

Enhanced Symmetries, Duality Invariance and Moduli Stabilization in String Gas Cosmology

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January 24, 2008

Submitted in partial fulfillment of the requirements for the degree of Doctor
or Philosophy at McGill University

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DEDICATION

This thesis is dedicated to those exceptional teachers I have had the honour and privilege of having learned from. Curiosity is a gift that the mundane all too easily beats out of us as we grow into this world, and for me to still be consumed by the love of the question, I owe a great debt that I can only hope to repay others one day. To my parents, Shubha and Prakash Patil, who planted all of the early seeds and watered them with unconditional love and support. To Jonathan Straker, who first piqued my curiosity in mathematics in primary school. To Roger Costin and John Allum who managed to inspire and captivate a wayward teenager with science. To Jane Elliot for encouraging a healthy intellectual rebelliousness. To Robert Richardson, under whom I first acquired the tools, knowledge and self belief to be where I am today as a physicist. Not least, this thesis is dedicated to Robert Brandenberger, whose continued inspiration, kindness, guru-like presence, encyclopedic knowledge and keen scientific mind were only matched by his enthusiasm and skilled guidance in my research. Any successes that I might chance upon will be in no small part due to these remarkable individuals.

ACKNOWLEDGEMENTS

During the third year of my PhD, incredibly difficult external circumstances nearly prevented me from being able to return to the United States, where this thesis work commenced at Brown University. I thank all of those who helped me during this period, foremost Robert Brandenberger and Dave Cutts at Brown University for their unconditional and untiring efforts on my behalf, Bill Spence at Queen Mary University of London and Sandip Trivedi and Rudra Pratap Jena at the Tata Institute in Mumbai for helping me to continue my work. I especially thank Camilla Schofield and my parents for being there for me in every way possible.

ABSTRACT

This manuscript based thesis explores the consequences of string gas cosmology for the moduli problem in string theory. We compile three published papers which set up the formalism and technique of massless string gas cosmology as developed by the author in collaboration with Robert Brandenberger, and explores the consequences of such a framework for late time cosmology. We find that when consistently coupled to the massless background of string theory (given a fixed dilaton), a fluid of massless strings can stabilize all radial and shape moduli of a toroidal compactification of bosonic or heterotic string theory. This stabilization mechanism is consistent with observational bounds coming from fifth force experiments and reproduces all of the desirable features of late time cosmology. We identify the open issues associated with the string gas framework as it stands, and propose directions for future work. We also investigate the possibility that string gas cosmology might offer a dynamical solution to the moduli problem.

ABRÉGÉ

Cette thèse basée par manuscrit explore les conséquences de la cosmologie de gaz des cordes pour le problème de modules dans la théorie des cordes. Nous compilons trois papiers publiés qui installent le formalisme et la technique de la cosmologie de gaz des cordes sans masse comme développée par l’auteur en collaboration avec Robert Brandenberger, et explorons les conséquences d’un tel cadre pour la cosmologie de temps tardives. Nous constatons qu’une fois uniformément couplé au fond sans masse de la théorie de corde (donnée un dilaton fixe), un fluide des cordes sans masse peut stabiliser tous les modules de radial et de forme d’un compactification toroidal de théorie bosonic ou hétérotique de corde. Ce mécanisme de stabilisation est conforme aux limites d’observation venant des expériences de force cinquièmes et reproduit tous les dispositifs souhaitables de cosmologie de temps tardives. Nous identifions les issues ouvertes liées au cadre de gaz de corde sans modification, et proposons des directions pour les travaux futurs. Nous étudions également la possibilité que la cosmologie de gaz de corde pourrait offrir une solution dynamique au problème de modules.

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CHAPTER 1

Introduction

1.1 Preliminaries

As a putative theory of quantum gravity, string theory implies radical reformulations of our notions of space and time at the smallest imaginable scales. In fact, our hitherto familiar conceptualizations of geometry no longer seem to apply once we take seriously the premise that all matter and force, and even spacetime itself, are generated by the dynamics of extended objects. From the string spacetime uncertainty relation [1], to the appearance of non-commutative geometries in non-trivial backgrounds [2], to the notion that spacetime itself is an emergent entity dynamically generated from some underlying quantum geometry or matrix model (see [3] for a review), we see that string theory implies radical departures from classical geometry in a wide variety of contexts.

Clearly, if cosmology is our domain of interest, then modifications forced by string theory upon our model of spacetime as a classical four dimensional Lorentzian manifold, poses at once many threats and many opportunities. Among the opportunities include new possibilities for resolving the initial singularity, new models in which to realize inflation, and new mechanisms to explain what the missing matter and energy components of the universe might be, and how they came to be arranged as they are today, to name but a few. The threats arise when we realize that the low energy consequences of string theory must not spoil the successes of standard big bang cosmology. One example of such a potentially troublesome feature is in the fact that string theory predicts a number of

extra dimensions, which upon compactification generate a plethora of scalar fields without potentials (known as moduli fields). These moduli fields will mediate gravitational strength fifth forces, which have not been observed in any experiment to date. This problem, which we will elaborate upon further is known as the moduli problem, and is the question confronting string theory that the subject of this thesis (string gas cosmology) hopes to address.

In addition to predicting the existence of extra dimensions, perhaps one of the more striking features of string theory, and one that is present even in its low energy effective limits is the existence of new symmetries (dualities) relating physics in distinct geometries (see [4] for a review). Having first been noticed by Buscher in the context of the low energy effective action for a massless string background [5], T-duality (as an example of such a duality) will be a full symmetry of any non-perturbative formulation of string theory. In a background with all anti-symmetric form fields set to zero, T-duality acts on the background metric as

$$G_{\mu\nu} \rightarrow G_{\mu\nu}^{-1}/\alpha'. \quad (1.1)$$

In a toroidal background with all dimensions compactified on a circle of radius R , T-duality implies that all string S-matrix elements map directly onto those calculated on a background of radius $1/R$, provided we account for the appropriate rearrangement of the degrees of freedom. The canonical example of such a rearrangement is that under the action of T-duality, winding states and momentum states map onto each other. Evidence for this is seen in the fact that the energy for a string wound w times around a cycle of the torus ($E \sim wR/\sqrt{\alpha'}$) is equivalent to the energy of a mode with

n quanta of momentum ($E \sim n/R$) under T-duality ($R \rightarrow \sqrt{\alpha'}/R$), provided we interchange w and n .

Inspired by the observation that T-duality effectively implies a minimum length scale to the universe (and as a result a maximal temperature), Brandenberger and Vafa (BV) [6] proposed a model of the early universe which began at this minimal length (the self dual radius— $R = \sqrt{\alpha'}$)¹ populated with a gas of strings excited close to the Hagedorn temperature [8]. Such a state initially loiters around the self dual radius due to the competing effects of winding modes, which like to halt expansion, and momentum modes, which like to source expansion. However, due to the tree level interaction wherein oppositely wound modes annihilate into unwound string states, a stable population of wound strings might not be possible unless certain conditions are met. As argued by BV, as one dimensional extended objects, wound strings are guaranteed to interact in all spatial dimensions up to three (ignoring for now the effects of long range forces between strings— see [9][10] for rather negative conclusions for the BV dimension counting argument once interactions are accounted for). Therefore in at most three dimensions can wound strings annihilate and so that the relevant dimensions become free to expand— that is, as a consequence of contact string interactions in the early universe, the big bang happens in at most three spatial dimensions.

Thus in the course of invoking T-duality in a model of the early universe, Brandenberger and Vafa proposed a dynamical mechanism to

¹ An assumption justified in [7] where a background solution starting in the Hagedorn phase, exiting to a radiation dominated phase was explicitly constructed.

generate a universe with three large spatial dimensions, and in doing so initiated what became known as the string gas cosmology program. Brandenberger and Vafa argued that instead of viewing the extra dimensions inevitable to string theory as having to be compactified, it may be more natural to view our macroscopic dimensions as having *decompactified*. In this way, the BV mechanism explains away the presence of extra dimensions as well as any supposed string compactification would. Recently however, doubts have emerged regarding the validity of the BV mechanism [9][10] when string exchange processes are accounted for in the interaction physics of wound strings². However, even if we ignore/ take for granted the details of the initial compactification/ decompactification mechanism, a more pressing problem looms imminent. All compactification/ decompactification models still have to subsequently address how these extra dimensions are kept at an unobservable scale at late times in a way that is consistent with fifth force experiments and consistent with the successes of big bang cosmology. Failure to do so would be the first and foremost place the low energy consequences of string theory would conflict with experiment. This is the moduli problem in the context of string theory in a nutshell.

² The key physics of these negative results appear to be the dynamics of the dilaton. If we fix the dilaton by hand, then we can evade such conclusions [11], this however, is a highly non-trivial assumption in most contexts. This can be partially understood in intuitive terms in the context of the long range forces mediated by massless fields, such as the dilaton. Such forces are of infinite range and certainly do not respect the dimension counting arguments of BV. For instance, we know in 3 dimensions that the scattering cross section of a Coulombic potential is infinite.

1.2 The Moduli Problem

As an introduction to the moduli problem, we consider it first in the simplest context possible. Consider the following action describing general relativity in $4 + 1$ dimensions with a compact 5^{th} dimension:

$$S = \int d^5x \sqrt{-G} \left(\frac{1}{16\pi G_5} R[G] + \mathcal{L}_M \right). \quad (1.2)$$

If our metric tensor takes the form $G_{44} = e^{2\sigma}$, $G_{4\mu} = 0$, $G_{\mu\nu} = g_{\mu\nu}$, with $\mu = \{0, 1, 2, 3\}$, and if there is no dependence of any metric quantity on x^4 , then we can readily integrate over the fifth dimension in the above to yield the following effective action:

$$S = \int d^4x \sqrt{-g} e^\sigma \left(\frac{1}{16\pi G_4} R[G] + \mathcal{V} \mathcal{L}_M \right), \quad (1.3)$$

with the four dimensional Newton's constant G_4 related to G_5 through the relation $G_4 = G_5/\mathcal{V}$, where $\mathcal{V} = \int dx^5$ is the volume of compact coordinate interval. This action now appears as a scalar tensor gravitational action. In order to understand how matter interacts with gravity in this compactification, it is necessary to perform the conformal transformation

$$g_{\mu\nu} = e^{-\sigma} \tilde{g}_{\mu\nu}, \quad (1.4)$$

under which the Ricci scalar transforms as:

$$R = e^\sigma [\tilde{R} - 3\tilde{\nabla}^2 \sigma - \frac{3}{2} \tilde{\nabla}_\mu \sigma \tilde{\nabla}^\mu \sigma]. \quad (1.5)$$

Substituting the above into (1.7) and integrating by parts leaves us with the resulting 4-d effective action

$$S = \int d^4x \sqrt{-\tilde{g}} \left(\frac{1}{16\pi G_4} (R[\tilde{g}] - \frac{3}{2} \tilde{\nabla}^\mu \sigma \tilde{\nabla}_\mu \sigma) + \mathcal{V} e^{-\sigma} \mathcal{L}_M \right). \quad (1.6)$$

We see that this action describes the usual Einstein gravity with a scalar sector which has gravitational strength coupling to the matter sector. To begin to see the problem induced by the presence of this modulus field, note that the Einstein equations now become:

$$G^\mu_\nu = (T_\sigma)^\mu_\nu + 8\pi G_4 e^{-\sigma} \mathcal{V}(T_M)^\mu_\nu, \quad (1.7)$$

where the subscripts on the energy-momentum tensors denote the constituent fields. Firstly, we note that the contribution of the matter sector is Planck suppressed compared to the contributions of the kinetic terms of the modulus field. Hence if the modulus field is rolling quickly at a large enough radius, the energy-momentum sourcing expansion will be dominated by a $\rho = +p$ fluid, which will yield a cosmological evolution inconsistent with any epoch of the known history of the universe.

If we now focus purely on the matter sector, we see that the prefactor $e^{-\sigma}$ induces Yukawa interactions between material particles, as any fermionic mass term is modified as

$$m_\psi \bar{\psi}\psi \rightarrow m_\psi e^{-\sigma} \bar{\psi}\psi. \quad (1.8)$$

Such a Yukawa coupling will in general mediate a long range force of gravitational strength between ψ particles:

$$F_{\psi\psi} \propto \frac{1}{r}, \quad (1.9)$$

unless of course, a mass term for the modulus field is generated, in which case the force law acquires a finite range:

$$F_{\psi\psi} \propto \frac{e^{-m_\sigma r}}{r}, \quad (1.10)$$

where m_σ is the mass of the modulus field. So unless a large enough mass is generated for the modulus field, we stand to not only spoil the successful predictions of big bang cosmology, but also to introduce interactions which violate the equivalence principle. Although this has not been ruled out as a matter of principle, all experiments to date have placed stringent bounds on such violations [12]. Thus it seems that if moduli fields exist, as necessarily implied at low energies by string theory, we must work out some mechanism which stabilizes them.

The standard approach in string theory (see [13] for a review) has been to turn on fluxes for the various form fields present at any corner of moduli space, which generate potentials for the various complex structure moduli (which describe the dilaton and the shape moduli of the internal manifold in question), and to consider non-perturbative effects such as gluino condensation to generate potentials for the overall volume modulus. Such a prescription has motivated various string cosmological scenarios for the early universe such as stringy models of inflation [14], as well as its logical corollary— the predicted existence of a vast landscape of vacua [15]. The focus of this thesis however, is on an unrelated approach to the moduli problem in string theory coming from string gas cosmology, which arises from studying the effects of massless strings on the evolution of the universe. Such an approach might allow for a natural (i.e. somewhat unconstructed) solution of the moduli problem as we wish to motivate presently.

Consider the bosonic sector of the low energy effective action of string theory:

$$S = \frac{1}{2\kappa_0} \int d^{D+1}x \sqrt{-g} e^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right), \quad (1.11)$$

where Φ is the dilaton, D is the number of spatial dimensions, $H_{\mu\nu\lambda}$ is the field strength associated with the two form field $B_{\mu\nu}$ and κ_0^2 is the dimensional normalization of the action. Such an action imposes as its equations of motion, the conditions that arise when we demand that the first order in α' truncation of the one loop beta function of string theory vanishes³. In general, higher orders in α' generate higher derivative corrections to the low energy effective action, but we will not be needing these. The reason for this is that the above describes a consistent background on which one can study the propagation of massless string modes, and only massless string modes, and as such is dubbed the ‘massless background’ of string theory. This action remains valid so long as we stay within curvature regimes such that

$$\alpha' R \ll 1, \tag{1.12}$$

where R is the Ricci scalar of our background at any spacetime point. Since we know that massive string states have an energy which scales as

$$m \sim \frac{N}{\sqrt{\alpha'}}, \tag{1.13}$$

where N is some integer, we see that provided that (1.12) is satisfied, massive string states will not be created by the curvature of the background, and whatever massless strings we have around, will not back-react on the geometry such that (1.12) is violated. Hence it is consistent only to couple (1.11) to a matter content constituted by massless strings in various states.

³ Thus ensuring us an anomaly free quantum theory. We assume that D is the necessary critical dimension.

This is the philosophy of string gas cosmology as espoused in [16][17][18]—if it is consistent to couple massless strings to (1.11), why don't we? As we shall see in later chapters, it turns out that when we couple a massless string gas to (1.11), we arrive at a minimally tuned and rather natural candidate for a solution to the moduli problem in string theory, certain features of which we advertise presently.

Firstly, we note that a gas of massless strings will only have one tunable parameter— the average energy of its component strings. The requirements of masslessness of its constituents, consistency with isotropy and overall charge conservation, cancel out all other quantum numbers which might be ascribed to the macrostate of the gas. Secondly, we note that the action associated with our string gas (which will be derived from microphysical considerations later on) appears as:

$$S = - \int d^{D+1}x \sqrt{-g_{00}} E[g_{ij}], \quad (1.14)$$

where E is the energy of the string gas and g_{ij} are the spatial components of our energy-momentum tensor, and that this is to be contrasted with

$$S = - \int d^{D+1}x \sqrt{-g} V(\phi), \quad (1.15)$$

where V is some potential generated for ϕ , which is some modulus field.

We immediately infer from (1.14) that such an action will not quite be seen by gravity the same way it sees (1.15). Indeed, the minimum of (1.15) will be seen by gravity as a cosmological constant, and will thus require some tuning to achieve consistency with observation, whereas no such tuning will be required of (1.14). In addition, we note that the cosmological energy density and the effective mass of the modulus field generated by (1.15) go as

$$m_\phi^2 \sim V''(\phi) \ , \ \rho \sim V(\phi). \quad (1.16)$$

Thus we see that we will typically require some tuning to simultaneously make m_ϕ^2 large enough so as to avoid fifth force constraints, without making $V(\phi)$ too large such that it would lead to unacceptable overclosure of the universe. Conversely, as we will soon find out, a massless string gas generates an effective mass for a given modulus field and has an associated energy density which scale as

$$m^2 \sim \frac{\mu_0}{E a^3} \ , \ \rho \sim \frac{\mu_0 E}{a^3}, \quad (1.17)$$

where E is the energy of the string gas and μ_0/a^3 is the number density of strings. In this case, it is relatively easy to tune the moduli masses to be large enough without introducing too much energy density into the universe.

A key aspect of the physics underlying moduli stabilization with string gases is the fact that strings are extended objects, and hence have the potential to feel out the metric tensor in a non-trivial manner, resulting in non-standard ‘potentials’ such as (1.14). Another crucial aspect of the relevant physics is that we are using only massless string states, as massive string states (aside from being inconsistent to use on our dilaton gravity background) do not appear to share similar properties as far as moduli stabilization is concerned [19][20]. In fact, until the importance of massless string states was emphasized in [16][17][18], such negative results for moduli stabilization using string gases were taken to be a general conclusion. The use of massless states for dynamical moduli stabilization was also emphasized in [21], and it appears at present that massless states which appear at specific enhanced symmetry points in various string compactifications might one day offer a window towards a dynamical

solution to the moduli problem in string theory, an issue we elaborate upon next.

1.3 Enhanced Symmetries and String Gas Cosmology

If we consider any particular uncompactified version of string theory, we notice that the string spectrum contains a massless sector corresponding to a rank two symmetric traceless tensor, a rank two anti-symmetric tensor and a scalar— the graviton, the two form field and the dilaton respectively. Setting the form fields to be vanishing, we first notice that compactification changes the spectrum of our theory. For instance, if we compactified on a d -dimensional torus, the possibility of non-trivial winding and momentum quantum numbers will alter the masses of certain states as the radius of compactification is changed. Although the string spectrum will always contain states that remain massless independent of the details of the compactification (corresponding to graviton excitations along the non-compact dimensions), it turns out that at special points, certain states that otherwise were massive become massless. This is seen in the context of Bosonic string theory by the mass formula for a string propagating on a manifold that has had d dimensions toroidally compactified [22]:

$$m^2 = (n, \gamma^{-1}n) + \frac{1}{\alpha'^2}(w, \gamma w) + \frac{1}{\alpha'}[2(n, w) + 4(N - 1)], \quad (1.18)$$

where γ_{ab} , $1 \leq a, b \leq d$, is the metric on the torus, n^a and w_a are respectively the momentum and winding quantum numbers along the a^{th} direction, $(,)$ is the real scalar product and N is the right moving oscillator level of the string state. Implicit in the above is that we have solved for the left moving oscillator level $\tilde{N}$ through the level matching constraint:

$$\tilde{N} = N + (n, w). \quad (1.19)$$

Clearly, for any form for the toroidal metric, we have massless states at $N = 1$, $n^a = w_a = 0$. This is the graviton sector of our theory (as well as the form field sector had we not set it to zero from the outset). However, we notice that at the self-dual⁴ radius ($\gamma_{ab} = \alpha' \delta_{ab}$), our mass formula becomes

$$\begin{aligned} \alpha' m^2 &= (n, n) + (w, w) + [2(n, w) + 4(N - 1)] \\ &= (n + w, n + w) + 4(N - 1), \end{aligned} \quad (1.20)$$

which indicates a whole bunch of massless states that are otherwise massive away from the self-dual radius. Such states satisfy $N = 1$, $n^a = -w_a$, or $(n + w, n + w) = 4$, subject of course to (1.19). These states will be systematically classified further on, but for now we simply note their appearance at the self dual radius. We arrive at the same conclusions in the case of Heterotic string theory, as the mass formula in this case is also given by (1.18)⁵.

If we now consider the energy-momentum tensor of a gas of strings that are massless at the self dual radius, we find that once coupled to gravity, a stabilizing force for all the radial moduli results (and as subsequently uncovered ([23][18], all shape moduli as well). These are the main results of the first two papers included in this thesis [16][17], and is the basis

⁴ Self dual in the sense of T-duality

⁵ Although the zero point contribution from the right movers vanishes in the case of the Heterotic string, there is a compensating term in the Heterotic level matching constraint such that (1.18) still results.

behind the claim that a gas of strings naturally offers us a solution to the moduli problem. Specifically, were we to begin near the self-dual radius, the dynamics of massless string states which are likely to condense at this point in moduli space will likely keep us there.

The results of [16] thus anticipated a part of the observations made in [21] that ‘beauty is attractive’, namely that points of enhanced symmetry might serve as dynamical attractors in moduli space. As shall become clear in the following chapters, the specific properties of massless string states (aside from being the a priori only consistent matter content for a string gas) appear to be so crucial in obtaining a stabilizing force for the moduli fields ⁶, that we might wonder if the results of string gas cosmology, like the results of [21] have touched upon a dynamical selection principle in moduli space, that enhanced symmetry points are the only places where our low energy effective universe might end up. As we shall see in the next section, the answer is only partly in the affirmative. By explicit construction, we can demonstrate the existence of enhanced symmetry points which are actually unstable points for the evolution of the moduli fields. Instead, it appears to us that the true principle that might account for an eventual mechanism for dynamical vacuum selection appears to be that of duality invariance.

1.4 Not Just Enhanced Symmetry Points

Even in the context of heterotic string theory, we know that there are points other than the self dual radius where extra massless states appear. There are in fact an infinite number of such points [22][24]. However, the majority of these will be rather uninteresting to us, as most of these do

⁶ In that massive states would cause destabilizing terms to appear once their energy-momentum tensor is derived properly.

not function as stable points for the dynamics of moduli fields from the perspective of string gas cosmology. For instance, consider for simplicity the case of one extra dimension compactified on the circle ⁷. According to the mass formula (1.18) applied to this case, there are an infinite number of special points where new string states become massless. Consider for instance, when the size of the circle is some multiple of the self-dual radius, $\gamma_{11} := b^2 = \alpha' \lambda^2$, so that the mass formula becomes

$$\alpha' m^2 = \left(\frac{n}{\lambda} + \lambda w\right)^2 + 4(N - 1), \quad (1.21)$$

where again, this is subject to the level matching constraint $nw + N \geq 0$. In the next chapter[16], we will classify all the enhanced symmetry points[16] and their associated massless states. We find that for any $m \in \mathbb{Z}$, massless states appear at the radii

$$\begin{aligned} b &= \frac{2}{|m|} ; w = \pm m, n = 0, N = 0 \\ b &= \frac{|m|}{2} ; n = \pm m, w = 0, N = 0. \end{aligned} \quad (1.22)$$

When coupled to gravity, these states turn out to generate a driving term for the scale factor b , which drives expansion of the scale factor away from the enhanced symmetry point if b is greater than the self dual radius, and drives contraction if b is below the self dual radius. Hence the self dual radius appears to be a specially preferred enhanced symmetry point as far as moduli stabilization with string gases is concerned. This is just as

⁷ In this way we avoid the complicated combinatorics associated with identifying and summing over all given massless modes.

well, for if any enhanced symmetry point were to be as good as any other as far as moduli stabilization is concerned, then string theory would be in big trouble, as to invoke such points as the preferred vacua of string theory would imply that string theory has lost *all* predictability due to the infinite number of these (compared to the astronomically large, but finite number of flux vacua [15]). Instead, it turns out that in the simple context we are considering, only the enhanced symmetry point which is also a duality fixed point is stable. In [25] this was explored in the context of toroidal heterotic compactifications, where other fixed points of the duality transformations were considered and were shown to be stable points in moduli space. Towards the conclusion of this thesis, we investigate the possibility that string gas cosmology suggests its own landscape with its own dynamical selection principle, namely that our low energy universe is likely to dynamically end up only on duality fixed points in moduli space. This is perhaps the most promising aspect of string gas cosmology, and one that should motivate continued research into its formalism and consequences for cosmology.

1.5 Progress and Problems— a Review

We presently offer a review of moduli stabilization in the context of string gas cosmology⁸. This is in fact a rather recent and somewhat small subset of the overall string gas cosmology program, whose focus until about 2003 was mainly on elaborating upon the BV mechanism [6] as the origin of our three large spatial dimensions. We note parenthetically that utilizing

⁸ In particular, we do not focus at all on brane gas cosmology and its associated applications. For a general review of string/brane gas cosmology, we refer the reader to [26].

string gases for the purposes of moduli stabilization at late times does not depend on the BV mechanism, although it does provide a very natural motivation for them. In fact, it was from noticing that the initial loitering phase of string cosmology in the Hagedorn phase arose from competing pressures resulting from momentum and winding modes, that led Watson and Brandenberger in 2003 [27] to wonder if they could be used in a late time setting to stabilize extra dimensions. It was discovered that a gas of winding modes, superimposed on a gas of momentum modes, both treated classically, could stabilize extra dimensions in the string frame. In [16], the energy-momentum tensor for a string gas was explicitly derived from microphysical considerations, and it was discovered that when treated quantum mechanically, a gas of massless non-interacting strings creates a stabilizing force for one extra dimension in a way that is consistent with observational bounds. That is, it is easy to make an all-pervasive fluid of massless strings wrapping all extra dimensions at any point in space consistent with big bang cosmology.

In general, as discussed in [16][17], it is difficult to make moduli stabilization in string gas cosmology consistent with inflation unless wound strings are created by some form of stringy reheating. The crucial role of massless states in moduli stabilizing would become clear as negative results concerning moduli stabilization using only massive modes in extra dimensions were uncovered in [19]. As was shown in [17], massless string gases can stabilize any number of toroidal dimensions at the string scale. The negative results of [19] are an artifact of using the classical energy-momentum tensor for a string gas (and hence restricting only to massive string states). Similar negative results for using string gases in conjunction with potentials to stabilize the dilaton simultaneous with the radial moduli of the extra

dimensions [20], were shown to disappear once we focus on massless string states [18]. Shortly after [27] and [16], Watson proposed an abstraction from the string gas program, wherein moduli stabilization at enhanced symmetry points arose in the low energy effective field theory description through the stringy Higgs mechanism [28]. In [23], and subsequently confirmed in [18] shape moduli (and consequently the background $B_{\mu\nu}$ field) were turned on and it was discovered that these are also stabilized around vanishing values by a massless string gas. In [18] the effects of adding a D-string gas was considered and it was found that if one could trust the background at the self S-dual and the self T-dual points, then string gases could stabilize all moduli fields. However the conclusions of this paper are to be qualified by the realization that in general, unless we are at weak coupling, the string spectrum is not to be trusted. To be at the self S-dual point is to be in a strong coupling regime. The issue of S-duality in conjunction with T-duality was also studied in [29][30]. As confidence grew in the ability of a gas of massless strings to stabilize extra dimensions in a toroidal context, wider issues of concern to cosmology were addressed in [31], where the effects of spatial inhomogeneities and metric perturbations on a cosmology sourced by massless string modes were studied (as [32][33] did for the case of massive string modes).

Since this flurry of work, moduli stabilization in string gas cosmology has stumbled upon its lack of success in stabilizing the dilaton [34][18][35], and this remains one of its outstanding challenges. In addition, string gas cosmology does not appear to naturally connect with any inflationary models. This last point however served to motivate the authors of [36] (see also [37][38][39]) to propose a string gas cosmology alternative to

structure formation⁹. In addition, questions concerning the consistency of the effective field theory treatment arise when we stabilize at the self-dual radius (even if we remain consistent with the α' expansion by not inducing dangerous curvatures). We will return to all of these outstanding issues after we have familiarized ourselves with string gas cosmology explicitly. We will also assess the future prospects of string gas cosmology in the face of these open issues and identify directions for further investigation.

1.6 Outline

We begin this manuscript based thesis with our first paper on string gas cosmology [16]. In this paper, we first derived the correct energy-momentum tensor for a string gas treated quantum mechanically. In this way, we derive the form of the energy-momentum tensor for massless states and discover their stabilizing properties at the self dual radius. We discuss towards the end of the paper consistency with observational bounds if this is to be the relevant moduli stabilization mechanism at work in the universe at present. Initially motivated by the negative results of [19] when considering massive string states (or string gases without fully accounting for the quantum nature of the energy-momentum tensor), our next paper [17] generalizes the mechanism to any number of extra dimensions, again exploring issues of consistency with observational bounds. In the third paper included in this thesis [31], we consider the implications of stabilizing extra dimensions with a string gas for metric perturbations. In particular, we show that the inhomogeneous fluctuations of extra dimensions stabilized with a string gas are stable, and that any initially scale invariant spectrum in five dimensions is undistorted by the transients of the stabilization

⁹ Although the flatness problem remains unanswered by such a model.

mechanism and results in a scale invariant spectrum in four dimensions. We also take a five dimensional universe with the fifth dimension stabilized with a string gas as a model for a non-singular bounce in the context of ekpyrotic/ cyclic universe scenarios, and show there that one does indeed obtain a controlled transfer of metric fluctuations accross the bounce. In our concluding chapter, we discuss open issues and promising future directions for research in string gas cosmology.

CHAPTER 2

Contributions of Authors

The first two papers included in this thesis were co-authored by the author and Robert Brandenberger (RB). The work contained therein was mostly done independently by the author under the guidance of RB. The author undertook the majority of the labour, and the initial ideas, although directly inspired by RB, were his. The third paper included in this thesis was done in collaboration with RB and Thorsten Battefeld (TB). Some of the work in this paper concerning the stability of extra dimensions to inhomogenous perturbations were uncovered by the author during the time in which [16] was in preparation, however the initial idea for this paper arose in a collaborative discussion and the labour was evenly divided between the author and TB under the guidance of RB. In addition to the work on moduli stabilization presented in this thesis, the author also worked on the string gas cosmology alternative to structure formation [37][38][39] in collaboration with RB, Ali Nayeri and Cumrun Vafa. We also take note that during his thesis the author has also worked on topics unrelated to string gas cosmology, such as perturbations in bouncing brane models of structure formation [40], $D0$ matrix mechanics as a model for topology changing D-brane dynamics [41][42], and kinetic inflation [43].

CHAPTER 3

Radion Stabilization by String Effects in General Relativity

3.1 Abstract

We consider the effects of a gas of closed strings (treated quantum mechanically) on a background where one dimension is compactified on a circle. After we address the effects of a time dependent background on aspects of the string spectrum that concern us, we derive the energy-momentum tensor for a string gas and investigate the resulting space-time dynamics. We show that a variety of trajectories are possible for the radius of the compactified dimension, depending on the nature of the string gas, including a demonstration within the context of General Relativity (i.e. without a dilaton) of a solution where the radius of the extra dimension oscillates about the self-dual radius, without invoking matter that violates the various energy conditions. In particular, we find that in the case where the string gas is in thermal equilibrium, the radius of the compactified dimension dynamically stabilizes at the self-dual radius, after which a period of usual Friedmann-Robertson-Walker cosmology of the three uncompactified dimensions can set in. We show that our radion stabilization mechanism requires a stringy realization of inflation as scalar field driven inflation invalidates our mechanism. We also show that our stabilization mechanism is consistent with observational bounds¹.

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3.2 Introduction

In the early days of string cosmology, it was realized that superstrings had an effect on space-time dynamics that was qualitatively quite different from that of particles or fields. In particular, it was realized that string winding modes could provide a confining mechanism for certain compact directions in such a way as to allow only three spatial dimensions to grow large [6]. Key to this realization are the T-duality of the spectrum of string states, and the fact that the background is described by Dilaton Gravity, and not by General Relativity with a fixed dilaton (this is crucial in order that the background equations obey the T-duality symmetry). The arguments of [6] were put on a firmer basis by the analysis of [7] (see also [44]).

Starting point of the considerations of [6] is the assumption that all spatial dimensions begin at close to the self-dual radius (the string scale), and that matter consists of a hot gas of string states. The considerations of [6] were more recently applied to “brane gas cosmology” [45, 46], a scenario in which the initial string gas is generalized to be a gas of all brane modes. It was shown that given the hot dense initial conditions assumed in [6], the string winding modes are the last modes to fall out of equilibrium and thus dominate the late time dynamics. Hence [45], the inclusion of brane degrees of freedom does not change the prediction that only three dimensions grow large. The dynamical equations describing the growth of the three dimensions which can become large were solved in [46] (see also [47]). In [48], it was shown that isotropy in these large dimensions is a consequence

by the American Physical Society. This article can be accessed online at <http://link.aps.org/abstract/PRD/v71/e103522>.

of the dynamics. In [27], it was found that if both the momentum and winding modes of the strings are included in the dynamical equations, the radius of the compactified dimensions is stabilized at the self-dual radius. More precisely, the expansion of the three large dimensions leads to damped oscillations in the “radion” about the self-dual value. Thus, in the context of a background described by Dilaton Gravity, radion stabilization is a natural consequence of brane gas cosmology ².

At the present time, however, the dilaton is most likely fixed (see, however, [56] for an alternate scenario). Thus, it is of interest to explore how the inclusion of string (and brane) winding and momentum modes influences the dynamical evolution of the radion in a background space-time described by General Relativity (GR). There is another motivation for studying this issue. Another corner of the M-theory moduli space is 11-d supergravity. In [57] it was found that a brane gas in this background also admits a region in the phase space of initial conditions in which only three spatial dimensions can become large, although this corner may not be consistent with holographic entropy bounds [58] (see also [59] where the considerations in this corner of M-theory moduli space was extended to spaces with more general topologies). Motivated by these considerations, we in this paper study a simplified problem, namely the questions of how a gas of winding and momentum modes of strings winding one compactified spatial dimension (taken to be a circle) effects the evolution of the radius (the radion). We start with initial conditions in which the three spatially non-compact

² See e.g. [49][50][51][52][53][54][32] for other recent papers on brane gas cosmology, and [55] for an early work on the cosmology of string winding modes.

dimensions are expanding. We find that the gas of string winding and momentum modes gives a natural radion stabilization mechanism. Our approach is to consider the effect of strings on 5D space-time dynamics (with the extra spatial dimension compactified to a circle) by adding the appropriate matter term to the standard Einstein-Hilbert action. We will derive this term shortly (see also [60] for a similar derivation). The resulting energy-momentum tensor leads to a novel behavior when inserted into the Einstein equations. We will find that we can generate a non-singular bouncing solution for the radius of the compactified dimension in the context of GR (without a dilaton) while respecting the Dominant Energy Condition for the matter content. Specifically, the radion performs damped oscillations about the self-dual radius. Initially, we study a pure state of matter with specific quantum numbers obeying the T-duality symmetry. However, we will find that we can rather naturally extend the analysis to a gas of these strings in thermal equilibrium (with a bath of gravitons and photons), with the result that the radius of the compact dimension is dynamically stabilized at the self dual radius $R = \sqrt{\alpha'}$, where $2\pi\alpha'$ is the inverse of the string tension (see also [61] for a study of string gases in thermal equilibrium).

In addition, we find that our model evolves according to standard Friedmann-Robertson-Walker (FRW) cosmology after the compact dimension has been stabilized, and that the resultant stabilization is incompatible with any subsequent inflationary epoch driven by a bulk scalar field (for string-specific ideas on how to generate inflation in brane gas cosmology see [62]). However this conclusion can be avoided if some form of stringy inflation is realised where strings are produced in re-heating.

Before we can turn to any of this, we will have to address a question of principle concerning the string spectrum in a cosmological context (this issue is also being studied in [63]). The question of formulating String Theory in a time-dependent background is a current and active area of research. However, we are primarily interested in the behavior of strings in a background that evolves on a cosmological time scale. As can be seen from the FRW equations, the cosmological time scale H^{-1} (where H is the Hubble expansion rate) is larger than the characteristic microscopic time σ^{-1} (where σ^4 is the matter energy density) by a factor of m_{pl}/σ , where m_{pl} is the Planck mass. Thus, away from singular epochs in the history of the Universe, the cosmological time scale is going to be many, many orders of magnitude longer than the characteristic time scale of the string dynamics, and hence we should be able to inherit many of the features of the string spectrum in a static space-time (with some obvious modifications). We justify this intuition more rigorously in the Appendix, but we feel that it might suffice at this point to remind the reader of the approximate irrelevance of a time dependent background for a much more familiar theory: Quantum Field Theory (QFT). Although quantum fields in curved spaces exhibit several qualitatively different features from quantum fields in flat spaces³, we still manage to do a lot of sensible (and spectacularly successful) flat space-time QFT calculations despite the persistent Hubble expansion of space-time. The reason for this is easy to see: the contributions to masses, to scattering amplitudes, to the structure of the Hilbert space of our theory, etc., that come from terms that depend on derivatives of the

³ Examples are particle creation, non-uniqueness of the vacuum, non-trivial issues concerning existence of asymptotic states.

metric are in the present epoch highly suppressed and irrelevant. This is partly captured by the Adiabatic Theorem, which is the statement that given two systems with Hamiltonians that can be continuously interpolated, then in a precise sense, the eigenstates of the initial system will evolve into the eigenstates of the final system if this interpolation takes place slowly enough. Slow enough in simple quantum systems usually means that the variation happens over much longer time scales than the characteristic time of the system (by which we mean the time associated with the typical energy of the system: $\tau \sim \frac{1}{E}$). Having said this, were we to study QFT in places where the metric varies a lot more rapidly (at the edges of black holes or in the very early Universe) we invariably have to account for the curvature of space. Thus, we can hope that the effects of a time dependent background on the closed string spectrum only require minor modifications to the flat space spectrum, if this time dependence is slow compared to the characteristic time of the string dynamics. We show in the Appendix that this is indeed the case, and in what follows we will stay within this regime.

The Outline of this paper is as follows: we first derive the energy-momentum tensor of a string gas (the derivation here is more general than the one given in [60]). We then insert this tensor into the Einstein equations and study the dynamics of the radius of the compact dimension, assuming that the three large spatial dimensions are in the expanding phase. First, we consider a pure state of matter. Next, we extend the discussion to a thermal state. In Section IV, we discuss the late time dynamics and show that the stabilization of the radion is not compatible with inflation in the three large spatial dimensions, assuming the simplified description of matter which we are using.

A few words on our notation: Greek indices typically stand for 5-dimensional space-time indices, Roman indices $i, j, \dots$ are associated with the non-compact spatial dimensions, and Roman indices $a, b, \dots$ are string world-sheet coordinates. The 5-dimensional Planck mass is denoted by M_{pl_5} (or M_5 in abbreviated form). We also work in natural units ($c = \hbar = k_B = 1$) where we pick energy to be measured in electron volts.

3.3 The Energy-Momentum Tensor

To study how a gas of strings affects space-time dynamics, we need to derive the energy-momentum tensor of such a gas. We begin by studying the energy-momentum tensor of a single closed string. Starting with the Nambu-Goto action

$$S_{NG} = \frac{-1}{2\pi\alpha'} \int d^2\sigma \sqrt{-h}, \quad (3.1)$$

where h_{ab} denotes the world sheet metric

$$h_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X) \quad (3.2)$$

(and h is its determinant), we see that any variation in the space-time metric $g_{\mu\nu}$ induces a variation in the induced world-sheet metric (where the unmatched indices indicate that we perturb only the $\lambda\beta$ component of the metric):

$$\begin{aligned} g_{\mu\nu}(X) &\rightarrow g_{\mu\nu}(X) + \underbrace{\delta_\mu^\lambda \delta_\nu^\beta \delta^D(X^\tau - y^\tau)}_{\delta g_{\mu\nu}} \\ h_{ab}(\sigma) &\rightarrow h_{ab} + \underbrace{\partial_a X^\lambda \partial_b X^\beta \delta^D(X^\tau - y^\tau)}_{\delta h_{ab}} \end{aligned}$$

Now, varying the Nambu-Goto action *with respect to the space-time metric* (performing a perturbation $\delta g_{\alpha\beta}$ which acts on the metric as given above) will give us the space-time energy-momentum tensor of a single

string:

$$\begin{aligned}\frac{\delta S_{NG}}{\delta g_{\lambda\beta}(y)} &= -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{ab} \delta h_{ab} \\ &= -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\lambda \partial_b X^\beta \delta^D(X^\tau - y^\tau).\end{aligned}$$

We must first discuss the meaning of the expression

$$\begin{aligned}\int d^2\sigma \delta^D(X^\tau - y^\tau) \\ = \int d^2\sigma \delta(X^0 - y^0) \delta(X^1 - y^1) \dots \delta(X^D - y^D).\end{aligned}$$

In order to change the variable of integration, we need to apply $d\sigma^a = \frac{dX^\lambda}{\partial_a X^\lambda}$ and sum over all the zeroes of $X^\lambda[\sigma] - y^\lambda$ when performing the integration. However, since we are considering modes winding one particular spatial direction, there are precisely two coordinates that are monotonic functions of the world-sheet parameters: X^0 being a monotonic function of σ^0 and X^D being a monotonic function of σ^1 (the D^{th} direction is taken to be compact). Thus,

$$\begin{aligned}d^2\sigma \delta^D(X^\tau[\sigma] - y^\tau) \\ = d\sigma^0 d\sigma^1 \delta(X^0[\sigma] - y^0) \delta(X^1[\sigma] - y^1) \dots \delta(X^D[\sigma] - y^D) \\ = \frac{dX^0}{|\dot{X}^0|} \frac{dX^D}{|X'^D|} \sqrt{-g_{00}g_{DD}} \\ \times \delta(X^0 - y^0) \delta(X^D - y^D) \delta^{D-2}(X^i - y^i),\end{aligned}$$

where we include the metric factors in the last line so that we can take the delta functions in X^0 and X^D to be properly normalized. With this result,

we get:

$$\begin{aligned}
\frac{\delta S_{NG}}{\delta g_{\lambda\beta}} &= -\frac{\delta^{D-2}(X^i - y^i)}{4\pi\alpha'\sqrt{-g^{00}g^{DD}}} \\
&\times \int \frac{dX^0}{|\dot{X}^0|} \frac{dX^D}{|X'^D|} \delta(X^0 - y^0) \delta(X^D - y^D) \sqrt{-h} h^{ab} \partial_a X^\lambda \partial_b X^\beta \\
&= -\frac{1}{4\pi\alpha'\sqrt{-g^{00}g^{DD}}} \\
&\times \frac{\delta^{D-2}(X^i - y^i)}{|\dot{X}^0| |X'^D|} \sqrt{-h} h^{ab} \partial_a X^\lambda \partial_b X^\beta \Big|_{X^0=y^0, X^D=y^D},
\end{aligned}$$

where we use the inverse metric to write the metric contributions in the denominator. Thus, the single string space-time energy-momentum tensor becomes

$$\begin{aligned}
T^{\lambda\beta} &= \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\lambda\beta}} \\
&= \frac{1}{2\pi\alpha'} \frac{\delta^{D-2}(X^i - y^i)}{|\dot{X}^0| |X'^D| \sqrt{-g}} \frac{\sqrt{-h} h^{ab} \partial_a X^\lambda \partial_b X^\beta}{\sqrt{-g^{00}g^{DD}}}.
\end{aligned} \tag{3.3}$$

Inserting the explicit form of the inverse world-sheet metric

$$h^{ab} = \frac{1}{h} \begin{pmatrix} h_{22} & -h_{12} \\ -h_{21} & h_{11} \end{pmatrix} = \frac{1}{h} \begin{pmatrix} X'^\mu X'_\mu & -\dot{X}^\mu X'_\mu \\ -\dot{X}^\mu X'_\mu & \dot{X}^\mu \dot{X}_\mu \end{pmatrix} \tag{3.4}$$

and using the constraints on the world-sheet fields ⁴

$$P_\mu X'^\mu = 0 \tag{3.5}$$

$$P_\mu P^\mu + X'_\mu X'^\mu = 0, \tag{3.6}$$

we can write (3.3) as

$$T^{\lambda\beta} = \frac{-1}{2\pi\alpha'} \frac{\delta^{D-2}(X^i - y^i)}{|\dot{X}^0| |X'^D| \sqrt{-\det g_{ij}}} [X'^\lambda X'^\beta - \dot{X}^\lambda \dot{X}^\beta] \tag{3.7}$$

⁴ Which come from working in conformal gauge $h_{ab} = \text{diag}(-1, 1)$

Next, we solve for $\dot{X}^0$ using the constraint (3.6) which becomes

$$\begin{aligned} 0 = & -\dot{X}^0\dot{X}^0 + \dot{X}^i\dot{X}_i + \dot{X}^D\dot{X}^D \\ & -X'^0X'^0 + X'^iX'_i + X'^DX'_D, \end{aligned}$$

where we have explicitly used the background metric

$$g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t), a^2(t), b^2(t)). \quad (3.8)$$

It is consistent with the equations of motion in a (slow enough) time varying background to set $X'^0 = 0$ ⁵, so that

$$\dot{X}^0{}^2 = P^iP_i + X'^iX'_i + P^DP_D + X'^DX'_D. \quad (3.9)$$

In addition (in a slowly time dependent background) the right hand side can be expressed in terms of the familiar oscillator expansion. Accounting for the zero mode operators explicitly, we get the center of mass momentum from the spatial zero modes and the winding energy from the zero mode terms in the compactified direction. The other modes give us the left and right moving oscillator terms (see [22] for details):

$$\dot{X}^0 = \sqrt{g^{ij}p_ip_j + \frac{2}{\alpha'}(N + \bar{N} - 2) + \left(\frac{n}{b}\right)^2 + \left(\frac{wb}{\alpha'}\right)^2}, \quad (3.10)$$

where n and w are the quantum numbers for momentum and winding in the compact direction, respectively, and N and $\bar{N}$ are the levels of the left- and right-moving oscillator modes of the string, respectively. The expression above is none other than the energy of the string. Using the level matching

⁵ See the Appendix for all statements made in this section concerning results that are valid in a time dependent background

constraints

$$N + nw - \bar{N} = 0, \quad (3.11)$$

we finally end up with ⁶

$$\dot{X}^0 = \sqrt{g^{ij}p_i p_j + \frac{4}{\alpha'}(N-1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2}. \quad (3.12)$$

Now, we are ready to evaluate (3.7) for a single wound string. We have an explicit expression for $\dot{X}^0$ and we know that $|X'^5| = |w|b$ in units of α' for a wound string, the factor of $|w|$ being canceled by the summation over all (w in total) zeroes of the argument of the delta function. Thus, component by component, we get:

$$\begin{aligned} T_0^0 &= -\rho \\ &= -\frac{1}{2\pi} \frac{\delta^3(X^i - y^i)}{a^3 b} \sqrt{p^i p_i + \frac{4}{\alpha'}(N-1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2} \end{aligned} \quad (3.13)$$

$$\begin{aligned} T_i^i &= p \\ &= \frac{1}{2\pi} \frac{\delta^3(X^i - y^i)}{a^3 b} \frac{p^i p_i}{\sqrt{p^i p_i + \frac{4}{\alpha'}(N-1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2}} \end{aligned} \quad (3.14)$$

$$\begin{aligned} T_5^5 &= r \\ &= \frac{1}{2\pi} \frac{\delta^3(X^i - y^i)}{a^3 b} \frac{\frac{n^2}{b^2} - \frac{w^2 b^2}{\alpha'^2}}{\sqrt{p^i p_i + \frac{4}{\alpha'}(N-1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2}} \end{aligned} \quad (3.15)$$

(note that we label the extra spatial coordinate by “5”) where we ignore off-diagonal components since we are about to apply these expressions to an isotropic gas of strings.

⁶ See the second footnote in the Appendix where we remind the reader why the momenta must be contracted with the inverse metric.

However, we wish to present at this point another derivation of this result which is rather more direct. Consider the energy of a single wound string:

$$E^2 = p^i p_i + \frac{4}{\alpha'}(N - 1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2. \quad (3.16)$$

A spatially uniform gas of such strings with the same quantum numbers would have a 5-dimensional energy density

$$\epsilon = \frac{\mu(t)}{2\pi b} \sqrt{p^i p_i + \frac{4}{\alpha'}(N - 1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2}, \quad (3.17)$$

where $\mu(t)$ is the number density of strings. We divide by $2\pi b$ since this energy will be uniformly distributed over the length of the string. The momentum that appears in this expression is now the momentum squared of a gas of strings whose momenta have identical magnitudes, but whose directions are distributed isotropically. To fully account for the metric factors in this expression, we write $\mu(t)$ as $\mu_0(t)/a^3(t)$ since this is how a number density explicitly depends on the metric. Now, realizing that this is an energy density, we can introduce this gas of strings as matter interacting with the gravitational field by just adding the following term to the gravitational part of the action:

$$S_{int} = - \int d^5x \sqrt{-g} \epsilon \quad (3.18)$$

(see e.g. Section 10.2 of [64]).

Realizing now that the metric factors in the denominator of the expression for the energy density can be written as $a^3 = \sqrt{\det(g_{ij})}$ and

$b = \sqrt{g_{55}}$, we can write the above equation as:

$$\begin{aligned}
S_{int} &= - \int \frac{d^5x \sqrt{-g}}{\sqrt{\det g_{ij}} \sqrt{g_{55}}} \frac{\mu_0(t)}{2\pi} \\
&\quad \times \sqrt{p^i p_i + \frac{4}{\alpha'}(N-1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2} \\
&= - \int d^5x \sqrt{-g_{00}} \frac{\mu_0(t)}{2\pi} \\
&\quad \times \sqrt{p^i p_i + \frac{4}{\alpha'}(N-1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2}
\end{aligned}$$

By our metric ansatz and the isotropy of the distribution of the momenta, we have that $p^i p_i = a^{-2}(\frac{p^2}{3} + \frac{p^2}{3} + \frac{p^2}{3})$. Using this fact, it is straightforward to show that the energy-momentum tensor derived from this interaction term is:

$$T_0^0 = -\rho = -\frac{1}{2\pi} \frac{\mu_0}{a^3 b} \sqrt{p^i p_i + \frac{4}{\alpha'}(N-1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2} \quad (3.19)$$

$$T_i^i = p = \frac{1}{2\pi} \frac{\mu_0}{a^3 b} \frac{p^2/3}{\sqrt{p^i p_i + \frac{4}{\alpha'}(N-1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2}} \quad (3.20)$$

$$T_5^5 = r = \frac{1}{2\pi} \frac{\mu_0}{a^3 b} \frac{\frac{n^2}{b^2} - \frac{w^2 b^2}{\alpha'^2}}{\sqrt{p^i p_i + \frac{4}{\alpha'}(N-1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2}} \quad (3.21)$$

which is exactly what we would get from (3.13), (3.14) and (3.15) were we to construct a hydrodynamical average with an isotropic momentum distribution.

We now investigate some simple aspects of our result. The first thing to note is that T_5^5 , which is the pressure along the compact direction, gets a negative contribution from the winding of our strings and a positive contribution from the momentum along this direction. The spatial pressure is always positive, and for the simple case $n = w = 0$, $N = 1$, which describes a gas of gravitons moving in the non compact directions, we obtain $r = 0, p = \rho/3$.

Since we are about to study the effects of this energy-momentum tensor on space-time, we should make sure that the energy-momentum tensor is covariantly conserved, or else it will not be consistent to equate it to the covariantly conserved Einstein tensor. The covariant conservation of T^μ_ν

$$0 = \nabla_\mu T^\mu_\nu ,$$

where ∇_μ is the covariant derivative operator, implies

$$\begin{aligned} 0 &= \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) + \frac{\dot{b}}{b}(\rho + r) \\ 0 &= \partial_i p \\ 0 &= \partial_5 r . \end{aligned}$$

It is straightforward to check that our energy-momentum tensor satisfies this as an identity. In the continuity equation, this is due to the metric factors contained in the energy density, which upon differentiation produce terms that exactly cancel the terms proportional to the Hubble factors. The remaining equations are trivially satisfied by our setup, which assumed an axis of symmetry along the compactified dimension (the Kaluza-Klein setup) with homogeneous and isotropic spatial sections.

One final point to note is that we have derived an energy-momentum tensor that exhibits positive pressures along the non-compact directions and positive or negative pressures along the compactified direction. We need to ensure that this negative pressure has a bounded equation of state as otherwise our theory would be unstable. The Dominant Energy Condition (DEC) of General Relativity [65] ensures the stability of the vacuum, and requires that the equation of state parameter $\omega = p/\rho$ be greater than or equal to -1 (see e.g. [66] for a recent discussion). Since the spatial pressures are always positive, we only need to check our equation of state for the

pressure along the compact direction:

$$\begin{aligned}
r &= \frac{1}{2\pi} \frac{\mu_0}{a^3 b} \frac{\frac{n^2}{b^2} - \frac{w^2 b^2}{\alpha'^2}}{\sqrt{p^i p_i + \frac{4}{\alpha'}(N-1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2}} \\
&= \rho \times \frac{\frac{n^2}{b^2} - \frac{w^2 b^2}{\alpha'^2}}{p^i p_i + \frac{4}{\alpha'}(N-1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2},
\end{aligned}$$

where the co-efficient of ρ in the above is our equation of state parameter.

Were we to consider states described by $n = \pm 1$, $w = -n$, $N = 1$ (which as we will see further on, turn out to be the relevant states that give us stabilization), this parameter remains bounded as b varies ⁷ :

$$-1 \leq \omega \leq 1 \quad (3.22)$$

Thus, we have verified that the spectrum of string states satisfies the DEC, and in doing so ensured ourselves of sensible space-time dynamics arising from the string gas, the topic we will turn our attention to next.

3.4 Space-Time Dynamics

We start with the Einstein tensor derived from the metric (3.8):

$$\begin{aligned}
G_0^0 &= -3 \frac{\dot{a}}{a} \left[\frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right] \\
G_j^i &= -\delta_j^i \left[2 \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \left(\frac{\dot{a}}{a} \right)^2 + 2 \frac{\dot{b}}{b} \frac{\dot{a}}{a} \right] \\
G_5^5 &= -3 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right].
\end{aligned}$$

⁷ Where key to this is the observation that as we approach the value $b = \sqrt{\alpha'}$, these states become massless, and acquire a non-zero momentum along the non-compact directions, which depends on the ambient temperature. If this momentum is large enough, we can be assured that (3.22) is satisfied.

Equating this to $\frac{1}{M_{pl_5}^3}T_\nu^\mu$ will give us the Einstein equations. However, let us focus on the equation that governs the evolution of the scale factor b . Starting with G_j^i and eliminating $\ddot{a}$ and $\dot{a}^2$ by adding the appropriate combinations of G_0^0 and G_5^5 , we get:

$$\ddot{b} + 3H \dot{b} + \frac{b}{M_{pl_5}^3} \left(p - \frac{2r}{3} - \frac{\rho}{3} \right) = 0, \quad (3.23)$$

where H is the 3-dimensional Hubble factor. This is a second order, nonlinear (because of the b dependence in the matter terms) differential equation with a damping term and a driving term. We will demonstrate further on that the Einstein equations admit expanding solutions for the non-compact dimensions ($H > 0$), and take it as a given for what follows. Thus, inspite of its non-linearity, we easily see that (3.23) describes an expanding or a contracting scale factor depending on the sign of the driving term.

The first thing to notice from this equation is that matter for which the quantity $p - \frac{2r}{3} - \frac{\rho}{3}$ vanishes will not contribute to the dynamics of the compact dimension. Thus, recalling that a gas of gravitons ($n = w = 0$, $N = 1$) has an equation of state $p = \frac{\rho}{3}$, $r = 0$, as does a gas of ordinary 4-dimensional photons, we see that such matter will not affect the dynamics of the scale factor b . In fact, such a background 4-dimensional gas provides an excellent candidate for a thermal bath which we will eventually couple our gas of winding modes to.

First, however, we will study this driving term as it is, for a gas consisting of strings with identical quantum numbers. Upon evaluating the

driving term, we find that:

$$\begin{aligned} \frac{b}{M_{pl_5}^3} \left(p - \frac{2r}{3} - \frac{\rho}{3} \right) &= \frac{\mu_0}{M_{pl_5}^3 a^3 2\pi} \\ &\times \frac{-\frac{n^2}{b^2} - \frac{2nw}{3\alpha'} + \frac{w^2 b^2}{3\alpha'^2} - \frac{4(N-1)}{3\alpha'}}{\sqrt{p^i p_i + \frac{4}{\alpha'}(N-1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2}}, \end{aligned} \quad (3.24)$$

from which we infer that momentum modes and oscillator modes lead to expansion of the scale factor, whereas the winding modes produce contraction. Exactly what happens, of course, depends on the values of the quantum numbers. It should be recalled that the quantum numbers are subject to the constraint $nw + N \geq 0$ coming from the level matching conditions (see Eq. 3.11).

Let us pick a particular set of quantum numbers. As we shall see later, the most interesting case is when $n = -w = \pm 1$, $N = 1$, in which case the driving term becomes:

$$\frac{b}{M_{pl_5}^3} \left(p - \frac{2r}{3} - \frac{\rho}{3} \right) = \frac{2\mu_0}{M_{pl_5}^3 a^3 2\pi \sqrt{\alpha'}} \frac{-\frac{1}{\bar{b}^2} + \frac{2}{3} + \frac{\bar{b}^2}{3}}{\sqrt{\alpha' p^i p_i + \left(\frac{1}{\bar{b}} - \bar{b}\right)^2}}, \quad (3.25)$$

where $\bar{b}$ is the scale factor in units of $\sqrt{\alpha'}$. Quite generically, we can explore features of the “potential” energy that will yield this driving term. We see that, since the denominator is strictly positive, and the driving term changes sign at $\bar{b} = 1$, this value will be a minimum of the potential energy, and hence a point of equilibrium. Numerical integration of the driving term

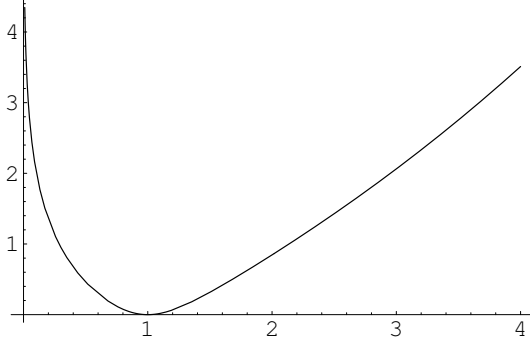


Figure 3–1: Potential term for $n = -w = \pm 1, N = 1$. The horizontal axis is b (in string units), the vertical axis gives the potential in units of $\mathcal{U}$, where $\mathcal{U}$ is the prefactor on the right hand side of (3.25).

yields the potential energy curve of Figure 1⁸ where the potential is plotted in units of $\frac{2\mu_0}{M_{pl_5}^3 a^3 2\pi\sqrt{\alpha'}}$ as a function of $\bar{b}$.

Because of the Hubble damping term in the equation of motion for $\bar{b}$ (which is obtained by dividing (3.23) through by a factor of $\sqrt{\alpha'}$), the scale factor will perform damped oscillations about the minimum of the potential to which it will evolve with rapidity depending on the value of the “spring constant” multiplying the driving term:

$$k = \frac{2\mu_0}{M_{pl_5}^3 a^3 2\pi\alpha'} . \quad (3.26)$$

Thus, we have established that a gas of string modes with non zero winding and momentum numbers in the compact direction will provide a dynamical

⁸ We model the momentum squared as a smooth function of $\bar{b}$ such that it takes on some non zero value at $\bar{b}=1$ and falls off on either side. This is because at $\bar{b}=1$ the state described by $n = -w = 1, N = 1$ becomes massless and should have a finite non zero momentum, but as the scale factor increases or decreases the state becomes more and more massive and hence the momentum becomes negligible in comparison. The generic feature of a minimum at $\bar{b}=1$ is robust, however, since as we mentioned earlier, the force term will always change sign at $\bar{b}=1$. One finds that the minimum is always fairly concave independent of the nature of the b dependence of the momentum.

stabilization mechanism for the radion, provided that the three non-compact dimensions are expanding (such a behavior was already found in an early study [55] of the dynamics of string winding modes - we thank Scott Watson for drawing our attention to this paper). We will address the phenomenology of this stabilization mechanism further on, simply stating for now that we can obtain a robust stabilization mechanism which is consistent with observational bounds.

At this point, we wish to mention that the “Quantum Gravity Effects” required to stabilize the extra dimensions in earlier attempts [67, 68] at Kaluza-Klein cosmology find a stringy realization here, in that all that was required for radius stabilization was matter that depended on the size of the extra spatial dimensions in a non-trivial way.

To round off the discussion, we wish to demonstrate that our assumption of an expanding scale factor $a(t)$ is consistent with an oscillating scale factor $b(t)$. Consider the two Einstein equations that do not contain second derivatives of b :

$$\begin{aligned}\frac{\rho}{3M_5^3} &= H^2 + H\frac{\dot{b}}{b}, \\ \frac{-r}{3M_5^3} &= \dot{H} + 2H^2.\end{aligned}$$

These equations imply that

$$\dot{H} - 2H\frac{\dot{b}}{b} = -\frac{1}{3M_5^3}(2\rho + r). \quad (3.27)$$

The resulting equation for H has the integrating factor $1/b^2$, and hence the solution:

$$H(t) = H_0\left(\frac{b(t)}{b_0}\right)^2 - \frac{b^2(t)}{3M_5^3} \int_0^t dt' \frac{(2\rho + r)}{b^2(t')}. \quad (3.28)$$

Now, from the discussion surrounding (3.22), we see that $\rho \geq r \geq -\rho$. Thus, the contribution to the integral in the above is strictly positive, and the

second term on the right hand side of (3.28) can at most take on the value:

$$\frac{b^2(t)}{M_5^3} \int_0^t dt' \frac{\rho(t')}{b^2(t')}. \quad (3.29)$$

Thus, we see that if we pick the initial conditions for H appropriately, the scale factor a can be taken to be expanding ($H > 0$) regardless of the detailed motion of b . In fact, if we assume that H_0 starts out positive, then $H(t)$ will remain so if

$$H_0 \geq \frac{1}{3M_5^3} \int_0^t dt' \frac{2\rho + r}{b^2(t')/b_0^2}, \quad (3.30)$$

where the eventual stabilization of b and the $1/Vol$ dependence of ρ and r will bound the integral, which implicitly depends on H itself. This implicit dependence works in our favor in that the larger we make H_0 , the smaller the integral becomes and so we can imagine picking an initial H_0 such that a persistent expansion of the non-compact dimensions results. Note, however that if in the spirit of brane gas cosmology, we assume that all spatial dimensions are starting out with the same size and instantaneously static, then it may not be possible to evolve to a situation in which three large spatial dimensions are expanding. This is, in fact, the result that emerges from the work of [58], at least in a certain region of phase space.

3.5 Thermal String Gases

In what we have done so far, we have just considered the behavior of the size of the extra dimension in a rather artificial setting, namely imposing a gas of strings with a fixed set of quantum numbers. One expects the early Universe to be in a state of thermal agitation, and it is inevitable that transitions between different energy levels will be induced in the string gas. Thus, to have any hope of realistically applying our setup to cosmology, we need to study the effects of placing the string gas in a thermal bath.

Referring to our expression for the energy density of a string (3.19), we see that a gas of strings with different quantum numbers will have the energy density:

$$\rho = \sum_{n,w,N,p^2} \frac{\mu_{n,w,N,p^2}}{a^3 b 2\pi} \sqrt{p^i p_i + \frac{4}{\alpha'} (N-1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2} \quad (3.31)$$

with densities μ_{n,w,N,p^2} for each given set of quantum numbers. The expressions for the pressure terms p and r are similarly modified. If we are in thermal equilibrium, the densities are given by the Boltzmann weight

$$\mu_{n,w,N,p^2} = e^{\beta E_{ref}} e^{-\beta E_{n,w,N,p^2}} \mu_{ref} , \quad (3.32)$$

where the subscript “ref” refers to some arbitrary reference energy level.

What constitutes the thermal bath to which the string gas is coupled to? We know from the discussion at the end of Section II that gravitons described by unwound strings propagate in the non-compact directions with an equation of state $p = \rho/3$. Introducing a gas of ordinary photons will also add a 4-dimensional component to the energy-momentum tensor with the same equation of state. Such particles offer us an ideal candidate for a thermal bath, for two reasons. Firstly, thermal equilibrium demands a coupling of some kind between the gas of winding modes and the gas of gravitons and photons. Such a coupling is readily provided by the tree-level reaction $w + \bar{w} \rightarrow h_{\mu\nu}$ via which winding modes of equal and opposite winding scatter to produce 4-d gravitons. This thermalization mechanism will, at non-zero temperatures, create an equilibrium where there will be an ever-present non-zero winding mode density due to gravitons scattering into winding modes (and vice-versa). This thermal bath has the further property that it does not affect the dynamics of the extra dimension other than through the Hubble factor (which it influences), since the driving term

is only sensitive to the combination $p - \rho/3 - 2r/3$ which vanishes for the graviton and photon components of the energy-momentum tensor.

With the above in mind, the driving term for the scale factor b becomes:

$$\begin{aligned} \frac{b}{M_{pl_5}^3} \left(p - \frac{2r}{3} - \frac{\rho}{3} \right) &= \frac{\mu_{ref} e^{\beta E_{ref}}}{M_{pl_5}^3 a^3 2\pi} \\ \times \sum_{n,w,N,p^2} e^{-\beta E} &\frac{-\frac{n^2}{b^2} - \frac{2nw}{3\alpha'} + \frac{w^2 b^2}{3\alpha'^2} - \frac{4(N-1)}{3\alpha'}}{\sqrt{p^i p_i + \frac{4}{\alpha'}(N-1) + (\frac{n}{b} + \frac{wb}{\alpha'})^2}}, \end{aligned} \quad (3.33)$$

where the Boltzmann weight in the summation depends on the values of the quantum numbers. We remind the reader that the sum is restricted by the level matching condition $N + nw \geq 0$. For completeness we also remind the reader of the resulting equation of motion for b :

$$\begin{aligned} 0 &= \ddot{b} + 3H \dot{b} + \frac{\mu_{ref} e^{\beta E_{ref}}}{M_{pl_5}^3 a^3 2\pi} \\ &\times \sum_{n,w,N,p^2} \frac{e^{-\beta E}}{\sqrt{E}} \left(-\frac{n^2}{b^2} - \frac{2nw}{3\alpha'} + \frac{w^2 b^2}{3\alpha'^2} - \frac{4(N-1)}{3\alpha'} \right) \end{aligned} \quad (3.34)$$

The summation which has to be performed in order to obtain the driving term is quite formidable, were it not for a rather special feature of string thermodynamics. Consider the argument of the exponential in the Boltzmann factor:

$$\begin{aligned} \beta E_{n,w,N,p^2} &= \beta \sqrt{p^i p_i + \frac{4}{\alpha'}(N-1) + (\frac{n}{b} + \frac{wb}{\alpha'})^2} \\ &= \frac{\beta}{\sqrt{\alpha'}} \sqrt{\alpha' p^i p_i + 4(N-1) + (\frac{n}{b} + \bar{b} w)^2} \end{aligned}$$

We see that when the energy is expressed in terms of dimensionless variables, we pull out a factor of $\sqrt{\alpha'}$. Thus, the argument of the exponential in the Boltzmann weight is $\beta/\sqrt{\alpha'}$ times a term of order unity. To be able to neglect all but the first few terms in the summation, we need the Boltzmann

factor to be considerably less than unity, i.e. that

$$e^{-\frac{\beta}{\sqrt{\alpha'}}} \ll 1.$$

Thus, if this condition is satisfied, then the terms which dominate the sum will be those whose quantum numbers render them nearly massless, since any state with even one of its quantum numbers being different from the nearly massless combination will produce a term of order unity times $\beta/\sqrt{\alpha'}$.

Let us take a closer look at the above condition. We know from string thermodynamics that there exists a limiting temperature – the Hagedorn temperature T_H [8]. Thus, for us to even be able to apply thermodynamics, we need to be well below this temperature, which for all the string theories is of the order of $T_H \sim 1/\sqrt{\alpha'}$. Thus, $\beta_H \sim \sqrt{\alpha'}$, and so if we are at a temperature much lower than the Hagedorn temperature, i.e. $T \ll T_H$ or equivalently $\beta_H \ll \beta$, then

$$\sqrt{\alpha'} \ll \beta, \tag{3.35}$$

which is exactly what we need for the Boltzmann weights of higher mass states to be negligible. So, even if the thermal bath has a temperature of only one order of magnitude below the Hagedorn temperature, then $e^{-\frac{\beta}{\sqrt{\alpha'}}} \sim 10^{-5}$ which clearly lets us ignore any term whose energy in dimensionless units $\sqrt{\alpha' p^i p_i + 4(N-1) + (\frac{n}{\bar{b}} + \bar{b} w)^2}$ is anything other than zero. This translates into us being able to *neglect all states other than those that are massless*. Thus, the summation now becomes very tractable, and we can also have faith in our truncation of the string spectrum to the lightest states all the way up to very high temperatures ($T \sim T_H/10$). Before we carry on we should remark that exactly massless states have a non-zero momentum given by the thermal expectation value of $E = |p| = 3/\beta$.

Let us then proceed to evaluate (3.33), so that we can evolve b in time using (3.34), recalling that now we only sum over the massless and near massless states subject to the level matching constraint. Let us begin near $\bar{b} = 1$, i.e. $\bar{b} = 1 + \Gamma$. Then for the case that $\Gamma \neq 1$, we only have one truly massless state: $n = w = 0$, $N = 1$. This term will not contribute to the driving force for b since

$$\frac{\left(-\frac{n^2}{b^2} - \frac{2nw}{3\alpha'} + \frac{w^2b^2}{3\alpha'^2} - \frac{4(N-1)}{3\alpha'}\right)}{|p|} = 0. \quad (3.36)$$

Thus, the next lightest state which has quantum numbers $N = 1, n = -w = \pm 1$ will dominate the evolution of b . The level matching constraints $N + nw \geq 0$ ensure that there are no more nearly massless states (Note we only consider states with positive mass squared - any tachyonic states are posited to be absent from our spectrum). Such states will contribute:

$$\frac{e^{\frac{-\beta}{\sqrt{\alpha'}}\sqrt{(\frac{1}{b}-\bar{b})^2+\alpha'p^2}}\left(-\frac{1}{b} + \frac{b^2}{3} + \frac{2}{3}\right)}{\sqrt{(\frac{1}{b}-\bar{b})^2+\alpha'p^2}}. \quad (3.37)$$

Expanding $\bar{b}$ as $1 + \Gamma$ and ignoring terms higher than quadratic in Γ results in a contribution to (3.37) of:

$$\frac{e^{-\beta|p|}}{\sqrt{\alpha'}|p|}\left(\frac{8\Gamma}{3}\right),$$

Since there are two such terms which yield identical contributions, the sum total of the contributions from the near massless states yields the equation of motion

$$\ddot{\Gamma} + 3H \dot{\Gamma} + \frac{\mu}{M_5^3 2\pi a^3 |p| \alpha'^{3/2}} \left(\frac{8\Gamma}{3}\right) = 0, \quad (3.38)$$

where the exponential factor gets cancelled if we use this massless state as our reference state, as in (3.32). The form of this equation clearly shows that Γ will tend to zero if it starts out on either side of this value.

However, to confirm that $\Gamma = 0$ is a genuine point of equilibrium, we need to confirm that the extra massless states that appear at this radius (8 in all) contribute in such a way so that their sum vanishes. This can be verified by a straightforward calculation ⁹.

However, we wish to point out that as long as we stay in thermal equilibrium with the graviton gas, this equilibrium is actually metastable. The reason for this is easy to see from the formula for the mass of a winding mode:

$$\alpha' m^2 = \left(\frac{n}{\bar{b}} + w \bar{b} \right)^2 + 4(N - 1). \quad (3.39)$$

In addition to the massless state given by $n = w = 0, N = 1$ (the graviton), and the 8 other massless states that appear at the self-dual radius (which are given by $N = 1, n = -w = \pm 1$; $N = 0, w = n = \pm 1$; $N = 0, w = 0, n = \pm 2$ and $N = 0, w = \pm 2, n = 0$), there are additional massless states at further special radii

$$\bar{b} = \frac{2}{|m|} \quad ; \quad w = \pm m, n = 0, N = 0 \quad (3.40)$$

$$\bar{b} = \frac{|m|}{2} \quad ; \quad n = \pm m, w = 0, N = 0 \quad (3.41)$$

$$m \in \mathbb{Z}.$$

Thus, at half-integer multiples and half integer fractions of the self-dual radius, two massless modes appear and will thus yield the dominant

⁹ The only non-zero contribution to the driving term comes from the states $n = 0, w = \pm 2, N = 0$ and $n = \pm 2, w = 0, N = 0$ which make equal and opposite contributions and hence cancel.

contributions to the driving term. These contributions again exactly cancel at twice the self dual radius, and at half the self dual radius. However in general, we will get a driving term that yields expansion at half integer points above twice the self dual radius and similarly, contraction at half integer fractions below half the self-dual radius. However since we posit that we begin at or near the self-dual radius, we are guaranteed to stay locked near it if our initial conditions satisfy

$$\begin{aligned} b(0) &\sim \sqrt{\alpha'} \\ \dot{b}(0) &\leq \sqrt{k} = \sqrt{\frac{2\mu_0}{M_3^3 a^3 2\pi\alpha'}} \end{aligned}$$

where the last condition constrains the initial “velocity” of the scale factor to be such that it cannot roll over the “hump” in the potential energy surrounding the metastable equilibrium at $b \sim \sqrt{\alpha'}$.

Thus, we have demonstrated in the context of GR how a string gas in thermal equilibrium with a bath of gravitons and photons will *dynamically stabilize the scale factor of the compact direction* if we begin close to that radius. Thermal equilibrium with the graviton bath ensures a persistent non-zero density of such winding modes. One can now imagine that, at some point, the winding mode gas becomes decoupled from the graviton gas, i.e. falls out of thermal equilibrium. In this situation, we are left with an unchanging driving term of the form (3.24), which yields the potential in Fig. 1, which will guarantee radion stabilization at the self dual radius for the remainder of the Universe’s dynamics. We now turn our attention to the possible connection between this mechanism and inflationary and standard Big Bang cosmology.

3.6 Late Time Evolution

Recall the Einstein tensor for our metric setup:

$$\begin{aligned} G_0^0 &= -3\frac{\dot{a}}{a}\left[\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right] \\ G_j^i &= -\delta_j^i\left[2\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\dot{b}}{b}\frac{\dot{a}}{a}\right] \\ G_5^5 &= -3\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]. \end{aligned}$$

We know that the dynamics of the scale factor b in the situations we studied above cause it to undergo damped oscillations around the self dual radius.

We demonstrated in a previous section how the “spring constant” of this evolution will lock in to this equilibrium fairly rapidly.

We can then study the evolution of the Universe after radius stabilization, which implies that $\dot{b}=\ddot{b}=0$ and $p - 2r/3 - \rho/3 = 0$. The resulting Einstein equations are:

$$\begin{aligned} G_0^0 = \frac{1}{M_5^3}T_0^0 &\rightarrow H^2 = \frac{\rho}{3} \\ G_i^i = \frac{1}{M_5^3}T_i^i &\rightarrow \dot{H} = -\frac{1}{2}(\rho + p) \\ G_5^5 = \frac{1}{M_5^3}T_5^5 &\rightarrow p - \frac{2r}{3} - \frac{\rho}{2} = 0 \\ \nabla_\mu T_\nu^\mu = 0 &\rightarrow \dot{\rho} + 3H(\rho + p) = 0, \end{aligned}$$

where the 55 equation is precisely the equilibrium condition. Thus, we recognize in the above the basic equations of FRW cosmology. We now consider how one achieves the two important epochs of late time FRW cosmology, namely the radiation dominated era and the matter dominated era.

3.6.1 Radiation Dominated Evolution

If we assume that the density of 4-d matter gas is far greater than the density of the winding mode gas, then this will be the dominant component

that drives the evolution of the macroscopic dimensions. If the 4-d matter has an equation of state parameter w , then the solutions to the Einstein equations become:

$$\rho(t) \propto a^{-3(1+w)} \quad (3.42)$$

or

$$a(t) \propto t^{\frac{2}{3(1+w)}}. \quad (3.43)$$

Thus, for a 4-d graviton and photon background, we get that $a(t) \propto t^{1/2}$, and so we reproduce a late time FRW evolution that is consistent with standard Big-Bang cosmology immediately after the end of inflation, whilst maintaining radius stabilization.

3.6.2 Matter Dominated Evolution

Reconsidering (3.23):

$$\ddot{b} + 3H \dot{b} + \frac{b}{M_{pl_5}^3} \left(p - \frac{2r}{3} - \frac{\rho}{3} \right) = 0,$$

We see that any matter with the equation of state of non-relativistic dust ($p = 0$), can only drive the expansion of the radion if it is of a 3-dimensional nature (i.e. $r \equiv 0$). This is surely to be a cause for concern when considering that at late times, one (naively) expects 3-dimensional non-relativistic dust to be the major driving component of the expansion of the universe, which would normally invalidate our stabilization mechanism in the present epoch.

However we wish to remind the reader that present day observations demand that a significant fraction of the energy density of the universe, which also drives the present day matter dominated expansion, be in the form of cold dark matter— whose nature is as of yet completely unknown. There is a significant amount of interest in the prospect that extra-dimensional

matter or extra dimensional effects might account for the ‘missing’ matter in the universe. In what follows, we find that the only way to make our stabilization mechanism consistent with a ‘matter dominated’ epoch is to introduce extra dimensional cold dark matter. We propose a candidate for this cold dark matter which is naturally contained in our framework, and discuss other possibilities which might have a natural realisation within the general brane gas framework (note that a similar proposal was made in [69]).

We see that in order for the driving term in (3.23) to correspond to a stable minimum at the self dual radius for matter which obeys the equation of state for non-relativistic dust ($p = 0$), we need to consider matter for which:

$$r = -\rho/2|_{b=\sqrt{\alpha'}}. \quad (3.44)$$

That is, we require the dominant component of the energy density which is driving the expansion of the universe be such that it preserves the stability of the radion at the self dual radius. Matter which exhibits such an equation of state will surely have to be massive (else there will be a non-zero pressure along the non-compact directions for any non-zero energy density). In addition, such matter will have to be something beyond presently supposed dark matter candidates (WIMPS, supersymmetric relics etc.) as it will necessarily have to be ‘extra-dimensional’ in nature. We now show that our model naturally contains such a candidate. Recalling the discussion surrounding (3.22), we see that the equation of state parameter for a gas of strings with a particular set of quantum numbers is obtained from the following equation:

$$r = \rho \times \frac{\frac{n^2}{b^2} - \frac{w^2 b^2}{\alpha'^2}}{p^i p_i + \frac{4}{\alpha'}(N-1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2}, \quad (3.45)$$

where the momentum will be set to zero (or is vanishingly small) in order to satisfy $p = 0$ (c.f. (3.20)). In particular, since we are looking for states which can satisfy (3.44) at the self dual radius, we need to find the appropriate quantum numbers which have an equation of state parameter $w = -1/2$ at $b = \sqrt{\alpha'}$, which reduces to the following condition:

$$3n^2 - w^2 + 4(N - 1) + 2nw = 0, \quad (3.46)$$

and we have to be mindful of the level matching constraint: $N + nw \geq 0$. As expected, it turns out that the massless states that satisfy these conditions have an energy density proportional to $|p|$, whereas the pressure is proportional to $|p|^2/3$, and hence one cannot have a non-zero pressure without having a vanishing energy density.

The first massive states which satisfy (3.46) are represented by the quantum numbers:

$$|p| = 0, \quad N = 2; \quad n = 0, \quad w = \pm 2. \quad (3.47)$$

These states contribute to the energy-momentum tensor as follows (c.f. (3.19) - (3.21)):

$$p = 0, \quad \rho = \frac{2\mu\sqrt{2}}{2\pi a^3 \alpha'}, \quad r = -\rho/2, \quad (3.48)$$

where the factor of α' in the denominator comes from two factors of $\sqrt{\alpha'}$ —one from the metric factor b which has stabilized at the self dual radius, and the other as the pre-factor of the non zero rest mass of this string state. As we will see in the next section, were we to look at fluctuations of the radion around the self dual radius, these states also provide a stable equilibrium at $b = \sqrt{\alpha'}$ in a phenomenologically acceptable manner. Thus, taking questions of consistency with observation for granted for the moment, we see if these particular string states are taken to dominate the present energy density

of the universe, then by the Einstein equations derived at the start of this section, we can admit an epoch of dust driven FRW expansion ($p = 0$, $a \propto t^{2/3}$) at late times, consistent with radius stabilization.

However, there are many further issues that will have to be resolved if we are to take this idea of stringy dark matter seriously, which we postpone to a future report. At present, however, we wish to state that there are indications that such stringy dark matter might have the right clustering properties at the level of first order perturbation theory, in that local overdensities of this stringy dark matter induces gravitational clustering in the remaining 3-dimensions.

We also wish to point out that in certain situations, non-relativistic p-branes might also be able to provide us with a matter content which satisfies (3.44) [60]. In the context of brane gas cosmology, this is an appealing idea as one might need higher dimensional branes to stabilize compact sub-manifolds that do not admit topological one-cycles (and hence do not admit winding modes). We will investigate this possibility further in a future report. Finally, we wish to address the effects of an intermediary epoch of scalar field driven inflation on our stabilization mechanism.

3.6.3 Intermediate (Non-Stringy) Inflation

We find that our mechanism for radius stabilization might not be compatible with an intermediate epoch of bulk scalar field driven inflation. To investigate this, we first consider the energy-momentum tensor of an almost homogeneous inflaton field:

$$T_{\nu}^{\mu} = diag\left(-\left[\frac{\dot{\phi}^2}{2} + V(\phi)\right], \left[\frac{\dot{\phi}^2}{2} - V(\phi)\right], \dots, \left[\frac{\dot{\phi}^2}{2} - V(\phi)\right]\right) \quad (3.49)$$

Adding this energy-momentum tensor to our string gas yields a non-trivial contribution to the driving term in the equation of motion for b . This

driving term then takes the form

$$\begin{aligned}
& - \frac{2b}{3M_5^3} V(\phi) + \frac{\mu_{ref} e^{\beta E_{ref}}}{M_{pl_5}^3 a^3 2\pi} \\
& \sum_{n,w,N,p^2} \frac{e^{-\beta E}}{\sqrt{E}} \left(-\frac{n^2}{b^2} - \frac{2nw}{3\alpha'} + \frac{w^2 b^2}{3\alpha'^2} - \frac{4(N-1)}{3\alpha'} \right)
\end{aligned} \tag{3.50}$$

from which it is easy to see that the inflaton contribution will drive expansion in the extra dimension. In general, this term will compete with the string gas contribution which, as we have seen, drives contraction, if we are above the self-dual radius. However, this competition is short lived, as the factor of a^3 in the denominator of the string gas driving term will quickly render it irrelevant and the scale factor will then expand according to

$$\ddot{b} + 3H \dot{b} - \frac{2b}{3M_5^3} V(\phi) = 0. \tag{3.51}$$

Recalling that during this (slow roll) inflation H , and by the time-time Einstein equation, also $V(\phi)$ are almost constant, we can solve the above equation, with the resulting two solutions:

$$\begin{aligned}
b(t) & \propto \exp^{-\frac{3H}{2} \left(1 + \sqrt{1 + \frac{8V(\phi)}{9H^2 3M_5^3}} \right) t} \\
b(t) & \propto \exp^{-\frac{3H}{2} \left(1 - \sqrt{1 + \frac{8V(\phi)}{9H^2 3M_5^3}} \right) t}.
\end{aligned}$$

Substituting in the Einstein equation $H^2 = V(\phi)/3M_5^3$ gives us

$$\begin{aligned}
b(t) & \propto \exp^{-\frac{3H}{2} \left(1 + \sqrt{1 + \frac{8}{9}} \right) t} \propto e^{-3.56Ht} \\
b(t) & \propto \exp^{-\frac{3H}{2} \left(1 - \sqrt{1 + \frac{8}{9}} \right) t} \propto e^{0.56Ht}.
\end{aligned}$$

Except for very special initial conditions, the growing mode will rapidly come to dominate. Thus, we conclude that b expands exponentially (though not as fast as a). After inflation has finished (and after reheating to a temperature smaller than the one required for the pair production of string

winding modes), a expands as $t^{1/2}$. The energy density in the string gas will have been exponentially suppressed by the inflationary evolution, and thus to good approximation the equation of motion for b will take the form

$$\ddot{b} + 3H \dot{b} = 0$$

leading to

$$\dot{b} = C a^{-3} \quad (3.52)$$

which implies

$$b(t) = b(1) + 2 \dot{b}(1) \left(1 - \frac{1}{\sqrt{t}}\right), \quad (3.53)$$

where ($1 \leq t$). Thus, b asymptotically expands to some limiting, and very large (due to the initial conditions that result at the end of inflation) value.

In conclusion, we have seen how our radion stabilization mechanism is consistent with the FRW evolution of the non-compact dimensions, but not with an intermediate inflationary period, with inflation driven by a bulk scalar matter field. Thus, in order for brane gas cosmology to make contact with the present cosmological observations, one either needs a different (maybe stringy) realization of inflation (see [62] for some ideas) where strings are produced in re-heating, or a non-inflationary mechanism to solve the cosmological problems of standard Big Bang cosmology and to produce a spectrum of almost scale-invariant cosmological perturbations. Finally, we turn to various phenomenological issues pertaining to this model.

3.7 Phenomenology

There are two potential phases of applicability of our considerations. The first is to the early phase of string gas cosmology before a period of inflation. In this case, there are no phenomenological constraints on the model since the number density of the particles (from the four dimensional perspective) which correspond to the string states wrapping the extra

dimension are diluted exponentially during inflation. However, in this case our considerations would no longer be relevant for the late-time stabilization of the extra dimension.

The second phase of potential applicability of our considerations is to the universe after inflation of our three spatial dimensions. We then need to assume that winding and momentum modes about the extra dimension can be regenerated in sufficient number, as discussed in [69]. In this case, there are important constraints on our model. We must ensure that the particles corresponding to our string states do not overclose the universe. In addition, there is a radion mass constraint. Since the radion corresponds to a scalar particle from our four-dimensional perspective, we must make sure that its mass is consistent with the experimental constraints (we thank the Referee for stressing this point to us).

An additional constraint comes from the string theoretical aspect of our model: we must ensure that the derivatives of the metric remain several orders of magnitude smaller than the worldsheet derivatives (see Appendix). This is to ensure that we can inherit aspects of the string spectrum and constraint algebra that we have used so far. From (3.38), which has the form:

$$\ddot{\Gamma} + 3H\dot{\Gamma} + k\Gamma = 0 \quad (3.54)$$

we see that the ‘spring constant’ which sets the scale for the how fast the metric factor b varies, is given by

$$k = \frac{8\mu_0}{3M_5^3\alpha'^{3/2}|p|a^3} \quad (3.55)$$

where the subscript on μ is to remind the reader that any explicit metric dependence has already been factored out (see section II). We have taken the stabilization to be provided by the massless states discussed earlier with

$|p|$ denoting the momentum in the non compact directions, $N = 1$, and $n = -w = \pm 1$. In order that our metric factors evolve much slower than the string scale, we require that $\partial_\mu g \ll 1/\sqrt{2\pi\alpha'}$. Since (3.38) is a second order ODE, this implies that $k \ll 1/2\pi\alpha'$. As discussed in the Appendix, we choose to be quite conservative and demand that $k \leq 10^{-6}/2\pi\alpha'$.

A second constraint comes from requiring that the stabilization mechanism be effective at all times. This leads to a lower bound on k . We take this bound to be given by the ‘critically damped’ value for k :

$$k_{crit} = 9H^2. \quad (3.56)$$

The two above constraints yield the following bounds:

$$9H^2 \leq \frac{8\mu_0}{3M_5^3\alpha'^{3/2}|p|a^3} \leq \frac{10^{-6}}{2\pi\alpha'}. \quad (3.57)$$

Since the winding states that stabilize the extra dimensions are massless at the self-dual radius, they will behave as hot dark matter – dark matter because they only interact gravitationally (through the tree level interactions $w + \bar{w} \rightarrow h_{\mu\nu}$) with other fields, hot because they are massless and have a radiative equation of state. We have to ensure that we do not introduce too many of these objects so that we can ensure consistency with observational bounds.

Next, we have to ensure that the massive string states that we propose as a candidate for the cold dark matter that is presently driving the ‘dust dominated’ expansion of the universe do not violate any observational bounds while preserving our stabilization mechanism.

Let us begin by considering the massless states which are presently stabilizing the radion. From (3.17) we see that post stabilization, for stringy

matter with $N = 1$, $n = -w = \pm 1$, we must have:

$$\rho = \frac{\mu_0 |p|}{a^3} \leq 10^{-4} \rho_{crit} \quad (3.58)$$

in order to ensure consistency with the nucleosynthesis bounds. The critical density of the universe is $\rho_{crit} = 10^{-29} g/cm^3$. We then find that (3.58) becomes

$$\mu_0 \leq 10^{-4} 10^{-37} \text{GeV}^4 |p|^{-1}. \quad (3.59)$$

Let us now parametrize the present value of $|p|$ as

$$|p| = 10^{-\gamma} \text{eV} \quad (3.60)$$

where γ is some constant determined by the initial conditions and the history of the universe, and this parametrization being motivated by the fact that $|p|$ is likely to be of the order of a few eV in the present epoch if it corresponds to an initial $|p|$ of the order of the Planck energy. Then, the above bound takes the form

$$\mu_0 \leq 10^{-41} \text{GeV}^3 10^\gamma. \quad (3.61)$$

On the other hand, the first inequality in (3.57) becomes:

$$\mu_0 \geq H^2(t_0) |p| \sim 10^{-93} 10^{-\gamma} \text{GeV}^3 \quad (3.62)$$

which is consistent with (3.61). In the above, we have made use of

$$M_5^{-3} = 8\pi G_5 = 8\pi G_4 \sqrt{\alpha'}. \quad (3.63)$$

Using (3.63), it can easily be checked that the upper bound on μ_0 which follows from the second inequality in (3.57) is much weaker than the bound (3.61). We conclude that one can easily arrange the number of string modes

such that stabilization of the extra dimension is ensured and at the same time the massless modes do not conflict with the nucleosynthesis bound.

Let us next turn to the radion mass constraint: Since the radion appears in four dimensions as a scalar field, its mass must be larger than

$$m_{crit} = 10^{-12}\text{GeV} \quad (3.64)$$

in order to avoid fifth force type constraints. Since the square of the radion mass is given by k , this constraint becomes

$$\mu_0 \geq M_5^3 \alpha'^{3/2} |p| m_{crit}^2 \sim 10^{-35-\gamma} \text{GeV}^3 \quad (3.65)$$

which is consistent with the upper bound (3.61) on μ_0 if $\gamma \geq 3$. Such a value of γ is not at all unreasonable and could easily arise from an additional suppression of the momentum during a short period of inflation.

It turns out to be crucial that we use massless modes to stabilize the extra dimensions, as more massive string states would bring down the upper bound, and as we are about to see, do not provide as effective a spring constant and hence bring up the lower bound, to the net effect that it is phenomenologically inconsistent to have these as the only strings that are stabilizing the radion. We arrive at this observation by considering the second aspect of our phenomenology— namely, that we would like the cold dark matter content of our present universe to consist of massive string modes (which satisfy (3.44), which as we have seen is require in order to maintain stabilization at the self dual radius).

Considering the contribution to the energy density by a string gas with quantum numbers $|p| = 0$, $N = 2$, $n = 0$, $w = \pm 2$ (3.48), and equating this

to the critical density of the universe, we see that:

$$\frac{2\sqrt{2}\mu_{dm}}{2\pi a^3 \alpha'^{1/2}} \approx \rho_{crit} \quad (3.66)$$

Where now the subscript on μ serves to indicate that this is our dark matter candidate. This requires

$$\mu_{dm} \sim 10^{-67} \text{GeV}^3. \quad (3.67)$$

Upon perturbing around a stabilized radius, we find that these string modes contribute to the stability of the radion with the spring constant:

$$k_{dm} = \frac{8\pi G_5 \mu_{dm}}{2\pi \alpha' a^3} \frac{\sqrt{8}}{3}. \quad (3.68)$$

Demanding that this value of k is consistent with the radion mass bound yield a lower bound on μ_{dm}

$$\mu_{dm} \geq m_{crit}^2 M_4 \sim 10^{-6} \text{GeV} \quad (3.69)$$

which is clearly inconsistent with (3.66) for values of γ which are extremely large.

Thus we see that if we introduce the correct amount of our dark matter candidate, it contributes too feebly to the dynamics of the radion (even though it does provide its own contribution to stabilization). However, this is of no concern to us, as we have already shown that the massless string states provide a robust stabilization mechanism that is consistent with observational bounds. Thus, if our string gas has a massive component that serves as cold dark matter, and a massless component that stabilizes the radion (and behaves like hot dark matter) in the right proportions, which as we have shown is quite easy to achieve, we can be assured of the

phenomenological consistency of our stabilization mechanism with late time FRW cosmology.

3.8 Conclusions

The analysis in this paper is motivated by brane gas cosmology [6, 45]. As a simplified problem, we have studied the effects of a gas of strings with non-vanishing momentum and winding modes about a single compact extra dimension taken to be a circle (the three large dimensions are taken to be spatially flat and isotropic) on the evolution of the radius of that dimension, assuming that the background space-time satisfies the equations of motion of General Relativity. We discovered that such a string gas leads to a dynamical stabilization mechanism for the radius of this dimension, the radion. Assuming initial conditions in which the three large dimensions are expanding, we found that the radion performs damped oscillations about the self-dual radius.

In a first step, we studied the effects of a gas of non-interacting strings, each string having the identical momentum and winding numbers. Key to the stabilization mechanism is the fact that winding modes and momentum modes contribute with opposite sign to the pressure of the string gas in the direction of the compact dimension, and that the winding modes generate a potential for the radion which favors contraction, whereas the momentum modes generate a potential favoring expansion. We then showed that the stabilization mechanism also holds for a gas of strings in thermal equilibrium.

We also showed that, after radion stabilization, the scale factor for the three large spatial dimensions obeys the usual FRW equations of standard Big Bang cosmology. Thus, our scenario leads in a natural way to a late time FRW Universe. However, we have also shown that the

radion stabilization mechanism is not compatible with a period of scalar field driven bulk inflation. Thus, in order for brane gas cosmology to make successful contact with present cosmological observations, one either needs to find a stringy mechanism for driving inflation where strings are produced in the re-heating epoch, or else one must provide an alternative to inflationary cosmology, both for solving the mysteries of standard Big Bang cosmology, and for explaining the origin of the observed large-scale fluctuations.

Note that we start with the assumption that three spatial dimensions are already much larger than the other ones (one other dimension in our case). Whether or not the dynamics of strings in the initial stages will indeed lead to this situation may depend on the corner of M-theory one is working in, i.e. on the specific form of the background equations of motion and initial conditions (see [6, 45, 46, 57, 58] for different angles on this issue). However, it should not be hard to generalize our analysis to a situation with more compact dimensions or different topologies and geometries of the extra dimensions, which will be the focus of our future work.

In future work we also plan to study the annihilation rate of the string modes which are central to this work, namely modes which have both winding and momentum in the compact direction. Since these modes interact only gravitationally just like gravitons, they will be out of thermal equilibrium at late times and hence will not decay.

3.9 Appendix: The String Spectrum in a Time Dependent Background

Let us then begin with the Polyakov action for a closed string:

$$S = \frac{-1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X). \quad (3.70)$$

Varying this action with respect to the world sheet metric gives us the equation of motion

$$\gamma_{ab} = \lambda h_{ab} = \lambda \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X), \quad (3.71)$$

where we can exploit the world-sheet diffeomorphism and Weyl invariance to make the world-sheet metric flat (conformal gauge):

$$\gamma_{ab} = \gamma^{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.72)$$

Varying the action with respect to the world-sheet fields, and imposing the gauge choice yields the equations

$$\partial_a(\partial^a X^\lambda g_{\lambda\mu}(X)) = \frac{1}{2} \partial^a X^\lambda \partial_a X^\nu g_{\lambda\nu,\mu}(X), \quad (3.73)$$

where the meaning of the derivatives of the metric should be clear. This equation translates into

$$\partial_a \partial^a X^\tau + \Gamma_{\lambda\nu}^\tau \partial_a X^\nu \partial^a X^\lambda = 0. \quad (3.74)$$

Now, we consider the case when our metric depends only on time, that is $g_{\mu\nu}(X) = g_{\mu\nu}(X^0)$, is diagonal and has -1 as its 00 entry (This last point is not essential to the argument, it only serves to simplify the equations).

Unpacking the above equation yields the equations

$$-\partial_\tau^2 X^0 + \partial_\sigma^2 X^0 = -\frac{1}{2} g_{\lambda\nu,0} (-\partial_\tau X^\nu \partial_\tau X^\lambda + \partial_\sigma X^\nu \partial_\sigma X^\lambda), \quad (3.75)$$

and

$$-\partial_\tau^2 X^i + \partial_\sigma^2 X^i = -g^{ii} g_{ii,0} (-\partial_\tau X^i \partial_\tau X^0 + \partial_\sigma X^i \partial_\sigma X^0). \quad (3.76)$$

On the right hand sides of the equations there is an overall factor containing the time derivative of the metric. To estimate the magnitude of either side

of the equations, we realize that world-sheet time derivatives will be of the order of the energy of the string, which is of the order of the square root of the string tension: $\partial_\tau \sim \frac{1}{\sqrt{\alpha'}}$. Similarly, the world-sheet spatial derivative will be of the order of the inverse of the string length l_s , $\partial_\sigma \sim \frac{1}{l_s}$, which is of the order of the square root of the string tension. On the right hand sides, the terms are of the same order as the terms on the left, except for the factors containing derivatives of the metric. As long as these metric derivatives are several orders of magnitude smaller than the string tension scale, we can safely ignore them ¹⁰. Assuming such a background, (3.75) and (3.76) reduce to the usual flat space wave equations, and we can proceed to expand the solutions in terms of plane waves. We then impose the canonical commutation relations on the expansion coefficients ¹¹.

To complete the analysis, we impose the constraints coming from our gauge choice:

$$h_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.77)$$

¹⁰ For instance, we are tempted to be conservative and to ask for them to be roughly 6-8 orders of magnitude smaller, in order to be certain of a consistent treatment.

¹¹ Recall that $\partial_\tau X^\mu := P_\mu$ is the canonical momentum, where we must view P_μ as having its index lowered which is to be raised with the inverse metric. Not realizing this will produce nonsensical results elsewhere (such as in the computation of the energy-momentum tensor of the string) in addition to making quantization very awkward. This fact is easier to understand if we recall that P_μ and X^μ are canonically conjugate world sheet fields and not 4-vectors. With this in mind, canonical quantization means imposing $[P_\mu(\tau, \sigma), X^\nu(\tau, \sigma')] = -i\delta_\mu^\nu \delta(\sigma - \sigma')$ which, as is, does not involve the metric. Hence we do not have to do anything different at this stage of the analysis.

$$\partial_\tau X^\mu \partial_\tau X^\nu g_{\mu\nu}(X) + \partial_\sigma X^\mu \partial_\sigma X^\nu g_{\mu\nu}(X) = 0, \quad (3.78)$$

$$\partial_\tau X^\mu \partial_\sigma X^\nu g_{\mu\nu}(X) = 0. \quad (3.79)$$

By implementing our plane wave expansion for X^μ in terms of the creation and annihilation operators, we obtain the spectrum by requiring that physical states are annihilated by the constraint operators. However, when we write the constraints in terms of the canonical variables, their metric dependence becomes manifest:

$$P_\mu X'^\mu = 0 \quad (3.80)$$

$$P_\mu P^\mu + X'_\mu X'^\mu = 0. \quad (3.81)$$

We see that (3.80) is independent of the metric. Since this constraint is the origin of the Virasoro algebra, we see that it remains valid in suitably time-dependent backgrounds. However (3.81) does depend on the metric since P^μ requires the inverse metric to raise its index and X'_μ requires the metric to lower its index. Now we choose to work in a 5-d background with metric:

$$g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t), a^2(t), b^2(t)), \quad (3.82)$$

where the 5'th dimension is taken to be compact with radius $2\pi b$ ¹². With this as our background metric, (3.81) becomes (for a string wound along the 5'th dimension):

$$\begin{aligned} -E^2 + g^{ij}p_i p_j &+ \frac{2}{\alpha'}(N + \bar{N} - 2) \\ &+ g^{55}P_5 P_5 + g_{55}X'^5 X'^5 = 0, \end{aligned} \quad (3.83)$$

where all we have done is expanded out (3.81), and realized that the terms coming from the non-compact X^μ and P_μ give us the center of mass momenta and the left and right oscillator pieces, and the terms coming from X^5 have been accounted for explicitly. We know that this part of the energy contributes [22]:

$$P_5 P_5 + X'_5 X'^5 = \frac{n^2}{b^2} + \frac{w^2 b^2}{\alpha'^2}, \quad (3.84)$$

so that combined with the level matching conditions

$$nw + N - \bar{N} = 0 \quad (3.85)$$

we get

$$E = \sqrt{g^{ij}p_i p_j + \frac{4}{\alpha'}(N - 1) + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2}, \quad (3.86)$$

¹² The results in our paper easily generalize to backgrounds of any number of non-compact dimensions so long as precisely one dimension is compactified on a circle. For consistency, we could state that our background is a compactified 10 dimensional space with six dimensions compactified on a Calabi-Yau (CY)space and one dimension compactified on a circle. Since in the prototypical compactification scenarios such as the Horava-Witten model [70][71] the radius of the CY is smaller than the radius of the circle, since we will be interested in string winding modes but Calabi-Yau spaces do not admit one-cycles, we can ignore the presence of the Calabi-Yau space if we work in an effective Lagrangian description valid at energies smaller than the energy scale of CY compactification. We could always go back and work in a $9 + 1 \times S^1$ or $24 + 1 \times S^1$ space-time where we would derive the same conclusions as we do here.

where the only remnant of the level matching condition is the requirement that

$$N + nw \geq 0. \tag{3.87}$$

Thus, we see that the only effect of working in a slowly varying background is to introduce time-dependent metric factors in the obvious places in (3.86) which is otherwise identical to the result we would obtain in a static background.

CHAPTER 4

Interlude I

Having seen that a gas of massless strings which condenses around the self dual point can stabilize the radion field in a way that is consistent with fifth force constraints and the predictions of big bang cosmology, the next logical thing to do is to see if this mechanism generalizes to any number of extra dimensions. This is motivated all the more since in [19], using only massive string states, Watson and Battefeld argued that it would be difficult to stabilize more than one extra dimension. We will see in the next chapter that this won't be the case if we focus only on massless string states, which results in a cosmology of massless string modes which is also consistent with the predictions of big bang cosmology, whilst evading fifth force constraints.

CHAPTER 5

The Cosmology of Massless String Modes

5.1 Abstract

We consider the spacetime dynamics of a gas of closed strings in the context of General Relativity in a background of arbitrary spatial dimensions. Our motivation is primarily late time String Gas Cosmology, where such a spacetime picture has to emerge after the dilaton has stabilized. We find that after accounting for the thermodynamics of a gas of strings, only string modes which are massless at the self-dual radius are relevant, and that they lead to a dynamics which is qualitatively different from that induced by the modes usually considered in the literature. In the context of an ansatz with three large spatial dimensions and an arbitrary number of small extra dimensions, we obtain isotropic stabilization of these extra dimensions at the self-dual radius. This stabilization occurs for fixed dilaton, and is induced by the special string states we focus on. The three large dimensions undergo a regular Friedmann-Robertson-Walker expansion. We also show that this framework for late-time cosmology is consistent with observational bounds¹.

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5.2 Introduction

The String Gas Cosmology (SGC) program, initiated in [6, 7] (see also [55]) is a string-motivated cosmological model within which it appears possible to obtain a nonsingular cosmology in which a universe with three large spatial dimensions is dynamically generated from initial conditions in which all spatial dimensions have a similar microphysical scale. The scenario is based on coupling a gas of string and brane matter degrees of freedom to a dilaton-gravity background geometry. As initial conditions, space is chosen to be compact and of string length in all directions, and matter is taken to be a hot gas with all string and brane degrees of freedom excited. The specific stringy degrees of freedom which have both winding and momentum quantum numbers play an important role in the dynamics of the early universe. There has been a steady stream of research on this scenario over recent years (see e.g. [9, 10, 11, 16, 19, 20, 23, 27, 28, 32, 33, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 57, 58, 59, 60, 61, 72, 73, 74, 75, 76, 77, 78]).

in spite of its above mentioned main successes, SGC has encountered important obstacles (some of them yet to be resolved) which stand in the way of SGC being a complete and testable model of the early universe. These obstacles arise in attempting to implement the ideas introduced in [6] and [45] in realistic string/M-theory settings. Chief among them, was the observation that a dynamical dilaton proved crucial in stabilizing the extra dimensions (radion stabilization was first considered in [27]), and as such posed a problem for stabilization in the present epoch (see e.g. [19, 20]). However, this was not so much a general conclusion rather than an observation about the particular role the dilaton played in the specific realizations studied. Several other outstanding issues concern the question of stability of the extra dimensions to fluctuations (addressed in [32, 33] at

the level of linear cosmological perturbation theory), the phenomenology of having a space-filling fluid of strings to maintain stabilization (we do not want to overclose the universe, whilst still maintaining an effective stabilization mechanism), and the consistency in using a low-energy effective theory at compactification radii comparable to the string scale. Also, a concern has recently been raised [9, 10] concerning the ability of string winding interactions to only annihilate in three large spatial dimensions.

It is the goal of this paper to report on recent work which not only hopes to resolve some of the outstanding difficulties faced by SGC, but also points to a resulting model of the universe which is surprisingly complete, and potentially testable by observation. As such, this report constitutes the first in a series of papers [79, 31], where this particular paper will primarily elaborate on a model of the universe where extra dimensions (not just one extra dimension as in [16]) are stabilized at late times (i.e. assuming the dilaton is fixed) by a confining potential induced by a string gas which contains modes which are massless at the self-dual radius. A subsequent report will further investigate the resulting phenomenology and point to potentially testable predictions arising from the model introduced here [79]. There are already preliminary indications that the model we are studying contains a stringy mechanism for the generation of metric fluctuations [79] and may even offer applications to the construction of a nonsingular realization [31] of the Ekpyrotic/cyclic universe scenario [80, 81, 82].

Key to the results of this paper (and the phenomenology which follows) is the special role played by massless string states, whose utility has been generally neglected - see, however, the discussions in [16, 28]. In the following introductory section we begin by clarifying the philosophy of SGC, and then give a preview of some of the special properties of the massless

string states which explains their vital role in SGC. After discussing the outstanding problems of SGC, we offer our ideas for resolving these problems within our framework. In Section III, we set up our model, study the effects of a gas of strings in a background described by Einstein gravity (fixed dilaton), and derive the resulting spacetime dynamics. We show that it is possible to stabilize any number of extra dimensions, making crucial use of the massless string states. In Sections IV and V, we demonstrate that this stabilization mechanism leads to a phenomenologically acceptable late-time cosmology (modulo the outstanding issue of the horizon problem of the three large dimensions). In particular, we show that Friedmann-Robertson-Walker (FRW) expansion of the universe results after the extra dimensions have stabilized.

5.3 SGC: Philosophy, Progress and Problems

String Gas Cosmology is a paradigm constructed along similar principles as the Standard Big Bang Cosmology (BBC). Beginning with the observation of spatial homogeneity and isotropy of the universe on large scales, BBC is the cosmology that results when one couples a theory of space-time (Einstein gravity) to matter described as a set of ideal gas fluids (or a homogeneous distribution of fields as in the Inflationary Universe scenario [83], an extension of BBC which solves some but not all of its problems), and assumes “hot” initial conditions, i.e. that all matter degrees of freedom are highly excited.

The philosophy of String Gas Cosmology is similar. Instead of a homogeneous set of ideal gases or fields, one takes matter to be a gas of all string and brane degrees of freedom which string theory admits, and the background space-time is described not necessarily in terms of Einstein gravity, but in terms of the particular gravity theory (e.g. dilaton gravity)

which depends on which corner of the string theory moduli space one picks to be in. Novel features of SGC compared to BBC include the existence of extra dimensions and extra fields (form fields and scalars) and the existence of new symmetries (dualities) relating the various corners of moduli space. The existence of extended objects in the matter sector allows for a radically different coupling between matter and geometry as these objects feel the metric tensor in a manner different from what the naive application of intuition from BBC would imply. The new matter degrees of freedom also generate their own new symmetries (e.g. target space (T) duality) and cause the dynamics of spacetime to be very different than in BBC. The peculiar features of string thermodynamics, such as the existence of a limiting Hagedorn temperature [8] also serve to ensure that any stringy cosmology implemented in the spirit of BBC will have qualitatively different features.

What then are the consequences of attempting to do cosmology in the framework outlined above? In addition to the fact that T duality raises the hope to be able to construct a nonsingular cosmology [6], the main result (and in fact the main motivation for the SGC program) is that it is possible to generate a 3+1 dimensional universe through the dynamical “decompactification” of an early universe where all directions are initially taken to be compact and of similar size [6, 45, 27]. The way this works can briefly be summarized as follows: suppose we begin with a 10-dimensional toroidal universe, where all dimensions start at or around the self dual radius ($R = \sqrt{\alpha'}$). Populate this universe with a gas of strings in thermal equilibrium. Due to the presence of closed strings wound around the various cycles of the 10-torus, there is an energy expenditure associated with any expansion of a particular cycle, given by the relation:

$$E^2 \sim w^2 R^2, \tag{5.1}$$

where R is the radius of the one-cycle and w is the winding number of a string wound along that cycle. Hence, the expansion of all of the cycles of the 10-torus should be held in check by the presence of these wound strings, so long as there are enough of them around. However, we know that oppositely wound closed strings like to scatter into unwound strings through the tree level interaction $w + \bar{w} \rightarrow \dots$ (where the ellipses denote unwound strings). If we begin near a state of thermal equilibrium, with the tree level string interactions maintaining a non-zero population of wound strings, we can then ask the question under what conditions is it possible to maintain this state of thermal equilibrium? A naive dimension counting argument [6] tells us that the scattering cross-sections that describe winding mode annihilation depend on the dimensionality of spacetime. For instance, on a two dimensional torus, wound strings are very likely to intersect at some point. The same is true in three dimensions. However, in more than three spatial dimensions, the subset in phase space of initial conditions for which two wound strings intersect is of measure zero: strings will generically miss each other (for some recent concerns with this argument see [9, 10]). Hence, in a background of any number of spatial dimensions, one finds that in at most 3 spatial dimensions these strings can meet each other, and hence unwind. It is in these dimensions that the universe is then free to expand if there is an initial expansion in place, the interactions having frozen out in the other dimensions. This stringy explanation for the dimensionality of spacetime [6] has been generalised [45] to a setting where in addition to strings, one has branes of various dimensions present in the initial state of the universe.

Implementing the above argument in realistic settings has been the focus of much of the literature on SGC to date. In addition to the successes of this program, there are many open issues which remain. Among the successes is the result that isotropy of the 3-dimensions which subsequently describe our universe is a consequence of the dynamics of SGC [48]. Progress has also been made in implementing SGC for more realistic compactifications of string theory (which do not necessarily admit topological 1-cycles— a pre-requisite for the existence of winding modes) [50]. However, a major obstacle to the success of this program is the fact that in the realizations of the SGC scenario to date, a dynamical dilaton proves to be crucial in obtaining stabilization of the extra dimensions [27]. An argument [7] for the necessity of a dynamical dilaton for the stabilization of the extra dimensions is that in general relativity, matter tends to cause monotonic expansion or contraction of the universe:

$$H^2 \propto \rho, \quad (5.2)$$

and hence obtaining stabilization ($H = 0$) would be problematic. However, this intuition is flawed since it is based on assuming that one has isotropy of all dimensions. If instead we assume isotropy in the non-compact and separate isotropy in the compact dimensions then the time-time Einstein equation becomes:

$$\rho = \frac{1}{16\pi G_D} [d(d-1)H^2 + p(p-1)\mathcal{H}^2 + 2pd\mathcal{H}H], \quad (5.3)$$

where G_D is Newton's gravitational constant in the full space-time, d is the number of spatial non-compact dimensions, p the number of compact dimensions, H the Hubble rate in the non-compact dimensions, and $\mathcal{H}$ the Hubble rate in the compact dimensions. Hence one could easily have

an oscillating $\mathcal{H}$ provided H is large enough. In the case of one extra dimension, this was shown explicitly in [16].

Another reason why a dynamical dilaton was included in most of the previous work on SGC is that in the Type II string theory corners of the M-theory moduli space the dilaton appears at the same footing as the graviton. Hence, in the absence of a mechanism which stabilizes the dilaton, this field should be taken to be dynamical. A dynamical dilaton also is crucial in order for T-duality to be manifest.

However, in the current universe there is no evidence for a dynamical dilaton, and hence, if SGC is to make successful connection with current observations, it must be assumed that the dilaton is stabilized (ideally, this should be a result of string cosmology itself). A crucial question for SGC is, therefore, whether the extra dimensions remain stable after the dilaton has been fixed.

The main goal of this paper is to show that any number of extra dimensions can be stabilized in the context of General Relativity (GR) (i.e. without a dynamical dilaton), and hence to extend the results of SGC to be applicable to late time cosmology. We find preliminary indications that the resulting cosmology has rich phenomenological implications (this will be discussed in more depth in a followup report [79]). In addition to being stable to radion fluctuations there is a potential non-inflationary mechanism for generating metric fluctuations [31].

The workhorse of this cosmology are string modes which are massless at the self-dual radius, and whose role was discussed in a previous paper where the stabilization of a single extra dimension was studied [16]. These modes are related to enhanced symmetries at the self-dual radius (see e.g. [28, 21] for more general discussions). We argue that these states must play a crucial

role in SGC, and neglecting them will lead to incorrect conclusions about the cosmological evolution.

The role of these special states is also important in addressing a common objection to SGC (and all other approaches to string cosmology), namely questioning the use a low energy effective theory description (dilaton gravity or GR) in a situation when characteristic lengths are around the string scale. The loop expansion breaks down once one reaches curvature regimes comparable to the string scale, and hence calls into question the consistency of the effective theory we are interested in. However, we will be studying the physics of certain massless modes which appear at enhanced symmetry points (i.e. the self dual radius), and as such describe new degrees of freedom that appear at these special points. Since the existence of these new degrees of freedom will transcend higher order corrections, we feel justified in using their perturbative properties. Perhaps an equivalent QFT intuition would be that just as the photon mass remains zero due to gauge invariance after higher order loop corrections are taken into account, we expect the properties of these states to similarly transcend higher order loop corrections due to the enhanced symmetries associated with them.

An open issue for SGC[84] is the concern that the states which keep the radii of the extra dimensions confined might overclose the universe. This concern would indeed be valid if these states were Planck mass objects (like string states with only winding number). However, if stabilization is achieved by a fluid of strings which are massless at the self-dual radius, and which from the point of view of the large spatial dimensions behave as a gas of massless states, the overclosure concern disappears, and one can achieve an acceptable cosmology. This has already been shown in a model with one

extra dimension [16], and we will demonstrate that the situation is the same in our model (though a detailed study shall form a separate report [79]).

However, it is important to stress that these massless modes which are so important for SGC to be viable appear naturally in our framework rather than as an ad hoc input. We now turn to the discussion of the model. We have left the total number of spacetime dimensions D and the number of compact dimensions p arbitrary, with different choices corresponding to various corners of the string theory moduli space. Later we will of course be studying the case where $D - p = d = 3$ (three non-compact spatial dimensions).

5.4 The Model

We will be doing Einstein gravity in the presence of string sources. This is the late time scenario that has to result in SGC after the dilaton has been fixed. We wish to determine whether it is still possible to stabilize compact extra dimension in SGC after dilaton stabilization. We refer to this as a ‘late time’ scenario. We assume that the dilaton is robustly stabilized, and can hence be taken as a constant in what follows. We start then by assuming that the universe has the topology of $R^d \times T^p$, where d is the number of non-compact dimensions and p is the number of toroidal dimensions, both of which we leave general for the moment. We make the metric ansatz:

$$ds^2 = -dt^2 + \sum_{i=1}^D a_i^2(t) dx_i^2, \quad (5.4)$$

where D is the total number of spatial dimensions. The resulting Einstein equations ($G_\nu^\mu = 8\pi G_D T_\nu^\mu$) can be recast in the form:

$$\ddot{a}_i + \dot{a}_i \left(\sum_{j \neq i} \frac{\dot{a}_j}{a_j} \right) = 8\pi G_D a_i \left[p_i - \frac{1}{D-1} \sum_{j=1}^D p_j + \frac{1}{D-1} \rho \right]. \quad (5.5)$$

One then has to determine what the energy-momentum tensor for a gas of strings (with a fixed set of quantum numbers) is, in order to proceed. One can obtain this directly from the Nambu-Goto action for a single string (after hydrodynamically averaging)[16]. We offer a derivation of this result in the appendix. However, it turns out that one obtains the same answer if one was to introduce the following ‘matter’ Lagrangian density:

$$S_{string} = - \int \sqrt{-G} \mu_{\vec{n}, \vec{w}, \vec{p}, N, \tilde{N}}(t) \epsilon_{\vec{n}, \vec{w}, \vec{p}, N, \tilde{N}}, \quad (5.6)$$

where G is the determinant of the full space-time metric, the subscripts indicate that the number density (μ) and the energy of a given string state (ϵ) depend on its particular quantum numbers. The notation is as follows: $\vec{n}$ describes the momentum quantum numbers along the compact directions, which have been organised into a p -dimensional vector. Similarly, $\vec{w}$ describes the winding quantum numbers, N and $\tilde{N}$ are the oscillator levels of the string state and $\vec{p}$ is the center of mass momentum along the non-compact directions, organised into a $d = D - p$ dimensional vector. We see that this naively corresponds to introducing the appropriate ‘interaction energy’ term to our action. If one (we shall drop the subscripts momentarily) factors out the metric dependence of the number density

$$\mu = \frac{\mu_0}{\sqrt{g}}, \quad (5.7)$$

where g is the determinant of the spatial part of the metric tensor, one is left with:

$$S_{string} = - \int \sqrt{-G_{00}} \mu_0(t) \epsilon, \quad (5.8)$$

which, once we consider the thermodynamics of the system, corresponds to the treatment given in [7].

The mass of a closed string in a background where p dimensions are compactified on a torus, described by the metric γ_{ab} , $1 \leq a, b \leq p$, is given by [22] ($n_a, w^a \in \mathbb{Z}$):

$$m_{\vec{n}, \vec{w}, \vec{p}, N, \tilde{N}}^2 = \frac{1}{R^2} \gamma^{ab} n_a n_b + \frac{R^2}{\alpha'^2} \gamma_{ab} w^a w^b + \frac{2}{\alpha'} (N + \tilde{N} - 2), \quad (5.9)$$

where this formula is subject to the level matching constraint:

$$\tilde{N} = n_a w^a + N \quad (5.10)$$

and R describes the coordinate interval for each cycle of the torus: $x^a = \theta^a R$, $0 \leq \theta \leq 2\pi$, which can be set to unity (the physical lengths of cycles being described by the metric γ). We can re-write the above as:

$$m_{\vec{n}, \vec{w}, \vec{p}, N}^2 = (n, \gamma^{-1} n) + \frac{1}{\alpha'^2} (w, \gamma w) + \frac{2}{\alpha'} (N + \tilde{N} - 2), \quad (5.11)$$

where $(n, w) = n_a w^a$ is the standard p -dimensional real scalar product.

We now use (5.10) to solve for $\tilde{N}$ in (5.11), which gives us the following expression for the energy of a closed string in such a background:

$$\epsilon_{\vec{n}, \vec{w}, \vec{p}, N} = \frac{1}{\sqrt{\alpha'}} \sqrt{\alpha' p_{n.c.}^2 + (n, \bar{\gamma}^{-1} n) + (w, \bar{\gamma} w) + 2(n, w) + 4(N - 1)}, \quad (5.12)$$

where $\bar{\gamma}_{ab} = \gamma_{ab}/\alpha'$, and $p_{n.c.}^2$ is the center of mass momentum along the non-compact directions.

Notice that we are not ignoring the oscillator modes, as is customary in the SGC literature. This allows us to utilise the level matching constraint in such a way that the above expression for the mass of a string mode contains a perfect square when all compact dimensions are at the self dual radius

($\bar{\gamma}_{ab} = \delta_{ab}$):

$$m^2 = \frac{1}{\alpha'} \left[(n + w, n + w) + 4(N - 1) \right]. \quad (5.13)$$

This should be contrasted to the expression obtained when oscillator modes are neglected at the outset:

$$m^2 = \frac{1}{\alpha'} \left[(n, n) + (w, w) \right] \quad (5.14)$$

which only vanishes for completely unwound strings, and hence fails to capture the wound states that become massless at enhanced symmetry points. These states prove to have very special properties and should not be ignored in any study of string gas cosmology.

Returning to the question of deriving the energy-momentum of a string gas, one can insert the expression (5.12) into (5.6) to arrive at:

$$\rho_{\vec{n}, \vec{w}} = \frac{\mu_{0, \vec{n}, \vec{w}}}{\epsilon_{\vec{n}, \vec{w}} \sqrt{-G}} \epsilon_{\vec{n}, \vec{w}}^2 \quad (5.15)$$

$$p_{\vec{n}, \vec{w}}^i = \frac{\mu_{0, \vec{n}, \vec{w}}}{\epsilon_{\vec{n}, \vec{w}} \sqrt{-G}} p_{n.c.}^2 / d \quad (5.16)$$

$$p_{\vec{n}, \vec{w}}^a = \frac{\mu_{0, \vec{n}, \vec{w}}}{\epsilon_{\vec{n}, \vec{w}} \sqrt{-G} \alpha'} \left(\frac{n_a^2}{\tilde{b}_a^2} - w_a^2 \tilde{b}_a^2 \right) \quad (5.17)$$

where $\tilde{b}_a = b_a / \sqrt{\alpha'}$ (where b_a^2 is the a^{th} diagonal component of the metric of the torus). Henceforth $\vec{n}, \vec{w}$ shall serve as an abbreviation for the full set of quantum numbers, which include N and p^2 as well. Note that the above can be obtained directly from the Nambu-Goto action [16], as shown in the appendix. We can immediately infer several qualitative conclusions from the above form of the energy-momentum tensor. We see that according to (5.17), winding quantum numbers contribute a negative pressure along the compact directions whereas momentum quantum numbers do the opposite. Along the non-compact directions, the fluid of strings exerts a positive pressure as is typical for a gas of point particles.

The equations of motion (5.5) for both the compact and the non-compact dimensions which result from the above are given by:

$$\begin{aligned}\ddot{\tilde{b}}_a + \dot{\tilde{b}}_a \left(\sum_j \frac{\dot{a}_j}{a_j} + \sum_{b \neq a} \frac{\dot{b}_c}{b_c} \right) &= \frac{8\pi G_D \mu_{0,\vec{n},\vec{w}}}{\alpha'^{3/2} \sqrt{\hat{G}_a \epsilon_{\vec{n},\vec{w}}}} \left[\frac{n_a^2}{\tilde{b}_a^2} - w_a^2 \tilde{b}_a^2 + \frac{2}{(D-1)} [(w, \bar{\gamma}w) + (n, w) + 2(N-1)] \right] \\ \ddot{a}_i + \dot{a}_i \left(\sum_{j \neq i} \frac{\dot{a}_j}{a_j} + \sum_b \frac{\dot{b}_c}{b_c} \right) &= \frac{8\pi G_D \mu_{0,\vec{n},\vec{w}}}{\sqrt{\hat{G}_i \epsilon_{\vec{n},\vec{w}}}} \left[\frac{p_{n.c.}^2}{d} + \frac{2}{\alpha'(D-1)} [(w, \bar{\gamma}w) + (n, w) + 2(N-1)] \right]\end{aligned}\quad (5.18)$$

where $\hat{G}_\mu$ is the determinant of the metric without the μ 'th diagonal element. In the above, the indices $i = 1, \dots, d$ run over the non-compact dimensions and $a = 1, \dots, p$ run over the compact ones. One thing to note from (5.18) is that unwound strings with $n_a = w_a = 0$ at the oscillator level $N = 1$ do not contribute to the driving term (the right hand side) of the equation of motion for the compact dimensions. These states correspond to gravitons, but in general from the form of (5.5), we see that any matter which is pressureless along the compact dimensions (such as ordinary matter) and satisfies the equation of state ($p = \rho/d$) also does not contribute to the driving term for the compact dimensions.

The above equations are somewhat artificial in that they apply to a universe filled with a string gas with a fixed set of quantum numbers. In general, one will have a string gas that consists of a superposition of strings with many different quantum numbers. In that case, the driving terms in the above equations of motion will become:

$$\begin{aligned}\ddot{\tilde{b}}_a + \dot{\tilde{b}}_a \left(\sum_j \frac{\dot{a}_j}{a_j} + \sum_{b \neq a} \frac{\dot{b}_c}{b_c} \right) &= \sum_{\vec{n}, \vec{w}} \frac{8\pi G_D \mu_{0,\vec{n},\vec{w}}}{\alpha'^{3/2} \sqrt{\hat{G}_a \epsilon_{\vec{n},\vec{w}}}} \left[\frac{n_a^2}{\tilde{b}_a^2} - w_a^2 \tilde{b}_a^2 + \frac{2}{(D-1)} [(w, \bar{\gamma}w) + (n, w) + 2(N-1)] \right] \\ \ddot{a}_i + \dot{a}_i \left(\sum_{j \neq i} \frac{\dot{a}_j}{a_j} + \sum_b \frac{\dot{b}_c}{b_c} \right) &= \sum_{\vec{n}, \vec{w}} \frac{8\pi G_D \mu_{0,\vec{n},\vec{w}}}{\sqrt{\hat{G}_i \epsilon_{\vec{n},\vec{w}}}} \left[\frac{p_{n.c.}^2}{d} + \frac{2}{\alpha'(D-1)} [(w, \bar{\gamma}w) + (n, w) + 2(N-1)] \right]\end{aligned}\quad (5.19)$$

The number densities $\mu_{0,\vec{n},\vec{w}}$ are determined by physical considerations.

In the spirit of SGC and following the treatment given in [16], we assume

thermal equilibrium at the initial time (when the number densities are determined). In this case, we have

$$\mu_{0,\vec{n},\vec{w}} = \mu_{0,ref} e^{-\beta\epsilon_{\vec{n},\vec{w}}} e^{\beta\epsilon_{ref}}, \quad (5.22)$$

where the subscript ‘ref’ refers to some reference state which we are free to pick at our convenience. One has to wonder what constitutes the thermal bath that is a pre-requisite for any thermodynamical treatment. We have already seen that gravitons and photons (matter which only exists in the non-compact dimensions) do not contribute to the driving term for the equations of motion for the b_a ’s, and are an ideal candidate for such a thermal bath. The tree level processes $h_{\mu\nu} \rightarrow w + \bar{w}$ provide the appropriate interactions that thermally couple our stringy matter to this bath. With this in mind, (5.22) implies that the driving term for the non compact dimensions becomes:

$$\frac{8\pi G_D \mu_{ref} e^{\beta\epsilon_{ref}}}{\alpha' \sqrt{\hat{G}_a}} \sum_{\vec{n}, \vec{w}, N, p^2} \frac{e^{-\beta\tilde{\epsilon}_{\vec{n}, \vec{w}, N, p^2}/\sqrt{\alpha'}}}{\tilde{\epsilon}_{\vec{n}, \vec{w}, N, p^2}} \left[\frac{n_a^2}{\tilde{b}_a^2} - w_a^2 \tilde{b}_a^2 + \frac{2}{(D-1)} [(w, \bar{\gamma}w) + (n, w) + 2(N-1)] \right] \quad (5.23)$$

Where $\tilde{\epsilon}$ is a dimensionless quantity (energy expressed in string units: $\epsilon = \tilde{\epsilon}/\sqrt{\alpha'}$), which according to (5.12) is a number of order unity for the lowest lying string states.

With this realization of thermal equilibrium, were we to define the partition function as

$$Z(\beta, a_i, b_a) := \sum_{\vec{n}, \vec{w}, N, p^2} e^{-\beta\epsilon(a_i, b_a)_{\vec{n}, \vec{w}, N, p^2}}, \quad (5.24)$$

then one could derive the components of the energy-momentum tensor as follows (c.f. [7]):

$$P_i = \frac{1}{\beta} a_i \frac{\partial Z}{\partial a_i} = \frac{1}{\beta} \frac{\partial Z}{\partial \lambda_i} ; \quad a_i = e^{\lambda_i} \quad (5.25)$$

$$P_a = \frac{1}{\beta} b_a \frac{\partial Z}{\partial b_a} = \frac{1}{\beta} \frac{\partial Z}{\partial \lambda_a} ; \quad b_a = e^{\lambda_a} \quad (5.26)$$

$$E = \frac{-1}{\beta} \frac{\partial Z}{\partial \beta}, \quad (5.27)$$

where P_μ (and in a similar way, E) is defined through the equation:

$$\frac{\mu_{ref} e^{\beta \epsilon_{ref}}}{\sqrt{G}} P_\mu = p_\mu \quad (5.28)$$

We can use this to rewrite the condition that the driving term for equations of motion vanishes for any particular direction, as the action of a linear differential operator on Z :

$$\left[a_\mu \frac{\partial}{\partial a_\mu} - \frac{1}{D-1} \left(\sum_\nu a_\nu \frac{\partial}{\partial a_\nu} + \frac{\partial}{\partial \beta} \right) \right] Z = 0. \quad (5.29)$$

One can then rephrase the question of whether or not stringy matter can stabilize extra dimensions in terms of the existence of solutions to this equation. Were we to find simultaneous solutions to this system of first order PDE's for all compact directions ($\mu = a$, for a given β), we will have determined the nature and existence of stabilized extra dimensions. Although we will show further on that, in the particular regime we are interested in, we will have a much simpler means of proceeding, this method is certainly more general (though less tractable). It might be necessary to resort to this method in high temperature regimes and in situations where the compactifications we study are not so straightforward. In fact, if all we are interested in is proving the existence of stabilized extra dimensions, then one could imagine reformulating this question in terms of the existence of solutions to this PDE, for which a great deal is already known.

However, since we are interested in late times (i.e. post dilaton stabilization), we find after inspection of (5.23) that the summation is actually quite tractable. Taking a closer look at the Boltzmann factor that weights all terms in the summation:

$$e^{-\frac{\beta}{\sqrt{\alpha'}}\tilde{\epsilon}}. \quad (5.30)$$

we see that the argument of the exponential is the factor $\beta/\sqrt{\alpha'}$ multiplied by a term that is of order unity. Now for all string theories there exists a limiting temperature, known as the Hagedorn temperature [8], which, independent of the particular theory, is always of the order $\sqrt{\alpha'}$ [85]:

$$\beta_H \sim \sqrt{\alpha'}. \quad (5.31)$$

If we set the string scale to be equal to the Planck scale, then we see that if the temperature is even slightly (let us say a factor of 10) below this scale, that is if:

$$\frac{\beta}{\sqrt{\alpha'}} \geq 10, \quad (5.32)$$

then any term in the summation (5.23) which corresponds to an $\tilde{\epsilon}$ which is anything other than zero will contribute vanishingly. Hence, the Boltzmann weight approaches a window function which projects out all but the massless modes in the summation. That is:

$$\sum_{\vec{n}, \vec{w}, N, p^2} \frac{e^{-\beta \tilde{\epsilon}_{\vec{n}, \vec{w}, N, p^2} / \sqrt{\alpha'}}}{\tilde{\epsilon}_{\vec{n}, \vec{w}, N, p^2}} [\dots] \rightarrow \lambda \sum_{m^2=0} [\dots] \quad (5.33)$$

and hence the sum becomes very easy to calculate. Note that the condition (5.32) gives us an operational definition of ‘late times’.

Turning to the question of evaluating the summation (5.23) in light of (5.33), we first need to know which quantum numbers correspond to massless states for a given form of the metric $\tilde{\gamma}$ of the torus. To do this,

suppose we start out with all toroidal dimensions compactified at the self-dual radius ($\tilde{\gamma} = I$). Then the formula for the mass of a closed string in this background is given by (5.13):

$$m^2 = \frac{1}{\alpha'} \left[(n+w, n+w) + 4(N-1) \right],$$

from which we read off that the massless states are those which simultaneously satisfy the following set of equations:

$$(n+w, n+w) = 4(1-N) \quad (5.34)$$

$$N + (n, w) \geq 0 \quad (5.35)$$

where the last equation is the level matching constraint. We immediately see that the only possibilities for massless states at this radius (and as it turns out all others) are those with oscillator levels 0 or 1. We catalogue the quantum numbers of all of these massless states below:

N	(n, n)	(w, w)	(n, w)
1	0	0	0
1	1	1	-1
0	1	1	1
0	2	2	0
0	1	3	0
0	3	1	0
0	4	0	0
0	0	4	0

It should be noted that the exact realisation of these massless modes depends on the number of extra dimensions available (e.g. for 1 or 2 extra dimensions, it is not possible to satisfy $(w, w) = 3$ etc.). Of most interest to us (for reasons to be made clear shortly) are the first four possibilities

mentioned in the above table. The first possibility ($N = 1, n = w = 0$) corresponds to unwound gravitons. These states do not contribute to the driving term for the compact directions (5.23) at all. The second possibility ($N = 1, n = \pm w, (w, w) = 1$) corresponds to singly wound strings with an equal and opposite quantum of momentum along the same dimension, at oscillator level 1. The third possibility ($N = 0, n = w, (w, w) = 1$) corresponds to a singly wound state at oscillator level zero, with one quantum of momentum of the same sign along the same dimension. The second class of states are all at oscillator level zero, and correspond to various multiply wound/unwound strings with/without motion along various cycles of our torus. It turns out that if we consider the masses of states with the quantum numbers tabulated above as the metric fluctuates around the self dual radius, we find that only the first class of states (the first four in the above table) remain massless to first order. We see this by perturbing the metric as follows:

$$\begin{aligned}\tilde{\gamma} &= I - \Delta \\ \tilde{\gamma}^{-1} &= I + \Delta + \sum_{k=2} \Delta^k \quad \quad \quad ||\Delta|| < 1\end{aligned}$$

where Δ is not necessarily small, but has a matrix norm of less than 1 (so that $\tilde{\gamma}^{-1}$ can be written thus). Expanding the formula for the mass of a closed string, we get

$$\begin{aligned}\alpha' m^2 &= (n, \tilde{\gamma}^{-1} n) + (w, \tilde{\gamma} w) + 2(n, w) + 4(N - 1) \\ &= (n + w, n + w) + 4(N - 1) - (w, \Delta w) + (n, \Delta n) + \sum_{k=2} (n, \Delta^k n) + (w, \Delta^k w)\end{aligned}\tag{5.36}$$

and hence, for the mass difference compared to the value for $\Delta = 0$,

$$\alpha' \delta m^2 = (n, \Delta n) - (w, \Delta w) + \sum_{k=2} (n, \Delta^k n) - (w, \Delta^k w).\tag{5.38}$$

We see that only the first class of states remain massless to first order. For the first three sets of states, where the quantum number vectors n, w are equal to each other up to a sign, this is easy to see. For the states with quantum numbers given by $N = 0$, $(n, n) = (w, w) = 2$, $(n, w) = 0$, only the states with entries in the same rows have vanishing fluctuations. This is because the condition $(n, w) = 0$, in conjunction with $(n, n) = (w, w) = 2$ implies that these states either have no, or both entries in common. The contribution from the metric fluctuations would be non-vanishing in the former case, but for the latter, states of the form below remain massless to first order:

$$\begin{aligned}
w &= (1, 1) \quad n = \pm(1, -1), \\
w &= (1, -1) \quad n = \pm(1, 1), \\
w &= (-1, 1) \quad n = \pm(1, 1), \\
w &= (-1, -1) \quad n = \pm(1, -1),
\end{aligned}$$

where we have only indicated the non-vanishing entries in the above. At late times, when $\beta/\sqrt{\alpha'} \gg 1$, all states which do not remain exactly massless close to the self-dual radius are projected out of the summation (5.23). To emphasize this point, let us consider an epoch when the temperature is two orders of magnitude below the Planck energy. Then the Boltzmann factor goes as e^{-100} for all states that are not exactly massless close to the self dual radius. We will show later that at exactly the self dual radius, the summation over all the remaining states tabulated above sums to zero, and hence their effects on the dynamics of the extra dimensions is vanishing.

Now that we have determined the properties of the string states which will enter the summation (5.23), we now have to compute their contribution

to the driving term and sum over all the possibilities. Recalling that the unwound graviton states ($N = 1, n = w = 0$) do not contribute to the driving term, we find that the contribution of the states with quantum numbers $N = 1, n = \pm w, (w, w) = 1$ to (5.23) for the a^{th} compact direction is:

$$\frac{8\pi G_D \mu_{0ref}}{\alpha'^{3/2} \sqrt{\hat{G}_a}} \frac{2}{|p_{n.c.}|} \left[\frac{1}{\tilde{b}_a^2} - \tilde{b}_a^2 + \frac{2}{D-1} \left(\sum_{c=1}^p \tilde{b}_c^2 - p \right) \right], \quad (5.39)$$

where if we take the reference energy to be precisely one of the massless states, the exponential prefactor in (5.23) cancels. The factor $2/|p_{n.c.}|$ comes from the overall degeneracy of the states which can appear either as, for example $w = (0, 0, 0 \dots 1, 0 \dots)$, $n = (0, 0, 0 \dots -1, 0 \dots)$ or with the opposite signs, with the factor $|p_{n.c.}|$ coming from the factor of energy in the denominator in the summand in (5.23). The states with quantum numbers $N = 0, n = w, (w, w) = 1$ sum to yield an identical contribution to the driving term. It is straightforward to show that the states within the class $N = 0, (n, n) = (w, w) = 2, (n, w) = 0$, that remain massless to first order also yield a similar driving term, but now with the prefactor $8(p-1)$, instead of 2. The factor $p-1$ has the combinatorial origin of being the number of ways one can pick two entries to be identical out of p choices, and 8 corresponds to the overall degeneracy of these states (as indicated above). One might be worried that introducing such states with $(w, w) = 2$ might force us to consider off-diagonal elements for our toroidal metric γ_{ab} , as we now have strings that diagonally wrap the torus. It is in fact true that for a *single* diagonally wound string, the stress energy tensor will have off-diagonal components and that these should be matched by off diagonal elements in our toroidal metric. However, recall that we are considering a string gas, which at the point in moduli space we begin in ($\gamma_{ab} = \delta_{ab}$), will consist of massless quantum numbers which will democratically wrap

along all cycles of the torus with equal probability. That is, singly wound strings will wind along all cycles of the torus with equal probability. In addition, strings wound around more than one cycle (such as the states with $(w, w) = 2$) will also wrap any given pair of cycles with equal probability. When we sum over all the quantum numbers, we will invariably encounter summing over states with winding number vectors that are opposite in sign, which results in the cancellation of off-diagonal entries in the net stress energy tensor (see appendix).²

Denoting $\sum_{c=1}^p \tilde{b}_c^2$ as $(\tilde{b}, \tilde{b})$, we then find that the total contribution to the driving term from states that remain exactly massless near the self dual radius is:

$$\frac{8\pi G_D \mu_{0ref}}{\alpha'^{3/2} \sqrt{\hat{G}_a}} \frac{(8p-4)}{|p_{n.c.}|} \left[\frac{1}{\tilde{b}_a^2} - \tilde{b}_a^2 + \frac{2}{D-1} [(\tilde{b}, \tilde{b}) - p] \right]. \quad (5.40)$$

The condition for this driving term to vanish yields the solution which corresponds to a stabilization of the extra dimensions. This can also be interpreted as a global minimum of the potential for the radion with vanishing amplitude (thus avoiding the “no-go theorem for radion stabilization by Giddings [87] which studies local minima of the radion potential with positive value). The condition that the driving term vanish is:

$$\frac{1}{\tilde{b}_a^2} - \tilde{b}_a^2 + \frac{2}{D-1} [(\tilde{b}, \tilde{b}) - p] = 0. \quad (5.41)$$

² However the general issue of off-diagonal elements of the toroidal metric (which correspond to complex structure moduli), is an important one, as one would also have to address how these are stabilized in this framework. It turns out that this same string gas that we have introduced also stabilizes the shape moduli of the torus. In [23] and [18], it was shown that the shape moduli decouple from the radial moduli, and are stabilized by the effects of the string gas in an analogous way to the radial moduli. A similar result was uncovered in [86].

We see that this can be recast as a quadratic equation for $\tilde{b}_a^2$, where the coefficients are formally the same for all a . Hence, if a solution exists, it must be the same for all a . Thus we find that the extra dimensions are isotropically stabilized if they are stabilized at all. We now insert the ansatz appropriate to this observation ($\tilde{b}_a = \lambda, \forall a$) into the above to obtain the condition:

$$\frac{1}{\lambda^2} - \lambda^2 + \frac{2p}{D-1}[\lambda^2 - 1] = 0. \quad (5.42)$$

Using the usual technique to solve a quadratic equation, we find a solution to be:

$$\lambda^2 = \frac{\frac{2p}{D-1} + \sqrt{(\frac{2p}{D-1})^2 - 4[\frac{2p}{D-1} - 1]}}{2[\frac{2p}{D-1} - 1]}. \quad (5.43)$$

Were we to define $c = 2p/(D-1)$, we find that the above simplifies into the formula:

$$\lambda^2 = \frac{c + (c-2)}{2(c-1)} = 1. \quad (5.44)$$

Thus, we have shown that the extra dimensions are stabilized isotropically at the self-dual radius.

To complete the analysis we need to show that the fixed point determined by (5.44) is indeed a stable equilibrium point. A quick way to see this is to observe that the potential which determines the radion dynamics is minimized at this point. This can also be seen by inserting the driving term (5.40) into (5.20), and expanding $\tilde{b}_a$ as $\tilde{b}_a = 1 + \Gamma_a$. Our equations of motion become:

$$\ddot{\Gamma}_a + \dot{\Gamma}_a(dH + \sum_{c \neq a} \dot{\Gamma}_c) + \frac{8\pi G_D 4(8p-4)\mu_0^{ref}}{a^d \alpha'^{\frac{3+(p-1)}{2}} |p_{n.c.}|(D-1)} [(D-2)\Gamma_a - \sum_{c \neq a} \Gamma_c] = 0, \quad (5.45)$$

where H is the Hubble factor in the d non-compact directions. Interpreting the driving term as $\partial_a V$ where the derivatives are taken with respect to Γ_a , we find that the Hessian matrix ($H_{ab} = \partial_a \partial_b V$), up to a (positive) factor, is

given by:

$$\begin{pmatrix} D-2 & -1 & -1 & \dots & -1 \\ -1 & D-2 & -1 & \dots & -1 \\ -1 & -1 & D-2 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & D-2 \end{pmatrix} \quad (5.46)$$

where we remind the reader that this is a p -dimensional matrix. The eigenvalues of this matrix are $D - 1 - p$ ($= d - 1$), and $D - 1$, the former appearing once and the latter with a degeneracy of $p - 1$. These are all clearly positive, and hence we conclude that the fluctuations around the self-dual radius in all directions are indeed stable.

Let us summarize the method and discuss the result. We determined the energy-momentum tensor of a gas of closed strings and observed that only the massless string modes will be present in any significant number if we start the evolution in a hot thermal state (an observation that will also be crucial for the success of the late time phenomenology). We find that, after accounting for all the relevant states, their quantum numbers and their degeneracies, we end up with a driving term (5.40) for the radion fields which stabilizes any number of extra dimensions at the self-dual radius. This result is a non-trivial result since it was obtained in the setting of general relativity, i.e. with fixed dilaton. Thus, we have shown that SGC provides a mechanism for stabilizing extra dimensions in the present epoch.

It is a natural question to ask whether or not it was necessary to introduce stringy states in order to affect this stabilization, as one could imagine any such states which become massless at special points of moduli space could have the same effect. This question will certainly occupy us in future work. However at present, we wish to point out that (as we will see

further on), it is a property peculiar to extended objects that we can obtain moduli stabilization in such a way as to be consistent with observational bounds from late times.

We wish to end this section with a comment on the states which are massless at the self-dual radius but whose energy is not minimized at this point (the second class of quantum numbers in the previous table). Since they are tachyonic either for small or for large radii, they should probably be excluded from consideration from the outset. Even if they are included, however, we note that their effects cancel in the driving term of the compact dimensions at the self-dual radius. This can easily be seen by realising that these states appear in T-dual combinations ($\vec{w} \leftrightarrow \vec{n}$). Considering the driving term at the self-dual radius:

$$\frac{8\pi G_D \mu_{ref}}{\alpha'^{3/2} \sqrt{\hat{G}_a} |p_{n.c.}|} \left[n_a^2 - w_a^2 + \frac{2}{(D-1)} [(w, w) + (n, w) + 2(N-1)] \right] \quad (5.47)$$

we see that T-duality will ensure that the first two terms cancel each other in summing over all states. It is easy to check, after accounting for the correct degeneracies of each set of quantum numbers, that the factors grouped in the inner square brackets also sum to zero. This completes the demonstration of stabilization at the self dual radius.

5.5 Spacetime Dynamics of the Non-Compact Dimensions

We would now like to consider the resulting cosmology for the non-compact dimensions after the extra dimensions have been stabilized (in this section, we will take $d = 3$ independent of the choice for D and p). Before we do this we would like to discuss several outstanding issues that should not be overlooked. Our first issue concerns an important consistency check concerning general relativity: we should check that our energy momentum

tensor is consistent with the covariant conservation of the Einstein tensor:

$$\nabla_\mu T^\mu_\nu = 0. \quad (5.48)$$

This condition yields the following series of equations (one for each value of the index ν in the above):

$$\dot{\rho} + \sum_{i=1}^d \frac{\dot{a}_i}{a_i}(\rho + p^i) + \sum_{a=1}^p \frac{\dot{b}_a}{b_a}(\rho + p^a) = 0 \quad (5.49)$$

$$\partial_i p^i = 0 \quad (5.50)$$

$$\partial_a p^a = 0. \quad (5.51)$$

The first condition is none other than the continuity equation, and is trivially satisfied by (5.15)-(5.17). This arises from the time derivative of ρ precisely cancelling the terms proportional to the Hubble factors. The remaining equations are also trivially satisfied as a consequence of the spatial homogeneity of our setup.

The second issue concerns the equation of state parameter for the pressure along the non-compact directions. We know from (5.16) that the pressure along the non-compact directions of this string gas is always positive. However, we see from (5.17) that the pressure along the compact directions can be either negative or positive. If we want to avoid violations of the dominant energy condition (DEC) we must ensure that the equation of state parameter $\omega = p/\rho$ remains bounded from below:

$$-1 \leq \omega. \quad (5.52)$$

Consider now the relationship between (5.15) and (5.17) for the states that we have found give us stabilization (those with quantum numbers

$\vec{n} = \pm \vec{w}, (w, w) = 1, N = 1$ or $\vec{n} = \vec{w}, (w, w) = 1, N = 0$). If $p^a = \omega^a \rho$, then

$$\omega^a = \frac{\tilde{b}_a^{-2} - \tilde{b}_a^2}{\tilde{b}_a^{-2} + \tilde{b}_a^2 - 2 + \alpha' p_{n.c.}^2}. \quad (5.53)$$

Since the string states we are considering are massless when the scale factor $\tilde{b}_a$ is at its self-dual value, they will have non-zero momentum along the non-compact directions, which will assume its thermal expectation value if we are in thermal equilibrium. If we are in a sufficiently hot regime, we can always ensure that ω^a satisfies

$$-1 \leq \omega \leq 1, \quad (5.54)$$

where, long after the stabilization has been achieved, and the ambient temperature has cooled down considerably, one has a robust stabilization mechanism that keeps the compact dimensions locked at the self dual radius, where the equation of state parameter vanishes. Hence we can easily arrange a situation where the DEC is not violated in our model even though the compact dimensions are undergoing damped bounces. This is a novel result in the context of GR and was uncovered first in our study of this model in the case where we only had one extra dimension [16].

Turning now to the issue of the resulting spacetime dynamics of the non-compact directions, which we take to be homogenous and isotropic, we remind the reader of the Einstein equations applied to our anisotropic metric, Eq. (5.5). We consider a situation where the dominant matter component of the universe as a whole is in the form of radiation; that is, matter which has no pressure along the compact directions and which satisfies the equation of state

$$p = \rho/3. \quad (5.55)$$

Considering the effect of this matter on the dynamics of the compact dimensions, we have

$$\ddot{b}_a + \dot{b}_a \left(3H + \sum_{c \neq a}^p \frac{\dot{b}_c}{b_c} \right) = 8\pi G_D b_a \left[-\frac{1}{D-1} dp + \frac{1}{D-1} \rho \right]. \quad (5.56)$$

Thus, we see that such matter does not contribute to the dynamics of the compact dimensions. Hence, the stringy matter studied in the previous section will be the only factor at play in the dynamics of these dimensions, except of course for the Hubble damping factor due to the expansion of the large dimensions. To put it in another way, it is consistent with the stabilization mechanism we have studied to have radiation like matter drive the expansion of the non-compact dimensions. We see through (5.3), (5.5) and (5.49), that the standard FRW expansion of the non-compact dimensions results:

$$\rho = \frac{3}{8\pi G_D} H^2 \quad (5.57)$$

$$\frac{\ddot{a}}{a} + 2H^2 = 8\pi G_D p \quad (5.58)$$

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (5.59)$$

Hence, it is easy to realise a post-stabilization radiation dominated FRW expansion of the universe. We can also obtain dust dominated evolution in our model, but this is not such a straightforward matter. In fact, obtaining a dust driven expansion results in a definite prediction of SGC: if string gases are indeed responsible for present day stabilization of extra dimensions in our model, then the dark matter will necessarily have to be extra dimensional in nature. This issue was studied in [16] and we repeat the argument here. Considering (5.5) for a compact dimension, any matter which only exists in the non-compact dimensions ($p^a = 0$) which satisfies the equation of state $p^i = 0$ (i.e. all pressures vanish) will necessarily lead

to expansion of the compact directions and will derail any stabilization mechanism we might have had in place. Hence, if we take the dominant matter content of the universe to have the equation of state of dust, then it must necessarily exert pressure along the compact directions which satisfies the following equation of state when the compact dimensions are all at the self-dual radius:

$$\left[p^a - \frac{1}{D-1} \sum_{b=1}^p p^b + \frac{1}{D-1} \rho \right]_{b_a=\sqrt{\alpha'}} = 0. \quad (5.60)$$

This implies, since we have isotropically stabilized at the self-dual radius, that this matter satisfies the equation of state

$$r = -\rho/(d-1) = -\rho/2, \quad (5.61)$$

where r is the pressure along any of the compact dimensions. There is a candidate within our framework for 3 large spatial dimensions (see [16]), namely the stringy states with the quantum numbers

$$|p_{n.c.}| = 0, N = 2; n^a = 0, w^a = \pm 2. \quad (5.62)$$

It can easily be checked that such matter satisfies the required equation of state. These states contribute to the stability of the extra dimensions rather feebly when phenomenological bounds are accounted for [16], but preserve the stability of the extra dimensions nevertheless (the massless states being dominant in their contribution to the stabilization mechanism). Hence we can take these states as a candidate for the dark matter responsible for our present FRW expansion ³

³ String winding states as candidates for dark matter were recently also considered in [69, 88].

The resulting FRW equations, once the extra dimensions have stabilized are given by:

$$\rho = \frac{3}{8\pi G_D} H^2 \quad (5.63)$$

$$\frac{\ddot{a}}{a} + 2H^2 = 8\pi G_D \frac{(p+2)}{2(D-1)} \rho \quad (5.64)$$

$$\dot{\rho} + 3H(\rho + p) = 0 \quad (5.65)$$

where p on the right hand side of the above refers to the number of compact dimensions. Hence we have shown that any epoch of late time FRW cosmology can result from our model post stabilization. We now turn to a brief discussion of the phenomenology of this model.

5.6 Phenomenology

Most of what is to appear in this section appears in [16], where the phenomenology of a string gas used to compactify one extra dimension at the self dual radius is discussed. Since the string modes used in this work are an immediate generalization of what was used in [16] for one extra dimension to the case of many extra dimensions, the resulting phenomenological bounds on the scenario from the point of view of late time cosmology are the same. Since we plan to give a detailed study of the phenomenology of this model in a future work [79], we here present only the briefest discussion.

There are three key aspects to our phenomenology that we need to discuss: the first being that we would like not to overclose the universe with the fluid of closed strings (which behaves like hot dark matter from the 4-d perspective). The second aspect, is that we do not want to have too few of these strings such that the stabilization mechanism is ineffective. In words, we would like to show that it is possible to introduce an effective stabilization mechanism without overclosing the universe. The third aspect

is that we do not want the dynamics of the extra dimensions from the 4-d perspective to introduce any long range scalar interactions (no fifth forces).

As a starting point, consider (5.45), which in normal coordinates has the form:

$$\ddot{\Gamma} + 3H\dot{\Gamma} + k\Gamma = 0. \quad (5.66)$$

The value for k (the spring constant) will differ by a factor of order unity depending on precisely which mode we are looking at, but for an order-of-magnitude estimate this factor is irrelevant. An upper and a lower bound on this spring constant result (in the case of the lower bound) from requiring that the stabilization mechanism be effective, and (in the case of the upper bound) from requiring that the metric varies on a time scale that is many orders of magnitude (let us say 10^6) longer than the string scale [16] (otherwise the effective field theory analysis would not be justified).

The lower bound is given by the value for k which yields critical damping , $k_c = 9H^2/4$ and the upper bound by the string tension. Thus, we require

$$9H^2/4 \leq k \leq 10^{-6}/(2\pi\alpha'). \quad (5.67)$$

From (5.45) this implies

$$H^2 \leq \frac{8\pi G\mu_0^{ref}}{a^3\alpha'^{\frac{3+(p-1)}{2}}|p_{n.c.}|} \leq \frac{10^{-6}}{2\pi\alpha'}, \quad (5.68)$$

neglecting factors of order unity. Using the relationship

$$G_D = G_3 \times Vol \ T^p = G_3(2\pi\sqrt{\alpha'})^{p/2} \quad (5.69)$$

between the higher dimensional gravitational constant G_D and Newton's constant G_3 in our $3 + 1$ dimensional space-time, and setting the string scale to the Planck scale implies

$$2\pi\alpha' = G_3. \quad (5.70)$$

Thus, (5.68) becomes:

$$H^2 \leq \frac{\mu_0}{a^3 |p_{n.c.}|} \leq \frac{10^{-6}}{2\pi\alpha'} . \quad (5.71)$$

Furthermore, from (5.15) we see that in order not to overclose the universe with these strings we require that the 4-dimensional energy density ($\rho_4 = \rho_D \times Vol T^p$) be several orders of magnitude less than the critical density:

$$\rho = \frac{\mu_0 |p_{n.c.}|}{a^3} \leq 10^{-4} \rho_{crit} . \quad (5.72)$$

Taking $\rho_{crit} = 10^{-29} g/cm^3$, we find that this bound translates into

$$\mu_0 \leq 10^{-4} 10^{-10} eV^4 |p_{n.c.}|^{-1} . \quad (5.73)$$

If we parametrize the momentum along the non-compact directions as

$$|p_{n.c.}| = 10^{-\gamma} eV = 10^{-\gamma} eV , \quad (5.74)$$

then (5.73) becomes:

$$\mu_0 \leq 10^{\gamma-41} GeV^3 , \quad (5.75)$$

which is stronger than the upper bound in (5.71), whereas the lower bound implies:

$$\mu_0 \geq H^2 |p_{n.c.}| \sim 10^{-93-\gamma} GeV^3 . \quad (5.76)$$

Hence, we conclude that:

$$10^{-93-\gamma} GeV^3 \leq \mu_0 \leq 10^{-32+\gamma} GeV^3 , \quad (5.77)$$

which is easily satisfied. The remaining constraint comes from requiring that from the perspective of the effective 4-d theory, the masses of the fluctuations are sufficiently high so as not to mediate long range ‘fifth forces’. Observational bounds require the mass of these scalars to be greater than $10^{-12} GeV$. Since the spring constant corresponds to the masses

squared of these fluctuations, (5.71) implies that:

$$\mu_0 \geq 10^{-33-\gamma} GeV^3, \quad (5.78)$$

which, in conjunction with (5.77), leads to the condition

$$10^{-33-\gamma} GeV^3 \leq \mu_0 \leq 10^{-41+\gamma} GeV^3. \quad (5.79)$$

This is easy to satisfy for any $\gamma \geq 4$.

Hence, we have demonstrated that the stabilization mechanism, because of some very novel aspects of the string gas energy-momentum tensor, not only offers a robust stabilization mechanism, but does so in a way that is phenomenologically consistent. That is, we can obtain this stabilization without overclosing the universe and violating any fifth force constraints. This is in marked difference with previous attempts at obtaining stabilizing effects for extra dimensions by introducing new fields—where the energy density that appears in the energy-momentum tensor and the mass of the resulting scalar field (or the spring constant for the radion fluctuations) in the effective field theory are usually proportional to each other. We see that for stringy matter, they are markedly different in that $\rho \propto |p_{n.c.}|$, whereas $k \propto |p_{n.c.}|^{-1}$. A more thorough study of the phenomenology of this scenario, where more issues (including a demonstration of the stability of this model to fluctuations) will be tackled, will be the subject of a future report [79].

5.7 Conclusions

In this report, we have studied the effects of a gas of closed strings on the dynamics of a homogeneous but anisotropic space-time described by General Relativity (fixed dilaton) where several dimensions are toroidally compactified. The modes that turn out to be relevant are the string modes which are massless at the self-dual radius. In the context of a hot early

universe we show, based on thermodynamical considerations, that these are the dominant modes. We demonstrated that these modes lead to the isotropic stabilization of the extra dimensions, thus generalizing the results of [16] which were derived in the case of only one extra dimension. We also showed that the dominance of these massless modes is crucial to the late-time phenomenological viability of the scenario, and leads to a scenario consistent with several observational bounds. We feel that this formalism is the beginning of a promising avenue of research, which we believe may result in a complete and testable model of the universe. We already have promising indications that this framework is capable of modelling a non-singular bouncing cosmology [31], and potentially provides a stringy mechanism for the generation of metric fluctuations. Our ultimate hope is to extract testable predictions of this formulation of string gas cosmology, the prospects for which seem very promising.

5.8 Appendix— The string gas energy-momentum tensor

In this appendix, we derive the energy-momentum tensor of a string gas from micro-physical considerations. That is to say, we first arrive at the space-time energy-momentum tensor of a single string, after which we perform a hydrodynamical averaging to obtain the result for a string gas. We wish to comment that there are several non-trivial issues to address when considering string propagation on a time dependent background. Intuitively however, one is tempted to conclude that provided the metric of space-time is varying on time scales much longer than the string scale, we should be able to proceed (as we do here) in the ‘adiabatic’ approximation. We shall take this for granted in what follows, referring the reader to [16] for a justification of this approximation.

To begin with, consider the Nambu-Goto action for a single string:

$$S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-h}, \quad (5.80)$$

with the worldsheet metric h_{ab} defined by:

$$h_{ab} = g_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu. \quad (5.81)$$

The space-time metric, which in the above context is generically a function of the worldsheet fields X^μ , is assumed to have the form:

$$g_{00} = g_{00}(X^0), \quad (5.82)$$

$$g_{ij} = \delta_{ij} a^2(X^0), \quad (5.83)$$

$$g_{ab} = \gamma_{ab}(X^0). \quad (5.84)$$

Variation of (5.80) with respect to $g_{\mu\nu}$ gives us the *spacetime* energy-momentum tensor through the expression:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}. \quad (5.85)$$

Arbitrary variations of the background metric induce variations of the worldsheet metric in the following manner:

$$\delta^{\lambda\beta} g_{\mu\nu} = \delta_\mu^\lambda \delta_\nu^\beta \delta^{D+1}(X^\tau - y^\tau) \rightarrow \delta^{\lambda\beta} h_{ab} = \partial_a X^\lambda \partial_b X^\beta \delta^{D+1}(X^\tau - y^\tau). \quad (5.86)$$

The unmatched indices λ and β mean that we perturb only these components of the metric tensor, and $\delta^{D+1}(X^\tau - y^\tau)$ is a delta function in $D + 1$ space-time dimensions. This variation results in an expression for the energy-momentum tensor for a single string:

$$T^{\mu\nu} = \frac{-1}{\sqrt{-g}2\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu \delta^{D+1}(X^\tau - y^\tau). \quad (5.87)$$

We now pick a gauge to work in. We choose to work in conformal gauge, defined by:

$$h_{ab} = \lambda \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (5.88)$$

where we keep this up to an arbitrary positive factor. From (5.81), we see that this gauge choice implies the conditions:

$$g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu + g_{\mu\nu} X'^\mu X'^\nu = 0 \quad (5.89)$$

$$g_{\mu\nu} \dot{X}^\mu X'^\nu = 0. \quad (5.90)$$

Even though we are in a (weakly) time-dependent background (in particular one that is not flat), it can be shown that one can still make this gauge choice simultaneous with the condition:

$$X'^0 = 0, \quad (5.91)$$

where the prime denotes differentiation with respect to the spacelike worldsheet co-ordinate. We will use these conditions repeatedly in what follows.

Upon examining (5.87), we see that in order to make use of the delta functions in the integrand, one has to use the following transformation:

$$d^2\sigma = \frac{dX^0 dX^a}{|\dot{X}^0| |X'^a|}, \quad (5.92)$$

where X^a is the string co-ordinate field along any wound compact direction (which one we pick will turn out to be insignificant). The terms in the denominator arise in the evaluation of the Jacobian of this transformation, subject to (5.91). Note that we picked the particular co-ordinates X^0 and X^a because they are monotonic functions of σ^0 and σ^1 respectively. Using the constraints (5.89) and (5.90), we see that:

$$\dot{X}^0 = \frac{2\pi\alpha'}{\sqrt{-g_{00}}} \sqrt{g^{ij}P_i P_j + \frac{1}{(2\pi\alpha')^2} g_{ij} X'^i X'^j}, \quad (5.93)$$

where we have used the fact that in conformal gauge, we have:

$$P_\mu = \frac{g_{\mu\nu} \dot{X}^\nu}{2\pi\alpha'}. \quad (5.94)$$

The expression in the square root in (5.93) is given by the L_0 constraint in our constraint algebra [22] (see also (5.9) and (5.10)), and is equal to the energy of the closed string:

$$\epsilon = \sqrt{|p_d|^2 + (n, \gamma^{-1}n) + \frac{1}{\alpha'^2}(w, \gamma w) + \frac{1}{\alpha'}[2(n, w) + 4(N-1)]}, \quad (5.95)$$

where the worldsheet zero modes give us the contributions $|p_d|$ for momentum along the non-compact directions, as well as the terms containing the winding and momentum quantum numbers for the compact dimensions. All the other Fourier modes give us the oscillator contributions. We can write the above as:

$$\dot{X}^0 = \frac{2\pi\alpha'}{\sqrt{-g_{00}}} \epsilon. \quad (5.96)$$

As for the second factor (5.92) entering the Jacobian, we see that for any string wound w times around the a^{th} direction:

$$X'^a = w^a. \quad (5.97)$$

Recall that one has to sum over all zeroes of the argument of the delta function along the a^{th} direction when evaluating the integral (5.87), after implementing the change of variables (5.92). Thus the contribution (5.96) is cancelled by the string winding w^a times around the a^{th} direction, as the argument of the delta function is zero precisely w^a times. Note that this is how the choice of which cycle we take in evaluating the change of variable ends up being inconsequential. However, the choice is residually implicit in which of the wound co-ordinates X^a remain in the $D - 2$ delta functions left over after integrating over the worldsheet. This will prove to be irrelevant after we hydrodynamically average to obtain the result for a gas of strings. Hence we evaluate (5.87)– using the gauge fixing conditions and the results just obtained (5.93) - (5.97)– as:

$$T_0^0 = \frac{\epsilon}{\sqrt{g_s}} \delta^{D-1}(X^\tau - y^\tau), \quad (5.98)$$

$$T_i^i = \frac{p^i p_i}{\epsilon \sqrt{g_s}} \delta^{D-1}(X^\tau - y^\tau), \quad (5.99)$$

$$T_a^a = \frac{1}{\epsilon \sqrt{g_s}} \left(\frac{n_a^2}{b_a^2} - \frac{w_a^2 b_a^2}{\alpha'^2} \right) \delta^{D-1}(X^\tau - y^\tau), \quad (5.100)$$

$$T_c^a = \frac{1}{\epsilon \sqrt{g_s}} \left(\frac{n_a n_c}{b_a b_c} - \frac{w_a w_c b_a b_c}{\alpha'^2} \right) \delta^{D-1}(X^\tau - y^\tau), \quad (5.101)$$

where g_s is the determinant of the spatial part of the metric, and ϵ is defined by (5.12). We now hydrodynamically average as follows: keeping the quantum numbers $p_d^2, \vec{w}, \vec{n}$ and N fixed, we sum the contributions over a distribution of such strings, where the momentum along the non-compact directions is distributed isotropically. We note that according to (5.12), a wound string with quantum number vectors $\vec{w}, \vec{n}$ will have the same

energy as a string with both vectors with the opposite sign. Moreover, for diagonally wound strings, this energy will be the same were we to keep the winding number around any given cycle fixed, whilst winding around any other cycle oppositely, provided that we also reverse the sign of the momentum quantum number corresponding to that cycle (this is so that the term (n, w) remain invariant). Hence, the off diagonal terms in the above will cancel out when summing over a gas of strings. In this way, our hydrodynamical averaging results in the term (5.101) dropping out, yielding (5.15)-(5.17):

$$\rho = \frac{\mu_0 \epsilon}{\sqrt{g_s}}, \quad (5.102)$$

$$p^i = \frac{\mu_0}{\sqrt{g_s} \epsilon} |p_d|^2 / d, \quad (5.103)$$

$$p^a = \frac{\mu_0}{\sqrt{g_s} \epsilon} \left[\frac{n_a^2}{b_a^2} - \frac{w_a^2 b_a^2}{\alpha'^2} \right]. \quad (5.104)$$

As is easily checked, we note that one would obtain this same result if one were to introduce the following action for the string gas:

$$S = - \int d^{D+1}x \sqrt{-g_{00}} \mu_0 \epsilon . \quad (5.105)$$

CHAPTER 6

Interlude II

Having seen that a gas of strings can stabilize any number of extra dimensions around the self dual radius in a manner that is consistent with various phenomenological bounds, we now investigate further the stability of such a gas against perturbations. Such an investigation is important if we are to find out the effects of stabilization on any primordial spectrum of metric fluctuations. Were this spectrum modified by the stabilization mechanism, then this would rule it out as a solution to the moduli problem. As we will see, string gas cosmology passes this test with flying colours, and even offers a mechanism to regularize a bouncing brane scenario whilst smoothly transferring metric fluctuations accross the branes. In this way, we shall see that string gas cosmology offers a rather complete framework to solve the moduli problem in a way that is consistent with our usual cosmology, and even extensions to our usual cosmology.

CHAPTER 7

On The Transfer of Metric Fluctuations when Extra Dimensions Bounce or Stabilize

7.1 Abstract

In this report, we study within the context of general relativity with one extra dimension compactified either on a circle or an orbifold, how radion fluctuations interact with metric fluctuations in the three non-compact directions. The background is non-singular and can either describe an extra dimension on its way to stabilization, or immediately before and after a series of non-singular bounces. We find that the metric fluctuations transfer undisturbed through the bounces or through the transients of the pre-stabilization epoch. Our background is obtained by considering the effects of a gas of massless string modes in the context of a consistent 'massless background' (or low energy effective theory) limit of string theory. We discuss applications to various approaches to early universe cosmology, including the ekpyrotic/cyclic universe scenario and string gas cosmology¹.

7.2 Introduction

The idea that the universe is comprised of any number of extra dimensions, in addition to the three non-compact spatial dimensions that we observe, goes back all the way to the work of Kaluza and Klein in the 1920's [89, 90], where extra dimensions were first proposed as a manner

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in which one could unify particle interactions with gravity. Since then, theories with extra dimensions have evolved from being a mathematical curiosity, albeit one with remarkable consequences for particle physics [91], to a theoretical necessity with the advent of string theory. Since the famous anomalies cancellation calculation of Green and Schwarz [92], the challenge has remained for cosmologists and phenomenologists alike to account for the presence of extra dimensions in physics. From a pessimist's point of view, one might view these extra dimensions as an unwanted surplus that has to be swept under the rug at presently accessible energies. In its most unimaginative form, one could view this as the gist of the moduli problem. From an optimist's point of view however, one could view the requirement of extra dimensions for the consistency of string theory as a new window of opportunity, through which one might be able to resolve some of the outstanding problems of particle physics and cosmology.

Taken as an essential part of the radically new picture of space-time that string theory offers us, extra dimensions, with strings and branes of various dimensions propagating through them, could offer us a potentially non-anthropic explanation for the observed dimensionality of space-time [6]. In addition to offering us potentially non-singular cosmologies [6, 7, 93], they could very well be at the heart of the resolution of the dark matter problem [19, 69, 88, 94]. Recently, several alternative stringy cosmologies have emerged as possible successors to the standard big bang model of the universe, such as the pre-big bang model [93], brane/string gas cosmology [6, 7, 26, 95, 96] , and the cyclic/ekpyrotic scenarios [31, 80, 81, 82, 97], all of which have the ultimate aim of becoming complete and testable models of the early universe.

A central question that arises in determining whether or not these models of the early universe reproduce observations is, how metric fluctuations in the presence of extra dimensions evolve in the backgrounds that these cosmologies propose. The importance of understanding this question cannot be understated in the context of the abundance of available experimental data, against which we must compare our eventual predictions. For instance, in the ekpyrotic/cyclic scenario the effects of branes colliding in a 5-dimensional bulk are explored ²

Key to the success of this program is being able to follow the evolution of metric fluctuations through the bounces of the extra dimension. The eventual goal of this is to be able to explain the observed inhomogeneities of the cosmic microwave background as being seeded by radion fluctuations generated through the motion of the branes towards each other, rather than through the quantum fluctuations of some putative inflaton field.

Similarly, in the context of the pre-big bang scenario, where an expanding and a contracting phase of the universe's evolution are naturally related to each other by the scale factor duality symmetry inherent to low energy effective string theory, a universe undergoing a big bang-big crunch transition is naturally implemented. In this context, the evolution of metric fluctuations through the bounce becomes a question of central importance in understanding the way the big bang phase of our universe's evolution happened the way it did. As it turns out, although at first seemingly unrelated, an associated and as yet unexamined issue is how any tentative solution

² Note that, as stressed in [98], none of these alternatives at the present stage solve the homogeneity and flatness problems of standard big bang cosmology without invoking a period of inflation (or something which effectively acts as inflation).

of the moduli problem will modify the spectrum of the metric fluctuations which we observe. Given that the universe is likely to contain many extra dimensions whose shape and volume moduli are dynamical in the early universe, one might expect that the dynamics of these moduli fields will lead to a highly nontrivial modification of any initial pre-stabilization spectrum.

It is the goal of this report to show that, in the context of a 5-d universe where an extra dimension undergoes non-singular bounces, or is en route to stabilization, the final spectrum for the Bardeen potential corresponding to (long wavelength) fluctuations of the scale factor for the non-compact dimensions, is identical to the initial spectrum of the five-dimensional fluctuations (in a sense which will be specified later). Specifically, if we had an initial scale invariant spectrum for the five-dimensional fluctuations, we would end up with a scale invariant spectrum for the Bardeen potential after the bounce, or after the transients of the stabilization had settled down. Furthermore, we show that the variable corresponding to metric fluctuations of the compact dimensions decays, which corresponds to the stability of the radion degree of freedom to fluctuations in our non-singular setup.

One application of our work concerns the evolution of fluctuations in ekpyrotic/cyclic type models. In the context of a four space-time dimensional effective field theory toy model of this scenario, the dynamics of perturbations has been investigated in detail. The initial analyses [80, 99] yielded the result that a scale-invariant spectrum before the bounce transfers to a scale-invariant spectrum after the bounce. These analyses, however, were questioned in [100, 101, 102, 103, 104, 105]. A serious complicating factor turned out to be the fact that the proposed background evolution was singular, thus requiring the use of “matching conditions” (such as those

derived in [106] in the context of an expanding cosmology undergoing a sudden phase transition) to compute the post-bounce spectrum of fluctuations. As discussed in [107], there is a very sensitive dependence of the final result on the choice of the matching surface. Independently, there has been recent work on the evolution of fluctuations through a nonsingular bounce in four space-time dimensional cosmologies, in which the bounce is constructed by adding extra terms to the standard Lagrangian. These analyses [108, 109, 110, 111, 112, 113, 114, 115, 116] yield results showing a sensitive dependence on the nature of the bounce. The bottom line of this work is that a correct analysis of fluctuations in the ekpyrotic/cyclic scenario needs to be done in a five space-time dimensional context, a context in which the nature of the bounce is unambiguous. A first important step in this direction was taken in [117], confirming the result that a scale-invariant spectrum passes through the bounce (in the five-dimensional context, the bounce means that the radius of the extra spatial dimension bounces, not that the four space-time dimensional scale factor bounces) without change in the spectral index. However, the analysis of [117] was done in the context of a singular background and assuming specific matching conditions for fluctuations applied at a point when the perturbations in fact blow up. Thus, the results are open to doubt. In this work, we study the transfer of fluctuations through a cosmology in which two boundary branes approach each other and bounce without encountering a singularity (see also [31] for previous work done in the context of a particular nonsingular ekpyrotic-type bounce proposed in [118]).

We begin by introducing a non-singular bouncing model of a 5-dimensional universe where one dimension is compactified on a circle or an orbifold. As we shall see shortly in detail, our non-singular background is

obtained by considering the effects of gas composed of massless string modes on the dynamics of space-time, in the context of a consistent low energy effective theory limit, or ‘massless background’, of string theory. We then set up the framework for studying cosmological perturbations in this model and study how these transfer through the various bounces that the extra dimension undertakes. Although our setup is seemingly specific to string gas cosmology, the essence of our framework is that we have a non-singular bounce/stabilization mechanism that is affected by degrees of freedom that become massless at a certain point. Hence, the hope is that the results obtained here can be generalized to other settings, an issue we will discuss in detail when we consider applications to different approaches to stringy cosmology. We now commence our paper with a few preliminaries.

7.3 The background – $R^4 \times S^1$

Consider a five dimensional space-time with the topology of $R^4 \times S^1$, described by the metric ³

$$g_{AB} = \text{diag}(-1, a^2(t), a^2(t), a^2(t), b^2(t)), \quad (7.1)$$

from which we derive the following components of the Einstein tensor:

$$G^t_t = -3\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right), \quad (7.2)$$

$$G^{x_i}_{x_j} = -\delta^i_j \left[2\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\dot{b}}{b}\frac{\dot{a}}{a} \right], \quad (7.3)$$

$$G^y_y = -3\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]. \quad (7.4)$$

³ The metric can always be cast into this form, such that t corresponds to cosmic time.

Here, the indices i and j run over the three large spatial dimensions and y denotes the extra dimension. The form of the energy-momentum tensor which will couple to the Einstein tensor is given by

$$T_B^A = \text{diag}(-\rho, p, p, p, r). \quad (7.5)$$

We can recast the Einstein equations $G_B^A = 8\pi G T_B^A$, where G is the five-dimensional Newton's constant, in the form:

$$\frac{\ddot{a}}{a} + H(2H + \mathcal{H}) + \frac{8\pi G}{3}[r - \rho] = 0, \quad (7.6)$$

$$\ddot{b} + 3H\dot{b} + 8\pi G b[p - \frac{2r}{3} - \frac{\rho}{3}] = 0, \quad (7.7)$$

$$\rho = \frac{3}{8\pi G}H(H + \mathcal{H}), \quad (7.8)$$

where $H = \dot{a}/a$ and $\mathcal{H} = \dot{b}/b$. The energy-momentum tensor for a string gas in this toroidally compactified background was derived in [17] to which we refer the reader if any of what follows is unfamiliar. It was found to be

$$\rho_{n,w} = \frac{\mu_{0,n,w}}{\epsilon_{n,w}\sqrt{-g}}\epsilon_{n,w}^2, \quad (7.9)$$

$$p_{n,w} = \frac{\mu_{0,n,w}}{\epsilon_{n,w}\sqrt{-g}}\frac{p_{n.c.}^2}{3}, \quad (7.10)$$

$$r_{n,w} = \frac{\mu_{0,n,w}}{\epsilon_{n,w}\sqrt{-g}\alpha'}\left(\frac{n^2}{\tilde{b}^2} - w^2\tilde{b}^2\right), \quad (7.11)$$

where g denotes the determinant of the metric, $\tilde{b} := b/\sqrt{\alpha'}$ (with $2\pi\alpha'$ being the inverse string tension) and the subscripts n, w refer to the momentum and winding quantum numbers of a closed string along the 5th dimension respectively, on which all of the subscripted quantities will depend. The density μ_0 is the number density of the string gas with the metric dependence factored out ($\mu(t) = \mu_0/\sqrt{-g}$), while $\epsilon_{n,w}$ is the energy of

a single closed string in this background

$$\epsilon_{n,w} = \sqrt{p_{n.c.}^2 + \left(\frac{n}{b} + \frac{wb}{\alpha'}\right)^2 + \frac{4}{\alpha'}(N-1)}, \quad (7.12)$$

where $p_{n.c.}$ denotes the center of mass momentum along the three non-compact directions, and N is the number of right-moving string vibrational modes. For reasons made clear in [16] and [17], string states that are massless at special symmetry points (i.e. the self-dual radius) should play a very distinguished role in any string gas cosmology (see also [27, 21, 28]). There are several reasons for this, among which are prominently the desire to obtain a viable late time phenomenology and a robust stabilization mechanism for the radion. However, the main motivation is that the low energy effective theory limits of string theory (such as general relativity and dilaton gravity) are also consistent backgrounds on which we can study the propagation of massless (and only massless) strings. For these reasons we do not further justify the focus on massless string modes, and refer the reader to [16, 17, 18] for more details. For a general review of string gas cosmology, we refer the interested reader to [95][96][26].

We wish to emphasize that the framework within which we chose to work assumes nothing other than the fact that nature is described by a string theory at high energies, and that the degrees of freedom that are likeliest to be excited (namely massless string modes) will be excited. The two particular geometries that we consider for the extra dimensions (i.e. a toroidal geometry in this section and an orbifold geometry in the next section), are necessitated by the fact that they are the unique compact backgrounds on which one stays within the approximation of the low energy effective limit of string theory, when considering extra dimensions that are similar in size to the string scale. This is because in general, were the

metric to depend on the extra dimension (as is the case for warped extra dimensions), one will introduce curvatures that approach the string scale when the extra dimensions themselves approach the string scale, and hence the low energy approximation will break down. We discuss this point further in the next section.

The particular states we are interested in are those for which $n = -w = \pm 1, N = 1$. That these states are massless at the self-dual radius ($b = \sqrt{\alpha'}$) is easily checked from (7.12). These states imply that (7.7) becomes

$$\ddot{\tilde{b}} + 3\frac{\dot{a}}{a}\dot{\tilde{b}} + \frac{8\pi G\mu_0}{a^3\alpha'} \frac{-\frac{1}{\tilde{b}^2} + \frac{\tilde{b}^2}{3} + \frac{2}{3}}{\sqrt{\alpha' p_{n.c.}^2 + \left(\frac{1}{\tilde{b}} - \tilde{b}\right)^2}} = 0. \quad (7.13)$$

We will see that this background stabilizes around the self-dual radius ($\tilde{b} = 1$), whilst the non-compact directions persistently expand as they would in a radiation dominated universe. Before we get to this, however, we wish to discuss a generalization of this background to the situation where the extra dimension is compactified on an orbifold.

7.3.1 Extension to $R^4 \times S^1/Z_2$

The consistency of the framework that we propose, namely that we remain within the low energy effective theory (or massless background) limit of string theory, requires the background we consider to satisfy the condition

$$R[g] \ll \frac{1}{\alpha'}. \quad (7.14)$$

That is, the Ricci scalar should be considerably bounded from above by the string tension. This requirement translates into the statement that the metric should not change very much on the string scale, which ensures that this background remains a consistent background for the propagation of massless string modes. Conversely, were one to consider backgrounds

which do not satisfy (7.14), not only would the approximations made in deriving the low energy limit of string theory break down, one would also expect massive modes to be created [22]. For our purposes, the requirement of (7.14) means that were we to consider compactifications of the extra dimension on scales comparable to the self-dual radius, then there should be absolutely no dependence of the metric along the compactified direction. This is a remarkable feature of using a toroidal compactification: one can in fact study string scale processes without invalidating the approximations inherent in the low energy limit of string theory. It was shown in [17] and [18] that the background we are about to derive, does in fact satisfy (7.14) throughout its dynamics.

Returning to the problem at hand, we see that since we are not allowed to consider metric dependencies along the extra dimension (by homogeneity and isotropy of the non-compact dimensions, the metric can then only depend on time), the components of the Einstein tensor (7.2)-(7.4) are unchanged after orbifolding the extra dimension. The only place where orbifolding might make a difference is in the energy momentum tensor of the string gas. However, because the massless states that we have focused on are in the so-called untwisted sector [22] (in general, twisted states are localized at the orbifold fixed points), it turns out that there is no difference in the energy-momentum tensor either (7.9)-(7.11). This is a consequence of the ‘inheritance’ principle of orbifold theories [22], and permits us to use the framework just presented in either situation, provided we only use the modes that we have indicated and we do not consider any variations in any metric quantities along the extra dimension. However, there is a caveat to this in that two orbifold fixed planes are present, which might have matter confined to live on them. The requirement not to induce any metric

variations along the extra dimension translates into, via the Israel junction conditions [119, 120], the condition (among others)

$$\left[\frac{da}{dy} \right] \propto 8\pi G \rho_{brane}, \quad (7.15)$$

where $[da/dy]$ is the jump of the derivative of the scale factor a along the extra dimension, evaluated around either of the orbifold fixed planes. Hence, we see that if we require there to be no dependence of the metric along the extra dimensions, then any matter localized on the branes must be sufficiently dilute to render the branes to behave as test branes. Bearing all of this in mind, we can now proceed to derive the background solution within our framework.

7.4 Background solution in 5D

7.4.1 The three large dimensions

We first note from (7.7) that any form of matter which satisfies a radiative equation of state ($p = \rho/3$, $r = 0$) for all times (as opposed to our wound string states which become massless, hence radiative, only at the self-dual radius) drops out of the driving term in the equation of motion for b (7.7). Hence, we can safely consider a situation where we have a radiation gas in addition to our gas of wound strings, which dominates the evolution of the Hubble factor H for the non-compact dimensions through (7.8) after the extra dimension has stabilized, or is close to stabilization. In fact, massless unwound closed string states (gravitons: $n = w = 0$, $N = 1$) can provide exactly such a bath. Henceforth, it is natural to include such a bath in our setup. We do not consider this matter further, simply taking as a given in what follows that the scale factor a for the non-compact dimensions expands at the background level as it would in a radiation dominated

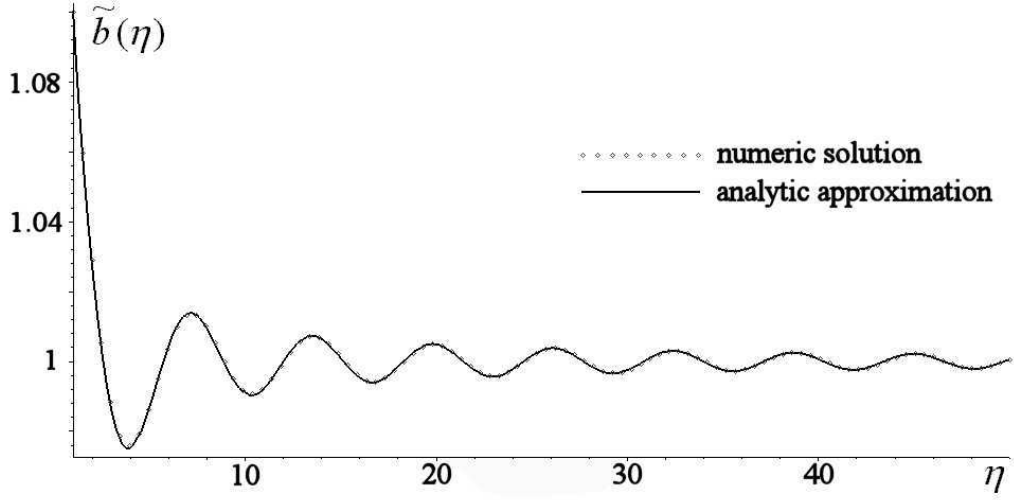


Figure 7–1: The analytic approximation (7.23) (solid line) is compared with the numeric solution of (7.20) (circles).

universe, that is

$$a(t) \propto t^{1/2}. \quad (7.16)$$

7.4.2 The extra dimension

Let us go back to the equation of motion (7.13) for $\tilde{b}$, which reads

$$0 = \ddot{\tilde{b}} + 3\frac{\dot{a}}{a}\dot{\tilde{b}} + \frac{8\pi G\mu_0}{a^3\alpha'} \frac{-\frac{1}{\tilde{b}^2} + \frac{\tilde{b}^2}{3} + \frac{2}{3}}{\sqrt{\alpha' p_{n.c.}^2 + \left(\frac{1}{\tilde{b}} - \tilde{b}\right)^2}}, \quad (7.17)$$

where $p_{n.c.} =: q/a$ with $q = \text{const}$ is the center of mass momentum along the three large dimensions. Introducing the characteristic time scale

$$t_0^{-1} := \sqrt{\frac{64\pi G\mu_0}{3\alpha'^{3/2}a_0^2q}} \quad (7.18)$$

and defining the dimensionless time variable

$$\eta := 2\sqrt{\frac{t}{t_0}}, \quad (7.19)$$

such that $a = a_0\eta$, we can simplify (7.17) to

$$\tilde{b}'' + \frac{2}{\eta}\tilde{b}' + \frac{3}{8} \frac{-\frac{1}{\tilde{b}^2} + \frac{\tilde{b}^2}{3} + \frac{2}{3}}{\sqrt{1 + \frac{\eta^2}{4}l^2 \left(\frac{1}{\tilde{b}} - \tilde{b}\right)^2}} = 0, \quad (7.20)$$

where a prime denotes a derivative with respect to η and we also introduced the free dimensionless parameter

$$l := \frac{a_0}{\sqrt{\alpha'q}}. \quad (7.21)$$

This equation can easily be integrated numerically, but it will turn out to be useful to have a simple analytic approximation.

If we are close to the self-dual radius, that is $\tilde{b} = 1 + \varepsilon$ with $\varepsilon \ll 1$, we can expand the driving term in (7.20) so that

$$\varepsilon'' + \frac{2}{\eta}\varepsilon' + \varepsilon = 0 \quad (7.22)$$

results. The general solution is given by

$$\varepsilon(\eta) = \frac{1}{\eta} (A \sin(\eta) + B \cos(\eta)), \quad (7.23)$$

with A and B constants. This solution may either describe the stabilization of the extra dimension or a series of non-singular bounces⁴. See Fig.7–1 for a comparison of the numeric solution of (7.20) with the analytic one. Since they are barely discernable we will use the analytic approximation in the following and switch freely between bounce/stabilization-language.

⁴ Note that this series of bounces differs from the one in e.g. the cyclic scenario, since ϵ is small.

7.5 Scalar perturbations in 5D

We will first perturb the metric, focusing only on scalar metric perturbations. The most general scalar perturbation in “generalized” longitudinal gauge [31, 64, 122] can be characterized by four scalar functions, Φ, Ψ, W and Γ . These functions can be viewed as a basis of gauge invariant variables. In this gauge, the metric is given by

$$ds^2 = -(1 - 2\Phi(t, \mathbf{x})) dt^2 + a(t)^2(1 - 2\Psi(t, \mathbf{x}))d\mathbf{x}^2 + b(t)^2(1 + 2\Gamma(t, \mathbf{x})) dy^2 - 2W(t, \mathbf{x}) dt dy, \quad (7.24)$$

where the signs in front of the perturbations are a mere convention⁵.

For reasons discussed in sections 7.3 and 7.3.1, we do not consider any y dependence, that is we assume homogeneity in the extra dimension (see also [26] for a discussion of this and other assumptions of SGC). The corresponding Einstein tensor can be computed to be [31, 122] (in the following Δ denotes the Laplacian of the three large spatial dimensions):

$$\delta G_{x_j}^{x_i} = \frac{1}{a^2} \partial_{x_i} \partial_{x_j} [\Psi + \Phi - \Gamma], \quad i \neq j, \quad (7.25)$$

$$\delta G_t^y = \left[\frac{3}{b^2} \left(\frac{\ddot{a}}{a} - \frac{\dot{a}\dot{b}}{ab} \right) + \frac{1}{2b^2 a^2} \Delta \right] W, \quad (7.26)$$

$$\delta G_{x_i}^y = \partial_{x_i} \left[\frac{1}{2b^2} \left(\frac{\dot{b}}{b} + \frac{\dot{a}}{a} + \partial_t \right) W \right], \quad (7.27)$$

⁵ In our notation $\Phi = -\Psi$ would be called the Newtonian potential in a four dimensional description.

$$\delta G_{x_i}^{x_i} = \frac{1}{a^2} [\partial_{x_i} \partial_{x_i} - \Delta] (-\Gamma + \Psi + \Phi) + 2 \left(\frac{\dot{b}}{b} + 3 \frac{\dot{a}}{a} + \partial_t \right) \partial_t \Psi \quad (7.28)$$

$$-2 \left(\frac{\ddot{b}}{b} + 2 \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{a}^2}{a^2} + 2 \frac{\ddot{a}}{a} + \frac{\dot{b}}{2b} \partial_t + \frac{\dot{a}}{a} \partial_t \right) \Phi - \left(2 \frac{\dot{a}}{a} + 2 \frac{\dot{b}}{b} + \partial_t \right) \partial_t \Gamma,$$

$$\delta G_y^y = \left[-\frac{2}{a^2} \Delta + 3 \left(\partial_t + 4 \frac{\dot{a}}{a} \right) \partial_t \right] \Psi + \left[-\frac{1}{a^2} \Delta - 6 \left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} + \frac{\dot{a}}{2a} \partial_t \right) \right] \Phi \quad (7.29)$$

$$\delta G_t^t = \left[3 \left(\frac{\dot{b}}{b} \partial_t + 2 \frac{\dot{a}}{a} \partial_t \right) - \frac{2}{a^2} \Delta \right] \Psi - 6 \left(\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{a}^2}{a^2} \right) \Phi - \left(\frac{3\dot{a}}{a} \partial_t - \frac{1}{a^2} \Delta \right) \Gamma \quad (7.30)$$

$$\delta G_t^{x_i} = \partial_{x_i} \left[\frac{2}{a^2} \partial_t \Psi - \frac{1}{a^2} \left(\frac{\dot{b}}{b} + 2 \frac{\dot{a}}{a} \right) \Phi - \frac{1}{a^2} \left(\frac{\dot{b}}{b} - \frac{\dot{a}}{a} + \partial_t \right) \Gamma \right]. \quad (7.31)$$

One can check that the equations of motion involving W decouple from the other ones for the matter content (7.5) we consider. Since W would appear only squared in a four dimensional effective theory, we will not need to compute it at all ⁶.

To write down the perturbed Einstein equations ⁷, we also need the perturbed energy-momentum tensor δT_B^A . It will include the thermal bath of radiation (denoted by the subscript r), and the stringy matter sources denoted by a tilde. To be specific, we have

$$(\delta^{(r)} T_B^A) = \begin{pmatrix} \delta \rho_{(r)} & -(\rho_{(r)} + p_{(r)}) V_{,i} & 0 \\ (\rho_{(r)} + p_{(r)}) V_{,i} & \delta p_{(r)} \delta_j^i & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

⁶ That the equations of motion for W decouple from the other ones is a direct consequence of the homogeneity in the y -direction.

⁷ Note that fluctuations in SGC (before dilaton stabilization) were considered in [32, 33].

(where $\rho_{(r)}$ and $p_{(r)}$ are the radiation energy density and pressure, respectively, and V is the radiation three velocity potential) and

$$(\delta \tilde{T}_B^A) = \begin{pmatrix} \delta \tilde{\rho} & -(\rho_{(r)} + p_{(r)})\tilde{V}_{,i} & 0 \\ (\rho_{(r)} + p_{(r)})\tilde{V}_{,i} & \delta \tilde{p} \delta_j^i & 0 \\ 0 & 0 & \delta \tilde{r} \end{pmatrix},$$

where $\tilde{\rho}$ and $\tilde{p}$ are the string gas energy density and pressure, respectively, and $\tilde{V}$ is the string gas three velocity potential. Note that anisotropic stress does not feature in our setup (see appendix), but we will keep $\tilde{V}$ around for the time being. We will also focus on adiabatic perturbations of the radiation fluid only, that is $\delta p_{(r)} = \delta \rho_{(r)}/3$, even though two ideal fluids are present so that iso-curvature perturbations could arise. The reason for neglecting those is simplicity. The $x_i - x_j$ ($i \neq j$) Einstein equations yield immediately

$$\Gamma = \Psi + \Phi, \quad (7.32)$$

and, after introducing

$$\xi := \Psi - \Phi \quad (7.33)$$

$$N := \frac{\delta \mu_0}{\mu_0}, \quad (7.34)$$

we infer from (7.9)-(7.11), that the perturbed energy-momentum tensor is of the form

$$\delta \tilde{\rho} = (N + \Gamma + 2\xi)\tilde{\rho}, \quad (7.35)$$

$$\delta \tilde{p} = (N + \Gamma + 2\xi)\tilde{p}, \quad (7.36)$$

$$\delta \tilde{r} = (N + \xi)\tilde{r} - 2\Gamma \left(\frac{1}{\tilde{b}^2} + \tilde{b}^2 \right) 3\tilde{p} \frac{a^2}{q^2 \alpha'}, \quad (7.37)$$

where the background quantities are given by

$$\tilde{\rho} = \frac{\mu_0}{\alpha'} \frac{1}{a^3 \tilde{b}} \sqrt{\frac{q^2 \alpha'}{a^2} + \left(\frac{1}{\tilde{b}} - \tilde{b}\right)^2}, \quad (7.38)$$

$$\tilde{p} = \frac{\mu_0}{\alpha'} \frac{1}{a^3 \tilde{b}} \frac{\frac{q^2 \alpha'}{3a^2}}{\sqrt{\frac{q^2 \alpha'}{a^2} + \left(\frac{1}{\tilde{b}} - \tilde{b}\right)^2}}, \quad (7.39)$$

$$\tilde{r} = \frac{\mu_0}{\alpha'} \frac{1}{a^3 \tilde{b}} \frac{\frac{1}{\tilde{b}^2} - \tilde{b}^2}{\sqrt{\frac{q^2 \alpha'}{a^2} + \left(\frac{1}{\tilde{b}} - \tilde{b}\right)^2}}. \quad (7.40)$$

In performing this calculation, we started with the source action for this string gas [16], and obtained the perturbed energy-momentum tensor through direct calculation. The string gas energy-momentum tensor only depends on the metric, the number density of strings μ_0 , and the center of mass momentum of these strings. However, the latter quantity is not perturbed, because for long wavelength perturbations (compared to the string scale), which we are restricted to if we are to remain within the limits of low energy effective theory, the center of mass motion of a string propagating on a perturbed spacetime is unaffected to first order. Hence the only quantities left to perturb are the metric dependencies, and the number density of the string gas.

Next, we can write the perturbed zero component of the conservation equation $\nabla^A \tilde{T}_{AB} = 0$ for the stringy matter as

$$\begin{aligned} 0 = & \delta \dot{\tilde{\rho}} + 3H(\delta \tilde{\rho} + \delta \tilde{p}) + \mathcal{H}(\delta \tilde{\rho} + \delta \tilde{r}) \\ & - \frac{3}{2} \dot{\xi}(\tilde{\rho} + \tilde{p}) - \frac{1}{2} \dot{\Gamma}(\tilde{\rho} - 3\tilde{p} + 2\tilde{r}) + \frac{\Delta \tilde{V}}{a^2}(\tilde{\rho} + \tilde{p}). \end{aligned} \quad (7.41)$$

For simplicity we will set $\tilde{V} = 0$ from now on, that is we neglect the scalar velocity potential of the string gas, consistent with the arguments of the

appendix. Combining the diagonal Einstein equations yields

$$\begin{aligned} \frac{1}{3M_5^3} (2\delta\tilde{r} + \delta\tilde{\rho} - 3\delta\tilde{p}) &= \ddot{\Gamma} + \dot{\Gamma}(3H + \mathcal{H}) - 2\mathcal{H}\dot{\xi} - \frac{\Delta}{a^2}\Gamma \\ &+ (\Gamma - \xi)(3H\mathcal{H} + \mathcal{H}^2 + \dot{\mathcal{H}}), \end{aligned} \quad (7.42)$$

$$\begin{aligned} \frac{1}{3M_5^3} (3\delta\tilde{p} - \delta\tilde{\rho}) &= \ddot{\xi} + \dot{\xi}(2\mathcal{H} + 5H) - \mathcal{H}\dot{\Gamma} - \frac{\Delta}{3a^2}\xi \\ &+ (\xi - \Gamma)(4H^2 + 3H\mathcal{H} + \mathcal{H}^2 + \dot{\mathcal{H}} + 2\dot{H}). \end{aligned} \quad (7.43)$$

The last three equations (7.41)-(7.43) are the dynamical ones for the perturbation variables Γ , ξ and N . The $(t - t)$ equation gives the radiation fluid perturbation $\delta\rho_{(r)}$ and the $(x_i - t)$ equation gives the scalar velocity potential V of the radiation fluid in terms of the other variables.

These equations simplify if we write them in terms of η defined in (7.19) and make use of the background equation $a = a_0\eta$. Introducing $h := \tilde{b}'/\tilde{b}$ we get

$$\begin{aligned} \Gamma_k'' + \Gamma_k' \left(h + \frac{2}{\eta} \right) - \xi_k' 2h + (\Gamma_k - \xi_k) \left[\frac{2h}{\eta} + h^2 + h' \right] \\ + \frac{k^{*2}}{4} \Gamma_k = \eta^2 C (2\delta r^* + \delta\rho^* - 3\delta p^*), \end{aligned} \quad (7.44)$$

$$\begin{aligned} \xi_k'' + \xi_k' 2 \left(h + \frac{2}{\eta} \right) - \Gamma_k' h + (\xi_k - \Gamma_k) \left[\frac{2h}{\eta} + h^2 + h' \right] \\ + \frac{k^{*2}}{12} \xi_k = \eta^2 C (3\delta p^* - \delta\rho^*), \end{aligned} \quad (7.45)$$

$$\begin{aligned} \delta\rho^{*'} + \frac{3}{\eta}(\delta\rho^* + \delta p^*) + h(\delta\rho^* + \delta r^*) - \frac{3}{2}\xi_k'(\rho^* + p^*) \\ - \frac{1}{2}\Gamma_k'(\rho^* - 3p^* + 2r^*) = 0. \end{aligned} \quad (7.46)$$

Here, we made the transition to Fourier space and we performed the rescaling $a^* := a/a_0$, $k^* := kt_0/a_0$. All other starred quantities are defined via

$$f^* := f_k a_0^3 \frac{\alpha'}{\mu_0}. \quad (7.47)$$

The two dimensionless constants left are

$$C := \frac{t_0^2}{M_5^3} \frac{1}{12a_0^3} \frac{\mu_0}{\alpha'}, \quad (7.48)$$

$$q^* := \frac{q\sqrt{\alpha'}}{a_0}, \quad (7.49)$$

which are not independent but related via

$$\frac{2^5 C}{q^*} = 1. \quad (7.50)$$

7.5.1 Analytic late time solution

After $h = h' = 0$ got approached (that is after the extra dimension got stabilized or, in the language of the ekpyrotic/cyclic scenario, the two approaching branes came to a halt), we can simplify the equations of the previous section to

$$\Gamma_k'' + \frac{2}{\eta} \Gamma_k' + \frac{k^{*2} + \tilde{k}^2}{4} \Gamma_k = 0, \quad (7.51)$$

$$\xi_k'' + \frac{4}{\eta} \xi_k' + \frac{k^{*2}}{12} \xi_k = 0, \quad (7.52)$$

where we introduced the constant

$$\tilde{k}^2 := \frac{2^5 C}{q^*} = 1. \quad (7.53)$$

We note that the source term for ξ_k vanishes because of the background equation of state for the gas of massless string modes. Also, the equations for ξ_k and Γ_k decouple at late times, just as they should do. The solutions are given by

$$\Gamma_k = \frac{1}{\eta} (C_1 \cos(\omega_\Gamma \eta) + C_2 \sin(\omega_\Gamma \eta)), \quad (7.54)$$

$$\begin{aligned} \xi_k = \frac{1}{\eta^3} & \left(C_3 (k^* \eta \cos(\omega_\xi \eta) - 2\sqrt{3} \sin(\omega_\xi \eta)) \right. \\ & \left. + C_4 (k^* \eta \sin(\omega_\xi \eta) + 2\sqrt{3} \cos(\omega_\xi \eta)) \right), \end{aligned} \quad (7.55)$$

with C_i constant and

$$\omega_\Gamma^2 := \frac{k^{*2} + \tilde{k}^2}{4}, \quad (7.56)$$

$$\omega_\xi^2 := \frac{3k^{*2}}{36}. \quad (7.57)$$

We would like to evaluate the spectrum

$$\mathcal{P}_k := k^3 \Phi_k^2 \propto k^{n_s-1} \quad (7.58)$$

when a long wavelength mode enters the Hubble radius again at $k^* \eta_r = 2$. If we Taylor expand (7.55) for small k^* we see that there is an approximately constant mode present for ξ_k . On the other hand, Γ_k from (7.54) is oscillating and decaying $\propto 1/\eta$. Therefore, whatever the spectrum for ξ_k was at the initial time, it should persist till re-entry and determine the spectrum for Φ_k because, by neglecting Γ_k , we have $\Phi_k \approx -\Psi_k \approx -\xi_k/2$. This is the same result one would conclude in a simple four dimensional universe, dominated by radiation. To be specific, we can approximate

$$|\Phi_k(\eta_r)| \approx \left| -\frac{\xi_k(\eta_i)}{2} + \frac{\Gamma_k(\eta_i) \cos(\tilde{k}\eta_r/2 + \beta)}{2\eta_r/\eta_i \cos(\tilde{k}\eta_i/2 + \beta)} \right|, \quad (7.59)$$

$$\approx \left| -\frac{\xi_k(\eta_i)}{2} \right|, \quad (7.60)$$

where β is some irrelevant phase. Of course this holds true only, if the transient stabilizing epoch leaves no strong imprints onto the spectrum; hence, we will examine this crucial issue in the next section.

Before we do that, let us have a quick look at the physics after a mode entered the Hubble radius: ξ_k will start to oscillate with a decaying amplitude $\propto 1/\eta^2$. If one waits long enough, Γ_k will catch up and become larger than ξ_k . Now $\Phi_k \approx \Psi_k \approx \Gamma_k/2$ and thus Γ_k will determine the

spectrum, because for $\eta \gg \eta_r$

$$|\Phi_k(\eta)| \approx \left| \frac{\Gamma_k(\eta_i)}{2\eta/\eta_i} \frac{\cos(\tilde{k}\eta/2 + \beta)}{\cos(\tilde{k}\eta_i/2 + \beta)} \right|. \quad (7.61)$$

There is of course a transition region where neither approximation holds. Nevertheless, this only occurs after Hubble radius crossing and shall not concern us in the following.

7.5.2 Numerical solution and contact with late times

The analytic solution of the previous section is valid after $\tilde{b}$ got stabilized (or the branes came to a halt). However, the main question we would like to address is the following: what effect on the perturbations, if any, does the process of trapping the modulus $\tilde{b}$ have? In order to address this question we have to solve (7.44)-(7.46) numerically. As we shall see, the late time solution is approached rapidly and long wavelength perturbations are not affected in any significant way. Hence, the naive intuition that the value of Φ_k for modes outside the Hubble radius stay frozen is indeed valid, with important consequences for cosmological model building, be it in the framework of the ekpyrotic/cyclic scenario, or within string gas cosmology (SGC) – we will focus on concrete models in the next section.

Firstly, let us specify constants and initial conditions: the only free parameter in our setup is q^* defined in (7.49), describing the initial momentum of the string gas in the three large dimensions. Being a dimensionless parameter, we choose the most natural value

$$q^* := 1. \quad (7.62)$$

Our results are not very sensitive to its exact value ⁸ . We start close to the self-dual radius with some arbitrary velocity (once again, its exact value does not affect our results - see Fig.7–6), that is

$$\tilde{b}(\eta_i) = 1.1, \quad (7.63)$$

$$\tilde{b}'(\eta_i) = -0.1. \quad (7.64)$$

Next, we have to specify initial conditions for ξ_k, Γ_k and N_k at $\eta_i = 1$ ⁹ . Since we did not compute how the universe came close to the self-dual radius (e.g. within the ekpyrotic/cyclic setup or a SGC setup), we have no way of deriving those. However, we can make an educated guess: if the universe underwent some phase of inflation in the three dimension before (we will make this point more concrete in the next section), one should expect similar values for all metric perturbations; hence we will set

$$\xi_k(\eta_i) = \Gamma_k(\eta_i) =: \epsilon_k, \quad (7.65)$$

with some small ϵ_k . For simplicity, we will use $\epsilon_k = 0.01$ for all k . Furthermore, since the long wavelength modes we are interested in should be frozen once they cross the Hubble radius, we set

$$\xi'_k(\eta_i) = \Gamma'_k(\eta_i) = 0. \quad (7.66)$$

⁸ Decreasing q induces a phase shift in Φ_k and an increase in N_k , while an increase in q leads to a decrease in N_k and no phase shift; however, the metric degrees of freedom decouple from N_k quickly and N_k itself is of no interest to us.

⁹ We start at $\eta_i = 1$, because this is the characteristic time scale of the background oscillations. This means, no oscillation could occur before η_i .

The initial string density perturbation $N_k(\eta_i)$ will be set to zero, because these massless modes just got produced. Naturally, N_k will get sourced by Γ_k and ξ_k . No conclusion in the following is sensitive to the chosen initial conditions, hence we chose the most simple ones above (other ones were of course also tested).

The only thing left to specify is the wave-number k^* : We are interested in long wavelength modes, that is modes outside the Hubble radius with small k^* . In addition, we shall require $k^* < \tilde{k}$, so that we can compare our results with (7.59). With all initial conditions specified, we can use the analytic solution (7.23) to solve (7.44)-(7.46) numerically.

Let us first compare $\Phi_k(\eta) = (\Gamma_k(\eta) - \xi_k(\eta))/2$ for different values of k^* , Fig. 7-2 (a)-(c): There Φ_k is plotted both numerically (grey) and analytic (black). We see that the analytic late time solution is approached pretty fast – in fact, the small difference at the beginning is not visible in this plot, but only if ξ_k is plotted alone as in Fig. 7-5 (a). The visible decaying oscillation of Φ_k is the decaying mode of Γ_k from (7.54) with frequency ω_Γ , which is plotted in Fig. 7-5 (b) (compare with Fig. 7-2 (b)). In addition, there are strongly damped oscillations on top of the constant mode of ξ_k , that carry the same frequency as $\tilde{b}(\eta)$, Fig. 7-4. As mentioned before, these are not visible in Fig. 7-2 but in Fig. 7-5 (a) where ξ_k is plotted alone. These oscillations are the impact of trapping the modulus $\tilde{b}$, an impact that can safely be ignored in the long wavelength regime: if one decreases k^* , all of these effects become less pronounced.

An other feature visible in Fig. 7-2 is a bending of the lower curve in comparison to the analytic solution (7.59). This bending is the beginning of an oscillation of ξ_k and hence an expected feature, since the constant mode is actually the beginning of an oscillation with a very small frequency due to

ω_ξ . All features are not sensitive to the initial values we choose for Γ_k and ξ_k .

Last but not least, N_k is plotted in Fig. 7–5 (c): it gets sourced quickly by Γ_k and ξ_k and oscillates with a nearly constant amplitude thereafter, with all frequencies entering. However, there is no mentionable back-reaction of N_k on the metric perturbations. Hence, one could give N_k a non zero initial value without changing the evolution of Γ_k and ξ_k in any significant way.

7.6 Consequences for ekpyrotic/cyclic models and SGC

In the end, we are interested in the spectrum of Φ_k . So far, there seems to be little to no impact of the transient stabilizing era of b onto Γ_k and ξ_k . This translates directly to the spectrum, where Φ_k should approach a horizontal line for small k^* . This is indeed the case, as can be seen in Fig. 7–3 (c), where the spectrum is evaluated at horizon crossing: the oscillations that are present for relatively large k^* get damped once the long wavelength regime is approached. This is expected, because long wavelength modes enter the Hubble radius later and henceforth, the decaying modes responsible for the oscillations in the spectrum get damped more. The frequency in the spectrum of Φ_k is proportional to $1/k^*$, in accordance with $\eta_r \sim 1/k^*$ and equation (7.59).

Note that no shift of the spectral index is induced, leaving the overall index unaltered. The oscillations in ξ_k in Fig. 7–3(a) are the main impact of the transient era of a dynamical b . They are clearly negligible, because the tiny amplitude gets even smaller with decreasing k^* .

To summarize, the main conclusion of our numerical study is that long wavelength metric perturbations quickly approach their asymptotic solution, leaving no trace of the many bounces b experiences on its way to

stabilization. More specifically, we have shown that the initial spectrum of the non-decaying mode of ξ_k is preserved, i.e. equals the spectrum of ξ_k at late times. At late times, the spectrum of ξ_k equals the spectrum of the Bardeen potential Φ_k , the potential which is relevant for late time observations. Speaking more loosely, we have shown that the initial pre-bounce spectrum of bulk perturbations is transferred to the late time cosmological perturbations without a change in the spectral index.

We saw in the previous section that there is no significant effect of trapping the modulus $\tilde{b}$ on the spectrum of long wavelength perturbations in the sense that the final spectrum of the Bardeen potential Φ equals the initial value of the non-decaying mode of the bulk perturbation ξ . This comes about since the value of ξ for small k^* modes which have a wavelength larger than the Hubble radius remains frozen during the phase of the trapping of $\tilde{b}$, and since the final spectrum of ξ coincides with the spectrum of Φ .

Even though our result agrees with a “naive” intuition coming from the analysis of fluctuations in expanding four space-time dimensions, it could be viewed as an unexpected result: The “naive” intuition turned out to be wrong in the case of bouncing cosmologies in four space-time dimensions, for example in pre-big-bang models where it was shown [121] that the growing mode of Φ in the contracting phase couples almost exclusively to the decaying mode of Φ in the expanding phase, leading to the result that there is a large change in the spectral index of the dominant mode between the contracting and the expanding phase. Other analyses performed within the context of four-dimensional general relativity (see [113, 114, 115, 116] and references therein) yielded a similar result. It is now also generally accepted that the four space-time dimensional toy models which were proposed to

describe the ekpyrotic/cyclic models have the same feature, namely that the dominant mode of Φ in the contracting phase matches predominantly to the decaying mode of Φ in the expanding phase (see e.g. [102]). Our analysis, however, confirms the analysis of [117], which showed, in the context of a singular bounce, that in the case of a bounce of boundary branes in five space-time dimensions the spectrum of Φ could be preserved.

Our result is an important step in constructing a viable alternative to standard scalar field-driven inflationary models (next to the remarkable but highly nontrivial KKLMNT construction, [14] and follow-up papers) in the context of cosmologies with extra dimensions. We will now outline two possible proposals which seem promising to us in that regard:

Firstly, one could envisage a modification of the ekpyrotic/cyclic scenario. In the ekpyrotic proposal [80], a test brane moves slowly through the bulk towards the boundary brane of an orbifold (the Horava-Witten [70] setup). In the cyclic scenario [82], it is the two boundary branes that approach each other. In either case, a scale-invariant spectrum of perturbations can be generated during the contracting phase given a suitable potential for the modulus field which describes the inter-brane distance [99]. During the collision of the branes (a singular event in the ekpyrotic/cyclic scenarios) a hot universe is supposed to emerge on the boundary brane we live on today. Our background construction can be viewed as a regularized version of the ekpyrotic/cyclic scenarios (ekpyrotic in the context of a single bounce, cyclic in the case we consider the evolution until $\tilde{b}$ has stabilized). Our work shows that in the context of such a regularized scenario, an initial scale-invariant spectrum could pass through the bounces and thus survive from the initial contracting phase to the final phase of expansion of our

three spatial dimensions (this scenario will be developed further in a follow-up paper). In our scenario, the branes do not actually hit each other, but come to a halt as a consequence of the appearance of new massless modes that get produced explosively at a certain brane-separation (that is at a certain value of b). Our results show that such dynamics does not spoil a scale invariant spectrum generated during the initial brane movement. In this framework, one can still reheat the Universe, since the “kinetic energy” of the branes will get transferred to the stabilizing massless modes and other light modes which act like radiation on the brane (corresponding to the radiation bath that we already included in our setup).

Secondly, an incorporation of inflation into brane/string gas cosmology will rely on the stabilization mechanism described in this article: if any mechanism of inflating three dimensions is found, the internal dimensions will most likely have to deflate in some way. For example, one could employ the idea of anisotropic inflation (studied in the context of vacuum solutions of higher dimensional general relativity in [123][124][125] and [126]). Once the internal dimensions get close to the self-dual radius, certain string modes become massless and get produced explosively along with a radiation bath, in close analogy to reheating after standard scalar field-driven inflation. These modes will then stabilize the internal dimensions, curing the graceful exit problem of anisotropic inflation, while the spectrum of fluctuations produced during inflation in the metric degrees of freedom remains unaltered. A concrete realization of this proposal is in preparation by the authors of this article.

7.7 Conclusions

In this report, we studied how radion, matter and metric fluctuations interact in a universe that exhibits a transient stabilizing epoch of its extra

dimension, which could also be viewed as a series of bounces of the extra dimension. We were primarily interested in the imprints of this epoch on the spectrum of the Bardeen potentials, motivated by the hope of combining an earlier phase, generating a scale invariant spectrum (e.g. in the framework of a modified ekpyrotic scenario, or via an incorporation of inflation within string gas cosmology), with a successful late time stabilizing mechanism, provided by a gas of massless string states.

We found that the spectrum of long wavelength perturbations remains unaltered by this epoch, which has important consequences for various approaches to stringy models of the early universe. In particular, specific realizations of the ekpyrotic/cyclic scenario or an incorporation of inflation into string gas cosmology become possible – we provided two proposals in this report, but many variations are viable.

7.8 Appendix– Anisotropic stress

In this appendix, we will provide arguments as to why no anisotropic stress arises in the setup of [16], which we examined at the perturbative level in this article. In the following $a, b, \dots$ denote string world-sheet coordinates.

First note that our unperturbed background is undergoing FRW expansion in the non-compact dimensions while the compact direction is stabilized at the self dual radius by a string gas. As a preliminary, we wish to show that long wavelength perturbations of the metric do not affect the motion of strings to first order in the perturbation variables. We begin with the Polyakov action for a closed string

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \partial^a X^A \partial_a X^B g_{AB}(X), \quad (7.67)$$

where we work in conformal gauge from the outset. Consider now perturbations around the background: $g_{AB} = g_{AB}^0 + h_{AB}$. It was shown in the appendix of [16] how one can inherit the string spectrum and the constraint algebra provided the derivatives of the background metric g_{AB}^0 (i.e. the Hubble factors) are several orders of magnitude smaller than the string energy scales. We want to make sure that we stay within this regime while studying the dynamical compactification of the extra dimension. Thus, when we consider the equations of motion for a closed string in this perturbed spacetime, we must neglect all derivatives of the unperturbed metric compared to the worldsheet derivatives of the string co-ordinates, as these will be of the order of the string scale whereas the metric derivatives are constrained to be much smaller [16].

Consider the equation of motion for a closed string

$$\partial_a \left(\partial^a X^A g_{CA}(X) \right) = \frac{1}{2} \partial_a X^A \partial^a X^B \partial_C g_{AB}(X),$$

from which follows

$$\begin{aligned} \frac{1}{2} \partial_a X^A \partial^a X^B \partial^C h_{AB} &= \partial_a \partial^a X^C + h_A^C \partial_a \partial^a X^A \\ &\quad + \partial_a X^A \partial^a X^B \partial_A h_{BD} g_0^{DC}. \end{aligned}$$

Now we expand X^A as $X^A = X_0^A + \delta X^A$, where X_0^A is the solution to the unperturbed equation of motion $\partial_a \partial^a X_0^A = 0$, see appendix of [16]. Our perturbed equation of motion then becomes

$$\begin{aligned} \frac{1}{2} \partial_a X_0^A \partial^a X_0^B \partial^C h_{AB} &= \partial_a \partial^a \delta X^C \\ &\quad + \partial_a X_0^A \partial^a X_0^B \partial_A h_{BD} g_0^{DC}. \end{aligned} \tag{7.68}$$

Notice that there are two vastly different scales in the equation above: the worldsheet derivatives and the spacetime derivatives. If we imagine

expanding h_{AB} in terms of fourier modes, the derivatives would bring down a factor of the 4-d wave-vector. Realizing that we need to stay within the domain of validity of our setup, namely that we cannot introduce fluctuations in the metric that would invalidate the conclusion in the appendix of [16] concerning the string spectrum, we need these fluctuations to vanish on energy scales several orders of magnitude smaller than the string scale. This still allows for metric fluctuations that are considerably short distance in nature, but nevertheless above the string scale. With this in mind, we can neglect all the terms in (7.68) that are multiplied by derivatives of the metric perturbation, since these will be orders of magnitude smaller than the other ones. Henceforth, we obtain the result

$$\partial_a \partial^a \delta X^A = 0, \quad (7.69)$$

that is, the string motion is unchanged by long wavelength perturbations.

Ordinarily this would signal the end of any hopes to study perturbative physics; this is not the case for us, since we are primarily interested in perturbing around a background which consists of a gas of such strings. As a result, if we consider the perturbed energy-momentum tensor for such a gas, it will contain no anisotropic stress. To see this, consider the off-diagonal spatial components which can be written as

$$\delta T_j^i \propto \langle p^i \delta p_j \rangle + \langle \delta p^i p_j \rangle, \quad (7.70)$$

where $i \neq j$ and $\langle \dots \rangle$ indicates an ensemble (gas) average. These components vanish, since we just determined that for individual strings $\delta p^i = 0$, which means that there is no anisotropic stress to deal with in our setup.

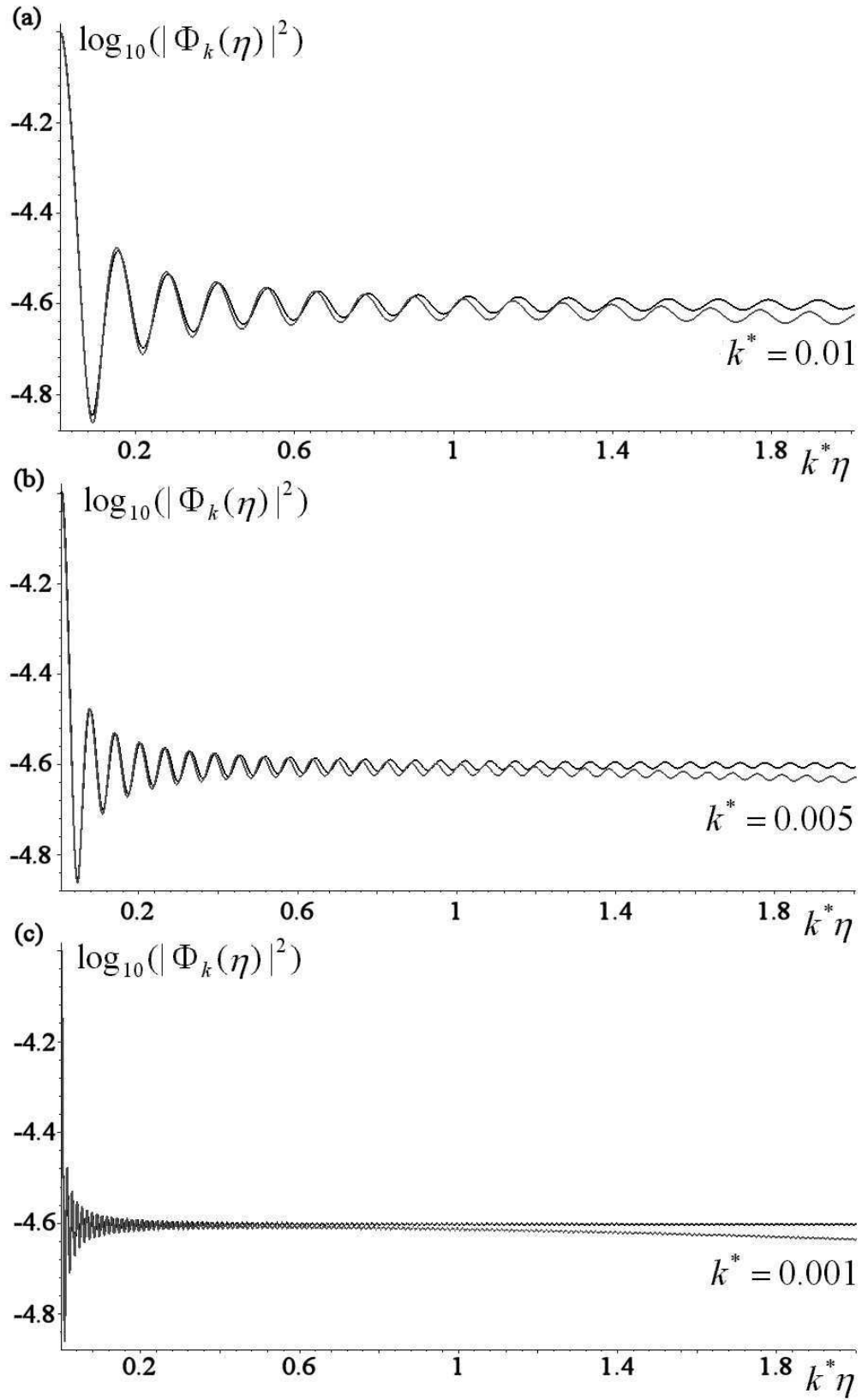


Figure 7-2: $\log(|\Phi_k^2|)$ is plotted for different values of k^* , with the initial conditions given in section 7.5.2. Black: analytic solution of (7.59); Grey (bending curve): numerical solution dB(7.44)-(7.46).

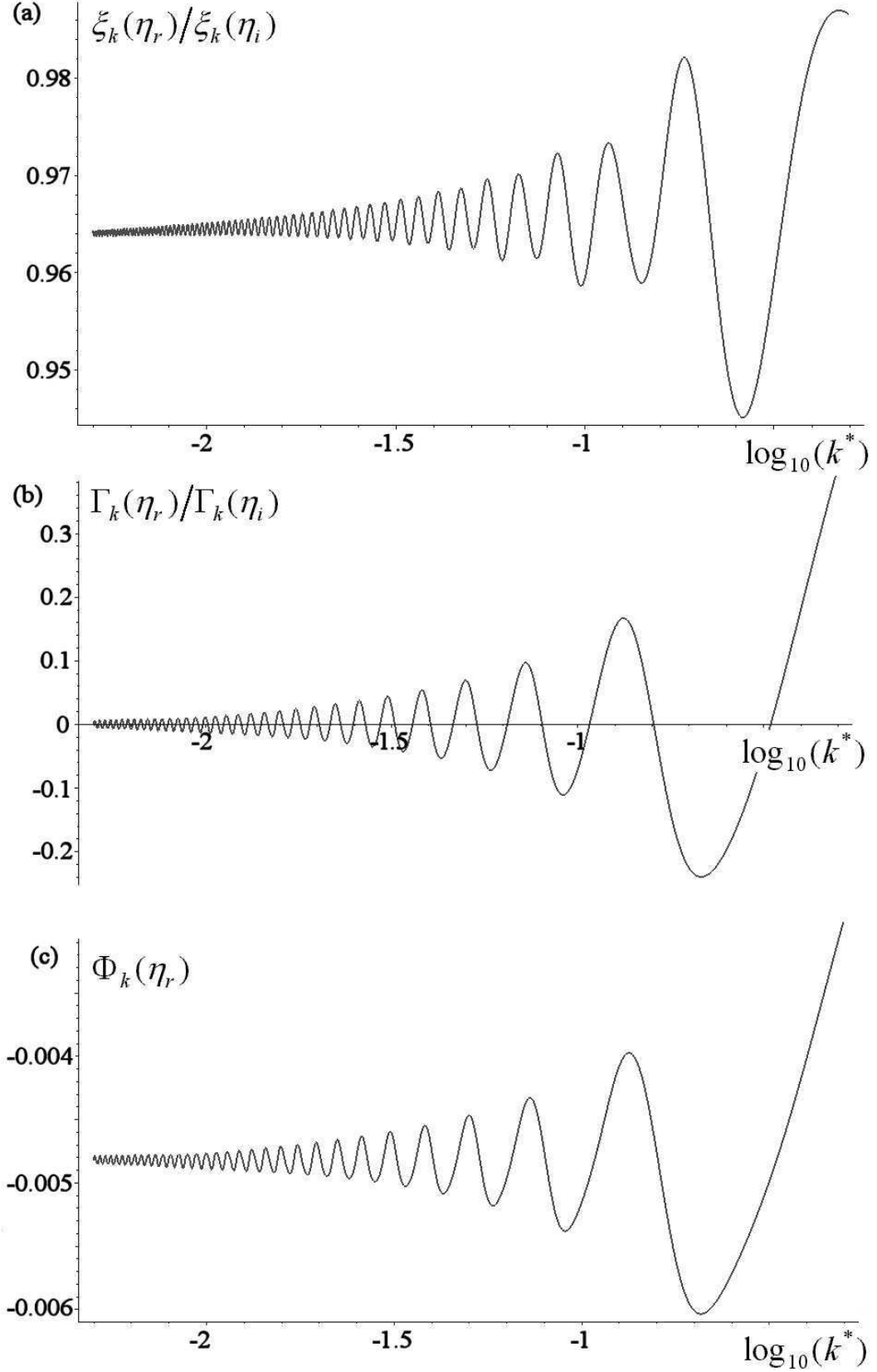


Figure 7-3: The spectrum of (a) ξ_k , (b) Γ_k and (c) $\Phi_k = (\Gamma_k - \xi_k)/2$ is evaluated at Hubble radius crossing $\eta_r = 2/k^*$, with the initial conditions given in section 7.5.2. If the long wavelength regime $k^* \ll 1$ is approached, all oscillations are damped away such that only the constant mode of ξ_k survives, as expected.

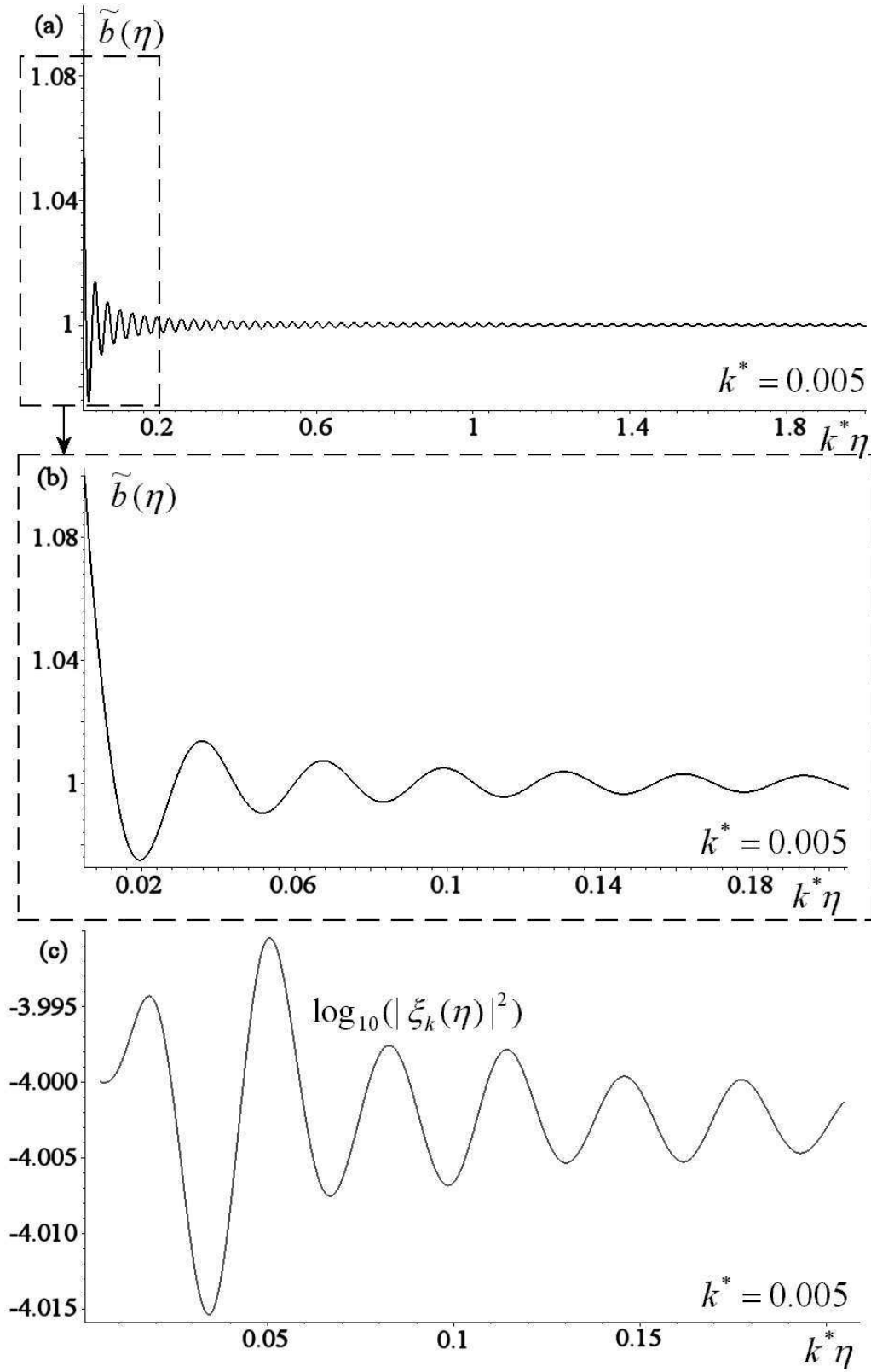


Figure 7-4: (a) and (b): $\tilde{b}$ over $k^*\eta$ is plotted for $k^* = 0.005$, with the initial conditions given in section 7.5.2. (c): ξ_k is plotted over the same time range as b ; Note how the oscillations in the background scale factor source transient oscillations in ξ_k .

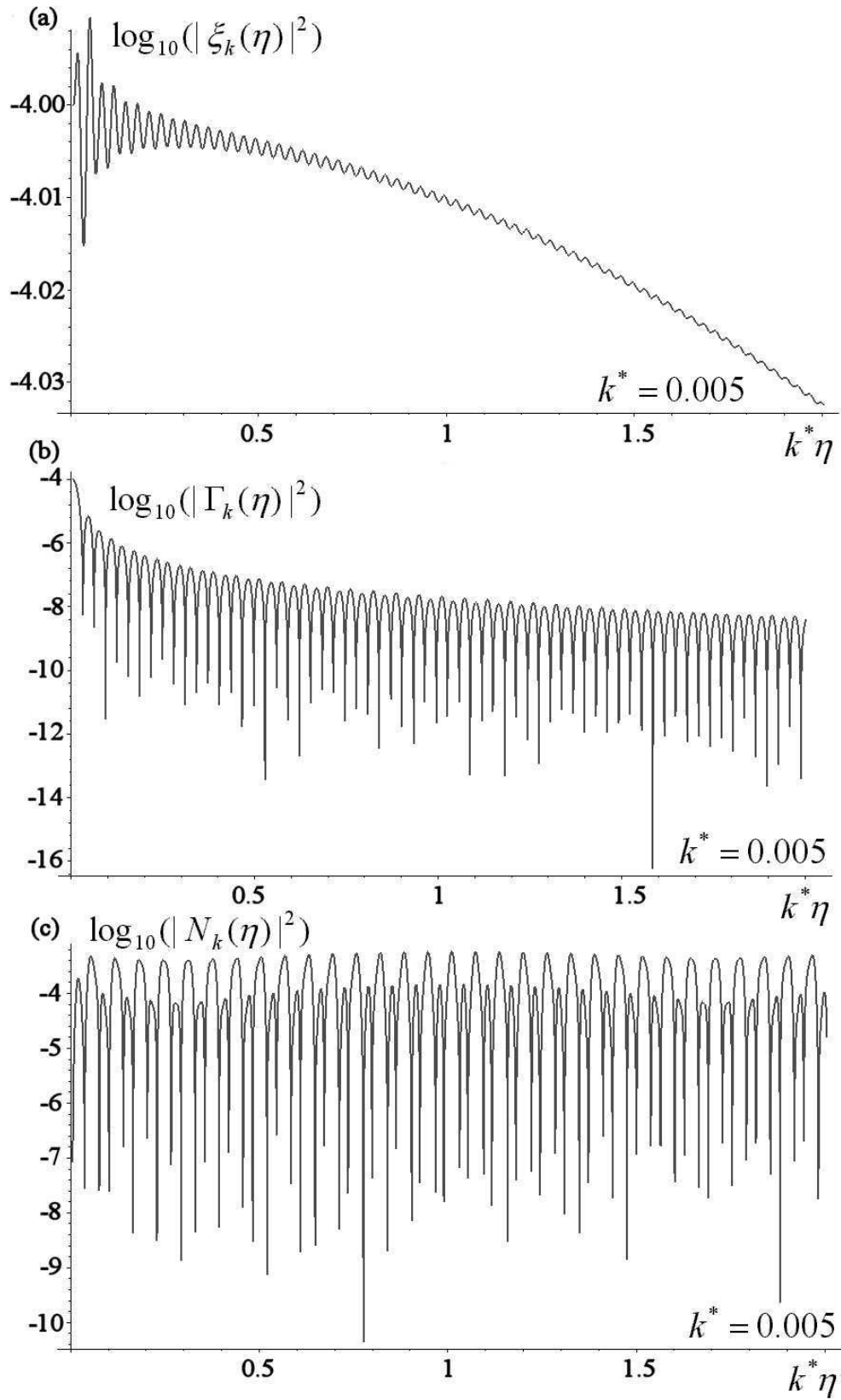


Figure 7–5: The perturbation variables ξ_k , Γ_k and N_k plotted for $k^* = 0.005$, with the initial conditions given in section 7.5.2.

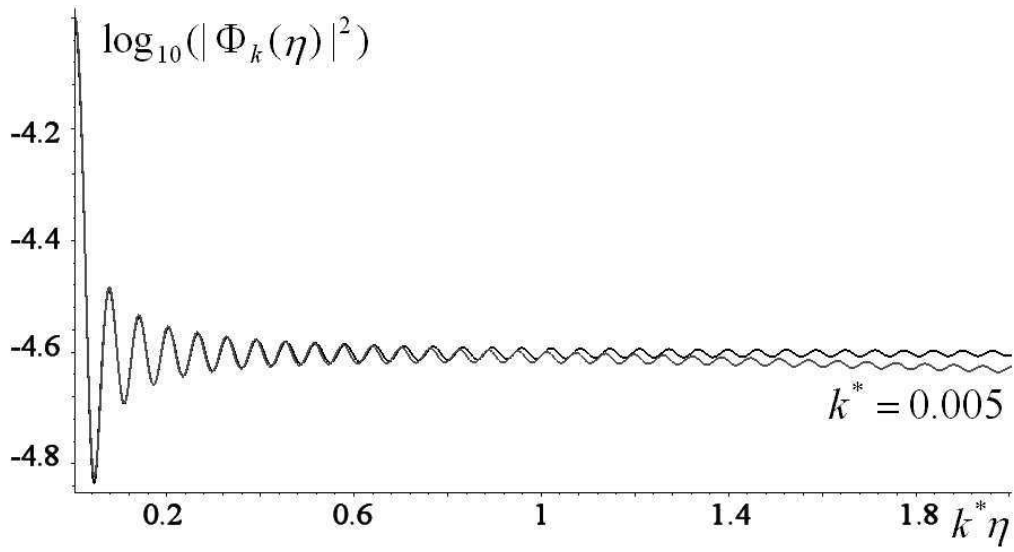


Figure 7-6: $\log(|\Phi_k^2|)$ is plotted for $k^* = 0.005$. This is the same plot as in Fig.7-2 (b), but with different initial conditions for $\tilde{b}$: $\tilde{b}(\eta_i) = 1$ and $\tilde{b}'(\eta_i) = 0.1$ are used instead of (7.63) and (7.64). Note that there is no discernable difference between this plot and Fig.7-2 (b).

CHAPTER 8

Conclusions and Future Prospects

As we hope to have demonstrated to the reader over the last few chapters, string gas cosmology in the simple context of toroidal compactifications, appears to allow us a solution to the moduli problem in string theory in a way that is consistent with various aspects of late time cosmology. However, there are certain key assumptions that underlie our results, and it is worth going through them presently. Firstly, we assumed that the dilaton has been stabilized at some earlier epoch and at a scale that is inaccessible to the energy regimes in which we consider moduli stabilization (see [18] for an explicit demonstration that it is possible to do so by some external dilaton potential). Secondly, we assumed that even though we stay within regimes such that $R \ll 1/\alpha'^{1/2}$, and hence respect the consistency of the α' truncation which makes the dilaton gravity action the consistent one to use, that no essentially stringy effects will take appear to call our treatment into question.

The third assumption we made is a subtle one, and one that the author is currently actively working on relaxing. This assumption involves the suitability of using the dilaton gravity action throughout our treatment

¹ This is demonstrated explicitly in [18], where it is noted that R is given by a multiple of the trace of the energy-momentum tensor, which can be computed explicitly for our gas of massless modes and is indeed shown to be much less than the string scale even through the transients of stabilization.

on moduli stabilization. We know that as a consistent truncation of string theory to the massless sector, the action

$$S = \int d^{D+1}x \, e^{-2\phi} (R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}), \quad (8.1)$$

describes the dynamics of all the massless fields present. As a low energy effective theory, we can consider this action as having been obtained from integrating out all massive modes. However we know from studying the string spectrum on various backgrounds, that the spectrum of our theory depends on which point in moduli space we are in. We exploited this very fact in the preceding chapters to obtain our results on moduli stabilization through the effects of new massless states condensing at the self dual point. However we should be concerned that the use of the action (8.1) might not be appropriate. The fact that we are interested in modes that only become massless at special points means that at precisely such points, these modes have to be integrated back in to the action. Hence we reason that the action (8.1) should change at the enhanced symmetry points, which calls into question the results that we have derived.

The effects of including in the effective action states which are massless at these enhanced symmetry points has been studied extensively by Giveon and Porrati [127]. We are currently in the process of investigating the ramifications of this work to the string gas program, and it is possible to conclude a priori that this avenue holds promise towards resolving some of the open issues concerning string gas cosmology.

The main result of [127] is that the extra massless fields appear as a non-linear sigma model over spacetime, with a potential for the various Higgs fields, whose masses co-incide with the massless spectrum of the heterotic string around enhanced symmetry points. Hence the string Higgs

effect is realized in this context, and in a similar vein to [28], would indicate moduli stabilization around these points purely from the perspective of the low energy effective field theory. Since the action proposed by Giveon and Porrati is known over all points of moduli space, one could conceivably study the effects of moduli stabilization at enhanced symmetry points away from the string scale, hence relaxing at once the first and third assumptions as outlined above. In fact, by turning on expectation values for the background form fields, one will also be able to stabilize the dilaton in such a framework. Such an investigation is presently underway, and the author hopes to have results to report on in the near future. In this way, we hope to have assuaged some of the readers suspicions of the assumptions underlying string gas cosmology so that we can proceed to assess what we have learned in the course of the last few chapters.

Starting with the assumption that it is consistent to study the effects of the propagation of massless strings on a background modelled by the action (8.1), we derived the energy-momentum tensor of a fluid of such strings. We then coupled this fluid to gravity, and discovered that stabilization of one extra dimension resulted [16], in a way that is consistent with fifth force constraints and the results of big bang cosmology. In [17] we considered the effects of general toroidal compactifications and found that similar conclusions held there. In [18], we considered the effects of inhomogeneities on the moduli stabilization mechanism, and found that any initial perturbation spectrum ² persists even though we have dynamically stabilized extra dimensions. We also consider applications towards modelling the transfer of metric fluctuations through non-singular bounces.

² whose origin we take for granted— see however [36][37][38][39]

We conclude this thesis with a discussion of the possibility that string gas cosmology suggests a dynamical selection principle in moduli space. Although we have up till now only seen evidence for this point in a very simple context, it is compelling enough to warrant a discussion of the possibility. As argued in [21], enhanced symmetry points are likely to be dynamical attractors in moduli space. However as discussed in [16], not all enhanced symmetry points are dynamically stable from the perspective of string gas cosmology. For instance, referring to (3.40) in the context of one compactified dimension, we take as an example the massless state $n = 4, w = 0, N = 0$ at the radius $b = 2\sqrt{\alpha'}$. We see from (3.23) and (3.19)-(3.21), that the driving term for the scale factor at this point becomes

$$\ddot{b} + 3H\dot{b} \sim \frac{1}{b^2}, \quad (8.2)$$

which clearly causes the scale factor to quickly move away from this enhanced symmetry point. However we have seen that the self-dual radius is a local attractor of the dynamics of the extra dimension, in that if we start close enough to it, we will be driven to this point and held there by the condensation of massless modes. This was also seen in a general toroidal compactification in [17]. In [25], the analysis was extended to other fixed points of the duality group where it was found that moduli stabilization using string gases also resulted.

If we extend the string gas formalism to include the effects of all fields which might become massless at any given point in moduli space in the low energy effective action, then one is ideally placed to explore the notion that string gases might offer a dynamical selection principle in moduli space. The dynamics of the moduli fields that we have uncovered so far indicate that string gases have an appetite for fixed points under T-duality (see [18][29][35]

for investigations into S-duality), and to confirm this in a more thorough analysis would be a very important development. This would imply that string gas cosmology would have its own landscape, complete with a vacuum selection principle such that the universe is likely to end up only at duality fixed points.

Thus we see that in spite of origins [6] as a mechanism in which we dynamically generate a $3 + 1$ dimensional universe, string gas cosmology has subsequently uncovered a mechanism to stabilize moduli fields in string theory. Although the string technology implemented in string gas studies has been quite primitive so far, we hope to have motivated further study into the subject. The prospect of a dynamical vacuum selection principle in moduli space is a rather appealing one, and one on which the author hopes to have more to report on in the near future.

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