) = 0$ (in particular, if it is Kähler), then it is clearly Gauduchon, and in fact the degree only depends on the first Chern class $c_1(\det \mathcal{F})$, making it a topological invariant of $\det \mathcal{F}$; for an arbitrary Gauduchon metric it is only a holomorphic invariant of $\det \mathcal{F}$. Note that, for an arbitrary Hermitian metric g on M, the definition of degree does not make sense, however, as shown in the following theorem proved in [19], the conformal class of g always contains a Gauduchon metric, which is essentially unique.

Theorem 2.3.7. *If M is compact, then for every Hermitian metric g on M there exists a positive function $\varphi \in C^\infty(M, \mathbb{R}^{>0})$ such that*

$$g' := \varphi \cdot g$$

is Gauduchon. If M is connected and $n \geq 2$, then g' is unique up to a positive constant.

We are now ready to define stability of torsion-free coherent sheaves on M.

Definition 2.3.8. Let g be a Gauduchon metric on M, and let $\mathcal{F}$ be a nontrivial torsion-free coherent sheaf. The *g-slope* of $\mathcal{F}$ is given by

$$\mu_g(\mathcal{F}) := \frac{\deg_g(\mathcal{F})}{\mathrm{rk}(\mathcal{F})},$$

denoted simply by $\mu(\mathcal{F})$ when the metric is clear from the context. The sheaf $\mathcal{F}$ is called *g-stable* (resp. *g-semi-stable*) if for every subsheaf $\mathcal{G} \subseteq \mathcal{F}$ with $0 < \mathrm{rk}(\mathcal{G}) < \mathrm{rk}(\mathcal{F})$ we have

$$\mu_g(\mathcal{G}) < \mu_g(\mathcal{F}) \ \ (\text{resp. } \mu_g(\mathcal{G}) \leq \mu_g(\mathcal{F}))\,.$$

$\mathcal{F}$ is called *g-polystable* if it is a direct sum of g-stable bundles of the same slope. It is called *irreducible* if it has no proper subsheaves of lower rank.

It's clear that an irreducible $\mathcal{F}$ is stable with respect to any metric on M. It's also clear that for a reflexive sheaf, $\mathcal{F}$ is irreducible if and only $\mathcal{F}^*$ is irreducible.

We would now like to give the definition of Hermitian-Einstein structures on a holomorphic vector bundle E over a Hermitian manifold (M, I, g), a concept which is intimately related to the notion of stability, in a sense that will be made precise later. Recall that the Hermitian structure on M defines a linear operator on the bundle of differential forms of M given by exterior multiplication with the Hermitian form ω of g:

$$\begin{array}{rccc} L_g & : & \Lambda^{p,q}M & \longrightarrow & \Lambda^{p+1,q+1}M \\ & & \alpha & \longmapsto & \alpha \wedge \omega \end{array}$$

We will denote the g-adjoint operator of L_g by $\Lambda_g : \Lambda^{p,q}M \to \Lambda^{p-1,q-1}M$. It can be shown that for a $(1,1)$-form α, $\Lambda_g(\alpha)$ satisfies the following identity:

$$\alpha \wedge \omega^{n-1} = \frac{1}{n}\Lambda_g(\alpha)\omega^n.$$

Definition 2.3.9. A Hermitian metric h on a holomorphic vector bundle E on M is called *g-Hermitian-Einstein* if the curvature $R^h \in \Gamma(\Lambda^{1,1}M \otimes \mathrm{End}(E))$ of its Chern connection satisfies the equation

$$\sqrt{-1}\Lambda_g R^h = \gamma \cdot \mathrm{id}_E,$$

or equivalently

$$\left(\sqrt{-1}R^h\right) \wedge \omega^{n-1} = \frac{\gamma}{n}\omega^n \cdot \mathrm{id}_E,$$

where γ is a real constant, called the *Einstein constant* of h.

In case the metric g is Gauduchon, the Einstein constant is proportional to the degree of the vector bundle E, as the following proposition shows.

Proposition 2.3.10. *If g is Gauduchon and h a Hermitian-Einstein metric on E with Einstein constant γ, then*

$$\gamma = \frac{2\pi}{(n-1)! \cdot \mathrm{Vol}_g(M)} \cdot \mu_g(E),$$

where

$$\mathrm{Vol}_g(M) = \int_M \frac{1}{n!} \cdot \omega^n$$

is the volume of M with respect to the metric g.

Proof. Lemma 2.1.8 in [32]. □

The following theorem is known as the Kobayashi vanishing theorem.

Theorem 2.3.11. *Let* (E, h) *be a* g*-Hermitian-Einstein vector bundle with Einstein constant* γ*. If* γ *is negative, then* E *has no nontrivial global holomorphic sections. If* $\gamma = 0$*, then every global holomorphic section of* E *is parallel with respect to the Chern connection of* (E, h)*.*

Proof. Theorem 2.2.1 in [32]. □

A consequence of the Kobayashi vanishing theorem is that Hermitian-Einstein metrics are essentially unique. Recall that a *simple* holomorphic vector bundle E is one whose only endomorphisms are homotheties, in other words, $\mathrm{Hom}(E, E) = \mathbb{C}$.

Proposition 2.3.12. *If* E *is simple, then a* g*-Hermitian-Einstein metric on* E *(if it exists) is unique up to a positive scalar.*

Proof. Proposition 2.2.2 in [32]. □

For examples of Hermitan-Einstein vector bundles, let M be a hyperkähler manifold. As in the previous section, we denote by g the hyperkähler metric on M and by $S_0 \subseteq S^2 \cong \mathbb{CP}^1$ the set of generic complex structures of M. We have the following lemma.

Lemma 2.3.13. *An* $SU(2)$*-invariant 2-form* β *on a hyperkähler manifold* M *satisfies*

$$\Lambda_I \beta = 0$$

for any induced complex structure I*, where by* Λ_I *we mean the operator* Λ_g *on the manifold* M_I*.*

Proof. Lemma 2.1 in [51]. □

It follows immediately from this lemma that any hyperholomorphic bundle E on a hyperkähler manifold M is Hermitian-Einstein with Einstein constant 0, since its hyperholomorphic connection has $SU(2)$-invariant curvature. Another consequence of the lemma is the following.

Corollary 2.3.14. *Let M be a compact hyperkähler manifold and $\mathrm{Tw}(M)$ its twistor space. The twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$ establishes a one-to-one correspondence between divisors on $\mathbb{CP}^1$ and those on $\mathrm{Tw}(M)$.*

Proof. It suffices to show that the only (irreducible) hypersurfaces on $\mathrm{Tw}(M)$ are the fibres of the twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$. Suppose this is not so, and $V \subseteq \mathrm{Tw}(M)$ is an irreducible hypersurface which is not a fibre of π. Using Remmert's proper mapping theorem (see [21], p. 34), we can conclude that $\pi(V) = \mathbb{CP}^1$, so that V intersects every fibre of π. We can choose a generic structure $I \in S_0$ so that the restriction $V \cap \pi^{-1}(I) = V \cap M_I$ is a divisor on M_I. Letting L be the line bundle corresponding to this divisor, the first Chern class $c_1(L) \in H^{1,1}(M_I) \cap H^2(M, \mathbb{Z})$ is $SU(2)$-invariant by genericity of I. Letting η denote the harmonic form representing $c_1(L)$, it's clear that η is $SU(2)$-invariant as a differential form. By Proposition II.2.23 in [30], there is a Hermitian metric h on L such that $c_1(L, h) = \eta$; in other words,

$$\frac{\sqrt{-1}}{2\pi} R^h = \eta,$$

where R^h is the curvature of ∇^h. Since R^h is $SU(2)$-invariant, (L, h) is hyperholomorphic, and it follows from Lemma 2.3.13 that (L, h) is Hermitian-Einstein with Einstein constant 0. But then by the Kobayashi vanishing theorem (Theorem 2.3.11) all global sections of L are parallel with respect to the Chern connection of h, which implies that they are either globally zero or globally non-vanishing which contradicts the construction of L as the line bundle of an effective divisor on M_I. In fact, the argument shows that M_I has no effective divisors and thus cannot be algebraic. In particular, $V \subseteq \mathrm{Tw}(M)$ as chosen above cannot exist. □

There is an intimate relationship between Hermitian-Einstein structures and stability. The following fundamental theorem, whose proof is the subject of the book [32], shows that the two notions are essentially equivalent.

Theorem 2.3.15. *If g is a Gauduchon metric and E is a holomorphic vector bundle on M, then E admits a Hermitian-Einstein metric if and only if it is polystable. In case the bundle is stable, this metric is unique up to a positive constant.*

This result is called the *Kobayashi-Hitchin correspondence*, after the people who conjectured it, and was proved in increasing generality by various mathematicians, among whom the greatest contributions were by Donaldson [13, 14, 15], Uhlenbeck and Yau [48, 49], Buchdahl [10] and Li and Yau [31].

As a first application of this theorem, we see that a holomorphic line bundle L on M, which is clearly stable in any metric, always admits a Hermitian-Einstein metric h, unique up to constant rescaling. Since the Chern connection stays the same when the metric is multiplied by a constant, we see that for an arbitrary Hermitian metric g on M, we can define the Einstein constant of L with respect to g by

$$\gamma_g(L) := \sqrt{-1}\Lambda_g R^h,$$

where R^h is the curvature of the Chern connection of (L, h). We then have the following result.

Proposition 2.3.16. *Let g' be a Gauduchon metric in the conformal class of g. Then there exists a positive constant c, depending only on g and g', such that*

$$\deg_{g'}(L) = c \cdot \gamma_g(L)$$

for all holomorphic line bundles L on M.

Proof. Let $g' = \varphi \cdot g$, where $\varphi \in C^\infty(M, \mathbb{R}^{>0})$. Then $\omega' = \varphi \cdot \omega$ for the corresponding Hermitian forms. Let h be a g-Hermitian-Einstein metric on L, and let R^h denote the curvature of the Chern connection of (L, h). We have

$$\deg_{g'}(L) = \int_M c_1(L,h) \wedge (\omega')^{n-1} = \frac{1}{2\pi} \int_M \sqrt{-1} R^h \wedge (\varphi \cdot \omega)^{n-1} =$$

$$= \frac{1}{2\pi n} \int_M \varphi^{n-1} (\sqrt{-1} \Lambda_g R_h) \omega^n = \frac{1}{2\pi n} \left(\int_M \varphi^{n-1} \omega^n \right) \cdot \gamma_g(L).$$

□

We close this section with one further notion of stability which is defined for a hyperkähler manifold M. Recall that M comes equipped with a twistor space $\mathrm{Tw}(M)$ and a holomorphic projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$, whose fibres parametrize the totality of the induced Kähler structures M_I on M.

Definition 2.3.17. A holomorphic vector bundle E on $\mathrm{Tw}(M)$ is called *fibrewise stable* if its restriction E_I to M_I is stable in the induced Kähler metric for each $I \in \mathbb{CP}^1$. E is called *generically fibrewise stable* if E_I is stable for all I in a nonempty Zariski open subset of $\mathbb{CP}^1$. Similarly, E is called *fibrewise simple* if all the restrictions E_I are simple, in the sense that $\mathrm{Hom}(E_I, E_I) = \mathbb{C}$, and *generically fibrewise simple* if E_I is simple for all I in a nonempty Zariski open subset of $\mathbb{CP}^1$.

CHAPTER 3
Balanced metrics on twistor spaces

Recall that the twistor space $\mathrm{Tw}(M)$ of a hyperkähler manifold M comes equipped with a natural Hermitian metric induced from the hyperkähler metric on M and the Fubini-Study metric on $\mathbb{CP}^1$. One would like to hope that this metric on $\mathrm{Tw}(M)$ is Kähler, but this need not be so (Corollary 3.2.4). However, as shown by Kaledin and Verbitsky [28], the metric on $\mathrm{Tw}(M)$ satisfies the weaker condition of being balanced. Thus, in a certain sense, there is a loss of metric structure when passing from M to $\mathrm{Tw}(M)$. In view of this, one would not think that for general hypercomplex M without any ambient metric, $\mathrm{Tw}(M)$ should enjoy any interesting metric properties. The surprising result presented in this chapter is that in fact $\mathrm{Tw}(M)$ is balanced for a general compact hypercomplex manifold M, showing that no metric assumptions on M are needed for the balancedness of $\mathrm{Tw}(M)$.

In Section 3.1, we present the argument of Kaledin and Verbitsky (perhaps in a bit more detail than in their original article [28]) that for a hyperkähler M, the induced metric on the twistor space $\mathrm{Tw}(M)$ is balanced. In Section 3.2, we show that the twistor space $\mathrm{Tw}(M)$ of a general compact hypercomplex manifold M admits a balanced metric. In this case, the construction of the metric is implicit: it is obtained by taking the "n^{th} root" of a certain closed strictly positive (n,n)-form on the twistor space $\mathrm{Tw}(M)$, where $n = \dim_{\mathbb{C}} M$. The content of this chapter is largely identical to the material in the article [47] by the author of the present thesis.

3.1 The hyperkähler case

Recall that for a hypercomplex manifold (M, I, J, K), the twistor space $\mathrm{Tw}(M)$ comes equipped with the natural projections

$$\begin{array}{ccc} & \mathrm{Tw}(M) & \\ {}^{\sigma}\swarrow & & \searrow^{\pi} \\ M & & \mathbb{CP}^1, \end{array}$$

and at a point $(m, A) \in \mathrm{Tw}(M)$, the tangent space decomposes as a direct sum $T_{(m,A)} \mathrm{Tw}(M) = T_m M \oplus T_A \mathbb{CP}^1$; we call vectors in $T_m M$ vertical and vectors in $T_A \mathbb{CP}^1$ horizontal, and similarly for the cotangent bundle. If g_M is a hyperkähler metric on M, then

$$g := \sigma^* (g_M) + \pi^* (g_{\mathbb{CP}^1})$$

is a Hermitian metric on $\mathrm{Tw}(M)$, where $g_{\mathbb{CP}^1}$ is the Fubini-Study metric on $\mathbb{CP}^1$; we write simply $g = g_M + g_{\mathbb{CP}^1}$. Similarly the Hermitian form ω of g on $\mathrm{Tw}(M)$ decomposes as

$$\omega = \omega_M + \omega_{\mathbb{CP}^1},$$

where $\omega_M \in \Lambda^2_m M$ and $\omega_{\mathbb{CP}^1} \in \Lambda^2_A \mathbb{CP}^1$ at the point $(m, A) \in \mathrm{Tw}(M)$; there is no component in $\Lambda^1_m M \otimes \Lambda^1_A \mathbb{CP}^1$. Note that $\omega_{\mathbb{CP}^1}$ is just the pullback of the Fubini-Study form on $\mathbb{CP}^1$ via the map π, while ω_M is not the pullback of any single form from M but rather it is assembled from all the Kähler forms ω_A of g_M on M for the various induced complex structures $A \in \mathbb{CP}^1$. More explicitly, for $(X, V), (X', V') \in T_m M \oplus T_A \mathbb{CP}^1 = T_{(m,A)} \mathrm{Tw}(M)$, we have:

$$\omega\left((X, V), (X', V')\right) = \omega_M(X, X') + \omega_{\mathbb{CP}^1}(V, V') = g_M(AX, X') + g_{\mathbb{CP}^1}(I_{\mathbb{CP}^1} V, V').$$

Theorem 3.1.1. *(Kaledin-Verbitsky) Let* (M, I, J, K, g_M) *be a hyperkähler manifold of complex dimension* n*. Then its twistor space* $\mathrm{Tw}(M)$ *with the Hermitian metric induced from the hyperkähler structure is balanced.*

Proof. We closely follow the argument laid out in Section 4.4 of [28]. In the notation used above, we need to show that $d(\omega^n) = 0$. This is clearly equivalent to showing

$$\omega^{n-1} \wedge d\omega = 0.$$

Observe that we have a decomposition of the differential operator $d = d_M + d_{\mathbb{CP}^1}$ according to the direct sum $T\,\mathrm{Tw}(M) = TM \oplus T\mathbb{CP}^1$. Since $\omega = \omega_M + \omega_{\mathbb{CP}^1}$, we have

$$d\omega = d_M\omega_M + d_{\mathbb{CP}^1}\omega_M + d_M\omega_{\mathbb{CP}^1} + d_{\mathbb{CP}^1}\omega_{\mathbb{CP}^1}.$$

The first term is zero by the hyperkähler condition on M, while the last two terms are zero because $\omega_{\mathbb{CP}^1}$ is a pullback of a closed form on $\mathbb{CP}^1$ to $\mathrm{Tw}(M)$. We need to investigate the second term. To simplify our argument, we will work over a fixed horizontal twistor line $\{m\} \times \mathbb{CP}^1 \subseteq \mathrm{Tw}(M)$.

Let

$$W := \mathrm{Im}\,\mathbb{H} = \{aI + bJ + cK\} \cong \mathbb{R}^3,$$

and let $\mathcal{W} = \mathbb{CP}^1 \times W$ be the corresponding trivial bundle. When we view $\mathbb{CP}^1$ as a parametrization of the complex structures on M, it's just the unit sphere $S^2 \subseteq W$, hence we can view $\mathcal{W}$ as the restriction $\mathcal{W} = TW|_{S^2}$. There is a canonical embedding of $\mathcal{W}$ into the (trivial) bundle of vertical 2-forms over the horizontal line $\{m\}\times\mathbb{CP}^1$:

$$\begin{aligned} \mathcal{W}=\mathbb{CP}^1\times W &\longrightarrow \{m\}\times\mathbb{CP}^1\times\Lambda^2_m M \\ (A, aI+bJ+cK) &\longmapsto (m, A, a\omega_I+b\omega_J+c\omega_K). \end{aligned}$$

Given an element $V=aI+bJ+cK$ of W, we denote by $\omega_V=a\omega_I+b\omega_J+c\omega_K$ its image under this mapping. In this way, we can think of $\mathcal{W}$ as a bundle of vertical 2-forms over $\{m\}\times\mathbb{CP}^1$ with global frame $\{\omega_I,\omega_J,\omega_K\}$. Since $d_{\mathbb{CP}^1}\omega_I=d_{\mathbb{CP}^1}\omega_J=d_{\mathbb{CP}^1}\omega_K=0$, we can think of the operator $d_{\mathbb{CP}^1}$ on $\mathcal{W}$ as a flat connection

$$\begin{aligned} d_{\mathbb{CP}^1}=\nabla:\Gamma(\mathcal{W}) &\longrightarrow \Gamma\left(\Lambda^1\mathbb{CP}^1\otimes\mathcal{W}\right) \\ f_1\omega_I+f_2\omega_J+f_3\omega_K &\longmapsto df_1\otimes\omega_I+df_2\otimes\omega_J+df_3\otimes\omega_K. \end{aligned}$$

Of course, this is just the usual Euclidean connection on $\mathbb{R}^3\cong\operatorname{Im}\mathbb{H}$ restricted to $S^2\cong\mathbb{CP}^1$. Note that $\mathcal{W}=TW|_{S^2}\cong T\mathbb{R}^3|_{S^2}=N\oplus TS^2$, where N is the normal bundle of the embedding $S^2\subseteq\mathbb{R}^3$ and TS^2 is the tangent bundle. At the point $A=(a_1,a_2,a_3)\in S^2\cong\mathbb{CP}^1$, we have

$$N_A=\{\lambda\, a_1\omega_I+\lambda\, a_2\omega_J+\lambda\, a_3\omega_K:\lambda\in\mathbb{R}\},$$

$$T_AS^2=\{v_1\,\omega_I+v_2\,\omega_J+v_3\,\omega_K:a_1v_1+a_2v_2+a_3v_3=0\}.$$

Thus, N is a trivial bundle with a global trivialization given by $\omega_M=x_1\omega_I+x_2\omega_J+x_3\omega_K$, while the almost complex structure $I_{\mathbb{CP}^1}:T\mathbb{CP}^1\to T\mathbb{CP}^1$ at the point $A\in\mathbb{CP}^1$ is given by the quaternion multiplication $V\mapsto AV$, where we once again think of $A\in N_A$, $V\in T_AS^2$ as elements of W. We want to compute

$$d_{\mathbb{CP}^1}\omega_M=\nabla\left(x_1\omega_I+x_2\omega_J+x_3\omega_K\right)=dx_1\otimes\omega_I+dx_2\otimes\omega_J+dx_3\otimes\omega_K.$$

Fix a point $A = (a_1, a_2, a_3)$ in $\{m\} \times \mathbb{CP}^1$ and look at the decomposition $d_{\mathbb{CP}^1}\omega_M = \partial_{\mathbb{CP}^1}\omega_M + \bar{\partial}_{\mathbb{CP}^1}\omega_M$. We claim that $\bar{\partial}_{\mathbb{CP}^1}\omega_M \in \Gamma\left(\Lambda^{0,1}\mathbb{CP}^1 \otimes \Lambda^{2,0}_m M\right)$, where the complex structure on $T_m M$ is understood to be A. To verify this, we use the description of $d_{\mathbb{CP}^1}$ as the connection ∇ and plug in an arbitrary vector $V + \sqrt{-1}I_{\mathbb{CP}^1}V = V + \sqrt{-1}AV \in T^{0,1}_A\mathbb{CP}^1$, where $V = (v_1, v_2, v_3) \in T_A\mathbb{CP}^1$ is real.

$$\nabla_{V+\sqrt{-1}AV}\omega_M = v_1\omega_I + v_2\omega_J + v_3\omega_K +$$

$$+\sqrt{-1}(a_2v_3 - a_3v_2)\omega_I + \sqrt{-1}(a_3v_1 - a_1v_3)\omega_J + \sqrt{-1}(a_1v_2 - a_2v_1)\omega_K =$$

$$= \omega_V + \sqrt{-1}\omega_{AV}.$$

Plugging into this form an arbitrary vector $X \in T_m M$ and a (0,1)-vector $Y \in T^{0,1}_m M$ (with respect to the complex structure A), we get

$$\omega_V(X,Y) + \sqrt{-1}\omega_{AV}(X,Y) = g(VX,Y) + \sqrt{-1}g(AVX,Y) =$$

$$= g(VX,Y) + \sqrt{-1}g(A(AV)X, AY) = g(VX,Y) + \sqrt{-1}g(-VX, -\sqrt{-1}Y) = 0.$$

Hence $\bar{\partial}_{\mathbb{CP}^1}\omega_M \in \Gamma\left(\Lambda^{0,1}\mathbb{CP}^1 \otimes \Lambda^{2,0}_m M\right)$ and $\partial_{\mathbb{CP}^1}\omega_M \in \Gamma\left(\Lambda^{1,0}\mathbb{CP}^1 \otimes \Lambda^{0,2}_m M\right)$, since ω_M is real and $\bar{\partial}_{\mathbb{CP}^1}\omega_M$ is the conjugate of $\partial_{\mathbb{CP}^1}\omega_M$.

We now examine the form $\omega^{n-1} \wedge d\omega$. We have

$$\omega^{n-1} \wedge d\omega = \left(\omega_M + \omega_{\mathbb{CP}^1}\right)^{n-1} \wedge d_{\mathbb{CP}^1}\omega_M = \omega_M^{n-1} \wedge \partial_{\mathbb{CP}^1}\omega_M + \omega_M^{n-1} \wedge \bar{\partial}_{\mathbb{CP}^1}\omega_M +$$

$$+(n-1)\omega_M^{n-2} \wedge \omega_{\mathbb{CP}^1} \wedge \partial_{\mathbb{CP}^1}\omega_M + (n-1)\omega_M^{n-2} \wedge \omega_{\mathbb{CP}^1} \wedge \bar{\partial}_{\mathbb{CP}^1}\omega_M.$$

Since $\omega_M^{n-1} \in \Lambda^{n-1,n-1}_m M$, the vertical bidegree of the first two terms is $(n-1, n+1)$, $(n+1, n-1)$, respectively, making them zero, since $\dim_{\mathbb{C}} M = n$. On the other hand, the degree of the horizontal part of the last two terms is $3 > 2 = \dim_{\mathbb{R}} \mathbb{CP}^1$, making them zero as well.

□

3.2 The hypercomplex case

We now would like to prove a generalization of Theorem 3.1.1 for compact hyperhermitian (and hence general hypercomplex) manifolds M. In contrast to the hyperkähler case, the product metric on $\mathrm{Tw}(M) = M \times \mathbb{CP}^1$ need not be balanced, so we need to approach the problem differently. We start with two lemmas of an essentially linear-algebraic nature. Recall from Section 2.1 that a real (1,1)-form η on a complex manifold (M, I) of complex dimension n is strictly positive if it satisfies the condition $\eta(X, IX) > 0$ for all nonzero $X \in TM$. Similarly, we say that a real $(n-1, n-1)$-form η is strictly positive if for any nonzero $\alpha \in \Lambda^1 M$ we have that $\eta \wedge \alpha \wedge I\alpha$ is a strictly positive multiple of (any) volume form on M compatible with the orientation determined by the complex structure. There is an intimate relationship between closed strictly positive $(n-1, n-1)$-forms on M and balanced metrics.

Lemma 3.2.1. *Let (M, I, g) be a Hermitian manifold of $\dim_{\mathbb{C}} M = n$. The existence of a closed strictly positive $(n-1, n-1)$-form on M is equivalent to the balancedness of M, not necessarily with respect to the given metric.*

Proof. (Cf. [34], pp. 279-280) Let $\eta \in \Gamma(\Lambda^{n-1,n-1} M)$ be a closed strictly positive form. The Riemannian volume form $\Omega \in \Gamma(\Lambda^{2n} M)$ induces an isomorphism of bundles $\Lambda^{n-1,n-1} M \cong \Lambda^{1,1} TM \cong T^{1,0} M \otimes T^{0,1} M$, whereas the metric g gives an isomorphism $\Lambda^{1,1} TM \cong \Lambda^{1,1} M$. Under these identifications, η can be thought of as a strictly positive $(1,1)$-form on M. By basic linear algebra, there exists a local orthonormal frame $\{e_1, Ie_1, \ldots, e_n, Ie_n\}$ of TM, such that $\eta \in \Gamma(\Lambda^{1,1}M)$ can be expressed as

$$\eta = \sum_{i=1}^{n} a_i\, e_i \wedge Ie_i,$$

where we think of e_i as sections of $\Lambda^1 M \cong TM$ and all $a_i > 0$. Since $\Omega = e_1 \wedge Ie_1 \wedge \ldots \wedge e_n \wedge Ie_n$, we have that, as a section of $\Lambda^{n-1,n-1}M$, η can be expressed in terms of this frame as

$$\eta = \sum_{i=1}^{n} a_i\, e_1 \wedge Ie_1 \wedge \ldots \wedge \widehat{e_i \wedge Ie_i} \wedge \ldots \wedge e_n \wedge Ie_n.$$

We are now looking for a strictly positive form $\omega \in \Gamma(\Lambda^{1,1}M)$ such that $\omega^{n-1} = \eta$. If we can establish the existence of such a form, our proof will be finished, since the condition $d\left(\omega^{n-1}\right) = 0$ will imply that the Hermitian metric on M induced by ω is balanced. If we write

$$\omega = \sum_{i=1}^{n} b_i\, e_i \wedge Ie_i,$$

we then have

$$\omega^{n-1} = \sum_{i=1}^{n} (n-1)!\, b_1 \ldots \widehat{b_i} \ldots b_n\, e_1 \wedge Ie_1 \wedge \ldots \wedge \widehat{e_i \wedge Ie_i} \wedge \ldots \wedge e_n \wedge Ie_n.$$

If $\omega^{n-1} = \eta$, observe that

$$\frac{a_i}{a_j} = \frac{(n-1)!\, b_1 \ldots \widehat{b_i} \ldots b_n}{(n-1)!\, b_1 \ldots \widehat{b_j} \ldots b_n} = \frac{b_j}{b_i}.$$

Writing

$$a_1 = (n-1)!\, b_2 \ldots b_n = (n-1)!\, \frac{b_2}{b_1} \ldots \frac{b_n}{b_1} \cdot b_1^{n-1} = (n-1)!\, \frac{a_1}{a_2} \ldots \frac{a_1}{a_n} b_1^{n-1},$$

we can solve for b_1 uniquely, since we know that $b_1 > 0$ and all the $a_i > 0$. Knowing b_1 clearly gives us all the other b_i. This shows that ω exists locally, while its global existence is a consequence of its uniqueness. □

Lemma 3.2.2. *Let (M, I) be a compact complex manifold and suppose that its tangent space TM decomposes into a direct sum $TM = E \oplus F$ of complex subbundles E and F. If ω, ω' are real (1,1)-forms on M such that ω is strictly positive when restricted to E, while ω' is strictly positive on F and $E \subseteq \ker \omega'$, there exists a number $T > 0$ such that $\omega + T\omega'$ is strictly positive on M.*

Proof. The problem is local in nature by compactness of M, since if $\{U_i\}$ is a cover of M such that $\omega + T_i\omega'$ is strictly positive on U_i, then taking a finite subcover and letting T be the maximum of the corresponding T_i's, we get a strictly positive form $\omega + T\omega'$ on the whole M.

Let $\omega = \omega_1 + \omega_2 + \omega_3$ be the decomposition of ω according to the direct sum

$$\Lambda^2 (E^* \oplus F^*) = \Lambda^2(E^*) \oplus (E^* \otimes F^*) \oplus \Lambda^2(F^*),$$

and observe that ω' lies entirely in the third summand. By assumption of strict positivity, ω_1 is a Hermitian form on E, hence comes from a Hermitian metric. Choosing a local orthonormal frame $\{e_1, Ie_1, \ldots, e_k, Ie_k\}$ for this metric, we can express ω_1 as

$$\omega_1 = \sum_{i=1}^{k} e_i \wedge Ie_i,$$

where we regard the e_i as sections of $E^* \cong E$. Similarly, ω' is a Hermitian form on F induced by some Hermitian metric. By simple linear algebra, there exists a local orthonormal frame $\{f_1, If_1, \ldots, f_l, If_l\}$ of F in which the two forms decompose as

$$\omega_3 = \sum_{j=1}^{l} a_j\, f_j \wedge If_j,\ \omega' = \sum_{j=1}^{l} f_j \wedge If_j,$$

where again we regard f_j as sections of $F^* \cong F$. Clearly, we can choose $T > 0$ such that on some neighborhood, $\omega_3 + T\omega'$ is strictly positive on F. This makes $\omega + T\omega'$ locally strictly positive when restricted to both E and F, so we only need to take care of the ω_2 term. For this, it is enough to show that we can choose T such that $\omega_1 + \omega_2 + T\omega'$ is locally strictly positive. Let

$$X = \sum_{i=1}^{k} \left(X_{2i-1}e_i + X_{2i}Ie_i\right),\ Y = \sum_{j=1}^{l} \left(Y_{2j-1}f_j + Y_{2j}If_j\right)$$

be arbitrary nonvanishing sections of E, F written in the above bases and let $t > 0$. We want to show that plugging in $(X + tY, I(X + tY))$ into the above form always gives a strictly positive number:

$$\omega_1(X, IX) + \omega_2(X, tIY) + \omega_2(tY, IX) + T\omega'(tY, tIY) > 0,$$

$$\omega_1(X, IX) + 2t\,\omega_2(X, IY) + t^2\, T\omega'(Y, IY) > 0.$$

Thinking of this as a quadratic equation in t, its strict positivity is equivalent to the discriminant being negative:

$$4\,\omega_2(X, IY)^2 - 4T\,\omega_1(X, IX)\omega'(Y, IY) < 0,$$

$$\omega_2(X, IY)^2 < T\,\omega_1(X, IX)\omega'(Y, IY).$$

Writing out the right hand side in the bases $\{e_i, Ie_i\}$, $\{f_j, If_j\}$, we get

$$T\left(\sum_{i=1}^{2k} X_i^2\right)\left(\sum_{j=1}^{2l} Y_j^2\right),$$

whereas

$$\omega_2(X, IY) = \sum_{i,j} c_{ij} X_i Y_j,$$

for some coefficients c_{ij}. Applying the Cauchy-Schwarz inequality to $\omega_2(X, IY)^2$, we get

$$\left(\sum_{i,j} c_{ij} X_i Y_j\right)^2 \leq \sum_{i,j} (c_{ij})^2 \sum_{i,j} X_i^2 Y_j^2 = \sum_{i,j} (c_{ij})^2 \left(\sum_{i=1}^{2k} X_i^2\right)\left(\sum_{j=1}^{2l} Y_j^2\right).$$

The sum $\sum_{i,j} (c_{ij})^2$ is clearly locally bounded by some $T > 0$, which gives the required inequality. □

Now let (M, I, J, K, g_M) be a hyperhermitian manifold, let n be the complex dimension of M, and let g denote the induced Hermitian metric on the twistor space $\mathrm{Tw}(M)$ with Hermitian form ω, following the notation in the previous section. If g_M is hyperkähler, the argument in the proof that $\mathrm{Tw}(M)$ is balanced in Theorem 3.1.1 consisted in showing that the (n,n)-form

$$\omega^n = (\omega_M + \omega_{\mathbb{CP}^1})^n = \omega_M^n + n\omega_M^{n-1} \wedge \omega_{\mathbb{CP}^1}$$

on $\mathrm{Tw}(M)$ is closed. For a general hyperhermitian g_M we will instead use Lemma 3.2.2 to show that a certain linear combination of forms

$$\alpha\, \omega_M^n + \beta\, dd^c \left(\omega_M^{n-1}\right)$$

is a closed strictly positive (n,n)-form on $\mathrm{Tw}(M)$, and then use Lemma 3.2.1 to conclude that $\mathrm{Tw}(M)$ is balanced. We will need compactness of M in order to apply Lemma 3.2.2.

Theorem 3.2.3. *Let (M, I, J, K, g_M) be a compact hyperhermitian manifold of complex dimension n. Then its twistor space $\mathrm{Tw}(M)$ is balanced.*

Proof. The volume form on $\mathrm{Tw}(M) = M \times \mathbb{CP}^1$ with respect to the product metric is given by

$$\Omega_{\mathrm{Tw}(M)} = \frac{\omega^{n+1}}{(n+1)!} = \frac{(\omega_M + \omega_{\mathbb{CP}^1})^{n+1}}{(n+1)!} = \frac{(n+1)\omega_M^n \wedge \omega_{\mathbb{CP}^1}}{(n+1)!} = \Omega_M \wedge \omega_{\mathbb{CP}^1},$$

where Ω_M denotes the pullback of the volume form on M via the projection $\sigma : \mathrm{Tw}(M) \to M$. Note that $\omega_M^n = n!\,\Omega_M$ and $dd^c(\omega_M^{n-1})$ are closed (n,n)-forms on $\mathrm{Tw}(M)$ and we can think of them as elements of $\Lambda^{1,1} T\,\mathrm{Tw}(M)$ via the isomorphism induced by the volume form $\Omega_{\mathrm{Tw}(M)}$. Because the metric on $\mathrm{Tw}(M)$ induces an isomorphism $T\,\mathrm{Tw}(M) \cong \Lambda^1\,\mathrm{Tw}(M)$, we will be able to apply Lemma 3.2.2 if we can show that ω_M^n is strictly positive on horizontal forms and vertical forms lie in its kernel, while $\pm dd^c(\omega_M^{n-1})$ (the sign will depend on the dimension of M) is strictly positive when restricted to vertical forms in $\Lambda^1 M$. The first statement is easy, since ω_M^n is a constant multiple of the vertical volume form Ω_M, and we know that $\Omega_{\mathrm{Tw}(M)} = \Omega_M \wedge \omega_{\mathbb{CP}^1}$. For the second statement, since we only need to establish strict positivity on vertical forms, it's enough to restrict to a horizontal twistor line $\{m\} \times \mathbb{CP}^1$ and consider the form

$$d_{\mathbb{CP}^1} d^c_{\mathbb{CP}^1}\left(\omega_M^{n-1}\right) = 2\sqrt{-1}(n-1)\left(\partial_{\mathbb{CP}^1}\bar{\partial}_{\mathbb{CP}^1}\omega_M\right) \wedge \omega_M^{n-2} +$$

$$+2\sqrt{-1}(n-1)(n-2)\left(\partial_{\mathbb{CP}^1}\omega_M\right)\wedge\left(\bar{\partial}_{\mathbb{CP}^1}\omega_M\right)\wedge\omega_M^{n-3}.$$

Note that if $n = 2$, the second term vanishes. We will now use our description of the $d_{\mathbb{CP}^1}$ operator as a connection from the proof of Theorem 3.1.1 to show that both of these terms are multiples of $\omega_{\mathbb{CP}^1}\wedge\omega_M^{n-1}$, which is strictly positive since products of positive forms are positive (see [12], Section III.1).

$$\partial_{\mathbb{CP}^1}\bar{\partial}_{\mathbb{CP}^1}\omega_M = \nabla^{1,0}\nabla^{0,1}\omega_M = \partial\bar{\partial}x_1\otimes\omega_I + \partial\bar{\partial}x_2\otimes\omega_J + \partial\bar{\partial}x_3\otimes\omega_K.$$

$$\partial_{\mathbb{CP}^1}\omega_M = \nabla^{1,0}\omega_M = \partial x_1\otimes\omega_I + \partial x_2\otimes\omega_J + \partial x_3\otimes\omega_K.$$

$$\bar{\partial}_{\mathbb{CP}^1}\omega_M = \nabla^{0,1}\omega_M = \bar{\partial}x_1\otimes\omega_I + \bar{\partial}x_2\otimes\omega_J + \bar{\partial}x_3\otimes\omega_K.$$

We will work in the local holomorphic coordinates coming from the stereographic projections on $\mathbb{CP}^1\cong S^2$. Recall that the sphere $S^2 = \left\{x_1^2+x_2^2+x_3^2 = 1\right\}$ has holomorphic charts

$$\begin{array}{cccccccc}
P_N : S^2\smallsetminus\{(0,0,1)\} & \longleftrightarrow & \mathbb{C} & & P_S : S^2\smallsetminus\{(0,0,-1)\} & \longleftrightarrow & \mathbb{C} \\
(x_1,x_2,x_3) & \longmapsto & \frac{x_1-\sqrt{-1}x_2}{1-x_3} & & (x_1,x_2,x_3) & \longmapsto & \frac{x_1+\sqrt{-1}x_2}{1+x_3} \\
\left(\frac{z+\bar{z}}{1+|z|^2},\frac{\sqrt{-1}(z-\bar{z})}{1+|z|^2},\frac{-1+|z|^2}{1+|z|^2}\right) & \longleftarrow & z & & \left(\frac{w+\bar{w}}{1+|w|^2},\frac{\sqrt{-1}(\bar{w}-w)}{1+|w|^2},\frac{1-|w|^2}{1+|w|^2}\right) & \longleftarrow & w
\end{array}$$

We will make the computation in the holomorphic coordinate $z = x+\sqrt{-1}y$ coming from the stereographic projection P_N from the point $(1,0,0)$. The computation in the other chart is completely analogous, and we will omit it. The Fubini-Study metric $\omega_{\mathbb{CP}^1}$ takes the form

$$\omega_{\mathbb{CP}^1} = \sqrt{-1}\partial\bar{\partial}\log\left(1+|z|^2\right) = \sqrt{-1}\partial\left(\frac{z\,d\bar{z}}{1+|z|^2}\right) = \frac{\sqrt{-1}dz\wedge d\bar{z}}{\left(1+|z|^2\right)^2}.$$

Calculating the various partial derivatives of x_1, x_2, x_3, we get

$$\partial x_1 = \partial\left(\frac{z+\bar{z}}{1+|z|^2}\right) = \frac{1-\bar{z}^2}{(1+|z|^2)^2}\,dz, \qquad \bar{\partial}x_1 = \bar{\partial}\left(\frac{z+\bar{z}}{1+|z|^2}\right) = \frac{1-z^2}{(1+|z|^2)^2}\,d\bar{z},$$
$$\partial x_2 = \partial\left(\frac{\sqrt{-1}(z-\bar{z})}{1+|z|^2}\right) = \frac{\sqrt{-1}(1+\bar{z}^2)}{(1+|z|^2)^2}\,dz, \quad \bar{\partial}x_2 = \bar{\partial}\left(\frac{\sqrt{-1}(z-\bar{z})}{1+|z|^2}\right) = \frac{-\sqrt{-1}(1+z^2)}{(1+|z|^2)^2}\,d\bar{z},$$
$$\partial x_3 = \partial\left(\frac{-1+|z|^2}{1+|z|^2}\right) = \frac{2\bar{z}}{(1+|z|^2)^2}\,dz, \qquad \bar{\partial}x_3 = \bar{\partial}\left(\frac{-1+|z|^2}{1+|z|^2}\right) = \frac{2z}{(1+|z|^2)^2}\,d\bar{z},$$

$$\partial\bar{\partial}x_1 = \frac{-2(z+\bar{z})\,dz\wedge d\bar{z}}{\left(1+|z|^2\right)^3},$$
$$\partial\bar{\partial}x_2 = \frac{-2\sqrt{-1}(z-\bar{z})\,dz\wedge d\bar{z}}{\left(1+|z|^2\right)^3},$$
$$\partial\bar{\partial}x_3 = \frac{-2\left(-1+|z|^2\right)\,dz\wedge d\bar{z}}{\left(1+|z|^2\right)^3}.$$

Thus,

$$\sqrt{-1}\partial_{\mathbb{CP}^1}\bar{\partial}_{\mathbb{CP}^1}\omega_M = -2\left(\frac{\sqrt{-1}dz\wedge d\bar{z}}{\left(1+|z|^2\right)^2}\right)\otimes\left(\frac{z+\bar{z}}{1+|z|^2}\,\omega_I + \frac{\sqrt{-1}(z-\bar{z})}{1+|z|^2}\,\omega_J + \frac{-1+|z|^2}{1+|z|^2}\,\omega_K\right) =$$
$$= -2\,\omega_{\mathbb{CP}^1}\wedge\omega_M,$$

from which we conclude that

$$2\sqrt{-1}(n-1)\left(\partial_{\mathbb{CP}^1}\bar{\partial}_{\mathbb{CP}^1}\omega_M\right)\wedge\omega_M^{n-2} = -4(n-1)\omega_{\mathbb{CP}^1}\wedge\omega_M^{n-1}.$$

If $n = 2$, then, as we noted above, this is equal to $d_{\mathbb{CP}^1}d^c_{\mathbb{CP}^1}\left(\omega_M^{n-1}\right)$, so taking the negative of $dd^c\left(\omega_M^{n-1}\right)$ gives a form that is strictly positive on vertical 1-forms, and we can apply Lemma 3.2.2 to conclude that $\exists\, T > 0$ such that

$$T\,\omega_M^n - dd^c\left(\omega_M^{n-1}\right)$$

is strictly positive. For the case $n > 2$, we also need to examine the other term. We know that at any point $A \in \mathbb{CP}^1$, for any $V \in T_A\mathbb{CP}^1$,

$$\partial_{\mathbb{CP}^1}\omega_M(V - \sqrt{-1}AV) = \omega_V - \sqrt{-1}\omega_{AV},$$

$$\bar{\partial}_{\mathbb{CP}^1}\omega_M(V + \sqrt{-1}AV) = \omega_V + \sqrt{-1}\omega_{AV}.$$

If we take $V = \frac{1}{2}\frac{\partial}{\partial x}$, then $V - \sqrt{-1}AV = \frac{\partial}{\partial z}$, $V + \sqrt{-1}AV = \frac{\partial}{\partial \bar{z}}$, and we conclude from the above that

$$\partial_{\mathbb{CP}^1}\omega_M = dz \wedge \omega_{\frac{\partial}{\partial z}} = dz \wedge \left(\omega_V - \sqrt{-1}\omega_{AV}\right),$$
$$\bar{\partial}_{\mathbb{CP}^1}\omega_M = d\bar{z} \wedge \omega_{\frac{\partial}{\partial \bar{z}}} = d\bar{z} \wedge \left(\omega_V + \sqrt{-1}\omega_{AV}\right).$$

Hence

$$\sqrt{-1}\,(\partial_{\mathbb{CP}^1}\omega_M) \wedge \left(\bar{\partial}_{\mathbb{CP}^1}\omega_M\right) = \sqrt{-1}\,dz \wedge \omega_{\frac{\partial}{\partial z}} \wedge d\bar{z} \wedge \omega_{\frac{\partial}{\partial \bar{z}}} =$$
$$= \frac{\sqrt{-1}dz \wedge d\bar{z}}{(1+|z|^2)^2} \wedge \left(1+|z|^2\right)\omega_{\frac{\partial}{\partial z}} \wedge \left(1+|z|^2\right)\omega_{\frac{\partial}{\partial z}} = \omega_{\mathbb{CP}^1} \wedge \Psi \wedge \bar{\Psi}.$$

We now compute the expression $\Psi \wedge \bar{\Psi} \wedge \omega_M^{n-3}$. To simplify things we only do the computation at the point $z = 1$, which corresponds to $I \in \mathbb{CP}^1$, where it takes the form

$$\left(\omega_K + \sqrt{-1}\omega_J\right) \wedge \left(\omega_K - \sqrt{-1}\omega_J\right) \wedge \omega_I^{n-3} = \left(\omega_J + \sqrt{-1}\omega_K\right) \wedge \left(\omega_J - \sqrt{-1}\omega_K\right) \wedge \omega_I^{n-3},$$

while at a general point $A \in \mathbb{CP}^1$ corresponding to $z \in \mathbb{C}$, an entirely analogous argument applies, except that (I, J, K) need to be replaced by $\left(A, \frac{1+|z|^2}{2}\frac{\partial}{\partial x}, \frac{1+|z|^2}{2}\frac{\partial}{\partial y}\right)$, which form a quaternionic triple in the space $W = N_A \oplus T_A\mathbb{CP}^1$.

The vertical tangent space T_mM to the point $(m, I) \in M \times \mathbb{CP}^1$ is a quaternionic vector space with respect to the triple (I, J, K), so we can identify it with $\mathbb{H}^k$, where

$k = \dim_{\mathbb{H}} M = \frac{1}{2}\dim_{\mathbb{C}} M = \frac{n}{2}$. The metric g restricted to T_mM is quaternionic-hermitian, hence we can find a quaternionic orthonormal basis $\{e_1, \ldots, e_k\}$ of T_mM; let $\left\{e_1^*, \ldots, e_k^*\right\}$ denote the dual basis of $\Lambda_m^1 M$. We define the following complex-valued 1-forms $\forall 1 \leq i \leq k$, which constitute a complex basis of $\Lambda_m^1 M \otimes \mathbb{C} = \Lambda_m^{1,0} M \oplus \Lambda_m^{0,1} M$, where the decomposition is relative to the complex structure I.

$$d\zeta := e_i^* + \sqrt{-1} I e_i^* \quad d\xi = J e_i^* + \sqrt{-1} K e_i^*$$

$$d\bar{\zeta} := e_i^* - \sqrt{-1} I e_i^* \quad d\bar{\xi} = J e_i^* - \sqrt{-1} K e_i^*$$

With respect to this basis, it's not hard to see that the forms ω_I, ω_J, ω_K decompose as follows:

$$\omega_I = \sum_{i=1}^{k} \left(\frac{\sqrt{-1}}{2} d\zeta_i \wedge d\bar{\zeta}_i + \frac{\sqrt{-1}}{2} d\xi_i \wedge d\bar{\xi}_i \right),$$

$$\omega_J = \sum_{i=1}^{k} \left(\frac{1}{2} d\zeta_i \wedge d\xi_i + \frac{1}{2} d\bar{\zeta}_i \wedge d\bar{\xi}_i \right),$$

$$\omega_K = \sum_{i=1}^{k} \left(-\frac{\sqrt{-1}}{2} d\zeta_i \wedge d\xi_i + \frac{\sqrt{-1}}{2} d\bar{\zeta}_i \wedge d\bar{\xi}_i \right).$$

Further computing,

$$\omega_J + \sqrt{-1}\omega_K = \sum_{i=1}^{k} d\zeta_i \wedge d\xi_i,$$

hence

$$\left(\omega_J + \sqrt{-1}\omega_K\right) \wedge \left(\omega_J - \sqrt{-1}\omega_K\right) \wedge \omega_I^{n-3} =$$

$$= \left(\sum_{i=1}^{k} d\zeta_i \wedge d\xi_i, \right) \wedge \left(\sum_{i=1}^{k} d\bar{\zeta}_i \wedge d\bar{\xi}_i, \right) \wedge \left\{ \sum_{i=1}^{k} \left(\frac{\sqrt{-1}}{2} d\zeta_i \wedge d\bar{\zeta}_i + \frac{\sqrt{-1}}{2} d\xi_i \wedge d\bar{\xi}_i \right) \right\}^{n-3} =$$

$$= -\left(\frac{\sqrt{-1}}{2}\right)^{n-3} (n-1)(n-3)! \sum_{i=1}^{k} \left(\bigwedge_{j=1}^{k} d\zeta_j \wedge d\bar{\zeta}_j \right) \wedge \left(\bigwedge_{j \neq i} d\xi_j \wedge d\bar{\xi}_j \right) -$$

$$-\left(\frac{\sqrt{-1}}{2}\right)^{n-3} (n-1)(n-3)! \sum_{i=1}^{k} \left(\bigwedge_{j \neq i} d\zeta_j \wedge d\bar{\zeta}_j \right) \wedge \left(\bigwedge_{j=1}^{k} d\xi_j \wedge d\bar{\xi}_j \right).$$

On the other hand,

$$\omega_I^{n-1} = \left\{ \sum_{i=1}^{k} \left(\frac{\sqrt{-1}}{2} \, d\zeta_i \wedge d\bar{\zeta}_i + \frac{\sqrt{-1}}{2} \, d\xi_i \wedge d\bar{\xi}_i \right) \right\}^{n-1} =$$

$$= \left(\frac{\sqrt{-1}}{2} \right)^{n-1} (n-1)! \sum_{i=1}^{k} \left(\bigwedge_{j=1}^{k} d\zeta_j \wedge d\bar{\zeta}_j \right) \wedge \left(\bigwedge_{j \neq i} d\xi_j \wedge d\bar{\xi}_j \right) +$$

$$+ \left(\frac{\sqrt{-1}}{2} \right)^{n-1} (n-1)! \sum_{i=1}^{k} \left(\bigwedge_{j \neq i} d\zeta_j \wedge d\bar{\zeta}_j \right) \wedge \left(\bigwedge_{j=1}^{k} d\xi_j \wedge d\bar{\xi}_j \right).$$

We conclude that

$$\left(\omega_J + \sqrt{-1} \omega_K \right) \wedge \left(\omega_J - \sqrt{-1} \omega_K \right) \wedge \omega_I^{n-3} = \frac{4}{n-2} \omega_I^{n-1}$$

at the point $z = 1$, and generally, $\Psi \wedge \bar{\Psi} \wedge \omega_m^{n-3} = \frac{4}{n-2} \omega_M^{n-1}$. We thus have

$$2\sqrt{-1}(n-1)(n-2) \left(\partial_{\mathbb{CP}^1} \omega_M \right) \wedge \left(\bar{\partial}_{\mathbb{CP}^1} \omega_M \right) \wedge \omega_M^{n-3} =$$

$$= 2(n-1)(n-2) \, \omega_{\mathbb{CP}^1} \wedge \Psi \wedge \bar{\Psi} \wedge \omega_M^{n-3} =$$

$$= 8(n-1) \, \omega_{\mathbb{CP}^1} \wedge \omega_M^{n-1},$$

and so if $n > 2$,

$$d_{\mathbb{CP}^1} d^c_{\mathbb{CP}^1} \left(\omega_M^{n-1} \right) = 4(n-1) \, \omega_{\mathbb{CP}^1} \wedge \omega_M^{n-1},$$

which is strictly positive on vertical forms, hence applying Lemma 2, we get a $T > 0$ such that

$$T \, \omega_M^n + dd^c \left(\omega_M^{n-1} \right)$$

is strictly positive.

Thus both in case $n = 2$ and $n > 2$, we are assured of the existence of a closed strictly positive (n, n)-form on $\mathrm{Tw}(M)$, which immediately implies that $\mathrm{Tw}(M)$ is balanced by Lemma 3.2.1. We are finished. □

We conclude with a short corollary demonstrating that the Kähler condition is too strong for twistor spaces, as opposed to balancedness.

Corollary 3.2.4. *Let* (M, I, J, K, g_M) *be a compact hyperkähler manifold of complex dimension* n*. Then its twistor space* $\mathrm{Tw}(M)$ *is never Kähler.*

Proof. In the notations of the previous proof, we have $d_M\omega_M = d^c_M\omega_M = 0$ by the hyperkähler condition on M, hence

$$dd^c\omega_M = \sqrt{-1}\partial_{\mathbb{CP}^1}\bar{\partial}_{\mathbb{CP}^1}\omega_M = -2\,\omega_{\mathbb{CP}^1} \wedge \omega_M,$$

hence if $\omega_{\mathrm{Tw}(M)}$ is any Kähler form on $\mathrm{Tw}(M)$,

$$-dd^c\omega_M \wedge \left(\omega_{\mathrm{Tw}(M)}\right)^{n-1}$$

is an exact strictly positive $(n+1, n+1)$-form on $\mathrm{Tw}(M)$, in the sense that it is a strict positive multiple of the volume form $\Omega_{\mathrm{Tw}(M)}$. But this is impossible, since then

$$-\int_{\mathrm{Tw}(M)} d\left(d^c\omega_M \wedge \left(\omega_{\mathrm{Tw}(M)}\right)^{n-1}\right) = -\int_{\mathrm{Tw}(M)} dd^c\omega_M \wedge \left(\omega_{\mathrm{Tw}(M)}\right)^{n-1} > 0$$

by compactness, whereas the first integral is zero by Stokes' theorem. □

CHAPTER 4
Fibrewise stable bundles on twistor spaces of hyperkähler manifolds

In this chapter, we work with compact simple hyperkähler manifolds M and their twistor spaces $\mathrm{Tw}(M)$. Our main topic of study is the relationship between irreducible and fibrewise stable bundles on $\mathrm{Tw}(M)$. The main result is Theorem 4.2.1, whose forward implication that a generically fibrewise stable bundle on $\mathrm{Tw}(M)$ is irreducible is due to Kaledin and Verbitsky [28]. It is the partial converse, namely the result that an irreducible vector bundlle E on $\mathrm{Tw}(M)$ is generically fibrewise stable provided it is of rank 2, 3, or generically fibrewise simple, which is the most difficult part. In contrast to the previous chapter, where the exposition is more differential-geometric in nature, many algebro-geometric and complex-analytic theorems and techniques are used here. The main references for these are the books [20], [21], [22] and [41]; other sources are referenced throughout the text.

In Section 4.1 we show that for a vector bundle E on $\mathrm{Tw}(M)$, viewed as a family of bundles on the fibres of the twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$, fibrewise stability and semi-stability are Zariski open conditions on the base $\mathbb{CP}^1$. The argument is basically that of Teleman from [46], where the result is established for families of bundles on the fibres of $X \times Y \to Y$, where X, Y are complex manifolds satisfying some conditions. Section 4.2 contains the main result described above. In Section 4.3 we explicitly construct an example of a stable but nowhere fibrewise stable bundle on $\mathrm{Tw}(M)$ for M a K3 surface.

4.1 Zariski openness of fibrewise stability

It is a general fact, which can be made precise, that given a morphism of spaces $f : X \to S$ and a vector bundle E on X, thought of as a family of vector bundles $\{E_s : s \in S\}$ on the fibres $\left\{f^{-1}(s) : s \in S\right\}$, the set

$$S^{\mathrm{st}} := \{s \in S : E_s \text{ is stable}\}$$

is open in S under some assumptions on the morphism $f : X \to S$. This holds true in the projective algebraic setting (see Proposition 2.3.1 in [25]), where the topology on S is understood to be the Zariski topology, as well as in the complex hermitian setting (see [32], Theorem 5.1.1), where the topology on S is the usual Euclidean manifold topology. We would like to study families of vector bundles on the fibres of the twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$ for a hyperkähler manifold M, and while it is natural to work in the Zariski topology on $\mathbb{CP}^1$, the twistor space $\mathrm{Tw}(M)$ is never projective (not even Kähler, see Corollary 3.2.4). We could apply the result of [32] and conclude that the stability condition is open in $\mathbb{CP}^1$ in the classical topology, but for our purposes we would like to have the stronger result of Zariski openness.

In the paper [46], Teleman proves, among other results, the following theorem.

Theorem 4.1.1. *Let Y be a compact connected Gauduchon manifold, S an arbitrary complex manifold, and E a holomorphic vector bundle on $Y \times S$, thought of as a family of vector bundles on Y parametrized by the projection $Y \times S \to S$. Then the sets*

$$S^{\mathrm{st}} = \{s \in S : E_s \text{ is stable}\}, \; S^{\mathrm{sst}} = \{s \in S : E_s \text{ is semi-stable}\}$$

are Zariski open provided the parameter manifold S is compact.

Although the twistor space $\mathrm{Tw}(M)$ is topologically a product $M \times S^2$, we cannot apply this theorem directly since it is not a complex-analytic product of M and $\mathbb{CP}^1$. We would thus like to extend Teleman's result in the slightly more general setting of the complex structure M_I varying on the fibres of the twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$.

Theorem 4.1.2. *Let M be a compact simple hyperkähler manifold with hyperkähler metric g and twistor space $\mathrm{Tw}(M)$, and E a holomorphic vector bundle on $\mathrm{Tw}(M)$ of rank r. Then the sets*

$$\left(\mathbb{CP}^1\right)^{\mathrm{st}} = \left\{I \in \mathbb{CP}^1 : E_I \textit{ is stable}\right\}, \left(\mathbb{CP}^1\right)^{\mathrm{sst}} = \left\{I \in \mathbb{CP}^1 : E_I \textit{ is semi-stable}\right\}$$

are Zariski open in $\mathbb{CP}^1$.

Our notation will follow that of [46] and our proof will essentially be a verification that Teleman's argument in [46] works for the twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$. We will start by defining the *relative Picard group* of the twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$, which will be an object $\mathrm{Pic}_{\mathbb{CP}^1} \mathrm{Tw}(M)$ parametrizing the Picard groups $\left\{\mathrm{Pic}\, M_I : I \in \mathbb{CP}^1\right\}$ of the fibres of π.

Fix an induced complex structure $I \in \mathbb{CP}^1$ on M. The exponential sheaf sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_I \xrightarrow{\exp} \mathcal{O}_I^* \longrightarrow 0$$

gives rise to a long exact sequence in cohomology, a portion of which looks like

$$0 \longrightarrow H^1(M, \mathbb{Z}) \longrightarrow H^1(M_I, \mathcal{O}_I) \longrightarrow \mathrm{Pic}\, M_I \xrightarrow{c_1} H^2(M, \mathbb{Z})$$

Since M is simply connected, $H^1(M,\mathbb{Z}) = 0$ and $H^2(M,\mathbb{Z})$ has no torsion. By Hodge theory, $H^1(M_I,\mathcal{O}_I) = 0$, and by the Lefschetz theorem on $(1,1)$-classes, the image of $\operatorname{Pic} M_I \xrightarrow{c_1} H^2(M,\mathbb{Z})$ is equal to $H^{1,1}(M_I) \cap H^2(M,\mathbb{Z})$. It follows from all this that $\operatorname{Pic} M_I$ is isomorphic to the Néron-Severi group $NS(M_I)$, and we can identify it with a subgroup of $H^2(M,\mathbb{Z})$:

$$\operatorname{Pic} M_I = H^{1,1}(M_I) \cap H^2(M,\mathbb{Z}) \subseteq H^2(M,\mathbb{Z}).$$

We now assemble the groups $\operatorname{Pic} M_I$ into one object, and define

$$\operatorname{Pic}_{\mathbb{CP}^1} \operatorname{Tw}(M) := \{(I,[\eta]) : [\eta] = c_1(L_I) \text{ for } L_I \in \operatorname{Pic} M_I\} \subseteq \mathbb{CP}^1 \times H^2(M,\mathbb{Z}).$$

To see that this is a closed analytic subset of $\mathbb{CP}^1 \times H^2(M,\mathbb{Z})$ (viewed as a disjoint union of copies of $\mathbb{CP}^1$, parametrized by $H^2(M,\mathbb{Z})$), observe that its connected components are either of the form $\mathbb{CP}^1 \times \{[\eta]\}$, where

$$[\eta] \in \bigcap_{I \in \mathbb{CP}^1} H^{1,1}(M_I) \cap H^2(M,\mathbb{Z})$$

(by Lemma 2.2.3, this is equivalent to $[\eta]$ being $SU(2)$-invariant), or singletons $\{(I,[\eta])\}$ for non-generic I. For such $[\eta]$, the proof of Proposition 2.2 in [51] shows that the intersection

$$\operatorname{Pic}_{\mathbb{CP}^1} \operatorname{Tw}(M) \cap \left(\{\mathbb{CP}^1 \times [\eta]\}\right)$$

is finite, hence Zariski closed in $\mathbb{CP}^1 \times \{[\eta]\}$, from which we can conclude that $\operatorname{Pic}_{\mathbb{CP}^1} \operatorname{Tw}(M)$ is a Zariski closed subset of $\mathbb{CP}^1 \times H^2(M,\mathbb{Z})$. It is clear from construction that the fibre of the natural projection $\operatorname{Pic}_{\mathbb{CP}^1} \operatorname{Tw}(M) \to \mathbb{CP}^1$ over $I \in \mathbb{CP}^1$ is just the Picard group $\operatorname{Pic} M_I$.

Recall that stability of a vector bundle E is defined as a condition on its subsheaves $F \subseteq E$. Equivalently, it can be defined as a condition on its quotient sheaves $E \twoheadrightarrow Q$ (see Theorem 1.2.2 of Chapter 2 in [41]). Given a vector bundle E on the twistor space $\mathrm{Tw}(M)$, we will think of it as a family $\{E_I\}$ of vector bundles over the manifolds $\{M_I\}$ parametrized by $\mathbb{CP}^1$. In order to study stability of bundles E_I, we would like to assemble all of their possible quotient sheaves into one geometric object. This is accomplished with the relative Douady Quot space construction [44], which we now introduce.

We will work in the category **Comp** of complex-analytic spaces and their morphisms. A *complex space* is a locally ringed space $(X, \mathcal{O}_X)$, where X is Hausdorff and $\mathcal{O}_X$ is a sheaf of local $\mathbb{C}$-algebras, locally modeled on $(V, \mathcal{O}_V)$, where $V \subseteq D$ is a complex analytic subset of some domain $D \subseteq \mathbb{C}^n$ with an ideal sheaf $\mathscr{I} \subseteq \mathcal{O}_D$ and $\mathcal{O}_V = (\mathcal{O}_D/\mathscr{I})|_V$ is the structure sheaf of V; for an introduction to complex spaces and their properties, see, for instance, Chapter 1 in [20]. We will fix a proper morphism of complex spaces $X \to S$, which we will think of as an object in the category $\mathbf{Comp}(S)$ of complex S-spaces, and a vector bundle E on X. To ease notation, given any other object $T \to S$ of $\mathbf{Comp}(S)$, we will denote by X_T the fibred product of X and T over S, while E_T will denote the pullback of E via the map $X_T \to X$. We define a contravariant functor

$$\mathcal{Q}uot_S(E) : \mathbf{Comp}(S)^{\mathrm{op}} \longrightarrow \mathbf{Set}.$$

as follows. Recall that for a morphism $\varphi : X \to Y$, a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called *flat at* $x \in X$ if $\mathcal{F}_x$ is flat as an $\mathcal{O}_{\varphi(x)}$-module, and simply *flat* if it is flat at every point

$x \in X$; we say that X is flat over Y if $\mathcal{O}_X$ is. Given an object $T \to S$ in $\mathbf{Comp}(S)$, we define

$$\mathcal{Q}uot_S(E)(T) := \{\text{quotient sheaves } E_T \to Q \to 0 \text{ on } X_T : Q \text{ is flat over } T\},$$

whereas for a morphism of S-spaces $f : T' \to T$, the corresponding map

$$\mathcal{Q}uot_S(E)(f) : \mathcal{Q}uot_S(E)(T) \longrightarrow \mathcal{Q}uot_S(E)(T')$$

is just taking the pullback along the map of fibred products $X_f : X_{T'} \to X_T$, where we use the fact that base change preserves flatness (Proposition III.9.2b in [22]). The following theorem is proved in [44]:

Theorem 4.1.3. *$\mathcal{Q}uot_S(E)$ is representable. In other words, there is an object $\mathrm{Quot}_S(E)$ in $\mathbf{Comp}(S)$ together with a quotient sheaf*

$$E_{\mathrm{Quot}_S(E)} \longrightarrow \mathcal{R}_S(E) \longrightarrow 0$$

on the fibred product space $X_{\mathrm{Quot}_S(E)} = X \times_S \mathrm{Quot}_S(E)$ that satisfies the following universal property:

(i) $\mathcal{R}_S(\mathcal{E})$ is flat over $\mathrm{Quot}_S(\mathcal{E})$;

(ii) given an object T in $\mathbf{Comp}(S)$, and a quotient sheaf

$$E_T \longrightarrow Q \longrightarrow 0$$

on the fibred product space $X_T = X \times_S T$ such that Q is flat over T, there exists a unique S-morphism $f : T \to \mathrm{Quot}_S(\mathcal{E})$ such that $(X_f)^*(\mathcal{R}_S(E)) = Q$, where

X_f is as in the diagram

$$\begin{array}{ccccc} X_T & \overset{X_f}{\dashrightarrow} & X_{\mathrm{Quot}_S(E)} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ T & \overset{f}{\dashrightarrow} & \mathrm{Quot}_S(\mathcal{E}) & \longrightarrow & S \end{array}$$

The space $\mathrm{Quot}_S(E)$ is called the *relative Quot space* of E with respect to S. Set-theoretically, we have

$$\mathrm{Quot}_S(E) = \{(s, \psi_s) : s \in S,\ \psi_s : E_s \to Q_s \to 0 \text{ quotient sheaf over } X_s\},$$

and the universal family $\mathcal{R}_S(E)$ represents these Q_s as a family of sheaves on the spaces X_s. Note that, because the universal family $\mathcal{R}_S(E)$ is flat over $\mathrm{Quot}_S(E)$, the rank of Q_s is constant on connected components of $\mathrm{Quot}_S(E)$, and also the set of points (s, ψ_s) of $\mathrm{Quot}_S(E)$ where $\psi_s : E_s \to Q_s \to 0$ has locally free kernel is open. We denote by $\mathrm{Quot}^1_{\mathrm{lf},S}(E)$ the set of elements (s, ψ_s) of $\mathrm{Quot}_S(E)$ where $\psi_s : E_s \to Q_s \to 0$ has invertible kernel; by the above, $\mathrm{Quot}^1_{\mathrm{lf},S}(E)$ is an open subspace of $\mathrm{Quot}_S(E)$. In the particular case $E = \mathcal{O}_X$, we denote $\mathrm{Quot}^1_{\mathrm{lf},S}(E)$ by $\mathrm{Dou}_S(X)$ and call it the *relative Douady space* of X with respect to S. Set-theoretically, it is just the collection of effective divisors D of the spaces X_s. The following properness result mentioned in [46] is a consequence of Bishop's compactness theorem [5].

Theorem 4.1.4. *Let h be a Hermitian metric on a complex manifold X, and let $X \to S$ be a proper map onto a complex manifold S. Then $\forall \varepsilon > 0$ the topological subspaces*

$$\mathrm{Dou}_S(X)_{\le \varepsilon} \coloneqq \{D \in \mathrm{Dou}_S(X) : \mathrm{Vol}_h(D) \le \varepsilon\} \subseteq \mathrm{Dou}_S(X)$$

are proper over S. Here, for an element $D \subseteq X_s$, $s \in S$, $\mathrm{Vol}_h(D)$ is the volume of D with respect to the restriction of the metric h.

We now come back to the special case of a vector bundle E over the twistor space $\mathrm{Tw}(M)$, thought of as an object of $\mathbf{Comp}(\mathbb{CP}^1)$ via the twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$. We will identify the relative Quot space $\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)$ with the set of equivalence classes of sheaf monomorphisms $\varphi_I : L_I \hookrightarrow E_I$, where $I \in \mathbb{CP}^1$ and L_I is a line bundle over M_I. We can define a map

$$\begin{array}{rccc} p & : & \mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E) & \longrightarrow & \mathrm{Pic}_{\mathbb{CP}^1}\mathrm{Tw}(M) \\ & & [\varphi_I : L_I \hookrightarrow E_I] & \longmapsto & (I, c_1(L_I)), \end{array}$$

where we think of $\mathrm{Pic}_{\mathbb{CP}^1}\mathrm{Tw}(M)$ as a subspace of $\mathbb{CP}^1 \times H^2(M,\mathbb{Z})$, identifying $L_I \in \mathrm{Pic}\, M_I$ with its image $c_1(L_I) \in H^2(M,\mathbb{Z})$ under the homomorphism $c_1 : \mathrm{Pic}\, M_I \to H^2(M,\mathbb{Z})$, whose injectivity follows from the fact that M is simply connected, as discussed previously. To see that this map is analytic, note that the flatness of the universal family $\mathcal{R}_{\mathbb{CP}^1}(E)$ over $\mathrm{Tw}(M) \times_{\mathbb{CP}^1} \mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)$ ensures that the first Chern class of $\mathrm{Ker}(\varphi_I : E_I \twoheadrightarrow Q_I)$ is locally constant on $\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)$. It's not hard to see that, given an element $L_I \in \mathrm{Pic}\, M_I$, the set-theoretical fibre of p over L_I is simply

$$p^{-1}(L_I) = \mathbb{P}\left(H^0(M_I, L_I^* \otimes E_I)\right).$$

Now let $Z = \mathbb{P}(E^*)$ be the projectivization of the dual bundle of E over $\mathrm{Tw}(M)$, thought of as a family of projectivizations $Z_I = \mathbb{P}(E_I^*)$ parametrized by $I \in \mathbb{CP}^1$, and let $\mathrm{Dou}_{\mathbb{CP}^1}(Z)$ be the relative Douady space of Z. As a first step in the proof of Theorem 4.1.2, we will identify $\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)$ with a certain subspace of $\mathrm{Dou}_{\mathbb{CP}^1}(Z)$. Just like for $\mathrm{Tw}(M)$, we can define the relative Picard group of Z with the natural projection $\mathrm{Pic}_{\mathbb{CP}^1} Z \to \mathbb{CP}^1$, with the fibre over $I \in \mathbb{CP}^1$ being $\mathrm{Pic}\, Z_I = \mathrm{Pic}\, \mathbb{P}(E_I^*)$. Since $\mathrm{Pic}\, \mathbb{P}(E_I^*)$ is canonically isomorphic to $\mathrm{Pic}\, M_I \times \mathbb{Z}$, with the $\mathbb{Z}$ summand generated by the line bundle $\mathcal{O}_{Z_I}(1)$, we conclude that

$$\mathrm{Pic}_{\mathbb{CP}^1} Z \cong \mathrm{Pic}_{\mathbb{CP}^1} \mathrm{Tw}(M) \times \mathbb{Z}.$$

There is a natural map

$$\begin{array}{rccc} n_Z & : & \mathrm{Dou}_{\mathbb{CP}^1}(Z) & \longrightarrow & \mathrm{Pic}_{\mathbb{CP}^1} Z \\ & & D_I \subseteq M_I & \longmapsto & (I, [\mathcal{O}_{D_I}]), \end{array}$$

which is analytic for the same reason as p is. Given an element $N_I \in \mathrm{Pic}\, Z_I$, the set-theoretic fibre of n_Z over N_I is just

$$n_Z^{-1}(N_I) = \mathbb{P}\left(H^0(Z_I, N_I)\right).$$

Let $q : Z \to \mathrm{Tw}(M)$ denote the natural projection, and let $q_I : Z_I \to M_I$ be the obvious restrictions. Given a line bundle L_I on M_I such that $H^0(M_I, L_I^* \otimes E_I) \neq 0$, we use the projection formula ([41], p. 6) and the fact that $q_*(\mathcal{O}_Z(1)) = E$, to obtain the following identifications:

$$H^0(M_I, L_I^* \otimes E_I) \cong H^0(M_I, q_{I*}(q_I^*(L_I^*) \otimes \mathcal{O}_{Z_I}(1))) \cong H^0(Z_I, q_I^*(L_I^*) \otimes \mathcal{O}_{Z_I}(1)).$$

Taking into consideration the set-theoretic identifications in the previous paragraphs, we have defined a bijection

$$\Phi : p^{-1}(L_I) \longleftrightarrow n_Z^{-1}(q_I^*(L_I^*) \otimes \mathcal{O}_{Z_I}(1)).$$

If we now let $a : \operatorname{Pic}_{\mathbb{CP}^1} \operatorname{Tw}(M) \to \operatorname{Pic}_{\mathbb{CP}^1} Z$ be the embedding

$$[L_I] \longmapsto [q_I^*(L_I^*) \otimes \mathcal{O}_{Z_I}(1)],$$

the mappings Φ between fibres of p and n_Z defined above assemble into a set-theoretic embedding $\Phi : \operatorname{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E) \to \operatorname{Dou}_{\mathbb{CP}^1}(Z)$ that makes the diagram

$$\begin{array}{ccc} \operatorname{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E) & \xrightarrow{\Phi} & \operatorname{Dou}_{\mathbb{CP}^1}(Z) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle n_Z} \\ \operatorname{Pic}_{\mathbb{CP}^1} \operatorname{Tw}(M) & \xrightarrow[a]{} & \operatorname{Pic}_{\mathbb{CP}^1} Z \end{array}$$

commute. We would like to verify that Φ is actually analytic.

Proposition 4.1.5. *The map*

$$\Phi : \operatorname{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E) \xrightarrow{\sim} n_Z^{-1}(a(\operatorname{Pic}_{\mathbb{CP}^1} \operatorname{Tw}(M))) \subseteq \operatorname{Dou}_{\mathbb{CP}^1}(Z)$$

is a complex-analytic isomorphism.

Proof. The proof closely follows the argument of Teleman in Proposition 2.3 of [46]. We will exhibit Φ as a morphism between the corresponding functors on the category $\mathbf{Comp}(\mathbb{CP}^1)$. Recall that the object $\operatorname{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)$ represents the functor

$$\mathcal{Q}uot^1_{\mathrm{lf},\mathbb{CP}^1}(E) : \mathbf{Comp}(\mathbb{CP}^1)^{\mathrm{op}} \longrightarrow \mathbf{Set},$$

which takes an object $T \to \mathbb{CP}^1$ in $\mathbf{Comp}(\mathbb{CP}^1)$ to

$$\mathcal{Q}uot^1_{\mathrm{lf},\mathbb{CP}^1}(E)(T) := \{\text{quotients } E_T \to Q \to 0 \text{ on } \operatorname{Tw}(M)_T = \operatorname{Tw}(M) \times_{\mathbb{CP}^1} T :$$

$$Q \text{ is flat over } T \text{ and } E_T \to Q \text{ has invertible kernel}\},$$

where E_T denotes the pullback of E via the projection $\mathrm{Tw}(M)_T = \mathrm{Tw}(M) \times_{\mathbb{CP}^1} T \to \mathrm{Tw}(M)$. On the other hand, the subspace $n_Z^{-1}(a(\mathrm{Pic}_{\mathbb{CP}^1} \mathrm{Tw}(M))) \subseteq \mathrm{Dou}_{\mathbb{CP}^1}(Z)$ represents the functor

$$\mathcal{D} : \mathbf{Comp}(\mathbb{CP}^1)^{\mathrm{op}} \longrightarrow \mathbf{Set},$$

which takes an object $g : T \to \mathbb{CP}^1$ in $\mathbf{Comp}(\mathbb{CP}^1)$ to

$$\mathcal{D}(T) := \{\text{divisors } D \subseteq Z_T = Z \times_{\mathbb{CP}^1} T :$$

$$\mathcal{O}_D \text{ is flat over } T \text{ and } \forall t \in T, \mathcal{O}(D_t) \in a(\mathrm{Pic}\, M_{g(t)})\},$$

where we identify $\mathrm{Pic}\, M_I$ as a subset of $\mathrm{Pic}_{\mathbb{CP}^1} \mathrm{Tw}(M)$.

Now fix $g : T \to \mathbb{CP}^1$. We want to construct a bijection

$$\Phi_T : \mathcal{Q}uot^1_{\mathrm{lf},\mathbb{CP}^1}(E)(T) \longleftrightarrow \mathcal{D}(T).$$

This will be essentially a generalization of our previous construction of the maps $\Phi : p^{-1}(L_I) \longleftrightarrow n_Z^{-1}(q^*(L_I^*) \otimes \mathcal{O}_{Z_I}(1))$. Indeed, identify $\mathcal{Q}uot^1_{\mathrm{lf},\mathbb{CP}^1}(E)(T)$ with the set of equivalence classes of sheaf monomorphisms $L \hookrightarrow E_T$ on $\mathrm{Tw}(M)_T$, where L is a line bundle on $\mathrm{Tw}(M)_T$. There is a canonical map

$$\begin{array}{cccc} p_T & : & \mathcal{Q}uot^1_{\mathrm{lf},\mathbb{CP}^1}(E)(T) & \longrightarrow & \mathrm{Pic}\,\mathrm{Tw}(M)_T \\ & & [L \hookrightarrow E_T] & \longmapsto & [L]. \end{array}$$

Note that, if $L \in \mathrm{Pic}\,\mathrm{Tw}(M)_T$, the fibre of p_T over L looks like

$$p_T^{-1}(L) = \{[\phi] \in \mathbb{P}\left(\mathrm{Hom}(L, E_T)\right) : \phi : L \to E_T \text{ is a sheaf monomorphism}$$

$$\text{and the quotient } Q \text{ is flat over } T\}.$$

Now let $q_T : Z_T \cong \mathbb{P}(E_T^*) \to \mathrm{Tw}(M)_T$ be the natural projection, and let n_{Z_T} be the map

$$\begin{aligned} n_{Z_T} \ : \ \mathcal{D}(T) &\longrightarrow \mathrm{Pic}\, Z_T \\ D \subseteq Z_T &\longmapsto [\mathcal{O}(D)]. \end{aligned}$$

Similarly to the above, if $N \in \mathrm{Pic}\, Z_T$, then

$$n_{Z_T}^{-1}(N) = \{[\psi] \in \mathbb{P}(\mathrm{Hom}(\mathcal{O}_{Z_T}, N)) : \psi : \mathcal{O}_{Z_T} \to N \text{ is a sheaf monomorphism,}$$

$$\text{the quotient } Q' \text{ is flat over } T \text{ and } \forall t \in T, N_t \in a(\mathrm{Pic}\, M_{g(t)})\Big\}.$$

Now, given $L \in \mathrm{Pic}\,\mathrm{Tw}(M)_T$, using the projection formula and $(q_T)_*(\mathcal{O}_{Z_T}(1)) = E_T$, we have

$$\mathrm{Hom}(L, E_T) = H^0(\mathrm{Tw}(M)_T, L^* \otimes E_T) \cong H^0(Z_T, q_T^*(L^*)(1)) = \mathrm{Hom}(\mathcal{O}_{Z_T}, q_T^*(L^*)(1)).$$

In order to conclude that this defines a bijection

$$\Phi_T : p_T^{-1}(L) \longleftrightarrow n_{Z_T}^{-1}(q_T^*(L^*)(1)),$$

we need to verify two things. First, we have to check that sheaf monomorphisms in $\mathrm{Hom}(L, E_T)$ correspond to sheaf monomorphisms in $\mathrm{Hom}(\mathcal{O}_{Z_T}, q_T^*(L^*)(1))$. Since this is a local statement on $\mathrm{Tw}(M)_T$, we can restrict our attention to a neighbourhood U of a point $x \in \mathrm{Tw}(M)_T$ where $L|_U \cong \mathcal{O}_U$ and E_T has a local frame $s_1, \dots, s_r \in \mathcal{O}(E_T)(U)$. Then, with our identification, both elements of $\mathrm{Hom}(L|_U, E_T|_U) \cong \mathrm{Hom}(\mathcal{O}_U, E_T|_U)$ and $\mathrm{Hom}(\mathcal{O}_{q_T^{-1}(U)}, q_T^*(L^*)(1)|_{q_T^{-1}(U)}) \cong \mathrm{Hom}(\mathcal{O}_{q_T^{-1}(U)}, \mathcal{O}_{q_T^{-1}(U)}(1))$ correspond to sections

$$a_1 s_1 + \dots a_r s_r,$$

where $a_1, \ldots, a_r \in \mathcal{O}_{\mathrm{Tw}(M)_T}(U)$. It's not hard to see that in either case we have a sheaf monomorphism at $x \in \mathrm{Tw}(M)_T \iff \mathrm{Ann}((a_1)_x) \cap \ldots \cap \mathrm{Ann}((a_r)_x) = \{0\}$ in $\mathcal{O}_x$.

The second verification we have to make is that the flatness conditions for $p_T^{-1}(L)$ and $n_{Z_T}^{-1}(q_T^*(L^*)(1))$ are equivalent. So let Q be the quotient of a monomorphism $L \to E_T$, and let Q' be the quotient of the corresponding monomorphism $\mathcal{O}_{Z_T} \to q_T^*(L^*)(1)$. Choosing a point $x = (m,t) \in \mathrm{Tw}(M)_T = \mathrm{Tw}(M) \times_{\mathbb{CP}^1} T$, by the local flatness criterion (see [16], Theorem 6.8), Q is T-flat at (m,t) if and only if $\mathrm{Tor}_1^{\mathcal{O}_t}(\mathbb{C}_t, Q_{(m,t)}) = 0$. Since $\pi_T : \mathrm{Tw}(M)_T \to T$ is a flat morphism (a consequence of the fact that $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$ is flat, and that base change preserves flatness), we have

$$\mathrm{Tor}_1^{\mathcal{O}_t}(\mathbb{C}_t, Q_{(m,t)}) = \mathrm{Tor}_1^{\mathcal{O}_{(m,t)}}(\mathbb{C}_t \otimes_{\mathcal{O}_t} \mathcal{O}_{(m,t)}, Q_{(m,t)}),$$

and the latter is just the stalk at (m,t) of the sheaf

$$\mathcal{T}or_1(\mathcal{O}_{M_{\pi_T^{-1}(t)}}, Q) = \mathcal{T}or_1(\mathcal{O}_{M_{g(t)}}, Q),$$

where the fibre $\pi_T^{-1}(t) \subseteq \mathrm{Tw}(M)_T$ is identified with $M_{g(t)}$ via the diagram

$$\begin{array}{ccccc} M_{g(t)} & \hookrightarrow & \mathrm{Tw}(M)_T & \xrightarrow{g'} & \mathrm{Tw}(M) \\ \downarrow & & \downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \pi} \\ \{t\} & \hookrightarrow & T & \xrightarrow[g]{} & \mathbb{CP}^1 \end{array}$$

So the flatness of Q is equivalent to the vanishing of the sheaves $\mathcal{T}or_1(\mathcal{O}_{M_{g(t)}}, Q)$ for every $t \in T$, which in turn is equivalent to the injectivity of the sheaf morphism

$$L_t := L|_{\pi_T^{-1}(t)} \longrightarrow (E_T)|_{\pi_T^{-1}(t)} \cong E_{g(t)}$$

for all t. By an entirely analogous argument, Q' is flat if and only if the induced sheaf morphism

$$\mathcal{O}_{Z_T}|_{\pi_T^{-1}(t)} \cong \mathcal{O}_{Z_{g(t)}} \longrightarrow q_T^*(L_t^*)(1)$$

is injective for all t. The equivalence of the two conditions is shown exactly as the corresponding statement for $\mathrm{Hom}(L, E_T)$ and $\mathrm{Hom}(\mathcal{O}_{Z_T}, q_T^*(L^*)(1))$.

Finally, if we let $a_T : \mathrm{Pic}\,\mathrm{Tw}(M)_T \to \mathrm{Pic}\,Z_T$ be the map $L \mapsto q_T^*(L^*)(1)$, and put the bijections $p_T^{-1}(L) \longleftrightarrow n_{Z_T}^{-1}(q_T^*(L^*)(1))$ together, we get a bijective map Φ_T that makes the diagram

$$\begin{array}{ccc} \mathcal{Q}uot^1_{\mathrm{lf},\mathbb{CP}^1}(E)(T) & \xrightarrow{\Phi_T} & \mathcal{D}(T) \\ \downarrow{\scriptstyle p_T} & & \downarrow{\scriptstyle n_{Z_T}} \\ \mathrm{Pic}\,\mathrm{Tw}(M)_T & \xrightarrow[a_T]{} & \mathrm{Pic}\,Z_T \end{array}$$

commute. □

We would now like to apply Proposition 4.1.5 to translate Theorem 4.1.4 from a statement about properness of subsets of $\mathrm{Dou}_{\mathbb{CP}^1}(Z)$ into a statement about properness of subsets of $\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)$.

Proposition 4.1.6. *Let g denote the hyperkähler metric on M. For any $d \in \mathbb{R}$, the subspaces*

$$\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)_{\geq d},\ \mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)_{>d} \subseteq \mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E),$$

defined by the inequalities $\deg_g(L_I) \geq d$, *resp.* $\deg_g(L_I) > d$, *are complex-analytic and proper over* $\mathbb{CP}^1$.

Proof. Recall that we have maps

$$\begin{array}{ccccc} \mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E) & \xrightarrow{p} & \mathrm{Pic}_{\mathbb{CP}^1}\mathrm{Tw}(M) & \xrightarrow{\deg_g} & \mathbb{R} \\ [\varphi : L_I \to E_I] & \longmapsto & (I, c_1(L_I)) & \longmapsto & \deg_g(L_I) \end{array}$$

By construction of $\mathrm{Pic}_{\mathbb{CP}^1}\mathrm{Tw}(M)$ and Lemma 2.3.13, we can easily see that the map $\deg_g$ is locally constant on $\mathrm{Pic}_{\mathbb{CP}^1}\mathrm{Tw}(M)$, hence it is also locally constant on $\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)$. It follows at once that both $\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)_{\geq d}$ and $\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)_{>d}$ are unions of connected components of $\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)$, hence they are analytic. It remains to show that they are compact.

The rest of the proof closely follows the argument of Teleman on page 9 of [46]. Let r denote the rank of E and n the complex dimension of M. Recall that the hyperkähler metric g induces a natural metric on the twistor space $\mathrm{Tw}(M)$, which we will also denote by g, abusing the notation slightly. Choose an arbitrary Hermitian metric h on E. The Chern connection of (E, h) induces an Ehresmann connection on the projective bundle $q : Z = \mathbb{P}(E^*) \to \mathrm{Tw}(M)$, that is, a subbundle $HZ \subseteq TZ$ such that there is a direct sum decomposition

$$TZ = HZ \oplus VZ,$$

where VZ is the vertical tangent bundle of $q : Z \to \mathrm{Tw}(M)$. With this decompostion, one can think of HZ as the horizontal tangent bundle with respect to q. Note that, because the Chern connection of (E, h) is compatible with the holomorphic structure of E, the distribution $HZ \subseteq TZ$ is preserved by the almost-complex structure of Z. On the other hand, note that the metric h on E induces a natural Hermitian metric on the vertical tangent bundle VZ on Z; this is just the Fubini-Study metric on the fibres $q^{-1}(x) = \mathbb{P}(E_x^*) \cong \mathbb{P}^{r-1}$ of the projection $q : \mathbb{P}(E^*) \to \mathrm{Tw}(M)$ induced by the metric h. We will denote by ω_{FS} the corresponding Hermitian form, thought of as a real vertical $(1,1)$-form on Z. It is now easy to see that if ω denotes the Hermitian form of g on $\mathrm{Tw}(M)$,

$$\Omega := q^*(\omega) + \omega_{FS}$$

is a real positive $(1,1)$-form on Z such that $TZ = HZ \oplus VZ$ becomes an orthogonal direct sum in the corresponding metric G on Z. Letting $I \in \mathbb{CP}^1$, the restriction of G to the submanifold $Z_I \subseteq Z$, as in the diagram

$$\begin{array}{ccc} Z_I = \mathbb{P}(E_I^*) & \hookrightarrow & Z = \mathbb{P}(E^*) \\ {\scriptstyle q_I}\downarrow & & \downarrow{\scriptstyle q} \\ M_I & \hookrightarrow & \mathrm{Tw}(M), \end{array}$$

will be denoted by G_I; the corresponding Hermitian form is

$$\Omega_I := q_I^*(\omega_I) + \omega_{FS}$$

with ω_I the Kähler form on M_I.

Now fix $I \in \mathbb{CP}^1$ and let L_I be a holomoprhic line bundle on M_I. We want to relate the degree of L_I with respect to g to the degree of $q_I^*(L_I)$ with respect to a Gauduchon metric in the conformal class of G_I. By Theorem 2.3.15, there exists a g-Hermitian-Einstein metric γ on L_I, and by Proposition 2.3.10, the curvature R^γ of the Chern connection on (L_I, γ) satisfies the equation

$$\left(\frac{\sqrt{-1}}{2\pi} R^\gamma\right) \wedge \omega_I^{n-1} = \frac{\deg_g(L_I)}{n!\,\mathrm{Vol}_g(M)} \omega_I^n.$$

We now verify that the metric $q_I^*(\gamma)$ on $q_I^*(L_I)$ is G_I-Hermitian-Einstein, and its Einstein constant is proportional to $\deg_g(L_I)$. We will use the fact that $\omega_{FS}^k = 0$ for $k \geq r$ on Z_I.

$$\left(\frac{\sqrt{-1}}{2\pi} q_I^*(R^\gamma)\right) \wedge \Omega_I^{n+r-2} = q_I^*\left(\frac{\sqrt{-1}}{2\pi} R^\gamma\right) \wedge \binom{n+r-2}{r-1} \left(q_I^*(\omega_I)\right)^{n-1} \wedge \omega_{FS}^{r-1} =$$

$$= \frac{(n+r-2)!}{(r-1)!(n-1)!} q_I^* \left(\frac{\sqrt{-1}}{2\pi} R^\gamma \wedge \omega_I^{n-1}\right) \wedge \omega_{FS}^{r-1} =$$

$$= \frac{(n+r-2)!}{(r-1)!(n-1)!} \frac{\deg_g(L_I)}{n! \operatorname{Vol}_g(M)} q_I^*(\omega_I^n) \wedge \omega_{FS}^{r-1} =$$

$$= \frac{1}{n+r-1} \left(\frac{\deg_g(L_I)}{(n-1)! \operatorname{Vol}_g(M)}\right) \Omega_I^{n+r-1},$$

where we have used the fact that

$$\Omega_I^{n+r-1} = \binom{n+r-1}{r-1} q_I^*(\omega_I^n) \wedge \omega_{FS}^{r-1}.$$

We have thus shown that the G_I-Einstein constant of the line bundle $q_I^*(L_I)$ on Z_I is proportional to $\deg_g(L_I)$ (and in fact the constant of proportionality does not depend on the complex structure I). If G_I' is a Gauduchon metric in the conformal class of G_I, it follows from Proposition 2.3.16 that

$$\deg_{G_I'}(q_I^*(L_I)) = C \cdot \deg_g(L_I)$$

for some positive constant $C > 0$. If we now let $I \in \mathbb{CP}^1$ vary, then since G_I depends smoothly on $I \in \mathbb{CP}^1$, we can choose a family of Gauduchon metrics $\{G_I'\}$ on $\{Z_I\}$ such that G_I' is in the conformal class of G_I and G_I' depends smoothly on $I \in \mathbb{CP}^1$; furthermore, using Proposition 1.3.5 of [32], we can choose the G_I' in such a way that for any nontrivial line bundle $N_I \in \operatorname{Pic} Z_I$, and any nonzero section $s \in H^0(Z_I, N_I)$, we have

$$\deg_{G'_I} N_I = \operatorname{Vol}_{G'_I} \{s = 0\},$$

where $\operatorname{Vol}_{G'_I} \{s = 0\}$ is the volume of the analytic subset $\{s = 0\} \subseteq Z_I$ with respect to the restriction of the metric G'_I. It follows from this that there exists a continuous function $C_1 : \mathbb{CP}^1 \to \mathbb{R}^{>0}$ such that for any $I \in \mathbb{CP}^1$ and $L_I \in \operatorname{Pic} M_I$,

$$\deg_{G'_I}(q_I^*(L_I)) = C_1(I) \cdot \deg_g(L_I).$$

Recall that we have a map

$$\Phi : \operatorname{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E) \longrightarrow \operatorname{Dou}_{\mathbb{CP}^1}(Z),$$

which is a complex-analytic isomorphism of $\operatorname{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)$ with a union of connected components in $\operatorname{Dou}_{\mathbb{CP}^1}(Z)$. We have, for an element $[\varphi_I : L_I \to E_I]$ of $\operatorname{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)$,

$$\operatorname{Vol}_{G'_I} \Phi([\varphi_I]) = \deg_{G'_I}(\mathcal{O}_{Z_I}(1) \otimes q_I^*(L_I^*)) = \deg_{G'_I}(\mathcal{O}_{Z_I}(1)) - C_1(I) \cdot \deg_g(L_I).$$

Using the continuity of the family $\{G'_I\}$, we have that the function C_2 on $\mathbb{CP}^1$ defined by

$$C_2(I) \coloneqq \deg_{G'_I}(\mathcal{O}_{Z_I}(1))$$

is continuous. In other words, we have that

$$\operatorname{Vol}_{G'_I} \Phi([\varphi_I]) = C_2(I) - C_1(I) \cdot \deg_g(L_I),$$

where $C_1 : \mathbb{CP}^1 \to \mathbb{R}^{>0}$, $C_2 : \mathbb{CP}^1 \to \mathbb{R}$ are continuous. Letting $\varepsilon > 0$, we know by Theorem 4.1.4 that the subset

$$\mathrm{Dou}_{\mathbb{CP}^1}(Z)_{\leq\varepsilon} = \left\{D \subseteq Z_I : \mathrm{Vol}_{G_I'}(D) \leq \varepsilon\right\} \subseteq \mathrm{Dou}_{\mathbb{CP}^1}(Z)$$

is proper over $\mathbb{CP}^1$, hence in particular compact. Accordingly, its preimage under the map Φ,

$$\Phi^{-1}\left(\mathrm{Dou}_{\mathbb{CP}^1}(Z)_{\leq\varepsilon}\right) = \left\{[\varphi : L_I \to E_I] : \deg_g(L_I) \geq \frac{C_2(I) - \varepsilon}{C_1(I)}\right\} \subseteq \mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)$$

is also compact. Since $\mathbb{CP}^1$ is compact and $C_1 > 0$, we can choose $\varepsilon \gg 0$ such that

$$\frac{C_2(I) - \varepsilon}{C_1(I)} \leq d \text{ for all } I \in \mathbb{CP}^1.$$

Then both $\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)_{\geq d}$ and $\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)_{>d}$ are subsets of $\Phi^{-1}\left(\mathrm{Dou}_{\mathbb{CP}^1}(Z)_{\leq\varepsilon}\right)$, hence both are compact. We are done. □

To go ahead with the proof of Theorem 4.1.2, we need one more technical result, whose proof is given in [46]. Recall that if V is an r-dimensional complex vector space, for any $1 \leq s \leq r-1$, the Grassmanian $\mathrm{Gr}(s,V)$ of s-dimensional complex subspaces of V can be embedded via the Plücker map

$$\begin{array}{ccc} \mathrm{Gr}(s,V) & \longrightarrow & \mathbb{P}\left(\Lambda^s V\right) \\ \langle v_1, \ldots, v_s\rangle & \longmapsto & [v_1 \wedge \ldots \wedge v_n] \end{array}$$

as a Zariski-closed subset of $\mathbb{P}\left(\Lambda^s V\right)$. We denote by $C_s(V)$ the cone in $\Lambda^s V$ over the image of $\mathrm{Gr}(S,V)$ in $\mathbb{P}(\Lambda^s V)$. Similarly, if E is a complex rank r vector bundle over a manifold Y, we denote by $C_s(E)$ the fibre subbundle of $\Lambda^s E$ consisting of exterior monomials in the fibres of $\Lambda^s E$. It turns out that to test E for stability we only have to consider line subsheaves of the various bundles $\Lambda^s E$ with values in the closed cone subbundle $C_s(E) \subseteq \Lambda^s E$.

Proposition 4.1.7. *Let Y be a complex manifold with a Gauduchon metric g, and let E be a holomorphic rank r vector bundle over Y. The following conditions are equivalent:*

(i) E is g-stable (g-semi-stable).

(ii) For every $1 \leq s \leq r-1$, and any non-trivial morphism $\varphi : L \to \Lambda^s E$, where L is a line bundle and $\mathrm{Im}(\varphi) \subseteq C_s(E)$, one has

$$\deg_g L < s \cdot \mu_g(E) \ (\textit{resp.}\ \deg_g L \leq s \cdot \mu_g(E)).$$

Proof. Proposition 2.15 in [46]. □

Proof of Theorem 4.1.2. Recall that we denote by $S_0 \subseteq S^2$ the set of generic complex structures on M. Note that if $I \in S_0$, $\mu_g(E_I) = 0$ by Lemma 2.3.13. Since S_0 is dense in S^2 (Proposition 2.2.5), by continuity, we have $\mu_g(E_I) = 0$ for all $I \in \mathbb{CP}^1$. By Proposition 4.1.6, the subpaces

$$\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)_{\geq d},\ \mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)_{> d} \subseteq \mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)$$

are analytic and proper over $\mathbb{CP}^1$, hence their projections in $\mathbb{CP}^1$

$$\mathbb{CP}^1_{\geq d} = \left\{ I : \exists L_I \in \mathrm{Pic}\, M_I \text{ and } \varphi_I : L_I \hookrightarrow E_I \text{ such that } \deg_g L_I \geq d \right\},$$

$$\mathbb{CP}^1_{> d} = \left\{ I : \exists L_I \in \mathrm{Pic}\, M_I \text{ and } \varphi_I : L_I \hookrightarrow E_I \text{ such that } \deg_g L_I > d \right\}$$

are Zariski closed. For any $1 \le s \le r-1$, let $\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)^s$ denote the closed analytic subspace of $\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(\Lambda^s E)$ consisting of equivalence classes of sheaf monomorphisms $[\varphi_I : L_I \to \Lambda^s E_I]$ with $\mathrm{Im}(\varphi_I) \subseteq C_s(E_I)$. Then the intersections of $\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(\Lambda^s E)_{\ge d}$ and $\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(\Lambda^s E)_{>d}$ with $\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)^s$ are again analytic and proper over $\mathbb{CP}^1$, hence their projections

$$\left(\mathbb{CP}^1_{\ge d}\right)^s = \left\{I : \exists \varphi_I : L_I \hookrightarrow \Lambda^s E_I \text{ with } \mathrm{Im}(\varphi_I) \subseteq C_s(E_I) \text{ and such that } \deg_g L_I \ge d\right\},$$

$$\left(\mathbb{CP}^1_{>d}\right)^s = \left\{I : \exists \varphi_I : L_I \hookrightarrow \Lambda^s E_I \text{ with } \mathrm{Im}(\varphi_I) \subseteq C_s(E_I) \text{ and such that } \deg_g L_I > d\right\}$$

are again Zariski closed. It only remains to observe that, by Proposition 4.1.7,

$$\left(\mathbb{CP}^1\right)^{\mathrm{st}} = \mathbb{CP}^1 \smallsetminus \bigcup_{1\le s\le r-1} \left(\mathbb{CP}^1_{\ge 0}\right)^s, \ \left(\mathbb{CP}^1\right)^{\mathrm{sst}} = \mathbb{CP}^1 \smallsetminus \bigcup_{1\le s\le r-1} \left(\mathbb{CP}^1_{>0}\right)^s.$$

□

4.2 Irreducible bundles and fibrewise stability

Recall that an irreducible vector bundle is one that does not have proper subsheaves of lower rank, while a generically fibrewise stable bundle E on the twistor space $\mathrm{Tw}(M)$ of a hyperkähler manifold M is one that stably restricts to all the fibres of the twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$, except perhaps finitely many. For an arbitrary induced complex structure $I \in \mathbb{CP}^1$, we denote by M_I the complex manifold (M, I) and by E_I the restriction of E to the fibre $\pi^{-1}(I) = M_I$. The main result of this chapter follows.

Theorem 4.2.1. *Let M be a compact simple hyperkähler manifold and let E be a holomorphic vector bundle on the twistor space* $\mathrm{Tw}(M)$. *If E is generically fibrewise* *stable, then it is irreducible. The converse is true for vector bundles of rank 2 and 3, as well as for bundles E of general rank that are generically fibrewise simple.*

Proof of forward implication and converse for the cases $\operatorname{rk} E = 2, 3$. The forward implication is due to Kaledin and Verbitsky ([28], Lemma 7.3). Suppose E is generically fibrewise stable. Since the set of generic induced complex structures S_0 is dense in $\mathbb{CP}^1$ by Proposition 2.2.5, we can always choose a complex structure $I \in S_0$ such that E_I is stable. In fact, by Lemma 2.3.13, it is irreducible, since any proper subsheaf of E_I of lower rank would destabilize E_I, both having slope 0. Given a subsheaf $\mathcal{F} \subseteq E$ on $\operatorname{Tw}(M)$, observe that $\mathcal{F}$ is torsion-free, being a subsheaf of the torsion-free E, hence its singularity set has codimension ≥ 2, so in particular its restriction to M_I is a subsheaf $\mathcal{F}_I \subseteq E_I$ of the same rank as $\mathcal{F}$. It follows that either $\operatorname{rk}(\mathcal{F}) = 0$, in which case $\mathcal{F} = 0$ since it's torsion-free, or $\operatorname{rk}(\mathcal{F}) = \operatorname{rk}(E)$. Thus E is irreducible. Observe that in our proof we have only used that E_I is stable for a single generic complex structure $I \in S_0 \subseteq \mathbb{CP}^1$, which is consistent with the results of the previous section.

We now prove the converse for the cases $\operatorname{rk} E = 2$ and 3. Let E be an irreducible rank 2 bundle on $\operatorname{Tw}(M)$, and suppose E is not generically fibrewise stable, i.e. there are infinitely many $I \in \mathbb{CP}^1$ such that E_I has a destabilising line subsheaf. Then, by Theorem 4.1.2, it actually follows that E_I is non-stable for all $I \in \mathbb{CP}^1$, i.e. the map

$$\operatorname{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)_{\geq 0} \longrightarrow \mathbb{CP}^1$$

is surjective. Since this map is analytic and proper, we conclude that there is a connected component in $\operatorname{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)_{\geq 0}$ which projects onto $\mathbb{CP}^1$. By the set-theoretic description of $\mathrm{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)$ in the previous section, this means that there is a complex line bundle L on M with $SU(2)$-invariant first Chern class $c_1(L)$ (which guarantees the existence of a unique hyperholomorphic structure on L, see proof of Corollary 2.3.14), such that $\forall I \in \mathbb{CP}^1$, if L_I denotes the induced holomorphic structure on L over M_I,

$$\dim \mathrm{Hom}_{M_I}(L_I, E_I) = \dim H^0(M_I, L_I^* \otimes E_I) \geq 1.$$

Let $\mathrm{Tw}(L)$ be the twistor transform of the hyperholomorphic bundle L. We will examine global sections of the vector bundle $\mathrm{Tw}(L^*) \otimes E$ on $\mathrm{Tw}(M)$. Applying the semicontinuity theorem ([41], p. 5, see also Theorem III.12.8 in [22] for the algebraic version) to the twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$, we see that there exists an integer $m \geq 1$ such that $\dim H^0(M_I, L_I^* \otimes E_I) = m$ on a non-empty Zariski open subset of $\mathbb{CP}^1$. Since the pushforward sheaf $\pi_*(\mathrm{Tw}(L^*) \otimes E)$ is torsion-free, and torsion-free sheaves on $\mathbb{CP}^1$ are locally free, we conclude that $\pi_*(\mathrm{Tw}(L^*) \otimes E)$ is a vector bundle of rank m on $\mathbb{CP}^1$. By the Birkhoff-Grothendieck theorem (Theorem 2.1.1 in Chapter 1 of [41]), $\pi_*(\mathrm{Tw}(L^*) \otimes E)$ splits as a direct sum of line bundles. It's clear that the evaluation map

$$\pi^*\left(\pi_*(\mathrm{Tw}(L^*) \otimes E)\right) \longrightarrow \mathrm{Tw}(L^*) \otimes E$$

is nonzero, hence for a suitable $d \in \mathbb{Z}$ we can find a line subbundle

$$0 \longrightarrow \mathcal{O}_{\mathbb{CP}^1}(d) \longrightarrow \pi_*(\mathrm{Tw}(L^*) \otimes E)$$

such that, when taking the pullback along π, we get a nonzero composition of morphisms

$$\mathcal{O}_{\mathrm{Tw}(M)}(d) := \pi^*\left(\mathcal{O}_{\mathbb{CP}^1}(d)\right) \longrightarrow \pi^*\left(\pi_*(\mathrm{Tw}(L^*) \otimes E)\right) \longrightarrow \mathrm{Tw}(L^*) \otimes E.$$

But tensoring with $\mathrm{Tw}(L)$, we get a sheaf monomorphism $\mathrm{Tw}(L)(d) \hookrightarrow E$. This contradicts the irreduciblility of E. It follows that E has to be generically fibrewise stable.

Now let E be an irreducible rank 3 bundle on $\mathrm{Tw}(M)$, and suppose E is not generically fibrewise stable. Again, by Theorem 4.1.2, E_I admits a destabilizing subsheaf $\forall I \in \mathbb{CP}^1$. In the notation of the proof of Theorem 4.1.2, we have

$$\mathbb{CP}^1 = \left(\mathbb{CP}^1_{\geq 0}\right)^1 \cup \left(\mathbb{CP}^1_{\geq 0}\right)^2,$$

and since these subsets are Zariski closed, it follows that one of them is equal to the whole $\mathbb{CP}^1$. If $\left(\mathbb{CP}^1_{\geq 0}\right)^1 = \mathbb{CP}^1$, then a repeat of the argument for the case $\operatorname{rk} E = 2$ gives a line subsheaf of E. In case

$$\left(\mathbb{CP}^1_{\geq 0}\right)^2 = \left\{I : \exists L_I \in \operatorname{Pic} M_I \text{ with } \deg L_I \geq 0 \text{ and } L_I \hookrightarrow C_2(E_I) \subseteq \Lambda^2 E_I\right\} = \mathbb{CP}^1,$$

observing that the cone subbundle of exterior monomials $C_2(E) \subseteq \Lambda^2 E$ is equal to the whole $\Lambda^2 E$ for a rank 3 vector bundle, we repeat the same argument as above with E replaced by $\Lambda^2 E$ to conclude the existence of a hyperholomorphic line bundle L and an integer $d \in \mathbb{Z}$ such that there exists a sheaf monomorphism $\mathrm{Tw}(L)(d) \hookrightarrow \Lambda^2 E$ on $\mathrm{Tw}(M)$. Since the image of this morphism clearly lies in $C_2(E) = \Lambda^2 E$, we can use it to construct a rank 2 subsheaf of E (see proof of Proposition 2.15 in

[46]), contradicting irreducibility. We thus conclude that E is generically fibrewise stable. □

Before proving the converse for the case that E is generically fibrewise simple, let us try to see if our proof for the case $\operatorname{rk} E = 3$ would generalize to irreducible bundles of a general rank r. Arguing by contradiction and assuming E is not generically fibrewise stable, we can again deduce from Theorem 4.1.2 that all E_I admit destabilizing subsheaves over M_I, from which it follows that

$$\mathbb{CP}^1 = \left(\mathbb{CP}^1_{\geq 0}\right)^1 \cup \left(\mathbb{CP}^1_{\geq 0}\right)^2 \cup \ldots \cup \left(\mathbb{CP}^1_{\geq 0}\right)^{r-1}.$$

Since this is a union of Zariski closed subsets of $\mathbb{CP}^1$, we must have

$$\mathbb{CP}^1 = \left(\mathbb{CP}^1_{\geq 0}\right)^s = \{I : \exists L_I \in \operatorname{Pic} M_I \text{ with } \deg L_I \geq 0 \text{ and } L_I \hookrightarrow C_s(E_I) \subseteq \Lambda^s E_I\}$$

for some $1 \leq s \leq r-1$, and so a connected component of the intersection of analytic spaces $\operatorname{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(E)^s \cap \operatorname{Quot}^1_{\mathrm{lf},\mathbb{CP}^1}(\Lambda^s E)_{\geq 0}$ projects onto $\mathbb{CP}^1$. Just as above, this connected component is associated to a hyperholomorphic line bundle L on M and we can conclude that $\pi_*(\mathrm{Tw}(L^*) \otimes \Lambda^s E)$ is a nonzero vector bundle on $\mathbb{CP}^1$. In fact, by Grauert's theorem (Theorem 10.5.5 in [20]), outside a finite set in $\mathbb{CP}^1$, the fibre of $\pi_*(\mathrm{Tw}(L^*) \otimes \Lambda^s E)$ at $I \in \mathbb{CP}^1$ has the form

$$\pi_*(\mathrm{Tw}(L^*) \otimes \Lambda^s E)_I \cong H^0(M_I, L_I^* \otimes \Lambda^s E_I) = \operatorname{Hom}_{M_I}(L_I, \Lambda^s E_I).$$

However, unless $s = 1$ or $r - 1$, it's no longer true that $C_s(E_I) = \Lambda^s E_I$, so that taking an arbitrary line subbundle

$$0 \longrightarrow \mathcal{O}_{\mathbb{CP}^1}(d) \longrightarrow \pi_*(\mathrm{Tw}(L^*) \otimes \Lambda^s E),$$

on $\mathbb{CP}^1$ no longer guarantees that the corresponding sheaf monomorphism

$$0 \longrightarrow \mathrm{Tw}(L)(d) \longrightarrow \Lambda^s E$$

on $\mathrm{Tw}(M)$ will take values in $C_s(E)$, so it will not in general give a rank s subsheaf of E. So while for every $I \in \mathbb{CP}^1$ there exist sheaf monomorphisms $\varphi : L_I \to \Lambda^s E_I$ over M_I taking values in $C_s(E_I)$, it's not apparent that they can be "glued" into a global monomorphism over $\mathrm{Tw}(M)$. Thus, in case $s \neq 1, r-1$, a direct generalization of the argument for the case $\mathrm{rk}\, E = 3$ fails, and we need to make further efforts to arrive at a contradiction.

To describe this problem slightly differently, take the projectivization of the vector bundle $N = \pi_*(\mathrm{Tw}(L^*) \otimes \Lambda^s E)$ on $\mathbb{CP}^1$, and note that there is a 1-to-1 correspondence between line subbundles of N and sections of the projection $v : \mathbb{P}(N) \to \mathbb{CP}^1$. By Grauert's theorem, the generic fibre of v looks like

$$v^{-1}(I) = \mathbb{P}(\mathrm{Hom}_{M_I}(L_I, \Lambda^s E_I)),$$

and since for an element $\varphi \in \mathrm{Hom}_{M_I}(L_I, \Lambda^s E_I)$ taking values in $C_s(E_I)$, every multiple $a\varphi$ for $a \in \mathbb{C}$ clearly also takes values in $C_s(E_I)$, we get a well-defined closed analytic subset

$$\begin{array}{ccc} Y = \{(I,[\varphi]) \mid \varphi : L_I \hookrightarrow \Lambda^s E_I \text{ takes values in } C_s(E_I)\} & \hookrightarrow & \mathbb{P}(N) \\ & \overset{u}{\searrow} & \downarrow{\scriptstyle v} \\ & & \mathbb{CP}^1, \end{array}$$

where the map u is surjective. The problem then reduces to finding a section of the map $u : Y \to \mathbb{CP}^1$, which would give a line subbundle $\mathcal{O}_{\mathbb{CP}^1}(d) \hookrightarrow \pi_*(\mathrm{Tw}(L^*) \otimes \Lambda^s E)$, from which one can construct a rank s subsheaf of E on $\mathrm{Tw}(M)$, as described in the previous paragraph. Note that at this point it becomes a purely algebraic problem, since $\mathbb{P}(N)$ is projective algebraic, being the projectivization of a vector bundle on $\mathbb{CP}^1$, and thus so is Y, by Chow's theorem ([21], p. 167). Unfortunately, we don't have any information about the structure of Y, so we cannot assume the existence of a section of u. However, we have the following algebraic result.

Lemma 4.2.2. *Let $u : Y \to C$ be a surjective morphism of complex projective varieties, where C is a smooth curve. There always exists a* multisection *of u, in other words, an algebraic curve X, which we can assume to be smooth and projective, together with a branched cover $f : X \to C$, and a morphism $s : X \to Y$, making the diagram*

$$\begin{array}{ccc} & & Y \\ & \overset{s}{\nearrow} & \downarrow{\scriptstyle u} \\ X & \xrightarrow[f]{} & C \end{array}$$

commute.

Proof. We use the language of schemes (see [22], Chapter II). Let $K(C)$ denote the function field of C, and let $\overline{K(C)}$ be its algebraic closure. We take the fibred product of $u : Y \to C$ with the composition of canonical morphisms $\operatorname{Spec} \overline{K(C)} \to \operatorname{Spec} K(C) \to C$:

$$\begin{tikzcd} Y \times_C \overline{K(C)} \arrow[r] \arrow[d] & Y \arrow[d, "u"] \\ \operatorname{Spec} \overline{K(C)} \arrow[r] & C \end{tikzcd}$$

By the scheme-theoretic version of the Nullstellensatz (see Proposition 1.8.3 and the discussion following it in [33]), there exists a $\overline{K(C)}$-rational point of $Y \times_C \overline{K(C)}$, i.e. a morphism $\operatorname{Spec} \overline{K(C)} \to Y \times_C \overline{K(C)}$ over $\overline{K(C)}$. From the diagram above, it's clear that such $\overline{K(C)}$-rational points are in a 1-to-1 correspondence with morphisms $\operatorname{Spec} \overline{K(C)} \to Y$ making

$$\begin{tikzcd} & Y \arrow[d, "u"] \\ \operatorname{Spec} \overline{K(C)} \arrow[ur] \arrow[r] & C \end{tikzcd}$$

commute. We fix such a morphism $\operatorname{Spec} \overline{K(C)} \to Y$, and choose open affine subschemes $V = \operatorname{Spec} B \subseteq Y$, $U = \operatorname{Spec} A \subseteq C$ such that $\operatorname{Spec} \overline{K(C)}$ maps into V and $u(V) \subseteq U$:

$$\begin{tikzcd} & V = \operatorname{Spec} B \arrow[d, "u"] \\ \operatorname{Spec} \overline{K(C)} \arrow[ur] \arrow[r] & U = \operatorname{Spec} A \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dl] \\ \overline{K(C)} & A \arrow[l, hook'] \arrow[u, hook, "\tilde{u}"'] \end{tikzcd} \tag{4.1}$$

Now observe that the subfield $L \subseteq \overline{K(C)}$ generated by the image of B contains $K(C)$, and in fact $K(C) \hookrightarrow L$ is a finite field extension, since B is a finitely-generated A-algebra. By the correspondence between smooth projective curves and finitely-generated extension fields of $\mathbb{C}$ of transcendence degree 1 (see Corollary I.6.12 in [22]), there exists a unique smooth projective curve X such that $K(X) = L$ and a unique dominant morphism $f : X \to C$ that induces our field extension $K(C) \hookrightarrow K(X) = L$ on the level of function fields. Let $W = \operatorname{Spec} R \subseteq X$ be any open subscheme such that $f(W) \subseteq U$, so that we have commutative diagrams

$$\begin{array}{ccc} W = \operatorname{Spec} R & \xrightarrow[f]{} & U = \operatorname{Spec} A \\ \uparrow & & \uparrow \\ \operatorname{Spec} L & \longrightarrow & \operatorname{Spec} K(C) \end{array} \qquad\qquad \begin{array}{ccc} R & \xleftarrow[\tilde{f}]{} & A \\ \downarrow & & \downarrow \\ L & \longleftarrow & K(C) \end{array} \tag{4.2}$$

By shrinking W if necessary, we can make sure that the homomorphism $B \to L \subseteq \overline{K(C)}$ in the diagram (4.1) can be written as the composition

$$B \overset{\tilde{s}}{\to} R \hookrightarrow L \hookrightarrow \overline{K(C)},$$

where the homomorphism $\tilde{s} : B \to R$ induces a morphism of affine schemes $s : W = \operatorname{Spec} R \to V = \operatorname{Spec} B$. Combining the diagrams (4.1) and (4.2), we get

$$\begin{array}{ccccc} & & & & V = \operatorname{Spec} B \\ & & & \overset{s}{\nearrow} & \downarrow u \\ \operatorname{Spec} \overline{K(C)} & \longrightarrow & W = \operatorname{Spec} R & \xrightarrow[f]{} & U = \operatorname{Spec} A \\ & \searrow & \uparrow & & \uparrow \\ & & \operatorname{Spec} L & \longrightarrow & \operatorname{Spec} K(C) \end{array} \qquad \begin{array}{ccccc} & & & & B \\ & & & \overset{\tilde{s}}{\swarrow} & \uparrow \tilde{u} \\ \overline{K(C)} & \longleftarrow & R & \xleftarrow[\tilde{f}]{} & A \\ & \nwarrow & \downarrow & & \downarrow \\ & & L & \longleftarrow & K(C) \end{array}$$

We have $\tilde{f} = \tilde{s} \circ \tilde{u}$ in the diagram on the right (and hence $f = u \circ s$ in the diagram on the left) since the upper and lower paths from A to $\overline{K(C)}$ are the same by virtue of (4.1), the lower triangle and square are commutative, and $R \hookrightarrow \overline{K(C)}$ is injective. It only remains to observe that the commutative diagram

$$\begin{array}{ccc} & & Y \\ & {}^{s}\nearrow & \big\downarrow u \\ W & \xrightarrow[f]{} & C \end{array} \quad \text{can be completed to} \quad \begin{array}{ccc} & & Y \\ & {}^{s}\nearrow & \big\downarrow u \\ X & \xrightarrow[f]{} & C \end{array}$$

since X is smooth and Y is projective (see Proposition I.6.8 in [22]). □

Going back to the map $u : Y \to \mathbb{CP}^1$ we have constructed previously and applying this lemma, we get a multisection of u over some branched cover $f : X \to \mathbb{CP}^1$. We proceed as follows to arrive at a contradiction.

1. We take the fibred product Z of $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$ and $f : X \to \mathbb{CP}^1$ as in the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & \mathrm{Tw}(M) \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ X & \xrightarrow[f]{} & \mathbb{CP}^1 \end{array}$$

   and use the multisection obtained above to construct a subsheaf $\mathcal{F} \subseteq \varphi^* E$ of rank s on Z.

2. We take the pushforward of $\mathcal{F}$ and $\varphi^* E$ along φ to obtain a rank s subsheaf $\varphi_*(\mathcal{F}) \subseteq \varphi_*(\varphi^* E)$ over $\mathrm{Tw}(M)$. We show that $\varphi_*(\varphi^* E)$ is a direct sum of copies of E twisted by some divisors on $\mathrm{Tw}(M)$:

$$\varphi_*(\varphi^* E) \cong E(D_1) \oplus \ldots \oplus E(D_d).$$

3. In view of the above direct sum decomposition of $\varphi_*(\varphi^* E)$ and the irreducibility of E, we show that the subsheaf $\varphi_*(\mathcal{F}) \subseteq E(D_1) \oplus \ldots \oplus E(D_d)$ is essentially

isomorphic (in a sense to be made precise) to a direct sum of some of the $E(D_1), \ldots, E(D_d)$, say $E(D_1) \oplus \ldots \oplus E(D_t)$ with $t < d$.

4. Identifying $\varphi_*(\varphi^* E)$ with $E(D_1) \oplus \ldots \oplus E(D_d)$ and $\varphi_*(\mathcal{F})$ with $E(D_1) \oplus \ldots \oplus E(D_t)$, we now use the generic fibrewise simplicity assumption on E to deduce that the sheaf monomorphism from $E(D_1) \oplus \ldots \oplus E(D_t)$ to $E(D_1) \oplus \ldots \oplus E(D_d)$ has a particularly simple form, namely that of a $d \times t$ matrix of meromorphic functions on $\mathbb{CP}^1$. From this we can get a contradiction to the fact that $\mathcal{F}$ is a proper subsheaf of $\varphi^* E$ of lower rank on Z.

To sum up the above in one sentence, the irreducibility of E and the fact that is is generically fibrewise simple put rigid conditions on subsheaves of direct sums of copies of E on $\mathrm{Tw}(M)$, from which one concludes that the pullback bundle $\varphi^* E$ on Z is irreducible as well, and this gives a contradiction to the fact that the multisection constructed in Lemma 4.2.2 can be used to obtain a proper subsheaf of $\varphi^* E$ of lower rank. We now proceed with the rest of the proof of the theorem.

Proof of converse of Theorem 4.2.1 for the case E generically fibrewise simple. Let E be a vector bundle of rank r on $\mathrm{Tw}(M)$ which is irreducible and generically fibrewise simple. Assume E_I is non-stable for infinitely many $I \in \mathbb{CP}^1$. By Theorem 4.1.2, E is non-stable for all $I \in \mathbb{CP}^1$, and by the proof of Theorem 4.1.2, there exists a number $1 \leq s \leq r-1$ and a hyperholomorphic line bundle L on M such that for all $I \in \mathbb{CP}^1$ there are non-trivial morphisms

$$L_I \longrightarrow C_s(E_I) \subseteq \Lambda^s E_I$$

over M_I, where as usual $C_s(E_I)$ denotes the cone subbundle of exterior monomials in $\Lambda^s E_I$. If $s = 1$ or $r - 1$, $C_s(E_I) = \Lambda^s E_I$, and an argument entirely analogous to the case $\operatorname{rk} E = 3$ shows that these morphisms can be assembled into a line subsheaf of $\Lambda^s E$ on $\mathrm{Tw}(M)$ taking values in $C_s(E)$, which in turn contradicts the irreducibility of E, proving that E is generically fibrewise stable. Assume $1 < s < r - 1$.

By a previous discussion, $\pi_*(\mathrm{Tw}(L^*) \otimes \Lambda^s E$ is a nonzero vector bundle on $\mathbb{CP}^1$ whose generic fibre is isomorphic to $\mathrm{Hom}_{M_I}(L_I, \Lambda^s E_I)$. Taking the corresponding projective bundle $\mathbb{P}\left(\pi_*(\mathrm{Tw}(L^*) \otimes \Lambda^s E\right)$, we have the closed algebraic subvariety

$$\begin{array}{ccc} Y = \{(I, [\varphi]) \mid \varphi : L_I \hookrightarrow C_s(E_I) \subseteq \Lambda^s E_I\} & \hookrightarrow & \mathbb{P}\left(\pi_*(\mathrm{Tw}(L^*) \otimes \Lambda^s E\right), \\ \downarrow u & \swarrow v & \\ \mathbb{CP}^1 & & \end{array}$$

where u is surjective. Applying Lemma 4.2.2, there exists a multisection

$$\begin{array}{ccc} & \nearrow^{s} & Y \\ & & \downarrow u \\ X & \xrightarrow{f} & \mathbb{CP}^1 \end{array}$$

where $f : X \to \mathbb{CP}^1$ is branched cover of degree d. Taking the fibred product

$$Y \times_{\mathbb{CP}^1} X \subseteq \mathbb{P}(f^*(\pi_*(\mathrm{Tw}(L^*) \otimes \Lambda^s E))),$$

our multisection $s : X \to Y$ gives a section of the morphism $Y \times_{\mathbb{CP}^1} X \to X$, so we obtain a line subbundle

$$0 \longrightarrow \mathcal{O}_X(\widetilde{D}) \longrightarrow f^*(\pi_*(\mathrm{Tw}(L^*) \otimes \Lambda^s E))$$

on X, where $\widetilde{D}$ is some divisor. By construction, over a generic point $P \in X$, this subbundle is taking values in $f^*(L^*_{f(P)} \otimes C_s(E_{f(P)}))$. Let Z denote the fibred product of $f : X \to \mathbb{CP}^1$ and the twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$, as in the diagram

$$\begin{CD} Z @>{\varphi}>> \mathrm{Tw}(M) \\ @V{\rho}VV @VV{\pi}V \\ X @>>{f}> \mathbb{CP}^1 \end{CD} \tag{4.3}$$

On X, there is a canonical morphism,

$$f^*(\pi_*(\mathrm{Tw}(L^*) \otimes \Lambda^s E)) \longrightarrow \rho_*(\varphi^*(\mathrm{Tw}(L^*) \otimes \Lambda^s E)),$$

which over a generic point of X is an isomorphism, since f is biholomorphic in a neighborhood of every $P \in X$ except at branch points, of which there are finitely many. Composing this with the morphism $\mathcal{O}_X(\widetilde{D}) \to f^*(\pi_*(\mathrm{Tw}(L^*) \otimes \Lambda^s E))$ constructed above, we get a line subsheaf

$$0 \longrightarrow \mathcal{O}_X(\widetilde{D}) \longrightarrow \rho_*(\varphi^*(\mathrm{Tw}(L^*) \otimes \Lambda^s E)) = \rho_*(\varphi^*(\mathrm{Tw}(L))^* \otimes \varphi^*(\Lambda^s E)),$$

and pulling back along ρ to Z, we take the composition

$$\mathcal{O}_Z(\widetilde{D}) \coloneqq \rho^*(\mathcal{O}_X(\widetilde{D})) \to \rho^*(\rho_*(\varphi^*(\mathrm{Tw}(L))^* \otimes \varphi^*(\Lambda^s E))) \to \varphi^*(\mathrm{Tw}(L))^* \otimes \varphi^*(\Lambda^s E),$$

which is a monomorphism since it is nonzero, and which takes values in $\varphi^*(\mathrm{Tw}(L))^* \otimes \varphi^*(C_s(E))$ by construction. Tensoring with $\varphi^*(\mathrm{Tw}(L))$, we get

$$0 \longrightarrow \varphi^*(\mathrm{Tw}(L))(\widetilde{D}) \longrightarrow \varphi^*(\Lambda^s E),$$

which takes values in $\varphi^*(C_s(E)) = C_s(\varphi^* E)$. The line subsheaf $\varphi^*(\mathrm{Tw}(L))(\widetilde{D}) \subseteq \varphi^*(\Lambda^s E)$ gives rise to a rank s subsheaf $\mathcal{F} \subseteq \varphi^* E$ on Z; replacing $\mathcal{F}$ by a normal extension in $\varphi^* E$ if needed (see [41], p. 80), we can assume that $\mathcal{F}$ is a normal subsheaf of $\varphi^* E$ over Z.

Taking the pushforward of the sheaf monomorphism $\mathcal{F} \hookrightarrow \varphi^* E$ along the map φ, we obtain by the left-exactness of φ_* a sheaf monomorphism $\varphi_*(\mathcal{F}) \hookrightarrow \varphi_*(\varphi^* E)$ on $\mathrm{Tw}(M)$, which we will denote by γ. Since we assumed that $\mathcal{F}$ was normal over Z, it's not hard to see that $\varphi_*(\mathcal{F})$ is normal on $\mathrm{Tw}(M)$ as well. As for $\varphi_*(\varphi^* E)$, it happens to be a vector bundle whose structure can be described nicely in terms of the original bundle E. In the diagram (4.3) we have an isomorphism

$$\varphi_*(\rho^*(\mathcal{O}_X)) \cong \pi^*(f_*(\mathcal{O}_X))$$

(see Theorem III.3.10 and Theorem III.3.4 in [1]). Using the Birkhoff-Grothendieck theorem, we can write

$$\varphi_*(\mathcal{O}_Z) = \varphi_*(\rho^*(\mathcal{O}_X)) \cong \pi^*(f_*(\mathcal{O}_X)) \cong \pi^*\left(\bigoplus_{l=1}^{d} \mathcal{O}_{\mathbb{CP}^1}(D_l)\right) = \bigoplus_{l=1}^{d} \mathcal{O}_{\mathrm{Tw}(M)}(D_l),$$

where $D_1, \ldots, D_d$ are some divisors on $\mathbb{CP}^1$; we use the same notation for the corresponding divisors on $\mathrm{Tw}(M)$, which should cause no confusion in view of Corollary 2.3.14. Using this decomposition and the projection formula ([41], p. 6), we have

$$\varphi_*(\varphi^* E) = \varphi_*(\varphi^* E \otimes \mathcal{O}_Z) \cong E \otimes \varphi_*(\mathcal{O}_Z) \cong E(D_1) \oplus \ldots \oplus E(D_d).$$

We thus have a sheaf monomorphism $\gamma : \varphi_*(\mathcal{F}) \hookrightarrow E(D_1) \oplus \ldots \oplus E(D_d)$ on $\mathrm{Tw}(M)$. For any subset $\{i_1, i_2, \ldots, i_t\} \subseteq \{1, 2, \ldots, d\}$, we have the usual projection

$$E(D_1) \oplus \ldots \oplus E(D_d) \longrightarrow E(D_{i_1}) \oplus \ldots \oplus E(D_{i_t}).$$

We would now like to show that there exists a choice of such a subset $\{i_1, i_2, \ldots, i_t\}$ so that the composition

$$\varphi_*(\mathcal{F}) \xrightarrow{\gamma} E(D_1) \oplus \ldots \oplus E(D_d) \longrightarrow E(D_{i_1}) \oplus \ldots \oplus E(D_{i_t})$$

is a monomorphism of sheaves with quotient being a torsion sheaf; this last condition is equivalent to the condition $\mathrm{rk}\,\varphi_*(\mathcal{F}) = \mathrm{rk}\,E(D_{i_1}) \oplus \ldots \oplus E(D_{i_t})$. Let $1 \leq j \leq d$ and look at the composition

$$\varphi_*(\mathcal{F}) \xrightarrow{\gamma} E(D_1) \oplus \ldots \oplus E(D_d) \longrightarrow \bigoplus_{l \neq j} E(D_l) = E(D_1) \oplus \ldots \oplus E(\widehat{D_j}) \oplus \ldots \oplus E(D_d).$$

Let K_j denote the kernel of this composition. We have the following diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_j & \longrightarrow & \varphi_*(\mathcal{F}) & \longrightarrow & \bigoplus_{l \neq j} E(D_l) & & \\
 & & \downarrow & & \downarrow \gamma & & \| & & \\
0 & \longrightarrow & E(D_j) & \longrightarrow & E(D_1) \oplus \ldots \oplus E(D_d) & \longrightarrow & \bigoplus_{l \neq j} E(D_l) & \longrightarrow & 0
\end{array}$$

It follows from the irreducibility of $E(D_j)$ that the induced morphism $K_j \to E(D_j)$ is either zero or $\mathrm{rk}\,K_j = \mathrm{rk}\,E(D_j)$. If the latter is true for every j from 1 to d, it's not hard to see that $\mathrm{rk}\,\varphi_*(\mathcal{F}) = \mathrm{rk}\,E(D_1) \oplus \ldots \oplus E(D_d)$, but this cannot be as $\varphi_*(\mathcal{F})$ is a subsheaf of $\varphi_*(\varphi^* E)$ of lower rank. Thus for some $j = j_1$, $K_{j_1} = 0$, and so the composition

$$\varphi_*(\mathcal{F}) \xrightarrow{\gamma} E(D_1) \oplus \ldots \oplus E(D_d) \longrightarrow \bigoplus_{l \neq j_1} E(D_l)$$

must be a monomorphism. If $\operatorname{rk}\varphi_*(\mathcal{F}) = \operatorname{rk}\bigoplus_{l\neq j_1} E(D_l)$, we stop here. If not, we repeat the argument above with $\{1,\ldots,d\}$ replaced by $\left\{1,\ldots,\widehat{j_1},\ldots,d\right\}$ to conclude the existence of an index $j_2 \in \left\{1,\ldots,\widehat{j_1},\ldots,d\right\}$ so that the composition

$$\varphi_*(\mathcal{F}) \longrightarrow \bigoplus_{l\neq j_1} E(D_l) \longrightarrow \bigoplus_{l\neq j_1, j_2} E(D_l)$$

is still a monomorphism. Continuing in this manner, we eventually arrive at a monomorphism $\varphi_*(\mathcal{F}) \hookrightarrow E(D_{i_1}) \oplus \ldots \oplus E(D_{i_t})$ with $\operatorname{rk}\varphi_*(\mathcal{F}) = \operatorname{rk} E(D_{i_1}) \oplus \ldots \oplus E(D_{i_t})$. We cannot have $t = 0$ as this would imply $\varphi_*(\mathcal{F}) = 0$, contrary to the construction of $\mathcal{F}$. Rearranging indices if necessary, we can assume that $E(D_{i_1}) \oplus \ldots \oplus E(D_{i_t}) = E(D_1) \oplus \ldots \oplus E(D_t)$. We denote the morphism that we just constructed by

$$\mu : \varphi_*(\mathcal{F}) \longrightarrow E(D_1) \oplus \ldots \oplus E(D_t),$$

and taking its quotient, we get the short exact sequence

$$0 \longrightarrow \varphi_*(\mathcal{F}) \xrightarrow{\mu} E(D_1) \oplus \ldots \oplus E(D_t) \longrightarrow \mathcal{T} \longrightarrow 0, \tag{4.4}$$

where $\mathcal{T}$ is a torsion sheaf because of rank considerations. Thus, the morphism $\mu : \varphi_*(\mathcal{F}) \to E(D_1) \oplus \ldots \oplus E(D_t)$ has an inverse on the open set $\operatorname{Tw}(M) \smallsetminus \operatorname{Supp}(\mathcal{T})$ with a complement of positive codimension, and we would like to extend this inverse to the whole $\operatorname{Tw}(M)$ in some way.

We start by observing that $\operatorname{Supp}(\mathcal{T})$ has pure codimension 1 in $\operatorname{Tw}(M)$, in other words, it lies on a divisor. Indeed, let $x \in \operatorname{Tw}(M)$, and suppose $\mathcal{T}_x$ contains a nonzero element ζ, defined over a neighborhood $U \ni x$. Suppose $A = \mathrm{Supp}(\zeta) \subseteq U$ has codimension ≥ 2 in U. Shrinking U if necessary, we can assume that $\zeta \in \mathcal{T}(U)$ comes from an element $\xi \in (E(D_1) \oplus \ldots \oplus E(D_t))(U)$. We have the following diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & \varphi_*(\mathcal{F})(U) & \xrightarrow{\mu(U)} & (E(D_1) \oplus \ldots \oplus E(D_t))(U) & \longrightarrow & \mathcal{T}(U) \\
& \downarrow \cong & & \downarrow \cong & & \downarrow \\
0 \longrightarrow & \varphi_*(\mathcal{F})(U \smallsetminus A) & \xrightarrow{\mu(U \smallsetminus A)} & (E(D_1) \oplus \ldots \oplus E(D_t))(U \smallsetminus A) & \longrightarrow & \mathcal{T}(U \smallsetminus A)
\end{array}$$

The first and second vertical arrows are isomorphisms because the corresponding sheaves are normal. As the restriction of ζ to $U \smallsetminus A$ is zero, it follows by the exactness of the second row that the restriction of ξ to $U \smallsetminus A$ comes from $\varphi_*(\mathcal{F})(U \smallsetminus A)$. But then the same must hold over U, so by the exactness of the first row, ζ must be zero over U, which is a contradiction. It follows from this that $\mathrm{Supp}(\zeta) \subseteq U$ cannot have components of codimension ≥ 2. Globally, it means that $\mathrm{Supp}(\mathcal{T})$ must have pure codimension 1.

In order to construct an inverse to the morphism $\mu : \varphi_*(\mathcal{F}) \to E(D_1) \oplus \ldots \oplus E(D_t)$ on $\mathrm{Tw}(M)$, we first need to pass to a different category. Let $\mathcal{M}$ denote the sheaf of meromorphic functions on $\mathbb{CP}^1$, and let $\pi^*\mathcal{M}$ be its pullback to $\mathrm{Tw}(M)$. As $\pi^*\mathcal{M}$ is a sheaf of rings on $\mathrm{Tw}(M)$ (in fact, a sheaf of $\mathcal{O}_{\mathrm{Tw}(M)}$-algebras), it gives rise to the corresponding abelian category $\pi^*\mathcal{M}$-**Mod**. Tensoring μ with $\pi^*\mathcal{M}$, we get a morphism in this category, which we will denote by the same letter:

$$\mu : \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M} \longrightarrow [E(D_1) \oplus \ldots \oplus E(D_t)] \otimes \pi^*\mathcal{M}.$$

Moreover, by tensoring the short exact sequence (4.4) with $\pi^*\mathcal{M}$, we get the sequence

$$0 \longrightarrow \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M} \xrightarrow{\mu} [E(D_1) \oplus \ldots \oplus E(D_t)] \otimes \pi^*\mathcal{M} \longrightarrow \mathcal{T} \otimes \pi^*\mathcal{M} \longrightarrow 0 \quad (4.5)$$

in $\pi^*\mathcal{M}$**-Mod**, which is also exact. To see this, note that the sheaf $\mathcal{M}$ is flat over $\mathbb{CP}^1$. Indeed, since the stalk of $\mathcal{O}_{\mathbb{CP}^1}$ over any point in $\mathbb{CP}^1$ is a PID, the flatness condition of $\mathcal{M}$ is equivalent to its stalks being torsion-free, which they certainly are. It then easily follows that $\pi^*\mathcal{M}$ is flat over $\mathrm{Tw}(M)$. In the above short exact sequence, we have $\mathcal{T} \otimes \pi^*\mathcal{M} = 0$. Indeed, for any $x \in \mathrm{Tw}(M)$ with $\mathcal{T}_x \neq 0$, we know that the support of every element $\zeta \in \mathcal{T}_x$ lies on the hypersurface $V = \pi^{-1}(\pi(x))$. Choosing a local coordinate z about $\pi(x) \in \mathbb{CP}^1$, the fact that $\mathcal{T}_x$ is a finitely-generated $\mathcal{O}_x$-module guarantees that a sufficiently high power of the local coordinate z^k annihilates $\mathcal{T}_x$. Then for any element $\zeta \otimes f \in \mathcal{T}_x \otimes (\pi^*\mathcal{M})_x$, we have

$$\zeta \otimes f = \zeta \cdot z^k \otimes z^{-k} f = 0.$$

Thus, $\mathcal{T} \otimes \pi^*\mathcal{M} = 0$, and so $\mu : \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M} \to [E(D_1) \oplus \ldots \oplus E(D_t)] \otimes \pi^*\mathcal{M}$ is an isomorphism of $\pi^*\mathcal{M}$-modules. So it has an inverse in the category $\pi^*\mathcal{M}$**-Mod**, which we denote by

$$\eta : [E(D_1) \oplus \ldots \oplus E(D_t)] \otimes \pi^*\mathcal{M} \longrightarrow \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}.$$

The $\mathcal{O}_{\mathrm{Tw}(M)}$-algebra structure on $\pi^*\mathcal{M}$ gives rise to the natural functor

$$\begin{array}{rcccl} - \otimes \pi^*\mathcal{M} & : & \mathcal{O}_{\mathrm{Tw}(M)}\textbf{-Mod} & \longrightarrow & \pi^*\mathcal{M}\textbf{-Mod}, \\ & & \mathcal{G} & \longmapsto & \mathcal{G} \otimes \pi^*\mathcal{M} \end{array}$$

which, as we have seen above, is exact. On the other hand, there is a natural forgetful functor in the other direction,

$$\begin{array}{rcccc} (-)_{\mathcal{O}_{\mathrm{Tw}(M)}} & : & \pi^*\mathcal{M}\textbf{-Mod} & \longrightarrow & \mathcal{O}_{\mathrm{Tw}(M)}\textbf{-Mod}, \\ & & \mathcal{H} & \longmapsto & \mathcal{H}_{\mathcal{O}_{\mathrm{Tw}(M)}} \end{array}$$

where by $\mathcal{H}_{\mathcal{O}_{\mathrm{Tw}(M)}}$ we simply mean the sheaf of $\pi^*\mathcal{M}$-modules $\mathcal{H}$ viewed as a sheaf of $\mathcal{O}_{\mathrm{Tw}(M)}$-modules; if the context is clear, we will usually denote $\mathcal{H}_{\mathcal{O}_{\mathrm{Tw}(M)}}$ by $\mathcal{H}$ as well. Just like for ring extensions, the tensor product functor and the forgetful functor described above are adjoint to each other; namely, for an arbitrary $\mathcal{O}_{\mathrm{Tw}(M)}$-module $\mathcal{G}$ and an arbitrary $\pi^*\mathcal{M}$-module $\mathcal{H}$ we have a one-to-one correspondence

$$\begin{array}{ccc} \mathrm{Hom}_{\pi^*\mathcal{M}}\left(\mathcal{G}\otimes\pi^*\mathcal{M},\mathcal{H}\right) & \overset{\sim}{\longleftrightarrow} & \mathrm{Hom}_{\mathcal{O}_{\mathrm{Tw}(M)}}\left(\mathcal{G},\mathcal{H}_{\mathcal{O}_{\mathrm{Tw}(M)}}\right), \\ \mathcal{G}\otimes\pi^*\mathcal{M}\longrightarrow\mathcal{H} & \longmapsto & \mathcal{G}\longrightarrow\mathcal{G}\otimes\pi^*\mathcal{M}\longrightarrow\mathcal{H} \\ \mathcal{G}\otimes\pi^*\mathcal{M}\longrightarrow\mathcal{H} & \longleftarrow & \mathcal{G}\longrightarrow\mathcal{H} \end{array}$$

where the natural morphism $\mathcal{G}\to\mathcal{G}\otimes\pi^*\mathcal{M}$ is obtained by tensoring the inclusion $\mathcal{O}_{\mathrm{Tw}(M)}\hookrightarrow\pi^*\mathcal{M}$ with $\mathcal{G}$. Taking $\mathcal{G}=E(D_1)\oplus\ldots\oplus E(D_t)$, $\mathcal{H}=\varphi_*(\mathcal{F})\otimes\pi^*\mathcal{M}$ in the above correspondence, the isomorphism of $\pi^*\mathcal{M}$-modules $\eta:[E(D_1)\oplus\ldots\oplus E(D_t)]\otimes\pi^*\mathcal{M}\longrightarrow\varphi_*(\mathcal{F})\otimes\pi^*\mathcal{M}$ that was constructed in the previous paragraph gets mapped to the composition

$$E(D_1)\oplus\ldots\oplus E(D_t)\longrightarrow[E(D_1)\oplus\ldots\oplus E(D_t)]\otimes\pi^*\mathcal{M}\longrightarrow\varphi_*(\mathcal{F})\otimes\pi^*\mathcal{M},$$

which we will denote by η as well. We would now like to show the existence of a divisor D on $\mathbb{CP}^1$ and a morphism of $\mathcal{O}_{\mathrm{Tw}(M)}$-modules $E(D_1)\oplus\ldots\oplus E(D_t)\to\varphi_*(\mathcal{F})(D)$

that completes the diagram

$$
\begin{array}{ccc}
 & & \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M} \\
 & \overset{\eta}{\nearrow} & \uparrow \\
E(D_1) \oplus \ldots \oplus E(D_t) & \dashrightarrow & \varphi_*(\mathcal{F})(D),
\end{array}
$$

where the vertical map is the natural morphism $\varphi_*(\mathcal{F})(D) \to \varphi_*(\mathcal{F})(D) \otimes \pi^*\mathcal{M} = \varphi_*(\mathcal{F}) \otimes \mathcal{O}_{\mathrm{Tw}(M)}(D) \otimes \pi^*\mathcal{M} \cong \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}$.

Our first goal is to show that for any divisor D the natural map $\varphi_*(\mathcal{F})(D) \to \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}$ is a monomorphism. We can combine the short exact sequences (4.4) and (4.5) into the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \varphi_*(\mathcal{F}) & \longrightarrow & E(D_1) \oplus \ldots \oplus E(D_t) & \longrightarrow & \mathcal{T} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M} & \xrightarrow{\cong} & [E(D_1) \oplus \ldots \oplus E(D_t)] \otimes \pi^*\mathcal{M} & \longrightarrow & 0 & &
\end{array}
$$

The second vertical arrow is a monomorphism since it is obtained by tensoring the inclusion $\mathcal{O}_{\mathrm{Tw}(M)} \hookrightarrow \pi^*\mathcal{M}$ with $E(D_1) \oplus \ldots \oplus E(D_t)$ which locally is just a direct sum of copies of $\mathcal{O}_{\mathrm{Tw}(M)}$. Then by commutativity and exactness it follows that the first vertical arrow $\varphi_*(\mathcal{F}) \to \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}$ is a monomorphism. Given any divisor D, twisting the above diagram by D and repeating the same argument shows that $\varphi_*(\mathcal{F})(D) \to \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}$ is a monomorphism as well. In this way, the collection $\{\varphi_*(\mathcal{F})(D) : D \text{ is a divisor on } \mathrm{Tw}(M)\}$ constitutes a hierarchy of subsheaves of $\varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}$, in the sense that for divisors $D \leq D'$ we have the commutative diagram

$$\begin{array}{ccc} \varphi_*(\mathcal{F})(D) & \hookrightarrow & \varphi_*(\mathcal{F})(D') \\ & \searrow \quad \swarrow & \\ & \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M} & \end{array}$$

In fact, it's not hard to see that $\varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}$ is the direct limit of the system $\{\varphi_*(\mathcal{F})(D)\}$.

Recall that we have the morphism $\eta : E(D_1) \oplus \ldots \oplus E(D_t) \to \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}$, which was obtained from the inverse to the morphism $\mu : \varphi_*(\mathcal{F}) \to E(D_1) \oplus \ldots \oplus E(D_t)$ in the category $\pi^*\mathcal{M}$-**Mod**. We would like to show that there exists a divisor D on $\mathrm{Tw}(M)$ such that the image of $\eta : E(D_1) \oplus \ldots \oplus E(D_t) \to \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}$ lies inside the subsheaf $\varphi_*(\mathcal{F})(D) \subseteq \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}$. Going back to the short exact sequence (4.4), it's clear that μ is an isomorphism outside $\mathrm{Supp}(\mathcal{T})$, so on the open set $\mathrm{Tw}(M) \setminus \mathrm{Supp}(\mathcal{T})$, η takes values in the subsheaf $\varphi_*(\mathcal{F}) \subseteq \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}$. On the other hand, let $x \in \mathrm{Supp}(\mathcal{T})$. We have previously shown that $\mathrm{Supp}(\mathcal{T})$ is a union of fibres of π; letting $V = \pi^{-1}(\pi(x))$, we can choose a local coordinate z about $\pi(x) \in \mathbb{CP}^1$ and a neighborhood $U \ni x$ in $\mathrm{Tw}(M)$ such that $U \cap \mathrm{Supp}(\mathcal{T}) \subseteq V$, and such that the vector bundle $E(D_1) \oplus \ldots \oplus E(D_t)$ trivializes on U with generating sections $\{s_1, s_2, \ldots, s_N\}$. The restriction of η to U is completely defined by its action on these generating sections, and shrinking U if necessary, we can write

$$\begin{array}{rcccc} \eta(U) & : & [E(D_1) \oplus \ldots \oplus E(D_t)](U) & \longrightarrow & [\varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}](U) \\ & & s_i & \longmapsto & u_i \otimes z^{k_i} \end{array}$$

for $1 \le i \le N$, where $u_i \in \varphi_*(\mathcal{F})(U)$ and z^{k_i} is some power of the local coordinate. Taking

$$k = -\min_{1 \le i \le N} k_i,$$

it's not hard to see that $\eta|_U$ takes values in the subsheaf $\varphi_*(\mathcal{F})(kV)|_U \subseteq \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}|_U$. Repeating the above argument for every point in V besides x, and invoking the compactness of V, we can conclude the existence of an integer $k_V \in \mathbb{Z}$ such that for an open neighborhood $W \supseteq V$ satisfying $W \cap \operatorname{Supp}(\mathcal{T}) \subseteq V$, we have

$$\operatorname{Im}(\eta|_W) \subseteq \varphi_*(\mathcal{F})(k_V V)|_W \subseteq \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}|_W.$$

Writing $\operatorname{Supp}(\mathcal{T}) = V_1 \cup \ldots V_n$ as a union of fibres of π, and repeating the above for every hypersurface $V_1, \ldots, V_n$, we can conclude that there exist integers $k_1, \ldots, k_n \in \mathbb{Z}$ such that, globally,

$$\operatorname{Im}(\eta) \subseteq \varphi_*(\mathcal{F})(k_1 V_1 + \ldots + k_n V_n) \subseteq \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}.$$

In other words, letting $D = k_1 V_1 + \ldots + k_n V_n$, the morphism $\eta : E(D_1) \oplus \ldots \oplus E(D_t) \to \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}$ factors through the subsheaf $\varphi_*(\mathcal{F})(D) \subseteq \varphi_*(\mathcal{F}) \otimes \pi^*\mathcal{M}$, giving us a morphism which we will denote by the same letter:

$$\eta : E(D_1) \oplus \ldots \oplus E(D_t) \longrightarrow \varphi_*(\mathcal{F})(D)$$

This morphism η is a partial inverse to the morphism $\mu : \varphi_*(\mathcal{F}) \to E(D_1) \oplus \ldots \oplus E(D_t)$ in the category $\mathcal{O}_{\mathrm{Tw}(M)}$-**Mod**, in the sense of the following commutative diagram:

$$\begin{array}{ccc} \varphi_*(\mathcal{F}) & \xrightarrow{\mu} & E(D_1) \oplus \ldots \oplus E(D_t) \\ \big\downarrow & \swarrow_{\eta} & \big\downarrow \\ \varphi_*(\mathcal{F})(D) & \xrightarrow[\mu(D)]{} & [E(D_1) \oplus \ldots \oplus E(D_t)](D) \end{array} \tag{4.6}$$

Note that outside $\mathrm{Supp}(D)$, μ and η are bona fide inverses of each other. Our next goal is to use the identification of $\varphi_*(\mathcal{F})$ with $E(D_1) \oplus \ldots \oplus E(D_t)$ via the above diagram to describe the sheaf monomorphism

$$\gamma : \varphi_*(\mathcal{F}) \longrightarrow \varphi_*(\varphi^* E) = E(D_1) \oplus \ldots \oplus E(D_d)$$

in a particularly simple form.

Let $1 \le j \le t$, $t + 1 \le i \le d$, and look at the composition

$$E(D_j - D) \hookrightarrow E(D_1 - D) \oplus \ldots \oplus E(D_t - D) \xrightarrow{\eta(-D)} \varphi_*(\mathcal{F}) \xrightarrow{\gamma} E(D_1) \oplus \ldots \oplus E(D_d) \twoheadrightarrow E(D_i),$$

where the first and last arrow are the usual direct sum inclusion and projection, respectively. To describe the structure of this morphism, we look at

$$\mathrm{Hom}\,(E(D_j - D), E(D_i)) = H^0(\mathrm{Tw}(M), E^*(-D_j + D) \otimes E(D_i)) =$$

$$= H^0(\mathrm{Tw}(M), \mathrm{End}(E)(-D_j + D + D_i)) = H^0(\mathbb{CP}^1, \pi_*\,[\mathrm{End}(E)(-D_j + D + D_i)]) =$$

$$H^0(\mathbb{CP}^1, \pi_*(\mathrm{End}\,E)(-D_j + D + D_i)),$$

where we have used the projection formula in the last line. At this point we examine the pushforward sheaf $\pi_*(\mathrm{End}\,E)$ on $\mathbb{CP}^1$. It is locally free since it is torsion-free, so it splits as a direct sum of line bundles by the Birkhoff-Grothendieck theorem. Since the vector bundle E on $\mathrm{Tw}(M)$ is irreducible, it is stable, hence it is simple (see [32], Proposition 1.4.5), so we have

$$\mathrm{Hom}(E,E) = H^0(\mathrm{Tw}(M), \mathrm{End}\, E) = H^0(\mathbb{CP}^1, \pi_*(\mathrm{End}\, E)) = \mathbb{C}.$$

Consequently, using the Bott formula (see [41], p. 4), we can conclude that in the Birkhoff-Grothendieck direct sum decomposition of $\pi_*(\mathrm{End}\, E)$, there is exactly one summand of the form $\mathcal{O}_{\mathbb{CP}^1}$, while all other summands (if any) are negative line bundles. On the other hand, by Grauert's theorem, the rank of $\pi_*(\mathrm{End}\, E)$ is equal to $\dim H^0(M_I, \mathrm{End}\, E_I) = \dim \mathrm{Hom}(E_I, E_I)$ for a generic point $I \in \mathbb{CP}^1$. But since we assume E to be generically fibrewise simple, this is equal to 1, so

$$\pi_*(\mathrm{End}\, E) = \mathcal{O}_{\mathbb{CP}^1}.$$

With this in mind, we have

$$\mathrm{Hom}\left(E(D_j - D), E(D_i)\right) = H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(-D_j + D + D_i)).$$

In other words, the composition

$$E(D_j - D) \hookrightarrow E(D_1 - D) \oplus \ldots \oplus E(D_t - D) \xrightarrow{\eta(-D)} \varphi_*(\mathcal{F}) \xrightarrow{\gamma} E(D_1) \oplus \ldots \oplus E(D_d) \twoheadrightarrow E(D_i)$$

is simply the pullback of a meromorphic function on $\mathbb{CP}^1$ to $\mathrm{Tw}(M)$. Let us denote this function by $a_{i,j}$. It now follows that the composition

$$E(D_1 - D) \oplus \ldots \oplus E(D_t - D) \xrightarrow{\eta(-D)} \varphi_*(\mathcal{F}) \xrightarrow{\gamma} E(D_1) \oplus \ldots \oplus E(D_d)$$

takes the form of a $d \times t$ matrix of meromorphic functions from $\mathbb{CP}^1$:

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ a_{t+1,1} & a_{t+1,2} & \cdots & a_{t+1,t} \\ a_{t+2,1} & a_{t+2,2} & \cdots & a_{t+2,t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d,1} & a_{d,2} & \cdots & a_{d,t} \end{pmatrix}$$

Here, the upper $t \times t$ submatrix is the identity matrix by the commutativity of the diagram (4.6). From now on, we restrict our attention to the open set $\mathrm{Tw}(M) \setminus \mathrm{Supp}(D)$. We know that on this set $\eta(-D)$ is an isomorphism while the bundle $E(D_1 - D) \oplus \ldots \oplus E(D_t - D)$ is naturally identified with $E(D_1) \oplus \ldots \oplus E(D_t)$, so that the above composition takes the form

$$E(D_1) \oplus \ldots \oplus E(D_t) \xrightarrow{\cong} \varphi_*(\mathcal{F}) \xrightarrow{\gamma} E(D_1) \oplus \ldots \oplus E(D_d)$$

In what follows, we will identify $\varphi_*(\mathcal{F})$ with $E(D_1) \oplus \ldots \oplus E(D_t)$ on the set $\mathrm{Tw}(M) \setminus \mathrm{Supp}(D)$.

We will now complete our argument by showing that the above description of the morphism $\gamma : \varphi_*(\mathcal{F}) \to \varphi_*(\varphi^* E)$ is incompatible with the fact that $\mathcal{F}$ is a proper subsheaf of $\varphi^* E$ of lower rank on Z. Recall that in the diagram (4.3) f and φ are branched coverings of degree d. We can choose a point $x \in \mathrm{Tw}(M)$ and an open neighborhood $x \ni U$ such that

(i) x lies outside the ramification locus of φ (equivalently, $\pi(x)$ lies outside the ramification locus of f) and the neighborhood U is evenly covered by φ, that is, $\varphi^{-1}(U) \subseteq Z$ is a disjoint union of open neighborhoods $U_1, \ldots, U_d$ such that $\forall 1 \leq i \leq d$, φ restricted to U_i is an isomorphism

$$\varphi|_{U_i} : U_i \xrightarrow{\cong} U.$$

$\forall 1 \leq i \leq d$, let $x_i \in U_i$ denote the point mapped to x by φ.

(ii) U does not intersect $\mathrm{Supp}(D) \cup \mathrm{Supp}(D_1) \cup \ldots \cup \mathrm{Supp}(D_d)$ in $\mathrm{Tw}(M)$.

(iii) $\varphi^{-1}(U) \subseteq Z$ does not intersect the singularity set $S(\mathcal{F})$ of $\mathcal{F}$. In other words, the restriction of $\mathcal{F}$ to $\varphi^{-1}(U)$ is locally free.

We first look at the sheaf $\varphi_*(\mathcal{O}_Z)$ on $\mathrm{Tw}(M)$. Since by choice of U the covering φ trivializes over this set, we have the natural trivialization

$$\alpha : \varphi_*(\mathcal{O}_Z)|_U \xrightarrow{\cong} \mathcal{O}_U^{\oplus d},$$

over U, where the i-th direct summand on the right corresponds to the sheet U_i over U. On the other hand, we have the decomposition

$$\varphi_*(\mathcal{O}_Z) \cong \mathcal{O}_{\mathrm{Tw}(M)}(D_1) \oplus \ldots \oplus \mathcal{O}_{\mathrm{Tw}(M)}(D_d).$$

Since U was chosen to lie outside $\mathrm{Supp}(D_1) \cup \ldots \cup \mathrm{Supp}(D_d)$, for each $1 \leq j \leq d$ the restriction of the direct summand $\mathcal{O}_{\mathrm{Tw}(M)}(D_j)$ to U is naturally isomorphic to the structure sheaf $\mathcal{O}_U$, giving us another trivialization

$$\beta : \varphi_*(\mathcal{O}_Z)|_U \xrightarrow{\cong} \mathcal{O}_U^{\oplus d}.$$

The transition function $\alpha \circ \beta^{-1} : \mathcal{O}_U^{\oplus d} \to \mathcal{O}_U^{\oplus d}$ takes the form of a $d \times d$ matrix B of holomorphic functions on U, everywhere non-singular. In particular, at the point $x \in U$, the matrix $B(x)$ represents the transition

$$\mathcal{O}_x(D_1) \oplus \ldots \oplus \mathcal{O}_x(D_d) \longrightarrow \mathcal{O}_{x_1} \oplus \ldots \oplus \mathcal{O}_{x_d}$$

between the two descriptions of the stalk $\varphi_*(\mathcal{O}_Z)_x$.

We now look at the restriction of the sheaf monomorphism $\gamma : \varphi_*(\mathcal{F}) \to \varphi_*(\varphi^* E)$ to U, in particular at the corresponding morphism of stalks at $x \in U$:

$$\gamma_x : \varphi_*(\mathcal{F})_x \longrightarrow \varphi_*(\varphi^* E)_x.$$

γ_x can be described in two different ways. Firstly, as U is evenly covered by $U_1, \ldots, U_d$ with $x_1, \ldots, x_d$ the corresponding preimages of x, we have

$$\varphi_*(\mathcal{F})_x \cong \mathcal{F}_{x_1} \oplus \ldots \oplus \mathcal{F}_{x_d}, \qquad \varphi_*(\varphi^* E)_x \cong (\varphi^* E)_{x_1} \oplus \ldots \oplus (\varphi^* E)_{x_d},$$

and γ_x is simply the direct sum of the monomorphisms $\mathcal{F}_{x_i} \hookrightarrow (\varphi^* E)_{x_i}$ defining $\mathcal{F}$ as a subsheaf of $\varphi^* E$ on Z:

$$\gamma_x : \mathcal{F}_{x_1} \oplus \ldots \oplus \mathcal{F}_{x_d} \longrightarrow (\varphi^* E)_{x_1} \oplus \ldots \oplus (\varphi^* E)_{x_d}.$$

In particular, since $\mathcal{F}$ was constructed as a proper subsheaf of $\varphi^* E$ of lower rank (and by choice of U, $\mathcal{F}$ is locally free in a neighborhood of each of $x_1, \ldots, x_d$), for any $1 \leq i \leq d$ the composition

$$\varphi_*(\mathcal{F})_x = \mathcal{F}_{x_1} \oplus \ldots \oplus \mathcal{F}_{x_d} \xrightarrow{\gamma_x} \varphi_*(\varphi^* E)_x = (\varphi^* E)_{x_1} \oplus \ldots \oplus (\varphi^* E)_{x_d} \longrightarrow (\varphi^* E)_{x_i}$$

of γ_x with the usual projection cannot be surjective. On the other hand, with the isomorphism $\varphi_*(\varphi^* E) \cong E \otimes \varphi_*(\mathcal{O}_Z) \cong E(D_1) \oplus \ldots \oplus E(D_d)$ and the identification of $\varphi_*(\mathcal{F})$ with $E(D_1) \oplus \ldots \oplus E(D_t)$ on $\mathrm{Tw}(M) \smallsetminus \mathrm{Supp}(D)$ (and hence on U), we have

$$\varphi_*(\mathcal{F})_x \cong E_x(D_1) \oplus \ldots \oplus E_x(D_t), \qquad \varphi_*(\varphi^* E)_x \cong E_x(D_1) \oplus \ldots \oplus E_x(D_d),$$

and γ_x can be described by the matrix $A(x)$ constructed previously:

$$\gamma_x : E_x(D_1) \oplus \ldots \oplus E_x(D_t) \xrightarrow{A(x)} E_x(D_1) \oplus \ldots \oplus E_x(D_d)$$

By the discussion in the previous paragraph, the transition between the two descriptions of the stalk $\varphi_*(\varphi^* E)_x$

$$E_x(D_1) \oplus \ldots \oplus E_x(D_d) \xrightarrow{B(x)} (\varphi^* E)_{x_1} \oplus \ldots \oplus (\varphi^* E)_{x_d}$$

is given by the matrix $B(x)$. Taking the composition of the above two morphisms, since the matrix $B(x)A(x)$ has rank $t > 0$, at least one of its entries, say (i, j), is nonzero. This means that the composition

$$E_x(D_j) \hookrightarrow E_x(D_1) \oplus \ldots \oplus E_x(D_t) \xrightarrow{B(x)A(x)} (\varphi^* E)_{x_1} \oplus \ldots \oplus (\varphi^* E)_{x_d} \twoheadrightarrow (\varphi^* E)_{x_i}$$

is an isomorphism, and in particular the composition

$$\varphi_*(\mathcal{F})_x = E_x(D_1) \oplus \ldots \oplus E_x(D_t) \xrightarrow{\gamma_x} \varphi_*(\varphi^* E)_x = (\varphi^* E)_{x_1} \oplus \ldots \oplus (\varphi^* E)_{x_d} \longrightarrow (\varphi^* E)_{x_i}$$

is surjective, contradicting our earlier statement. We conclude that our original assumption that E is not generically fibrewise stable is wrong, finishing the proof of the theorem.

□

4.3 A counterexample: stable but nowhere fibrewise stable bundle on $\mathrm{Tw}(M)$

A *K3 surface* is a compact simply-connected complex surface M with trivial canonical bundle $\Lambda^{2,0}M$ (see, for example, [21], pp. 590-594 for basic properties of K3 surfaces). A nonzero section of the canonical bundle $\Lambda^{2,0}M$ is a holomorphic symplectic form on M; as a consequence of the Calabi-Yau theorem [53], a K3 surface is hyperkähler. It is a simple hyperkähler manifold as $h^{2,0}(M) = \dim_{\mathbb{C}} H^{2,0}M = 1$. Its Picard group is discrete; the *Picard number* $\rho(M)$ of a K3 surface is the rank of $\operatorname{Pic} M$, a number between 0 and $20 = h^{1,1}(M)$.

Let M be a K3 surface of Picard number $\rho(M) \geq 2$. Then the degree map $\deg : \operatorname{Pic}(M) \to \mathbb{Z}$ has a nontrivial kernel, and any $L \in \operatorname{Pic}(M)$ of degree zero is actually hyperholomorphic (see Theorem 2.4 in [51]). We claim that such L can be chosen to satisfy $h^1(M, L^*) \neq 0$. Indeed, the Riemann-Roch formula (see [21], p. 472) for a line bundle L on a K3 surface reads

$$h^0(M,L) - h^1(M,L) + h^2(M,L) = \frac{c_1(L)^2}{2} + 2.$$

If L is a nontrivial line bundle of degree zero, then $h^0(M,L) = 0$, since the fact that L is hyperholomorphic guarantees that the Einstein constant of L is zero, and then by the Kobayashi vanishing theorem (Theorem 2.3.11), any nonzero global section of L must be parallel, hence nonvanishing. Similarly, $h^2(M,L) = h^0(M,L^*) = 0$, so for such line bundle L,

$$h^1(M,L) = -\frac{c_1(L)^2}{2} - 2.$$

By the Hodge-Riemann bilinear relations (see [21], p. 123), $c_1(L)^2 < 0$, since $c_1(L)$ is primitive with respect to any induced Kähler form on M (see the proof of Theorem 2.4 in [51]), hence replacing L by its multiple if necessary, we get $h^1(M, L) \neq 0$, and similarly $h^1(M, L^*) \neq 0$.

So let L be a nontrivial degree zero line bundle on M with $h^1(M, L^*) \neq 0$. Since L is hyperholomorphic, we can take the corresponding twistor transform $\mathrm{Tw}(L)$ and look at extensions

$$0 \longrightarrow \mathcal{O}_{\mathrm{Tw}(M)}(-1) \longrightarrow E \longrightarrow \mathrm{Tw}(L) \longrightarrow 0$$

on the twistor space $\mathrm{Tw}(M)$. These are parametrized by $\mathrm{Ext}^1(\mathrm{Tw}(L), \mathcal{O}_{\mathrm{Tw}(M)}(-1))$. We have

$$\mathrm{Ext}^1(\mathrm{Tw}(L), \mathcal{O}_{\mathrm{Tw}(M)}(-1)) \cong H^1(\mathrm{Tw}(L^*) \otimes \mathcal{O}_{\mathrm{Tw}(M)}(-1)) = H^1(\mathrm{Tw}(L^*)(-1)).$$

Recall that we have the twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$. Using the projection formula and Proposition 2.2.8, we have

$$R^1\pi_*(\mathrm{Tw}(L^*)(-1)) \cong R^1\pi_*(\mathrm{Tw}(L^*)) \otimes_{\mathcal{O}_{\mathbb{CP}^1}} \mathcal{O}_{\mathbb{CP}^1}(-1) \cong \mathcal{O}_{\mathbb{CP}^1} \otimes_{\mathbb{C}} H^1(M, L^*).$$

Thus,

$$H^1(M, L^*) = \Gamma\left(\mathbb{CP}^1, R^1\pi_*(\mathrm{Tw}(L^*)(-1))\right) = H^1(\mathrm{Tw}(L^*)(-1)) =$$

$$= \mathrm{Ext}^1(\mathrm{Tw}(L), \mathcal{O}_{\mathrm{Tw}(M)}(-1)).$$

By the choice of L, this space is nonzero, hence we can choose a nontrivial extension

$$0 \longrightarrow \mathcal{O}_{\mathrm{Tw}(M)}(-1) \longrightarrow E \longrightarrow \mathrm{Tw}(L) \longrightarrow 0 \tag{4.7}$$

E is actually a vector bundle: given any coherent sheaf $\mathcal{G}$ on $\mathrm{Tw}(M)$, taking the long exact sequence of the above with respect to the functor $\mathscr{H}om(-,\mathcal{G})$, since

$$\mathscr{E}xt^p(\mathcal{O}_{\mathrm{Tw}(M)}(-1),\mathcal{G}) = \mathscr{E}xt^p(\mathrm{Tw}(L),\mathcal{G}) = 0$$

for any $p > 0$, it follows that likewise $\mathscr{E}xt^p(E,\mathcal{G}) = 0$, hence E is locally free.

We now show that the vector bundle E is stable on $\mathrm{Tw}(M)$. Firstly, note that

$$\deg E = \deg \mathcal{O}_{\mathrm{Tw}(M)}(-1) + \deg \mathrm{Tw}(L) = -1 + 0 = -1 \implies \mu(E) = -\frac{1}{2}.$$

Given any horizontal twistor line $\{x\} \times \mathbb{CP}^1 \subseteq \mathrm{Tw}(M)$, the short exact sequence (4.7) restricts to

$$0 \longrightarrow \mathcal{O}_{\mathbb{CP}^1}(-1) \longrightarrow E|_{\{x\}\times\mathbb{CP}^1} \longrightarrow \mathcal{O}_{\mathbb{CP}^1} \longrightarrow 0$$

Since $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{CP}^1},\mathcal{O}_{\mathbb{CP}^1}(-1)) = H^1(\mathcal{O}_{\mathbb{CP}^1}(-1)) = 0$, $E|_{\{x\}\times\mathbb{CP}^1} \cong \mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)$. It follows that the only possibility for the degree of a destabilizing line subsheaf of E is zero. Suppose we have such a subsheaf, which by Theorem 2.2.7 must necessarily have the form $\mathrm{Tw}(\tilde{L}) \hookrightarrow E$, where $\tilde{L}$ is an autodual line bundle on M. We look at the composition θ of $\mathrm{Tw}(\tilde{L}) \hookrightarrow E$ with the map $E \to \mathrm{Tw}(L)$, as in the diagram

$$\begin{array}{ccccccccc} & & & & \mathrm{Tw}(\tilde{L}) & & & & \\ & & & & \downarrow & \overset{\theta}{\searrow} & & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathrm{Tw}(M)}(-1) & \longrightarrow & E & \longrightarrow & \mathrm{Tw}(L) & \longrightarrow & 0 \end{array}$$

If $\theta = 0$, then by the exactness of the sequence $\mathrm{Tw}(\tilde{L}) \hookrightarrow E$ lifts to a sheaf monomorphism $\mathrm{Tw}(\tilde{L}) \hookrightarrow \mathcal{O}_{\mathrm{Tw}(M)}(-1)$. This, however, is impossible, because, restricting to any horizontal twistor line, the only possible morphism $\mathcal{O}_{\mathbb{CP}^1} \to \mathcal{O}_{\mathbb{CP}^1}(-1)$ is zero. Thus, θ represents a nonzero section in

$$\mathrm{Hom}(\mathrm{Tw}(\tilde{L}), \mathrm{Tw}(L)) \cong H^0(\mathrm{Tw}(\tilde{L})^* \otimes \mathrm{Tw}(L)) \cong H^0(\mathrm{Tw}(\tilde{L}^* \otimes L)),$$

but since $\mathrm{Tw}(\tilde{L}^* \otimes L)$ has degree zero, by the Kobayashi vanishing theorem (Theorem 2.3.11), this section is parallel, hence in particular nonvanishing. It follows that $\theta : \mathrm{Tw}(\tilde{L}) \to \mathrm{Tw}(L)$ is an isomorphism, so from the above diagram we see that there is a splitting $E \cong \mathrm{Tw}(L) \oplus \mathcal{O}_{\mathrm{Tw}(M)}(-1)$, contrary to the choice of E. It follows that E must be stable.

Finally, we notice that restricting the short exact sequence (4.7) to any fibre $\pi^{-1}(I) = M_I \subseteq \mathrm{Tw}(M)$, we get

$$0 \longrightarrow \mathcal{O}_{M_I} \longrightarrow E_I \longrightarrow L_I \longrightarrow 0,$$

where both $\mathcal{O}_{M_I}$ and L_I have degree zero, hence E_I also has degree and slope zero. It follows from this that E_I is not stable for any $I \in \mathbb{CP}^1$.

CHAPTER 5
Conclusion

The results presented in this thesis can be generalized and investigated further in many ways. Just as the result of Theorem 3.1.1 is used in [28] to establish correspondences between certain moduli spaces on a hyperkähler manifold M and its twistor space $\mathrm{Tw}(M)$, a natural question to ask would be whether the result of Theorem 3.2.3 could help one to establish similar correspondences in the case of a hypercomplex M. While one cannot hope for the overall picture to be as nice for general hypercomplex manifolds as it is for hyperkähler manifolds as too much structure is lost in the absence of a metric on M, one can still try to establish a twistor correspondence for hypercomplex manifolds in one way or another. Perhaps the right procedure would be to investigate the case of HKT-manifolds. These are hyperhermitian manifolds that satisfy the property $\partial_I \Omega_I = 0$, where $\Omega_I = \omega_J + \sqrt{-1}\omega_K$ (see Section 2.2). Being a generalization of hyperkähler manifolds, they retain enough structure and have a rich and developing theory (see [2] for a survey of HKT-geometry). One can ask whether in the case of existence of an HKT-structure on M Theorem 3.2.3 can be proved in a simpler way, perhaps giving the balanced metric explicitly. While the canonical Hermitian metric on $\mathrm{Tw}(M)$ obtained from the HKT-structure fails to be balanced, it could perhaps be altered to produce a balanced metric.

Another question to investigate is whether the twistor space $\mathrm{Tw}(M)$ of a compact hypercomplex manifold M admits other types of non-Kähler metrics in addition to a balanced one. For example, an astheno-Kähler metric on a complex manifold is a Hermitian metric such that its Hermitian form ω satisfies the condition $\partial\overline{\partial}\,(\omega^{n-2}) = 0$, where n is the dimension of the manifold. It was shown in [18] that for a compact hyperkähler M the twistor space $\mathrm{Tw}(M)$ doesn't admit astheno-Kähler metrics. In fact, balanced and astheno-Kähler metrics are somewhat adverse to each other: a longstanding opinion (referred to as "folklore conjecture" in [17]) stated that a complex manifold cannot admit both a balanced and an astheno-Kähler metric unless it is Kähler. Although this was proven wrong in [18], it would still be interesting to see whether or not the twistor space $\mathrm{Tw}(M)$ of a compact hypercomplex manifold M admits an astheno-Kähler metric.

For the results of Chapter 4, it is strongly suspected that the full converse to Theorem 4.2.1 in fact holds for bundles E of arbitrary rank, without any further conditions. One way of approaching this is apparent: knowing that the converse is true for generically fibrewise simple E, we can try to show that this condition is always satisfied for any irreducible bundle E on $\mathrm{Tw}(M)$. An idea for the proof is to argue by contradiction and assume the existence of a morphism $F : E \to E(D)$ for some divisor D, which doesn't come from a meromorphic function on $\mathbb{CP}^1$, and look at the eigenvalues of this morphism. Clearly, no such eigenvalues exist over $\mathbb{CP}^1$ (else, their eigenspaces would contradict the irreducibility of E), but one can take a branched covering $f : X \to \mathbb{CP}^1$ over which they do exist, and try to argue similarly as in the proof of Theorem 4.2.1 by taking the fibred product of $f : X \to \mathbb{CP}^1$ and $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$, and using the eigendecomposition of the pullback of E to this fibred product to arrive at a contradiction to the irreducibility of E.

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