Stability of Solutions to Stochastic Differential Equations

Bryden Cheong

Department of Mathematics and Statistics
McGill University, Montreal

April 2023

A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of Master of Science

Acknowledgements

I would like to express my gratitude to Linan Chen for her endless patience and encouraging guidance throughout my degree. I would also like to thank my parents for their support. I would have starved some time ago without their home-cooked meals.

Abstract

Kiyoshi Itô's formulation of stochastic calculus is a key component to modern stochastic analysis. From this framework comes stochastic differential equations (SDEs) which have seen important applications in various disciplines such as finance, biology, and physics. While ordinary differential equations (ODEs) are often used to study the motions of a particle under the influence of a deterministic vector field, SDEs are models of such motions when the particle is also subject to a random 'diffusing' effect. The question as to whether the trajectory of a particle in an SDE system is 'close' to that of the same particle in the corresponding ODE system has long been considered under the context of *stability*. This thesis surveys concepts and methods that were developed to investigate the stability of solutions to SDEs. Similarly as in the case of ODEs, Lyapunov functions, upon adapted to the stochastic setting, play important roles in the study of stability for SDEs. But different from the case of ODEs, since solutions to SDEs are stochastic processes, the notion of stability has richer and more flexible interpretations from a probabilistic point of view. Aiming to extend the study of stability to certain classes of SDEs, this thesis also proposes a new notion of stability, known as *stability in ratio*, which captures the relative growth rate of the SDE solution to the corresponding ODE solution. The relations between this new notion of stability and existing definitions are examined, and some techniques for determining the stability in ratio are illustrated through examples as well.

Abstracte

La formulation de calcul stochastique de Kiyoshi Itô est très importante dans la theorie d'analyse stochastique modernes. Les équations différentielles stochastiques (EDS) ont été utilisées dans divers disciplines, notamment la finance, la biologie, et la physique. Tandis que les équations différentielles ordinaires (EDO) sont souvent utilisées pour étudier les mouvements d'une particule sous l'influence d'un champ vectoriel déterministe, les EDS sont des modèles de ces mouvements lorsque la particule est influencée par un effet de 'diffusion'. On se demande si la trajectoire d'une particule dans une système EDS est 'proche' de celle de la même particule dans le système EDO correspondant. Ce question a été considérée par plusieurs auteurs dans le contexte de la *stabilité* de la solution d'une EDS. Cette thèse examine les méthodes développées pour étudier la stabilité de la solution d'une EDS. Parmi les outils les plus efficaces pour étudier la stabilité des EDS sont les fonctions de Lyapunov qui ont été introduites à l'origine il y a plus d'un siècle pour étudier la dynamique des systèmes d'EDO. Mais contrairement au cas des EDO, les solutions des EDS étant des processus stochastiques, la notion de stabilité a des interprétations plus riches et plus flexibles d'un point de vue probabiliste. Cette thèse propose également une nouvelle notion de stabilité qui élargit l'étude de la stabilité. Nommé *stabilité en ratio*, cette notion permet de saisir le taux de croissance de la solution EDS par rapport à la solution EDO correspondante. Cette thèse examine également comment cette nouvelle notion de stabilité se rapporte aux définitions existantes. Certaines techniques pour établir la stabilité en ratio sont illustrées par des exemples.

Contents

List of Figures

1 Introduction

Stochastic differential equations (SDEs) can be seen as 'perturbations' of ordinary differential equations (ODEs) when the deterministic system is perturbed by a stochastic diffusing force. In this sense, SDEs are the probabilistic extension to ODEs [13]. It is natural then to wonder how the behavior of the stochastic solution differs from that of the deterministic counterpart. This is the main question investigated in this thesis. The study of the connections and the comparisons between the solution to an SDE and that to the corresponding ODE in which the diffusing force is removed gives rise to the notion of *stability* of solutions to SDEs. Stability of solutions to deterministic dynamical systems, including ODEs and partial differential equations (PDEs), has long been considered, for which a rich literature of theories and techniques has been developed. However, to study stability for a stochastic system, we face new challenges and need to acquire new tools. Since the solution to an SDE, if it exists, is a stochastic process, there are different notions of stability with different levels of 'strength' from a probabilistic point of view, which leads to diverse stability conditions and results. This thesis gives a comprehensive review of the existing literature on stability of solutions to SDEs, including numerous notions of stability, standard methods, and general results. In general, sample paths of solutions to an SDE have very different properties from trajectories driven by a deterministic vector field. In addition to showing contrasting local properties such as regularity, the two paths also often exhibit different behaviors in the long run. The discrepancy between the long-term asymptotic behavior of the SDE solution and that of the ODE solution is especially prominent when the ODE solution is unbounded. We zoom in on these types of SDEs (and ODEs) and propose a new approach towards stability named *stability in ratio*. This new approach is adopted to examine specific classes of SDEs and generates interesting and promising results.

Below we give a detailed description of the structure of the thesis.

Chapter 2 begins with a review of the notion of stability in the deterministic setting [15, 18]. Given a dynamical system $\dot{x}(t) = f(t, x(t))$ with an equilibrium point x_e (that is to say $f(t, x_e) = 0$ for all $t \geq 0$), a solution $x(t)$ starting from some point close to x_e is stable if $x(t)$ remains close to x_e for all time. There are multiple ways of interpreting 'close' mathematically, which give rise to multiple definitions of stability. In the late nineteenth century, Russian mathematician A. M. Lyapunov introduced a framework [23], known as *Lyapunov's method* or the *direct method*, for analyzing the stability of $x(t)$ without necessarily determining the explicit expression for $x(t)$. This is done by constructing a non-negative *Lyapunov* function $V(t, x)$ which can be viewed as a generalized representation of the energy of $x(t)$ at time t at position x, and which is uniquely minimized at x_e. Under some mild conditions, the existence of a Lyapunov function itself is sufficient for establishing stability. Intuitively speaking, to determine $x(t)$ being stable with respect to x_e, it is enough to verify that $x(t)$ does not gain energy as time progresses. Similar conditions exist for the other types of stability.

Chapter 2 continues with a review of concepts in probability theory that are necessary for understanding SDEs [2, 4, 24, 26]. We start with an introduction to Brownian motion, arguably the most important stochastic process, which was first observed by botanist Robert Brown in 1827 and then formalized by Einstein [10] several decades later. In the 1940s, Itô created a new scope in the study of Brownian motion by introducing Itô's integral and Itô's calculus [16], which in turn allowed for the formal definition of an SDE. We review basic concepts and results involving existence, uniqueness, and other properties of solutions to SDEs, as well as useful tools such as the comparison theory for SDEs.

Chapter 3 gives an overview of the theory of stability for solutions to SDEs, following mostly the comprehensive work of Khaminskii [19]. To summarize, for a given SDE $\mathrm{d}X(t) = b(t, X(t))\,\mathrm{d}t + \sigma(t, X(t))\,\mathrm{d}B(t)$, we want to investigate the relation between $X(t)$, the solution to the SDE, and $\tilde{X}(t)$, the solution to the ODE $\mathrm{d}X(t) = b(t, X(t))\,\mathrm{d}t$ obtained by removing Itô's differential $\mathrm{d}B(t)$ from the SDE. Compared with the stability previously introduced for ODEs, in the probabilistic setting there are considerably more ways in which we can interpret $X(t)$ as being stable or unstable with respect to the deterministic counterpart. In addition, SDEs are generally intractable, and solutions to SDEs, as stochastic processes, are expected to have low regularity (for example, lack of bounded variation), fractal structures, and other properties that are intrinsically different from the deterministic trajectories. To this end, several rather restrictive assumptions on the coefficients of SDEs are imposed to keep the solutions accessible and manageable. However, despite of all the new challenges associated with stochastic systems, similarly as in the deterministic systems, one can define a stochastic analogue of Lyapunov function which can be used to study stability. Efforts to extend Lyapunov's method to the random setting had already been carried out in 1960s, not only for SDEs but also for more general stochastic systems. An influential work was conducted by Bucy [5] in which he argued that the stochastic Lyapunov function should be constructed as a super-martingale. Other early works on this topic include Gikhman [14], Kats [17], and Kushner [21] to name a few.

Most of the stability results reviewed in Chapter 3 are established under rather stringent conditions. In particular, one important assumption requires that the diffusive force $\sigma(t, X(t))$ is weak when $X(t)$ is close to $\tilde{X}(t)$, and $\sigma(t, X(t))$ vanishes completely when $X(t)$ agrees with $\tilde{X}(t)$. This assumption certainly puts a strong restriction on the type of SDEs to which the results apply, and dropping this assumption would allow us to consider more general SDEs. However, when $\sigma(t, X(t))$ is no longer 'sensitive' to the relation between $X(t)$ and $\tilde{X}(t)$, the current definitions of stability would require substantial modifications. In addition, the existing theory of stability mostly concerns SDEs and ODEs whose solutions remain bounded in time, in which case the difference between $X(t)$ and $\tilde{X}(t)$ is captured in a straightforward way by the distance $\left|X(t) - \tilde{X}(t)\right|$. However, this boundedness assumption also greatly limits the scope of SDEs to which the results apply, and in the case when $X(t)$ and $\tilde{X}(t)$ grow unboundedly, $\left|X(t) - \tilde{X}(t)\right|$ may not be the most appropriate quantifier of the proximity of $X(t)$ to $\tilde{X}(t)$. As an attempt to circumvent the restrictions mentioned above, in Chapter 4 we propose a new form of stability. Given an SDE with the solution $X(t)$ and the deterministic counterpart $\tilde{X}(t)$, we consider a new stochastic process, referred to as the ratio process, defined as $Y(t) = X(t)/\tilde{X}(t)$. We say that $X(t)$ is *stable in ratio* if almost surely $Y(t)$ and $1/Y(t)$ stay bounded for all time. This stability captures the relative growth rate of $X(t)$ compared to $\tilde{X}(t)$. We apply the framework of stability in ratio to study specific classes of SDEs, particularly those with coefficients being power functions in the spatial variable, and obtain a set of diverse outcomes on stability in ratio, even among SDEs that have similar structures. In this process, we also experiment with various techniques, including Lyapunov's method, martingale method and transformation method, to treat $Y(t)$ for more general SDEs.

2 Preliminaries

This chapter is devoted to setting up the necessary ideas and notation required to properly define the various concepts of stability. We begin with an overview of results in the deterministic case in Section 2.1. Lyapunov's method is introduced and used to determine the stability for dynamical systems not influenced by random diffusive forces, namely $\dot{x}(t) = f(t, x(t))$. The idea of a Lyapunov function is formalized and the criteria it must satisfy to achieve the various types of stability are presented.

The basic measure-theoretic probability notation is introduced. The notion of a stochastic process is presented, with emphasis on the process known as Brownian motion. Many facts on Brownian motion is presented. We build Itô integrals by approximations constructed with simple functions, which in turn are used to formally define SDEs. We discuss what it means for a stochastic process $X(t)$ to be a solution to an SDE. Many results on the properties of SDEs are given, including Itô's lemma, conditions for existence and uniqueness, and comparison results.

2.1 Dynamical Systems and Stability

In this section, we review some concepts related to dynamical systems from [15] and [18]. A dynamical system describes motion under the influence of a deterministic vector field. Formally, a **dynamical system** in l-dimensions is governed by the ordinary differential equation

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(0) = x_0 \in \mathbb{R}^l \end{cases} . \tag{2.1}$$

Here, the quantity $x(t) \in \mathbb{R}^l$ for every $t \geq 0$ is the **state**. In many physical systems, the state represents the position of a particle. Here and throughout, we set $I = [0, \infty)$. Letting $D \subseteq \mathbb{R}^l$ be a domain containing the initial value x_0, the function $f : I \times D \to \mathbb{R}^l$ describes how the state changes with time. Understanding the behavior of the function f is critical to understanding how $x(t)$ behaves. For this reason, f is often classified into various categories, such as 'linear' or 'non-linear' (whether or not f is a linear function of $x(t)$), 'time-homogeneous' or 'time-inhomogenous' (whether or not f is independent of t), etc. In this section, we assume f is piecewise continuous in time and Lipschitz in space (or at least in some neighborhood of x_0), that is to say that for every $t \geq 0$, there exists a constant B so that

$$\big| f(t, x) - f(t, y) \big| \leq B|x - y| \tag{2.2}$$

holds for every $x, y \in D$. The existence of a unique solution is given by the following theorem:

Theorem 2.1 (Picard-Lindelöf Theorem)**.** *Let $U \subseteq I \times \mathbb{R}^l$ be a closed rectangle containing (t_0, x_0). Suppose that $f : U \to \mathbb{R}^l$ is continuous in time and Lipschitz in space. Then for some $\epsilon > 0$, the system (2.1) with initial condition replaced with $x(t_0) = x_0$ has a unique solution on the interval $[t_0 - \epsilon, t_0 + \epsilon]$.*

2.1.1 Stability

It is useful to consider the long term asymptotic behavior of dynamical systems. One particularly useful concept is the idea of stability. The idea of stability materializes in many different forms.

A point x_e said to be an **equilibrium point** or a **critical point** of system (2.1) if $f(t, x_e) = 0$ for every $t \geq 0$. It is clear that if the state ever reaches the equilibrium point, it will remain there in perpetuity. That is, if $x(t_0) = x_e$, then $x(t) = x_e$ for every $t \geq t_0 \geq 0$. An equilibrium point x_e is said to be **stable** (for $t \geq t_0$) if for every $\epsilon > 0$, there exists some $\delta = \delta(\epsilon, t_0) > 0$ such that if

$$\left|x(t_0) - x_e\right| < \delta, \tag{2.3}$$

then for every $t \geq t_0 \geq 0$ we have that

$$\left|x(t) - x_e\right| < \epsilon. \tag{2.4}$$

Conceptually, this means that if the initial state begins sufficiently close to an equilibrium point, then a solution $x(t)$ will remain close to said equilibrium point forever. By the estimate (2.2) and Theorem 2.1, this solution exists and is unique. An equilibrium point that is not stable is said to be **unstable**. This definition is a local concept as the stability of a point x_e only relies on the behavior of solutions near it. In the definition of stability, one could also replace x_e in (2.4) by a solution $x_0(t)$ of (2.1) and x_e in (2.3) by $x_0(t_0)$ to instead have the definition of stability for a solution to the system rather than just one individual point.

The concept of stability can be refined even further into stronger definitions. An equilibrium point of the system (2.1) is said to be **uniformly stable** if it is stable and the choice of δ does not depend on t_0. An equilibrium point x_e is said to be **asymptotically stable** (for $t \geq t_0$) if, in addition to being stable, we have that

$$x(t) \to x_e \tag{2.5}$$

as $t \to \infty$ provided $x(t_0)$ satisfies (2.3). Being asymptotically stable requires the trajectory to not deviate from its initial point as before and additionally demands the state also approaches the equilibrium point as time progresses. The point x_e is **globally asymptotically stable** if (2.5) holds for every $x(t_0) \in \mathbb{R}^l$. Clearly this definition only applies if the equilibrium point x_e is unique. Note that condition (2.5) alone is not sufficient to conclude the stability of the system as this condition is only a control on the long-term behavior and says nothing about the short-term behavior of $x(t)$.

An equilibrium point x_e is said to be **exponentially stable** (for $t \geq t_0$) if for every $t \geq t_0$ and every starting state $x(t_0)$ satisfying (2.3) it holds that

$$\left|x(t) - x_e\right| \leq A \exp\left(-\alpha(t - t_0)\right)\left|x(t_0)\right|,$$

for some constants $A > 0$ and $\alpha > 0$. Not only does the distance between $x(t)$ and x_e shrink, but the rate at which $x(t)$ converges to x_e is at least exponentially fast. It is clear that exponential stability implies asymptotic stability, which in turn implies stability. Haddad [15] provides clear visuals for most of these types of stability.

We remark that we will use the terminology of stability in an exchangeable manner between an equilibrium point and a solution. Namely, whenever one of the above mentioned stability conditions is satisfied by a solution $x(t)$ and a point x_e, we will also say that $x(t)$ is stable (in the way according to the occurred condition) with respect to x_e.

Finally, observe that by performing a translation if necessary, we can always assume that the equilibrium point is $x_e = 0$ for another system. Therefore, it is sufficient to study stability with respect to the trivial constant 0 solution. To see this formally, consider the dynamical system governed by

$$\frac{\mathrm{d}}{\mathrm{d}s} y(s) = g(s, y(s)) \tag{2.6}$$

and defined for $s \geq a \in \mathbb{R}$. Let $y_0(s)$ and $y(s)$ be two solutions to this system, and define the transformation

$$x(t) = y(s) - y_0(s)$$

for $t = s - a$. Then we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = g(s, y(s)) - \frac{\mathrm{d}}{\mathrm{d}t} y_0(s) = g(t + a, x(t) + y_0(t + a)) - \frac{\mathrm{d}}{\mathrm{d}t} y_0(t + a).$$

Using the fact that y_0 is a solution to (2.6), is clear that

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = g(t + a, x(t) + y_0(t + a)) - g(t + a, y_0(t + a)).$$

If $x(t) = 0$ for all $t \geq 0$, the right hand side is also zero and hence $x_e = 0$ is an equilibrium point for the transformed system. As such, to study the stability of $y(s)$ with respect to $y_0(s)$, we only need to consider the cases when the equilibrium point is the trivial constant solution.

2.1.2 Lyapunov's Second Method

It is generally not difficult to analyze the stability if an explicit form the solution $x(t)$ to (2.1) is available. However, it is very often the case that this form is not available. As alluded to previously, due to the work of A. M. Lyapunov there are ways to obtain results on stability without having access to $x(t)$ itself. This method is known as Lyapunov's second method or Lyapunov's direct method [23].

Let $D \subseteq \mathbb{R}^l$ be some domain containing $x = 0$. A function $V : I \times D \to \mathbb{R}$ is said to be **positive definite** on D if for every $t \geq 0$ we have that

$$\forall x \in D : \quad V(t, x) \geq 0, \tag{2.7}$$

and

$$V(t,x) = 0 \quad \Leftrightarrow \quad x = 0, \tag{2.8}$$

and

$$V(t,x) \to \infty \quad \text{as} \quad |x| \to \infty. \tag{2.9}$$

Together, conditions (2.7) and (2.8) imply there must exist a unique minimum of $V(t,x)$, namely $x = 0$. The function V is said to be **uniformly positive definite** if additionally there exists a pair of continuous positive definite functions $W_1, W_2 : \mathbb{R}^l \to \mathbb{R}$ so that

$$W_1(x) \leq V(t,x) \leq W_2(x). \tag{2.10}$$

This definition allows the dependence on t to be removed while maintaining properties of positive definite functions. In particular, for every $t \geq 0$ the function V is bounded away from 0 provided $x \neq 0$. Figure 1 gives an example of this.

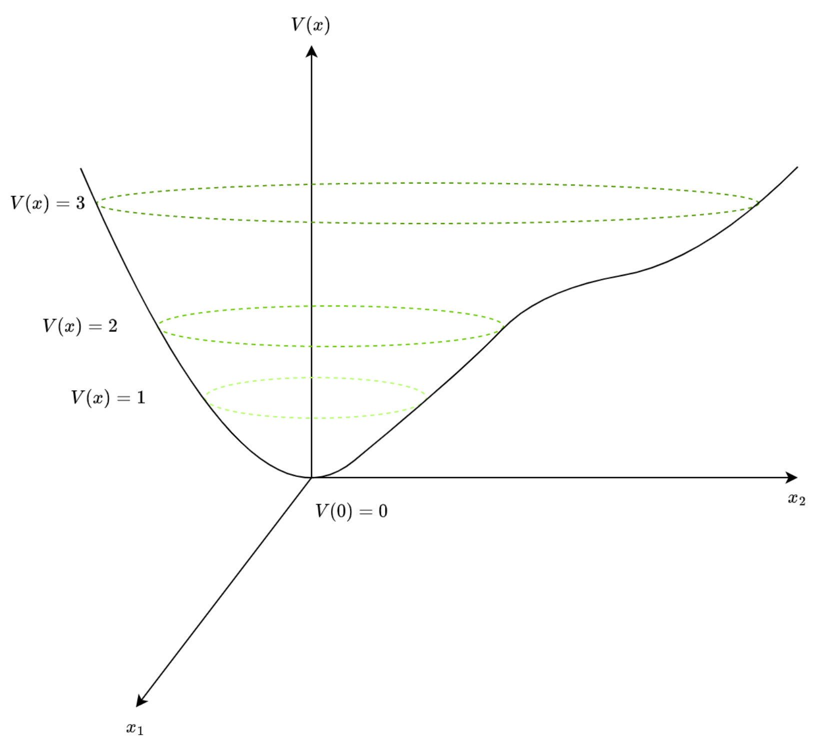

Figure 1: An example of a time-homogeneous Lyapunov function $V(x)$ for $D \subseteq \mathbb{R}^2$. The dotted green lines depict various level sets of $V(x)$.

If V is differentiable in both arguments, we define the derivative of V along the trajectory of the solution $x(t)$ to (2.1) as

$$\dot{V}(t,x) := \frac{\partial V(t,x)}{\partial t} + \left(\tfrac{\partial V(t,x)}{\partial x}, f(t,x) \right)_{\mathbb{R}^l} = \frac{\partial V(t,x)}{\partial t} + \sum_{i=1}^{l} f_i(t,x) \frac{\partial V(t,x)}{\partial x_i}. \tag{2.11}$$

Often, we seek some condition on $\dot{V}$ in order to conclude the stability of the solution $x(t)$ with respect to the zero solution. It is usually the case that we require $\dot{V}(t,x)$ to be non-positive (or some slight variation of this). This is known as the monotonicity requirement.

The function $V(t,x)$ can be interpreted as a generalized energy function for the dynamical system (2.1) with equilibrium point $x_e = 0$. For every $x \in \mathbb{R}^l$, the quantity $V(t,x)$ represents the amount of energy the solution $x(t)$ has at time t. The further $x(t)$ is from the equilibrium point, the more energy it has. The quantity $\dot{V}$ represents the change in energy of the solution $x(t)$. The monotonicity requirement therefore represents the restriction that the solution does not gain energy. Since the unique point where the energy is null is the equilibrium point at $x_e = 0$, this implies that the state cannot exceed some distance from the equilibrium point x_e. This is formalized as Theorem 4.8 in [18] or Theorem 3.1 of [15], which we present here as:

Theorem 2.2 (Lyapunov's Stability Theorem)**.** *Let $U \subseteq \mathbb{R}^l$ be a neighborhood of the origin. Suppose there exists on $I \times U$ a continuously differentiable uniformly positive definite function V such that $\dot{V}(t,x) \leq 0$ for all $t \in I$ and all $x \in U$, except possibly at the origin. Then $x_e = 0$ is a uniformly stable point of the dynamical system (2.1).*

By considering variations on the condition imposed on $\dot{V}$, we can achieve different results. If instead $\dot{V} < 0$, one interpretation is that instead of simply requiring that the energy of the system is non-increasing, we require that it is always decreasing. This leads to the following:

Theorem 2.3 (Lyapunov's Asymptotic Stability Theorem)**.** *Let $U \subseteq \mathbb{R}^l$ be a neighborhood of the origin. For some positive definite function $W(x)$, suppose there exists on $I \times U$ a continuously differentiable uniformly positive definite function V such that $\dot{V}(t,x) < -W(x)$ for all $t \in I$ and all $x \in U$, except possibly at the origin. Then the system (2.1) is uniformly asymptotically stable at the point $x_e = 0$.*

One more improvement we can make is to require that not only does the system lose energy as in Theorem 2.3, but it also loses energy at a certain rate. The following theorem discusses the case where loss of energy proportional to the amount of energy remaining.

Theorem 2.4 (Lyapunov's Exponential Stability Theorem)**.** *Let $U \subseteq \mathbb{R}^l$ be a neighborhood of the origin. Suppose there exists on $I \times U$ a continuously differentiable function V such that, for some positive constants k_1, k_2, k_3, and $p \geq 1$, it holds that $k_1|x|^p \leq V(t,x) \leq k_2|x|^p$ and $\dot{V}(t,x) \leq -k_3|x|^p$ for all $t \in I$ and all $x \in U$, except possibly at the origin. Then $x_e = 0$ is exponentially stable.*

These results, whose proofs can also be found in [15] or [18], give methods of determining the stability of a system without requiring an explicit form for the solution. This is very powerful since the dynamical system (2.1) can be difficult to solve. It will be seen in Chapter 3 that these results can be generalized to the stochastic setting where having access to an explicit solution is even more difficult to achieve.

2.2 Probability Basics

In this section we review some important concepts in probability theory. Let the triple $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A function $X : \Omega \to \mathbb{R}^l$ is said to be a **random variable** if X is $\mathcal{F}$-measurable, that is to say

$$X^{-1}(B) = \big\{\omega \in \Omega : X(\omega) \in B\big\} \in \mathcal{F}$$

for every $B \in \mathcal{B}\left(\mathbb{R}^l\right)$, the Borel σ-algebra. Whenever $l > 1$, we also call X a **random vector**. Given $(\Omega, \mathcal{F}, \mathbb{P})$, we say $\{\mathcal{F}_t : t \geq 0\}$ is a **filtration** if $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \geq 0$ and $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $0 \leq s \leq t$. We call the space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ a **filtered probability space**.

Denote by $\mathbb{R}^l(I)$ the collection of all $\mathbb{R}^l$-valued functions defined on I. Define by $T_t : \mathbb{R}^l(I) \to \mathbb{R}^l$ by $T_t(f) = f(t)$, which is the coordinate projection map. Define by $\Sigma^I_{\mathbb{R}^l} = \sigma\left(\{T_t : t \in I\}\right)$ the σ-algebra generated by the projection maps at every time $t \geq 0$. A **stochastic process** is a function $X : \Omega \to \mathbb{R}^l(I)$ that is measurable with respect to $\Sigma^I_{\mathbb{R}^l}$. Such a process is written as $\big\{X(t) : t \geq 0\big\}$ or simply just $X(t)$ and is called a process with values in $\mathbb{R}^l$ or simply l-dimensional. For a given time $s \geq 0$, the value $X(s)$ is itself a random variable that depends on $\omega \in \Omega$ and returns the value of the process at the specific time s. Viewed holistically, the process $\big\{X(t) : t \geq 0\big\}$ defines a path in $\mathbb{R}^l$ that evolves with time. The process is **adapted** to the filtration $\{\mathcal{F}_t : t \geq 0\}$ if $X(t)$ is measurable with respect to $\mathcal{F}_t$ for every $t \geq 0$. The process is called **progressively measurable** if $(s, \omega) \in [0, t] \times \Omega \to X(s, w) \in \mathbb{R}^l$ is $\mathcal{B}\left([0, t]\right) \times \mathcal{F}_t$ measurable for every $t \geq 0$. It is observed that the latter implies the former but the converse is only guaranteed when the sample paths of X are right-continuous and have left limits.

A stochastic process $\big\{X(t) : t \geq 0\big\}$ on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ that is progressively measurable with respect to $\{\mathcal{F}_t : t \geq 0\}$ is called a **martingale** if $X(t) \in L^1$ and

$$\mathbb{E}\left[X(t)|\mathcal{F}_s\right] = X(s)$$

for all times $0 \leq s \leq t$. It is a **supermartingale** if the equality is replaced with $\leq$. Martingales are a general class of stochastic processes for which many behaviors can be identified.

A function $f : I \to \mathbb{R}$ is said to be **cádlág** (or RCLL) if, for every $x \in I$, we have that $\lim_{s\nearrow x} f(s)$ exists and also $\lim_{t\searrow x} f(t)$ exists and is equal to $f(x)$. The following version of a well known result of Doob (Theorem 5.2.15 in [26]) will be useful in Chapter 4:

Theorem 2.5 (Doob's Martingale Convergence Theorem)**.** *Let $\big\{X(t) : t \geq 0\big\}$ be a martingale with cádlág paths. If for some $p > 1$, $\sup_{t\geq 0} \mathbb{E}\left[\big|X(t)\big|^p\right] < \infty$, then there exists a random variable $X_\infty \in L^p$ such that $X(t) \to X_\infty$ almost surely and in L^p.*

2.2.1 Brownian Motion

One of the most important and widely studied stochastic processes is Brownian motion. A stochastic process $\{B(t) : t \geq 0\}$ is said to be a standard **one-dimensional Brownian motion** on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ if

1. $B(0) = 0$ almost surely.
2. the process has Normal increments, that is $B(t) - B(s)$ has distribution $\mathcal{N}(0, t-s)$ for all $0 \leq s \leq t$, where $\mathcal{N}$ denotes the univariate Gaussian distribution.
3. the process has independent increments, that is for all $n \geq 1$ and $0 < t_1 < t_2 < \cdots < t_n$, the random variables $B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$ are independent.

The following result is due to Weiner and is Theorem 1.3 in [24]:

Theorem 2.6 (Existence of Brownian Motion)**.** *Standard one-dimensional Brownian motion exists.*

Observe that the above conditions imply that $B(t)$ is a martingale. An **l-dimensional Brownian motion** is defined by $B(t) = \left(B_j(t)\right)_{j=1}^{l}$ where $B_j(t)$ are independent one-dimensional Brownian motions for $1 \leq j \leq l$.

An important property of Brownian motion is that it satisfies the law of the iterated logarithm, a general result for sums of independent, identically distributed random variables with zero mean and unit variance [26]. This gives insight into the asymptotic behavior of Brownian motion. See Theorem 3.2 of [2] for a proof.

Theorem 2.7 (Law of the Iterated Logarithm for Brownian Motion)**.** *If $B(t)$ is a standard one-dimensional Brownian motion, then*

$$\limsup_{t\to\infty} \frac{B(t)}{\sqrt{2t\ln\ln t}} = 1 = -\liminf_{t\to\infty} \frac{B(t)}{\sqrt{2t\ln\ln t}}$$

almost surely.

This shows that for almost every ω, the path $\omega \to B(t)$ is eventually bounded by the envelope functions $\pm\sqrt{2t\ln\ln t}$. It is possible to show that a direct consequence of the above is

$$\limsup_{t\searrow 0} \frac{B(t)}{\sqrt{2t\ln\ln(1/t)}} = 1 = -\liminf_{t\searrow 0} \frac{B(t)}{\sqrt{2t\ln\ln(1/t)}}$$

almost surely. This describes the instantaneous behavior of Brownian motion.

Let $\{X(t) : t \geq 0\}$ be a stochastic process on some domain D. Consider $U_0 \subseteq D$ and let U be the complement of U_0. The process $\{X(t) : t \geq 0\}$ is said to be **recurrent relative to** U_0 if, whenever $X(s) = x$ for some $s \geq 0$ and $x \in U$, the exit time of U after s, denoted by τ_U, is finite almost surely. Intuitively, a recurrent process returns to the set U_0 in finite time regardless of the starting position. Additionally, the process is **point recurrent** if for every $x_0 \in U_0$, there exists a sequence $\{t_n : n \in \mathbb{N}\}$ with $t_n \nearrow \infty$ so that $X(t_n) = x_0$ for all $n \in \mathbb{N}$. It is **neighborhood recurrent** if we have the weaker condition that for every $\epsilon > 0$, the sequence satisfies $X(t_n) \in B(x_0, \epsilon)$, the ball of radius ϵ centered at x_0. A process $\big\{X(t) : t \geq 0\big\}$ said to be **transient** if $\big|X(t)\big| \nearrow \infty$ almost surely as $t \to \infty$. In one-dimension, we can specify that $\big\{X(t) : t \geq 0\big\}$ is transient at $+\infty$ if $X(t) \nearrow +\infty$ as $t \to \infty$. We define $\big\{X(t) : t \geq 0\big\}$ to be transient at $-\infty$ similarly. The following reuslt (Theorem 3.20 in [24]) describes the behavior of Brownian motion in relation to these notions.

Theorem 2.8 (Recurrence and Transience of Brownian Motion)**.** *Let $\big\{B(t) : t \geq 0\big\}$ be an $\mathbb{R}^l$-valued Brownian motion. Then*

- *if $l = 1$, then $\big\{B(t) : t \geq 0\big\}$ is point recurrent*
- *if $l = 2$, then $\big\{B(t) : t \geq 0\big\}$ is neighborhood recurrent*
- *if $l \geq 3$, then $\big\{B(t) : t \geq 0\big\}$ is transient*

The previous results show that Brownian motion behaves somewhat nicely. This is not always the case. Below is Theorem 1.35 of [24] which shows that Brownian motion also behaves erratically.

Theorem 2.9 (Brownian Motion has Locally Unbounded Variation)**.** *Almost surely, for every $t \geq 0$, $s \in [0, t] \to B(s) \in \mathbb{R}^l$ has unbounded variation.*

More specifically, the theorem says that almost surely

$$\sup_{k \in \mathbb{N}, P \in \mathcal{P}_k} \sum_{j=1}^{k} \big|B(t_j) - B(t_{j-1})\big| = \infty$$

where $\mathcal{P}_k$ denotes the set of partitions with $k + 1$ elements of the interval $[0, t]$ (that is to say of the form $\mathcal{P}_k \supseteq P = \{0 = t_0 < t_1 < \cdots < t_k = t\}$). Let $\mathcal{P}$ be the set of all partitions of $[0, t]$ and let $P^{(n)} \subseteq \mathcal{P}$ be a sequence of nested partitions with mesh vanishing as $n \to \infty$. The **quadratic variation** of a process $\big\{X(t) : t \geq 0\big\}$ is defined to be

$$\langle X \rangle_t = \lim_{n \to \infty} \sum_{j=1}^{k(n)} \left(X\left(t_j^{(n)}\right) - X\left(t_{j-1}^{(n)}\right)\right)^2$$

for every $t \geq 0$, provided the limit exists. Here, $k(n)$ is the number so that the sum ranges over all elements of the n-th partition. Applying this to Brownian motion, we have that $\langle B \rangle_t = t$ for every $t \geq 0$. This gives the heuristic that '$\big(\mathrm{d}B(t)\big)^2 \approx \mathrm{d}t$', which will be useful in subsequent discussions. By applying Kolmogorov's continuity theorem, a result on the regularity of Brownian motion is obtained (see Corollary 1.20 in [24]).

Theorem 2.10 (Brownian Motion Continuity)**.** *Brownian motion is almost surely α-Hölder continuous for $0 < \alpha < 1/2$. Brownian motion is almost surely nowhere α-Hölder continuous for $\alpha > 1/2$.*

In particular, this shows that $\big\{B(t) : t \geq 0\big\}$ is almost surely nowhere differentiable.

2.2.2 Itô's Integral

The following construction is taken from [13]. Let $B(t)$ be a standard one-dimensional Brownian motion. Let $P = \{0 = t_0 < t_1 < \cdots < t_m = t\}$ be a partition of $[0,t]$ and $g : [0,t] \to \mathbb{R}$ be a simple function with $g(s) = g_k$ if $t_k \leq s < t_{k+1}$. We define

$$\int_0^t g(s)\,\mathrm{d}B(s) = \sum_{k=0}^{m} g_k \left(B(t_{k+1}) - B(t_k)\right).$$

Now let φ be a stochastic process defined on $[0,t]$ such that

$$\|\varphi\|^2_{L^2(I\times\Omega)} = \int_0^t \mathbb{E}\left[\varphi^2(s)\right] \mathrm{d}s < \infty$$

and let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of simple functions with the same domain as φ so that $\varphi_n \to \varphi$ under the $L^2(I\times\Omega)$ norm. The **Itô stochastic integral** as in [13] is defined as

$$\mathcal{I}_\varphi(t) = \int_0^t \varphi(s)\,\mathrm{d}B(s) = \lim_{n\to\infty} \int_0^t \varphi_n(s)\,\mathrm{d}B(s)$$

for every $t \geq 0$. Below are two important properties regarding the expectation of this integral for all $t \geq 0$. The first asserts that the expectation of an Itô integral is zero. The second relates the second moment of the stochastic integral to the expectation of a standard integral as is known as the **Itô isometry**. They are stated as, for every $t \geq 0$,

$$\mathbb{E}\left[\int_0^t \varphi(s)\,\mathrm{d}B(s)\right] = 0 \tag{2.12}$$

and

$$\mathbb{E}\left[\left(\int_0^t \varphi(s)\,\mathrm{d}B(s)\right)^2\right] = \mathbb{E}\left[\int_0^t \varphi^2(s)\,\mathrm{d}s\right]. \tag{2.13}$$

In the case that φ is a deterministic function, the Itô integral $\{\mathcal{I}_\varphi(t) : t \geq 0\}$ is a Gaussian process and hence it has the same distribution as

$$\left\{B\left(\int_0^t \varphi^2(s)\,\mathrm{d}s\right) : t \geq 0\right\}$$

where B is a standard Brownian motion. The process behaves similarly to a Brownian motion except that it is running on its own 'clock' determined by φ. This can be viewed as a time-transformed Brownian motion.

2.3 Stochastic Differential Equations

Let $\{B(t) : t \geq 0\}$ be a m-dimensional Brownian motion defined on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ where, for every $t \geq 0$, we have that

$$\mathcal{F}_t = \sigma\left(\{B(s) : 0 \leq s \leq t\}\right)$$

is the filtration generated by the Brownian motion up to time t. For some $T > 0$, suppose we have the deterministic functions

$$b : [0,T] \times \mathbb{R}^l \to \mathbb{R}^l$$

$$\sigma : [0,T] \times \mathbb{R}^l \to \mathbb{M}^{l\times m}$$

where $\mathbb{M}^{l\times m}$ denotes the space of $l \times m$ symmetric non-negative definite matrices. The notation b_i is the one-dimensional function that represents the i-th component of b. Similarly, $\sigma_{i,j}$ represents the (i,j) entry of σ.

Let $\{B(t) : t \geq 0\}$ be a m-dimensional Brownian motion. We define a $\mathbb{R}^l$-valued stochastic process $X(t)$ to be a **weak solution** to the Itô stochastic differential equation (SDE)

$$\begin{cases} \mathrm{d}X(t) = b(t, X(t))\,\mathrm{d}t + \sigma(t, X(t))\,\mathrm{d}B(t) \\ X(0) = x_0 \in \mathbb{R}^l \end{cases} \tag{2.14}$$

over $[0,T]$ if it satisfies that

1. $\{X(t) : t \geq 0\}$ is progressively measurable with respect to $\{\mathcal{F}_t : 0 \leq t \leq T\}$.
2. For every $1 \leq i \leq l$, $F_i = b_i(t, X(t)) \in L^1(I \times \Omega)$. That is to say
$$\int_0^T \mathbb{E}\left[b_i(t, X(t))\right] \mathrm{d}t < \infty.$$
3. For every $1 \leq i \leq l$ and $1 \leq j \leq m$, $G_{i,j} = \sigma_{i,j}(t, X(t)) \in L^2(I \times \Omega)$. That is to say
$$\int_0^T \mathbb{E}\left[\sigma_{i,j}^2(t, X(t))\right] \mathrm{d}t < \infty.$$
4. For every $0 \leq t \leq T$, almost surely
$$X(t) = x_0 + \int_0^t b(s, X(s))\,\mathrm{d}s + \int_0^t \sigma(s, X(s))\,\mathrm{d}B(s). \tag{2.15}$$

Here, the latter integral
$$\int_0^t \sigma(s, X(s))\,\mathrm{d}B(s)$$
is a multidimensional Itô integral and is understood to be a $\mathbb{R}^l$-valued random variable whose i-th component is the sum of one-dimensional Itô integrals given by
$$\sum_{j=1}^m \int_0^t G_{i,j}\,\mathrm{d}B_j(s)$$
for every $1 \leq i \leq l$. Often we deal with one-dimensional SDEs ($l = m = 1$) in which case b and σ are simply functions from $[0,T] \times \mathbb{R}^l$ to $\mathbb{R}$.

In the equation (2.14), the term b is called the **drift** coefficient which represents the movement of the process $\{X(t) : t \geq 0\}$ with respect to time as a function of the current time and its current position. The term σ is the **diffusion** coefficient which captures the intensity of the random movements that $\{X(t) : t \geq 0\}$

experiences due to increments of the Brownian motion $\{B(t) : t \geq 0\}$. In the case when $\sigma = 0$ everywhere, the system experiences no random forces and (2.14) reduces to an ODE whose solution, if it exists, is a deterministic function. On the other hand, if it is the case that $b = 0$ everywhere, then the solution $\{X(t) : t \geq 0\}$, if it exists, is driven solely by Brownian motion and hence is a martingale by (2.12).

For notational convenience, we write the initial condition of the stochastic process in superscript as $X^{s,x_0}(t)$ which denotes $X(s) = x_0$. If $s = 0$, it will be omitted in the superscript. Hence, we will also write the differential expression (2.14) as

$$\mathrm{d}X^{x_0}(t) = F\,\mathrm{d}t + G\,\mathrm{d}B(t). \tag{2.16}$$

The right hand side of (2.16) is called the **stochastic differential** of $X^{x_0}(t)$. This can also be written coordinate-wise with the i-th component being equal to

$$\mathrm{d}X_i^{x_0}(t) = F_i\,\mathrm{d}t + \sum_{j=1}^{m} G_{i,j}\,\mathrm{d}B_j(t)$$

2.3.1 Itô's Lemma

The following result of Itô is a way to convert between the stochastic differentials of two stochastic processes related by a sufficiently differentiable function. The following is presented as Theorem 3.3 in [19].

Theorem 2.11 (Itô's Formula)**.** *Let $\{X(t) : t \geq 0\}$ be a stochastic process associated with the stochastic differential $\mathrm{d}X(t) = F\,\mathrm{d}t + G\,\mathrm{d}B(t)$ where F and G depend only on t. Suppose $u : [0,T] \times \mathbb{R}^l \to \mathbb{R}^p$ is a continuous function with continuous partial derivatives $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 u}{\partial x_i \partial x_j}$ for $1 \leq i,j \leq l$, and denote its k-th component by u_k. Let $Y(t) = u(t, X(t))$ be a stochastic process with values in $\mathbb{R}^p$. Then $\{Y(t) : t \geq 0\}$ has the stochastic differential*

$$\begin{aligned}\mathrm{d}Y(t) = &\left(\frac{\partial}{\partial t}u(t,X(t)) + \sum_{i=1}^{l} b_i(t,X(t))\frac{\partial}{\partial x_i}u(t,X(t)) + \frac{1}{2}\sum_{i,j=1}^{l} a_{i,j}(t,X(t))\frac{\partial^2}{\partial x_i \partial x_j}u(t,X(t))\right)\mathrm{d}t \\ &+ \sum_{i=1}^{l}\frac{\partial}{\partial x_i}u(t,X(t))\left(\sum_{r=1}^{m}\sigma_{i,r}(t)\,\mathrm{d}B_r(t)\right),\end{aligned} \tag{2.17}$$

where $a_{i,j}(t,X(t))$ is the (i,j) entry of the matrix $\sigma(t,X(t))\left(\sigma(t,X(t))\right)^T$.

In the one-dimensional case $l = m = p = 1$ where $\mathrm{d}X(t) = b(t,X(t))\,\mathrm{d}t + \sigma(t,X(t))\,\mathrm{d}B(t)$ is the SDE of interest, this takes the cleaner form

$$\begin{aligned}\mathrm{d}Y(t) = &\left(\frac{\partial}{\partial t}u(t,X(t)) + b(t,X(t))\frac{\partial}{\partial x}u(t,X(t)) + \frac{\sigma^2(t,X(t))}{2}\frac{\partial^2}{\partial x^2}u(t,X(t))\right)\mathrm{d}t \\ &+ \sigma(t,X(t))\frac{\partial}{\partial x}u(t,X(t))\,\mathrm{d}B(t).\end{aligned}$$

The last term in front of $\mathrm{d}t$ is the only difference between the standard calculus and stochastic calculus. The appearance of this term can be attributed to the previous argument that '$\left(\mathrm{d}B(t)\right)^2 \approx \mathrm{d}t$'.

It is often the case that an explicit formula for the solution to an SDE is unavailable. However, in some situations it is possible to determine such an expression. The following example illustrates this using Theorem 2.11.

Example 2.12. *Let $l = m = 1$. Consider the SDE given by*

$$\begin{cases} \mathrm{d}Y(t) = g(t)Y(t)\,\mathrm{d}B(t) \\ Y(0) = y_0 \end{cases} \tag{2.18}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous. Then we verify that

$$Y(t) = y_0 \exp\left(-\frac{1}{2}\int_0^t g^2(s)\,\mathrm{d}s + \int_0^t g(s)\,\mathrm{d}B(s)\right)$$

is a solution to (2.18) over $[0, T]$ for every $T > 0$. To see this, let $X(t)$ be the argument in the exponential. Then $X(t)$ has stochastic differential

$$\mathrm{d}X(t) = -\frac{1}{2}g^2(t)\,\mathrm{d}t + g(t)\,\mathrm{d}B(t).$$

Let $u(t, x) = e^x$. Then by Itô's formula (2.17), we have

$$\begin{aligned} \mathrm{d}Y(t) &= \left(\tfrac{\partial}{\partial t}u(t, X(t)) + b(t, X(t))\tfrac{\partial}{\partial x}u(t, X(t)) + \frac{1}{2}a(t, X(t))\tfrac{\partial^2}{\partial x_i \partial x_j}u(t, X(t))\right)\mathrm{d}t \\ &\quad + \sigma(t, X(t))\tfrac{\partial}{\partial x}u(t, X(t))\,\mathrm{d}B(t) \\ &= \left(0 - \frac{1}{2}g^2(t)e^{X(t)} + \frac{1}{2}g^2(t)e^{X(t)}\right)\mathrm{d}t + g(t)e^{X(t)}\,\mathrm{d}B(t) \\ &= g(t)Y(t)\,\mathrm{d}B(t). \end{aligned}$$

In particular, we observe that $\{Y(t) : t \geq 0\}$ is a martingale.

If $g \in L^2(I)$, then almost surely $Y(t)$ converges to a random variable as $t \to \infty$. Otherwise, by the law of the iterated logarithm (Theorem 2.7), $\left|\int_0^t g(s)\,\mathrm{d}B(s)\right|$ grows more slowly than $\frac{1}{2}\int_0^t g^2(s)\,\mathrm{d}s$ as $t \to \infty$, so $Y(t) \to 0$ almost surely as $t \to \infty$.

2.3.2 Existence and Uniqueness Conditions

It is important to study when a solution to the SDE (2.14) exists and, if it does, when such a solution is unique. The definitions of existence and uniqueness in the stochastic setting are slightly more nuanced than those in the deterministic setting.

A process $\{X(t) : t \geq 0\}$ is said to be a **strong solution**, or simply just a solution, to (2.14) if (2.15) holds for every standard Brownian motion. It is also shown [9] that, equivalently, $\{X(t) : t \geq 0\}$ is a strong solution if it is $\{\mathcal{F}_t^B : t \geq 0\}$ adapted, where for every $t \geq 0$, $\mathcal{F}_t^B$ is the σ-algebra generated by $\sigma(\{B(s) : s \le t\})$ and by the subsets of null sets from $\sigma(\{B(s) : s \ge 0\})$. In contrast, $\{X(t) : t \ge 0\}$ is a weak solution to (2.14) if there exists a Brownian motion such that $\{X(t) : t \ge 0\}$ satisfies (2.15) but $\{X(t) : t \ge 0\}$ is not necessarily measurable with respect to the σ-algebra generated by that chosen Brownian motion. Hence, a strong solution is automatically a weak solution. Additionally, weak solutions are sometimes denoted as $\{(X(t), B(t)) : t \ge 0\}$, so as to specify the specific Brownian motion for which $\{X(t) : t \ge 0\}$ solves the SDE.

Solutions to the SDE (2.14) are said to be **unique in law** if for any weak solutions $\{(X_1(t), B_1(t)) : t \ge 0\}$ and $\{(X_2(t), B_2(t)) : t \ge 0\}$, the laws of $\{X_1(t) : t \ge 0\}$ and $\{X_2(t) : t \ge 0\}$ are the same. The solutions are **pathwise unique** if they are indistinguishable (that is to say $\mathbb{P}\left(\forall t \ge 0 : X_1(t) = X_2(t)\right) = 1$). It is clear that the latter implies the former. The following result depicted as Figure 1.1 in [9] describes how these relate to each other.

Proposition 2.13. *There exists a pathwise unique strong solution to the SDE (2.14) if any of the following holds:*

- *there exists a weak solution and pathwise uniqueness holds for the SDE*
- *there exists a strong solution and uniqueness in law holds for the SDE*

General results are possible given some regularity conditions on the drift and the diffusion terms. In particular, we require $b : [0,T] \times \mathbb{R}^l \to \mathbb{R}^l$ and $\sigma_r : [0,T] \times \mathbb{R}^l \to \mathbb{R}^l$ (σ_r is the r-th column of σ for every $1 \le r \le m$) to be functions that satisfy the following two conditions which control their growth and regularity:

$$\left|b(t,x)\right| + \sum_{r=1}^{m}\left|\sigma_r(t,x)\right| \le B\left(1 + |x|\right), \tag{2.19}$$

$$\left|b(t,x) - b(t,y)\right| + \sum_{r=1}^{m}\left|\sigma_r(t,x) - \sigma_r(t,y)\right| \le B|x-y|. \tag{2.20}$$

These conditions must hold for some $B > 0$ and every $x, y \in \mathbb{R}^l$ and every $0 \le t \le T$. Condition (2.19) states that the functions b and σ_r do not grow faster than a linear function in space. Condition (2.20) requires that b and σ_r are spatially Lipschitz, uniformly in a given finite time interval. An SDE whose coefficients satisfy these conditions is guaranteed to have a unique solution [2].

Theorem 2.14 (Existence and Uniqueness of Solutions to SDEs)**.** *Consider the SDE (2.14). For every* $T > 0$*, if (2.19) and (2.20) hold on* $[0,T]$*, then there exists a pathwise unique strong solution to the SDE over* $[0,T]$*.*

In the case of one-dimensional homogeneous SDEs ($b(t,x) = b(x)$ and $\sigma(t,x) = \sigma(x)$ for every $t \ge 0$), the Lipschitz condition can be relaxed. The following is attributed to Engelbert and Schmidt [11].

Theorem 2.15 (Engelbert-Schmidt). *Consider the one-dimensional SDE*

$$\mathrm{d}X(t) = b(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}B(t) \tag{2.21}$$

with b/σ^2 being locally integrable in $\mathbb{R}$. Assume that there exists a constant $C > 0$ such that for every $x, y \in \mathbb{R}$

$$\left|b(x)\right| + \left|\sigma(x)\right| \leq C(1 + |x|)$$

and

$$\left|\sigma(x) - \sigma(y)\right| \leq C\sqrt{|x-y|}$$

Then there exists a pathwise unique strong solution to the SDE.

It is worth mentioning that, also for one-dimensional homogeneous SDEs, Yamada and Watanabe proved pathwise uniqueness under further relaxed conditions on b and σ [28].

Theorem 2.16 (Yamada-Watanabe). *Consider the one-dimensional SDE (2.21). Assume that there exists a positive increasing function ρ and a positive concave function κ on $(0, \infty)$ such that for every $x, y \in \mathbb{R}$*

$$\left|b(t,x) - b(t,y)\right| \leq \kappa(|x-y|)$$
$$\left|\sigma(t,x) - \sigma(t,y)\right| \leq \rho(|x-y|)$$

and

$$\int_0^\infty \frac{1}{\rho^2(u)}\,\mathrm{d}u = \infty \tag{2.22}$$

and

$$\int_0^\infty \frac{1}{\kappa(u)}\,\mathrm{d}u = \infty.$$

Then, if there exists a solution to the SDE, the solution is pathwise unique.

The following example which relates to the 'model equation' in [6] that is discussed in Section 3.5 is an example where the Yamada-Watanabe theorem, rather than Theorem 2.14, gives the uniqueness of solutions.

Example 2.17. *Consider the one-dimensional SDE*

$$\mathrm{d}X(t) = 2\sqrt{X(t)}\,\mathrm{d}B(t)$$

with $X(0) = x_0 > 0$. The coefficients satisfy the Yamada-Watanabe theorem conditions. Combined with Proposition 2.13, we know that if a weak solution exists, then it is pathwise unique, and hence there exists a strong solution that is pathwise unique. We will revisit this example after introducing the next theorem.

The theorems presented above concern the existence and the uniqueness in the strong sense, but often in practice we only need the existence and uniqueness in the weak sense. In many cases, the SDEs we are interested in studying fall out of the scope of Theorems 2.14, 2.15, and 2.16.

We state some results (specifically, Theorems 4.1, 4.2, and 4.3 in [9]) on the existence and the uniqueness in the weak sense for time-homogeneous SDEs. These results also discuss the recurrence or the transience of the unique solution. The following theorems rely on the assumption that there exists a $\delta > 0$ so that

$$\int_{\theta-\delta}^{\theta+\delta} \frac{1+\left|b(x)\right|}{\sigma^2(x)}\,\mathrm{d}x < \infty$$

for every $\theta \in (a, \infty)$ for some $a \in \mathbb{R}$. We also define, for every $x \in [a, \infty)$, the quantities

$$\rho(x) = \exp\left(-\int_a^x \frac{2b(t)}{\sigma^2(t)}\,\mathrm{d}t\right)$$

and

$$s(x) = -\int_x^\infty \rho(t)\,\mathrm{d}t.$$

Theorem 2.18. *Suppose that*

$$\int_a^\infty \rho(x)\,\mathrm{d}x = \infty.$$

Given $x_0 \in [a, \infty)$, there exists a weak solution starting from x_0 defined up to τ_a, where τ_a is the hitting time of a (that is to say $\tau_a = \inf_{s\geq 0}\left\{X(s) = a\right\}$), and the solution is unique in law. Further, $\tau_a < \infty$ almost surely.

In fact, under the condition of Theorem 2.18, a global solution exists. Now we revisit Example 2.17. The coefficient of the SDE also satisfies the condition of Theorem 2.18 with $a = 0$, which implies that a weak solution exists up to the time it hits 0 (it can be shown that this happens in finite time). Following our previous arguments, we know that a strong solution exists up to the time it hits 0 and the solution is pathwise unique. The next two theorems supplement Theorem 2.18 and concern the behavior of the solution near ∞.

Theorem 2.19. *Suppose that*

$$\int_a^\infty \rho(x)\,\mathrm{d}x < \infty, \qquad \int_a^\infty \frac{\left|s(x)\right|}{\rho(x)\sigma^2(x)}\,\mathrm{d}x = \infty.$$

Given $x_0 \in [a, \infty)$, there exists a weak solution starting from x_0 defined up to τ_a, and the solution is unique in law. Further, if $x_0 > a$, $\tau_a = \infty$ with positive probability and the solution tends to ∞ almost surely as $t \to \infty$ conditioning on $\tau_a = \infty$.

Theorem 2.20. *Suppose that*

$$\int_a^\infty \rho(x)\,\mathrm{d}x < \infty, \qquad \int_a^\infty \frac{\left|s(x)\right|}{\rho(x)\sigma^2(x)}\,\mathrm{d}x < \infty.$$

Given $x_0 \in (a, \infty)$, there exists a weak solution starting from x_0 defined up to τ_∞, where τ_∞ is the time of explosion, and the solution is unique in law. Furthermore, $\tau_\infty < \infty$ with positive probability.

In fact, under the conditions of Theorem 2.18 or 2.19, a unique in law global solution exists. The following example illustrates the use of Theorems 2.18, 2.19, and 2.20 in the case where the drift and the diffusion are power functions and hence do not satisfy (2.19) or (2.20) except in the case where b and σ are linear functions of x.

Example 2.21. *Consider the SDE*

$$\mathrm{d}X(t) = \mu X^{\alpha}(t)\,\mathrm{d}t + \nu X^{\beta}(t)\,\mathrm{d}B(t)$$

with $X(0) = x_0 = 0$. Set $\lambda = \mu/\nu^2$ and $\gamma = \alpha - 2\beta$. Then, by Theorem 2.18 the solution $X(t)$ is recurrent relative to $|x| < \epsilon$ if one of

$$\gamma < -1, \qquad \gamma = -1 \text{ and } \lambda \leq 1/2, \qquad \gamma > -1 \text{ and } \lambda \leq 0 \tag{2.23}$$

is true. By Theorem 2.19, it is transient if none of the conditions for recurrence holds and one of

$$\gamma = -1 \text{ and } \beta \leq 1, \qquad \gamma > -1 \text{ and } \beta \leq \frac{1-\gamma}{2} \tag{2.24}$$

is true. Further, if $X(t)$ is recurrent or transient, then a unique in law solution exists. If neither (2.23) nor (2.24) hold, then $X(t)$ explodes in the sense that it reaches $+\infty$ in finite time with positive probability by Theorem 2.20. In this case, a unique in law solution exists up until the time of explosion.

2.3.3 Comparison Theory

When solutions to SDEs are not easily available, it may be sufficient to relate one SDE to another. For example, if the solution $X_1(t)$ to a given SDE is not available but $X_1(t)$ is related to the solution $X_2(t)$ to another SDE whose solution is better studied, then one may leverage knowledge of $X_2(t)$ to determine some useful properties of $X_1(t)$. Theorem 3 of Yamada [27] is often useful.

Theorem 2.22 (Almost Sure Comparison of Solutions to SDEs)**.** *Let $X_1(t)$ and $X_2(t)$ be the solutions to*

$$\mathrm{d}X_1(t) = b_1(t, X_1(t))\,\mathrm{d}t + \sigma(t, X_1(t))\,\mathrm{d}B(t) \tag{2.25}$$

and

$$\mathrm{d}X_2(t) = b_2(t, X_2(t))\,\mathrm{d}t + \sigma(t, X_2(t))\,\mathrm{d}B(t) \tag{2.26}$$

respectively. Suppose that $b_1(t,x) < b_2(t,x)$ and $\left|\sigma(t,x) - \sigma(t,y)\right| \leq \rho(|x-y|)$ where ρ satisfies condition (2.22). Then if $X_1(0) \leq X_2(0)$, then $X_1(t) \leq X_2(t)$ almost surely for all $t \geq 0$. If in addition the pathwise uniqueness holds for both equations, then the previous condition may be relaxed to $b_1(t,x) \leq b_2(t,x)$.

This result requires that the diffusion term is the same for both SDEs. Thus, unfortunately, it is rather restrictive. In the case when two given SDEs have different diffusion terms, one may attempt to apply a transformation to the SDEs to match the diffusion terms. This is done later in Section 4.3.3 for other reasons. Alternatively, results on the expectation of the solutions can be obtained instead (see Theorem 2.2 and Table 1 in [3]).

Theorem 2.23 (Comparison of Solutions to SDEs in Expectation)**.** *Let $X_1(t)$ and $X_2(t)$ be as in (2.25) and (2.26) again with drift $b_1(t,x) \leq b_2(t,x)$ but now diffusion $\sigma_1(t,x) \leq \sigma_2(t,x)$. Then if $X_1(0) \leq X_2(0)$, then $\mathbb{E}\left[X_1(t)\right] \leq \mathbb{E}\left[X_2(t)\right]$ for all $t \geq 0$.*

One particularly nice aspect of this theorem is that it does not require the coefficients to satisfy the Lipschitz conditions.

3 Stability of Solutions to Stochastic Differential Equations

We now turn to the main topic of interest. Previously, it was investigated when the solutions to dynamical systems were stable. To do this, multiple definitions of stability were considered. In order to establish the stability of these solutions, Lyapunov type methods were used. Now, we turn our attention to when solutions to SDEs are stable when compared to their deterministic counterparts. In the probabilistic setting, there are more ways in which we can define stability. Fortunately, the Lyapunov theorems that was used in the deterministic case translates without too much difficulty to the stochastic setting.

3.1 Problem Statement

This section prepares some of the concepts that will be required to discuss stability of solutions to SDEs. We begin by considering the SDE in $\mathbb{R}^l$ given by

$$\mathrm{d}X(t) = b(t, X(t))\,\mathrm{d}t + \sigma(t, X(t))\,\mathrm{d}B(t) \tag{3.1}$$

defined on $t \in [s, T]$ for some $s \leq T$. Throughout the rest of this thesis, unless specified otherwise, the term solution (to an SDE) always refers to a strong solution, and similarly for existence and uniqueness. Suppose that for some $x \in \mathbb{R}^l$, solutions to this SDE exist and are unique whenever the starting position $X(s)$ is sufficiently close to x. As before, we denote by $X^{s,x_1}(t)$ the unique solution to the SDE (3.1) subject to the initial condition $X(s) = x_1$. Sometimes this will be written simply as $X(t)$ if no ambiguity exists.

3.1.1 Deterministic Solutions

To consider stability, first we need to choose a reference with respect to which the solution to (3.1) should be stable. Let $X^{s,x_1}(t)$ be the unique solution to the system (3.1). We refer to $\tilde{X}^{s,x_0}(t)$ as the **deterministic solution** of the system if it is the unique solution to the ODE system given by

$$\begin{cases} \mathrm{d}\tilde{X}(t) = b(t, \tilde{X}(t))\,\mathrm{d}t \\ \tilde{X}(s) = x_0 \in \mathbb{R}^l \end{cases} \tag{3.2}$$

for $t \geq s$. Similarly, this will sometimes be written simply as $\tilde{X}(t)$ if no ambiguity exists.

The deterministic solution is how the stochastic process $X^{s,x_1}(t)$ would behave in the absence of the random pertubative force (that is to say when $\sigma = 0$), provided their respective starting positions x_0, x_1 equate. In this sense, $\tilde{X}^{s,x_0}(t)$ serves as a good candidate with what $X^{s,x_1}(t)$ should be compared: if $X^{s,x_1}(t)$ does not deviate too far from $\tilde{X}^{s,x_0}(t)$ when the starting points x_0 and x_1 are sufficiently close, then it is natural to consider $X^{s,x_1}(t)$ as 'stable'.

3.1.2 Lyapunov Functions

Many results in this section are stated in terms of Lyapunov functions as in Section 2.1.2. To this end we define the operator L associated with the SDE (3.1) that acts on V as

$$LV(t,x) = \frac{\partial V(t,x)}{\partial t} + \sum_{i=1}^{l} b_i(t,x)\frac{\partial V(t,x)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{l} a_{i,j}(t,x)\frac{\partial^2 V(t,x)}{\partial x_i \partial x_j}. \tag{3.3}$$

Here, b_i and $a_{i,j}$ correspond to the drift and diffusion coefficients as per (2.17). The operator L is also known as the **generator** of the process. This form is very similar to (2.11). By adding the term involving $a_{i,j}$, this quantity can be viewed as the stochastic analogue of $\dot{V}$. Just as before, we may view a positive definite function V as a generalized energy function for the solution to (3.1).

The quantity LV in 3.3 therefore represents the change of energy just as $\dot{V}$ did in the deterministic case. The additional term captures changes in energy caused by random perturbative forces. The following result (Lemma 3.2 in [19]) best expresses this relationship.

Lemma 3.1 (Expected Energy Change)**.** *Let $X^{s,x_1}(t)$ be a solution to (3.1), U be a bounded domain with $X^{s,x_1}(s) \in U$, and τ_U be the first exit time of U. Set $\tau_U(t) = \min(\tau_U, t)$. Then if V is twice continuously differentiable, then*

$$\mathbb{E}\left[V(\tau_U(t), X^{s,x_1}(\tau_U(t))) - V(s,x_1)\right] = \mathbb{E}\left[\int_s^{\tau_U(t)} LV(u, X^{s,x_1}(u))\,\mathrm{d}u\right].$$

This result says that the expected change in energy from time s to $\tau_U(t)$ is given by the expectation of the integral of the LV term over the same time interval. Heuristically, Lemma 3.1 allows us to quantify energy changes of the process $X^{s,x_1}(t)$ on average. This is visualized in Figure 3. In Section 2.1.2, Theorems 2.2, 2.3, and 2.4 asserted that if $\dot{V}$ satisfied some non-positivity condition, then the stability of the solution was guaranteed. In the stochastic setting, we will see that if LV satisfies similar conditions, then the stability of the solution to the SDE (3.1) is attained.

3.2 Notions of Stability

Here, we consider stability of a solution to an SDE with respect to the deterministic solution. In order to obtain results involving the operator L and the Lyapunov function V, definitions of stability are required. As in the deterministic case, there exist several notions of stability for stochastic systems.

Due to the inherent probabilistic nature of the stochastic solution, there is another dimension of depth to consider. The goal is to compare $X^{s,x_1}(t)$ and $\tilde{X}^{s,x_0}(t)$ and how their distance converges to 0. Since the former is a random variable, we may consider all the different ways in which random variables converge (convergence in distribution, in probability, almost sure, and in L^p).

3.2.1 Weak Stability

This section considers weak notions of stability as given in Chapter 1.5 of Khaminskii [19]. These notions of stability deal with the process $X^{s,x_1}(t)$ for a given moment t. This is in contrast to strong definitions in Section 3.2.2 where the whole path of the solution will be considered. The definitions presented in this section do not constitute all possible definitions, but are the ones that are most studied.

Definition 3.2 (Weakly Stable in Probability)**.** *A solution $X^{s,x_1}(t)$ of (3.1) is said to be weakly stable in probability with respect to the deterministic solution $\tilde{X}^{s,x_0}(t)$ for $t \geq t_0$ if for every $\epsilon > 0$, $\delta > 0$, there exists $r \geq 0$ such that if $|x_1 - x_0| < r$, then*

$$\mathbb{P}\left(\left|X^{s,x_1}(t) - \tilde{X}^{s,x_0}(t)\right| > \epsilon\right) < \delta.$$

for every $t \geq t_0$.

This definition requires that for sufficiently close starting points the processes are arbitrarily close to one another with probability arbitrarily close to 1 at any time past t_0. This is alternatively stated as

$$\lim_{x_1 \to x_0} \sup_{t \geq t_0} \mathbb{P}\left(\left|X^{s,x_1}(t) - \tilde{X}^{s,x_0}(t)\right| > \epsilon\right) \to 0. \tag{3.4}$$

If, in addition to the the conditions in Definition 3.2, it holds that

$$\lim_{t \to \infty} \mathbb{P}\left(\left|X^{s,x_1}(t) - \tilde{X}^{s,x_0}(t)\right| > \epsilon\right) \to 0$$

for $|x_1 - x_0| < r$, then we say that the solution $X^{s,x_1}(t)$ to the SDE (3.1) is **asymptotically stable** with respect to the deterministic solution. For convenience, the stability of the solution $X^{s,x_1}(t)$ to the SDE will sometimes be referred to as the stability of the SDE itself. Stronger types of stability consider the distance between the stochastic solution $X^{s,x_1}(t)$ and the deterministic solution $\tilde{X}^{s,x_0}(t)$ under L^p.

Definition 3.3 (Weakly p-stable)**.** *A solution $X^{s,x_1}(t)$ of (3.1) is said to be weakly p-stable with respect to the deterministic solution $\tilde{X}^{s,x_0}(t)$ for $t \geq t_0$ if for every $\epsilon > 0$, there exists $r \geq 0$ such that if $|x_1 - x_0| < r$, then*

$$\mathbb{E}\left[\left|X^{s,x_1}(t) - \tilde{X}^{s,x_0}(t)\right|^p\right] < \epsilon \tag{3.5}$$

for every $t \geq t_0$.

By Markov's inequality, we see that that weak p-stability implies weak stability in probability. Similarly, if the quantity in (3.5) converges to 0 as $t \to \infty$, then we say the solution is **asymptotically p-stable**. Finally, $X^{s,x_1}(t)$ is said to be **exponentially p-stable** with respect to $\tilde{X}^{s,x_0}(t)$ if there exists constants $A > 0$ and $\alpha > 0$ such that

$$\mathbb{E}\left[\left|X^{s,x_1}(t) - \tilde{X}^{s,x_0}(t)\right|^p\right] \leq A|x_1 - x_0|^p \exp(-\alpha t)$$

for every $t \geq t_0$.

Another way to strengthen the stability is to remove the localized dependence of x_1 on x_0. That is, for arbitrary x_0 and x_1, the above conditions hold so long as t is sufficiently large. The following definition is stated for stability in probability but can be adapted to p-stability or asymptotic stability as well.

Definition 3.4 (Weakly Stable in Probability in the Large)**.** *A solution $X^{s,x_1}(t)$ of (3.1) is said to be weakly stable in probability in the large with respect to the deterministic solution $\tilde{X}^{s,x_0}(t)$ if for every $\epsilon > 0$, $\delta > 0$, and every x_0, there exists some $T = T(x_1 - x_0, \epsilon, \delta)$ so that*

$$\mathbb{P}\left(\left|X^{s,x_1}(t) - \tilde{X}^{s,x_0}(t)\right| > \epsilon\right) < \delta$$

for $t > T$.

3.2.2 Strong Stability

In this section we consider stability in the strong sense. Compared to weak forms of stability, these concepts are more widely studied in literature. Many definitions in this section translate from the weak setting with little difficultly. When a solution $X^{s,x_1}(t)$ is strongly stable, it is often simply referred to as stable. Later in Section 3.4, we investigate sufficient conditions for stability of the solution to (3.1) in terms of the restrictions imposed on the coefficients.

Definition 3.5 (Strongly Stable in Probability)**.** *A solution $X^{s,x_1}(t)$ of (3.1) is said to be (strongly) stable in probability with respect to the deterministic solution $\tilde{X}^{s,x_0}(t)$ for $t \geq 0$ if for every $\epsilon > 0$ and $s \geq 0$*

$$\lim_{x_1 \to x_0} \mathbb{P}\left(\sup_{t \geq s}\left|X^{s,x_1}(t) - \tilde{X}^{s,x_0}(t)\right| > \epsilon\right) = 0.$$

The solution $X^{s,x_1}(t)$ is said to be **unstable** if it is not stable. We additionally say the solution is **stable uniformly** if the limit tends to 0 uniformly in $s \geq 0$. Definition 3.5 requires that the uniform distance between the stochastic solution and the deterministic solution converges to 0 as the initial points converge to one another. Clearly this definition implies (3.4), and hence strong stability implies weak stability. Here and throughout the remainder of this thesis, the term ‘stable’ refers to strong stability unless otherwise specified.

As in the weak case, we extend the strong definition of stability by imposing some additional requirements.

Definition 3.6 (Strongly Asymptotically Stable in Probability)**.** *A solution $X^{s,x_1}(t)$ to (3.1) is said to be asymptotically stable with respect to the deterministic solution $\tilde{X}^{s,x_0}(t)$ if it is strongly stable and additionally*

$$\lim_{x_1 \to x_0} \mathbb{P}\left(\lim_{t \to \infty}\left|X^{s,x_1}(t) - \tilde{X}^{s,x_0}(t)\right| = 0\right) = 1. \tag{3.6}$$

If the statement (3.6) holds without taking the outermost limit, we say the solution is **asymptotically stable in the large**. The following definition is the analogue of Definition 3.3 for strong stability.

Definition 3.7 (p-stability)**.** *A solution $X^{s,x_1}(t)$ to (3.1) is said to be p-stable for $p > 0$ with respect to the deterministic solution $\tilde{X}^{s,x_0}(t)$ if*

$$\lim_{\delta\to 0}\left(\sup_{|x_1-x_0|<\delta, t\geq s}\mathbb{E}\left[\left|X^{s,x_1}(t)-\tilde{X}^{s,x_0}(t)\right|^p\right]\right)\to 0$$

If additionally the expectation term vanishes as $t \to \infty$, then the solution is asymptotically p-stable. Finally, if there exists constants $A > 0$ and $\alpha > 0$ so that

$$\mathbb{E}\left[\left|X^{s,x_1}(t)-\tilde{X}^{s,x_0}(t)\right|^p\right]\leq A|x_1-x_0|^p\exp\left(-\alpha t\right)$$

for every $t \geq s$, then the solution is exponentially p-stable.

3.3 Assumptions

In order to obtain meaningful results, certain assumptions on the drift term b and the diffusion term σ must be made. We assume these assumptions hold throughout the remainder of this chapter. The first assumption controls the behavior of the deterministic solution by requiring it to be bounded. Formally, we require that there exists $M > 0$ so that

$$\sup_{t\geq s}\left|\tilde{X}^{s,x}(t)\right|\leq M \tag{3.7}$$

for all x in some neighborhood of x_0. This requirement alleviates some of the issues that could arise when considering unbounded solutions in the context of definitions introduced in Section 3.2. We later drop this assumption in Chapter 4 when considering a different notion of stability.

The next assumption controls the effect of the random perturbations on $X^{s,x_1}(t)$ along the trajectory of the deterministic solution. We assume that if $X^{s,x_1}(t_0) = \tilde{X}^{s,x_0}(t_0)$ for some $t_0 \geq s$, then $X^{s,x_1}(t) = \tilde{X}^{s,x_0}(t)$ for all $t \geq t_0$. This is to say the effect of diffusion disappears along the trajectory of the deterministic solution. Intuitively, the stochastic process 'sticks' to the deterministic solution. This vanishing assumption is written as

$$\sigma(t, \tilde{X}^{s,x_0}(t)) = 0. \tag{3.8}$$

If a continuity condition (such as the Lipschitz condition (2.20)) is placed on σ, then this assumption controls the strength of random forces in a neighborhood of the deterministic solution. Note that this assumption is necessary for studying the current notions of stability. To see this, observe that even when $x_0 = x_1$, if the diffusion coefficient is non-zero for some interval of time, then with positive probability $X^{s,x_1}(t)$ will move away from $\tilde{X}^{s,x_0}(t)$. In general, there would be no reason to expect the stochastic solution and the deterministic solution to be close. Despite this, they may still may behave similarly as will be examined in Chapter 4.

3.4 Lipschitz Coefficients

We consider now some results when the coefficients of the SDE (3.1) satisfy conditions (2.19) and (2.20). In this case, some properties of the sample paths of the solution can be derived. These properties allow for general Lyapunov type methods to be applicable in the stochastic setting. The stability theorems presented here are mostly from [19] but are also presented in earlier works such as [1, 14, 17, 21].

3.4.1 Lyapunov Stability Theorems

Let us first establish a couple of definitions. First, we say that a process $\{X(t) : t \geq 0\}$ is **regular** if $\mathbb{P}(\tau = \infty) = 1$ where τ is the exit time of every bounded domain. That is to say, a process is regular if it almost surely does not explode in finite time. Second, we formally define the **trajectory** of the deterministic solution to (3.2) as $\mathcal{T} = \left\{(t, x) : t \geq s \text{ with } \tilde{X}^{s,x_0}(t) = x\right\}$. Finally, a closed set Γ is said to be **inaccessible** to a process $\{X(t) : t \geq 0\}$ if $\mathbb{P}\left(\tau^\Gamma < \infty\right) = 0$, where τ^L is the hitting time of the set Γ (that is to say $\tau^\Gamma = \inf_{s\geq 0} \{X(s) \in \Gamma\}$). With these definitions, we have the following lemma.

Lemma 3.8 (Inaccessibility of the Deterministic Solution)**.** *Suppose the solution $\{X^{s,x_1}(t) : t \geq s\}$ of the SDE (3.1) has coefficients satisfying the Lipschitz condition (2.20) in every spatially bounded domain, as well as (3.8). Then if $x_0 \neq x_1$, the trajectory of the deterministic solution $\tilde{X}^{s,x_0}(t)$ is inaccessible to $X^{s,x_1}(t)$.*

Beyond guaranteeing the uniqueness of a solution, Lemma 3.8 shows that the Lipschitz condition also allows for better understanding of the behaviors of $X^{s,x_1}(t)$. This in turn leads to Lyapunov type stability theorems for the stochastic case similar to those seen in Section 2.1 for deterministic systems.

Suppose $x : I \to \mathbb{R}^l$ is a function. We say that a function $V : I \times D \to \mathbb{R}$ is **positive definite about the path** $x(t)$ on D if, in addition to (2.7) and (2.9), we have that

$$\forall t \in I : \quad V(t, x(t)) = 0.$$

As before, this definition can be upgraded to be **uniform** if the additional condition (2.10) holds where now W_1 and W_2 are continuous positive definite functions about the path $x(t)$. When the function V is viewed as representing an energy, the above definition asserts that the trajectory of $x(t)$ is where the energy is minimized. The previous interpretation as in Section 2.1.2 still applies. With this setup, we are nearing our first result regarding the stability of SDEs. We first need the next lemma which justifies the supermartingale idea first suggested by [5].

Lemma 3.9 (Lyapunov Process is a Supermartingale)**.** *Let $V(t, x)$ be a function continuously differentiable with respect to t and twice continuously differentiable with respect to x on the set $I \times \{U \setminus \Gamma\}$ where U is a bounded domain in $\mathbb{R}^l$ and $\Gamma \subset U$ is a set inaccessible to a solution $X^{s,x_1}(t)$ of (3.1). Assume that $LV(t, x) \leq 0$ on the set $I \times \{U \setminus \Gamma\}$. Then the process $\{V(\tau_U(t), X(\tau_U(t))) : t \geq s\}$ is a supermartingale, where we recall τ_U is the exit time of U.*

Finally, we get the stochastic analogue of Theorem 2.2. The proof is slightly modified from that in [19].

Theorem 3.10 (Lyapunov Strong Stability). *Suppose the assumptions stated in Section 3.3 hold. Suppose also that there exists a function $V(t,x)$, positive definite about the path $\tilde{X}^{s,x_0}(t)$, that is continuously differentiable with respect to t and twice continuously differentiable with respect to x. Assume $LV(t,x) \leq 0$ everywhere in a domain U except possibly along the trajectory of $\tilde{X}^{s,x_0}(t)$. Then the solution $X^{s,x_1}(t)$ of the SDE (3.1) is strongly stable in probability with respect to the deterministic solution $\tilde{X}^{s,x_0}(t)$.*

Proof. Let U be a bounded domain containing the path of $\tilde{X}^{s,x_0}(t)$, which is possible because $\tilde{X}^{s,x_0}(t)$ is bounded by assumption. Let $V(t,x)$ be sufficiently differentiable on $I \times U$ as in the assumption of the theorem. Let r be the number so that

$$A_r = \left\{(t,x) : \exists t \geq s \text{ such that } \left|\tilde{X}^{s,x_0}(t) - x\right| < r\right\}$$

is contained in the closure of $I \times U$. Then $V_r = \inf_{(t,x)\in(I\times U)\setminus A_r} V(t,x) > 0$ by assumption.

Now set $Y^{s,x_1,x_0}(t) = X^{s,x_1}(t) - \tilde{X}^{s,x_0}(t)$. Write $\Omega_1 = \left\{\omega \in \Omega : \sup_{s\leq u\leq t}\left|Y^{s,x_1,x_0}(u,\omega)\right| > r\right\}$. Observe that the following chain of inequalities

$$\begin{aligned}
\mathbb{E}\left[V(\tau_{A_r}(t), X^{s,x_1}(\tau_{A_r}(t)))\right] &= \int_\Omega V(\tau_{A_r}(t), X^{s,x_1}(\tau_{A_r}(t)))\,\mathrm{d}\mathbb{P} \\
&\geq \int_{\Omega_1} V(\tau_{A_r}(t), X^{s,x_1}(\tau_{A_r}(t)))\,\mathrm{d}\mathbb{P} \\
&\geq \inf_{(t,x)\in(I\times U)\setminus A_r} V(t,x) \int_{\Omega_1} \mathrm{d}\mathbb{P} \\
&= \inf_{(t,x)\in(I\times U)\setminus A_r} V(t,x) \cdot \mathbb{P}\left(\sup_{s\leq u\leq t}\left|Y^{s,x_1,x_0}(u,\omega)\right| > r\right)
\end{aligned}$$

implies that

$$\mathbb{P}\left(\sup_{s\leq u\leq t}\left|Y^{s,x_1,x_0}(u,\omega)\right| > r\right) \leq \frac{\mathbb{E}\left[V(\tau_{A_r}(t), X^{s,x_1}(\tau_{A_r}(t)))\right]}{V_r}. \tag{3.9}$$

By Khaminskii's Theorem 3.5 (see Khaminskii's Example 3.2), the regularity of the process $X^{s,x_1}(t)$ follows from continuity of its coefficients and the Lipschitz conditions. By Lemma 3.8, the trajectory $\mathcal{T}$ of the deterministic solution is inaccessible to the process $X^{s,x_1}(t)$ whenever $x_0 \neq x_1$. In particular, by Lemma 3.9, $\left\{V(\tau_{A_r}(t), X^{s,x_1}(\tau_{A_r}(t))) : t \geq s\right\}$ is a supermartingale, so

$$\mathbb{E}\left[V(\tau_{A_r}(t), X^{s,x_1}(\tau_{A_r}(t)))\right] \leq V(s,x_1)$$

for every $t \geq s$. Combining this with (3.9), we obtain

$$\mathbb{P}\left(\sup_{s\leq u\leq t}\left|Y^{s,x_1,x_0}(u,\omega)\right| > r\right) \leq \frac{\mathbb{E}\left[V(\tau_{A_r}(t), X^{s,x_1}(\tau_{A_r}(t)))\right]}{V_r} \leq \frac{V(s,x_1)}{V_r}.$$

Then taking $t \to \infty$, we obtain

$$\mathbb{P}\left(\sup_{s\leq u}|Y^{s,x_1,x_0}|(u) > r\right) \leq \frac{V(s,x_1)}{V_r}.$$

Since $V_r > 0$ and by the continuity of $V(s,x_1)$, taking $x_1 \to x_0$, the right hand side goes to 0. ☐

If, in addition to the hypotheses of Theorem 3.10, we also have

$$\lim_{x \to x_0} \sup_{t>0} V(t, x) = 0,$$

then the stability is uniform. In this case $V(t, x)$ is said to have **infinitesimal upper limit**.

Naturally, imposing stricter conditions will yield stronger forms of stability. We could expect that if we require that $LV < 0$ then asymptotic stability will hold. The following results, Theorems 5.5 and 5.11 in [19], assume that the deterministic solution is $\tilde{X}^{s,x_0}(t) = 0$. It also assumes that the vanishing diffusion condition (3.8) holds.

Theorem 3.11 (Lyapunov Asymptotic Stability). *Suppose that there exists a function $V(t, x)$, positive definite about the path $\tilde{X}^{s,x_0}(t)$, continuously differentiable with respect to t and twice continuously differentiable with respect to x. Assume further that $V(t, x)$ has infinitesimal upper limit and $LV(t, x) < 0$ everywhere in a domain U except possibly along the trajectory of $\tilde{X}^{s,x_0}(t)$. Then the solution $X^{s,x_1}(t)$ of the SDE (3.1) is strongly asymptotically stable in probability with respect to the deterministic solution $\tilde{X}^{s,x_0}(t)$.*

Theorem 3.12 (Lyapunov Exponential Stability). *Suppose that there exists a function $V(t, x)$, positive definite about the path $\tilde{X}^{s,x_0}(t)$, continuously differentiable with respect to t and twice continuously differentiable with respect to x. Further assume that there exists positive constants k_1, k_2, and k_3 so that*

$$k_1|x|^p \leq V(t, x) \leq k_2|x|^p$$

and

$$LV(t, x) \leq -k_3|x|^p$$

for every $t \geq s$ and $x \in \mathbb{R}^l$. Then the solution $X^{s,x_1}(t)$ of the SDE (3.1) is exponentially p-stable with respect to the deterministic solution $\tilde{X}^{s,x_0}(t)$.

In addition, Theorem 5.7 of [19] will be useful in Section 4.3.1.

Theorem 3.13 (Lyapunov Asymptotic Stability in the Large). *Assume that the constant 0 is a solution to (3.1). If $X^{s,x_1}(t)$ is a solution to (3.1) that is uniformly stable in probability with respect to 0 and is recurrent to $\left\{x \in \mathbb{R}^l : |x| < \epsilon\right\}$ for any $\epsilon > 0$, then $X^{s,x_1}(t)$ is asymptotically stable in the large.*

3.4.2 Examples

The following examples illustrate the use of the theorems presented in Section 3.4.1. By choosing the appropriate Lyapunov function V, we can obtain results on stability. It is occasionally the case that stability can be shown without the use of Lyapunov functions, which will also be demonstrated.

One class of SDEs which are particularly nice to work with are those with linear coefficients. We use the notation of Evans [13]. Assuming $s = 0$, the SDE (3.1) is said to be **linear** if the drift and diffusion coefficients has the form

$$b(t, x) = c(t) + d(t)x \tag{3.10}$$

$$\sigma(t, x) = e(t) + f(t)x \tag{3.11}$$

for appropriate functions c, d, e, and f. The solution to a one-dimensional linear SDE is

$$\begin{aligned} X^{x_1}(t) &= \exp\left(\int_0^t d(s)\,\mathrm{d}s - \int_0^t \frac{f^2(s)}{2}\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}B(s)\right) \\ &\quad\cdot \left(x_1 + \int_0^t \exp\left(-\int_0^s d(r)\,\mathrm{d}r + \int_0^s \frac{f^2(r)}{2}\,\mathrm{d}r - \int_0^s f(r)\,\mathrm{d}B(r)\right)\big(c(s) - e(s)f(s)\big)\,\mathrm{d}s \right. \\ &\quad\left. + \int_0^t \exp\left(-\int_0^s d(r)\,\mathrm{d}r + \int_0^s \frac{f^2(r)}{2}\,\mathrm{d}r - \int_0^s f(r)\,\mathrm{d}B(r)\right) e(s)\,\mathrm{d}B(s)\right) \end{aligned} \tag{3.12}$$

for every $t \geq 0$.

The linear SDE is said to be **homogeneous** if $c = e = 0$. In the case that only $e = 0$, we say it is **semi-homogeneous of first kind**. If, alternatively, we have that only $c = 0$ then it is said to be **semi-homogeneous of second kind**. Is it clear that condition (3.8) fails to hold in the semi-homogeneous cases. The next example examines the stability of the solution to the homogeneous linear SDE.

Example 3.14. *Consider the one-dimensional SDE*

$$\mathrm{d}X(t) = b(t)X(t)\,\mathrm{d}t + \sigma(t)X(t)\,\mathrm{d}B(t) \tag{3.13}$$

with initial condition $X(0) = x_1 > 0$. We assume that $b(t)$ and $\sigma(t)$ are bounded so as to satisfy conditions (2.19) and (2.20). By (3.12),

$$X^{x_1}(t) = x_1 \exp\left(\int_0^t \left(b(s) - \frac{\sigma^2(s)}{2}\right)\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B(s)\right) \tag{3.14}$$

solves this SDE. Set

$$\eta(t) = \int_0^t \left(b(s) - \frac{\sigma^2(s)}{2}\right)\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B(s).$$

It can be seen directly that if $\eta(t) \to -\infty$ as $t \to \infty$ almost surely, then $X^{x_1}(t)$ is asymptotically stable with respect to the constant function $\tilde{X}(t) = 0$. If $\limsup_{t\to\infty} \eta(t) < \infty$ almost surely, we have stability. If instead $\limsup_{t\to\infty} \eta(t) = \infty$ with non-zero probability, we have instability.

This result can also be achieved by the method of Lyapunov. To see stability, fix $\epsilon > 0$ so that

$$\int_0^t \left(b(s) - \frac{\sigma^2(s)}{2} + \epsilon\right)\mathrm{d}s < k$$

holds for some $k > 0$ and all $t > 0$. This condition limits the effects of drift and so the Lyapunov function

$$V(t,x) = |x|^{\alpha} \exp\left(-\alpha \int_0^t \left(b(s) - \frac{\sigma^2(s)}{2} + \epsilon\right) \mathrm{d}s\right)$$

satisfies the conditions of Theorem 3.10 for small enough $\alpha > 0$. Apply the generator

$$L = \frac{\partial}{\partial t} + b(t)|x| \frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t)|x|^2 \frac{\partial^2}{\partial x^2}$$

to obtain

$$LV(t,x) = \alpha |x|^{\alpha} \exp\left(-\alpha \int_0^t \left(b(s) - \frac{\sigma^2(s)}{2} + \epsilon\right) \mathrm{d}s\right) \left(-\epsilon + \alpha \cdot \frac{\sigma^2(t)}{2}\right).$$

Taking α to be sufficiently small so that the last term is negative, the Lyapunov function satisfies $LV(t,x) \leq 0$ everywhere. By Theorem 3.10, this shows that the system is stable.

Example 3.15. *Let us consider again (3.13) but with time-homogeneous coefficients. That is to say the drift is $b(t) = b \in \mathbb{R}$ and the diffusion is $\sigma(t) = \sigma \in \mathbb{R}$. Setting $V(t,x) = |x|^p$, we have that*

$$LV(t,x) = p|x|^{p-1} \left(b + \frac{\sigma^2}{2}(p-1)\right).$$

Hence if $b + \frac{\sigma^2}{2}(p-1) < 0$, then we have p-stability by Theorem 3.12.

Example 3.16. *Consider the one-dimensional SDE*

$$\mathrm{d}X(t) = g'(t)\,\mathrm{d}t + \big|X(t) - g(t)\big|\,\mathrm{d}B(t) \tag{3.15}$$

for some differentiable function g such that (3.7) holds. If $x_0 = g(0)$, we claim that $g(t)$ is the deterministic solution with respect to which $X^{x_1}(t)$ is stable. Choose the Lyapunov function $V(t,x) = \big|x - g(t)\big|$ which satisfies Theorem 3.10. Apply the generator to obtain

$$LV(t,x) = \frac{\partial}{\partial t}\big|x - g(t)\big| + g'(t)\frac{\partial}{\partial x}\big|x - g(t)\big| = -g'(t) + g'(t) = 0$$

in a neighborhood around but excluding the path of $g(t)$. Then again by Theorem 3.10, stability is obtained.

3.4.3 Numerical Visualizations

The core idea introduced in this section has been that, through the appropriate choice of Lyapunov function $V(t,x)$, the solution $X^{s,x_1}(t)$ to the system (3.1) is stable in some sense with respect to $\tilde{X}^{s,x_0}(t)$. Now, we visualize these ideas through a numerical example that helps us understand the interaction between the solution $X(t)$ and the Lyapunov results.

Consider the two-dimensional SDE given by

$$\begin{cases} \mathrm{d}X_1(t) = \frac{1}{4}X_1(t)\,\mathrm{d}t + X_1(t)\,\mathrm{d}B_1(t) \\ \mathrm{d}X_2(t) = \frac{1}{4}X_2(t)\,\mathrm{d}t + X_2(t)\,\mathrm{d}B_2(t) \end{cases} \tag{3.16}$$

and, for $x_1, x_2 > 0$, let $X^{(x_1,x_2)}(t)$ be its solution with satisfying $X^{(x_1,x_2)}(0) = (x_1, x_2)$. Each dimension of the two-dimensional process is a independent process that is stable with respect to 0 as seen in Example 3.14. It can then be similarly observed that the two-dimensional process $X^{(x_1,x_2)}(t)$ is stable with respect to the origin.

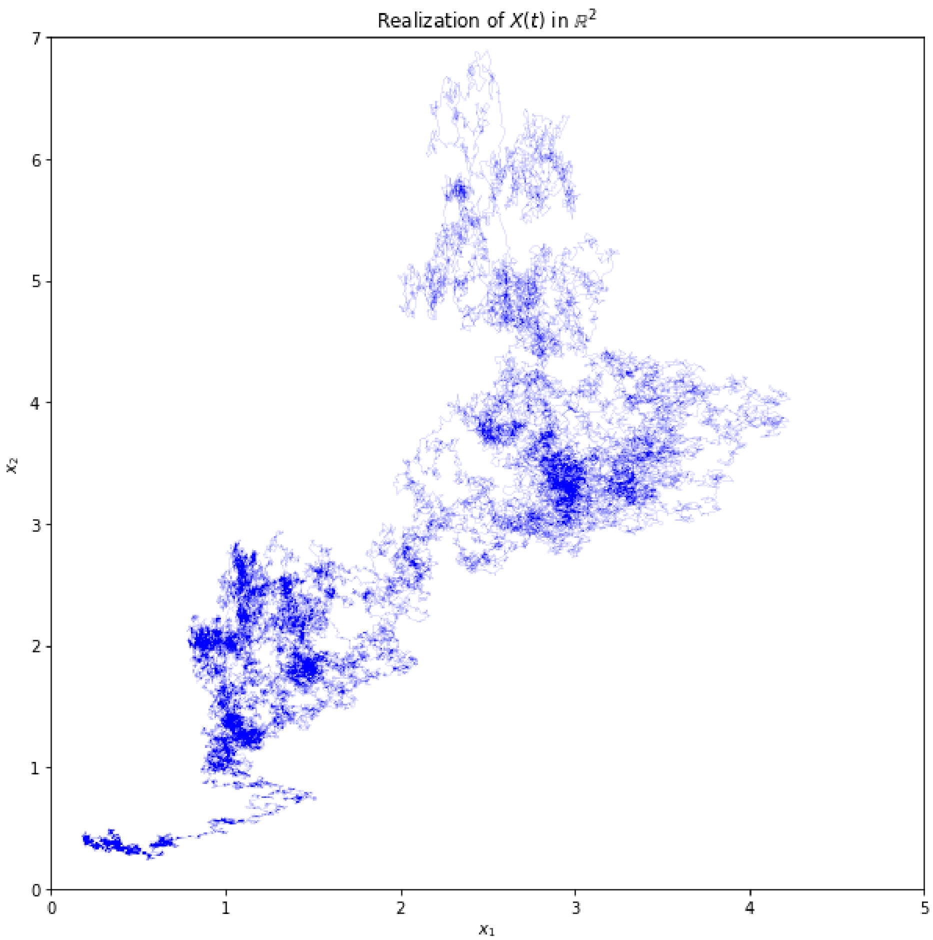

Figure 2: One realization of $X(t)$
which is the solution to the SDE (3.16) for $0 \leq t \leq 3$.

In order to simulate this, we use the explicit order 1 strong scheme (also called a stochastic Runge-Kutta scheme) in [20] to obtain a numerical solution to (3.16). This scheme is given by

$$\begin{aligned} X(n+1) &= X(n) + b(n, X(n))\Delta t + \sigma(n, X(n))\Delta B(t) \\ &\quad + \frac{1}{2\sqrt{\Delta t}} \left(\sigma(n, K(n)) - \sigma(n, X(n))\right) \left(\left(\Delta B(t)\right)^2 - \Delta t\right) \end{aligned}$$

where n is the time discretization, Δt is the time increment, $\Delta B(t)$ is the corresponding increment of Brownian motion, and

$$K(n) = X(n) + b(n, X(n))\Delta t + \sigma(n, X(n))\sqrt{\Delta t}.$$

A sample path of $X^{(x_1,x_2)}(t)$ for $0 \le t \le 3$ is seen in Figure 2 with $x_1 = x_2 = 1$. All figures in this section were created using the Python programming language.

We choose the Lyapunov function $V(t,x) = |x|^{1/4}$. Then we may visualize Figure 2 in three dimensions where the solution $X^{(1,1)}(t)$ 'lives' on the Lyapunov function. This is illustrated in Figure 3. The orange contour lines depict $z = V(t,x)$ with deeper colors indicating a higher energy level. The same sample path of $X^{(1,1)}(t)$ traverses this energy function. Since $LV(t,x) \le 0$, we expect that the trajectory in blue eventually falls into the energy well located at the origin.

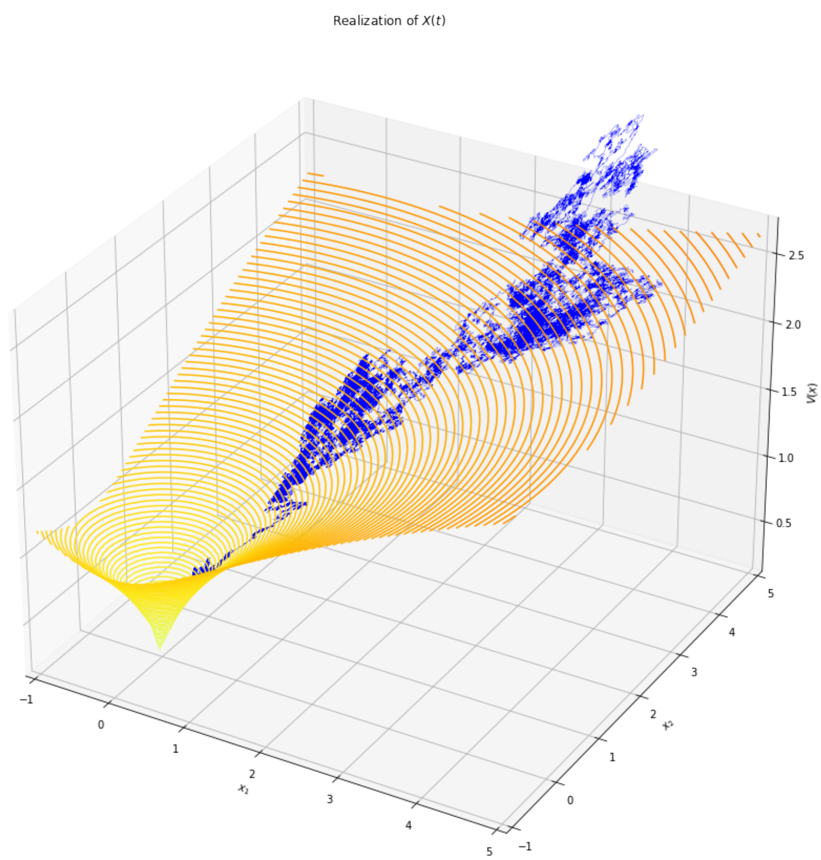

Figure 3: Stochastic process on the Lyapunov function $V(t,x) = |x|^{1/4}$ for $0 \le t \le 3$.

A video animation can be found on *GitHub*.

3.5 Non-Lipschitz Coefficients

In the setting of non-Lipschitz coefficients, the methods of Lyapunov no longer apply. However, it should not be surprising that the definitions of stability continue to hold nevertheless. This thesis demonstrates a couple of methods to obtain stability when condition (2.20) fails to hold. The methods used in this section work more directly with the properties of the specific SDE in comparison with the Lyapunov method in Section 3.4.1.

Currently, no general theory as applicable as the Lyapunov stability theorems exists for SDEs with non-Lipschitz coefficients. Even when the drift and the diffusion terms satisfy Hölder continuity conditions, problems still occur. In Example 2.17, the diffusion coefficient $\sigma(t,x) = 2\sqrt{x}$ is only α-Hölder continuous for $\alpha \leq 1/2$. It was also discovered that the hitting time at $x = 0$ of this process is finite almost surely and consequently the conditions and the assertion of Lemma 3.8 fail to hold. As a result, Theorem 3.10 does not apply even if the stability of this SDE can be established by other means (as will be seen in greater detail in Section 3.5.2). As such, the Lipschitz condition is indispensable for the general Lyapunov framework to apply. Despite this, some partial results are still obtainable (as evidenced by [22] for example).

3.5.1 Stability by Comparison

It could be the case that drift does not satisfy the Lipschitz condition (2.20), yet it is sufficiently well behaving in a neighborhood of the deterministic solution. Under such conditions, the construction of a related process which is stable is sufficient to conclude the stability of the original solution via the comparison theory stated in Theorem 2.22. The following example illustrates this idea.

Example 3.17. *Consider the SDE given by*

$$\mathrm{d}X(t) = X(t)\ln(X(t))\,\mathrm{d}t + X(t)\,\mathrm{d}B(t) \tag{3.17}$$

with $x_1 > 0$. By Theorem 2.19 there exists a unique solution. We cannot apply any of the Lyapunov results directly since the drift fails to be Lipschitz near 0. Instead, we seek to apply Lyapunov results to a related process. Consider

$$\mathrm{d}Z(t) = (Z^2(t) - Z(t))\,\mathrm{d}t + Z(t)\,\mathrm{d}B(t).$$

By Theorem 2.22, we have that $X(t) \leq Z(t)$ almost surely. Take the Lyapunov function $V(t,z) = |z|$ to obtain $LV(t,z) \leq 0$ in a neighborhood of the origin. Then by Theorem 3.10, $Z(t)$ is stable with respect to the constant 0. Since $X(t)$ is non-negative, $X(t)$ is also stable with respect to 0.

3.5.2 Square Bessel Processes

In this section, we consider the square Bessel processes. The unique solution to the SDE

$$\mathrm{d}X(t) = n\,\mathrm{d}t + 2\sqrt{X(t)}\,\mathrm{d}B(t) \tag{3.18}$$

with $X(0) = x_1$ is referred to as the square of the n-dimensional Bessel process for every $n \geq 0$. In this case the diffusion coefficient $\sigma(t,x) = 2\sqrt{x}$ does not satisfy the Lipschitz condition (2.20) at $x = 0$. As a result, Theorem 3.10 does not apply. Regardless, in some cases the definitions of stability still apply to this stochastic process. For $n \geq 3$, the solution $X(t)$ of (3.18) has the property that $X(t) \to \infty$ as $t \to \infty$ [25], so any possible hope for stability is dashed.

One case which is easy to solve is if $n = 1$, the square of the 1-dimensional Bessel process. In fact, given $x_1 > 0$, by Itô's formula (2.17) we have the explicit formula for $X^{x_1}(t)$ given by

$$X^{x_1}(t) = \left(\sqrt{x_1} + B(t)\right)^2. \tag{3.19}$$

Clearly, neither the assumption (3.7) nor (3.8) are satisfied by the deterministic solution. As such, the current notion of stability is not applicable to this SDE.

A less trivial and more interesting case is when $n = 0$, the square of the 0-dimensional Bessel process, which was already examined in Example 2.17. This process is linked to the Kimura equation and has been studied extensively in [6, 7, 8, 12] to name a few sources. Here we can consider stability with respect to the deterministic solution with starting point $x_0 = 0$. Then the deterministic path is simply the constant $c = 0$ and assumption (3.8) holds. Chen and Stroock [6] showed that in this case the solution $X^{x_1}(t)$, up to the hitting time of 0, has the transition density given by

$$q(x,y,t) = \frac{1}{y}\exp\left(-\frac{x+y}{2t}\right)\sum_{n=1}^{\infty}\frac{\left(\frac{xy}{4t^2}\right)^n}{n!(n-1)!}.$$

Let τ_0 be the first time that the solution $X^{x_1}(t)$ of (3.18) hits $\{0\}$. If we impose that the solution gets absorbed at 0 upon hitting, it follows that $X^{x_1}(t_1) = 0 \Rightarrow X^{x_1}(t_2) = 0$ whenever $t_1 \leq t_2$. Then

$$\mathbb{P}(\tau_0 = \infty) = \lim_{t\to\infty}\mathbb{P}\left(X^{x_1}(t) > 0\right) = \lim_{t\to\infty}\int_0^\infty q(x,y,t)\,\mathrm{d}y.$$

It is easy to verify that the last limit is 0, thereby proving the previous statement that $\tau_0 < \infty$ almost surely. Alternatively, another way to see that $\tau_0 < \infty$ is through Theorem 2.22 by comparing this solution to that when $b(x) = 1$, where the latter solution (3.19) reaches $x = 0$ in finite time by the recurrence of $B(t)$. As such, τ_0 must be finite almost surely.

These observations are in contrast to the assertion of Lemma 3.8. For SDEs that satisfy (3.8) and the Lipschitz condition, the diffusion near the deterministic solution is sufficiently damped so that the process cannot reach the path. However, if this assumption is dropped, the trajectory of the deterministic solution may be accessible to the solution to the SDE.

In fact, a stronger sense of stability holds. Since when $n = 0$ we are considering the model equation with no drift, $X^{x_1}(t)$ is a martingale. In particular

$$\mathbb{E}\left[X^{x_1}(t)\right] = x_1 \tag{3.20}$$

for all times $t \geq 0$. As $x_1 \to 0$ this quantity vanishes, hence the process is strongly 1-stable with respect to 0. By using the density, it is shown in [6] that

$$\mathbb{E}\left[\left|X^{x_1}(t)\right|^p\right] = \int_0^\infty y^p q(x_1, y, t)\,\mathrm{d}y = \sum_{j=1}^{p} C_{p,j} t^{p-j} x_1^j$$

for appropriate constants $C_{p,j}$ with $p \in \mathbb{N}$ and $1 \leq j \leq p$. For every $t \geq 0$, the quantity on the right hand side is a polynomial in x_1 with no constant term. Sending $x_1 \to 0$ establishes that the model equation with no drift is actually strongly p-stable for every $p \geq 1$. On the other hand, since (3.20) has no dependence on t, sending $t \to \infty$ does not yield 0 and so the process is not asymptotically 1-stable, and hence not asymptotically p-stable for every $p > 1$ also. Evidently it is not exponentially p-stable either.

3.6 Relative Stability of Solutions to Different SDEs

Hitherto, we have considered the stability of the solution $X(t)$ with respect to the deterministic solution $\tilde{X}(t)$. It is natural to then ask if we may consider the stability of a solution to an SDE with respect to the solution to another SDE sharing the same drift. In this section we show that this is essentially the same as what has been already discussed throughout this chapter.

Let $X_1^{s,x_1}(t)$ and $X_2^{s,x_2}(t)$ be the solutions to the SDE (3.1) but with the diffusion term $\sigma(t,x)$ replaced by $\sigma_1(t,x)$ and $\sigma_2(t,x)$, respectively. Assume $|x_1 - x_2| < \delta$. We consider what happens to the trajectories of these solutions as the starting points get close. That is, we wish to study what happens to the quantity

$$\sup_{t\in[s,T]} \left|X_1^{s,x_1}(t) - X_2^{s,x_2}(t)\right|$$

as $\delta \to 0$. To examine this, consider the following chain of inequalities. For appropriate starting point x_0, let $\tilde{X}^{s,x_0}(t)$ be the deterministic solution shared by $X_1^{s,x_1}(t)$ and $X_2^{s,x_2}(t)$. For any $\epsilon > 0$ we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[s,T]} \left|X_1^{s,x_1}(t) - X_2^{s,x_2}(t)\right| > 2\epsilon\right) &\leq \mathbb{P}\left(\sup_{t\in[s,T]} \left|X_1^{s,x_1}(t) - \tilde{X}^{s,x_0}(t)\right| + \left|\tilde{X}^{s,x_0}(t) - X_2^{s,x_2}(t)\right| > 2\epsilon\right) \\
&\leq \mathbb{P}\left(\sup_{t\in[s,T]} \left|X_1^{s,x_1}(t) - \tilde{X}^{s,x_0}(t)\right| > \epsilon\right) \\
&\quad + \mathbb{P}\left(\sup_{t\in[s,T]} \left|\tilde{X}^{s,x_0}(t) - X_2^{s,x_2}(t)\right| > \epsilon\right).
\end{aligned} \tag{3.21}$$

Observe that a result similar to (3.21) can be obtained under L^p norm. It implies that showing the processes $X^{s,x_1}(t)$ and $X^{s,x_2}(t)$ being individually stable with respect to $\tilde{X}^{s,x_0}(t)$ is sufficient to showing that they are stable with respect to each other. In the other direction, one can similarly show that if $X_1^{s,x_1}(t)$ is stable with respect to both $X_2^{s,x_2}(t)$ and $\tilde{X}^{s,x_0}(t)$, then $X_2^{s,x_2}(t)$ is stable with respect to $\tilde{X}^{s,x_0}(t)$.

4 Stability in Ratio

Chapter 3 considered rather strict forms of stability. In all the senses of stability previously considered, it was essential that the solution to the SDE did not deviate from the deterministic solution by more than an arbitrarily small amount either in probability or pathwise. In order to obtain results for this, it was necessary to impose the rather strict assumptions in Section 3.3.

In this chapter, an attempt to generalize the notion of stability is made. This thesis considers the *ratio process* which is defined to be the quotient of the solution to the stochastic equation and the solution to the deterministic equation. Using the ratio process, a new definition of stability for the SDE (3.1) is considered. Known as *stability in ratio*, this new definition of stability allows us to relax the previously imposed assumptions in Section 3.3 and hence it can be used to study a broader class of SDEs. We will discuss the relation between stability in ratio and the previous definitions studied in Section 3.2. Although these new definitions and the methods for studying them are weaker than those considered in Chapter 3, a close look at them leads to observations of interesting behaviors that were not visible under the framework previously adopted in Chapter 3. This analysis is restricted to one-dimensional processes.

4.1 The Ratio Process and Stability in Ratio

The method to study stability in ratio that we propose here is to consider the ratio of the stochastic solution $X^{s,x_1}(t)$ to (3.1) to its deterministic solution $\tilde{X}^{s,x_0}(t)$. That is, for the one-dimensional process $X^{s,x_1}(t)$ with deterministic solution $\tilde{X}^{s,x_0}(t) > 0$ for all $t \geq 0$, we define the **ratio process** $Y^{s,x_0,x_1}(t)$ as

$$Y^{s,x_0,x_1}(t) = \frac{X^{s,x_1}(t)}{\tilde{X}^{s,x_0}(t)} \tag{4.1}$$

for every $0 \leq s \leq t$ and $x_0, x_1 \in \mathbb{R}$. For the majority of this chapter, we restrict ourselves to one-dimensional SDEs which have unique solutions in the strong sense. We will assume $s = 0$ and denote the ratio process by $Y^{x_0,x_1}(t)$ or simply $Y(t)$ when no ambiguity exists. This quantity attempts to capture how the stochastic process $X^{x_1}(t)$ grows in relation to the deterministic process $\tilde{X}^{x_0}(t)$. If the process $X^{x_1}(t)$ is strongly asymptotically stable with respect to $\tilde{X}^{x_0}(t)$ as defined in Section 3.2 and at the same time $\tilde{X}^{x_0}(t)$ is bounded away from 0 in the sense that $m = \inf_{t\geq 0}\left|\tilde{X}^{s,x_0}(t)\right| > 0$, then for every $\epsilon > 0$

$$\begin{aligned}\lim_{x_1\to x_0} \mathbb{P}\left(\sup_{t\geq 0}\left|Y^{x_0,x_1}(t) - 1\right| > \epsilon\right) &= \lim_{x_1\to x_0} \mathbb{P}\left(\sup_{t\geq 0}\left|\frac{X^{x_1}(t)}{\tilde{X}^{x_0}(t)} - 1\right| > \epsilon\right) \\ &\leq \lim_{x_1\to x_0} \mathbb{P}\left(\sup_{t\geq 0}\left|X^{s,x_1}(t) - \tilde{X}^{s,x_0}(t)\right| > m\epsilon\right) = 0,\end{aligned} \tag{4.2}$$

and

$$\lim_{x_1\to x_0} \mathbb{P}\left(\lim_{t\to\infty} Y^{x_0,x_1}(t) = 1\right) = \lim_{x_1\to x_0} \mathbb{P}\left(\lim_{t\to\infty}\left|X^{x_1}(t) - \tilde{X}^{x_0}(t)\right| = 0\right) = 1. \tag{4.3}$$

We can similarly examine the behavior of $Y^{x_0,x_1}(t)$ under other stability conditions introduced in Chapter 3, such as weakly (asymptotically) stable or weakly p-stable. All the outcomes will point towards the fact that if $X^{x_1}(t)$ is stable with respect to $\tilde{X}^{x_0}(t)$ in any reasonable sense, $Y^{x_0,x_1}(t)$ will stay near 1 for all time with high (or even 1) probability. Based on these observations, we are inspired to adopt the converse perspective and use the proximity of $Y^{x_0,x_1}(t)$ to 1 as a new criterion for stability. Below we state our first stability definition in this spirit.

Definition 4.1 (Strictly Stable in Ratio)**.** *The solution $X^{x_1}(t)$ to (3.1) is said to be strictly stable in ratio with respect to $\tilde{X}^{x_0}(t)$ if, for every x_0, x_1, the ratio process $Y^{x_0,x_1}(t)$ satisfies that*

$$\lim_{x_1 \to x_0} \lim_{t \to \infty} Y^{x_0,x_1}(t) = 1 \tag{4.4}$$

almost surely.

Intuitively, this says that the stochastic solution and the deterministic solution grow at the same rate provided they start from sufficiently close positions. There are multiple motivations for this definition:

1. This definition only considers the long term behavior of $X^{x_1}(t)$ and $\tilde{X}^{x_0}(t)$. Under this definition, it is possible for the process to be stable in ratio even though there might be considerable variations between the stochastic and the deterministic processes in any finite time interval. Stability in ratio only requires that the asymptotic growth rates of the two processes are close. In a sense, this definition of stability does not penalize the processes for short periods of erratic behaviors so long as the eventual behavior of the processes are consistent.

2. When considering this ratio process, we may drop the assumption (3.7). For two quantities that both grow unboundedly, their difference may lose significance or even become indeterminate. For a given SDE, due to the compounded effects of random movements over long periods of time, the absolute difference between $X^{x_1}(t)$ and $\tilde{X}^{x_0}(t)$ may be large, but their ratio may remain controlled. Hence, the ratio process $Y^{x_0,x_1}(t)$ awards partial credit when the processes $X^{x_1}(t)$ and $\tilde{X}^{x_0}(t)$ move apart but still grow at similar rates.

3. The previous assumption (3.8) can also be dropped. This condition required that the stochastic solution experienced no diffusion when it matched exactly the deterministic solution. This would only be satisfied by very specific SDEs. Without this assumption, we allow $X^{x_1}(t)$ to evolve under a general diffusive force that is not necessarily tied to the deterministic vector field that drives $\tilde{X}^{x_0}(t)$. This would be a more realistic model of physical phenomena. This is in contrast to situations such as Example 3.15 where the diffusion coefficient is made to purposely vanish around the deterministic solution.

It should be noted that dropping these assumptions, the notions of stability in this chapter relax the idea that a stochastic solution must be arbitrarily close to its deterministic solution. In this sense, the concept of stability in ratio is much more forgiving. While this is weaker than the stability definitions considered in Section 3.2, it allows for a larger class of SDEs to be studied. It may also reveal new insights into solutions to SDEs which are stable under the previous definitions.

As we will see shortly, the condition (4.4) is rather strict, and it fails even in very simple examples that were stable under the notions in Section 3.2. The following two definitions handle these cases.

Definition 4.2 (Stable in Ratio). *The solution $X^{x_1}(t)$ to (3.1) is said to be stable in ratio with respect to $\tilde{X}^{x_0}(t)$ if, for every x_0, x_1, the ratio $Y^{x_0,x_1}(t)$ satisfies that*

$$\lim_{x_1 \to x_0} \liminf_{t \to \infty} Y^{x_0,x_1}(t) > 0 \tag{4.5}$$

and

$$\lim_{x_1 \to x_0} \limsup_{t \to \infty} Y^{x_0,x_1}(t) < \infty \tag{4.6}$$

almost surely.

This definition allows more flexibility in the relative growth rate between the stochastic solution to (3.1) and the corresponding deterministic solution. It is no longer required that $X^{x_1}(t)$ and $\tilde{X}^{x_0}(t)$ grow at exactly the same rate, but their growth rates should still be comparable in terms of the order of magnitude. If Definition 4.1 or 4.2 holds even without taking the limit $x_1 \to x_0$, then the solution is said to be **strictly stable in ratio in the large** and **stable in ratio in the large**, respectively.

Definition 4.3 (Unstable in Ratio). *The solution $X^{x_1}(t)$ to (3.1) is said to be unstable in ratio with respect to $\tilde{X}^{x_0}(t)$ if, for every $x_0 \neq x_1$, the ratio process Y^{x_0,x_1} satisfies that*

$$\lim_{x_1 \to x_0} \liminf_{t \to \infty} Y^{x_0,x_1}(t) = 0$$

or

$$\lim_{x_1 \to x_0} \limsup_{t \to \infty} Y^{x_0,x_1}(t) = \infty$$

with positive probability.

4.2 Illustrations with Examples

Before attempting to establish general properties on $Y^{x_0,x_1}(t)$, we will study the following illustrative examples to gain some intuitions.

4.2.1 Linear Coefficients

This section revisits linear SDEs which we recall are SDEs whose drift and diffusion coefficients satisfy (3.10) and (3.11). It will be seen that, even in this class of SDEs, the results on stability in ratio vary considerably.

Example 4.4. *Consider the one-dimensional linear SDE with $d = f = 0$ and $e = 1$, that is*

$$\mathrm{d}X(t) = c(t)\,\mathrm{d}t + \mathrm{d}B(t).$$

This SDE is in the class of semi-homogeneous of second kind. It is easy to see that for every $x_0, x_1 \in \mathbb{R}$

$$X^{x_1}(t) = x_1 + \int_0^t c(s)\,\mathrm{d}s + B(t) := x_1 + C(t) + B(t)$$

and

$$\tilde{X}^{x_0}(t) = x_0 + C(t),$$

hence

$$Y^{x_0,x_1}(t) = \frac{x_1 + C(t) + B(t)}{x_0 + C(t)}. \tag{4.7}$$

First suppose that $c(t) = 1$ and hence $C(t) = t$. By Theorem 2.7 we know that $B(t)$ has sub-linear growth. Therefore, we have that (4.7) converges to 1 as $t \to \infty$ almost surely, which means in this case $X^{x_1}(t)$ is strictly stable in ratio with respect to $\tilde{X}^{x_0}(t)$. Furthermore, this applies to any choice of x_0 and x_1, so it is strictly stable in the large. By the law of the iterated logarithm for $B(t)$, we know that the same conclusion holds whenever $C(t) \nearrow \infty$ faster than $\sqrt{2t \ln \ln t}$.

This is no longer the case if $c(t)$ vanishes quickly. For example, if $C(t) = 2\sqrt{2t \ln \ln t}$ (for sufficiently large t), then again by the law of the iterated logarithm, $Y^{x_0,x_1}(t)$ almost surely remains within $[1/2, 3/2]$ eventually and hence $X^{x_1}(t)$ is stable in ratio with respect to $\tilde{X}^{x_0}(t)$. If $c(t)$ is integrable, then the main contributor to the ratio in (4.7) is $B(t)$, and $X^{x_1}(t)$ is unstable in ratio with respect to $\tilde{X}^{x_0}(t)$. This falls into the case where $\tilde{X}^{x_0}(t)$ remains bounded, which is discussed in Section 4.2.3.

The last case of the previous example showed that the process $X^{x_1}(t)$ is unstable in ratio with respect to $\tilde{X}^{x_0}(t)$ since the behavior $Y^{x_0,x_1}(t)$ was mostly driven by $B(t)$ as the drift term fizzled out. The following example shows that, even in the presence of drift, the stability in ratio can be violated.

Example 4.5. *Let $x_0, x_1 > 0$. Let $X^{x_1}(t)$ be the solution to the linear homogeneous SDE (3.13), again with the boundedness assumptions on the coefficients. The deterministic system is given by*

$$\mathrm{d}\tilde{X}(t) = b(t)\tilde{X}(t)\,\mathrm{d}t$$

and is therefore solved by

$$\tilde{X}^{x_0}(t) = x_0 \exp\left(\int_0^t b(s)\,\mathrm{d}s\right).$$

Then by (3.14) we have

$$Y^{x_0,x_1}(t) = \frac{x_1}{x_0} \exp\left(-\int_0^t \frac{\sigma^2(s)}{2}\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B(s)\right). \tag{4.8}$$

First assume that $\sigma \in L^2$. Then by Example 2.12, $Y^{x_0,x_1}(t)$ converges to a random variable as $t \to \infty$. In particular, this limit random variable, upon rescaling, has the log-normal distribution with parameters

$$\mu = -\int_0^\infty \frac{\sigma^2(s)}{2}\,\mathrm{d}s, \qquad s = \int_0^\infty \sigma^2(s)\,\mathrm{d}s.$$

This is true for all choices of x_0 and x_1, so $X^{x_1}(t)$ is stable in ratio with respect to $\tilde{X}^{x_0}(t)$ in the large.

Now suppose that $\sigma \notin L^2$. By the recurrence of Brownian motion in one-dimension, this process visits all of I infinitely often almost surely. However, again by Example 2.12, the process goes to 0 almost surely as $t \to \infty$. In this situation the process is unstable in ratio.

Example 4.5, in contrast to Example 3.14, highlights the new phenomenon that has occurred with stability in ratio. Previously in Example 3.14, stability (in the traditional sense) depended on the drift and the diffusion coefficients. If the drift was sufficiently large relative to the diffusion, the solution $X^{s,x_1}(t)$ was unstable. If the drift was relatively small (or negative), then stability was achieved. In the case of stability in ratio, the drift does not appear in (4.8) at all. Related to this observation, we remark that when the drift coefficient is negative and not integrable, $\tilde{X}^{x_0}(t) \to 0$ as $t \to \infty$, and hence the estimate (4.2) does not apply. Indeed, it is possible for $X^{x_1}(t)$ to be strongly asymptotically stable but not stable in ratio with respect to $\tilde{X}^{x_0}(t)$. This means that, although the newly introduced notion of stability in ratio is generally more relaxed than the classical stability notions previously introduced in Section 3.2, it is not a weaker condition in the rigorous sense and there is no implication relation from the latter to the former.

The next example shows that the semi-homogeneous of first kind case behaves similarly to the homogeneous case under some assumptions on the coefficients.

Example 4.6. *Consider the linear SDE of semi-homogeneous of first kind given by*

$$\mathrm{d}X(t) = (c(t) + d(t)X(t))\,\mathrm{d}t + f(t)\,\mathrm{d}B(t).$$

We assume that $c(t)$, $d(t)$, and $f(t)$ are all bounded. Given $x_0, x_1 > 0$, by (3.12) the ratio process is given by

$$Y^{x_0,x_1}(t) = \exp\left(-\int_0^t \frac{f^2(s)}{2}\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}B(s)\right)$$
$$\times \frac{x_1 + \int_0^t \exp\left(-\int_0^s d(r)\,\mathrm{d}r + \int_0^s \frac{f^2(r)}{2}\,\mathrm{d}r - \int_0^s f(r)\,\mathrm{d}B(r)\right) c(s)\,\mathrm{d}s}{x_0 + \int_0^t \exp\left(-\int_0^s d(r)\,\mathrm{d}r\right) c(s)\,\mathrm{d}s}.$$

We assume that $c(t)$ is non-negative so that $\tilde{X}(t) \neq 0$ for $t \geq 0$. Let

$$u(t) = \int_0^t f^2(s)\,\mathrm{d}s$$

for every $t \geq 0$*. Then* $Y^{x_0,x_1}(t)$ *has the same distribution as the process*

$$N(t) = \exp\left(-\frac{u(t)}{2} + B(u(t))\right) \times \frac{x_1 + \int_0^t \exp\left(-\int_0^s d(r)\,\mathrm{d}r + \frac{u(s)}{2} - B(u(s))\right) c(s)\,\mathrm{d}s}{x_0 + \int_0^t \exp\left(-\int_0^s d(r)\,\mathrm{d}r\right) c(s)\,\mathrm{d}s}$$

$$= \frac{x_1 \exp\left(-\frac{u(t)}{2} + B(u(t))\right)}{x_0 + \int_0^t \exp\left(-\int_0^s d(r)\,\mathrm{d}r\right) c(s)\,\mathrm{d}s}$$

$$+ \frac{\int_0^t \exp\left(-\int_0^s d(r)\,\mathrm{d}r - \frac{u(t)-u(s)}{2} + B(u(t)) - B(u(s))\right) c(s)\,\mathrm{d}s}{x_0 + \int_0^t \exp\left(-\int_0^s d(r)\,\mathrm{d}r\right) c(s)\,\mathrm{d}s}.$$

When $f \in L^2$*, as we have seen in Example 4.5,* $\exp\left(-\frac{u(t)}{2} + B(u(t))\right)$ *converges to a non-trivial random variable almost surely. It is easy to see that* $N(t)$ *remains bounded and at the same time bounded away from* 0 *for all time almost surely. Thus, we conclude that in this case* $X^{x_1}(t)$ *is stable in ratio with respect to* $\tilde{X}^{x_0}(t)$*.*

Now assume $f \notin L^2$*. For simplicity, we will assume that* $f = 1$ *in which case* $u(t) = t$*. The case with general* f *can be treated similarly. Under this assumption* $\exp\left(-\frac{t}{2} + B(t)\right)$ *converges to* 0 *as* $t \to \infty$ *almost surely. So, to study the limit of* $N(t)$*, it is sufficient to consider*

$$\frac{\int_0^t \exp\left(-\int_0^s d(r)\,\mathrm{d}r - \frac{t-s}{2} + B(t) - B(s)\right) c(s)\,\mathrm{d}s}{x_0 + \int_0^t \exp\left(-\int_0^s d(r)\,\mathrm{d}r\right) c(s)\,\mathrm{d}s}$$

which further has the same distribution as

$$M(t) = \frac{\int_0^t \exp\left(-\int_0^s d(r)\,\mathrm{d}r - \frac{t-s}{2} + B(t-s)\right) c(s)\,\mathrm{d}s}{x_0 + \int_0^t \exp\left(-\int_0^s d(r)\,\mathrm{d}r\right) c(s)\,\mathrm{d}s}$$

$$= \frac{\int_0^t \exp\left(-\int_0^{t-q} d(r)\,\mathrm{d}r - \frac{q}{2} + B(q)\right) c(t-q)\,\mathrm{d}q}{x_0 + \int_0^t \exp\left(-\int_0^{t-q} d(r)\,\mathrm{d}r\right) c(t-q)\,\mathrm{d}q}.$$

By the law of the iterated logarithm, we know that almost surely $\exp\left(-\frac{q}{2} + B(q)\right)$ *is bounded by, say,* $\exp\left(-\frac{q}{4}\right)$*, for all* q *sufficiently large. Therefore, if the denominator above is unbounded, that is*

$$\lim_{t\to\infty} \int_0^t \exp\left(-\int_0^s d(r)\,\mathrm{d}r\right) c(s)\,\mathrm{d}s = \infty,$$

then it is easy to see that almost surely $M(t) \to 0$ *as* $t \to \infty$*, in which case* $X^{x_1}(t)$ *is unstable in ratio with respect to* $\tilde{X}^{x_0}(t)$*. However, if the denominator stays bounded as* $t \to \infty$*, then for all* $t \geq 1$*,*

$$M(t) \geq \frac{\int_0^1 \exp\left(-\int_0^{t-q} d(r)\,\mathrm{d}r - \frac{q}{2} + B(q)\right) c(t-q)\,\mathrm{d}q}{x_0 + \int_0^\infty \exp\left(-\int_0^{t-q} d(r)\,\mathrm{d}r\right) c(t-q)\,\mathrm{d}q}$$

$$\geq \frac{\exp\left(-\frac{1}{2} + \min_{s\in[0,1]} B(s)\right) \int_0^1 \exp\left(-\int_0^{t-q} d(r)\,\mathrm{d}r\right) c(t-q)\,\mathrm{d}q}{x_0 + \int_0^\infty \exp\left(-\int_0^{t-q} d(r)\,\mathrm{d}r\right) c(t-q)\,\mathrm{d}q},$$

where the second line is a random variable that is almost surely positive. This means that almost surely $M(t)$ *is bounded away from* 0 *for all time, and hence in this case* $X^{x_1}(t)$ *is stable in ratio with respect to* $\tilde{X}^{x_0}(t)$.

The last example in this section that we present shows that different choices of diffusion lead to different asymptotic behaviors of the ratio process even if the drift terms are the same.

Example 4.7. *Consider two one-dimensional linear SDEs given by*

$$\begin{aligned} \mathrm{d}X_1(t) &= \frac{X_1(t)}{t+1}\,\mathrm{d}t + X_1(t)\,\mathrm{d}B(t) \\ \mathrm{d}X_2(t) &= \frac{X_2(t)}{t+1}\,\mathrm{d}t + (t+1)\,\mathrm{d}B(t) \end{aligned}$$

with initial condition $x_0 = x_1 = x_2 = 1$. *Applying (3.12) to each of these yields*

$$\begin{aligned} X_1^{x_1}(t) &= (t+1)\exp\left(-\frac{t}{2} + B(t)\right) \\ X_2^{x_2}(t) &= (t+1)(B(t)+1) \end{aligned}$$

with the deterministic solution

$$\tilde{X}^{x_0}(t) = t+1.$$

Therefore, the respective ratio processes are

$$\begin{aligned} Y_1^{x_0,x_1}(t) &= \exp\left(-\frac{t}{2} + B(t)\right) \\ Y_2^{x_0,x_2}(t) &= B(t)+1 \end{aligned}$$

which behave differently. The former SDE is a homogeneous linear SDE which has been studied in Example 4.5. The latter case is analogous to Example 4.4. In either case, however, we have instability in ratio.

4.2.2 Non-linear Coefficients

In this section we consider an SDE with nonlinear coefficients. Unsurprisingly, the behavior of the ratio process in this case is more erratic.

Example 4.8. *Consider the SDE as in (3.18) with* $n = 1$. *Given* $x_0, x_1 > 0$, *the ratio process is given by*

$$Y^{x_0,x_1}(t) = \frac{(\sqrt{x_1} + B(t))^2}{x_0 + t}. \tag{4.9}$$

Applying the law of the iterated logarithm, since $\left|B(t)\right|^2$ *grows faster than any line, the solution is therefore unstable in ratio. Hence, by having a diffusion term with stronger degeneracy near* 0 *compared to the linear case in Example 4.5, the ratio process behaves more erratically. As already seen in Section 3.5.2, the hitting time of* $x = 0$ *is finite almost surely, although the drift in this case 'revives' the process afterwards.*

More generally, consider the SDE

$$\mathrm{d}X(t) = \frac{n(n-1)}{2}(X(t))^{\frac{n-2}{n}}\,\mathrm{d}t + n(X(t))^{\frac{n-1}{n}}\,\mathrm{d}B(t). \tag{4.10}$$

For $x_0, x_1 > 0$, we have that

$$X^{x_1}(t) = \left(x_1^{1/n} + B(t)\right)^n \tag{4.11}$$

and

$$\tilde{X}^{x_0}(t) = \left(x_0^{2/n} + (n-1)t\right)^{n/2}.$$

When $n \geq 2$, the diffusive force is proportional to $(X(t))^\alpha$ for some $\alpha \in [1/2, 1)$. It should be expected that the ratio process in this case has the same recurrence property as (4.9). However, if $n > 2$, then the solution is not guaranteed to be unique. For example, when $n = 4$ then the SDE (4.10) with initial condition $x_1 = 1$ is solved by $X^{x_1}(t) = (1 + B(t))^4$ as per (4.11), but also by $X_a^{x_1}(t) = (1 + B(t))^4\mathbf{1}_{[t\leq\tau_0]}$ where τ_0 is the first hitting time of the set $\{-1\}$ of $B(t)$. This corresponds to the process that 'dies' upon reaching 0. As a consequence, we obtain two ratio processes with contrasting long-term behaviors: one becoming unbounded as time progresses and the other being constantly zero after a finite amount of time, justifying the necessity of requiring the solution to the SDE to be unique.

4.2.3 Bounded Deterministic Solution

Nearly all of the previous examples considered $\tilde{X}^{x_0}(t)$ which was unbounded. This appears to be the correct class of SDEs to study since, in the case where the deterministic solution is bounded, the problem of studying the ratio process is analogous to studying the behavior of $X^{x_1}(t)$. The one exception to this is when both the stochastic and deterministic solutions vanish (see the discussion involving Example 4.5). The following two examples illustrate the idea that bounded solutions can be made to behave in any which way we wish.

Example 4.9. *Let $x_0, x_1 \in (1, 2)$ and consider the SDE (3.1) with*

$$b(t, x) = \min\left\{(x-1)(x-2), K\right\}$$

for some $K > 0$, and

$$\sigma(t, x) = \mathbf{1}_{[x<2]} + (3-x)\mathbf{1}_{[2\leq x\leq 3]}.$$

These functions satisfy Theorem 2.14, so a unique solution exists. On the open interval $(1, 2)$, $b(x)$ is negative, and hence $\tilde{X}^{x_0}(t) \to 1$ as $t \to \infty$.

Meanwhile, $X^{x_1}(t)$ hits $x = 3$ in finite time with probability one. From here, the SDE has positive drift (of at least the constant $\min\{1, K\} > 0$) and is no longer under the influence of random forces, so $X^{x_1}(t) \to \infty$ as $t \to \infty$. Hence the $X^{x_1}(t)$ is unstable in ratio with respect to $\tilde{X}^{x_0}(t)$.

The diffusion term in Example 4.9 vanishes on $x \geq 3$ and, after a random finite time, $X^{x_1}(t)$ becomes deterministic. The following example shows the same behavior where the diffusion vanishes on a set of Lebesgue measure 0.

Example 4.10. *Consider the SDE*

$$\mathrm{d}X(t) = \sin^2(\pi X(t))\,\mathrm{d}t + \big|\sin(\pi X(t))\big|\,\mathrm{d}B(t).$$

For any $k \in \mathbb{Z}$, the diffusion coefficient $\big|\sin(\pi x)\big|$ vanishes at $x = k$ (that is to say k is a degenerate point for the diffusion). Furthermore, when x approaches k, $\big|\sin(\pi x)\big|$ is well approximated by either $\big|\pi(x-k)\big|$. In other words, near every degenerate point k, the diffusion coefficient degenerates at the linear order. On the other hand, for every $k \in \mathbb{Z}$, although the drift coefficient $\sin^2(\pi x)$ is positive within $(k-1,k)$, as x approaches $k-1$ or k, $\sin^2(\pi x)$ decays to 0 at the quadratic order in terms of the distance between x and either $k-1$ or k. This is a case where every integer becomes a 'natural' boundary for the solution (see [8]). Therefore, for every $x_1 \in \mathbb{R}$, if $x \in [k-1,k)$ for some $k \in \mathbb{Z}$, then almost surely $X^{x_1}(t) \in [k-1,k)$ for all $t \geq 0$.

As for the deterministic solution, it is easy to verify that for every $x_0 \in \mathbb{R}$, if $x_0 \in [k-1,k)$ for some $k \in \mathbb{Z}$, then

$$\tilde{X}^{x_0}(t) = -\frac{1}{\pi}\operatorname{arccot}\big(\pi t + \cot(\pi x_0)\big) + k,$$

and hence $\tilde{X}^{x_0}(t) \to k$ as $t \to \infty$. Therefore, we have for any x_0 and x_1 outside of $(-1,1)$, $X^{x_1}(t)$ is stable in ratio with respect to $\tilde{X}^{x_0}(t)$.

4.3 Methods for Determining Stability in Ratio

This section will consider some techniques that can be used to study the ratio processes of various SDEs. Due to the fact that these definitions of stability are inherently more relaxed than those considered in Section 3.2 and admit a wider class of SDEs, it should not be surprising that the results obtained are less powerful. Nevertheless, relatively general techniques can be applied in studying the stability (or instability) in ratio.

4.3.1 Lyapunov Method for Ratio Processes

Consider again the linear homogeneous SDE in Example 4.5. In this case, we had that $Y^{x_0,x_1}(t) \to 0$ almost surely as $t \to 0$, regardless of the choice of x_0 and x_1. So $X^{x_1}(t)$ is unstable in ratio. However, if we consider the process $Y^{x_0,x_1}(t)$, then it is asymptotically stable with respect to the constant 0 in the large as defined in Section 3.2.1. It is unlikely that the linear homogeneous case is the sole example in which the ratio process vanishes as $t \to \infty$.

It is natural to investigate the class of SDEs whose ratio processes also behave in this manner. To this end, we can apply Lyapunov type stability results directly to the ratio process to obtain sufficient conditions for $X^{x_1}(t)$ to be unstable in ratio. In order to use this approach, the point of stability of the ratio process must be known a priori (in this case $y = 0$). This point serves as the deterministic process $\tilde{Y}(t)$ for which the stability of $Y^{x_0,x_1}(t)$ can be established.

Let $X^{x_1}(t)$ be the unique solution to (3.1) with time-homogeneous coefficients $b(t, x) = b(x)$ and $\sigma(t, x) = \sigma(x)$. Assume the stochastic solution and the deterministic solution are positive for all $t \geq 0$. Applying Itô's formula (2.17) to (4.1), one readily obtains

$$\mathrm{d}Y^{x_0,x_1}(t) = \frac{b(\tilde{X}^{x_0}(t))}{\tilde{X}^{x_0}(t)} \left(\frac{b(X^{x_1}(t))}{b(\tilde{X}^{x_0}(t))} - Y^{x_0,x_1}(t) \right) \mathrm{d}t + \frac{\sigma(X^{x_1}(t))}{\tilde{X}^{x_0}(t)} \, \mathrm{d}B(t). \tag{4.12}$$

This SDE obtained is not necessarily time-homogeneous, nor is there any guarantee that it satisfies the Lipschitz conditions (2.19) and (2.20), so care must be taken when working with this SDE. In the case that the above mentioned properties hold, we may apply the Lyapunov stability results obtained in Section 3.4.

There are many cases in which (4.12) turns out to behave nicely. We explore some examples here which give sufficient conditions for the ratio process to be asymptotically stable in the large with respect to 0. This in turn gives the instability in ratio of $X^{x_1}(t)$ with respect to $\tilde{X}^{x_0}(t) \neq 0$.

Example 4.11. *Consider the SDE*

$$\mathrm{d}X(t) = \mu X(t) \, \mathrm{d}t + \nu X^{\beta}(t) \, \mathrm{d}B(t) \tag{4.13}$$

for $\mu > 0$ and $\nu \neq 0$. While a unique solution may exist for values as small as $\beta = 1/2$, we force $\beta \geq 1$ to enforce the Lipschitz condition (2.20). Set $x_0 = x_1 > 0$. By (4.12), the stochastic differential for the ratio process $Y^{x_0,x_1}(t) = Y(t)$ becomes

$$\mathrm{d}Y(t) = \nu Y^{\beta}(t) \tilde{X}^{\beta-1}(t) \, \mathrm{d}B(t).$$

Since the drift term does not appear, we conclude $Y(t)$ is a martingale. For the process $Y(t)$, assumptions (3.7) and (3.8) hold. Although the condition (2.19) does not hold, this is not required for Theorem 3.10. Take the Lyapunov function $V(t, y) = |y|$ and obtain that $LV = 0$. This implies that $Y(t)$ is stable with respect to $y = 0$. By Example 2.21, the process $X^{x_1}(t)$ is recurrent to every neighborhood of the origin whenever $\beta > 1$. Since $\tilde{X}^{x_0}(t)$ is strictly positive, the same recurrent property can be said about $Y(t)$. By Theorem 3.13, we conclude that $Y(t)$ is asymptotically stable in the large with respect to 0.

The asymptotic stability for the case $\beta = 1$ is already solved by Example 4.5. Note that this method does not require the explicit form of $Y(t)$. Previously, when $b(t) = \mu$ and $\sigma(t) = \nu$, it was shown that $Y(t) \to 0$ almost surely. When using the Lyapunov method, this only applies if $\lambda = \mu/\nu^2 \leq 1/2$ (as shown in Example 2.21). This is because we are making use of the recurrence conditions for $X^{x_1}(t)$ rather than $Y(t)$ directly.

We also remark that in this example the ratio process is a martingale. This is true whenever $X(t)$ is linear. This is useful in Section 4.3.2. If in (4.13), the drift was replaced with $X^\alpha(t)$ for $\alpha < 1$, the solution may not be unique. The next example comments briefly on the case when $\alpha > 1$ in a more general setting.

Example 4.12. *Consider the SDE*

$$\mathrm{d}X(t) = g(t)X^\alpha(t)\,\mathrm{d}t + h(t)X^\beta(t)\,\mathrm{d}B(t). \tag{4.14}$$

First consider $g(t) = \mu > 0$ and $h(t) = \nu \neq 0$. Assume also that $\alpha > 1$ and $x_0, x_1 > 0$. If β is chosen such that the SDE falls under the scope of Theorem 2.18 or 2.19 as in Example (2.21), then since $\tilde{X}^{x_0}(t)$ explodes in finite time, say at time t_0, we have that $\lim_{t\nearrow t_0} Y(t) = 0$.

The phenomenon occurs in the general case too. From (4.14), the stochastic differential of $Y(t)$ given by (4.12) is

$$\mathrm{d}Y(t) = g(t)\tilde{X}^{\alpha-1}(t)(-Y(t) + Y^\alpha(t))\,\mathrm{d}t + h(t)\tilde{X}^{\beta-1}(t)Y^\beta(t)\,\mathrm{d}B(t). \tag{4.15}$$

For all values $y \in (0,1)$, since $-y + y^\alpha < 0$, the drift is negative. Then, unless $g(t)$ 'corrects' the SDE in some sense, the explosion of $\tilde{X}(t)$ at t_0 translates directly to the drift term of the SDE (4.15) for $Y(t)$ being arbitrarily large close to t_0 (times right before t_0). This effect causes $\lim_{t\nearrow t_0} Y(t) = 0$ again.

Example 4.13. *Consider the SDE*

$$\mathrm{d}X(t) = X(t)\ln(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}B(t).$$

Interestingly, the stochastic differential of the ratio process given by (4.12) is of a very similar form to $\mathrm{d}X(t)$ since

$$\mathrm{d}Y(t) = Y(t)\ln(Y(t))\,\mathrm{d}t + \frac{\sigma(Y(t)\tilde{X}(t))}{\tilde{X}(t)}\,\mathrm{d}B(t)$$

for $x_0, x_1 > 0$. This is similar to Example 3.17. As in that example, if $\sigma(x)$ is chosen such that $X(t)$ has a unique solution, then we may apply Theorem 2.22 with the SDE

$$\mathrm{d}Z(t) = (Z^2(t) - Z(t))\,\mathrm{d}t + \frac{\sigma(Z(t)\tilde{X}(t))}{\tilde{X}(t)}\,\mathrm{d}B(t)$$

and obtain that $Y(t) \leq Z(t)$ almost surely. The choice of Lyapunov function $V(t,z) = |z|$ works in this case and yields $LV \leq 0$ in a neighborhood of the origin. If the recurrence of the process $Y(t)$ can be obtained, then again by Theorem 3.13 the ratio process $Y(t)$ is asymptotically stable in the large.

4.3.2 Martingale Convergence Methods

In Example 4.11 we have that the ratio process $Y^{x_0,x_1}(t)$ is a martingale and accordingly we choose the value of β large in order to apply the Lyapunov stability result. The following example considers $\beta = 1/2$. Although we can no longer apply Theorem 3.10, using martingale convergence methods allows us to gain knowledge on how the ratio process behaves.

Example 4.14. *Consider the SDE*

$$\mathrm{d}X(t) = \mu X(t)\,\mathrm{d}t + \nu\sqrt{X(t)}\,\mathrm{d}B(t)$$

with $x_1 > 0$, $x_0 \neq 0$, $\mu \neq 0$, and $\nu \neq 0$. By Theorem 2.16, there is a unique solution $X^{x_1}(t)$. We multiply this SDE by $e^{-\mu t}$ to obtain

$$d(e^{-\mu t}X(t)) = e^{-\mu t}\,\mathrm{d}X(t) - e^{-\mu t}\mu X(t)\,\mathrm{d}t = e^{-\mu t}\nu\sqrt{X(t)}\,\mathrm{d}B(t).$$

Hence

$$e^{-\mu t}X(t) = x_1 + \nu\int_0^t e^{-\mu t}\sqrt{X(s)}\,\mathrm{d}B(s). \tag{4.16}$$

Using property (2.12) we can calculate that $\mathbb{E}\left[X(t)\right] = x_1 e^{\mu t}$ or $\mathbb{E}\left[Y(t)\right] = x_1/x_0$. By squaring (4.16), taking expectation, using (2.12) and (2.13), and applying Fubini's theorem, we get that

$$\begin{aligned}\mathbb{E}\left[X^2(t)\right]e^{-2\mu t} &= x_1^2 + \mathbb{E}\left[\nu^2\int_0^t e^{-2\mu s}X(s)\,\mathrm{d}s\right]\\ &= x_1^2 + \nu^2\int_0^t e^{-2\mu s}\mathbb{E}\left[X(s)\right]\mathrm{d}s\\ &= x_1^2 + x_1\nu^2\frac{1-e^{-\mu t}}{\mu},\end{aligned} \tag{4.17}$$

so

$$\mathbb{E}\left[Y^2(t)\right] = \left(\frac{x_1}{x_0}\right)^2 + \frac{\nu^2}{\mu}\frac{x_1}{x_0^2}(1-e^{-\mu t}).$$

Since the second moment is bounded, by Theorem 2.5 there exists a non-trivial random variable Y_∞ such that $Y(t) \to Y_\infty$ as $t \to \infty$ almost surely, as well as $\mathbb{E}\left[\left|Y(t) - Y_\infty\right|^2\right] \to 0$ as $t \to \infty$. Hence $X^{x_1}(t)$ is stable in ratio. This behavior can also be seen from the SDE of $Y(t)$ given by (4.12). Using this, we obtain

$$\mathrm{d}Y(t) = \nu e^{-\mu t/2}\sqrt{Y(t)}\,\mathrm{d}B(t).$$

The effects of diffusion are decaying exponentially fast in time. This is similar to Example 2.12, where a choice of $g \in L^2$ leads to a log-normal distribution for $Y(t)$ as per Example 4.5.

One of the reasons this example works well is that the second moment of $Y(t)$ is straightforward to compute. The expectation of the stochastic integral in (4.17) is reduced to a standard integral since the expectation in the integrand is already known. If we did not have $\beta = 1/2$, then, barring a few particular cases, the computation would not be as simple.

4.3.3 Transformations on Solutions

By applying a suitable transformation to $X^{x_1}(t)$, we can obtain a new SDE for the transformed process whose behavior is better understood. For example, at the end of Section 4.3.2, we discussed why $\beta = 1/2$ worked nicely. It would be helpful to relate the SDE we want to study to one having the same diffusion as in Example 4.14. The following example does exactly this.

Example 4.15. *Consider the SDE*

$$\mathrm{d}X(t) = X^{\alpha}(t)\,\mathrm{d}t + X^{\beta}(t)\,\mathrm{d}B(t)$$

for $\beta \in (0,1)$, and $\beta \neq 1/2$. Let $x_0, x_1 > 0$. Assume α is such that a unique solution exists as per Example 2.21. Take the function $G(x) = x^{2-2\beta}$ whose first derivative is

$$G'(x) = 2(1-\beta)x^{1-2\beta} = 2(1-\beta)x^{-\beta}G^{1/2}(t).$$

Hence, if $Z(t) = G(X(t))$, then by Itô's formula (2.17), we have

$$\begin{aligned}
\mathrm{d}Z(t) &= 0\,\mathrm{d}t + G_x\,\mathrm{d}X(t) + \frac{1}{2}G_{xx}\sigma^2\,\mathrm{d}t \\
&= 2(1-\beta)X^{-\beta}(t)Z^{1/2}(t)\,\mathrm{d}X(t) + \frac{1}{2}(2-2\beta)(1-2\beta)X^{-2\beta}(t)(X^{\beta}(t))^2\,\mathrm{d}t \\
&= 2(1-\beta)X^{-\beta}(t)Z^{1/2}(t)\left(X^{\alpha}(t)\,\mathrm{d}t + X^{\beta}(t)\,\mathrm{d}B(t)\right) + (1-\beta)(1-2\beta)\,\mathrm{d}t \\
&= \left(2(1-\beta)X^{\alpha-\beta}(t)Z^{1/2}(t) + (1-\beta)(1-2\beta)\right)\mathrm{d}t + 2(1-\beta)Z^{1/2}(t)\,\mathrm{d}B(t) \\
&= \left(2(1-\beta)Z^{\frac{1}{2}\left(1+\frac{\alpha-\beta}{1-\beta}\right)}(t) + (1-\beta)(1-2\beta)\right)\mathrm{d}t + 2(1-\beta)Z^{1/2}(t)\,\mathrm{d}B(t)
\end{aligned}$$

Note that if $\alpha = 1$, then the power of $Z(t)$ in the drift is simply 1. The process $Z(t)$ has a drift consistent of the constant $(1-\beta)(1-2\beta)$.

For the remainder of the example, set $\alpha = 1$ and $\beta = 1/4$. Then

$$\mathrm{d}Z(t) = \left(\frac{3}{2}Z(t) + \frac{3}{8}\right)\mathrm{d}t + \frac{3}{2}Z^{1/2}(t)\,\mathrm{d}B(t)$$

with $z_1 = x_1^{3/2}$. The solution does not have an explicit formula. This example is only different from Example 4.14 by the constant in the drift term introduced by the transformation $G(x) = x^{3/2}$. By the comparison result 2.22, we have that $U(t) \leq Z(t)$ where $U(t)$ is the solution to

$$\mathrm{d}U(t) = \frac{3}{2}U(t)\,\mathrm{d}t + \frac{3}{2}U^{1/2}(t)\,\mathrm{d}B(t)$$

with the same initial condition z_1. The SDE for $U(t)$ matches exactly the one examined in Example 4.14. Therefore, we know that $W(t)$, the ratio process induced by $U(t)$, converges to some non-trivial random variable W_∞ as $t \to \infty$. Then

$$Y^{x_0,x_1}(t) = \frac{X^{x_1}(t)}{x_0 e^t} = \frac{Z^{2/3}(t)}{x_0\left(e^{3t/2}\right)^{2/3}} = \frac{1}{x_0}\left(\frac{Z(t)}{e^{3t/2}}\right)^{2/3} \geq \frac{1}{x_0}\left(\frac{U(t)}{e^{3t/2}}\right)^{2/3} \tag{4.18}$$

Since $\tilde{U}(t) = z_0 e^{3t/2}$, taking the limit of (4.18) as $t \to \infty$, we see that $Y^{x_0,x_1}(t) \to Y_\infty$ where

$$Y_\infty \geq \frac{z_0^{2/3}}{x_0}W_\infty^{2/3}$$

almost surely. Although this computation does not give us the stability in ratio of $X^{x_1}(t)$ directly, it does illustrate that it cannot be unstable in ratio because of condition (4.5). Then it only remains to show that (4.6) cannot hold in order to obtain the stability in ratio of $X^{x_1}(t)$.

More generally, for time-homogeneous SDEs, the transformation $G(x)$ that satisfies

$$G(x) = \int_0^x \sigma^{-1}(s)\psi(G(s))\,\mathrm{d}s$$

transforms the SDE with diffusion $\sigma(X(t))$ to a new SDE $Z(t)$ with diffusion $\psi(Z(t))$. In theory, it is possible to convert any unwieldy SDE to one with a more familiar diffusion term at the cost of a cumbersome drift term. Note that this is possible only if, for $x \in \mathbb{R}$, $\sigma(x) \neq 0$. Hence, dropping assumption (3.8) is essential in this situation.

4.4 Future Extensions

This section briefly considers how definitions in this chapter can be extended in directions that this thesis did not discuss but might be promising to investigate further.

4.4.1 Ratio Process in Higher Dimensions

The first idea is to extend Definitions 4.1 and related definitions to higher dimensions. We therefore require a higher-dimensional version of the ratio process that is reasonable since (4.1) only applies to one-dimensional processes. Consider the solution $X^{x_1}(t)$ to (3.1) in $\mathbb{R}^l$ and again let $\tilde{X}^{x_0}(t)$ be its deterministic solution. One proposal is to consider $\xi \in \mathbb{R}^l$ and then set

$$Y^{x_0,x_1}(t,\xi) = \frac{\left\langle X^{x_1}(t), \xi \right\rangle_{\mathbb{R}^l}}{\left\langle \tilde{X}^{x_0}(t), \xi \right\rangle_{\mathbb{R}^l}} \tag{4.19}$$

provided $\left\langle \tilde{X}^{x_0}(t), \xi \right\rangle_{\mathbb{R}^l} \neq 0$. Then $Y^{x_0,x_0}(t,\xi)$ is the **multi-dimensional ratio process** along vector ξ.

This definition allows us to consider the ratio along any direction. For example, if $\xi = e_j$ (where $1 \leq j \leq l$), the j-th element in the standard basis for $\mathbb{R}^l$, then (4.19) simply reduces to the ratio process (4.1) when considering the one dimensional process given by the j-th element of the random vector $X^{x_1}(t)$.

In terms of stability, to extend Definition 4.1 to the multi-dimensional case, we propose that $X^{x_1}(t)$ is said to be **strictly stable in ratio** if

$$\lim_{x_1 \to x_0} \lim_{t \to \infty} Y^{x_0,x_1}(t,\xi) = 1$$

almost surely for all $\xi \in \mathbb{R}^l$ such that $|\xi| = 1$. Then, in order for the multi-dimensional process $X^{x_1}(t)$ to be strictly stable in ratio, it is required that every constituent one-dimensional process of $X^{x_1}(t)$ is strictly stable in ratio. As an extension of the one-dimensional case, this requirement is natural. Similar extensions can be created for Definitions 4.2 and 4.3.

4.4.2 Ratio Process in L^p

In Section 3.2, we considered the p-stability of the solution to (3.1). We can attempt to do the same for the ratio process. In the same way that (4.3) was a desirable trait, we now wish that

$$\lim_{x_1 \to x_0} \mathbb{E}\left[Y^{x_0,x_1}(t)\right] = 1$$

for all times $t \geq 0$. This is the case in Example 4.5 since we know $Y^{x_0,x_1}(t)$ is a martingale. However, if $\sigma \notin L^2$, then $Y^{x_0,x_1}(t) \to 0$ almost surely as $t \to \infty$. So we do not have that

$$\lim_{t \to \infty} \mathbb{E}\left[\left|Y^{x_0,x_1}(t) - 1\right|\right] = 0.$$

This is to say that asymptotic p-stability would fail for any $p \geq 1$. An investigation of this could prove insightful.

5 Conclusion

This thesis investigated the stability of stochastic differential equations. Following a review of methods in the deterministic case as well as probability prerequisites in Chapter 2, the results that establish stability are extended to solutions to SDEs in Chapter 3. The key results in both the stochastic and the deterministic settings rely on the Lyapunov framework. Section 3.5 of this thesis considers SDEs failing the Lipschitz requirement and shows that stability definitions could be applied to these models too. Such results are much less general and have not been significantly considered in the literature.

In order to extend the results to more general settings, the thesis proposes the ratio process and the concept of stability in ratio. This definition of stability focuses on the asymptotic behavior (in particular the growth rate) of the solution to an SDE relative to that to the corresponding ODE. This definition of stability is more forgiving in the sense that the local behavior of the solution to the SDE does not impact the stability of the system. Without the burden of restrictive assumptions, this notion can be applied more generally compared to the traditional definitions of stability. This definition has shed new insight into some previously considered examples, but it should be noted that it does not replace the existing definitions. Although the provided examples demonstrate various techniques to study the ratio process, as of yet no general approach has been established in this setting and usually considerable knowledge on the behavior of the solution to the SDE is required in order to get meaningful results.

References

[1] Ludwig Arnold. *Stochastic Differential Equations: Theory and Applications*. Wiley, 1974. ISBN: 9780471033592.

[2] Paolo Baldi. *Stochastic Calculus: An Introduction Through Theory and Exercises*. Universitext. Springer, 2017. ISBN: 9783319622262.

[3] Jan Bergenthum and Ludger Rüschendorf. "Comparison of semimartingales and Lévy processes". In: *The Annals of Probability* 35.1 (2007), pp. 228–254.

[4] Andrei N. Borodin and Paavo Salminen. *Handbook of Brownian Motion - Facts and Formulae*. Probability and Its Applications. Birkhäuser Basel, 2002. ISBN: 978-3-7643-6705-3.

[5] R. S. Bucy. "Stability and positive supermartingales". In: *Journal of Differential Equations* 1.2 (1965), pp. 151–155.

[6] Linan Chen and Daniel W. Stroock. "The Fundamental Solution to the Wright–Fisher Equation". In: *SIAM Journal on Mathematical Analysis* 42.2 (2010), pp. 539–567.

[7] Linan Chen and Ian Weih-Wadman. "Fundamental solution to 1D degenerate diffusion equation with locally bounded coefficients". In: *Journal of Mathematical Analysis and Applications* 510.2 (2022), p. 125979.

[8] Linan Chen and Ian Weih-Wadman. "The fundamental solution to 1D degenerate diffusion equation with one-sided boundary". In: *Journal of Mathematical Analysis and Applications* 492.1 (2020), p. 124368.

[9] Alexander S. Cherny and Hans-Jürgen Engelbert. *Singular Stochastic Differential Equations*. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 2004. ISBN: 978-3-540-24007-5.

[10] Albert Einstein. "On the movement of small particles suspended in stationary liquids required by the molecular-kinetic theory of heat". In: *Annals of Physics* 322 (1905), pp. 549–560.

[11] H. J. Engelbert and W. Schmidt. "On one-dimensional stochastic differential equations with generalized drift". In: *Stochastic Differential Systems Filtering and Control* 69 (1985), pp. 143–155.

[12] Charles L. Epstein and Rafe Mazzeo. "Wright–Fisher Diffusion in One Dimension". In: *SIAM Journal on Mathematical Analysis* 42.2 (2010), pp. 568–608.

[13] Lawrence C. Evans. *An Introduction to Stochastic Differential Equations*. American Mathematical Society, 2013. ISBN: 9781470410544.

[14] I.I. Gikhman and A. V. Skorokhod. *Stochastic Differential Equations*. Springer-Verlag, New York, Heidelberg, 1972. ISBN: 978-3-642-88264-7.

[15] Wassim M. Haddad and VijaySekhar Chellaboina. *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*. Princeton University Press, 2008. ISBN: 9780691133294.

[16] Kiyosi Itô. "Stochastic Integral". In: *Proceedings of the Imperial Academy* 20.8 (1944), pp. 519–524.

[17] I.Ia. Kats and N.N. Krasovskii. "On the stability of systems with random parameters". In: *Journal of Applied Mathematics and Mechanics* 24.5 (1960), pp. 1225–1246.

[18] Hassan K. Khalil. *Nonlinear Systems*. Prentice Hall, 1996. ISBN: 9780132280242.

[19] Rafail Khasminskii. *Stochastic Stability of Differential Equations*. Stochastic Modelling and Applied Probability. Springer Berlin, Heidelberg, 2011. ISBN: 978-3-642-23279-4.

[20] Peter E. Kloeden and Eckhard Platen. *Numerical Solution of Stochastic Differential Equations*. Stochastic Modelling and Applied Probability. Springer Berlin, Heidelberg, 1992. ISBN: 978-3-540-54062-5.

[21] Harold J. Kushner. "On the construction of stochastic Liapunov functions". In: *IEEE Transactions on Automatic Control* 10.4 (1965), pp. 477–478.

[22] Harold J. Kushner. *Stochastic Stability and Control*. Academic Press, 1967. ISBN: 9780080955407.

[23] A. M. Lyapunov. "The General Problem of the Stability of Motion". In: *Kharkov Mathematical Society* (1892).

[24] Peter Mörters and Yuval Peres. *Brownian Motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2010. ISBN: 9780511750489.

[25] Daniel Revuz and Marc Yor. *Continuous Martingales and Brownian Motion*. Grundlehren der mathematischen Wissenschaften. Springer Berlin, Heidelberg, 1991. ISBN: 978-3-662-21726-9.

[26] Daniel W. Stroock. *Probability Theory An Analytic View*. Cambridge University Press, 2010. ISBN: 9780521132503.

[27] Toshio Yamada. "On a comparison theorem for solutions of stochastic differential equations and its applications". In: *Journal of Math Kyoto University* 13.3 (1973), pp. 497–512.

[28] Toshio Yamada and Shinzo Watanabe. "On the uniqueness of solutions of stochastic differential equations". In: *Journal of Math Kyoto University* 11.1 (1971), pp. 55–167.