The Ekedahl-Oort stratification of unitary Shimura varieties

Amy Wai Ling Jane Wooding

Doctor of Philosophy

Department of Mathematics and Statistics

McGill University

Montreal,Quebec

July 25, 2016

A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of Doctor of Philosophy.

ACKNOWLEDGMENTS

Though there is only name on the cover, this thesis could not have come to be without the help, support, and guidance of a great number of people.

First, I would like to thank my supervisor, Eyal Goren, for all that he has taught me mathematically, and for his overall guidance, patience, and advice. I thank him for encouraging me to persevere, and to always try one more time than I thought I had in me. The lessons that I have learned from him apply not just to mathematics, but also to life.

Next, I would like to thank Jayce Getz for what he taught me about automorphic representations. Though my research eventually took a different direction, his time and guidance were invaluable. I would also like to thank John Proos for valuable combinatorial discussions related to this work.

I would like to thank the Mathematics and Statistics Department at McGill for their financial support during my tenure there.

Research can't happen alone. As such, I wish to thank the students, postdocs, and faculty at McGill, Concordia, U de M, UQAM, and the CRM for creating and fostering such an incredible learning environment, and the researchers at the Tutte Institute for making me a better researcher and mathematician.

I need to thank my husband, Kjell for making me breakfast.

Finally, I would like to thank my friends and family for the numerous ways they have supported, loved, and encouraged me through this process. None of this would have been possible without them.

ABSTRACT

In this thesis, we explore various aspects of the Ekedahl-Oort (E-O) stratification of unitary Shimura varieties for any signature (m_1, m_2). We begin with a detailed study of the E-O strata themselves, describing models for the corresponding p-torsion group schemes, computing standard invariants of the strata and exploring relationships between the E-O and Newton strata. We then explicitly derive the E-O stratum of the reduction of a CM point from its CM type, hence providing, under suitable conditions, concrete examples of abelian varieties lying in given E-O strata. Following the techniques of Ekedahl and van der Geer in the Siegel case, we show that the cycle classes of the E-O strata in the Chow group can be written in terms of Chern classes of the Hodge bundle, and thus lie in the tautological ring. Finally, we use calculations of Hasse-Witt matrices over certain E-O strata to give new results on the geometry of these Shimura varieties.

ABRÉGÉ

Dans cette thèse, on étudie plusieurs aspects de la stratification d'Ekedahl-Oort (E-O) des variétés de Shimura associées aux groupes unitaires pour toute signature (m_1, m_2). On commence par une étude détaillée des strates d'Ekedahl-Oort elles-mêmes. Cette étude repose sur la description de certains modèles pour les schémas en groupes associés à la p-torsion des variétés abéliennes paramétrées par les différentes strates, sur le calcul des invariants de ces strates, et sur l'analyse des relations entre les strates d'E-O et les strates de Newton. On montre ensuite comment déduire explicitement la strate d'E-O de la réduction d'un point CM en utilisant son type CM, fournissant ainsi des exemples concrets de variétés abéliennes se trouvant dans une strate d'E-O donnée. Après les techniques de Ekedahl et van der Geer dans le cas Siegel, on montre que les classes de cycle des strates d'E-O dans le groupe Chow peuvent être écrits en termes de classes de Chern du fibré de Hodge, et sont ainsi dans l'anneau tautologique. Enfin, on utilise des calculs de matrices Hasse-Witt sur certaines strates d'E-O pour donner de nouveaux résultats sur la géométrie de ces variétés de Shimura.

TABLE OF CONTENTS

Chapter 1
Introduction

Let $\mathcal{M}$ denote a unitary Shimura variety coming from the group $\mathrm{GU}(m_1, m_2)$ in characteristic $p > 0$. These Shimura varieties correspond to moduli problems of abelian schemes with polarization and endomorphism structure coming from a quadratic imaginary field K. The p-torsion group scheme $A[p]$ and the p-divisible group $A(p)$ of points A on $\mathcal{M}$ can be used to partition the moduli space into locally closed subschemes called strata. In particular, the Ekedahl-Oort (E-O) stratification on $\mathcal{M}$ is the stratification based on the isomorphism classes of the p-torsion group schemes $A[p]$, where the isomorphism classes take into account extra structures coming from the endomorphism and polarization structures on A.

The E-O stratification is a particularly refined stratification and is a powerful tool for understanding the geometry of Shimura varieties in positive characteristic. Additionally, previous study of the E-O stratification has been used to solve problems outside of number theory. For example, the Hecke correspondences away from the prime p respect the E-O stratification and lead to representations of the Hecke algebra. Representations of the Hecke algebra arising from the zero-dimensional E-O strata alone have been shown to contain fascinating arithmetic and combinatorial structures in the case of Siegel and Hilbert modular varieties [Ghi03, Nic05], with applications to expander graph theory and cryptography [CLG09, CGL09].

The Ekedahl-Oort stratification was first studied extensively in the Siegel case by Oort and others (see for instance [Oor01], [LO98]). A method for studying the E-O strata by constructing a flag space over the Siegel moduli space was carried out by Ekedahl and van der Geer in [EvdG09]—a technique we will apply to the unitary case. Our approach to studying the Hasse-Witt matrices is due to Norman [Nor75], Norman-Oort [NO80], and Koblitz [Kob75] in the Siegel case. The E-O strata have been described for PEL Shimura varieties in general [Moo01, MW04, VW13], and we build on this work to recover more detailed results in the unitary case.

Previous study of the E-O strata of unitary Shimura varieties when $m_2 = 1$ was done by Bultel-Wedhorn [BW06], Vollaard [Vol05], Vollaard-Wedhorn [VW11] in the case when p is inert in K and Harris-Taylor [HT01] for the case when p is split in K. More recently, Howard and Pappas [HP13] gave extensive results in the inert case of signature $(2, 2)$. Related results concerning the Hasse-invariants and modular forms can be found in work of Goldring-Nicole [GN16], Reduzzi [Red12], Boxer [Box15], and de Shalit-Goren [dSG15].

In this thesis we take up the study of unitary Shimura varieties in general with no conditions on the signature, and in both cases where p is unramified in K. This allows us to study two moduli problems that are in a sense opposites on the moduli space of principally polarized abelian varieties. One the one hand, we will show that the generic E-O stratum is almost never ordinary and the zero-dimensional stratum is always superspecial when p is inert in K. When p is split in K the generic E-O stratum is always ordinary and the zero-dimensional stratum is almost never supersingular, let alone superspecial. Despite these differences, the E-O strata in both cases are parametrized by the same Weyl group coset allowing for a unified treatment of the subject.

1.1 Structure of the thesis

Chapter 2 provides a definition of unitary Shimura varieties and an overview of fundamental tools that can be used to study these varieties in characteristic p.

In Chapter 3, we describe specifics of the Ekedahl-Oort stratification based on the classification of the strata via a Weyl group coset ${}^J W$. Using this classification, we obtain bases for the Dieudonné modules corresponding to each E-O stratum and derive invariants of the associated p-torsion group schemes such as the a-number, f-number and the minimal power of F that kills the p-torsion. We show that there are four special strata of particular interest that arise from the description of the E-O strata in terms of ${}^J W$: the unique strata of dimensions 0, 1, $m_1 m_2 - 1$ and $m_1 m_2$. We exhibit models for the p-torsion group schemes of these strata and make use of them throughout the rest of the thesis.

In Chapter 4 we construct CM points on unitary Shimura varieties in characteristic zero that reduce to a given E-O stratum mod p. To this end, we explicitly derive the E-O stratum of the reduction of a CM point from its CM type. This leads to a constructive proof that under certain suitable conditions the zero-dimensional E-O stratum is non-empty (even when it is not supersingular).

Chapter 5 introduces the Newton stratification—a stratification whose strata can be classified by a poset $B(G, \mu)$—and explores its relationship with the Ekedahl-Oort stratification. In the case where p is split in the field K, we effectively compute the map $B(G, \mu) \hookrightarrow {}^J W$ that takes a Newton stratum to the minimal E-O stratum contained within it. This allows us to give a nearly complete description of the Newton strata corresponding to the special E-O strata.[1] While we have not yet established as nice of a description for the relationship between the Newton and E-O strata in the case where p is inert in K, we are still able to give a complete description of the Newton strata corresponding to the special E-O strata.

In Chapter 6, we adapt the techniques of Ekedahl and van der Geer from the Siegel case [EvdG09] to the study of the E-O strata. In particular, we construct a flag variety over $\mathcal{M}$ in such a way that the strata of the flag variety (coming from its Schubert cells) map onto the E-O strata via finite étale surjective maps. Therefore, the cycle classes of the closed E-O strata in the Chow group can be written in terms of Chern classes of the Hodge bundle, and thus lie in the tautological ring.

Finally, in Chapter 7 we calculate the Hasse-Witt matrices over the special E-O strata from Chapter 3. These calculations show that the (partial) Hasse-invariants vanish to order one on the non-ordinary stratum. Furthermore, in the case where p is split in K, we show that the non-ordinary locus is not only connected, but irreducible, with the corollary that the non-ordinary locus for a Shimura variety coming from the group $\mathrm{GU}(2,1)$ is a smooth, irreducible curve.

---

[1] The exception is in the case of the one-dimensional E-O stratum when $m_1 - m_2 > 1$ and $m_2 > 1$.

Chapter 2
Background

We begin this chapter by defining unitary Shimura varieties and the associated moduli problem of abelian schemes in Section 2.1. We then introduce the basic tools, techniques and definitions for studying abelian schemes in characteristic p throughout the rest of the chapter. In Sections 2.2.1 and 2.2.2 we review the definitions of the p-torsion group schemes and p-divisible groups arising from abelian schemes (with extra structures corresponding to the unitary moduli problem). These are the primary objects of study in this thesis. However, it is easier to study them through linear algebraic data coming through Dieudonné theory which we recall in Section 2.2.3. The classification of isocrystals by slope sequence is given in Section 2.2.4 so that we can interpret the results on the Newton stratification in Chapter 5 using the classical approach to slope sequences. We also briefly review the theory of displays in Section 2.2.5 which will be used for studying the local deformations of the Hasse-Witt matrices of points on $\mathcal{M}$ in Chapter 7.

2.1 Unitary Shimura varieties with good reduction at p

Shimura varieties, according to Deligne's formulation [Del72], are defined by so-called Shimura data. We begin by defining Shimura data with additional structure at a fixed prime p according to Kottwitz [Kot92] in order to ensure that we obtain models for the resulting Shimura varieties that have good reduction at the given prime p.

2.1.1 The Shimura datum

Fix a prime $p > 2$. Consider the following data.

- $(B, *) = (K, *)$ is a quadratic imaginary extension of $\mathbb{Q}$ in which p is unramified, and $*$ is the non-trivial automorphism of $K/\mathbb{Q}$.
- $(V, \psi) = (K^g, \psi)$ where $\psi : V \times V \to \mathbb{Q}$ is a non-degenerate alternating form such that $\psi(bu, v) = \psi(u, b^* v)$ for all $u, v \in V$ and $b \in B$. Furthermore, letting $\mathbf{G}$ denote the group of B-linear symplectic similitudes of (V, ψ), assume that $\mathbf{G}_{\mathbb{R}}$ is isomorphic to the group $GU(m_1, m_2)$ of unitary similitudes of the diagonal matrix $(1^{m_1}, -1^{m_2})$.
- $\mathcal{O}_B = \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, an $*$-invariant $\mathbb{Z}_{(p)}$-order of B such that $\mathcal{O}_B \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ is a maximal $\mathbb{Z}_p$-order of $B_{\mathbb{Q}_p}$.
- $\Lambda = (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^g \subset V_{\mathbb{Q}_p} = (K \otimes_{\mathbb{Q}} \mathbb{Q}_p)^g$, an $\mathcal{O}_B$-invariant $\mathbb{Z}_p$ lattice on which ψ is a perfect pairing.
- $h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ is the homomorphism of algebraic groups over $\mathbb{R}$ defined on $\mathbb{R}$-points by taking $z \in \mathbb{C}^\times$ to the diagonal matrix $(z^{m_1}, \bar{z}^{m_2})$. Note that h is a Hodge structure of type $(-1, 0), (0, -1)$ on $V \otimes_{\mathbb{Q}} \mathbb{R}$ making $(\mathbf{G}, h)$ a **Shimura datum** as in Section 1.5 of Deligne [Del72].

Then $\mathscr{D} = (B, *, V, \psi, \mathcal{O}_B, \Lambda, h)$ is a **PEL Shimura datum**[1] with good reduction at p and in particular, we call $\mathscr{D}$ a **unitary PEL Shimura datum of signature** (m_1, m_2) **with good reduction at** p.

[1] PEL in this context refers to the polarization, endomorphism and level structure that appear in the corresponding moduli problems of abelian schemes.

Not only is $\mathbf{G}_\mathbb{R}$ a unitary group, but $\mathbf{G}$ itself is a unitary group for $K/\mathbb{Q}$. To see this, begin by writing $K = \mathbb{Q}(\sqrt{d})$ where $-d > 0$ is a non-square integer. Then ψ can be written as

$$\psi(u, v) = \mathrm{Tr}_{K/\mathbb{Q}}(\sqrt{d}\Psi(u, v))$$

for a unique non-degenerate $*$-Hermitian form $\Psi : V \times V \to K$ (see [Del81] Lemma 4.6). The involution σ_ψ on $\mathrm{End}_K(V)$ induced by ψ is the same as the involution σ_Ψ induced by Ψ, and the group $\mathbf{G}$ defined by the involution σ_ψ on $\mathrm{End}_K(V)$ is isomorphic as an algebraic group over $\mathbb{Q}$ to the group

$$\begin{aligned}\mathrm{GU}(V, \Psi)(R) &:= \left\{ g \in \mathrm{GL}_{K\otimes_\mathbb{Q} R}(V \otimes_\mathbb{Q} R) \;\middle|\; \Psi(gu, gv) = c(g)\Psi(u, v), c(g) \in R^\times \right\} \\ &= \left\{ g \in (\mathrm{End}_K(V) \otimes R)^\times \;\middle|\; \sigma_\Psi(g)g \in R^\times \right\}.\end{aligned}$$

Over $\mathbb{R}$, $\mathbf{G}_\mathbb{R}$ becomes a unitary group with respect to $\mathbb{C}/\mathbb{R}$ and such a group can be uniquely identified up to isomorphism by its signature, (m_1, m_2).

There are two cocharacters of interest associated with $h : \mathbb{S} \to \mathbf{G}_\mathbb{R}$. In order to describe them, first make the identification of $(\mathbb{C} \otimes_\mathbb{R} \mathbb{C})^\times$ with $\mathbb{C}^\times \times \mathbb{C}^\times$ via $a \otimes z \mapsto (az, \bar{a}z)$, and then the canonical identification

$$\begin{aligned}\mathbf{G}(\mathbb{C}) = \mathrm{GU}(m_1, m_2) \otimes_\mathbb{R} \mathbb{C} &\subseteq \mathrm{GL}_g(\mathbb{C}) \times \mathrm{GL}_g(\mathbb{C}) \\ M \otimes z &\mapsto (Mz, (\sigma_\Psi(M)z)^t)\end{aligned}$$

taking the involution $(\sigma_\Psi)_\mathbb{C}$ to the involution $(M_1, M_2) \mapsto (M_2^t, M_1^t)$. Pre-composing $h(\mathbb{C})$ with the map $\mathbb{G}_m(\mathbb{C}) \to \mathbb{S}(\mathbb{C})$ where $z \mapsto (z, 1)$ then gives

$\mu_h : \mathbb{G}_{m/\mathbb{C}} \to \mathbf{G}_\mathbb{C}$ where

$$\mu_h : \mathbb{C}^\times \to \mathrm{GU}(m_1, m_2)_\mathbb{C} \subseteq \mathrm{GL}_g(\mathbb{C}) \times \mathrm{GL}_g(\mathbb{C})$$
$$z \mapsto (\mathrm{diag}(z^{m_1}, 1^{m_2}), \mathrm{diag}(1^{m_1}, z^{m_2})).$$

On the other hand, pre-composing h with the map $\mathbb{G}_m \to \mathbb{S}$ taking $r \mapsto r^{-1}$ for $r \in \mathbb{R}^\times$ gives the **weight homomorphism**

$$w_h : \mathbb{R}^\times \to \mathrm{GU}(m_1, m_2)$$
$$r \mapsto r^{-1} I_g.$$

By the definition of $h : \mathbb{S} \to \mathbf{G}_\mathbb{R}$, the cocharacter μ_h induces a decomposition of $V_\mathbb{C}$ into weight spaces, $V_\mathbb{C} = V_0 \oplus V_1$. The **reflex field** of a PEL datum $\mathscr{D}$ is the field of definition $E_\mathscr{D}$ of the isomorphism class of the complex representation V_1 of K. When $\mathscr{D}$ is a unitary PEL Shimura datum of signature (m_1, m_2), then either $E_\mathscr{D} = \mathbb{Q}$ when $m_1 = m_2$ or $E_\mathscr{D} = K$ when $m_1 \neq m_2$.

As we will be mostly working with structures related to p, it is useful to let G be the $\mathbb{Z}_p$-group of $\mathcal{O}_B$-linear symplectic similitudes of Λ. Note that this is a quasi-split reductive group scheme with generic fibre $\mathbf{G}_{\mathbb{Q}_p}$. Let $\bar{G}$ denote the special fibre of G.

2.1.2 The moduli problem

Before stating the moduli problem associated to the Shimura datum $\mathscr{D}$, we begin with some necessary definitions. Let $\chi_i : K \hookrightarrow \mathbb{C}$ denote the two embeddings of K into $\mathbb{C}$ and let E be the reflex field of $\mathscr{D}$. Let S be a locally noetherian scheme over $\mathcal{O}_E \otimes_\mathbb{Z} \mathbb{Z}_{(p)}$, and let A be an abelian scheme over S of dimension

$g = m_1 + m_2$ together with a $\mathbb{Z}_{(p)}$-algebra homomorphism $\iota : \mathcal{O}_B \to \mathrm{End}_S(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Then (A, ι) is said to satisfy the **determinant condition** if

$$\mathrm{charpol}(b, \mathrm{Lie}(A)) = (X - \chi_1(b))^{m_1}(X - \chi_2(b))^{m_2} \in \mathcal{O}_S[X], \forall b \in \mathcal{O}_B. \tag{2.1}$$

Suppose that $f : A_1 \to A_2$ is a prime-to-p isogeny of abelian schemes over S. Then the $\mathbb{Z}_{(p)}^{\times}$**-equivalence class** of f is defined by the equivalence relation $f \sim \alpha f$ if $\alpha : S \to \mathbb{Z}_{(p)}^{\times}$ is a locally constant map.

Let $\mathscr{D} = (B, *, V, \psi, \mathcal{O}_B, \Lambda, h)$ be a unitary PEL Shimura datum with good reduction at p and let $E_{\mathscr{D}}$ be the reflex field of $\mathscr{D}$. Let $\mathbf{G}$ be the group associated with $\mathscr{D}$ and let $C^p < \mathbf{G}(\mathbb{A}_f^p)$ be an open compact subgroup (here $\mathbb{A}_f^p$ denotes the finite adeles that are trivial at p). Then as in Kottwitz [Kot92, Section 5], $\mathscr{D}$ and C^P define a moduli problem in the following way.

Denote by $\mathcal{M}_{(m_1,m_2),C^P}$ the set-valued contravariant functor from the category of locally Noetherian schemes S over $\mathcal{O}_{E_{\mathscr{D}}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ taking S to the set of isomorphism classes of tuples $(A, \iota, \lambda, \eta)$ where

- A is an abelian scheme over S,
- $\iota : \mathcal{O}_B \to \mathrm{End}_S(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a $\mathbb{Z}_{(p)}$-algebra homomorphism,
- λ is a $\mathbb{Z}_{(p)}^{\times}$-equivalence class of an $\mathcal{O}_B$-linear $\mathbb{Z}_{(p)}^{\times}$-polarization of A with respect to the $\mathcal{O}_B$-action induced by ι on $A^{\vee}$ *i.e.* $b \in \mathcal{O}_B$ acts on $A^{\vee}$ through $\iota(b^*)^{\vee}$,
- η is a C^p level structure in the sense of [Kot92],

such that (A, ι) satisfies the determinant condition defined in (2.1). Two tuples $(A_1, \iota_1, \lambda_1, \eta_1)$ and $(A_2, \iota_2, \lambda_2, \eta_2)$ are isomorphic if there exists an $\mathcal{O}_B$-linear quasi-isogeny $f : A_1 \to A_2$ with prime-to-p degree taking λ_1 to λ_2 and η_1 to η_2.

Let $s \in S$ be a geometric point, and suppose that S is connected. Then a **level structure**, η, of type C^p for (A, ι, λ) is a C^p-orbit of isomorphisms of $V_{\mathbb{A}_f^p} \to H_1(A_s, \mathbb{A}_f^p)$ of skew-Hermitian B-modules (up to a scalar multiple) such that the C^p-orbit η is fixed by $\pi_1(S, s)$. Because η is fixed by $\pi_1(S, s)$, a choice of level structure is independent of the choice of geometric point $s \in S$ (see [Kot92, Section 5]).

When C^p is sufficiently small, $\mathcal{M}_{(m_1,m_2),C^P}$ is representable by smooth quasi-projective scheme over $\mathcal{O}_{E_{\mathscr{D}}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ [Kot92]. We will assume that C^p is "small enough" throughout the entirety of this thesis.

Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let $\mathfrak{p}$ be the prime of the reflex field E lying over the rational prime p that corresponds to the chosen embedding. Let $\kappa(\mathfrak{p})$ denote the residue field of $E_{\mathfrak{p}}$. Our main object of study will be the special fibre of $\mathcal{M}_{(m_1,m_2),C^P}$ at $\mathfrak{p}$,

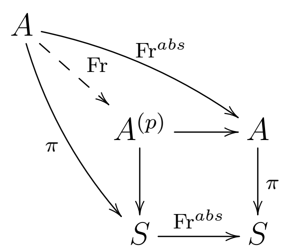
$$\mathcal{M} := \mathcal{M}_{(m_1,m_2),C^p} \otimes_{O_{E,(p)}} \kappa(\mathfrak{p}).$$

To simplify notation, the fixed choices on which $\mathcal{M}$ depends—C^p, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, $\mathscr{D}$, and hence (m_1, m_2)—will be taken to be implied throughout.

2.2 Abelian schemes in positive characteristic

One of the most important features of varieties in positive characteristic is the existence of the Frobenius morphism. To be precise, we begin by noting that for any scheme X of characteristic p, there is an **absolute Frobenius morphism**

$\mathrm{Fr}^{abs} : X \to X$ coming from the p^{th}-power map $x \mapsto x^p$ on the structure sheaf $\mathcal{O}_X$. In case of an abelian scheme $\pi : A \to S$, the absolute Frobenius morphisms on A and S give rise to the **relative Frobenius morphism** $\mathrm{Fr} : A \to A^{(p)}$, a degree p isogeny where $A^{(p)}$ and Fr are defined by the diagram



where the square is cartesian. The Frobenius morphism has a dual **Verschiebung morphism** $\mathrm{Ver} : A^{(p)} \to A$ satisfying $\mathrm{Fr} \circ \mathrm{Ver} = [p]_A$ and $\mathrm{Ver} \circ \mathrm{Fr} = [p]_{A^{(p)}}$ where $[p]$ denotes the multiplication-by-p isogeny. This special feature of the multiplication-by-p map underlies all of the tools we will use for studying abelian schemes in positive characteristic.

The remainder of this section will be spent reviewing the basic theory of "mod-p structures" that are arise on abelian schemes in characteristic p, and especially structures that encode information about the Frobenius and Verschiebung morphisms. We are particularly interested in the structures that arise when studying geometric points on $\mathcal{M}$. For each of the mod-p structures that we introduce, we will begin with a brief general introduction over a perfect field k of characteristic p, to be followed with a description of that mod-p structure on k-points of $\mathcal{M}$ together with the contribution of the polarization and endomorphism structure determined by the moduli problem for $\mathcal{M}$. To do so, it will be necessary to also make sure that $\mathcal{O}_K/(p) \otimes_{\mathbb{F}_p} k \cong k_1 \oplus k_2$. In other words, we need k to be any perfect extension of κ where $\kappa = \mathbb{F}_p$ when p is split in K and $\kappa = \mathbb{F}_{p^2}$ when p is inert in K. Unless otherwise indicated, we will simply let k be an algebraically closed field of characteristic p when considering structures arising from k-points of $\mathcal{M}$ as this is all that will be needed in the sequel.[2]

2.2.1 p-torsion group schemes

Let $\pi : A \to S$ be an abelian scheme of relative dimension g. Then let the **n-torsion group scheme** $A[n]$ denote the kernel of the multiplication-by-n map where $n \in \mathbb{Z}_{>0}$. As the kernel of a proper flat morphism, $A[n]$ is a finite flat group scheme of over S of rank n^{2g}. When S is the spectrum of a perfect field of characteristic p, the group scheme $A[p]$ is particularly interesting and encodes significant information pertaining to A.

Let G be a commutative algebraic group scheme over k where k is an algebraically closed field of characteristic p . Let α_p be the local-local p-torsion group scheme of rank one, and let $\underline{\mathbb{Z}/p\mathbb{Z}}$ be the constant group scheme of the group $\mathbb{Z}/p\mathbb{Z}$. Then, the **f-number**[3] of G, $f(G)$, is defined as the positive integer such that $f(G) = \dim_k \mathsf{Hom}_k(\underline{\mathbb{Z}/p\mathbb{Z}}, G)$, and the **$a$-number** of G, $a(G)$, is given by $a(G) := \mathsf{Hom}_k(\alpha_p, G)$. In case of an abelian variety A/k, the f-number and

[2] even though the results and arguments often remain valid over perfect extensions of κ.

[3] The f-number of G is also known as the *p-rank* of G in the literature.

a-number of A are defined to be the f-number and a-number of $A[p]$ and have alternate descriptions as

- $p^{f(A)} = \# A[p](k)$,
- $a(A)$ is such that the order of the maximal α_p-elementary subgroup of $A[p]$ is $p^{a(A)}$, or such that the rank of $\alpha(A) := \ker(\mathrm{Ver} : A^{(p)} \to A) \cap \ker(\mathrm{Fr} : A^{(p)} \to A^{(p^2)})$ is $p^{a(A)}$.

The f-number and a-number of A both range from 0 to g where g is the dimension of A. An abelian variety with maximal a-number is called **superspecial** and an abelian variety with maximal f-number is called **ordinary**.

Let $(A, \iota, \lambda, \eta) \in \mathcal{M}(k)$ where k is an algebraically closed field of characteristic p. The polarization λ makes $A[p]$ self-dual as a group scheme as $\lambda : A[p] \to A^t[p]$ is an isomorphism and $A^t[p] \cong A[p]^\vee$ where $A[p]^\vee$ denotes the dual of $A[p]$ as a finite commutative group scheme. Furthermore, the endomorphism structure ι induces an action of $\mathcal{O}_K/(p)$ on $A[p]$. This determines a decomposition of $A[p]$ via the characters $\chi_1, \chi_2 : \mathcal{O}_K/(p) \to k$.

Let I be an ideal in $\mathcal{O}_K$. Then we define $A[I]$ to be the group scheme $\cap_{t\in I}\ker(t)$, which is closed under the induced $\mathcal{O}_K$-action. Note that if $I \mid J$ then $A[I] \subseteq A[J]$. Furthermore, for $(p) \subseteq \mathcal{O}_K$, $A[(p)] = A[p]$ as $\iota(p) = [p]$ so that if $\mathfrak{p} \mid (p)$, then $A[\mathfrak{p}] \subset A[p]$ is an $\mathcal{O}_K$-subgroup scheme of $A[p]$.

Lemma 2.2.1. *Suppose that p splits as $(p) = \mathfrak{p}_1\mathfrak{p}_2$ in K where* $\ker(\chi_i) = \mathfrak{p}_i$. *Then $A[p]$ decomposes under the $\mathcal{O}_K/(p) \cong \mathcal{O}_K/\mathfrak{p}_1 \oplus \mathcal{O}_K/\mathfrak{p}_2$ action as*

$$A[p] = A[\mathfrak{p}_1] \oplus A[\mathfrak{p}_2]$$

where $A[\mathfrak{p}_i] = \cap_{t\in\mathfrak{p}_i}\ker(t)$ *for* $i = 1, 2$. *Furthermore,* $A[\mathfrak{p}_1] \cong A^\vee[\mathfrak{p}_2] \cong A[\mathfrak{p}_2]^\vee$ *and vice versa.*

Proof. Write $A[p] = A_1 \oplus A_2$ as the decomposition of $A[p]$ under the action of $\mathcal{O}_K/\mathfrak{p}_1 \oplus \mathcal{O}_K/\mathfrak{p}_2$ and let $i = 1, 2$. Suppose that B is an $\mathcal{O}_K/(p)$-submodule of $A[p]$ contained in A_i. Then for all $t \in \mathfrak{p}_i$, $\chi_i(t) = 0$ so that $B \subseteq A[\mathfrak{p}_i]$. On the other hand, $A[\mathfrak{p}_i]$ is an $\mathcal{O}_K/(p)$ submodule of $A[p]$ and decomposes as $A[\mathfrak{p}_i]_1 \oplus A[\mathfrak{p}_i]_2$ under the action of $\mathcal{O}_K/\mathfrak{p}_1 \oplus \mathcal{O}_K/\mathfrak{p}_2$. However, $A[\mathfrak{p}_i]_j \subseteq A[\mathfrak{p}_j]$, and $A[\mathfrak{p}_1] \cap A[\mathfrak{p}_2] = A[\mathfrak{p}_1 + \mathfrak{p}_2] = \cap_{t\in\mathcal{O}_K}\ker(t) = 0$ so that if $i \neq j$, $A[\mathfrak{p}_i]_j = 0$. It follows that $A_i = A[\mathfrak{p}_i]$.

Cartier duality gives $A[p]^\vee = A_1^\vee \oplus A_2^\vee$, where r in $\mathcal{O}_K$ acts through the dual map $r^\vee : A^\vee \to A^\vee$. Note that $A[r]^\vee = A^\vee[r^\vee]$ for all $r \in \mathcal{O}_K$ so that $r^\vee : A^\vee \to A^\vee$ restricted to $A[p]^\vee = A^\vee[p]$ is same as $r^\vee$ induced by Cartier duality.

By the condition on the Rosati involution, $r^\vee = \lambda \circ \bar{r} \circ \lambda^{-1}$ for $r \in \mathcal{O}_K$, λ is an $\mathcal{O}_K$-module isomorphism $A \to A^\vee$ if we twist the $\mathcal{O}_K$ action on $A^\vee$; that is, if $r \in \mathcal{O}_K$ acts as $\bar{r}^\vee$ on $A^\vee$ where $\bar{r}^\vee$ is the dual of the endomorphism $\bar{r} : A \to A$. Therefore, λ is an isomorphism $A_1 \cong A_2^\vee$. □

Note that his implies that the rank of $A[\mathfrak{p}_1]$ is equal to the rank of $A[\mathfrak{p}_2]$ giving them both rank p^g.

Remark. The fact that the p-torsion group scheme $A[p]$ splits as a group scheme under the $\mathcal{O}_K$-action whenever p splits in K will come up repeatedly. In many cases, this results in simpler proofs for the case when p is split in K than for the case where p is inert in K (for example see Proposition 7.1.1).

2.2.2 p-divisible groups

A p**-divisible group** (also **Barsotti-Tate group**) H over a scheme S is an inductive system of finite flat commutative group schemes over S and closed immersions

$$0 \hookrightarrow H_1 \hookrightarrow H_2 \hookrightarrow \dots \hookrightarrow H_i \hookrightarrow \dots$$

such that for every $i \geq 1$, the sequence

$$0 \longrightarrow H_1 \longrightarrow H_{i+1} \xrightarrow{[p]} H_i \longrightarrow 0$$

is exact. If S is connected, there exists an integer $\mathrm{ht}(H)$ called the **height** of H such that $\mathrm{rank}(H_i) = p^{h(H)i}$ for all $i \geq 1$. The **dimension** of H is its dimension as a formal scheme.

A p-divisible group H, has an associated **dual** p-divisible group (or **Serre dual**) H^t defined by taking the inductive system obtained from the Cartier duals $[p]^D : H_i^D \hookrightarrow H_{i+1}^D$.

A **homomorphism** $f : H \to H'$ between p-divisible groups is a family of homomorphisms $f_i : H_i \to H_i'$ such that the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & \dots & \longrightarrow & H_i & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & & \downarrow{\scriptstyle f_i} & & \\ 0 & \longrightarrow & H_1' & \longrightarrow & H_2' & \longrightarrow & \dots & \longrightarrow & H_i' & \longrightarrow & \dots \end{array}$$

commutes. An **isogeny** $f : H \to H'$ is a homomorphism that is also an epimorphism with finite kernel. For example, for every $n \geq 0$ and p-divisible group H, the multiplication-by-p^n map given by the family of maps $[p^n] : H_i \to H_i$ is an isogeny with kernel H_n. A **quasi-isogeny** $f : H \to H'$ is a homomorphism such that $[p^n] \circ f$ is an isogeny for some $n \geq 0$.

Let k be algebraically closed. Given a point $\underline{A} = (A, \iota, \lambda, \eta) \in \mathcal{M}(k)$ corresponding to a PEL datum $\mathscr{D}$, there is an associated p-divisible group $A(p)$ coming from $A[p^i] \hookrightarrow A[p^{i+1}]$ for all $i \geq 0$. Since the construction of p-divisible groups is functorial, $A(p)$ comes with induced extra structures:

- an action $\iota(p) : \mathcal{O}_B \hookrightarrow \mathrm{End}\,(A(p))$,
- an isomorphism $\lambda(p) : A(p) \to A^t(p) = A(p)^t$ respecting $\iota(p)$ which is defined up to a scalar in $\mathbb{Z}_p^\times$ such that $\lambda(p)^t = \lambda(p)$.

The association of the data $(A(p), \iota(p), \lambda(p))$ to the point $\underline{A}$ is well-defined up to isomorphism.

In order to account for the extra structures on p-divisible groups that arise from (unitary) PEL moduli problems in a more general setting, we define a **p-divisible group with $\mathscr{D}$-structure over** S, (H, ι, λ), for a PEL datum $\mathscr{D} = (B, *, V, \psi, \mathcal{O}_B, \Lambda, h)$ where S is a $\mathbb{Z}_p$-scheme as follows:

- H is a p-divisible group over S of height $\dim_{\mathbb{Q}}(V)$
- an action $\iota : \mathcal{O}_B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \hookrightarrow \mathrm{End}_S\,(H)$,
- an $\mathcal{O}_B$-linear isomorphism $\lambda : H \to H^t$ up to $\mathbb{Z}_p^\times$-equivalence such that $\lambda^t = \lambda$,
- $\mathrm{Lie}(H)$ together with the $\mathcal{O}_B$-action satisfies a determinant condition (see below).

The **determinant condition** for $\mathrm{Lie}(H)$ is the analogue of the determinant condition for an abelian scheme defined in Section 2.1.2. We require that the characteristic polynomial of the action of $b \in \mathcal{O}_B$ on Lie(H) to be $(X - \chi_1(b))^{m_1}(X - \chi_2(b))^{m_2}$.

2.2.3 Dieudonné theory

Let k be a perfect field of characteristic p, let $W(k)$ be the Witt-vectors over k with Frobenius morphism $\sigma : W(k) \to W(k)$, and let $Q(k)$ denote the fraction field of $W(k)$. Let $\mathcal{D}_k$ denote the non-commutative ring generated by F and V over $W(k)$ subject to the relations:

$$FV = VF = p, F\lambda = \lambda^\sigma F, \lambda V = V\lambda^\sigma \qquad \forall \lambda \in W(k).$$

Then a **Dieudonné module** is a $\mathcal{D}_k$-module that is a finitely generated $W(k)$-module.

Given a Dieudonné module M of finite length over $W(k)$, let $M^\vee$ denote the **dual** of M as a Dieudonné module where $M^\vee = \mathsf{Hom}_{W(k)}(M, Q(k)/W(k))$ with F, V-structure given by:

$$(F\phi)(x) = \phi(Vx)^\sigma, (V\phi)(x) = \phi(Fx)^{\sigma^{-1}}$$

for all $x \in W(k)$ and $\phi \in \mathsf{Hom}_{W(k)}(M, Q(k)/W(k))$. Similarly, let M be a Dieudonné module that is free over $W(k)$. Then let $M^\vee = \mathsf{Hom}_{W(k)}(M, W(k))$ denote the **dual** of M as a Dieudonné module, where the $\mathcal{D}_k$-module structure on M is given by

$$(F\phi)(x) = \phi(Vx)^\sigma, (V\phi)(x) = \phi(Fx)^{\sigma^{-1}}$$

for all $x \in W(k)$ and $\phi \in \mathsf{Hom}_{W(k)}(M, W(k))$.

Proposition 2.2.2 (Dieudonné)**.** *There is an equivalence of categories* $\mathfrak{D}$ *between the category of finite commutative group schemes over* k *of* p*-power order and the category of Dieudonné modules of finite length as a* $W(k)$*-module. Furthermore,* $\mathfrak{D}$ *satisfies the following properties:*

1. *If* k' *is a perfect extension of* k*, then there exists a functorial isomorphism* $\mathfrak{D}(H \otimes_k k') \cong \mathfrak{D}(H) \otimes_{W(k)} W(k')$*. In particular,*

$$\mathfrak{D}(H^{(p)}) \cong \mathfrak{D}(H) \otimes_{W(k),\sigma} W(k). \tag{2.2}$$

2. $\mathrm{Fr} : H \to H^{(p)}$ *corresponds with* $V : \mathfrak{D}(H) \to \mathfrak{D}(H)$ *and* $\mathrm{Ver} : H \to H^{(1/p)}$ *correspond with* $F : \mathfrak{D}(H) \to \mathfrak{D}(H)$ *through 2.2. It follows that* H *is étale if and only if* $V_{\mathfrak{D}(H)}$ *is an isomorphism and* H *is connected if and only if* $V_{\mathfrak{D}(H)}$ *is nilpotent.*
3. *The rank of* H *is equal to* p^r *where* r *is the length of* $\mathfrak{D}(H)$ *as a* $W(k)$*-module.*
4. *There is a functorial isomorphism:*

$$\mathfrak{D}(H^D) = \mathfrak{D}(H)^\vee$$

where H^D *denotes the Cartier dual of* H*.*

Through the duality $\mathfrak{D}(H^D) = \mathfrak{D}(H)^\vee$, we can define the contravariant Dieudonné functor $\mathcal{D}$ by $\mathcal{D}(H) := \mathfrak{D}(H^D)$. It is worth noting that Fr becomes $F_{\mathcal{D}(H)}$ and Ver becomes $V_{\mathcal{D}(H)}$ when applying $\mathcal{D}$ instead of $\mathfrak{D}$.

On the level of p-divisible groups there is the following equivalence of categories.

Proposition 2.2.3 (Dieudonné)**.** *There is an equivalence of categories* $\mathfrak{D}$ *between the category of* p*-divisible groups over* k *and the category of Dieudonné modules that are free and finite rank as* $W(k)$*-modules. Furthermore,* $\mathfrak{D}$ *satisfies the following properties:*

1. *If* k' *is a perfect extension of* k*, then there exists a functorial isomorphism* $\mathfrak{D}(H \otimes_k k') \cong \mathfrak{D}(H) \otimes_{W(k)} W(k')$*.*
2. *The height of* H *is equal to the dimension of* $\mathfrak{D}(H)$ *as a* $W(k)$*-module.*
3. *There is a functorial isomorphism:*

$$\mathfrak{D}(H^t) = \mathfrak{D}(H)^\vee$$

where H^t *denotes the Serre dual of* H*.*

Under this equivalence of categories, a p-divisible group (H, ι, λ) with $\mathscr{D}$-structure over k induces the following structure on its associated Dieudonné module $\mathfrak{D}(H)$ (see for instance [Red12, Prop2.6]):

- $\mathfrak{D}(H)$ is free of rank $\dim_{\mathbb{Q}} V$ as an $W(k)$-module.
- $\mathfrak{D}(H)$ has an induced $\mathcal{O}_B$-action and sympletic form $\langle\ ,\ \rangle$ coming from ι and λ respectively such that $\langle bm, n\rangle = \langle m, b^* n\rangle$ for all $b \in \mathcal{O}_B$ and $m, n \in \mathfrak{D}(H)$.
- F is an injective $W(k) \otimes \mathcal{O}_B$-linear map preserving the symplectic form $\langle\ ,\ \rangle$ up to the scalar $p \in W(k)$.

Remark. A Dieudonné module with the above properties is called a $\mathscr{D}$-module in [VW11].

Now let k be algebraically closed and let $\chi_i : \mathcal{O}_K/(p) \to k$ such that $\bar{\chi}_1 = \chi_2$. Let $\underline{A} = (A, \iota, \lambda, \eta) \in \mathcal{M}(k)$ and $\mathfrak{D}$ be the covariant Dieudonné module of $A(p)$.

The extra structures on $\mathfrak{D}$ induced by ι and λ are described in Proposition 2.2.4. Write $\mathfrak{D} = \mathfrak{D}_1 \oplus \mathfrak{D}_2$ for the decomposition of $\mathfrak{D}$ as a $\mathcal{O}_K \otimes W(k) = W(k)_1 \oplus W(k)_2$-module.

Proposition 2.2.4. *Let $\gamma : C_2 \to C_2$ be the identity when p is split in K and let $\gamma(i) = i+1$ when p is inert in K. Then*

(a) $F(\mathfrak{D}_i) \subseteq \mathfrak{D}_{\gamma(i)}$ and $V(\mathfrak{D}_i) \subseteq \mathfrak{D}_{\gamma(i)}$;

(b) writing $F_i[p], V_i[p] := F|_{\mathfrak{D}_i[p]}, V|_{\mathfrak{D}_i[p]} : \mathfrak{D}_i[p] \to \mathfrak{D}_{\gamma(i)}[p]$ we get that

$$\ker(F_i[p]) = \operatorname{im}(V_{\gamma(i)}[p]),\ \ker(V_i[p]) = \operatorname{im}(F_{\gamma(i)}[p]);$$

(c) $\dim_k \left(\mathfrak{D}_i/V(\mathfrak{D}_{\gamma(i)})\right) = m_i$, $\dim_k (\mathfrak{D}_i[p]) = m_1 + m_2$, and $\dim_k \left(V(\mathfrak{D}_{\gamma(i)})\right) = \dim_k (\ker(F_i)) = m_{i+1}$ for $i \in \{1, 2\}$;

(d) under the pairing $\langle\ ,\ \rangle : \mathfrak{D} \times \mathfrak{D} \to k$ induced by $\lambda : \mathfrak{D} \to \mathfrak{D}^\vee$, $\mathfrak{D}_i \perp \mathfrak{D}_i$;

(e) when $\gamma(i) = i$, $\mathfrak{D}_i[p] \cong \mathfrak{D}_{i+1}[p]^\vee$ as Dieudonné modules.

Proof. (a) This is immediate as $\sigma \circ \chi_i = \chi_{\gamma(i)}$.

(b) By combining $\ker(F)[p] = \operatorname{im}(V)[p]$ with (a) we have that $F = F_1 + F_2$ and $V = V_1 + V_2$, and hence

$$\ker(F_1[p]) \oplus \ker(F_2[p]) = \ker(F[p]) = \operatorname{im}(V[p]) = \operatorname{im}(V_1[p]) \oplus \operatorname{im}(V_2[p]).$$

This necessarily corresponds to the decomposition of $\mathfrak{D}[p]$ as an $W(k)_1 \oplus W(k)_2$-module. Therefore $\ker(F_i[p]) = \operatorname{im}(V_{\gamma(i)}[p])$ and $\ker(V_i[p]) = \operatorname{im}(F_{\gamma(i)}[p])$.

(c) The identity

$$\operatorname{Lie}(A(p)) = \mathfrak{D}/V\mathfrak{D} = \mathfrak{D}_1/V(\mathfrak{D}_{\gamma(1)}) \oplus \mathfrak{D}_2/V(\mathfrak{D}_{\gamma(2)})$$

corresponds to the decomposition of Lie $(A(p))$ as an $\mathcal{O}_K \otimes W(k) = W(k)_1 \oplus W(k)_2$-module. By the determinant condition for Lie $(A(p))$, $\dim_k \left(\mathfrak{D}_i / V(\mathfrak{D}_{\gamma(i)})\right) = m_i$ for $i \in \{1, 2\}$. Combining the determinant condition with (b) gives the desired result.

(d) For any $r \in \mathcal{O}_K$, we have that

$$\begin{aligned} \langle r \cdot x, y \rangle &= \lambda(y)(r \cdot x) \\ &= r^\vee \circ \lambda(y)(x) \\ &= \lambda \circ \bar{r}(y)(x) \\ &= \lambda(\bar{r} \cdot y)(x) \\ &= \langle x, \bar{r} \cdot y \rangle \end{aligned}$$

for all $x, y \in \mathfrak{D}$. In particular, if we take $r \in \mathcal{O}_K$ such that $r \otimes 1 \mapsto (1, 0)$ under $\chi_1 : \mathcal{O}_K \otimes W(k) \to W(k)_1 \oplus W(k)_2$, then

$$\langle x, y \rangle = \langle r \cdot x, r \cdot y \rangle = \langle x, \bar{r} r \cdot y \rangle = 0, \qquad \forall x, y \in \mathfrak{D}_1.$$

Therefore $\mathfrak{D}_1 \perp \mathfrak{D}_1$. A similar argument shows that $\mathfrak{D}_2 \perp \mathfrak{D}_2$.

(e) This is a restatement of Lemma 2.2.1. Note that the pairing induced by λ on $\mathfrak{D}$ becomes a perfect paring $\langle \ , \ \rangle$ on $\mathfrak{D}_1[p] \times \mathfrak{D}_2[p]$. □

2.2.4 Isocrystals

Let k be a perfect field of characteristic p. A ($Q(k)$-)**isocrystal**, (P, F), is a finite dimensional $Q(k)$-vector space P together with a σ-linear automorphism F. The **height** of an isocrystal (P, F) is the dimension of P a $Q(k)$-vector space. An **isomorphism** of isocrystals is an isomorphism of vector spaces preserving F.

If H is a p-divisible group then its Dieudonné module gives rise to the associated $Q(k)$-isocrystal $(P, F) = (\mathfrak{D}(H) \otimes_{W(k)} Q(k), F \otimes \mathrm{id})$.

Given an isogeny of p-divisible groups, $f : H \to H'$, applying the Dieudonné functor gives rise to an injective morphism of Dieudonné modules $\mathfrak{D}(f) : \mathfrak{D}(H) \to \mathfrak{D}(H')$. Since $\mathfrak{D}(H)$ and $\mathfrak{D}(H')$ have the same rank (as f is an isogeny), tensoring with $Q(k)$ produces an isomorphism of the resulting isocrystals. On the other hand, if $\phi : \mathfrak{D}(H) \otimes_{W(k)} Q(k) \to \mathfrak{D}(H') \otimes_{W(k)} Q(k)$ is an isomorphism, there exists an $m \in \mathbb{N}$ such that $\phi(\mathfrak{D}(H)) \subset p^{-m}\mathfrak{D}(H')$, making $p^m\phi : \mathfrak{D}(H) \to \mathfrak{D}(H')$ correspond to an isogeny from H to H'. In this way, we see that the category of isocrystals up to isomorphism is equivalent to the category of p-divisible groups up to isogeny.

Furthermore, if (H, ι, λ) is a p-divisible group with $\mathscr{D}$-structure, then its associated isocrystal comes with the extra structures that simply come from tensoring with $Q(k)$, and hence,

- P has height $\dim_{\mathbb{Q}}(V)$.
- P has an induced $\mathcal{O}_B$-action $*$ and sympletic form $\langle \ , \ \rangle$ coming from ι and λ respectively such that $\langle bm, n\rangle = \langle mb^*n\rangle$ for all $b \in \mathcal{O}_B$ and $m, n \in \mathfrak{D}(H)$.
- F is a $Q(k) \otimes \mathcal{O}_B$-linear isomorphism respecting the symplectic form.

2.2.4.1 Classification of isocrystals using slope sequences

There is a combinatorial approach to classifying isogeny classes of p-divisible groups via isomorphism classes of isocrystals using what will be called *slope sequences*. This idea will be generalized to p-divisible groups and isocrystals with $\mathscr{D}$-structure in Section 5. For now, we give the classical theorem by Manin classifying isogeny classes of p-divisible groups *without* additional structures. For this section let k be algebraically closed.

Let $\lambda = \frac{s}{r} \in \mathbb{Q}_{\geq 0}$, $r \geq 1$ and $(r, s) = 1$. Define an associated isocrystal $P^{\lambda} := (Q(k)^r, M \cdot \sigma)$ where M is the $r \times r$ matrix of the form

$$\begin{pmatrix} 0 & \dots & \dots & 0 & p^s \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

For $\lambda \in \mathbb{Q} \leq 0$, define P^{λ} to be the dual of $P^{-\lambda}$.

Theorem 2.2.5. *[Dem72, p. 85] The category of isocrystals over k is semi-simple with simple objects given by P^{λ} where $\lambda = \frac{s}{r} \in \mathbb{Q}$ and $(r, s) = 1$.*

A $\lambda \in \mathbb{Q}$ appearing in the decomposition of P is called a **slope**. For any isocrystal P, the **multiplicity** of $\lambda \in \mathbb{Q}$ for P is the $Q(k)$-dimension of isotypic component of P with slope λ. For example, the isocrystal P^{λ} where $\lambda = \frac{s}{r}$ has slope λ with multiplicity r. The **slope sequence** of an isocrystal $P \cong \oplus_{i=1}^{h} P^{\lambda_i}$ is the sequence

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_h$$

such that each slope appears according to its multiplicity. Note that h will be equal to the dimension of P over $Q(k)$. The **slope sequence** of a p-divisible group is the slope sequence of its isocrystal.

The **Newton polygon** of P with slope sequence

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_h$$

is the polygon formed from the coordinates $(0,0), \ldots, (i, \lambda_1 + \ldots + \lambda_i)$ for $1 \leq i \leq h$. Notice that the polygon has integral break points and the slopes of its segments are the λ_is. Furthermore, the Newton polygon of an isocrystal is uniquely determined by its slope sequence and vice versa.

Newton polygons give rise to a poset structure on the set of isocrystals up to isomorphism by setting $P \geq P'$ if and only if $\sum_{i=1}^{r} \lambda_i \leq \sum_{i=1}^{r} \lambda'_i$ for all $1 \leq r \leq h$ if and only if the Newton polygon of P is below the Newton polygon of P' (that is, no point of the Newton polygon of P is strictly above the Newton polygon of P'). The Newton polygon of a p-divisible group is the Newton polygon of its isocrystal.

The slopes of an isocrystal arising from a p-divisible group fall in the interval $[0,1]$ [Dem72, p. 88] giving the following corollary of 2.2.5.

Corollary 2.2.6. *Every p-divisible group H over k is isogenous to a p-divisible group of the form $\oplus_{\lambda \in \mathbb{Q} \cap [0,1]} (H^{\lambda})^{m_{\lambda}}$ where $m_{\lambda} \in \mathbb{N}$ and H^{λ} denotes the p-divisible group whose isocrystal is P^{λ}.*

In other words, the isogeny classes of p-divisible groups over k of a fixed height h can by classified by the set of slope sequences of the form

$$0 \leq \lambda_1 \leq \ldots \leq \lambda_h \leq 1.$$

Equivalently, isogeny classes of p-divisible groups correspond to Newton polygons P such that

- P starts at $(0,0)$ and end at $(h, h-d)$ where h is the height and d is the dimension of the corresponding p-divisible groups,
- P is lower convex,

- every slope λ of P is between 0 and 1.

Furthermore, if H has slope sequence $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_h$, then its Serre dual H^t has slope sequence $1-\lambda_h \leq 1-\lambda_{h-1} \leq \ldots \leq 1-\lambda_1$.

Since the p-divisible group of an abelian variety $A(p)$ is in the same isogeny class as $A(p)^t$, the isogeny classes of p-divisible groups coming from abelian varieties correspond to Newton polygons P such that

- P starts at $(0,0)$ and end at $(2g, g)$ where g is the dimension of the abelian variety,
- P is lower convex,
- every slope λ of P is between 0 and 1,
- the Newton polygon is **symmetric** (*i.e.* $\lambda_i = 1-\lambda_{2g-i+1}$ for all $1 \leq i \leq 2g$).

There are two extremes in terms of possible slope sequences for $A(p)$: at one extreme, all the slopes are equal to $1/2$ and A is called **supersingular**; at the other extreme, $A(p)$ has slope sequence $(0^g, 1^g)$ and A is called **ordinary**.

2.2.5 Deformation theory via displays

Let k be a field of characteristic p. There is a well-known equivalence of categories between local deformations in characteristic p of an abelian variety A/k and local deformations of its p-divisible group $A(p)$ by the Serre-Tate theorem. In this context a local deformation is taken to mean deformations over local artinian rings $(R, \mathbf{m}_R)$ that are $W(k)$-algebras together with a fixed isomorphism $R/\mathbf{m}_R \cong k$. The local deformation functor for a point $\underline{A} \in \mathcal{M}(k)$ where k is field of characteristic p is pro-representable by the completed local ring $\hat{\mathcal{O}}_{\mathcal{M},\underline{A}}$, its **universal deformation ring**. We have already seen that there is an equivalence of categories between p-divisible groups and Dieudonné modules, and we would like to study the deformations of structures on A via the deformations of the corresponding Dieudonné modules. Techniques of Norman [Nor75], Norman-Oort [NO80] and Zink [Zin02] allow one to study the local deformations of A via the theory of so-called displays through a particular choice of basis for the Dieudonné module. We now outline a simplified version of displays that is suitable for our purposes.

Let G be a p-divisible group over a perfect ring R of characteristic p, and let $\mathfrak{D} = \mathfrak{D}(G)$ be its covariant Dieudonné module. Then a **displayed basis** for $\mathfrak{D}$ is a set of generators $\{e_1, \ldots, e_{d+n}\}$ for $\mathfrak{D}$ as a $W(R)$-module such that there exist $\alpha_{ij} \in W(R)$ satisfying:

- $F(e_j) = \sum_{i=1}^{d+n} \alpha_{ij} e_i, j \in \{1, \ldots, d\}$,
- $e_j = V(\sum_{i=1}^{d+n} \alpha_{ij} e_i), j \in \{d+1, \ldots, n\}$.

The matrix

$$(\alpha_{ij}) := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is called a **displayed matrix** for $\mathfrak{D}$. The matrix (α_{ij}) is invertible and F is given by

$$\begin{pmatrix} A & pB \\ C & pD \end{pmatrix}.$$

For an abelian variety over a field k, the matrix $A \pmod p$ from displayed matrix of its Dieudonné module is called the **Hasse-Witt matrix**, and it is the matrix of F on $\mathfrak{D}/V\mathfrak{D}$.

The universal local deformation ring for a p-divisible group over k of height h dimension and d is isomorphic to $k[[t_{ij} : 1 \leq i \leq d, 1 \leq j \leq h-d]]$, and its universal displayed matrix has the form

$$\begin{pmatrix} A+TC & B+TD \\ C & D \end{pmatrix}$$

where $T = (t_{ij})_{1 \leq i,j \leq g}$. The deformation of F on $\mathfrak{D}/V\mathfrak{D}$ is then captured by the universal Hasse-Witt matrix $A + TC \pmod{p}$, which will be studied in Chapter 7.

In case G is a p-divisible group coming from an abelian variety of dimension g, the universal deformation ring respecting the polarization structure has the form $k[[t_{ij} : 1 \leq i < j \leq g]]$ and the condition on the universal display is that T is symmetric.

We now consider what happens if we want to incorporate endomorphism structure coming from $\mathcal{O}_K$ as well. Let k be algebraically closed, and let the Dieudonné modules and displays below be considered over k.

Lemma 2.2.7. *Suppose p is split in K. Then there exists a displayed basis for the Dieudonné module $\mathfrak{D}$ of $A(p)$ of the form $\{e_1, \ldots, e_g; f_1, \ldots, f_g\}$ such that*

- $\mathcal{B}_1 = \{e_1, \ldots, e_{m_1}, f_{m_1+1}, \ldots, f_g\}$ *is a basis for* $\mathfrak{D}_1$*, and* $\mathcal{B}_2 = \{e_{m_1+1}, \ldots, e_g, f_1, \ldots, f_{m_1}\}$ *is a basis for* $\mathfrak{D}_2$
- $V(\mathfrak{D}) = \operatorname{span}\{f_1, \ldots, f_g\}$,
- $\langle e_i, f_j \rangle = \delta_{ij} = -\langle f_j, e_i \rangle$

and the displayed matrix for $\mathfrak{D}$ has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & 0 & 0 & B_1 \\ 0 & A_2 & B_2 & 0 \\ 0 & C_2 & D_2 & 0 \\ C_1 & 0 & 0 & D_1 \end{pmatrix}$$

with respect to the basis $\{e_1, \ldots, e_g; f_1, \ldots, f_g\}$, and satisfies the relation

$$\begin{pmatrix} A & pB \\ C & pD \end{pmatrix} \begin{pmatrix} -pD^t & pB^t \\ C^t & -A^t \end{pmatrix} = pId_{2g \times 2g}. \tag{2.3}$$

Furthermore, the universal display of $\mathfrak{D}$ preserving $\mathcal{O}_K$-action and prime-to-p polarization has the form

$$\begin{pmatrix} A+TC & B+TD \\ C & D \end{pmatrix}$$

where

$$T = \begin{pmatrix} 0_{m_1} & M \\ {}^tM & 0_{m_2} \end{pmatrix}.$$

Proof. The construction of a basis satisfying the desired properties is immediate from Proposition 2.2.4. Since we have a symplectic basis for $\mathfrak{D}$, we can obtain $V = \begin{pmatrix} -pD^t & pB^t \\ C^t & -A^t \end{pmatrix}$ as a σ^{-1}-linear operator from $F = \begin{pmatrix} A & pB \\ C & pD \end{pmatrix}$ (a σ-linear operator) via the relation $\langle Fx, y\rangle = \langle x, Vy\rangle^\sigma$ so that $F \circ V = [p]$ gives relation (2.3).

Consider now the universal display. In order to preserve polarization, we require that $T = T^t$ by work of Norman [Nor75] and Norman Oort [NO80]. Furthermore, in order to preserve the $\mathcal{O}_K$-structure we require that the lifted

$\mathcal{O}_K$-action commutes with F; *i.e.* we need $F\Sigma(a)^\sigma = \Sigma(a)F$ for all $a \in \mathcal{O}_K$ where

$$F = \begin{pmatrix} A & pB \\ C & pD \end{pmatrix}$$

and

$$\Sigma(a) = \mathrm{diag}(\overbrace{\sigma_1(a), \ldots \sigma_1(a)}^{m_1}, \overbrace{\sigma_2(a), \ldots, \sigma_2(a)}^{g}, \overbrace{\sigma_1(a), \ldots, \sigma_1(a)}^{m_2}).$$

Since $\mathcal{M}$ is known to be smooth of dimension $m_1 m_2$ the result follows from the calculation. □

Lemma 2.2.8. *Suppose p is inert in K. Then there exists a displayed basis for the Dieudonné module $\mathfrak{D}$ of $A(p)$ of the form $\{e_1, \ldots, e_g; f_1, \ldots, f_g\}$ such that*

- $\mathcal{B}_1 = \{e_1, \ldots, e_{m_1}, f_{m_1+1}, \ldots, f_g\}$ *is a basis for $\mathfrak{D}_1$, and* $\mathcal{B}_2 = \{e_{m_1+1}, \ldots, e_g, f_1, \ldots, f_{m_1}\}$ *is a basis for $\mathfrak{D}_2$,*
- $V(\mathfrak{D}) = \mathrm{span}\,\{f_1, \ldots, f_g\}$,
- $\langle e_i, f_j \rangle = \delta_{ij} = -\langle f_j, e_i \rangle$

and the displayed matrix for $\mathfrak{D}$ has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & A_1 & B_1 & 0 \\ A_2 & 0 & 0 & B_2 \\ C_2 & 0 & 0 & D_2 \\ 0 & C_1 & D_1 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, \ldots, e_g; f_1, \ldots, f_g\}$ and satisfies the relation

$$\begin{pmatrix} A & pB \\ C & pD \end{pmatrix} \begin{pmatrix} -pD^t & pB^t \\ C^t & -A^t \end{pmatrix} = pId_{2g\times 2g}. \tag{2.4}$$

Furthermore, the universal display of $\mathfrak{D}$ preserving $\mathcal{O}_K$-action and prime-to-p polarization has the form

$$\begin{pmatrix} A + TC & B + TD \\ C & D \end{pmatrix}$$

where

$$T = \begin{pmatrix} 0_{m_1} & M \\ {}^tM & 0_{m_2} \end{pmatrix}.$$

Proof. This result follows from Proposition 2.2.4 in a similar manner to the proof of the case where p is split in K given above (Lemma 2.2.7). □

We are now ready to begin our examination of the Ekedahl-Oort stratification of unitary Shimura varieties.

Chapter 3
The Ekedahl-Oort stratification

The Ekedahl-Oort stratification is way to decompose $\mathcal{M}$ using the isomorphism type of the p-torsion group schemes $A[p]$ that arise from $\underline{A} \in \mathcal{M}(k)$, where k is an algebraically closed field of characteristic p. It is important to note that the isomorphism type takes into account the extra structures on $A[p]$ coming from the endomorphism and polarization structures on A as described in Section 2.2.1. In order to be precise, let a **stratification** be a partition of a scheme into locally closed subschemes called **strata**. In this chapter, we will define the strata that arise in the Ekedahl-Oort stratification, their connection to Weyl group cosets, and explore the structure of the Ekedahl-Oort stratification in detail for the unitary Shimura varieties that come from the Shimura datum described in Section 2.1.1.

Throughout the remainder of this thesis, we will be most interested in k-points of $\mathcal{M}$ where k is an algebraically closed field of characteristic p. As such, let k be an algebraically closed field of characteristic p throughout the rest of this chapter and even thesis, except where otherwise indicated (Section 6.4).

3.1 Combinatorics of Coxeter systems

As the Ekedahl-Oort stratification will be parametrized by Weyl group cosets, it is necessary to begin by covering some combinatorial details of Coxeter systems. This section only states basic facts; proofs of these statements and further details may be found in Chapters 1 and 2 of [BB05].

A **Coxeter system** is a pair (W, S) such that W is a group with a presentation

$$\langle s_1, s_2, \ldots, s_n \mid (s_i s_j)^{r_{ij}} = 1 \rangle ,$$

where $S = \{s_1, \ldots s_n\} \subset W$ and $r_{ij} \geq 2$ for $i \neq j$ and $r_{ii} = 1$ for all $i = 1, \ldots, n$. We call W the **Coxeter group** and S the set of **simple reflections**. The set

$$T = \left\{ wsw^{-1} \;\middle|\; s \in S, w \in W \right\}$$

is called the set of **reflections**. In general, Coxeter groups may be infinite, however, all of the Coxeter groups that we will consider will be finite.

We define the **length**, $\ell(w)$, of an element $w \in W$ to be the length of a shortest expression for w as an S-word. If $w = s_1 \ldots s_{\ell(w)}$ then the word $s_1 \ldots s_{\ell(w)}$ is called a **reduced word** for w. Note that a reduced word representation need not be unique.

There is an partial order on W called the **Bruhat order** defined as follows: for $u, w \in W$, $u \leq w$ if there exists a sequence

$$u_0 = u, u_1, \ldots, u_m = w$$

of $u_i \in W$ such that $\ell(u_{i-1}) \leq \ell(u_i)$ and $u_{i-1}^{-1} u_i$ is a reflection (*i.e.* an element of T) for $i = 1, \ldots, m$. It immediately follows that if $u \leq w$ but $w \not\leq u$, then $\ell(u) < \ell(w)$. The identity element e is the unique minimal element under the Bruhat order. To see this, write any w as one of its reduced word representations, $w = s_1 \ldots s_m$. Then the sequence $u_i = s_1 \ldots s_i$ is such that $\ell(u_i) = \ell(u_{i-1}) + 1$ and $u_{i-1}^{-1} u_i = s_i \in S \subseteq T$ for all $i = 1, \ldots, m$. The order makes W a directed poset with a unique maximal element which we denote by w_0 [BB05, Proposition 2.3.1]. We now highlight some of the properties of w_0 as they will be used extensively as facts throughout the rest of this thesis.

Proposition 3.1.1. *The element w_0 has the following properties:*

- $w_0^2 = e$,
- $\ell(w_0) = \# T$,
- $\ell(w_0 w) = \ell(w w_0) = \ell(w_0) - \ell(w)$ *for all* $w \in W$,
- $\ell(w_0 w w_0) = \ell(w)$ *for all* $w \in W$,
- *the maps $w \mapsto w_0 w$ and $w \mapsto w w_0$ are automorphisms of Coxeter groups that reverse the Bruhat order,*
- *the map $w \mapsto w_0 w w_0$ is an automorphism of the Coxeter group (W, S) that respects the Bruhat order.*

Proof. [BB05] Propositions 2.3.2 and 2.3.4. □

Let $J \subseteq S$. Let W_J denote the subgroup of W generated by J and let $w_{0,J}$ be its unique maximal element. Each coset of $W_J \backslash W$ has a unique minimal length representative which gives rise to a bijection

$$W_J \backslash W \leftrightarrow {}^J W := \{w \in W \mid w < sw \ \forall s \in J\}$$

under the identification taking a coset $W_J w$ to its minimal length representative [BB05, Proposition 2.4.4].

The set ${}^J W$ inherits a poset structure from W as ${}^J W \subset W$. The poset ${}^J W$ is also a directed poset. This follows from the facts that W is directed and the projection map $W \to W_J \backslash W \to {}^JW$ is order preserving [BB05, Proposition 2.5.1]. Call the induced order on ${}^JW$ the Bruhat order denoted by $u \leq w$.

Example 3.1.2. Let $W = S_g$ be the symmetric group on g elements, and $S = \{s_i = (i, i+1) \mid i = 1, \ldots, g-1\}$ its subset of simple reflections. Then (W, S) is a Coxeter group. Writing permutations as $x = \left(\begin{smallmatrix} 1 & 2 & \cdots & g \\ x(1) & x(2) & \cdots & x(g) \end{smallmatrix}\right) \in S_g$ or more compactly, $x = [x(1)\, x(2) \cdots x(g)]$, the maximal element of (W, s) is given by the reversal permutation $w_0 = [g\, g-1 \ldots 1]$.

There is a convenient description of ${}^JW$ in case $J = S \setminus \{s_j\}$ for $s_m \in S$. In particular,

$$ {}^JW = \left\{ x \in S_g \;\middle|\; x^{-1}(1) < \cdots < x^{-1}(m) \text{ and } x^{-1}(m+1) < \cdots < x^{-1}(g) \right\}. $$

Note that in the extreme case where $m = 1$,

$$ {}^JW = \left\{ x \in S_g \;\middle|\; x^{-1}(m+1) < \cdots < x^{-1}(g) \right\} $$

and when $m = g - 1$,

$$ {}^JW = \left\{ x \in S_g \;\middle|\; x^{-1}(1) < \cdots < x^{-1}(m) \right\}. $$

Therefore, the set ${}^JW$ can be identified with the set of **shuffles** of the sequences $1, \ldots, m$ and $m+1, \ldots, g$ (*i.e.* any linear order of $1, \ldots, g$ that preserves the order of $1, \ldots, m$ and $m+1, \ldots, g$). It follows that the size of ${}^JW$ is $\binom{g}{m}$. Furthermore, the length of an element $x \in {}^JW$ is given by

$$ \ell(x) = \sum_{a=1}^{m} (x^{-1}(a) - a). $$

For example, if $g = 3$ and $m = 2$, the shuffles of the sequence $1, 2$ with the single element sequence 3 are $[123], [132], [312]$. These give precisely the elements of ${}^JW$ as $\left(\begin{smallmatrix}1&2&3\\1&2&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right)$. The lengths of these elements are $0, 1, 2$ respectively.

We will also need to use another partial order on ${}^JW$ derived from the Bruhat order. Let $\psi : (W, S) \to (W, S)$ be an automorphism of Coxeter systems. Then for $w, w' \in {}^JW$

$$w' \preccurlyeq_\psi w \iff \exists y \in W_J \text{ such that } yw'x\psi(y^{-1})x^{-1} \leq w \qquad (3.1)$$

where $x = w_0 w_{0,\psi(J)}$. This is a partial order by Corollary 6.3 of [PWZ11] (see also [He07, Proposition 3.13]).

Example 3.1.3. For (W, S) coming from the symmetric group as described above in Example 3.1.2, there are two standard automorphisms of the Coxeter system: the identity automorphism, and conjugation by the reverse permutation w_0. We now give examples showing how $\preccurlyeq_\psi$ differs from the Bruhat order in each of these cases.

If $\psi = 1$, then $\preccurlyeq_\psi$ is given by

$$w' \preccurlyeq_\psi w \iff \exists y \in W_J \text{ such that } yw'w_0 w_{0,J} y^{-1} w_{0,J} w_0 \leq w. \qquad (3.2)$$

In the example above, when $g = 3$ and $m = 2$, $\preccurlyeq_\psi$ is same as the Bruhat order. However, this is not always the case. For example, when $g = 6$ and $m = 3$, the length 2 element $w' = [124536]$ is less than the length 3 element $w = [412356]$

under $\preccurlyeq_\psi$, as $yw'w_0w_{0,J}y^{-1}w_{0,J}w_0 = [14325] \leq w$ when $y = [13245]$. But [124536] and [412356] are unrelated under the Bruhat order.

Suppose that $\psi(w) = w_0ww_0$, then for $J = S \setminus \{s_m\}$, $\psi(J) = S \setminus \{s_{g-m}\}$, and $w_0w_{0,J}w_0 = w_{0,\psi(J)}$. It follows that $w' \preccurlyeq_\psi w$ if and only if there exists a $y \in W_J$ such that

$$yw'(w_0w_{0,\psi(J)})(w_0y^{-1}w_0)(w_{0,\psi(J)}w_0) = yw'w_{0,J}y^{-1}w_{0,J} \leq w. \tag{3.3}$$

Again, in the example when $g = 3$ and $m = 2$, $\preccurlyeq_\psi$ is same as the Bruhat order. However, when $g = 5$ and $m = 3$, $[41235] \preccurlyeq_\psi [14523]$ but $[41235] \not\leq [14523]$. Here $\ell([41235]) = 3$ and $\ell([14523]) = 4$.

3.2 The relative position of two parabolics

Coxeter groups arise naturally in the theory of connected reductive groups, especially in the context of describing the *relative positions* of parabolic subgroups. We review some of these connections following [Bor91] Section IV.14.

Recall that k is an algebraically closed field, and let G be a connected reductive algebraic group k. Fix a maximal torus T and a Borel B containing T. Then the Weyl group $W = W(T,G) := N_G(T)/Z_G(T)$ of G relative to T can be generated by a subset of simple reflections S corresponding to B—together, (W,S) is a Coxeter system.

Every subset $J \subseteq S$ corresponds to a parabolic containing B by letting $P_J = BW_JB$ for $J \neq \emptyset$ where W_J is the subgroup of W generated by J, and $P_\emptyset = B$. Every parabolic subgroup of G containing B corresponds to a subset $J \subseteq S$ in this way—such a parabolic P_J is called a **standard parabolic**. Recall that every parabolic subgroup of G is conjugate to a unique parabolic subgroup of G that contains B. A parabolic has **type** J if it is conjugate to the standard parabolic P_J. Furthermore, for $I, J \subseteq S$ there is a canonical bijection

$$W_I \backslash W / W_J \to P_I \backslash G / P_J$$

obtained by sending $W_I \cdot w \cdot W_J$ to $P_I \cdot w \cdot P_J$ induced by the Bruhat decomposition $G = \coprod_{w \in W} BwB$.

Let $\mathscr{B}$ denote the set of Borels of G, and let $\mathscr{B}^T$ denote the set of Borel subgroups containing T. Recall that the group of inner automorphisms of G acts transitively on the set of pairs (B', T') consisting of a Borel subgroup and a maximal torus $T' \subseteq B'$, and that the Weyl group acts simply transitively on $\mathscr{B}^T$ by conjugation by $N_G(T)$ [Bor91, Proposition IV.11.19]. For $g \in G$ and $H < G$, let ${}^gH$ denote gHg^{-1}.

The following definition plays a key role in the sequel. Consider $G \backslash (\mathscr{B} \times \mathscr{B})$ where $g \in G$ acts on $\mathscr{B} \times \mathscr{B}$ by $({}^gB, {}^gB)$. Let $(B_1, B_2) \in G \backslash (\mathscr{B} \times \mathscr{B})$ and let T' be a torus contained in both B_1 and B_2. Then by conjugacy of tori, there is a $g \in G$ such that ${}^gT' = T$. Thus ${}^gB_1$ and ${}^gB_2$ are both in $\mathscr{B}^T$ and there exists an $h \in W$ such that ${}^{hg}B_1 = B$. It follows that $[B_1, B_2] = [{}^{hg}B_1, {}^{hg}B_2] = [B, {}^wB]$ for some $w \in W$. By the Bruhat decomposition, this gives a bijection $W \to G \backslash (\mathscr{B} \times \mathscr{B})$ via the map $w \mapsto (B, {}^wB)$. The inverse of this map will be denoted by $w(B_1, B_2)$, and the resulting Weyl group element $w(B_1, B_2)$ is called the **relative position** of B_1 and B_2.

Similarly, given two standard parabolics P_1, P_2 of types I, J respectively, we define an element $w(P_1, P_2) \in W_I \backslash W / W_J$, the **relative position** of P_1, P_2, by setting $w(P_1, P_2)$ to be the class of $w(B_1, B_2) \in W_I \backslash W / W_J$ where B_i is a Borel contained in P_i. Alternately, if ${}^g P_1 = P_I$ and ${}^h P_2 = P_J$, then $w(P_1, P_2)$ is the image of gh^{-1} under the bijection $P_I \backslash G / P_J \to W_I \backslash W / W_J$. See Section 3 of [Moo01] for more details.

3.3 The Ekedahl-Oort stratification

Let $\underline{A} = (A, \iota, \lambda, \bar{\eta}) \in \mathcal{M}(k)$ and let $\mathcal{D} = \mathcal{D}(A[p])$ be the contravariant Dieudonné module of $A[p]$. Then recall from Section 2.2.3 that by functoriality of the Dieudonné functor, $\mathcal{D}$ has the structure of a k-vector space with a symplectic form $\langle \ , \ \rangle$ coming from λ and a compatible $\mathcal{O}_K/(p)$-module structure with involution induced by ι (*i.e.* $\langle bm, n \rangle = \langle m, \bar{b}n \rangle$ for all $b \in \mathcal{O}_K/(p)$, $m, n \in \mathcal{D}$). Let G be the group of symplectic similitudes of $\mathcal{D} \cong k^{2g}$ respecting the $\mathcal{O}_K/(p)$-module structure.

By applying F, V^{-1} to $(0) \subset \mathcal{D}$ until it stabilizes, we obtain an F, V^{-1}-stable flag of $\mathcal{D}$,

$$\mathcal{C}_\bullet : \mathcal{C}_0 = (0) \subset \ldots \subset \mathcal{C}_g = \mathcal{D}[V] = F(\mathcal{D}) \subset \ldots \subset \mathcal{C}_{2g} = \mathcal{D}$$

where $\dim \mathcal{C}_i = i$, called the **canonical flag (of $\underline{A}$)**. This fact will be explored in more detail in Section 6.3, and follows from Lemma 6.3.1 when $S = \mathrm{Spec}(k)$. Observe that the canonical filtration is an $\mathcal{O}_K$-invariant symplectic flag. Refer to [Moo01, Sections 2.5, 4.4, 6.3] for more details on the canonical filtration.

Let any extension of $\mathcal{C}_\bullet$ to a complete $\mathcal{O}_K$-invariant symplectic flag of $\mathcal{D}$ be called a **conjugate flag (of $\underline{A}$)**.[1] Let $\mathcal{C}_\bullet$ denote a conjugate filtration of $\underline{A}$ and let $Q = \mathrm{Stab}(\mathcal{C}_\bullet) \subset G$. Since $\mathcal{C}_\bullet$ is a maximal flag, Q is a Borel, and its type is the empty set.

Let J be the type of the parabolic $P = \mathrm{Stab}(\mathcal{D}[F] \subset \mathcal{D}) \subset G$ (typically $\mathcal{D}[F]$ is not part of $\mathcal{C}_\bullet$). The type J of P is determined by the moduli problem and does not depend on the choice of $\underline{A}$ as $\mathcal{D}[F]$ is a maximal isotropic subspace of $\mathcal{D}$, and the $\mathcal{O}_K/(p)$-structure on $\mathcal{D}[F]$ in relation to $\mathcal{D}$ is fixed by the signature condition. We can associate to $\underline{A}$ an element $w(\underline{A}) := w(P,Q) \in W_J\backslash W/W_\emptyset = W_J\backslash W$. By construction, the element $w(\underline{A})$ measure the relative position of $\mathcal{D}[F]$ with respect to the canonical filtration $\mathcal{C}_\bullet$.

Proposition 3.3.1 (Theorem 6.7 [Moo01])**.** *The element $w(\underline{A}) \in W_J\backslash W$ is well-defined and does not depend on the choice of refinement of the canonical flag of $\underline{A}$. Furthermore, the element $w(\underline{A}) \in W_J\backslash W$ determines the isomorphism class of $A[p]$ as a p-torsion group scheme with polarization and endomorphism structure.*

From now on, identify $W_J\backslash W$ with ${}^JW$ as in Section 3.1. The **Ekedahl-Oort (E-O) stratum** associated to $w \in {}^JW$ is the locally-closed reduced subscheme V^w of $\mathcal{M}$ with geometric points given by

$$V^w := \{\underline{A} \in \mathcal{M} \mid w(\underline{A}) = w\}.$$

[1] A choice of such an extension is equivalent to choosing a Borel contained in the parabolic subgroup stabilizing the canonical flag.

The following Theorem combines Theorems 2 and 3 of [VW13]. Note that parts of this theorem were known prior to [VW13]. For instance, the result on dimension was previously shown by Moonen in [Moo04a] under the assumption of non-emptiness of the strata. Furthermore, non-singularity was shown by Vasiu in [Vas08].

Theorem 3.3.2. *[VW13] The Frobenius map on G given by the p^{th}-power map induces an automorphism of the Coxeter system, $\psi : (W, S) \to (W, S)$. Let $\preccurlyeq_\psi$ denote the partial order on ${}^JW$ as defined by Equation (3.1).*

- *For all $w \in {}^JW$ the E-O stratum V^w is non-empty and equidimensional of dimension $\ell(w)$ (where $\ell(w)$ is the length of w as a Weyl group element of W).*
- *The E-O strata are non-singular and quasi-affine.*
- *The closure of an E-O stratum is a union of E-O strata with respect to a partial order $\preccurlyeq_\psi$ on ${}^JW$. That is,*

$$\overline{V}^w = \coprod_{w' \preccurlyeq_\psi w} V^{w'}.$$

The unique 0-dimensional E-O stratum corresponding to the identity element of the Weyl group will be called the **core locus** in this thesis. This is non-standard notation. Note that in [VW13], the 0-dimensional E-O stratum is called the superspecial stratum; we choose to call the 0-dimensional stratum the core stratum as the underlying abelian varieties may or not be superspecial or even supersingular. The unique E-O stratum corresponding to the maximal element of the Weyl group is called the **μ-ordinary stratum**.

Remark. In [VW13] the language of so-called D-Zips is used to encode the isomorphism class of $\mathcal{D}$ with extra structure to produce an element of $w(\underline{A}) \in {}^J W$ corresponding to $\underline{A}$. In the case we are considering the procedures for obtaining $w(\underline{A})$ are equivalent.

3.4 E-O stratifications of unitary Shimura varieties

The k-points of the group G that appears from the unitary PEL datum of type (m_1, m_2) (see Section 2.1) is the group $GU(k^{2g}, \Psi)$ where Ψ is a Hermitian form. Such a group is isomorphic to $\mathrm{GL}_g(k) \times \mathbb{G}_m(k)$, and can be identified with the subgroup of $\mathrm{GL}_g(k) \times \mathrm{GL}_g(k)$,

$$GU(k^{2g}, \Psi) \cong \{(M, aM^\vee) \in \mathrm{GL}_g(k) \times \mathrm{GL}_g(k) \mid a \in k^*\}$$

where $M^\vee = (M^t)^{-1}$. Let B be the Borel subgroup of G given by the subset of elements of the form $(b, ab^\vee) \in G(k) \subseteq \mathrm{GL}_g(k) \times \mathrm{GL}_g(k)$ where b is upper triangular and $a \in k^*$. This isomorphism identifies the action of $b \in \mathcal{O}_K/(p)$ on $\mathcal{D}$ with the matrix $(\chi_1(b) I_g, \chi_2(b) I_g)$.

Using the description of $G(k) \subseteq \mathrm{GL}_g(k) \times \mathrm{GL}_g(k)$, the Weyl group of G is $W = \{(w_1, w_2) \subseteq S_g \times S_g \mid w_2 = w_0 w_1 w_0\}$ where $w_0 = [g \;\; g-1 \ldots 2\, 1]$. The set of simple reflections corresponding to B is then identified with

$$S_B := \{(s_i, w_0 s_i w_0) \in S_g \times S_g \mid s_i = (i \; i+1)\}.$$

Observe that by projection onto the first coordinate $W \cong S_g$ (the Weyl group of $\mathrm{GL}_g(k)$) and S_B corresponds to the standard choice of simple reflections of S_g.

Lemma 3.4.1. *Let S denote the set of simple reflections of S_g given by $s_i = (i\ i+1)$ for $i = 1, \ldots, g-1$. The parabolic P stabilizing the flag $\mathcal{D}[F] \subset \mathcal{D}$ corresponds to the set of the simple reflections*

$$J = \{(s,t) \in S \setminus \{s_{m_1}\} \times S \setminus \{s_{m_2}\} \mid s = w_0 t w_0\} \subseteq S_B.$$

Proof. Let $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$ be the decomposition of $\mathcal{D}$ under its $\mathcal{O}_K/(p)$-action—note that this corresponds to the identification of $G(k)$ as a subset of $\mathrm{GL}_g(k) \times \mathrm{GL}_g(k)$. The elements $(g_1, g_2) \in P$ satisfy the property that g_i is in the stabilizer of the flag $\mathcal{D}_i[F] \subset \mathcal{D}_i$ for $i = 1, 2$. The stabilizer of $\mathcal{D}_1[F] \subset \mathcal{D}_1$ as a subgroup of $\mathrm{GL}(\mathcal{D}_1) \cong \mathrm{GL}_g(k)$ is a maximal parabolic P_1. Then $W(S_g, S)$ is the Coxeter group of $\mathrm{GL}(\mathcal{D}_1) \cong \mathrm{GL}_g(k)$ with respect to the upper triangular Borel of $\mathrm{GL}_g(k)$, and the type of P_1 (with respect to the set of simple reflections S) is $S \setminus \{s_d\}$ where $d = \dim \mathcal{D}_1[F]$. Similarly for $\mathcal{D}_2[F] \subset \mathcal{D}_2$, and since $\dim \mathcal{D}_i[F] = m_i$,

$$J \subseteq (S \setminus \{s_{m_1}\} \times S \setminus \{s_{m_2}\}) \cap S_B = \{(s,t) \in S \setminus \{s_{m_1}\} \times S \setminus \{s_{m_2}\} \mid s = w_0 t w_0\}.$$

On the other hand, since the standard parabolic subgroup of G with type

$$\{(s,t) \in S \setminus \{s_{m_1}\} \times S \setminus \{s_{m_2}\} \mid t = w_0 s w_0\}$$

preserves the flags $\mathcal{D}_i[F] \subset \mathcal{D}_i$, it also preserves $\mathcal{D}[F] = \mathcal{D}_1[F] \oplus \mathcal{D}_2[F] \subset \mathcal{D}$. Therefore, J is the type of the parabolic stabilizing the flag $\mathcal{D}[F] \subset \mathcal{D}$. □

Corollary 3.4.2. *${}^JW$ can be presented as the set*

$$\left\{(w_1, w_2) \in {}^{J_1}S_g \times {}^{J_2}S_g \;\middle|\; w_2 = w_0 w_1 w_0\right\}$$

where $J_i = S \setminus \{s_{m_i}\}$ *and* $w_0 = [g \;\; g-1 \ldots 2\, 1]$.

Recall from Example 3.1.2 that ${}^{J_i}S_g$ where $J_i = S \setminus \{s_{m_i}\}$ can be identified with the set of shuffles of $1, 2, \ldots, m_i$ with $m_i + 1, \ldots, g$ by writing the permutation $\left(\begin{smallmatrix} 1 & 2 & \cdots & g \\ x(1) & x(2) & \cdots & x(g) \end{smallmatrix}\right)$ as the shuffle $[x(1)x(2)\cdots x(g)]$. For $w = (w_1, w_2) \in {}^{J}W$, the length of w as an element in $W \subseteq S_g \times S_g$ is equal to $\ell(w_1) = \ell(w_0 w_1 w_0) = \ell(w_2)$ as (W, S_B) is isomorphic to (S_g, S) by projecting onto one of the coordinates. Therefore, for $w \in {}^{J}W$,

$$\ell(w) = \sum_{a=1}^{m_2} (w_1^{-1}(a) - a) = \sum_{a=1}^{m_1} (w_2^{-1}(a) - a) \tag{3.4}$$

by Example 3.1.2.

We now explore the poset structure on ${}^{J}W$ as defined in Theorem 3.3.2 that describes the closure relations on the E-O strata associated with the elements of ${}^{J}W$. In this case, $\bar{G}$ is defined over $\mathbb{F}_p$ and Frobenius is an automorphism of $\bar{G}$, that takes a k-point of $\bar{G}$ to its p^{th} power. Recall that the isomorphism of $G(k)$ with a subset of $\mathrm{GL}(k) \times \mathrm{GL}(k)$ identifies the action of $b \in \mathcal{O}_K/(p)$ on $\mathcal{D}$ with the matrix $B = (\chi_1(b)I_g, \chi_2(b)I_g)$. When p is split in K, $B^p = B$, but when p is inert in K, $B^p = (\chi_2(b)I_g, \chi_1(b)I_g)$. Therefore, when p is split, $\psi : (W, S_B) \to (W, S_B)$ is simply the identity, but when p is inert, $\psi : (W, S_B) \to (W, S_B)$ is given by the non-trivial automorphism $\psi(w) = w_0 w w_0$ which when $w = (w_1, w_2)$ means that $\psi(w) = (w_2, w_1)$. Thus, $\preccurlyeq_\psi$ can be described by using the two examples for ψ given in Example 3.1.3. From now on we will suppress the ψ in the notation, and use $\preccurlyeq$ for $\preccurlyeq_\psi$.

Example 3.4.3 (G has signature $(m_1, m_2) = (2, 1)$)**.** In this case:

- $S = \{s_1, s_2\} \subset S_3$
- $J_1 = S \setminus \{s_2\}, J_2 = S \setminus \{s_1\}$
- $w_0 = [321] = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right)$

Therefore ${}^{J_1}S_3$ is the set of shuffles of the sequence $1, 2$ with 3:

$$ {}^{J_1}S_3 = \{[123], [132] = s_2, [312] = s_2 s_1\}, $$

and similarly, ${}^{J_2}S_3$ is the set of shuffles of 1 with the sequence $2, 3$:

$$ {}^{J_2}S_3 = \{[123], [213] = s_1, [231] = s_1 s_2\}. $$

Now, ${}^{J}W = \left\{(w_1, w_2) \in {}^{J_1}S_3 \times {}^{J_2}S_3 \,\middle|\, w_2 = w_0 w_1 w_0\right\}$. Therefore,

$$ {}^{J}W = \{(312, 231), (132, 213), (123, 123)\}. $$

Remark. The use of braces is dropped for elements of ${}^{J}W$ so as to keep the notation from becoming too cumbersome. For instance, $(312, 231)$ means $([312], [231])$.

By Equation 3.4,

$$ \ell((312, 231)) = 2,\ \ell((132, 213)) = 1,\ \ell((123, 123)) = 0 $$

which implies that ${}^{J}W$ has minimal length S_B-word representatives of the form

$$ (312, 231) = (s_2 s_1, s_1 s_2), (132, 213) = (s_2, s_1), (123, 123) = (1_W, 1_W). $$

Thus the Bruhat order on ${}^{J}W$ is

$$ (123, 123) \leq (132, 213) \leq (312, 231). $$

Now, $W_J = \{(1_W, 1_W), (s_1, s_2)\}$, and $W_{w_0 J w_0} = \{(1_W, 1_W), (s_2, s_1)\}$ so that

$$x_1 = w_0 w_{0,J} = (231, 312), x_2 = w_0 w_{0,w_0 J w_0} = (312, 231).$$

Then for $y = (s_1, s_2) \in W_J$,

w'	$yw'x_1y^{-1}x_1^{-1}$	$yw'x_2w_0y^{-1}w_0x_2^{-1}$
$(123, 123)$	$(231, 312) = (s_2s_1, s_1s_2)$	$(123, 123)$
$(132, 213)$	$(213, 132) = (s_1, s_2)$	$(321, 321) = (s_2s_1s_2, s_1s_2s_1)$
$(312, 231)$	$(312, 231) = (s_2s_1, s_1s_2)$	$(231, 312) = (s_1s_2, s_2s_1)$

Therefore, for both p split and p inert, the partial order giving the closure relations on the E-O strata is given by

$$(123, 123) \preccurlyeq (132, 213) \preccurlyeq (312, 231)$$

Using the description of ${}^JW$ as shuffles, the following figures represent the poset structure on ${}^JW$ and give an idea of how the E-O stratification works in certain low dimensional examples. Here, the minimal element (core stratum) appears at the bottom of each diagram. The black edges represents $\leq$ in the Bruhat order, the red edges represent $\preccurlyeq$ but not $\leq$ where ψ is the identity (as in the case where p is split in K), and the blue edges represent $\preccurlyeq$ where ψ is $\mathrm{Int}(w_0)$ but not $\leq$ (the case where p is inert in K). The braces in the notation for permutations $[w(1) \dots w(g)]$ are dropped so as to make the diagrams easier to read.

In general, the problem of determining whether two elements are related under the Bruhat order is not clear; however, there exist effective methods for checking this relationship (for example, by application of Theorem 2.1.5 or Theorem 2.6.3 of [BB05]). The diagrams below were calculated by brute force: first by calculating the Bruhat order using the above method, then calculating $yww_0w_{0,\psi(J)}y^{-1}w_{0,\psi(J)}w_0$ for all $y \in W_J$ and $w \in {}^JW$, and finally checking the Bruhat order for new relations coming from $\preccurlyeq_\psi$.

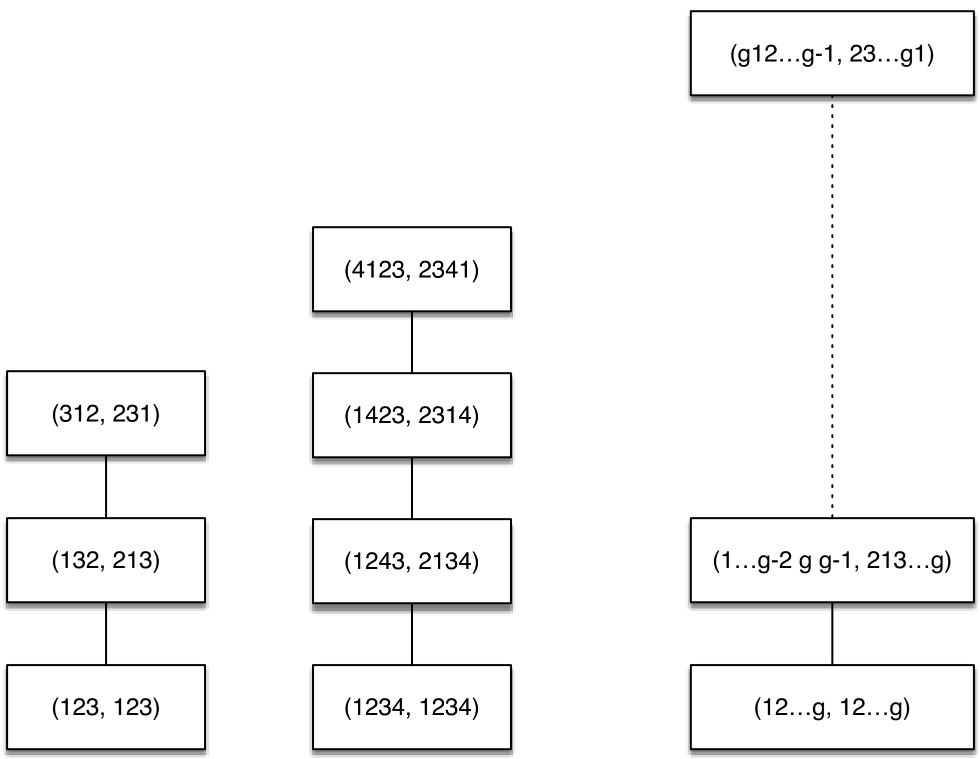


Figure 3–1: $\mathrm{GU}(2,1), \mathrm{GU}(3,1), \mathrm{GU}(g,1)$

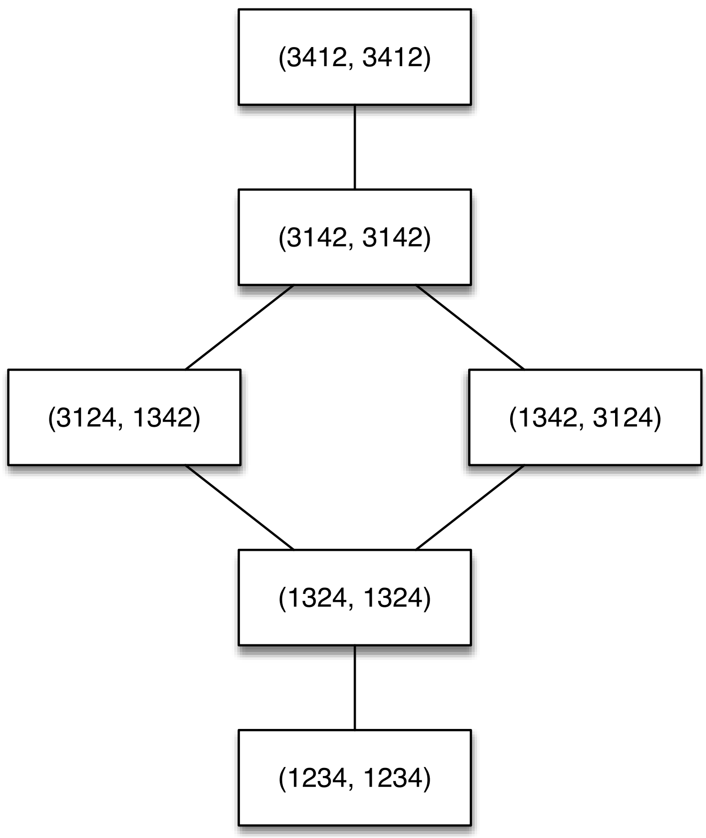


Figure 3–2: GU(2, 2)

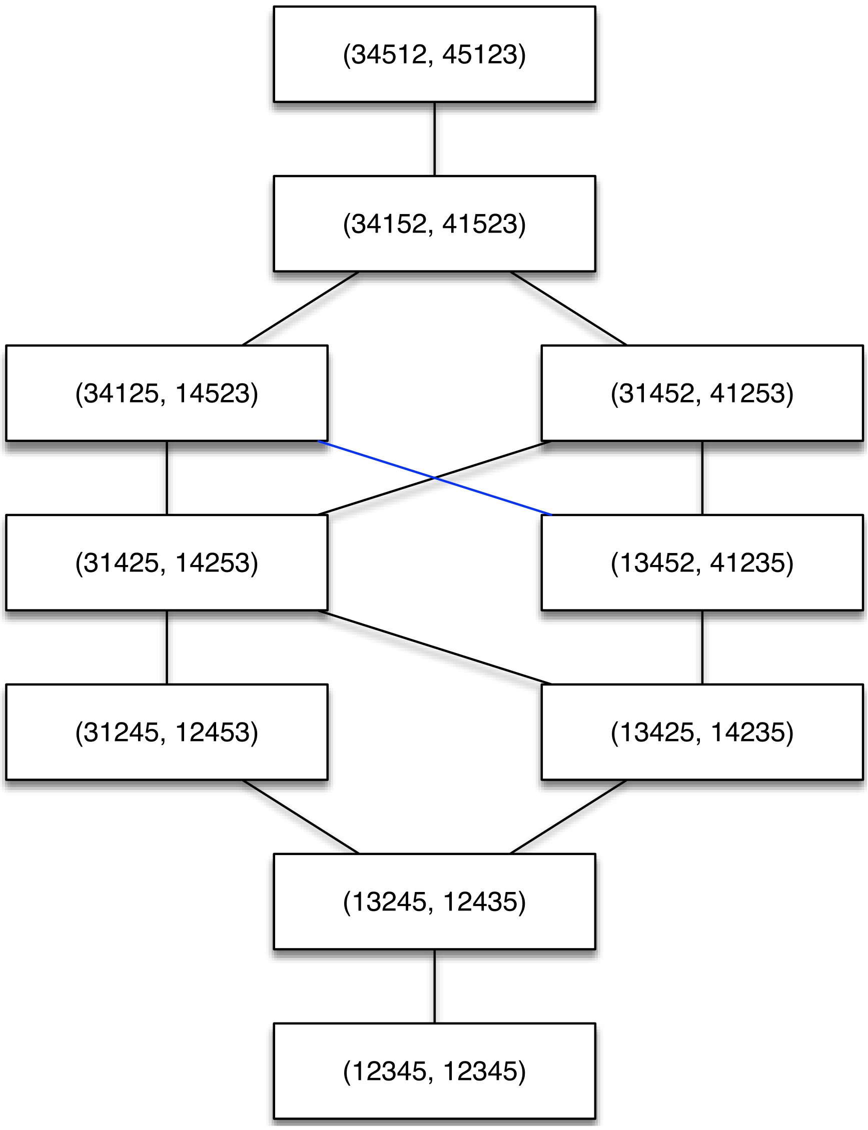


Figure 3–3: GU(3, 2)

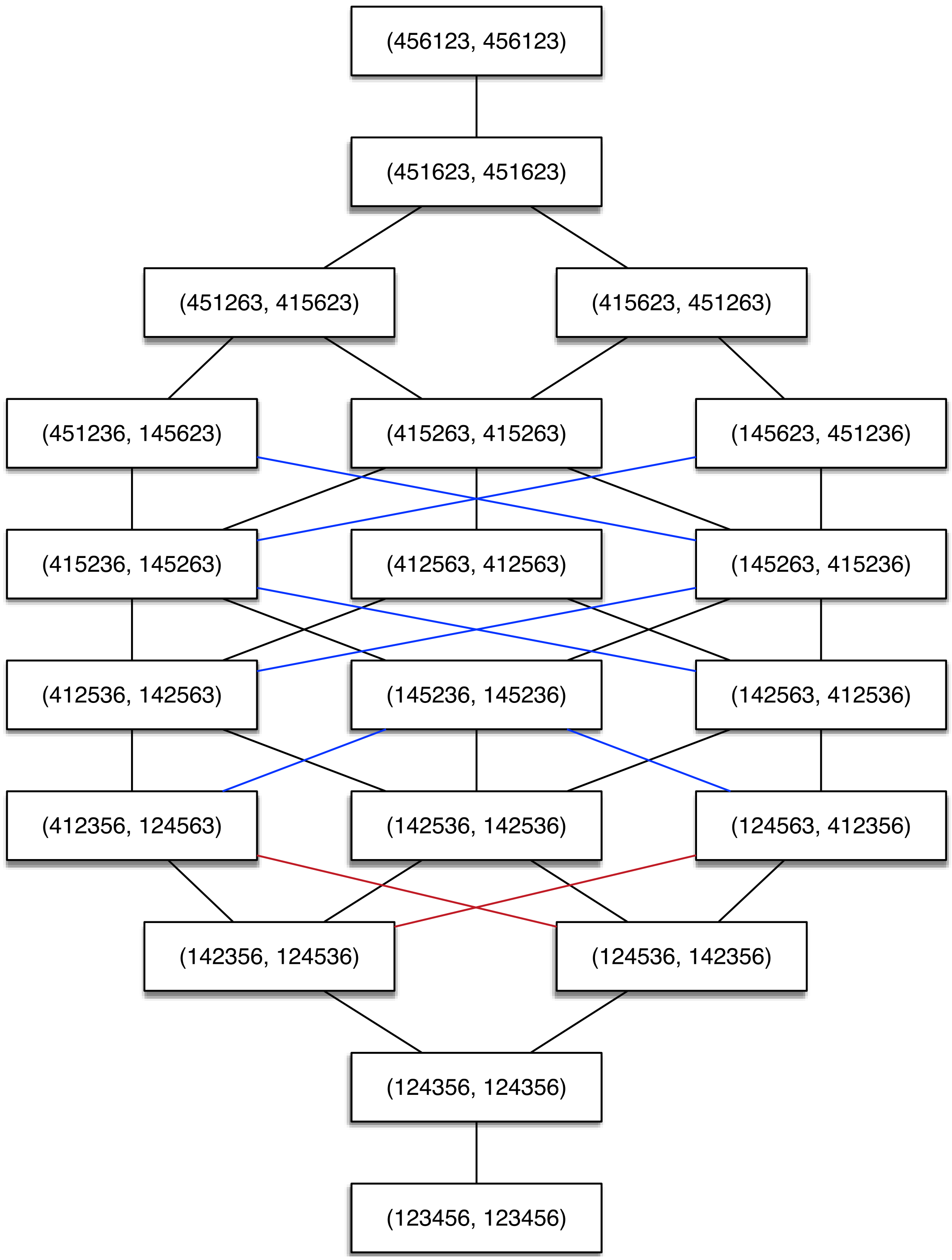


Figure 3–4: GU(3, 3)

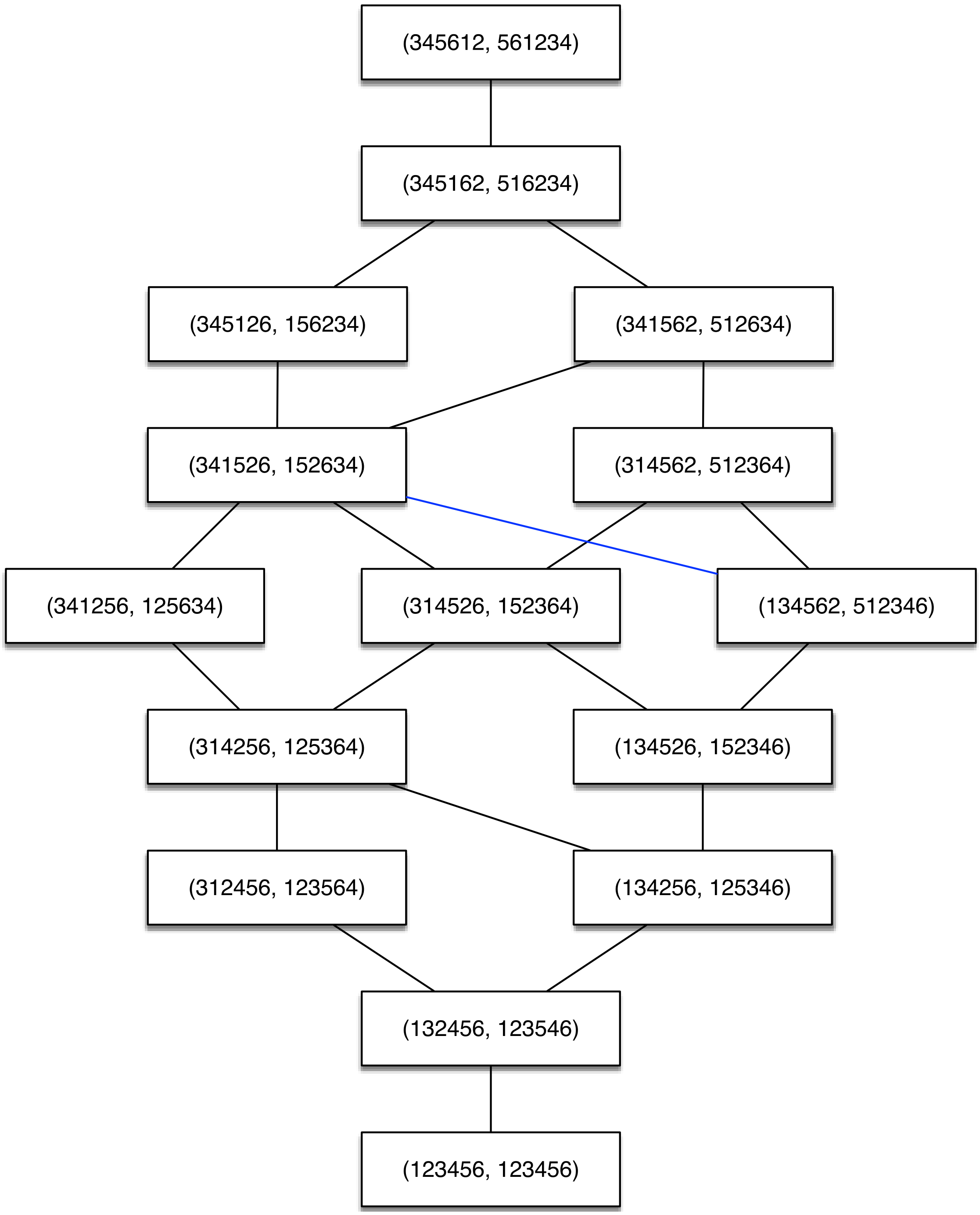


Figure 3–5: GU(4, 2)

Proposition 3.4.4. *The E-O stratification of a unitary Shimura variety of signature* (m_1, m_2) *has the following properties:*

1. *There are* $\binom{g}{m_1}$ *strata.*
2. *The number of strata of a given dimension* d *is equal to* $f(m, n, g)$ *where*

$$\sum f(m,n,g)s^m q^n = (1+sq)(1+sq^2)\dots(1+sq^g)$$

   $n = d + m_1(m_1+1)/2$, $g = m_1 + m_2$ *and* $m = m_1$.

Proof. Observe that a m-shuffle is completely determined by the choice of m positions for the first m elements since the $1, \dots, m$ and $m+1, \dots, g$ must be linearly ordered.[2] The first statement now follows immediately from the fact that there are $\binom{g}{m_1} = \binom{g}{m_2}$ shuffles of $1, 2, \dots, m_i$ with $m_i + 1, \dots, g$ for $i = 1, 2$.

The function

$$\sum f(m,n,g)s^m q^n = (1+sq)(1+sq^2)\dots(1+sq^g)$$

is the generating series for the number of partitions of n into m distinct parts that are less than or equal to g. In other words, we need to show that the number of strata of a given dimension d is equal to the number of partitions of $d + m_1(m_1 + 1)/2$ into m_1 distinct parts that are less than or equal to g.

Recall that for $w = (w_1, w_2) \in {}^JW$, the length of w as an element in $W \subseteq S_g \times S_g$ is equal to $\ell(w_1) = \ell(w_2)$ by considering w_1 and w_2 as elements in S_g.

[2] As a permutation, a choice of m distinct values between 1 and g for $w^{-1}(a)$ where $a = 1, \dots, m$ determines the element w in ${}^{S\backslash\{s_m\}}S_g$ since w is an m-shuffle.

Therefore, the number of strata of dimension d is equal to the number of elements of ${}^JW$ of length d which, as noted above, is equal to the number of elements of ${}^{J_i}S_g$ of length d. If $\ell(w) = d$, then for $i = 1, 2$,

$$d = \ell(w_i) = \sum_{a=1}^{m_i}(w_i^{-1}(a) - a) = \sum_{a=1}^{m_i} w_i^{-1}(a) - \frac{m_i(m_i+1)}{2}.$$

In other words, the number of elements of length d is the number of elements of S_g such that

$$d + \frac{m_i(m_i+1)}{2} = w_i^{-1}(1) + w_i^{-1}(2) + \ldots + w_i^{-1}(m_i)$$

where $0 < w_i^{-1}(1) < w_i^{-1}(2) < \ldots < w_i^{-1}(m_i) < g$. This is the number of partitions of $d + m_i(m_i+1)/2$ into m_i distinct parts that are less than or equal to g. $\square$

Corollary 3.4.5. *There are unique elements in ${}^JW$ of length $0, 1, m_1m_2 - 1$ and m_1m_2. In particular,*

1. *there is a unique 0-dimensional stratum, the* ***core locus****, corresponding to the identity element of ${}^JW$;*
2. *there is a unique 1-dimensional stratum corresponding to the element (w_1, w_2) where*

$$w_i = [1\ 2 \ldots m_i - 1\ m_i + 1\ m_i\ m_i + 2 \ldots g]$$

called the ***almost-core locus****;*

3. *there is a unique codimension 1 stratum corresponding to the element (w_1, w_2) where*

$$w_i = [m_i + 1\ m_i + 2\ \ldots g - 1\ 1\ g\ 2 \ldots m_i]\,,$$

called the ***almost-ordinary locus***[3]

4. *the* μ***-ordinary locus*** *has dimension* $m_1 m_2$ *and corresponds the element* $w_{0,J} = (w_1, w_2)$ *where*

$$w_i = [m_i + 1 \; m_i + 2 \ldots g \; 1 \; 2 \ldots m_i] \,.$$

Proof. As ${}^J W$ is a directed poset under the Bruhat order, it follows directly from Proposition 3.4.4 that there is a unique 0-dimensional stratum (the core locus), and a unique maximal-dimensional stratum, the μ-ordinary locus.

Furthermore, from Proposition 3.4.4 the elements in ${}^J W$ of length d are related to partitions of $d + m_i(m_i+1)/2$. In case $d = 0$, the number of partitions of $m_i(m_i+1)/2$ into m_i distinct parts is exactly 1 as $m_i(m_i+1)/2 = \sum_{a=1}^{m_i} a$, which corresponds to the identity permutation as previously expected. When $d = 1$, the only way to partition $1+\sum_{a=1}^{m_i} a$ into m_i distinct parts is by taking $1+2+\ldots+(m_i-1)+(m_i+1)$. This corresponds with the shuffle $[12 \cdots m_i - 1 \; m_i + 1 \; m_i \; m_i + 2 \cdots g]$.

By the requirement that each part in the partition is less than or equal to g, we see that the maximum possible length of an element in ${}^{J_i} S_g$ is $\sum_{a=g-m_i+1}^{g} a - \sum_{a=1}^{m_i} a = m_1 m_2$, and even more, the sum $(g - m_i + 1) + (g - m_i + 2) + \ldots + g$ is the only way to partition $m_1 m_2 + m_i(m_i + 1)/2$ into m_i distinct parts with each part less than or equal to g. Similarly, there is only one way to partition

[3] The use of almost-ordinary is meant to convey the notion of *almost μ-ordinary* in the sense of E-O strata and not *almost ordinary* in the sense of abelian varieties as in the Siegel case.

$m_1m_2 - 1 + m_i(m_i + 1)/2$ into m_i distinct parts with each part less than or equal to g: $(g - m_i) + (g - m_i + 2) + (g - m_i + 3) \ldots + g$ which corresponds to the desired element in ${}^JW$. □

Proposition 3.4.6. *The poset diagram of the E-O strata for* $\mathrm{GU}(m_1, m_2)$ *has a line of vertical symmetry when* $m_1 = m_2$*.*

Proof. To show the poset diagram of the E-O strata has a vertical axis of symmetry when $m_1 = m_2$, it suffices to show that ${}^JW$ has an order preserving automorphism. Consider the isomorphism ${}^{J_1}S_g \times {}^{J_2}S_g \to {}^{J_2}S_g \times {}^{J_1}S_g$ given by $(w_1, w_2) \mapsto (w_2, w_1)$ or rather $w \mapsto \hat{w} := w_0ww_0$. Since $J = J_1 = J_2$, this is an automorphism preserving the Bruhat order.

It remains to show that $w \mapsto \hat{w}$ preserves $\preccurlyeq$ for $\psi = 1$ and $\psi(w) = w_0ww_0$. Let $x = w_0w_{0,\psi(J)}$. Again, since $J = J_1 = J_2$, and $w \mapsto \hat{w}$ is a Bruhat order preserving automorphism, x is the same for both choices of ψ, namely, $x = (x_1, x_2) = w_0w_{0,J}$ and $x = \hat{x}$.

Suppose that $w \preccurlyeq w'$. Then there exists a $y \in W_J \times W_J$ such that

$$ywx\psi(y)^{-1}x^{-1} \leq w'.$$

For either choice of ψ, $\psi(\hat{y}) = \widehat{\psi(y)}$. Then,

$$\hat{y}\hat{w}x\psi(\hat{y})^{-1}x^{-1} = \hat{y}\hat{w}x\widehat{\psi(y)}^{-1}x^{-1} = w_0(ywx\psi(y)^{-1}x^{-1})w_0 \leq \hat{w}'.$$

Since W_J is preserved by $w \mapsto \hat{w}$, it follows that $\hat{y} \in W_J$ and $\hat{w} \preccurlyeq \hat{w}'$. Therefore $\preccurlyeq$ is preserved. □

Remark. Note that the map $w \mapsto w_0 w w_0$ on strata comes from the automorphism on the unitary Shimura variety of signature (m, m) that arises by twisting the $\mathcal{O}_K$-action by complex conjugation.

It is also natural to ask if and when there may be a horizontal line of symmetry to the diagrams. The map $\alpha : w \to w_{0,J} w w_0$ is an anti-automorphism of ${}^J W$ with respect to $\leq$ [BB05, Prop 2.5.4]. Therefore, when the Bruhat order is the same as $\preccurlyeq_\psi$, if m_1 and m_2 are odd, or when every element of length $\frac{m_1 m_2 + 1}{2}$ is fixed under α, then there is a horizontal line of symmetry. These are necessary conditions as the diagram for signature $(4, 3)$ under the Bruhat order does not have a horizontal line of symmetry. In general there is no horizontal line of symmetry, as α is *not* an anti-automorphism of ${}^J W$ with respect to $\preccurlyeq_\psi$. As examples, see Figures 3–4 and 3–5.

3.5 Models for mod p Dieudonné modules

The explicit presentations of ${}^J W$ obtained above can also be used to construct models for the Dieudonné modules of the p-torsion corresponding to each given stratum using the proof of [Moo01, Theorem 4.7]. Throughout this section assume that $m_1 \geq m_2 \geq 1$.

Let $(w_1, w_2) \in {}^J W$ and let $\mathcal{D} = \mathcal{D}(w_1, w_2)$ be the contravariant Dieudonné module of the p-torsion group scheme (up to isomorphism) corresponding to the E-O stratum of (w_1, w_2). Recall from Proposition 2.2.4, that $\mathcal{D}$ decomposes as $\mathcal{D}_1 \oplus \mathcal{D}_2$ under the $\mathcal{O}_K/(p)$-action on $\mathcal{D}$. By [Moo01, Theorem 4.7] there is a model for $\mathcal{D}$ such that each $\mathcal{D}_i$ has a basis $e_{i,1}, \ldots, e_{i,g}$ and F, V act on $\mathcal{D}$ as follows:

$$F(e_{i,j}) = \begin{cases} 0 & w_i(j) \leq m_i \\ e_{\gamma(i),a} & w_i(j) = m_i + a \end{cases} \tag{3.5}$$

$$V(e_{i,j}) = \begin{cases} 0 & j \leq m_{\gamma(i+1)} \\ e_{\gamma(i),b} & j = m_{\gamma(i+1)} + w_{\gamma(i)}(b) \end{cases} \tag{3.6}$$

where $\gamma(i) = i$ for p split and $\gamma(i) = i + 1$ (considered as the cyclic group of order 2) for p inert. This choice of basis has the property that $\mathcal{D}$ is self-dual as a Dieudonné module under the transformation taking $e_{i,j}$ to $e_{i+1,w_0(j)}$.

3.5.1 Invariants of p-torsion group schemes

Some basic invariants of commutative p-torsion group schemes are their a-number, f-number, and the minimal power of Frobenius that kills them as introduced in Section 2.2.1. The latter is finite if the group scheme has no étale part and is equal to the power of Verschiebung that kills the group scheme if it is a self-dual group scheme (as in this case). For example, let A be a g-dimensional principally polarized abelian variety. If $a(A) = g$ then $A[p]$ is uniquely determined, and by a theorem of Oort, A is superspecial [Oor75, Theorem 2]. At the other extreme if $f(A) = g$ then $A[p]$ is again uniquely determined and A is ordinary. Likewise, if $f(A) = g - a(A)$, then $A[p]$ is uniquely determined by its a-number and f-number. Or, in case $g = 2$, $a = 1$ and $A[p]$ is killed by F^2, the E-O stratum of A is uniquely determined by the a-number and the minimal power of Frobenius that kills $A[p]$. In other words, there are circumstances where one can determine the E-O strata by other invariants. This motivates the following discussion.

Proposition 3.5.1. *Let $w = (w_1, w_2) \in {}^J W$. Then the E-O stratum associated to w has a-number*

$$a(w) = \begin{cases} \# \{(i,j) \mid i \in \{1,2\}, 1 \le j \le m_{i+1}, 1 \le w_i(j) \le m_i\} & p \text{ is split,} \\ \# \{(i,j) \mid i \in \{1,2\}, 1 \le j \le m_i, 1 \le w_i(j) \le m_i\} & p \text{ is inert,} \end{cases}$$

Proof. The a-number of the E-O stratum of w is equal to $\dim \mathcal{D}/(F\mathcal{D} + V\mathcal{D})$ where $\mathcal{D} = \mathcal{D}(w)$ is the contravariant Dieudonné module (up to isomorphism) corresponding to the E-O stratum of w. When p is split,

$$F(e_{i,j}) = \begin{cases} 0 & w_i(j) \le m_i \\ e_{i,a} & w_i(j) = m_i + a \end{cases} \qquad V(e_{i,j}) = \begin{cases} 0 & j \le m_{i+1} \\ e_{i,b} & j = m_{i+1} + w_i(b). \end{cases}$$

Therefore,

$$F\mathcal{D} = \operatorname{span}\left\{e_{i,1}, \dots, e_{i,m_{i+1}}\right\}_{i=1,2}, V\mathcal{D} = \operatorname{span}\left\{e_{i,w_i^{-1}(a)} \;\middle|\; 1 \le a \le m_i\right\}_{i=1,2}.$$

For a set S, let $w_i(S) := \{w_i(s) \mid s \in S\}$. Now, w_i is a bijection so for a fixed i,

$$\begin{aligned} &\left\{a \in \{1, \dots, m_i\} \;\middle|\; 1 \le w_i^{-1}(a) \le m_{i+1}\right\} \\ &= w_i\left(\left\{w_i^{-1}(1), \dots, w_i^{-1}(m_i)\right\} \cap \{1, \dots, m_{i+1}\}\right) \\ &= \{1, \dots, m_i\} \cap \{w_i(1), \dots, w_i(m_{i+1})\} \\ &= \{j \in \{1, \dots, m_{i+1}\} \mid 1 \le w_i(j) \le m_i\}. \end{aligned}$$

Therefore the dimension of $F\mathcal{D} + V\mathcal{D}$ is given by

$$\begin{aligned}\dim F\mathcal{D} + V\mathcal{D} &= \dim F\mathcal{D} + \dim V\mathcal{D} - \dim(F\mathcal{D} \cap V\mathcal{D}) \\ &= 2g - \#\left\{a \in \{1, \ldots, m_i\} \;\middle|\; 1 \leq w_i^{-1}(a) \leq m_{i+1}, i = 1, 2\right\} \\ &= 2g - \#\left\{j \in \{1, \ldots, m_{i+1}\} \mid 1 \leq w_i(j) \leq m_i, i = 1, 2\right\},\end{aligned}$$

and $\dim \mathcal{D}/(F\mathcal{D} + V\mathcal{D}) = \#\left\{j \in \{1, \ldots, m_{i+1}\} \mid 1 \leq w_i(j) \leq m_i, i = 1, 2\right\}$.

On the other hand, when p is inert, $F\mathcal{D} = \operatorname{span}\left\{e_{i,j} \mid 1 \leq j \leq m_i\right\}_{i=1,2}$ and $V\mathcal{D} = \operatorname{span}\left\{e_{i,w_i^{-1}(b)} \;\middle|\; 1 \leq b \leq m_i\right\}_{i=1,2}$. Thus, the dimension of $F\mathcal{D} + V\mathcal{D}$ is

$$\begin{aligned}\dim F\mathcal{D} + V\mathcal{D} &= \dim F\mathcal{D} + \dim V\mathcal{D} - \dim(F\mathcal{D} \cap V\mathcal{D}) \\ &= 2g - \#\left\{b \in \{1, \ldots, m_i\} \;\middle|\; 1 \leq w_i^{-1}(b) \leq m_i, i = 1, 2\right\} \\ &= 2g - \#\left\{j \in \{1, \ldots, m_i\} \mid 1 \leq w_i(j) \leq m_i, i = 1, 2\right\},\end{aligned}$$

and $\dim \mathcal{D}/(F\mathcal{D} + V\mathcal{D}) = \#\left\{j \in \{1, \ldots, m_i\} \mid 1 \leq w_i(j) \leq m_i, i = 1, 2\right\}$. □

Proposition 3.5.2. *Let $w = (w_1, w_2) \in {}^J W$. Then the E-O stratum associated to w has f-number*

$$f(w) = \begin{cases} \#\left\{(i, j) \mid w_i(j) = j + m_i\right\} & p \text{ is split,} \\ \#\left\{(i, j) \mid w_{i+1}(w_i(j) - m_i) = j + m_{i+1}\right\} & p \text{ is inert.} \end{cases}$$

Proof. Begin with the following observation. Recall that since w_i is a shuffle, $w_i^{-1}(m_i + 1) < w_i^{-1}(m_i + 2) < \ldots < w_i^{-1}(g)$. In particular,

$$a \leq w_i^{-1}(a + m_i) \qquad a \in \{1, \ldots, m_{i+1}\} \tag{3.7}$$

as $w_i^{-1}(m_i+1), w_i^{-1}(m_i+2), \ldots, w_i^{-1}(m_i+a-1)$ must all be smaller than $w_i^{-1}(m_i+a)$; that is,

$$1 \leq w_i^{-1}(m_i+1) < w_i^{-1}(m_i+2) < \ldots < w_i^{-1}(m_i+a).$$

The identity (3.7) will be used repeatedly in what follows.

The f-number can be calculated by finding the dimension of $F^N\mathcal{D}$ where N is large enough to kill everything except for the étale part of $\mathcal{D}$. Recall that when p is split,

$$F(e_{i,j}) = \begin{cases} 0 & w_i(j) \leq m_i \\ e_{i,a} & w_i(j) = m_i + a \end{cases}$$

and $F(e_{i,j}) = e_{i,j}$ if and only if $w_i(j) = j + m_i$. It then suffices to show that

$$F^N\mathcal{D} = \operatorname{span}\{e_{i,j} \mid F(e_{i,j}) = e_{i,j}\}_{i=1,2}, \qquad N >> 0.$$

We begin by showing that if $F(e_{i,j}) = e_{i,a} \neq 0$, then $a \leq j$. Suppose that $F(e_{i,j}) = e_{i,a} \neq 0$ where $a = w_i(j) - m_i$. Then $w_i(j) = a + m_i$ where $1 \leq a \leq m_{i+1}$, and by identity (3.7) above,

$$a \leq w_i^{-1}(a + m_i) = j.$$

Fix $i \in \{1,2\}$. Now we show that $F^N\mathcal{D}_i = \operatorname{span}\{e_{i,j} \mid F(e_{i,j}) = e_{i,j}\}$ for $N >> 0$. Trivially, the span of $\{e_{i,j} \mid F(e_{i,j}) = e_{i,j}\}$ is a subset of $F^N\mathcal{D}_i$. On the other hand, suppose that $e_{i,j} \in F^N\mathcal{D}$. Then there must be an m such that $F^m(e_{i,j}) = e_{i,j}$. But if $F(e_{i,j}) = e_{i,a} \neq 0$, then $a \leq j$, and in particular, if $a < j$, $F^m(e_{i,j}) = e_{i,\ell}$ where $\ell \leq a < j$ for all $m \geq 1$. Therefore $F(e_{i,j}) = e_{i,j}$.

The proof in the inert case is like that of the split case, only F^2 will need to be considered instead of F. Indeed, when p is inert,

$$F(e_{i,j}) = \begin{cases} 0 & w_i(j) \leq m_i \\ e_{i+1,a} & w_i(j) = m_i + a \end{cases} \tag{3.8}$$

and if $F^2(e_{i,j}) \neq 0$,

$$F^2(e_{i,j}) = F(e_{i+1,w_i(j)-m_i}) = e_{i,w_{i+1}(w_i(j)-m_i)-m_{i+1}}.$$

Therefore, $F^2(e_{i,j}) = e_{i,j}$ if and only if $w_{i+1}(w_i(j) - m_i)) = j + m_{i+1}$. It remains to show that

$$F^N\mathcal{D} = \operatorname{span}\left\{e_{i,j} \;\middle|\; F^2(e_{i,j}) = e_{i,j}\right\}_{i=1,2}, \qquad N >> 0.$$

Similar to the proof of the split case, this can be done by showing that if $F^2(e_{i,j}) = e_{i,b} \neq 0$, then $b \leq j$. Suppose that $F^2(e_{i,j}) = e_{i,b} \neq 0$ so that $w_{i+1}(w_i(j) - m_i) - m_{i+1} = b$. Setting $a = w_i(j) - m_i$, (3.7) gives

$$b \leq w_{i+1}^{-1}(b + m_{i+1}) = a \leq w_i^{-1}(a + m_i) = j$$

completing the proof. □

When $f(w)$ is zero, there is a minimal power of Frobenius that kills $A[p]$. Denote this invariant by $\operatorname{minF}(w)$, and let $\operatorname{minF}_i(w)$ be the minimal power of F that kills $\mathcal{D}_i$ where $\mathcal{D}(A[p]) = \mathcal{D}_1 \oplus \mathcal{D}_2$. If S is a set of integers, then for $a \in \mathbb{Z}$ write

$$S - a := \{s - a \mid s \in S\}.$$

Proposition 3.5.3. *Suppose $m_1 \geq m_2$. When p is split,* $\mathrm{minF}(w) = \mathrm{minF}_2(w)$ *and can be calculated as follows. Let $S_1 = \{m_2 + 1, \ldots, g\}$ and for $i \geq 2$, let*

$$S_i = S_1 \cap w_2(S_{i-1} - m_2).$$

Then $\mathrm{minF}(w)$ *is the smallest i such that S_i is empty.*

On the other hand, when p is inert, let

$$S_1 = \{m_1 + 1, \ldots, g\}, \qquad T_1 = \{m_2 + 1, \ldots, g\},$$

and inductively for $i \geq 2$

$$S_i = S_1 \cap w_2(T_{i-1} - m_2), \qquad T_i = T_1 \cap w_1(S_{i-1} - m_1).$$

Then $\mathrm{minF}(w)$ *is the smallest i such that $S_i \cup T_i$ is empty.*

Proof. The calculation of $\mathrm{minF}_2(w)$ (and likewise $\mathrm{minF}_1(w)$) in the split case and $\mathrm{minF}(w)$ in the inert case follows directly from Equation 3.5. Since $m_1 \geq m_2$, when p is split, $\mathrm{minF}_1(w) \leq \mathrm{minF}_2(w)$ so that $\mathrm{minF}(w) = \mathrm{minF}_2(w)$. □

When $m_1 = m_2 = 1$, there are two strata, the core stratum and the μ-ordinary stratum which have the following invariants both when p is split and when p is inert:

- μ-ordinary: $f = 2$, $a = 0$
- core: $f = 0$, $a = 2$, $\mathrm{minF} = 2$.

As a further application of Propositions 3.5.1, 3.5.2, and 3.5.3, the Tables 3–1 and 3–2 give the a-numbers, f-numbers and smallest powers of F killing $A[p]$ for the

Table 3–1: Invariants of E-O strata when p is split

p is split		$a(w)$	$f(w)$	$\text{minF}(w)$
Core		$2m_2$	0	$\lceil \frac{g}{m_2} \rceil$
Almost-core	$m_1 = m_2 = m$	$2m - 2$	0	3
	$m_1 - m_2 \geq 1,\ m_2 = 1$	$2m_2 = 2$	1	—
	$m_1 - m_2 \geq 1,\ m_2 > 1,\ m_2 \mid m_1$	$2m_2$	0	$\frac{g}{m_2} + 1$
	$m_1 - m_2 \geq 1,\ m_2 > 1,\ m_2 \nmid m_1$	$2m_2$	0	$\lceil \frac{g}{m_2} \rceil$
Almost-ordinary		2	$g - 2$	—
μ-ordinary		0	g	—

Table 3–2: Invariants of E-O strata when p is inert

p is inert		$a(w)$	$f(w)$	$\text{minF}(w)$
Core		g	0	2
Almost-core	$m_1 \geq m_2 > 1$	$g - 2$	0	3
	$m_1 > m_2 = 1$	$g - 2$	0	4
Almost-ordinary	$m_1 = m_2 = m$	2	$2m - 2$	—
	$m_1 - m_2 = 1, m_2 = 1$	$m_1 - m_2$	$2m_2 - 2 = 0$	4
	$m_1 - m_2 = 1, m_2 > 1$	$m_1 - m_2$	$2m_2 - 2$	—
	$m_1 - m_2 > 1$	$m_1 - m_2$	$2m_2 - 2$	—
μ-ordinary		$m_1 - m_2$	$2m_2$	—

E-O strata of particular interest in all cases where $m_1 \geq m_2 \geq 1$ (excluding the case where $m_1 = m_2 = 1$).

Note that in the case where p is split, the core locus consists of superspecial abelian varieties if and only if $m_1 = m_2$, but the μ-ordinary locus is always ordinary. At the other extreme, when p is inert the core locus always consists of superspecial abelian varieties, but the μ-ordinary locus is ordinary if and only if $m_1 = m_2$. This result for the μ-ordinary locus is consistent with [Moo04a, 1.3.10].

For the most part, we see that these invariants can be used to distinguish the almost-core locus from the core locus and the almost-ordinary locus from the μ-ordinary locus. However, it is worth pointing out that when p is split even the combination of the a-number and the minimal power of Frobenius that kills the group scheme is *not* sufficient to distinguish the core locus from the almost-core locus in the most generic case.

3.5.2 p inert in K

Recall from Corollary 3.4.5 that there are 4 strata of particular interest, the core stratum, the almost-core stratum, the almost-ordinary stratum and the μ-ordinary stratum. In this section we give models for the p-torsion of these strata when p is inert by computing models for the covariant Dieudonné modules.

The signature $(n, 0)$ and $(1, 1)$ cases are special cases in which there are 1 and 2 strata respectively. The results in these cases are well-known and follow from Deuring's Theorem. They are recorded in Table 3–3 for completeness. Here $\mathscr{G}$ denotes the p-torsion group scheme of a supersingular elliptic curve, and subscripts are used to indicate the action of K (*i.e.* $\mathscr{G}_1$ has signature $(1, 0)$ and $\mathscr{G}_2$ has

Table 3–3: E-O strata when p is inert

Signature	Stratum	$A[p]$
$(m, 0)$	Core	$\mathscr{G}_1^m$
$(0, m)$	Core	$\mathscr{G}_2^m$
$(1, 1)$	μ-ordinary	$(\mathcal{O}_K \otimes \mu_p) \oplus \underline{\mathcal{O}_K/(p)}$
$(1, 1)$	Core	$\mathscr{G}_1 \oplus \mathscr{G}_2$

signature $(0, 1)$). Having taken care of the $(m, 0)$ and $(1, 1)$ cases, for the rest of this section, assume that $m_1 \geq m_2 \geq 1$ and if $m_2 = 1$, $m_1 > 1$.

Proposition 3.5.4 (The core locus)**.** *The p-torsion group scheme of a point in the core locus is isomorphic to*

$$A[p] \cong \mathscr{G}_1^{m_1} \oplus \mathscr{G}_2^{m_2}.$$

Proof. Equations (3.5) and (3.6) give that

i	j	$F(e_{i,j}) = V(e_{i,j})$
1	$1 \leq j \leq m_1$	0
1	$m_1 + 1 \leq j \leq g$	$e_{2,j-m_1}$
2	$1 \leq j \leq m_2$	0
2	$m_2 + 1 \leq j \leq g$	$e_{1,j-m_2}$

and in particular, for $1 \leq j \leq m_i$,

$$e_{i+1,j+m_{i+1}} \underset{V}{\overset{F}{\rightrightarrows}} e_{i,j} \underset{V}{\overset{F}{\rightrightarrows}} 0\,.$$

Therefore $\mathcal{D}(\mathscr{G}_i^{m_i}) \cong \operatorname{span}\{e_{i,1}, \ldots, e_{i+1,m_i}, e_{i+1,m_i+1}, \ldots, e_{i+1,g}\}$, and

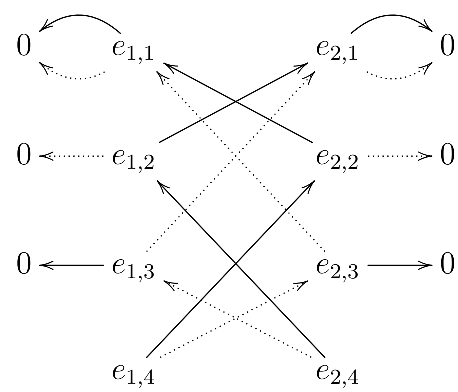
$$A[p] \cong \mathscr{G}_1^{m_1} \oplus \mathscr{G}_2^{m_2}.$$

$\square$

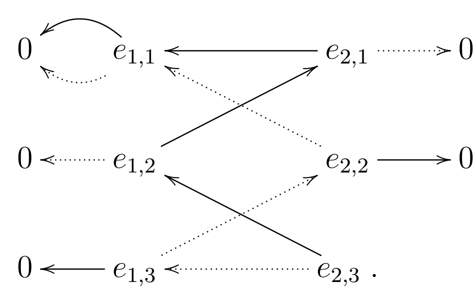
Proposition 3.5.5 (The almost-core locus)**.** *Let $AC(m_1, m_2)$ denote a model for the p-torsion group scheme of the almost-core locus of signature (m_1, m_2). Then $AC(m_1, m_2)$ is isomorphic to the following group schemes*

- $m_1 \geq m_2 > 1$*:* $AC(2,2) \oplus \mathscr{G}_1^{m_1-2} \oplus \mathscr{G}_2^{m_2-2}$
- $m_1 > m_2 = 1$*:* $AC(2,1) \oplus \mathscr{G}_1^{g-3}$.

Letting $\longrightarrow$ *denote* F *and* $\dashrightarrow$ *denote* V*,* $\mathcal{D}(AC(2,2))$ *has a basis where*



and $\mathcal{D}(AC(2,1))$ *has a basis where*



Proof. The almost-core locus corresponds to the element (w_1, w_2) where

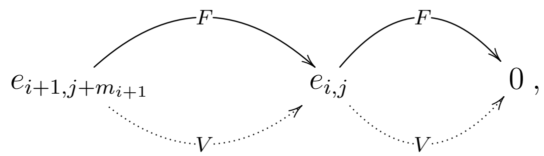
$$w_i(j) = \begin{cases} m_i + 1 & j = m_i \\ m_i & j = m_i + 1 \\ j & \text{otherwise.} \end{cases}$$

By equations (3.5) and (3.6), bases for $\mathcal{D}(w)$ are described in the following tables—there are two cases depending on whether $m_2 = 1$ or $m_2 > 1$.

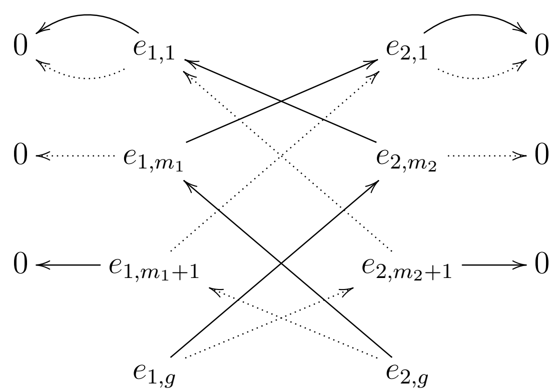
p is inert and $m_1 \geq m_2 > 1$

i	j	$F(e_{i,j})$	$V(e_{i,j})$
1	$1 \leq j \leq m_1 - 1$	0	0
1	m_1	$e_{2,1}$	0
1	$m_1 + 1$	0	$e_{2,1}$
1	$m_1 + 2 \leq j \leq g - 1$	$e_{2,j-m_1}$	$e_{2,j-m_1}$
1	g	e_{2,m_2}	e_{2,m_2+1}
2	$1 \leq j \leq m_2 - 1$	0	0
2	m_2	$e_{1,1}$	0
2	$m_2 + 1$	0	$e_{1,1}$
2	$m_2 + 2 \leq j \leq g - 1$	$e_{1,j-m_2}$	$e_{1,j-m_2}$
2	g	e_{1,m_1}	e_{1,m_1+1}

For $2 \leq j \leq m_i - 1$,



corresponding to $m_i - 2$ copies of $\mathcal{D}(\mathscr{G}_i)$, and the remainder consists of

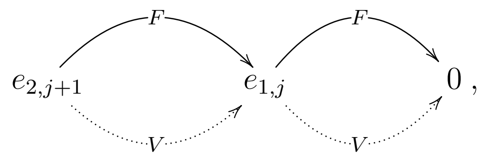


which is a basis for $\mathcal{D}(AC(2,2))$ by setting $m_1 = m_2 = 2$ and $g = 4$ in the diagram.

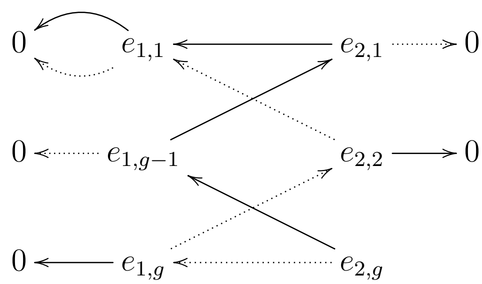
p is inert and $m_1 > m_2 = 1$

i	j	$F(e_{i,j})$	$V(e_{i,j})$
1	$1 \leq j \leq m_1 - 1$	0	0
1	$m_1 = g - 1$	$e_{2,1}$	0
1	$m_1 + 1 = g$	0	$e_{2,2}$
2	$m_2 = 1$	$e_{1,1}$	0
2	$m_2 + 1 = 2$	0	$e_{1,1}$
2	$m_2 + 2 \leq j \leq g - 1$	$e_{1,j-1}$	$e_{1,j-1}$
2	g	e_{1,m_1}	$e_{1,g}$

For $2 \leq j \leq m_1 - 1$,



which gives $m_1 - 2 = g - 3$ copies of $\mathcal{D}(\mathscr{G}_1)$ and



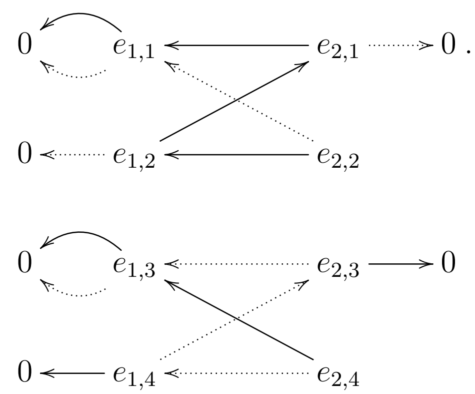
which is isomorphic to $\mathcal{D}(AC(2,1))$ by setting $g = 3$ in the diagram. $\square$

Remark. Note this is consistent with the results of [BW06] in the $m_2 = 1$ case as $AC(2,1)$ is the dual of a braid of length 3.

Proposition 3.5.6 (The almost-ordinary locus)**.** *Let $AO(m_1, m_2)$ be a model for the p-torsion group scheme of the almost-ordinary stratum with signature (m_1, m_2). Then, depending on the difference between m_1 and m_2, $AO(m_1, m_2)$ is isomorphic to:*

- $m_1 - m_2 > 1$*:* $AO(3,1) \oplus (\mathcal{O}_K \otimes \mu_p)^{m_2-1} \oplus (\underline{\mathcal{O}_K/(p)})^{m_2-1} \oplus \mathscr{G}_1^{m_1-m_2-2}$
- $m_1 - m_2 = 1$*:* $AC(2,1) \oplus (\mathcal{O}_K \otimes \mu_p)^{m_2-1} \oplus (\underline{\mathcal{O}_K/(p)})^{m_2-1}$
- $m_1 = m_2 = m$*:* $(\mathcal{O}_K \otimes \mu_p)^{m-1} \oplus (\underline{\mathcal{O}_K/(p)})^{m-1} \oplus \mathscr{G}_1 \oplus \mathscr{G}_2$.

Letting $\longrightarrow$ *denote* F *and* $\dashrightarrow$ *denote* V, $\mathcal{D}(AO(3,1))$ *has a basis of the form*



Proof. The locus with codimension 1 corresponds to the element (w_1, w_2) where

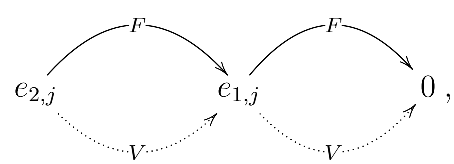
$$w_i(j) = \begin{cases} j + m_i & 1 \leq j \leq m_{i+1} - 1 \\ j - m_{i+1} & m_{i+1} + 2 \leq j \leq g \\ 1 & j = m_{i+1} \\ g & j = m_{i+1} + 1. \end{cases}$$

When $m_1 - m_2 > 1$, $\mathcal{D}(w)$ has a basis as described in the following table.

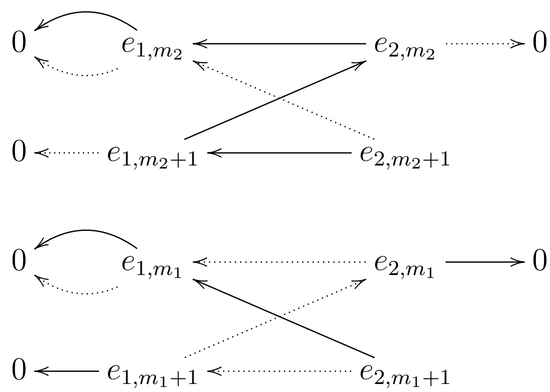
p is inert $m_1 - m_2 > 1$

i	j	$F(e_{i,j})$	$V(e_{i,j})$
1	$1 \leq j \leq m_2 - 1$	$e_{2,j}$	0
1	m_2	0	0
1	$m_2 + 1$	e_{2,m_2}	0
1	$m_2 + 2 \leq j \leq m_1 - 1$	0	0
1	m_1	0	0
1	$m_1 + 1$	0	e_{2,m_1}
1	$m_1 + 2 \leq j \leq g$	0	$e_{2,j}$
2	$1 \leq j \leq m_2$	$e_{1,j}$	0
2	$m_2 + 1$	$e_{1,j}$	e_{1,m_2}
2	$m_2 + 2 \leq j \leq m_1 - 1$	$e_{1,j}$	$e_{1,j}$
2	m_1	0	$e_{1,j}$
2	$m_1 + 1$	e_{1,m_1}	$e_{1,j}$
2	$m_1 + 2 \leq j \leq g$	0	$e_{1,j}$

For $m_2 + 2 \leq j \leq m_1 - 1$ (when such a j exists),



which gives $\mathscr{G}^{m_1-m_2-2}$, $\mathcal{D}((\mathcal{O}_K \otimes \mu_p)^{m_2-1}) \cong \operatorname{span}\{e_{i,j} \mid m_i + 2 \le j \le g\}_{i=1,2}$, and $\mathcal{D}(\underline{\mathcal{O}_K/(p)}^{m_2-1}) \cong \operatorname{span}\{e_{i,j} \mid 1 \le j \le m_2 - 1\}_{i=1,2}$. What remains is

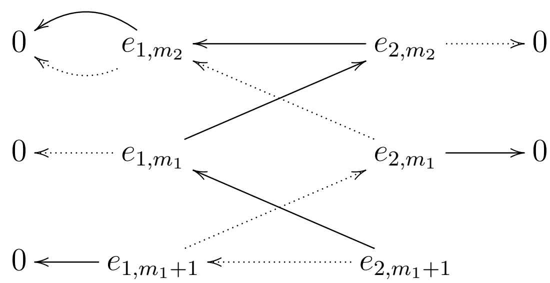


corresponding to $\mathcal{D}(AO(3,1))$.

p is inert $m_1 - m_2 = 1$

i	j	$F(e_{i,j})$	$V(e_{i,j})$
1	$1 \le j \le m_2 - 1$	$e_{2,j}$	0
1	m_2	0	0
1	$m_1 = m_2 + 1$	e_{2,m_2}	0
1	$m_1 + 1$	0	e_{2,m_1}
1	$m_1 + 2 \le j \le g$	0	$e_{2,j}$
2	$1 \le j \le m_2$	$e_{1,j}$	0
2	$m_1 = m_2 + 1$	0	e_{1,m_2}
2	$m_1 + 1$	e_{1,m_1}	e_{1,m_1+1}
2	$m_1 + 2 \le j \le g$	0	$e_{1,j}$

When $m_1 - m_2 = 1$, $\mathcal{D}((\mathcal{O}_K \otimes \mu_p)^{m_2-1}) \cong \text{span}\,\{e_{i,j} \mid m_1 + 2 \leq j \leq g\}_{i=1,2}$, and $\mathcal{D}(\underline{\mathcal{O}_K/(p)}^{m_2-1}) \cong \text{span}\,\{e_{i,j} \mid 1 \leq j \leq m_2 - 1\}_{i=1,2}$. This leaves

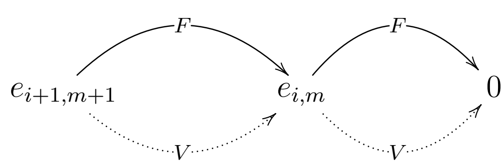


which is isomorphic to $\mathcal{D}(AC(2,1))$ as desired.

p is inert $m = m_1 = m_2 \geq 2$

j	$F(e_{i,j})$	$V(e_{i,j})$
$1 \leq j \leq m - 1$	$e_{i+1,j}$	0
m	0	0
$m + 1$	$e_{i+1,m}$	$e_{i+1,m}$
$m + 2 \leq j \leq g$	0	$e_{i+1,j}$

Finally, when $m = m_1 = m_2 \geq 2$, $\mathcal{D}((\mathcal{O}_K \otimes \mu_p)^{m-1})$ is isomorphic to $\text{span}\,\{e_{i,j} \mid m + 2 \leq j \leq g\}_{i=1,2}$, and $\mathcal{D}(\underline{\mathcal{O}_K/(p)}^{m-1})$ is isomorphic to $\text{span}\,\{e_{i,j} \mid 1 \leq j \leq m - 1\}_{i=1,2}$ which leaves



for $i \in \{1, 2\}$ giving rise to $\mathscr{G}_1 \oplus \mathscr{G}_2$. □

Proposition 3.5.7 (The μ-ordinary locus)**.** *The μ-ordinary locus corresponds to*

$$A[p] \cong (\mathcal{O}_K \otimes \mu_p)^{m_2} \oplus (\underline{\mathcal{O}_K/(p)})^{m_2} \oplus \mathscr{G}_1^{m_1-m_2}.$$

Proof. The μ-ordinary locus corresponds to the element (w_1, w_2) where

$$w_i(j) = \begin{cases} j + m_i & 1 \leq j \leq m_{i+1} \\ j - m_{i+1} & m_{i+1} + 1 \leq j \leq g \end{cases}$$

Then $\mathcal{D}(w)$ has a basis such F and V are given by

i	j	$F(e_{i,j})$	$V(e_{i,j})$
1	$1 \leq j \leq m_2$	$e_{2,j}$	0
1	$m_2 + 1 \leq j \leq m_1$	0	0
1	$m_1 + 1 \leq j \leq g$	0	$e_{2,j}$
2	$1 \leq j \leq m_2$	$e_{1,j}$	0
2	$m_2 + 1 \leq j \leq m_1$	$e_{1,j}$	$e_{1,j}$
2	$m_1 + 1 \leq j \leq g$	0	$e_{1,j}$

$\mathcal{D}((\mathcal{O}_K \otimes \mu_p)^{m_2})$ is isomorphic to span $\{e_{i,j} \mid m_1 + 1 \leq j \leq g\}_{i=1,2}$, $\mathcal{D}(\underline{\mathcal{O}_K/(p)}^{m_2})$ is isomorphic to span $\{e_{i,j} \mid 1 \leq j \leq m_2\}_{i=1,2}$, and $\mathcal{D}(\mathscr{G}_1^{m_1-m_2}) \cong$ span $\{e_{i,j} \mid m_2 + 1 \leq j \leq m_1\}_{i=1,2}$. □

Table 3–4: E-O strata when p is split

Signature	Stratum	$A[p]$
$(m,0)$	Core	$(\mathcal{O}_K/\mathfrak{p}_1 \otimes \mu_p)^m \oplus (\underline{\mathcal{O}_K/\mathfrak{p}_2})^m$
$(0,m)$	Core	$(\underline{\mathcal{O}_K/\mathfrak{p}_1})^m \oplus (\mathcal{O}_K/\mathfrak{p}_2 \otimes \mu_p)^m$
$(1,1)$	μ-ordinary	$\underline{\mathcal{O}_K/(p)} \otimes (\mu_p \oplus \underline{\mathbb{Z}/p\mathbb{Z}})$
$(1,1)$	Core	$\mathscr{G}_1 \oplus \mathscr{G}_2$

3.5.3 p split in K

As in the previous section, the cases where $m_2 = 0$ or $m_1 = m_2 = 1$ are treated separately and are given in Table 3–4. For what follows in this section, assume that $m_1 \geq m_2 \geq 1$ and if $m_2 = 1$, $m_1 > 1$.

Proposition 3.5.8 (The core locus)**.** *Let $C(m_1, m_2)$ denote the p-torsion group scheme of the core locus of signature (m_1, m_2). Then, if $d = \gcd(m_1, m_2)$,*

$$C(m_1, m_2) \cong \oplus_{i=1}^{d} C\left(\frac{m_1}{d}, \frac{m_2}{d}\right).$$

In particular, when $m_1 = m_2 = m$,

$$C(m, m) \cong \oplus_{i=1}^{m} C(1, 1) \cong (\mathscr{G}_1 \oplus \mathscr{G}_2)^m.$$

Proof. When p is split, Equations (3.5) and (3.6) give a basis for the Dieudonné module of the core stratum as in the table below.

i	j	$F(e_{i,j})$	$V(e_{i,j})$
1	$1 \leq j \leq m_2$	0	0
1	$m_2 + 1 \leq j \leq m_1$	0	$e_{1,j-m_2}$
1	$m_1 + 1 \leq j \leq g$	$e_{1,j-m_1}$	$e_{1,j-m_2}$
2	$1 \leq j \leq m_2$	0	0
2	$m_2 + 1 \leq j \leq m_1$	$e_{2,j-m_2}$	0
2	$m_1 + 1 \leq j \leq g$	$e_{2,j-m_2}$	$e_{2,j-m_1}$

Let $d = \gcd(m_1, m_2)$. Then for a fixed $t \in \{g - d + 1, \ldots, g\}$

$$\operatorname{span}\{e_{i,t-sd} \mid s \in \{0, \ldots, g/d - 1\}\}_{i=1,2}$$

is a submodule of $\mathcal{D}(w)$ that is stable under F and V. By sending $e_{i,t-sd} \mapsto e_{i,g/d-s}$ we obtain an isomorphism of

$$\operatorname{span}\{e_{i,t-sd} \mid s \in \{0, \ldots, g/d - 1\}\}_{i=1,2}$$

with $\mathcal{D}\left(C\left(\frac{m_1}{d}, \frac{m_2}{d}\right)\right)$ respecting F, V and the $\mathcal{O}_K$-action. Since this is true for every $t \in \{g - d + 1, \ldots, g\}$, it follows that

$$C(m_1, m_2) \cong \oplus_{i=1}^{d} C\left(\frac{m_1}{d}, \frac{m_2}{d}\right).$$

□

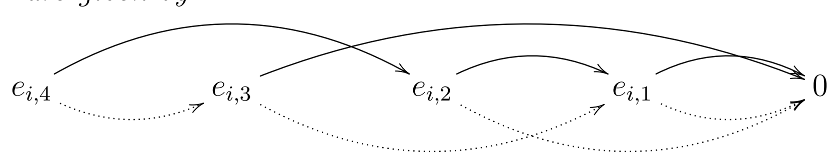
Recall that $AC(m_1, m_2)$ denotes a model for the p-torsion group scheme of a the almost-core E-O stratum.

Proposition 3.5.9 (The almost-core locus)**.** *The group scheme $AC(m_1, m_2)$ is of the form:*

- $m_1 = m_2 = m$*:* $AC(2,2) \oplus (\mathscr{G}_1 \oplus \mathscr{G}_2)^{m-2}$
- $m_1 - m_2 \geq 1, m_2 = 1$*:* $(\mathcal{O}_K/\mathfrak{p}_1 \otimes \mu_p) \oplus (\underline{\mathcal{O}_K/\mathfrak{p}_2}) \oplus C\,(m_1 - 1, 1)$
- $m_1 - m_2 = 1, m_2 > 1$*:* $C\,(m_2, m_2 - 1) \oplus \mathscr{G}_1 \oplus \mathscr{G}_2$
- $m_1 - m_2 > 1, m_2 > 1$*: There is no obvious way to consistently describe the models in this case. For example,*

(m_1, m_2)	$AC(m_1, m_2)$
$(4,2)$	$AC(4,2)$
$(5,2)$	$C\,(3,1) \oplus C\,(2,1)$
$(5,3)$	$C\,(3,2) \oplus C\,(2,1)$
$(6,2)$	$AC(6,2)$
$(6,3)$	$AC(4,2) \oplus C\,(2,1)$
$(6,4)$	$AC(6,4)$
$(7,2)$	$C\,(4,1) \oplus C\,(3,1)$
$(7,3)$	$C\,(5,2) \oplus C\,(2,1)$
$(7,4)$	$C\,(5,3) \oplus C\,(2,1)$
$(7,5)$	$C\,(4,3) \oplus C\,(3,2)$

Note that $\mathcal{D}(AC(2,2))$ has a basis $\{e_{i,j} \mid 1 \leq j \leq 4\}_{i=1,2}$ where for $i \in \{1,2\}$, F and V are given by



Proof. The almost-core locus corresponds to the element (w_1, w_2) where

$$w_i(j) = \begin{cases} m_i + 1 & j = m_i \\ m_i & j = m_i + 1 \\ j & \text{else.} \end{cases}$$

There are different cases depending on the difference between m_1 and m_2.

p is split $m_1 - m_2 \geq 1$

i	j	$F(e_{i,j})$	$V(e_{i,j})$
1	$1 \leq j \leq m_2$	0	0
1	$m_2 + 1 \leq j \leq m_1 - 1$	0	$e_{1,j-m_2}$
1	m_1	$e_{1,1}$	e_{1,m_1-m_2}
1	$m_1 + 1$	0	e_{1,m_1-m_2+1}
1	$m_1 + 2 \leq j \leq g - 1$	$e_{1,j-m_1}$	$e_{1,j-m_2}$
1	g	$e_{1,j-m_1}$	e_{1,m_1+1}
2	$1 \leq j \leq m_2 - 1$	0	0
2	m_2	$e_{2,1}$	0
2	$m_2 + 1$	0	0
2	$m_2 + 2 \leq j \leq m_1$	$e_{2,j-m_2}$	0
2	$m_1 + 1 \leq j \leq g - 1$	$e_{2,j-m_2}$	$e_{2,j-m_1}$
2	g	$e_{2,j-m_2}$	e_{2,m_2+1}

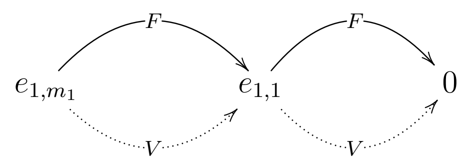
First consider the case where $m_1 - m_2 \geq 1$ and $m_2 = 1$. Then $F(e_{1,m_1}) = e_{1,1} = e_{m_1-(m_1-1)}$ and $V(e_{2,g}) = e_{2,m_2+1} = e_{2,g-(m_1-1)}$. Since $j - m_2 = j - 1$,

$$\operatorname{span}\{e_{1,j} \mid 1 \leq j \leq g-1\} \oplus \operatorname{span}\{e_{2,j} \mid 2 \leq j \leq g\}$$

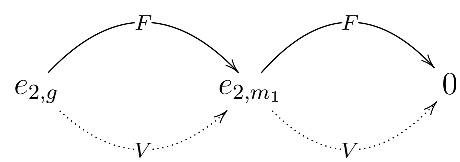
is isomorphic to $\mathcal{D}(C(m_1 - 1, 1))$. This leaves the one-dimensional submodules $\operatorname{span}\{e_{1,g}\}$ and $\operatorname{span}\{e_{2,1}\}$ corresponding to $(\mathcal{O}_K/\mathfrak{p}_1 \otimes \mu_p)$ and $(\underline{\mathcal{O}_K/\mathfrak{p}_2})$ respectively. Therefore,

$$AC(m_1, m_2) \cong (\mathcal{O}_K/\mathfrak{p}_1 \otimes \mu_p) \oplus (\underline{\mathcal{O}_K/\mathfrak{p}_2}) \oplus C(m_1 - 1, 1).$$

Now suppose that $m_1 - m_2 = 1$ and $m_2 > 1$. Observe that



and



giving rise to $\mathscr{G}_1 \oplus \mathscr{G}_2$. By relabelling:

- $e_{1,j} \mapsto f_{1,j-1}$ for $2 \leq j \leq m_2 = m_1 - 1$ and $m_1 + 2, \leq j \leq g$,
- $e_{1,m_1+1} \mapsto f_{1,m_2}$

then F and V on $\operatorname{span}\{f_{1,j} \mid 1 \leq j \leq 2m_2 - 1 = g - 2\}$ are determined by,

$$F(f_{1,j}) = \begin{cases} 0 & 1 \leq j \leq m_2 \\ f_{1,j-m_2} & m_2 + 1 \leq j \leq g-2 \end{cases}$$

and

$$V(f_{1,j}) = \begin{cases} 0 & 1 \le j \le m_2 - 1 \\ f_{1,j-(m_2-1)} & m_2 \le j \le g-2. \end{cases}$$

Similarly, for $i = 2$, relabelling by:

- $e_{2,j} \mapsto f_{2,j}$ for $1 \le j \le m_2 = m_1 - 1$
- $e_{2,j} \mapsto f_{2,j-1}$ for $m_1 + 1 = m_2 + 2 \le j \le g - 1$

means that F and V on span $\{f_{2,j} \mid 1 \le j \le 2m_2 - 1 = g - 2\}$ are determined by,

$$F(f_{2,j}) = \begin{cases} 0 & 1 \le j \le m_2 - 1 \\ f_{2,j-(m_2-1)} & m_2 \le j \le g-2 \end{cases}$$

and

$$V(f_{2,j}) = \begin{cases} 0 & 1 \le j \le m_2 \\ f_{2,j-m_2} & m_2 + 1 \le j \le g-2. \end{cases}$$

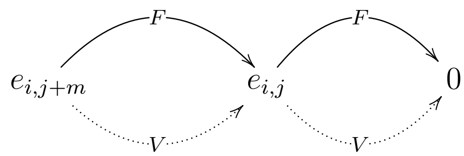
Therefore,

$$AC(m_1, m_2) \cong C\left(m_2, m_2 - 1\right) \oplus \mathscr{G}_1 \oplus \mathscr{G}_2.$$

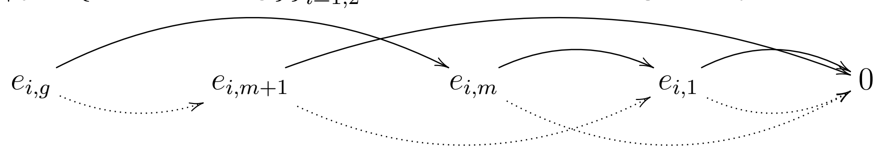
p is split $m_1 = m_2 = m$

j	$F(e_{i,j})$	$V(e_{i,j})$
$1 \le j \le m-1$	0	0
m	$e_{i,1}$	0
$m+1$	0	$e_{i,1}$
$m+2 \le j \le g-1$	$e_{i,j-m}$	$e_{i,j-m}$
g	$e_{i,j-m}$	$e_{i,m+1}$

Finally, consider the case when $m = m_1 = m_2$. For $2 \leq j \leq m-1$,



giving $m - 2$ copies of $\mathscr{G}_1 \oplus \mathscr{G}_2$. This leaves the submodule span $\{e_{i,j} \mid j \in \{1, m, m+1, g\}\}_{i=1,2}$. Here F and V are given by



which means that span $\{e_{i,j} \mid j \in \{1, m, m+1, g\}\}_{i=1,2}$ is isomorphic to $\mathcal{D}(AC(2,2))$. Therefore

$$AC(m, m) \cong AC(2,2) \oplus (\mathscr{G}_1 \oplus \mathscr{G}_2)^{m-2}.$$

$\square$

Proposition 3.5.10 (The almost-ordinary locus)**.** *There is a model for the p-torsion group scheme of the almost-ordinary locus of the form*

$$(\mathcal{O}_K/\mathfrak{p}_1 \otimes \mu_p)^{m_1-1} \oplus (\mathcal{O}_K/\mathfrak{p}_1)^{m_2-1} \oplus \mathscr{G}_1 \oplus (\mathcal{O}_K/\mathfrak{p}_2 \otimes \mu_p)^{m_2-1} \oplus (\mathcal{O}_K/\mathfrak{p}_2)^{m_1-1} \oplus \mathscr{G}_2.$$

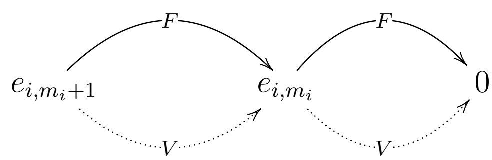
Proof. The locus with codimension 1 corresponds to the element (w_1, w_2) where

$$w_i(j) = \begin{cases} j + m_i & 1 \leq j \leq m_{i+1} - 1 \\ j - m_{i+1} & m_{i+1} + 2 \leq j \leq g \\ 1 & j = m_{i+1} \\ g & j = m_{i+1} + 1 \end{cases}$$

thus giving a basis

j	$F(e_{i,j})$	$V(e_{i,j})$
$1 \leq j \leq m_i - 1$	$e_{i,j}$	0
m_i	0	0
$m_i + 1$	e_{i,m_i}	e_{i,m_i}
$m_i + 2 \leq j \leq g$	0	$e_{i,j}$

Then span $\{e_{i,j} \mid 1 \leq j \leq m_i - 1\}$ is isomorphic to $\mathcal{D}(\mathcal{O}_K/\mathfrak{p}_i \otimes \mu_p)^{m_i - 1}$ and span $\{e_{i,j} \mid m_i + 2 \leq j \leq g\}$ is isomorphic to $\mathcal{D}(\underline{\mathcal{O}_K/\mathfrak{p}_i})^{m_{i+1}-1}$. Finally,



for $i \in \{1, 2\}$, and

$$\begin{aligned} AO(m_1, m_2) \cong & (\mathcal{O}_K/\mathfrak{p}_1 \otimes \mu_p)^{m_1 - 1} \oplus (\underline{\mathcal{O}_K/\mathfrak{p}_1})^{m_2 - 1} \oplus \mathscr{G}_1 \\ & \oplus (\mathcal{O}_K/\mathfrak{p}_2 \otimes \mu_p)^{m_2 - 1} \oplus (\underline{\mathcal{O}_K/\mathfrak{p}_2})^{m_1 - 1} \oplus \mathscr{G}_2. \end{aligned}$$

□

Proposition 3.5.11 (The μ-ordinary locus)**.**

$$A[p] = (\mathcal{O}_K/\mathfrak{p}_1 \otimes \mu_p)^{m_1} \oplus (\mathcal{O}_K/\mathfrak{p}_1)^{m_2} \oplus (\mathcal{O}_K/\mathfrak{p}_2 \otimes \mu_p)^{m_2} \oplus (\mathcal{O}_K/\mathfrak{p}_2)^{m_1}$$

Proof. The μ-ordinary locus corresponds to the element (w_1, w_2) where

$$w_i(j) = \begin{cases} j + m_i & 1 \leq j \leq m_{i+1} \\ j - m_{i+1} & m_{i+1} + 1 \leq j \leq g \end{cases}$$

The basis in this case is particularly straightforward as the μ-ordinary locus consists of ordinary abelian varieties. In other words,

j	$F(e_{i,j})$	$V(e_{i,j})$
$1 \leq j \leq m_{i+1}$	$e_{i,j}$	0
$m_{i+1} + 1 \leq j \leq g$	0	$e_{i,j}$

hence span $\{e_{i,j} \mid 1 \leq j \leq m_{i+1}\}$ corresponds to the $\underline{\mathcal{O}_K/\mathfrak{p}_i}$ part and span $\{e_{i,j} \mid m_{i+1} + 1 \leq j \leq g\}$ corresponds to the $\mathcal{O}_K/\mathfrak{p}_i \otimes \mu_p$ part. $\square$

Chapter 4
CM points on unitary Shimura varieties

4.1 CM points over the complex numbers

Let E be a **CM field**—that is, E is a quadratic imaginary extension of a totally real field E^+. Let ρ denote the non-trivial automorphism of E over E^+. A **CM type** of E is a subset Φ of $\mathsf{Hom}\,(E, \mathbb{C})$ such that

$$\mathsf{Hom}\,(E, \mathbb{C}) = \Phi \sqcup \rho\Phi$$

where $\rho\Phi := \{\rho \circ \varphi \mid \varphi \in \Phi\}$. A pair (E, Φ) for which Φ is a CM-type for E is called a **CM pair**. A CM-pair (E, Φ) is said to be **primitive** is there are no proper CM subfields E' of E for which $\Phi|_{E'}$ is a CM-type for E'.

Let A be a simple abelian variety over $\mathbb{C}$ of dimension $g = [E^+ : \mathbb{Q}]$, and suppose that there is an embedding $\iota : E \hookrightarrow \mathrm{End}^0(A)$. Then A has **complex multiplication (by E)**. Furthermore, ι induces an action of E on Lie(A), and

$$\mathrm{Lie}(A) \cong \bigoplus_{i=1}^{g} \mathrm{Lie}_{\varphi_i}$$

where $\varphi_i : E \to \mathbb{C}$ and $a \in E$ acts on Lie_{φ_i} via $\varphi_i(a)$. Then $\Phi = \{\varphi_1, \ldots, \varphi_g\}$ is a CM-type of E, and A is said to have **CM type** (E, Φ). This CM type is necessarily primitive as A is simple [Lan83, Theorem I.3.5].

Given a primitive CM-type (E, Φ), and a lattice $\mathfrak{a}$ in E, define an associated complex torus $A_\Phi := \mathbb{C}_\Phi / \Phi(\mathfrak{a})$ where

$$\mathbb{C}_\Phi := \bigoplus_{\varphi \in \Phi} \mathbb{C}_\varphi = \bigoplus_{i=1}^{g} \mathbb{C}_{\varphi_i}$$

such that $\mathbb{C}_\varphi$ denotes a copy of $\mathbb{C}$ indexed by φ, and

$$\Phi(\mathfrak{a}) := \{(\varphi_1(a), \varphi_2(a), \dots, \varphi_g(a)) \in \mathbb{C}_\Phi \mid a \in \mathfrak{a}\}.$$

By construction A_Φ comes with an injective homomorphism $\iota_\Phi : E \to \operatorname{End}^0(A)$, which is in fact an isomorphism, by letting E act on A_Φ through φ_i on the $\mathbb{C}_{\varphi_i}$ component. When the choice of lattice $\mathfrak{a}$ is relevant, we will write $(A_\Phi; \mathfrak{a})$ for the abelian variety $\mathbb{C}_\Phi / \Phi(\mathfrak{a})$.

Proposition 4.1.1. *The complex torus A_Φ is a simple abelian variety with CM-type (E, Φ). The map $(E, \Phi) \to (A_\Phi, \iota_\Phi)$ is a bijection between primitive CM types of E up to isomorphism and simple abelian varieties with CM by E up to isogeny. Furthermore, for $(A_\Phi, \iota_\Phi; \mathfrak{a})$, the subring $R = \{\alpha \in E \mid \alpha\mathfrak{a} \subset \mathfrak{a}\}$ satisfies*

$$\iota_\Phi(R) = \iota_\Phi(E) \cap \operatorname{End}(A_\Phi).$$

Proof. [Lan83, Theorem I.4.1, I.4.2]. □

It is worth noting that a simple CM abelian variety (A_Φ, ι_Φ) with CM type (E, Φ) is not only defined over $\mathbb{C}$, but can be defined over any field containing the reflex field of the CM type (E, Φ) (see [Lan83, Theorem3.1.1]). As such, unless otherwise specified, we fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and will consider all the CM abelian varieties that follow to be defined over $\bar{\mathbb{Q}}$.

Polarizations on $(A_\Phi; \mathfrak{a})$ correspond to Riemann forms $\psi : \mathfrak{a} \times \mathfrak{a} \to \mathbb{Z}$ such that $\psi_\mathbb{R}$ is symmetric and positive definite. Polarizations that respect ι_Φ are the Riemann forms for which the associated Rosati involution induces complex conjugation on E. The elements $\lambda \in E^\times$ such that

- $\rho\lambda = -\lambda$
- $\Im(\varphi(\lambda)) > 0$ for all $\varphi \in \Phi$

give rise to non-degenerate $\mathbb{Q}$-bilinear forms:

$$\begin{aligned} \psi_\lambda : E \times E &\to \mathbb{Q} \\ (x, y) &\mapsto \mathrm{Tr}_{E/\mathbb{Q}}(\lambda x \rho(y)) \end{aligned}$$

with the properties that

- the involution on E induced by ψ_λ is ρ,
- ψ_λ is alternating,
- the associated form $\psi_\lambda(x, J_\Phi y)$ on $E \otimes \mathbb{R}$ is symmetric and positive definite where J_Φ is the isomorphism $\mathfrak{a} \otimes_\mathbb{Z} \mathbb{R} \to \mathbb{C}_\Phi$ induced by Φ.

Furthermore, any non-degenerate $\mathbb{Q}$-bilinear form $E \times E \to \mathbb{Q}$ satisfying the above properties has the form ψ_λ for some $\lambda \in E^\times$ such that $\rho\lambda = -\lambda$ and $\Im(\varphi(\lambda)) > 0$ for all $\varphi \in \Phi$ (see [Mil06, 2.9] or [Lan83, Theorem I.4.5]). Given a $\lambda \in E^\times$ such that $\rho\lambda = -\lambda$ and $\Im(\varphi(\lambda)) > 0$ for all $\varphi \in \Phi$, every other element of $E^\times$ satisfying these properties is equal to $a\lambda$ where a is a totally positive element of E^+. Even more, for any such ψ_λ there exists an $m \in \mathbb{N}$ such that $m\psi_\lambda$ restricted to $\mathfrak{a} \times \mathfrak{a}$ is a Riemann form. This gives a description of the polarizations on A_Φ in terms of elements of $E^\times$. Note that such an element always exists so A_Φ is always polarizable, making it an abelian variety.

The condition that ψ_λ is integral on $\mathfrak{a} \times \mathfrak{a}$ corresponds to the condition that $\lambda \in (\mathcal{D}_{E/\mathbb{Q}}\mathfrak{a}\rho(\mathfrak{a}))^{-1}$ where $\mathcal{D}_{E/\mathbb{Q}}$ denotes the different ideal of $E/\mathbb{Q}$. In case $\mathfrak{a} = \mathcal{O}_E$, the degree of the polarization associated with λ is $\mathrm{Nm_{E/\mathbb{Q}}}(\lambda)d_E$ where d_E is the discriminant of the field E; for p unramified in E, finding a prime-to-p polarization when $\mathfrak{a} = \mathcal{O}_E$ amounts to finding a $\lambda \in \mathcal{D}_{E/\mathbb{Q}}^{-1}$ as above where $\mathrm{Nm_{E/\mathbb{Q}}}(\lambda)$ is prime-to-p.

For example, suppose that E^+ has strict class number 1 and $\mathcal{D}_{E/E^+} = (\alpha)$. Then $\mathcal{D}_{E^+/\mathbb{Q}} = (\beta)$ for a totally positive element β so that $\alpha\beta \in \mathcal{D}_{E/\mathbb{Q}}$ and $\overline{\alpha\beta} = -\alpha\beta$. Furthermore, since E^+ has strict class number 1, there are units with all the signs in $\mathcal{O}_{E^+}$, and there exists an $\epsilon \in \mathcal{O}_{E^+}^\times$ such that $\lambda = 1/\epsilon\alpha\beta$ gives a principal polarization on any $(A_\Phi, \iota_E; \mathcal{O}_E)$ with CM type (E, Φ).

Recall that K denotes quadratic imaginary field used in the definition of a unitary Shimura datum $\mathscr{D}$ in Section 2.1.2, and χ_1, χ_2 denote the distinct embeddings of K into $\mathbb{C}$. We assume that p is unramified in K.

Lemma 4.1.2. *Fix a prime p. Let E be a CM field containing K such that p is unramified in E, $\mathcal{O}_E$ is a free $\mathcal{O}_K$-module of rank $g = [E : K]$, and $\lambda \in \mathcal{D}_{E/\mathbb{Q}}^{-1}$ of prime-to-p norm. Let (E, Φ) be a CM type and let $m_i = \#\{\phi \in \Phi \mid \phi|_K = \chi_i\}$. Then there exists a complex abelian variety with CM by (E, Φ) on a unitary Shimura variety with good reduction at p with level structure C^p corresponding to the group $\mathrm{GU}(m_1, m_2)$ as in Section 2.1.2.*

Proof. First consider the case where (E, Φ) is simple. Let $(A_\Phi, \iota_\Phi, \lambda; \mathcal{O}_E)$ be a corresponding abelian variety where $\lambda \in E^\times$ represents a primte-to-p polarization on $(A_\Phi, \iota_\Phi; \mathcal{O}_E)$ coming from the construction above. Then let ι_K be the restriction of ι_Φ to $\mathcal{O}_K$ as

$$\iota_K : \mathcal{O}_K \subseteq \mathcal{O}_E \xhookrightarrow{\iota_\Phi} \mathrm{End}\,(A_\Phi)\,.$$

Since the Rosati involution induces ρ on E/E^+, it restricts to the non-trivial Galois automorphism of $K/\mathbb{Q}$, and therefore λ respects the $\mathcal{O}_K$-action coming from ι_K. Let $\chi_i : K \to \mathbb{C}$ be the two embeddings of K into $\mathbb{C}$. It follows that

$$\det(a; \mathrm{Lie}(A_\Phi)) = (T - \chi_1(a))^{m_1}(T - \chi_2(a))^{m_2}$$

for $a \in \mathcal{O}_K$ where $m_i = \#\{\phi \in \Phi \mid \phi|_K = \chi_i\}$.

By assumption, there exists an $\mathcal{O}_K$-module isomorphism $f : \mathcal{O}_K^g \to \mathcal{O}_E$. Write $\psi_2(v, w) := \mathrm{Tr}_{E/\mathbb{Q}}(\lambda f(v)\overline{f(w)})$ for $v, w \in K^g$ (f is the induced isomorphism $K^g \to E$). Then ψ_2 is an alternating form on K^g over $\mathbb{Q}$, corresponding to $\mathrm{GU}(m_1, m_2)$ over $\mathbb{R}$. Suppose that $\mathscr{D}$ is a unitary Shimura datum of signature (m_1, m_2) where $\psi : V \times V \to \mathbb{Q}$. Then from the description of the complex points of Shimura varieties in Section 8 of [Kot92], f will produce a level structure η of type C^p for any $\mathscr{D}$ of signature (m_1, m_2) as long as ψ and ψ_2 are equivalent up to a scalar multiple after tensoring with $\mathbb{Q}_q$ for all primes q. Then $(A_\Phi, \iota_K, \lambda, \eta)$ is a $\mathbb{C}$-point of $\mathcal{M}_{(m_1,m_2),C^p}$.

Suppose that (E, Φ) is a lift of a simple CM type (E', Φ') such that $[E : E'] = d$ and $\mathcal{O}_{E'}$ is a free $\mathcal{O}_K$-module of rank g/d. Then $\lambda' = \mathrm{Tr}_{E/E'}(\lambda)$ is an element in $\mathcal{D}_{E'/\mathbb{Q}}^{-1}$ such that $\mathrm{Nm}_{\mathrm{E'}/\mathbb{Q}}(\lambda')$ is prime-to-p, $\rho(\lambda') = -\lambda'$, and $\Im(\phi(\lambda')) > 0$

for all $\phi \in \Phi'$. Then E' contains K and by the preceding argument, there is an abelian variety with CM by (E', Φ') of the form $(A_{\Phi'}, \iota'_K, \lambda')$ together with a symplectic isomorphism $f' : \mathcal{O}_K^{g/d} \to \mathcal{O}'_E$ for some unitary Shimura datum with signature $(m_1/d, m_2/d)$. Then the product $B = (A^d_{\Phi'}, (\iota')^d_K, \lambda'^d)$ is an abelian variety with CM type (E, Φ) and f' induces an isomorphism $f : \mathcal{O}_K^g \to \mathcal{O}'^d_E$ that induces a level structure on B for some unitary Shimura datum with signature (m_1, m_2). Therefore B gives rise to a $\mathbb{C}$-point on a unitary Shimura variety with good reduction at p coming from the group $\mathrm{GU}(m_1, m_2)$. □

4.2 Reduction of CM points

Fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$ allowing the identifications $\mathsf{Hom}\,(E, \mathbb{C}) = \mathsf{Hom}\,(E, \overline{\mathbb{Q}}_p)$ and $\mathsf{Hom}\,(K, \mathbb{C}) = \mathsf{Hom}\,(K, \overline{\mathbb{Q}}_p)$. For the remainder of this chapter, let $\mathcal{M} := \mathcal{M}_{(m_1,m_2),C^p}$, the unitary Shimura variety of signature (m_1, m_2) and level structure C^p with good reduction at p as defined in Section 2.1.2, and let $\bar{\mathcal{M}}$ denote its special fibre defined over $\kappa(\mathfrak{p})$. Furthermore, let $k = \overline{\mathbb{F}}_p$.

Suppose that $\underline{A} = (A, \iota, \lambda, \eta)$ is a CM point of type $(E, \Phi) \in S$ in $\mathcal{M}(\mathbb{C})$. Then $\underline{A}$ reduces mod p to a point $\mathcal{M}(k)$. We now determine the Ekedahl-Oort stratum in which the reduction of $\underline{A}$ lies.

Let $\underline{A} \in \mathcal{M}(\mathbb{C})$ be a CM point of type (E, Φ). Then write $(A_\Phi, \iota_E, \lambda; \mathfrak{a})$ for an abelian variety with CM by (E, Φ) such that $(A_\Phi, \iota_E|_{\mathcal{O}_B}, \lambda, \eta)$ is a representative for the class $\underline{A}$ where ι_E denotes an E-structure on A_Φ that restricts to ι on $\mathcal{O}_B = \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

Recall that the E-O strata of $\bar{\mathcal{M}}(k)$ are in bijection with the elements of the Weyl group coset ${}^J W$ as in Section 3.4.

Proposition 4.2.1. *Let $m_1 \geq m_2 \neq 0$, and let p be a prime that is unramified in K. Let $S(E, m_1, m_2)$ be the set of CM types (E, Φ) such that*

- *E is a CM field containing K such that $[E : K] = g = m_1 + m_2$ and p is unramified in E,*
- *Φ is a CM type of E such that $m_i = \# \{\phi \in \Phi \mid \phi|_K = \chi_i\}$.*

Let $\underline{A} \in \mathcal{M}(\mathbb{C})$ be a CM point of type $(E, \Phi) \in S(E, m_1, m_2)$ and let (A_Φ, ι_E) be a corresponding abelian variety with CM by (E, Φ). Then there is a map $\nu : S(E, m_1, m_2) \to {}^J W$ with the property that if $\mathcal{O} := \iota_E^{-1}(\iota_E(E) \cap \mathrm{End}\,(A_\Phi))$ is maximal at p, the reduction of $\underline{A}$ is in the Ekedahl-Oort stratum associated to $\nu((E, \Phi))$. Moreover, the map ν can be calculated explicitly.

Proof. Let $\underline{A} \in \mathcal{M}(\mathbb{C})$ be as in the proposition and let $(A_\Phi, \iota_E, \lambda; \mathfrak{a})$ be a corresponding abelian variety with CM type (E, Φ). Then A_Φ is defined over a number field L where A_Φ has good reduction at a prime $\mathfrak{p}|p$ of L. Let $(\bar{A}_\Phi, \bar{\iota}_K, \bar{\lambda}; \mathfrak{a})$ be the reduction of $\underline{A}$ mod $\mathfrak{p}$.

Let Φ_0 be the image of Φ under the identification

$$\mathsf{Hom}\,(E, \mathbb{C}) \to \mathsf{Hom}\,(E, \mathbb{Q}_p^{un}) \to \mathsf{Hom}\,(\mathcal{O}/p\mathcal{O}, k) = \prod_{\mathcal{P}|p} \mathsf{Hom}\,(\mathcal{O}/\mathcal{P}, k),$$

and let σ denote the Frobenius element of k. Then there is a model for the Dieudonné module of $\bar{A}_\Phi[p]$ with a k-basis $\{e_\phi \mid \phi \in \mathsf{Hom}\,(\mathcal{O}/p\mathcal{O}, k)\}$ where

- $F(e_\phi) = e_{\sigma\circ\phi}$ if $\phi \notin \Phi_0$, and $F(e_\phi) = 0$ otherwise.
- $V(e_{\sigma\circ\phi}) = e_\phi$ if $\phi \in \Phi_0$, else $V(e_{\sigma\circ\phi}) = 0$.

Additionally, the element $x \in \mathcal{O}/p\mathcal{O}$ acts on e_ϕ as multiplication by $\phi(x)$. This means that $\mathcal{O}_K/p\mathcal{O}_K$ acts on e_ϕ through χ_i where $\phi|_K = \chi_i$. Therefore, $\mathcal{D}_i$ has a basis $\{e_\phi \mid \phi \in \Phi_0, \phi|_K = \chi_i\}$.

We can now use the methods from Chapter 3 to identify the E-O stratum of $\underline{A}$. As in Section 3.3, applying F and V^{-1} to the flag $(0) \subset \mathcal{D} = \oplus_{\phi \in \Phi_0} k e_\phi$ gives rise to a canonical flag. This can be completed to a conjugate flag, which is a complete, $\mathcal{O}_K$-invariant symplectic flag, using the basis elements given above in a way that *only* depends on Φ and w. The relative position of the conjugate flag and the Hodge flag gives an element $\nu((E, \Phi))$ of ${}^J W$ by Proposition 3.3.1 and $(A_\Phi, \iota_K, \lambda; \mathfrak{a})$ reduces mod $\mathfrak{p}$ to a point in the E-O stratum associated with $\nu((E, \Phi))$.

In Sections 3.4 and 3.5, a presentation for ${}^J W$ was given corresponding to a particular choice of basis for $\mathcal{D}$. We now explicitly describe the procedure for obtaining $\nu((E, \Phi)) = (w_1, w_2) \in {}^J W$ with respect to that presentation.

The conjugate flag decomposes as two flags, one of $\mathcal{D}_1$ and another of $\mathcal{D}_2$. Now, make an identification between the labels for the basis of $\mathcal{D}_i$, $\{\phi \mid \phi|_K = \chi_i\}$ with $\{1, \ldots, g\}$, so that the conjugate flag has the form:

$$(0) \subset \{e_{i,1}\} \subset \{e_{i,1}, e_{i,2}\} \subset \ldots, \{e_{i,1}, \ldots, e_{i,g}\} = \mathcal{D}_i,$$

and the Hodge flag takes the form

$$(0) \subset \{e_{i,j} \mid e_{i,j} = e_\phi, \phi \in \Phi_0\} \subset \mathcal{D}.$$

Then define $\nu((E,\Phi))$ to be (w_1, w_2) where w_i is the image of any permutation taking $\{j \mid e_{i,j} = e_\phi, \phi \in \Phi_0\}$ to $\{1, 2, \ldots, g\}$ under the map $S_g \to {}^{J_i}S_g$.

Observe that the procedure from obtaining $\nu((E,\Phi)) \in {}^J W$ from a Dieudonné module with basis $\{e_\phi \mid \phi \in \mathsf{Hom}\,(\mathcal{O}/p\mathcal{O}, k)\}$ as constructed above can be carried out completely formally. In this way one defines $\nu : S \to {}^J W$ independently from any reference to a point in $\mathcal{M}(\mathbb{C})$. □

Corollary 4.2.2. *Let (E,Φ) be a CM type as in Proposition 4.2.1 and let Φ_0 be the image of Φ under the map $\mathsf{Hom}\,(E,\mathbb{C}) \to \mathsf{Hom}\,(\mathcal{O}_E/p\mathcal{O}_E, k)$. Then*

$$a\left(\nu(E,\Phi)\right)) = \#\,\sigma(\Phi_0^c) \cap \Phi_0$$

and

$$f\left(\nu((E,\Phi))\right) = \#\left\{\phi \in \Phi_0^c \cap \sigma(\Phi_0^c) \;\middle|\; \sigma^2 \circ \phi = \phi\right\}$$

where $\sigma(S) = \{\sigma \circ s \mid s \in S\}$.

Proof. From the proof of Proposition 4.2.1, the size of $\ker(F) \cap \ker(V)$ is $\#\,\{e_\phi \mid \phi \notin \Phi_0, \sigma \circ \phi \in \Phi_0\}$ giving the a-number. As for the f-number, the proof of Proposition 3.5.2 showed that for $w \in {}^J W$, $f(w) = \#\,\{e_\phi \mid F^2(e_\phi) = e_\phi\}$. The result follows. □

4.3 Examples

Example 4.3.1. Let (E,Φ) be as Proposition 4.2.1 such that p splits completely in E. This implies that p also splits in K. Then F acts as 0 on e_ϕ when $e_\phi \in \Phi_0$ and as the identity otherwise, and V acts as 0 on e_ϕ when $e_\phi \notin \Phi_0$ and the identity otherwise. Therefore, $\nu((E,\Phi))$ corresponds to the ordinary locus (see Example 3.5.11).

Example 4.3.2. Let E be a CM field and suppose further that

- $E = KE^+$,
- p splits in K,
- p is inert in E^+,
- E^+ is cyclic Galois of order g,
- Φ is a CM type of E as in Proposition 4.2.1.

Then $E/\mathbb{Q}$ is Galois with Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/g\mathbb{Z}$ where $\mathbb{Z}/2\mathbb{Z}$ corresponds with complex conjugation on E so that the complex conjugate of an element (a,b) is $(a+1,b)$ and $(0,b)$ restricts to χ_1 on K. Then the CM type Φ of E has the form

$$\{(a_i, i) \mid 0 \le i \le g-1\}$$

where $a_i \in \{0,1\}$. For convenience, denote the CM type Φ of E as the binary string $a = a_0 a_1 \dots a_{g-1}$. Observe that m_1 is equal to the number of 0's in a and m_2 is equal to the number of 1's in a.

Since σ acts on Φ as right shift, by Corollary 4.2.2

$$a\left(\nu((E,\Phi))\right) = \#\left\{a_i \mid a_i + 1 = a_{i+1} \pmod 2\right\}$$

and

$$f\left(\nu((E,\Phi))\right) = 0.$$

In particular, $\nu((E,\Phi))$ is never almost-ordinary (unless $g = 2$) and never μ-ordinary. Furthermore, if the CM type is chosen so that a has the maximal number of alternating 0s and 1s, then $a(\nu((E,\Phi))) = 2m_2$ and $\nu((E,\Phi))$ has maximal a-number among E-O strata of signature (m_1, m_2). Note that this does *not* mean that $\nu((E,\Phi))$ corresponds to the core locus (see Section 3.5.1 and the examples below).

Table 4–1 gives $\nu((E,\Phi)) = (w_1, w_2)$ for all CM types up to signature $(5,2)$. Observe that for every signature in the table, there is a CM type Φ such that $\nu((E,\Phi))$ is the identity Weyl element; that is, $\nu((E,\Phi))$ corresponds to the core locus. These CM types have the property that the 1's are as *evenly spaced* among the 0's as possible. This pattern continues, and was checked[1] for $m_1, m_2 \leq 200$, supporting the following conjecture.

Conjecture 4.3.3. *Let E be a CM field such that*

- $E = KE^+$,
- *p splits in K,*
- *p is inert in E^+,*
- *E^+ is cyclic Galois of order g.*

Then for every signature (m_1, m_2) such that $m_1 + m_2 = g$, there exists a CM type Φ for E such that $\nu((E,\Phi))$ is the identity Weyl element.

Remark. One may ask why we have yet to obtain a proof of this conjecture. Constructing Φ with the property that the 1's and 0's are as evenly spaced as

[1] This check was done using a Python implementation of the map ν following the proof of Proposition 4.2.1.

Table 4–1: E-O strata by CM type

CM type	(m_1, m_2)	w_1	w_2	$\ell(w)$	$a(w)$	Type of Stratum
001	(2, 1)	123	123	0	2	Core
0001	(3, 1)	1234	1234	0	2	Core
0011	(2, 2)	1324	1324	1	2	Almost-core
0101	(2, 2)	1234	1234	0	4	Core
00011	(3, 2)	14235	13425	2	2	—
00101	(3, 2)	12345	12345	0	4	Core
000111	(3, 3)	145236	145236	4	2	—
001011	(3, 3)	142356	124536	2	4	—
010011	(3, 3)	124536	142356	2	4	—
010101	(3, 3)	123456	123456	0	6	Core
000011	(4, 2)	152346	134526	3	2	—
000101	(4, 2)	123546	132456	1	4	Almost-core
001001	(4, 2)	123456	123456	0	6	Core
0000111	(4, 3)	1562347	1456237	6	2	—
0001011	(4, 3)	1523647	1425637	4	4	—
0100011	(4, 3)	1256347	1452367	4	4	—
0010011	(4, 3)	1253467	1245367	2	4	—
0100101	(4, 3)	1234567	1234567	0	6	Core
0000011	(5, 2)	1623457	1345627	4	2	—
0000101	(5, 2)	1236457	1342567	2	4	—
0001001	(5, 2)	1234567	1234567	0	4	Core

possible is quite subtle, which means that the interaction between F and V that give rise to $\nu(E, \Phi)$ is difficult to characterize in general. For instance, when the signature is $(12, 5)$, the choice of $\Phi = (10001000100100100)$ does not correspond to the core locus and instead corresponds to $\nu((E, \Phi)) = (w_1, w_2) \in {}^J W$ where $\ell(w) = 2$ and

$$w_1 = [1234567891011131412151617] .$$

The core locus for $(12, 5)$ corresponds to $\Phi = (10001001000100100)$.

This conjecture has the following application: under the additional assumptions that E has a relative integral basis over K (*i.e.* $\mathcal{O}_E$ is a free $\mathcal{O}_K$-module of rank g) and that there exists an element $\lambda \in \mathcal{D}_{E/\mathbb{Q}}^{-1}$ satisfying:

- $\rho(\lambda) = -\lambda$,
- $\Im \phi(\lambda) > 0$ for all $\phi \in \Phi$,
- $\mathrm{Nm_{E/\mathbb{Q}}}(\lambda)$ is prime-to-p

then by Lemma 4.1.2 there exists an abelian variety with CM by (E, Φ) giving rise to a point in the core E-O stratum of $\bar{\mathcal{M}}$ by Proposition 4.2.1.

Remark. The previous examples show that CM points occur in the most extreme strata for $\bar{\mathcal{M}}$ when p is split (and when CM fields with the appropriate properties exist). It follows that on the level of CM points of type (E, Φ) (where E has degree $2(m_1 + m_2)$ over $\mathbb{Q}$), the restriction on the splitting behaviour of p in the CM field E coming from the prescribed splitting behaviour of p in the CM subfield K is reflected in the possibilities for the types of the extreme E-O strata for $\bar{\mathcal{M}}$: since p cannot be completely inert, this dictates *how close to superspecial* a point in the core locus of $\bar{\mathcal{M}}$ may be. However, p can split completely in E, and this is reflected by the fact that the μ-ordinary stratum is always ordinary.

Example 4.3.4. Let E be a CM field and suppose that

- $E = KE^+$,
- p is inert in K,
- E^+ is cyclic Galois of order g
- Φ is a CM type of E as in Proposition 4.2.1.

Write D_E (or D_K, D_{E^+}) for the decomposition group of E (resp. K and E^+) over p. Since $K \cap E^+ = \mathbb{Q}$,

$$\mathrm{Gal}(\mathbb{F}_{p^f}/\mathbb{F}_p) \cong D_E = D_K \times D_{E^+} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/h\mathbb{Z}$$

where $h = f/2$. It follows that h is an odd number such that $\mathbb{Z}/h\mathbb{Z}$ is a subgroup of $\mathbb{Z}/g\mathbb{Z}$. Then under

$$\mathsf{Hom}\,(E, \mathbb{C}) \mapsto \prod_{\mathcal{P}|p} \mathsf{Hom}\,(\mathcal{O}_E/\mathcal{P}\mathcal{O}_E, k) \cong \prod_{i=1}^{g/h} \mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$$

complex conjugation maps to σ^h.

If $h = 1$, then p splits completely in E^+. Furthermore, $D_E \cong \mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$, and every $\mathcal{P}|p$ is fixed by complex conjugation. Under

$$\mathsf{Hom}\,(E, \mathbb{C}) \mapsto \prod_{\mathcal{P}|p} \mathsf{Hom}\,(\mathcal{O}_E/\mathcal{P}\mathcal{O}_E, k) \cong \prod_{i=1}^{g} \mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$$

complex conjugation maps to σ. Then $\phi \in \Phi_0$ if and only if $\sigma \circ \phi \notin \Phi_0$ and $\sigma^2 \circ \phi = \phi$. Then $\sigma(\Phi_0^c) = \Phi_0$ and $a\,(\nu(E, \Phi))) = \#\,\sigma(\Phi_0^c) \cap \Phi_0 = g$ by Corollary

Table 4–2: E-O strata by CM type

CM type	(m_1, m_2)	w_1	w_2	$\ell(w)$	$a(w)$	Type of Stratum
001	(2, 1)	132	213	1	1	Almost-core
00011	(3, 2)	14235	13425	2	3	—
00101	(3, 2)	14523	34125	4	1	—
0000111	(4, 3)	1253647	1425367	3	5	—
0010011	(4, 3)	1256374	4152367	5	3	—
0001011	(4, 3)	5162347	1456273	7	3	—
0100011	(4, 3)	1562374	4156237	7	3	—
0100101	(4, 3)	1567234	4561237	9	1	—
0000011	(5, 2)	1263457	1345267	3	5	—
0000101	(5, 2)	1627345	3451627	7	3	—
0001001	(5, 2)	1263745	3415267	5	3	—

4.2.2. It follows that $\nu(E, \Phi)$ is the identity Weyl element corresponding to the core locus which is also superspecial when p is inert.

At the other extreme, suppose that $h = g$ and hence p is inert in E^+. Then the image Φ_0 of a CM type corresponds to a choice between σ^i and σ^{g+i} for each $i = 1, \ldots, g$. Since g is odd, exactly one of i and $g + i$ is even for each i. Then Φ_0 can be represented as binary string of length g, $a = a_1 a_2 \ldots a_g$ where $\sigma^i \in \Phi_0$ if and only if $i \equiv a_i \pmod 2$. We may assume that, the signature of the CM type is $(\#a_i = 0, \#a_i = 1)$ (as opposed to $(\#a_i = 1, \#a_i = 0)$). Since $\sigma^2 \circ \phi \neq \phi$ for all

$\phi \in \mathrm{Gal}(\mathbb{F}_{p^g}/\mathbb{F}_p)$, it follows that the f-number of a CM type of this form is always 0. Low dimensional examples of these CM types are listed in Table 4–2.

Chapter 5
The Newton stratification

Chapter 3 gave rise to a stratification of the moduli space $\mathcal{M}$ by isomorphism classes of p-torsion group schemes with extra structure. We now study another stratification of the moduli space that can be obtained by considering p-divisible groups with extra structure up to isogeny—the Newton stratification. We will ultimately demonstrate the relationship between the Newton stratification and the E-O stratification of unitary Shimura varieties—reflecting the often subtle relationship between the isogeny class of a p-divisible group and the isomorphism class of its p-torsion part.

5.1 Group theoretic classification of isocrystals

In Section 2.2.4, we introduced isocrystals, and recalled how they can be used to classify the isogeny classes of p-divisible groups of abelian varieties with prime-to-p polarization using slope sequences. However, we will see that this is not always enough to classify p-divisible groups with additional endomorphism structure up to isogeny.[1] In order to obtain a classification of p-divisible groups with $\mathscr{D}$-structure—where $\mathscr{D}$ is a unitary PEL Shimura datum—we require the

[1] For example, this can be easily inferred from the diagram in Section 5.3.1.

classification of isocrystals with additional structures due to Kottwitz [Kot85]. This is the subject of this section.

Let $\mathscr{D} = (B, *, V, \langle \ , \ \rangle, \mathcal{O}_B, \Lambda, h)$ be a PEL Shimura datum with good reduction at p, and let G be the $\mathbb{Z}_p$-group of $\mathcal{O}_B$-linear symplectic similitudes of Λ with Borel B and maximal torus T as defined in Section 2.1.1. Let $(X^*(T), \Phi, X_*(T), \Phi^\vee, \Delta)$ be the corresponding based root system (*i.e.* Δ is an ordered basis for Φ corresponding to the choice of Borel B). Furthermore, recall from Section 2.1.1 that there is a cocharacter μ_h of $\mathbf{G}$ over $\mathbb{C}$ coming from h, such that $\mu_h(z) = h(\mathbb{C})(z, 1)$ and the $\mathbf{G}(\mathbb{C})$-conjugacy class of μ_h is defined over the reflex field of $\mathscr{D}$, a finite extension of $\mathbb{Q}$. Then by fixing an embedding of $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ where $\overline{\mathbb{Q}} \subseteq \mathbb{C}$, the conjugacy class of μ_h can be considered a $G(\overline{\mathbb{Q}}_p)$-conjugacy class $[\mu]$ of cocharacters of G. In turn, $[\mu]$ can be thought of as a $W(G, T)$-orbit of $X_*(T)$, where $W(G, T)$ is the Weyl group of G with respect to T. Define $\mu \in X_*(T)$ to be the representative for the class $[\mu]$ that is dominant with respect to $\Delta^\vee$.

There is way to classify p-divisible groups with $\mathscr{D}$-structure using the group G of $\mathcal{O}_B$-linear sympletic similitudes of Λ by passing through its associated Dieudonné module and isocrystal. We follow the presentation of [VW13] closely throughout this section, and more details can be found in [RR96].

Recall that k is an algebraically closed field of characteristic p and that $Q(k)$ denotes the fraction field of $W(k)$. Consider the following observation, given the Dieudonné module $\mathfrak{D}$ coming from a p-divisible group with $\mathscr{D}$ structure, there exists an isomorphism $\alpha : \mathfrak{D} \to \Lambda \otimes_{\mathbb{Z}_p} W(k)$ where Λ is the $\mathbb{Z}_p$-lattice from $\mathscr{D}$ [RZ96, Theorem 3.16]. Then F corresponds to $b(\mathrm{id} \otimes \sigma)$ for some $b \in G(Q(k))$, and for any other choice of isomorphism $\alpha' : P \to \Lambda \otimes_{\mathbb{Z}_p} W(k)$, there exists a unique $g \in G(W(k))$ such that F corresponds to $b'(\mathrm{id} \otimes \sigma)$ on $\Lambda \otimes_{\mathbb{Z}_p} W(k)$ through α' where $b' = gb\sigma(g)^{-1}$. It follows that the set of isomorphism classes of p-divisible groups with $\mathscr{D}$-structure up to isomorphism injects into the set $G(Q(k))/\sim$ where $b \sim b'$ if $b' = gb\sigma(g)^{-1}$ for some $g \in G(W(k))$. Thus, considering p-divisible groups with $\mathscr{D}$-structure up to *isogeny* requires taking elements $b \in G(Q(k))$ up to $G(Q(k))$-σ-conjugacy instead of $G(W(k))$-σ-conjugacy.

Let the $G(Q(k))$-σ-conjugacy class of $b \in G(Q(k))$ be denoted by $[b]$, *i.e.*

$$[b] = \left\{ gb\sigma(g)^{-1} \;\middle|\; g \in G(Q(k)) \right\}.$$

Let $B(G)$ denote the set of all $G(Q(k))$-σ-conjugacy classes of $G(Q(k))$. The considerations above give an injection

$$\left\{ \begin{array}{c} p\text{-divisible group with} \\ \mathscr{D}\text{-structure up to isogeny} \end{array} \right\} \hookrightarrow B(G). \tag{5.1}$$

Finally, let $B(G, \mu)$ be defined to be the image of this map taking p-divisible groups to $B(G)$.

5.1.1 Description of $B(G, \mu)$

In order to properly define the Newton stratification, we need a poset structure on $B(G, \mu)$ that corresponds to the closure relations on the Newton strata. The poset structure on $B(G, \mu)$ is not immediately accessible by the definition given in the previous section. Work by [RR96, KR03, Luc04, Gas10] shows that there is another description of $B(G, \mu)$ that can be used to endow it with a poset structure that will reflect the closure relations on the Newton strata as desired.

Following Kottwitz [Kot85], define $\mathcal{N}(G)$ to be the set of σ-invariants in the set of conjugacy classes of homomorphisms $\mathbf{D}_{Q(k)} \to G_{Q(k)}$:

$$\mathcal{N}(G) = (\mathrm{Int}(G(Q(k)))\backslash \mathsf{Hom}_{Q(k)}(\mathbf{D}, G))^{\langle\sigma\rangle}.$$

As in [RR96], we make the identifications

$$\begin{aligned}\mathcal{N}(G) &= (\mathrm{Int}(G(Q(k)))\backslash \mathsf{Hom}_{Q(k)}(\mathbf{D}, G))^{\langle\sigma\rangle} \\ &= (X_*(T)_{\mathbb{Q}})^{\Gamma}_{dom}\end{aligned}$$

where $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $(X_*(T)_{\mathbb{Q}})_{dom}$ is defined by as the set of representatives from $X_*(T)_{\mathbb{Q}}$ that are dominant with respect to the Borel B.

Theorem 5.1.1. *[Kot85, Section 4] Let $b \in G(Q(k))$. Then there exists a unique $\nu(b) \in \mathsf{Hom}(\mathbf{D}, G)$ for which there exists $s \in \mathbb{Z}_{>0}$ and $c \in G(Q(k))$ such that*

- $s\nu(b) \in \mathsf{Hom}(\mathbb{G}_m, G)$,
- $\mathrm{Int}(c) \circ s\nu(b)$ *is defined over the fixed field of* σ^s *in* $Q(k)$
- $cb\sigma(b)\ldots\sigma^s(b)\sigma^s(c)^{-1} = c(s\nu(b)(p))c^{-1}$.

The morphism $\nu(b)$ satisfying the above criterion satisfy the following properties:

- $gb\sigma(g)^{-1} \mapsto \mathrm{Int}(g) \circ \nu(b)$ *for all* $g \in G(Q(k))$,
- $\sigma(b) = \sigma(\nu(b))$,
- $\nu(b) = \mathrm{Int}(b) \circ \sigma(\nu(b))$.

Succinctly, there is a map called the **Newton map**, ν_G, given by

$$\nu_G : B(G) \to \mathcal{N}(G).$$

The image under ν of an element $b \in B(G)$ is called the **slope** or **Newton polygon** of b. These terms originate from the terminology for $G = \mathrm{GL}_h$ where $\mathcal{N}(G)$ corresponds to the classification of isocrystals of height h by slope sequence and Newton polygons as in Section 2.2.4.1. As an example, we now show how ν_G captures the classical information of the slope sequence for an isocrystal of height h (as in Section 2.2.4.1).

In particular, any isocrystal (P, F) of height h, can be written as

$$(P, F) = (V \otimes_{\mathbb{Q}_p} Q(k), b(\mathrm{id}_V \otimes \sigma))$$

where $b \in \mathrm{GL}_h(Q(k))$. Suppose that (P, F) has slope sequence

$$\lambda_1 < \ldots < \lambda_r$$

and isotypic components P_i corresponding to each λ_i. Then

$$\nu_{\mathrm{GL}_h}(b) = \oplus_{i=1}^{r} \lambda_i, \qquad \lambda_i : \mathbf{D} \to \mathbb{G}_m \hookrightarrow \mathrm{GL}(P_i).$$

The Newton map for GL_h is injective by the classification of isocrystals by slope sequence.

While the slope morphism ν_G captures all of the information necessary to classify the elements of $B(\mathrm{GL}_h)$, this is not the case in general. Therefore, an additional map is required to classify the elements of $B(G)$ for other reductive groups G. There is a map $\kappa_G : B(G) \to \pi_1(G)_\Gamma$ where $\pi_1(G) := X_*(T)/\Phi^\vee$[2] and

[2] recall that $\Phi^\vee$ is the coroot lattice of the root system for (G, T)

$\pi_1(G)_\Gamma$ denotes taking the independent of $\pi_1(G)$ under the Galois group Γ. See [Kot85] and [RR96] for the general description of κ_G. Denote by $\lambda^\flat$ the image of $\lambda \in X_*(T)$ in $\pi_1(G)$. Given a $b \in G(Q(k))$ there exists a unique $\lambda \in X_*(T)_{dom}$ such that $b \in G(W(k))\lambda(p)G(W(k))$ by the Bruhat decomposition of G. Then κ_G has the simple description:

$$\kappa_G : B(G) \to \pi_1(G)_\Gamma$$
$$[b] \mapsto \lambda^\flat.$$

In particular, when $b = \lambda(p)$ for some $\lambda \in X_*(T)_{dom}$, $\kappa_G([\lambda(p)]) = \lambda^\flat$. Together, $\nu_G(b)$ and $\kappa_G(b)$ determine an element $b \in B(G)$ uniquely.

Now, $(X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q})_{dom}$ has a partial order given by $\nu \leq \nu'$ if and only if $\nu' - \nu$ is a non-negative linear combination of positive coroots. This induces a partial order on $B(G)$ via $b \leq b'$ if and only if $\nu(b) \leq \nu(b')$ and $\kappa_G(b) = \kappa_G(b')$. In the case when $G = \mathrm{GL}_h$, the condition that $\kappa_G(b) = \kappa_G(b')$ corresponds to the Newton polygons (in the sense of Section 2.2.4.1) of b and b' having the same endpoints, and if $\kappa_G(b) = \kappa_G(b')$, then $\nu(b) \leq \nu(b')$ if and only if the Newton polygon of b lies above the Newton polygon of b' [RR96, Proposition 2.4]. In other words, the partial order on $B(\mathrm{GL}_h)$ corresponds to the familiar partial order on Newton polygons as described in Section 2.2.4.1.

Let Γ_μ be the stabilizer of μ in Γ and set

$$\bar{\mu} := [\Gamma : \Gamma_\mu]^{-1} \sum_{\tau \in \Gamma/\Gamma_\mu} \tau(\mu) \in (X_*(T)_{\mathbb{Q}})^{\Gamma}_{dom},$$

then

$$B(G,\mu) = \left\{ b \in B(G) \;\middle|\; \nu(b) \leq \bar{\mu}, \kappa(b) = \mu^\flat \right\}.$$

By [RR96, Proposition 2.4], $B(G,\mu)$ is finite. This last characterization of $B(G,\mu)$ will be used to construct the stratification of $\mathcal{M}$ by isogeny classes of p-divisible groups.

5.2 The Newton stratification

We are now in a position to define the Newton stratification. Recall that $\mathcal{M}$ denotes the special fibre at $\kappa(\mathfrak{p})$ of the Shimura variety corresponding to the PEL datum $\mathscr{D}$ with level structure $C^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$ as defined in Section 2.1.2. Following [VW13, Section 8], define

$$\mathrm{Nt} : \mathcal{M} \to B(G,\mu)$$

by taking a point $\underline{A} = (A, \iota, \lambda, \eta) \in \mathcal{M}(k)$ to the image of its p-divisible group $A(p)$ with $\mathscr{D}$-structure in $B(G,\mu)$ under (5.1). $\mathrm{Nt}(\underline{A})$, is called the **Newton point** of $\underline{A}$.

Let $b \in B(G,\mu)$. Then by [RR96, Section 3], [VW13, Theorem 11.1], [Ham14, Theorem 1.1] the set

$$\mathcal{N}_b = \{\underline{A} \in \mathcal{M}(k) \mid \mathrm{Nt}(\underline{A}) = b\}$$

is a non-empty locally-closed subset of $\mathcal{M}(k)$, and

$$\overline{\mathcal{N}_b} = \bigcup_{b' \leq b} \mathcal{N}_{b'}. \tag{5.2}$$

The **Newton stratum** of $b \in B(G,\mu)$ is then $\mathcal{N}_b$ endowed with its corresponding reduced subscheme structure.

There are two extreme strata that come from the maximal and minimal elements of $B(G,\mu)$. In particular, $B(G,\mu)$ has a unique maximal element, b_μ, by construction, and a unique minimal element, b_{basic}, by [RR96, Proposition 2.4]. The element b_μ is characterized by the property that $\nu(b_\mu) = \bar{\mu}$, and the element b_{basic} is characterized by the property that $\nu(b_{\text{basic}}) \in X_*(Z)_{\mathbb{Q}}$ where Z is the centre of G. The stratum $\mathcal{N}_{b_\mu}$ is called the **μ-ordinary stratum** and $\mathcal{N}_{b_{\text{basic}}}$ is called the **basic stratum**. By the closure relation in (5.2), the basic stratum is closed and the μ-ordinary stratum is open in $\mathcal{M}$.

Let $\mathcal{N}(G)_\mu$ be the image of $B(G,\mu)$ under $\nu_G : B(G) \to \mathcal{N}(G)$. By construction, every element $\nu \in \mathcal{N}(G)_\mu$ satisfies $\nu \leq \bar{\mu}$. Suppose that $\nu' \leq \nu$ for $\nu, \nu' \in \mathcal{N}(G)_\mu$. Then a **chain** between ν' and ν is a sequence:

$$\nu' = \nu_0 \leq \nu_1 \leq \ldots \leq \nu_n = \nu$$

where $\nu_i \in \mathcal{N}(G)_\mu$. A chain is called **maximal** if it is not a proper subsequence of another chain between ν' and ν. By work of Chai [Cha00, Theorem 7.4] and Hamacher [Ham14], maximal chains exist between ν' and ν, and every maximal chain has the same length (which is independent of $\bar{\mu}$). We write $\text{length}([\nu',\nu])$ for the length of a maximal chain between $\nu \leq \nu'$ in $\mathcal{N}(G)_\mu$.

Theorem 5.2.1 ([Ham14, Theorem1.1])**.** *Let $b \in B(G,\mu)$. The Newton stratum $\mathcal{N}_b$ is equidimensional and*

$$\dim \mathcal{N}_b = m_1 m_2 - \text{length}([\nu(b), \bar{\mu}]).$$

Remark. For our purposes, this description of the dimension of $\mathcal{N}_b$ is sufficient, although there are explicit formulae for how to calculate the dimension of $\mathcal{N}_b$. See [Ham14] for details.

5.3 Newton stratifications of unitary Shimura varieties

In this section, we calculate $B(G, \mu)$ and its poset structure for a unitary PEL Shimura datum $\mathscr{D}$ with signature (m_1, m_2) and good reduction at p. In the end, we will give a description of $B(G, \mu)$ together with its poset structure using classical slope sequences by embedding G into general linear groups, making it easier to visualize and understand the results.

We begin by describing the structure of the group G arising from $\mathscr{D}$. In a similar manner to Section 3.4, the group $G(Q(k))$ can be identified with the subgroup of $\mathrm{GL}_g(Q(k)) \times \mathrm{GL}_g(Q(k))$, given by

$$GU(Q(k)^{2g}, \Psi) \cong \{(M, aM^\vee) \in \mathrm{GL}_g(Q(k)) \times \mathrm{GL}_g(Q(k)) \mid a \in Q(k)^*\} \qquad (5.3)$$

where $M^\vee = (M^t)^{-1}$. Under this isomorphism, the involution σ_Ψ corresponds to the involution $\varepsilon : (M_1, M_2) \mapsto (M_2^t, M_1^t)$. Let T be the diagonal torus of G; *i.e.* the torus whose $Q(k)$-points are given by pairs $(M, aM^\vee)$ where M is a diagonal matrix and $a \in Q(k)^\times$. Let B be the Borel subgroup of G corresponding to the subset of elements of the form $(M, aM^\vee)$ where M is upper triangular and $a \in Q(k)^*$.

Let ϕ_i denote the cocharacter given by sending t to $(M_i, M_i^\vee)$ where M_i is the matrix with t in the i^{th} position on the diagonal and 1's on the rest of the diagonal, and let ϕ be the cocharacter taking t to $(1,t)$. It follows that

$$X_*(T) = \langle \phi_i, \phi \mid i = 1, \ldots, g \rangle \cong \mathbb{Z}^g \times \mathbb{Z}.$$

On the other hand, $\Phi^\vee = \langle \phi_i - \phi_j, \phi | 1 \leq i \neq j \leq g \rangle$ and $(X_*(T)_{\mathbb{Q}})_{dom}$ corresponds to situation where $\phi_i - \phi_j$ is positive if $i < j$. Furthermore, $\pi_1(G) \cong \mathbb{Z}$ and

$$\begin{aligned}(-)^\flat &: \mathbb{Z}^g \times \mathbb{Z} \to \mathbb{Z}\\ (a_1, \ldots, a_g, a) &\mapsto \sum_{i=1}^{g} a_i.\end{aligned}$$

For a unitary PEL datum with signature (m_1, m_2), μ_h is the cocharacter

$$z \mapsto (\mathrm{diag}(z^{m_1}, 1^{m_2}), \mathrm{diag}(1^{m_1}, z^{m_2})).$$

Therefore, μ is the character $(1^{m_1}, 0^{m_2}, 1) \in X_*(T)$ or $(1^{m_1}, 0^{m_2}), (0^{m_1}, 1^{m_2})$ if viewed inside $\mathrm{GL}_g(\overline{\mathbb{Q}}_p) \times \mathrm{GL}_g(\overline{\mathbb{Q}}_p)$ with respect to the standard $2g$ cocharacters. So far, the Galois action on $X_*(T)$ has not been taken into account. The next two sections consider the two cases arising from when p is either split or inert in K.

5.3.1 p split in K

In the situation where p splits in K, even the $\mathbb{Q}_p$-points of G can be viewed as a subgroup of $\mathrm{GL}_g(\mathbb{Q}_p) \times \mathrm{GL}_g(\mathbb{Q}_p)$ via (5.3) and σ acts trivially on $X_*(T)$. It follows that $\bar{\mu} = \mu$ and $\mu^\flat = m_1$. Therefore,

$$B(G, \mu) = \{ b \in B(G) \mid \nu(b) \leq \mu, \kappa(b) = m_1 \}.$$

$B(G, \mu)$ can be described in terms of its embedding into $B(\mathrm{GL}_g) \times B(\mathrm{GL}_g)$. Using the description of the $B(\mathrm{GL}_g)$ in terms of Newton polygons to describe the first factor of the embedding, b_μ can be recognized as corresponding to the Newton polygon having endpoint (g, m_1) and slope sequence $(0, 1)$ with multiplicities (m_2, m_1). On the other hand, b_{basic} corresponds to the Newton polygon with endpoint (g, m_1) and all slopes equal, that is, slope m_1/g with multiplicity g. The rest of the elements on $B(G, \mu)$ correspond to Newton polygons with integral breakpoints and endpoint (g, m_1) lying between b_{basic} and b_μ. The second factor in the embedding into $B(\mathrm{GL}_g) \times B(\mathrm{GL}_g)$ can be obtained by taking the dual of the Newton polygon of the first factor. This discussion leads to the following corollary.

Corollary 5.3.1. *When p is split in K, the total Newton polygon (as embedded in $B(\mathrm{GL}_{2g})$) of b_{basic} has slopes $(m_2/g, m_1/g)$ each with multiplicity g, and b_μ is the ordinary Newton polygon.*

The following figures represent $B(G, \mu)$ as embedded in $B(\mathrm{GL}_g) \times B(\mathrm{GL}_g)$. In order to better explain the diagrams, suppose that $A \in \mathcal{M}(k)$ lies in the Newton stratum defined by $b \in B(G, \mu)$. Then the embedding of b into $B(\mathrm{GL}_g) \times B(\mathrm{GL}_g)$ corresponds to the decomposition of the Dieudonné module of A into two Dieudonné modules $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$ under the $\mathcal{O}_K/(p)$-action as in Proposition 2.2.4. The Newton polygon with height equal to m_i is the Newton polygon for $\mathcal{D}_i$. Each slope b corresponding to a Dieudonné module $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$ is depicted in such a way that $\mathcal{D}_1$ and $\mathcal{D}_2$ are depicted using the same colour on both diagrams. The blue lines represent the Newton polygons for the μ-ordinary locus, and the red lines represent the Newton polygons of the basic locus.

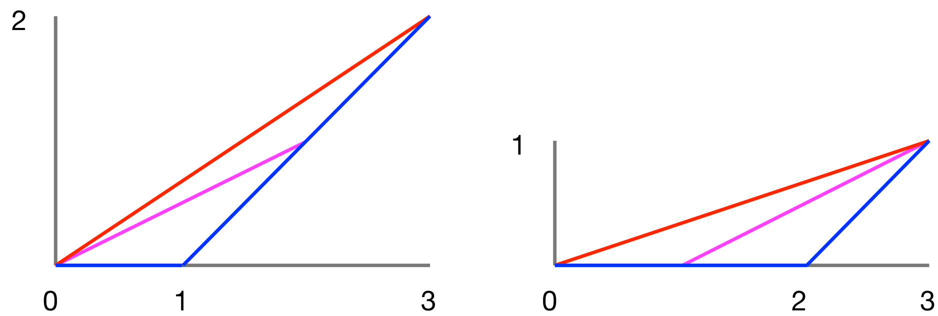


Figure 5–1: GU(2, 1)

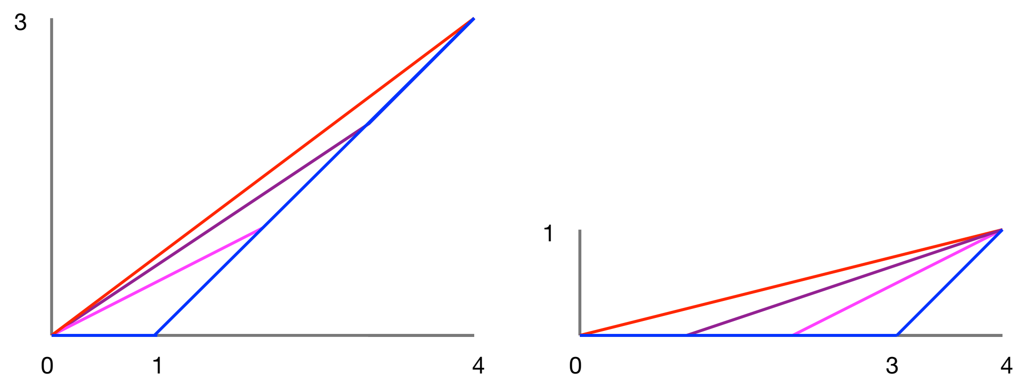


Figure 5–2: GU(3, 1)

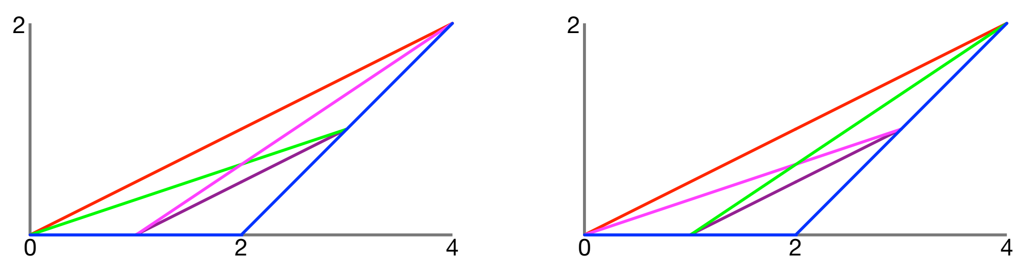


Figure 5–3: GU(2, 2)

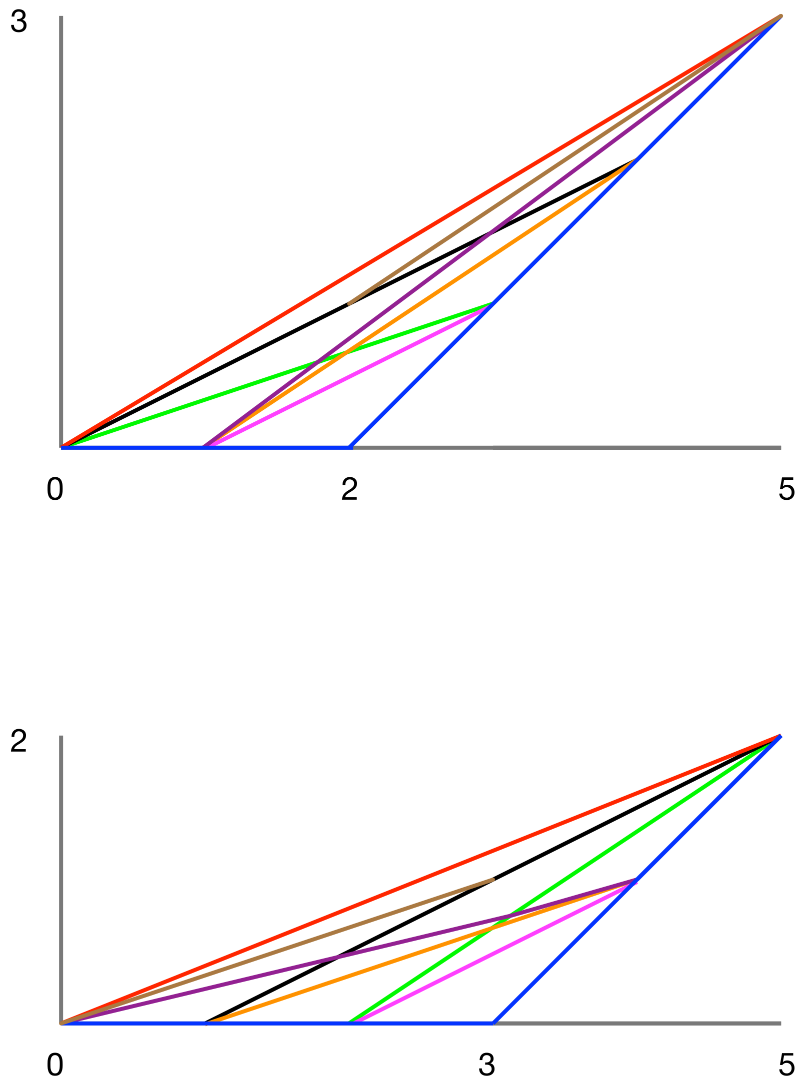


Figure 5–4: GU(3, 2)

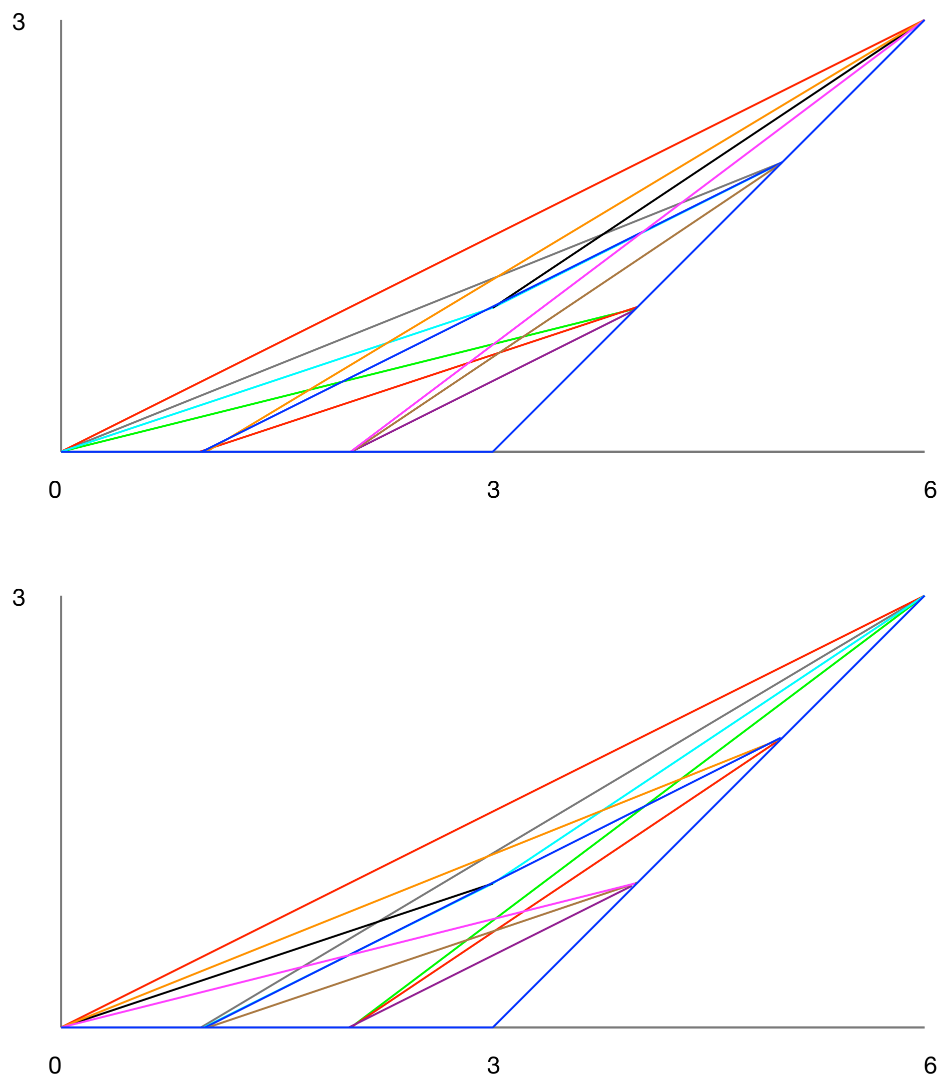


Figure 5–5: GU(3, 3)

5.3.2 p inert in K

In this case, only the L-points of G can be embedded into a product of GL_g where L is field containing $Q(\mathbb{F}_{p^2})$, and σ acts on $X_*(T)$ as $\phi_i \mapsto \phi_{g-i+1}^{-1}\phi$ and $\phi \mapsto \phi_1\phi_2 \dots \phi_g$. Then,

$$\sigma(\mu) = (1^{m_2}, 0^{m_1}, 1)$$

and

$$\bar{\mu} = (1^{m_2}, 1/2^{m_1-m_2}, 0^{m_2}, 1).$$

In this case it is easiest to describe the Newton polygons that arise via the representation of $\rho : G \hookrightarrow \mathrm{GL}_{2g/\mathbb{Q}_p}$ taking the character $(a_1, \dots, a_g, a) \in X_*(T)$ to

$$(a_1, \dots, a_g, a - a_1, \dots, a - a_g) \in X_*(D)$$

where D is the standard diagonal torus and dominant coroots of $X_*(D)$ are taken with respect to the upper triangular Borel. The corresponding action of σ on $(d_1, \dots, d_{2g}) \in X_*(D)$ is given by $d_i \mapsto d_{2g-i+1}$.

Under this description,

$$\bar{\mu} = (1^{m_2}, 1/2^{m_1-m_2}, 0^{m_2}, 0^{m_2}, 1/2^{m_1-m_2}, 1^{m_2})$$

and $\mu^\flat = g$. The conditions for $b \in B(\mathrm{GL}_{2g})$ to be in the subset $B(G, \mu)$ become:

- b lies above b_μ, the Newton polygon with slopes $(0, 1/2, 1)$ and multiplicities $(2m_2, 2(m_1 - m_2), 2m_2)$;
- every slope of b has even multiplicity;

- if b has slope sequence $(\lambda_1, \ldots, \lambda_r)$ with multiplicities $(2h_1, \ldots, 2h_r)$, the polygon of $(\lambda_1, \ldots, \lambda_r)$ with multiplicities $(h_1, \ldots, h_r)$ is symmetric in the sense that $\lambda_i = 1 - \lambda_{r+1-i}$ and $h_i = h_{r-1+i}$.

In this case b_{basic} is always supersingular, as the Newton polygon with slope 1/2 to multiplicity $2g$ will always satisfy the above criteria and factors through the centre of G. This is consistent with results for $m_2 = 1$ that can be found in [BW06] and [VW11]. The figures below give a visual representation of the Newton polygons of low dimensional examples. As in the previous section, the blue lines represent the Newton polygon for the μ-ordinary locus, and the red lines represent the Newton polygon of the basic locus.

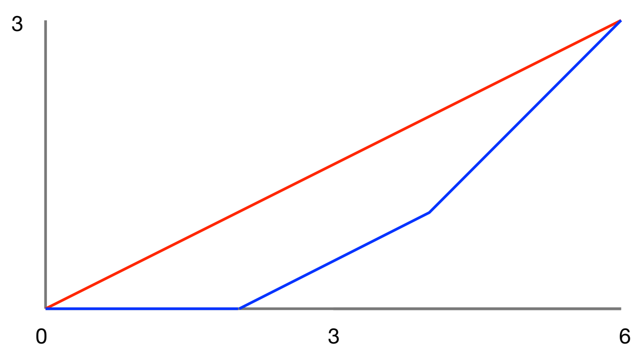


Figure 5–6: GU(2, 1)

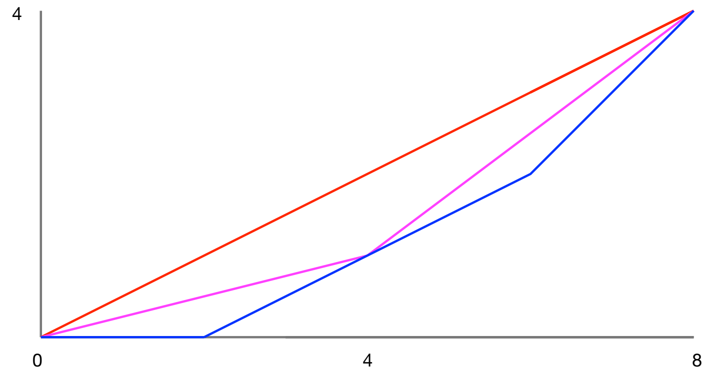


Figure 5–7: GU(3, 1)

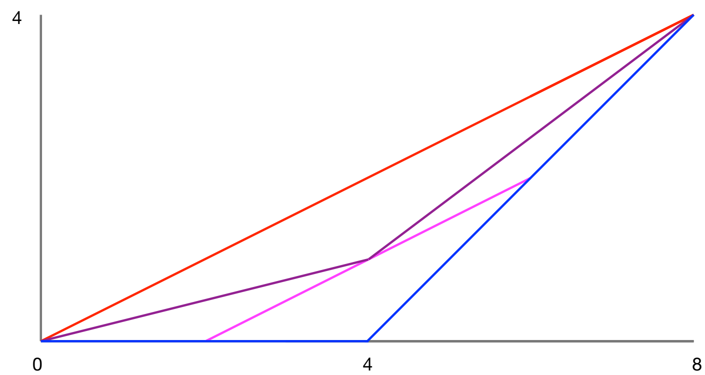


Figure 5–8: GU(2, 2)

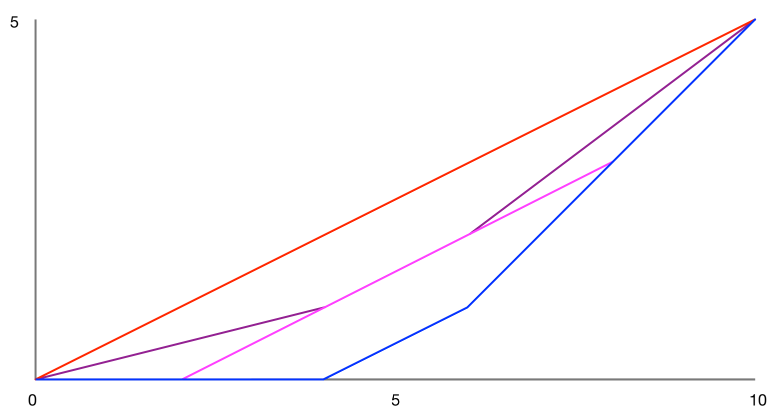


Figure 5–9: GU(3, 2)

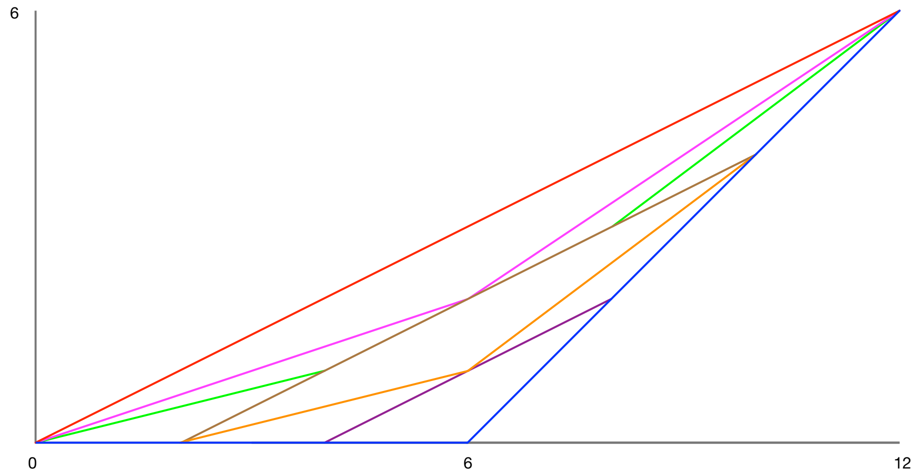


Figure 5–10: GU(3, 3)

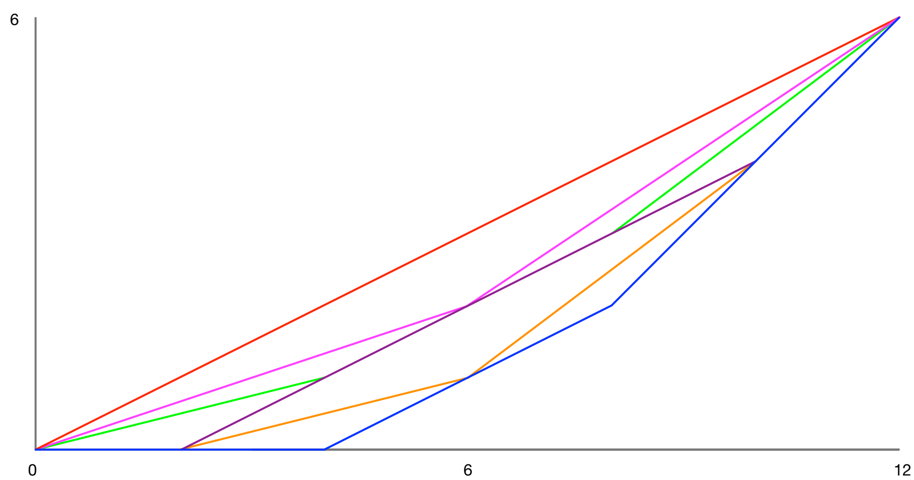


Figure 5–11: GU(4, 2)

5.4 Relationship with E-O strata

The Newton and E-O stratifications have an interesting relationship. Given a point $\underline{A} = (A, \iota, \lambda, \eta) \in \mathcal{M}(k)$, its Newton stratum corresponds to the isogeny class of its p-divisible group, and its E-O stratum corresponds to the isomorphism class of its truncated p-divisible group—that is, its p-torsion part. For abelian varieties in general, this relationship is rather subtle (see for instance [Oor05]) and we will see that this is no different for abelian varieties with extra structures on $\mathcal{M}$.

In particular, a p-divisible group (H, ι, λ) with $\mathscr{D}$-structure is called **distinguished** if the isomorphism class of $(H, \iota, \lambda)[p]$ determines its isomorphism class as a p-divisible group. A point $\underline{A} \in \mathcal{M}(k)$ is called distinguished if its p-divisible group is distinguished. Likewise an element $w \in {}^J W$ representing an E-O stratum is called distinguished if there exists a point $\underline{A}$ in the E-O stratum V^w that is distinguished.

Remark. Note that we are using the term distinguished p-divisible group in place of the more common *minimal* p-divisible group (see [Oor05] or [VW13]) so as to not confuse the use of minimal in the context of p-divisible groups with the minimal elements with respect to the partial orders we're discussing.

Proposition 5.4.1 ([VW13, Proposition 8.17] [Moo04b, Theorem 0.3])**.** *The core E-O stratum is distinguished and contained in the basic Newton stratum. Likewise the μ-ordinary E-O stratum is distinguished and the same as the μ-ordinary Newton stratum.*

Observe that this means that there is a consistent notion of a μ-ordinary stratum between the Newton and Ekedahl-Oort stratifications.

5.4.1 p split in K

When p is split in K, the group G itself is split and results from [VW13, Section 9] and [Oor05] can be combined with the explicit models of E-O strata in Section 3.5.3 to give a thorough description of how the Newton strata and E-O strata are related.

Theorem 5.4.2 ([VW13, 9.10, 9.20, 9.22])**.** *There is an injective map of posets:*

$$w : B(G, \mu) \hookrightarrow {}^J W$$

such that $V^{w(b)} \subseteq \mathcal{N}_b$ and $w(b)$ is distinguished. Furthermore, $w(b)$ is the unique minimal element in the set

$${}^J W_b := \left\{ w \in {}^J W \;\middle|\; V^w \cap \mathcal{N}_b \neq \emptyset \right\}.$$

Corollary 5.4.3. *When $m_2 = 1$, there is as a one-to-one correspondence between Newton strata and E-O strata. Furthermore, the map $B(G, \mu) \hookrightarrow {}^J W$ takes $b = (b_1, b_2) \in B(G, \mu)$ where b_1 has slopes $\left(\frac{d}{d+1}, 1\right)$ with multiplicities $(d+1, g-d-1)$ for $0 \leq d \leq g-1$ to the unique element $w \in {}^J W$ such that $\ell(w) = g - d$.*

Proof. In this case there are exactly $g+1$ Newton strata and $g+1$ E-O strata, so the one-to-one correspondence follows immediately from Theorem 5.4.2. The rest follows from Theorem 5.2.1 by a comparison of dimensions. □

This map $b \mapsto w(b)$ can be made explicit. Given a slope $\lambda = \frac{r}{s}$ such that $(r, s) = 1$ and $r, s \in \mathbb{N}$, define the p-divisible group $H_{s-r,r}$ as in [dJO00, 5.3] by giving a model for its covariant Dieudonné module. It has a basis $\{e_0, e_1, ..., e_{s-1}\}$

over $W(k)$. For $j \geq s$, write $e_j = p^a e_i$, for $j = i + as$. Then define $F(e_i) = e_{i+s-r}$ and $V(e_i) = e_{i+r}$. The p-divisible group $H_{s-r,r}$ has dimension $s - r$, its Serre-dual has dimension r, it is isosimple with slope r/s, its endomorphism ring is a maximal order in its endomorphism algebra (over k), and these properties characterize it completely as a p-divisible group over k [dJO00, 5].

Furthermore, for an element $b \in B(\mathrm{GL}_g)$ with slope sequence $(\lambda_1, \ldots, \lambda_h)$ and multiplicities $(n_1, \ldots, n_h)$, where $\lambda_i = \frac{r_i}{s_i}$ such that $(r_i, s_i) = 1$, define $H(b)$ to be the p-divisible group with covariant Dieudonné module,

$$\oplus_{i=1}^{r} (H_{s_i - r_i, r_i})^{n_i/s_i}.$$

In [Oor05], Oort shows that the isomorphism class of $H(b)[p]$ determines the isomorphism class of $H(b)$.

Write $b = (b_1, b_2) \in B(G, \mu) \subseteq B(\mathrm{GL}_g) \times B(\mathrm{GL}_g)$, and let $w(b_i)$ be the isomorphism class of $H(b_i)[p]$ in ${}^{J_i}W$. Then $w(b) = (w(b_1), w(b_2))$. Since the model for the covariant Dieudonné module of $H(b_i)$ is given, $w(b)$ can calculated directly from the models for $H(b_i)[p]$. The following Figures represent the map $b \mapsto w(b)$ in low dimensional examples. The underlying poset diagram corresponds to the E-O stratification as derived in Section 3.4, and the blue ovals represent the map $b = (b_1, b_2) \mapsto w(b)$ in the sense that b_1 is the blue label for the oval around the element $w(b)$. In many cases, the diagrams imply more relationships between the E-O and Newton strata. For example, Figure 5–13 implies that the two smallest

E-O strata comprise the basic Newton stratum and in all other cases the Newton and E-O strata agree.

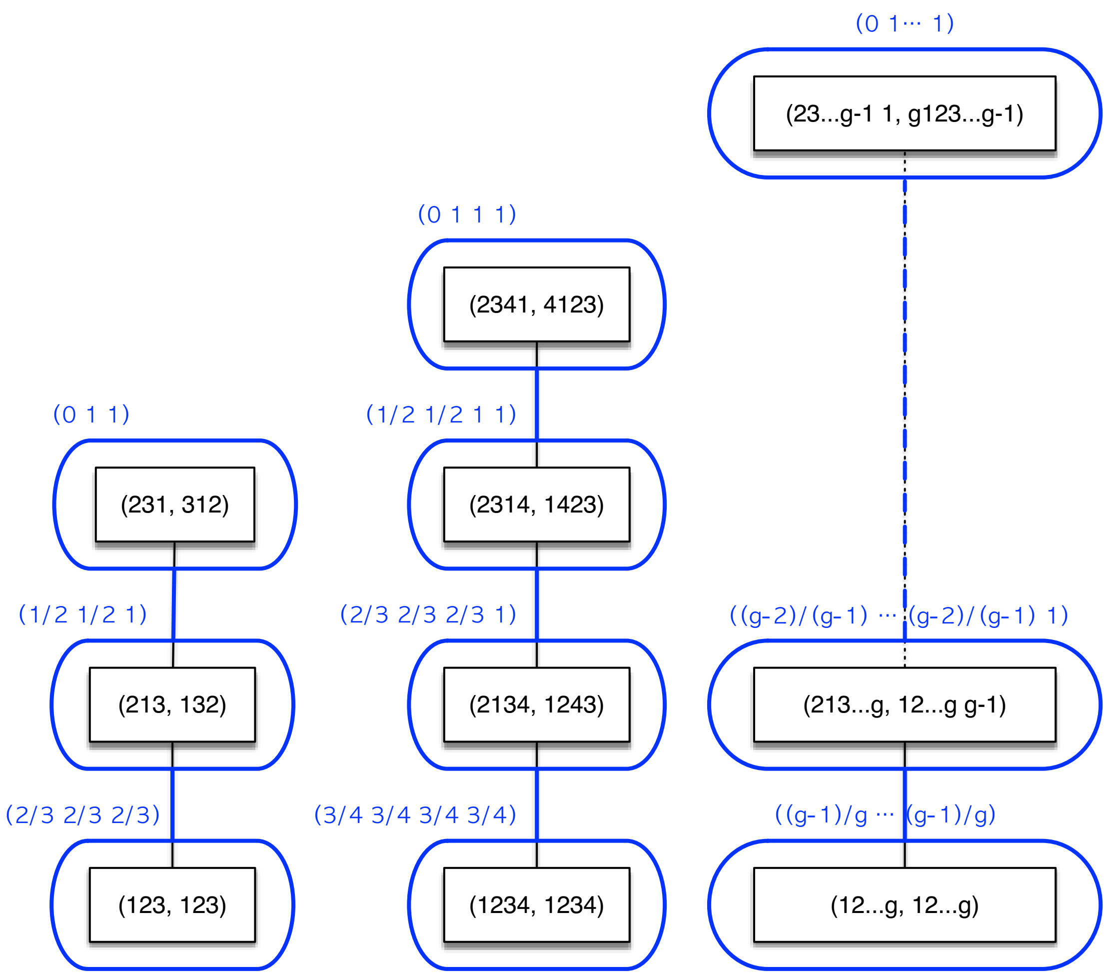


Figure 5–12: $\mathrm{GU}(2,1), \mathrm{GU}(3,1), \mathrm{GU}(g-1,1)$

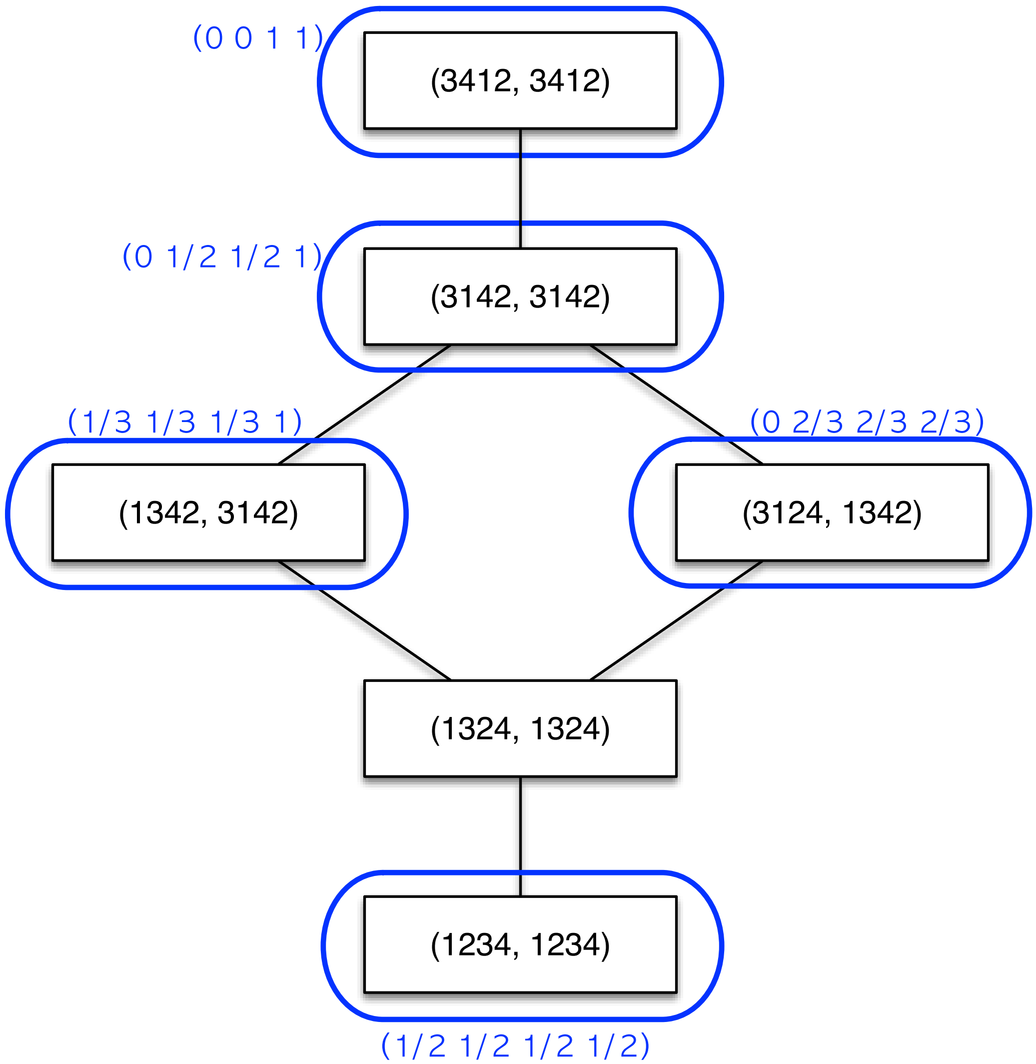


Figure 5–13: GU(2, 2)

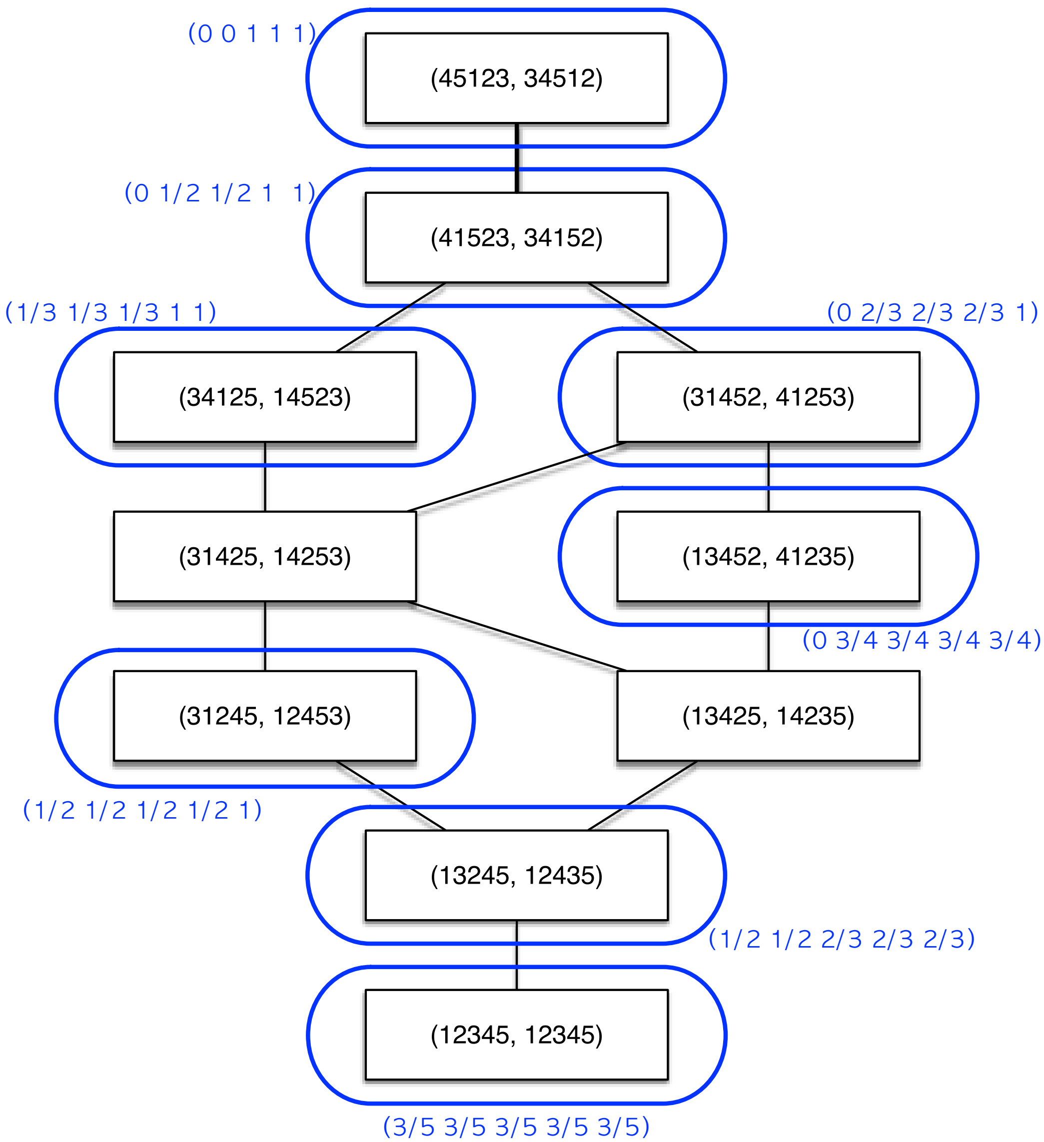


Figure 5–14: GU(3, 2)

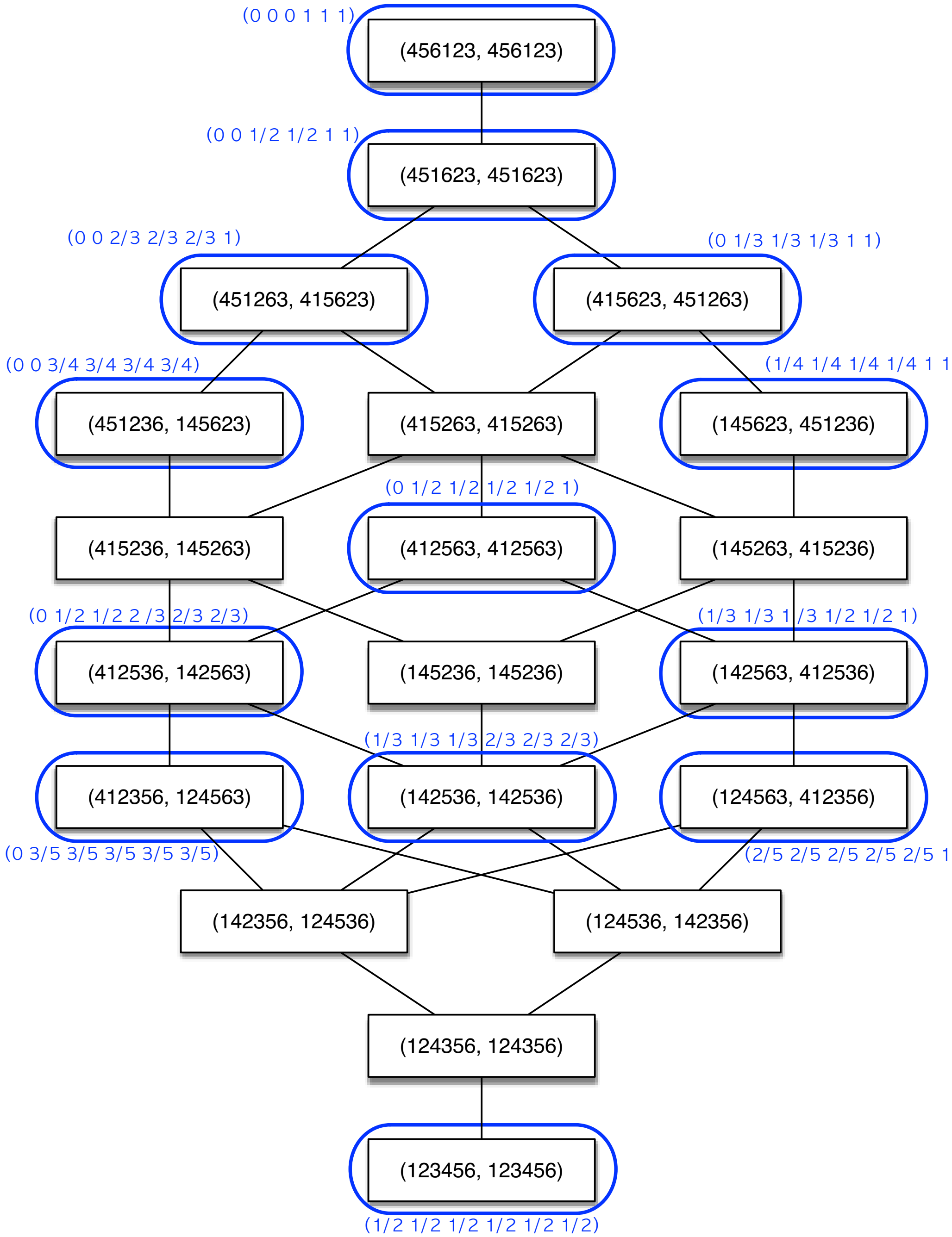


Figure 5–15: GU(3, 3)

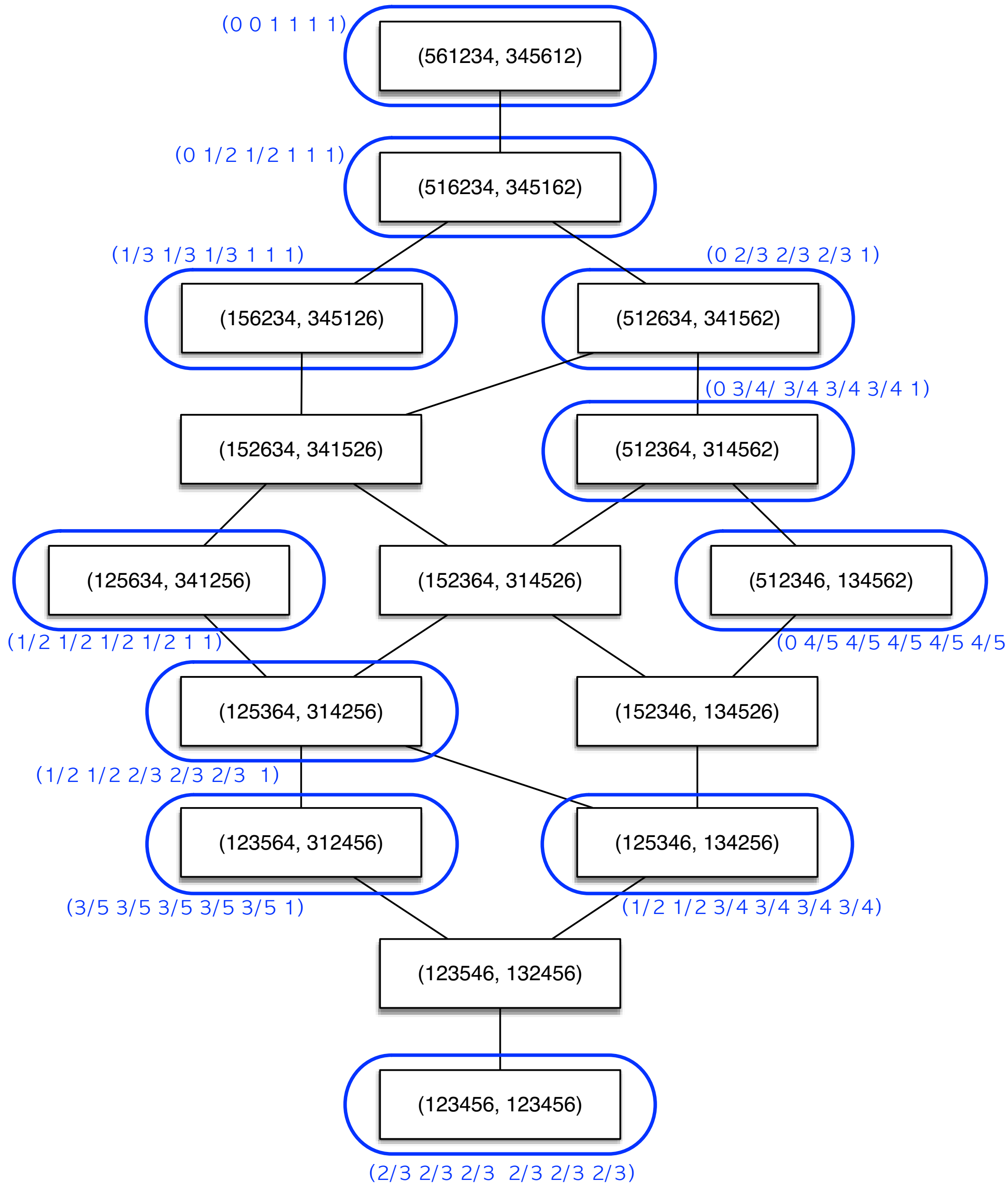


Figure 5–16: GU(4, 2)

5.4.1.1 The Split Case: Special E-O Strata

Proposition 5.4.1 describes the Newton strata of the core and μ-ordinary E-O strata, but what are the Newton strata corresponding to the almost-ordinary and almost-core E-O strata?

Proposition 5.4.4. *There exists a unique codimension one Newton stratum, and it is the same as the almost-ordinary E-O stratum.*

Proof. Consider the Newton stratum associated $b = (b_1, b_2)$ such that b_1 has slope sequence $(0, 1/2, 1)$ with multiplicities $(m_2 - 1, 2, m_1 - 1)$. This is the unique codimension one Newton stratum. By Theorem 5.4.2 it suffices to show that $w(b)$ corresponds to the almost-ordinary E-O stratum.

This is straightforward as $w(b_1)$ corresponds to $H_{0,1}[p]^{m_1-1} \oplus H_{1,0}[p]^{m_2-1} \oplus H_{1,1}[p]$, and this is isomorphic to $\mu_p^{m_1-1} \oplus \mathbb{Z}/p\mathbb{Z}^{m_2-1} \oplus \mathscr{G}_1$. Therefore $w(b)$ corresponds to the almost-ordinary E-O stratum by Example 3.5.10. □

Proposition 5.4.5. *Either the almost-core E-O stratum is distinguished or it is contained in the basic Newton stratum. Furthermore, if the signature (m_1, m_2) satisfies*

- $m_1 = m_2 \geq 2$*: the almost-core E-O stratum is basic,*
- $m_1 - m_2 \geq 1, m_2 = 1$ *or* $m_1 - m_2 = 1, m_2 > 1$*: the almost-core E-O stratum is distinguished.*

Proof. Let $w \in {}^JW$ be the element corresponding to the almost-core stratum. Suppose that $V^w \cap \mathcal{N}_b \neq \emptyset$ for some $b \in B(G, \mu)$ where b is *not* basic. But $w \leq w'$ for all $w' \in {}^JW \setminus \mathrm{id}$. Therefore w is the minimal element in ${}^JW_b :=$

$\left\{w \in {}^J W \;\middle|\; V^w \cap \mathcal{N}_b \neq \emptyset\right\}$ and by Theorem 5.4.2, $w = w(b)$ and the almost-core stratum is distinguished.

Suppose that $m = m_1 = m_2 \geq 2$. Then by Example 3.5.9, the almost-core E-O stratum has a model of the form $AC(2,2) \oplus (\mathscr{G}_1 \oplus \mathscr{G}_2)^{m-2}$. In the case where $m_1 = m_2 = 2$, Figure 5–13 together with Theorem 5.4.2 shows that $AC(2,2)$ is contained in the basic locus and is isotypic of slope 1/2. On the other hand, $(\mathscr{G}_1 \oplus \mathscr{G}_2)^{m-2}$ is the core locus for $(m-2, m-2)$ and is therefore distinguished and isotypic of slope 1/2. It follows that the almost-core E-O stratum is contained in the basic Newton stratum.

Next, suppose that $m_1 - m_2 \geq 1, m_2 = 1$. Then the fact that the almost-core E-O stratum is distinguished follows immediately from Corollary 5.4.3.

Finally, suppose that $m_1 - m_2 = 1, m_2 > 1$ so that the almost-core E-O stratum has a model of the form $C(m_2, m_2 - 1) \oplus C(1,1)$. Then since the core stratum is distinguished for $C(m_2, m_2 - 1)$ and $C(1,1)$, it follows that the almost-core E-O stratum is distinguished, and furthermore, it has slope sequence $\left(\frac{1}{2}, \frac{m_2}{2m_2 - 1}\right)$ with multiplicities $(2, 2m_2 - 1)$. □

In the remaining cases, where $m_1 - m_2 > 1, m_2 > 1$, determining which choices of (m_1, m_2) correspond to a distinguished almost-core E-O stratum is not clear. Recall from Example 3.5.9 that even determining a model for the almost-core E-O stratum in this case is subtle. Even so, observe that if there exists a $b \in B(G, \mu)$ such that $w(b)$ corresponds to the almost-core locus, then necessarily $b \leq b'$ for all $b' \in B(G, \mu)$ such that b' is not basic.

Necessary conditions for the existence of a $b \in B(G, \mu)$ such that $w(b)$ is the almost-core E-O stratum are that either

- $m_2 \nmid m_1$ and $m_2 \nmid m_1 - 1$ and $m_1 + 1 \neq \gcd(m_1, m_2) + \gcd(m_1, m_2 - 1)$, or
- $m_2 \mid m_1 - 1$.

Interestingly, even under these conditions $w(b)$ may not correspond to the almost-core stratum. For example, if $(m_1, m_2) = (6, 4)$, the element $b \in B(G, \mu)$ such that b_1 has slope sequence $(4/7, 2/3)$ with multiplicities $(7, 3)$ is less than or equal to every element in $B(G, \mu)$ except for the basic element. Here $w(b_1) = [1\,2\,3\,4\,5\,7\,8\,6\,9\,10]$ is a 2-dimensional stratum and does not correspond to the almost-core locus. This demonstrates, once again, the intricate relationship between the signature (m_1, m_2) and the possibilities for the almost-core E-O stratum as was previously encountered in Section 3.5.

5.4.2 p inert in K

In the inert case, much less can be shown as the analogue of Theorem 5.4.2 is not true for general signatures (m_1, m_2). However, we are still able to determine the relationship between the E-O strata and the Newton strata on a case-by-case basis. We give results for the almost-core and almost-ordinary E-O strata in this section.

In the case where the signature is $(g - 1, 1)$, the distinguished strata were determined in [BW06], leading to the following lemma:

Lemma 5.4.6. *Let p be a prime that is inert in K, and let $(m_1, m_2) = (g - 1, 1)$. Then*

1. *the basic locus is supersingular and corresponds to the closure of the Weyl element $w \in {}^JW$ such that $\ell(w) = \lfloor (g-1)/2 \rfloor$. That is,*

$$w_1 = \left[1\,2 \ldots \left\lceil \frac{g-1}{2} \right\rceil - 1 \;\; g \;\; \left\lceil \frac{g-1}{2} \right\rceil \ldots g-1 \right]$$
$$w_2 = \left[2\,3 \ldots \left\lfloor \frac{g-1}{2} \right\rfloor \;\; 1 \;\; \left\lfloor \frac{g-1}{2} \right\rfloor + 1 \ldots g \right];$$

2. *every non-basic E-O stratum is distinguished.*

Proof. (1) is simply a combination of [BW06, 5.4] and [BW06, Proposition 5.5] with the observation that there is a unique element $w \in {}^JW$ (the one given above) such that $\ell(w) = \lfloor (g-1)/2 \rfloor$. Claim (2) is exactly [BW06, Theorem 5.3]. □

Another case that follows from [BW06] is when $m_2 = 0$:

Lemma 5.4.7 ([BW06, Lemma 3.4])**.** *If (H, ι, λ) is a p-divisible group of signature $(m, 0)$ or $(0, m)$ whose slopes are all $1/2$, then H is superspecial, that is $F\mathfrak{D}(H) = V\mathfrak{D}(H)$, and $\bar{H} \cong \mathscr{G}_1^m$ (resp. $\mathscr{G}_2^m$).*

We now extend these results to get an understanding of the relationship between the Newton stratification and the Ekedahl-Oort stratification in other cases.

Proposition 5.4.8. *If $m_1 = m_2$, or $m_1 - m_2 > 1$, then the almost-ordinary E-O stratum is distinguished and is equal to the codimension 1 Newton stratum. If $m_1 - m_2 = 1$, the almost-ordinary E-O stratum is contained in the Newton stratum having slopes $(0, 1/2, 1)$ with multiplicities $(2(m_2 - 1), 6, 2(m_2 - 1))$.*

Proof. Begin with the situation where $m = m_1 = m_2$. Then the slope sequence of the μ-ordinary stratum is $(0^g, 1^g)$ and the codimension 1 Newton stratum has slope sequence $(0^{g-2}, 1/2^2, 1^{g-2})$. Let (H, ι, λ) be the Dieudonné module of the p-divisible group of a point in the codimension 1 Newton stratum. Then H decomposes into étale-local, local-étale, and local-local pieces, $H^{ét\text{-}0} \oplus H^{0\text{-}ét} \oplus H^{0,0}$. Therefore, $H^{ét\text{-}0} \oplus H^{0\text{-}ét}$ corresponds to the slopes $(0^{g-2}, 1^{g-2})$, and this p-divisible group has signature $(m-1, m-1)$. The p-torsion of $H^{ét\text{-}0} \oplus H^{0\text{-}ét}$ then must have the form $\mathcal{O}_K/(p)^{m-1} \oplus (\mathcal{O}_K/(p) \otimes \mu_p)^{m-1}$. This leaves the self-dual p-divisible group $H^{0,0}$ with signature $(1,1)$, which corresponds to the basic locus for the signature $(1,1)$ case. The only possibility for the p-torsion of $H^{0,0}$ is then $\mathscr{G}_1 \oplus \mathscr{G}_2$ where $\mathscr{G}_1$ (resp. $\mathscr{G}_2$) is the p-torsion of a supersingular elliptic curve with signature $(1,0)$ (resp. $(0,1)$) as in Section 3.5.2. Therefore, there is only one isomorphism class of p-torsion group schemes with extra structure corresponding to the Newton stratum of H, which corresponds by Proposition 3.5.6 to the almost-ordinary E-O stratum.

Conversely, suppose that decomposition of $(H[p], \iota, \lambda)$ is a model for the almost-ordinary E-O stratum as in Proposition 3.5.6. Then the decomposition of $H[p]$ into étale-local, local-étale, and local-local pieces and its decomposition under the $\mathcal{O}_K/(p)$-action correspond to the respective decompositions of the isogeny class of any lift of $H[p]$ to a p-divisible group with extra structure, H. The étale-local and local-étale parts correspond to $\mathcal{O}_K/(p)^{m-1} \oplus (\mathcal{O}_K/(p) \otimes \mu_p)^{m-1}$ which lift uniquely to p-divisible groups with slopes $(0^{g-2}, 1^{g-2})$, which leaves $\mathscr{G}_1 \oplus \mathscr{G}_2$ which lifts uniquely to the local-local isogeny class with slope sequence $(1/2^2)$. Therefore the almost-ordinary E-O stratum is contained in the codimension 1 Newton stratum and it is distinguished. This proves the desired result for the case when $m_1 = m_2$.

The argument for the case where $m_1 - m_2 > 1$ is similar to the previous argument. In this case the codimension 1 Newton stratum has slope sequence $(0^{m_2-1}, 1/4, 1/2^{(m_1-m_2-2)}, 3/4, 1^{m_2-1})$. By separating off the étale-local and local-étale parts, we are left with slopes $(1/4, 1/2^{(m_1-m_2-2)}, 3/4)$ and $H^{0\text{-}0}$ has signature $(m_1 - m_2 + 1, 1)$. On the other hand, the analogous local-local part of the almost-ordinary E-O stratum has the form

$$AO(m_1 - m_2 + 1, 1) \cong AO(3, 1) \oplus \mathscr{G}_1^{m_1-m_2-2}$$

by Proposition 3.5.6. As $H^{0\text{-}0}$ is not basic, Lemma 5.4.6 completes the proof.

Now consider the case $m_1 - m_2 = 1$. The almost-ordinary E-O stratum has a model of the form

$$H[p] = AC(2, 1) \oplus (\mathcal{O}_K \otimes \mu_p)^{m_2-1} \oplus (\underline{\mathcal{O}_K/(p)})^{m_2-1}.$$

There exists a unitary p-divisible group with slope sequence 1/2 whose reduction mod p is $AC(2, 1)$ by the $(m_1, m_2) = (2, 1)$ case. Again, since the étale-local and local-étale parts of $H[p]$ lift uniquely, it follows that the almost-ordinary E-O stratum has non-empty intersection with the Newton stratum with slope sequence $(0^{2(m_2-1)}, 1/2^6, 1^{2(m_2-1)})$. However, the Newton stratum with slope sequence $(0^{2(m_2-1)}, 1/2^6, 1^{2(m_2-1)})$ is the unique codimension one stratum, and since the μ-ordinary Newton stratum is equal to the μ-ordinary E-O stratum, it follows that the almost-ordinary E-O stratum is contained in the codimension one Newton stratum. In this case, the almost-ordinary E-O stratum is not equal to the codimension 1 Newton stratum as the E-O stratum with model:

$$\mathscr{G}_1^2 \oplus \mathscr{G}_2^1 \oplus (\mathcal{O}_K \otimes \mu_p)^{m_2-1} \oplus (\underline{\mathcal{O}_K/(p)})^{m_2-1}$$

is also contained in the codimension 1 Newton stratum. □

Our next goal is to find the Newton stratum corresponding to the almost-core E-O stratum in general (Proposition 5.4.10); however, we begin with the specific situation of signature $(2,2)$.

Lemma 5.4.9. *The almost-core E-O stratum* $AC(2,2)$ *of signature* $(2,2)$ *is contained in the basic Newton stratum.*

Proof. Recall that the E-O stratification for $\mathrm{GU}(2,2)$ was given in Figure 3–2. By Proposition 5.4.8, the almost-ordinary E-O stratum and the codimension 1 Newton stratum are the same. Passing to the next codimension, we find two E-O strata of dimension 3 whose p-torsion only differ in that their $\mathcal{O}_K$-action differs by the non-trivial automorphism of $\mathcal{O}_K$. Dimension considerations from Proposition 5.2.1 as applied to Figure 5–8 show that the two 3-dimensional E-O strata are in the Newton stratum with slopes $(1/4, 3/4)$ and multiplicities $(4,4)$. Since the basic Newton stratum is one dimensional, it must contain the almost-core E-O stratum. □

We can now prove the general case:

Proposition 5.4.10. *The almost-core E-O stratum is contained in the basic Newton stratum.*

Proof. Recall from Example 3.5.5 that the almost-core E-O stratum has the form:

- $m_1 \geq m_2 > 1$: $AC(2,2) \oplus \mathscr{G}_1^{m_1-2} \oplus \mathscr{G}_2^{m_2-2}$
- $m_1 > m_2 = 1$: $AC(2,1) \oplus \mathscr{G}_1^{g-3}$

The case where $m_2 = 1$ was already proved in Lemma 5.4.6. Suppose that $m_1 \geq m_2 > 1$. Then $AC(2,2) \oplus \mathscr{G}_1^{m_1-2} \oplus \mathscr{G}_2^{m_2-2}$ has all slopes equal to 1/2 as a consequence of Lemma 5.4.9. This proves the result since the basic locus has slope sequence 1/2 by Section 5.3.2. □

Chapter 6
Cycle classes of E-O strata

This chapter develops a relationship between the Schubert cells of flag spaces of a g-dimensional vector space and the flag spaces over $\mathcal{M}$ that come from extending the Hodge filtration $\mathbb{E} \subseteq \mathbb{H}$. We will ultimately show that these are locally isomorphic in the étale topology. This gives finite étale covering maps from strata in a flag space lying over $\mathcal{M}$ (coming from extending the Hodge filtration), onto the Ekedahl-Oort strata of $\mathcal{M}$. One consequence of this result (to be pursued in future work) is that the cycle classes of the closed E-O strata in the Chow group can be studied by pushing down cycle classes of strata from the overarching flag space. A significant intellectual debt is owed to Ekedahl and van der Geer for their work in the Siegel case. Many of the proofs that follow are closely related to those in [EvdG09], and complete details are included for the purpose of being self-contained. There are two salient features that allow us to take advantage of their approach: the interaction of the $\mathcal{O}_K$-action on the de Rham cohomology $\mathbb{H}$ of the universal abelian variety $\mathcal{A} \to \mathcal{M}$ with the symplectic pairing on $\mathbb{H}$ induced by a prime-to-p polarization; and a combinatorial description of the map on strata from $\mathcal{M}$ to the Siegel space (see Proposition 6.1.4).

6.1 Combinatorial preliminaries

Recall from Example 3.1.2, that $(S_g,\, S)$ where S_g is the symmetric group on g elements and S is the set of reflections $S = \{s_i = (i, i+1) | i = 1, \dots, g-1\}$ is a Coxeter group, and in case $J = S \setminus \{s_m\}$ is a subset of simple reflections, the quotient group $S_{g_J} \backslash S_g$ has a choice of minimal coset representatives with respect to the Bruhat order on S_g;

$$^J S_g = \left\{ x \in S_g \;\middle|\; x^{-1}(1) < \cdots < x^{-1}(m) \text{ and } x^{-1}(m+1) < \cdots < x^{-1}(g) \right\}.$$

Let W_J denote the subgroup of $W = S_g$ generated by J. Then there is a projection map $S_g \to W_J \backslash S_g \cong {}^J S_g$ taking $w \mapsto {}^J w$ where ${}^J w$ denotes the minimal length representative of the coset $W_J w$, and another projection map $S_g \to S_g / W_J \cong S_g^J$ taking $w \mapsto w^J$, where w^J denotes the minimal length representative of the coset wW_J. The first map, $S_g \to {}^J S_g$ consists of taking the element $w = [w_1 w_2 \dots w_g]$ and permuting the order of the elements $w_1, w_2, \dots, w_g$ so that the set of $w_j \in \{1, \dots, m\}$ still appear in the same places, but now appear in ascending order, and likewise for the set of $w_j \in \{m+1, \dots, g\}$. That is,

$$\begin{aligned} \left\{ {}^J w^{-1}(a) \leq m \;\middle|\; 1 \leq a \leq g \right\} &= \left\{ w^{-1}(a) \leq m \;\middle|\; 1 \leq a \leq g \right\} \\ &= \{ w_j \mid 1 \leq w_j \leq m \} \end{aligned}$$

and

$$\begin{aligned} \left\{ {}^J w^{-1}(a) \geq m+1 \;\middle|\; 1 \leq a \leq g \right\} &= \left\{ w^{-1}(a) \geq m+1 \;\middle|\; 1 \leq a \leq g \right\} \\ &= \{ w_j \mid m+1 \leq w_j \leq g \}, \end{aligned}$$

but now the elements of these sets appear in ascending order in ${}^J w = \left[{}^J w_1 \dots {}^J w_g \right]$ so that ${}^J w^{-1}(1) < \ldots < {}^J w^{-1}(m)$ and ${}^J w^{-1}(m+1) < \ldots < {}^J w^{-1}(g)$. On the other hand, $S_g \to S_g^J$ consists of taking $w = [w_1 w_2 \dots w_g]$ and putting each of the first

m-entries $w_1, \ldots, w_m$ and the last $(g-m)$-entries $w_{m+1}, \ldots w_g$ into ascending order. For example, if $J = S \setminus \{s_4\}$, then in the first case

$$w = \left[\boxed{341}\,5\,\boxed{2}\,76\right] \mapsto \left[\boxed{123}\,5\,\boxed{4}\,67\right] = {}^J w,$$

and in the second case,

$$w = \left[\boxed{3415}\,276\right] \mapsto \left[\boxed{1345}\,267\right] = w^J.$$

Let

$$W = \{(w_1, w_2) \subseteq S_g \times S_g \mid w_2 = w_0 w_1 w_0\},$$

where $w_0 = [g \;\; g-1 \ldots 2\,1]$. Recall that this is the Weyl group of the unitary group of signature (m_1, m_2) where $m_1 + m_2 = g$. From Section 3.4, (W, S) is a Coxeter group where

$$S = \{(s_i, w_0 s_i w_0) \in S_g \times S_g \mid s_i = (i, i+1)\}.$$

For

$$J = \{(s, t) \in S \setminus \{s_{m_1}\} \times S \setminus \{s_{m_2}\} \mid s = w_0 t w_0\},$$

one has

$${}^J W = \left\{(w_1, w_2) \in {}^{J_1} S_g \times {}^{J_2} S_g \;\middle|\; w_2 = w_0 w_1 w_0\right\}$$

where $J_i = S \setminus \{s_{m_i}\}$. Furthermore, $(w_1, w_2) \leq (w_1', w_2')$ under the Bruhat order if and only if $w_1 \leq w_1'$ if and only if $w_2 \leq w_2'$. The maps $W \to {}^J W$ and $W \to W^J$ arise from taking $S_g \to {}^{J_i} S_g$ (resp. $S_g \to S_g^{J_i}$) on each component.

For a permutation $w \in S_g$, and integers $1 \leq j, n \leq g$, define

$$r_w(j,n) := \#\{a \leq j \mid w(a) \leq n\}.$$

This function will give the analogue of a *final type* in the Siegel case.[1] For example in the Siegel case, $\nu(j) = j - r_w(j,g)$ is the final type associated to the Weyl group element $w \in {}^{S\setminus\{s_g\}}S_{2g}$, and there is a one-to-one correspondence between final types and elements in ${}^{S\setminus\{s_g\}}S_{2g}$ [EvdG09, Section 1].

Lemma 6.1.1. *Let $w \in S_g$ and $w_0 = [g\ \ g-1 \ldots 2\,1]$. For all $1 \leq j, n \leq g$,*

$$r_{w_0 w w_0}(j,n) = j + n - g + r_w(g-j, g-n).$$

In particular, for $(w_1, w_2) \in W$,

$$r_{w_1}(j, m_1) = j - m_2 + r_{w_2}(g-j, m_2).$$

[1] in the language of Oort

Table 6–1: $w = ([1235467], [1243567])$

j	0	1	2	3	4	5	6	7
$r_{w_1}(j,4)$	0	1	2	3	3	4	4	4
$r_{w_2}(j,3)$	0	1	2	2	3	3	3	3
$r_{w_1}(j,4) - r_{w_2}(7-j,3)$	-3	-2	-1	0	1	2	3	4

Proof.

$$\begin{aligned}
r_{w_0ww_0}(j,n) &= \#\{w_0ww_0(1),\ldots,w_0ww_0(j)\} \cap \{1,\ldots,n\} \\
&= \#\{w_0w(g-j+1),\ldots,w_0w(g)\} \cap \{1,\ldots,n\} \\
&= \#\{w(g-j+1),\ldots,w(g)\} \cap \{g-n+1,\ldots,g\} \\
&= j - \#\{w(g-j+1),\ldots,w(g)\} \cap \{1,\ldots,g-n\} \\
&= j - (g-n-\#\{w(1),\ldots,w(g-j)\} \cap \{1,\ldots,g-n\}) \\
&= j+n-g+r_w(g-j,g-n).
\end{aligned}$$

□

As an example consider the element $(w_1, w_2) \in W$ for $(m_1, m_2) = (4, 3)$ of length 1 where

$$w_1 = [1235467], \qquad w_2 = [1243567].$$

Then Table 6–1 shows the values of $r_{w_i}(j, m_i)$ for $i = 1, 2$ and the difference $r_{w_1}(j, m_1) - r_{w_2}(g - j, m_2)$.

Proposition 6.1.2. *Let $w \in S_g$. Then*

1. $r_w(i,j) = r_{w^{-1}}(j,i)$

2. *Let* $J = S_g \setminus \{s_m\}$. *Then for* $w \in S_g$, $r_{^J w}(j,m) = r_w(j,m)$ *for all* $1 \le j \le g$ *and* $r_{w^J}(m,j) = r_w(m,j)$ *for all* $1 \le j \le g$.
3. $0 \le r_w(s,n) - r_w(t,n) \le s-t$ *for* $0 \le t < s \le g$, $n \in \{0,\dots,g\}$.

Proof. First, $r_w(i,j) = r_{w^{-1}}(j,i)$. That is,

$$\begin{aligned} r_w(i,j) &= \#\{a \le i \mid w(a) \le j\} \\ &= \#(\{w(1), w(2), \dots, w(i)\} \cap \{1, 2 \dots, j\}) \\ &= \#(\{1, 2 \dots, i\} \cap \{w^{-1}(1), w^{-1}(2), \dots, w^{-1}(j)\}) \\ &= \#\{a \le j \mid w^{-1}(a) \le i\} \\ &= r_{w^{-1}}(j,i). \end{aligned}$$

As the map $S_g \to {}^J S_g$ consists of taking $w = [w_1 w_2 \dots w_g]$ and linearly ordering $1 \le w_i \le m$ and $m+1 \le w_j \le g$ on the same places, it follows that $w(i) \le m$ if and only if ${}^J w(i) \le m$. Therefore,

$$r_w(i,m) = \#\{a \le i \mid w(a) \le m\} = \#\{a \le i \mid {}^J w(a) \le m\} = r_{^J w}(i,m).$$

On the other hand, the map $S_g \to S_g^J$ consists of taking $w = [w_1 w_2 \dots w_g]$ and linearly ordering $w_1, \dots, w_m$ and $w_{m+1}, \dots, w_g$. In particular,

$$\{w(1), \dots, w(m)\} = \{w^J(1), \dots, w^J(m)\}.$$

Therefore,

$$\begin{aligned} r_w(m,j) &= \#\left\{a \le m \mid w(a) \le j\right\} \\ &= \#(\{w(1), w(2), \dots, w(m)\} \cap \{1, 2 \dots, j\}) \\ &= \#(\left\{w^J(1), w^J(2), \dots, w^J(m)\right\} \cap \{1, 2 \dots, j\}) \\ &= r_{w^J}(m,j). \end{aligned}$$

Finally, it follows from the definition that $r_w(s+1,n)$ is either $r_w(s,n)$ or $r_w(s,n)+1$ depending on whether $w(s+1)$ is greater than n or not. In other words, $0 \le r_w(s,n) - r_w(t,n) \le s-t$ for $0 \le t < s \le g$, $n \in \{0, \dots, g\}$. □

For $1 \le t \le g$, let

$$\mu_{w_i}(t) := \min\left\{0 \le j \le m_{i+1} \mid r_{w_i}(t, m_i + j) = t\right\}$$

and

$$\nu_{w_i}(t,u) := t - r_{w_i}(t,u). \tag{6.1}$$

Lemma 6.1.3. *The inequality $\mu_{w_i}(t) \ge \nu_{w_i}(t, m_i)$ holds for all $1 \le t \le g$. Furthermore, the following are equivalent:*

- $\mu_{w_i}(t) = \nu_{w_i}(t, m_i)$,
- $w = (w_1, w_2) \in {}^J W$,
- $r_{w_i}(t, m_i + \nu_{w_i}(t, m_i)) = t$.

Proof. Let $1 \le t \le g$. Then $\mu_{w_i}(t) = s$ implies that

$$\{w_i(1), \dots, w_i(t)\} \subseteq \{1, \dots, m_i + s\}$$

and

$$r_{w_i}(t, m_i) = \# \{w_i(1), \ldots, w_i(t)\} \cap \{1, \ldots, m_i\} \le t - s.$$

In other words, $\mu_{w_i}(t) \ge t - r_{w_i}(t, m_i) = \nu_{w_i}(t, m_i)$.

Recall that $w \in {}^{J_i}W_i$ if and only if $w_i^{-1}(1) < \ldots < w_i^{-1}(m_i)$ and $w_i^{-1}(m_i+1) < \ldots < w_i^{-1}(g)$. Hence if $\mu_{w_i}(t) = s$,

$$\# \{w_i(1), \ldots, w_i(t)\} \cap \{1, \ldots, m_i + s\} = \# \{1, \ldots, t\} \cap \left\{w_i^{-1}(1), \ldots, w_i^{-1}(m_i + s)\right\} = t$$

and there exists $1 \le r \le t$ such that $w_i(r) = m_i + s$. Thus,

$$\# \{1, \ldots, t\} \cap \left\{w_i^{-1}(1), \ldots, w_i^{-1}(m_i + s)\right\} = t.$$

Since $w_i \in {}^{J_i}W_i$, $w_i^{-1}(m_i + 1) < \ldots < w_i^{-1}(m_i + s) = r \le t$. Therefore

$$\# \{1, \ldots, t\} \cap \left\{w_i^{-1}(m_i + 1), \ldots, w_i^{-1}(m_i + s)\right\} = s$$

and

$$\# \{1, \ldots, t\} \cap \left\{w_i^{-1}(1), \ldots, w_i^{-1}(m_i)\right\} = t - s,$$

giving $\mu_{w_i}(t) = \nu_{w_i}(t, m_i)$.

On the other hand, suppose that $\mu_{w_i}(t) = \nu_{w_i}(t, m_i)$ for all $1 \le t \le g$. It is enough to show that $w_i^{-1}(m_i + 1) < \ldots < w_i^{-1}(g)$ for $i = 1, 2$ as $w_i^{-1}(m_i + 1) < \ldots < w_i^{-1}(g)$ if and only if $w_{i+1}^{-1}(1) < \ldots < w_{i+1}i^{-1}(m_{i+1})$.

By the definition of μ_{w_i}, for each t, either $w_i(t) \le m_i$ and $\nu_{w_i}(t, m_i) = \nu_{w_i}(t-1)$ or $w_i(t) = m_i + \mu_{w_i}(t)$ and $\nu_{w_i}(t, m_i) = \nu_{w_i}(t-1) + 1$. Since $\mu_{w_i}(t) = \nu_{w_i}(t, m_i)$, either $w_i(t) \le m_i$ or $t = w_i^{-1}(m_i + \nu_{w_i}(t, m_i))$. But ν_{w_i} is non-decreasing, (either $\nu_{w_i}(t+1) = \nu_{w_i}(t, m_i)$ or $\nu_{w_i}(t+1) = \nu_{w_i}(t, m_i) + 1$), and hence, $w_i^{-1}(m_i + 1) < \ldots < w_i^{-1}(g)$ for $i = 1, 2$. Therefore, $w \in {}^J W$. This also shows that if $\mu_{w_i}(t) = \nu_{w_i}(t, m_i)$ for all $1 \leq t \leq g$, then $r_{w_i}(t, m_i + \nu_{w_i}(t, m_i)) = t$ for $1 \leq t \leq g$.

Finally, if $r_{w_i}(t, m_i + \nu_{w_i}(t, m_i)) = t$, then $\nu_{w_i}(t, m_i) \geq \mu_{w_i}(t)$ by the definition of μ_{w_i}. But we always have that $\nu_{w_i}(t, m_i) \leq \mu_{w_i}(t)$, so equality holds. □

Define $\gamma : C_2 \to C_2$ (where C_2 is the cyclic group of order two) by either $\gamma(i) = i + 1$ for $i = 1, 2$ or $\gamma(i) = i$. These two choices will again allow us to differentiate between the two different cases for unitary Shimura varieties that we are considering.

Proposition 6.1.4. *Fix a choice of γ as above. Let $(w_1, w_2) \in W$ (i.e. $w_i \in S_g$ and $w_2 = w_0 w_1 w_0$). Then there exists a choice of functions*

$$d_i : \{0, \ldots, 2g\} \to \{0, \ldots, g\}$$

satisfying

1. $d_1(j) + d_2(j) = j$,
2. $0 \leq d_i(j) - d_i(n) \leq j - n$ *for* $j < n$;
3. $d_i(2g - j) = g - d_{i+1}(j)$.

and a unique $w \in S_{2g}$ such that

$$w^{-1}(1) < \ldots < w^{-1}(g), w^{-1}(g + 1) < \ldots < w^{-1}(2g),$$

$$r_w(j, g) = r_{w_1}(d_1(j), m_1) + r_{w_2}(d_2(j), m_2),$$

and

$$d_{\gamma(i)}(\nu_w(j, g)) = \nu_{w_i}(d_i(j), m_i)$$

for all $0 \le j \le 2g$.

Proof. Begin by defining functions d_i and ν inductively on a subset of $\{0, \ldots, 2g\}$. Let $\nu(0) = d_i(0) = 0$ and $d_i(2g) = \nu(2g) = g$ for $i = 1, 2$ (this is forced by properties (1) and (3) of the d_i. Let T be the subset of $\{0, \ldots, 2g\}$ on which the d_i are defined. Then, for $j \in T$ such that $\sum \nu_{w_i,i}(d_i(j), m_i) \notin T$, set

$$\nu(j) := \sum \nu_{w_i}(d_i(j), m_i),$$

$$d_{\gamma(i)}(\nu(j)) := \nu_{w_i}(d_i(j), m_i),$$

$$d_i(2g - \nu(j)) := g - d_{i+1}(\nu(j)),$$

and

$$\nu(2g - \nu(j)) := \sum_{i=1,2} \nu_{w_i}(d_i(2g - \nu(j), m_i)$$

for $i = 1, 2$. Repeat until $\sum \nu_{w_i}(d_i(j), m_i) \in T$ for all $j \in T$.

Then $d_i : T \to \{0, \ldots, g\}$ satisfies the following properties:

1. $d_1(j) + d_2(j) = j$,
2. $0 \le d_i(j) - d_i(n) \le j - n$ for $j < n$;
3. $d_i(2g - j) = g - d_{i+1}(j)$.

Properties (1) and (3) of d_i are immediate from the definition. For (2), we prove it by induction on the size of T (starting with $T = \{0, g, 2g\}$ and increasing the size as new elements are added to T by the above construction, in particular, if $j \in T$, then so it $2g - j$).

Suppose first that $0 \le n < j \le g$. Then there exists some $j', n' \in T$ such that $j = \nu(j')$ and $n = \nu(n')$. Since $\nu(n') < \nu(j')$, $\nu_{w_i}(d_i(n'), m_i) < \nu_{w_i}(d_i(j'), m_i)$ for at least one of $i = 1, 2$. We may assume that it is true for $i = 1$. Then $d_1(n') < d_1(j')$ as ν_{w_i} is non-decreasing by Proposition 6.1.2. By the inductive hypothesis, $n' < j'$ and $d_2(n') \leq d_2(j')$. Therefore,

$$\nu_{w_i}(d_i(n'), m_i) \leq \nu_{w_i}(d_i(j'), m_i)$$

for $i = 1, 2$, and

$$\begin{aligned} 0 &\leq \nu_{w_i}(d_i(j'), m_i) - \nu_{w_i}(d_i(n'), m_i) \\ &= d_{\gamma(i)}(\nu(j')) - d_{\gamma(i)}(\nu(n')) \\ &= d_{\gamma(i)}(j) - d_{\gamma(i)}(n) \\ &\leq \sum_{i=1,2} \nu_{w_i}(d_i(j'), m_i) - \nu_{w_i}(d_i(n'), m_i) \\ &= \nu(j') - \nu(n') = j - n. \end{aligned}$$

On the other hand, suppose that $g \leq n < j \leq 2g$. Then there exists some $j', n' \in T$ such that $j = 2g - \nu(j')$ and $n = 2g - \nu(n')$. Then, without loss of generality, $\nu_{w_1}(d_1(j'), m_1) < \nu_{w_1}(d_1(n'), m_1)$ and $d_1(j') < d_1(n')$. By the inductive hypothesis, $j' < n'$ and

$$\nu_{w_i}(d_i(j'), m_i) \leq \nu_{w_i}(d_i(n'), m_i)$$

for $i = 1, 2$. Therefore,

$$\begin{aligned}
0 &\leq \nu_{w_i}(d_i(n'), m_i) - \nu_{w_i}(d_i(j'), m_i) \\
&= (g - d_{\gamma(i)}(\nu(j')) - (g - d_{\gamma(i)}(\nu(n')) \\
&= d_{\gamma(i)}(j) - d_{\gamma(i)}(n) \\
&\leq \sum_{i=1,2} \nu_{w_i}(d_i(n'), m_i) - \nu_{w_i}(d_i(j'), m_i) \\
&= \nu(n') - \nu(j') = j - n.
\end{aligned}$$

Finally suppose that $0 \leq n < g < j \leq 2g$. By the previous two cases,

$$0 \leq d_i(g) - d_i(n) \leq g - n$$

and

$$0 \leq d_i(j) - d_i(g) \leq j - g$$

which combine to give the desired result. Therefore $d_i : T \to \{0, \ldots, g\}$ satisfies the given properties for $i = 1, 2$.

Now, the stopping condition implies that $\nu : T \to T \cap \{0, \ldots, g\}$. Furthermore, ν and T satisfy the following properties:

1. T is stable under $t \mapsto 2g - t$;
2. $\nu(2g) = g$ and $\nu(0) = 0$;
3. $0 \leq \nu(j) - \nu(n) \leq j - n$ for $n < j \in T$;
4. for any two consecutive elements in T, $n < j$, then $\nu(j) - \nu(n) = j - n$ implies that $\nu(2g - j) = \nu(2g - n)$;
5. ν is surjective.

Now, items (1), (2) and (5) are immediate by definition. For (3),

$$\nu(j) - \nu(n) = \sum_{i=1,2} (\nu_{w_i,i}(d_i(j)) - \nu_{w_i,i}(d_i(n))),$$
$$= \sum_{i=1,2} d_i(j) - d_i(n) - (r_{w_i}(d_i(j), m_i) - r_{w_i}(d_i(n))).$$

But $d_i(j) \geq d_i(n)$, $0 \leq r_{w_i}(s, m_i) - r_{w_i}(t, m_i) \leq s - t$ for $0 \leq s \leq t \leq g$ and $d_1(j) + d_2(j) = j$. Thus

$$0 \leq \nu(j) - \nu(n) \leq \sum_{i=1,2} d_i(j) - d_i(n) = j - n.$$

Finally, for (4), suppose that $n < j$ such that there is no k such that $n < k < j$ and $\nu(j) - \nu(n) = j - n$. Then

$$\begin{aligned}
\nu(2g - j) - \nu(2g - n) &= \sum_{i=1,2} \nu_{w_i}(d_i(2g - j), m_i) - \nu_{w_i}(d_i(2g - n), m_i) \\
&= \sum_{i=1,2} \nu_{w_i}(g - d_{i+1}(j), m_i) - \nu_{w_i}(g - d_{i+1}(n), m_i) \\
&= j - n - \sum_{i=1,2} r_{w_i}(g - d_{i+1}(j), m_i) - r_{w_i}(g - d_{i+1}(n), m_i) \\
&= j - n - \sum_{i=1,2} r_{w_i}(g - d_{i+1}(j), g - m_{i+1}) \\
&= j - n - \sum_{i=1,2} (r_{w_i}(d_i(j), m_i) - d_i(j) - m_i + g) \\
&\quad + \sum_{i=1,2} (r_{w_i}(d_i(n), m_i) - d_i(n) - m_i + g) \\
&= j - n - \nu(j) + \nu(n) \\
&= j - n - (j - n) = 0.
\end{aligned}$$

The properties $(1)-(5)$ of ν and T are exactly the properties needed to satisfy [EvdG09, Lemma 2.11], which in particular, means that either $\nu(n) = \nu(n')$ or $\nu(n) - \nu(n') = n - n'$ for all $n, n' \in T$, and that $\nu : T \to \{0, \ldots, g\} \cap T$ can be extended linearly to a non-decreasing surjective function $\nu : \{0, \ldots, 2g\} \to \{0, \ldots, g\}$ satisfying $\nu(2g - j) = \nu(j) - j + g$ for $1 \leq j \leq g$. Furthermore, by [EvdG09, Corollary 2.13], there exists a unique element $w \in S_{2g}$ such that

$$w^{-1}(1) < \ldots < w^{-1}(g), w^{-1}(g+1) < \ldots < w^{-1}(2g),$$

and $\nu(j) = \nu_w(j, g)$ for all $0 \leq j \leq 2g$.

Note that the statement $r_w(j, g) = r_{w_1}(d_1(j), m_1) + r_{w_2}(d_2(j), m_2)$ is equivalent to $\nu_w(j, g) = \sum_{i=1,2} \nu_{w_i}(d_i(j), m_i)$. Therefore, d_i and w satisfy the requirements of the proposition for all $j \in T$. It remains to extend d_i to the rest of $\{0, \ldots, 2g\}$.

Claim. Let n, n' be two consecutive elements in T where $n + 1 < n'$. Then either

- $\nu_w(n, g) = \nu_w(n', g)$ and $1 \leq w_i(j) \leq m_i$ for all $d_i(n) + 1 \leq j \leq d_i(n')$, or
- $\nu_w(n, g) - \nu_w(n', g) = n - n'$ and $m_i + 1 \leq w_i(j) \leq g$ for all $d_i(n') + 1 \leq j \leq d_i(n)$.

First suppose that $\nu_w(n, g) = \nu_w(n', g)$. Then,

$$\nu_{w_i}(d_i(n), m_i) = d_{\gamma(i)}(\nu_w(n, g)) = d_{\gamma(i)}(\nu_w(n', g)) = \nu_{w_i}(d_i(n'), m_i).$$

But $\nu_{w_i}(-, m_i)$ is non-decreasing, so $\nu_{w_i}(j, m_i) = \nu_{w_i}(d_i(n), m_i)$ for all $d_i(n) \leq j \leq d_i(n')$. By definition of $\nu_{w_i}(j, m_i)$, this means that $1 \leq w_i(j) \leq m_i$ for all $d_i(n) + 1 \leq j \leq d_i(n')$.

On the other hand, suppose that $\nu_w(n,g) - \nu_w(n',g) = n - n'$. Then $\nu_w(2g-n,g) = \nu_w(2g-n',g)$ so that for $d_i(2g-n)+1 \leq j \leq d_i(2g-n')$, $1 \leq w_i(j) \leq m_i$. But

$$g - d_{i+1}(n) + 1 = d_i(2g-n) + 1 \leq j \leq d_i(2g-n') = g - d_{i+1}(n')$$

and $d_{i+1}(n')+1 \leq g-j+1 \leq d_{i+1}(n)$. Then $m_{i+1}+1 \leq w_{i+1}(g-j+1) \leq g$ proving the desired result.

Now, we can define d_i on the rest of $\{0,\dots,2g\}$. Set T_0 to be the original set on which ν and d_i were defined. Let S be the set on which d_i has yet to be defined. Then as long as S is not empty, let j be the smallest element in S and let

$$n = \max\{t \in T_0 \mid t < j\}, \qquad n' = \min\{t \in T_0 \mid t > j\}.$$

In particular, $n < j < n'$ where n, n' are two consecutive elements in T_0. Then either $\nu_w(n,g) = \nu_w(j,g)$ or $\nu_w(j,g) - \nu_w(n,g) = j - n$.

First, suppose that $\nu_w(n,g) = \nu_w(n',g)$. Then $\nu_w(n,g) = \nu_w(j-1,g) = \nu_w(j,g)$. Since j was the smallest element for which d_i was not yet defined, d_i is defined for $j-1$. Fix i such that $d_i(n') - d_i(j-1) > 0$. Then by the claim,

$$\nu_{w_i}(d_i(n),m_i) = \nu_{w_i}(d_i(j-1),m_i) = \nu_{w_i}(d_i(j-1)+1,m_i) = \nu_{w_i}(n',m_i).$$

Set $d_i(j) := d_i(j-1)+1$, $d_{i+1}(j) = d_{i+1}(j-1)$, $d_i(2g-j) = g - d_{i+1}(j)$ and $d_{i+1}(2g-j) = g - d_i(j)$.

Now, suppose that $\nu_w(n', g) - \nu_w(n, g) = n' - n$, and hence $\nu_w(n', g) - \nu_w(j - 1, g) = n' - (j - 1) > 0$. Fix i so that

$$d_{\gamma(i)}(\nu_w(n', g)) - d_{\gamma(i)}(\nu_w(j - 1, g)) > 0.$$

Then set $d_i(j) := d_i(j - 1) + 1$, $d_{i+1}(j) = d_{i+1}(j - 1)$, $d_i(2g - j) = g - d_{i+1}(j)$ and $d_{i+1}(2g - j) = g - d_i(j)$. Observe that $d_i(n) + 1 \leq d_i(j - 1) + 1 = d_i(j) \leq d_i(n')$, and by the claim,

$$\nu_{w_i}(d_i(j), m_i) = \nu_{w_i}(d_i(j-1), m_i) + 1, \nu_{w_{i+1}}(d_{i+1}(j), m_{i+1}) = \nu_{w_{i+1}}(d_{i+1}(j-1), m_{i+1}).$$

By induction on the construction, at each step d_i satisfies

1. $d_1(j) + d_2(j) = j$,
2. $0 \leq d_i(j') - d_i(n') \leq j' - n$ for $j' < n'$, $j', n' \in \{0, \ldots, 2g\} \setminus S$,
3. $d_i(2g - j) = g - d_{i+1}(j)$, and
4. $d_{\gamma(i)}(\nu_w(j)) = \nu_{w_i}(d_i(j), m_i)$,

completing the proof. □

Remark. Lemma 6.1.4 provides a canonical map from ${}^J W$ to ${}^{J'} W'$ where W' is the Weyl group of the symplectic group with $2g$ elements and J' is the subset of standard simple reflections for W' with s_g removed.

6.1.0.1 Examples

We now give examples of how Proposition 6.1.4 works for the element $([1235467], [1243567]) \in {}^J W$ and $(m_1, m_2) = (4, 3)$ for both choices of $\gamma : C_2 \to C_2$. First consider the case where $\gamma(i) = i$. The first step of the construction of

Table 6–2: $w = ([1235467], [1243567]), \gamma(i) = i$

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$d_1(j)$	0	0	0	0		1	2	3	3	3		4	5	6	7
$d_2(j)$	0	1	2	3		4	4	4	5	6		7	7	7	7
$\nu(j)$	0	0	0	1		1	1	1	2	3		5	5	6	7
$\nu_{w_1}(j, m_1)$	0	0	0	0		0	0	0	0	0		1	1	2	3
$\nu_{w_2}(j, m_2)$	0	0	0	1		1	1	1	2	3		4	4	4	4

Table 6–3: First choice of d_1, d_2

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$d_1(j)$	0	0	0	0	0	1	2	3	3	3	3	4	5	6	7
$d_2(j)$	0	1	2	3	4	4	4	4	5	6	7	7	7	7	7
$\nu(j)$	0	0	0	1	1	1	1	1	2	3	4	5	5	6	7
$\nu_{w_1}(j, m_1)$	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3
$\nu_{w_2}(j, m_2)$	0	0	0	1	1	1	1	1	2	3	4	4	4	4	4

Table 6–4: Second choice of d_1, d_2

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$d_1(j)$	0	0	0	0	1	1	2	3	3	3	4	4	5	6	7
$d_2(j)$	0	1	2	3	3	4	4	4	5	6	6	7	7	7	7
$\nu(j)$	0	0	0	1	1	1	1	1	2	3	4	5	5	6	7
$\nu_{w_1}(j, m_1)$	0	0	0	0	0	0	0	0	0	0	1	1	1	2	3
$\nu_{w_2}(j, m_2)$	0	0	0	1	1	1	1	1	2	3	3	4	4	4	4

Table 6–5: $w = ([1235467], [1243567]), \gamma(i) = i+1$

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$d_1(j)$	0		1			3		4		5			6		7
$d_2(j)$	0		1			2		3		4			6		7
$\nu(j)$	0		0			0		2		2			5		7
$\nu_{w_1}(j, m_1)$	0		0			0		1		1			2		3
$\nu_{w_2}(j, m_2)$	0		0			0		1		1			3		4

Table 6–6: First choice of d_1, d_2

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$d_1(j)$	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7
$d_2(j)$	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7
$\nu(j)$	0	0	0	0	0	0	1	2	2	2	3	4	5	6	7
$\nu_{w_1}(j, m_1)$	0	0	0	0	0	0	0	1	1	1	1	2	2	3	3
$\nu_{w_2}(j, m_2)$	0	0	0	0	0	0	1	1	1	1	2	2	3	3	4

Table 6–7: Second choice of d_1, d_2

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$d_1(j)$	0	0	1	2	2	3	4	4	5	5	5	6	6	6	7
$d_2(j)$	0	1	1	1	2	2	2	3	3	4	5	5	6	7	7
$\nu(j)$	0	0	0	0	0	0	1	2	2	2	3	4	5	6	7
$\nu_{w_1}(j, m_1)$	0	0	0	0	0	0	1	1	1	1	1	2	2	2	3
$\nu_{w_2}(j, m_2)$	0	0	0	0	0	0	0	1	1	1	2	2	3	4	4

Proposition 6.1.4 partially defines the functions ν, d_1 and d_2, and this is exhibited in Table 6–2 as determined immediately from (w_1, w_2). As expected, the partial function $\nu(j)$ in Table 6–2 extends uniquely to a non-decreasing function. In this example, ν corresponds to the element of length 9 in S_{2g},

$$w = [1\ 2\ 8\ 3\ 4\ 5\ 6\ 9\ 10\ 11\ 12\ 7\ 13\ 14]\,.$$

Since Table 6–2 is not complete, extending d_1 and d_2 to all of $\{1, \ldots, 14\}$ involves making choices, and in this example there are two possible choices for the pair of functions d_1 and d_2, as in Tables 6–3 and 6–4 respectively.

For the other choice of $\gamma : C_2 \to C_2$, the first step of Proposition 6.1.4 partially defines the functions ν, d_1 and d_2 as in Table 6–5. The partial function $\nu(j)$ in Table 6–5 extends uniquely to a non-decreasing function $\nu : \{1, \ldots, 14\} \to \{1, \ldots, 7\}$. This time, the element $w \in S_{2g}$ corresponding to ν is the element of length 4,

$$w = [1\ 2\ 3\ 4\ 5\ 8\ 9\ 6\ 7\ 10\ 11\ 12\ 13\ 14]$$

and there are again two choices for the pair d_1, d_2 satisfying the desired conditions. Tables 6–6 and 6–7 give the two choices for d_1 and d_2.

6.1.1 Complementary elements and canonical domains

Let $(w_1, w_2) \in {}^J W$. Then the **canonical domain** for (w_1, w_2) is the smallest pair of sets (D_1, D_2) where $\{0, g\} \subseteq D_i \subseteq \{0, 1, \ldots, g\}$ for $i = 1, 2$ such that if $j \in D_i$ then $g - j \in D_{i+1}$ and $\nu_{w_i}(j, m_i) \in D_{\gamma(i)}$.

Lemma 6.1.5. *Let $w \in S_{2g}$ and $d_1, d_2 : \{0, \ldots, 2g\} \to \{0, \ldots, g\}$ be the resulting element and functions associated to (w_1, w_2) from Proposition 6.1.4. Let D be the* *smallest set containing $\{0, 2g\}$ such that if $j \in D$, so is $\nu_w(j, g)$ and $2g - j$. Then $d_i(D) = D_i$ for $i = 1, 2$.*

Proof. This follows by induction from the construction of D and (D_1, D_2) together with the properties that follow from Proposition 6.1.4 that

$$\nu_{w_i}(d_i(j), m_i) = d_{\gamma(i)}(\nu_w(j, g))$$

and

$$d_i(2g - j) = g - d_{i+1}(j).$$

□

Corollary 6.1.6. *If $t_i < s_i$ are successive elements in D_i, then either:*

$$\nu_{w_i}(s_i, m_i) = \nu_{w_i}(t_i, m_i)$$

or

$$\nu_{w_i}(s_i, m_i) - \nu_{w_i}(n, m_i) = s_i - n$$

for all $t_i \leq n \leq s_i$.

Proof. Suppose $\nu_{w_i}(s_i, m_i) \neq \nu_{w_i}(t_i, m_i)$. Since $D_i = d_i(D)$, there exist elements s, t in D such that $d_i(s) = s_i$ and $d_i(t) = t_i$. Let s be the smallest element in D such that $d_i(s) = s_i$ and let t be the largest element in D such that $d_i(t) = t_i$, so that $t < s$ are consecutive elements in D. Then by Lemma-Definition 2.11 of [EvdG09], $\nu_w(s, g) - \nu_w(t, g) = s - t$. By Proposition 6.1.4,

$$\sum_{i=1,2} \nu_{w_i}(d_i(s), m_i) - \nu_{w_i}(d_i(t), m_i) = \nu_w(s, g) - \nu_w(t, g) = s - t = \sum_{i=1,2} d_i(s) - d_i(t)$$

but by Proposition 6.1.2,

$$0 \le \nu_{w_i}(d_i(s), m_i) - \nu_{w_i}(d_i(t), m_i) \le \sum_{i=1,2} d_i(s) - d_i(t),$$

and hence $\nu_{w_i}(s_i, m_i) - \nu_{w_i}(t_i, m_i) = s_i - t_i$. $\square$

For $J \subseteq S_g$, write $w_0(J)$ for the longest element of the subgroup of S_g generated by the elements of J. For an element $w_i \in {}^{J_i}S_g$, define its **complementary element** $v_i \in {}^{J_{i+1}}S_g$ to be the element $v_i = w_0 w_0(J_i) w_i$, and let the **complementary element** of $(w_1, w_2) \in {}^J W$ be (v_1, v_2) where v_i is the complementary element of $w_i \in {}^{J_i}S_g$.

Remark. The map taking w_i to it complementary element v_i is the composition of two maps. First, the order reversing automorphism [BB05, Prop 2.5.4]

$$\begin{aligned} {}^{J_i}S_g &\to {}^{J_i}S_g \\ w_i &\mapsto w_0(J_i) w_i w_0 \end{aligned}$$

and secondly,

$$\begin{aligned} {}^{J_i}S_g &\to {}^{J_{i+1}}S_g \\ w_i &\mapsto w_0 w_i w_0. \end{aligned}$$

Lemma 6.1.7. *Let $w = (w_1, w_2) \in {}^J W$ and let (v_1, v_2) be the complementary element of w. Then*

1. *$1 \le w_i(a) \le m_i$ if and only if $m_{i+1} + 1 \le v_i(a) \le g$, and*
2. *if $\nu_{w_i}(j, m_i) \neq \nu_{w_i}(j-1, m_i)$ then $v_i(j) = \nu_{w_i}(j, m_i)$, otherwise, if $\nu_{w_i}(j, m_i) = \nu_{w_i}(j-1, m_i)$ then $v_i(j) = g - \nu_{w_{i+1}}(g-j, m_{i+1})$.*

Proof. First, note that $w_0(J_i) = m_i\ m_i - 1\ \ldots 1\ g\ g-1 \ldots m_i + 1$ implies: $1 \leq w_i(a) \leq m_i$ if and only if $1 \leq w_0(J_i)w_i(a) \leq m_i$ if and only if $g \geq w_0 w_0(J_i)w_i(a) \geq m_{i+1} + 1$. Therefore, $1 \leq w_i(a) \leq m_i$ if and only if $m_{i+1} + 1 \leq v_i(a) \leq g$, as

Suppose that $\nu_{w_i}(j, m_i) \neq \nu_{w_i}(j-1, m_i)$. Then $j = w_i^{-1}(n)$ for some $n \geq m_i + 1$. Since $w_i \in {}^{J_i}S_g$, $w_i^{-1}(m_i+1) < \ldots < w_i^{-1}(n) = j$ so that $\nu_{w_i}(j, m_i) = n - m_i$. On the other hand,

$$w_0 w_0(J_i) w_i(j) = w_0 w_0(J_i)(n) = w_0(g - (n - m_i) + 1) = n - m_i$$

and $v(j) = \nu_{w_i}(j, m_i)$.

Finally, suppose that $\nu_{w_i}(j, m_i) = \nu_{w_i}(j-1, m_i)$. Then $j = w_i^{-1}(n)$ for some $n \leq m_i$, and

$$w_i^{-1}(1) < \ldots < w_i^{-1}(n) = j.$$

Since $w_i = w_0 w_{i+1} w_0$,

$$w_{i+1}^{-1}(m_{i+1} + 1) < \ldots < w_{i+1}^{-1}(g - n) \leq g - j$$

and $\nu_{w_{i+1}}(g - j, m_{i+1}) = m_i - n$. But

$$v_i(j) = w_0(m_i - n + 1) = g - m_i + n = m_{i+1} + n$$

and the desired result holds. □

Example 6.1.8. The complementary element of (w_1, w_2) where $w_1 = [1235467]$ and $w_2 = [1243567]$ is (v_1, v_2) where $v_1 = [4561723]$ and $v_2 = [5617234]$. The canonical domain of w is (D_1, D_2) depending on $\gamma : C_2 \to C_2$. Therefore,

$$D_1 = \{0, 1, 2, 3, 4, 5, 6, 7\}, D_2 = \{0, 1, 2, 3, 4, 5, 6, 7\}, \gamma(i) = i$$

and

$$D_1 = \{0, 1, 3, 4, 5, 6, 7\}, D_2 = \{0, 1, 2, 3, 4, 6, 7\}, \gamma(i) = i + 1.$$

Let $v_i \in {}^{J_{i+1}}S_g$ and $v_{i+1} = w_0 v_i w_0$. Then a **canonical fragment** for v_i is a maximal interval $(j, j'] \subseteq \{1, \ldots, g\}$ such that $v_i^n((j, j'])$ where $v_i^n := v_{\gamma^{n-1}(i)} \circ \ldots \circ v_{\gamma(i)} \circ v_i$ stays an interval for all n. We will use $(i, (n, n'])$ to denote a canonical fragment of v_i when we want to keep track of whether $(n, n']$ is a canonical fragment for v_i or v_{i+1}. In that case, we will let $v((i, I)) := (\gamma(i), v_i(I))$ to simplify notation.

Proposition 6.1.9. *Let $w \in {}^J W$, and $v = (v_1, v_2)$ be the complementary element of w. Then*

1. *$\{1, 2, \ldots, g\}$ is a disjoint union of the canonical fragments for v_i, and the canonical fragments of v_1 and v_2 are permuted by v_i^n for $i = 1, 2$.*
2. *If $(j, j']$ is a canonical fragment for v_i, then $(i+1, (g-j', g-j])$ is a canonical fragment for v_{i+1}.*
3. *The set of upper endpoints of the canonical fragments of v_i together with 0 is exactly D_i where (D_1, D_2) is the canonical domain of (w_1, w_2).*

Proof. For every $j \in \{1, 2, \ldots, g\}$, $v_i^n((j-1, j])$ stays an interval for all n, so every element in $\{1, 2, \ldots, g\}$ is in a canonical fragment. Now, suppose that two different canonical fragments I and J for v_i have non-empty intersection. Then

$v_i^n(I \cup J) = v_i^n(I) \cup v_i^n(J)$ is an interval for all n, so by maximality $I = J$, proving part (1).

Part (2) follows from the observation that $v_{i+1}^n = w_0 v_i^n w_0$ as $\gamma^n(i+1) = \gamma^n(i) + 1$ for all n. Then for any interval $(j, j']$,

$$v_{i+1}^n((g - j', g - j]) = g - v_i^n((j, j']) + 1 \tag{6.2}$$

so that $v_i^n((j, j'])$ is an interval for all n if and only if $v_{i+1}^n((g - j', g - j])$ is an interval for all n.

Let R_i be the set of upper endpoints of the canonical fragments of v_i together with 0. In order to show that $R_i \supseteq D_i$, we need to show that if $j \in R_i$, then $g - j \in R_{i+1}$ and $\nu_{w_i}(j, m_i) \in R_{\gamma(i)}$. By (1), if $(j', j]$ is a canonical fragment for v_i, then both j and j' are in R_i. Therefore, (2) implies that if $j \in R_i$, then $g - j \in R_{i+1}$.

Let $j' \in R_i$. Then j' is the upper endpoint of a canonical fragment $(j, j']$ for v_i. If $\nu_{w_i}(j', m_i) \neq \nu_{w_i}(j' - 1, m_i)$, then by Lemma 6.1.7 $v_i(j) = \nu_{w_i}(j, m_i)$ so that $v_i((j, j']) = (v_i(j') - (j' - j), v_i(j')]$ as $v_i((j, j']))$ is an interval. Therefore, $v_i(j') \in R_{\gamma(i)}$.

Now, suppose that $\left\{a \in R_i \;\middle|\; \nu_{w_i}(a, m_i) \notin R_{\gamma(i)}\right\}$ is non-empty, and let j' be the smallest such element. Then $\nu_{w_i}(j', m_i) = \nu_{w_i}(j' - 1, m_i) \neq 0$. Let $j \in \{1, \ldots, g\}$ be such that $\nu_{w_i}(j, m_i) = \nu_{w_i}(j', m_i)$ but $\nu_{w_i}(j - 1, m_i) \neq \nu_{w_i}(j, m_i)$. Then by Lemma 6.1.7, j and j' cannot be in the same canonical fragment, and there exists an $j \leq n < j'$ such that $(n, j']$ is a canonical fragment. But the $n \in R_i$ and

$\nu_{w_i}(n, m_i) = \nu_{w_i}(j', m_i) \notin R_{\gamma(i)}$, but j' was the smallest such element by definition. Therefore $\nu_{w_i}(j, m_i) \in R_i$ for all $j \in R_i$.

To show the reverse inclusion, it suffices to show that if $j < j'$ are successive elements in D_i, then $v_i^n((j, j'])$ is an interval and $v_i^n(j')$ is an endpoint of $v_i^n((j, j'])$ for all n. This would mean that $I = (j, j']$ is contained in a canonical fragment for v_i and j' is the only element of D_i that can be in R_i. Therefore $R_i \subseteq D_i$.

We now show that if $j < j'$ are successive elements in D_i, then $v_i^n((j, j'])$ is an interval and $v_i^n(j')$ is an endpoint of $v_i^n((j, j'])$ for all n. First, observe that if $j \in D_i$, then $v_i^n(j) \in D_{\gamma^{n-1}(i)}$ and even more, $v_i^n((j, j']) \cap D_{\gamma^{n-1}(i)} = v_i^n(j)$ by Lemma 6.1.7 and the definition of the canonical domain. Suppose that $v_i^n((j, j'])$ is an interval and $v_i^n(j')$ is an endpoint of $v_i^n((j, j'])$ for some n. Then since $v_i^n((j, j']) \cap D_{\gamma^{n-1}(i)} = v_i^n(j)$, it follows that either $\nu_{w_i}(-, m_i)$ is constant on $v_i^n((j, j'])$ or $\nu_{w_i}(v_i(j'), m_i) - \nu_{w_i}(a, m_i) = v_i(j') - a$ for $a \in v_i^n((j, j'])$ by Corollary 6.1.6. If $\nu_{w_i}(-, m_i)$ is constant on $v_i^n((j, j'])$, and $(a', v_i^n(j')] = v_i^n((j, j'])$, then $\nu_{w_i}(v_i^n(j'), m_i) = \nu_{w_i}(a, m_i)$ and $\nu_{w_i}(g - v_i^n(j'), m_i) - \nu_{w_i}(g - a, m_i) = v_i^n(j') - a$ for all $a' \leq a \leq v_i^n(j')$ as $v_i^n(j') \in D_{\gamma^{n-1}(i)}$ but a is larger than the next element to $v_i^n(j')$ in $D_{\gamma^{n-1}(i)}$.

On the other hand, if $\nu_{w_i}(v_i^n(j'), m_i) - \nu_{w_i}(a, m_i) = v_i^n(j') - a$ for $a \in v_i^n((j, j'])$, then by Lemma 6.1.7, $v_i(a) = \nu_{w_i}(a, m_i)$ for $a' < a \leq v_i^n(j')$ and $v_i^{n+1}((j, j']) = (\nu_{w_i}(a', m_i), v_i^{n+1}(j')]$. □

Let (i, I) be a canonical fragment for v_i. Then the **orbit** O of I is defined to be

$$O := \{(\gamma^n(i), v_i^n(I)) \mid n \in \mathbb{N}\}.$$

Note that $v_i^n(I)$ is a canonical fragment for $v_{\gamma^n(i)}$, as the first coordinate keeps track of whether $v_i^n(I)$ is a canonical fragment for v_i or v_{i+1}. For a canonical fragment $(i, I) \in O$, define its **dual** to be $(\widetilde{i, (n, n']}) := (i+1, (g-n', g-n])$. By (6.2), for $(i, I) \in O$, $(\widetilde{\gamma(i), v_i}(I)) = v((\widetilde{i, I}))$, and taking duals commutes with applying v. Therefore,

$$\tilde{O} := \left\{ \widetilde{(i, I)} \;\middle|\; (i, I) \in O \right\}.$$

is another orbit under v, called the **dual orbit** of O. An orbit O is said to be **self-dual** if $O = \tilde{O}$. If O is *not* self-dual, then $\left\{O, \tilde{O}\right\}$ is said to be a **pair of dual orbits**.

Example 6.1.10. Recall that the complementary element of (w_1, w_2) where $w_1 = [1235467]$ and $w_2 = [1243567]$ is (v_1, v_2) where $v_1 = [4561723]$ and $v_2 = [5617234]$. When $\gamma(i) = i$, there is one orbit of canonical fragments for v_1 that is dual to the one orbit of canonical fragments for v_2. All of the fragments are singletons (*i.e.* $(j, j+1]$ for some $j \in \{0, \ldots, 6\}$). This is consistent with the fact that the canonical domain for w is $D_1 = \{0, 1, 2, 3, 4, 5, 6, 7\}, D_2 = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

When $\gamma(i) = i + 1$, the orbits of canonical fragments are

$$O_1 : (0, 1] \xrightarrow{v_1} (3, 4] \xrightarrow{v_2} (6, 7] \xrightarrow{v_1} (2, 3] \xrightarrow{v_2} (0, 1]$$

$$O_2 : (3, 4] \xrightarrow{v_1} (0, 1] \xrightarrow{v_2} (4, 5] \xrightarrow{v_1} (6, 7] \xrightarrow{v_2} (3, 4]$$

$$O_3 : (1, 3] \xrightarrow{v_1} (4, 6] \xrightarrow{v_2} (1, 3]$$

$$O_4 : (5, 6] \xrightarrow{v_1} (1, 2] \xrightarrow{v_2} (5, 6].$$

Then $\tilde{O}_1 = O_2$ and the orbits O_3 and O_4 are self-dual.

6.2 Hodge flags

Let $\pi : \mathcal{A} \to \mathcal{M}$ be the structure map of the universal abelian variety $\mathcal{A}$ over $\mathcal{M}$. Recall that in general $\mathcal{M}$ can be taken to be a scheme over residue field of the reflex field $E_{\mathscr{D}}$ at a prime lying above p, $\kappa(\mathfrak{p})$, as in Section 2.1.2 (in particular, over $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$ depending on the choice of (m_1, m_2)). Throughout this chapter, we will additionally take $\mathcal{M}$ to be defined over κ where $\kappa = \mathbb{F}_p$ or $\kappa = \mathbb{F}_{p^2}$ according to whether p is split or inert in K to ensure that the $\mathcal{O}_K/(p)$-action on $\mathcal{A}$ can be decomposed via the two characters $\chi_i : \mathcal{O}_K/(p) \to \kappa$ for $i \in \{1, 2\}$. Recall that there is an exact sequence of locally free sheaves on $\mathcal{M}$:

$$0 \to \pi_*(\Omega^1_{\mathcal{A}/\mathcal{M}}) \to \mathcal{H}^1_{dR}(\mathcal{A}/\mathcal{M}) \to R^1\pi_*(\mathcal{O}_{\mathcal{A}}) \to 0$$

which we will denote by

$$0 \to \mathbb{E} \to \mathbb{H} \to \mathbb{E}^\vee \to 0$$

where $\mathbb{E} = \pi_*(\Omega^1_{\mathcal{A}/\mathcal{M}})$ is the **Hodge bundle** and $\mathbb{H} = \mathcal{H}^1_{dR}(\mathcal{A}/\mathcal{M})$. The Frobenius map $F : \mathcal{A} \to \mathcal{A}^{(p)}$ and Verschiebung map $V : \mathcal{A}^{(p)} \to \mathcal{A}$ are isogenies and induce linear maps on the cohomology groups:

$$F : \mathbb{H}^{(p)} \to \mathbb{H}, \qquad V : \mathbb{H} \to \mathbb{H}^{(p)}.$$

Furthermore, $\mathrm{im}(V) = \ker(F) = \mathbb{E}^{(p)}$, $\mathcal{O}_K$ acts on $\mathbb{E}$ and $\mathbb{H}$, and both FV and VF are multiplication by p. Even more, F and V commute with the $\mathcal{O}_K$-action. Under the $\mathcal{O}_K \otimes_{\mathbb{F}_p} \kappa = \kappa_1 \oplus \kappa_2$-action, $\mathbb{H}$ and $\mathbb{E}$ decompose as

$$\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2, \qquad \mathrm{rank}(\mathbb{E}_i) = m_i,$$

$$\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2, \qquad \mathrm{rank}(\mathbb{H}_i) = g.$$

The prime-to-p polarization on $\mathcal{A}$ induces a perfect pairing $\mathbb{H}_1 \times \mathbb{H}_2 \to \mathcal{O}_{\mathcal{M}}$. Furthermore, let $\gamma : C_2 \to C_2$ be defined so that $V(\mathbb{H}_i) \subseteq \mathbb{H}_{\gamma(i)}^{(p)}$; that is $\gamma(i) = i + 1$ when p is inert in K and $\gamma(i) = i$ when p is split in K.

Fix $i = 1, 2$ and consider the flag space $\mathcal{F}_i$ of complete flags of $\mathbb{H}_i$,

$$\mathcal{E}_{i,\bullet} : 0 = \mathcal{E}_{i,0} \subsetneq \mathcal{E}_{i,1} \subsetneq \ldots \subsetneq \mathcal{E}_{i,m_i} = \mathbb{E}_i \subsetneq \ldots \subsetneq \mathcal{E}_{i,g} = \mathbb{H}_i$$

such that the m_i^{th} term is equal to $\mathbb{E}_i$. Flags in $\mathcal{F}_i$ are called **Hodge flags**. Given a Hodge flag $\mathcal{E}_{i,\bullet}$, construct an associated flag $\mathcal{E}_{i+1,\bullet}$ by setting

$$\mathcal{E}_{i+1,j} = \mathcal{E}_{i,g-j}^{\perp} \cap \mathbb{H}_{i+1}.$$

In particular, $\mathcal{E}_{i+1,\bullet}$ is a flag in $\mathcal{F}_{i+1}$ as $\mathcal{E}_{i,j_1} \supsetneq \mathcal{E}_{i,j_2}$ implies that $\mathcal{E}_{i+1,g-j_1} \subsetneq \mathcal{E}_{i+1,g-j_2}$ and $\mathbb{E}^{\perp} = \mathbb{E}$ implies that $\mathcal{E}_{i+1,m_{i+1}} = \mathbb{E}_{i+1}$. Call $\mathcal{E}_{i+1,\bullet}$ the **complementary** Hodge flag to $\mathcal{E}_{i,\bullet}$, and a pair $(\mathcal{E}_{1,\bullet}, \mathcal{E}_{2,\bullet})$ such that

$$\mathcal{E}_{i+1,j} = \mathcal{E}_{i,g-j}^{\perp} \cap \mathbb{H}_{i+1}, i \in \{1, 2\}$$

a **complementary pair** of Hodge flags.

Given a complementary pair of Hodge flags $(\mathcal{E}_{i,\bullet})$, define another pair of flags as follows. For $0 \leq j \leq m_{\gamma(i)}$, set

$$\mathcal{D}_{i,m_{\gamma(i)+1}+j} := V^{-1}(\mathcal{E}_{\gamma(i),j}^{(p)}) \cap \mathbb{H}_i$$

and then let

$$\mathcal{D}_{i,j} := \mathcal{D}_{i+1,g-j}^{\perp} \cap \mathbb{H}_i$$

for $0 \le j \le m_{\gamma(i)+1}$. $\mathcal{D}_{i,\bullet}$ is called the **conjugate flag** to $\mathcal{E}_{i,\bullet}$ and the pair $(\mathcal{D}_{i,\bullet})$ is called the **pair of conjugate flags** to $(\mathcal{E}_{i,\bullet})$. It is a complementary pair as well.

Note that these constructions can be made in families over any scheme S in characteristic p. That is, for $A \to S$ an abelian scheme over S coming from the specialization of $\mathcal{A} \to \mathcal{M}$ over $S \to \mathcal{M}$, a **Hodge flag** for $A \to S$ is any complete flag $\mathcal{E}_\bullet$ of $\mathbb{H}_i$ such that $\mathcal{E}_{m_i}$ is $\mathbb{E}_i$. A Hodge flag for $A \to S$ is then the same as a lifting of the classifying map $S \to \mathcal{M}$ to a morphism $S \to \mathcal{F}_i$.

Let $\mathcal{G}_i$ be the flag space of complete flags of $\mathbb{H}_i$ over $\mathcal{F}_i$ (*sic*). Then $\mathcal{G}_i \to \mathcal{F}_i$ is a G/B-bundle where G is the group $\mathrm{GL}(\mathbb{H}_i)$ and B is the Borel stabilizing $\mathcal{E}_{i,\bullet}$. Let $s : \mathcal{F}_i \to \mathcal{G}_i$ be the tautological section and let t be the section defined by taking $\mathcal{E}_{i,\bullet}$ to its conjugate flag $\mathcal{D}_{i,\bullet}$. Let S_g be the Weyl group of $\mathrm{GL}(\mathbb{H}_i)$. As in Section 4.1 of [EvdG09], locally choose a trivialization of $\mathcal{G}_i$ such t is a constant section, so that the trivialization s corresponds to a map $\mathcal{F}_i \to G/B$. Let $\mathcal{U}^w$ be the inverse image of BwB under s, and let $\overline{\mathcal{U}}^w$ be the inverse image of $\overline{BwB}$. These definitions are independent of the choice of trivializations and give rise to global subschemes of $\mathcal{F}_i$. Then a Hodge flag $\mathcal{E}_{i,\bullet}$ is said to have **relative position** w if $\mathcal{E}_{i,\bullet} \in \mathcal{U}^w$ and is said to have **relative position** $\le w$ if $\mathcal{E}_{i,\bullet} \in \overline{\mathcal{U}}^w$. Combinatorially, the condition that a Hodge flag $\mathcal{E}_{i,\bullet}$ has relative position $w \in S_g$ then corresponds to the condition that

$$\mathrm{rank}(\mathcal{D}_{i,j} \cap \mathcal{E}_{i,n}) = r_w(j,n), \forall 1 \le j,n \le g$$

and a Hodge flag $\mathcal{E}_{i,\bullet}$ has relative position $\le w$ if and only if

$$\mathrm{rank}(\mathcal{D}_{i,j} \cap \mathcal{E}_{i,n}) \ge r_w(j,n), \forall 1 \le j,n \le g.$$

Proposition 6.2.1. *A Hodge flag $\mathcal{E}_{i,\bullet}$ has relative position $(\leq)w$ if and only if its associated Hodge flag $\mathcal{E}_{i+1,\bullet}$ has relative position $(\leq)w_0ww_0$.*

Proof. Let $1 \leq j \leq n$. Then $\mathcal{D}_{i+1,j} = \mathcal{D}_{i,g-j}^{\perp} \cap \mathbb{H}_{i+1}$ and $\mathcal{E}_{i+1,n} = \mathcal{E}_{i,g-n}^{\perp} \cap \mathbb{H}_{i+1}$. Therefore,

$$\mathcal{D}_{i+1,j} \cap \mathcal{E}_{i+1,n} = \mathcal{D}_{i,g-j}^{\perp} \cap \mathcal{E}_{i,g-n}^{\perp} \cap \mathbb{H}_{i+1} = (\mathcal{D}_{i,g-j} \cap \mathcal{E}_{i,g-n})^{\perp} \cap \mathbb{H}_{i+1}.$$

Since $\mathcal{E}_{i,\bullet}$ has relative position $(\leq)w$, it follows from Lemma 6.1.1 that

$$\begin{aligned}
\operatorname{rank}(\mathcal{D}_{i+1,j} \cap \mathcal{E}_{i+1,n}) &= 2g - \operatorname{rank}(\mathcal{D}_{i,g-j} \cap \mathcal{E}_{i,g-n})^{\perp} - g \\
&= g - (2g - j - n - \operatorname{rank}(\mathcal{D}_{i,j} \cap \mathcal{E}_{i,n})) \\
&= j + n - g + \operatorname{rank}(\mathcal{D}_{i,j} \cap \mathcal{E}_{i,n}) \\
(\geq) &= j + n - g + r_w(j,n) \\
&= j + n - g + r_w(j,n) \\
&= r_{w_0ww_0}(j,n).
\end{aligned}$$

□

Now, consider the Weyl group,

$$W = \{(w_1, w_2) \subseteq S_g \times S_g \mid w_2 = w_0w_1w_0\}.$$

A complementary pair of Hodge flags $(\mathcal{E}_{i,\bullet})$ is said to have **relative position** $(\leq)(w_1, w_2)$ if $\mathcal{E}_{i,\bullet}$ has relative position $(\leq)w_i$ for $i = 1, 2$. By Proposition 6.2.1, the relative position of a complementary pair of Hodge flags is an element in W.

Proposition 6.2.2. *Let $X \in \mathcal{M}(S)$ and let $(\mathcal{E}_{i,\bullet})$ be a complementary pair of Hodge flags for X/S with conjugate flags $(\mathcal{D}_{i,\bullet})$ and relative position $\leq (w_1, w_2)$ such that (w_1, w_2) is the smallest element in W with that property. Then*

1. *for $1 \leq n \leq m_{\gamma(i)}$ and $1 \leq j \leq m_{\gamma(i+1)}$, $r_{w_i}(m_{\gamma(i+1)} + n, j) = j$ if and only if $V(\mathcal{E}_{i,j}) \subseteq \mathcal{E}^{(p)}_{\gamma(i),n}$ with $i = 1, 2$;*
2. *for $1 \leq n \leq m_{i+1}$ and $1 \leq j \leq m_i + n$, $r_{w_i}(j, m_i + n) = j$ implies that $F(\mathcal{D}_{i,j}) \subseteq \mathcal{D}^{(p)}_{\gamma(i),n}$, and the converse is true when S is reduced.*

Proof. Since $\mathbb{E}_\bullet$ has relative position $\leq w$, it follows that

$$j \geq \operatorname{rank}(\mathcal{D}_{i,m_{\gamma(i+1)}+n} \cap \mathcal{E}_{i,j}) \geq r_{w_i}(m_{\gamma(i+1)} + n, j) = j,$$

and $\mathcal{E}_{i,j} \subseteq \mathcal{D}_{i,m_{\gamma(i+1)}+n} = V^{-1}(\mathcal{E}_{\gamma(i),n}) \cap \mathbb{H}_i$ as $1 \leq n \leq m_{\gamma(i)}$. This happens if and only if $V(\mathcal{E}_{i,j}) \subseteq \mathcal{E}^{(p)}_{\gamma(i),n}$.

Suppose that $r_{w_i}(j, m_i + n) = j$. Then

$$j = r_{w_i}(j, m_i + n) \leq \operatorname{rank}(\mathcal{D}_{i,j} \cap \mathcal{E}_{i,m_i+n}) \leq j$$

and $\mathcal{D}_{i,j} \subseteq \mathcal{E}_{i,m_i+n}$. But then,

$$\mathcal{D}^{(p)}_{i,j} \subseteq \mathcal{E}^{(p)}_{i,m_i+n} = (\mathcal{E}^{(p)}_{i+1,m_{i+1}-n})^{\perp} \cap \mathbb{H}^{(p)}_i = V(\mathcal{D}_{\gamma(i+1),g-n})^{\perp} \cap \mathbb{H}^{(p)}_i.$$

In other words, for all $u \in \mathcal{D}^{(p)}_{i,j}$ and $v \in \mathcal{D}_{\gamma(i+1),g-n}$, $\langle u, Vv \rangle = 0$, and

$$\langle Fu, v \rangle = \langle u, Vv \rangle = 0.$$

Therefore,

$$F(\mathcal{D}^{(p)}_{i,j}) \subseteq \mathcal{D}^{\perp}_{\gamma(i+1),g-n} = \mathcal{D}_{\gamma(i),n} \oplus \mathcal{D}_{\gamma(i+1),g}$$

and hence $F(\mathcal{D}_{i,j}^{(p)}) \subseteq \mathcal{D}_{\gamma(i),n}$ as desired. The converse holds by reversing the argument when S is reduced. □

6.3 Canonical flags

Similar to [EvdG09, Section 3.2], we now construct the analogue of the canonical filtration of the p-torsion group scheme of $A[p]$ for vector bundles. See [Box15, Section 4.2.3] for the construction of the canonical filtration on the level of p-torsion group schemes.

Let S be a Noetherian scheme in characteristic p. For any abelian scheme A/S coming from $S \to \mathcal{M}$, let $\mathbb{D}_g$ denote $\ker(V : \mathcal{H}^1_{dR}(\mathcal{A}/\mathcal{M}) \to \mathcal{H}^1_{dR}(\mathcal{A}/\mathcal{M})^{(p)})$ or its specialization over $S \to \mathcal{M}$. This is the same as $V^{-1}(0)$.We will also use $\mathbb{H}$ to denote $\mathcal{H}^1_{dR}(A/S)$. Under the $\mathcal{O}_K/(p)$-action,

$$\mathbb{D}_g = \mathbb{D}_1 \oplus \mathbb{D}_2, \qquad \operatorname{rank}(\mathbb{D}_i) = m_{\gamma(i)+1}.$$

For simplicity of notation, for any $\mathbb{D} \subseteq \mathbb{H}$, write $F(\mathbb{D})$ for $F(\mathbb{D}^{(p)})$ and $V^{-1}(\mathbb{D})$ for $V^{-1}(\mathbb{D}^{(p)})$ as necessary so that for any finite word R in F and V^{-1}, the resulting $R(\mathbb{D}) \subseteq \mathbb{H}$. We also need to ensure that we get subbundles of $\mathbb{H}$ at every step in order to check for equality. Now, if $\mathbb{D}$ is a subbundle of $\mathbb{H}$ over a base T, then the rank of $F(\mathbb{D}^{(p)})$ is locally constant over T, and there exists a unique minimal and canonical decomposition of T into locally closed subschemes such that the rank of $F(\mathbb{D}^{(p)})$ is constant on each subscheme. Similarly, since $V^{-1}(\mathbb{D}^{(p)}) = (F(\mathbb{D}^{(p)})^{\perp})^{\perp}$, the analogous property holds for V^{-1}. Therefore, for every word R, there exists a minimal decomposition of S into locally closed subschemes such that over each subscheme the rank of $R(\mathbb{D})$ is constant. For what follows, suppose that the base scheme S has been replaced by the requisite disjoint union of subschemes so that the comparison between $R_1(\mathbb{D}_g)$ and $R_2(\mathbb{D}_g)$ for two different words R_1 and R_2 in F, V^{-1} makes sense.

Lemma 6.3.1. *Let $\mathbb{D}_g = V^{-1}(0)$ for some A/S. Suppose that R_1, R_2 are two words in F, V^{-1} of finite length. Then either $R_1(\mathbb{D}_g) \subseteq R_2(\mathbb{D}_g)$ or vice versa.*

Proof. This is the vector bundle analogue of [Box15, Lemma 4.2.2] and [Moo01, Section 2.5]. First, note that for any $\mathbb{D} \subseteq \mathbb{H}$, $F(\mathbb{D}) \subseteq \mathbb{D}_g \subseteq V^{-1}(\mathbb{D})$, and this also settles the case where one of the R_i is the empty word. Now, when R_1 and R_2 both start with the same letter, the result follows by induction on the length of R_1, R_2. Otherwise, we may assume that $R_1 = FR_1'$ and $R_2 = V^{-1}R_2'$. Then

$$R_1(\mathbb{D}_g) = FR_1'(\mathbb{D}_g) \subseteq \mathbb{D}_g \subseteq V^{-1}R_2'(\mathbb{D}_g) = R_2(\mathbb{D}_g)$$

and the result follows. □

Let $\mathcal{R}$ be the set of all finite words in F and V^{-1}. Then by Lemma 6.3.1, the set $\{R(\mathbb{D}_g) \mid R \in \mathcal{R}\}$ is finite, and can be ordered by inclusion giving a filtration:

$$\mathcal{C}_\bullet : 0 = \mathcal{C}_0 \subsetneq \mathcal{C}_1 \subsetneq \ldots \subsetneq \mathcal{C}_c = \mathbb{D}_g \subsetneq \ldots \subseteq \mathcal{C}_n = \mathbb{H}.$$

This is called the **canonical flag** of A/S.

Lemma 6.3.2. *The canonical flag $\mathcal{C}_\bullet$ is the coarsest F, V^{-1}-stable symplectic flag of $\mathbb{H}$ such that each $\mathcal{C}_j$ is $\mathcal{O}_K$-stable. Even more, $\mathcal{C}_j^\perp = \mathcal{C}_{2c-j}$ for $0 \leq j \leq c$ where $\mathcal{C}_c = \mathbb{D}_g$.*

Proof. $\mathcal{C}_\bullet$ is the coarsest F, V^{-1}-stable flag of $\mathbb{H}$ by construction, and the $\mathcal{C}_j$ are $\mathcal{O}_K$-stable as F and V commute with the $\mathcal{O}_K$-action. Furthermore, the length of the filtration is $2c$ where $\mathcal{C}_c = \mathbb{D}_g$.

It remains to show that $\mathcal{C}_\bullet$ is symplectic. This is analogous to [Box15, Proposition 4.2.14]. Let $\mathcal{C} \subseteq \mathbb{H}$. Under the pairing $\langle\ ,\ \rangle$ induced on $\mathbb{H}$ by the polarization λ on A, $F(\mathcal{C})^\perp = V^{-1}(\mathcal{C}^\perp)$ and $F(\mathcal{C}^\perp) = V^{-1}(\mathcal{C})^\perp$. Furthermore, $\mathcal{C}_c = \mathbb{D}_g = \mathbb{D}_g^\perp = \mathcal{C}_c^\perp$. For $R \in \mathcal{R}$ and let R' be the **opposite** of R in that F and V^{-1} are exchanged. Therefore, for any $\mathcal{C} \subseteq \mathbb{H}$, $R'(\mathcal{C}) = R(\mathcal{C})^\perp$ so that $\mathcal{C}_\bullet$ is a symplectic flag and $\mathcal{C}_j^\perp = \mathcal{C}_{2c-j}$ for $0 \leq j \leq c$ as desired. □

Since each $\mathcal{C}_j$ in $\mathcal{C}_\bullet$ is $\mathcal{O}_K$-stable, $\mathcal{C}_\bullet$ decomposes under the $\mathcal{O}_K \otimes_{\mathbb{F}_p} k = k_1 \oplus k_2$ action into two separate flags $\mathcal{C}_{i,\bullet}$ for $i \in \{1, 2\}$. Let the **canonical decomposition** of S be the minimal decomposition of S into locally closed subschemes $S = \coprod S_\alpha$ such that each $\mathcal{C}_{i,j}$ in the flag $\mathcal{C}_{i,\bullet}$ for $i \in \{1, 2\}$ has constant rank over S_α (see [Box15, Thm 4.2.18] for more details). The S_α are called the **strata** of S. For what follows, the **pair of canonical flags** $(\mathcal{C}_{1,\bullet}, \mathcal{C}_{2,\bullet})$ will be viewed as being defined over the canonical decomposition of S. It will be convenient to index the elements of the of the canonical flags by their ranks; that is,

$$\mathcal{C}_{i,\bullet} : 0 = \mathcal{C}_{i,j_1} \subsetneq \mathcal{C}_{i,j_2} \subsetneq \mathcal{C}_{i,m_{\gamma(i)+1}} = \mathbb{D}_i \subsetneq \ldots \subsetneq \mathcal{C}_{i,g} = \mathbb{H}_i$$

for $i = 1, 2$ where $\mathbb{D}_g = \mathbb{D}_1 \oplus \mathbb{D}_2$ is the decomposition of $\mathbb{D}_g$ under $k_1 \oplus k_2$ and $j_n = \mathrm{rank}(\mathcal{C}_{i,j_n})$.

Recall the definition from (6.1) that for $w_i \in S_g$, $\nu_{w_i}(t, u) = t - r_{w_i}(t, u) = \#\{a \leq t \mid w(a) > u\}$.

Proposition 6.3.3. *Let $A \in \mathcal{M}(S)$, let $\mathcal{C}_\bullet$ be the canonical flag of A/S and let $(\mathcal{C}_{1,\bullet}, \mathcal{C}_{2,\bullet})$ be its associated pair of canonical flags. For each stratum S_α in the canonical decomposition of S, there exists an element $w = (w_1, w_2) \in {}^J W$ such that*

$$F(\mathcal{C}_{i,j}^{(p)}) = \mathcal{C}_{\gamma(i),\nu_{w_i}(j,m_i)}, \qquad V^{-1}(\mathcal{C}_{i,j}^{(p)}) = \mathcal{C}_{\gamma(i),g-\nu_{w_{i+1}}(g-j,m_{i+1})}$$

for all $\mathcal{C}_{i,j}$ in $\mathcal{C}_{i,\bullet}$. The element $w \in {}^J W$ is called the ***canonical type*** *of S_α, and the induced locally constant function $S \to {}^J W$ is the* ***canonical type*** *of A/S.*

Proof. Let T be the set of ranks that appear in the flag $\mathcal{C}_\bullet$ over S_α. Then by Section 3.2 of [EvdG09], there exists an element $w \in S_{2g}$ satisfying

$$w^{-1}(1) < \ldots < w^{-1}(g), w^{-1}(g+1) < \ldots < w^{-1}(2g),$$

with the properties that for $j \in T$, $F : \mathcal{C}_j^{(p)} \to \mathcal{C}_{\nu_w(j,g)}$. Let $d_i : T \to \{0, \ldots, g\}$ be the maps defined by the decomposition of $\mathcal{C}_\bullet$ into the pair of flags $\mathcal{C}_{i,\bullet}$, that is, if $\mathcal{C}_j$ has rank t, then $\mathcal{C}_j = \mathcal{C}_{1,d_1(t)} \oplus \mathcal{C}_{2,d_2(t)}$ (recall that the flags $\mathcal{C}_{i,\bullet}$ are indexed by their ranks). By construction of the d_i and the canonical flag, there exists a unique $(w_1, w_2) \in {}^J W$ satisfying the conditions

$$\nu_w(j,g) = \nu_{w_1}(d_1(j), m_1) + \nu_{w_2}(d_2(j), m_2), \tag{6.3}$$

and

$$d_{\gamma(i)}(\nu_w(j,g)) = \nu_{w_i}(d_i(j), m_i) \tag{6.4}$$

for all $j \in T$. Together, these statements imply that $F(\mathcal{C}_{i,j}^{(p)}) = \mathcal{C}_{\gamma(i),\nu_{w_i}(j,m_i)}$. On the other hand, $\mathcal{C}_j$ is self-dual, $F(\mathcal{C}^\perp) = V^{-1}(\mathcal{C})^\perp$, $\mathcal{C}_{i,j}^\perp = \mathcal{C}_{i+1,g-j}$, (6.3) and (6.4) together imply that $V^{-1}(\mathcal{C}_{i,j}^{(p)}) = \mathcal{C}_{\gamma(i),g-\nu_{w_{i+1}}(g-j,m_{i+1})}$. □

Proposition 6.3.4. *Let $A \in \mathcal{M}(S)$, and let $(\mathcal{E}_{i,\bullet})$ be a complementary pair of Hodge flags of relative position ${}^J w = (w_1, w_2) \in {}^J W$. The conjugate flags $(\mathcal{D}_{i,\bullet})$ are refinements of the canonical flags $(\mathcal{C}_{i,\bullet})$. That is, ${}^J w$ is determined by A/S and*

$$\operatorname{rank}(\mathcal{D}_{i,j} \cap \mathcal{E}_{i,n}) = r_{w_i}(j, n)$$

for $i = 1, 2$, $1 \leq j, n \leq g$. In particular, the canonical decomposition of S with respect to A/S consists of a single stratum and the canonical type of A/S is ${}^J w$.

Proof. By Proposition 6.1.4, let $w \in S_{2g}$ and $d_i : \{0, \ldots, 2g\} \to \{0, \ldots, g\}$ for $i = 1, 2$ satisfy

1. $d_1(j) + d_2(j) = j$,
2. $0 \leq d_i(j) - d_i(n) \leq j - n$ for $j < n$,
3. $d_i(2g - j) = g - d_{i+1}(j)$,
4. $\nu_w(j, g) = \nu_{w_1}(d_1(j), m_1) + \nu_{w_2}(d_2(j), m_2)$, and
5. $d_{\gamma(i)}(\nu_w(j, g)) = \nu_{w_i}(d_i(j), m_i)$.

Let $\mathbb{D}_\bullet$ be the flag constructed from $\mathcal{D}_{i,\bullet}$ by setting

$$\mathbb{D}_j := \mathcal{D}_{1,d_1(j)} \oplus \mathcal{D}_{2,d_2(j)}.$$

Properties (1), (2) and (3) imply that

$$0 = \mathbb{D}_0 \subsetneq \mathbb{D}_1 \subsetneq \ldots \subsetneq \mathbb{D}_{2g} = \mathbb{H}_1 \oplus \mathbb{H}_2 = \mathbb{H}$$

is a complete symplectic flag where $\operatorname{rank}(\mathbb{D}_j) = d_1(j) + d_2(j) = j$. Showing that $\mathbb{D}_\bullet$ is F, V^{-1}-stable means that $\mathbb{D}_\bullet$ is a refinement of the canonical flag $\mathcal{C}_\bullet$ for A/S and hence $\mathcal{D}_{i,\bullet}$ is a refinement of $\mathcal{C}_{i,\bullet}$ for $i = 1, 2$. Therefore it remains to show that $\mathbb{D}_\bullet$ is F, V^{-1}-stable.

Define $e_i : \{0, \ldots, 2g\} \to \{0, \ldots, g\}$ for $i = 1, 2$ by

$$e_i(j) := m_i - d_{\gamma(i)+1}(g - j) = d_{\gamma(i)}(g + j) - m_{i+1}$$

and

$$e_i(2g - j) := g - e_i(j)$$

for all $0 \leq j \leq g$. Then setting

$$\mathbb{E}_j := \mathcal{E}_{1,e_1(j)} \oplus \mathcal{E}_{2,e_2(j)}$$

gives a complete symplectic flag of $\mathbb{H}$ with $\mathbb{E}_g = \mathbb{E}$. Furthermore, by the construction of $(\mathcal{D}_{i,\bullet})$, for $0 \leq j \leq g$,

$$\begin{aligned} V^{-1}(\mathbb{E}_j^{(p)}) &= V^{-1}(\mathcal{E}_{1,e_1(j)}^{(p)}) \oplus V^{-1}(\mathcal{E}_{2,e_2(j)}^{(p)}) \\ &= \mathcal{D}_{\gamma(1),m_2+e_1(j)} \oplus \mathcal{D}_{\gamma(2),m_1+e_2(j)} \\ &= \mathcal{D}_{\gamma(1),d_{\gamma(1)(g+j)}} \oplus \mathcal{D}_{\gamma(2),d_{\gamma(2)(g+j)}} \\ &= \mathbb{D}_{g+j}. \end{aligned}$$

On the other hand, since $r_{w_i}(j, m_i + \nu_{w_i}(j, m_i)) = j$ for all $1 \leq j \leq m_i$ by Lemma 6.1.3,

$$\begin{aligned} F(\mathbb{D}_j^{(p)}) &= \oplus_{i=1,2} F(\mathcal{D}_{i,d_i(j)}) \\ &= \oplus_{i=1,2} \mathcal{D}_{\gamma(i),\nu_{w_i}(d_i(j),m_i)} \\ &= \mathbb{D}_{\nu_w(j,g)} \end{aligned}$$

by Proposition 6.2.2. □

Corollary 6.3.5. *Let D_i be the set of ranks that appear in the canonical flag $\mathcal{C}_{i,\bullet}$. Then (D_1, D_2) is the canonical domain of (w_1, w_2). If (v_1, v_2) is the complementary element of (w_1, w_2), then for $j < n$ successive elements in D_i either*

$$F(\mathcal{C}_{i,n}^{(p)}/\mathcal{C}_{i,j}^{(p)}) = \mathcal{C}_{\gamma(i),v_i(n)}/\mathcal{C}_{\gamma(i),v_i(j)} \tag{6.5}$$

if $\nu_{w_i}(n, m_i) \neq \nu_{w_i}(n-1, m_i)$ or

$$V^{-1}(\mathcal{C}_{i,n}^{(p)}/\mathcal{C}_{i,j}^{(p)}) = \mathcal{C}_{\gamma(i),v_i(n)}/\mathcal{C}_{\gamma(i),v_i(j)} \tag{6.6}$$

if $\nu_{w_i}(n, m_i) = \nu_{w_i}(n-1, m_i)$. In other words, if I is a canonical fragment for v_i, then there is an isomorphism

$$\mathcal{C}_{i,I}^{(p)} \to \mathcal{C}_{\gamma(i),v_i(I)}$$

induced by F or V^{-1} where $\mathcal{C}_{i,(j,n]} := \mathcal{C}_{i,n}/\mathcal{C}_{i,j}$.

Proof. Recall that the canonical domain for (w_1, w_2) is the smallest pair of sets (D_1, D_2) where $\{0, g\} \subseteq D_i \subseteq \{0, \ldots, g\}$ for $i = 1, 2$ such that if $j \in D_i$ then $g - j \in D_{i+1}$ and $\nu_{w_i}(j, m_i) \in D_{\gamma(i)}$. But the construction of the canonical flags are the result of applying F and V^{-1} repeatedly to $\mathbb{D}_1 \oplus \mathbb{D}_2$. By Proposition 6.3.3, $F(\mathcal{C}_{i,j}^{(p)}) = \mathcal{C}_{\gamma(i),\nu_{w_i}(j,m_i)}$ and $V^{-1}(\mathcal{C}_{i,j}^{(p)}) = \mathcal{C}_{\gamma(i),g-\nu_{w_{i+1}}(g-j,m_{i+1})}$ when $\mathcal{C}_{i,\bullet}$ has canonical type (w_1, w_2), so it follows that the canonical domain for (w_1, w_2) is the set of ranks that appear in the canonical flags.

The map $F : \mathcal{C}_{i,n}^{(p)}/\mathcal{C}_{i,j}^{(p)} \to \mathcal{C}_{\gamma(i),\nu_{w_i}(n,m_i)}/\mathcal{C}_{\gamma(i),\nu_{w_i}(j,m_i)}$ is always surjective. By Corollary 6.1.6, if $\nu_{w_i}(n, m_i) \neq \nu_{w_i}(j, m_i)$ then the dimension of $\mathcal{C}_{\gamma(i),\nu_{w_i}(n,m_i)}/\mathcal{C}_{\gamma(i),\nu_{w_i}(j,m_i)}$ is equal to the dimension of $\mathcal{C}_{i,n}^{(p)}/\mathcal{C}_{i,j}^{(p)}$ and F is an isomorphism. Otherwise, F is the zero map and $\nu_{w_{i+1}}(g-j, m_{i+1}) - \nu_{w_{i+1}}(g-n, m_{i+1}) = n - j$ so that V^{-1} is an isomorphism on $\mathcal{C}_{i,n}^{(p)}/\mathcal{C}_{i,j}^{(p)}$. Then the result follows from Lemma 6.1.7. □

As a converse to Proposition 6.3.4, consider the following.

Proposition 6.3.6. *Let $A \in \mathcal{M}(S)$. Suppose that S is reduced and the canonical decomposition of S with respect to A/S consists of a single stratum. Let $w = (w_1, w_2) \in {}^JW$ be the canonical type of the canonical flag. Let $(\mathcal{E}_{i,\bullet})$ be a pair of complementary Hodge flags with relative position $w' = (w'_1, w'_2) \in W$ and associated pair of conjugate flags $(\mathcal{D}_{i,\bullet})$. Then if the conjugate flags $(\mathcal{D}_{i,\bullet})$ are refinements of the canonical flags $(\mathcal{C}_{i,\bullet})$ for A/S such that $F(\mathcal{D}_{i,j}^{(p)}) \subseteq \mathcal{D}_{\gamma(i),\nu_{w_i}(j,m_i)}$ for all $1 \leq j \leq g$, then $w = w'$. In particular, w' is in ${}^JW$.*

Proof. Let (D_1, D_2) be the canonical domain of w. Since $\mathcal{D}_{i,\bullet}$ is an extension of $\mathcal{C}_{i,\bullet}$ it follows that

$$\nu_{w_i}(j, m_i) = \nu_{w'_i}(j, m_i)$$

for all $j \in D_i$. Therefore, $w' \mapsto w$ under the map $W \to {}^J W$. The condition that $F(\mathcal{D}_{i,j}^{(p)}) \subseteq \mathcal{D}_{\gamma(i),\nu_{w_i}(j,m_i)} = \mathcal{D}_{\gamma(i),\nu_{w_i'}(j,m_i)}$ implies that $r_{w_i'}(j, m_i + \nu_{w_i'}(j, m_i)) = j$ for all $j \in \{1, \ldots, g\}$. Then by Lemma 6.1.3, $w' \in {}^J W$ so that $w = w'$. □

6.4 Local structure of the E-O strata

By the definitions given in the previous section, the condition that A/S has canonical type $w \in {}^J W$ specializes when S is the spectrum of an algebraically closed field k of characteristic p to the condition that A/k is in the E-O stratum of w, denoted V^w, as in Chapter 3. By the theory of degeneracy loci of flag varieties, the Schubert cells of the flag variety $\mathcal{F}_i$ give a stratification of $\mathcal{F}_i$ where the strata correspond to elements in S_g. We now consider what happens to the strata of $\mathcal{F}_i$ under the natural map $\mathcal{F}_i \to \mathcal{M}$. In particular, we will see that if $w = (w_1, w_2) \in {}^J W$, then $\mathcal{F}_i \to \mathcal{M}$ restricted to $\mathcal{U}^{w_i}$ is a finite surjective étale covering map $\mathcal{U}^{w_i} \to V^w$. Note that Proposition 6.2.1 and Lemma 6.4.1 show that the choice of $i \in \{1, 2\}$ does not matter in this context. This section closely follows Section 8 of [EvdG09], as the results of the previous section allow us to apply their techniques to the unitary Shimura varieties under consideration.

6.4.1 Extensions of the canonical flags

We would like to find the number of extensions of a pair of canonical flags $(\mathcal{C}_{i,\bullet})$ of type $(w_1, w_2) \in {}^J W$ to a pair of conjugate flags $(\mathcal{D}_{i,\bullet})$ of type $(w_1, w_2) \in {}^J W$ for $i = 1, 2$ over an algebraically closed field k of characteristic p. This will eventually be the degree of the maps $\mathcal{F}_i \to \mathcal{M}$ when restricted to $\mathcal{U}^{w_i} \to V^w$. To this end, recall the following definitions and results from Section 6.1.1. Let $v = (v_1, v_2)$ be the complementary element of w. Then the canonical fragments of v_1 and v_2 can grouped into distinct orbits by letting the orbit of a canonical fragment (i, I) of v_i be its orbit under v for all n (this is possible by Proposition 6.1.9). Note if $\gamma(i) = i$, then the orbit of a canonical fragment of v_i only contains canonical fragments of v_i, but if $\gamma(i) = i + 1$, the orbit of a canonical fragment contains canonical fragments of both v_1 and v_2.

Let $N_n^e(m)$ be the number of complete $\mathbb{F}_{p^m}$-flags in $\mathbb{F}_{p^m}^n$, and let $N_n^o(m)$ be the number of complete $\mathbb{F}_{p^{2m}}$-flags in $\mathbb{F}_{p^2m}^n$ that are self-dual under the unitary form

$$\langle (u_j), (v_j) \rangle = \sum_{j=1}^{n} u_j v_j^{p^m}. \tag{6.7}$$

Finally, let

$$N(w) := \begin{cases} \prod_{O_1} N^e_{\# I(O_1)}(\# O_1) & \gamma(i) = i \\ \prod_{O=\tilde{O}} N^o_{\# I(O)}(\# O/2) + \prod_{\{O,\tilde{O}\}} N^e_{\# I(O)}(\# O) & \gamma(i) = i + 1 \end{cases} \tag{6.8}$$

where the products are taken over the orbits O_1 of the canonical fragments of v_1 in the first case, and over the orbits O of the canonical fragments of v_i for $i = 1$ and 2 in the second case; in both cases $\# I(O)$ is the size of the fragments in O.

Lemma 6.4.1. *Let $\underline{A} \in \mathcal{M}(k)$, $w \in {}^J W$ be the canonical type of $\underline{A}/k$, and $(\mathcal{C}_{i,\bullet})$ be its canonical flags. Then $N(w)$ is the number of extensions of $\mathcal{C}_{i,\bullet}$ to a complete flag $\mathcal{D}_{i,\bullet}$ such that $(\mathcal{D}_{i,\bullet})$ are the conjugate flags for a pair of Hodge flags of relative position w.*

Proof. The proof is similar to that of [EvdG09, Lemma 4.6], except that we need to account for the $\mathcal{O}_K$-action on the flags. Since we are working over a perfect field k, Proposition 6.3.6 implies that a pair of conjugate flags for a Hodge flag of relative position w is the same as a pair of conjugate flags $(\mathcal{D}_{i,\bullet})$ extending $\mathcal{C}_{i,\bullet}$ such that $F(\mathcal{D}_{i,j}^{(p)}) \subseteq \mathcal{D}_{\gamma(i),\nu_{w_i}(j,m_i)}$ for all $1 \leq j \leq g$. But then by Proposition 6.3.4, $F(\mathcal{D}_{i,j}^{(p)}) = \mathcal{D}_{\gamma(i),\nu_{w_i}(j,m_i)}$ for all $1 \leq j \leq g$.

Now, since $\mathcal{D}_{i,\bullet}$ is an extension of $\mathcal{C}_{i,\bullet}$ for $i \in \{1, 2\}$, it is determined by the flags of $\mathcal{D}_{i,I}$ for all of the canonical fragments I of v_i. However, by Corollary 6.3.5 there is an isomorphism $\mathcal{D}_{i,I}^{(p)} \to \mathcal{D}_{v(i,I)}$, so that a choice of flag for $\mathcal{D}_{i,I}$ determines the flags for $\mathcal{D}_{i',I'}$ for all (i', I') in the orbit O of (i, I). Therefore the problem can be considered on an orbit-by-orbit basis. For any (i, I) in an orbit O of length m, the condition that $F(\mathcal{D}_{i,j}^{(p)}) = \mathcal{D}_{\gamma(i),\nu_{w_i}(j,m_i)}$ for all $1 \leq j \leq g$ translates into the condition that the flag for $\mathcal{D}_{i,I}$ is stable under the isomorphism $\mathcal{D}_{i,I}^{(p^m)} \to \mathcal{D}_{v(i,I)}$. The notation has been set up in such a way that the proof now follows the proof of [EvdG09, Lemma 4.6] essentially verbatim. However, we will continue the rest of the proof for completeness.

Suppose that O is not equal to its dual $\tilde{O}$. Then the flags for $(i, I) \in O$ determine the flags for $\widetilde{(i, I)} \in \tilde{O}$ by duality under the symplectic form ($i.e.$ $\mathcal{D}_{i+1,g-j} = \mathcal{D}_{i,j}^{\perp} \cap \mathbb{H}_{i+1}$). Therefore, for each orbit pair, $\left\{O, \tilde{O}\right\}$, the flag for $\mathcal{D}_{i,I}$ for a single $(i, I) \in O$ determines the flags for all of the blocks corresponding to the canonical fragments in O and $\tilde{O}$. The only remaining condition is that the flag of $\mathcal{D} = \mathcal{D}_{i,I}$ is stable under the isomorphism $t_m : \mathcal{D}^{(p^m)} \to \mathcal{D}$. As $\mathcal{D}$ is a vector bundle over an algebraically closed field k, $\mathcal{D}_p = \{v \in \mathcal{D} \mid t_m(v) = v\}$ is an $\mathbb{F}_{p^m}$-vector space such that the $\mathcal{D}_p \subseteq \mathcal{D}$ induces an isomorphism $\mathcal{D}_p \otimes k \to \mathcal{D}$. Therefore, the number of flags of $\mathcal{D}_{i,I}$ that are stable under t_m is equal to the number of complete flags of $\mathcal{D}_p$. This completes the proof in the case when p is split.

When p is inert, we also have to consider the case where $O = \tilde{O}$. The size of the orbit O is even, say $2m$, and we require that $\mathcal{D} = \mathcal{D}_{i,i}$ is stable under $t_{2m} : \mathcal{D}^{(p^{2m})} \to \mathcal{D}$, and that $t_m : \mathcal{D}^{(p^m)} \to \mathcal{D}_{\widetilde{i,I}}$. Therefore, the number of possible flags for $\mathcal{D}_{i,I}$ corresponds to the number of complete $\mathbb{F}_{p^{2m}}$-flags of $\mathbb{F}_{p^{2m}}^{\# I}$ that are self-dual under the unitary form of equation (6.7). $\square$

6.4.2 Stratified spaces

In order to show that the maps from $\mathcal{F}_i \to \mathcal{M}$ are locally like maps from $\mathrm{GL}_g/B \to \mathrm{GL}_g/Q$ where B is a Borel and Q is a parabolic of GL_g, we introduce the notion of diagrams of stratified spaces.

A **stratified space** is a scheme with a partition into locally closed subschemes; that is, a scheme together with a stratification on it. A **morphism of stratified spaces** is a morphism of schemes taking strata to strata in the sense that the pre-image of a stratum is a union of strata. Let P be a poset. Then a **diagram of stratified space** is a contravariant functor from the category of the poset P to the category of stratified spaces.

There are two particular diagrams of stratified spaces that we will be concerned with. Let $\mathcal{F}\ell_i^\bullet$ to be the following diagram of stratified spaces. Let P be the set of all subsets of $\{1, \ldots, g-1\}$ containing m_i and choose a vector space W of dimension g. For $T \in P$, let $\mathcal{F}\ell_i^\bullet(T)$ be the flag space of partial flags of W whose dimensions are the values that appear in T. Then $\mathcal{F}\ell_i^\bullet(T)$ has the structure of a stratified space by defining its strata to be its locally closed Schubert cells with respect to a fixed complete flag of W. For $T \subseteq T'$, there is a map from $\mathcal{F}\ell_i^\bullet(T') \to \mathcal{F}\ell_i^\bullet(T)$ that comes from forgetting elements in the flag of dimensions $T' \setminus T$. This map takes strata to strata.

Likewise, let $\mathcal{F}_i^\bullet$ to be the diagram over the same P, where $\mathcal{F}_i^\bullet(T)$ is the space of partial flags $\mathcal{E}_{i,\bullet}$ of $\mathbb{H}_i$ extending the flag $\mathbb{E}_i \subseteq \mathbb{H}_i$ with dimensions in T. To a partial flag $\mathcal{E}_{i,\bullet}$, we can associate a partial conjugate flag $\mathcal{D}_{i,\bullet}$ by the same method use for complete Hodge flags. That is, first construct a partial complementary flag $\mathcal{E}_{i+1,\bullet}$ by setting $\mathcal{E}_{i+1,j} := \mathcal{E}_{i,g-j}^{\perp} \cap \mathbb{H}_i$, and then use V^{-1} and take duals to get another partial flag $\mathcal{D}_{i,\bullet}$ of $\mathbb{H}_i$ as in Section 6.2. Then $\mathcal{F}_i^\bullet(T)$ has the structure of stratified space by considering the relative positions of $\mathcal{E}_{i,\bullet}$ and $\mathcal{D}_{i,\bullet}$. For $T \subseteq T'$, there is again a forgetful map from $\mathcal{F}_i^\bullet(T') \to \mathcal{F}_i^\bullet(T)$ of stratified spaces that comes from forgetting elements in the partial flags $\mathcal{E}_{i,\bullet}$ in $\mathcal{F}_i^\bullet(T')$ of dimensions $T' \setminus T$.

Let k be a perfect field of characteristic p. We will need to introduce the notion of a *height 1-neighbourhoods* of a k-point. Over such a neighbourhood, the deformation theory of a k-point A in the moduli space of principally polarized abelian schemes corresponds to the local deformation of a Grassmanian variety coming from the variation of the Hodge filtration $\mathbb{E} \subseteq \mathbb{H}$ of A by Grothendieck-Messing theory. Ekedahl and van der Geer show that these ideas can be extended to certain flag spaces over $\mathcal{M}$ by considering the variation symplectic flags of $\mathbb{H}$ containing $\mathbb{E} \subseteq \mathbb{H}$ ([EvdG09]). In Theorem 6.4.2, we show that this can be done incorporating $\mathcal{O}_K$-action as well. We now make this precise.

Let R be a local ring in characteristic p with maximal ideal $\mathfrak{m}_R$, and let $\mathfrak{m}_R^{(p)}$ denote the ideal generated by the p^{th}-powers of elements in $\mathfrak{m}_R$. Following [EvdG09], the **height 1-hull** of the local ring R is the quotient $R/\mathfrak{m}_R^{(p)}$. If S is a characteristic p scheme and x a k-point of S, then the **height 1-neighbourhood** of x is the spectrum of the height 1-hull of its completed local ring $\mathrm{Spec}(\hat{\mathcal{O}}_{S,x}/\mathfrak{m}_x^{(p)})$. For instance, if x is a smooth k-point of S, then the height 1-neighbourhood of x has the form $\mathrm{Spec}(k[[t_1,\ldots,t_n]]/(t_1,\ldots,t_n)^{(p)})$. The operation of taking height 1-neighbourhoods of k-points is functorial in the sense that if $f : S \to S'$ is a morphism of schemes such that $f(x) = y$ on k-points x and y, then f induces a map from height 1-neighbourhood of x to the height 1-neighbourhood of y. The height 1-neighbourhoods of two k-points x and y are said to be **height 1-isomorphic** if the height 1-hulls of their completed local rings are isomorphic and are *also* isomorphic to $k[[t_1,\ldots,t_n]]/(t_1,\ldots,t_n)^{(p)}$ for some n.

A k-point of a diagram of stratified spaces $\mathcal{F}^\bullet$ over P is a k-point of $\mathcal{F}^\bullet(T)$ for some $T \in P$ together with its diagram of restrictions to $\mathcal{F}^\bullet(T')$ for all $T' \leq T$. For a k-point x of $\mathcal{F}^\bullet$, its height 1-neighbourhood (resp. Hesenlization) is taken to be the height 1-neighbourhood (resp. Hesenlization) as a point of $\mathcal{F}^\bullet(T)$ together with its diagram of compatible restrictions to the height 1-neighbourhoods (resp. Hesenlization) of x as a point in $\mathcal{F}^\bullet(T')$ for all $T' \leq T$.

Theorem 6.4.2. *For each perfect field k of characteristic p and each k-point x of $\mathcal{F}_i^\bullet$, there is a k-point y of $\mathcal{F}\ell_i^\bullet$ such that the height 1-neighbourhood of x is isomorphic to the height 1-neighbourhood of y by a stratified isomorphism of diagrams.*

Proof. A point $x \in \mathcal{F}_i^\bullet$ corresponds to a partial flag $\mathcal{E}_{i,\bullet}$ of $\mathbb{H}_i$ extending $\mathbb{E}_i \subseteq \mathbb{H}_i$, together with all restrictions $\mathcal{F}_i(T) \to \mathcal{F}_i(T')$ for $T' \subseteq T$, where T is the set dimensions appearing in $\mathcal{E}_{i,\bullet}$. Let $X^\bullet$ be the height 1-neighbourhood of $x \in \mathcal{F}_i^\bullet$. Now, the de Rham cohomology has a canonical trivialization over the height 1-neighbourhood X of x: $\mathbb{H}_X \cong X \times W'$ which is horizontal with respect to the Gauss-Mannin connection. Furthermore, $\mathbb{H}_X \cong X \times W'$ induces isomorphisms $(\mathbb{H}_i)_X \to X \times W_i$ where $W_1 \oplus W_2 = W'$. The absolute Frobenius map factors through the closed point, so that $\mathcal{E}_{i,\bullet}^{(p)}$ is constant. Even more, applying V^{-1} and taking duals are horizontal operations, so that the conjugate flag $\mathcal{D}_{i,\bullet}$ is also horizontal.

Choose an isomorphism of W_i with the fixed reference space W for $\mathcal{F}\ell_i$ so that the flag $\mathcal{D}_{i,\bullet}$ gets mapped into the fixed complete reference flag of W. Then $\mathcal{E}_{i,\bullet}$ gets taken to a partial flag of W corresponding to a point $y \in \mathcal{F}\ell_i(T)$. Furthermore, this map preserves strata as the relative position of $\mathcal{E}_{i,\bullet}$ and $\mathcal{D}_{i,\bullet}$ corresponds to the relative position of the partial flag corresponding to y with respect to the reference flag of W. The isomorphism W_i with W also induces morphisms on all the partial flag spaces lying below x in the diagram by restriction, so we get a map from $x \in \mathcal{F}_g^\bullet$ to $y \in \mathcal{F}\ell_i^\bullet$. By Grothendieck-Messing theory, this gives an isomorphism on height 1-neighbourhoods of the diagrams. □

Remark. It is worth noting that the reason we can work with Schubert cells of $\mathcal{F}_i^\bullet$ instead of the product $(\mathcal{F}_1 \times \mathcal{F}_2)^\bullet$ is because $\mathbb{H}_1$, $\mathbb{H}_2$, and $\mathbb{E}$ are all maximal isotropic subspaces under the symplectic form on $\mathbb{H}$ induced by the polarizations of the underlying abelian schemes. This means that the flag $\mathbb{E}_i \subseteq \mathbb{H}_i$ completely determines both $\mathbb{E} \subseteq \mathbb{H}$ and $\mathbb{D}_g \subseteq \mathbb{H}$. Furthermore, a (partial) Hodge flag $\mathcal{E}_{i,\bullet}$ of $\mathbb{H}_i$ completely determines a complementary pair of (partial) Hodge flags and hence the pair of (partial) conjugate flags $\mathcal{D}_{i,\bullet}$ as well. In other words, the relative position of $\mathcal{E}_{i,\bullet}$ with respect to $\mathcal{D}_{i,\bullet}$ determines the relative position of $\mathcal{E}_{i+1,\bullet}$ with respect to $\mathcal{D}_{i+1,\bullet}$. This was shown in Proposition 6.2.1.

Theorem 6.4.3. *For each perfect field k of characteristic p and each k-point x of $\mathcal{F}_i^\bullet$, there is a k-point y of $\mathcal{F}\ell_i^\bullet$ such that the Henselization of x is isomorphic to the Henselization of y by a stratified isomorphism of diagrams.*

Proof. Let $X^\bullet$ and $Y^\bullet$ be the height-1 hulls of x and y respectively. Then Theorem 6.4.2 gives a stratified isomorphism $X^\bullet \to Y^\bullet$ between the height-1 hulls which can be extended to an isomorphism between the local rings $\tilde{X}^\bullet \to \tilde{Y}^\bullet$ of x and y by extending the trivialization of $\mathbb{H}_X \to X \times W'$ to a trivialization of $\mathbb{H}$ over $\mathcal{O}_{\mathcal{F}_i,x}$ so that the isomorphism between W_i and the fixed reference space W for $\mathcal{F}\ell_i$ takes the trivialization of $\mathcal{D}_{i,\bullet}$ to the fixed flag of W over y (see [EvdG09] or [DP94] for more details). This gives a isomorphism on tangent spaces, and both $\mathcal{F}_i^\bullet(T)$ and $\mathcal{F}\ell_i^\bullet(T)$ are smooth of the same dimensions for all $T \in P$. Therefore this trivialization gives an isomorphism on the Henselizations and proves the theorem. □

Lemma 6.4.4. *Let $\underline{A} \in \mathcal{M}(k)$ where k is an algebraically closed field and suppose that $\underline{A}$ has canonical type $w = (w_1, w_2) \in {}^JW$. Then if $(\mathcal{E}_{i,\bullet})$ is a pair of Hodge flags for $\underline{A}$ with type $w' = (w_1', w_2') \in W$ such that $w' \leq w$, then $w = w' \in {}^JW$.*

Proof. Let $(\mathcal{D}_{i,\bullet})$ be the pair of conjugate flags corresponding to $(\mathcal{E}_{i,\bullet})$. First, the condition that $w' \leq w$ means that $\nu_{w'_i}(j,n) \leq \nu_{w_i}(j,n)$ for all $1 \leq j,n \leq g$, and hence

$$F(\mathcal{D}_{i,j}^{(p)}) \subseteq \mathcal{D}_{\gamma(i),\nu_{w_i}(j,m_i)}$$

for all $1 \leq j \leq g$.

Define $J_i \subseteq \{1,\ldots,g\}$ to be the set of indices such that $\mathcal{D}_{i,j}$ is part of the canonical flag. Trivially $\{0,g\} \subseteq J_i$ and if $j \in J_i$ then $g-j \in J_{i+1}$. Furthermore, for $j \in J_i$, $F(\mathcal{D}_{i,j}^{(p)}) = \mathcal{D}_{\gamma(i),\nu_{w_i}(j,m_i)}$, so that $\nu_{w_i}(j,m_i) \in J_{\gamma(i)}$. By the definition of the canonical domain of w, (J_1,J_2) contains the canonical domain for w, and the conjugate flags $(\mathcal{D}_{i,\bullet})$ are refinements of the canonical flags for A/k. Proposition 6.3.6 then gives the desired result. □

Corollary 6.4.5. *For each $w_i \in {}^{J_i}S_g$, the stratum $\mathcal{U}^{w_i}$ of $\mathcal{F}_i$ is smooth of dimension $\ell(w_i)$. Furthermore, the closed stratum $\overline{\mathcal{U}}^{w_i}$ (as defined in Section 6.2) is the closure of $\mathcal{U}^{w_i}$ in $\mathcal{F}_i$ and is Cohen-Macaulay, reduced and normal of dimension $\ell(w_i)$.*

Proof. As in the proof of [EvdG09, Corollary 8.4], the facts that $\mathcal{U}^{w_i}$ is smooth of dimension $\ell(w_i)$ and $\overline{\mathcal{U}}^{w_i}$ is Cohen-Macaulay, reduced and normal of dimension $\ell(w_i)$, follow from the analogous results for the Schubert cells of the flag varieties $\mathcal{F}\ell_i$ as these are all properties that pass from the Henselization of the local ring at a point to the local ring itself.

To show that $\overline{\mathcal{U}}^{w_i}$ is the closure of $\mathcal{U}^{w_i}$, consider the following. By construction, $\overline{\mathcal{U}}^{w_i}$ is a closed set containing $\mathcal{U}^{w_i}$, in fact, $\overline{\mathcal{U}}^{w_i} = \cup_{w' \leq w_i} \mathcal{U}^{w'}$. But when

$w' \leq w_i$, $\ell(w') = \ell(w_i)$ if and only if $w' = w_i$. Since $\ell(w')$ is the dimension of each $\mathcal{U}^{w'}$, the closed stratum $\overline{\mathcal{U}}^{w_i}$ must be the closure of $\mathcal{U}^{w_i}$ in $\mathcal{F}_i$. □

Proposition 6.4.6. *For $w = (w_1, w_2) \in {}^J W$, the restriction to $\mathcal{U}^{w_i}$ of the projection $\mathcal{F}_i \to \mathcal{M}$ is a finite surjective étale covering from $\mathcal{U}^{w_i}$ to V^w of degree $N(w)$ where $N(w)$ is given in (6.8).*

Proof. First of all, Proposition 6.3.4 shows that a conjugate flag $\mathcal{D}_{i,\bullet}$ for w_i is as refinement of the canonical flag $\mathcal{C}_{i,\bullet}$ for w, and hence the image of $\mathcal{U}^{w_i}$ under $\mathcal{F}_i \to \mathcal{M}$ is V^w. It follows from Theorem 6.4.3 that the map $\mathcal{U}^{w_i} \to V^w$ is unramified, as the analogous map on Schubert cells is unramified by [BGG73, Proposition 5.1].

Next we show that $\mathcal{U}^{w_i} \to V^w$ is proper. Let R be a DVR, then $x : \mathrm{Spec}(R) \hookrightarrow V^w$ corresponds to an abelian scheme over R with extra structure such that the canonical type of both its special and generic points is w, and hence its canonical decomposition is a single stratum, $\mathrm{Spec}(R)$. Suppose that we have a Hodge flag $\mathcal{E}_{i,\bullet}$ of type w_i over the generic point of x. Then its corresponding pair of Hodge flags $(\mathcal{E}_{i,\bullet})$ has relative position $w \in {}^J W$ and the associated pair of conjugate flags are refinements of the canonical flags of the generic point of x. $\overline{\mathcal{U}}^{w_i}$ is proper over $\mathcal{M}$ since $\mathcal{F}_i \to \mathcal{M}$ is proper. Therefore, the flag $\mathcal{E}_{i,\bullet}$ extends to a Hodge flag and therefore a pair of Hodge flags $(\mathcal{E}_{i,\bullet})$ over $\mathrm{Spec}(R)$. But then the pair $(\mathcal{E}_{i,\bullet})$ gives Hodge flags over the special point of $\mathrm{Spec}(R)$ with relative position $w' \leq w$. Lemma 6.4.4 shows that $w' = w$. Therefore, the Hodge flag $\mathcal{E}_{i,\bullet}$ over the generic point of x can be extended uniquely to a Hodge flag over $\mathrm{Spec}(R)$ with relative position w_i, thus proving that $\mathcal{U}^{w_i} \to V^w$ is proper.

Finally, Lemma 6.4.1 shows that the fibres of the map $\mathcal{U}^{w_i} \to V^w$ are finite of constant rank $N(w)$. It follows that $\mathcal{U}^{w_i} \to V^w$ is finite and flat, thus completing the proof that $\mathcal{U}^{w_i} \to V^w$ is a finite surjective étale covering of degree $N(w)$. $\square$

Chapter 7
Hasse-invariants and Hasse-Witt matrices

This chapter introduces the (partial) Hasse-invariants on unitary Shimura varieties. In the case of elliptic modular curves, the Hasse-invariant is a mod p modular form of weight $p-1$ that vanishes precisely on the complement of the μ-ordinary stratum, the supersingular points. We show that a similar phenomena occurs with unitary Shimura varieties, in particular, the Hasse-invariants are mod p modular forms and they vanish on the complement of the μ-ordinary stratum, the **non-ordinary locus** (*i.e.* the closure of the almost-ordinary E-O stratum). We then use deformations of Hasse-Witt matrices to obtain further geometric information about the E-O strata.

In order to keep this chapter as self-contained as possible, we begin by recalling the definitions of the Hodge filtration and related vector bundles from Section 6.2. Let $\pi : \mathcal{A} \to \mathcal{M}$ be the structure map of the universal abelian variety $\mathcal{A}$ over $\mathcal{M}$ defined over the finite field $\kappa(\mathfrak{p})$. Let k be the algebraically closed field extension of κ where $\kappa = \mathbb{F}_p$ when p is split and $\mathbb{F}_{p^2}$ when p is inert. There is an exact sequence of locally free sheaves on $\mathcal{M}$:

$$0 \to \pi_*(\Omega^1_{\mathcal{A}/\mathcal{M}}) \to \mathcal{H}^1_{dR}(\mathcal{A}/\mathcal{M}) \to R^1\pi_*(\mathcal{O}_{\mathcal{A}}) \to 0$$

which we will denote by

$$0 \to \mathbb{E} \to \mathbb{H} \to \mathbb{E}^{\vee} \to 0$$

where $\mathbb{E} = \pi_*(\Omega^1_{\mathcal{A}/\mathcal{M}})$ is the **Hodge bundle** and $\mathbb{H} = \mathcal{H}^1_{dR}(\mathcal{A}/\mathcal{M})$.

Over a k-point (A, ι, λ) of $\mathcal{M}$ this specializes to the sequence of vector spaces over k,

$$0 \to H^0(\Omega^1_A) \to H^1_{dR}(A) \to H^1(\mathcal{O}_A) \to 0$$

which can be identified through contravariant Dieudonné theory with

$$0 \to \mathcal{D}[F] \otimes_{k,\sigma^{-1}} k \to \mathcal{D} \to \mathcal{D}/V\mathcal{D} \to 0$$

where $\mathcal{D} = \mathcal{D}(A)$ denotes the contravariant Dieudonné module of A and $\mathcal{D}[F]$ denotes the kernel of F on $\mathcal{D}$ [Oda69, Corollary 5.11].

Let R be an κ-algebra such that $\mathbb{E}_R$ and $\mathbb{H}_R$ are locally free over R. The bundles $\mathbb{E} = \mathbb{E}_R$ and $\mathbb{H} = \mathbb{H}_R$ split under the $\mathcal{O}_K/(p) \otimes_{\mathbb{F}_p} R = R_1 \oplus R_2$-action as $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2$ and $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2$ via the characters $\chi_1, \chi_2 : \mathcal{O}_K \to k$. Therefore, $\det(\mathbb{E}) = \det(\mathbb{E}_1) \otimes \det(\mathbb{E}_2)$. Let $j = 1$ if $\gamma(i) = i$ (p is split) and $j = 2$ if $\gamma(i) \neq i$ (p is inert). Then a **unitary** (m_1, m_2) **modular form mod** p **of weight** χ_i^b **over** R is a global section of $\det(\mathbb{E}_i)^{\otimes b} \otimes R$ over $\mathcal{M} \otimes_{\mathbb{F}_p} R$.

7.1 Hasse invariants

The morphisms $\mathrm{Fr} : \mathcal{A} \to \mathcal{A}^{(p)}$ and $\mathrm{Ver} : \mathcal{A}^{(p)} \to \mathcal{A}$ induce linear maps in cohomology $\mathrm{Fr}^* : \mathbb{H}^{(p)} \to \mathbb{H}$ and $\mathrm{Ver}^* : \mathbb{H} \to \mathbb{H}^{(p)}$ (these maps correspond to the σ-linear map F and σ^{-1}-linear map V respectively on the contravariant Dieudonné modules) such that $\mathrm{Fr}^*_i : \mathbb{H}^{(p)}_i \to \mathbb{H}_{\gamma(i)}$ and $\mathrm{Ver}^*_i : \mathbb{H}_i \to \mathbb{H}^{(p)}_{\gamma(i)}$.

Let $j = 1$ if p splits in K and $j = 2$ if p is inert in K. Then

$$(\mathrm{Ver}^*_i)^j : \mathbb{H}_i \to \mathbb{H}_i^{(p^j)}$$

and the resulting map

$$\wedge^{m_i}(\mathrm{Ver}_i^*)^j : \det(\mathbb{E}_i) \to \det(\mathbb{E}_i)^{(p^j)}$$

gives rise to a global section

$$h_i \in H^0(\det(\mathbb{E}_i)^{\otimes(p^j-1)}).$$

The h_i have the following vanishing properties depending on the splitting behaviour of p in K.

Proposition 7.1.1 ([GN16, 4.1, 4.2])**.** *Let $\underline{A} \in \mathcal{M}(k)$ and let p be split in K. Then $h_1(\underline{A}) \neq 0$ if and only if $h_2(\underline{A}) \neq 0$ if and only if $\underline{A}$ is μ-ordinary.*

Proof. For completeness, we include a simple proof of this fact. We actually show that $h_1(\underline{A}) = 0$ if and only if $h_2(\underline{A}) = 0$ if and only if $\underline{A}$ is *not* μ-ordinary. As was seen in Section 3.5, the μ-ordinary locus is actually ordinary in this case, and $a(A) = 0$ if and only if $\underline{A}$ is in the μ-ordinary locus. On the other hand, $h_i(\underline{A}) = 0$ if and only if the a-number of A_i is non-zero, so it suffices to show that $a(A_1[p]) > 0$ if and only if $a(A_2[p]) > 0$. But for any finite commutative group scheme G that is killed by p, $a(G) > 0$ means that there is an embedding $\alpha_p \hookrightarrow G$, and since α_p is self-dual, there is a surjection $G^\vee \to \alpha_p^\vee \cong \alpha_p$. Thus $G^\vee$ has a non-trivial local-local part and $a(G^\vee) > 0$. Since $A_1[p] \cong A_2[p]^\vee$ as group schemes, the result follows. $\square$

On the other hand, when p is inert in K, we have the following.

Proposition 7.1.2. *Let $\underline{A} \in \mathcal{M}(k)$ and let p be inert in K. If $m_1 = m_2$, then $h_1(\underline{A}) \neq 0$ if and only if $h_2(\underline{A}) \neq 0$ if and only if $\underline{A}$ is μ-ordinary. If $m_1 > m_2$, then h_1 vanishes everywhere, and $h_2(\underline{A}) \neq 0$ if and only if $\underline{A}$ is μ-ordinary.*

Proof. By Proposition 3.5.7, the p-torsion group scheme of the μ-ordinary locus has the form

$$(\mathcal{O}_K \otimes \mu_p)^{m_2} \oplus (\underline{\mathcal{O}_K/(p)})^{m_2} \oplus \mathscr{G}_1^{m_1-m_2}.$$

Since Ver^2 is an isomorphism on $(\mathcal{O}_K \otimes \mu_p)^{m_2}$ and zero on $(\underline{\mathcal{O}_K/(p)})^{m_2} \oplus \mathscr{G}_1^{m_1-m_2}$, $h_2 \neq 0$ and $h_1 = 0$ if and only if $m_1 > m_2$ on the μ-ordinary locus.

By Proposition 3.5.6, the p-torsion group scheme of the almost-ordinary locus has the form

- $m_1 - m_2 > 1$: $AO(3,1) \oplus (\mathcal{O}_K \otimes \mu_p)^{m_2-1} \oplus (\underline{\mathcal{O}_K/(p)})^{m_2-1} \oplus \mathscr{G}_1^{m_1-m_2-2}$
- $m_1 - m_2 = 1$: $AC(2,1) \oplus (\mathcal{O}_K \otimes \mu_p)^{m_2-1} \oplus (\underline{\mathcal{O}_K/(p)})^{m_2-1}$
- $m_1 = m_2 = m$: $(\mathcal{O}_K \otimes \mu_p)^{m-1} \oplus (\underline{\mathcal{O}_K/(p)})^{m-1} \oplus \mathscr{G}_1 \oplus \mathscr{G}_2$.

Now, Ver is nilpotent on $AO(3,1)$, $AC(2,1)$, $\mathscr{G}_1$ and $\mathscr{G}_2$. Therefore, $h_1 = h_2 = 0$ on the almost-ordinary E-O stratum and its closure, the complement of the μ-ordinary stratum. □

When p is split in K, the h_i for $i \in \{1,2\}$ are called the **partial Hasse invariants** and are modular forms of weight χ_i^{p-1} over k. The product $h := h_1h_2$ is called the **(total) Hasse invariant**. When p is inert in K, $h := h_2$ is called the **Hasse invariant**, and it is a modular form of weight $\chi_2^{p^2-1}$ over k.

7.2 Hasse-Witt matrices of E-O strata

In this section, we will work with covariant Dieudonné theory. To that end, let $\mathfrak{D} = \mathfrak{D}(A[p])$ be the covariant Dieudonné module of $A[p]$. Now, $\mathrm{Ver}^* : \mathbb{E} \to \mathbb{E}$ corresponds to F on $\mathfrak{D}(A[p])/V\mathfrak{D}(A[p])$, so the vanishing of h_i can be studied in more detail by studying the Hasse-Witt matrix of A as introduced in Section 2.2.5. In particular, we will be interested in studying the Hasse-Witt matrices of various Ekedahl-Oort strata as a tool for understanding the deformation of V^* on the Hodge bundle.

Let $w = (w_1, w_2)$ be an element in the Weyl group coset ${}^J W$ as described in Section 3.3. By Section 3.5, we know how to associate models for the contravariant Dieudonné module (modulo p) $\mathcal{D} = \mathcal{D}(w)$ of the Ekedahl-Oort stratum associated to w. However, in order to compute the Hasse-Witt matrices of the E-O strata, we need to find a displayed basis for the covariant Dieudonné modules.

The general strategy we take for this is as follows. Recall that for $\underline{A} = (A, \iota, \lambda, \eta) \in \mathcal{M}(k)$, the prime-to-$p$ polarization $\lambda : A \to A^\vee$ induces an isomorphism of Dieudonné modules $\mu : \mathcal{D} \to \mathcal{D}^\vee$ that is conjugate-linear with respect to the $\mathcal{O}_K/(p)$ action on $\mathcal{D}$ coming from ι on A. The isomorphism μ gives rise to a non-degenerate alternating bilinear form:

$$\begin{aligned}\Psi : \mathcal{D} \times \mathcal{D} &\to k \\ (d_1, d_2) &\mapsto \mu(d_1)(d_2)\end{aligned}$$

such that $\Psi(bd_1, d_2) = \Psi(d_1, \bar{b}d_2)$ for $b \in \mathcal{O}_K/(p)$ and $\Psi(Fd_1, d_2) = \Psi(d_1, Vd_2)^\sigma$ for all $d_1, d_2 \in \mathcal{D}$. By [Moo01, Theorem 6.7], there exists a unique such $\mu : \mathcal{D}(w) \to \mathcal{D}(w)^\vee$ for a given $w \in {}^J W$ up to isomorphism.

The proof of [Moo01, Theorem 6.7] is constructive and shows how to find a bilinear form $\Psi : \mathcal{D} \times \mathcal{D} \to k$ corresponding to μ. Then by finding a symplectic basis for $\mathcal{D}$ from the bases as described in Section 3.5, applying $\mu(d) = \Psi(d, -)$ gives a symplectic basis for $\mathcal{D}^\vee = \mathfrak{D}$. Furthermore, μ takes $\mathcal{D}_i$ to $\mathfrak{D}_{i+1}$, so by keeping track of the $\mathcal{O}_K$-action on the basis, we can choose a symplectic basis for $\mathcal{D}$ such that its image in $\mathfrak{D}$ is the reduction modulo p of a displayed basis for $\mathfrak{D}(A)$. Finally, applying the results of Section 3.5 allow us to find the action of F on the displayed basis, giving the Hasse-Witt matrices for $\mathfrak{D}(w)$.

Recall from Section 2.2.5 that a displayed basis for $\mathfrak{D}(A)$ is a basis of the form $\mathcal{B} = \{e_1, \dots, e_g; f_1, \dots, f_g\}$ where

- $\mathcal{B}_1 = \{e_1, \dots, e_{m_1}, f_{m_1+1}, \dots, f_g\}$ is a basis for $\mathfrak{D}(A)_1$, and
  $\mathcal{B}_2 = \{e_{m_1+1}, \dots, e_g, f_1, \dots, f_{m_1}\}$ is a basis for $\mathfrak{D}(A)_2$,
- the set $\{e_1, \dots, e_g\}$ spans $\mathfrak{D}(A)/V\mathfrak{D}(A)$ and $\{f_1, \dots, f_g\}$ spans $\mathfrak{D}(A)[F]$,
- $\mathcal{B}$ is a standard symplectic basis for $\mathfrak{D}(A)$; *i.e.* $\Psi(e_i, f_j) = \delta_{ij} = -\Psi(f_j, e_i)$.

We will also call a basis for $\mathfrak{D} = \mathfrak{D}(A[p])$ a **displayed basis** if it satisfies the analogues of the above properties for $\mathfrak{D}$. Implicit in our methods is the fact that a displayed basis for $\mathfrak{D}(A[p])$ can be lifted to a displayed basis for the full Dieudonné module $\mathfrak{D}(A)$. Since we are primarily interested in the Hasse-Witt matrices, we will work directly with a basis for $\mathfrak{D}$ and all the matrices in this chapter will be taken modulo p.

7.2.1 p split in K

Recall that K denotes the quadratic imaginary field coming from the Shimura datum $\mathscr{D}$ for the Shimura variety $\mathcal{M}$. When p is split in K, the decomposition of $\mathcal{D}$ as $\mathcal{D}_1 \oplus \mathcal{D}_2$ as an $\mathcal{O}_K/(p)$-module actually makes $\mathcal{D}_1$ and $\mathcal{D}_2$ sub-Dieudonné modules of $\mathcal{D}$ as in Proposition 2.2.4. Therefore, finding an isomorphism of $\mu_1 : \mathcal{D}_1 \to \mathcal{D}_2^\vee$ of Dieudonné modules gives $\mu : \mathcal{D} \to \mathcal{D}^\vee$ as desired by taking $\mu = \mu_1 \oplus (-\mu_1^\vee)$.

Observe that the models for the Dieudonné modules of the p-torsion given in 3.5.3 have built in isomorphisms $\mu_i : \mathcal{D}_i \to \mathcal{D}_{i+1}^\vee$ that come from taking $e_{i,j}$ to $e_{i+1,w_0(j)}^\vee$. Therefore,

$$\mu(e_{i,j}) = (-1)^{i+1} e_{i+1,w_0(j)}^\vee$$

gives the desired self-duality for $\mathcal{D}$. Setting

$$e_{w_1(j)} = \begin{cases} -\mu(e_{2,w_0(j)}) & w_1(j) \leq m_1 \\ \mu(e_{1,j}) & w_1(j) \geq m_1 + 1 \end{cases} \tag{7.1}$$

$$f_{w_1(j)} = \begin{cases} \mu(e_{1,j}) & w_1(j) \leq m_1 \\ \mu(e_{2,w_0(j)}) & w_1(j) \geq m_1 + 1 \end{cases} \tag{7.2}$$

gives a standard symplectic basis for $\mathfrak{D}$ with respect to Ψ induced by μ such that $\mathfrak{D}_1 = \{e_1, \ldots, e_{m_1}, f_{m_1+1}, \ldots, f_g\}$ and $\mathfrak{D}_2 = \{f_1, \ldots, f_{m_1}, e_{m_1+1}, \ldots, e_g\}$.

Recall the notation from Section 2.2.5; *i.e.* the matrix of the display of $\mathfrak{D}$ when p is split in K has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & 0 & 0 & B_1 \\ 0 & A_2 & B_2 & 0 \\ 0 & C_2 & D_2 & 0 \\ C_1 & 0 & 0 & D_1 \end{pmatrix}.$$

Let $M = (t_{ij})$ for $1 \le i \le m_1$ and $1 \le j \le m_2$, and let $T = \begin{pmatrix} 0 & M \\ M^t & 0 \end{pmatrix}$. Then the matrix

$$A + TC = \begin{pmatrix} A_1 + MC_1 & 0 \\ 0 & A_2 + M^t C_2 \end{pmatrix} \pmod{p}$$

is the Hasse-Witt matrix. Letting $A_1 = (a_{ij}), C_1 = (c_{ij}), A_2 = (a'_{ij}), C_2 = (c'_{ij})$, it follows that

$$A_1 + MC_1 = \left(a_{ij} + \sum_{k=1}^{m_2} t_{ik} c_{kj} \right)_{1 \le i,j \le m_1}$$

and

$$A_2 + M^t C_2 = \left(a'_{ij} + \sum_{k=1}^{m_2} t_{ki} c'_{kj} \right)_{1 \le i,j \le m_2}.$$

Writing F in terms of the new basis gives

$$F(e_j) = \begin{cases} e_{w_1(j+m_2)} & 1 \le j \le m_1, w_1(m_2 + j) \le m_1 \\ f_{w_1(j+m_2)} & 1 \le j \le m_1, w_1(m_2 + j) \ge m_1 + 1 \\ e_{w_1(j-m_1)} & m_1 + 1 \le j \le g, w_1(j - m_1) \ge m_1 + 1 \\ f_{w_1(j-m_1)} & m_1 + 1 \le j \le g, w_1(j - m_1) \le m_1. \end{cases}$$

Note that $F(f_j) = 0$ for $1 \leq j \leq g$ by construction. With this basis, $a_{ij} \equiv 1 \pmod p$ for $i = w_1(m_2 + j)$ and 0 otherwise, and $c_{ij} \equiv 1 \pmod p$ for $i = w_1(m_2 + j) - m_1$ and 0 otherwise. We get similar conditions for A_2 and C_2. Therefore, $A_1 + MC_1 = (m_{ij})_{1 \leq i,j \leq m_1}$ where

$$m_{ij} = \begin{cases} 1 & i = w_1(m_2 + j) \\ t_{ik} & k = w_1(m_2 + j) - m_1 \end{cases}$$

and $A_2 + M^t C_2 = (m'_{ij})_{1 \leq i,j \leq m_2}$ where

$$m'_{ij} = \begin{cases} 1 & i = w_1(j) - m_1 \\ t_{ki} & k = w_1(j). \end{cases}$$

Example 7.2.1 (The almost-ordinary stratum)**.** Consider the case where $\ell(w) = m_1 m_2 - 1$. Then,

$$A_1 = \begin{pmatrix} 0_{1\times 1} & \\ & I_{m_1 - 1} \end{pmatrix} \qquad A_2 = \begin{pmatrix} I_{m_2 - 1} & \\ & 0_{1\times 1} \end{pmatrix},$$

and

$$C_1 = \begin{pmatrix} & 0_{m_2 - 1 \times m_1 - 1} \\ 1 & \end{pmatrix} \qquad C_2 = \begin{pmatrix} & 1 \\ 0_{m_1 - 1 \times m_2 - 1} & \end{pmatrix}.$$

It follows that

$$A_1 + MC_1 = \begin{pmatrix} t_{1m_2} & 0 \\ \vdots & I_{m_1 - 1} \\ t_{m_1 m_2} & \end{pmatrix}$$

$$A_2 + M^t C_2 = \begin{pmatrix} & t_{11} \\ I_{m_2-1} & \vdots \\ 0 & t_{1m_2} \end{pmatrix}.$$

and hence $\det(A_1 + MC_1) = \det(A_2 + M^t C_2) = t_{1m_2}$.

Example 7.2.2 (The codimension 2 strata)**.** Consider the case where $\ell(w) = m_1 m_2 - 2$. Then either

$$w_1(j) = \begin{cases} 1 & j = m_2 - 1 \\ j - m_2 & m_2 + 2 \leq j \leq g \\ j + m_1 & 1 \leq j \leq m_2 - 2 \\ g - 1 & j = m_2 \\ g & j = m_2 + 1 \end{cases} \tag{7.3}$$

or

$$w_1(j) = \begin{cases} 1 & j = m_2 \\ 2 & j = m_2 + 1 \\ j - m_2 & m_2 + 3 \leq j \leq g \\ j + m_1 & 1 \leq j \leq m_2 - 1 \\ g & j = m_2 + 2. \end{cases} \tag{7.4}$$

Note that w_1 can only have the form of (7.3) if $m_2 > 1$. Suppose that w_1 is as in (7.3). Then

$$A_1 + MC_1 = \begin{pmatrix} t_{1m_2} & 0 \\ \vdots & I_{m_1-1} \\ t_{m_1 m_2} & \end{pmatrix}$$

$$A_2 + M^tC_2 = \begin{pmatrix} & t_{11} & 0 \\ I_{m_2-2} & \vdots & \vdots \\ & t_{1m_2-1} & 1 \\ 0 & t_{1m_2} & 0 \end{pmatrix}.$$

and $\det(A_1 + MC_1) = -\det(A_2 + M^tC_2) = t_{1m_2}$.

Similarly, suppose that w_1 is defined as in (7.4). Then

$$A_1 + MC_1 = \begin{pmatrix} 0 & t_{1m_2} & 0 \\ 1 & t_{2m_2} & 0 \\ \vdots & \vdots & I_{m_1-2} \\ 0 & t_{m_1m_2} & \end{pmatrix}$$

$$A_2 + M^tC_2 = \begin{pmatrix} & t_{11} \\ I_{m_2-1} & \vdots \\ 0 & t_{1m_2} \end{pmatrix}.$$

and hence $\det(A_1 + MC_1) = -\det(A_2 + M^tC_2) = -t_{1m_2}$.

Example 7.2.3 (The codimension 3 strata)**.** Assuming that $m_1m_2 \geq 3$, there exists at least a codimension 3 stratum. For $w \in {}^JW$ such that $\ell(w) = m_1m_2 - 3$, either

$$w_1(j) = \begin{cases} 1 & j = m_2 - 2 \\ j - m_2 & m_2 + 2 \leq j \leq g \\ j + m_1 & 1 \leq j \leq m_2 - 3 \\ g - 2 & j = m_2 - 1 \\ g - 1 & j = m_2 \\ g & j = m_2 + 1 \end{cases} \tag{7.5}$$

giving

$$A_1 + MC_1 = \begin{pmatrix} t_{1m_2} & 0 \\ \vdots & I_{m_1-1} \\ t_{m_1m_2} & \end{pmatrix}, \qquad A_2 + M^tC_2 = \begin{pmatrix} & t_{11} & 0 & 0 \\ I_{m_2-3} & \vdots & \vdots & \vdots \\ & t_{1m_2-2} & 1 & 0 \\ 0_{3\times m_2-3} & t_{1m_2-1} & 0 & 1 \\ & t_{1m_2} & 0 & 0 \end{pmatrix},$$

and $\det(A_1 + MC_1) = \det(A_2 + M^tC_2) = t_{1m_2}$;

$$w_1(j) = \begin{cases} 1 & j = m_2 - 1 \\ 2 & j = m_2 + 1 \\ j - m_2 & m_2 + 3 \leq j \leq g \\ j + m_1 & 1 \leq j \leq m_2 - 2 \\ g - 1 & j = m_2 \\ g & j = m_2 + 2 \end{cases}, \tag{7.6}$$

giving

$$A_1 + MC_1 = \begin{pmatrix} 0 & t_{1m_2} & 0 \\ 1 & \vdots & \\ 0 & & I_{m_1-2} \\ 0 & t_{m_1m_2} & \end{pmatrix}, \qquad A_2 + M^tC_2 = \begin{pmatrix} & t_{11} & 0 \\ I_{m_2-2} & \vdots & \vdots \\ & t_{1m_2-1} & 1 \\ 0 & t_{1m_2} & 0 \end{pmatrix},$$

and $\det(A_1 + MC_1) = \det(A_2 + M^tC_2) = -t_{1m_2}$; or

$$w_1(j) = \begin{cases} 1 & j = m_2 \\ 2 & j = m_2 + 1 \\ 3 & j = m_2 + 2 \\ j - m_2 & m_2 + 4 \leq j \leq g \\ j + m_1 & 1 \leq j \leq m_2 - 1 \\ g & j = m_2 + 3 \end{cases}. \tag{7.7}$$

giving

$$A_1 + MC_1 = \begin{pmatrix} 0 & 0 & t_{1m_2} & 0 \\ 1 & 0 & t_{2m_2} & 0 \\ 0 & 1 & t_{3m_2} & 0 \\ \vdots & 0 & \vdots & I_{m_1-3} \\ 0 & 0 & t_{m_1m_2} & \end{pmatrix}, \qquad A_2 + M^tC_2 = \begin{pmatrix} & t_{11} \\ I_{m_2-1} & \vdots \\ 0 & t_{1m_2} \end{pmatrix}$$

and $\det(A_1 + MC_1) = \det(A_2 + M^tC_2) = t_{1m_2}$.

Note that the w in situation (7.5) can only occur if $m_2 \geq 3$, (7.6) can only occur if $m_2 \geq 2$, and (7.7) always occurs as long as $m_1m_2 \geq 3$.

Proposition 7.2.4. *When p splits in K, the partial Hasse-invariants h_1, h_2 over $\mathcal{M}$ both vanish to order one on the non-ordinary locus, and the intersection of a connected component of $\mathcal{M}$ with the non-ordinary locus is irreducible.*

Proof. Example 7.2.1 immediately gives vanishing to order one.

Let C be the intersection of a connected component of $\mathcal{M}$ with the almost-ordinary E-O stratum, and let Z be the closure of C. Then Z is precisely the zero locus of h_1 and h_2 on C. Indeed, it is the zero locus of h_1h_2, a global section of $\det(\mathbb{E})^{p-1}$. But $\det(\mathbb{E})$ is ample, so Z is connected [Har77, Cor. III.7.9].

Suppose that Z decomposes into irreducible components $Z_1, \ldots, Z_n$, where $n > 1$. We may assume that $Z_1 \cap Z_2 \neq \emptyset$. Since the almost-ordinary (open) E-O stratum is smooth over k by Theorem 3.3.2, $Z_1 \cap Z_2$ is codimension one in Z.

Let z be a generic point in this intersection—then there is a $w \in {}^JW$ such that $z \in V^w$ where $\ell(w) = m_1m_2 - 2$. By Example 7.2.2, the local equation defining Z in $k[|t_{ij}, 1 \leq i \leq m_1, 1 \leq j \leq m_2|]$ at z is $\pm t_{1m_2}$, and Z is smooth at z. But then it follows that $Z_1 \cap Z_2$ has codimension at least 2 and therefore must be empty. Therefore Z is irreducible. $\square$

Example 7.2.5 (The almost-core stratum)**.** Here, the matrices depend on the relationship between m_1 and m_2. When $m_1 > m_2$, it follows that

$$A_1 + MC_1 = \begin{pmatrix} & t_{11} & 0 & t_{12} & \ldots & t_{1m_2} \\ 0_{m_2 \times m_1 - m_2 - 1} & t_{21} & 0 & t_{22} & \ldots & t_{2m_2} \\ I_{m_1 - m_2 - 1} & \vdots & \vdots & \vdots & & \vdots \\ 0 & t_{m_1 1} & 1 & t_{m_1 2} & \ldots & t_{m_1 m_2} \end{pmatrix}.$$

$$A_2 + M^t C_2 = (t_{ij})^t_{1 \le i,j \le m_2}.$$

Then

$$\begin{aligned}\det(A_1 + MC_1) &= (-1)^{m_2(m_1-m_2)+1} \det(A_2 + M^t C_2) \\ &= (-1)^{m_2(m_1-m_2)+1} \det (t_{ij})_{1 \le i,j \le m_2}.\end{aligned}$$

On the other hand, when $m_1 = m_2 = m$,

$$A_1 + MC_1 = \begin{pmatrix} 0 & t_{12} & \dots & t_{1m} \\ 0 & t_{22} & \dots & t_{2m} \\ \vdots & \vdots & & \vdots \\ 1 & t_{m2} & \dots & t_{mm} \end{pmatrix} \qquad A_2 + M^t C_2 = \begin{pmatrix} t_{11} & \dots & t_{m-11} & 1 \\ t_{12} & \dots & t_{m-12} & 0 \\ \vdots & & \vdots & \vdots \\ t_{1m} & \dots & t_{m-1m} & 0 \end{pmatrix}$$

and

$$\begin{aligned}\det(A_1 + MC_1) &= \det(A_2 + M^t C_2) \\ &= (-1)^{m-1} \det (t_{ij})_{\substack{1 \le i \le m-1 \\ 2 \le j \le m}}.\end{aligned}$$

Example 7.2.6 (The core stratum)**.** In this case, $w_1 = w_2$ is the identity element, and it follows that there is no $m_1 + 1 \le k \le g$ such that $w_1(k - m_1) = k - m_1 \le m_1 + 1$. Thus $A_2 = 0$. On the other hand,

$$A_1 = \begin{pmatrix} 0_{m_2 \times m_1 - m_2} & 0_{m_2 \times m_2} \\ I_{m_1 - m_2} & 0_{m_1 - m_2 \times m_2} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0_{m_2 \times m_1 - m_2} & I_{m_2} \end{pmatrix},$$

$$C_2 = \begin{pmatrix} I_{m_2} \\ 0_{m_1-m_2\times m_2} \end{pmatrix}.$$

It follows that

$$A_1 + MC_1 = \begin{pmatrix} 0_{m_2\times m_1-m_2} & \\ & M \\ I_{m_1-m_2} & \end{pmatrix},$$

and

$$A_2 + M^tC_2 = (t_{ij})^t_{1\leq i,j\leq m_2}.$$

Then,

$$\begin{aligned} \det(A_1 + MC_1) &= (-1)^{(m_1-m_2)m_2} \det(A_2 + M^tC_2) \\ &= (-1)^{(m_1+1)m_2} \det{(t_{ij})}_{1\leq i,j\leq m_2}. \end{aligned}$$

These last two examples can be summed up by the following corollary.

Corollary 7.2.7. *The vanishing locus of the partial Hasse-invariants on the core and the almost-core locus are given locally formally by the same equation,* $\det(t_{ij})_{1\leq i,j\leq m_2}$*. Thus, the non-ordinary locus is locally irreducible on the core and almost-core strata. Furthermore, it is smooth at core or almost-core points if and only if* $m_2 = 1$*.*

The previous corollary together with Proposition 7.2.4 shows that the closed almost-ordinary stratum is a smooth irreducible curve in the case of $\mathrm{GU}(2,1)$.

Remark. In the low-dimensional situations that we've calculated, the determinants of the Hasse-Witt matrices have been determinants of square matrices in the t_{ij} for all of the E-O strata (except for the ordinary stratum). It seems probable that this phenomenon continues which would imply that the non-ordinary locus becomes nonsingular by moving down the poset diagram of E-O strata until the E-O strata are reached where $\det(H_i)$ becomes the determinant of a two-by-two matrix. This will not happen when $m_2 = 1$, so we expect that the non-ordinary locus is also smooth when $m_2 = 1$.

7.2.2 p inert in K

In the case when p is inert in K, the construction of μ is more involved as F and V on $\mathcal{D}$ take $\mathcal{D}_i \to \mathcal{D}_{i+1}$, so that neither $\mathcal{D}_i$ is ever a sub-Dieudonné module of $\mathfrak{D}$. In order to obtain $\mu : \mathcal{D} \to \mathcal{D}^\vee$, we'll follow the general notion of [Moo01, Section 5.7] as adapted to this context.

Recall the terminology of Section 6.1.1 on complementary elements and canonical fragments. In particular, the complementary element v of $w = (w_1, w_2) \in {}^J W$, is the permutation defined by $v_i = w_0 w_0(J_i) w_i$, and a canonical fragment for v_i is a maximal interval $(j, j'] \subseteq 1, \ldots, g$ such that $v_i^n((j, j'])$ where $v_i^n := v_{\gamma^{n-1}(i)} \circ \ldots \circ v_{\gamma(i)} \circ v_i$ stays an interval for all n. For a fixed $i = 1, 2$, $\{1, \ldots, g\}$ is a disjoint union of the canonical fragments of v_i by Proposition 6.1.9.

Let

$$\mathscr{A} = \{(i, I) \mid i \in \{1, 2\}, I \text{ is a canonical fragment for } v_i\}.$$

For $a = (i, I) \in \mathscr{A}$ then $v(a) := (i+1, v_i(I))$ and $\tilde{a} := (i+1, \tilde{I})$ where $\widetilde{(n, n']} := (g - n', g - n]$. By Proposition 6.1.9, $v(a)$ and $\tilde{a}$ are also elements of $\mathscr{A}$, and $\mathscr{A}$ decomposes into a disjoint union of orbits under v. If O is an orbit of $\mathscr{A}$, then recall that O is called self-dual if $O = \tilde{O} := \{\tilde{a} \mid a \in O\}$.

Write

$$C_{i,\bullet} : 0 = C_{i,j_1} \subsetneq C_{i,j_2} \subsetneq \ldots \subsetneq C_{i,g} = \mathcal{D}_i$$

for the canonical filtration of $\mathcal{D}$, that is, the decomposition of the coarsest F, V^{-1}-stable symplectic $\mathcal{O}_K$-flag of $\mathcal{D}$. Then for $a = (i, (n, n']) \in \mathscr{A}$, define

$$B_a := C_{i,n'}/C_{i,n+1}$$

which has a naturally ordered basis $\{e_{i,n+1}, \ldots, e_{i,n'}\}$. Observe that since $a = (i, I)$ where I is a canonical fragment for υ_i, there is a σ-linear isomorphism

$$B_a \to B_{\upsilon(a)}$$

induced by either F or V^{-1} that preserves the ordering of the basis elements as υ_i is order preserving on canonical fragments (see Corollary 6.3.5).

We can now define a form Ψ on $\mathcal{D}$ by specifying its values "block-by-block" as $\Psi_a : B_a \times B_{\tilde{a}} \to k$ for all $a \in \mathscr{A}$. For each self-dual orbit O of $\mathscr{A}$, let $2s$ be the length of the orbit O. Then define constants $c(a) \in \mathbb{F}_{p^{2s}}$ such that $c(a)^{p^s} = -c(a)$ and $c(\upsilon(a)) = c(a)^p$, so that $c(\tilde{a}) = -c(a)$ for all $a \in O$. If $a = (i, I)$ is not in a self-dual orbit, then set $c(a) := (-1)^{i+1}$. Finally let Ψ be the direct sum of forms $\Psi_a : B_a \times B_{\tilde{a}} \to k$ defined by the matrix

$$\begin{pmatrix} c(a) & & \\ & \ddots & \\ & & c(a) \end{pmatrix}.$$

These matrices are given in terms of the ordered bases for B_a and $B_{\tilde{a}}$. By [Moo01, Theorem 6.7], Ψ induces the desired self-duality $\mu : \mathcal{D} \to \mathcal{D}^\vee$ uniquely up to isomorphism. If $(i,j) \in a = (i, (n, n'])$ set $j^\flat := g - n - n + j$. Then

$$\Psi(e_{i,j}, e_{i+1,j^\flat}) = c(a)$$

and hence

$$\mu(e_{i,j}) = c(a) e^\vee_{i+1,j^\flat}.$$

Next we need to a find a transformation of the basis $\{e_{i,j} \mid 1 \le i,j \le g\}$ for $\mathcal{D}$ into a symplectic basis for $\mathcal{D}$ with respect Ψ. This can by done orbit-by-orbit.

Suppose that O is a self-dual orbit of $\mathscr{A}$ of length $2s$. Fix $a_0 = (1, I) \in O$ as a base point for the orbit. Then there exists as $x(a_0) = x \in k$ such that

$$x^{p^{2s}} = -x, \qquad (-1)^s x x^{p^s} c(a_0) = 1 \tag{7.8}$$

(potentially adjusting $c(a_0)$ and $c(a)$ for $a \in O$). Indeed, starting from an x that satisfies $x^{p^{2s}} = -x$,

$$c^{-1} := ((-1)^s x x^{p^s} c(a_0))^{p^s} = (-1)^s x x^{p^s} c(a_0) \in \mathbb{F}_{p^s}$$

as $c(a_0)^{p^s} = -c(a_0)$. Replacing $c(\upsilon^n(a_0))$ by $c^{-p^n} c(\upsilon^n(a_0))$ and x by $cx(a_0)$ gives the desired result. Define constants for all of $a \in O$ by setting

$$x(a) := (-1)^n x(a_0)^{p^n}$$

where $a = \upsilon^n(a_0)$ for $0 \le n \le 2s - 1$.

Then for $1 \leq n \leq s$,

$$x(\upsilon^n(a_0))x(\widetilde{\upsilon^n(a_0)})c(\upsilon^n(a_0)) = (x(a_0)x(\tilde{a}_0)c(a_0))^{p^n} = 1.$$

and for $s+1 \leq n \leq 2s$, $\upsilon^{n+s}(a_0) = \upsilon^{n-s}(a_0)$, and

$$x(\upsilon^n(a_0))x(\widetilde{\upsilon^n(a_0)})c(\upsilon^n(a_0)) = -(x(a_0)x(\tilde{a}_0)c(a_0))^{p^n} = -1.$$

Finally for $a = (1, I) \in \mathscr{A}$, such that $a = \upsilon^n(a_0)$ define

$$\varepsilon(a) = \begin{cases} 1 & 0 \leq n \leq s-1 \\ -1 & s \leq n \leq 2s-1 \end{cases}.$$

Set $\varepsilon(a) = 1$ for all $a = (2, I) \in \mathscr{A}$.

If O is not self-dual, set $\varepsilon(a) = x(a) = x(\tilde{a}) = 1$ for all $a \in O$.

Lemma 7.2.8. *The basis* $\{b_{i,j} = \varepsilon(a)x(a)e_{i,j} \mid (i,j) \in a, a \in \mathscr{A}\}$ *is a symplectic basis for* Ψ *such that*

$$\Psi(b_{1,j}, b_{2,j^\flat}) = 1 = -\Psi(b_{2,j^\flat}, b_{1,j})$$

for $1 \leq j \leq g$.

Proof. By definition, for $(i,j), (i,k) \in a$,

$$\begin{aligned}
\Psi(\varepsilon(a)x(a)e_{i,j}, \varepsilon(\tilde{a})x(\tilde{a})e_{i+1,k^\flat}) &= \Psi_a(\varepsilon(a)x(a)e_{i,j}, \varepsilon(\tilde{a})x(\tilde{a})e_{i+1,k^\flat}) \\
&= \varepsilon(a)\varepsilon(\tilde{a})x(a)x(\tilde{a})c(a)\delta_{jk} \\
&= -\varepsilon(a)\varepsilon(\tilde{a})x(a)x(\tilde{a})c(\tilde{a})\delta_{jk} \\
&= -\Psi_{\tilde{a}}(x(\tilde{a})e_{i+1,k^\flat}, x(a)e_{i,j}) \\
&= -\Psi(x(\tilde{a})e_{i+1,k^\flat}, x(a)e_{i,j}).
\end{aligned}$$

Therefore, we need to show that

$$\varepsilon(a)\varepsilon(\tilde{a})x(a)x(\tilde{a})c(a) = 1$$

for all $a \in \mathscr{A}$ such that $a = (1, I)$. This is immediate if a is not in a self-dual orbit as

$$\varepsilon(a) = \varepsilon(\tilde{a}) = x(a) = x(\tilde{a}) = 1 = (-1)^2 = c(a).$$

On the other hand, suppose that a is in a self-dual orbit O with base point a_0 and length $2s$. Then $a = \upsilon^n(a_0)$ for some n. If $0 \leq n \leq s-1$, then,

$$\varepsilon(a)\varepsilon(\tilde{a})x(a)x(\tilde{a})c(a) = x(a)x(\tilde{a})c(a) = x(\upsilon^n(a_0))x(\widetilde{\upsilon^n(a_0)})c(\upsilon^n(a_0)) = 1.$$

Likewise if $s \leq n \leq 2s-1$

$$\varepsilon(a)\varepsilon(\tilde{a})x(a)x(\tilde{a})c(a) = -x(a)x(\tilde{a})c(a) = -x(\upsilon^n(a_0))x(\widetilde{\upsilon^n(a_0)})c(\upsilon^n(a_0)) = 1.$$

□

The preceding discussion and lemma can be summed up in the following proposition.

Proposition 7.2.9. *The basis* $\mathcal{B} = \{e_1, \dots, e_g; f_1, \dots, f_g\}$ *for* $\mathfrak{D}$ *defined by*

$$e_{w_1(j)} := \begin{cases} -\mu(x(\tilde{a})e_{2,j^\flat}) & w_1(j) \leq m_1 \\ \mu(\varepsilon(a)x(a)e_{1,j}) & w_1(j) \geq m_1 + 1 \end{cases} \tag{7.9}$$

$$f_{w_1(j)} := \begin{cases} \mu(\varepsilon(a)x(a)e_{1,j}) & w_1(j) \leq m_1 \\ \mu(x(\tilde{a})e_{2,j^\flat}) & w_1(j) \geq m_1 + 1 \end{cases} \tag{7.10}$$

where $(1,j) \in a$ is a standard symplectic basis for $\mathfrak{D}$ such that $\mathfrak{D}_1 = \{e_1, \ldots, e_{m_1}, f_{m_1+1}, \ldots, f_g\}$ and $\mathfrak{D}_2 = \{f_1, \ldots, f_{m_1}, e_{m_1+1}, \ldots, e_g\}$. In other words, $\mathcal{B}$ is a displayed basis for $\mathfrak{D}$.

Finally, recall the notation from Section 2.2.5 where the matrix of the display of $\mathfrak{D}$ with respect to $\mathcal{B}$ has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & A_1 & B_1 & 0 \\ A_2 & 0 & 0 & B_2 \\ C_2 & 0 & 0 & D_2 \\ 0 & C_1 & D_1 & 0 \end{pmatrix}.$$

Setting $M = (t_{ij})$ for $1 \leq i \leq m_1$ and $1 \leq j \leq m_2$, so that $T = \begin{pmatrix} 0 & M \\ M^t & 0 \end{pmatrix}$ gives the Hasse-Witt matrix,

$$H = A + TC = \begin{pmatrix} 0 & A_1 + MC_1 \\ A_2 + M^tC_2 & 0 \end{pmatrix} \pmod p.$$

The matrix $H \cdot H^p$ corresponds to F^2 and

$$H \cdot H^p = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} (A_1 + MC_1)(A_2 + M^tC_2)^p & 0 \\ 0 & (A_2 + M^tC_2)(A_1 + MC_1)^p \end{pmatrix}.$$

Then the vanishing of $\det(H_i)$ coincides with the vanishing of the partial Hasse-invariant h_i.

Example 7.2.10 (The almost-ordinary stratum)**.** The almost-ordinary locus corresponds to the element (w_1, w_2) where

$$w_i(j) = \begin{cases} j + m_i & 1 \le j \le m_{i+1} - 1 \\ j - m_{i+1} & m_{i+1} + 2 \le j \le g \\ 1 & j = m_{i+1} \\ g & j = m_{i+1} + 1. \end{cases}$$

Recall that $w_0(J_i)$ is the element

$$w_0(J_i) = [m_i \ \dots \ 1 \ g \ \dots \ m_i + 1] \, .$$

Its complementary element is

$$v_i = w_0 w_0(J_i) w_i = [1 \dots m_{i+1} - 1 \ m_{i+1} + 1 \ m_{i+1} \ m_{i+2} \dots g] \, ,$$

with the appropriate meaning when $m_2 = 1$.

As was seen in Chapter 3, there are 3 different situations to consider depending on the difference between m_1 and m_2:

- $m_1 - m_2 > 1$,
- $m_1 = m_2 = 1$,
- $m = m_1 = m_2$.

Each of these cases will be considered in turn.

Begin with the case when $m_1 - m_2 > 1$. There are 5 orbits of canonical fragments if $m_2 > 1$ and 3 orbits when $m_2 = 1$. The orbits

$$(0, m_2 - 1] \xrightarrow{v_1} (0, m_2 - 1] \xrightarrow{v_2} (0, m_2 - 1]$$

and

$$(m_1+1,g]\xrightarrow{v_1}(m_1+1,g]\xrightarrow{v_2}(m_1+1,g]$$

only appear when $m_2 > 1$. These two orbits are dual to each other. The next two orbits have length 4 and are also dual to each other:

$$(m_2-1,m_2]\xrightarrow{v_1}(m_2,m_2+1]\xrightarrow{v_2}(m_2,m_2+1]\xrightarrow{v_1}(m_2-1,m_2]\xrightarrow{v_2}(m_2-1,m_2]$$

$$(m_1-1,m_1]\xrightarrow{v_1}(m_1-1,m_1]\xrightarrow{v_2}(m_1,m_1+1]\xrightarrow{v_1}(m_1,m_1+1]\xrightarrow{v_2}(m_1-1,m_1].$$

Finally, there is an single self-dual orbit, given by:

$$(m_2+1,m_1-1]\xrightarrow{v_1}(m_2+1,m_1-1]\xrightarrow{v_2}(m_2+1,m_1-1].$$

Let $x, c \in k$ satisfy

- $c^p = -c$
- $x^{p^2} = -x$
- $-xx^pc = 1$.

Then letting $a_0 = (1,(m_2+1,m_1-1])$ to be the base point of its orbit, set $x(a_0) = x$, $x(\tilde{a}) = -x^p$, $c(a_0) = c$ and $c(\tilde{a}_0) = -c$. For all $a = (i, I)$ in the other orbits, set $c(a) = (-1)^{i+1}$ and $x(a) = x(\tilde{a}) = 1$.

Then $\{e_1, \ldots, e_g; f_1, \ldots, f_g\}$ gives a displayed basis for $\mathfrak{D}$ where

$$e_j = \begin{cases} -\mu(e_{2,m_1+1}) & j \in (0,1] \\ \mu(x^p e_{2,j+m_2}) & j \in (1, m_1 - m_2 - 1] \\ -\mu(e_{2,m_2+1}) & j \in (m_1 - m_2 - 1, m_1 - m_2] \\ -\mu(e_{2,m_2}) & j \in (m_1 - m_2, m_1 - m_2 + 1] \\ -\mu(e_{2,j+m_2-m_1-1}) & j \in (m_1 - m_2 + 1, m_1] \\ \mu(e_{1,j-m_1}) & j \in (m_1, g-1] \\ \mu(e_{1,m_2+1}) & j \in (g-1, g] \end{cases}$$

$$f_j = \begin{cases} \mu(e_{1,m_2}) & j \in (0,1] \\ \mu(x e_{1,j+m_2}) & j \in (1, m_1 - m_2 - 1] \\ \mu(e_{1,m_1}) & j \in (m_1 - m_2 - 1, m_1 - m_2] \\ \mu(e_{1,m_1+1}) & j \in (m_1 - m_2, m_1 - m_2 + 1] \\ \mu(e_{1,j+m_2}) & j \in (m_1 - m_2 + 1, m_1] \\ \mu(e_{2,j+1}) & j \in (m_1, g-1] \\ \mu(e_{2,m_1}) & j \in (g-1, g]. \end{cases}$$

$F(e_j)$ is given by

$$F(e_j) = \begin{cases} -f_{m_1-m_2} & j \in (0,1] \\ -f_j & j \in (1, m_1-m_2-1] \\ -e_g & j \in (m_1-m_2-1, m_1-m_2] \\ -f_j & j \in (m_1-m_2, m_1-m_2+1] \\ e_{j+m_2-1} & j \in (m_1-m_2+1, m_1] \\ -e_{j-m_2+1} & j \in (m_1, g-1] \\ -e_{m_1-m_2+1} & j \in (g-1, g]. \end{cases}$$

Writing $M = (t_{ij})$, the Hasse-Witt matrix for the almost-ordinary locus has the form

$$A_1 + MC_1 = A_1 = \begin{pmatrix} & 0_{(m_1-m_2)\times m_2} & & \\ 0 & \cdots & 0 & -1 \\ & & & 0 \\ & -I_{m_2-1} & & \vdots \\ & & & 0 \end{pmatrix}$$

and

$$A_2 + M^t C_2 = \begin{pmatrix} -t_{(m_1-m_2)1} & & 0 & -t_{(m_1-m2+1)1} & \\ \vdots & (t_{ij})^t_{\substack{1\le i\le m_2 \\ 2\le j\le m_1-m_2-1}} & \vdots & \vdots & I_{m_2-1} \\ \vdots & & 0 & \vdots & \\ -t_{(m_1-m_2)m_2} & & -1 & -t_{(m_1-m2+1)m_2} & 0 \end{pmatrix}.$$

It follows that

$$H_1 = \begin{pmatrix} & & 0_{(m_1-m_2)m_1} & & & \\ t^p_{(m_1-m_2)m_2} & -t^p_{2m_2} & \dots & -t^p_{(m_1-m_2-1)m_2} & 1 & 0 \\ t^p_{(m_1-m_2)1} & & -(t^p_{ij})^t_{\substack{1\le i\le m_2-1 \\ 2\le j\le m_1-m_2-1}} & & 0 & \\ \vdots & & & & \vdots & -I_{m_2-1} \\ t^p_{(m_1-m_2)(m_2-1)} & & & & 0 & \end{pmatrix}$$

and

$$H_2 = \begin{pmatrix} & t_{(m_1-m_2+1)1} \\ -I_{m_2-1} & \vdots \\ 0 \quad \dots & t_{(m_1-m_2+1)m_2} \end{pmatrix}.$$

When $m_2 = 1$, this simplifies to

$$H_1 = \begin{pmatrix} & & 0_{(m_1-1)\times m_1} & & & \\ t^p_{(m_1-1)1} & -t^p_{21} & \dots & -t^p_{(m_1)1} & 1 & 0 \end{pmatrix}, \qquad H_2 = (t_{m_1 1}).$$

In both cases for m_2, $\det H_2 = (-1)^{m_2-1} t_{(m_1-m_2+1)m_2}$ and there are zero rows in H_1. This is consistent with the result that $h_1 = 0$.

Now suppose that $m_1 = m_2+1$. Then there are 3 orbits of canonical fragments in general and one orbit if $(m_1, m_2) = (2, 1)$. The orbits

$$(0, m_2-1] \xrightarrow{v_1} (0, m_2-1] \xrightarrow{v_2} (0, m_2-1]$$

and

$$(m_1+1, g] \xrightarrow{v_1} (m_1+1, g] \xrightarrow{v_2} (m_1+1, g]$$

do not appear when $m_2 = 1$. These two orbits are dual to each other. The third orbit is the single self-dual orbit, given by:

$$(m_2-1, m_2] \xrightarrow{v_1} (m_2, m_1] \xrightarrow{v_2} (m_1, m_1+1] \xrightarrow{v_1} (m_1, m_1+1]$$

$$\xrightarrow{\qquad v_2 \qquad} (m_2, m_1] \xrightarrow{v_1} (m_2-1, m_2] \xrightarrow{v_2} (m_2-1, m_2].$$

Let $x, c \in k$ satisfy

- $c^{p^3} = -c$
- $x^{p^6} = -x$
- $-xx^{p^3}c = 1.$

Take $a_0 = (1, (m_2 - 1, m_2])$ to be the base point of the self-dual orbit. Set $x(a_0) = x$ and $c(a_0) = c$. Then set $x(a)$, $c(a)$ and $\varepsilon(a)$ for the rest of the orbit accordingly. Note that $\varepsilon\left((1, (m_2, m_1])\right) = -1$. For all $a = (i, I)$ in the other orbits, set $c(a) = (-1)^{i+1}$ and $\varepsilon(a) = x(a) = x(\tilde{a}) = 1$.

Then a displayed basis $\mathcal{B} = \{e_1, \ldots, e_g; f_1, \ldots, f_g\}$ for $\mathfrak{D}$ can be defined by

$$e_j = \begin{cases} \mu(x^{p^3} e_{2,m_1+1}) & j \in (0,1] \\ \mu(x^{p^5} e_{2,m_2}) & j \in (1,2] \\ -\mu(e_{2,j-2}) & j \in (2,m_1] \\ \mu(e_{1,j-m_1}) & j \in (m_1, g-1] \\ -\mu(x^{p^4} e_{1,m_1}) & j \in (g-1, g] \end{cases}$$

$$f_j = \begin{cases} \mu(x e_{1,m_2}) & j \in (0,1] \\ \mu(x^{p^2} e_{1,m_1+1}) & j \in (1,2] \\ \mu(e_{1,j+m_2}) & j \in (2,m_1] \\ \mu(e_{2,j+1}) & j \in (m_1, g-1] \\ -\mu(x^{p} e_{2,m_1}) & j \in (g-1, g] \end{cases},$$

and $F(e_j)$ is

$$F(e_j) = \begin{cases} -e_g & j \in (0,1] \\ -f_1 & j \in (1,2] \\ -e_{j+m_2-1} & j \in (2,m_1] \\ -e_{j-m_1+2} & j \in (m_1, g-1] \\ -e_2 & j \in (g-1, g]. \end{cases}$$

Writing $M = (t_{ij})$, the Hasse-Witt matrix for the almost-ordinary locus has the form

$$A_1 + MC_1 = A_1 = \begin{pmatrix} 0 & \dots & \dots & 0 \\ 0 & \dots & 0 & -1 \\ & & & 0 \\ & -I_{m_2-1} & & \vdots \\ & & & 0 \end{pmatrix}$$

and

$$A_2 + M^t C_2 = \begin{pmatrix} 0 & -t_{21} & \\ \vdots & \vdots & -I_{m_2-1} \\ 0 & \vdots & \\ -1 & -t_{2m_2} & 0 \end{pmatrix}.$$

It follows that

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & t_{2m_2}^p & 0 & \dots & 0 \\ 0 & t_{21}^p & & & \\ \vdots & \vdots & & I_{m_2-1} & \\ 0 & t_{2(m_2-1)}^p & & & \end{pmatrix}$$

and

$$H_2 = \begin{pmatrix} & & t_{21} \\ & I_{m_2-1} & \vdots \\ 0 & \dots & t_{2m_2} \end{pmatrix}.$$

Again, H_1 has a zero row, and in this case $\det H_2 = t_{2m_2}$.

Finally suppose that $m = m_1 = m_2$. Then there are 4 orbits. The first two are dual to each other:

$$(0, m-1] \xrightarrow{v_1} (0, m-1] \xrightarrow{v_2} (0, m-1]$$

and

$$(m+1, g] \xrightarrow{v_1} (m+1, g] \xrightarrow{v_2} (m+1, g].$$

The other two orbits are self-dual and given by:

$$(m-1, m] \xrightarrow{v_1} (m, m+1] \xrightarrow{v_2} (m-1, m]$$

and

$$(m, m+1] \xrightarrow{v_1} (m-1, m] \xrightarrow{v_2} (m, m+1].$$

Let $x, c \in k$ satisfy

- $c^p = -c$
- $x^{p^2} = -x$
- $-xx^{p^2}c = 1.$

Take $a_0 = (1, (m-1, m])$ and $(1, (m, m+1])$ to be the base points of their respective self-dual orbits. Set $x(a_0) = x$ and $c(a_0) = c$. Then set $x(a)$, $c(a)$ and $\varepsilon(a)$ for the rest of the orbit accordingly. For all $a = (i, I)$ in the other orbits, set $c(a) = (-1)^{i+1}$ and $\varepsilon(a) = x(a) = x(\tilde{a}) = 1$.

Then $\mathfrak{D}$ has a displayed basis where $\mathcal{B} = \{e_1, \ldots, e_g; f_1, \ldots, f_g\}$ is defined by

$$e_j = \begin{cases} \mu(e_{1,j-m}) & j \in (m, g-1] \\ \mu(x^p e_{2,m+1}) & j \in (0,1] \\ \mu(x e_{1,m+1}) & j \in (g-1, g] \\ -\mu(e_{2,j-1}) & j \in (1, m] \end{cases} \tag{7.11}$$

$$f_j = \begin{cases} \mu(e_{2,j+1}) & j \in (m, g-1] \\ \mu(x e_{1,m}) & j \in (0,1] \\ -\mu(x^p e_{2,m}) & j \in (g-1, g] \\ \mu(e_{1,j_m}) & j \in (1, m]. \end{cases} \tag{7.12}$$

$F(e_j)$ is then given by

$$F(e_j) = \begin{cases} -f_1 & j \in (0,1] \\ -e_{j+m-1} & j \in (1, m] \\ -e_{j-m+1} & j \in (m, g-1] \\ -f_g & j \in (g-1, g]. \end{cases}$$

The Hasse-Witt matrix for the almost-ordinary locus when $m = m_1 = m_2$ has the form

$$A_1 + MC_1 = \begin{pmatrix} 0 & \ldots & 0 & -t_{1m} \\ & -I_{m-1} & & \vdots \\ & & & -t_{mm} \end{pmatrix}$$

and

$$A_2 + M^t C_2 = \begin{pmatrix} -t_{11} & & & \\ \vdots & & -I_{m-1} & \\ -t_{1m} & 0 & \dots & 0 \end{pmatrix}.$$

Finally,

$$H_1 = \begin{pmatrix} t_{1m}^{p+1} & 0 & \dots & 0 \\ t_{11}^p + t_{1m}^p t_{2m} & & & \\ \vdots & & & I_{m-1} \\ t_{1(m-1)}^p + t_{1m}^p t_{(m-1)m} & & & \\ t_{1m}^p t_{mm} & & & \end{pmatrix}$$

and

$$H_2 = \begin{pmatrix} 0 & \dots & 0 & t_{11}t_{1m}^p + t_{2m}^p \\ & & & t_{12}t_{1m^p} + t_{3m}^p \\ & I_{m-1} & & t_{13}t_{1m^p} + t_{4m}^p \\ & & & \vdots \\ & & & t_{1(m-1)}t_{1m^p} + t_{2m}^p \\ & & & t_{1m}^{p+1} \end{pmatrix}.$$

In this case, $\det H_1 = (-1)^{m-1} \det H_2 = (-1)^{m-1} t_{1m}^{p+1}$.

The results from the previous example can be summed up by the following proposition.

Proposition 7.2.11. *When $m_1 > m_2$ the Hasse-invariant vanishes to order one on the almost-ordinary E-O stratum, however, when $m_1 = m_2$, the Hasse-invariant vanishes to order $p + 1$.*

Example 7.2.12 (The almost-core stratum)**.** The almost-core locus corresponds to the element (w_1, w_2) where

$$w_i(j) = \begin{cases} m_i + 1 & j = m_i \\ m_i & j = m_i + 1 \\ j & \text{otherwise.} \end{cases}$$

Then the complementary element v where $v_i = w_0 w_0(J_i) w_i$ of w is given by

$$v_i = [m_{i+1}+1\ m_{i+1}+2\ \ldots g-1\ 1\ g\ 2 \ldots m_{i+1}]\,.$$

There are two distinct situations to cover, when $m_2 > 1$ and when $m_2 = 1$. First, suppose that $m_2 > 1$. In this case there are four orbits and two different kinds of orbits:

$$(0,1] \xrightarrow{v_i} (m_{i+1}, m_{i+1}+1] \xrightarrow{v_{i+1}} (g-1, g] \xrightarrow{v_i} (m_{i+1}-1, m_{i+1}] \xrightarrow{v_{i+1}} (0,1]$$

and

$$(1, m_i - 1] \xrightarrow{v_i} (m_{i+1}+1, g-1] \xrightarrow{v_{i+1}} (1, m_i - 1]$$

where we are starting from a canonical fragment of v_i. The two orbits of length four are dual to each other, and the orbits of length two are self-dual.

Let

$$\mathscr{A}_i^{(1)} = \{(i, I) \mid I \in \{(0,1], (m_i - 1, m_i], (m_i, m_i + 1], (g-1, g]\}\}$$

be the canonical fragments for υ_i whose orbits are *not* self-dual. Then for $a \in \mathscr{A}_i^{(1)}$, $\tilde{a} \in \mathscr{A}_{i+1}^{(1)}$ and $\upsilon(a) \in \mathscr{A}_{i+1}^{(1)}$. Let

$$\mathscr{A}_i^{(2)} = \{(i, I) \mid I \in \{(1, m_i - 1], (m_i + 1, g - 1]\}\}$$

be the canonical fragments for υ_i whose orbits are self-dual.

For $a \in \mathscr{A}_1^{(1)}$ set $c(a) = 1$, $c(\tilde{a}) = -1$, and $\varepsilon(a) = x(a) = x(\tilde{a}) = 1$. For $a \in \mathscr{A}_1^{(2)}$, set $c(a) = c$ and $x(a) = x$ where x and c satisfy,

- $c^p = -c$
- $x^{p^2} = -x$
- $x^p x c = -1$.

Then $c(a) = c^p = -c$ and $x(a) = -x^p$ for $a \in \mathscr{A}_1^{(2)}$. Choose base points $a_0 \in \{(1, (1, m_1 - 1]), (1, (m_2 + 1, g - 1])\}$ for the two orbits of length two. Then there are no elements $a = (1, I) \in \mathscr{A}$ such that $\varepsilon(a) = -1$.

The following gives a displayed basis for $\mathfrak{D}$. For $(1, j) \in a = (1, I)$,

$$e_{w_1(j)} = \begin{cases} -\mu(e_{2,g}) & j \in (0, 1] \\ \mu(x^p e_{2,j+m_2}) & j \in (1, m_1 - 1] \\ -\mu(e_{2,m_2}) & j \in (m_1, m_1 + 1] \\ \mu(e_{1,m_1}) & j \in (m_1 - 1, m_1] \\ \mu(x e_{1,j}) & j \in (m_1 + 1, g] \\ \mu(e_{1,g}) & j \in (g - 1, g] \end{cases}$$

$$f_{w_1(j)} = \begin{cases} \mu(e_{1,1}) & j \in (0, 1] \\ \mu(x e_{1,j}) & j \in (1, m_1 - 1] \\ \mu(e_{1,m_1+1}) & j \in (m_1, m_1 + 1] \\ \mu(e_{2,m_2+1}) & j \in (m_1 - 1, m_1] \\ \mu(-x^p e_{2,j-m_1}) & j \in (m_1 + 1, g - 1] \\ \mu(e_{2,1}) & j \in (g - 1, g]. \end{cases}$$

This means that

$$F(e_j) = \begin{cases} e_{m_1+1} & j \in (0, 1] \\ -f_j & j \in (1, m_1 - 1] \\ -f_1 & j \in (m_1 - 1, m_1] \\ f_g & j \in (m_1, m_1 + 1] \\ -f_j & j \in (m_1 + 1, g - 1] \\ -e_{m_1} & j \in (g - 1, g]. \end{cases}$$

It follows that H is given by

$$H_1 = \begin{pmatrix} t_{1m_2} & & \Big| \\ \vdots & M_1 & \vec{y}_1 \\ \vdots & & \Big| \\ t_{m_1m_2} \;\text{———}\; & \vec{x}_1 & \;\text{———}\; z_1 \end{pmatrix}$$

where

$$M_1 = \left(\left(\sum_{n=2}^{m_1-1} t_{in} t_{jn}^p \right) - t_{im_2} t_{j1}^p \right)_{\substack{1 \le i \le m_1-1 \\ 2 \le j \le m_1-1}}$$

$$\vec{x}_1 = \left(\sum_{n=2}^{m_1-1} t_{m_1 n} t_{jn}^p \right) - t_{m_1 m_2} t_{j1}^p + t_{jm_2}^p,$$

$$\vec{y}_1 = \left(\sum_{n=2}^{m_1-1} t_{in}^{p+1} \right) - t_{im_2} t_{11}^p,$$

$$z_1 = \left(\sum_{n=2}^{m_1-1} t_{m_1 n}^{p+1} \right) - t_{1m_2} t_{11}^p + t_{1m_2}^p.$$

and

$$H_2 = \begin{pmatrix} z_2 \;\text{———}\; & \vec{x}_2 & \;\text{———}\; t_{11} \\ \Big| & & \vdots \\ \vec{y}_2 & M_2 & \vdots \\ \Big| & & \vdots \\ & & t_{1m_2} \end{pmatrix}$$

where

$$M_2 = \left(\left(\sum_{n=2}^{m_1-1} t_{ni} t_{nj}^p \right) + t_{1i} t_{m_1 j}^p \right)_{\substack{1 \le i \le m_1 - 1 \\ 2 \le j \le m_1 - 1}}$$

$$\vec{x}_2 = \left(\sum_{n=2}^{m_1-1} t_{n1} t_{nj}^p \right) + t_{11} t_{m_1 j}^p - t_{1j}^p,$$

$$\vec{y}_2 = -\left(\sum_{n=2}^{m_1-1} t_{ni} t_{nm_2}^p \right) - t_{1i} t_{m_1 m_2}^p,$$

$$z_2 = \left(\sum_{n=2}^{m_1-1} t_{n1} t_{nm_2}^p \right) + t_{1m_2}^p - t_{11} t_{m_1 m_2}^p.$$

Now consider the case where $m_2 = 1$. Then

$$v_1 = [2\ 3\ \ldots g-1\ 1\ g \ldots m_{i+1}],$$

$$v_2 = [1\ g\ 2 \ldots g-1],$$

There are two different orbits of canonical fragments here:

$$(0,1] \xrightarrow{v_1} (1,2] \xrightarrow{v_2} (g-1,g] \xrightarrow{v_1} (g-1,g] \xrightarrow{v_2} (g-2,g-1] \xrightarrow{v_1} (0,1] \xrightarrow{v_2} (0,1]$$

and

$$(1,g-2] \xrightarrow{v_1} (2,g-1] \xrightarrow{v_2} (1,g-2]$$

where in each case we are starting from a canonical fragment of v_1. Both orbits are self-dual. Let $y, d \in k$ such that

- $d^{p^3} = -d$

- $y^{p^6} = -y$
- $-yy^{p^3}d = 1$

and let $x, c \in k$ such that

- $c^p = -c$
- $x^{p^2} = -x$
- $-xx^pc = 1$.

For $a_0 = (1,(0,1])$ set $x(a_0) = y$, $c(a_0) = d$ and $x(\upsilon^n(a_0)) = (-1)^n y^{p^n}$ for $1 \leq n \leq 5$. Similarly, $c(\upsilon^n(a_0)) = d^{p^n}$ for $1 \leq n \leq 5$. Likewise, for $a_0 = (1,(1,g-2])$ set $x(a_0) = x$, $c(a_0) = c$, which determines the values of $x(\upsilon(a)) = x(\tilde{a}) = -x^p$ and $c(\upsilon(a)) = c(\tilde{a}) = -c$.

The following gives a displayed basis for $\mathfrak{D}$.

$$e_j = \begin{cases} \mu(y^{p^3}e_{2,g}) & j \in (0,1] \\ \mu(x^p e_{2,j+1}) & j \in (1,g-2] \\ \mu(y^{p^4}e_{1,g-1}) & j \in (g-1,g] \\ \mu(y^{p^5}e_{2,1}) & j \in (g-2,g-1] \end{cases}$$

$$f_j = \begin{cases} \mu(ye_{1,1}) & j \in (0,1] \\ \mu(xe_{1,j}) & j \in (1,g-2] \\ \mu(-y^p e_{2,2}) & j \in (g-1,g] \\ \mu(y^{p^2}e_{1,g}) & j \in (g-2,g-1]. \end{cases}$$

Consequently,

$$F(e_j) = \begin{cases} e_g & j \in (0,1] \\ -f_j & j \in (1, g-2] \\ e_{g-1} & j \in (g-1, g] \\ f_1 & j \in (g-2, g-1]. \end{cases}$$

Since $m_2 = 1$, the matrix $M = (t_{ij})$ consists of a single column, so write $t_j := t_{1j}$. Then

$$A_1 + MC_1 = (0, \ldots, 0, 1)^t$$

and

$$A_2 + M^t C_2 = (1, -t_2, -t_3, \ldots, -t_{g-1}, t_1).$$

Therefore,

$$H_1 = \begin{pmatrix} 0 & & & & \\ \vdots & & 0_{m_1-1} & & \\ 0 & & & & \\ 1 & -t^p_{m_2+1} & \ldots & -t^p_{m_1} & t^p_1 \end{pmatrix}, \qquad H_2 = (t_1).$$

Example 7.2.13 (The core stratum)**.** Here w_i is the identity element, and

$$v_i = w_0 w_0(J_i) = [m_{i+1}+1 \ \ldots \ g \ 1 \ \ldots m_{i+1}] \, .$$

The orbits of the canonical fragments of v_i are given by:

$$(0, m_1] \xrightarrow{v_1} (m_2, g] \xrightarrow{v_2} (0, m_1]$$

$$(m_1, g] \xrightarrow{v_1} (0, m_2] \xrightarrow{v_2} (m_1, g]$$

meaning that the canonical fragments of v_i are $(0, m_i]$ and $(m_i, g]$. Both orbits have length 2 and are self-dual.

Let $\mathcal{A}_i = \{a \in \mathscr{A} \mid a = (i, I)\}$ and choose the base points to be the elements of $\mathcal{A}_1$. Then set $c(a) = c$ and $x(a) = x$ for $a \in \mathcal{A}_1$ where x and c satisfy,

- $c^p = -c$
- $x^{p^2} = -x$
- $x^p x c = -1$.

This determines the values of $c(a)$ and $x(a)$ for $a \in \mathcal{A}_2$. Since the orbits are both of length 2, $\varepsilon(a) = 1$ for all $a \in \mathscr{A}$. Thus, $\mathfrak{D}$ has a displayed basis where

$$e_j = \begin{cases} \mu(x^p e_{2,j+m_2}) & j \leq m_1 \\ \mu(x e_{1,j}) & j \geq m_1 + 1 \end{cases} \tag{7.13}$$

$$f_j = \begin{cases} \mu(x e_{1,j}) & j \leq m_1 \\ -\mu(x^p e_{2,j-m_1}) & j \geq m_1 + 1 \end{cases} \tag{7.14}$$

and

$$F(e_j) = \begin{cases} -\mu(x e_{1,j}) & j \leq m_1 \\ \mu(x^p e_{1,j-m_1}) & j \geq m_1 + 1 \end{cases} = \begin{cases} -f_j & j \leq m_1 \\ -f_j & j \geq m_1 + 1 \end{cases}.$$

Therefore, the Hasse-Witt matrix has the form: $H = A + TC = -T$. It follows that

$$H = T(T^p) = \begin{pmatrix} M(M^t)^p & 0 \\ 0 & M^t(M)^p \end{pmatrix}.$$

Since $M = (t_{ij})_{1\leq i\leq m_1, 1\leq j\leq m_2}$ there is not a particularly simple description in general for the determinants of the H_i. However, when $m_1 = m_2$, the matrix M is square and we see directly from the matrices that $\det H_1 = \det H_2$.

Fortunately, the case where $m_2 = 1$ is easy enough to describe, as M is a single column. Writing $t_j := t_{1j}$ gives

$$H_1 = (t_i t_j^p)_{1\leq i,j\leq m_1}, \qquad H_2 = \left(\sum_{1\leq j\leq m_1} t_j^{p+1}\right).$$

Now, all the rows of H_1 are linearly dependant as dividing the i^{th} row of H_1 by t_i gives the same result for all $1 \leq i \leq m_1$, once again confirming that $\det H_1 = 0$, and locally formally the non-ordinary locus is cut out by the equation of a Fermat hypersurface

$$t_1^{p+1} + t_2^{p+1} + \ldots + t_{m_1}^{p+1} = 0.$$

Chapter 8
Conclusion and future work

In this thesis, we have extensively studied the Ekedahl-Oort stratification of unitary Shimura varieties. By describing the Weyl group coset ${}^JW$ that classifies the E-O strata as a subset of $S_g \times S_g$, we were able to characterize many properties of the stratification combinatorially. This had the immediate consequence of allowing us to demonstrate the symmetry of the stratification under complex conjugation on $\mathcal{M}$ in case $m_1 = m_2$, and to count the number of strata of a given dimension. It then followed that not only are there unique strata of dimension 0 and m_1m_2 (the core stratum and μ-ordinary stratum respectively), but there are also unique strata of dimension and codimension-one, corresponding to the *almost-core* and *almost-ordinary* strata. By giving models for the Dieudonné modules based on the corresponding element in ${}^JW$ (following [Moo01]), we calculated invariants such as the a-number and f-number. This showed that generally, when p is split in K, the core stratum is not superspecial, and there is no way to distinguish the core stratum from the almost-core stratum based on standard invariants. We also gave models for the p-torsion groups schemes of the special strata of interest, which allowed us to describe in detail both the most generic and most degenerate behaviours of the strata throughout the course of the thesis.

As the points in the core strata where p is split were unlike the core strata of other Shimura varieties examined in previous work, we had no immediate models for the core points in terms of well-understood abelian varieties. In order to tackle this problem, we showed how to explicitly derive the E-O stratum of the reduction of a CM point from its CM type in a general setting. This allowed us to give concrete examples of abelian varieties that lie in given E-O strata, including the mysterious core stratum of the Shimura variety when p is split.

The study of the E-O stratification would not be complete without also comparing it to the Newton stratification—that is, comparing the p-divisible groups and p-torsion group schemes of the abelian varieties on the moduli space. In the case where p is inert, we demonstrated the relationship of the special E-O strata with the Newton stratification using the models for their p-torsion group schemes. In the case where p is split, we effectively computed the map $B(G, \mu) \hookrightarrow {}^J W$ from [VW13] that takes a Newton stratum to the minimal E-O stratum contained within it. This showed that the core stratum is not even supersingular (unless $m_1 = m_2$). In low dimensional examples we saw that the calculation of the map $B(G, \mu) \hookrightarrow {}^J W$ and the closure relations on E-O strata completely determined the relationship between the Newton and E-O strata.

In order to understand the cycle classes of the E-O strata, we then constructed a flag space $\mathcal{F}_i$ over $\mathcal{M}$ using the de Rham cohomology. A crucial combinatorially calculation showed how to construct a complete symplectic flag extending the Hodge filtration (together with its corresponding Weyl group element) from a flag extending $\mathbb{E}_i \subseteq \mathbb{H}_i$, which allowed us to take advantage of the techniques of Ekedahl and van der Geer from the Siegel case [EvdG09]. We showed that the map $\mathcal{F}_i \to \mathcal{M}$ takes strata to strata and is isomorphic to a map from

$\mathrm{GL}_g/B \to \mathrm{GL}_g/P$ locally in the étale topology. As a consequence, we showed that the maps between strata are finite étale surjective maps. In the future, this will allow us to study the cycles classes of E-O strata in the Chow group by pushing-down cycles of closed strata in $\mathcal{F}_i$. Furthermore, the cycle classes of the closed E-O strata can be written in terms of Chern classes of the Hodge bundle and consequently lie in the tautological ring. This is a topic that we intend to pursue further in future work.

Finally, we calculated the deformation of the Hasse-Witt matrices over the special E-O strata in order to study the Hasse-invariants via local equations giving results on the geometry of its vanishing locus—the closure of the almost-ordinary stratum. These calculations show that the (partial) Hasse-invariants vanish to order one on the almost-ordinary stratum. When p is inert in K we show that the vanishing locus of the Hasse-invariant is locally formally cut-out at the core points by the equation of a Fermat hypersurface when $m_2 = 1$. Furthermore, in the case where p is split in K we show that the non-ordinary locus is not only connected, but irreducible, with the corollary that the non-ordinary locus for a Shimura variety coming from the group $\mathrm{GU}(2,1)$ is a smooth, irreducible curve.

There are two main directions we intend to pursue in future work. First, we would like to continue extending the program of Ekedahl-van der Geer in the Siegel case to both unitary Shimura varieties and Hilbert modular varieties. We hope to obtain similar results such as irreducibility of many of the E-O strata. We will also provide formulae for the cycle classes and expect to be able to use these calculations to determine the number of core points on unitary Shimura varieties.

Another direction is to examine the action of the prime-to-p Hecke operators on the E-O stratification. We expect that phenomena similar to the Siegel and Hilbert modular case will arise when studying the effect on the core stratum, with applications to mod p modular forms as well as expander graph theory. This thesis gives a good foundational understanding of the core E-O stratum of unitary Shimura varieties—one previously missing from the literature—enabling the study the Hecke action on the core stratum of unitary Shimura varieties. Along the same lines, we will also study the prime-to-p Hecke operators on the entire stratification, as in the work of Goren-Oort on Hilbert modular varieties [GO00].

Index

List of Notation

Bibliography

[BB05] Anders Bjorner and Francesco Brenti, *Combinatorics of Coxeter groups*, Springer, 2005.

[BGG73] I.N. Bernstein, Israel M. Gel'fand, and Sergei I. Gel'fand, *Schubert cells and cohomology of the spaces G/P*, Russian Mathematical Surveys **28** (1973), no. 3, 1.

[Bor91] Armand Borel, *Linear algebraic groups*, vol. 126, Springer-Verlag New York, 1991.

[Box15] George Boxer, *Torsion in the coherent cohomology of Shimura varieties and Galois representations*, Ph.D. thesis, Harvard University, 2015.

[BW06] Oliver Bultel and Torsten Wedhorn, *Congruence relations for Shimura varieties associated to some unitary groups*, Journal-Institute of Mathematics of Jussieu **5** (2006), no. 2, 229.

[CGL09] Denis X. Charles, Eyal Z. Goren, and Kristin E. Lauter, *Families of Ramanujan graphs and quaternion algebras*, Groups and Symmetries, CRM Proc. Lecture Notes **47** (2009), 53–80.

[Cha00] Ching-Li Chai, *Newton polygons as lattice points*, American Journal of Mathematics **122** (2000), no. 5, 349–376.

[CLG09] Denis X. Charles, Kristin E. Lauter, and Eyal Z. Goren, *Cryptographic hash functions from expander graphs*, Journal of Cryptology **22** (2009), no. 1, 93–113.

[Del72] Pierre Deligne, *Travaux de Shimura*, Séminaire Bourbaki vol. 1970/71 Exposés 382–399 (1972), 123–165.

[Del81] ———, *Hodge cycles on abelian varieties*, Hodge cycles, motives, and Shimura varieties, Springer, 1981, pp. 9–100.

[Dem72] Michel Demazure, *Lectures on p-divisible groups*, Springer-Verlag, 1972.

[dJO00] A.J. de Jong and Frans Oort, *Purity of the stratification by Newton polygons*, Journal of the American Mathematical Society **13** (2000), no. 1, 209–241.

[DP94] Pierre Deligne and Georgios Pappas, *Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant*, Compositio Math **90** (1994), no. 1, 59–79.

[dSG15] Ehud de Shalit and Eyal Z. Goren, *Supersingular curves on Picard modular surfaces modulo an inert prime*, 2015, Submitted.

[EvdG09] Torsten Ekedahl and Gerard van der Geer, *Cycle classes of the EO stratification on the moduli of abelian varieties*, Algebra, Arithmetic, and Geometry, Springer, 2009, pp. 567–636.

[Gas10] Qëndrim R. Gashi, *On a conjecture of Kottwitz and Rapoport*, Ann. Sci. École Norm. Sup **43** (2010), 1017–1038.

[Ghi03] Alexandru Ghitza, *Siegel modular forms (mod p) and algebraic modular forms*, Ph.D. thesis, Massachuesetts Institute of Technology, 2003.

[GN16] Wushi Goldring and Marc-Hubert Nicole, *The μ-ordinary Hasse invariant of unitary Shimura varieties*, Journal für die reine und angewandte Mathematik (Crelles Journal) (2016), To appear.

[GO00] Eyal Z. Goren and Frans Oort, *Stratifications of Hilbert modular varieties*, Journal of Algebraic Geometry **9** (2000), no. 1, 111–154.

[Ham14] Paul Hamacher, *The geometry of Newton strata in the reduction modulo p of Shimura varieties of PEL type*, Ph.D. thesis, Technische Universität München, 2014.

[Har77] Robin Hartshorne, *Algebraic geometry*, vol. 52, Springer Science & Business Media, 1977.

[He07] Xuhua He, *The G-stable pieces of the wonderful compactification*, Transactions of the American Mathematical Society **359** (2007), no. 7, 3005–3024.

[HP13] Benjamin Howard and George Pappas, *On the supersingular locus of the* $\mathrm{GU}(2,2)$ *Shimura variety*, 2013.

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, vol. 151, Princeton University Press, 2001.

[Kob75] Neal Koblitz, *p-adic variation of the zeta-function over families of varieties defined over finite fields*, Compositio Mathematica **31** (1975), no. 2, 119–218.

[Kot85] Robert E. Kottwitz, *Isocrystals with additional structure*, Compositio Mathematica **56** (1985), no. 2, 201–220.

[Kot92] ______, *Points on some Shimura varieties over finite fields*, Journal of the American Mathematical Society (1992), 373–444.

[KR03] Robert E. Kottwitz and Michael Rapoport, *On the existence of F-crystals*, Commentarii Mathematici Helvetici **78** (2003), no. 1, 153–184.

[Lan83] Serge Lang, *Complex multiplication*, vol. 255, Springer Science, 1983.

[LO98] Ke-Zheng Li and Frans Oort, *Moduli of supersingular abelian varieties*, vol. 1680, Springer, 1998.

[Luc04] Catherine Lucarelli, *A converse to Mazur's inequality for split classical groups*, Journal of the Institute of Mathematics of Jussieu **3** (2004), no. 02, 165–183.

[Mil06] Joseph S. Milne, *Complex multiplication*, 2006.

[Moo01] Ben Moonen, *Group schemes with additional structures and weyl group cosets*, Moduli of abelian varieties, Springer, 2001, pp. 255–298.

[Moo04a] ______, *A dimension formula for Ekedahl-Oort strata*, Annales de l'Institut Fourier, vol. 54, 2004, pp. 666–698.

[Moo04b] ______, *Serre–Tate theory for spaces of PEL type*, Annales Scientifiques de l'École Normale Supérieure, vol. 37, 2004, pp. 223–269.

[MW04] Ben Moonen and Torsten Wedhorn, *Discrete invariants of varieties in positive characteristic*, International Mathematics Research Notices **2004** (2004), no. 72, 3855–3903.

[Nic05] Marc-Hubert Nicole, *Superspecial abelian varieties, theta series and the Jacquet-Langlands correspondence*, Ph.D. thesis, McGill University, 2005.

[NO80] Peter Norman and Frans Oort, *Moduli of abelian varieties*, Annals of Mathematics (1980), 413–439.

[Nor75] Peter Norman, *An algorithm for computing local moduli of abelian varieties*, Annals of Mathematics (1975), 499–509.

[Oda69] Tadao Oda, *The first de Rham cohomology group and Dieudonné modules*, Annales Scientifiques de l'École Normale Supérieure, vol. 2, 1969, pp. 63–135.

[Oor75] Frans Oort, *Which abelian surfaces are products of elliptic curves?*, Mathematische Annalen **214** (1975), no. 1, 35–47.

[Oor01] ———, *A stratification of a moduli space of abelian varieties*, Progress in Mathmatics-Boston **195** (2001), 345–416.

[Oor05] ———, *Minimal p-divisible groups*, Annals of Mathematics (2005), 1021–1036.

[PWZ11] Richard Pink, Torsten Wedhorn, and Paul Ziegler, *Algebraic zip data*, Documenta Mathematica **16** (2011), 253–300.

[Red12] Davide A. Reduzzi, *Hecke eigensystems for (mod p) modular forms of PEL type and algebraic modular forms*, International Mathematics Research Notices (2012), rns114.

[RR96] Michael Rapoport and Melanie Richartz, *On the classification and specialization of F-isocrystals with additional structure*, Compositio Mathematica **103** (1996), no. 2, 153–181.

[RZ96] Michael Rapoport and Thomas Zink, *Period spaces for p-divisible groups*, no. 141, Princeton University Press, 1996.

[Vas08] Adrian Vasiu, *Level m stratifications of versal deformations of p-divisible groups*, Journal of Algebraic Geometry **17** (2008), no. 4, 599–641.

[Vol05] Inken Vollaard, *The supersingular locus of the Shimura variety of* $\mathrm{GU}(1,s)$, Ph.D. thesis, Universitäts-und Landesbibliothek Bonn, 2005.

[VW11] Inken Vollaard and Torsten Wedhorn, *The supersingular locus of the Shimura variety of* $\mathrm{GU}(1, n-1)$ *II*, Inventiones Mathematicae **184** (2011), no. 3, 591–627.

[VW13] Eva Viehmann and Torsten Wedhorn, *Ekedahl–Oort and Newton strata for Shimura varieties of PEL type*, Mathematische Annalen **356** (2013), no. 4, 1493–1550.

[Zin02] Thomas Zink, *The display of a formal p-divisible group*, Astérisque (2002), 127–248.