The Spectral Theory of Ordinary Derivations of $K[X, Y]$ and Weaker Forms of the Jacobian Conjecture

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Abstract

Let K be a field of characteristic zero, $K[x, y]$ be the polynomial ring in two variables. Let Δ_f denote the K-derivation of $K[x, y]$ given by $\Delta_f(g) = J(f, g) = f_x g_y - f_y g_x$, the Jacobian determinant of f, g with respect to the coordinate system x, y. The derivation Δ_f is a differential operator on $K[x, y]$. The main objective of this thesis is to develop the spectral theory of the differential operator Δ_f. More precisely, we not only determine the eigenvalues but also the structure of the eigenfunctions of Δ_f. In developing this spectral theory, we prove two weaker forms of the Jacobian Conjecture and establish some relations between the Jacobian Conjecture and our spectral theory.

Résumé

Soient K un corps de caractéristique zéro et $K[x, y]$ l'anneau des polynômes à deux variables. De plus, dénotons par Δ_f la K-dérivé de $K[x, y]$, défini par $\Delta_f(g) = J(f, g) = f_x g_y - f_y g_x$, le déterminant du Jacobian de f, g par rapport au système de coordonnées x, y. Notons que la dérivée Δ_f est un opérateur différentiel sur $K[x, y]$. L'objectif principal de cette thèse est de développer la théorie spectrale de l'opérateur différentiel Δ_f. En plus de calculer les valeurs propres, nous déterminons la strucure des fonctions propres de l'opérateur Δ_f. En développant cette théorie, nous démontrons deux formes plus faibles de *la Conjecture de la Jacobienne* et nous établissons quelques rélations entre *la Conjecture de la Jacobienne* et notre théorie spectrale.

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Contents

Notations and conventions

In this paper the letter K is reserved for fields. $\overline{K}$ is the algebraic closure of K. L is a field extension of K. $trans.deg_K L$ denote the transcendence degree of L over K for any field extension L of K.

$K^\times$ is the multiplicative group of the field K, K^+ is the additive group of the field K..

$\mathcal{Q}, \mathcal{R}, \mathcal{C}$ are the fields of rational, real and complex numbers, respectively.

$K^{(n)}$ is the pure transcendental field extension of K with transcendence degree n.

A is a commutative K-algebra, $A^\times$ is the multiplicative group of A for any K-algebra A, $Qt(A)$ denotes the quotient field of A for any integral domain A and $A^{[n]}$ the polynomial algebra in n variables over A. Moreover, $trans.degree_K A = trans.degree_K(Qt(A))$.

$\mathcal{Z}$ is the ring of rational integers. $\mathcal{N}$ is the set of positive integers. $\mathcal{Z}_+ = \{0\} \cup \mathcal{N}$.

ζ_N is a primitive $N-$th root of unity, ω_N is the set of $N-$th roots of unity.

$\mathcal{G}_a$ is the additive group scheme, $\mathcal{G}_m$ is the multiplicative group scheme.

$K[x,y]$ is the polynomial algebra of two variables x, y over K. Except in chapter 4, section 1, the letter R is reserved for $K^{[2]}$ or $K[x,y]$.

$R_m := K[x, y^{1/m}, y^{-1/m}]$ is the K-algebra generated by $x, y^{1/m}, y^{-1/m}$ with $m \geq 1$. $R_m \cong K[x, \eta, \eta^{-1}]$. Then R contains the subalgebra $\Re = K[xy, y]$ and R_m contains the subalgebra $\Re^{(m)} = K[xy, y^{1/m}]$. We put $\Re(n) = K[xy]y^n, \Re^{(m)}(n) = K[xy]y^{n/m}$ for every $n \geq 1$.

For $f \in K[x,y]$, we use f_x, f_y to denote the partial differential of f with respect to x and y, respectively. That is $f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}$.

Given an element $f \in K[x,y]$, $f = \sum_{i,j} a_{i,j} x^i y^j, a_{i,j} \in K$, we will use the following notation: $Supp(f) = \{(i,j) : a_{i,j} \neq 0\}$ is the *support set* of f. $deg(f) = max_{(i,j) \in Supp(f)}(i+j)$ is called the (total) degree of f.

A pair of two rational numbers $w = (w_1, w_2)$ induces a $\mathcal{Z}$-grading of $K[x,y]$, and $deg_w(f)$ is the w-degree of f. In particular, $deg(f) = deg_{w_0}(f)$, where $w_0 = (1,1)$.

Except in section 3 of chapter 5, the letter δ is reserved for an arbitrary K-derivation on a K-algebra A, and Δ_f is used usually if a Jacobian type derivation is considered, that is $\Delta_f(g) = J(f,g)$ and $J(f,g)$ is the Jacobian determinant. In section 3 of chapter 5, where the field K has characteristic $p > 0$, we use the symbol D to denote a K-derivation of a K-algebra while δ denotes a certain standard differential operator.

$E(\delta), N(\delta), T(\delta), Ker(\delta)$, and $\Lambda(\delta)$ are defined in 1.1 and used throughout the rest of this paper.

For any K-algebra A, $Aut_K A$ is the group of K-automorphisms of A, and we use $\Phi, \Psi, \phi, \psi, \ldots$ to denote the elements of $Aut_K A$.

$\mathcal{F}$ is a filtration of a K-algebra A.

Γ is a $\mathcal{Z}$−grading of a K-algebra A.

$Gr_{\mathcal{F}} A$ is the associated graded algebra of the filtration $\mathcal{F}$.

For any real number x, $[x]$ denote the least integer which is less than or equal to x. We use the symbol $n >> 0$ to denote a sufficiently large integer n.

Chapter 1

Introduction and Summary

What today is called "Spectral Theory" has deep roots in the history of mathematics, going back to J. Von Neumann, D. Hilbert, and even A.Fourier. As far as the author knows, the term "spectral" was coined by D.Hilbert in his study of integral equations. [1] Briefly speaking, two objects were given at first: a function space and an operator on this function space. The purpose of the so-called "spectral theory" is two-fold as well: one problem is to study the eigenvalues (*spectrum*) of a given linear operator and another one is to classify the eigenfunctions (*eigenfunction problem*) and to see whether the general element in the original function space has an expansion in terms of those eigenfunctions (*spectral expansion or Fourier expansion*). In analysis, to prove interesting results along this line, the function spaces usually are assumed to be *complete* and operators are assumed to be *bounded.*

We discuss a quite different situation on function spaces arising from algebraic geometry. We consider a field K, an affine algebraic K-variety X, the regular functions $A = \Gamma(X, 0_X)$ on X, and a K-derivation δ of A. δ can be treated as a differential operator on the space A. In this dissertation we are interested in the simplest example where X is the affine plane and $A = K^{[2]}$, the polynomial algebra in two variables

[1]It comes from two different directions: Fourier theory and matrix theory. A number λ_0 is called an eigenvalue of the linear operator T on a finite dimensional vector space if there exists a vector $x_0 \neq 0$ such that $Tx_0 = \lambda_0 x_0$. The terms "proper value", "characteristic value", "secular value" and "latent roots" were used by various authors at different times. The term "spectrum" is due to Hilbert and the term "spectral" came from the term "spectrum".

over K. Since the affine space is non-compact, to develop the spectral theory in this framework is a challenge as will be shown in this dissertation even in this simplest case.

Here is a brief introduction and summary to the main results proved in this dissertation.

1.1. Let K be a field of characteristic zero. Let A be a commutative K-algebra,[2] $\delta : A \to A$ a K-derivation on A, that is a K-linear map satisfying the Lebiniz rule. We first recall some conventions on notations and terminology. Denote by

$$\Lambda(\delta) = \{\lambda \in K : \exists g \in A, \delta(g) = \lambda g\} \tag{1.1}$$

the set of all *eigenvalues* of the operator δ on A. The function $g \in A$ is called a δ-*eigenfunction* if $\delta(g) = \lambda g$ for some $\lambda \in K$. Let

$$E(\delta) = \{\sum_i a_i g_i : a_i \in K, g_i \in A, \exists \lambda_i \in K, \delta(g_i) = \lambda_i g_i\} \tag{1.2}$$

denote the K-linear vector space generated by the δ-eigenfunctions. Let

$$Ker(\delta) = \{g \in A : \delta(g) = 0\} \tag{1.3}$$

denote the ring of constants of δ. Define

$$N(\delta) = \{g \in A : \exists n \geq 0, \delta^n(g) = 0\}. \tag{1.4}$$

If $N(\delta) = A$, δ is called a *locally nilpotent derivation* of A.

Regard A as a $K[\delta]$-module via the action $(\sum_i a_i \delta^i) f = \sum_i a_i (\delta^i f)$, $a_i \in K$, $f \in A$. Let

$$T(\delta) = \{g \in A : \exists 0 \neq p[T] \in K[T], p(\delta) g = 0\}, \tag{1.5}$$

that is, $T(\delta)$ denote the torsion $K[\delta]$-submodule of A. We call δ a *locally finite*

[2]In this thesis, we always assume that the K-algebras are commutative. We should mention that our framework is also suitable for non-commutative K-algebras. See [13].

derivation if $T(\delta) = A$.

At last, we call δ a *fully spectral derivation* of A if $E(\delta) = A$.

In particular, for $R = K[x,y] \simeq K^{[2]}$ and $f \in R$, $\Delta_f^{(x,y)} : R \to R, \Delta_f^{(x,y)}(g) = J(f,g)$ for $g \in R$, is a K-derivation of R, where

$$J(f,g) = \det \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} \tag{1.6}$$

is the Jacobian determinant of f, g with respect to x, y. We shall write $\Delta_f = \Delta_f^{(x,y)}$ to simplify the notation if x, y are already specified. Hence $Ker(\Delta_f), E(\Delta_f), \Lambda(\Delta_f)$ and $T(\Delta_f)$ are defined and will be used throughout this paper.

1.2. Let us first explain the main result about the structure of $T(\delta)$.

Suppose $f \in K[x,y]$. We say that f satisfies the *weak Jacobian condition* if

$$T(\Delta_f) \neq Ker(\Delta_f).$$

If so, then $trans.degree_K T(\Delta_f) = 2$. We say f satisfies the *Jacobian condition* if $\Delta_f(g) \in K^\times$ for some $g \in K[x,y]$. The Jacobian Conjecture in two variables is:[3]

Suppose f satisfies the Jacobian condition. Then $T(\Delta_f) = K[x,y]$.

Here, we are interested in whether $T(\Delta_f) \cong K^{[2]}$ if f satisfies the weak Jacobian condition. This question, in a sense, is analogous to the Jacobian Conjecture in two variables.

To explain our solution for this question, we need to explain first the relation between $T(\Delta_f), E(\Delta_f)$ and $N(\Delta_f)$ as follows.

If K is an algebraically closed field, we prove (Proposition 2.1.9) that either $T(\Delta_f) = E(\Delta_f)$ or $T(\Delta_f) = N(\Delta_f)$. Therefore, to discuss the question about the structure of $T(\Delta_f)$, it is enough to discuss the structure of $E(\Delta_f)$, and $N(\Delta_f)$ independently.

We say that a K-derivation δ of $K^{[2]}$ is an *ordinary* derivation if $trans.deg_K(Ker(\delta)) =$

[3]This is just one of the many equivalent forms of this famous conjecture. We shall give its general form in 2.1.4 below.

1. For example, the derivation Δ_f is an ordinary derivation if f is any non-constant polynomial. Recall that a K-algebra A is *geometrically factorial* over K if $A \otimes_K \overline{K}$ is an unique factorization domain (UFD for simplicity). Among other things, we have the following result, the first part of which was proved by Miyanishi-Nakai([33], [36] Theorem 1) in case $K = \overline{K}$.

Theorem A.

(A.1). Let K be a field of characteristic zero and δ an ordinary K-derivation of $K[x,y]$. Assume that $N(\delta) \neq Ker(\delta)$ and $N(\delta)$ is a finitely generated K-algebra and geometrically factorial over K. Then $N(\delta) \cong K^{[2]}$.

(A.2). There exists an ordinary K-derivation δ of $K[x,y]$ with $N(\delta) \neq Ker(\delta)$ and $N(\delta) \not\cong K^{[2]}$.

In view of the examples of 2.2.7, the problem to determine the algebraic structure of $N(\Delta_f)$, in general, is not solved in this paper (See conjecture 2.2.11). When $T(\Delta_f) = E(\Delta_f)$, we can determine its algebraic structure explicitly in the spectral theory (theorem B.1) below. Here we just give a reason why the discussion about $N(\Delta_f)$ seems more difficult from the partial differential equation point of view. Note that $E(\Delta_f) \neq Ker(\Delta_f)$ is equivalent to the solvability of the partial differential equation

$$J(f,g) = \lambda g \tag{1.7}$$

for some $g \in K[x,y]$ and $\lambda \in K^{\times}$, and $N(\Delta_f) \neq Ker(\Delta_f)$ is equivalent to the solvability of the partial differential equation

$$J(f, J(f,g)) = 0, J(f,g) \neq 0 \tag{1.8}$$

for some $g \in K[x,y]$. Although the equation (1.8) looks more complicated than the equation (1.7), we observe that, from the partial differential equation point of view, the condition $N(\Delta_f) \neq Ker(\Delta_f)$ is more *flexible* than the condition $E(\Delta_f) \neq Ker(\Delta_f)$ (see Prop. 5.4.3 and Prop. 5.4.4 below). Hence, it is harder, in general, to examine the structure of $N(\Delta_f)$ under the condition $N(\Delta_f) \neq Ker(\Delta_f)$.

Theorem A and several related results about locally nilpotent (finite) derivations will be proved in chapter 2.

1.3. We now summarize the spectral theory for the affine plane.

We first explain our solution to the *eigenvalue problem.*

One of the main results proved in this dissertation is Theorem 5.1.1 :$\Lambda(\Delta_f) = \mathcal{Z}\rho_f$ *for some element* $\rho_f \in K$. Clearly, ρ_f is uniquely determined up to sign. Moreover, we determine exactly what ρ_f is (*the least eigenvalue*) in Proposition 5.4.2, and thereby solve the *eigenvalue problem* for Δ_f.

Theorem 5.1.1 is used in an essential way to solve the *eigenfunction problem.* More precisely, we have

Theorem B.

(B.1). Let f be a non-constant polynomial in $K[x, y]$. Assume $E(\Delta_f) \neq Ker(\Delta_f)$. Then $E(\Delta_f) \cong K[X, Y, Z]/(XY - a(Z))$, as a K-algebra, for some polynomial $a(T) \in K[T]$.

(B.2). $E(\Delta_f) \cong K^{[2]}$ if and only if there exists Δ_f-eigenfunctions $g, h \in K[x, y]$ and $c \in K$ such that $f + c = gh$ with $J(g, h) \in K^{\times}$.

(B.3). There exist a polynomial f such that $E(\Delta_f) \neq Ker(\Delta_f)$ and $E(\Delta_f) \ncong K^{[2]}$.

The *eigenfunction problem* is solved by Theorem B completely.

As we said before, it is natural to study the *Fourier expansion problem* in the spectral theory, i.e., whether every polynomial $f \in R$ can be expanded as a linear combination of the eigenfunctions. Theorem B tells us this problem is, more or less, equivalent to the Jacobian Conjecture in two variables. We do not aim to solve this famous conjecture in this thesis. Instead we shall treat two weaker forms of this conjecture in proving Theorem B.

1.4. We are going to explain the weaker forms of the Jacobian Conjecture proved in this thesis and why they are closely related to Theorem B.

In order to explain this, we employ the following notation. Let $\Re = K[xy, y] \subseteq K[x, y]$. For $m \geq 1$, let $K[x, y^{1/m}, y^{-1/m}]$ be the K-algebra generated by $x, y^{1/m}, y^{-1/m}$. Then every element of $K[x, y^{1/m}, y^{-1/m}]$ has the form $\sum_{i \geq 0, i \in \mathcal{Z}, j \in \mathcal{Z}} a_{ij} x^i y^{j/m}$, $a_{ij} \in K$. The analogue of $\Re$ in $K[x, y^{1/m}, y^{-1/m}]$ is defined by $\Re^{(m)} := K[xy, y^{1/m}]$

The essential ingredient in proving Theorem 5.1.1 and Theorem B in our present study of Spectral Theory is:

Suppose $f \in K[x,y]$, *and assume* $E(\Delta_f) \neq Ker(\Delta_f)$. *Then either there exists an automorphism* $\Psi \in Aut_K K[x,y]$ *such that* $\Psi(f) \in \Re$, *or there exists* $\Psi \in Aut_K K[x,y]$, *a positive integer* $m \geq 1$ *and an automorphism* $\Psi_m \in Aut_K(K[x, y^{1/m}, y^{-1/m}])$ *of the form* :$\Psi_m(y^{-1/m}) = y^{-1/m}, \Psi_m(y^{1/m}) = y^{1/m}, \Psi_m(x) = x + h(y^{-1/m})$ $(h(T) \in K[T])$ *such that* $\Psi_m(\Psi(f)) \in \Re^{(m)}$. [4]

The connection of this result with the Jacobian Conjecture can now be explained as follows.

Let $(JC.f)$ denote the statement: [5]

If $\Delta_f(g) \in K^\times$ *for some* $g \in K[x,y]$, *then* $K[x,y] = K[f,g]$.

We prove (Theorem 4.3.14): *Suppose* $f \in \Re$. *Then* $(JC.f)$ *holds.* Therefore the Jacobian Conjecture in two variables, in a sense, is reduced to prove the following statement:

Suppose $f \in K[x,y]$, *and* $T(\Delta_f) \neq Ker(\Delta_f)$. *Then there exists an automorphism* $\Psi \in Aut_K(K[x,y])$ *such that* $\Psi(f) \in \Re$.

According to Theorem 4.3.14, the Reduction Theorem 4.3.1 can be considered as a weaker form of the Jacobian Conjecture.

Another *weaker form of the Jacobian Conjecture* proved in this thesis is Theorem 5.3.14: *If* $J(f,g) \in K^\times$, *then* $E(\Delta_{fg}) = K[f,g]$. Theorem 5.3.14 follows immediately from the proof of Theorem B.2.

Theorem B and the above *weaker forms of the Jacobian Conjecture* will be proved in chapter 5. The technical part needed will be developed in chapter 3 and chapter 4.

1.5. Besides the spectral theory and the weaker forms of the Jacobian Conjecture the following subject is also considerably studied in this dissertation.

In order to explain these results, we give a close examination of $E(\Delta_f)$ from the $\mathcal{G}_m$-action point of view.

Let A be a K-algebra with a $\mathcal{G}_m$-action. As is well known, A has a $\mathcal{Z}$−grading,

[4]This is a weaker form of the Reduction Theorem 4.3.1,

[5]This is another equivalent form of the Jacobian Conjecture in two variables.

that is, a decomposition $A = \bigoplus_{n \in \mathcal{Z}} A_n$.[6] Define by $Supp(A) = \{n \in \mathcal{Z} : A_n \neq 0\}$. Then $Supp(A)$ is a sub-semigroup of $\mathcal{Z}$. If $Supp(A)$ is a non-trivial subgroup of $\mathcal{Z}$, we call this action a *mixed* $\mathcal{G}_m$-action. Then Theorem 5.1.1 essentially asserts that there is a mixed $\mathcal{G}_m$-action on $E(\Delta_f)$ if Δ_f has non-zero eigenvalues. But, we will see there is more algebraic structure on $E(\Delta_f)$ along this line. To see this, we define $\mathcal{G}_m - \Delta_f-$domains and quasi $\mathcal{G}_m - \Delta_f-$domains, as follows.

Let A be a K-subalgebra of $K[x, y]$ endowed with a mixed $\mathcal{G}_m$-action with the associated $\mathcal{Z}-$grading decomposition $A = \bigoplus_{n \in \mathcal{Z}} A_n$. We say that f is a *closed polynomial* in $K[x, y]$ if f is not a polynomial of degree ≥ 2 in another polynomial g. Let f be a non-constant closed polynomial in $K[x, y]$. If $A_0 = K[f]$ and $trans.degree_K A = 2$, we call A a *quasi* $\mathcal{G}_m - \Delta_f-$*domain.* Given three K-algebras $A_1 \subseteq A_2 \subseteq A_3$, we say A_2 is *factorially closed* in A_3 relative to A_1, if for any $a_3 \in A_3, a_1 \in A_1, a_1 a_3 \in A_2$ implies that $a_3 \in A_2$. A quasi $\mathcal{G}_m - \Delta_f-$domain is a $\mathcal{G}_m - \Delta_f-$*domain* if A is factorially closed in R relative to $K[f]$. In particular, we can prove (Prop. 5.3.15) that $E(\Delta_f)$ is a $\mathcal{G}_m - \Delta_f-$domain when Δ_f has non-zero eigenvalues. Moreover, we prove (Theorem 6.1.2) that a $\mathcal{G}_m - \Delta_f-$domain is a finitely generated K-algebra.

In this thesis we will also consider the converse problem, i.e., to classify all $\mathcal{G}_m - \Delta_f-$domains. Recall that a polynomial f has a *multiple factor* if there exists an irreducible polynomial P and $n \geq 2$ such that $f = P^n Q$ for some polynomial Q. When for any $c \in K$, $f + c$ has no multiple factor, the converse problem is solved completely by the following theorem.

Theorem C. *Let $K = \overline{K}, R = K[x, y]$, and $f \in R$ a non-constant closed polynomial.*

(C.1). Let $A = \bigoplus_{n \in \mathcal{Z}} A_n$ be a $\mathcal{G}_m - \Delta_f-$domain. Then there exists $\lambda \in K[x, y]$ such that $A_n = \{g \in K[x, y] : \Delta_f(g) = n\lambda g\}$ for all $n \in \mathcal{Z}$.

(C.2). Let A be a $\mathcal{G}_m - \Delta_f-$domain. Assume that $f + c$ has no multiple factor, for any $c \in K$. Then $A \cong K[X, Y, Z]/(XY - a(Z))$, as a K-algebra, for some polynomial $a(T) \in K[T]$.

Conversely, we have

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[6]See section 3.1 for the definition.

(C.3). Let $\lambda \in K[x, y] - \{0\}$, and assume that f_x, f_y have no common factor. Put

$$B_n = \{g \in K[x, y] : \Delta_f(g) = n\lambda g\},$$

and $B = \bigoplus_{n \in Z} B_n$. Suppose $B \neq B_0$. Then B is a quasi $\mathcal{G}_m - \Delta_f-$domain.

Moreover:

(C.4). Assume that $a(T)$ is a polynomial of $K[T]$ which is not a power of another polynomial. Then there exists a closed polynomial $f \in K[x, y]$ and a quasi $\mathcal{G}_m - \Delta_f-$domain A such that $A \cong K[X, Y, Z]/(XY - a(Z))$, as a K-algebra.

Theorem C.1 gives the structure of a $\mathcal{G}_m - \Delta_f-$domain in terms of one element $\lambda \in K[x, y]$. It is relatively simple to prove (Lemma 5.3.5) that any $f + c$ has no multiple factors if Δ_f has non-zero eigenvalues. Then Theorem C.2 is a slight generalization of theorem B.1 as stated above. C.3 and C.4 are, in a sense, converses of Theorem C.1 and C.2, respectively.

Theorem C will be proved in chapter 5.

1.6. In the general case when $f + c$ has a multiple factor for some $c \in K$, the structure of a $\mathcal{G}_m - \Delta_f-$domain is more complicated. We have not solved this problem *completely* so far. We prove that

Theorem D. *Let $K = \overline{K}$, and $f \in R = K[x, y]$ be a non-constant closed polynomial. Then*

(D.1). Any $\mathcal{G}_m - \Delta_f-$domain is an affine rational surface.

(D.2). Suppose $\zeta(f) = 1$ (defined in section 5.3). Then $A \cong (K^{[2]})^{\omega_N}$ as a K-algebra, where ω_N is a cyclic group of order N acting on $K^{[2]}$.

(D.3). Given a K-algebra of the form $(K^{[2]})^{\omega_N}$, where ω_N acts on $K^{[2]}$, there exists $f \in K[x, y]$ and a $\mathcal{G}_m - \Delta_f$-domain A such that $A \cong (K^{[2]})^{\omega_N}$ as a K-algebra.

Theorem D is proved in chapter 6.

Chapter 2

Locally Nilpotent Derivations and Theorem A

Our main purpose in this chapter is to study locally nilpotent derivations and to prove theorem A.

2.1 Ordinary Derivations

Since ordinary derivations are our main object in this paper, it is appropriate to devote the first section to a collection of observations concerning them. The main result of this section is Prop. 2.1.9, which establishes the relation between $T(\delta)$, $N(\delta)$ and $E(\delta)$ for an ordinary derivation δ. Prop. 2.1.7, which has independent interest, is used heavily to prove Prop. 2.1.9.

2.1.1. Ordinary derivations. Let K be an arbitrary field of characteristic zero and $K^{[n]} = K[x_1, \ldots, x_n]$ (the polynomial ring in n variables over K). An *ordinary* derivation of $K^{[n]}$ is a non-zero K-derivation of $K^{[n]}$ such that $Ker(\delta)$, the ring of constants of δ, has transcendence degree $n-1$ over K. For an ordinary derivation δ, there exists polynomials $f_1, \ldots, f_{n-1} \in Ker(\delta)$ such that $Ker(\delta)$ is algebraic over $K[f_1, \ldots, f_{n-1}]$, that is, $f_1, \ldots, f_{n-1}, g$ are algebraically dependent over K for any $g \in Ker(\delta)$. Moreover, if $Ker(\delta)$ is an affine K-domain, we may choose $f_1, \ldots, f_{n-1}$ such that $Ker(\delta)$ is a finitely generated $K[f_1, \ldots, f_{n-1}]$-module by Noether's normalization theorem.

Suppose δ is an ordinary K-derivation of $K[x_1, \ldots, x_n]$, and $Ker(\delta)$ is algebraic over $K[f_1, \ldots, f_{n-1}]$ for some $f_1, \ldots, f_{n-1} \in K[x_1, \ldots, x_n]$, as above. By [31], Lemma 2, then there exists a rational function $h \in K(x_1, \ldots, x_n)$ such that $\delta = h\Delta_{f_1,\ldots,f_{n-1}}$, where $\Delta_{f_1,\ldots,f_{n-1}}(g) = J(f_1, \ldots, f_{n-1}, g)$, the Jacobian determinant of $f_1, \ldots, f_{n-1}, g$ with respect to $x_1, \ldots, x_n$, for all $g \in K[x_1, \ldots, x_n]$.

2.1.2. Ring property. Let δ be a K-derivation of $K[x_1, \ldots, x_n]$. By the formula

$$\delta^n(a+b) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\delta^k(a)\delta^{n-k}(b), \tag{2.1}$$

we see that $N(\delta)$ is a ring. Write, for $\lambda \in R$,

$$E(\delta, \lambda) = \{g \in K[x_1, \ldots, x_n] : (\delta - \lambda)g = 0\}. \tag{2.2}$$

Then

$$E(\delta, \lambda_1) \bigotimes_K E(\delta, \lambda_2) \subseteq E(\delta, \lambda_1 + \lambda_2), \tag{2.3}$$

In particular, $E(\delta)$ (see 1.1) is a subring of $K[x_1, \ldots, x_n]$. Define

$$T(\delta, \lambda) = \{g \in K[x_1, \ldots, x_n] : \exists m \geq 0, (\delta - \lambda)^m g = 0\}. \tag{2.4}$$

$T(\delta)$ is a subring of $K[x_1, \ldots, x_n]$ by its definition (in 1.1).

2.1.3. $Ker(\delta)$. The structure of $Ker(\delta)$ is very complicated in general. For example, there are examples with $Ker(\delta)$ not even finitely generated as a K-algebra. In fact, To determine $Ker(\delta)$ is closely related to Hilbert's fourteenth problem as many authors have shown to us (see [37], [42]). Precisely, let $G \subseteq GL_n(K)$ be a connected algebraic group. Then there exists a K-derivation δ of $K[x_1, x_2, \ldots, x_n]$ such that $Ker(\delta) = K[x_1, x_2, \ldots, x_n]^G$ (see [37], Theorem 6.4). In particular, the counterexamples, such as Nagata's, to the fourteenth problem of Hilbert provide us examples of derivation δ with $Ker(\delta)$ not even finitely generated as a K-algebra. More recently, Freudenburg, Roberts, and Miyanishi have shown us such examples of locally nilpotent derivations δ. See [18], [22] and [42].

2.1.4. The Jacobian Conjecture and the Weak Jacobian Question. Let δ be a K-derivation of $K[x_1, \ldots, x_n]$. If $\delta(g) \in K^\times$ for some $g \in K[x_1, \ldots, x_n]$, g is called a *slice* of δ. If $\delta(g) \neq 0$ and $\delta^2 g = 0$, we call g a *local slice* of δ. See [16]. If δ has a slice g, then $N(\delta) = Ker(\delta)[g] \cong Ker(\delta)^{[1]}$ ([51], Prop. 2.1).

Now consider $\delta = \Delta_f, f = (f_1, \ldots, f_{n-1})$. If $\Delta_f(g) = J(f_1, \ldots, f_{n-1}, g) \in K^\times$ for some $g \in K[x_1, \ldots, x_n]$, and if $K[f_1, \ldots, f_{n-1}]$ is *factorially closed* in $K[x_1, \ldots, x_n]$,[1] then by [10], Corollary 2.4, $Ker(\Delta_f) = K[f_1, \ldots, f_{n-1}]$, and $N(\Delta_f) = K[f_1, \ldots, f_{n-1}, g]$. Recall that the famous Jacobian conjecture:

JC: Suppose $f_1, \ldots, f_n \in K[x_1, \ldots, x_n]$ satisfy $J(f_1, \ldots, f_{n-1}, f_n) \in K^\times$. Then $K[x_1, \ldots, x_n] = K[f_1, \ldots, f_n]$.

Therefore proving the Jacobian Conjecture is equivalent to showing that Δ_f is a locally nilpotent derivation. We put forward a related question in this paper. We called it the Weak Jacobian Question, WJQ in short. [2]

WJQ: If δ is an ordinary K-derivation of $K[x_1, \ldots, x_n]$, and $T(\delta) \neq Ker(\delta)$, when is $T(\delta) \cong K^{[n]}$?

For general n, this question seems not attackable because it relates to many open questions, such as giving an *algebraic-geometric* characterization of affine space, the classification of locally finite derivations, and in particular, the classification of locally nilpotent derivations. Those questions are still mysterious at present. As a matter of fact, only very recently some definite results about locally nilpotent derivations on $K^{[3]}$ were described in [9], [11], [16], [17], and [12]. The main purpose of this paper is to give the solution of WJQ when $n = 2$.

In the remains of this section we shall prove several general results which will be useful in the sequel.

2.1.5. Proposition. *Suppose L is an algebraic extension of K and δ a K-*

[1] A subring A of an integral domain B is called factorially closed if for all $x, y \in B$ we have $xy \in A - \{0\}$ implies that $x, y \in A$.

[2] We state it as a question, not as a conjecture since we don't have strong evidence to believe it is true or not, even if we assume the Jacobian Conjecture. Here, "*weak*" means: both the conditions and the claims of WJQ are weaker than those of JC.

derivation of $K[x_1, \ldots, x_n]$*. Let* δ_L *be the* L*-derivation of* $L[x_1, \ldots, x_n]$ *extending* δ *and* $T(\delta_L)$ *the* $L[\delta_L]$*−torsion submodule of* $L[x_1, \ldots, x_n]$*. Then* $T(\delta) \otimes_K L \cong T(\delta_L)$*.*

Proof. We first prove that $T(\delta_L)$ is the torsion $K[\delta_L]$-submodule of $L[x_1, \ldots, x_n]$. In fact, for any polynomial $p(T) \in L[T]$, its coefficients generate a finite algebraic extension K_1 of K. Let $q(T)$ denote the product of all the conjugates of $p(T)$ in the Galois closure of K_1. Then $q(T) \in K[T]$. Moreover, for $g \in L[x_1, \ldots, x_n]$, $p(\delta_L)g = 0$ implies that $q(\delta_L)g = 0$. Hence, $T(\delta_L)$ is the torsion $K[\delta_L]$-submodule of $L[x_1, \ldots, x_n]$.

For any $g \in T(\delta_L)$, we choose a polynomial $p(t) = t^m + c_1 t^{m-1} + \ldots + c_m \in K[t]$ such that $p(\delta_L)g = 0$. Write $\delta_L = \sum \delta_L(x_i)\frac{\partial}{\partial x_i}, g = \sum g_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}$ with $\delta_L(x_i) \in K[x_1, \ldots, x_n]$ by assumption. Then, by rewriting the equation $(\delta_L^m + c_1\delta_L^{m-1} + \ldots + c_m)g = 0$, we have a linear system of equations V for the coefficients $g_{i_1,\ldots,i_n}$ defined over K. The L-zero space [3] of this linear system is a tensor product of L with the K-zero space of this linear system, by linear algebra. Then the stated assertion follows.

The following lemma is well known. We state it here for easy reference.

2.1.6. Lemma. *Let* A *be a* K*-algebra,* δ *a* K*-derivation of* A*, and* $\lambda \in K$*. Then the sum* $\sum_{n \in \mathbb{Z}} T(\delta, n\lambda)$ *is direct, that is, for any* $m \geq 2$*,* $\sum_{i=1}^m a_i = 0$ *with* $a_i \in T(\delta, n_i\lambda)$ *and* $n_1 < \ldots < n_m, n_i \in \mathbb{Z}$ *implies that each* $a_i = 0, i = 1, \ldots, m$*.*

2.1.7. Proposition. *Let* A *be a* K*-algebra and an integral domain,* δ *a* K*-derivation of* A*. Suppose there exists* $\lambda \in K^\times$ *and* $h \in A$ *such that* $(\delta - \lambda)^2 h = 0, (\delta - \lambda)h \neq 0$*. Let* $g = (\delta - \lambda)h$*. Then* g, h *are algebraically independent over* $Ker(\delta)$*.*

Proof. For any positive integer $m \geq 1$, we first prove that

$$(\delta - m\lambda)^k h^m = m(m-1)\ldots(m-k+1)g^k h^{m-k}, 1 \leq k \leq m. \tag{2.5}$$

To show this, we note that (2.5) is true when $k = 1$ since $(\delta - m\lambda)h^m = \delta(h^m) - m\lambda h^m = mh^{m-1}(\delta(h) - \lambda h) = mgh^{m-1}$. Suppose the formula (2.5) holds for any

[3] L-zero space is the solutions of this linear system in L, for any field L.

$k < m$. Then

$$\begin{aligned}(\delta - m\lambda)^{k+1}h^m &= m(m-1)\dots(m-k+1)[(\delta - m\lambda)g^k h^{m-k}] \\ &= m(m-1)\dots(m-k+1)[\delta(g^k)h^{m-k} + g^k\delta(h^{m-k}) - m\lambda g^k h^{m-k}] \\ &= m(m-1)\dots(m-k)g^{k+1}h^{m-k-1}.\end{aligned}$$

Therefore (2.5) is proved. In particular, for any positive integer $m \geq 1$, we have (when $k = m$)

$$(\delta - m\lambda)^m h^m = m! g^m \neq 0 \tag{2.6}$$

and then

$$(\delta - m\lambda)^{m+1} h^m = m!(\delta - m\lambda)g^m = 0. \tag{2.7}$$

We next prove that, for any integers $i, j \in \mathcal{Z}$,

$$E(\delta, i\lambda) \bigotimes_K T(\delta, j\lambda) \subseteq T(\delta, (i+j)\lambda). \tag{2.8}$$

It suffices to prove that

$$(\delta - \lambda_1 - \lambda_2)^n (ab) = 0 \tag{2.9}$$

if $(\delta - \lambda_1)a = 0$, and $(\delta - \lambda_2)^n b = 0, n \geq 1$. We go by induction on n. When $n = 1$, (2.9) is true. Suppose (2.9) is true for n, and suppose that $(\delta - \lambda_2)^{n+1} b = 0$. Let $c = (\delta - \lambda_2)b$. Then $(\delta - \lambda_1 - \lambda_2)^n (ac) = 0$ by induction hypothesis. On the other hand, it is readily seen that $(\delta - \lambda_1 - \lambda_2)ab = ac$. Hence $(\delta - \lambda_1 - \lambda_2)^{n+1}(ac) = 0$. We have thus proved (2.8).

Now we are in the position to prove Prop. 2.1.7.

Given a relation

$$\sum_{i,j} a_{i,j} g^i h^j = 0, a_{i,j} \in Ker(\delta), \tag{2.10}$$

to prove Prop. 2.1.7, it suffices to show that each $a_{i,j} = 0$. Since $a_{i,j} g^i \in E(\delta, i\lambda)$, $a_{i,j} g^i h^j \in T(\delta, (i+j)\lambda)$ by (2.7) and (2.8). By Lemma 2.1.6, for each positive integer

m, we have

$$\sum_{i+j=m} a_{i,j} g^i h^j = 0. \tag{2.11}$$

Write $b_k = a_{k,m-k} g^k, k = 0, \ldots, m$. Then $b_0 h^m + b_1 h^{m-1} + \ldots + b_m = 0$. For any positive integer k, m with $k \leq m$, we shall show that

$$(\delta - m\lambda)^l (b_{m-k} h^k) = k(k-1)\ldots(k-l+1) b_{m-k} g^l h^{k-l}, l = 1, \ldots, k. \tag{2.12}$$

When $l = 1$, (2.12) is evident since $b_{m-k} \in E(\delta, (m-k)\lambda)$. Suppose the formula (2.12) holds for $i < k$. Then

$$\begin{aligned}
(\delta - m\lambda)^{i+1} b_{m-k} h^k &= (\delta - m\lambda)(k(k-1)\ldots(k-i+1)) b_{m-k} g^i h^{k-i} \\
&= k \ldots (k-i+1) a_{m-k,k} [\delta(g^{m-k+i} h^{k-i}) - m\lambda g^{m-k+i} h^{k-i}] \\
&= k(k-1)\ldots(k-i+1)(k-i) b_{m-k} g^i h^{k-i-1} (\delta(h) - \lambda h) \\
&= k(k-1)\ldots(k-i) b_{m-k} g^{i+1} h^{k-i-1}.
\end{aligned}$$

We have thus proven (2.12). In particular, we have (when $l = k$)

$$(\delta - m\lambda)^k (b_{m-k} h^k) = k! b_{m-k} g^k \tag{2.13}$$

and then

$$(\delta - m\lambda)^{k+1} (b_{m-k} h^k) = 0. \tag{2.14}$$

Therefore, acting with $(\delta - m\lambda)^m$ on both sides of the equation $b_0 h^m + \ldots + b_m = 0$, we obtain $m! b_0 h^m = 0$. Hence $b_0 = 0$. Acting with $(\delta - m\lambda)^{k-1}$ on the equation $b_1 h^{m-1} + \ldots + b_m = 0$, we have $b_1 = 0$. By repeating this procedure, we see $b_k = 0$ if $m \geq k$. Hence $a_{i,j} = 0$, for all i, j. So the proposition is proved.

2.1.8. Remark. Suppose δ is a K-derivation of $R = K[x, y]$, and $Ker(\delta) = K$. Assume $T(\delta, \lambda) \neq E(\delta, \lambda)$ for some $\lambda \in K^\times$. We claim that the derivation δ is determined by $T(\delta, \lambda)$. In fact, choose $h \in T(\delta, \lambda) - E(\delta, \lambda)$ with $(\delta - \lambda)^2 h = 0$. Let $g = (\delta - \lambda)h$. Then g, h are algebraically independent over $Ker(\delta) = K$ by Prop.

2.1.7. Hence $K(x, y)$ is an algebraic extension of $K(g, h)$. Since $charK = 0$, any K-derivation on $K(g, h)$ can be extended uniquely to a K-derivation on $K(x, y)$. In particular, the derivation δ on $K[x, y]$ is determined by $\delta(g) = \lambda g$ and $\delta(h)$.

When $A = K[x_1, \ldots, x_n]$, and δ is an ordinary K-derivation on A, Prop. 2.1.7 can be used to determine the relation between $T(\delta), E(\delta)$ and $N(\delta)$.

2.1.9. Proposition. *Let K be an algebraically closed field and δ an ordinary derivation of $K[x_1, \ldots, x_n]$. Then either $T(\delta) = N(\delta)$ or $T(\delta) = E(\delta)$.*

Proof. By [8], Theorem 3.2 (*The spectral decomposition theorem*),

$$T(\delta) = \bigoplus_{\lambda \in K} T(\delta, \lambda) \tag{2.15}$$

and

$$E(\delta) = \bigoplus_{\lambda \in K} E(\delta, \lambda). \tag{2.16}$$

We first prove that

$$T(\delta, \lambda) = E(\delta, \lambda) \tag{2.17}$$

for all $\lambda \in K^{\times}$. Suppose that equality is not true. Then there exists a $\lambda \in K^{\times}$, $h \in K[x_1, \ldots, x_n]$ such that $(\delta - \lambda)h \neq 0, (\delta - \lambda)^2 h = 0$. Let $g = (\delta - \lambda)h$. Then by Prop. 2.1.7, g, h are algebraically independent over $Ker(\delta)$, which is impossible since $Ker(\delta)$ contains $n - 1$ algebraically independent elements. Then $T(\delta, \lambda) = E(\delta, \lambda)$ for all $\lambda \in K^{\times}$.

By (2.15) and (2.17), $T(\delta) = N(\delta) + E(\delta)$. If both $T(\delta) \neq N(\delta)$ and $T(\delta) \neq E(\delta)$ hold, then $Ker(\delta)$ is strictly contained in $N(\delta)$ and in $E(\delta)$ because $E(\delta) \cap N(\delta) = Ker(\delta)$. So there exists non-constant polynomials $g_1, g_2 \in K[x_1, \ldots, x_n], \lambda \in K^{\times}$ such that $(\delta - \lambda)g_1 = 0, \delta(g_2) \neq 0$, and $\delta^2(g_2) = 0$. Thus

$$(\delta - \lambda)(g_1 g_2) = g_1(\delta g_2) \neq 0, (\delta - \lambda)^2(g_1 g_2) = 0. \tag{2.18}$$

The last formula contradicts the fact that $T(\delta, \lambda) = E(\delta, \lambda)$ as we have just proved. Therefore, Prop. 2.1.9 is proved.

2.1.10. Remark. Suppose A is a finitely generated K-domain with Krull-dimension 1 and $K = \overline{K}$. Let δ be a non-zero K-derivation of A. Then for all $\lambda \in K^{\times}, T(\delta, \lambda) = E(\delta, \lambda)$, by Prop. 2.1.7. Hence $T(\delta) = E(\delta) + N(\delta)$. Then either $T(\delta) = N(\delta)$, or $T(\delta) = E(\delta)$.

Recall that a subring A of B is *integrally closed* in B if $b \in A$, for any $b \in B$ which satisfies $b^n + a_1 b^{n-1} + \ldots + a_n = 0$ for some $a_i \in A, i = 1, \ldots, n, n \geq 1$. If an integral domain A is integrally closed in the quotient field of A, we say that A is *normal.*

2.1.11. Proposition. *Let δ be a K-derivation of $K[x_1, \ldots, x_n]$. Then $N(\delta)$ is a normal domain.*

Proof. Let $f, g \in N(\delta), g \neq 0$, with

$$(\frac{f}{g})^n + c_1(\frac{f}{g})^{n-1} + \ldots + c_n = 0, c_i \in N(\delta). \tag{2.19}$$

Since $K[x_1, \ldots, x_n]$ is normal, it follows that $\frac{f}{g} \in K[x_1, \ldots, x_n]$. Consider the ring homomorphism $\Psi_\delta : K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n][[T]]$ defined by $\Psi_\delta(f) = \sum_{n=0}^{\infty} \frac{\delta^n(f)}{n!} T^n$ corresponding to δ. Extend Ψ_δ to $qt(K[x_1, \ldots, x_n])$ by $\Psi_\delta(\frac{f}{g}) = \frac{\Psi_\delta(f)}{\Psi_\delta(g)}$. Then, by (2.19)

$$(\frac{\Psi_\delta(f)}{\Psi_\delta(g)})^n + \Psi_\delta(c_1)(\frac{\Psi_\delta(f)}{\Psi_\delta(g)})^{n-1} + \ldots + \Psi_\delta(c_n) = 0, \tag{2.20}$$

where $\Psi_\delta(f), \Psi_\delta(g), \Psi_\delta(c_i) \in K[x_1, \ldots, x_n][T], i = 1, \ldots, n$. Since $K[x_1, \ldots, x_n][T]$ is normal, $\frac{\Psi_\delta(f)}{\Psi_\delta(g)} \in K[x_1, \ldots, x_n][T]$. Therefore $\frac{f}{g} \in N(\delta)$ by the definition of $N(\delta)$.

2.2 Locally Nilpotent Derivations

We work in the polynomial ring in two variables starting from this section. Let δ be a non-zero K-derivation of $R = K[x, y]$. Our main objective in this section is to prove Theorem A.

2.2.1. $Ker(\delta)$. We begin by reviewing some facts about $Ker(\delta)$, as follows.

Let δ be a non-zero K-derivation of R. Then $Ker(\delta) = K[f]$ for a polynomial $f \in K[x, y]$. Moreover, $Ker(\delta)$ is integrally closed in R ([40], Theorem 2.8). Therefore δ is an ordinary derivation if and only if $Ker(\delta)$ strictly contains K. Conversely, let

A be a subring of $K[x, y]$ containing K such that A is integrally closed in $K[x, y]$ and Krull-dim(A) ≤ 1. Then $A = Ker(\delta)$ for an ordinary K-derivation δ ([40], Th. 3.4). The polynomial f is studied in detail in [38]. In particular, f is a *closed* polynomial. Note that f is a closed polynomial if and only if $Ker(\Delta_f) = K[f]$.

The classification of locally nilpotent derivations on R follows from Rentschler's theorem [41]. For an extensive study of the automorphism of $K^{[2]}$, see [35].

2.2.2. Theorem. *Let δ be a locally nilpotent derivation of R. Then there exists $\Phi \in Aut_K K[x, y], p(T) \in K[T]$ such that $\delta = p(\Phi(x))\Delta_{\Phi(x)}$.*

2.2.3. Rank of ordinary derivations. Suppose δ is a locally nilpotent derivation. Then by 2.2.2, δ is ordinary, $rank(\delta) = 1$, [4] and $Ker(\delta) = K[f]$ for some *variable* f. We say that f is a variable in $K[x, y]$ if there exists another polynomial $g \in K[x, y]$ such that $K[x, y] = K[f, g]$.

Conversely, suppose δ is a K-ordinary derivation of $K[x, y]$, and $rank(\delta) = 1$. Then $K[X] \subseteq Ker(\delta)$ for some variable $X \in R$. By 2.2.1, $K[X] \subseteq K[f]$ for some $f \in R$, and $X = p(f)$ for some polynomial $p(T) \in K[T]$. Write $K[X, Y] = K[x, y]$ for some polynomial $Y \in R$. Since $J(p(f), Y) \in K^{\times}$, $deg(p(T)) = 1$.[5] So $f \in K[X]$. Hence $Ker(\delta) = K[X]$. This proves that $rank(\delta) = 1$ if and only if f is a variable. Hence $rank(\delta) = 2$ for most ordinary derivations δ.

2.2.4. Let δ be an ordinary K-derivation on R. Then by 2.1.1, $\delta = h\Delta_f$ for some $h \in K(x, y)$ and $f \in K[x, y]$ with $Ker(\delta) = K[f]$. It is natural to ask when h is in $K[x, y]$.

Write $h = a(x, y)/b(x, y)$, where $a, b \in K[x, y]$ and a, b have no common factor. Then $b \mid f_x$ and $b \mid f_y$. Hence, for $\delta = h\Delta_f$, if f_x, f_y have no common factor, then $h \in K[x, y]$.

Moreover, $Ker(\delta) = Ker(\Delta_f) = K[f]$ by assumption. Then f is a closed polynomial by 2.2.1.

[4]if δ is a K-derivation of $K[x_1, \ldots, x_n]$, $rank(\delta)$ is defined in the sense of [12], that is $rank(\delta)$ is the least integer $r \geq 0$ for which there exists $X_1, \ldots, X_n \in K[x_1, \ldots, x_n]$ satisfying $K[x_1, \ldots, x_n] = K[X_1, \ldots, X_n]$ and $K[X_1, \ldots, X_{n-r}] \subseteq Ker(\delta)$.

[5]When we write $deg(p(T)), p(T) \in K[T]$, we mean the degree of the one-variable polynomial $p(T)$.

2.2.5. Special derivations. A K-derivation δ of R is called *special* if $\delta = h\Delta_f$ for some $h \in K[x,y]$ with $\delta(h) = 0$. If δ is special, then $K[f] \subseteq Ker(\delta) = K[g]$ for some polynomial g. Then both f and h are polynomials in g. Say, $f = \alpha(g)$, and $h = \beta(g)$ for polynomials $\alpha(T), \beta(T) \in K[T]$. Then $\delta = \Delta_u$ where $u = \gamma(g)$, $\gamma(T) \in K[T]$ with $\gamma(T)' = \alpha(T)'\beta(T)$. We have thus shown that δ is special if and only if $\delta = \Delta_u$ for some non-constant polynomial $u \in K[x,y]$.

2.2.6. Proof of A.1.

By 2.2.1, $Ker(\delta) = K[f]$ for some $f \in K[x,y]$ by 2.2.1. Write $A = N(\delta)$ and set $S = K[f] - \{0\}$. Note that

(i) $S^{-1}A = K(f)^{[1]}$.

(ii) $K(f) \cap A = K[f]$.

(iii) A is geometrically factorial over K (by assumption).

In fact, choose a local slice $g \in A$ of δ. Then for every element $a \in A$, there exists elements $b_0, \ldots, b_k \in Ker(\delta)$ with $b_0 a = b_1 + b_2 g + \ldots b_k g^{k-1}$. Thus (i) follows. (ii) follows from [12] 1.1 (2), since δ is a locally nilpotent derivation of A.

Then by [45], Th. 2.4.2, or [36], Th. 1, we have $N(\delta) = K[f]^{[1]} \cong K^{[2]}$. Then, we have completed the proof of theorem A.1.

2.2.7. Example. We shall prove (A.2) by giving an example.

Suppose $\delta = \Delta_{x^m y^n}$, with $GCD(m,n) = 1$, and $1 \leq m < n$. We assert that

$$N(\delta) = K[x^m y^n, xy, x^{n-m}]. \tag{2.21}$$

To show this, for a K-derivation δ, we define a function deg_δ on $N(\delta)$ by

$$deg_\delta(f) = min\{n : \delta^n(f) = 0\} - 1 \tag{2.22}$$

if $f \in N(\delta)$ is non-zero, and we put $deg_\delta(0) = -\infty$. It is well known that deg_δ is a *degree* function, i.e.,

$$deg_\delta(a+b) \leq deg_\delta(a) + \deg_\delta(b), \tag{2.23}$$

and

$$deg_\delta(ab) = deg_\delta(a)deg_\delta(b), \tag{2.24}$$

and

$$deg(0) = -\infty, deg(1) = 0, deg(f) \in \mathcal{Z}. \tag{2.25}$$

for any $f \neq 0$ (see [31], Lemma 2). Since $GCD(m,n) = 1$, then $Ker(\delta) = K[x^m y^n]$. Define a map $\psi : \mathcal{Z}_+^2 \to \mathcal{Z}_+^2$ by $\psi(i,j) = (m+i-1, n+j-1)$. Let $A_0 = \{(mk, nk) : k \in \mathcal{Z}_+\}$ where $\mathcal{Z}_+ = \{0, 1, 2, \ldots, \}$. For any $f \in K[x,y]$, by induction on deg_δ, we have

$$deg_\delta(f) = k \Longleftrightarrow Supp(f) \subseteq \bigcup_{l=0}^{k} \Psi^{-l}(A_0). \tag{2.26}$$

Therefore $N(\delta)$ is generated by monomials $x^{m(k-r)+r}y^{n(k-r)+r}$, where $k, r \in \mathcal{N}$, and $m(k-r)+r \geq 0, n(k-r)+r \geq 0$. *If* $k \geq r$, then $x^{m(k-r)+r}y^{n(k-r)+r} = (x^m y^n)^{k-r}(xy)^r$; *if* $k < r$, then

$$x^{m(k-r)+r}y^{n(k-r)+r} = (x^{n-m})^{r-k}(xy)^{n(k-r)+r}.$$

Hence $N(\delta) = K[x^m y^n, xy, x^{n-m}]$. As a surface, $N(\delta)$ is the hypersurface $uv = w^n$ with a singularity. Thus, $N(\delta) \ncong K^{[2]}$. We have thus shown (A.2).

2.2.8. $\mathcal{G}_a$-Action. Let X be an affine K-variety and $A = K[X]$. As is well known, an algebraic $\mathcal{G}_a$-action $\sigma : \mathcal{G}_a \times X \to X$ induces an algebra homomorphism $\phi : A \to A[t], \phi(p) = p(\sigma(t,x)) \in A[t]$. Then $\delta = \frac{d}{dt}|_{t=0}(p(\sigma(t,x)))$ is a locally nilpotent K-derivation on A. Conversely, a locally nilpotent K-derivation δ on A defines an algebra homomorphism $\phi_\delta : A \to A[t]$ by

$$\phi_\delta(a) = exp(t\delta a) = \sum_{n=0}^{\infty} \frac{t^n \delta^n a}{n!} \tag{2.27}$$

and this yields an algebraic $\mathcal{G}_a$-action on $Spec(A)$. See [53].

By the definition of $N(\delta)$, δ is a locally nilpotent derivation on $N(\delta)$. But $N(\delta)$ may not be a maximal subring of $K[x,y]$ with a $\mathcal{G}_a$-action such that the ring of invariants is $Ker(\delta)$, as is shown by the following example: Let $\delta = \Delta_{xy^3}$. Then $N(\delta) = K[xy^3, x^2, xy]$ and $Ker(\delta) = K[xy^3]$ by 2.2.7. Let $\partial = \frac{1}{y}\Delta_{xy^3}$. Note that

$xy^2 \in N(\partial)$ and $xy^2 \notin N(\delta)$.[6] Hence, $N(\delta)$ is not a maximal subring of $K[x,y]$ with a $\mathcal{G}_a$-action with the same ring of invariants as $Ker(\delta)$.

We recall a beautiful theorem about the characterization of $K^{[2]}$ for completeness. See [34], [43] and [49].[7]

2.2.9. Algebraic characterization of the affine plane. *Let $K = \overline{K}$, A a regular, factorial affine K-domain with trans.$deg_K A = 2$. If $A \subseteq K[x_1, \ldots, x_n] \cong K^{[n]}$ for some n, then $A \cong K^{[2]}$.*

2.2.10. Algorithm to find $N(\delta)$. We give an algorithm to compute $N(\delta)$ as follows. Suppose $N(\delta) \neq Ker(\delta)$, and $Ker(\delta) = K[f]$ as above. Choose and fix a local slice $g \in N(\delta)$. Let $d = \delta(g)$. Put $R_{(0)} = K[f,g]$ and define, by induction on m, that $R_{(m)} = \{h \in K[x,y] : dh \in R_{(m-1)}\}$. By induction on deg_δ, we see that $deg_\delta b = n$ implies $b \in R_{(n-1)}$. Therefore

$$N(\delta) = \bigcup_{m=0}^{\infty} R_{(m)}. \tag{2.28}$$

Moreover, each $R_{(m)}$ is a finitely generated K-algebra. If $N(\delta)$ is a finitely generated K-algebra, then $N(\delta) = R_{(m)}$ for some m. On the other hand, it is hard to see whether or not $N(\delta)$ is a finitely generated K-algebra, even if δ is a special derivation. By the same argument as in [40], we obtain that $K(f,g) \bigcap K[x,y]$ is a finitely generated K-algebra, where $g \in N(\delta)$ is an arbitrary local slice of δ. By (2.28), we have $N(\delta) = K[x,y] \bigcap K(f)[g]$. It is not clear at all whether $K[x,y] \bigcap K(f)[g]$ is finitely generated or not.[8]

We conclude this section with a conjecture about the structure of $N(\Delta_f)$. By the examples in 2.2.7 and our following results about the structure of $E(\Delta_f)$ (see Theorem B), it seems reasonable to make the following:

2.2.11. Conjecture. *For any non-constant polynomial $f \in K[x,y]$ such that*

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[6]In fact, by a method as above in 2.2.7, $N(\partial) = K[x, xy, xy^2, xy^3]$.

[7]Miyanishi proved, by a similar method in proving 2.2.9 in [34], that $Ker(\delta) \cong K^{[2]}$ if δ is a non-zero locally nilpotent derivation of $K[x,y,z]$. There exists an extensive recent study of locally nilpotent derivations of $K^{[3]}$ by Daigle and Freudenburg [11], [17] and [12]. It would be very interesting to extend the results of this section to the locally nilpotent derivations on $K^{[3]}$.

[8]Let $\overline{\delta}$ be the derivation of $K(x,y)$ that extends δ. The point here is that $N(\overline{\delta})$ may not be a field.

$N(\Delta_f) \neq Ker(\Delta_f)$, $N(\Delta_f) \cong K[X,Y,Z]/(XY - a(Z))$, as a K-algebra, for some $a(T) \in K[T]$.

Surfaces of the forms $XY = a(Z)$ and their automorphism groups have been studied extensively. See [30].

2.3 Locally Finite Derivations

In this short section we derive a criterion for $T(\Delta_f)$ to equal $Ker(\Delta_f)$. We shall examine the condition $T(\Delta_f) \neq Ker(\Delta_f)$ more closely in chapter 3 and chapter 4.

2.3.1. Classification of Locally finite derivations. A.Van den Essen ([15]) classifies the locally finite derivations δ of R as follows:

There exists $P, Q \in K[x, y]$ such that $K[x, y] = K[P, Q]$ and δ is one of the following:

(i) $\delta = (aP + bQ)\Delta_P + (cP + dQ)\Delta_Q, a, b, c, d \in K$;

(ii) $\delta = \Delta_Q + aQ\Delta_P, a \in K$;

(iii) $\delta = aP\Delta_Q + (amQ + P^m)\Delta_P, m \in \mathcal{N}$;

(iv) $\delta = f(P)\Delta_P$.

In the case of $K = \mathcal{R}$, the real number field, and $K = \mathcal{C}$, the complex number field, these results were proved in [6] and [8] before by a different method.

Let $f = \sum_{i,j} a_{i,j}x^iy^j \in R = K[x, y]$. We define $f^+ = \sum_{(i,j)\in Supp(f), i+j=deg(f)} a_{i,j}x^iy^j$.

We shall study under what condition $T(\delta) = Ker(\delta)$.

For this purpose, we need

2.3.2. Lemma. *Let δ be an ordinary derivation of $K[x, y]$. Suppose, for any $g, h \in T(\delta), deg(g) = deg(h)$ implies that $g^+ = ah^+$ for some $a \in K^\times$. Then $T(\delta)$ is a finitely generated $Ker(\delta)$-module.*

Proof. Let $Ker(\delta) = K[f]$ and $n = deg(f)$. Write $W_f = \{deg(g) : g \in T(\delta)\}$. Then W_f is closed under addition. Let $\overline{W_f} = \{m \bmod n : m \in W_f\}$. Then $\overline{W_f}$ is an subgroup of $\mathcal{Z}/n\mathcal{Z}$. So $\overline{W_f}$ is a cyclic group of some order s. Write $\overline{W_f} = \{0 = \overline{m_1}, \ldots, \overline{m_s}\}$, and choose m_i as the least number in W_f whose mod n class is equal to $\overline{m_i}$. Choose $g_i \in T(\delta)$ with $deg(g_i) = m_i$. By induction on $n = deg(g)$, we shall show that $g \in K[f]g_1 + \ldots + K[f]g_s$ for every $g \in T(\delta)$. In fact, let $deg(g) = k, k \equiv m_i \mod n$ for some i. Since $k \geq m_i, k = m_i + nl$ for some $l \geq 0$. Both g and $f^l g_i$, with the same degree, belong to $T(\delta)$, and have, by assumption, $g^+ = a(f^l g_i)^+$ for some $a \in K^\times$. By induction hypothesis, $g - af^l g_i \in K[f]g_1 + \ldots + K[f]g_s$, whence $g \in K[f]g_1 + \ldots + K[f]g_s$. Therefore $T(\delta) = K[f]g_1 + \ldots + K[f]g_s$.

2.3.3. Proposition. *Suppose $K = \overline{K}$. Let δ be an ordinary derivation of $K[x, y]$. Then the following assertions are equivalent:*

(i) $T(\delta) = Ker(\delta)$.

(ii) For every $g, h \in T(\delta)$, either $(g^+)^n = a(h^+)^m$, or $(h^+)^n = a(g^+)^m$ for some $a \in K^\times, m \geq 1, n \geq 1$.

(iii) For every $g, h \in T(\delta)$, if $deg(g) = deg(h)$, then $g^+ = ah^+$ for some $a \in K^\times$.

Proof. Let $Ker(\delta) = K[f]$ as given in 2.2.1. Clearly $(i) \Rightarrow (ii) \Rightarrow (iii)$. We shall prove the part $(iii) \Rightarrow (i)$. Suppose (iii) holds. Then $T(\delta)$ is a finitely generated $K[f]$-module, with generators $g_1, \ldots, g_s$ by Lemma 2.3.2. If $T(\delta) = N(\delta)$, choose m such that $\delta^m(g_i) = 0, i = 1, \ldots, s$. Then $\delta^m N(\delta) = 0$. If $N(\delta) \neq Ker(\delta)$, choose a local slice g of δ). Then $\delta^n g^n \neq 0$ for any n. Therefore, $N(\delta) = Ker(\delta)$. If $T(\delta) = E(\delta)$, and $E(\delta, \lambda) \neq 0$ for some $\lambda \in K^\times$. Then $E(\delta, n\lambda) \neq 0$ for any $n \in \mathcal{Z}$, and $E(\delta)$ is an infinite direct sum of non-zero $Ker(\delta)$-modules. In particular, $E(\delta)$ is not finitely generated as a $Ker(\delta)$-module. Thus $E(\delta) = Ker(\delta)$. By Prop. 2.1.9, (i) is proved.

We call $T(\delta) \neq Ker(\delta)$ *the weak Jacobian condition.* We shall investigate the weak Jacobian condition from now on.

Chapter 3

The Weak Jacobian Condition (I)

This chapter, together with the next chapter, is devoted to developing the preliminary results for proving Theorem B. For this purpose, we shall investigate in detail the weak Jacobian condition. We shall assume that $K = \overline{K}$ throughout this chapter and the next chapter. The purpose of this chapter is to study the leading forms of polynomials under the weak Jacobian condition. We shall first discuss the leading forms of polynomials under some special conditions in section 3.1. Then we prove theorem 3.2.2 in section 3.2, which gives us the relation between these conditions and the weak Jacobian condition. Prop. 3.1.10, Lemma 3.1.11, Theorem 3.2.2 , and Lemma 3.3.3 will be used in proving Corollary 3.2.4, 3.2.5, Prop. 3.3.5 and Prop. 3.3.8, which in turn will be used in proving Theorem 4.3.1.

3.1 $\mathcal{Z}$−Grading of $K[x, y]$

The purpose of this section is to study standard $\mathcal{Z}$−gradings on $K^{[2]}$. We divide the $\mathcal{Z}$−gradings into three cases: Elliptic, Parabolic, and Hyperbolic $\mathcal{Z}$−gradings. Proposition 3.1.8, 3.1.12 and 3.1.13 are somewhat technical results of this section, which provide us with the precise forms of the w-homogeneous polynomials f in $K[x, y]$ (see 3.1.5 for its definition) under the condition that Δ_f *has a w-homogeneous local slice in* $K[x, y]$. Proposition 3.1.10 and Lemma 3.1.11 are useful to prove these results and other results in subsequent discussions. The proof of Prop. 3.1.10 and

Lemma 3.1.11 rests on a careful study of polynomial solutions of certain ordinary differential equations.

3.1.1. Filtrations. Let us first briefly recall the basic properties of a filtration of a K-algebra A.

Let A be a K-algebra. By a *Filtration* of A we mean a sequence of K-linear subspaces of A, $\mathcal{F} = \{F^iA : i \in \mathcal{Z}\}$, satisfying:

(f.1). $F^iA \subseteq F^{i+1}A$ (ascending).

(f.2). $A = \bigcup_{i\in\mathcal{Z}} F^iA$ (exhaustive), and $\bigcap_{i\in\mathcal{Z}} F^iA = \{0\}$, $1 \in F^0A - F^{-1}A$.

(f.3). For all $i, j \in \mathcal{Z}$, $(F^iA - F^{i-1}A)(F^jA - F^{j-1}A) \subseteq (F^{i+j}A - F^{i+j-1}A)$.

Define $deg : A \to \mathcal{Z} \cup \{-\infty\}$ by $deg(a) = i$ if and only if $a \in F^iA - F^{i-1}A$, and $deg(0) = -\infty$. Then deg is a degree function in the sense of 2.2.8. Conversely, given a degree function $deg : A \to \mathcal{Z} \cup \{-\infty\}$, let

$$F^iA = \{a \in A : deg(a) \leq i\}. \tag{3.1}$$

Then $\mathcal{F} = \{F^iA : i \in \mathcal{Z}\}$ is a filtration of A.

Given a filtration $\mathcal{F} = \{F^iA : i \in \mathcal{Z}\}$ of A, the *associated graded algebra* is $Gr_{\mathcal{F}}A = \bigoplus_{i\in\mathcal{Z}} Gr^i_{\mathcal{F}}A$, where $Gr^i_{\mathcal{F}}A = F^iA/F^{i-1}A$. $Gr_{\mathcal{F}}A$ can be identified with the algebra of Laurent polynomials $\{\sum_k^{k+l} \hat{f}_i u^i\}$, where $\hat{f}_i$ is either zero or is equal to $gr_i f := f_i + F^{i-1}A \in Gr^i_{\mathcal{F}}A$ for some $f_i \in F^iA$ (See [53], section 7.2). Let $gr : A \to A$ be the homomorphism of multiplicative semigroups defined by $grf = \hat{f}$.

Suppose A is a finitely generated K-algebra and δ a K-derivation on A. Then for any filtration $\mathcal{F} = \{F^iA : i \in \mathcal{Z}\}$ of A, there exists an integer $k \in \mathcal{N}$ such that $\delta(F^iA) \subseteq F^{i+k}A$ for all $i \in \mathcal{Z}$. [1] Denote by $deg\delta = k_0$ the minimal such k. Define $\hat{\delta}_{\mathcal{F}} = gr\delta : Gr_{\mathcal{F}}A \to Gr_{\mathcal{F}}A$ by: $\hat{\delta}_{\mathcal{F}}(\hat{f}) = \delta(f) + F^{i+k_0-1}A$ for all $f \in F^iA - F^{i-1}A$, and then naturally extend $\hat{\delta}_{\mathcal{F}}$ to the whole algebra $Gr_{\mathcal{F}}A$. $\hat{\delta}_{\mathcal{F}}$ is a K-derivation of $Gr_{\mathcal{F}}A$ (See [31]). We may omit the symbol $\mathcal{F}$ if the filtration $\mathcal{F}$ is already specified.

3.1.2. $\mathcal{Z}-$Gradings. We shall study a subclass of filtrations, that is $\mathcal{Z}-$gradings. Let A be a finitely generated K-algebra, let Σ be an additive semigroup. By a

[1] One may prove this fact by induction on the number of generators. See [53], Exercise 7.10.

Σ−grading of A we mean a decomposition:

$$A = \bigoplus_{\gamma \in \Sigma} A_\gamma \tag{3.2}$$

with $A_\gamma A_\delta \subseteq A_{\gamma+\delta}$ and each A_γ a K-subspace of A. The Σ−grading is called non-trivial if $A \neq A_0$. If Σ is isomorphic to a sub-semigroup of $\mathcal{Z}$, a Σ−grading is called a $\mathcal{Z}$−grading. In this section we shall discuss non-trivial $\mathcal{Z}$−gradings.

A $\mathcal{Z}$−grading of A induces naturally a filtration of A, as follows. Let Γ be a $\mathcal{Z}$−grading of A. Write $A_{\Gamma,i} = \sum_{\delta \leq i} A_\delta$. Then $\{A_{\Gamma,i}, i \in \mathcal{Z}\}$ is a filtration of A. The associated graded algebra corresponding with this filtration, that is $Gr_\Gamma(A) = \sum_{i \in \mathcal{Z}} (A_{\Gamma,i}/A_{\Gamma,i-1})$ is called the associated graded algebra with the $\mathcal{Z}$−grading Γ. The induced K-derivation $\hat{\delta}_{\mathcal{F}}$ on $Gr_\Gamma(A)$ is denoted by $\hat{\delta}_\Gamma$. Below, $Gr_\Gamma A$ always means the associated graded algebra of A with respect to a $\mathcal{Z}$−grading Γ, $Gr_{\mathcal{F}} A$ means the associated graded algebra of A with respect to a filtration $\mathcal{F}$. Moreover, $gr(f)$ denote the corresponding element of $Gr_{\mathcal{F}}(A)$ and f_Γ^+ the corresponding element of $Gr_\Gamma A$, for every $f \in A$. Note that $Gr_\Gamma(A)$ is naturally isomorphic to A as a graded ring, but $\hat{\delta}_\Gamma$ may differ from δ.

The next lemma tells us the relation between δ and its induced derivations $\hat{\delta}_{\mathcal{F}}, \hat{\delta}_\Gamma$ on the associated graded algebras.

3.1.3. Lemma. *(1). Suppose δ is a locally nilpotent derivation of A. Then $\hat{\delta}_{\mathcal{F}}$ is a locally nilpotent derivation of $Gr_{\mathcal{F}}(A)$.*

(2). Suppose δ is a locally finite derivation of A of degree $\neq 0$. Then $\hat{\delta}_\Gamma$ is a locally nilpotent derivation of A.

Proof. (1) was proved in [31], Lemma 4. (2) was proved in [15]. See [15] for the definition of the degree of derivations.

3.1.4. Standard $\mathcal{Z}$−gradings. For the purpose of this paper, and by the result to be explained later (theorem of Bialynicki-Birula), we will study the *standard $\mathcal{Z}$-gradings*.

Let $A \cong K^{[n]}$ be a polynomial algebra with n variables on K. A *standard $\mathcal{Z}$−grading* of A is a $\mathcal{Z}$−grading such that there exists variables $x_i, 1 \leq i \leq n$ with

$A = K[x_1, \ldots, x_n]$ and given by making each variables x_i a homogeneous element of certain weight $w_i \in \mathcal{Z}$. See [19], [44]. When the choice of $x_1, \ldots, x_n$ is clear, we denote these standard $\mathcal{Z}$−gradings by $\Gamma = (w_1, \ldots, w_n)$. A standard $\mathcal{Z}$−grading of $K^{[n]}$ is the same as the *weight degree function* in the sense of Zaidenberg [53].[2]

For $n = 2$, Bialynicki-Birula ([3]) proved that any $\mathcal{Z}$−grading on $K^{[2]}$ is a standard one. It is highly remarkable that ([23], [24]) this is true also for the case $n = 3$, which is equivalent to a well known conjecture that every $\mathcal{G}_m$-action on $\mathcal{C}^3$ is linearizable.

3.1.5. $\mathcal{Z}$−gradings on $K^{[2]}$. Now, we more closely examine the $\mathcal{Z}$−gradings of $K^{[2]}$.

Suppose $n = 2$ and $R = K^{[2]}$. Any $\mathcal{Z}$−grading Γ of R comes from a weight $w = (w_1, w_2) \in \mathcal{Z}^2$ and a choice of coordinate system $\{x, y\}$, that is $R = K[x, y]$. Write $deg_\Gamma(x) = deg_w(x) = w_1$, and $deg_\Gamma(y) = deg_w(y) = w_2$. We define the *w-weight* of any monomial $x^i y^j$ to be $d_w(x^m y^n) = mw_1 + nw_2$ and $d_w(f) = max\{d_w(x^m y^n) : (m, n) \in Supp(f)\}$. Write $f_w^+ = f_\Gamma^+ = \sum_{w_1 m + w_2 n = d_w(f)} a_{m,n} x^m y^n$ to denote the *leading terms* of f under the $\mathcal{Z}$−grading Γ. f_w^+ is also called the *w-leading form* of f. When $w = (1, 1)$ we shall write $f^+, deg(f)$ to denote $f_w^+, d_w(f)$ respectively.

Recall that f is *w-homogeneous* if $f = f_w^+$ ([1]). The polynomials f and g are called *w-dependent* if $J(f_w^+, g_w^+) = 0$. See [1]. In our whole discussion below we shall assume that $GCD(|w_1|, |w_2|) = 1$ since the corresponding leading terms are the same for the weights $w = (w_1, w_2)$ and $nw = (nw_1, nw_2), n \geq 1$.

Define $deg(w) = w_1 + w_2$. We shall restrict ourselves to the case that $deg(w) \neq 0$ and divide it into the three following cases.

Elliptic : $w_1 w_2 > 0$

Parabolic : $w_1 w_2 = 0$

Hyperbolic : $w_1 w_2 < 0$

The following easy while very useful lemma belongs to S.S.Abhyankar [1]. [3]

3.1.6. Lemma. *(1). Suppose $w = (w_1, w_2) \in \mathcal{Z}^2$ and assume that f and g are*

[2] A weight degree function on $K^{[n]}$ is a degree function d such that $d(p) = max\{d(m)\}$, where $p \in K^{[n]}$ is a non-zero polynomial, and m runs over the set of all monomials appearing in p.

[3] When we write $J(f, g)$ we mean the coordinate system of $K^{[2]}$ has been given and the Jacobian determinant is computed with respect to the given coordinate system.

two w-homogeneous polynomials. Then $J(f,g)=0$ *if and only if* $f^{d_w(g)}=ag^{d_w(f)}$ *for some* $a\in K^\times$*. In particular, if* $d_w(f)=d_w(g)\neq 0$*, then* $f=ag$ *for some* $a\in K^\times$*.*

(2). Given $w=(w_1,w_2)\in \mathcal{Z}^2$ *and* $f,g\in R$*, suppose* $J(f_w^+,g_w^+)\neq 0$*. Then* $J(f,g)_w^+=J(f_w^+,g_w^+)$*.*

3.1.7. Definition. A polynomial $f\in R=K[x,y]$ is a *stable* polynomial if for every elliptic $\mathcal{Z}-$grading Γ on R, f_Γ^+ is a monomial. We note that $f\in K[x,y]$ is a monomial if and only if f_Γ^+ is a monomial for every standard $\mathcal{Z}-$grading Γ on R.

We shall study $\mathcal{Z}-$gradings on R in this section and the next section. We shall also study, for technical reasons, the hyperbolic $\mathcal{Z}-$gradings on $K[x,y^{1/m},y^{-1/m}]$ in the last section of this chapter.

The elliptic $\mathcal{Z}-$gradings were studied in detail by Abhyankar [1] in the following proposition.

3.1.8. Proposition. *(Elliptic). Suppose both* w_1 *and* w_2 *are positive integers,* $w=(w_1,w_2)$*. Suppose* f *and* $J(f,g)$ *are w-dependent and* $J(f,g)\neq 0$*. Then* $f_w^+=au_1^{i_1}u_2^{i_2}, i_1,i_2\geq 0, i_1+i_2>0, a\in K^\times$*, where*

(i) if $w_1=w_2$*, then* $u_1=a_{11}x+a_{12}y, u_2=a_{21}x+a_{22}y, a_{ij}\in K, a_{11}a_{22}-a_{12}a_{21}\in K^\times$*;*

(ii) if $w_1>w_2$*, then* $u_1=x+ay^{w_1/w_2}, u_2=y, a\in K$*; moreover, if* $a\neq 0, w_2\mid w_1$*;*

(iii) if $w_2>w_1$*, then* $u_1=x, u_2=y+ax^{w_1/w_2}, a\in K$*; moreover, if* $a\neq 0, w_1\mid w_2$*.*

Proof. This is a well known result of S.S.Abhyankar. Although it was stated under the Jacobian condition that $J(f,g)\in K^\times$ for some $g\in R$, its proof works under the weak condition that f and $J(f,g)$ are w-dependent. See [1], Th. 18.13. In particular, if $w_1,w_2>1$, and if they are coprime, then f_w^+ is a monomial.

3.1.9. A question.

In order to study the Parabolic and Hyperbolic cases, we need to solve the following question.

Given two Γ*-homogeneous polynomials* $f,g\in R$*, if* g *is a local slice of* Δ_f*, what is* f*?*

The following proposition is the first, but most important step to solve this question.

3.1.10. Proposition. *Let f, g be two Γ-homogeneous polynomials in $K[x, y]$, where $\Gamma = (m, n) \in \mathcal{Z}^2$ with $m + n > 0$. Suppose g is a local slice of Δ_f, and assume that $s = d_\Gamma(f) > 0$. Let $h = fg/\Delta_f(g)$. Then $J(f, h) = f$ and $h \in K[x, y]$.*

Proof. Since $J(f, J(f, g)) = 0$, $J(f, \frac{1}{\Delta_f(g)}) = 0$. Then

$$\begin{aligned} J(f,h) &= fJ(f, \frac{g}{\Delta_f(g)}) \\ &= f[\frac{J(f,g)}{J(f,g)} + gJ(f, \frac{1}{\Delta_f(g)})] \\ &= f. \end{aligned}$$

Moreovrr, h is a Γ-homogeneous function of Γ-degree $m + n$. The crucial point of this proposition is to show that $h \in K[x, y]$. To show this, by the Euler equations for Γ-homogeneous functions f, h, we have

$$sf = mx\frac{\partial f}{\partial x} + ny\frac{\partial f}{\partial y}, \tag{3.3}$$

$$(m+n)h = mx\frac{\partial h}{\partial x} + ny\frac{\partial h}{\partial y}. \tag{3.4}$$

Then we obtain (by using the assumption that $m + n > 0$ and $s > 0$)

$$\frac{\partial}{\partial x}(\frac{f^{m+n}}{h^s}) = ny(\frac{f^{m+n}}{h^{s+1}}), \tag{3.5}$$

and

$$\frac{\partial}{\partial y}(\frac{f^{m+n}}{h^s}) = -mx(\frac{f^{m+n}}{h^{s+1}}). \tag{3.6}$$

We only prove the first formula (3.5). The second one is done similarly. The left hand of (3.5) is

$$\frac{\partial}{\partial x}(\frac{f^{m+n}}{h^s}) = \frac{f^{m+n-1}}{h^{s+1}}[(m+n)\frac{\partial f}{\partial x}h - sf\frac{\partial h}{\partial x}]. \tag{3.7}$$

Calculating (3.4) $\times \frac{\partial f}{\partial x}$ − (3.3) $\times \frac{\partial h}{\partial x}$, we find

$$(m+n)\frac{\partial f}{\partial x}h - sf\frac{\partial h}{\partial x} = nyJ(f,h) = nyf. \tag{3.8}$$

Hence (3.5) is proved.

Consider h^{-1} as a rational function of y over $K(x)$. Suppose $h \notin K(x)[y]$. Let $\alpha(x) \in \overline{K(x)}$ be a root of h^{-1} of order $v > 0$ and a root of f of order $v' \geq 0$. Then by formula (3.6), $(m+n)v' + sv - 1 = (m+n)v' + (s+1)v$ if $\alpha(x) \neq 0$, and $(m+n)v' + sv - 1 = (m+n)v' + (s+1)v + 1$ if $\alpha(x) = 0$. Hence $v = -1$ or $v = -2$. This contradiction shows that $h \in K(x)[y]$. Similarly, $h \in K(y)[x]$. Hence $h \in K(x)[y] \cap K(y)[x] = K[x,y]$. The proposition is proved.

The next Lemma will be useful in the subsequent discussion.

3.1.11. Lemma. *Suppose $G(t) \in K(t), H(t) \in K[t]$ satisfy*

$$\frac{dG(t)}{dt} = \frac{cG(t)}{H(t)}, c \in K^{\times}$$

Then we have

(i) If $G(t) \in K[t]$, then $H = a(t-\alpha), G = b(t-\alpha)^m$ for some $a, b \in K^{\times}, \alpha \in K, m \geq 1$.

(ii) Each root of G is a root of H.

(iii) Each root of H is a simple root.[4]

Proof. Write $G(t) = \beta \prod_{i=1}^{k} (t-\alpha_i)^{u_i}, u_i \in \mathcal{Z}, u_i \neq 0, \alpha_i \neq \alpha_j$ and $\beta \neq 0$. Then

$$\frac{dG}{dt} = \sum_{i=1}^{k} \frac{u_i}{t-\alpha_i} G = \frac{cG}{H}. \tag{3.9}$$

So

$$H(t) = c\frac{(t-\alpha_1)\ldots(t-\alpha_k)}{p(t)}, \tag{3.10}$$

where $p(t) = u_1(t-\alpha_2)\ldots(t-\alpha_k) + \ldots + u_k(t-\alpha_1)\ldots(t-\alpha_{k-1})$. We shall show that $p(t)$ is a non-zero constant in K. In fact, *Suppose* $p(t)$ *has degree* ≥ 1. Then $p(t)$ divides $(t-\alpha_1)\ldots(t-\alpha_k)$ since $H(t) \in K[t]$. But $p(\alpha_i) \neq 0$ for each $i = 1, \ldots, k$. This is a contradiction. Therefore $p(t) \in K^{\times}$ and $H(t) = d(t-\alpha_1)\ldots(t-\alpha_k)$ for

[4]For a rational function $G(T) \in K(T)$, by the roots of G we mean the union of its zeros and poles.

some $d \in K^\times$. So both (ii) and (iii) are proved. For (i), suppose $G(t) \in K[t]$, and assume that G has $k \geq 2$ roots. Then $G(t) = \prod_{i=1}^{k}(t-\alpha_i)^{u_i}, u_i \geq 1$, and $k \geq 2$. Therefore, the leading term of $p(t)$ is $(u_1 + \ldots + u_k)t^{k-1} \neq 0$. This contradicts the fact that $p(t) \in K^\times$. Hence $G(t)$ has only one zero $\alpha \in K$. So $H(t) = a(t-\alpha)$, for some $a \in K^\times, \alpha \in K$. The proof of (i) is completed.

The next two propositions solve the problem in 3.1.9 for parabolic $\mathcal{Z}$−gradings and hyperbolic $\mathcal{Z}$−gradings.

3.1.12. Proposition. *(Parabolic). Suppose* $w = (w_1, w_2) = (1, 0)$. *Let* f, g *be two w-homogeneous polynomials such that* $\Delta_f(g) \neq 0$ *and* $\Delta_f^2(g) = 0$. *Assume that* $r = d_w(f) \geq 1$. *Then either* $f = ax^r(y-\alpha)^i, a \in K^\times, \alpha \in K, i < r$, *or* $f = x^r(y-\alpha)^i F(y), \alpha \in K, i > r, F(y) \in K[y]$ *with* $F(\alpha) \neq 0$.

The crucial point in this proposition is that $i \neq r$ in both cases. Hence under an automorphism Φ of R, the degree of $\Phi(f)$ respect to x is strictly less than the degree of $\Phi(f)$ with respect to y.

Proof. Write $f = x^r F_1(y)$ and $h = fg/\Delta_f g = xH(y)$. Then by Prop. 3.1.10, $F_1, H \in K[y]$. Put $G = H^r/F_1 \in K(y)$. Then by (3.6), we have

$$dG/dy = G/H. \tag{3.11}$$

We consider the following two cases.

Case 1. *Suppose* $G(y) \in K[y]$. Then $H = a(y-\alpha), a \in K^\times$ by Lemma 3.1.11. Therefore $F_1 = b(y-\alpha)^i, b \neq 0$, and $i < r$ because $G = \frac{H^r}{F_1} \in K[y]$.

Case 2. *Suppose* $G(y) \notin K[y]$. Then there exists a root α of order m of F_1 and of order n of H such that $m > rn$. Since each root of H is a simple root, by Lemma 3.1.11, we have $n = 1$. So $m > r$. Thus $F_1 = (y-\alpha)^i F(y)$ with $i > r, F(\alpha) \neq 0$. The proof is finished.

3.1.13. Proposition. *(Hyperbolic). Suppose* $w_2 < 0, w_1 + w_2 > 0$. *Let* f, g *be two w-homogeneous polynomials in* $K[x, y]$ *such that* $\Delta_f(g) \neq 0$ *and* $\Delta_f^2(g) = 0$. *Assume that* $r = d_w(f) > 0$. *Then either* f *is a monomial, or* $f = (x - \alpha y^{w_1/w_2})^i y^{(r-w_1 i)/w_2} F(z)$, *where* $\alpha \in K, z = xy^{-w_1/w_2}, i \in \mathcal{N}, 1 \leq i < (r - w_1 i)/w_2$

and $F(z) \in K[z], F(\alpha) \neq 0$. *Note that it is possible that* $w_1, r - w_1 i \notin \mathcal{Z} w_2$.

This proposition asserts that either f is a monomial, or in the enlarged algebra $R_{-w_2} = K[x, y^{1/w_2}, y^{-1/w_2}]$, and after a "restricted" automorphism Φ of R_{-w_2} (to be defined later in 3.3), the degree of $\Phi(f)$ with respect to x is strictly less than the degree of $\Phi(f)$ with respect to y.

Proof. Write $z = xy^{-w_1/w_2}, f = y^{r/w_2}F(z)$, and let $h = y^{1+(w_1/w_2)}H(z)$. Then $F, H \in K[z]$ by Prop. 3.1.10. Put $G = H^r/F^{w_1+w_2} \in K(z)$. Then (3.6) implies that

$$dG/dz = -w_2 G/H. \tag{3.12}$$

By Lemma 3.1.11, each root of H is a simple root and each root of G is a root of H. Since $w_1 + w_2 > 0$, each root of F is a root of H.

The proof consists of the following three cases.

Case 1. *Suppose* $G \in K[z]$. Then by Lemma 3.1.11, $G = a(z-\alpha)^s$, and $H = b(z-\alpha)$ for some $\alpha \in K, a, b \in K^\times, s \geq 1$. Since $h = bxy - b\alpha y^{1+(w_1/w_2)} \in K[x, y]$ by prop 3.1.10, we obtain $\alpha = 0, h = bxy$. By using $\Delta_f(h) = f$, we see that f is a monomial.

Case 2. *Suppose* $F(z) = a(z-\alpha)^m$ *for some* $m \geq 1$. Then $f = ay^{r/w_2}(xy^{-w_1/w_2} - \alpha)^m \in K[x, y]$. Since $\alpha^m y^{r/w_2}$ and $\alpha^{m-1}(xy^{-w_1/w_2})y^{r/w_2}$ belong to $K[x, y]$, we obtain $\alpha = 0$. Thus f is a monomial.

Case 3. *At last, suppose none of above holds.* Then F possesses at least two distinct roots α_1, α_2 of multiplicity $m_1, m_2 > 0$ and (since $G \notin K[z]$) there exists one, α_1, say, that satisfies $m_1(w_1 + w_2) > r$ since each root of H is a simple root. Write $F(z) = \beta \prod_{i=1}^{s}(z - \alpha_i)^{m_i}, s \geq 2, m_i \geq 1, \alpha_i \neq \alpha_j$ and $\beta \neq 0$. So $f = (x - \alpha y^{w_1/w_2})^i y^{(r-w_1 i)/w_2} F(z), \alpha = \alpha_1$, with $1 \leq i < (r - w_1 i)/w_2$, $F(z) \in K[z]$, and $F(\alpha) \neq 0$. Thus the proposition is proved.

3.2 The Weak Jacobian Condition on $K[x, y]$

The purpose of this section is to study the relation between *the weak Jacobian condition on f and the condition that $\Delta_{f_w^+}$ has a homogeneous local slice* for the grading given by $w = (w_1, w_2)$ studied in detail in the last section. The main result of this section is Theorem 3.2.2, which asserts roughly that if f satisfies the weak Jacobian condition, then $\Delta_{f_w^+}$ has a homogeneous local slice. Together with the results in the last section, Theorem 3.2.2 yields Corollary 3.2.4 and Corollary 3.2.5. We note that Corollary 3.2.5 is essentially used in the proof of theorem 4.3.1.

Consider a subalgebra $B \subseteq K^{(n)} = K(x_1, \ldots, x_n)$, and a K-derivation δ of $K^{(n)}$ with $\delta(B) \subseteq B$. Then δ induces a K-derivation of B. By 3.1.2, a $\mathcal{Z}-$grading Γ on $K^{(n)}$ induces a filtration of $K^{(n)}$, and then a filtration $\mathcal{F}_B = \{F^i K^{(n)} \cap B\}$ of B. The associated graded algebra is denoted by $Gr_{\mathcal{F},\Gamma}(B)$, and δ induces a K-derivation of $Gr_{\mathcal{F},\Gamma}(B)$ by 3.1.1. The corresponding statement is false, in the case of $\mathcal{Z}-$gradings, i.e. a $\mathcal{Z}-$grading on $K^{(n)}$ doesn't always induce a $\mathcal{Z}-$grading on B.

Before proving theorem 3.2.2, we first recall a highly interesting result of Makar-Limanov [31] Lemma 6, [21] Lemma 7.2.

3.2.1. Proposition. *Let $K^{(n)} = K(x_1, \ldots, x_n)$, K a field of characteristic zero. Assume we have a standard $\mathcal{Z}-$grading $\Gamma = (w_1, \ldots, w_n)$ of $K^{(n)}$, given by $deg_\Gamma(x_i) = w_i \in Z$. Suppose B is a subalgebra of $K^{(n)}$ which contains m algebraically independent elements. Then $Gr_{\mathcal{F},\Gamma}(B)$ contains m algebraically independent elements.*

Let $n = 2$, $B = N(\Delta_f)$. If $N(\Delta_f) \neq Ker(\Delta_f)$, then B contains 2 algebraically independent elements. By Prop. 3.2.1, for any (standard) $\mathcal{Z}-$grading $\Gamma = (w_1, w_2)$, $Gr_{\mathcal{F},\Gamma}(B)$ contains 2 algebraically independent elements. On the other hand, if $\Delta_f(B) \subseteq B$, then Δ_f induces a K-derivation $\hat{\Delta}_f$ on $Gr_{\mathcal{F},\Gamma}(B)$ by 3.1.1. *Suppose $\hat{\Delta}_f$ is non-zero.* Then $\hat{\Delta}_f$ has a local slice in $Gr_{\mathcal{F},\Gamma}(B)$.

Theorem 3.2.2 give us more information about $\Delta_{f_w^+}$ under the weak Jacobian condition.

3.2.2. Theorem. *Given a $\mathcal{Z}-$grading Γ on $K[x, y]$ induced by $w = (w_1, w_2) \in \mathcal{Z}^2$, let f be a non-constant polynomial of $K[x, y]$ with $d_w(f) > 0$, and assume one of* *the following two conditions holds:*

(i) $N(\Delta_f) \neq Ker(\Delta_f)$.

(ii) $E(\Delta_f) \neq Ker(\Delta_f)$ *with* $d_w(f) > deg(w)$.

Then $\Delta_{f_w^+}$ *has a w-homogeneous local slice in* $K[x,y]$.

. **Proof.** We first consider the case (i). Choose and fix a local slice g of Δ_f. Define, for any positive integer n, $M(n) =$ the linear-span of $\{f^k g^l : k, l = 0, 1, \ldots, n\}$. Since $\Delta_f^{n+1}(f^k g^l) = 0$ for all $k, l = 0, 1, \ldots$, we have $M(n) \subseteq N(\Delta_f)$.

We may prove that the elements $f^k g^l$ are K-linear independent (as in Prop. 2.1.10). Hence

$$dim_K M(n) \geq n(n+1)/2. \tag{3.13}$$

We shall show that there exists an element $h \in M(n)$ with $\Delta_{f_w^+}(h_w^+) \neq 0$ when $n >> 0$. To show this, by considering all possible monomials in $M(n)$, we note that

$$L_2(n) \leq d_\Gamma(\gamma) \leq L_1(n) \tag{3.14}$$

for every $\gamma \in M(n)$, where $L_1(n), L_2(n)$ are two linear forms in n depending only on f and g. For every $i \in [L_2(n), L_1(n)]$, choose and fix one $\gamma_i \in M(n)$ with $d_w(\gamma_i) = i$, if such a element exists. Then for any $\gamma \in M(n), d_w(\gamma) = d_w(\gamma_i)$ for some i.

Suppose $\Delta_{f_w^+}(h_w^+) = 0$ *for every* $h \in M(n)$. Then, $\Delta_{f_w^+}(\gamma_w^+) = \Delta_{f_w^+}((\gamma_i)_w^+) = 0$. By Lemma 3.1.6.(i), $(f_w^+)^{d_w(\gamma_w^+)} = \alpha(\gamma_w^+)^{d_w(f_w^+)}$ for some $\alpha \in K^\times$. Since $d_w(f_w^+) > 0$ by assumption, we find that $d_w(\gamma_w^+) > 0$. Similarly, $d_w((\gamma_i)_w^+) > 0$. On the other hand, $J(\gamma_w^+, (\gamma_i)_w^+) = 0$. So $\gamma_w^+ = a(\gamma_i)_w^+$ for some $a \in K^\times$ by Lemma 3.1.6.(i). Then $d_w(\gamma - a\gamma_i) < d_w(\gamma)$. Repeating this procedure, we may express γ as a K-linear combinations of elements γ_i. Thus $dim_K M(n) \leq L_1(n) - L_2(n) + 1$, which is impossible for $n >> 0$ since $dim_K M(n) \geq n(n+1)/2$ by (3.13).

So we may choose $h \in M(n)$ with $\Delta_{f_w^+}(h_w^+) \neq 0$. Note that $\Delta_f^r(h) = 0$ for some positive integer r since $M(n) \subseteq N(\Delta_f)$. Let r be the least integer with $\Delta_f^r(h) = 0$. Then $r \geq 2$ and by Lemma 3.1.6.(ii), $J(f_w^+, (\Delta_f^{r-1}h)_w^+) = 0$. If $J(f_w^+, (\Delta_f^{r-2}h)_w^+) \neq 0$, then by Lemma 3.1.6.(ii), $(\Delta_f^{r-1}(h))_w^+ = J(f_w^+, (\Delta_f^{r-2}h)_w^+)$, and we put $g_1 =$

$(\Delta_f^{r-2}h)_w^+$. If $J(f_w^+, (\Delta_f^{r-2}h)_w^+) = 0$, since $\Delta_f^r(h) = 0$, there exists some k such that $J(f_w^+, (\Delta_f^{r-k}h)_w^+) \neq 0$, and $J(f_w^+, (\Delta_f^{r-k+i}h)_w^+) = 0$ for any $i > 0$, and we set $g_1 = (\Delta_f^{r-k}h)_w^+$. It follows readily that g_1 is a local slice of $\Delta_{f_w^+}$. This settles the theorem in the first case that $N(\Delta_f) \neq Ker(\Delta_f)$.

Next, consider the case (ii). Let $\lambda \in K^\times$ such that $\Delta_f g = \lambda g$ for some $g \in R$. Define, for any positive integer n, $N(n) = K$-linear span of $\{f^k g^l : k, l = 0, 1, \ldots, n\}$. Again, $dim_K N(n) \geq n(n+1)/2$. Then as in the first case, there exists some $h \in N(n)$ such that $\Delta_{f_w^+}(h_w^+) \neq 0$. We shall show that $\Delta_{f_w^+}^r(h_w^+) = 0$ for some positive integer r. Suppose $\Delta_{f_w^+}^r(h_w^+) \neq 0$, for every $r \geq 2$. Then $d_w(\Delta_f^r h) = r(d_w(f) - deg(w)) + d_w(h) \to \infty$ when $r \to \infty$, since $d_w(f) > deg(w)$ by assumption. On the other hand, $\Delta_f^r h \in N(n)$ and $d_w(\Delta_f^r h) \leq L_1(n)$, where $L_1(n)$ is a linear form in n. This contradiction proves that $\Delta_{f_w^+}^r(h_w^+) = 0$ for some $r \geq 1$ and $\Delta_{f_w^+}(h_w^+) \neq 0$. The remaining proof is similar to the first case. The proof of this theorem is finished.

We remark that the result holds when $d_w(f) > deg(w)$ is replaced by $d_w(f) \neq deg(w)$ in case (ii).

3.2.3. Remark. If $\Delta_{f_w^+}$ has a w-homogeneous local slice in $K[x, y]$, then $N(\Delta_{f_w^+}) \neq Ker(\Delta_{f_w^+})$.[5] Then in 3.2.2.(i), $N(\Delta_f) \neq Ker(\Delta_f)$ implies that $N(\Delta_{f_w^+}) \neq Ker(\Delta_{f_w^+})$. On the other hand, by Prop. 2.1.9, $N(\Delta_{f_w^+}) \neq Ker(\Delta_{f_w^+})$ implies that $E(\Delta_{f_w^+}) = Ker(\Delta_{f_w^+})$. Then in 3.2.2.(ii), $E(\Delta_f) \neq Ker(\Delta_f)$ with $d_w(f) > deg(w)$ implies that $E(\Delta_{f_w^+}) = Ker(\Delta_{f_w^+})$. In particular, if $E(\Delta_f) \neq Ker(\Delta_f)$ with $d_w(f) > deg(w)$, then $f \neq f_w^+$.

3.2.4. Corollary. *(Elliptic). Let f be a non-constant polynomial, and let $w = (w_1, w_2) \in \mathcal{N}^2$ with $d_w(f) > 0$. Assume that $N(\Delta_f) \neq Ker(\Delta_f)$, or $E(\Delta_f) \neq Ker(\Delta_f)$ with $d_w(f) > deg(w)$. Then $f_w^+ = a u_1^{i_1} u_2^{i_2}, i_1, i_2 \geq 0, i_1 + i_2 > 0, a \in K^\times$, where*

(i) if $w_1 = w_2$, then $u_1 = a_{11}x + a_{12}y, u_2 = a_{21}x + a_{22}y$ with $a_{ij} \in K$ and $a_{11}a_{22} - a_{12}a_{21} \in K^\times$;

[5] If $g \in K[x, y]$ is a local slice of $\Delta_{f_w^+}$ with $J(f_w^+, g_w^+) \neq 0$, then by Lemma 3.1.6. (ii), g_w^+ is a w-homogeneous local slice of $\Delta_{f_w^+}$.

(ii) if $w_1 > w_2$, *then* $u_1 = x + ay^{w_1/w_2}, u_2 = y, a \in K$. *Moreover, if* $a \neq 0, w_2 \mid w_1$;

(iii) If $w_2 > w_1$, *then* $u_1 = x, u_2 = y + ax^{w_1/w_2}, a \in K$. *Moreover, if* $a \neq 0, w_1 \mid w_2$

Proof. Prop. 3.1.8, Theorem 3.2.2.

3.2.5. Corollary. *Let* $w = (w_1, w_2) \in \mathcal{Z}^2, w_2 \leq 0$, *and* $w_1 + w_2 > 0$ *with* $d_w(f) > 0$. *Assume that* $N(\Delta_f) \neq Ker(\Delta_f)$, *or* $E(\Delta_f) \neq Ker(\Delta_f)$ *with* $r = d_w(f) > \dot{deg}(w)$. *Then*

(1) *(Parabolic) Suppose* $w_2 = 0$, *we may assume that* $w = (1, 0)$. *Then either* $f_w^+ = ax^r(y - \alpha)^i, a \in K^\times, i < r$, *or* $f_w^+ = x^r(y-\alpha)^i F(y), \alpha \in K, i > r, F(y) \in K[y]$ *with* $F(\alpha) \neq 0$.

(2) *(Hyperbolic) Suppose* $w_2 < 0$. *Then either* f_w^+ *is a monomial, or* $f_w^+ = (x - \alpha y^{w_1/w_2})^i y^{(r-w_1 i)/w_2} F(z)$, *where* $\alpha \in K, z = xy^{-w_1/w_2}, i \in \mathcal{N}, 1 \leq i < (r - w_1 i)/w_2$ *and* $F(z) \in K[z], F(\alpha) \neq 0$.

Proof. Prop. 3.1.12, Prop. 3.1.13 and Theorem 3.2.2.

3.3 The Weak Jacobian Condition on $K[x, y^{1/m}, y^{-1/m}]$

We shall generalize the results of the previous sections to the larger K-algebra $K[x, y^{1/m}, y^{-1/m}]$. The idea to extend $K[x, y]$ to $K[x, y^{1/m}, y^{-1/m}]$ and to discuss the weak Jacobian condition on $K[x, y^{1/m}, y^{-1/m}]$ is crucial to prove Theorem 4.3.1. Prop. 3.3.5 and Prop. 3.3.8 are the main results of this section.

3.3.1. Recall that (see 1.1) $R_m := K[x, y^{1/m}, y^{-1/m}]$, the K-algebra generated by $x, y^{1/m}$ and $y^{-1/m}$ with $m \geq 1$. Note that every element f of R_m can be uniquely written in the form $f = \sum_{i \geq 0, j \in \mathcal{Z}} a_{ij} x^i y^{j/m}$. Moreover, $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \in R_m$ for all $f \in R_m$. Then, for any two elements $f, g \in R_m$, $J(f, g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \in R_m$, and we can define a K-derivation Δ_f of R_m by $\Delta_f(g) = J(f, g)$. Therefore, $Ker(\Delta_f), N(\Delta_f), E(\Delta_f)$ are defined (as in 1.1). Given $w = (w_1, w_2) \in \mathcal{Z}^2$, there is a standard $\frac{1}{m}\mathcal{Z}$-grading on R_m given by $(R_m)_n = \sum_{w_1 i + w_2 j/m = n/m} a_{i,j} x^i y^{\frac{j}{m}}$, where $n \in \frac{1}{m}\mathcal{Z}$. Since $\frac{1}{m}\mathcal{Z} \cong \mathcal{Z}$, we have a $\mathcal{Z}-$grading on R_m. We may also define $d_w(f), f_w^+$, for all $f \in R_m$. An element of the form $ax^iy^{j/m}, a \in K^\times$, is called a *monomial* of R_m. Write $\Re^{(m)}(n) = K[xy]y^{n/m}$, and $\Re(n) = K[xy]y^n$. Then $\Re^{(m)} = \bigoplus_{n=0}^{\infty} \Re^{(m)}(n)$, and $\Re = \bigoplus_{n=0}^{\infty} \Re(n)$. Clearly, $R_m = \bigoplus_{n=-\infty}^{\infty} \Re^{(m)}(n)$.

As a ring, $R_m \cong K[t, t^{-1}][x]$, with $t^m = y$. The automorphism group of R_m is easily determined. We are interested in the following subgroup of the automorphism group:

$$\begin{aligned} G^{(m)} &= \{\psi \in Aut(R_m) : \psi(y^{1/m}) = y^{1/m}, \psi(y^{-1/m}) = y^{-1/m}, \\ &\quad \psi(x) = x + \psi(y^{-1/m}), \psi(T) \in K[T]\}. \end{aligned}$$

We say that an automorphism in $G^{(m)}$ is a "restricted" automorphism of R_m. For $R = K[x, y]$, define

$$G = \{\phi \in Aut(R) : J(\phi(x), \phi(y)) = 1\}. \tag{3.15}$$

Put

$$G^{(m)}G = \{\psi \circ \phi : \psi \in G^{(m)}, \phi \in G\}.$$

By using the "chain rule", we obtain

$$\Phi(J(f, g)) = J(\Phi(f), \Phi(g)), f, g \in R_m \tag{3.16}$$

for any $\Phi \in G^{(m)}G$. Let m and n be positive integers with $m \mid n$. Consider the chain of K-algebras $R \subseteq R_m \subseteq R_n$. Then any automorphism in $G^{(m)}$ can be extended to an automorphism in $G^{(n)}$.

3.3.2. A question.

For the purpose of the next chapter, to prove Theorem 4.3.1, we only consider the case that $w = (w_1, w_2) \in \mathcal{Z}^2, w_2 < 0, deg(w) > 0$, and $GCD(|w_1|, |w_2|) = 1$ in this section.[6]

Similar to the discussions in section 3.1 and section 3.2, we shall study the following

[6]One may discuss other cases by similar methods.

question first.

Given $w = (w_1, w_2) \in \mathcal{Z}^2, w_2 < 0$, and $deg(w) = w_1 + w_2 > 0$. Let $f \in R_m$ with $d_w(f) > 0$. Suppose f satisfies

(i) $N(\Delta_f) \neq Ker(\Delta_f)$, or

(ii) $E(\Delta_f) \neq Ker(\Delta_f)$ with $d_w(f) > deg(w)$.

What is f_w^+?

The next Lemma is an analogue of Prop. 3.1.10.

3.3.3. Lemma. *Suppose $f, g \in R_m$ are two w-homogeneous elements with $\Delta_f(g) \neq 0$, and $\Delta_f^2(g) = 0$. Let $h = \frac{fg}{\Delta_f(g)}$. Assume $r = d_w(f) > 0$. Then $\Delta_f(h) = f$ and $h \in R_m$.*

Proof. The first part that $\Delta_f(h) = f$ is exactly the same as in Prop. 3.1.10. It is enough to prove that $h \in R_m$.

To show this, we note that the only difference between this lemma and Prop. 3.1.10 is that $d_w(f) \in \frac{1}{m}\mathcal{Z}$ in the present situation. We find that, if $h \notin R_m$, then

$$h^{-1} = y^{s/m} \prod (y^{1/m} - \alpha_i(x))^{s_i}, s_i \in \mathcal{Z}, \alpha_i(x) \in \overline{K(x)}, \tag{3.17}$$

where $\overline{K(x)}$ is an algebraic closure of $K(x)$, and for at least one i, $s_i > 0$. Pick one $s_i > 0$ and write $s = s_i, \alpha(x) = \alpha_i(x) \neq 0$. By the same argument as in Prop. 3.1.10. since $deg(w) > 0$, and $d_w(f) > 0$ by assumption, we have

$$\frac{\partial}{\partial y}(\frac{f^{w_1+w_2}}{h^r}) = -w_1 x(\frac{f^{w_1+w_2}}{h^{r+1}}). \tag{3.18}$$

Let v denote the valuation of the algebraic function field $\overline{K(x)}(y^{1/m})$ with constant field $\overline{K(x)}$ which corresponds to the irreducible polynomial $y^{1/m} - \alpha(x)$. Since $t = v(f) \geq 0$, we compute the v-valuations of both sides of the above formula (3.18) and obtain

$$t(w_1 + w_2) + sr - 1 = t(w_1 + w_2) + (r+1)s. \tag{3.19}$$

This is clearly impossible. Hence $h \in R_m$ as desired.

We shall later make use of the next proposition in the inductive step of the proof of Theorem 4.3.1.

3.3.4. Proposition. *Let $f, g \in R_m$ be two w-homogeneous elements with $\Delta_f(g) \neq 0$ and $\Delta_f^2(g) = 0$. Let $r = d_w(f) > 0$. Then either there exists $\psi \in G^{(m)}$ (as in 3.3.1) such that $\psi(f)$ is a monomial in R_m, or there exists a positive integer n with $m \mid n$ and $\psi \in G^{(n)}$ such that $\psi(f) = x^v y^{(r-iw_1)/w_2} F(xy^{-w_1/w_2}), F(T) \in K[T], deg(F(T)) \geq 1$, and $1 \leq v < (r - iw_1)/w_2$. Here v is a positive integer, but $(r - iw_1)/w_2$ may be not an integer.*

This proposition asserts that either f becomes a monomial under an automorphism, or by extending the algebra R_m to a larger algebra R_n and under a restricted automorphism Φ in the larger algebra R_n, the degree of $\Phi(f)$ with respect to x is strictly less than the degree of $\Phi(f)$ with respect to y in some suitable gradings. By using Prop. 3.1.13 and Prop. 3.3.4 repeatedly, we shall reduce f to some *standard* form as explained in next chapter.

Proof. Let $h = \frac{fg}{\Delta_f(g)}$. By assumption and Lemma 3.3.3, we may write $f = y^{r/w_2} F(z), h = y^{1+\frac{w_1}{w_2}} H(z)$, where $z = xy^{-w_1/w_2}$ and $F(z), H(z) \in K[z]$. Define

$$G = (\frac{H^r}{F^{w_1+w_2}})^m.$$

Note that $G(z) \in K(z)$ since $mr \in \mathcal{Z}$. By the same proof as in Prop. 3.1.10, we have

$$\frac{dG}{dz} = -w_2 m \frac{G}{H}. \tag{3.20}$$

Then by Lemma 3.1.11, each root of H is a simple root and every root of G is a root of H.

The proof consists of the following three cases. (as in the proof of Prop. 3.1.13)

Case 1. *Suppose $G \in K[z]$.* By Lemma 3.1.11, $H(z) = \beta z + \gamma, \beta \neq 0$. Hence $h = xyz^{-1}(\beta z + \gamma) = \beta xy + \gamma y^{1+w_1/w_2}$.

1.1. *Suppose $\gamma = 0$.* Then $h = \beta xy$. By using $\Delta_f(h) = f$, we see that f is a monomial.

1.2. *Suppose $\gamma \neq 0$.* Note that $y^{1+w_1/w_2} \in R_m$ since $h \in R_m$. Then $y^{w_1/w_2} \in R_m$

and $h = \beta y(x + \frac{\gamma}{\beta} y^{w_1/w_2})$. In this case $\frac{w_1}{w_2} \in \frac{1}{m}\mathcal{Z}$.

Define $\psi \in G^{(m)}$ by $\psi(x) = x - \frac{\gamma}{\beta} y^{w_1/w_2}$. Then $\psi(h) = \beta xy$. From $\Delta_{\psi(f)}(\psi(h)) = \psi(f)$ we deduce that $\psi(f)$ is a monomial in R_m.

Case 2. *Suppose $F(z) = \alpha(z+\gamma)^v, v > 0, \alpha \neq 0$, i.e., F has only one root.* Then $f = \alpha y^{r/w_2}(xy^{-w_1/w_2} + \gamma)^v$.

2.1. *Suppose $\gamma = 0$.* Then f is a monomial in R_m as desired.

2.2. *Suppose $\gamma \neq 0$.* Since $f \in R_m$, we have $y^{r/w_2}, y^{r/w_2}(xy^{-w_1/w_2}) \in R_m$. Therefore $y^{\frac{w_1}{w_2}} \in R_m$ and $\frac{w_1}{w_2} \in (\frac{1}{m})\mathcal{Z}$. Define $\psi \in G^{(m)}$ by $\psi(x) = x - \gamma y^{w_1/w_2}$. Then $\psi(f)$ is a monomial of R_m.

Case 3. *Suppose $G \notin K[z]$ and F has at least two distinct roots.* Since each root of H is a simple root, there exists a root α of F with order $v > 0$ such that $v(w_1 + w_2) > r$. Write

$$F(z) = (z-\alpha)^v P(z) \tag{3.21}$$

with $P(z) \in K[z], deg(P) \geq 1$ and $P(\alpha) \neq 0$. Then

$$f = (x - \alpha y^{w_1/w_2})^v y^{\frac{r-w_1 v}{w_2}} P(xy^{-w_1/w_2}). \tag{3.22}$$

Since $v(w_1 + w_2) > r$ and $w_2 < 0$, we have $1 \leq v < \frac{r-w_1 v}{w_2}$. Let $n = -w_2 m$ and define $\psi \in G^{(n)}$ given by $\psi(x) = x + \alpha y^{w_1/w_2}$. Then $\psi(f) = x^v y^{(r-w_1 v)/w_2} Q(xy^{-w_1/w_2}) \in R_n$. where $Q(T) \in K[T]$ and $deg(Q) = deg(P) \geq 1, Q(0) \neq 0$. The proof is completed.

Actually, the above argument provides us with more information.

3.3.5. Proposition. *Let $f, g \in R_m$ be two w-homogeneous elements with $\Delta_f(g) = \lambda g$, where $\lambda \in K^\times$. Assume that both $d_w(f)$ and $d_w(g)$ are positive real numbers. Then one of the three following possibilities holds:*

(i) $f = axy, a \in K^\times$,

(ii) $w_2 \mid m$,

(iii) $f = axy(z+\alpha_1)\ldots(z+\alpha_s), s \geq 1, a \in K^\times, \alpha_i \neq \alpha_j$, *for* $i \neq j$, *and* $\alpha_i \neq 0$ *for each* $i = 1, \ldots, s$, *where* $z = xy^{-w_1/w_2}$.

Proof. Keep the notations as in the proof of Prop. 3.3.4 with f and g interchanged. Note that $h = \frac{fg}{J(f,g)} = \frac{1}{\lambda}f$. It suffices to figure out the form of h.

To do this, write $h = ay^{1+\frac{w_1}{w_2}}(z+\alpha_1)\dots(z+\alpha_n)$, where $n \geq 1, \alpha_i \neq \alpha_j$.

Suppose $n = 1$ and $\alpha_1 = 0$. Then $h = axy$.

Suppose $n = 1$ and $\alpha_1 \neq 0$. Then $y^{1+w_1/w_2} \in R_m$. Then $\frac{mw_1}{w_2} \in \mathcal{Z}$. Since $GCD(w_1, w_2) = 1$, $w_2 \mid m$.

Suppose $n \geq 2$ and each $\alpha_i \neq 0$. Then $a\prod_{i=1}^{s}\alpha_i y^{1+w_1/w_2} \in R_m$. So $\frac{mw_1}{w_2} \in \mathcal{Z}$, and then $w_2 \mid m$.

At last, suppose $n \geq 2$ and some $\alpha_i = 0$. By Lemma 3.1.11, only one $\alpha_i = 0$. Say $\alpha_n = 0$. Then $f = axy(z+\alpha_1)\dots(z+\alpha_{n-1}), \alpha_i \neq 0$ for $i = 1, \dots, s = n-1 > 0$. This reduces to (iii). This completes the proof of Prop. 3.3.5.

3.3.6. After extending the results in section 3.1 to the case $R_m = K[x, y^{1/m}, y^{-1/m}]$, we shall generalize Theorem 3.2.2 to R_m.

Actually, the proofs are completely parallel to those in Theorem 3.2.2. For example, for all $f, g \in R_m, w = (w_1, w_2)$, we have that

$$L_2(n)/m \leq d_w(f^k g^l) \leq L_1(n)/m, k, l = 0, 1, \dots, n, \tag{3.23}$$

where $L_1(n), L_2(n)$ are two linear forms in n that depend only on f, g. Therefore, one shows the following

3.3.7. Proposition. *Suppose $m \geq 1$ and $f \in R_m$. Let $w = (w_1, w_2) \in \mathcal{Z}^2$ where $w_2 < 0, deg(w) = w_1 + w_2 > 0$, and $GCD(|w_1|, |w_2|) = 1$. Let $r = d_w(f) > 0$. If $N(\Delta_f) \neq Ker(\Delta_f)$, or $E(\Delta_f) \neq Ker(\Delta_f)$ with $d_w(f) > deg(w)$, then $\Delta_{f_w^+}$ has a w-homogeneous local slice in R_m.*

By Prop. 3.3.4 and Prop. 3.3.7, the following proposition follows immediately.

3.3.8. Proposition. *Suppose $m \geq 1$ and $f \in R_m$. Let $w = (w_1, w_2) \in \mathcal{Z}^2$ where $w_2 < 0, deg(w) = w_1 + w_2 > 0$, and $GCD(|w_1|, |w_2|) = 1$. Let $r = d_w(f) > 0$. Assume that $N(\Delta_f) \neq Ker(\Delta_f)$, or $E(\Delta_f) \neq Ker(\Delta_f)$ with $d_w(f) > deg(w)$. Then one of the following three possibilities holds.*

(i) f is a monomial in R_m,

(ii) $f_w^+ = \alpha y^{r/w_2}(xy^{-w_1/w_2} - \beta)^v, v \geq 0, \alpha \neq 0$ *and* $\beta \neq 0$. *Moreover, there exists* $\psi \in G^{(m)}$ *such that* $\psi(f_w^+)$ *is a monomial in* R_m,

(iii) $f_w^+ = (x - \alpha y^{w_1/w_2})^v y^{\frac{r-w_1v}{w_2}} F_1(z), z = xy^{-w_1/w_2}, F_1(z) \in K[z], deg(F) \geq 1$ *and* $F_1(\alpha) \neq 0$. *Here* $1 \leq v < \frac{r-w_1v}{w_2}$ *and* v *is a positive integer. Moreover, there exists a positive integer* n *with* $m \mid n$ *and* $\psi \in G^{(n)}$ *such that* $\psi(f_w^+) = x^v y^{\frac{r-w_1v}{w_2}} F_2(xy^{-w_1/w_2}), F_2(T) \in K[T], deg(F_2) = deg(F_1)$, *and* $1 \leq v < (r - w_1v)/w_2$. *Note that* v *is a positive integer, but* $(r - w_1v)/w_2$ *may be not a rational integer.*

3.3.9. Observation. In Prop. 3.3.8.(ii), from $y^{r/w_2}, (y^{r/w_2})(xy^{-w_1/w_2}) \in R_m$, we have that $y^{w_1/w_2} \in R_m$. Therefore, $w_1/w_2 \in \frac{1}{m}\mathcal{Z}$. Since w_1, w_2 are co prime, $(-w_2)$ is a divisor of m. In particular, $-w_2 \leq m$. This observation will be very useful in the proof of Theorem 4.3.1 below.

Chapter 4

The Weak Jacobian Condition (II)

The most important result of this chapter is the reduction theorem 4.3.1. In section 4.2 we shall show some results related to the Jacobian condition which will be used to establish a relation between theorem 4.3.1 and the Jacobian Conjecture in two variables.

4.1 Divisor Theorem

In this section, we shall study further the weak Jacobian condition. Although the main result, the divisor theorem, is not used in proving Theorem B, we will make substantial use of its method of proof later. Moreover, the divisor theorem can be regarded as an analogue of the epimorphism theorem (see [2]) under the weak Jacobian condition. Hence it is worthwhile to state it here.

Note that, by an automorphism in $K[x, y]$, we may assume and fix the following form of f

$$f = y^j + \sum_{(k,l)\in S(f)} a_{kl}x^k y^l, \tag{4.1}$$

where $k + l < j$ for all $(k, l) \in S(f)$, and $S(f)$ is a non-empty subset of $Supp(f)$. Define $r(f) = sup\{k|(k, l) \in S(f) \text{ for some } l\}$, $s(f) = sup\{l|(r(f), l) \in S(f)\}$. Assume that $j = deg(f) \geq 3$.[1]

[1]Suppose $deg(f) = 1$. Then $f = y + c, c \in K$. Suppose $deg(f) = 2$. Then $f = y^2 + bx + cy + d$,

4.1.1. Divisor Theorem. *Keep the notations as above. Suppose $T(\Delta_f) \neq Ker(\Delta_f)$. Then there exists a positive integer $M \geq 2$ such that $j = s(f) + Mr(f)$.*

This theorem provides very strong necessary conditions on the polynomial f for $T(\Delta_f) \neq Ker(\Delta_f)$, and it is often relatively easy to see this is not the case by using this theorem.

Proof. Define $A = min\{r/(j-s), (r-k)/(l-s) | (k,l) \in S(f), k < r, l > s\}$, where $r = r(f), s = s(f)$. Then in view of (4.1),we have $A < 1$, and $f^+_{(1,\sigma)} = a_{rs}x^r y^s$ for every $\sigma \in (0, A)$. Choose and fix a rational number $n/m \in (0, A)$. Then $f^+_{(m,n)} = a_{rs}x^r y^s$. Further, choose a $\tau > \frac{m}{n}\frac{r}{j-s} \geq \frac{mA}{n} > 1$. Then we have $n\tau j > mr + ns\tau$. Hence $f^+_{(m,n\tau)} \neq f^+_{(m,n)}$. Define $\gamma = \inf\{\tau > 1 | f^+_{(m,n\tau)} \neq f^+_{(m,n)}\}$.

Since we will refer to the following assertion and its method of proof several times, we state it as a lemma.

4.1.2. Lemma. *Keep the notations in 4.1.1. Then $f^+_{(m,n\gamma)} \neq f^+_{(m,n)}$, $f^+_{(m,n\gamma)}$ is not a monomial and $\gamma = p/q \in Q, p > q, (p,q) = 1$.*

Proof of Lemma 4.1.2.

Let $B = \frac{m}{n}\inf\{\frac{r}{j-s}, \frac{r-k}{s-l} | (k,l) \in S(f) \setminus (r,s)\}$. Then $f^+_{(m,n\tau)} = f^+_{(m,n)}$ if and only if $\tau < B$. We first prove that $f^+_{(m,n\gamma)} \neq f^+_{(m,n)}$. Suppose not. Then $\gamma < B$. Choose a sequence γ_n with limit γ such that $f^+_{(m,n\gamma_n)} \neq f^+_{(m,n)}$ and $\gamma_n < B$. Clearly, this is impossible by the definition of B. Hence $f^+_{(m,n\gamma)} \neq f^+_{(m,n)}$. If $f^+_{(m,n\gamma)}$ is a monomial, there exists $(k_1, l_1) \in S(f)$ such that $mk_1 + l_1 n\gamma > nj\gamma$, and $mk_1 + l_1 n\gamma > mk + nl\gamma$, for all $(k,l) \in S(f) - (k_1, l_1)$. Equivalently, $\gamma \in (C, D)$ for some two constants $C, D > 0$. Choose a $\gamma_0 < \gamma$, with $\gamma_0 \in (C, D)$. Then $f^+_{(m,n\gamma_0)} = f^+_{(m,n\gamma)} \neq f^+_{(m,n)}$, which contradicts the definition of γ. Therefore, $f^+_{(m,n\gamma)}$ is not a monomial. Finally, $\gamma \in Q$ (since $f^+_{(w_1,w_2)}$ is a monomial if $w_1/w_2 \notin Q$ for any two integers w_1, w_2). Write $\gamma = p/q \in Q$ with $(p,q) = 1, p > q$. The proof is completed.

Let us come back to the proof of theorem 4.1.1.

Note that $mq, np > 1$, and $T(\Delta_f) \neq Ker(\Delta_f)$. Suppose $E(\Delta_f) \neq Ker(\Delta_f)$ and $d_w(f) \leq deg(w)$ with $w_1 = mq, w_2 = np$. Then we have $jw_2 \leq w_1 + w_2, kw_1 + lw_2 \leq w_1 + w_2$ for all $(k,l) \in S(f)$. Then f is of the form $f = axy + b(y) + c(x), a \in K$, and

and f becomes y or y^2 under an automorphism.

$b(T), c(T) \in K[T]$. We show that 4.1.1 follows for this kind of polynomial. *Suppose* $r(f) = 1$. Then 4.1.1 is evident since $j \geq 3$ by assumption. *Suppose* $r(f) \geq 2$. Then $s(f) = 0$ by (4.1). Considering the $w = (j, r(f))$-homogeneous decomposition of f, we note that $jr(f) > r(f) + s(f)$ since $r(f), j \geq 3$. By Corollary 3.2.4, one may assume that $f_w^+ = a(x + by^{j/r(f)})^{i'} y^{j'}$ for some $a \in K^\times, b \in K, i', j' \geq 0$. Since f_w^+ is not a monomial (it contains y^j and $cx^{r(f)}$ for some $c \in K^\times$), we have $b \neq 0$, $i' \geq 1$ and $r(f)$ divides j. Hence 4.1.1 follows. Therefore, we may assume that for $w = (mq, np)$, either $N(\Delta_f) \neq Ker(\Delta_f)$, or $E(\Delta_f) \neq Ker(\Delta_f)$ with $d_w(f) > deg(w)$. Then by Corollary 3.2.4, we obtain that either $mq \mid np$, or $np \mid mq$. Consider the following two cases.

Case 1. *Suppose* $np = mqM$ *for some* $M \geq 1$. Since $f^+_{(m,n\gamma)} = f^+_{(1,M)}$ is not a monomial, we know by Corollary 3.2.4 that $f^+_{(m,n\gamma)} = bx^{i'}(y + cx^M)^{j'}$ for some $b, c \in K^\times$, and $j' \geq 1, i' \geq 0$. Then

$$f = bx^{i'}(y + cx^M)^{j'} + \sum_{k+Ml<i'+Mj'} c_{k,l} x^k (y + cx^M)^l \tag{4.2}$$

is the $(1, M)$-homogeneous decomposition of f. This is impossible, because $f^+ = y^j$ by (4.1).

Case 2. *Suppose* $mq = npM$ *for some* $M \geq 2$. By Corollary 3.2.4, we have $f^+_{(m,n\gamma)} = f^+_{(M,1)} = b(x + cy^M)^{i'} y^{j'}$ for some $b, c \in K^\times, i' \geq 1$ and $j' \geq 0$. We express the $(M, 1)$-homogeneous decomposition of f as follows:

$$f = b(x + cy^M)^{i'} y^{j'} + \sum_{(k',l') \in S_1(f)} b_{k'l'} (x + cy^M)^{k'} y^{l'}, \tag{4.3}$$

where $Mk' + l' < Mi' + j'$ for $(k', l') \in S_1(f)$, and $S_1(f)$ is a finite set. We shall determine $f^+_{(m,n)}$. Let $w = (m, n)$. Then $d_w(x^s y^{M(i'-s)}) = m(1 - \frac{q}{p})s + \frac{i'mq}{p}$ for every $0 \leq s \leq i'$. Thus, we obtain

$$((x + cy^M)^{i'} y^{j'})^+_w = x^{i'} y^{j'}, ((x + cy^M)^{k'} y^{l'})^+_w = x^{k'} y^{l'} \tag{4.4}$$

for $(k', l') \in S_1(f)$.

Case 2.1. *Suppose $k' \leq i'$ for $(k', l') \in S_1(f)$.* Since $Mk' + l' < Mi' + j$, and $M = \frac{mq}{np} < \frac{m}{n}$, we have $m(i' - k') > n(l' - j'), i.e., mi' + j' > mk' + l'$. Hence $f^+_{(m,n)} = bx^{i'}y^{j'}$. Therefore, $i' = r(f), j' = s(f)$ from the choice of m and n. Further, by expanding f in the above $(M, 1)$-decomposition (4.3), we have $f^+ = dy^{Mi'+j'}$ for some $d \in K^\times$. Therefore $j = Mr + s$.

Case 2.2. *Suppose there exists some $(k'_0, l'_0) \in S_1(f)$ such that $k'_0 > i'$.* We shall show that this is impossible by getting a contradiction.

Put $\varphi : (x, y) \to (x - cy^M, y), \psi = \varphi^{-1}, g = \psi(f)$. Then

$$g = bx^{i'}y^{j'} + \sum_{(k',l') \in S_1(f)} b_{k'l'}x^{k'}y^{l'} \tag{4.5}$$

is the $(M, 1)$-decomposition of g and $deg(g) < deg(f)$ (since $M \geq 2$), $T(\Delta_g) \neq Ker(\Delta_g)$. Moreover, there exists $\theta \in (0, 1)$ with $Mk'_0 + \theta l'_0 > Mi' + \theta j'$. For example, take any $\theta < \frac{M(k'_0 - i')}{j' - l'_0} < 1$. Therefore $g^+_{(M,\theta)} \neq g^+_{(M,1)}$. Put $\delta = Sup\{\theta \in (0, 1) | g^+_{(M,\theta)} \neq g^+_{(M,1)}\}$. By the same argument as in Lemma 4.1.2, we know that $g^+_{(M,\delta)} \neq g^+_{(M,1)}$, $\delta \in (0, 1) \cap \mathcal{Q}$, and $g^+_{(M,\delta)}$ is not a monomial. Write $\delta = u/v \in \mathcal{Q} \cap (0, 1), (u, v) = 1$. Suppose $E(\Delta_g) \neq Ker(\Delta_g)$ with $d_w(g) \leq deg(w)$, where $w = (Mv, u)$, we have $Mvi' + uj', Mvk' + ul' \leq Mv + u$ for all $(k', l') \in S_1(f)$. Then $f = a(x + cy^M)y + b(x + cy^M) + c(y), a \in K$ and $b(T), c(T) \in K[T]$. As before, one can prove 4.1.1 directly for f. Hence we assume that for $w = (Mv, u)$, either $N(\Delta_g) \neq Ker(\Delta_g)$, or $E(\Delta_g) \neq Ker(\Delta_g)$ with $d_w(g) > deg(w)$. By Corollary 3.2.4, we obtain that u must divides Mv. Hence $Mv = Nu$ for some natural numbers N, M and $N > M$ since $u < v$. Let $g^+_{(M,\delta)} = g^+_{(N,1)} = b'(x + c'y^N)^{i''}y^{j''}$ for some $b', c' \in K^\times$, and $i'' \geq 1, j'' \geq 0$. We express the $(N, 1)$-decomposition of g as follows:

$$g = b'(x + c'y^N)^{i''}y^{j''} + \sum_{(k'',l'') \in S(g)} b_{k''l''}(x + c'y^N)^{k''}y^{l''}, \tag{4.6}$$

where $Ni'' + j'' > Nk'' + l''$ for all $(k'', l'') \in S(g)$ and $S(g)$ is a finite set depending on g. Since $f = \psi(g), \psi : (x, y) \to (x + cy^M, y)$, we have

$$f = b'(x + cy^M + c'y^N)^{i''} y^{j''} + \sum_{(k'',l'') \in S(g)} b_{k''l''}(x + c'y^M + c''y^N)^{k''} y^{l''}. \qquad (4.7)$$

Since $N > M$, we obtain

$$((x + cy^M + c'y^N)^{i''} y^{j''})^+_{(M,1)} = c'^{i''} y^{Ni''+j''}, \qquad (4.8)$$

and

$$((x + cy^M + c'y^N)^{k''} y^{l''})^+_{(M,1)} = c'^{k''} y^{Nk''+l''} \qquad (4.9)$$

for $(k'', l'') \in S(g)$. Hence $f^+_{(M,1)} = dy^{Ni''+j''}$ for some $d \in K^\times$ by (4.7). But this contradicts the fact that $f^+_{(M,1)} = b(x+cy^M)^{i'} y^{j'}$ is not a monomial, as we have shown in (4.3). This contradiction, therefore, completes the proof of the divisor theorem.

4.2 Elementary Polynomials

In this section, we derive some results concerning the Jacobian condition, instead of the weak Jacobian condition. By combining the results of this section with theorem 4.3.1, we will prove theorem 4.3.14 and see that theorem 4.3.1 can be seen as a weak form of the Jacobian Conjecture. The results of this section are true for arbitrary integral domain of characteristic zero. In this section we shall use R to denote an integral domain of characteristic zero.

4.2.1. Let $B = R[x, y]$ denote the polynomial ring of 2 variables over R. For a polynomial $f \in B$, define Δ_f by $\Delta_f(g) = J(f, g)$ for $g \in B$. Δ_f is a R-derivation of B. Some results in chapter 1 may be extended to this more general case. For example, if R is a UFD containing $\mathcal{Q}$, then any locally nilpotent derivation of B has the form $a\Delta_P$ for some variable P in B, and $a \in R[P]$ (as in 2.2.2) [12] . Moreover, a locally nilpotent derivation δ has a slice if and only if $\delta(B) = B$.

4.2.2. Definition. (X, Y) is a *coordinate system* of $B = R[x, y]$ if $B = R[X, Y]$. X is a *variable* of B if (X, Y) is a coordinate system of B for some $Y \in B$.

Prop. 4.2.11 is our main result in this section. To show it, we shall prove two propositions first.

4.2.3. Proposition. *Let X, Y be a coordinate system of B. Suppose $f = aX + Y^2 b(X,Y) \in B, a \in R^\times$, and $b(X,Y) \in R[X,Y]$. Then $f = aX + u(Y)$ for some polynomial $u(Y) \in R[Y]$ if and only if Δ_f has a slice.*

4.2.4. Proposition. *Let (X,Y) be a coordinate system of B. Suppose $f = aX + X^2 b(X,Y) \in B, a \in R^\times, b(X,Y) \in R[X,Y]$ and Δ_f has a slice $g \in B$. Then $f = aX$.*

4.2.5. Remark. The above two propositions 4.2.3 and 4.2.4 are not new and can be proved by using the similarity of the Newton diagram of a Jacobian pair.[2] See [47], section 3, proof of theorem 1.3. Here we give another proof of Prop. 4.2.3, and Prop. 4.2.4 without using the properties of Newton diagrams. It would be interesting to study the Newton diagram of Jacobian pairs in R_m or to extend our argument here to R_m.

4.2.6. Proof of Prop. 4.2.4. We follow the argument of Czerniakiewicz [7].

4.2.7. Step 1.

In the following proof, we always use a fixed *coordinate system* (X, Y). Suppose $\Delta_f(g) \in R^\times$ with $g \in K[X,Y]$. We assume that $g(0,0) = 0$, and change g by a multiple of f, one may write: $f = aX + f_2 + \ldots + f_{deg(f)}, g = bY + g_2 + \ldots + g_{deg(g)}$, where $a, b \in R^\times$, and f_i, g_j are the homogeneous components of f of degree i, and of g of degree j, respectively. For simplicity, we assume that $a = b = 1$ and $J(f,g) = 1$. Write $f_r = \sum_{i=0}^{r} a_{i,r} X^i Y^{r-i}$ and define $f_r^i = a_{i,r} X^i Y^{r-i}$ for $i \le r \le deg(f)$. Similarly, we define g_s^j for any $j \le s \le deg(g)$.

By comparing the coefficients of the monomial $x^{s-1} y^{t-s-1}$ on both sides of the formula $J(f,g) = 1$ with $t \ge 3, 1 \le s \le t-1$, we obtain:

$$\Upsilon_t^s : J(f_{t-1}^s, y) + \sum_{k=0}^{t-2} \sum_{h=0}^{min(k,s)} [J(f_{t-k}^{s-h}, g_k^h) + J(x, g_{t-1}^{s-1})] = 0. \tag{4.10}$$

At first, we consider Υ_r^1, for all r. By assumption on f, $f_r^0 = f_r^1 = 0$, and we have

[2] Given two polynomials $f, g \in K[x,y]$, f, g is a Jacobian pair if $J(f,g) \in K^\times$.

$J(X, g^0_{r-1}) = 0$. Hence $g^0_s = 0$ for $s = 2, \ldots, deg(g)$. Let $M = max(deg(f), deg(g))$. We may suppose that $M > 1$ [3]. We shall prove that:

$$f^2_M = g^1_M = 0. \tag{4.11}$$

In fact, Υ^2_{M+1} implies that $J(f^2_M, y) + J(x, g^1_M) = 0$. On the other hand, Υ^3_{2M} implies that $J(f^2_M, g^1_M) = 0$. Therefore (4.11) follows from Lemma 3.1.6.

4.2.8. Step 2.

We shall prove that for any $M \geq m \geq n \geq 2$,

$$(P_{m,n}) : f^n_m = g^{n-1}_m = 0. \tag{4.12}$$

If $(P_{m,n})$ is proved, for every $M \geq m \geq n \geq 2$, the proof of Prop. 4.2.4 follows immediately.

To show $(P_{m,n})$, for every $M \geq m \geq n \geq 2$, we employ an inductive argument with respect to a well ordering. Let $\nabla = \{(m, n) : M \geq m \geq n \geq 2\}$. We define $(m_1, n_1) \succ (m_2, n_2)$ in ∇ if $\frac{n_1-1}{m_1-1} > \frac{n_2-1}{m_2-1}$. Clearly, the first element of ∇ is $(M, 2)$, and $(P_{M,2})$ has been proved by formula (4.11). Note that ∇ is not a well ordering under $\succ$. Define an equivalence relation on ∇ by making all elements (m, n) with the same quotient $\frac{n-1}{m-1}$ belong to the same equivalent class. Then $\succ$ induces a well ordering on the set of equivalence classes. We shall make use of an induction argument on the set of equivalent classes.

Given $(a, b) \in \nabla$ with $GCD(a-1, b-1) = 1$, suppose that $f^n_m = g^{n-1}_m = 0$ for any $(m, n) \in \nabla$ with $\frac{n-1}{m-1} < \frac{b-1}{a-1}$. It is enough to prove:[4]

$$(\nabla)_r : f^{r(b-1)+1}_{r(a-1)+1} = g^{r(b-1)}_{r(a-1)+1} = 0 \tag{4.13}$$

for $r \geq 1$.

[3]When $M = 1$, both f and g are linear functions of X and Y, and the results are clear.

[4]Since for any two integers c, d, $\frac{d-1}{c-1} = \frac{b-1}{a-1}$ implies that $c = 1 + (b-1)r, d = 1 + (a-1)r$ for some integer $r \geq 1$.

To prove $(\nabla)_r$ for all $r \geq 1$, we need:

4.2.9. Lemma. *Suppose $a, b \geq 2$ and h, k, r are positive integer satisfying $k < 1 + r(a-1)$. Then either $\frac{r(b-1)-h}{r(a-1)+1-k} \leq \frac{b-1}{a-1}$, or $\frac{h}{k-1} < \frac{b-1}{a-1}$.*

Proof of Lemma 4.2.9.

Suppose

$$\frac{r(b-1)-h}{r(a-1)+1-k} > \frac{b-1}{a-1}. \tag{4.14}$$

Then

$$(a-1)(r(b-1)-h) > (b-1)(r(a-1)+1-k). \tag{4.15}$$

since $a-1>0$ and $r(a-1)+1-k>0$. Hence

$$h(a-1) < (b-1)(k-1). \tag{4.16}$$

Since both h and $a-1$ are positive numbers, and $k \geq 2$, we have $\frac{h}{k-1} < \frac{b-1}{a-1}$.

4.2.10. Step 3.

We come back to the proof of $(\nabla)_r$.

Consider the formula $\Upsilon^{r(b-1)+1}_{r(a-1)+2}$. By Lemma 4.2.9, it reduces to:

$$\Upsilon_r : J(f^{r(b-1)}_{r(a-1)+1}, y) + J(x, g^{r(b-1}_{r(a-1)+1}) + \sum_{n=1}^{r-1} J(f^{(r-n)(b-1)+1}_{(r-n)(a-1)+1}, g^{n(b-1)}_{n(a-1)+1}) = 0. \tag{4.17}$$

In proving $(\nabla)_r$ for every r, we use another induction argument as follows. As $(\nabla)_r$ is obviously true for sufficiently large r (since both f and g are polynomials), we suppose that $(\nabla)_k$ holds for all $k > N$, and then prove that $(\nabla)_N$ holds. After this is proven, the proof of this proposition is completed. We now prove $(\nabla)_N$. Consider the formula Υ_{2N}. By the assumption that $(\nabla)_k$ holds for any $k > N$, Υ_{2N} reduces to:

$$J(f^{N(b-1)+1}_{N(a-1)+1}, g^{N(b-1)}_{N(a-1)+1}) = 0. \tag{4.18}$$

By Lemma 3.1.6, (4.18) yields either $f^{N(b-1)+1}_{N(a-1)+1} = 0$ or $g^{N(b-1)}_{N(a-1)+1} = 0$.

Suppose that $g^{N(b-1)}_{N(a-1)+1} = 0$ and $f^{N(b-1)+1}_{N(a-1)+1} \neq 0$. We shall show the following formula by induction on h:

$$g^{(N-h)(b-1)}_{(N-h)(a-1)+1} = 0 \tag{4.19}$$

for $0 \le h \le N-1$. In fact, formula (4.19) holds for $h = 0$ by the assumption that $g^{N(b-1)}_{N(a-1)+1} = 0$. Suppose it holds for any $h < h_0 \le N-1$. Consider the formula Υ_{2N-h_0}. By the induction hypothesis that $(\nabla)_k$ for $k > N$, and the assumption that $g^{(N-h)(b-1)}_{(N-h)(a-1)+1} = 0$ for $h < h_0$, Υ_{2N-h_0} yields

$$J(f^{N(b-1)+1}_{N(a-1)+1}, g^{(N-h_0)(b-1)}_{(N-h_0)(a-1)+1}) = 0. \tag{4.20}$$

By Lemma 3.1.6 again and the assumption that $f^{N(b-1)+1}_{N(a-1)+1} \neq 0$, we obtain

$$g^{(N-h_0)(b-1)}_{(N-h_0)(a-1)+1} = 0. \tag{4.21}$$

Then the formula (4.19) is proved. Let us consider Υ_N finally. By the formula (4.19), Υ_N implies that $J(f^{N(b-1)+1}_{N(a-1)+1}, Y) = 0$. By Lemma 3.1.6, $f^{N(b-1)+1}_{N(a-1)+1} = 0$. Hence $f^{N(b-1)+1}_{N(a-1)+1} = g^{N(b-1)}_{N(a-1)+1} = 0$.

The argument is similar if $f^{N(b-1)+1}_{N(a-1)+1} = 0$. This completes the proof of proposition 4.2.4.

4.2.10. Proof of Proposition 4.2.3. Suppose that $f = aX + Y^2 b(X, Y)$. Then $f^r_r = f^{r-1}_r = 0$. By the formula Υ^s_{s+1} above, we have $J(X, g^{s-1}_s) = 0$. Hence $g^{s-1}_s = 0$. Then the formula Υ^s_s reduces to $J(x, g^{s-1}_{s-1}) = 0$. Hence $g^{s-1}_{s-1} = 0$. Therefore, $g \in a^{-1}Y + Y^2 R[x, y]$. Since Δ_g has a slice f, then by proposition 4.2.4, with interchange of f and g in notations, we have $f^{n-1}_m = 0$ for every $M \ge m \ge n \ge 2$. Then, we have $f = aX + u(Y)$ for some $u(T) \in R[T]$.

We shall next prove Prop. 4.2.11, which will be used in proving Theorem 4.3.14 in the next section.

4.2.11. Proposition. *Let K be a field of characteristic zero and f a polynomial of the form $f = ax + bxy + x^2 c(x, y)$, where $a, b \in K, c(x, y) \in K[x, y]$ and $deg(c) > 1$. Assume that Δ_f has a slice in $K[x, y]$. Then $f = ax, a \in K^\times$.*

Proof. Let g be a slice of f. Then $a \in K^\times$ by direct calculation, and after changing g by a multiple of f, one may write that $g = y + g_2 + \ldots + g_n$, where each g_i is a homogeneous polynomial of weight i. By Prop. 4.2.4, it suffices to show that $b = 0$. To show this, we use the same notations as that in the proof of Prop. 4.2.4. In this case, we have that $f_r^0 = 0$ for $r \geq 2$, $f_2^1 = bxy$, $f_2^2 = 0$, and $f_s^1 = 0$, for $s \geq 3$. Moreover, $g_1 = g_1^0 = y$. The formula Υ_t^1 gives us

$$J(f_{t-1}^1, y) + \sum_{k=0}^{t-2}[J(f_{t-k}^1, g_k^0) + J(f_{t-k}^0, g_k^1)] + J(x, g_{t-1}^0) = 0. \tag{4.22}$$

Suppose that $b \neq 0$. Write $g_n^0 = d_n y^n$ for $n \geq 2$. For any $t \geq 4$, the above formula (4.22) reduces to :

$$J(f_2^1, g_{t-2}^0) + J(x, g_{t-1}^0) = 0. \tag{4.23}$$

Then we obtain

$$bd_{n-2}(n-2) + d_{n-1}(n-1) = 0 \tag{4.24}$$

for any $n \geq 4$. Since g is a polynomial, $d_n = 0$ for $n >> 0$, so by the assumption $b \neq 0$, we obtain $d_n = 0$ for $n \geq 2$. In particular, $g_2^0 = 0$. We shall prove this is impossible. In fact, consider the formula Υ_3^1:

$$J(f_2^1, y) + J(f_2^1, g_1^0) + J(x, g_2^0) = 0. \tag{4.25}$$

The formula (4.25) implies that $2by = 0$, so $b = 0$. Therefore, we have shown that $b = 0$, and $f = ax + x^2c(x, y)$. Then this proposition follows from Prop. 4.2.4.

We may rewrite Prop. 4.2.4 and Prop. 4.2.11 in the following geometrical form.

4.2.12. Definition. Let K be a field of characteristic zero. A polynomial $f \in K[x, y]$ is a *line* if f is irreducible and $K[x, y]/(f) \cong K^{[1]}$ (polynomial ring in one variable). A polynomial f contains a line L as a factor if $f = Lg$ for some $g \in K[x, y]$ and L is a line.

4.2.13. Proposition. *Let $f \in K[x, y]$ be a non-constant polynomial and suppose f has a line L as a factor. Assume f satisfies the Jacobian condition, i.e., $J(f, g) \in K^\times$ for some $g \in K[x, y]$. Then f is a line.*

Proof. We may write the line L as a variable X for some coordinate system X, Y by the theorem of Abhyankar-Moh and Suzuki [2] and [50]. Then $f = Xh(X, Y)$ for some $h(X, Y) \in K[X, Y] = K[x, y]$. The result then follows from Prop 4.2.4. Indeed, if f satisfies the Jacobian condition, then the curve $\{f = 0\}$ is smooth. It follows that the line $\{X = 0\}$ and the curve $\{h(X, Y) = 0\}$ do not meet. In particular, $h(X, Y) = a + Xh_1(X, Y)$ for some $a \in K, h_1(X, Y) \in K[X, Y]$. Then Prop. 4.2.13 follows from Prop. 4.2.4 directly.

4.2.14. Remark. S.Kaliman [20] has shown that to prove the Jacobian Conjecture, it is enough to do it with the assumption that for every $c \in K$ the fiber $\{(x, y) : f(x, y) = c\}$ is irreducible.

4.3 Reduction Theorem

We preserve the notations of section 3.3.1. Recall that $\Re^{(m)}(n) = K[xy]y^{n/m}$, $\Re(n) = K[xy]y^n$, $\Re^{(m)} = \bigoplus_{n=0}^{\infty} \Re^{(m)}(n)$, and $\Re = \bigoplus_{n=0}^{\infty} \Re(n)$. Note that

$$J(\Re^{(m)}(i), \Re^{(m)}(j)) \subseteq \Re^{(m)}(i + j) \tag{4.26}$$

for $i, j \in \mathcal{Z}$.

The following Reduction Theorem plays a central role in proving Theorem 5.1.1 and Theorem B.

4.3.1. Reduction Theorem. *Let f be a non-constant polynomial in $K[x, y]$. Suppose $T(\Delta_f) \neq Ker(\Delta_f)$. Then either there exists $\Psi \in G$ such that $\Psi(f) \in \Re$, or there exists $m \geq 1, \Psi \in G, \Psi_m \in G^{(m)}$ such that $\Psi_m(\Psi(f)) \in \Re^{(m)}$.*

Proof.

We assume that $deg(f) \geq 3$.[5]

The proof of this reduction theorem is quite involved while elementary. It consists of several steps.

[5]The reduction theorem is obvious when $deg(f) \leq 2$, because f has the form x, x^2 or xy under an automorphism when $deg(f) \leq 2$.

4.3.2. Step 1.

We first prove by induction on $deg(f)$ that *there exists a* $\psi \in G$ *such that* $\psi(f)$ *is a stable polynomial* (in the sense of 3.1.7). This is a well known fact, and we just explain its main point.

Suppose this is true for polynomial of degree less than $k-1$ and let $deg(f) = k$. Since $T(\Delta_f) \neq Ker(\Delta_f)$, then by Corollary 3.2.4, with $w = (1,1)$ (since $deg(f) > deg(w) = 2$), there exists a $\psi_1 \in G$, such that $\psi_1(f)^+$ is a monomial, and $deg(\psi_1(f)) \leq deg(f)$. We express the standard homogeneous decomposition of $\psi_1(f)^+$ as follows:

$$\psi_1(f)^+ = ax^iy^j + \sum_{(k,l)\in S_0(f)} a_{kl}x^ky^l, \tag{4.27}$$

where $k+l < i+j$ for all $(k,l) \in S_0(f)$. Let $g = \psi_1(f)$.

Suppose $k \leq i, l \leq j$ *for all* $(k,l) \in S_0(f)$. Then g is stable, and we may choose $\psi = \psi_1$.

Suppose g *is not stable.* Then there exists some $(k_0, l_0) \in S_0(f)$ with either $k_0 > i$ or $l_0 > j$. Say $k_0 > i$ holds.

Choose a $\sigma > 0$ with $\sigma k_0 + l_0 > \sigma i + j$. Then $g^+_{(1,\sigma)} \neq g^+$. Put $\tau = \inf\{\sigma > 1 | g^+_{(1,\sigma)} \neq g^+\}$. Then exactly as in the proof of Lemma 4.1.2, we obtain $\tau = p/q > 1, (p,q) = 1$ and $g^+_{(1,\tau)} = g^+_{(q,p)}$ is not a monomial. We show that it is impossible that $deg_w(g) \leq deg(w) = q+p$ with $w = (q,p)$. Suppose the contrary. Then $qi+pj \leq q+p$. *Suppose both* i *and* j *greater than* 1. Then $i = j = 1$. So $deg(f) = 2 < 3$. *Suppose* $i = 0$. Then $pj \leq q+p$. But $j = deg(f) \geq 3$ implies that $p < q$ which contradicts $p > q$. *Suppose* $j = 0$. Then $i = deg(f)$. So there exists no $k_0 > i$. Therefore, by Corollary 3.2.4 with $w = (q,p)$, q must be 1 and $g^+_{(1,\tau)} = ay^j(y+bx^p)^i$ for some $a, b \in K^\times, j \geq 1$. Then there exists a $\psi_2 \in G, deg(\psi_2(g)) < deg(g) \leq k$. By induction hypothesis, we may find a $\psi \in G$ as desired.

4.3.2. Step 2.

From now on, we will assume that f is *stable* and write

$$f = ax^iy^j + \sum_{(k,l)\in S(f)} a_{kl}x^ky^l, \tag{4.28}$$

where $k+l<i+j, k\leq i, l\leq j$ for $(k,l)\in S(f)$, and $S(f)$ is a subset of $Supp(f)$. Our next task is *to reduce f to the special form (4.29) below.*

From the assumption $deg(f)=i+j\geq 3$, we know that either $i\geq 2$, or $j\geq 2$. Without loss of generality, we may assume that $i\geq 2$. By Corollary 3.2.5, (1) with $w=(1,0)$, and using $d_w(f)=i>deg(w)=1$, one knows that $i\neq j$. We may assume that $j>i$ [6]. Then by Corollary 3.2.5 (1) again, $f^+_{(1,0)}=x^i(y-\alpha)^{j'}F(y)$, where $F(y)\in K[y], deg(F)=j-j', i<j'\leq j$, and $F(\alpha)\neq 0$. Thus

$$f=x^i(y-\alpha)^{j'}F(y)+\sum_{(k,l)\in S_1(f)} a_{kl}x^ky^l, \tag{4.29}$$

where $S_1(f)$ is a subset of $S(f)$ and $k<i$ for $(k,l)\in S_1(f)$.

Therefore, one may assume that f is in the above form (4.29) and $j'>i\geq 2$.

4.3.3. Step 3.

Define $\psi_1\in G:(x,y)\to(x,y+\alpha)$, and let $f_1=\psi_1(f)$. Then by (4.29) we obtain

$$f_1=x^iy^{j'}F_1(y)+\sum_{(k,l)\in S_1(f)} a_{kl}x^k(y+\alpha)^l, \tag{4.30}$$

where $k<i$ for all $(k,l)\in S_1(f)$, $F_1(y)=F(y+\alpha)\in K[y]$, and $F_1(0)\neq 0$. We begin to *study f_1 in this step.*

Note that there is a positive real number $A>1$ such that for each $\rho\geq A$, $(f_1)^+_{(\rho,-1)}=bx^iy^{j'}$, where $b=F_1(0)$. For every $\rho>0$, write $d_\rho(f_1)=d_{(\rho,-1)}(f_1)$.

To study the form of $(f_1)^+_{(\rho,-1)}$, we need the following Lemma.

4.3.4. Lemma. *In either of the following two cases we have $f_1\in\Re$:*

(i) there exists $\rho>1$ such that $d_\rho(f_1)\leq\rho-1$;

(ii) there exists $\rho>A$ with $d_\rho(f_1)>\rho-1$, but $(f_1)^+_{(\rho_1,-1)}=(f_1)^+_{(\rho,-1)}$ for every $\rho_1\in(1,\rho)$.

Proof of 4.3.4. Choose and fix a $\rho_0>A$. Write the $(\rho_0,-1)$-homogeneous

[6] Otherwise, use the linear transformation $x\to y, y\to -x$.

decomposition of f_1 as follows:

$$f_1 = bx^i y^{j'} + \sum_{(k,l)\in S_2(f)} b_{kl}x^k y^l, \tag{4.31}$$

where $\rho_0 k - l < \rho_0 i - j' = d_{\rho_0}(f_1)$ for $(k,l) \in S_2(f)$ and $S_2(f) \subseteq Supp(f_1)$.

In case (i), $\rho k - l \le \rho - 1$ for $(k,l) \in S_2(f)$, and $\rho(k-1) < l - 1$. *If* $k = 0$, we have $l \ge 0$ since $f_1 \in K[x,y]$; *If* $k \ge 1$, then $k - 1 \le \rho(k-1) < l - 1$. So $k \le l$ for all $(k,l) \in S_2(f)$. Hence $f_1 \in \Re$.

In case (ii), (4.31) is the $(\rho,-1)$-homogeneous decomposition of f_1. For $(k,l) \in S_2(f)$, we have $\rho_1 k - l < \rho_1 i - j'$ for $\rho_1 \in (1,\rho)$. Let $\rho_1 \to 1$. We get $k - l \le i - j' < 0$. Hence $f_1 \in \Re$ also. Then Lemma 4.3.4 is proved.

Let us come back to the proof of Theorem 4.3.1-Step 3.

By 4.3.4, we can assume the existence of $\rho_0 > A$ with the following properties:

$$(f_1)^+_{(\rho_0,-1)} = bx^i y^{j'}, i < j', \tag{4.32}$$

$$d_{\rho_0}(f_1) > \rho_0 - 1, \tag{4.33}$$

and

$$(f_1)^+_{(\rho,-1)} \ne (f_1)^+_{(\rho_0,-1)} \tag{4.34}$$

for some $\rho \in (1,\rho_0)$. Put

$$\rho_1 = Sup\{\rho \in (1,\rho_0) | (f_1)^+_{(\rho,-1)} \ne (f_1)^+_{(\rho_0,-1)}\}.$$

By the same argument as in Lemma 4.1.2, we obtain

$$\rho_1 \in \mathcal{Q} \cap (1,\rho_0), \tag{4.35}$$

$$(f_1)^+_{(\rho_1,-1)} \ne (f_1)^+_{(\rho_0,-1)}, \tag{4.36}$$

and $(f_1)^+_{(\rho_1,-1)}$ is not a monomial.

4.3.5. Step 4.

We shall *study the relationship between certain invariants of f and of f_1*. The following Lemma is helpful for this purpose.

4.3.6. Lemma. Keep the notations as above. Then $(f_1)^+_{(\rho_1,-1)}$ contains the monomial $bx^iy^{j'}$.

Proof of Lemma 4.3.6. By the definition of ρ_1, we know that

$$(f_1)^+_{(\rho,-1)} = (f_1)^+_{(\rho_0,-1)} \tag{4.37}$$

for all $\rho \in (\rho_1, \rho_0]$. If $(f_1)^+_{(\rho_1,-1)}$ doesn't contains the monomial $bx^iy^{j'}$, then we may choose $\rho \in (\rho_1, \rho_0)$ close to ρ_1 such that $(f_1)^+_{(\rho,-1)}$ doesn't contain $bx^iy^{j'}$. This is impossible by (4.37).

We now continue the proof of Theorem 4.3.1-Step 4.

By Lemma 4.3.4, we may assume that $d_{\rho_1}(f_1) > \rho_1 - 1$. Then by Corollary 3.2.5, we obtain

$$(f_1)^+_{(\rho_1,-1)} = (x - \alpha y^{-\rho_1})^{v_1} y^{w_1} F_2(z), \tag{4.38}$$

where $\alpha \in K^\times, z = xy^{r_1/s_1}, \rho_1 = r_1/s_1, v_1 \in \mathcal{N}, 1 \leq v_1 < w_1$, and $F_2(z) \in K[z], F_2(\alpha) \neq 0$, with $deg(F_2) > 0$. Let $F_2(z) = \prod_{j=1}^n (z + \alpha_j)^{u_j}, n \geq 1$, and each $u_j \geq 1$. Then by Lemma 4.3.6, $v_1 + \sum u_j = i$. So

$$v_1 < i. \tag{4.39}$$

Note that w_1 may fail to be an integer.

We then rewrite f_1 in term of its $(\rho_1, -1)$-decomposition as follows:

$$f_1 = (x - \alpha y^{-r_1/s_1})^{v_1} y^{w_1} F_2(xy^{\rho_1}) + \sum_{(k,l)\in S_3(f)} b_{kl}(x - \alpha y^{-\rho_1})^k y^l, \tag{4.40}$$

where $l \in \frac{1}{s_1}\mathcal{Z}, \rho_1 k - l < \rho_1 v_1 - w_1$ for all $(k,l) \in S_3(f)$ and $S_3(f)$ is a finite set.

Now choose $\psi_2 \in G^{(s_1)}$ with $\psi_2(x) = x + \alpha y^{-r_1/s_1}$. Let $f_2 = \psi_2(f_1) \in R_{s_1}$. Then by the above formula (4.40),

$$f_2 = x^{v_1} y^{w_1} F_3(xy^{\rho_1}) + \sum_{(k,l)\in S_3(f)} b_{kl} x^k y^l \tag{4.41}$$

is the $(\rho_1, -1)$-decomposition of f_2, where $F_3(T) = F_2(T+\alpha) \in K[T]$, and $F_3(0) \neq 0$.

Before continuing the proof, we say a few words about the method of proof of this reduction theorem. The general idea is to decrease, recursively, the $(1,0)$-degree of the $(\rho, -1)$-leading term of f for some $\rho > 1$, under some automorphisms in $G^{(m)}$. We first reduce f to f_1 in the form (4.30) with $i < j' < j$ in Step 2, then reduce f_1 to f_2 as in (4.40) with $1 \leq v_1 < i$, and $v_1 < w_1$. Note that we had to expand $K[x,y]$ to R_m of section 3.3 in the reduction procedure from f_1 to f_2.

We will reduce f_2 in the following three steps.

4.3.7. Step 5.

Choose $1 < \tau_1 < \rho_1$, but close enough to ρ_1 such that $\tau_1 k - l < \tau_1 v_1 - w_1$ for $(k,l) \in S_3(f)$ and $d_{\tau_1}(f_2) > \tau_1 - 1$. We can do this since $d_{\rho_1}(f_2) = d_{\rho_1}(f_1) > \rho_1 - 1$. Then

$$(f_2)^+_{(\tau_1,-1)} = cx^{v_1}y^{w_1}, c = F_3(0) \neq 0. \tag{4.42}$$

Moreover, by the same argument as in Lemma 4.3.4 and since $v_1 < w_1$, one may assume the existence of some $\rho \in (1, \tau_1)$ with [7]

$$(f_2)^+_{(\rho,-1)} \neq (f_2)^+_{(\tau_1,,-1)}. \tag{4.43}$$

Define

$$\rho_2 = Sup\{\rho \in (1,\tau_1) | (f_2)^+_{(\rho,-1)} \neq (f_2)^+_{(\tau_1,-1)}\}.$$

By the same argument as in Lemma 4.1.2, we obtain

$$\rho_2 \in \mathcal{Q} \cap (1,\tau_1), \rho_2 = r_2/s_2, \tag{4.44}$$

$$(f_2)^+_{(\rho_2,-1)} \neq (f_2)^+_{(\tau_1,-1)}, \tag{4.45}$$

and $(f_2)^+_{(\rho_2,-1)}$ is not a monomial. Moreover, $(f_2)^+_{(\rho_2,-1)}$ contains the term $cx^{v_1}y^{w_1}$ by the same proof as in 4.3.6.

We shall use Prop. 3.3.8 in the present situation to determine the form of

[7] Otherwise $f_1 \in \Re^{(s_1)}$.

$(f_2)^+_{(\rho_2,-1)}$. Remember that in the case $E(\Delta_f) \neq Ker(\Delta_f)$ we have to check the condition $d_{\rho_2}(f_2) > \rho_2 - 1$.

4.3.8. Step 6.

Suppose for the moment that $N(\Delta_f) \neq Ker(\Delta_f)$, *or* $E(\Delta_f) \neq Ker(\Delta_f)$ *with* $d_{\rho_2}(f) > \rho_2 - 1$. [8] By Prop. 3.3.8, there are only two possibilities for $(f_2)^+_{(\rho_2,-1)}$ since $(f_2)^+_{(\rho_2,-1)}$ is not a monomial.

Case 1. $(f_2)^+_{(\rho_2,-1)} = dy^{t/s_2}(x - \beta y^{-r_2/s_2})^{v_1}$ for some $\beta \in K^\times$. In this case, $s_2 \mid s_1$ by 3.3.9. Therefore

$$(r_2, s_2) \in L(\rho_1), \tag{4.46}$$

where $L(\rho_1) = \{(r,s) : GCD(r,s) = 1, r, s \geq 1, \frac{r}{s} < \rho_1, s \mid s_1\}$ is a *finite* set with at most $[\rho_1 s_1^2]$ elements. Hence we can define $\psi \in G^{(s_1)}$ with $\psi(x) = x + \beta y^{-r_2/s_2}$ such that

$$(\psi(f_2))^+_{(\rho_2,-1)} = dx^{v_2}y^{w_2}, \tag{4.47}$$

where $v_2 = v_1, 1 \leq v_2 < w_2$. Here $w_2 = t/s_2 = w_1$. Put $f_3 = \psi(f_2)$. Note that $f_3 \in K_{s_1}[x, y]$.

Case 2. $(f_2)^+_{(\rho_2,-1)} = dy^{w_2}(x - \beta y^{-r_2/s_2})^{v_2}F_4(xy^{r_2/s_2})$ for some $d, \beta \in K^\times, 1 \leq v_2 < w_2$, $F_4[T] \in K[T]$ with $F_4(\beta) \neq 0$, and $deg(F_4(T)) \geq 1$.

In this case, we can define $\psi \in G^{(s_1 s_2)}$ with $\psi(x) = x - \beta y^{-r_2/s_2}$ such that

$$(\psi(f_2))^+_{(\rho_2,-1)} = dx^{v_2}y^{w_2}F_5(xy^{r_2/s_2}), \tag{4.48}$$

where $1 \leq v_2 < w_2$, and $v_2 < v_1$ since $deg(F_5) = deg(F_4) \geq 1$. Put $f_3 = \psi(f_2)$. Note that $f_3 \in K_{s_1 s_2}[x, y]$.

4.3.9. Step 7.

We consider the case $E(\Delta_f) \neq Ker(\Delta_f)$ and $d_{\rho_2}(f_2) \leq \rho_2 - 1$ in this step. In order to handle this situation, we use the following fact.

4.3.10. Lemma. *Let* f, g *be two elements in* R_m *with* $J(f, g) = \lambda g, \lambda \neq 0$. *Then for any* $\rho > 1, d_\rho(f) \geq \rho - 1$. *Moreover, if* $d_\rho(f) = \rho - 1$, *then* $J(f^+_{(\rho,-1)}, g^+_{(\rho,-1)}) =$

[8] Note that $T(\Delta_f) \neq Ker(\Delta_f)$ implies that $T(\Delta_{f_2}) \neq Ker(\Delta_{f_2})$.

$\lambda g^+_{(\rho,-1)} \neq 0$.

Proof of Lemma 4.3.10. Since $d_\rho(J(f,g)) \leq d_\rho(f) + d_\rho(g) - (\rho - 1)$, the first part is clear. Suppose $d_\rho(f) = \rho - 1$ and $J(f^+_{(\rho,-1)}, g^+_{(\rho,-1)}) = 0$. Then

$$d_\rho(J(f,g)) < d_\rho(f) + d_\rho(g) - (\rho - 1) = d_\rho(g), \tag{4.49}$$

which is impossible because $J(f,g) = \lambda g$. Hence $J(f^+_{(\rho,-1)}, g^+_{(\rho,-1)}) = \lambda g^+_{(\rho,-1)} \neq 0$.

We are now ready to finish the reduction step for f_2.

Suppose $E(\Delta_f) \neq Ker(\Delta_f)$ and $d_{\rho_2}(f_2) \leq \rho_2 - 1$. Then $d_{\rho_2}(f_2) = \rho_2 - 1$, and $J((f_2)^+_{(\rho_2,-1)}, g^+_{(\rho_2,-1)}) = \lambda g^+_{(\rho_2,-1)}$ for some $g \in R_{s_1}$ by 4.3.10. Note that $d_{\rho_2}(f) > 0$, and we may find such a g with $d_{\rho_2}(g) > 0$ because $d_{\rho_2}(f^n g) = n d_{\rho_2}(f) + d_{\rho_2}(g)$ for any Δ_f–eigenfunction g. By Prop. 3.3.5, we know that there are three possibilities for $(f_2)^+_{(\rho_2,-1)}$.

The first case in 3.3.5 for $(f_2)^+_{(\rho_2,-1)}$ is impossible because $(f_2)^+_{(\rho_2,-1)}$ is not a monomial by the choice of ρ_2. The second case in 3.3.5 for $(f_2)^+_{(\rho_2,-1)}$ implies that $s_2 \mid s_1$. Then $(r_2, s_2) \in L(\rho_1)$, and we are reduced to the above **Case 1.**

In case 3.3.5. (iii), we have $(f_2)^+_{(\rho_2,-1)} = axy(z+\alpha_1)\ldots(z+\alpha_s), \alpha_i \neq \alpha_j$, for $i \neq j, \alpha_i \neq 0$ for all i, and $z = xy^{\rho_2}$. We express the $(\rho_2, -1)$-homogeneous decomposition of f_2 as follows:

$$f_2 = axy(z+\alpha_1)\ldots(z+\alpha_s) + \sum_{(k,l)\in S_4(f)} c_{k,l} x^k y^l, \tag{4.50}$$

where $\rho_2 k - l < \rho_2 - 1$ for $(k,l) \in S_4(f)$, and $S_4(f)$ is a finite set.

To figure out the leading form of f_2, we consider the following two situations.

Suppose $k \geq 1$ for all $(k,l) \in S_4(f)$. We have

$$k - 1 \leq \rho_2(k-1) < l - 1 \tag{4.51}$$

for $(k,l) \in S_4(f)$. Then $f_2 \in \Re^{(s_1)}$, and we are done.

Suppose there exists $(k,l) \in S_4(f)$ *with* $k = 0$. Choose a τ_2 close enough to ρ_2 [9] such that $(f_2)^+_{(\tau_2,-1)} = cxy, c = a\prod_i \alpha_i$. Arguing as in Lemma 4.3.4, we may assume that there exists $\rho \in (1, \tau_2)$ with $(f_2)^+_{(\rho,-1)} \neq (f_2)^+_{(\tau_2,-1)}$. Define

$$\rho_3 = Sup\{\rho \in (1, \tau_2) | (f_2)^+_{(\rho,-1)} \neq (f_2)^+_{(\tau_2,-1)}\}.$$

Then by the same proof as in Lemma 4.1.2, we see that $(f_2)^+_{(\rho_3,-1)}$ is not a monomial. Moreover, $d_{\rho_3}(f_2) \geq \rho_3 - 1$ by 4.3.10.

If $d_{\rho_3}(f_2) > \rho_3 - 1$, we can apply the procedure as in the above proof in step 6, with ρ_3 replacing ρ_2.

Suppose $d_{\rho_3}(f_2) = \rho_3 - 1$. Then by Prop. 3.3.5, we have two possibilities for $(f_2)^+_{(\rho_3,-1)}$ (note that 3.3.5 (i) is impossible because $(f_2)^+_{(\rho_3,-1)}$ is not a monomial).

In case 3.3.5 (ii) for $(f_2)^+_{(\rho_3,-1)}$, we have $(r_3, s_3) \in L(\rho_1)$, where $\rho_3 = r_3/s_3$. Then we can reduce the number of the elements in the finite set $L(\rho_1)$.

We shall prove that the case 3.3.5 (iii) for $(f_2)^+_{(\rho_3,-1)}$ is impossible. To do it, we use a contradiction argument. Suppose $(f_2)^+_{(\rho_3,-1)} = bxy(\hat{z}+\beta_1)\dots(\hat{z}+\beta_t), \hat{z} = xy^{\rho_3}, \beta_j \neq 0, t \geq 1$ and $\beta_i \neq \beta_j$ for $i \neq j$. Consider the above $(\rho_2, -1)$-decomposition of f_2 in (4.50). For every $(k,l) \in S_4(f)$, if $k \geq 1$, we see that $\rho_2 k - l < \rho_2 - 1$. Hence

$$\rho_3 k - l < \rho_3 - 1 \tag{4.52}$$

because $\rho_3 < \rho_2$. It means that the monomials $c_{k,l}x^k y^l, (k,l) \in S_4(f)$, contained in the leading form of $(f_2)^+_{(\rho_3,-1)}$ satisfy $k = 0$, i.e., the x-degrees equal zero. On the other hand, among the leading terms of $(f_2)^+_{(\rho_2,-1)} = axy(z+\alpha_1)\dots(z+\alpha_s)$, we find

$$((f_2)^+_{(\rho_2,-1)})^+_{(\rho_3,-1)} = cxy. \tag{4.53}$$

Hence $(f_2)^+_{(\rho_3,-1)} = cxy+$ some terms of form $c_{0,l}y^l, (0,l) \in S_4(f)$. But, each monomial in $bxy(\hat{z}+\beta_1)\dots(\hat{z}+\beta_t)$ has x-degree at least 1. Therefore $(f_2)^+_{(\rho_3,-1)} = cxy$, and then 3.3.5. (iii) for $(f_2)^+_{(\rho_3,-1)}$ is impossible.

[9] For example, choose $\tau_2 < \rho_2$ and such that $\tau_2 k - l < \tau_2 - 1$ for $(k,l) \in S_4(f)$.

We have thus shown that for f_2, either $(r_2, s_2) \in L(\rho_1)$ and there exists $\psi \in G^{(s_1)}$ such that $\psi(f_2)^+_{(\rho_2,-1)} = dx^{v_2}y^{w_2}, 1 \leq v_2 = v_1, v_2 < w_2$, or there exists $\psi \in G^{(s_1 s_2)}$ with $\psi(f_2)^+_{(\rho_2,-1)} = dx^{v_2}y^{w_2}F_5(xy^{\rho_2}), 1 \leq v_2 < w_2, v_2 < v_1$. In the first case, $f_3 = \psi(f_2) \in R_{s_1}$. Hence we continue to reduce f_3 as above with f_2 replaced by f_3. Since $L(\rho_1)$ is a finite set, after finite many steps, we can reduce to the second case, that is, $1 \leq v_2 < v_1, v_2 < w_2$. At that time, we finish the reduction step for f_2. More precisely, for $f_2 \in R_{s_1}$, there exists $s_2 \geq s_1$ and $\psi_2 \in G^{(s_2)}$ such that either $\psi_2(f_2) \in \Re^{(s_2)}$ or there exists $\rho_2 > 1$ with

$$(\psi_2(f_2))^+_{(\rho_2,-1)} = a_2x^{v_2}y^{w_2}, a_2 \in K^\times, \tag{4.54}$$

and $1 \leq v_2 < v_1, v_2 < w_2$. Let $f_3 = \psi_2(f_2) \in R_{s_2}$.

4.3.11. Step 8.

Repeating the same procedure, for any $n \geq 2$, we can find $s_n \geq s_{n-1}$ and $\psi_n \in G^{(s_n)}$ such that either $\psi_n(f_n) \in \Re^{(s_n)}$ or there exists $\rho_n > 1$ with

$$(\psi_n(f_n))^+_{(\rho_n,-1)} = a_nx^{v_n}y^{w_n}, a_n \in K^\times, \tag{4.55}$$

and $1 \leq v_n < v_{n-1}, v_n < w_n$.

Note that each v_n is positive integer, and only finitely many positive integer less than i. Then the above reduction procedure must be stop after finitely many steps. That is, there exists $m \geq 1, \psi \in G^{(m)}$ such that $\psi(f) \in \Re^{(m)}$ as desired.

Then the theorem 4.3.1 is proved.

Since $\Re \subseteq \Re^{(m)}$ for all m, then $\Psi(f) \in \Re$ implies that $\Psi_m(\Psi(f)) \in \Re^{(m)}$ for $\Psi_m = identity$. In particular, we have shown that there exists $\Phi \in G^{(m)}G$ (defined in 3.3.1) such that $\Phi(f) \in \Re^{(m)}$.

We are going to study the structure of $\Lambda(\Delta_f)$. It is not difficult to see (see [8]) that $\Lambda(\Delta_f)$ is a semigroup with rank at most 2. Our following result asserts that $\Lambda(\Delta_f)$ is a semigroup with rank at most 1.

4.3.12. Proposition. *For any non-constant polynomial f, there exists $\rho_f \in K$ such that $\Lambda(\Delta_f) \subseteq \mathcal{Z}\rho_f$.*

Proof. If $E(\Delta_f) = Ker(\Delta_f)$, let $\rho_f = 0$. Otherwise, $E(\Delta_f) \neq Ker(\Delta_f)$. By Theorem 4.3.1, there exists a $m \geq 1, \Phi \in G^{(m)}G$ such that $\Phi(f) \in \Re^{(m)}$. Choose $g \in K[x,y]$ and $\lambda \in K^\times$ such that $\Delta_f(g) = \lambda g$. Let k denote the smallest integer such that $\Phi(f)$ has a non-zero component, say $\Phi(f)_1$, in $\Re^{(m)}(k)$. Let l denote the smallest integer such that $\Phi(g)$ has a non-zero component, say $\Phi(g)_1$, in $\Re^{(m)}(l)$. Then $k \geq 0$. By (4.26), we find

$$J(\Phi(f)_1, \Phi(g)_1) = \lambda \Phi(g)_1. \tag{4.56}$$

Then $k = 0$ and $\Phi(f)_1 \in K[x,y]$. Moreover, $\lambda \neq 0$ implies that $\Phi(f)_1 = \alpha xy + \beta$ for some $\alpha \in K^\times, \beta \in K$. We define $\phi \in G^{(1)}$ by $\phi(x) = x + \frac{\beta}{\alpha y}$, and $\Psi = \phi \circ \Phi \in G^{(m)}G$. Then

$$\Psi(f) = \alpha xy + \sum_{i=r}^{s} F_i(xy) y^{i/m}, \tag{4.57}$$

for some $\alpha \in K^\times, s \geq r \geq 1$, and $F_i(T) \in K[T]$.

For every $g \in E(\Delta_f, \lambda)$, $\lambda \in \Lambda(\Delta_f) - \{0\}$, let l be the smallest integer such that $\Phi(g)$ has a component $\Phi(g)_1 \in \Re^{(m)}(l)$. Then by (4.26), $J(\alpha xy, \Phi(g)_1) = \lambda \Phi(g)_1$. Therefore, $\lambda = \alpha l/m$. Put $\rho_f = \alpha/m$. Then Prop. 4.3.12 is proved.

4.3.13. Proposition. *Suppose $f \in K[x,y]$, and $\Lambda(\Delta_f) \neq 0$. Then $Ker(\Delta_f) = K[f]$, i.e., f is a closed polynomial.*

Proof. By Theorem 4.3.1, and the first part of the proof of Prop. 4.3.12, there exists $m \geq 1, \Psi \in G^{(m)}G$

$$\Psi(f) = \alpha xy + \sum_{i=r}^{s} F_i(xy) y^{i/m}, \tag{4.58}$$

for some $\alpha \in K^\times, s \geq r \geq 1$, and $F_i(T) \in K[T]$. Let $g \in Ker(\Delta_f)$. let l be the smallest integer such that $\Psi(g)$ has a component $\Psi(g)_1 \in \Re^{(m)}(l)$. By (4.26), we have $l = 0, \Psi(g)_1 = F(\alpha xy)$ for some $F(T) \in K[T]$. Put $h = g - F(f) \in Ker(\Delta_f)$. If $h \neq 0$, let l' be the smallest integer such that $\Psi(h)$ has a component $\Psi(h)_1 \in \Re^{(m)}(l')$. Then $l' > 0$ from the choice of $F(T)$. But $h \in Ker(\Delta_f)$ implies that $l' = 0$, as in the argument for g. This contradiction proves that $h = 0, g = F(f) \in K[f]$.

In view of the following Theorem 4.3.14, Theorem 4.3.1 is a weaker form of the Jacobian Conjecture.

4.3.14. Theorem. *If* $f \in \Re$, *and* Δ_f *has a slice* $g \in K[x, y]$, *then* $f = ay + b, a \in K^\times, b \in K$ *and* $K[x, y] = K[f, g]$.

Proof. Write $f = f_0(xy) + f_1(xy)y + \ldots + f_n(xy)y^n$. Suppose that $\Delta_f(g) = 1$. Considering the linear part of f. We know $f_1(T) \in a + TK[T], a \in K^\times$. Moreover, we may assume that $f_0(T) \in TK[T]$. Say, $f_0(T) \in bT + T^2K[T]$. So f has the form that $f = ay + bxy + y^2c(x, y)$. By Prop. 4.2.11, we know that $f = ay$. This completes the proof.

4.3.15. Remark. By Theorem 4.3.14, to prove the Jacobian Conjecture it suffices to prove that there exists $\psi \in G$ such that $\psi(f) \in \Re$ under the Jacobian condition. Hence, Theorem 4.3.1 in this section can be seen as solving a very special case of the Jacobian Conjecture.

Chapter 5

The Spectral Theory of Ordinary Derivations

The aim of this chapter is to develop the *spectral theory of ordinary derivations* by using results proved in chapter 3 and chapter 4. We shall prove theorem B and theorem C. We use freely the conventions and results of chapter 3 and chapter 4.

5.1 $\mathcal{G}_m$-Action on Eigenfunctions

The main purpose of this section is to prove

5.1.1. Theorem. *Let f be a non-constant polynomial in $K[x, y]$. Then there exists $\rho_f \in K$ such that $\Lambda(\Delta_f) = \mathcal{Z}\rho_f$.*

We recall the following theorem of Zariski for easy reference (See [54] or [40]).

5.1.2. Zariski's Theorem. *Let K be a field of characteristic zero, and let L be a subfield of $K(x_1, \ldots, x_n)$ containing K. If trans.deg$_K(L) \leq 2$, then the ring $L \cap K[x_1, \ldots, x_n]$ is finitely generated over K.*

We start by giving necessary materials for the proof of Theorem 5.1.1. The first and the most crucial is the following Prop. 5.1.3.

5.1.3. Proposition. *Let K be field of characteristic zero, A a K-domain and finitely generated K-algebra with Krull-dimension 1. Let δ be a non-zero K-derivation on A. Then $Ker(\delta)$ is a finite dimensional K-vector space.*

Proof. Since A is a finitely generated K-algebra, its Krull dimension equals the transcendence degree of its quotient field $Qt(A)$ over K. Hence

$$dim_K Qt(Ker\delta) \leq dim_K Qt(A) = 1, \tag{5.1}$$

where $Qt(Ker\delta)$ is the quotient field of $Ker(\delta)$. On the other hand, $Ker(\delta) = A \cap Qt(Ker(\delta))$. Then $Ker(\delta)$ is a finitely generated K-algebra by Theorem 5.1.2. There are two possibilities of $dim_K Qt(Ker\delta)$.

(i). *Suppose* $dim_K Qt(Ker\delta) = 1$. Then $Qt(A)$ is an algebraic extension of $Qt(Ker\delta)$. Since $\delta = 0$ on $Qt(Ker\delta)$, $\delta = 0$ on $Qt(A)$. This is impossible because δ is non-zero by assumption.

(ii). *Suppose* $dim_K Qt(Ker\delta) = 0$. Then $Qt(Ker\delta)$ is an algebraic extension of K. Since $Ker(\delta)$ is a finitely generated K-algebra, we know that $Ker(\delta)$ is a finitely generated K-module. Then 5.1.3 is proved.

Note that $Ker(\delta)$ is a subfield of $\overline{K}$ from the proof of Prop. 5.1.3.

5.1.4. Corollary. *Let δ be a non-zero K-derivation of $R = K[x, y]$ and $(g) = gK[x, y]$ a δ-invariant principal ideal of $K[x, y]$. Suppose that g is irreducible and $(\delta(R)) \not\subseteq (g)$, where $(\delta(R))$ is the ideal generated by the image of δ. Then*

$$dim_K(Ker(\delta) + (g)/(g)) < \infty.$$

Proof. *Suppose* $Ker(\delta) = K$. Then the conclusion is obvious.

Suppose δ is ordinary. Let $W = K[x, y]/(g)$. Then δ induces a K-derivation ∂ on W and $(Ker(\delta) + (g)/(g))) \subseteq Ker(\partial)$. W is a K-domain since g is irreducible, and ∂ is non-zero because $(\delta(R)) \not\subseteq (g)$. Then 5.1.4 follows from Prop. 5.1.3 immediately.

5.1.5. Proposition. *Let δ be a non-zero ordinary K-derivation of $R = K[x, y]$ that satisfies $(\delta(R)) \not\subseteq (h)$ for all non-constant polynomials $h \in R$. Let (g) be a δ-invariant principal ideal. Then, $Ker(\delta) \cap (g) \neq 0$.*

Proof. We may write $\delta(g) = ag$, for some $a \in K[x, y]$.

Suppose g is irreducible. By assumption, $(\delta(R)) \not\subseteq (g)$. Hence by Corollary 5.1.4,

$$dim_K(Ker(\delta)/(Ker(\delta) \cap (g))) < \infty. \tag{5.2}$$

But $dim_K Ker(\delta) = \infty$ since δ is ordinary. Hence $Ker(\delta) \cap (g) \neq 0$.

In the general case, we may write $g = g_1^{m_1} \dots g_n^{m_n}$, where the g_i are irreducible. coprime, and $m_i \geq 1$. Since $\delta(g) = ag$, we have $\delta(g_i) = a_i g_i$ for some $a_i \in K[x, y]$.[1] Then by the proof in the first part, there exists some h_i such that $g_i h_i \in Ker(\delta)$ for each i. Let $h = h_1^{m_1} \dots h_n^{m_n}$. Then $gh \in Ker(\delta)$. Then Prop. 5.1.5 is proved.

Proposition 5.1.6. *Suppose Δ_f has a non-zero eigenvalue. Let $R = K[x, y]$. Then $\Delta_f(R) \not\subseteq (h)$ for all non-constant polynomials $h \in R$.*

Proof. Write $\Delta_f(g) = \lambda g$ for some $\lambda \in K^\times$ and $g \in R$. Suppose $\Delta_f(R) \subseteq (h)$ for some polynomial $h \in K[x, y]$. We shall show that it is impossible by getting a contradiction.

We may write $f_x = ah, f_y = bh$, for some $a, b \in R$. Since $(f_x)_y = (f_y)_x$, we obtain

$$ah_y - bh_x = (b_x - a_y)h \in (h). \tag{5.3}$$

Since $J(f, g) = \lambda g$,

$$\lambda g = f_x g_y - f_y g_x = (ag_y - bg_x)h. \tag{5.4}$$

Hence $g = c_1 h$ where $\lambda c_1 = ag_y - bg_x$. We shall prove that $g = c_2 h^2$ for some $c_2 \in R$.

To do it, we note that $g = c_1 h$ implies that $\lambda c_1 = ag_y - bg_x = (a(c_1)_y - b(c_1)_x)h + c_1(ah_y - bh_x) \in (h)$. Then $g = c_2 h^2$ for some $c_2 \in R$.

Suppose $g = ch^m$, for some $c \in R$ for some $m \geq 2$. Note that $\lambda g = (ag_y - bg_x)h$. Since (from $g = ch^m$) $ag_y - bg_x = a(c_y h^m + mh^{m-1}h_y) - b(c_x h^m + mh^{m-1}h_x) = (ac_y - bc_x)h^m + mh^{m-1}(ah_y - bh_x) \in (h^m)$, we have $ag_y - bg_x = \lambda dh^m$, for some $d \in R$. Then $g = dh^{m+1}$. We have thus proved that, for any $n \geq 1, g = c_n g^n$ for some $c_n \in R$. This is impossible because g is a polynomial. Prop. 5.1.6 is proved.

[1] Because $\delta(gh) = agh, a \in R$, $g(\delta(h) - ah) = h\delta(g)$. Since g, h are coprime, $\delta(g) = bg, \delta(h) = ch$ for some $b, c \in R$.

5.1.7. Remark. If $f = xy^2$, then $\Delta_f(R) \subseteq (y)$. Hence Δ_f has no non-zero eigenvalues.

Now we are ready to prove Theorem 5.1.1.

5.1.8. Proof of theorem 5.1.1.

If Δ_f has no non-zero eigenvalues, put $\rho_f = 0$. From now on, we assume that Δ_f has non-zero eigenvalues.

Suppose $K = \overline{K}$. Let $\lambda \in \Lambda(\Delta_f)$ (see the notation in 1.1), with $\lambda \neq 0$. Assume $g \in E(\Delta_f, \lambda)$. Then (g) is a Δ_f-invariant principal ideal. By Prop. 5.1.5 and Prop. 5.1.6, $K[f] \cap (g) \neq 0$, i.e., there exists $h \in K[x,y]$ with $gh \in K[f]$. Then $\Delta_f h = -\lambda h, -\lambda \in \Lambda(\Delta_f)$. Therefore, $\Lambda(\Delta_f)$ is a subgroup of K^+. By Prop. 4.3.12, $\Lambda(\Delta_f) \cong \mathcal{Z}$.

In general, $\Lambda(\Delta_f) \subseteq \Lambda(\Delta_f, \overline{K})$, where $\Lambda(\Delta_f, \overline{K})$ is the set of Δ_f-eigenvalues on $\overline{K}$. By Prop. 5.1.5 and Prop. 5.1.6, $\Lambda(\Delta_f)$ is a subgroup of $\Lambda(\Delta_f, \overline{K})$. Since $\Lambda(\Delta_f, \overline{K}) \cong \mathcal{Z}$ by the first part, we have $\Lambda(\Delta_f) \cong \mathcal{Z}$. Hence $\Lambda(\Delta_f) = \mathcal{Z}\rho_f$ for some $\rho_f \in K$. Thus Theorem 5.1.1 is proved.

5.1.9. Definition. ρ_f in Theorem 5.1.1 is called *the least eigenvalue* of Δ_f. There are only two least eigenvalues, unique up to sign. We shall determine ρ_f in Prop. 5.4.2.

We shall turn to proving Theorem B and Theorem C. First, let us explain the relation between $\mathcal{G}_m$-actions and $\mathcal{Z}$-gradings, following closely the account given in [44].

5.1.10. $\mathcal{G}_m$-Action. Let X be an affine K-variety endowed with a $\mathcal{G}_m$-action τ. Then τ induces a homomorphism

$$\alpha = \tau^* : A \to A[t, t^{-1}] \tag{5.5}$$

with $\alpha(f) = \sum_{n \in \mathcal{Z}} f_n t^n$. Thus τ introduces a $\mathcal{Z}$-grading on $A = K[X] = \bigoplus_{n \in \mathcal{Z}} A_n$ of regular functions on X, where $A_n = \{f_n | f \in A\}$ consists of the quasi-invariants of weight n of τ. Vice versa, given a grading $A = \bigoplus_{n \in \mathcal{Z}} A_n$ of $A = K[X]$, one can define a $\mathcal{G}_m$-action τ on A by setting $\tau_\lambda(f_n) = \lambda^n f_n$ for $f_n \in A_n, n \in \mathcal{Z}$, and extending it to the whole algebra A in a natural way.

Therefore, Theorem 5.1.1 asserts that there is a $\mathcal{G}_m$-action on $E(\Delta_f)$ given by $t.g = t^n g$, if $g \in E(\Delta_f, n\rho_f)$. Moreover, this action is mixed (see the notation in 1.5), because $A_n \neq 0$ if and only if $A_{-n} \neq 0$ for all $n \in \mathcal{Z}$. To determine all eigenfunctions, we shall consider this problem in a framework considered by Miyanishi [32].

5.1.11. We recall briefly the main points in Miyanishi's results. The exposition follows closely Miyanishi's original paper [32].

Let δ be an ordinary K-derivation on $K[x, y]$. Recall that an element $u \in R = K[x, y]$ is a *δ-integral factor* in the sense of Miyanishi [32] if there exists an element $g \in R$ such that $\delta(g) = ug$. Such a g is called a *δ-integral element.* Write X_δ for the set of all δ-integral factors. Similarly, $g \in Qt(R) = K(x, y)$ is a δ-integral element in $K(x, y)$ if $\frac{\delta(g)}{g} \in R$, and then $\chi(g) = \frac{\delta(g)}{g}$ is called a δ-integral factor w.r.t. $K(x, y)$. The set $\overline{X_\delta}$ of all δ-integral factors w.r.t. $K(x, y)$ in $K(x, y)$ is an abelian group under addition. It is not hard to see that $\overline{X_\delta} = X_\delta - X_\delta$ [2], see [32], Lemma 1.3. Let A_δ be the subalgebra of $K[x, y]$ generated by all δ-integral elements. Then by [32], Lemma 1.3, A_δ is generated by invertible elements of R and those δ-integral elements which are prime in R (called *irreducible δ-integral elements*). When $\delta = \Delta_f$, we write A_f, X_f to denote $A_{\Delta_f}, X_{\Delta_f}$, respectively.

Let $\phi : V = SpecR \to C$ be a rational mapping onto a smooth algebraic curve C. Then ψ is defined outside a (possibly empty) finite set $\sum$ of V, that is, $\psi^0 := \psi|_{V-\sum} : V - \sum \to C$ is a morphism. We say that ψ is a *pencil* if ψ^0 is surjective and general fibres of ψ are irreducible and reduced. Let δ be a K-derivation of $R = K[x, y]$. If there exists a pencil $\psi : V \to C$ such that $\delta(I_F) \subseteq I_F$, where I_F signifies the defining ideal of some generic fiber F of ψ, we say δ is a *derivation of fibred type* in the sense of [32].

The structure of X_f follows from the next proposition.

5.1.12. Proposition. *Let f be a closed polynomial. Assume $\Delta_f(R) \not\subseteq (h)$ for all non-constant polynomials $h \in R = K[x, y]$. Then $\overline{X_f} = X_f$ is a finitely generated free abelian group and A_f is a finitely generated K-algebra.*

---

[2] $A - B = \{a - b : a \in A, b \in B\}$ for any two sets A and B.

Proof. Since f is closed, $Ker(\Delta_f) = K[f]$, and $K(f)$ is algebraically closed in $K(x, y)$ by 2.2.1. Therefore, δ is a derivation of fibred type in the sense of [32]. By [32], Prop. 2.8, $\overline{X_f}$ is finitely generated based on the observation that only finitely many polynomials of the form $f+c, c \in R$ are reducible if f is closed.[3] Since A_δ is generated by those finitely many irreducible δ-integral elements up to the multiplication by invertible elements of $K[x, y]$, A_δ is finitely generated as a K-algebra. It remains to prove that X_f is a group and hence $X_f = \overline{X_f}$. Suppose $\Delta_f(g) = tg$ for some $t, g \in R$. $I = (g)$ is a Δ_f-invariant principal ideal. By Prop. 5.1.5, $Ker(\Delta_f) \cap I \neq 0$. Hence there exists $h \in R$ such that $\Delta_f(gh) = 0$. Hence $\Delta_f(h) = -th, -t \in X_f$. Moreover, X_f is a subgroup of K^+. Hence X_f is free.

5.1.13. Remark. $\Lambda(\Delta_f)$ is a subgroup of X_f, and $rank(\Lambda(\Delta_f)) \leq 1$ by Theorem 5.1.1. On the other hand, we will see that $rank(X_f)$ can be arbitrarily large by 5.5.8 below.

Regard $E(\Delta_f)$ as a $Ker[\Delta_f]$-module, via the action $a.b = ab$, for all $a \in Ker[\Delta_f], b \in E(\Delta_f)$. Then the spectral decomposition in case $K = \overline{K}$(see (2.16) in 2.1.10)

$$E(\Delta_f) = \bigoplus_{\lambda \in K} E(\Delta_f, \lambda)$$

is a $Ker[\Delta_f]$-module decomposition of $E(\Delta_f)$. Before proceeding, let us examine $E(\Delta_f)$ as a $Ker[\Delta_f]$-module and ask whether the above spectral decomposition of $E(\Delta_f, \lambda)$ is an irreducible decomposition, and if so, to calculate the multiplicity of the irreducible $Ker[\Delta_f]$-modules. This question is solved by the following Multiplicity One Theorem.

5.1.14. Multiplicity One Theorem. *Suppose f is a closed polynomial, and $K = \overline{K}$. Then for any $\lambda \in \Lambda(\Delta_f), E(\Delta_f, \lambda)$ is a free $K[f]$-module of rank 1.*

Since we shall need a slightly generalization of this theorem in proving Theorem C later, we prove a general result first.

5.1.15. Theorem. *Suppose A_1, A_{-1} are two non-zero linear subspaces of $K[x, y]$ with $A_1 A_{-1} \subseteq A_0 := K[f]$, where f is a non-constant polynomial of $K[x, y]$. Assume*

[3] We shall give a precise form of this fact in Prop. 5.3.2 below.

that $A_0A_1 \subseteq A_1, A_0A_{-1} \subseteq A_{-1}$. Then there exists $u \in A_1, v \in A_{-1}$ such that $A_1 = A_0u$ and $A_{-1} = A_0v$.[4]

Proof of Theorem 5.1.15. Choose and fix $u \in A_1 - \{0\}$ with the minimal total degree as a polynomial in $K[x, y]$ [5]. Similarly, choose and fix $v \in A_{-1} - \{0\}$ with the least degree property in A_{-1}. We shall show that $A_1 = A_0u$ and $A_{-1} = A_0v$. To show this, let $w \in A_1 - \{0\}$. Then $wv \in A_1A_{-1} = A_0 = K[f]$. So $wv = b(f)$ for some polynomial $b(T) \in K[T], b(T) \neq 0$. Moreover, there exists a polynomial $a(T) \in K[T]$ such that $uv = a(f)$ by the same reason. By the Euclidean algorithm, there are two polynomials $c(T), r(T) \in K[T]$ with

$$b(T) = c(T)a(T) + r(T), \tag{5.6}$$

where $deg(r(T)) < deg(a(T))$ or $r(T) = 0$. *Suppose* $r(T) \neq 0$. Let $u_1 = w - c(f)u \in A_1$. Since $vu_1 = vw - c(f)vu = b(f) - c(f)a(f) = r(f) \neq 0$, $u_1 \neq 0$. But

$$deg(v)deg(u_1) = deg(r(f)) = deg(f)deg(r(T)) \tag{5.7}$$

and

$$deg(v)deg(u) = deg(a(f)) = deg(f)deg(a(T)) \tag{5.8}$$

imply that $deg(u_1) < deg(u)$, which contradicts the choice of u. Hence $r(T) = 0$ and $b(T) = c(T)a(T)$. We have thus proved that $w = c(f)u \in K[f]u$. Then $A_1 \subseteq K[f]u$. On the other hand, $K[f]u \subseteq A_0A_1 \subseteq A_1$ by assumption. Thus $A_1 = K[f]u$. Exactly in the same way, one may prove that $A_{-1} = K[f]v$. This completes the proof.

5.1.16. Proof of Theorem 5.1.14. This result is clear if $\lambda = 0$ since f is a closed polynomial. Suppose $\lambda \in \Lambda(\Delta_f)$ and $\lambda \neq 0$. Then by Theorem 5.1.1, $-\lambda \in \Lambda(\Delta_f)$. Put $A_1 = E(\Delta_f, \lambda), A_{-1} = E(\Delta_f, -\lambda)$ and $A_0 = Ker[\Delta_f]$. By Prop. 4.3.13, $A_0 = K[f]$ since λ is a non-zero eigenvalue of Δ_f. Hence Theorem 5.1.14 follows immediately from Theorem 5.1.15.

[4] For any two sets A, B, let $AB = \{ab : a \in A, b \in B\}$.

[5] We say that u has the minimal (or least) degree property in A.

5.2 Spectral Theory on $K(x,y)$

Before discussing the spectral theory of polynomial functions, we discuss the spectral theory of rational function in this section, because the main results of this section follows from the results in last section directly.

Let $f \in R = K[x,y]$. The derivation Δ_f extends naturally to a K-derivation $\overline{\Delta_f}$ of $Qt(R) = K(x,y)$. We define $Ker(\overline{\Delta_f}), \Lambda(\overline{\Delta_f})$, and $E(\overline{\Delta_f})$ as in 1.1.

5.2.1. Proposition. *Suppose f_x, f_y have no common factor. Then* $Ker(\overline{\Delta_f}) = Qt(Ker(\Delta_f))$.

Proof. Suppose $u/v \in Ker(\overline{\Delta_f})$, and $u, v \in R$ with no common factor. Then $\Delta_f(u)v = \Delta_f(v)u$. Then $\Delta_f(u) = tu, \Delta_f(v) = tv$ for some $t \in R$. By Prop. 5.1.5, there exists a $w \in R$ with $\Delta_f(w) = -tw$. Now both uw and vw belong to $Ker(\Delta_f)$. We have thus proved Prop. 5.2.1.

5.2.2. Proposition. *Suppose f_x, f_y have no common factor. Then* $\Lambda(\overline{\Delta_f}) = \Lambda(\Delta_f) = \mathcal{Z}\rho_f$ *and* $E(\overline{\Delta_f}) = E(\Delta_f) \otimes_{Ker(\Delta_f)} Qt(Ker(\Delta_f))$.

Proof. Let $\lambda \in \Lambda(\overline{\Delta_f})$ such that $\overline{\Delta_f}(u/v) = \lambda(u/v)$, and u, v have no common factor. Then $\Delta_f(u) = tu, \Delta_f(v) = (t-\lambda)v$ for some $t \in R$. By Prop. 5.1.5, there exists $w \in R$ with $\Delta_f(w) = (\lambda - t)w$. Then $\Delta_f(uw) = \lambda(uw)$. So $\lambda \in \Lambda(\Delta_f)$, and $\frac{u}{v} = \frac{uw}{vw} \in Qt(Ker(\Delta_f))E(\Delta_f, \lambda)$. Then the proposition follows from Theorem 5.1.1 directly.

5.2.3. Proposition. *Suppose Δ_f has a non-zero eigenvalue and let ρ_f denote the least eigenvalue of Δ_f as in theorem 5.1.1. Then $E(\overline{\Delta_f}) = K(f)[u, u^{-1}]$, for some $u \in E(\Delta_f, \rho_f), u \neq 0$.*

Proof. By Theorem 5.1.14, $E(\Delta_f, \rho_f) = K[f]u, E(\Delta_f, -\rho_f) = K[f]v$ for some polynomials $u, v \in K[x,y]$ and $uv \in K[f]$. For any $w \in E(\Delta_f, n\rho_f), n \in \mathcal{Z}$, either $wu^n \in K[f]$, or $wv^n \in K[f]$. Then $E(\overline{\Delta_f})$ is a $K(f)$-algebra generated by u, v. Thus $E(\overline{\Delta_f}) = K(f)[u, u^{-1}]$.

5.3 Spectral Theory of $K^{[2]}$-Proof of Theorem B

The main purpose of this section is to prove theorem B. Since $\Lambda(\Delta_f) \neq \{0\}$ (see 1.1 for notation), f is a closed polynomial by Prop. 4.3.13 under the condition that $K = \overline{K}$.

5.3.1. Closed Polynomials. We first recall several useful facts about closed polynomials.

Suppose $K = \overline{K}$. Define $\sigma(f) = \{c \in K : f + c \text{ is reducible }\}$. For each $c \in \sigma(f)$, write

$$f + c = \prod_{i}^{n(f,c)} P_i^{t_i}, \tag{5.9}$$

where $t_i \geq 1$ and the P_i are irreducible polynomials in $K[x, y]$. $n(f, c)$ is called the *reducibility order* of f at c. Let

$$\zeta(f) = \sum_{c \in \sigma(f)} (n(f, c) - 1). \tag{5.10}$$

$\zeta(f)$ is called the *total reducibility order* of f. $\zeta(f)$ is closely related to the group X_f defined in 5.1.11 by 5.3.6 below.

The first result asserts that if f is a closed polynomial, only finitely many polynomials $f + c, c \in K$ are irreducible polynomials in $K[x, y]$.

5.3.2. Proposition. *Suppose f is a closed polynomial in $K[x, y]$. Then $\zeta(f) < \infty$.*

Proof. This was proved by Bertini in 1882 for $K = C$ and by Krull in general. See [46], Theorem 18.

This result is improved by the following theorem of Y.Stein (see [48]).

5.3.3. Proposition. *Suppose f is a closed polynomial. Then $\zeta(f) < deg(f)$.*

In proving his theorem, Y.Stein essentially uses the following fact. We shall review the main point of the proof of this fact because we shall need it in proving Theorem B and Theorem C later.

5.3.4. Lemma. *Let f be a closed polynomial. For $c_i \in \sigma(f), i = 1, \ldots, m$, write $f + c_i = \prod_{j=1}^{n_i} P_{i,j}^{t_{i,j}}, t_{i,j} \geq 1, P_{i,j}$ irreducible in $K[x, y]$, $n_i = n(f, c_i)$. Define*

$a_{i,j} \in K(x,y)$ by $\Delta_f(P_{i,j}) = a_{i,j}P_{i,j}$ for $c_i \in \sigma(f)$. Then $a_{i,j} \in K[x,y]$. Moreover, suppose $\sum_{i=1}^{m}\sum_{j=1}^{n_i} a_{i,j}c_{i,j} = 0$, where $c_{i,j} \in \mathcal{Z}$. Then there exists $d_i \in \mathcal{Z}$ for all $i = 1, \ldots, m$ such that $c_{i,j} = d_i t_{i,j}$.

Proof. We recall the main point of the proof. It is evident that $a_{i,j} \in K[x,y]$. Moreover, $\sum_{j=1}^{n_i} a_{i,j}t_{i,j} = 0$ for $i = 1, \ldots, m$.

We can write $c_{i,j} = s_{i,j}t_{i,j} + r_{i,j}$, for some $s_{i,j}$ and $r_{i,j}$ with $0 \le r_{i,j} < t_{i,j}$. For each i, one may assume that s_{i,n_i} is the least number in $\{s_{i,j}, j = 1, \ldots, n_i\}$. Then we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n_i-1}(s_{i,j} - s_{i,n_i})t_{i,j}a_{i,j} + \sum_{i=1}^{m}\sum_{j=1}^{n_i} r_{i,j}a_{i,j} = 0. \tag{5.11}$$

Therefore,

$$H := \prod_{i=1}^{m}\prod_{j=1}^{n_i-1} P_{i,j}^{t_{i,j}(s_{i,j}-s_{i,n_i})}\prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{r_{i,j}} \in K[f]. \tag{5.12}$$

Clearly, $H = \prod_{i=1}^{m}(f + c_i)^{t_i}$ for some $t_i \ge 0$. By calculating the $P_{i,j}$-valuation value of H we find that $r_{i,j} \equiv 0 \mod t_{i,j}$. So $r_{i,j} = 0$ and $s_{i,j} = s_{i,n_i}$ for all $j = 1, \ldots, n_i$. This proves Lemma 5.3.4.

The next Lemma is needed as well in proving Theorem B.

5.3.5. Lemma. *Suppose Δ_f has non-zero eigenvalues. Then $f+c$ has no multiple factors, for all $c \in K$.*

Proof. We may assume that $c = 0$ because $\Delta_{f+c} = \Delta_f$ for any $c \in K$. Fix $g \in K[x,y]$ such that $\Delta_f(g) = \lambda g, \lambda \in K^\times$ and suppose that f has a multiple factor. Write $f = P^2Q$, where P is an irreducible polynomial in $K[x,y]$. Since

$$\Delta_f(g) = -\Delta_g(f) = -\Delta_g(P^2Q) = -(\Delta_g(Q)P + 2\Delta_g(P)Q)P \tag{5.13}$$

and $\Delta_f(g) = \lambda g$, P divides g. Suppose that P^n divides g in R, for some $n \ge 1$. Write $g = P^n g_1$ for some $g_1 \in K[x,y]$. By using the formula

$$\Delta_f(g) = J(P^2Q, P^n g_1) = P^{n+1}(J(Q,g_1)P + nJ(Q,P)g_1 + 2QJ(P,g_1)), \tag{5.14}$$

we see that P^{n+1} divides g. Clearly, this is impossible. Hence $f + c$ has no multiple factor.

Proof of Theorem B.

We first assume that $K = \overline{K}$.

The proof consists of several steps.

5.3.6. Step 1. We shall prove that $rank(X_f) = \varsigma(f)$ in this step.

By Lemma 5.3.5, we can write the decomposition into irreducible factors as $f+c_i = P_{i,1}P_{i,2}\dots P_{i,n_i}, i = 1,\dots,m$. Define $a_{ij} \in K[x,y]$ by $\Delta_f(P_{i,j}) = a_{i,j}P_{i,j}$. By 5.1.12. X_f is generated by $a_{ij}, j = 1,\dots,n_i, i = 1,\dots,m$. Moreover,

$$\sum_{j=1}^{n_i} a_{i,j} = 0 \tag{5.15}$$

for $i = 1,\dots,m$. By Lemma 5.3.4, (5.15) for $i = 1,\dots,m$ are the only relations between $a_{i,j}$, that is, if $\sum_{i=1}^{m}\sum_{j=1}^{n_i} c_{i,j}a_{i,j} = 0, c_{i,j} \in \mathcal{Z}$, then $c_{i,j} = c_i$ for all $j = 1,\dots,n_i$, where $c_i \in \mathcal{Z}$ depends only on i, for all i. Hence $rank(X_f) = \sum_{i=1}^{m}(n_i - 1) = \varsigma(f)$.

5.3.7. Step 2.

By Theorem 5.1.14, $E(\Delta_f, \lambda) = K[f]g$ if $g \in E(\Delta_f, \lambda)$ is an element with the least degree property. We shall study the generator g of the $K[f]$-module $E(\Delta_f, \lambda)$ in this step. The assumption $K = \overline{K}$ is essential in this step.

To do this, first note that there exists a polynomial $h \in K[x,y]$ such that $gh = a(f)$ for some polynomial $a(T) \in K[T]$ by Theorem 5.1.1 and Prop. 4.3.13. Since $K = \overline{K}$, we can write $a(T) = \prod_i (T + \alpha_i)$ with finitely many $\alpha_i \in K$. Then all irreducible factors of g are among in the irreducible factors of the polynomials $f + c, c \in K$.

If g has an irreducible factor $f+c, c \notin \sigma(f)$, i.e., $g = (f+c)g_1$ for some $g_1 \in K[x,y]$, then $g_1 \in E(\Delta_f, \lambda)$ and $deg(g_1) < deg(g)$, which contradicts the choice of g with the least degree property. Hence any irreducible factors of g are among the irreducible factors of the polynomials $f + c_i, i = 1,\dots,m$. So, we can write $g = \prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{m_{i,j}}$ for some $m_{i,j} \geq 0$. Then $\lambda = \sum_{i=1}^{m}\sum_{j=1}^{n_i} m_{i,j}a_{i,j}$.

Choose $u \in E(\Delta_f, \rho_f), v \in E(\Delta_f, -\rho_f)$ with the least degree property such that

$E(\Delta_f, \rho_f) = K[f]u, E(\Delta_f, -\rho_f) = K[f]v$. Then

$$u = \prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{m_{i,j}}, v = \prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{s_{i,j}} \tag{5.16}$$

and

$$\rho_f = \sum_{i=1}^{m}\sum_{j=1}^{n_i} m_{i,j}a_{i,j}, -\rho_f = \sum_{i=1}^{m}\sum_{j=1}^{n_i} s_{i,j}a_{i,j} \tag{5.17}$$

for some $m_{i,j} \geq 0, s_{i,j} \geq 0$.

5.3.8. Step 3. Keep the notation as in 5.3.7. We shall prove that *$E(\Delta_f)$ is generated by u, v and f, as a K-algebra.*

To show this, let $n \geq 2$ and $w \in E(\Delta_f, n\rho_f)$ with the least degree property. Then by theorem 5.1.14, $E(\Delta_f, n\rho_f) = K[f]w$, and by 5.3.7, w is of the form $w = \prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{n_{i,j}}$, for some $n_{i,j} \geq 0$. Then by (5.17), we obtain

$$\sum_{i=1}^{m}\sum_{j=1}^{n_i} n_{i,j}a_{i,j} = \sum_{i=1}^{m}\sum_{j=1}^{n_i} nm_{i,j}a_{i,j}. \tag{5.18}$$

By Lemma 5.3.4, there exist integers $d_i, i = 1, \ldots, m$, such that

$$n_{i,j} - m_{i,j}n = d_i t_{i,j}, i = 1, \ldots, m, j = 1, \ldots, n_i. \tag{5.19}$$

Therefore

$$w = \prod_{i=1}^{m}(f + c_i)^{d_i}u^n. \tag{5.20}$$

We shall show that, for any $i = 1, \ldots, m$, there exists some j_1 such that $m_{i,j_1} = 0$. *Suppose the contrary.* Then $m_{i,j} \geq 1$, for all $j = 1, \ldots, n_i$. Then u has a factor $f + c_i$ and then u is not the element of $E(\Delta_f, \rho_f)$ with the least degree. Hence $d_i = t_{i,j_1} \geq 0$. On the other hand, $t_{i,j_2} = 0$ for some j_2 by the same reason because w has the least property in $E(\Delta_f, n\rho_f)$. Hence $d_i = -m_{i,j_2}n \leq 0$. We have thus proved that $d_i = 0$ for every i. Then $w = \mu u^n$ for some $\mu \in K^\times$, i.e., $E(\Delta_f, n\rho_f) = K[f]u^n, n \geq 2$. Similarly, $E(\Delta_f, -n\rho_f) = K[f]v^n, n \geq 2$. Hence $E(\Delta_f)$ is generated by u, v, f as a

K-algebra.[6]

5.3.9. Step 4. In this step we shall prove Theorem B.1 under the condition $K = \overline{K}$.

Consider the K-morphism $\Phi : K[X, Y, Z] \to E(\Delta_f)$ by $\Phi(X) = u, \Phi(Y) = v, \Phi(Z) = f$. Then Φ is surjective by 5.3.8. Moreover, $Ker(\Phi)$ is a prime ideal, containing the irreducible polynomial $XY - a(Z)$, where $a(f) = uv$ for some polynomial $a(T) \in K[T]$. Hence $I = (XY - a(Z))$ by Krull's principal ideal theorem. Then, $E(\Delta_f) \cong K[X, Y, Z]/(XY - a(Z))$ as a K-algebra. We have thus proved Theorem B.1 with the condition $K = \overline{K}$.

5.3.10. Step 5. We shall prove Theorem B.3 under the condition $K = \overline{K}$.

Let $f = xy(xy^2 + 1)$. Then f is a closed polynomial because the leading term of f is x^2y^3 and $GCD(2, 3) = 1$. So $Ker(\Delta_f) = K[f]$. For any $c \in K^\times$, we see that $f + c$ is irreducible by direct calculation. Let $P_1 = x, P_2 = y, P_3 = xy^2 + 1, u = P_1^2P_2^3, v = P_1P_3^3$. Then $\Delta_f(u) = u, \Delta_f(v) = -v$. Note that P_3 doesn't divide u. By the proof in 5.3.8 above, u is the element of $E(\Delta_f, 1)$ with the least degree property. So $E(\Delta_f, 1) = K[f]u$. Similarly, $E(\Delta_f, -1) = K[f]v$. Now we shall show that ± 1 is the least eigenvalue of Δ_f. *Suppose not*, and let $\pm 1 = n\rho_f$ for some $n \geq 2$, and choose $w_1 \in E(\Delta_f, \rho_f), w_2 \in E(\Delta_f, -\rho_f)$ with the least degree. Then by the proof of Step 3, either $u = \mu w_1^n$ or $v = \mu w_2^n$ for some $\mu \in K^\times$. This is impossible by the above irreducible decompositions of u and v. Thus 1 is the least eigenvalues of Δ_f and then $E(\Delta_f) \cong K[X, Y, Z]/(XY - Z^3)$ by 5.3.9. In particular, $E(\Delta_f) \not\cong K^{[2]}$.

For a *general field of characteristic zero* we first have

5.3.11. Step 6.

Suppose $\Lambda(\Delta_f, K) \neq \{0\}$. We shall prove that $\Lambda(\Delta_f, K) = \Lambda(\Delta_f, \overline{K}) = \mathcal{Z}\rho_f$ for some $\rho_f \in K$ in this step, where for any extension field L of K, $\Lambda(\Delta_f, L) = \{\lambda \in L : \Delta_f(g) = \lambda(g) \text{ for some } g \in L[x, y]\}$.

[6]The fact that $f + c$ has no multiple factor is crucial in the proof of 5.3.8. As we will see later in Theorem C.2, if $f + c$ has no multiple factor, any $\mathcal{G}_m - \Delta_f$−domain has the same form as $E(\Delta_f)$ by the same proof. If $f + c$ has multiple factor, we can not show that the $\mathcal{G}_m - \Delta_f$−domain is generated by three elements. Instead we can prove that any $\mathcal{G}_m - \Delta_f$−domain is a finitely generated K-algebra in Theorem D.1.

In fact, $\Lambda(\Delta_f, K) \subseteq \Lambda(\Delta_f, \overline{K}) = \mathcal{Z}\rho_f$ for some $\rho_f \in \overline{K}$ by Theorem 5.1.1. Since $\Lambda(\Delta_f, K)$ is a group by Prop. 5.1.5 and Prop. 5.1.6, we have $\Lambda(\Delta_f, K) = n\mathcal{Z}\rho_f$ for some $n \geq 1$. So $\rho_f \in K$. We shall prove that $\rho_f \in \Lambda(\Delta_f, K)$. To show this, note that $\rho_f \in \Lambda(\Delta_f, \overline{K})$ which equivalent to the existence of $g \in \overline{K}[x, y]$ such that $J(f, g) = \rho_f g$. By writing the equation $J(f, g) = \rho_f g$ as a sequence of linear equations V of the coefficients of g, we know that V has non-trivial solution in $\overline{K}$. Since $f \in K[x, y]$ and $\rho_f \in K$, V is defined over K. Then V has non-trivial solution in K. In other words, there exists non-zero polynomial $g \in K[x, y]$ such that $J(f, g) = \rho_f g$. Hence $\rho_f \in \Lambda(\Delta_f, K)$. Then $\mathcal{Z}\rho_f \subseteq \Lambda(\Delta_f, K)$. Hence $\Lambda(\Delta_f, K) = \Lambda(\Delta_f, \overline{K}) = \mathcal{Z}\rho_f$.

5.3.12. Step 7. We shall show Theorem B.1 and B.3 for an arbitrary field of characteristic zero in this step.

For any $n \in \mathcal{Z}$, choose $u_n \in E(\Delta_f, n\rho_f)$ of the least degree. Then $E(\Delta_f, n\rho_f) = K[f]u_n$ by Theorem 5.1.15 since $\Lambda(\Delta_f, K) \neq \{0\}$. We shall prove that $u_n \in K[x, y]$ is also the least degree element in $E(\Delta_f, n\rho_f; \overline{K})$, which is the set of Δ_f-eigenfunctions on $\overline{K}[x, y]$ with eigenvalue $n\rho_f$. To prove this, choose a least degree element $w \in E(\Delta_f, n\rho_f; \overline{K})$. So $E(\Delta_f, n\rho_f; \overline{K}) = \overline{K}[f]w$ by Theorem 5.1.14. Then $u_n = a(f)w$ for some polynomial $a(T) \in \overline{K}[T]$. For any $\sigma \in Gal(\overline{K}/K)$, we have that

$$\frac{a(f)}{a(f)^\sigma} = \frac{w^\sigma}{w}. \tag{5.21}$$

Since $\Delta_f(w^\sigma) = (\Delta_f(w))^\sigma = n\rho_f w^\sigma$ and $deg(w) = deg(w^\sigma)$, $w^\sigma = \mu_\sigma w$ for some $\mu_\sigma \in \overline{K}^\times$. Note that $\sigma \to \mu_\sigma = \frac{w^\sigma}{w}$ is a 1-cocycle. By Hilbert's theorem 90, there exists $b \in \overline{K}^\times$ such that $\mu_\sigma = \frac{b}{b^\sigma}$. Then $\frac{(bw)^\sigma}{bw} = 1$, for all $\sigma \in Gal(\overline{K}/K)$. Hence $E(\delta, n\rho_f, \overline{K}) = \overline{K}[f](bw)$, and $bw \in E(\Delta_f, n\rho_f)$. Then bw must be a least degree element of $E(\Delta_f, n\rho_f)$. Hence $bw = cu_n$ for some $c \in K^\times$. So u_n is also the least degree element of $E(\Delta_f, n\rho_f, \overline{K})$ and $E(\Delta_f, n\rho_f, \overline{K}) = \overline{K}[f]u_n$. Therefore, we may choose $u, v \in K[x, y]$ and $E(\Delta_f, n\rho_f) = K[f]u^n$, or $K[f]v^n$ depending on $n \geq 1$ or $n \leq -1$. Then Theorem B.1 is proved by a similar argument in 5.3.9. Moreover. 5.3.10 also gives us an example with $E(\Delta_f) \cong K[X, Y, Z]/(XY - Z^3) \ncong K^{[2]}$ for $f = xy(xy^2 + 1)$, for any field K of characteristic zero.

Then $(B.1)$ and $(B.3)$ are proved. Our next theorem proves $(B.2)$.

5.3.13. Theorem. *Suppose $f + c = uv$ for some $u, v \in K[x, y], c \in K$ with $\Delta_f(u) = \lambda u, \lambda \in K^\times$. Then $E(\Delta_f) = K[u, v] \cong K^{[2]}, \lambda$ is the least eigenvalue of Δ_f, and $J(u, v) = -\lambda$. Conversely, if $E(\Delta_f) \cong K^{[2]}$, then $E(\Delta_f) = K[u, v]$ for some eigenfunctions $u, v \in K[x, y]$ with $J(u, v) \in K^\times$.*

Proof. By 5.3.9 (we use the same notations), $E(\Delta_f) \cong K^{[2]}$ if and only if $deg(a(T)) = 1$. If so, there exists $u \in E(\Delta_f, \rho_f), v \in E(\Delta_f, -\rho_f)$ with $uv = f + c$ for some constant $c \in K$. Hence $E(\Delta_f) = K[f, u, v] = K[u, v]$, where the first equality follows from 5.3.8 and the second one follows from $uv = f + c$. Moreover, by direct calculation, $J(u, v) = -\rho_f \in K^\times$.

On the other hand, if $f + c = uv$ with $\Delta_f(u) = \lambda u$, we shall show that λ is the least eigenvalue of Δ_f. *Suppose not.* Then by 5.3.8 in the above proof, $u = b(f)w^n$ for some $b(T) \in K[T], n \geq 2$ and $w \in E(\Delta_f, \rho_f)$ has the minimal degree property. By comparing the total degrees of both sides in the equation $u = b(f)w^n$, $b(f)$ has to be a constant. We may assume that $u = w^n$. By Lemma 5.3.5, there is no multiple factor in the decomposition of u because $f + c$ has no multiple factor. Then $n = 1$. This is a contradiction. Therefore $\lambda = \rho_f$, and u, v have the least degree property in $E(\Delta_f, \rho_f), E(\Delta_f, -\rho_f)$ respectively. Hence $E(\Delta_f)$ is a K-algebra generated by f, u, v (by 5.3.8) with $f + c = uv$. Thus $E(\Delta_f) = K[u, v] \cong K^{[2]}$. By calculation, $J(u, v) = -\lambda \in K^\times$.

Therefore, the proof of Theorem B is completed.

As an immediate corollary of Theorem 5.3.13, we obtain [7]

5.3.14. Theorem. *Suppose $f, g \in K[x, y]$ with $J(f, g) = \lambda \in K^\times$. Then $E(\Delta_{fg}) = K[f, g]$.*

We prove that $E(\Delta_f)$ is a $\mathcal{G}_m - \Delta_f$–domain (see 1.5 for notation) to conclude this section.

5.3.15. Proposition. *Suppose Δ_f has a non-zero eigenvalue. Then $E(\Delta_f)$ is a*

[7] To prove this fact was our initial motivation to develop the spectral theory in this paper. This result essentially means that the Jacobian Conjecture in two variables is equivalent to the assertion that every polynomial of $K[x, y]$ can be expressed as the linear combinations of eigenfunctions (Fourier expansion problem) under the Jacobian condition.

$\mathcal{G}_m - \Delta_f$*-domain.*

Proof. By Prop. 4.3.13, f is a closed polynomial. It suffices to check that $E(\Delta_f)$ is factorially closed in $K[x, y]$ with respect to $K[f]$. To check this, let $g \in K[x, y], a \in K[f]$ with $ag \in E(\Delta_f)$. We want to prove that $g \in E(\Delta_f)$.

By Theorem 5.1.1 and 5.3.11-5.3.12, we can write

$$ag = g_1 + \ldots + g_m \tag{5.22}$$

for some $m \geq 2, g_i \in E(\Delta_f, n_i\rho_f), i = 1, \ldots, m$ with $n_1 < n_2 < \ldots < n_m$. Then $\Delta_f^k(g_i) = (n_i\rho_f)^{k-1}g_i, \Delta_f^k(ag) = a\Delta_f^k(g)$ for all $k \geq 1, i = 1, \ldots, m$. Acting with Δ_f^k on (5.22) with $k = 1, 2, \ldots, m$, we obtain

$$a\Delta_f^k(g) = (n_1\rho_f)^{k-1}g_1 + \ldots + (n_m\rho_f)^{k-1}g_m. \tag{5.23}$$

By Vandermonde determinant, we can find some $h_i \in K[x, y]$ such that $g_i = ah_i$ for each $i = 1, \ldots, m$. Hence by (5.22), we obtain

$$g = h_1 + \ldots + h_m. \tag{5.24}$$

Moreover, we see that $\Delta_f(h_i) = n_i\rho_f h_i$ by using $g_i = ah_i$. Then $g \in E(\Delta_f)$ as desired.

5.4 The Least Eigenvalue of Δ_f

In this section, we determine the least eigenvalue of Δ_f by proposition 5.4.2. We also discuss the structure of $N(\Delta_f)$ in this section.

5.4.1. Let f be a non-constant polynomial of $K[x, y]$. After determining the eigenfunctions of Δ_f completely in the last section, we now determine its least eigenvalue. In geometry, when a differential operator acts on some function spaces on a compact manifold, its least eigenvalue, usually, has important geometric meaning.

5.4.2. Proposition. *ρ is the least eigenvalue of Δ_f if and only if there exists elements $h_1 \in E(\Delta_f, \rho), h_2 \in E(\Delta_f, -\rho)$ such that $deg(J(h_1, h_2))$ is minimal among*

$\{deg(J(g_1, g_2)) : g_1 \in E(\Delta_f, \lambda), g_2 \in E(\Delta_f, -\lambda), \lambda \in \Gamma(\Delta_f)\}$.

Proof. We may suppose that $K = \overline{K}$. By the same proof as in 5.3.12, the result then holds in the general case.

First of all, for any $g_1 \in E(\Delta_f, \lambda), g_2 \in E(\Delta_f, -\lambda)$, we have $J(g_1, g_2) \in Ker(\Delta_f) = K[f]$ by Jacobi's identity

$$J(f, J(g, h)) + J(g, J(h, f)) + J(h, J(f, g)) = 0 \tag{5.25}$$

for any $f, g, h \in K[x, y]$. Moreover, $J(g_1, g_2) = -a'(f)\lambda$ if $g_1 g_2 = a(f), a(T) \in K[T]$, where $a'(T)$ denote the derivative of the polynomial $a(T)$. Since the least degree element in $E(\Delta_f, n\rho)$ is the $|n|$-power of either of the least degree element in $E(\Delta_f, \rho_f)$ or the least degree element of $E(\Delta_f, -\rho_f)$ by 5.3.8, the stated assertion follows.

Now we give two results about the structure of $N(\Delta_f)$.

Theorem B asserts that if $E(\Delta_f) \neq Ker(\Delta_f)$, we can give a precise structure theorem about $E(\Delta_f)$. This means that the condition $E(\Delta_f) \neq Ker(\Delta_f)$ is a very strong restriction on the polynomial f (for example, f has to be a closed polynomial by Prop. 4.3.13). On the other hand, there are many (non-closed polynomials) $f \in K[x, y]$ with $N(\Delta_f) \neq Ker(\Delta_f)$ by the following proposition.

5.4.3. Proposition. *Suppose $f \in \Re(n)$ (we keep the notation of 3.3. and 4.3). Then $N(\Delta_f) \neq Ker(\Delta_f)$.*

Proof. We can write $f(x, y) = F(xy)y^n$, for some $F(T) \in K[T]$. Let $g = -\frac{xyf}{n}$. Then

$$\Delta_f(g) \neq 0, \Delta_f^2(g) = 0. \tag{5.26}$$

Hence $N(\Delta_f) \neq Ker(\Delta_f)$. Then Prop 5.4.3 is proved.

Suppose $E(\Delta_f) \neq Ker(\Delta_f)$, that is, $\Delta_f(g) = \lambda g$ for some $g \in K[x, y]$ and $\lambda \in K^{\times}$. Then $\Delta_g(f) = -\lambda g$, $\Delta_g^2(f) = 0$. Our next result tells us that $N(\Delta_g)$ strictly contains $K[f, g]$.

5.4.4. Proposition. *Let f, g be polynomials of $K[x, y]$ with $\Delta_f(g) = \lambda g$ and $\lambda \in K^{\times}$. Then $K[f, g]$ is strictly contained in $N(\Delta_g)$.*

Proof. If $g \in E(\Delta_f, \lambda), h \in E(\Delta_f, -\lambda r)$, $r \in \mathcal{Z}$, then

$$\Delta_f(\Delta_g(h)) = (1-r)\lambda\Delta_g(h) \tag{5.27}$$

by Jacobi's identity. Hence

$$\Delta_g(E(\Delta_f, -\lambda r)) \subseteq E(\Delta_f, (1-r)\lambda). \tag{5.28}$$

Hence, by using (5.28) repeatedly, we have

$$\Delta_g^n(E(\Delta_f, -\lambda n)) \subseteq E(\Delta_f, 0) = K[f]. \tag{5.29}$$

Since $K[f] \subseteq N(\Delta_g)$ (note that f is a local slice of Δ_g), we have

$$\sum_{n\geq 0} E(\Delta_f, -\lambda n) \subseteq N(\Delta_g). \tag{5.30}$$

On the other hand, $K[f,g] \subseteq \sum_{n\geq 0} E(\Delta_f, \lambda n)$. Therefore, $K[f,g] = N(\Delta_g)$ implies that $\sum_{n\geq 0} E(\Delta_f, -\lambda n) \subseteq \sum_{n\geq 0} E(\Delta_f, \lambda n)$. So $-\lambda \notin \Lambda(\Delta_f)$, which contradicts Theorem 5.1.1.

5.5 Mixed $\mathcal{G}_m$-Action on Subalgebras of $K[x,y]$-Proof of Theorem C

Let f be a closed polynomial of $K[x,y]$. Our purpose in this section is to classify those K-subalgebras A of $K[x,y]$ which can be endowed with a mixed $\mathcal{G}_m$-action with ring of invariant functions $K[f]$, and moreover are factorially closed in $K[x,y]$ relative to $K[f]$. As mentioned in 1.5, those K-subalgebras are called $\mathcal{G}_m - \Delta_f-$domains.

Proof of Theorem C.

We preserve the notations of 5.3 and suppose that $K = \overline{K}$.

5.5.1. First observation. Suppose A is a quasi $\mathcal{G}_m - \Delta_f-$domain. We can write $A = \oplus_{n\in\mathcal{Z}} A_n \subseteq K[x,y]$, with $A_0 = K[f]$. Moreover $\{n : A_n \neq 0\}$ is a subgroup of $\mathcal{Z}$

by assumption. So we may assume that each $A_n \neq 0$. For any $n \neq 0$, choose $u_n \in A_n$ with the least degree property. Since $A_{-n} \neq 0$, we see that $A_n = K[f]u_n$ by Theorem 5.1.15.

5.5.2. Second Observation. Now we suppose that A is a $\mathcal{G}_m - \Delta_f$-domain. By an argument as in 5.3.7, each element u_n in 5.5.1 is of the form $\prod_{i,j} P_{i,j}^{m_{i,j}}$ for some integers $m_{i,j}$. Let $B_t = \{\sum_{i=1}^{m}\sum_{j=1}^{n_i} a_{i,j}m_{i,j} \in X_f : m_{i,j} \geq 0, \prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{m_{i,j}} \in A_n \text{ for some } n \in \mathcal{Z}\}$. For any $b = \sum_{i=1}^{m}\sum_{j=1}^{n_i} a_{i,j}m_{i,j} \in B_t$, define $\Phi(b) = n$ if $\prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{m_{i,j}} \in A_n$. We shall prove that Φ is well defined and $B_t \cong \mathcal{Z}$ under Φ.

Suppose $b = \sum_{i=1}^{m}\sum_{j=1}^{n_i} a_{i,j}m_{i,j} = \sum_{i=1}^{m}\sum_{j=1}^{n_i} a_{i,j}m'_{i,j}$. We obtain

$$m_{i,j} = m'_{i,j} + d_i t_{i,j}, \forall i, j \tag{5.31}$$

for some integers d_i by Lemma 5.3.4. Then

$$\prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{m_{i,j}} = \prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{m'_{i,j}} \prod_{i=1}^{m}(f + c_i)^{d_i}. \tag{5.32}$$

If $\prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{m_{i,j}} \in A_n$, then

$$\prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{m'_{i,j}} \prod_{d_i>0} (f + c_i)^{d_i} \in A_n. \tag{5.33}$$

Since A is factorially closed in $K[x,y]$ relative to $K[f]$, $\prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{m'_{i,j}} \in A$. By using $A_0 = K[f]$ and the direct sum decomposition of $\prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{m'_{i,j}}$ in A, we know that $\prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{m'_{i,j}} \in A_n$. Conversely, if $\prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{m'_{i,j}} \in A_n$, we can prove that $\prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{m_{i,j}} \in A_n$ by the same argument. Hence Φ is well defined.

Φ is onto by the assumption in 5.5.1. Moreover, Φ is injective.[8] We have thus prove that Φ is an isomorphism of groups. Then $B_t \cong \mathcal{Z}$.

5.5.3. Proof of C.1.

[8] Suppose $\prod_{i=1}^{m}\prod_{j=1}^{n_i} P_{i,j}^{m_{i,j}} \in A_0 = K[f]$. We have $\sum_{i=1}^{m}\sum_{j=1}^{n_i} a_{i,j}m_{i,j} = 0$.

There exists $\lambda \in K[x,y]$ with $\mathcal{A} = \mathcal{Z}\lambda$ by 5.5.2. We shall show that

$$A_n = \{g : \Delta_f(g) = n\lambda g\}.$$

Let $\lambda = \sum_i \sum_j a_{i,j} m_{i,j}(0)$. This means that $\prod_i \prod_j P_{i,j}^{m_{i,j}(0)} \in A_m$ for some m. We may assume that $m \geq 0$. We shall show that $m = 1$. Suppose $m \geq 2$. Choose $\prod_{i,j} P_{i,j}^{m_{i,j}} \in A_1$ by the assumption that each $A_n \neq 0$. Then $\prod_{i,j} P_{i,j}^{mm_{i,j}} \in A_m$. Since $\prod_{i,j} P_{i,j}^{m_{i,j}(0)} \in A_m$ by the definition of λ, we have (by Theorem 5.1.15)

$$a(f) \prod_{i,j} P_{i,j}^{mm_{i,j}} = b(f) \prod_{i,j} P_{i,j}^{m_{i,j}(0)} \tag{5.34}$$

for two polynomials $a(T), b(T) \in K[T]$. Hence $\sum_{i,j} a_{i,j} mm_{i,j} = \sum_{i,j} a_{i,j} m_{i,j}(0) = \lambda$. But $\sum_{i,j} a_{i,j} m_{i,j} \in \mathcal{A} = \mathcal{Z}\lambda$. This is impossible. Hence we have shown that $m = 1$.

For any $g \in A_n, g \in K[f]u_n$ by 5.5.1, and $u_n = \prod_{i,j} P_{i,j}^{u_{i,j}}$ by the same argument as in 5.3.7 (note that $K = \overline{K}$). Then $\Delta_f(u_n) = n\lambda$ by the definition of λ. So $\Delta_f(g) = n\lambda g$, i.e., $A_n \subseteq E(\Delta_f, n\lambda), \forall n \in \mathcal{Z}$. Conversely, if $g \in E(\Delta_f, n\lambda)$, we want to prove that $g \in A_n$. To prove this, choose w_n as the least degree element of $E(\Delta_f, n\lambda)$. Note that $A_{-n} \subseteq E(\Delta_f, -n\lambda)$. Then by Theorem 5.1.15, $E(\Delta_f, n\lambda) = K[f]w_n$. Moreover, we may also write $w_n = \prod_i \prod_j P_{i,j}^{m_{i,j}}$ for some integers $m_{i,j}$. Therefore

$$m_{i,j} = nm_{i,j}(0) + d_i t_{i,j}, d_i \in \mathcal{Z}. \tag{5.35}$$

Put $a(f) = \prod_{d_i<0} (f + c_i)^{d_i}$. Then $a(f)^{-1} w_n \in K[f]A_n = A_n$. Since $a(f)^{-1} \in A_0$, and by the assumption that A is factorially closed in $K[x,y]$ relative to A_0, we have $w_n \in A$. Hence $w_n \in A_n$ by considering the direct sum decomposition of w_n in A. So $E(\Delta_f, n\lambda) \subseteq A_n$. Therefore $A_n = E(\Delta_f, n\lambda)$ for all $n \in \mathcal{Z}$. Hence Theorem C.1 is proved.

5.5.4. Proof of C.2.

For $n \in \mathcal{Z}$, choose $u_n \in A_n$ with the least degree property, and write

$$u_n = \prod_{i=1}^{m} \prod_{j=1}^{n_i} P_{i,j}^{m_{i,j}(n)}, m_{i,j} \geq 0. \tag{5.36}$$

Let $u = u_1, v = v_1$, and $uv = a(f)$ for some polynomial $a(T) \in K[T]$. For any $n \geq 2$, it suffices to prove that $u_n/u^n \in K^\times, u_{-n}/v^n \in K^\times$, this is similar to the proof of 5.3.8. Then $A \cong K[X, Y, Z]/(XY - a(Z))$ for some $a(T) \in K[T]$, as a K-algebra. Hence the proof of Theorem C.2 is completed.

5.5.5. Proof of C.3.

This is almost obvious. Let f be a closed polynomial, $B_0 = Ker(\Delta_f) = K[f]$. Since $B \neq B_0$, there exists $n \in \mathcal{Z}$, and a $g \in K[x, y]$ such that $\Delta_f(g) = n\lambda g$. Then as in the proof of Theorem 3.2.2 (ii), f, g are algebraically independent over K. Hence $trans.deg_K(B) = 2$ because $B \subseteq K[x, y]$. It remains to show that there exists a mixed $\mathcal{G}_m$-action τ on B with invariant functions equal $K[f]$. To show this, note that f_x and f_y have no common factor. Then, if $n \in \mathcal{Z}, n \neq 0$ such that $\Delta_f(g) = n\lambda g$, (g) is a Δ_f-invariant principal ideal of $R = K[x, y]$. By Prop. 5.1.5, $Ker(\Delta_f) \cap (g) \neq 0$, i.e., there exists $h \in K[x, y]$ such that $gh \in Ker(\Delta_f) = K[f]$. Hence $\Delta_f(h) = -n\lambda h$. We have thus proved the theorem C.3.

In order to complete the proof of Theorem C, we recall the following result of Lorenzini [29], Corollary 2.

5.5.6. Proposition. *Let $f(x, y) = \prod_{i=1}^{n} L_i(x, y)^{r_i}$, where the $L_i(x, y)$ are coprime linear polynomials, $n \geq 2, r_i \geq 1$. Assume at least two, say L_1 and L_2, have a common root and $GCD(r_1, r_2, \ldots, r_n) = 1$. Then $f + c$ is irreducible for any $c \in K^\times$.*

5.5.7. Proof of C.4.

Let $A \subseteq K[x, y]$ such that $A \cong K[X, Y, Z]/(XY - a(Z))$ as a K-algebra, where $a(T) \in K[T]$. We want to find a closed polynomial $f \in K[x, y]$ such that A is isomorphic to a quasi $\mathcal{G}_m - \Delta_f-$domain.

After a linear transformation, we may write $a(T) = T^{r_0} \prod_{i=1}^{m} (T + c_i)^{r_i}, c_i \in$

$K^\times, c_i \neq c_j$ for $i \neq j$ and $r_i \geq 1, GCD(r_0, r_1, \ldots, r_m) = 1$. Define

$$f(x,y) = xy\prod_{i=1}^{m}(x+c_i) + x.$$

Then by Prop. 5.5.6, we obtain: $f = xP_0, f + c_i = (x+c_i)P_i$, for some irreducible polynomials $P_i, i = 0, 1, \ldots, m$. Then $\zeta(f) \geq m+1$. On the other hand, $\zeta(f) \leq d\dot{e}g(f) - 1 = m+1$ by Prop. 5.3.3. Hence $\zeta(f) = m+1$. Hence,

(1) For any $c \neq \{0, c_1, \ldots, c_m\}$, $f + c$ is irreducible.

(2) f is a closed polynomial.

(3) f_x, f_y have no common factor.

Define, for $i = 0, 1, \ldots, m$,

$$u_i = \begin{cases} (x+c_i)^{r_i}, & \text{if i is even} \\ P_i^{r_i} & \text{otherwise} \end{cases} \tag{5.37}$$

and

$$v_i = \begin{cases} (x+c_i)^{r_i}, & \text{if i is odd} \\ P_i^{r_i} & \text{otherwise} \end{cases} \tag{5.38}$$

where $c_0 = c$. Let $u = \prod_{i=0}^{m} u_i, v = \prod_{i=0}^{m} v_i$. Then $uv = a(f)$ by definition of u and v. Since u, v have no common factor, $\Delta_f(u) = \lambda u$ with $\lambda \in K[x,y]$. Then $\Delta_f(v) = -\lambda v$. For $n \in \mathcal{Z}$, define $B_n = \{g \in K[x,y], \Delta_f(g) = n\lambda g\}$ and put $B = \bigoplus B_n$. Then u is an element of B_1 with the minimal degree property by the choice of u. Similarly, v is an element of B_{-1} with the minimal degree property. Hence B is generated by u, v, f as a K-algebra with proof as in 5.3.8. Then $B \cong A$. By Theorem C.3, B is a quasi $\mathcal{G}_m - \Delta_f-$domain.

Therefore, we have completed the proof of Theorem C.

5.5.8. Remark. Let $f = x(x+1)\ldots(x+d-2)y + x, d \geq 2$. Then by the same argument as in 5.5.7, we have $\zeta(f) = rank(X_f) = d-1$. Thus $rank(X_f)$ can be arbitrarily large.

5.5.9. Corollary. *Suppose $f + c$ is irreducible for every $c \in K$. Then Δ_f has no non-zero eigenvalues.*

Proof. Suppose $f + c$ is irreducible for every $c \in K$. Then by the proof of Theorem C, there exists no $\mathcal{G}_m - \Delta_f$—domain. Then by theorem B, Δ_f has no non-zero eigenvalues.

5.5.10. Example. We have found out polynomials $f \in K[x, y]$ such that $E(\Delta_f)$ is isomorphic to the surface $XY = P(Z)$, where $P(Z)$ has simple roots (e.g. $P(Z) = Z(Z+1)$) with $f = xy(x+1) + x$), and where $P(Z) = Z^n, n \geq 2$ (see 5.3.10 for $n = 3$.). But we don't know whether there exists a polynomial $f \in K[x, y]$ such that $E(\Delta_f)$ isomorphic to the surface $XY = a(Z)$, where $a(T)$ is any given polynomial in $K[T]$.

It is worthwhile to put forward the following problem for further study about closed polynomials

5.5.11. Existence Problem.

Given any m sequences of positive integers:

$$\alpha_1 = (a_{11}, \ldots, a_{1n_1}), \ldots, \alpha_m = (a_{m1}, \ldots, a_{mn_m})$$

with $GCD(a_{i1}, \ldots, a_{in_i}) = 1$ for $i = 1, \ldots, m$, to find a closed polynomial $f \in K[x, y]$ and distinct elements $c_1, \ldots, c_m \in K$ with

$$f + c_i = \prod_{j=1}^{n_i} P_{ij}^{a_{ij}}, i = 1, \ldots, m$$

where each P_{ij} and any $f + c, c \notin \{c_1, \ldots, c_m\}$, are irreducible.

5.5.12. Remark. The structure theorem about the eigenfunctions can be used to prove some derivations have no non-zero eigenfunctions. We explain our method by discussing Zaidenberg's polynomials to conclude this chapter.

Let $\sigma = xy + 1, \rho = x\sigma + 1$. Define, for $n \geq 1$,

$$f_n = 4q_n\rho^{n+1} + 1, g_n = 4q_n\rho^n + 1, \tag{5.39}$$

where $q_n = y\rho^n + \sigma(1 + \rho + \ldots + \rho^{n-1})$. We claim that for $n \geq 1$,

$$E(\Delta_{f_n}) = K[f_n], E(\Delta_{g_n}) = K[g_n]. \tag{5.40}$$

In fact, it is well known ([52], [4]) that for $n \geq 1$ and $c \neq -1$, $q_n, f_n + c, g_n + c$ are irreducible. Moreover,

$$\Delta_{f_n}(q_n) = (n+1)\rho^n J(\rho, q_n) q_n, \Delta_{f_n}(\rho) = -\rho^n J(\rho, q_n)\rho \tag{5.41}$$

Suppose $E(\Delta_{f_n}) \neq K[f_n]$. Then by Theorem B, there are two polynomials u, v such that $uv = a(f_n)$ for some polynomial $a(T) \in K[T]$, where u, v have the least degree properties. Then $uv = (f_n - 1)^r$ for some integer r. Therefore each irreducible factor of u is either q_n or p. Suppose $\Delta_{f_n}(u) = \lambda u$. By using (5.41), λ contains the factor $\rho^n J(\rho, q_n)$. It is impossible that $\lambda \in K^\times$. Hence $E(\Delta_{f_n}) = K[f_n]$. By a similar argument we have $E(\Delta_{g_n}) = K[g_n]$.

Chapter 6

Theorem D, Positive Characteristic and Questions

In this chapter we will prove Theorem D and give several results for fields of characteristic $p > 0$.

6.1 Finite Generation Properties

The aim of this section is to prove $(D.1)$.

We assume that $K = \overline{K}$ and preserve the conventions of the last chapter.

Let f be a closed polynomial of $R = K[x, y]$, and let A be a $\mathcal{G}_m - \Delta_f$-domain such that $A = \bigoplus_{n \in \mathcal{Z}} A_n$, with $A_0 = K[f]$. Then by 5.5.1, $A_n = K[f]u_n$, where u_n is an element of A_n with the least degree property. We shall prove (Theorem 6.1.2) that a $\mathcal{G}_m - \Delta_f$-domain is a finitely generated K-algebra. To prove it, we first examine the relations among $\{u_n, n \in \mathcal{Z}\}$.

We keep the notations as in 5.3.1. Write $\sigma(f) = \{c_i : 1 \leq i \leq m\}$, and

$$f + c_i = \prod_{j=1}^{n(f,c_i)} P_{i,j}^{t_{i,j}}$$

as the irreducible decomposition of $f + c_i$ for each $1 \leq i \leq m$. Then (see 5.5.2)we have $u_1 = \prod_{i,j} P_{i,j}^{s_{i,j}}$ for some $s_{i,j} \geq 0$. We shall determine u_n for $n \geq 2$ in terms of u_1.

6.1.1. Proposition. *Let* $d_{n,i} = min\{[\frac{ns_{i,j}}{t_{i,j}}]; j = 1, \ldots, n(f, c_i)\}$ *for each* $1 \leq i \leq m$. *Then* $u_n = u_1^n \prod_{i=1}^m (f + c_i)^{-d_{n,i}}$. u_{-n} *is determined from* u_{-1} *, similarly.*

Proof. Suppose we have integers $r_{i,j}, r'_{i,j}$ satisfying

$$\prod_{i,j} P_{i,j}^{r_{i,j}} = \prod_{i,j} P_{i,j}^{r'_{i,j}}, r_{i,j}, r'_{i,j} \geq 0 \qquad (6.1)$$

and that for each $i = 1, \ldots, m$, there exists some j with $r_{i,j} < t_{i,j}$, and $r'_{i,j} < t_{i,j}$. Then we prove that $r_{i,j} = r'_{i,j}$.

Actually, by Lemma 5.3.4, there exists integers e_i, $1 \leq i \leq m$, with $r_{i,j} = r'_{i,j} + e_i t_{i,j}$. By the assumptions on $r_{i,j}, r'_{i,j}$, we know that each $e_i = 0$. Hence $r_{i,j} = r'_{i,j}$ for all i, j.

Define $u'_n = u_1^n \prod_{i=1}^m (f + c_i)^{-d_{n,i}}$. Then $u'_n \in K[x, y]$ by the definition of $d_{n,i}$. By theorem C.1, there exists $\lambda \in R$ such that for all $n \in \mathcal{Z}$, each $A_n = E(\Delta_f, n\lambda)$. Note that each $f + c_i \in A_0 = E(\Delta_f, 0)$. Since $u_1 \in E(\Delta_f, \lambda)$, we find that $u'_n \in E(\Delta_f, n\lambda)$. By the first paragraph, u'_n has the least degree property in A_n.

Therefore, the elements of $A_n, n \geq 2$ with the minimal degree property are determined by u_1, unique up to a non-zero factor in K. Similarly, $u_{-n}, n \geq 2$ is determined by u_{-1} in the same way. We have thus proved Prop 6.1.1.

6.1.2. Theorem. *Let* A *be a* $\mathcal{G}_m - \Delta_f-$*domain. Then* A *is a finitely generated* K*-domain.*

Proof. By Prop. 6.1.1, A_n is generated by $\{u_n : n \in \mathcal{Z}\}$ with $u_0 = f$. It is enough to choose finitely many generators from $\{u_n, n \in \mathcal{Z}\}$. For each i, fix one j_0 such that $s_{i,j_0}/t_{i,j_0} \leq s_{i,j}/t_{i,j}$ for all $j \neq j_0$ and define $T_i = t_{i,j_0}$. Then $d_{n,i} = [ns_{i,j_0}/T_i]$ for each $n \geq 2$. Let $T = \prod_{i=1}^m T_i$. For any $n \geq 1$, write $n = kT + \alpha, \alpha = 0, \ldots, T-1$ for some integer k. Then

$$d_{n,i} = [(kT + \alpha)\frac{s_{i,j_0}}{T_i}] = kT_{i,0}s_{i,j_0} + [\alpha\frac{s_{i,j_0}}{T_i}], \qquad (6.2)$$

where $T_{i,0} = \frac{T}{T_i}$. So [1]

$$\begin{aligned} ord_{P_{i,j}}(u_n) &= ns_{i,j} - d_{n,i}t_{i,j} \\ &= k(Ts_{i,j} - T_{i,0}s_{i,j_0}t_{i,j}) + (\alpha s_{i,j} - [\alpha \frac{s_{i,j_0}}{T_i}]t_{i,j}). \end{aligned}$$

By the choice of T_i, we have $Ts_{i,j} - T_{i,0}s_{i,j_0}t_{i,j} \geq 0$, and $\alpha s_{i,j} - [\alpha \frac{s_{i,j_0}}{T_i}]t_{i,j} \geq 0$. Hence $\{u_n, n \geq 1\}$ is in the algebra generated by the elements:

$$\prod_{i,j} P_{i,j}^{Ts_{i,j} - T_{i,0}s_{i,j_0}t_{i,j}}, \prod_{i,j} P_{i,j}^{\alpha s_{i,j} - [\alpha \frac{s_{i,j_0}}{T_i}]}, \tag{6.3}$$

where $\alpha = 1, \ldots, T-1$. Moreover, for each $n \geq 1$,

$$u_n = c(\prod_{i,j} P_{i,j}^{Ts_{i,j} - T_{i,0}s_{i,j_0}t_{i,j}})^k (\prod_{i,j} P_{i,j}^{\alpha s_{i,j} - [\alpha \frac{s_{i,j_0}}{T_i}]}) \in A \tag{6.4}$$

for some $c \in K^\times$. For each $n \in \{1, 2, \ldots, T-1\}$, we have $k = 0$, and $\alpha = n$. Hence

$$\prod_{i,j} P_{i,j}^{\alpha s_{i,j} - [\alpha \frac{s_{i,j_0}}{T_i}]} \in A \tag{6.5}$$

On the other hand, for $n = T$, we have $k = 1$, and $\alpha = 0$. So

$$\prod_{i,j} P_{i,j}^{Ts_{i,j} - T_{i,0}s_{i,j_0}t_{i,j}} \in A \tag{6.6}$$

Then the plus part of A is contained in the subalgebra of A generated by the T elements in (6.3). The proof is similar for the minus part. Therefore A is generated by at most $2T+1$ generators as a K-algebra. Theorem 6.1.2 is then proved.

6.1.3. Proof of D.1. Let A be a $\mathcal{G}_m - \Delta_f$–domain. By Theorem C.1, there exists $\lambda \in K[x,y]$ such that for all $n \in \mathcal{Z}$, $A_n = E(\Delta_f, n\lambda)$. As in the proof of Prop. 5.2.3, we have that $A \otimes_{Ker(\Delta_f)} Qt(Ker(\Delta_f)) \cong K(f)[u, u^{-1}]$. Then $Qt(A) \cong K(f, u)$. Since A is a finitely generated K-algebra by Theorem 6.1.2, we know that A is then

[1] ord_P is the P–valuation of $K[x,y]$.

an affine rational surface. Theorem D.1 is proved. [2]

6.1.4. Remark. Suppose $A = \bigoplus_{n\in \mathcal{Z}} A_n$ is a finitely generated K-subalgebra of $K[x,y]$, where $A_0 \neq K, A_0 \neq A$, A_0 is normal and A is factorially closed in $K[x,y]$ relative to A_0. Moreover, we assume the associated $\mathcal{G}_m$-action on A is mixed. We shall show that A is a $\mathcal{G}_m - \Delta_f$—domain for some polynomial f. In fact, $Qt(A_0) \cap A = A_0$. Since A is finitely generated by assumption and $dim_K Qt(A_0) \leq 2$, A_0 is a finitely generated K algebra by Theorem 5.1.2. Moreover, every element of $A_n, n \neq 0$ is algebraically independent over A_0, so $dim_K Qt(A_0) = 1$, and $dim_K Qt(A) = 2$. Then by the same proof as in [40], Theorem 2.8, $A_0 = K[f]$ for some closed polynomial f. Hence A is a $\mathcal{G}_m - \Delta_f$—domain by definition.

6.2 $\zeta(f) = 1$

It is hard to classify, all $\mathcal{G}_m - \Delta_f$—domains. In this section we shall give an algebraic-characterization of all $A^2_K//\omega_N$, that is affine planes A^2_K divided by the action of a cyclic group ω_N, in terms of $\mathcal{G}_m - \Delta_f$—domains, where ω_N denotes the set of N—th roots of unity. As we shall prove in this section, those domains correspond with the situation when $\zeta(f) = 1$. Recall that $\zeta(f) = 1$ means that $\sigma(f) = \{c\}, c \in K$ and $f + c = P^\alpha Q^\beta$ where $\alpha, \beta \geq 1$ and both P and Q are irreducible polynomials of $K[x,y]$. Without loss of generality we assume that $c = 0$.

6.2.1. Proof of Theorem D.2.

Let $f = P^\alpha Q^\beta$, where $\alpha, \beta \geq 1$ and P, Q are irreducible polynomials. Since f is closed, $GCD(\alpha, \beta) = 1$.

Let A be a $\mathcal{G}_m - \Delta_f$—domain. Then A is finitely generated by 6.1.2. Note that

$$J(f,P) = -\beta P^{\alpha-1}Q^{\beta-1}J(P,Q)P, J(f,Q) = \alpha P^{\alpha-1}Q^{\beta-1}J(P,Q)Q. \tag{6.7}$$

Then by 5.5.2, there is $\lambda \in R$ such that for all $n \in \mathcal{Z}$ $A_n = E(\Delta_f, n\lambda)$. By easy calculation, $\lambda = (b\alpha - a\beta)P^{\alpha-1}Q^{\beta-1}J(P,Q)$ for some $a, b \in \mathcal{Z}$. We may assume that

[2]Since $K = \overline{K}$, we could also see it by Castelnovo's theorem.

$N := b\alpha - a\beta > 0$ because $A \neq A_0$. Since for any integer $t \geq 1$,

$$(b + \beta t)\alpha - (a + \alpha t)\beta = N, \tag{6.8}$$

we may assume that $a, b \geq 0$. Then $u = P^a Q^b \in A_1$. We may also assume that $0 \leq a \leq \alpha - 1$. This means that $u = P^a Q^b$ has the minimal degree property in A_1.

· We shall determine the generators of A. By Prop. 6.1.1, we know that $u_n = u^n / f^{d_n}$ is an element of A_n with the minimal degree property for $n \geq 2$, where $d_n = [na/\lambda]$. Write $b = m\beta + c, m \geq 0, 0 \leq c \leq \beta - 1$. Then $v = P^{(m+1)\alpha - a} Q^{\beta - c} \in A_{-1}$ has the minimal degree property. Then $v_n = v^n / f^{e_n}$ is an element in A_{-n} with the minimal degree property for all $n \geq 2$, where $e_n = [n(\beta - c)/\beta]$. Therefore A is generated by the following elements:

$$P^N, Q^N, P^\alpha Q^\beta, P^{ak - \alpha[ka/\alpha]} Q^{bk - \beta[ka/\alpha]}, P^{((m+1)\alpha - a)l - \alpha[l(\beta - c)/\beta]} Q^{(\beta - c)l - \beta[l(\beta - c)/\beta]},$$

where $k = 1, \ldots, \alpha - 1; l = 1, \ldots, \beta - 1$. Write $B = K[P, Q] \subseteq K[x, y]$, and define an ω_N-action on B as follows:

$$\zeta_N . P = \zeta_N^\beta P, \zeta_N . Q = \zeta_N^{-\alpha} Q,$$

where ζ_N is a primitive N-th root of unity. Then it is straightforward to see that all the above generators of A belong to B^{ω_N}. Hence $A \subseteq B^{\omega_N}$. Conversely, we shall prove that $B^{\omega_N} \subseteq A$. As a matter of fact, for any $g = \sum_{i,j} a_{i,j} P^i Q^j \in B, a_{i,j} \in K$, if $g \in B^{\omega_N}$, we have $\beta i - \alpha j \equiv 0 \mod N$. Therefore, $g \in \sum_{n \in \mathbb{Z}} E(\Delta_f, n\lambda) = A$. Hence $B^{\omega_N} \subseteq A$ and then $A = B^{\omega_N}$. So $(D.2)$ is proved.

6.2.2. Proof of D.3.

Let σ be an action of ω_N over $K[X, Y]$. Then [3] there exists $X_1, Y_1 \in K[X, Y]$ such

[3]This is a well known fact. For example, it is a consequence of the amalgamated product structure of the automorphism group of A_K^2. See [25] section 2.

that $K[X,Y] = K[X_1, Y_1]$ and the action σ is determined by

$$\zeta_N.X_1 = \zeta_N^{\gamma} X_1, \zeta_N.Y_1 = \zeta_N^{\delta} Y_1$$

for some γ, δ whose mod N classes are determined by σ. If σ has no isolated fixed points, then either $\gamma \equiv 0 \bmod N$ or $\delta \equiv 0 \bmod N$, and $K[X,Y]^{\omega_N} \cong K^{[2]}$. So we may assume that $1 \le \gamma \le N-1, 1 \le \delta \le N-1$. By the above proof of (D.2), this K-algebra $K[X,Y]^{\omega_N}$ is isomorphic to a $\mathcal{G}_m - \Delta_f -$domain [4] for $f = X_1^{\alpha} Y_1^{\beta} \in K[X,Y]$ with $\alpha = \delta, \beta = N - \gamma$. Hence the proof of Theorem D is completed.

6.2.3. Remark. It is not true that, in general, that any quasi $\mathcal{G}_m - \Delta_f -$domain has the form of $K[X,Y]^G$, where G is a finite (abelian) group acting on $K[X,Y]$ as the following example shows. Let $A = K[X,Y,Z]/(XY - Z(Z-1))$. Then A is a quasi $\mathcal{G}_m - \Delta_f -$domain by 5.5.8. Suppose that $Spec(A) \cong A_K^2//G$. Since A is smooth and any smooth surface of the form $A_K^2//G$ is the affine plane A_K^2, $Spec(A) \cong A_K^2$. This is impossible since $Pic(A) \cong \mathcal{Z}$.

6.3 Positive Characteristic Situation

We prove several related results in the positive characteristic case. Let K be a field of characteristic $p > 0$ and let D be a non-zero derivation of $K[x,y]$. Then the ring of constants $Ker(D)$ is a free $K[x^p, y^p]$-module of rank 1 or p. (See [27], [28] or [40]). We may also define $T(D), E(D)$ and $N(D)$ in this framework. We are interested in the case that $Ker(D)$ strictly contains $K[x^p, y^p]$, and we say D is an *ordinary derivation* under this condition. If so, $Ker(D)$ is a free $K[x^p, y^p]$-module of rank p. [5]

6.3.1. Proposition. *Let D be an ordinary derivation of $K[x,y]$ and let A be a K-subalgebra of $K[x,y]$ which contains $Ker(D)$. Assume that A is a normal domain. Then either $A = Ker(D)$ or $A = K[x,y]$.*

[4] Note that $ag \in K[X,Y]^{\omega_N}$ and $a \in K[f]$ imply that $g \in K[X,Y]^{\omega_N}$.

[5] In the positive characteristic case, the correct analogue of a locally nilpotent derivation should be locally finite iterated higher derivation in the sense of [33].

Proof. Let $B = Ker(D)$. Since

$$K[x^p, y^p] \subseteq B \subseteq A \subseteq K[x, y], \tag{6.9}$$

we know that either $Qt(A) = Qt(K[x, y])$ or $Qt(A) = Qt(B)$. Then either $A = B$ or $A = K[x, y]$ because both A and B are normal domains.

· **6.3.2. Remark.** For $A = T(D), E(D)$ or $N(D)$, it is enough to prove the normality of A to show that $A = K[x, y]$ in case D is ordinary.

In view of theorem 5.3.6, we make the following

6.3.3. Conjecture. *Suppose* $char(K) = p > 0$ *and* $f, g \in K[x, y]$ *with* $J(f, g) = \lambda \in K^\times$. *Then* $E(\Delta_{fg}) = K[x^p, y^p, f, g]$.

We recall a result of Nousiainen (see [5]).

6.3.4. Proposition. *Suppose* $char(K) = p > 0$ *and* $f, g \in K[x, y]$ *with* $J(f, g) = \lambda \in K^\times$. *Then* $K[x^p, y^p, f, g] = K[x, y]$.

By 6.3.4, the conjecture 6.3.3 gives the following positive characteristic analogue of the Jacobian Conjecture.

6.3.5. Conjecture. *Suppose* $char(K) = p > 0$ *and* $f, g \in K[x, y]$ *with* $J(f, g) = \lambda \in K^\times$. *Then* $E(\Delta_{fg}) = K[x, y]$.

We shall prove a positive characteristic analogue of theorem 5.1.1 to finish this thesis.

Define the differential operator δ by

$$\delta = \frac{\partial^{2(p-1)}}{\partial x^{p-1} \partial y^{p-1}} \tag{6.10}$$

For any $f \in K[x, y] \setminus K[x^p, y^p]$, let $a_f = \sum_{i=0}^{p-1} f^i \delta(f^{p-i-1})$. See [26] for its properties.

6.3.6. Theorem. *Suppose* Δ_f *has a non-zero eigenvalue. Then* $a_f \in K^\times$ *and* $\Lambda(\Delta_f)$ *(the set of all eigenvalues) is a finite cyclic group. Moreover* $\Lambda(\Delta_f) \cong \mathcal{Z}/p\mathcal{Z}$ *if* K *contains the* $(p-1)$*-th roots of unity.*

Proof. By using *Ganong's formula* [26], we have

$$\Delta_f^p = a_f \Delta_f \tag{6.11}$$

If $\Delta_f(g) = \lambda g$ for some $\lambda \in K^\times$, then $\Delta_f^p(g) = \lambda^p g$. So $a_f = \lambda^{p-1} \in K^\times$. Note that $\Lambda(\Delta_f)$ is a semigroup. Then the theorem follows.

Bibliography

[1] S.S.Abhyankar, Expansion techniques in algebraic geometry, Tata Institute Fundamental Research, Bombay, No 57, 1977.

[2] S.S.Abhyankar.,T.T.Moh, Embeddings of the line in the plane, J.Reine Angew.Math.276 (1975), 148-166.

[3] A.Bialynicki-Birula, Remarks on the action of an algebraic torus on k^n I and II, Bull.Acad.Pol.Sci.14 (1966), 177-181, 15 (1967), 123-125.

[4] E.A.Bartolo.,P.Cassou-Nouges.,I.L.Velasco, On polynomials whose fibres are irreducible with no critical points, Math.Ann.299 (1994), 477-490.

[5] H.Bass.,H.Connell.,D.Wright, The Jacobian Conjecture: Reduction of degree and formal expansion of the inverse, Bull.A.M.S.7 (1982), 287-330.

[6] H.Bass.H.,G.H.Meister, Polynomial flows in the plane, Adv.Math.55 (1978), 173-203.

[7] A.Czerniakiewicz, Automorphisms of a free associative algebra of rank 2. I, Trans.Amer.Math.Soc.160 (1971), 393-401.

[8] B.Coomes.,V.Zurkowski, Linearization of polynomial flows and spectra of derivations, J. Dynamics and Differential Equations 3 (1991), 29-66.

[9] D.Daigle, A necessary and sufficient condition for triangulability of derivations of $k[X, Y, Z]$, J. of Pure and Appl. Algebra 113 (1996), 297-305.

[10] D.Daigle, On some properties of locally nilpotent derivations, J. of Pure and Appl. Algebra 114 (1997), 221-230.

[11] D.Daigle, Homogeneous locally nilpotent derivations of $K[X, Y, Z]$, J. of Pure and Appl. Algebra 128 (1998), 109-132.

[12] D.Daigle.,G.Freudenburg, Locally nilpotent derivations over a UFD and an application to rank two locally nilpotent derivations of $k[X_1, \ldots, X_n]$, J. Algebra 204 (1998), 353-371.

[13] J.Dixmier, Sur les algébres de Weyl, Bull.Soc.Math.France 96 (1968), 209-242.

[14] A. Van den Essen, An algorithm to compute the invariant ring of a $\mathcal{G}_a$-action on an affine variety, J.Symbol.Comput.16 (1993), 551-555.

[15] A. Van den Essen, Locally finite and locally nilpotent derivations with applications to polynomial flows and polynomial morphisms, Proc.Amer.Math.Soc.116 (1992), 861-871.

[16] G.Freudenburg, Triangulability criteria for additive group actions on affine space, J. of Pure and Appl. Algebra 105 (1995), 267-275.

[17] G.Freudenburg, Actions of $\mathcal{G}_a$ on A^3 defined by homogeneous derivations, J. of Pure and Appl. Algebra 126 (1998), 169-181.

[18] G.Freudenburg, A counterexample to Hilbert's Fourteenth problem in dimension six, preprint, 1997.

[19] T.Kambayashi.,P.Russell, On linearizing algebraic torus actions, J. of Pure and Appl. Algebra 23 (1982), 243-250.

[20] S.Kaliman, On the Jacobian Conjecture, Proc.of A.M.S.117-1 (1993), 45-51.

[21] S.Kaliman.,L.Makar-Limanov, On the Russell-Koras contractible threefolds, J. Algebraic Geometry 6 (1997), 247-268.

[22] H.Kojima.,M.Miyanishi, On P.Roberts' counterexample to the fourteenth problem of Hilbert, J. of Pure and Appl. Algebra 122 (1997), 277-292.

[23] M.Koras.,P.Russell, $\mathcal{C}^*$−actions on $\mathcal{C}^3$: the smooth locus is not of hyperbolic type, CICMA reports, 1996-06.

[24] S.Kaliman.,M.Koras.,L.Makar-Limanov.,P.Russell, $\mathcal{G}_m$-Actions on $\mathcal{C}^3$ are linearizable, Electronic Research Announcements of AMS, 3 (1997), 63-71.

[25] H.Kraft.,G.Schwarz, Finite automorphisms of affine n-space, preprint, 1997.

[26] J.Lang, The divisor classes of the hypersurface $Z^{p^m} = G(X_1, \ldots, X_n)$ in characteristic $p > 0$, Trans.Amer.Math.Soc.278 (1983), 613-634.

[27] W.Li, Remarks on rings of constants of derivations, Proc.Amer.Math.Soc.107 (1989), 337-340.

[28] W.Li, Remarks on rings of constants of derivations II, Communication in Algebra 20-8 (1992), 2191-2194.

[29] D.Lorenzini, Reducibility of polynomials in two variables. J. Algebra 156 (1993). 65-75.

[30] L.Makar-Limanov, On groups of automorphisms of a class of surfaces, Israel Math.J.69-2 (1990), 250-256.

[31] L. Makar-Limanov, On the hypersurface $x + x^2y + z^2 + t^3 = 0$ in $\mathcal{C}^4$ or a $\mathcal{C}^3$-like threefold which is not $\mathcal{C}^3$, Israel Math.J.96 (1996), 419-430.

[32] M.Miyanishi, Vector Fields on Factorial Schemes, J. Algebra 173 (1995), 144-165.

[33] M.Miyanishi.,Y.Nakai, Some remarks on strongly invariant rings, Osaka J.Math.12 (1977), 1-17.

[34] M.Miyanishi.,T.Sugie, Affine surfaces containing cylinderlike open sets, J.Math.Kyoto Univ.20 (1980), 11-42.

[35] M.Nagata, On automorphism group of $k[X,Y]$, Lectures in Mathematics, Kyoto University, No.5, Kinokuniya Book Store Co.Ltd. Toyko, 1972.

[36] Y.Nakai, On locally finite iterative higher derivations, Osaka J.Math.15 (1978), 655-662.

[37] A.Nowicki, Rings and fields of constants for derivations in characteristic zero, J. of Pure and Appl. Algebra 96 (1994), 47-55.

[38] A.Nowicki, On the Jacobian equations $J(f,g) = 0$ for polynomials in $K[x,y]$, Nagoya Math.J.109 (1988), 151-157.

[39] A.Nowicki, Commutative basis of derivations in polynomial and power series rings, J. of Pure and Appl. Algebra 40 (1986), 279-302.

[40] A.Nowicki.,M.Nagata, Rings of constants for k-derivation in $k[x_1,\dots,x_n]$, J.Math.Kyoto Univ.28-1 (1988), 111-118.

[41] R.Rentschler, Opérations du groupe additif sur le plan affine, C.R.Acad.Sc.Paris.267 (1968), 384-387.

[42] P.Roberts, An infinitely generated symbolic blow-up in a power series ring and a new counterexample to Hilbert's Fourteen problem, J. Algebra 132 (1990), 461-473.

[43] P.Russell, On affine-ruled rational surfaces, Math.Ann.255 (1981), 287-302.

[44] P.Russell, Gradings of Polynomial Rings, Algebraic Geometry and its Application - In honour of S.Abhyankar (C.L.Bajaj ed.), Springer, 1994.

[45] P.Russell.,A.Sathaye, On Finding and Cancelling variables in $k[x,y,z]$, J. Algebra 57 (1979), 151-166.

[46] A.Schinzel, Selected Topics on Polynomials, The University of Michigan Press, 1982.

[47] V.Shpilrain.,J.Yu, Polynomial retracts and the Jacobian conjecture, MSRI Preprint, 1997-005.

[48] Y.Stein, The total reducibility order of a polynomial in two variables, Israel Math.J.68-1 (1989), 109-122.

[49] T.Sugie, Algebraic characterization of the affine plane and the affine 3-spaces, Topological methods in algebraic transformation groups, Progress in Math.80 (1989), 177-190.

[50] M.Suzuki, Propriétées topologiques des polynômes de deux variables complexes et automorphismes algébriques de l'espace C^2, J.Math.Soc.Japan 26 (1974), 241-257.

[51] D.Wright, On the Jacobian Conjecture, Illinois Journal of Mathematics 25 (1981), 423-440.

[52] M.Zaidenberg, On Ramanujam surface, C^{**}-families and exotic algebraic structures on $C^n, n \geq 3$, Trans. Moscow Math. Soc. 55 (1994), 1-56.

[53] M.Zaidenberg, Lectures on Exotic Algebraic Structures on Affine Spaces. math.AG/9801075, 10 June 1998.

[54] O.Zariski, Interprétations algébrico-géometriques du quatorzième probleme de Hilbert, Bull. Sci. Math.78 (1954), 155-168.