Relative Endoscopy for Unitary Symmetric Spaces

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Abstract

We study a class of symmetric space orbital integrals important for relating distinction of automorphic representations on unitary groups to distinction on general linear groups. In the first part, we verify a fundamental lemma for $\mathrm{U}_2 \times \mathrm{U}_2 \hookrightarrow \mathrm{U}_4$ via an explicit calculation, showing strong evidence that there is a general theory of endoscopy lurking here. In the second part, we determine a formula for the dimension of a family of affine Springer fibers associated to a symmetric space arising from the block diagonal embedding $\mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_{2n}$. These dimensions ought to be related to the transfer factors in a conjectural fundamental lemma.

Résumé

Nous étudions une classe d'intégrales orbitales pour les espaces symétriques qui sont utilisées pour relier a distinction de représentations automorphiques sur les groupes unitaires à distinction sur GL_n. Dans la première partie, nous vérifions un lemme fondamental pour $\mathrm{U}_2 \times \mathrm{U}_2 \hookrightarrow \mathrm{U}_4$ par un calcul explicite, montrant évidence qu'il cettes une théorie générale de l'endoscopie qui se cache ici. Dans la deuxième partie, nous déterminons une formule pour la dimension de fibres de Springer affines qui sont associées à un espace symétrique découlant du plongement diagonale par blocs $\mathrm{GL}_n \times \mathrm{GL}_n \hookrightarrow \mathrm{GL}_{2n}$. Ces dimensions doivent être liées à des facteurs de transfert d'un lemme fondamental conjecturale.

Dedicated to my grandmother Alenka Paquet.

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Chapter 1

Introduction

1.1 Distinction of Automorphic Representations

Let $H \leq G$ be reductive algebraic group over a number field $F/\mathbb{Q}$ and let $\mathbb{A}_F$ be the adeles of F. Roughly, we say that a cuspidal automorphic representation π of $G(\mathbb{A})$ is ***H*-distinguished** if there exists a cuspidal φ in the space of π such that the period integral

$$\int_{H(F)\backslash(H(\mathbb{A}_F)\cap {}^1G(\mathbb{A}_F))} \varphi(h)\mathrm{d}h$$

is nonzero. Here, ${}^1G(\mathbb{A}_F)$ is the **Harish-Chandra subgroup** defined by the kernel of the map $\mathrm{HC} : G(\mathbb{A}_F) \to \mathrm{Hom}(X^*(G)_{\mathbb{Q}}, \mathbb{R})$ given by $\mathrm{HC}(g)(\chi) = |\log(\chi(g))|$, which is introduced to make the integral converge; see [GW13] and [AGR93] for a lengthier discussion of convergence.

The spectral side of the Arthur-Selberg trace formula sums over isomorphism classes of automorphic representations. Jacquet and his school have introduced the *relative trace formula* whose spectral side is restricted to *distinguished automorphic representations*. Distinction by certain H is closely related to functorial images of automorphic representations from another group. Here is a classic:

1.1.1 Theorem ([Jac05])**.** *If E/F is a quadratic extension of number fields then a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_E)$ is a base change from $\mathrm{GL}_n(\mathbb{A}_F)$ if and only if it is distinguished by a unitary subgroup.*

We now describe a more recent result of a similar flavour. Let M/F be a quadratic extension of number fields and let τ the generator of $\mathrm{Gal}(M/F)$. Let H be a connected reductive group over F together with an automorphism $\sigma : H \to H$ and H^σ its fixed points under σ. We assume that H^σ is connected, as it will be in all cases we discuss.

Define $G = \mathrm{Res}_{M/F}H$. In particular, the automorphism $\tau \in \mathrm{Gal}(M/F)$ also acts on the R-points of G for any F-algebra R and the automorphism σ induces an automorphism on G that we also call σ. Let G^σ be the fixed points of σ and for $\theta = \sigma \circ \tau$, we let G^θ be the fixed points of θ.

Let π be a cuspidal automorphic representation of $H(\mathbb{A}_F)$, and suppose there exists a weak base change Π of π to $G(\mathbb{A}_F)$. Then [GW13] proposes the following:

General Principle. If Π is distinguished by G^σ and G^θ then there should exist an automorphic representation π' of $H(\mathbb{A}_F)$ nearly equivalent to π, and such that π' is distinguished by H^σ.

Here, we say that $\pi = \otimes'\pi_v$ and $\pi' = \otimes'\pi'_v$ are nearly equivalent if $\pi_v \cong \pi'_v$ for almost all places v. The main result of *loc. cit.* is a precise theorem following the form of this general principal as follows. Define U^σ to be a quasisplit unitary group in n variables attached to M/F and define $U = \mathrm{Res}_{E/F}U^\sigma$. For an F-algebra R we observe that

$$\begin{aligned} G(R) &= \mathrm{Res}_{M/F}\mathrm{Res}_{E/F}(U^\sigma)(R) \\ &= \mathrm{Res}_{E/F}(U^\sigma)(M \otimes_F R) \\ &= U^\sigma(E \otimes_F M \otimes_F R). \end{aligned}$$

Hence, we see that $G = \mathrm{Res}_{ME/F}\mathrm{GL}_n$.

1.1.2 Theorem ([GW13], Theorem 1.1)**.** *Suppose E/F is a totally real extension of number fields and M/F is a CM-extension of number fields. Let π be an automorphic* *representation of $U(\mathbb{A}_F)$ and let Π be a weak base change of π to $\mathrm{GL}_n(\mathbb{A}_{ME})$ satisfying the following conditions:*

1. *There exists a finite-dimensional representation V of U_{F_∞} such that π has nonzero cohomology with coefficients in V.*
2. *There exists a finite place v_1 of F totally split in ME/F such that π_{v_1} is supercuspidal.*
3. *There is a finite place $v_2 \neq v_1$ of F totally split in ME/F such that π_{v_2} is the discrete series.*
4. *For all places v of F such that ME/F is ramified and $M/F, E/F$ are both nonsplit at v the weak base change Π has the property that Π_v is relatively τ-regular.*

In particular we assume that there exists a finite-dimensional representation V of U_{F_∞} such that π has nonzero cohomology with coefficients in V. If the partial Asai L-function $L^S(s,\Pi,r)$ has a pole at $s=1$ then there exists a cuspidal automorphic representation π' of $U(\mathbb{A}_F)$ nearly equivalent to π and such that π' is U^σ-distinguished. Moreover, π' may be chosen to have nonzero cohomology with coefficients in V.

The importance of the pole of the Asai L-function for us is that it implies that Π is distinguished by G^σ and G^θ, so that Theorem 1.1.2 is indeed a particular case of the General Principle. Although the conditions on π and Π may look daunting at first, they are fairly typical if one wants to use simple trace formula.

We initiate the study of a *new* form of this general principal, with the following input: let $H = \mathrm{U}_{2n}$ be a quasisplit unitary group in $2n$-variables attached to the extension M/F and let $\sigma : \mathrm{U}_{2n} \to \mathrm{U}_{2n}$ be an automorphism such that $\mathrm{U}_{2n}^\theta \cong \mathrm{U}_n \times \mathrm{U}_n$. For an explicit description of how we define these unitary groups and automorphisms, we refer to Section 2.2.

In this case, $G := \mathrm{Res}_{M/F}(\mathrm{GL}_{2n})$ and $G^\sigma = \mathrm{Res}_{M/F}\mathrm{GL}_n \times \mathrm{Res}_{M/F}\mathrm{GL}_n$. Also, the group G^θ will be some form of $\mathrm{U}_n \times \mathrm{U}_n$. The relative trace formula comparison should then be between the two quotients

$$\mathrm{U}_n \times \mathrm{U}_n\backslash\mathrm{U}_{2n}/\mathrm{U}_n \times \mathrm{U}_n \leftrightsquigarrow \mathrm{Res}_{M/F}\mathrm{GL}_n \times \mathrm{Res}_{M/F}\mathrm{GL}_n\backslash\mathrm{Res}_{M/F}\mathrm{GL}_{2n}/\mathrm{Res}_{M/F}\mathrm{GL}_{2n}^\theta.$$

For a fixed F-algebra R, the elements of $\mathrm{U}_n(R)^2\backslash \mathrm{U}_{2n}(R)/\mathrm{U}_n(R)^2$ are called relative classes, and the same procedure on the right-hand side gives the τ-relative classes; these two sets of relative classes for the relatively semisimple elements roughly should correspond under a variant of the norm map as in [GW13], which together of matching of functions for the Hecke algebras and relative endoscopy should lead to a comparison of the geometric sides of the relevant trace formulas. This matching of relative classes will be addressed precisely in a future work.

We mention briefly that distinction for $\mathrm{GL}_n \times \mathrm{GL}_n \hookrightarrow \mathrm{GL}_{2n}$ can be expressed in terms of L-functions:

1.1.3 Theorem ([FJ93])**.** *Let F be a number field. A cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ is $\mathrm{GL}_n \times \mathrm{GL}_n$ distinguished if and only if the partial exterior square L-function for π has a pole at $s = 1$ and $L\left(\frac{1}{2}, \pi\right) \neq 0$.*

At any rate, we do not attempt to derive such a comparison in this thesis, but rather initiate the study of an important technical tool that should be necessary for such a comparison: the theory of relative endoscopy. Let us explain this term. At some point during the comparison of trace formulas, one encounters expressions known as *orbital integrals*. Endoscopy is essentially the process of replacing certain sums of these orbital integrals by integrals on smaller *endoscopic* spaces. In the case of $\mathrm{U}_n \times \mathrm{U}_n \hookrightarrow \mathrm{U}_{2n}$, we expect the endoscopic spaces to be a product of two spaces of the form $\mathrm{U}_{n_1} \times \mathrm{U}_{n_2} \hookrightarrow \mathrm{U}_{n_1+n_2}$, analogous to the endoscopy theory for unitary groups as in [LN08].

In the next section, we will explain exactly how sums of orbital integrals weighted by characters (called κ-orbital integrals) may be replaced by similar integrals on endoscopic spaces, which at this time are still conjectural.

1.2 Orbital Integrals

Up until now, we have been talking of the global problem of comparing trace formula for distinguished automorphic representations. For the remainder of the thesis we will focus on the local problem at the nonarchimedean places. We remark that the problem for archimedean places is also important and perhaps crucial, but it also will need quite different techniques compared to the nonarchimedean case, and so we leave that problem for another day. So, we let F be a nonarchimedean local field with residue field $\mathbb{F}_q$ and ring of integers $\mathfrak{o}$.

Let G be a smooth linear algebraic group scheme over $\mathfrak{o}$. The valuation $v : F^\times \to \mathbb{Z}$ induces an absolute value $|x| = q^{-v(x)}$ on F with the convention that $|0| = 0$. This absolute value induces a metric, which gives F a totally disconnected, locally compact Hausdorff topology consisting of a neighbourhood basis of compact open additive subgroups of F.

We give a topology on $G(F)$ via an embedding $G(F) \to \mathrm{GL}_n(F) \to F^{n^2}$, which can be shown to be independent of the embedding and is closed. For these and other facts, we refer to [Con12]. If V is an affine $\mathfrak{o}$-variety with a G-action $\rho : G \to \mathrm{Aut}(V)$ given by some $G \times V \to V$, we consider also the space $V(F)$ also with the induced topology of affine F-space.

We let $C_c^\infty(V(F))$ denote the $\mathbb{C}$-vector space of complex-valued, smooth (i.e. locally constant), compactly supported functions on $V(F)$. The space $C_c^\infty(G(F))$ is an algebra under the convolution product, though we will not actually need this algebra structure.

A **distribution** is a linear functional on $C_c^\infty(V(F))$. An **invariant distribution** is a distribution invariant under the action of $G(F)$ on $V(F)$. Perhaps the most important class of invariant distributions for us are the orbital integrals, which form part of the harmonic analysis initiated largely by Harish-Chandra.

1.2.1 Definition. Let $\gamma \in V(F)$ and I_γ be its stabiliser in G, which we assume to be reductive. Assume either that G is reductive or that I_γ is trivial. We define the **orbital integral** to be the distribution $\mathcal{O}_\gamma : C_c^\infty(V(F)) \to \mathbb{C}$ given by

$$\begin{aligned}\mathcal{O}_\gamma : C_c^\infty(V(F)) &\longrightarrow \mathbb{C}\\ f &\longmapsto \int_{I_\gamma(F)\backslash G(F)} f(\rho(g)^{-1}\gamma)\frac{\mathrm{d}g}{\mathrm{d}g_\gamma}.\end{aligned}$$

The measure $\mathrm{d}g$ is the unique Haar measure such that $\mathrm{vol}(G(\mathfrak{o}), \mathrm{d}g) = 1$ and similarly with $\mathrm{d}g_\gamma$. When I_γ is nontrivial, the measure $\mathrm{d}g/\mathrm{d}g_\gamma$ is the quotient measure, which exists since both $G(F)$ and $I_\gamma(F)$ are unimodular.

1.2.2 Example. Let $F = \mathbb{F}_q((t))$ be the field of Laurent series over a finite field $\mathbb{F}_q$, let $\mathfrak{o} = \mathbb{F}_q[[t]]$ and let $G = \mathrm{GL}_2$. Let G act on $V := G$ via the adjoint action. Let $\gamma = \mathrm{diag}(x, y) \in G(F)$ with $x \neq y \in \mathfrak{o}$. This element is regular semisimple and has as a stabiliser the maximal diagonal torus. Let f be the characteristic function of $V(\mathfrak{o})$. Then

$$\mathcal{O}_\gamma(f) = q^{v(x-y)}.$$

More generally, if $\gamma = \mathrm{diag}(x_1, \dots, x_n) \in V(F)$ is such that $x_i \neq x_j$ for $i \neq j$ and $x_i \in \mathfrak{o}$ for all i, then $\mathcal{O}_\gamma(f) = q^{\sum_{i<j} v(x_i - x_j)}$. We note that we just used $F = \mathbb{F}_q((t))$ as an example: the same calculation works also over p-adic fields.

1.3 κ-Orbital Integrals and Stable Classes

We now discuss stable classes. If $\rho : G \to \mathrm{Aut}(V)$ is an algebraic action of G on an affine variety V.

1.3.1 Definition. Assume I_γ is a torus.

1. If γ and γ' are elements of $V(F)$ and there exists a $g \in G(\overline{F})$ such that $\rho(g)\gamma = \gamma'$ then we say that γ and γ' are in the same **stable class** and write $\gamma \sim_{\mathrm{st}} \gamma'$.

2. For any $\gamma \in V(F)$, we define the stable class of γ to be

$$\mathrm{St}(\gamma) = \{\gamma' \in V(F) : \gamma' \sim_{\mathrm{st}} \gamma\}.$$

The condition that the stabiliser of γ is a torus simplifies the exposition. If I_γ is not a torus, then a more complicated definition must be used, which reduces to the one we gave under the assumption that I_γ is a torus. However, our actual results fall under this assumption so we will not go into the general definitions.

In order to do a kind of Fourier analysis for orbital integrals, we observe that the classes in the stable class of a $\gamma \in V(F)$ may be classified by a cohomology set. For a scheme

X defined over F, we use the standard notation $H^1(F, X)$ to denote the nonabelian Galois cohomology *set* $H^1(\mathrm{Gal}(\overline{F}/F), X(F))$.

1.3.2 Proposition. *Let G act on a variety V and $\gamma \in V(F)$. Then the classes in the stable class of γ are in bijection with the set*

$$\mathcal{D} = \mathcal{D}_\gamma := \ker[H^1(F, I_\gamma) \to H^1(F, G)]$$

via the map

$$\begin{aligned} \psi : \mathrm{St}(\gamma) &\longrightarrow \mathcal{D} \\ [\rho(g)\gamma] &\longmapsto [\sigma \mapsto g^{-1}\sigma(g)]. \end{aligned}$$

Here, the map $H^1(F, I_\gamma) \to H^1(F, G)$ is induced by the inclusion $I_\gamma \to G$ and ρ denotes the action of G on V.

Proof. The element $\rho(g)\gamma$ is an element of $V(F)$, and so is fixed by any element σ of $\mathrm{Gal}(\overline{F}/F)$. Hence for any such σ, we have

$$\rho(\sigma(g))\gamma = \rho(g)\gamma.$$

Thus, the element $g^{-1}\sigma(g) \in I_\gamma(\overline{F})$, and by definition $\sigma \mapsto g^{-1}\sigma(g)$ is trivial in $H^1(F, G)$. Next, we show that $\psi[\rho(g)\gamma]$ is independent of the representative g chosen. Indeed, suppose that $[\rho(g)\gamma]$ and $[\rho(g')\gamma]$ are the same $G(F)$ class. Then there exists an $h \in G(F)$ such that $\rho(h)\rho(g)\gamma = \rho(g')\gamma$. Hence $y = g^{-1}hg' \in I_\gamma(\overline{F})$ is such that

$$g'^{-1}\sigma(g') = y^{-1}g^{-1}\sigma(y)$$

for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$, and so the cocycles $\sigma \mapsto g^{-1}\sigma(g)$ and $\sigma \mapsto g'^{-1}\sigma(g')$ represent the same element in $H^1(F, I_\gamma)$.

Let us show injectivity. Suppose that $\sigma \mapsto g^{-1}\sigma(g)$ and $\sigma \mapsto g'^{-1}\sigma(g')$ are cohomologous cocycles with values in $I_\gamma(\overline{F})$. Then, there exists an element $y \in I_\gamma(\overline{F})$ such that $g^{-1}\sigma(g) =$

$y^{-1}g'^{-1}\sigma(g')\sigma(y)$. Moving everything with a σ to one side we get the equality

$$g'yg^{-1} = \sigma(g'yg^{-1})$$

and so $z = g'yg^{-1}$ is an element in $G(F)$ such that $\rho(z)\rho(g)\gamma = \rho(g')\gamma$. Thus $\rho(g)\gamma$ and $\rho(g')\gamma$ are in the same class and hence represent the same $G(F)$ class. Thus the map ψ is injective.

Finally, we show that ψ is surjective. Recall [Ser97, Section 5.4] that to the inclusion I_γ there corresponds an exact sequence of pointed sets

$$(G/I_\gamma)(F) \to H^1(F, I_\gamma) \to H^1(F, G).$$

Hence for each element $a_\sigma \in \mathcal{D}$ there exists a $gI_\gamma(\overline{F}) \in (G/I_\gamma)(F)$ such that $g^{-1}\sigma(g) = a_\sigma$. By definition, $\sigma(g)I_\gamma(\overline{F}) = gI_\gamma(\overline{F})$ and so $[\rho(g)\gamma]$ is the required class such that $\psi([\rho(g)\gamma]) = a_\sigma$. ■

Of course, so far we have only a pointed set $\mathcal{D}$, which is not very helpful for doing harmonic analysis. In the case where F is a nonarchimedean local field, however, $\mathcal{D}$ is an abelian group. Indeed, we have the following which is very useful for computation:

1.3.3 Theorem ([Kot84, Section 6]). *Let F be a complete nonarchimedean local field and $\Gamma = \mathrm{Gal}(\overline{F}/F)$. For a and any torus T, one may identify the map $H^1(F,T) \to H^1(F,G)$ with the map*

$$[\pi_0(\check{T}^\Gamma)]^D \to [\pi_0(Z(\check{G})^\Gamma]^D.$$

Here π_0 denotes the functor that takes an algebraic group to its group of connected components, $X \mapsto \check{X}$ denotes taking the dual group, $Z(-)$ denotes the center, and D denotes taking characters. Here, by dual group we mean the complex points of the complex reductive group obtained by switching the character and cocharacter lattice, sometimes referred to as the connected Langlands dual. For example, if T is a torus over F, then its dual is $\mathrm{Hom}(T_{\overline{F}}, \mathbb{G}_{m,\overline{F}}) \otimes_{\mathbb{Z}} \mathbb{C}^\times$, which has a natural action of $\mathrm{Gal}(\overline{F}/F)$.

Alternatively, still assuming that F is nonarchimedean, one may also identify $\mathcal{D}$ with $\mathrm{Im}[H^1(F,T^{\mathrm{sc}}) \to H^1(F,T)]$. Here, T^{sc} is the preimage under the map $G^{\mathrm{sc}} \to [G,G]$ of the image of T by the map $G \to [G,G]$, where G^{sc} denotes the simply connected cover of $[G,G]$, though this result is less useful for computation.

Given a character $\kappa : \mathcal{D} \to \mathbb{C}$, we define the **$\kappa$-orbital integral** of γ to be the distribution

$$\mathcal{O}_\gamma^\kappa(f) = \sum_{\gamma'} \kappa(\gamma')\mathcal{O}_{\gamma'}(f).$$

Here, $\kappa(\gamma')$ denotes the evaluation of κ on the element of A associated to the class of γ' in the stable class of γ. For a moment let $V = \mathfrak{g}$ be the adjoint representation. The following theorem is usually referred to as the fundamental lemma.

1.3.4 Theorem ([Ngô10]). *Let F be a complete nonarchimedean local field of sufficiently large positive characteristic. If $\kappa : \mathcal{D}_\gamma \to \mathbb{C}^\times$ is a character, then there exists an endoscopic group H such that for matching regular semisimple classes represented by $\gamma \in \mathfrak{g}(F)$ and $\gamma_H \in \mathfrak{h}(F)$, there is an equality*

$$\mathcal{O}_\gamma^\kappa(\mathbf{1}_{\mathfrak{g}(\mathfrak{o})}) = \Delta\mathcal{O}_{\gamma_H}^{\kappa=1}(\mathbf{1}_{\mathfrak{h}(\mathfrak{o})}).$$

Here Δ is some power of q up to a root of unity and depends only on γ and γ_H.

For instance, if κ is the trivial character then the statement reduced to the trivial statement with $G = H$. To actually deduce a theorem that can be applied to the trace formula from this one, a number of reductions are required, e.g. [LS90] to go from groups to Lie algebras, [Wal06] to go from positive characteristic to characteristic zero, and [Hal95] to go from unit elements to the full Hecke algebra, among other things. Suffice it to say, this reduction of the original fundamental lemma as written by Langlands [Lan83] to the statement Ngô proved is a somewhat involved process that is outside the scope of the current work, and the reader is advised to consult [Hal12] for more details. We are aware that similar theorems are needed for a general relative fundamental lemma. However, since we are presently interested in unitary groups, we expect that proving a version of the fundamental lemma usable in arithmetic applications should be significantly easier as in [LN08], where the authors take a more direct approach.

At any rate, as we have mentioned, for *distinguished* automorphic representations, one can establish relative trace formulas, and our goal is to develop a fundamental lemma suitable for use in such relative trace formulas. We remark that there already exists fundamental lemmas in some special cases. However, none of these are *endoscopic* in the sense that they use a relation between κ-orbital integrals and stable orbital integrals on smaller groups. Such an endoscopic fundamental lemma would be an important tool for the comparison of relative trace formulas. In this thesis, we begin the study of such a problem by studying the Lie algebra version of the problem.

1.4 A Summary of Our Results on Symmetric Spaces

We will initiate the study of relative endoscopy starting out with the local version on Lie algebras since the explicit computations are a little easier in this setting. And, the Lie algebraic version of orbital integrals for the quotient $\mathrm{U}_n \times \mathrm{U}_n \backslash \mathrm{U}_{2n}$ turn out to belong to the well-studied notion of symmetric space, the basic properties of which were established by Kostant and Rallis [KR71] in characteristic zero and Levy [Lev07] in positive characteristic.

Let G be a reductive algebraic group over a field F. Let $\theta : G \to G$ be an automorphism of order two, and let $G_0 := (G^\theta)^\circ$ denote the connected component of the fixed point group G^θ. The group G_0 is reductive [Vus74] and acts on the -1-eigenspace

$$\mathfrak{g}_1 := \{x \in \mathfrak{g} : \theta(x) = -x\}$$

via the adjoint action. Let $\gamma \in \mathfrak{g}_1(F)$ be semisimple, and additionally **regular**, which means that the stabiliser I_γ of γ in G_0 has minimal dimension. As usual, we assume that I_γ is a torus. If F is a complete nonarchimedean local field, we wish to understand the orbital integrals of the form

$$\begin{aligned} C_c^\infty(\mathfrak{g}_1(F)) &\longrightarrow \mathbb{C} \\ f &\longmapsto \mathcal{O}_\gamma(f) := \int_{I_\gamma(F)\backslash G_0(F)} f(\mathrm{Ad}(g)^{-1}\gamma)\frac{\mathrm{d}g}{\mathrm{d}g_\gamma}. \end{aligned}$$

Establishing a relative fundamental lemma for κ-orbital integrals is our goal.

The next two chapters of this thesis corresponds to the results of two papers that give partial progress towards this goal. In Chapter 2, we actually verify a relative fundamental lemma on Lie algebras, the precise statement of which is Theorem 2.1.2. This is the first example of a relative *endoscopic* fundamental lemma that has appeared in the literature, and hence is the first piece of evidence that a phenomenon of endoscopy is at work for symmetric spaces. The entire chapter is devoted to its verification, which is somewhat technical but completely elementary. The results have since been published in [Pol15b].

In Chapter 3, we study a geometric interpretation of orbital integrals via *affine Springer fibers*. We compute the dimension of this ind-scheme for the symmetric spaces appearing in Chapter 2. These dimensions are important for understanding the transfer factor that appear in fundamental lemmas. It can also be found as a preprint in [Pol15a].

Chapter 2

A Lie Algebra Fundamental Lemma

2.1 Introduction

In this chapter (which is a slightly modified version of the published paper [Pol15b]) we study κ-orbital integrals for a pair (G, θ) where G is a connected reductive algebraic group over a complete nonarchimedean local field and $\theta : G \to G$ is an involution. The orbital integrals will then be integrals on orbits for the action of $(G^\theta)^\circ$ on $\mathfrak{g}_1 = \{x \in \mathfrak{g} : \theta(x) = -x\}$. We first introduce a little notation.

Let F be a complete nonarchimedean local field of zero or odd positive characteristic with algebraic closure $\overline{F}$ and residue field $\mathbb{F}_q$. Let E/F an unramified quadratic extension. We denote the nontrivial action of the Galois group by $x \mapsto \overline{x}$. Since E/F is fixed throughout, we simply use $N(x) = x\overline{x}$ to denote the norm of x. We fix once and for all a $\delta \in E$ such that $\overline{\delta} = -\delta$ and $v(\delta) = 0$ so that $F\delta = \{x \in E : \overline{x} = -x\}$. Let $\mathfrak{o} \subset F$ be the ring of integers with maximal ideal $\mathfrak{m}$, and let $\mathfrak{o}_E$ be the ring of integers of E. We write $q = |\mathfrak{o}/\mathfrak{m}|$, the cardinality of the residue field. We fix once and for all a uniformiser π, and denote the resulting valuation on F by $v : F^\times \to \mathbb{Z}$ so that $v(\pi) = 1$.

In our computations we will consider various Haar measures on locally compact groups of the form $G(F)$ where G is a linear algebraic group over $\mathfrak{o}$. These are always normalised so that $G(\mathfrak{o})$ has volume 1. We will need the following proposition which follows from $\mathfrak{o}_E$ being stable under $\mathrm{Gal}(E/F)$.

2.1.1 Proposition. *If $ax + b \in \mathfrak{o}_E$ where $a, b \in F$ and $x \in F\delta$ then $ax \in \mathfrak{o}_E$ and $b \in \mathfrak{o}_E$.*

Let $\gamma \in \mathfrak{g}_1(F)$ be regular semisimple and let I_γ be its stabiliser, which we assume is a torus. We write $\mathcal{D}(I_\gamma)$ for the abelian group that parametrises classes in the stable class of γ. Let $\kappa : \mathcal{D}(I_\gamma) \to \mathbb{C}^\times$ be a character. In this setting we have a κ-orbital integral

$$\mathcal{O}^\kappa_\gamma(\mathbf{1}) = \sum_{\gamma'} \kappa(\gamma') \int_{I_{\gamma'}(F)\backslash G_0(F)} \mathbf{1}(\mathrm{Ad}(g)^{-1}\gamma') \frac{\mathrm{d}g}{\mathrm{d}t} \tag{2.1}$$

In this chapter, we prove a fundamental lemma for $(\mathrm{U}(4), \theta)$ where θ is an involution such that $\mathrm{U}(4)^\theta \cong \mathrm{U}(2) \times \mathrm{U}(2)$ and when γ is of the form $\gamma = \mathrm{diag}(x, y, -y, -x)$ with $x \neq \pm y \in F^\times$. Motivated by the usual fundamental lemma for unitary groups, we define for the nontrivial $\kappa : \mathcal{D}(I_\gamma) \to \mathbb{C}^\times$ the endoscopic symmetric space to be $(H, \theta_H) = (\mathrm{U}_2, \sigma_H) \times (\mathrm{U}_2, \sigma_H)$ where $\sigma_H : \mathrm{U}_2 \to \mathrm{U}_2$ is such that $\mathrm{U}_2^{\sigma_H} \cong \mathrm{U}_1 \times \mathrm{U}_1$. We then set $\gamma_H = \mathrm{diag}(x, -x) \times \mathrm{diag}(y, -y) \in \mathfrak{h}_1$. This gives a transfer $\gamma \mapsto \gamma_H$.

2.1.2 Theorem. *Assume that $\gamma = \mathrm{diag}(x, y, -y, -x)$ satisfies $v(x + y) > v(x - y)$. Then κ-orbital as defined in (2.1) satisfies*

$$\mathcal{O}^\kappa_\gamma(\mathbf{1}_{\mathfrak{g}_1(\mathfrak{o})}) = \Delta(\gamma, \gamma_H)\mathcal{SO}_{\gamma_H}(\mathbf{1}_{\mathfrak{h}_1(\mathfrak{o})}).$$

where $\Delta(\gamma, \gamma_H) \in \mathbb{C}$ can be calculated explicitly and is a simple power of q up to a root of unity.

Even though relative orbital integrals have been considered previously in the literature, this is the *first known example* of endoscopy in this setting, and will be helpful in formulating more general relative endoscopic fundamental lemmas.

The Lie subalgebra $\mathfrak{g}_0 = \mathfrak{g}^\theta$ of fixed points also plays an important role. For $x \in \mathfrak{g}_1$, if $\dim \mathfrak{z}_{\mathfrak{g}_0}(x) \leq \dim \mathfrak{z}_{\mathfrak{g}_0}(y)$ for all $y \in \mathfrak{g}_1$, then we say that x is **regular**. This is equivalent to the orbit $G_0 \cdot x$ having minimal dimension. For this and further facts, the reader is referred to the paper [Lev07].

2.2 $\mathrm{U}_2 \times \mathrm{U}_2 \hookrightarrow \mathrm{U}_4$ and Regular Elements

Define the $n \times n$ matrix

$$J_n = \begin{pmatrix} & & 1 \\ & \diagup & \\ 1 & & \end{pmatrix}.$$

The $n \times n$ unitary group functor is defined for all F-algebras R by

$$\mathrm{U}_n(R) = \{g \in \mathrm{GL}_n(R \otimes_F E) : J_n \bar{g}^{-t} J_n = g\}$$

and its Lie algebra is the functor given by

$$\mathfrak{g} := \mathfrak{u}_n(R) = \{x \in \mathfrak{gl}_n(R \otimes_F E) : -J_n \overline{x}^t J_n = x\}.$$

From now on we consider the case $n = 4$. We let $\theta : \mathrm{Res}_{E/F}\mathrm{GL}_4 \to \mathrm{Res}_{E/F}\mathrm{GL}_4$ be conjugation by

$$\theta = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$$

which by abuse of notation we also call θ. Since $\theta J = J\theta$, the involution θ on $\mathrm{Res}_{E/F}\mathrm{GL}_4$ gives a well-defined involution on U_4. For computational purposes, it is necessary to write down explicitly the F-points of the -1-eigenspace $\mathfrak{g}_1$ in terms of matrices. This can be most easily done by observing that

$$\mathfrak{gl}_4(E)(-1) = \{x \in \mathfrak{gl}_4(E) : \theta(x) = -x\} = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{24} & -x_{23} & -x_{22} & x_{21} \\ -x_{14} & x_{13} & x_{12} & -x_{11} \end{pmatrix} : x_{ij} \in E \right\}$$

so that $\mathfrak{g}_1$ is then the fixed points under $x \mapsto -J_4\overline{x}^t J_4$, and the F-points are then

$$\mathfrak{g}_1(F) = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ -\overline{x}_{12} & x_{22} & x_{23} & -\overline{x}_{13} \\ -\overline{x}_{13} & -x_{23} & -x_{22} & -\overline{x}_{12} \\ -x_{14} & x_{13} & x_{12} & -x_{11} \end{pmatrix} : \begin{matrix} x_{11}, x_{22} \in F \\ x_{14}, x_{23} \in F\delta \\ x_{12}, x_{13} \in E \end{matrix} \right\}. \tag{2.2}$$

The fixed-point group $G_0 := \mathrm{U}_4^\theta \cong \mathrm{U}_2 \times \mathrm{U}_2$ acts on $\mathfrak{g}_1$. Inside $\mathfrak{g}_1(F)$ there is $\mathfrak{a}$, a maximal subspace of commuting semisimple elements such that

$$\mathfrak{a}(F) = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & -x \end{pmatrix} : x, y \in F \right\}$$

We only consider regular semisimple elements in $\mathfrak{a}(F)$. If $\gamma = \mathrm{diag}(x, y, -y, -x) \in \mathfrak{a}(F)$, then γ is regular if and only if $x, y \in F^\times$ and $x \neq \pm y$. For any regular semisimple element $\gamma \in \mathfrak{g}_1(F)$ with stabiliser I_γ in $\mathrm{U}_2 \times \mathrm{U}_2$, and for any compactly supported complex-valued smooth function f on $\mathfrak{g}_1(F)$, we define the orbital integral

$$\mathcal{O}_\gamma(f) := \int_{I_\gamma(F)\backslash \mathrm{U}_2\times\mathrm{U}_2(F)} f(\mathrm{Ad}(g^{-1})\gamma)\frac{\mathrm{d}g}{\mathrm{d}g_\gamma}.$$

2.3 Stable Conjugacy

Fix a regular $\gamma = \mathrm{diag}(x, y, -y, -x) \in \mathfrak{a}(F)$. The stable class of γ in $\mathfrak{g}_1(F)$ by definition is $G_0(\overline{F})\gamma \cap \mathfrak{g}_1(F)$. When using cohomology, it is useful to express the stable class of γ as

$$\{\mathrm{Ad}(g)\gamma : g \in G_0(\overline{F}) \text{ and } g^{-1}\sigma(g) \in I_\gamma(\overline{F}) \text{ for all } \sigma \in \mathrm{Gal}(\overline{F}/F)\}.$$

In this section, we explicitly decompose the stable class of γ into $G_0(F)$-classes. As before, we denote by I_γ the stabiliser of γ in G_0. The inclusion $I_\gamma \to G_0$ gives rise to a long exact sequence of pointed sets

$$1 \to I_\gamma(F) \to G_0(F) \to (G_0/I_\gamma)(F) \to H^1(F, I_\gamma) \to H^1(F, G_0)$$

in nonabelian cohomology. One easily checks that the map $\mathrm{Ad}(g)\gamma \mapsto (\sigma \mapsto g^{-1}\sigma(g))$ is a well-defined bijection from the set of classes of γ in the stable class to $\mathcal{D} := \ker[H^1(F, I_\gamma) \to H^1(F, G_0)]$. Using this correspondence, we can compute explicit representatives for the classes within the stable class of γ.

We first compute

$$I_\gamma = \left\{ \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix} : a_{ii}\overline{a}_{ii} = 1 \right\}.$$

In other words, $I_\gamma \cong \mathrm{U}_1 \times \mathrm{U}_1$ (which suggests our choice of an endoscopic symmetric space). We can compute the Galois cohomology over the finite extension E/F which splits I_γ. Doing this we find that

$$H^1(F, I_\gamma) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Let us apply Tate-Nakayama-Kottwitz duality to show that

$$\mathcal{D} = \ker[H^1(F, I_\gamma) \to H^1(F, G_0)]$$

is isomorphic to $\mathbb{Z}/2$. Indeed, both $Z(\hat{G})$ and $\hat{T}$ split over the extension E/F so we may calculate the corresponding cohomology as $\mathrm{Gal}(E/F)$-cohomology. The corresponding map on dual tori $Z(\hat{G}) \to \hat{T}$ is the homomorphism of $\mathrm{Gal}(E/F)$-modules:

$$\begin{aligned} \mathbb{C}^\times &\longrightarrow \mathbb{C}^\times \times \mathbb{C}^\times \\ a &\longmapsto (a, a) \end{aligned}$$

where the $\mathbb{Z}/2 \cong \mathrm{Gal}(E/F)$ acts via the inverse map. Taking fixed points under this action and then character groups gives the map

$$\begin{aligned} \mathbb{Z}/2 \oplus \mathbb{Z}/2 &\longrightarrow \mathbb{Z}/2 \\ (a,b) &\longmapsto a+b \end{aligned}$$

whose kernel is evidently $\mathbb{Z}/2$. For the purposes of computations, we will need to find a generator of $\mathcal{D}$. Fix the isomorphism of algebras $E \otimes_F E \xrightarrow{\sim} E \oplus E$ given on pure tensors by $(a \otimes b) \mapsto (ab, \overline{a}b)$, where the multiplication on $E \oplus E$ is pointwise and the Galois action on the left factor of $E \otimes_F E$ translates to $\overline{(a,b)} = (b,a)$ in $E \oplus E$.

Let $\sigma \in \mathrm{Gal}(E/F)$ be the nontrivial element. The nontrivial element in $\mathcal{D}$ is then represented (for example) by the cocycle

$$\sigma \mapsto (\pi I_4, \pi^{-1} I_4) \in I_\gamma(E)$$

The element in $(I_\gamma \backslash G_0)(E)$ that maps to the corresponding cohomology class is represented by $(B, \pi B) \in G_0(E)$ where

$$B = \frac{1}{2}\begin{pmatrix} \pi^{-1}+1 & -\pi^{-1}+1 & 0 & 0 \\ -\pi^{-1}+1 & \pi^{-1}+1 & 0 & 0 \\ 0 & 0 & \pi^{-1}+1 & \pi^{-1}-1 \\ 0 & 0 & \pi^{-1}-1 & \pi^{-1}+1 \end{pmatrix}.$$

Indeed, it maps to $\sigma \mapsto g^{-1}\sigma(g) = (\pi I_4, \pi^{-1} I_4)$. We remark to the unwary reader that σ here acts on the *right* of $E \otimes_F E$, which on $E \oplus E$ is the same $(a,b) \mapsto (\overline{b}, \overline{a})$. Now $H^1(F, I_\gamma) \cong H^1(\mathbb{Z}/2, I_\gamma(E))$ by Hilbert's Theorem 90 and the Lyndon-Hochschild-Serre spectral sequence, and $I_\gamma(E) \cong \{\mathrm{diag}((a,a^{-1}),(b,b^{-1}),(b,b^{-1}),(a,a^{-1})) : a,b \in E^\times\}$, so we are just computing group cohomology of a cyclic group. We get

$$\begin{aligned} H^1(\mathbb{Z}/2, I_\gamma(E)) &\cong (E^\times/(E^\times)^2) \times (E^\times/(E^\times)^2) \\ &\cong \mathbb{Z}/2 \times \mathbb{Z}/2. \end{aligned}$$

Hence $\sigma \mapsto (\pi I_4, \pi^{-1} I_4)$ is a nontrivial cocycle because π is not a square in E. We set

$\gamma_{\rm stc} = \mathrm{Ad}(B)\gamma$. Now let $\kappa : \mathcal{D} \to \{\pm 1\}$ be a character. We have for any compactly supported smooth function f on $\mathfrak{g}_1(F)$ the κ-orbital integral

$$\mathcal{O}_\gamma^\kappa(f) = \int_{I_\gamma(F)\backslash G_0(F)} f(\mathrm{Ad}(g^{-1})\gamma)\mathrm{d}g + \kappa(-1)\int_{I_{\gamma_{\rm stc}}(F)\backslash G_0(F)} f(\mathrm{Ad}(g^{-1})\gamma_{\rm stc})\mathrm{d}g.$$

We note that we can omit the stabilisers of γ and of $\gamma_{\rm stc}$ since their F-points are compact. We would like to compute this integral when κ is the nontrivial character and when $f = \mathbf{1}$, the characteristic function of $\mathfrak{g}_1(\mathfrak{o})$:

$$\mathcal{O}_\gamma^\kappa(\mathbf{1}) = \int_{G_0(F)} \mathbf{1}(\mathrm{Ad}(g^{-1})\gamma)\mathrm{d}g - \int_{G_0(F)} \mathbf{1}(\mathrm{Ad}(g^{-1})\gamma_{\rm stc})\mathrm{d}g.$$

This is the goal of the remainder of the chapter.

2.4 Preliminaries on Integration

We choose a parabolic P so that we get an Iwasawa decomposition

$$G_0(F) = P(F)G_0(\mathfrak{o}) = M(F)U(F)G_0(\mathfrak{o})$$

where M is a Levi subgroup of P and U is the unipotent radical of P, so we reduce the computation to one on $P(F) = M(F)U(F)$. Although this does simplify matters, since the stabiliser of γ is compact, we unfortunately *cannot* use the method of descent that would otherwise make the computation significantly easier. Now, to specify a parabolic of G_0 is the same thing as giving a cocharacter $\lambda : \mathbb{G}_m \to G_0$. We use the cocharacter

$$\begin{aligned} \mathbb{G}_m &\longrightarrow G_0 \\ r &\longmapsto \frac{1}{2}\begin{pmatrix} r+r^{-1} & 0 & -r+r^{-1} & 0 \\ 0 & r+r^{-1} & 0 & r-r^{-1} \\ -r+r^{-1} & 0 & r+r^{-1} & 0 \\ 0 & r-r^{-1} & 0 & r+r^{-1} \end{pmatrix}. \end{aligned}$$

One verifies easily that λ is both θ-fixed and actually does land in U_4. This cocharacter uniquely specifies the parabolic $P = \{g \in G_0 : \lim_{r\to 0} \lambda(r) g \lambda(r^{-1}) \text{ exists }\}$. The unipotent radical of this parabolic is given by $U = \{g \in G_0 : \lim_{r\to 0} \lambda(r) g \lambda(r^{-1}) = 1\}$ and we calculate the F-points of the unipotent radical to be isomorphic to $F\delta \times F\delta$ via

$$\begin{aligned} F\delta \times F\delta &\longrightarrow U(F) \\ (c,d) &\longmapsto \begin{pmatrix} c+1 & d & c & -d \\ -d & -c+1 & -d & c \\ -c & -d & -c+1 & d \\ -d & -c & -d & c+1 \end{pmatrix}. \end{aligned} \tag{2.3}$$

For integrating, we use the product Haar measure on $F\delta \times F\delta$ where on each factor $F\delta$ we choose a Haar measure so that $\mathfrak{o}\delta$ has unit volume. A Levi subgroup of P is the subgroup that centralises the cocharacter λ. We calculate its F-points to be isomorphic to $E^\times \times E^\times$ through the isomorphism

$$\begin{aligned} E^\times \times E^\times &\longrightarrow M(F) \\ (r_1, r_2) &\longmapsto \frac{1}{4} \begin{pmatrix} r_1+r_2+\overline{r}_1^{-1}+\overline{r}_2^{-1} & r_1+r_2-\overline{r}_1^{-1}-\overline{r}_2^{-1} & r_1-r_2-\overline{r}_1^{-1}+\overline{r}_2^{-1} & -r_1+r_2-\overline{r}_1^{-1}+\overline{r}_2^{-1} \\ r_1+r_2-\overline{r}_1^{-1}-\overline{r}_2^{-1} & r_1+r_2+\overline{r}_1^{-1}+\overline{r}_2^{-1} & r_1-r_2+\overline{r}_1^{-1}-\overline{r}_2^{-1} & -r_1+r_2+\overline{r}_1^{-1}-\overline{r}_2^{-1} \\ r_1-r_2-\overline{r}_1^{-1}+\overline{r}_2^{-1} & r_1-r_2+\overline{r}_1^{-1}-\overline{r}_2^{-1} & r_1+r_2+\overline{r}_1^{-1}+\overline{r}_2^{-1} & -r_1-r_2+\overline{r}_1^{-1}+\overline{r}_2^{-1} \\ -r_1+r_2-\overline{r}_1^{-1}+\overline{r}_2^{-1} & -r_1+r_2+\overline{r}_1^{-1}-\overline{r}_2^{-1} & -r_1-r_2+\overline{r}_1^{-1}+\overline{r}_2^{-1} & r_1+r_2+\overline{r}_1^{-1}+\overline{r}_2^{-1} \end{pmatrix} \end{aligned} \tag{2.4}$$

Again, for integration, we use the product Haar measure on $E^\times \times E^\times$ so that $\mathfrak{o}_E^\times$ in $E^\times$ has unit volume.

We note that multiplying the matrix in 2.4 either on the left or the right by the matrix that represents the cocycle is the same matrix but with $\overline{r}_i$ replaced with $\pi\overline{r}_i$ for $i = 1, 2$. Thus, in any expressions that are a function of $m\gamma m^{-1}$ for $m \in M$, making this replacement gives us the equations for $m\gamma_{\mathrm{stc}}m^{-1}$.

In making these reductions, we are left to evaluate the integral

$$\begin{aligned} \mathcal{O}_\gamma^\kappa(\mathbf{1}) = &\int_{M(F)} \int_{U(F)} \mathbf{1}(\mathrm{Ad}(u^{-1})\,\mathrm{Ad}(m^{-1})\gamma) \mathrm{d}u \mathrm{d}m \\ &- \int_{M(F)} \int_{U(F)} \mathbf{1}(\mathrm{Ad}(u^{-1})\,\mathrm{Ad}(m^{-1})\gamma_{\mathrm{stc}}) \mathrm{d}u \mathrm{d}m \end{aligned}$$

where the Haar measures are chosen so that $U(\mathfrak{o})$ and $M(\mathfrak{o})$ each have unit volume. We now examine an element of the form $Y = \mathrm{Ad}(u^{-1})\,\mathrm{Ad}(m^{-1})\gamma$, where u is a matrix as in (2.3) and m is a matrix as in (2.4). Since $Y \in \mathfrak{g}_1$, by the explicit form of $\mathfrak{g}_1$ in (2.2), we see that $Y \in \mathfrak{g}_1(\mathfrak{o})$ exactly when $v = [Y_{11}, Y_{12}, Y_{13}, Y_{14}, Y_{22}, Y_{23}]^t \in \mathfrak{o}^6$. And, $v \in \mathfrak{o}^6$ exactly when $Av \in \mathfrak{o}^6$ for any $A \in \mathrm{GL}_6(\mathfrak{o})$. In particular, we take

$$A = \begin{pmatrix} 0 & -1 & 1 & 0 & -1 & 1 \\ 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 \end{pmatrix}, \qquad Av = \begin{pmatrix} -Y_{12} + Y_{13} - Y_{22} + Y_{23} \\ Y_{11} - Y_{12} - Y_{13} - Y_{14} \\ Y_{12} + Y_{13} - Y_{22} - Y_{23} \\ Y_{11} + Y_{12} - Y_{13} + Y_{14} \\ Y_{11} + Y_{12} + Y_{13} - Y_{14} \\ Y_{11} + Y_{13} - Y_{12} + Y_{14} \end{pmatrix}.$$

We calculate that $\det(A) = -32$, so that indeed $A \in \mathrm{GL}_6(\mathfrak{o})$. Now it is simply a matter of calculating $Y = \mathrm{Ad}(u^{-1})\,\mathrm{Ad}(m^{-1})\gamma$, and each of the quantities in Av. Straightforward but tedious computations, and making the harmless change of variables ($d - c \rightsquigarrow c, d + c \rightsquigarrow d$) show that

$$-Y_{12} + Y_{13} - Y_{22} + Y_{23} = \tfrac{r_2}{r_1}(x+y)(-c-\tfrac{1}{2}) + \tfrac{1}{2}(x-y)\tfrac{1}{r_1\overline{r}_2} \tag{2.5}$$
$$\overline{Y}_{11} - \overline{Y}_{12} - \overline{Y}_{13} - \overline{Y}_{14} = \tfrac{r_2}{r_1}(x+y)(-c+\tfrac{1}{2}) + \tfrac{1}{2}(x-y)\tfrac{1}{r_1\overline{r}_2} \tag{2.6}$$
$$\overline{Y}_{12} + \overline{Y}_{13} - \overline{Y}_{22} - \overline{Y}_{23} = \tfrac{r_2}{r_1}(x+y)(-d-\tfrac{1}{2}) + \tfrac{1}{2}(x-y)\overline{r}_1 r_2 \tag{2.7}$$
$$Y_{11} + Y_{12} - Y_{13} + Y_{14} = \tfrac{r_2}{r_1}(x+y)(-d+\tfrac{1}{2}) + \tfrac{1}{2}(x-y)\overline{r}_1 r_2 \tag{2.8}$$
$$\begin{aligned}\overline{Y}_{11} + \overline{Y}_{12} + \overline{Y}_{13} - \overline{Y}_{14} &= 2\tfrac{r_2}{r_1}cd(x+y) - d\left[\tfrac{r_2}{r_1}(x+y) + \tfrac{1}{r_1\overline{r}_2}(x-y)\right] \\ &\quad - \overline{r}_1 r_2 c(x-y) + \tfrac{1}{2}\left[\overline{r}_1 r_2(x-y) + \tfrac{\overline{r}_1}{\overline{r}_2}(x+y)\right]\end{aligned} \tag{2.9}$$
$$\begin{aligned}Y_{11} + Y_{13} - Y_{12} + Y_{14} &= 2\tfrac{r_2}{r_1}cd(x+y) - c\left[\tfrac{r_2}{r_1}(x+y) + \overline{r}_1 r_2(x-y)\right] \\ &\quad - \tfrac{1}{r_1\overline{r}_2}d(x-y) + \tfrac{1}{2}\left[\tfrac{\overline{r}_1}{\overline{r}_2}(x+y) + \tfrac{1}{r_1\overline{r}_2}(x-y)\right]\end{aligned} \tag{2.10}$$

We note that we have also harmlessly replaced some terms by their conjugates. We now simplify these terms even further, preserving their status of integrality. Subtracting (2.5) from (2.6) shows that the integrality of these implies that $\frac{r_2}{r_1}(x+y)$ is integral. In particular, it thus follows that if (2.5) is integral then so is

$$\tfrac{r_2}{r_1}(x+y)(-c)+\tfrac{1}{2}(x-y)\tfrac{1}{r_1\overline{r}_2}. \tag{2.11}$$

Multiplying this expression by the valuation-zero term $\pi^{-(v(r_1)+v(r_2))}r_1\overline{r}_2$ and applying Proposition 2.1.1 shows that each term of this expression is in fact integral exactly when the entire expression is integral. We can of course apply the same reasoning to (2.7) and (2.8), which shows that (2.5)-(2.8) being integral is equivalent to the following expressions being integral:

$$E_1 = \tfrac{r_2}{r_1}(x+y) \qquad E_4 = (x-y)\tfrac{1}{r_1\overline{r}_2}$$
$$E_2 = \tfrac{r_2}{r_1}(x+y)c \qquad E_5 = (x-y)\overline{r}_1 r_2$$
$$E_3 = \tfrac{r_2}{r_1}(x+y)d$$

We observe also that if we subtract (2.8) from (2.9), we get the same thing as subtracting (2.6) from (2.10), both differences being equal to

$$2\tfrac{r_2}{r_1}cd(x+y)-(x-y)\left[c\overline{r}_1 r_2+\tfrac{d}{r_1\overline{r}_2}\right]+\tfrac{1}{2}\tfrac{\overline{r}_1}{\overline{r}_2}(x+y) \tag{2.12}$$

so we might as well replace (2.9) and (2.10) by (2.12). We can again apply Proposition 2.1.1 to (2.12) multiplied by $\pi^{-(v(r_1)+v(r_2))}r_1\overline{r}_2$, which tells us that the integrality of (2.12) is actually equivalent to the integrality of these two:

$$E_6 = 4\tfrac{r_2}{r_1}cd(x+y)+\tfrac{\overline{r}_1}{\overline{r}_2}(x+y)$$
$$E_7 = (x-y)\left[c\overline{r}_1 r_2+\tfrac{d}{r_1\overline{r}_2}\right]$$

We have come to the end of our simplifications on the conditions that determine whether $Y = \operatorname{Ad}(u^{-1})\operatorname{Ad}(m^{-1})\gamma$ is integral.

2.5 Elimination of Fiendish Cases

We have determined in §2.4 expressions $E_1, E_2, \ldots, E_7$ that are integral exactly when the element

$$\operatorname{Ad}(u^{-1})\operatorname{Ad}(m^{-1})\gamma$$

is integral. We denote by $E_1^{\mathrm{stc}}, E_2^{\mathrm{stc}}, \ldots, E_7^{\mathrm{stc}}$ the corresponding equations for γ_{stc}. Recall that to get the conditions for γ_{stc}, we just replace $\overline{r}_i$ by $\pi\overline{r}_i$ for $i = 1, 2$ in $E_1, \ldots, E_7$. For the remainder of the chapter, we set

$$\begin{aligned} h &= v(r_1) + v(r_2), \\ V_m &= v(x - y) \\ V_p &= v(x + y). \end{aligned}$$

To aid the reader, we list $E_1, \dots, E_7$ and their stable versions, along with their valuations:

$$
\begin{aligned}
E_1 &= \tfrac{r_2}{r_1}(x+y) & E_1^{\mathrm{stc}} &= \tfrac{r_2}{r_1}(x+y)\\
v(E_1) &= h + V_p - 2v(r_1) & v(E_1^{\mathrm{stc}}) &= h + V_p - 2v(r_1)\\
E_2 &= \tfrac{r_2}{r_1}(x+y)c & E_2^{\mathrm{stc}} &= \tfrac{r_2}{r_1}(x+y)c\\
v(E_2) &= h + V_p + v(c) - 2v(r_1) & v(E_2^{\mathrm{stc}}) &= h + V_p + v(c) - 2v(r_1)\\
E_3 &= \tfrac{r_2}{r_1}(x+y)d & E_3^{\mathrm{stc}} &= \tfrac{r_2}{r_1}(x+y)d\\
v(E_3) &= h + V_p + v(d) - 2v(r_1) & v(E_3^{\mathrm{stc}}) &= h + V_p + v(d) - 2v(r_1)\\
E_4 &= (x-y)\tfrac{1}{r_1\overline{r}_2} & E_4^{\mathrm{stc}} &= (x-y)\tfrac{1}{\pi r_1\overline{r}_2}\\
v(E_4) &= V_m - h & v(E_4^{\mathrm{stc}}) &= V_m - h - 1\\
E_5 &= (x-y)\overline{r}_1 r_2 & E_5^{\mathrm{stc}} &= (x-y)\pi\overline{r}_1 r_2\\
v(E_5) &= V_m + h & v(E_5) &= V_m + h + 1\\
E_6 &= 4\tfrac{r_2}{r_1}cd(x+y) + \tfrac{\overline{r}_1}{\overline{r}_2}(x+y) & E_6^{\mathrm{stc}} &= 4\tfrac{r_2}{r_1}cd(x+y) + \tfrac{\overline{r}_1}{\overline{r}_2}(x+y)\\
v(E_6) &\geq \min\left\{ \begin{smallmatrix} h+V_p+v(c)+v(d)-2v(r_1), \\ V_p-h+2v(r_1) \end{smallmatrix} \right\} & v(E_6^{\mathrm{stc}}) &\geq \min\left\{ \begin{smallmatrix} h+V_p+v(c)+v(d)-2v(r_1), \\ V_p-h+2v(r_1) \end{smallmatrix} \right\}\\
E_7 &= (x-y)\left[c\overline{r}_1 r_2 + \tfrac{d}{r_1\overline{r}_2}\right] & E_7^{\mathrm{stc}} &= (x-y)\left[c\pi\overline{r}_1 r_2 + \tfrac{d}{\pi r_1\overline{r}_2}\right]\\
v(E_7) &\geq \min\left\{ \begin{smallmatrix} V_m+v(c)+h, \\ V_m+v(d)-h \end{smallmatrix} \right\} & v(E_7^{\mathrm{stc}}) &\geq \min\left\{ \begin{smallmatrix} V_m+v(c)+h+1, \\ V_m+v(d)-h-1 \end{smallmatrix} \right\}
\end{aligned}
$$

We see that only the fourth, fifth, and seventh expressions differ between γ and γ_{stc}. We observe that the difficulties will occur mainly with the sixth and seventh, since they are sums. *In this section*, we eliminate some of the more fiendish difficulties to prepare the way for the main calculation in §2.6. In order to lessen the wordiness and symbolism in the sequel, the *summands of* E_6 will refer to the two terms $4\frac{r_2}{r_1}cd(x+y)$ and $\frac{\overline{r}_1}{\overline{r}_2}(x+y)$. Similarly, the *summands of* E_7 will refer to the two terms: $(x-y)c\overline{r}_1 r_2$ and $(x-y)\frac{d}{r_1\overline{r}_2}$. We also use this terminology, suitably modified, for the stable versions. For example, if E_6 is integral, then we know that either both summands are integral or neither are. These possibilities for E_6 and E_7 break down the computation into various cases, and the next proposition shows that the worst of these actually cannot occur.

2.5.1 Lemma. *If none of the summands in* E_6 *and* E_7 *are integral, then* E_6 *and* E_7 *cannot* *be simultaneously integral.*

Proof. We proceed by contradiction, assuming that none of the summands in E_6 and E_7 are integral, but that both E_6 and E_7 are integral. In this case, the valuation of the first summand must be equal to the valuation of the second in E_6, and the same is true of E_7. Thus, we get a pair of equations:

$$2[v(r_2) - v(r_1)] + v(c) + v(d) = 0,$$
$$2[v(r_2) + v(r_1)] + v(c) - v(d) = 0.$$

In particular, $v(c) = -2v(r_2)$ and $v(d) = 2v(r_1)$. Hence

$$v\left(\tfrac{r_2}{r_1}d(x+y)\pi^{-(h+V_p+V_m)}\right) = -V_m.$$

Multiply E_6 by the inverse of the expression in $v(-)$ to get

$$4c\pi^{h+V_p+V_m} + \frac{N(r_1)\pi^{h+V_p+V_m}}{dN(r_2)} \in \pi^{V_m}\mathfrak{o}. \tag{2.13}$$

Using the same procedure on E_7 gives

$$4c\pi^{h+V_m+V_p} + \frac{4d\pi^{h+V_m+V_p}}{N(r_1)N(r_2)} \in \pi^{V_p}\mathfrak{o}. \tag{2.14}$$

We take the difference between (2.13) and (2.14), obtaining

$$\frac{\pi^{h+V_m+V_p}}{dN(r_1)N(r_2)}(N(r_1) - 2d)(N(r_1) + 2d). \tag{2.15}$$

We note that $N(r_1) \in F$ whereas $2d \in F\delta$. Hence, the valuation of $N(r_1) \pm 2d$ is precisely $v(d)$. Thus, the valuation of (2.15) is $V_m + V_p + v(d) - h$. There are two cases to consider: either $V_p \geq V_m$ or $V_p < V_m$.

Case 1: $V_p \geq V_m$**.** Then (2.15) lies in $\pi^{V_m}\mathfrak{o}$, or in other words, $v(d) + V_p + V_m - h \geq V_m$. Simplifying, we get $v(d) + V_p - h \geq 0$. On the other hand, the first summand of E_6 also has valuation $v(d) + V_p - h$, showing that this summand is integral, which is a contradiction.

Case 2: $V_p < V_m$. Then (2.15) lies in $\pi^{V_p}\mathfrak{o}$, or in other words, $v(d) + V_p + V_m - h \geq V_p$. Hence $v(d) + V_m - h \geq 0$, but this is the valuation of the second summand of E_7, which is again a contradiction. ■

2.5.2 Remark. In this thesis, Case 2 in the above proof does not actually occur since we will assume for the actual computation that $V_p > V_m$, but we have included the more general statement for completeness.

The reader will have no trouble applying the same argument to prove the stable version.

2.5.3 Lemma. *If none of the summands in E_6^{stc} and E_7^{stc} are integral, then E_6^{stc} and E_7^{stc} cannot be simultaneously integral.*

The next lemma allows us a significant simplification if we stick with a "limiting case" for γ.

2.5.4 Lemma. *Suppose that $V_p > V_m$, that $E_1, E_2, \ldots, E_7$ are integral, and that not all the E_i^{stc} are integral (the last condition being equivalent to: E_7^{stc} is not integral). Under these conditions, if the summands of E_7 are integral, then the summands of E_6 are integral as well.*

Proof. We suppose by contradiction that we have a solution that makes E_1 to E_7 integral, that the summands of E_7 are integral, but that the summands of E_6 are not integral. Before we list the inequalities in this case, let us make three observations.

1. From E_2, we get $h + V_p + v(c) - 2v(r_1) \geq 0$. However, the first summand of E_6 not being integral is equivalent to $h + V_p + v(c) + v(d) - 2v(r_1) < 0$. Hence $v(d) < 0$. Repeating the argument with E_3 in place of E_2 shows $v(c) < 0$.

2. The valuation of the first summand of E_6 is equal to the valuation of the second. This implies that $v(c) + v(d) = 4v(r_1) - 2h$.

3. Since $E_4 = E_4^{\mathrm{stc}}$, any solution of $E_1, \ldots, E_7$ will also be a solution of the stable versions unless $v(d) = h - V_m$, so we evaluate only under this additional condition, and this implies based on our second observation that $v(c) = 4v(r_1) - 3h + V_m$.

Hence we have the following inequalities, by using the substitutions in (1)-(3) in E_2, E_3, either term of E_6, and the first term of E_7 being integral:

$$\begin{aligned} 2v(r_1) &\geq 2h - V_p - V_m \\ 2h + V_p - V_m &\geq 2v(r_1) \\ h - V_p &> 2v(r_1) \\ 2v(r_1) &\geq h - V_m \end{aligned}$$

Or, more succinctly,

$$\min\{2h + V_p - V_m, h - V_p - 1\} \geq 2v(r_1) \geq \max\{2h - V_p - V_m, h - V_m\}.$$

We see that $2h + V_p - V_m \geq h - V_p - 1$ is equivalent to $h + 2V_p - V_m + 1 \geq 0$. Since $V_p > V_m$, so that $h + 2V_p - V_m + 1 \geq h + V_m + 1 \geq 1$. Similarly, $h - V_m \geq 2h - V_m - V_m$ is equivalent to $V_p \geq h$, which is true again since $V_p > V_m$. Hence we have that,

$$h - V_p - 1 \geq 2v(r_1) \geq h - V_m.$$

So $h - V_p - 1 \geq h - V_m$, or equivalently, $V_m \geq V_p + 1$, which is absurd. ■

Again, the same argument will apply to the stable version.

2.5.5 Lemma. *Suppose that $V_p > V_m$, and that $E_1^{\mathrm{stc}}, E_2^{\mathrm{stc}}, \ldots, E_7^{\mathrm{stc}}$ are integral, but that at least one of $E_1, \ldots, E_7$ is not integral. If the summands of E_7^{stc} are integral, then the summands of E_6^{stc} are integral as well.* ■

2.6 Brute Force Calculations

We have introduced the orbital integral

$$\mathcal{O}_\gamma^\kappa = \int_{G_0(F)} \mathbf{1}(\mathrm{Ad}(g)^{-1}\gamma)\mathrm{d}g - \int_{G_0(F)} \mathbf{1}(\mathrm{Ad}(g)^{-1}\gamma_{\mathrm{stc}})\mathrm{d}g. \tag{2.16}$$

The integral $\int_{G_0(F)} \mathbf{1}(\mathrm{Ad}(g^{-1}\gamma)\mathrm{d}g$ is the same as the measure of the set

$$\{(c, d, r_1, r_2) \in F\delta \times F\delta \times E^\times \times E^\times : E_1, \ldots, E_7 \text{ are integral } \},$$

and the analogous statement holds for the stable version. To evaluate the orbital integral, we do as follows: first, we find the measure of the subset of $F\delta \times F\delta \times E^\times \times E^\times$ such that all the E_* are integral but at least one of E_*^{stc} is not integral. We refer to this as the $(1,0)$-case. Similarly, the $(0,1)$-case is when all of E_*^{stc} are integral but at least one of E_* is not. We then take the measure of the $(1,0)$ case and subtract the measure of the $(0,1)$ case.

In this section we carry out the calculation, evaluating the integral. Our strategy is to fix $h = v(r_1) + v(r_2)$, write down an expression for the measure of the solution set, and then sum over all the possibilities for h:

2.6.1 Lemma. *The integrality of E_4 and E_5 is equivalent to the inequality $V_m \geq h \geq -V_m$. Similarly, the integrality of E_4^{stc} and E_5^{stc} is equivalent to $V_m - 1 \geq h \geq -V_m - 1$. In either case, if $V_m < 0$, then neither of this equalities can be satisfied and hence the κ-orbital integral vanishes.*■

2.6.2 Definition. We say that γ is *nearly singular* if $V_p > V_m$.

For the rest of this chapter, we assume that γ is nearly singular, which is relatively harmless since our calculation under this assumption still should give us the correct transfer factor, assuming that there is a sane version of endsocopy operating in the midst. At any rate, in view of Lemmas 2.5.1,2.5.3,2.5.4, and 2.5.5, we then have to consider two possibilities: all summands in E_6 and in E_7 are integral, and the summands of E_6 are integral but the summands of E_7 are not.

2.7 Integer Arithmetic and Measures

Here we state the properties of floor and ceiling functions we use. For any $r \in \mathbb{R}$ we write $\lfloor r \rfloor$ and $\lceil r \rceil$ for the floor of r and the ceiling of r respectively. If $a, b \in \mathbb{Z}$, then we will frequently need the following facts that are easy to verify, but included for convenience in following lengthy computations:

$$\left|\{x \in \mathbb{Z} : a \geq 2x \geq b\}\right| = \left\lfloor \frac{a}{2} \right\rfloor - \left\lceil \frac{b}{2} \right\rceil + 1 = \begin{cases} \frac{a-b}{2} + 1 & \text{if } a, b \text{ are even} \\ \frac{a-b}{2} & \text{if } a, b \text{ are odd} \\ \frac{a-b+1}{2} & \text{if } a, b \text{ have opposite parity} \end{cases}$$

$$\left\lfloor \frac{a}{2} \right\rfloor = \left\lceil \frac{a-1}{2} \right\rceil$$

$$\left\lfloor \frac{a}{2} \right\rfloor + 1 = \left\lfloor \frac{a+2}{2} \right\rfloor$$

$$\left\lceil \frac{a}{2} \right\rceil + 1 = \left\lceil \frac{a+2}{2} \right\rceil$$

In all sections, $(c,d) \in F\delta \times F\delta$, and $F\delta$ has the Haar measure so that $\mathfrak{o}\delta$ has unit volume. Then $F\delta \times F\delta$ has the product measure. Moreover, $r_1 \in E^\times$, and $E^\times$ has the Haar measure so that $\mathfrak{o}_E^\times$ has unit volume.

2.8 Computations: All Summands Integral

In this section, we evaluate case $(1,0)$ (resp. $(0,1)$) when all summands of E_6 and E_7 (resp. E_6^{stc} and E_7^{stc}) are integral. We consider two cases: one where $h = V_m - 1, \ldots, -V_m$ for both integrals, and the other where $h = V_m$ for $\mathcal{O}_\gamma$ and $h = -V_m - 1$ for $\mathcal{O}_{\gamma_{\text{stc}}}$. In order to make reading this section easier, here are the inequalities that must be satisfied in this case:

2.8.1 Lemma. *Suppose all summands are integral. Then the inequalities defining the set* *that we must determine the measure of are:*

For $\mathcal{O}_\gamma$	*For* $\mathcal{O}_{\gamma_{\text{stc}}}$
$h + V_p - 2v(r_1) \geq 0$	
$h + V_p + v(c) - 2v(r_1) \geq 0$	
$h + V_p + v(d) - 2v(r_1) \geq 0$	*Same as for* $\mathcal{O}_\gamma$
$h + V_p + v(c) + v(d) - 2v(r_1) \geq 0$	
$V_p - h + 2v(r_1) \geq 0$	
$V_m + v(c) + h \geq 0$	$V_m + v(c) + h + 1 \geq 0$
$V_m + v(d) - h \geq 0$	$V_m + v(d) - h - 1 \geq 0$

Proof. We take the terms listed at the beginning of §2.5 and set the valuation of each of them to be greater than or equal to zero for E_1 to E_5 and E_1^{stc} to E_5^{stc}, and set the valuation of each summand to be greater than or equal to zero for $E_6, E_7, E_6^{\text{stc}}$ and E_7^{stc}. We note that we have not written down the inequalities for $E_4, E_4^{\text{stc}}, E_5$, or E_5^{stc} because these will automatically be integral given our assumptions on h. ■

Case $(1, 0)$: The Integral $\mathcal{O}_\gamma$ for $h = V_m$

Our starting inequalities at the start of §2.8 reduce to the following:

$$V_m + V_p - 2v(r_1) \geq 0 \tag{2.17}$$

$$V_m + V_p + v(c) - 2v(r_1) \geq 0 \tag{2.18}$$

$$V_m + V_p + v(d) - 2v(r_1) \geq 0 \tag{2.19}$$

$$V_m + V_p + v(c) + v(d) - 2v(r_1) \geq 0 \tag{2.20}$$

$$V_p - V_m + 2v(r_1) \geq 0 \tag{2.21}$$

$$2V_m + v(c) \geq 0 \tag{2.22}$$

$$v(d) \geq 0 \tag{2.23}$$

We note that since $h = V_m$, the expressions E_*^{stc} cannot all be integral since in that case we must have $V_m - 1 \geq h$. Since $v(d) \geq 0$, we see that (2.19) and (2.20) are redundant, so we can eliminate them. In subsequent calculations, we shall frequently eliminate the obvious redundant inequalities without note. There are two cases to consider: $v(c) \geq 0$ and $v(c) < 0$.
Case 1: $v(c) \geq 0$. The remaining inequalities are

$$\begin{gathered} V_m + V_p \geq 2v(r_1) \geq V_m - V_p \\ v(c) \geq 0 \\ v(d) \geq 0. \end{gathered}$$

At this point, the reader may wish to review §2.7 containing various identities with floor and ceiling functions. Using these we see that the measure of the corresponding set of solutions is

$$\begin{cases} V_p + 1 & \text{if } V_m + V_p \text{ is even} \\ V_p & \text{if } V_m + V_p \text{ is odd} \end{cases}$$

Case 2: $v(c) < 0$. Now the relevant inequalities are

$$V_m + V_p + v(c) \geq 2v(r_1) \geq V_m - V_p \tag{2.24}$$

$$0 > v(c) \geq -2V_m \tag{2.25}$$

$$v(d) \geq 0 \tag{2.26}$$

We see that (2.24) implies that $v(c) \geq -2V_p$, which we would have to use instead of $v(c) \geq -2V_m$ if we did not assume $V_p > V_m$. We have the measure

$$(1 - q^{-1}) \sum_{v(c)=-2V_m}^{-1} q^{-v(c)} \left(\left\lfloor \tfrac{V_m+V_p+v(c)}{2} \right\rfloor - \left\lceil \tfrac{V_m-V_p}{2} \right\rceil + 1 \right)$$

Case $(1,0)$: The Integral $\mathcal{O}_\gamma$: $h = V_m - 1, \ldots, -V_m$

The assumptions for h are equivalent to $E_4, E_5 \in \mathfrak{o}$ and $E_4^{\rm stc}, E_5^{\rm stc} \in \mathfrak{o}$. Hence, any solution that makes E_i integral will make $E_i^{\rm stc}$ integral except when E_7 is integral but $E_7^{\rm stc}$ is not, which is equivalent to $V_m + v(d) - h \geq 0$ but $V_m + v(d) - h - 1 < 0$. In other words, $v(d) = -V_m + h$. Since $h \leq V_m - 1$ by assumption, this implies $v(d) < 0$. Under this additional requirement, we reduce to the following.

$$\begin{aligned} 2h + V_p - V_m &\geq 2v(r_1) \\ 2h + V_p - V_m + v(c) &\geq 2v(r_1) \\ 2v(r_1) &\geq h - V_p \\ v(c) &\geq -h - V_m \\ v(d) &= h - V_m \end{aligned}$$

We see again that there are two cases: $v(c) \geq 0$ and $v(c) < 0$.
Case 1: $v(c) \geq 0$. Then the inequalities reduce to the product set defined by

$$\begin{aligned} 2h + V_p - V_m \geq 2v(r_1) &\geq h - V_p \\ v(c) &\geq 0 \\ v(d) &= h - V_m \end{aligned}$$

Hence the measure here is

$$q^{V_m - h}(1 - q^{-1})\left(\left\lfloor \tfrac{2h+V_p-V_m}{2} \right\rfloor - \left\lceil \tfrac{h-V_p}{2} \right\rceil + 1\right)$$

Case 2: $v(c) < 0$. The relevant inequalities are

$$\begin{aligned} 2h + V_p - V_m + v(c) \geq 2v(r_1) &\geq h - V_p \\ 0 > v(c) &\geq -h - V_m \\ v(d) &= h - V_m \end{aligned}$$

Hence the measure of the corresponding set is

$$q^{V_m-h}(1-q^{-1})^2\sum_{v(c)=-h-V_m}^{-1} q^{-v(c)}\left(\left\lfloor\frac{2h+V_p-V_m+v(c)}{2}\right\rfloor-\left\lceil\frac{h-V_p}{2}\right\rceil+1\right)$$

Case $(0,1)$: The Integral $\mathcal{O}_{\gamma_{\rm stc}}$: $h=-V_m-1$

Since $h=-V_m-1$, the expressions E_* cannot all be integral. Because $V_m+v(c)+h+1\geq 0$, putting $h=-V_m-1$ into this gives $v(c)\geq 0$. We are left with

$$\begin{aligned}
-V_m-1+V_p-2v(r_1)&\geq 0\\
-V_m-1+V_p+v(d)-2v(r_1)&\geq 0\\
V_p+V_m+1+2v(r_1)&\geq 0\\
v(c)&\geq 0\\
v(d)&\geq -2V_m
\end{aligned}$$

We do two cases: $v(d)\geq 0$ and $v(d)<0$.
Case 1: $v(d)\geq 0$. We have:

$$\begin{aligned}
V_p-V_m-1\geq 2v(r_1)&\geq -V_p-V_m-1\\
v(c)&\geq 0\\
v(d)&\geq 0
\end{aligned}$$

The measure of this set is

$$\begin{cases} V_p & \text{if } V_m+V_p \text{ is even}\\ V_p+1 & \text{if } V_m+V_p \text{ is odd}\end{cases}$$

Case 2: $v(d) < 0$.

$$V_p - V_m - 1 + v(d) \geq 2v(r_1) \geq -V_p - V_m - 1 \tag{2.27}$$
$$v(c) \geq 0 \tag{2.28}$$
$$0 > v(d) \geq -2V_m \tag{2.29}$$

Transitivity in (2.27) shows that $v(d) \geq -2V_p$, but this is already satisfied under our hypothesis $V_p > V_m$. We see that the measure of this set is

$$(1 - q^{-1}) \sum_{v(d)=-2V_m}^{-1} q^{-v(d)} \left(\left\lfloor \tfrac{V_p - V_m - 1 + v(d)}{2} \right\rfloor - \left\lceil \tfrac{-V_p - V_m - 1}{2} \right\rceil + 1 \right)$$

Case $(0,1)$: The Integral $\mathcal{O}_{\gamma_{\text{stc}}}$: $h = V_m - 1, \ldots, -V_m$

We just need to evaluate under the conditions that each E_i^{stc} is integral but at least one of E_i is not. The only way this can happen is when $v(c) = -V_m - h - 1$. In particular, this implies that $v(c) < 0$.

We start with the following (in)equalities:

$$h + V_p + v(c) - 2v(r_1) \geq 0 \tag{2.30}$$
$$h + V_p + v(c) + v(d) - 2v(r_1) \geq 0 \tag{2.31}$$
$$V_p - h + 2v(r_1) \geq 0 \tag{2.32}$$
$$V_m + v(c) + h + 1 = 0 \tag{2.33}$$
$$V_m + v(d) - h - 1 \geq 0 \tag{2.34}$$

We do two cases: $v(d) \geq 0$ and $v(d) < 0$.

Case 1: $v(d) \geq 0$. Then, taking the above inequalities, eliminating the redundant ones

((2.31) and (2.34)), and putting $v(c) = -V_m - h - 1$ gives

$$\begin{aligned} V_p - V_m - 1 \geq 2v(r_1) &\geq h - V_p \\ v(d) &\geq 0 \\ v(c) &= -V_m - h - 1. \end{aligned}$$

The measure of this set is then

$$q^{V_m+h+1}(1-q^{-1})\left(\left\lfloor \tfrac{V_p-V_m-1}{2} \right\rfloor - \left\lceil \tfrac{h-V_p}{2} \right\rceil + 1\right)$$

Case 2: $v(d) < 0$. We again take (2.30)-(2.34), eliminate the redundant (2.30) and make the substitution $v(c) = -V_m - h - 1$ to get:

$$\begin{aligned} V_p - V_m - 1 + v(d) \geq 2v(r_1) &\geq h - V_p \\ 0 > v(d) &\geq h + 1 - V_m \\ v(c) &= -V_m - h - 1 \end{aligned}$$

Giving us the measure

$$q^{V_m+h+1}(1-q^{-1})^2 \sum_{v(d)=h+1-V_m}^{-1} q^{-v(d)} \left(\left\lfloor \tfrac{V_p-V_m-1+v(d)}{2} \right\rfloor - \left\lceil \tfrac{h-V_p}{2} \right\rceil + 1\right)$$

2.9 Computation: All Summands Integral: Taking the Difference

In this section, we take the measure we have found so far for case $(1,0)$ and subtract from it the measure for case $(0,1)$.

Extreme Cases: $h = V_m$ for $\mathcal{O}_\gamma$ and $h = -V_m - 1$ for $\mathcal{O}_{\gamma_{\text{stc}}}$

Here, we subtract from the measure for $h = V_m$ in $\mathcal{O}_\gamma$ the measure for $h = -V_m - 1$ in $\mathcal{O}_{\gamma_{\text{stc}}}$, for all summands being integral. We find the difference to be:

$$(-1)^{V_m+V_p} + (1-q^{-1})\sum_{j=1}^{2V_m} q^j \left(\left\lfloor \tfrac{V_m+V_p-j}{2} \right\rfloor - \left\lceil \tfrac{V_m-V_p}{2} \right\rceil - \left\lfloor \tfrac{V_p-V_m-1-j}{2} \right\rfloor + \left\lceil \tfrac{-V_p-V_m-1}{2} \right\rceil \right)$$

In the summation, we see that the terms where j is odd vanish, leaving us with:

$$(-1)^{V_m+V_p} + (1-q^{-1})\sum_{j=1}^{V_m} q^{2j} \left(\left\lfloor \tfrac{V_m+V_p-2j}{2} \right\rfloor - \left\lceil \tfrac{V_m-V_p}{2} \right\rceil - \left\lfloor \tfrac{V_p-V_m-1-2j}{2} \right\rfloor + \left\lceil \tfrac{-V_p-V_m-1}{2} \right\rceil \right)$$

Simplifying the floor and ceiling functions gives $(-1)^{V_m+V_p}$, so that we get

$$(-1)^{V_m+V_p}\left(1 + (1-q^{-1})\sum_{j=1}^{V_m} q^{2j}\right) = (-1)^{V_m+V_p}\left(1 + (q^{2V_m}-1)\frac{q}{q+1}\right).$$

$h = V_m - 1, \ldots, -V_m$

There were two cases here: the first, where $v(c) \geq 0$ for $\mathcal{O}_\gamma$ and $v(d) \geq 0$ for $\mathcal{O}_{\gamma_{\text{stc}}}$, and the second (reverse the inequalities).

Case 1: When $v(c) \geq 0$ for $\mathcal{O}_\gamma$ and $v(d) \geq 0$ for $\mathcal{O}_{\gamma_{\text{stc}}}$. In this case we had for $\mathcal{O}_\gamma$ the measure:

$$q^{V_m-h}(1-q^{-1})\left(\left\lfloor \tfrac{2h+V_p-V_m}{2} \right\rfloor - \left\lceil \tfrac{h-V_p}{2} \right\rceil + 1 \right)$$

and for $\mathcal{O}_{\gamma_{\text{stc}}}$:

$$q^{V_m+h+1}(1-q^{-1})\left(\left\lfloor \tfrac{V_p-V_m-1}{2} \right\rfloor - \left\lceil \tfrac{h-V_p}{2} \right\rceil + 1 \right)$$

We have to sum both over $h = -V_m, \ldots, V_m - 1$, and subtract the second from the first. However, in placing this in the summation, we may replace h in the second expression with $-h-1$, a transformation which preserves the summation range. We do this since then we will get pairs of nicely paired terms in the sum. So we get:

$$q^{V_m}(1-q^{-1})\sum_{h=-V_m}^{V_m-1} q^{-h}\left(\left\lfloor \tfrac{2h+V_p-V_m}{2}\right\rfloor - \left\lceil \tfrac{V_p-V_m-2}{2}\right\rceil + \left\lfloor \tfrac{-h-V_p}{2}\right\rfloor - \left\lceil \tfrac{h-V_p}{2}\right\rceil\right)$$

where we have converted the floor to ceiling and vice-versa in the second equation to make following the computations with §2.7 easier. We break the summation into two sums: one over $h=-V_m,-V_m-2,\dots,V_m-2$, and the other over $h=-V_m+1,\dots,V_m-1$. In other words, the first is over integers of the same parity as V_m and the second opposite parity. However, if we look at the opposite-parity case, we see that $2h+V_p-V_m$ and V_p-V_m-2 have the same parity, which is the opposite parity of both $-h-V_p$ and $h-V_p$. Hence, the sum vanishes. So we just have sum over $h=-V_m,-V_m+2,\dots,V_m-2$. In this case, we get

$$\begin{aligned} &(-1)^{V_m+V_p}q^{V_m}(1-q^{-1})(q^{V_m}+q^{V_m-2}+\cdots+q^{-V_m+2})\\ &=(-1)^{V_m+V_p}q^2(1-q^{-1})(1+q^2+\cdots+(q^2)^{V_m-1})\\ &=(-1)^{V_m+V_p}(q^{2V_m}-1)\frac{q}{q+1}\end{aligned}$$

Case 2: When $v(c)<0$ for $\mathcal{O}_\gamma$ and $v(d)<0$ for $\mathcal{O}_{\gamma_{\mathrm{stc}}}$ This is a little more lengthy, but not terribly so. We recall the two terms. The first for $\mathcal{O}_\gamma$ is

$$q^{V_m-h}(1-q^{-1})^2\sum_{v(c)=-h-V_m}^{-1} q^{-v(c)}\left(\left\lfloor \tfrac{2h+V_p-V_m+v(c)}{2}\right\rfloor - \left\lceil \tfrac{h-V_p}{2}\right\rceil + 1\right)$$

The one for $\mathcal{O}_{\gamma_{\mathrm{stc}}}$ is

$$q^{V_m+h+1}(1-q^{-1})^2\sum_{v(d)=h+1-V_m}^{-1} q^{-v(d)}\left(\left\lfloor \tfrac{V_p-V_m-1+v(d)}{2}\right\rfloor - \left\lceil \tfrac{h-V_p}{2}\right\rceil + 1\right)$$

The first thing we do is replace h by $-h-1$ in the γ_{stc}-version, and use j as the index of summation over positive instead of negative numbers. After doing this, converting the appropriate floors to ceilings and vice-versa, *and* subtracting the second from the first, we get:

$$q^{V_m-h}(1-q^{-1})^2\sum_{j=1}^{V_m+h} q^j\left(\left\lfloor\tfrac{2h+V_p-V_m-j}{2}\right\rfloor-\left\lceil\tfrac{V_p-V_m-2-j}{2}\right\rceil+\left\lfloor\tfrac{-h-V_p}{2}\right\rfloor-\left\lceil\tfrac{h-V_p}{2}\right\rceil\right)$$

Of course, we have still *not* summed over h yet, and we also note that if $h=-V_m$, the sum is actually empty. Anyways, to make sense of this chaos we write down two separate summations again: one for $h=-V_m,-V_m+2,\dots,V_m-2$ and one for $h=-V_m+1,-V_m+3,\dots,V_m-1$.

Case 2.1: $h=-V_m,-V_m+2,\dots,V_m-2$**.** Here, the upper limit of the summation is even. We also split the summation into two sums, depending on whether j is even or odd:

$$\begin{aligned}q^{V_m-h}(1-q^{-1})^2\Bigg[&\sum_{j=1}^{\frac{V_m+h}{2}} q^{2j}\left(\left\lfloor\tfrac{2h+V_p-V_m-2j}{2}\right\rfloor-\left\lceil\tfrac{V_p-V_m-2-2j}{2}\right\rceil+\left\lfloor\tfrac{-h-V_p}{2}\right\rfloor-\left\lceil\tfrac{h-V_p}{2}\right\rceil\right)\\&+\sum_{j=1}^{\frac{V_m+h}{2}} q^{2j-1}\left(\left\lfloor\tfrac{2h+V_p-V_m-2j+1}{2}\right\rfloor-\left\lceil\tfrac{V_p-V_m-1-2j}{2}\right\rceil+\left\lfloor\tfrac{-h-V_p}{2}\right\rfloor-\left\lceil\tfrac{h-V_p}{2}\right\rceil\right)\Bigg]\end{aligned}$$

We again see that the summation where j is odd vanishes, and we simplify the rest to get

$$\begin{aligned}(-1)^{V_m+V_p}q^{V_m-h}(1-q^{-1})^2\sum_{j=1}^{\frac{V_m+h}{2}} q^{2j}&=(-1)^{V_m+V_p}q^{V_m-h}(1-q^{-1})^2q^2\frac{q^{V_m+h}-1}{q^2-1}\\&=(-1)^{V_m+V_p}\frac{q-1}{q+1}(q^{2V_m}-q^{V_m-h})\end{aligned}$$

As a sanity check, putting in $h=-V_m$ gives zero. Good! Let's sum over h now to get:

$$\begin{aligned}&(-1)^{V_m+V_p}\left[\frac{q-1}{q+1}q^{2V_m}(V_m)-\frac{q-1}{q+1}q^{V_m}(q^{V_m}+q^{V_m-2}+q^{V_m-4}+\cdots+q^{-V_m+2})\right]\\&=(-1)^{V_m+V_p}\left[\frac{q-1}{q+1}q^{2V_m}(V_m)-\frac{q^2}{(q+1)^2}(q^{2V_m}-1)\right]\end{aligned}\tag{2.35}$$

Case 2.2: $h=-V_m+1,-V_m+3\cdots,V_m-1$**.** This time, the upper limit V_m+h is odd.

We again split the summation into two sums, depending on whether j is odd or even:

$$q^{V_m-h}(1-q^{-1})^2\Bigg[\sum_{j=1}^{\frac{V_m+h-1}{2}} q^{2j}\left(\left\lfloor\tfrac{2h+V_p-V_m-2j}{2}\right\rfloor-\left\lceil\tfrac{V_p-V_m-2-2j}{2}\right\rceil+\left\lfloor\tfrac{-h-V_p}{2}\right\rfloor-\left\lceil\tfrac{h-V_p}{2}\right\rceil\right)$$
$$+\sum_{j=1}^{\frac{V_m+h+1}{2}} q^{2j-1}\left(\left\lfloor\tfrac{2h+V_p-V_m-2j+1}{2}\right\rfloor-\left\lceil\tfrac{V_p-V_m-1-2j}{2}\right\rceil+\left\lfloor\tfrac{-h-V_p}{2}\right\rfloor-\left\lceil\tfrac{h-V_p}{2}\right\rceil\right)\Bigg]$$

This time, the opposite happens: in other words, the summation with even powers of q vanishes, and we are left with:

$$-(-1)^{V_m+V_p}q^{V_m-h}(1-q^{-1})^2\sum_{j=1}^{\frac{V_m+h+1}{2}} q^{2j-1}=-(-1)^{V_m+V_p}\frac{q-1}{q+1}(q^{2V_m}-q^{V_m-h-1})$$

Summing over $h=-V_m+1,-V_m+3,\cdots,V_m-1$ gives

$$-(-1)^{V_m+V_p}\left[\frac{q-1}{q+1}q^{2V_m}(V_m)-\frac{1}{(q+1)^2}(q^{2V_m}-1)\right]$$

We add this to (2.35) to get

$$(-1)^{V_m+V_p}\left[\frac{1}{(q+1)^2}(q^{2V_m}-1)-\frac{q^2}{(q+1)^2}(q^{2V_m}-1)\right]=(q^{2V_m}-1)\frac{1-q}{1+q}$$

Gathering of Terms

We have now collected all the terms in our integral for the "all summands positive" case. We add them together:

$$(-1)^{V_m+V_p}\left[1+(q^{2V_m}-1)\frac{q}{q+1}+(q^{2V_m}-1)\frac{q}{q+1}+(q^{2V_m}-1)\frac{1-q}{q+1}\right]=(-1)^{V_m+V_p}q^{2V_m}.$$

2.10 Computation: Summands of E_6 Integral Only

The last case is the case of the summands of E_6 being integral only. This case is a little different because here, it will be impossible that both E_7 and E_7^{stc} will be simultaneously satisfied.

2.10.1 Lemma. *For the summands of $E_6 = E_6^{\mathrm{stc}}$ to be integral and the summands of E_7 (resp. E_7^{stc}) to be not integral, the following inequalities have to be satisfied:*

For $\mathcal{O}_\gamma$	*(resp. For $\mathcal{O}_{\gamma_{\mathrm{stc}}}$)*
$h + V_p - 2v(r_1) \geq 0$	
$h + V_p + v(c) - 2v(r_1) \geq 0$	
$h + V_p + v(d) - 2v(r_1) \geq 0$	
$h + V_p + v(c) + v(d) - 2v(r_1) \geq 0$	
$V_p - h + 2v(r_1) \geq 0$	
$V_m + v(c) + h < 0$	$V_m + v(c) + h + 1 < 0$
$V_m + v(d) - h < 0$	$V_m + v(d) - h - 1 < 0$

Proof. We set the valuations of the expressions at the beginning of §2.5 to be greater than or equal to zero for E_1 to E_5, and we do the same for the summands of E_6. We also set the summands of E_7 (resp. E_7^{stc}) to have valuation less than zero. As usual, we have omitted the inequalities for $E_4, E_4^{\mathrm{stc}}, E_5, E_5^{\mathrm{stc}}$ since these are equivalent to our assumptions on h. ■

Case $(1, 0)$: The Integral $\mathcal{O}_\gamma$

Here, we come up with an expression for $h = -V_m, \ldots, V_m$. We make three straightforward observations:

1. Since $V_m + h + v(c) < 0$ and $V_m + h \geq 0$, we must have $v(c) < 0$, and similarly, $V_m - h + v(d) < 0$ implies that $v(d) < 0$

2. Since the summands of E_7 are not integral, we must have the valuations of these terms equal. Hence $v(d) - 2h = v(c)$.

3. In addition, since we are solving a congruence in E_7, once $v(d)$ is chosen, the measure of $\{c : v(E_7) \geq 0\}$ is q^{V_m+h}.

Making the substitution $v(c) = v(d) - 2h$ and eliminating redundancies gives the inequalities

$$V_p - h + 2v(d) \geq 2v(r_1) \geq h - V_p$$
$$h - V_m > v(d)$$

We see from the first that $v(d) \geq h - V_p$, which gives a lower limit for $v(d)$. Thus, the measure of this solution set is

$$q^{V_m+h}(1-q^{-1}) \sum_{v(d)=h-V_p}^{h-V_m-1} q^{-v(d)} \left(\left\lfloor \frac{V_p-h+2v(d)}{2} \right\rfloor - \left\lceil \frac{h-V_p}{2} \right\rceil + 1 \right). \tag{2.36}$$

Case $(0,1)$: The Integral $\mathcal{O}_{\gamma_{\mathrm{stc}}}$

Here we have essentially the same three observations as in §2.10, suitably modified.

1. Since $V_m + h + v(c) + 1 < 0$ and $V_m + h + 1 \geq 0$, we must have $v(c) < 0$, and similarly, $V_m - h + v(d) - 1 < 0$ implies that $v(d) < 0$

2. Since the summands of E_7^{stc} are not integral, we must have the valuations of these terms equal. Hence $v(d) - 2h - 2 = v(c)$.

3. In addition, since we are solving a congruence in E_7^{stc}, once $v(d)$ is chosen, the measure of $\{c : v(E_7) \geq 0\}$ is q^{V_m+h+1}.

We have the inequalities:

$$V_p - h + 2v(d) - 2 \geq 2v(r_1) \geq h - V_p$$
$$h + 1 - V_m > v(d) \geq h - V_p + 1$$

where the lower limit for $v(d)$ comes from the first inequality. We obtain the measure

$$q^{V_m+h+1}(1-q^{-1})\sum_{v(d)=h-V_p+1}^{h-V_m} q^{-v(d)}\left(\left\lfloor\tfrac{V_p-h+2v(d)-2}{2}\right\rfloor-\left\lceil\tfrac{h-V_p}{2}\right\rceil+1\right) \tag{2.37}$$

Gathering it Together

In this case, we see that after shifting the index of summation in (2.37) so that $v(d)$ starts at $h-V_p$, we get exactly (2.36), so that the two cancel.

2.11 Results and Interpretations

The calculations of §2.6, shown particularly in §2.9 give

$$\mathcal{O}^\kappa_\gamma(\mathbf{1}) = (-1)^{V_m+V_p}q^{2V_m}$$

We recall that $\gamma = \operatorname{diag}(x,y,-y,-x)$ and $V_m = v(x-y)$. We have suggested that the corresponding endoscopic space (H,θ_H) is two copies of $\mathrm{U}(1)\times\mathrm{U}(1)\hookrightarrow\mathrm{U}(2)$, given as follows. Set $J_2 = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. We have $\mathrm{U}(2) := \{g\in\mathrm{GL}_2 : J\overline{g}^{-t}J = g\}$ and $\mathrm{U}_2^{J_2}\cong\mathrm{U}(1)\times\mathrm{U}(1)$, so that $\theta_H = (J_2,J_2)$. A trivial computation shows that for $\gamma_H = (\operatorname{diag}(x,-x),\operatorname{diag}(y,-y))$, the corresponding stable orbital integral is just one. Although this does not suggest a way to define endoscopic pairs (H,θ_H) in general, it is the likely choice given the situation with the adjoint case. Hence:

2.11.1 Theorem. *For* (U_4,θ) *and* $(\mathrm{U}_2\times\mathrm{U}_2,\theta_H)$ *with* $\gamma=\operatorname{diag}(x,y,-y,-x)$ *nearly singular and* $\gamma_H=\operatorname{diag}(x,-x)\times\operatorname{diag}(y,-y)$, *we have the identity*

$$\mathcal{O}^\kappa_\gamma(\mathbf{1}_{\mathfrak{g}_1(\mathfrak{o})}) = (-1)^{V_m+V_p}q^{2V_m}\mathcal{SO}_{\gamma_H}(\mathbf{1}_{\mathfrak{h}_1(\mathfrak{o})}).$$

The factor of $(-1)^{V_m+V_p}$ is not terribly mysterious, and one could likely eliminate it by using the relative Kostant-Weierstrass section [Lev07], so we concentrate on the power of q. We offer the following tentative interpretation of the power of q.

We define $\widetilde{\gamma} \in \mathfrak{u}^\theta(F)$ by $\lambda\,\mathrm{diag}(x,y,y,x)$ where $\lambda \in E$ is such that $v(\lambda) = 0$ and $\overline{\lambda} = -\lambda$. The map $\gamma \mapsto \widetilde{\gamma}$ gives an F-linear isomorphism between $\mathrm{Lie}(I_\gamma)(F)$ and $\mathfrak{a}(F)$. But the stabiliser I_γ of γ in G_0 is $I_\gamma = \{\mathrm{diag}(a_{11}, a_{22}, a_{22}, a_{11}) : a_{ii}\overline{a}_{ii} = 1\}$. Its roots, or nonzero weights, of its action on $\mathfrak{g}_1$ (and on $\mathfrak{g}_0$), are given (in terms of homomorphisms to $\mathrm{Res}_{E/F}(\mathbb{G}_m)$, using adjointness of the restriction of scalars) by

$$\mathrm{diag}(a_{11}, a_{22}, a_{22}, a_{11}) \mapsto a_{11}a_{22}^{-1},$$
$$\mathrm{diag}(a_{11}, a_{22}, a_{22}, a_{11}) \mapsto a_{11}^{-1}a_{22}.$$

Each root space being two-dimensional. Let $D : \mathrm{Lie}(I_\gamma) \to F\delta$ be the discriminant function $\prod_\alpha (d\alpha)^{r_\alpha}$ where r_α is the F-dimension of the corresponding root space. Naïvely using the formula in [Ngô10] with this discriminant function on $\widetilde{\gamma}$ gives

$$\begin{aligned} D(\widetilde{\gamma})/2 &= \frac{2v(x-y) + 2v(x-y)}{2} \\ &= 2V_m. \end{aligned}$$

We shall attempt a reasonable explanation in a future work.

Chapter 3

Dimensions of Relative Affine Springer Fibers for GL_n

We note that this chapter is a slightly modified version of the preprint [Pol15a].

3.1 Introduction

Let G be a connected algebraic group over a finite field k. Let $F = k((t))$ be the Laurent series field over k and let $\mathfrak{o} = k[[t]]$ be the ring of integers of F. Write $v : F^\times \to \mathbb{Z}$ for the valuation on F corresponding to the uniformiser t. For any representation ρ of G on a vector space V defined over $\mathfrak{o}$, and $\gamma \in V(F)$ the set

$$X(G,\gamma)(k) = \{g \in G(F)/G(\mathfrak{o}) : \rho(g)^{-1}\gamma \in V(\mathfrak{o})\},$$

is the set of k-points of an ind-scheme $X(G,\gamma)$ over k called the affine Springer fiber with respect to γ. We have left V out of the notation since the representation shall be clear from context. Finding a formula for the dimension of $X(G,\gamma)$ when its dimension is actually finite is an important and intriguing problem in representation theory. What is this dimension? For simplicity, we shall assume that I_γ, the stabiliser of γ is split. The cocharacter group $X_*(I_\gamma)$ acts on $X(G,\gamma)$ via the formula $\lambda * [g] = [\lambda(t)g]$ and in favourable circumstances, $X_*(I_\gamma)\backslash X(G,\gamma)$ is actually a projective variety over k. We define the **dimension** of the affine Springer fiber $X(G,\gamma)$ to be the dimension of this projective variety.

When ρ is the adjoint representation, this was done by Kazhdan and Lusztig for G split over F with γ regular and in the Lie algebra of a split maximal torus [KL88], and by Bezrukavnikov for any connected reductive G and any regular semisimple γ [Bez96].

Let $G = \mathrm{GL}_{2n}$ and consider the involution $\theta : G \to G$ defined by

$$\theta(x) = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} x \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.$$

Then $G_0 := G^\theta \cong \mathrm{GL}_n \times \mathrm{GL}_n$ sits inside GL_{2n} block-diagonally. Write $\mathfrak{g}$ for the Lie algebra of G. We consider the representation of G_0 on $\mathfrak{g}_1 := \{x \in \mathfrak{g} : \theta(x) = -x\}$. Let $\gamma \in \mathfrak{g}_1(F)$ be semisimple, and also *regular*, which means that the orbit $G_0\gamma$ has maximal dimension. In this thesis, we consider the affine Springer fibers $X(G_0,\gamma)$, and compute their dimension when γ has a relatively simple form, analogous to the formula proved in §5 of [KL88]. More precisely:

3.1.1 Theorem. *Let* $\gamma = \left(\begin{smallmatrix} 0 & 1 \\ \beta & 0 \end{smallmatrix}\right) \in \mathfrak{g}_1(F)$ *where* $\beta = \mathrm{diag}(x_1,\dots,x_n)$ *and* $x_i \neq x_j$ *for* $i \neq j$. *Then*

$$\dim_k X(G_0,\gamma) = \sum_{i<j} v(x_i - x_j).$$

The proof of this theorem has two steps: first, we show that there is a well-defined map $X(G_0,\gamma) \to X(T,\gamma)$ where T is the diagonal maximal torus, and use the method of Kazhdan-Lusztig [KL88] to determine the dimensions of the fibers over each $t \in X(T,\gamma)$. Second, we show that the dimension of the fibers is independent of t, which is not immediately apparent in our case but was obvious in Kazhdan-Lusztig.

We are motivated by the application of our formula to the unitary symmetric spaces given by $\mathrm{U}_{2n}^\theta = \mathrm{U}_n \times \mathrm{U}_n$. The dimension we compute here should relate to the transfer factor that we computed in Chapter 2, but currently this relationship is unknown and is work in progress. Unfortunately, our result here does not directly apply to that case, as one still needs to know the so-called *Galois defect*.

Notations and Conventions

We assume that all reductive groups are connected. We write **Scheme** for the category of schemes and **Set** for the category of sets. If $a_1, \dots, a_n$ are elements of a ring then $\mathrm{diag}(a_1, \dots, a_n)$ denotes the corresponding $n \times n$ diagonal matrix. Any element of $\mathfrak{g}_1$ is of the form

$$\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$$

where X and Y are $n \times n$ matrices; for brevity we write this element as (X, Y).

3.2 Ind-Schemes

For any directed system $Y_1 \hookrightarrow Y_2 \hookrightarrow Y_3 \hookrightarrow \cdots$ of schemes where each morphism $Y_i \hookrightarrow Y_{i+1}$ is a closed embedding, we define a functor $Y = \varinjlim Y_i : \mathbf{Scheme}^{\mathrm{op}} \to \mathbf{Set}$ by first setting $Y(S) = \varinjlim \mathrm{Hom}_{\mathbf{Scheme}}(S, Y_i)$ for every affine scheme S, and then extend Y to a contravariant functor on all schemes by taking its Zariski sheafification. We shall call any functor naturally equivalent to one of this form an **ind-scheme**.

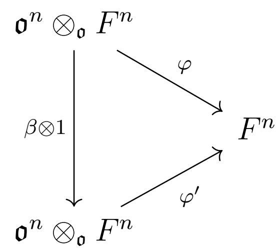
From now on we consider schemes over a finite field $k = \mathbb{F}_q$. We write $F = k((t))$ and $\mathfrak{o} = k[[t]]$. For us, the prototypical example of an ind-scheme over k is the affine Grassmanian whose set of k-points is $G(F)/G(\mathfrak{o})$ where G is a reductive algebraic group over k. As a functor the affine Grassmanian for G, denoted by $\mathrm{Gr}(G)$, is usually defined to be the fpqc-sheafification of the functor defined for each k-algebra R by

$$R \longmapsto G\Big(R((t))\Big)/G\Big(R[[t]]\Big).$$

One can recast this definition into more concrete language:

3.2.1 Proposition ([Ric14], Lemma 1.1)**.** *The affine Grassmannian represents the functor that assigns to every k-algebra R the set of isomorphism classes of pairs $(\mathcal{F}, \varphi)$ where $\mathcal{F}$ is a G-torsor over $\mathrm{Spec}(R[[t]])$ and φ is a trivialisation of $\mathcal{F}[t^{-1}]$ over $\mathrm{Spec}(R((t))$)*

When $G = \mathrm{GL}_n$, a G-torsor over $\mathrm{Spec}(R[[t]])$ is the same thing as an algebraic vector bundle of rank n over $\mathrm{Spec}(R[[t]])$, which is in turn the same thing as $R[[t]]$-projective module of rank n. So the k-points of the affine Grassmannian for GL_n can be described as the set isomorphisms $\varphi : \mathfrak{o}^n \otimes_{\mathfrak{o}} F \to F^n$ under the following equivalence relation: $\varphi \simeq \varphi'$ if and only if there exists an isomorphism $\beta : \mathfrak{o}^n \to \mathfrak{o}^n$ making the diagram



commute. In other words, $\mathrm{Gr}(\mathrm{GL}_n)(k) \cong \mathrm{GL}_n(F)/\mathrm{GL}_n(\mathfrak{o})$. The following Proposition 3.2.2 is a description of the ind-scheme structure on $\mathrm{Gr}(\mathrm{GL}_n)$ due to Gaitsgory [Gai01], generalising the ind-scheme structure for semisimple simply connected groups due to Kazhdan and Lusztig.

3.2.2 Proposition. *On k-schemes, the functor* $\mathrm{Gr}(\mathrm{GL}_n)$ *is naturally equivalent to the Zariski sheafification of the direct limit* $\mathrm{Gr}^1 \hookrightarrow \mathrm{Gr}^2 \hookrightarrow \cdots$ *where* $\mathrm{Gr}^m(R)$ *is the set of all R-flat t-stable submodules of $R \otimes_k t^{-m}V[[t]]/t^{m+1}V[[t]]$.*

One can give an ind-scheme structure on the affine Grassmannian for all reductive groups G by showing that a closed embedding $G \hookrightarrow \mathrm{GL}_n$ induces a closed embedding $\mathrm{Gr}(G) \to \mathrm{Gr}(\mathrm{GL}_n)$ of affine Grassmannians, and in fact one can even drop the reductive hypothesis (see [Gai01] for details). Let us now take a look at Example 3.2.3, which is the starting point for our investigations into the dimensions of affine Springer fibers.

3.2.3 Example (Dimension for Tori). To get an intuition for the geometric structure of the affine Grassmanian, consider $G = \mathbb{G}_m$. Then $G(F)/G(\mathfrak{o}) = F^\times/\mathfrak{o}^\times$ which is isomorphic to $\mathbb{Z}$ as groups. As described in Proposition 3.2.2, the functor $\mathrm{Gr}(\mathbb{G}_m)$ can be written as a colimit of schemes $Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots$ where for each m and for any field extension k'/k, the set $Y_m(k')$ is the set of all t-stable k'-subspaces of $t^{-m}k'((t))/t^{m+1}k'((t))$. This space is just isomorphic to k'^{2m+2} where multiplication by t is the *right-shift* linear operator $(a_1, a_2, \ldots, a_{2m+2}) \mapsto (0, a_1, \ldots, a_{2m+1})$. For a fixed integer $0 \leq d \leq 2m+2$, there is a unique d-dimensional subspace that is t-invariant.

Hence $Y_m(k')$ consists of one point in each irreducible component of the disjoint union over the $2m+2$ Grassmannians for $t^{-m}k'((t))/t^{m+1}k'((t))$ and so $\mathrm{Gr}(\mathbb{G}_m)$ is zero-dimensional. The same argument also shows that $\mathrm{Gr}(T)$ is zero-dimensional whenever $T \cong \mathbb{G}_m^n$.

3.3 The Fibration

The formula of Kazhdan and Lusztig was proved by first considering elements in the Lie algebra of a Levi subgroup, and in particular, the Lie algebra of a split maximal torus to obtain an explicit formula. The analog here is the following type of element: let $\gamma \in \mathfrak{g}_1(F)$ be a regular semisimple element of the form

$$\gamma = \begin{pmatrix} 0 & I_n \\ \beta & 0 \end{pmatrix} \tag{3.1}$$

where $\beta = \mathrm{diag}(x_1, \ldots, x_n)$ with $x_i \neq x_j$ for $i \neq j$ and I_n is the $n \times n$ identity (cf. [JR96, Proposition 2.1]). Incidentally, the choice of such a γ determines a Cartan subspace, the analogue of a Cartan subalgebra for symmetric spaces, and the formula we will derive gives the dimension of the corresponding affine Springer fiber for any element in such a subspace containing γ; see Remark 3.4.5 for further details.

We let $T \subset G_0$ be the diagonal maximal split torus and $B \subset G_0$ be the Borel subgroup of upper triangular matrices in each block. Then we have an Iwasawa decomposition $G_0(F) = T(F)U(F)G_0(\mathfrak{o})$ where U is the unipotent radical of B.

3.3.1 Proposition. *For each $g \in G_0(F)/G_0(\mathfrak{o})$, fix a decomposition $g = t_g u_g \in G_0(F)/G_0(\mathfrak{o})$ as in the Iwasawa decomposition. Then the map*

$$\begin{aligned} p : G_0(F)/G_0(\mathfrak{o}) &\longrightarrow T(F)/T(\mathfrak{o}) \\ g &\longmapsto t_g \end{aligned}$$

is well-defined.

Proof. It suffices to observe that if $t_1u_1 = t_2u_2 \in G_0(\mathfrak{o})$ then $u_2^{-1}t_2^{-1}t_1u_1 \in G_0(\mathfrak{o})$ and so $t_2^{-1}t_1 \in G_0(\mathfrak{o}) \cap T(F) = T(\mathfrak{o})$. ■

Similarly, we can prove:

3.3.2 Proposition. *If $g \in X(G_0, \gamma)$ then $t = p(g) \in X(T, \gamma)$.*

Proof. Write $u = \operatorname{diag}(v, w)$ where $v, w \in \mathrm{GL}_n(F)$ and similarly for $t = \operatorname{diag}(r, s)$ for $r, s \in \mathrm{GL}_n(F)$. Let $g = tu \in X(G_0, \gamma)$, so that $\operatorname{Ad}(g)^{-1}\gamma = \operatorname{Ad}(u)^{-1}\operatorname{Ad}(t)^{-1}\gamma \in \mathfrak{g}_1(\mathfrak{o})$. This is the same as saying that the matrices

$$v^{-1}r^{-1}sw, w^{-1}s^{-1}\beta rv$$

have integral entries. In particular, this implies that $r^{-1}s$ and $s^{-1}\beta r$ have integral entries since β is also a diagonal matrix. ■

3.4 The Dimension of the Fiber

We wish to determine the dimension of $X(G, \gamma)$. Since the dimension of $X(T, \gamma)$ is zero (Example 3.2.3), it suffices to determine the dimension of a fiber $p^{-1}(t)$, and show that this dimension is independent of t. To do this, we will use:

3.4.1 Proposition ([GW10], Corollary 14.119)**.** *Let X and Y be finite type k-schemes and $f : X \to Y$ be a dominant morphism of finite type. If all the nonempty fibers of f have dimension r then* $\dim X = \dim Y + r$.

Let Φ^+ be the set of positive roots of T with respect to B. For each $m \geq 1$, we define the set Φ^+_m to be the set of roots in Φ^+ that can be written as a sum of at least m roots of Φ^+. This gives us a finite filtration $\Phi^+ = \Phi^+_1 \supseteq \Phi^+_2 \supseteq \cdots \supseteq \Phi^+_n = 0$ with $\Phi_{n-1} \neq 0$. We let $\mathfrak{u}_m$ be the sum of the root spaces corresponding to the roots in Φ^+_m. For each m, let V_m be the sum of root spaces corresponding to the roots in Φ^+_m but not in Φ^+_{m+1}. Put more plainly for our specific situation, V_m consists of those block diagonal matrices $\operatorname{diag}(X, Y)$ where X and Y are $n \times n$ matrices whose nonzero entries are only on the mth upper off-diagonal.

Let U_m denote the root group corresponding to $\mathfrak{u}_m$. It is the unique unipotent subgroup of the unipotent radical U of B such that $\mathrm{Lie}(U_m) = \mathfrak{u}_m$. In particular, $U_1 = U$. Moreover, $U_m/U_{m+1} \cong V_m$ as abelian groups.

Fix a $t \in T(F)$ and a $g \in p^{-1}(t)$. Write $g = tu$ via the Iwasawa decomposition. We consider the linear map $\varphi = [\mathrm{Ad}(t)^{-1}\gamma, -] : \mathfrak{g} \to \mathfrak{g}$. By assumption, we have $\mathrm{Ad}(t)^{-1}\gamma \in \mathfrak{g}_1(\mathfrak{o})$, and so φ is defined over $\mathfrak{o}$ and its restriction to $\mathfrak{g}_0(\mathfrak{o}) = \mathrm{Lie}(G_0)$ gives a homomorphism

$$\varphi : \mathfrak{g}_0(\mathfrak{o}) \to \mathfrak{g}_1(\mathfrak{o})$$

of free $\mathfrak{o}$-modules. We let $\mathfrak{v}$ be the subspace of $\mathfrak{g}_1$ consisting of elements of the form

$$\begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix}$$

where M and N are strictly upper triangular. The next theorem expresses the dimensions of the fibers of $X(G_0, \gamma) \to X(T, \gamma)$ over each $t \in X(T, \gamma)$ in terms of the linear map φ. The proof is similar to that of [KL88, §5, Proposition 1], modified to work with our representation.

3.4.2 Theorem. *The dimension of the fiber of the map $X(G_0, \gamma) \to X(T, \gamma)$ at a point $t \in X(T, \gamma)$ is $v(\det(\varphi))$, where $\varphi : \mathfrak{u}(\mathfrak{o}) \to \mathfrak{v}(\mathfrak{o})$ is the map of free $\mathfrak{o}$-modules given by $[\mathrm{Ad}(t)^{-1}\gamma, -]$.*

Proof. Let $g \in X(G_0, \gamma)$ be an element in $Y_1 = p^{-1}(t)$, and write $g = tu$. Writing $\tilde{u}$ for the image of u in $U_1/U_2 \cong V_1$, we see that $[\mathrm{Ad}(t)^{-1}\gamma, \tilde{u}] \in \mathfrak{v}(\mathfrak{o})$ since $\mathrm{Ad}(u)^{-1}\,\mathrm{Ad}(t)^{-1}\gamma \in \mathfrak{g}_1(\mathfrak{o})$. We claim that this gives a well-defined map $p_1 : Y_1 \to Z_1 = \mathfrak{v}(\mathfrak{o})/\varphi(V_1(\mathfrak{o}))$. Indeed, if we choose some other u' such that $g = tu'$ in $X(G_0, \gamma)$, then we can write $u = u'v$ where $v \in U(\mathfrak{o})$, and so $[\mathrm{Ad}(t)^{-1}\gamma, \tilde{u}] - [\mathrm{Ad}(t)^{-1}\gamma, \tilde{u}']$ lies in $\varphi(V_1(\mathfrak{o}))$. Then the surjective map $p_1 : Y_1 \to Z_1$ is such that the dimension of $p_1^{-1}(z)$ is independent of $z \in Z_1$ (see Example 3.4.3).

Let $z_1 \in Z_1(k)$. We then reiterate the procedure but now with the fiber $p_1^{-1}(z_1)$ and since all the fibers have the same dimension it suffices just to take z_1 corresponding to $\widetilde{u} = 0$. Then for any $g \in q_1^{-1}(z_1) =: Y_2$ we can write $g = mu_2 \in X(G_0, \gamma)$ where $u_2 \in U_2(F)$ and the coset $u_2U_2(F)/U_2(\mathfrak{o})$ is well-defined. We let $p_2 : Y_2 \to Z_2 =: \mathfrak{v}(\mathfrak{o})/\varphi(V_2(\mathfrak{o}))$ be the map defined by $p_2(g) = \varphi(\tilde{u_2})$. Proceeding inductively, we have defined a sequence of surjective morphisms, with each morphism having constant-dimension fibers

$$\begin{array}{ccccccc} Y_{n-1} & \hookrightarrow & Y_{n-2} & \hookrightarrow & \cdots & \hookrightarrow & Y_1 \\ \downarrow & & \downarrow & & & & \downarrow \\ Z_{n-1} & & Z_{n-2} & & \cdots & & Z_1 \end{array}$$

where $Y_{n-1} \to Z_{n-1}$ is an isomorphism, and so the dimension of the fiber over t is just $\sum \dim(Z_i)$, which is the same as $v(\det(\varphi))$. ■

3.4.3 Example. Lest the general concept of this proof be lost in murky darkness, the following example for $n = 3$, i.e. $G = \mathrm{GL}_6$ should help. In this case, for $u = (v, w), t = (r, s), r = \mathrm{diag}(r_1, r_2, r_3), s = \mathrm{diag}(s_1, s_2, s_3), \beta = \mathrm{diag}(x_1, x_2, x_3)$, and

$$v = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix}, w = \begin{pmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

we have

$$v^{-1}r^{-1}sw = \begin{pmatrix} s_1r_1^{-1} & b_{12}s_1r_1^{-1}-a_{12}s_2r_2^{-1} & -a_{12}b_{23}s_2r_2^{-1}+a_{12}a_{23}s_3r_3^{-1}+b_{13}s_1r_1^{-1}-a_{13}s_3r_3^{-1} \\ 0 & s_2r_2^{-1} & b_{23}s_2r_2^{-1}-a_{23}s_3r_3^{-1} \\ 0 & 0 & s_3r_3^{-1} \end{pmatrix},$$

$$w^{-1}s^{-1}\beta rv = \begin{pmatrix} x_1r_1s_1^{-1} & a_{12}x_1r_1s_1^{-1}-b_{12}x_2r_2s_2^{-1} & -a_{23}b_{12}x_2r_2s_2^{-1}+b_{12}b_{23}x_3r_3s_3^{-1}+a_{13}x_1r_1s_1^{-1}-b_{13}x_3r_3s_3^{-1} \\ 0 & x_2r_2s_2^{-1} & a_{23}x_2r_2s_2^{-1}-b_{23}x_3r_3s_3^{-1} \\ 0 & 0 & x_3r_3s_3^{-1} \end{pmatrix}.$$

We see that the choice of fixing a_{12}, b_{12}, a_{23}, and b_{23} does not affect the dimension of the k-variety in the coordinates a_{13} and b_{13} since the coefficients of π^ℓ for $\ell < 0$ in the expressions $b_{13}s_1r_1^{-1} - a_{13}s_3r_3^{-1}$ and $a_{13}x_1r_1s_1^{-1} - b_{13}x_3r_3s_3^{-1}$ are uniquely determined. This follows because $x_i \neq x_j$ for $i \neq j$.

We next determine a formula for $v(\det(\varphi))$. For any $1 \le i, j \le n$ denote by e_{ij} the matrix whose only nonzero entry is 1 at i, j. Then an F-basis of $\mathfrak{u}(F)$ is

$$\{(e_{ij}, \mathbf{0}) : i < j\} \cup \{(\mathbf{0}, e_{ij}) : i < j\}$$

where $\mathbf{0}$ is the zero matrix. We also use this notation for a basis of $\mathfrak{v}(F)$. For example, in $\mathfrak{u}(F)$, the basis element $(e_{ij}, \mathbf{0})$ corresponds to the block diagonal matrix $\operatorname{diag}(e_{ij}, \mathbf{0})$, whereas used to denote a basis element of $\mathfrak{v}(F)$, it corresponds to the block antidiagonal matrix

$$\begin{pmatrix} \mathbf{0} & e_{ij} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

These bases are also bases for $\mathfrak{u}(\mathfrak{o})$ and $\mathfrak{v}(\mathfrak{o})$ as free $\mathfrak{o}$-modules. We need to write down the $n(n-1) \times n(n-1)$ matrix of φ. Recall that we have fixed $t \in X(T, \gamma)$. Write $t = (r, s)$ where $r = \operatorname{diag}(r_1, \ldots, r_n)$ and $s = (s_1, \ldots, s_n)$. Let us agree to use the notation $(A, B) \in \mathfrak{g}_1$ to denote the block antidiagonal matrix

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}.$$

Then $\operatorname{Ad}(t)^{-1}\gamma$ is the matrix $(\operatorname{diag}(r_1^{-1}s_1, \ldots, r_n^{-1}s_n), \operatorname{diag}(s_1^{-1}r_1x_1, \ldots, s_n^{-1}r_nx_n)) \in \mathfrak{g}_1(F)$. We compute the matrix with respect to the ordered basis

$$(e_{11}, \mathbf{0}), (e_{12}, \mathbf{0}), \ldots, (e_{n-1,n}, \mathbf{0}), (\mathbf{0}, e_{11}), \ldots, (\mathbf{0}, e_{n-1,n})$$

for $\mathfrak{u}(F)$ and the same notation denotes the ordered basis we use for $\mathfrak{v}(F)$. Then it is somewhat trivial to write down the matrix corresponding to φ:

$$\varphi = \begin{pmatrix}
-a_2 & & & & & a_1 & & & & \\
& \ddots & & & & & \ddots & & & \\
& & -a_n & & & & & a_1 & & \\
& & & \ddots & & & & & \ddots & \\
& & & & -a_n & & & & & a_{n-1} \\
b_1x_1 & & & & & -b_2x_2 & & & & \\
& \ddots & & & & & \ddots & & & \\
& & b_1x_1 & & & & & -b_nx_n & & \\
& & & \ddots & & & & & \ddots & \\
& & & & b_{n-1}x_{n-1} & & & & & -b_nx_n
\end{pmatrix} \tag{3.2}$$

Here, $a_i = r_i^{-1}s_i$ and $b_i = s_i^{-1}r_i$. Although it is a bit difficult to see from the typesetting, the only nonzero elements are on the main diagonals of each of the four $n(n-1)/2 \times n(n-1)/2$ blocks. For example, if $n = 2$, the matrix of φ is the 2×2 matrix

$$\begin{pmatrix} -r_2^{-1}s_2 & r_1^{-1}s_1 \\ s_1^{-1}r_1x_1 & -s_2^{-1}r_2x_2 \end{pmatrix}$$

Then $v(\det(\varphi)) = v(x_1 - x_2)$, and does not depend on t. This holds true in general:

3.4.4 Theorem. *The determinant* $\det(\varphi)$ *does not depend on the chosen* $t \in X(T, \gamma)$*, and*

$$v(\det(\varphi)) = \sum_{i<j} v(x_i - x_j).$$

In particular, since $X(T, \gamma)$ *is zero-dimensional (Example 3.2.3), this is also the dimension of the affine Springer fiber* $X(G_0, \gamma)$*.*

Proof. Let us abuse notation and write φ for the matrix of φ as in (3.2). First, multiply φ

on the left by the diagonal matrix

$$D = \mathrm{diag}(-b_2b_1x_1, -b_3b_1x_1, \dots, -b_nb_1x_1, -b_3b_2x_2, \dots, -b_nb_2x_2, \dots, -b_nb_{n-1}x_{n-1}, 1, 1, \dots, 1)$$

In other words, D corresponds to multiplying each of the first $n(n-1)/2$ rows of φ by nonzero elements in such a way so that the rows of the upper left $n(n-1)/2 \times n(n-1)/2$ block are the same as the rows of the corresponding lower left block. By subtracting the kth row from the $[n(n-1)/2+k]$th row for $k = 1, 2, \dots, n(n-1)/2$, we get a new matrix that is upper triangular, and whose diagonal entries are $\mathrm{diag}(A, B)$, where

$$\begin{aligned} A &= \mathrm{diag}(b_1x_1, b_1x_1, \dots, b_2x_2, b_2x_2, \dots, b_2x_2, \dots, b_{n-1}x_{n-1}), \\ B &= \mathrm{diag}(b_2(x_1 - x_2), b_3(x_1 - x_3), \dots, b_n(x_{n-1} - x_n)). \end{aligned}$$

Hence, the determinant of φ is $\det(\mathrm{diag}(A, B))/\det(D)$, which is

$$\det(\varphi) = \frac{\prod_{i<j} b_ib_jx_i(x_i - x_j)}{\prod_{i<j} -(b_ix_ib_j)} = (-1)^{n(n-1)/2}\prod_{i<j}(x_i - x_j).$$

Taking the valuation of this gives the desired result. ■

3.4.5 Remark. By definition, a *Cartan subspace* is a subspace $\mathfrak{a}$ of $\mathfrak{g}_1$ that is maximal with respect to being commutative and consistchapterntirely of semisimple elements. One easily calculates that the stabiliser of γ is of the form

$$\{(X, X\beta) : X\beta = \beta X\}.$$

Since $\gamma = \mathrm{diag}(x_1, \dots, x_n)$ is regular, which is the same thing as saying $x_i \neq x_j$ for all $i \neq j$, we see that the stabiliser of γ is just the commutative subspace

$$\mathfrak{a}(R) = \left\{ \big(\mathrm{diag}(c_1, \dots, c_n), \mathrm{diag}(c_1x_1, \dots, c_nx_n)\big) : c_i \in R \right\},$$

which consists entirely of semisimple elements. Since

$$\gamma' = \big(\mathrm{diag}(c_1, \ldots, c_n), \mathrm{diag}(c_1 x_1, \ldots, c_n x_n)\big) \in \mathfrak{a}(k)$$

is conjugate to $\big(I_n, \mathrm{diag}(c_1^2 x_1, \ldots, c_n^2 x_n)\big)$, we can also use our formula to compute that $\dim X_{\gamma'} = \sum_{i<j} v(c_i^2 x_i - c_j^2 x_j)$ whenever γ' is regular. This follows since the two affine Springer fibers for two elements in the same class are isomorphic.

3.4.6 Remark. One can define a map

$$\begin{aligned} X(G_0, \gamma) &\longrightarrow \mathbb{G}_m(F)/\mathbb{G}_m(\mathfrak{o}) \cong \mathbb{Z} \\ (A, B) &\longmapsto v(\det(A^{-1}B)) \end{aligned}$$

which has nonempty fibers over a finite set of points in $\mathbb{Z}$. The fiber over the point $0 \in \mathbb{Z}$ is the affine Springer fiber $X(\mathrm{GL}_n, \beta)$, which by Kazhdan-Lusztig's formula also has dimension $\sum_{i<j} v(x_i - x_j)$. Theorem 3.4.4 then says that all the other fibers also have this dimension.

Chapter 4

Final Remarks

We have seen two results: the computation of a relative endoscopic fundamental lemma of the form $\mathcal{O}_\gamma^\kappa(\mathbf{1}) = (-1)^* q^r \mathcal{O}_{\gamma_H}^{\kappa=1}(\mathbf{1}_H)$, and a computation of the dimension of the GL_n-version of the affine Springer fiber that corresponds to the orbital integral $\mathcal{O}_\gamma$. How do these two results relate?

The exponent r depends on the geometric properties of the two affine Springer fibers: in the usual Fundamental Lemma of Ngô, $r = \dim_k X(G,\gamma) - \dim_k X(H,\gamma_H)$ where H is the associated endoscopic group for κ. For symmetric spaces, either this result or something very close to it should hold. The result we computed in Chapter 3 applies to the closely related affine Springer fiber for GL_n. However, to write down the correct formula for unitary groups, one should combine the result in Chapter 3 with the ideas in [Bez96]. Such a formula for unitary groups should give precisely the transfer factor obtained in §2, and in general, give the transfer factor in a relative fundamental lemma for unitary groups.

After these results, it seems quite reasonable that a relative fundamental lemma for unitary groups exists. To derive it and use it in a comparison of relative trace formulas will be quite interesting.

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