ABSTRACT

NUCLEON ISOBAR PRODUCTION

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A formalism for Reggeizing helicity amplitudes, from which kinematic singularities have been removed previously, is developed and applied to quasi two body inelastic processes. This formalism is then applied to the reactions $\pi^- p \longrightarrow \pi^0 n$ $\pi^- p \longrightarrow \pi^0 N^{*++}$, and $NN \longrightarrow NN^*$, assuming dominance of these processes by either the π or ρ trajectories. The factorization theorem, the knowledge of the t channel density matrix elements obtained from the decay of the final state particles, and the data on the differential cross sections are then used to attempt to determine fits to the data which are not trivially constrained. Although the experimental data is still rather scanty, it is tentatively concluded that the ρ trajectory generates a fit to $\pi N \longrightarrow \pi N^*$ and $\pi^- p \longrightarrow \pi^0 n$. However, it is the pion trajectory which gives an adequate fit of the $NN \longrightarrow NN^*$ data.

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by

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CHAPTER 1

INTRODUCTION

The Regge pole model has been used for some time to make predictions about high energy elastic scattering[17] with a reasonable amount of success. However, these processes do not provide a strict test of the theory because of the large number of parameters and the lack of knowledge of the spin dependence of the amplitudes. Recently, this model has been applied to elastic change exchange processes with startling success.[27] However, these processes also yield information on the spin dependence of the scattering amplitudes only very grudgingly, although requiring fewer parameters.

In this thesis the extension of the Regge pole model to quasi two body processes will be considered. Specifically, the production of nucleon isobars in quasi two body processes such as $\pi N \rightarrow \pi N^*$ and $NN \rightarrow NN^*$ (1238) will be calculated using a formulation of the Regge pole model for processes which contain intrinsic spin. However, for such processes, the decay of the isobars in the final state yields a great amount of information about the spin dependence of the cross section. This information is provided by the crossed channel density matrix elements.[20] A complete knowledge of the dependence on momentum transfer of these density matrix elements will provide stringent tests of the theory. These tests can be expected to be particularly severe in those cases where a single Regge pole is expected to dominate the amplitude.

In the present work an attempt to fit the inelastic processes $\pi p \rightarrow \pi N^*$ (1238), $\pi^- p \rightarrow \pi^0 N$, $pp \rightarrow pN^*$ (1238) and $pp \rightarrow N^*(1238)N^*$ (1238) has been made under the assumption that the only isospin one trajectories contributing are the π and ρ trajectories. The constraints due to the factorization theorem for Regge residues have been fulfilled. Moreover, it has usually been assumed that the Regge residues are independent of momentum transfer. While no attempt has been made to include the effects of the $R(A_2)$ trajectory, it should be noted that the factorization theorem and fits to the processes $\pi p \rightarrow \eta N$ and $\pi p \rightarrow \eta N^*(1238)$ could be used to determine, in principle, the effect of this trajectory on the processes $NN \rightarrow NN^*$ (1238) and $NN \rightarrow N^*(1238)N^*(1238)$. Moreover, the more ephemeral isospin one trajectories, such as the A_1, B, and B' are also neglected. The effects of the threshold constraints on the helicity amplitudes noted by Franklin have also been discussed and fits to the experimental data which satisfy these constraints have been sought.

The general plan of the thesis can be summarized as follows. The second and third chapters contain a derivation of the Regge pole formalism and of the kinematical singularities contained in the helicity amplitudes. The fourth chapter provides detailed discussions of the constraints on and prescriptions for the Regge residues. The fifth chapter contains the comparison to experimental data while the conclusions follow in the sixth chapter.

CHAPTER 2

GENERAL FORMALISM FOR REGGEIZING SCATTERING AMPLITUDES

A general formalism for the "Reggeizing" of scattering amplitudes involving particles with arbitrary spins has evolved through the work of Gellman et al[1] and Wang.[2,3] In this chapter this formalism will be developed in detail with particular attention being paid to the analyticity properties and other assumptions necessary for the use of the Sommerfeld-Watson transformation.

This formalism is based upon the helicity amplitude expansion of the scattering amplitude given in the beautiful work of Jacob and Wick.[4] This expansion deals explicitly with S matrix elements between helicity states instead of the usual invariant amplitudes. For this reason these helicity amplitudes contain certain kinematical singularities and kinematical zeros, the detailed form of which will be discussed in the third chapter. However, it is to be understood that such singularities have been explicitly factored out of our amplitudes before we attempt Reggeization. These singularities and zeros will thus appear explicitly in the final form of the full amplitudes, since the Reggeized expressions must be multiplied by these factors to regenerate the full helicity amplitude.

The starting point for this development is the crossed or t channel partial wave helicity expansion of Jacob and Wick. The state of total angular momentum J, z component of angular momentum M, and helicities

λ_c, λ_d is labelled $|JM; \lambda_c \lambda_d\rangle$. Then the parity operation P produces the effect

$$P|JM; \lambda_c \lambda_d\rangle = \eta_c \eta_d (-1)^{J-S_c-S_d} |JM; -\lambda_c -\lambda_d\rangle \tag{2.1}$$

where S is spin, η is intrinsic parity.

Then the state

$$|JM; \lambda_c \lambda_d\rangle_{\pm} = 2^{-1/2}\left\{|JM; \lambda_c \lambda_d\rangle \pm \eta_c \eta_d (-1)^{S_c+S_d-\nu} |JM -\lambda_c -\lambda_d\rangle\right\} \tag{2.2}$$

is such that

$$P|JM \lambda_c \lambda_d\rangle_{\pm} = (\pm)(-1)^{J-\nu} |JM \lambda_c \lambda_d\rangle_{\pm} \tag{2.3}$$

where $\nu = 1/2$ for J half integer

$\nu = 0$ for J integer.

Then if a matrix F is defined from the S matrix such that

$$F_{fi} \equiv \left(S_{fi} - \delta_{fi}\right)(2i)^{-1}\ k_f^{+1/2}\ k_i^{-1/2} \tag{2.4}$$

where k_f and k_i are the barycentric momenta in the final and initial states respectively, it follows that

$$F^{J\pm}_{\lambda_c\lambda_d, \lambda_a\lambda_b} \equiv {}_{\pm}\langle JM \lambda_c \lambda_d | F | JM \lambda_a \lambda_b\rangle_{\pm}$$

$$= \langle JM \lambda_c\lambda_d | F | JM \lambda_a \lambda_b\rangle$$
$$\pm \eta_c \eta_d (-1)^{S_c+S_d-\nu} \langle JM -\lambda_c -\lambda_d | F | JM \lambda_a \lambda_b\rangle \tag{2.5}$$

However, the Jacob and Wick scattering amplitude can be expressed as

$$f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) = k_f^{1/2}\, k_i^{-1/2} \sum_J (2J+1) \langle \lambda_c\lambda_d | F^J | \lambda_a\lambda_b \rangle\, d^J_{\lambda\mu}(z) \tag{2.6}$$

where $\lambda = \lambda_a - \lambda_b$ and $\mu = \lambda_c - \lambda_d$,

where the function

$$d^J_{\lambda\mu}(z) = \pm \left[\frac{(J+M)!\,(J-M)!}{(J+N)!\,(J-N)!}\right]^{1/2} (\cos\theta/2)^{|\lambda+\mu|} (\sin\theta/2)^{|\lambda-\mu|}\, P_{J-M}^{(|\lambda-\mu|,|\lambda+\mu|)}(\cos\theta) \tag{2.7}$$

$$M = \max(|\lambda|, |\mu|) \qquad N = \min(|\lambda|, |\mu|)$$

and $P_{J-M}^{(|\lambda-\mu|,|\lambda+\mu|)}(\cos\theta)$ are Jacobi polynomials[2] in cos θ. An important property is the factorizability of the $d^J_{\lambda\mu}(\theta)$ into a factor independent of J multiplied by a polynomial in cos θ. It is these polynomials which will be Reggized since all dependence on J is contained in them. Therefore the following amplitudes are defined:

$$f^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) \equiv \left[\sqrt{2}\cos\tfrac{\theta}{2}\right]^{-|\lambda+\mu|}\left[\sqrt{2}\sin\tfrac{\theta}{2}\right]^{-|\lambda-\mu|} f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) \pm$$

$$(-1)^{\lambda+\lambda_m}\eta_c\eta_d(-1)^{s_c+s_d-\nu}\left[\sqrt{2}\sin\tfrac{\theta}{2}\right]^{-|\lambda+\mu|}\left[\sqrt{2}\cos\tfrac{\theta}{2}\right]^{-|\lambda-\mu|} f_{-\lambda_c-\lambda_d;\lambda_a\lambda_b}(z) \tag{2.8}$$

where $\lambda_m = \max(|\lambda|,|\mu|)$, $z = \cos\theta$.

If we define

$$e^{J\pm}_{\lambda\mu}(z) \equiv \frac{1}{2}\left\{\left[\sqrt{2}\cos\tfrac{\theta}{2}\right]^{-|\lambda+\mu|}\left[\sqrt{2}\sin\tfrac{\theta}{2}\right]^{-|\lambda-\mu|} d^J_{\lambda\mu}(z) \pm (-1)^{\lambda+\lambda_m}\left[\sqrt{2}\sin\tfrac{\theta}{2}\right]^{-|\lambda+\mu|}\left[\sqrt{2}\cos\tfrac{\theta}{2}\right]^{-|\lambda-\mu|} d^J_{\lambda,-\mu}(z)\right\} \tag{2.9}$$

It follows that

$$f^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,z) = \frac{k_f^{1/2}}{k_i^{1/2}} \sum_{J=\lambda_m}^{\infty} (2J+1)\left[e^{J+}_{\lambda\mu}(z)\, F^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t) + e^{J-}_{\lambda\mu}(z)\, F^{J\mp}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t)\right] \tag{2.10}$$

This formula can be inverted to obtain

$$F^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t) = 2^{-1/2}\left(\frac{k_i}{k_f}\right)^{1/2} \int_{-1}^{1} dz \left[c^{J+}_{\lambda\mu}(z)\, f^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,z) + c^{J-}_{\lambda\mu}(z)\, f^{\mp}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,z)\right] \tag{2.11}$$

where

$$2\, c^{J\pm}_{\lambda\mu}(z) \equiv \left(\sqrt{2}\cos\frac{\theta}{2}\right)^{|\lambda+\mu|}\left(\sqrt{2}\sin\frac{\theta}{2}\right)^{|\lambda-\mu|} d^{J}_{\lambda\mu}(z) \pm (-1)^{\lambda+\lambda_m-1}\left(\sqrt{2}\sin\frac{\theta}{2}\right)^{|\lambda+\mu|}\left(\sqrt{2}\cos\frac{\theta}{2}\right)^{|\lambda-\mu|} d^{J}_{\lambda,-\mu}(z) \tag{2.12}$$

Both $c^{J\pm}_{\lambda\mu}$ and $e^{J\pm}_{\lambda\mu}$ are polynomials in the Legendre functions of the first kind. Gellman et al[(1)] have listed these functions.

It is this representation of the helicity amplitudes which will be Reggeized. The first step in this process is the continuation to complex J of $e^{J\pm}_{\lambda\mu}$ and $F^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}$ in a manner that will permit a Sommerfeld-Watson transformation.

This continuation of the rotation functions $e^{J\pm}_{\lambda\mu}(z)$ is accomplished by replacing the Legendre polynomials P_n of cos θ by the function

$$\begin{aligned}\mathcal{P}_J &= -Q_{-J-1}\,\pi^{-1}\tan J\pi \\ &= \frac{\Gamma(J+1/2)}{\Gamma(J+1)\,\pi^{1/2}}(2z)^J\, F\left(-\tfrac{J}{2}, \tfrac{1}{2}-\tfrac{J}{2}, \tfrac{1}{2}-J; \tfrac{1}{z^2}\right)\end{aligned} \tag{2.13}$$

where F is the hypergeometric function and is suitable for continuation in J. $\mathcal{P}_J$ has the following properties:

$$\text{(a)}\quad \mathcal{P}_J = P_J \qquad \text{for } J = 0, 1, 2, \ldots \tag{2.14}$$

This equality follows from the identity

$$P_\ell(z)\big/\sin\pi\ell - \tfrac{1}{\pi}\,\frac{Q_\ell(z)}{\cos\pi\ell} = -\frac{1}{\pi}\,\frac{Q_{-\ell-1}}{\cos\pi\ell}\,^{(5)} \tag{2.15}$$

$$\text{(b)}\quad \mathcal{P}_J = 0 \qquad \text{for } J = -1, -2, -3, -4, \ldots \tag{2.16}$$

$$\text{(c)}\quad \mathcal{P}_J \text{ contains a pole at all half integral } J \tag{2.17}$$

The rotation functions will be designated by $E^{J\pm}_{\lambda\mu}$ rather than $e^{J\pm}_{\lambda\mu}$ when $\mathcal{P}_J$ is their argument. In order to continue the partial wave amplitudes an analogue to the usual Froissart-Gribov continuation is defined. This continuation requires a fixed t dispersion relation in s and u or, equivalently, in Z. It will be shown in the next chapter that $f^{\pm}_{\{\lambda\}}$ contains only factorizable kinematic t singularities. Then defining $f'^{\pm}_{\{\lambda\}}$ such that

$$f^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,z) = K^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t)\, f'^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,z)$$

where $k_{\{\lambda\}}(t)$ represents all kinematic singularities, it follows that

$$f'^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,z) = \frac{1}{\pi}\int_{z_s^0}^{\infty} \frac{\omega^{s\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,s(z'))\,dz'}{z'-z} + \frac{1}{\pi}\int_{z_u^0}^{\infty}\frac{\omega^{u\pm}(t,u(-z'))}{z'+z}\,dz' \tag{2.18}$$

We can now insert this dispersion relation in definition (2.11). If we interchange the order of integrations the first integral can be done analytically. It follows that

$$\frac{F^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}}{K^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t)} = 2^{-1/2}\left(\frac{k_i}{k_f}\right)^{1/2}\frac{2}{\pi}\int_{z^0(s)}^{\infty}dz'\; \omega^{s\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,s(z'))\; C^{J+}_{\lambda\mu}(z')$$

$$+\,2^{-1/2}\left(\frac{k_i}{k_f}\right)^{1/2}\frac{2}{\pi}\int_{z^0(u)}^{\infty}dz'\; \omega^{u\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,u(-z'))\; C^{J+}_{\lambda\mu}(-z')$$

$$+\,2^{-1/2}\left(\frac{k_i}{k_f}\right)^{1/2}\frac{2}{\pi}\int_{z^0(s)}^{\infty}dz'\; \omega^{s\mp}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,u(-z'))\; C^{J-}_{\lambda\mu}(z')$$

$$+\,2^{-1/2}\left(\frac{k_i}{k_f}\right)^{1/2}\frac{2}{\pi}\int_{z^0(u)}^{\infty}dz'\; \omega^{u\mp}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,u(-z'))\; C^{J-}_{\lambda\mu}(-z')$$

(2.19)

where we have used

$$-Q_\ell(z)=\frac{1}{2}\int\frac{P_\ell(z')}{z'-z}\,dz' \qquad \ell = \text{positive integer} \qquad (2.20)$$

and the $C^{J\pm}_{\lambda\mu}(z)$ are the small $c^{J\pm}_{\lambda\mu}(z)$ with P_n replaced by Q_n.

Now

$$C^{J\pm}_{\lambda\mu}(-z) = \pm(-1)^{J-\lambda_m} C^{J\pm}_{\lambda\mu}(z) \tag{2.21}$$

This implies

$$C^{J\pm}_{\lambda\mu}(-z) = \mp(-1)^{J-\lambda_m} C^{J\mp}_{\lambda\mu}(z) \tag{2.22}$$

In addition we define $F'^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} = F^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} / K^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}$

If the integral in expression (2.18) for the scattering amplitude is to converge, it is necessary that the helicity amplitudes be bounded for large Z by Z^{ϵ}, ϵ being some finite real number. Then the amplitudes $f'^{\pm}$ are bounded by $Z^{\epsilon-\lambda_{max}}$ and the weight functions by the same $Z^{\epsilon-\lambda_{max}}$. Since $Q_\ell(z) \xrightarrow[z\to\infty]{} z^{-\ell-1}$, $C^{J+}_{\lambda\mu} \xrightarrow[z\to\infty]{} z^{-J-1+\lambda_{max}}$ and $C^{J-}_{\lambda\mu} \xrightarrow[z\to\infty]{} z^{-J-2+\lambda_{max}}$, it is obvious that our definition (2.19) is convergent for real (J) $>\epsilon$. An examination of subtraction terms shows that they do not contribute to F in this region. This definition of the partial wave amplitudes will be heuristically extended to the left of real (J) = ϵ under the assumption that the only singularities in J are the Regge poles.

In order to use the Sommerfeld-Watson transformation, it is necessary that the background integral be equal to zero or at least negligible. Now if t is greater than the t channel threshold $z_0(s)>1$, $z_0(u)>1$ and $C^{J\pm}_{\lambda\mu} \xrightarrow[|J|\to\infty]{} \exp(-\epsilon|J-\lambda_m|)$. This last property follows from the asymptotic expression given by Squires[6] for $Q_\ell(z)$:

$$Q_\ell(z) \xrightarrow[|\ell|\to\infty]{} \ell^{-1/2} \exp\left((\ell+1/2)\log\left(z-(z^2-1)^{1/2}\right)\right). \tag{2.23}$$

However, the use of equation (2.22) forces the definition of the usual two continuations of F^J for positive and negative signature because of the factor $(-1)^J$. Moreover, it is interesting to note that the relation

$$d^J_{\lambda,\mu}(-z) = (-1)^{J-\mu}\, d^J_{\lambda,-\mu}(z)$$

indicates that if the $f^{\pm}_{\{\lambda\}}$ amplitudes of equation (2.10) had not been adopted as a starting point the introduction of signature would suggest these combinations in any case. So we define

$$F'^{\,J\pm(\pm)^S}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t)$$

$$= 2^{-1/2}\left(\frac{k_i}{k_f}\right)^{1/2}\frac{2}{\pi}\int_{z^0}^{\infty} dz'\left[W^{S\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,s(z')) \pm^S (-1)(-1)^{-\lambda_m} W^{u\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,u(-z'))\right] C^{J+}_{\lambda\mu}(z')$$

$$+\, 2^{-1/2}\left(\frac{k_i}{k_f}\right)^{1/2}\frac{2}{\pi}\int_{z^0}^{\infty} dz'\left[W^{S\mp}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,s(z')) \pm^S (-1)^{-\lambda_m} W^{u\mp}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,u(-z'))\right] C^{J-}_{\lambda\mu}(z')$$

(2.24)

Where $$F'^{\,J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t) = F'^{\,J\pm(\pm)^S}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t) \quad \text{for} \quad (\pm)^S = (-1)^J \quad J = \text{integer} \tag{2.25}$$

These "good signature amplitudes" have exponentially decreasing continuations as $|J| \to \infty$. Then

$$f'^{\,\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,z) = f'^{\,\pm(+)^S}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,z) + f'^{\,\pm(-1)^S}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,z) \tag{2.26}$$

where

$$f'^{\pm(\pm)^S}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,z) = \left(\frac{k_f}{k_i}\right)^{1/2} \sum_{\substack{J=\lambda_m \\ J=\binom{\text{even}}{\text{odd}}}}^{\infty} (2J+1)\left[F^{J\pm(\pm)^S}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t)\, E^{J+}_{\lambda\mu}(z) + F^{J\mp(\pm)^S}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t)\, E^{J-}_{\lambda\mu}(z)\right] \tag{2.27}$$

The notation $\pm^S$ has been introduced to differentiate between signature and J parity.

However,

$$E^{J\pm}_{\lambda\mu}(-z) = \mp(-1)^{J-\lambda_m} E^{J\pm}_{\lambda\mu}(z) \tag{2.28}$$

Therefore,

$$\begin{aligned} \tfrac{1}{2}\left(E^{J\pm}_{\lambda\mu}(z) \pm^S (\mp)(-1)^{\lambda_m} E^{J\pm}_{\lambda,\mu}(-z)\right) &= E^{J\pm}_{\lambda\mu}(z) \quad \text{if} \quad (\pm)^S = (-1)^J \quad J = \text{integer} \\ &= 0 \quad \text{if} \quad (\pm)^S = -(-1)^J \quad J = \text{integer} \end{aligned} \tag{2.29}$$

and placing this expression in (2.27) it follows

$$\begin{aligned} f'^{\pm(\pm)^S}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,z) = \frac{1}{2}\left(\frac{k_f}{k_i}\right)^{1/2} \sum_{J=\lambda_m}^{\infty}(2J+1)\Bigg[& F'^{J\pm(\pm)^S}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t)\left(E^{J+}_{\lambda\mu}(z) \pm^S (-1)^{\lambda_m} E^{J+}_{\lambda\mu}(-z)\right) \\ & + F'^{J\mp(\pm)^S}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t)\left(E^{J-}_{\lambda\mu}(z) \pm^S (-1)(-1)^{\lambda_m} E^{J-}_{\lambda\mu}(-z)\right)\Bigg] \end{aligned} \tag{2.30}$$

Now the summations in (2.30) will be extended to values of J less than λ_m. For $J = -\lambda_m - 1, -\lambda_m - 2 \ldots\ldots$ $E^{J\pm}_{\lambda\mu} = 0$ because of (2.16). There are assumed to be no fixed poles in $F^{J\pm}$ at these values. This assumption, that there are not fixed poles in $F'^{J\pm}$ at $J = -J_o$, demands

$$\int_{z^o}^{\infty} dz' \left\{\left[\omega^{s\pm}_{\{\lambda\}}(t,s(z')) - (-1)^{\lambda_m}\omega^{u}_{\{\lambda\}}(t,u(-z'))\right] C^{(J_o-1)+}_{\lambda\mu} + \left[\omega^{s\mp}_{\{\lambda\}}(t,s(z')) \pm^S (-1)^{-\lambda_m}\omega^{u\mp}_{\{\lambda\}}(t,u(-z))\right] C^{(J_o-1)-}_{\lambda\mu}\right\} = 0 \tag{2.31}$$

since the residue of a fixed pole at $J = -J_o$ in $C_{\lambda\mu}^{(-J_o)\pm}$ is $c_{\lambda\mu}^{(J_o-1)\pm}$. Trueman and Mueller[7] have pointed out that the relation (2.31) is a superconvergent dispersion relation. If the validity of such relations is assumed, the partial wave sum can be extended to $J < -\lambda_m$. In fact, the above assumptions are unnecessarily stringent since ~~a clear~~ further examination reveals that the assumption of no fixed poles is only necessary at the even integers for the even signature amplitude and at the odd integers for the odd signature amplitude.

There remain the terms with $-\lambda_m \leq J \leq \lambda_m - 1$ not included in the partial wave summations. These terms can be divided into pairs about $J = -1/2$ which satisfy the following identities:

$$F'^{(-\frac{1}{2}+\frac{m}{2})\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} = F'^{(-\frac{1}{2}-\frac{m}{2})\mp}_{\lambda_c\lambda_d;\lambda_a\lambda_b} \quad \text{for} \quad m = 1, 3, \ldots\ldots 2\lambda_m - 1 \tag{2.32}$$

$$E^{(-\frac{1}{2}+\frac{m}{2})\pm}_{\lambda\mu} = E^{(-\frac{1}{2}-\frac{m}{2})\mp}_{\lambda\mu} \quad \text{for} \quad m = 1, 3 \ldots\ldots 2\lambda_m - 1 \tag{2.33}$$

All the quantities in these identities must be defined in a limiting sense that is, in the limit as $m \to$ integer. Then, because the factor $(2J+1)$ takes on equal values and has opposite signs at $J = -1/2 \pm m/2$, these terms cancel in the partial wave series. Since the two functions differ in the order of the highest power of Z which each contains, this cancellation might appear surprising. However, on closer examination it is found that the coefficients of the higher powers of Z of the leading E function vanish at these integers, allowing the cancellation to take place.

This behaviour of E^J will be evident in the large Z expansions given in Chapter 4.

There are also poles in the functions $\wp_J$ at the half integers. For J = half integer $\leqslant -1/2$ these poles arise from the Γ function in the definition $\wp_J$; for $J > -1/2$ they arise from the hypergeometric function. These poles have residues which may be grouped as follows: at J = - 1/2 the residue is multiplied by a zero due to the (2J+1) in the partial wave expansion while at the remaining half integers the residues are equal in magnitude and opposite in sign at the points J_o and $-J_o-1$ because of the relations

$$F'^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} = F'^{(-J-1)\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} \qquad \text{for J half integer} \tag{2.34}$$

$$\lim_{J\to J_o} (J-J_o)\, E^{J\pm}_{\lambda\mu} = \lim_{J\to J_o} (J-J_o)\, E^{(-J-1)\pm}_{\lambda\mu} \qquad \text{for } J_o \text{ half integer} \tag{2.35}$$

Therefore if the entire real axis is included in the contour these half integral poles give no net contribution. Furthermore, as a Regge trajectory passes through J_o = half integer greater than - 1/2, the asymptotic behaviour in Z is not changed since the cancellation does not occur in the leading power of Z. This behaviour can be verified using (2.13).

It follows that

$$f'^{\pm(\pm)^S}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) = \frac{1}{2\pi i}\oint_C \frac{dJ\,(2J+1)(\mp)(-1)^{\lambda_m}}{2\sin\pi J}\left(\frac{k_f}{k_i}\right)^{1/2} \times$$

$$\times\left[F'^{J\pm(\pm)^S}_{\lambda_c\lambda_d;\lambda_a\lambda_b}\left(E^{J+}_{\lambda\mu}(-z) \pm^S (-1)^{\lambda_m} E^{J+}_{\lambda\mu}(z)\right) + F'^{J\mp(\pm)^S}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t)\left(E^{J-}_{\lambda\mu}(-z) \pm^S (-1)^{\lambda_m+1} E^{J-}_{\lambda\mu}(z)\right)\right] \tag{2.36}$$

The contour C contains the entire real axis. It was indicated previously that the partial wave amplitudes $F'^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t)$ decrease exponentially with $|J| \to \infty$ if $t >$ t - channel threshold.

The function $\wp_J$ is bounded by a polynomial as $|J| \to \infty$. Therefore, the contour can be expanded to the ∞ circle, picking up the singularities in the J plane. The contour at infinity gives no contribution for Z large. It is assumed that the Regge poles are the only singularities so encountered.

Then

$$f'^{\pm(\pm)^S}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) = \frac{1}{2}\,\frac{\left(2\alpha^{\pm S}_{\pm}+1\right)}{\sin \pi\alpha^{\pm S}_{\pm}}\left(k_f/k_i\right)^{1/2} \beta^{\pm S}_{\pm\lambda_c\lambda_d;\lambda_a\lambda_b}(t)\, E^{\alpha+}_{\lambda\mu}(z)\, K^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t)\left[1(\pm)^S e^{-i\pi\alpha^{\pm S}_{\pm}}\right] \tag{2.37}$$

where the inequality $E^{\alpha+}_{\lambda\mu}(z) >> E^{\alpha-}_{\lambda\mu}(z)$, for large Z, has been used to drop the second term. If this procedure, of including the entire real axis in the contour integral is found objectionable, it can obviously be circumvented by including the conventional background integral along some line $\operatorname{Re} J = -M$ where $M > 1/2$ and $M > \lambda_m$. Then it is necessary to explicitly include, as well as the background integral, the half integral poles for $J \geq M - 1$. However, precisely because these poles do not occur in the leading term in Z of $E^{J\pm}_{\lambda\mu}(Z)$, but in that term whose large Z behaviour is identical to that of half integral pole reflected through $J = -1/2$, as discussed earlier, the contribution of these poles will fall off at large Z at least as rapidly as the background integral. This integral is bounded by Z^{-M}, as is the usual background integral of the "Mandlestam-Sommerfeld-Watson"[8] transformation.

The kinematical region of interest in high energy inelastic scattering is that of large s with t negative. The derivation of the Regge pole expansion presented here has assumed that t > physical threshold in the t channel, that is, $t > t_o > 0$. The problem of continuation of the expansion to negative t can be resolved in either of two ways: with the use of fixed s dispersion relations in t[9] or with the use of a slightly different representation for the partial wave series.[10] Since the amplitudes $f'^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}$ and $F'^{\sigma\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}$ have had all kinematic singularities removed, they can be continued using either of the methods mentioned above.

It is well known that the phase of the Regge pole amplitudes is given by the signature factor in the spinless case. This property implies that the Regge residues are real. In order to prove that the residues are real in the case where the particles carry spin, a trivial generalization of a proof for the spinless case due to C. S. Lam[11] is presented.

If $f'^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,s)$ is the t channel helicity amplitude with all the kinematic singularities removed, then, as before

$$f'^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,z) = \frac{1}{\pi}\int_{z_s^0}^{\infty} dz' \frac{\omega^{s\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,s(z'))}{z'-z} + \frac{1}{\pi}\int_{-\infty}^{\infty} dz' \frac{\omega^{u}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,u(z'))}{z'-z} \qquad (2.38)$$
$$= f_1(z) + f_2(z).$$

Now the complete t channel helicity amplitudes are assumed to satisfy the Schwartz reflection principle, $f^*(Z) = f(Z^*)$. Since the kinematic factors also satisfy this identity,

$$f'^{\pm}_{\{\lambda\}}(t,z^*) = f'^{\pm\,*}_{\{\lambda\}}(t,z)$$

This identity requires

$$\omega_t(t,z') = \mathrm{Im}\, f'^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t, z'+i\delta) \quad \text{for} \quad z' > z_{0_s}$$

$$\omega_u(t,z') = \mathrm{Im}\, f'^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t, z'+i\delta) \quad \text{for } z' < -z_0 \tag{2.39}$$

The amplitude $f'^{\pm}$ is assumed to be given by a single Regge pole for large Z. Then $f'^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,z) = a(t)\, z^{-\epsilon(t)}$ for $Z + i\,\delta$, $Z \gg 0$, $\delta > 0$. It is also assumed that the high energy behaviour is not oscillatory, that is, that $\epsilon(t)$ is real in the scattering region.

Then $$f_1(z) = \frac{1}{\pi}\int_{z_{0_s}}^{\infty} dz' \; \omega^{s\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t, z')\Big/ z'-z$$

$$= \frac{1}{\pi}\int_{z_{0_s}}^{z^M} \frac{dz'\, \omega^{s\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,z')}{z'-z} + \int_{z_M}^{\infty} \frac{\mathrm{Im}\left(a(t)\, z'^{-\epsilon(t)}\right)\, dz'}{z'-z} \tag{2.40}$$

For large Z the first integral vanishes like $\dfrac{1}{Z - Z_M}$.

The second integral has the following properties:

(a) it has a cut from 0 to ∞ along the real axis,

(b) $\mathrm{Im}\, f_1(Z) = \mathrm{Im}\,(a(t))\; Z^{-\epsilon}$

(c) $f_1^{\,*}(Z) = f_1(Z^*)$

Therefore $$f_1(z) = \frac{e^{+i\pi\epsilon}\, z^{-\epsilon}}{\sin \pi\epsilon}\, \mathrm{Im}\, a \qquad \text{for Z large} \tag{2.41}$$

Now $$f_2(z) = \frac{1}{\pi}\int_{-z_{0_u}}^{-\infty} \frac{\omega^{u\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t, z'+i\delta)\, dz'}{z'-z} \tag{2.42}$$

$$= \frac{1}{\pi} \int_{z_{0u}}^{\infty} \frac{W^{u\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t, -z'+i\delta)}{(z'-z)} \, dz' \tag{2.43}$$

But $W^{u\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(t,-z'-i\delta) = \pm^{S} \, \mathrm{Im}[a(t)] \;\; z^{-\epsilon}$

depending on whether a $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ signature trajectory dominates the amplitude.

However $\mathrm{Im}\, f'^{\pm}_{\{\lambda\}}(t,-z'+i\delta) = -\mathrm{Im}\, f'^{\pm}_{\{\lambda\}}(t,-z'-i\delta)$

Therefore
$$f_2(z) = \mp^{S} \frac{1}{\pi} \mathrm{Im}\, a(t) \int_{z^M}^{\infty} \frac{z'^{-\epsilon}}{z'+z} \, dz' \quad \text{for Z large} \tag{2.44}$$

$$= \pm^{S} \frac{z^{-\epsilon} \, \mathrm{Im}\, a(t)}{\sin \pi\epsilon} \tag{2.45}$$

It follows that $f'^{\pm}_{\{\lambda\}}(t,z) = \pm\left[1 \pm e^{+i\pi\epsilon}\right] \frac{z^{-\epsilon}}{\sin\pi\epsilon} \, \mathrm{Im}\, a$

and
$$a(t) = \pm\left[1 \pm e^{i\pi\epsilon}\right] \mathrm{Im}\, a(t)/\sin\pi\epsilon \tag{2.46}$$

This relation implies that the phase of the Regge pole amplitude for $f'^{\pm}$ is given exactly by the signature factor. In addition, a careful consideration of all the kinematic factors shows these factors to be real in both the s and t channel physical regions. Therefore, the phase of the entire helicity amplitude is just that of the Regge pole amplitude. Moreover, for a process dominated by a single Regge pole all helicity amplitudes have the same phase.

CHAPTER 3

CONSTRUCTION OF THE KINEMATIC SINGULARITY FREE HELICITY AMPLITUDES

In order to Reggeize the helicity amplitudes using the procedure given in Chapter 1, it is necessary to separate the amplitude into a product of two terms, the first containing kinematic singularities and zeros and the second dynamical singularities. In the present theory, the dynamical singularities appear when the continuation of the t channel partial wave expansion, which is extended outside of its usual Lehmann ellipse of convergence by means of a Sommerfeld-Watson transformation, diverges. The kinematical singularities, on the other hand, arise directly out of the kinematical properties of the helicity amplitudes as implied by angular momentum conservation and the crossing relations, and depends only on the masses and spins of the initial and final state particles.

We now will sketch Wang's[2] developments, presenting in some detail the particular arrangement of masses that is needed for processes such as $\pi N \longrightarrow \pi N^*$ and $NN \longrightarrow NN^*$.

We consider the general helicity amplitude for

$$a + b \longrightarrow c + d \quad \text{with} \quad s = (p_a + p_b)^2$$
$$\text{and} \quad t = (p_a - p_c)^2 \tag{3.1}$$

Note that the letters a, b, c, d, etc. are used to indicate both the particle itself and its helicity, the meaning being implied by the context. Furthermore, the antiparticles are represented by A, B, C, D.

Then the t channel process can be represented as

$D + b \longrightarrow c + A$ where $t = (p_c + p_A)^2 = (p_D + p_b)^2$

The s channel helicity amplitudes which we use are related to those of Jacob and Wick by

$$f^{(S)}_{cd;ab}(s,t) = 2\pi \left(\frac{s\, p_{ab}}{p_{cd}}\right)^{1/2} f^{J.W}_{cd;ab}(s,t) \tag{3.2}$$

The overall normalization of these amplitudes is determined by the relation

$$\left(\frac{d\sigma}{d\Omega}\right)_{C.M.} = \left| f^{J.W.}_{cd;ab}(s,t)\right|^2 \tag{3.3}$$

The partial wave expansion for the helicity amplitude can then be written

$$f^{(S)}_{cd;ab}(s,t) = \sum_J (2J+1)\; F^J_{cd;ab}(S)\; d^J_{\lambda\mu}(\theta_s) \qquad \begin{array}{l}\lambda = a-b\\ \mu = c-d\end{array} \tag{3.4}$$

where θ_s = angle between $\bar{p}_a$ and $\bar{p}_c$ in the s channel barycentric system. The expressions in terms of s and t for cos θ_s, sin θ_s, etc. are given in Appendix A. An examination of the expression given there for cos θ_s reveals that it is an analytic function of the variable t. The rotation functions have the representation

$$d^J_{\lambda\mu}(\theta_s) = \pm\left[\frac{(J+M)!\,(J-M)!}{(J+N)!\,(J-N)!}\right]^{1/2} \left(\cos\frac{\theta_s}{2}\right)^{|\lambda+\mu|} \left(\sin\frac{\theta_s}{2}\right)^{|\lambda-\mu|} P^{(|\lambda-\mu|,|\lambda+\mu|)}_{J-M}(\cos\theta_s) \tag{3.5}$$

where $P_{J-M}^{(|\lambda-\mu|,|\lambda+\mu|)}(\cos\theta_s)$ is the Jacobi polynomial and

$M = \text{Max}\,(|\lambda|,|\mu|)$

$N = \text{Min}\,(|\lambda|,|\mu|)$

It is therefore evident that the only possible t singularities come from the $\sin(\theta_s/2)$, and $\cos(\theta_s/2)$ terms. However, these terms will factor out of the partial wave expansion since they are independent of J. Therefore the amplitude $\bar{f}^s_{\{\lambda\}}$ is defined such that

$$\bar{f}^s_{cd;ab} = f_{cd;ab}\left[\cos\tfrac{1}{2}\theta_s\right]^{-|\lambda+\mu|}\left[\sin\tfrac{1}{2}\theta_s\right]^{-|\lambda-\mu|}$$

$$= \sum_J (2J+1)\,\tilde{F}^J_{cd;ab}(s)\quad P_{J-M}^{(|\lambda-\mu|,|\lambda+\mu|)}(\cos\theta_s) \tag{3.6}$$

Now $\bar{f}^s$ contains no kinematic t singularities. Any t singularities of $\bar{f}^s$ are due to the divergence of this expansion and are said to be dynamical. Furthermore, it is evident that some of the factors arising from $d^J_{\lambda\mu}(\theta_s)$ have been absorbed into $F^J_{cd;ab}$. These factors may contain branch cuts in the complex J plane which must be considered in the Reggeization procedure.

The analogous t channel amplitude is defined

$$\bar{f}^t_{c'A';D'B'} = \left(\sin\tfrac{1}{2}\theta_t\right)^{-|\lambda'-\mu'|}\left(\cos\tfrac{1}{2}\theta_t\right)^{-|\lambda'+\mu'|} f^t_{c'A';D'B'} \qquad \begin{aligned}\lambda' &= D'-B'\\ \mu' &= c'-A'\end{aligned} \tag{3.7}$$

where $P^{(|\lambda-\mu|,|\lambda+\mu|)}_{J-M}(\cos\theta_s)$ is the Jacobi polynomial and

$M = \text{Max}\,(|\lambda|,|\mu|)$

$N = \text{Min}\,(|\lambda|,|\mu|)$

It is therefore evident that the only possible t singularities come from the $\sin(\theta_s/2)$, and $\cos(\theta_s/2)$ terms. However, these terms will factor out of the partial wave expansion since they are independent of J. Therefore the amplitude $\bar{f}^s_{\{\lambda\}}$ is defined such that

$$\bar{f}^s_{cd;ab} = f_{cd;ab}\left[\cos\tfrac{1}{2}\theta_s\right]^{-|\lambda+\mu|}\left[\sin\tfrac{1}{2}\theta_s\right]^{-|\lambda-\mu|}$$

$$= \sum_J (2J+1)\,\tilde{F}^J_{cd;ab}(s)\; P^{(|\lambda-\mu|,|\lambda+\mu|)}_{J-M}(\cos\theta_s) \tag{3.6}$$

Now $\bar{f}^s$ contains no kinematic t singularities. Any t singularities of $\bar{f}^s$ are due to the divergence of this expansion and are said to be dynamical. Furthermore, it is evident that some of the factors arising from $d^J_{\lambda\mu}(\theta_s)$ have been absorbed into $F^J_{cd;ab}$. These factors may contain branch cuts in the complex J plane which must be considered in the Reggeization procedure.

The analogous t channel amplitude is defined

$$\bar{f}^t_{c'A';D'B'} = \left(\sin\tfrac{1}{2}\theta_t\right)^{-|\lambda'-\mu'|}\left(\cos\tfrac{1}{2}\theta_t\right)^{-|\lambda'+\mu'|} f^t_{c'A';D'B'} \qquad \begin{matrix}\lambda' = D'-B' \\ \mu' = c'-A'\end{matrix} \tag{3.7}$$

This amplitude is free of kinematic s singularities. Now the Trueman and Wick[12] crossing relations are used to obtain crossing relations for these amplitudes

$$\bar{f}^{s}_{cd;ab}(s\,t) = \sum_{A'b'c'D'} \mathcal{M}^{cd;ab}_{c'A';D'b'}(s,t)\, \bar{f}^{t}_{c'A';D'b'} \tag{3.8}$$

Where

$$\mathcal{M}^{cd;ab}_{c'A';D'b'}(s,t) = \left[\sin\tfrac{1}{2}\theta_s\right]^{-|\lambda-\mu|}\left[\cos\tfrac{1}{2}\theta_s\right]^{-|\lambda+\mu|}\left[\sin\tfrac{1}{2}\theta_t\right]^{|\lambda'-\mu'|}\left[\cos\tfrac{1}{2}\theta_t\right]^{|\lambda'+\mu'|} d^{J_A}_{A'a}(\chi_a)\, d^{J_b}_{b'b}(\chi_b)\, d^{J_c}_{c'c}(\chi_c)\, d^{J_D}_{D'd}(\chi_d) \tag{3.9}$$

and

$$\cos\chi_a = \left[-(s+m_a^2-m_b^2)(t+m_a^2-m_c^2)-2m_a^2(m_c^2-m_a^2+m_b^2-m_d^2)\right]/S_{ab}\,T_{ac}$$

$$\cos\chi_b = \left[+(s+m_b^2-m_a^2)(t+m_b^2-m_d^2)-2m_b^2(m_c^2-m_a^2+m_b^2-m_d^2)\right]/S_{ab}\,T_{bd}$$

$$\cos\chi_c = \left[(s+m_c^2-m_d^2)(t+m_c^2-m_a^2)-2m_c^2(m_c^2-m_a^2+m_b^2-m_d^2)\right]/S_{cd}\,T_{ac}$$

$$\cos\chi_d = \left[-(s+m_d^2-m_c^2)(t+m_d^2-m_b^2)-2m_d^2(m_c^2-m_a^2+m_b^2-m_d^2)\right]/S_{cd}\,T_{bd}.$$

(3.10)

and

$$\sin\chi_a = 2m_a\left[\phi(s,t)\right]^{1/2}/S_{ab}\,T_{ac} \qquad \sin\chi_b = 2m_b\left[\phi(s,t)\right]^{1/2}/S_{ab}\,T_{bd}.$$

$$\sin\chi_c = 2m_c\left[\phi(s,t)\right]^{1/2}/S_{cd}\,T_{ac} \qquad \sin\chi_d = 2m_d\left[\phi(s,t)\right]^{1/2}/S_{cd}\,T_{bd}$$

(3.11)

The polynomials $\phi(s,t)$, S_{ab}, and T_{ac} are defined in the first Appendix.

Since $\bar{f}^t$ (s,t) is free of kinematic s singularities, all kinematic s singularities of $\bar{f}^s$ (s,t) must come from the crossing matrices. Wang has shown that there are pure s and pure t singularities at s = 0, $s_{ab} = 0$, $s_{cd} = 0$, t = 0, $\tau_{ac} = 0$, $\tau_{bd} = 0$ and that there are no mixed s and t singularities at $\phi(s,t) = 0$. If the pure s singularities of each M are factorizable and all M's have the same type of pure s singularity, it is then evident that this $\bar{f}^s$ has factorizable singularities.

There exists the relation

$$\bar{f}^t_{c'A';D'b'}(s,t) = \eta_t \bar{f}^t_{-c'-A';-D'-b'}(s,t)$$

from parity symmetry where

$$\eta_t = \frac{\eta_A \eta_c}{\eta_D \eta_b} (-1)^{J_c+J_a-J_d-J_b} (-1)^{\lambda'-\mu'} \tag{3.12}$$

Using the above relation, the linear combination

$$\bar{f}^{s\pm}_{cd;ab} = \bar{f}^s_{cd;ab} \pm \bar{f}^s_{-c-d;ab}$$

then satisfies the crossing relations

$$\bar{f}^{s\pm}_{cd;ab} = \sum_{\substack{D'>0 \\ A'c'b'}} \bar{M}^{\pm\, cd;ab}_{c'A';D'b'} \bar{f}^t_{c'A';D'b'} \tag{3.13}$$

where

$$\bar{M}^{\pm\, cd;ab}_{c'A';D'b'} = M^{cd;ab}_{c'A';D'b'} + \eta_t M^{cd;ab}_{-c'-A';-D'-b'} \pm \left(M^{-c-d;ab}_{c'A'D'b'} + \eta_t M^{-c-d;ab}_{-c'-A'-D'-b'} \right) \tag{3.14}$$

We will also need the relations

$$d^J_{\lambda\mu}(\pi-\theta) = (-1)^{J-\mu} d^J_{-\lambda\mu}(\theta) \tag{3.15}$$

and

$$d^J_{\lambda\mu}(\theta) = \pm \left(\frac{1-\cos\theta}{\sin\theta}\right)^{|\lambda-\mu|} \left(\cos\tfrac{1}{2}\theta\right)^{\nu} \mathcal{P}^{J-\nu/2}(\cos\theta) \tag{3.16}$$

where $v = 1$ for J = half integer

$v = 0$ for J = integer

and

$\mathcal{P}^{(J-\nu/2)}(\cos\theta)$ is a polynomial in $\cos\theta$ of order $J - v/2$

The particular set of external masses which interests us is given by: $m_a = m_b$, $m_c \neq m_d$. This case gives results identical to those of $m_a = m_b = m_c \neq m_d$.

Then, at the points $s = (m_a \pm m_b)^2$ the quantities $S_{ab}\sin x_a$, $S_{ab}\cos x_a$, $S_{ab}\sin x_b$ and $S_{ab}\cos x_b$ are free from singularities in s. Of course, the expression S_{ab} has a branch point in the variable s at each of these points.

As a next step it is necessary to consider the quantities

$$\begin{Bmatrix} \sin\frac{1}{2}\chi_a \\ \cos\frac{1}{2}\chi_a \end{Bmatrix} = \left[\tfrac{1}{2}\left(1 \mp \cos\chi_a\right)\right]^{1/2}$$

The cut in the s plane in the functions S_{ab}, $\cos x_a$ and $\sin x_a$ is chosen to run along the real axis and join $S = (m_a + m_b)^2$ and $S = (m_a - m_b)^2$. We then define

$$\begin{Bmatrix} s-(m_a+m_b)^2 \equiv r e^{i\phi} \\ s-(m_a-m_b)^2 \equiv r' e^{i\phi'} \end{Bmatrix} \quad \text{where} \quad \begin{Bmatrix} \pi \geq \phi \geq -\pi \\ 0 \leq \phi' \leq 2\pi \end{Bmatrix}$$

defines the first sheet. This definition implies $S_{ab} = R'\, e^{i/2(\phi+\phi')}$

and $\cos x_a = R''_a e^{-i(\phi+\phi')/2}$. Further, in the complex $\cos x_a$ plane, $\sin \frac{1}{2} x_a$ has a branch point at $\cos x_a = 1$ and we choose the cut to run from 1 to ∞ along the real axis. Similarly $\cos \frac{1}{2} x_a$ is chosen to be cut from -1 to $-\infty$ in $\cos x_a$. The angle x_b is treated analogously, defining cuts in $\sin \frac{x_b}{2}$ and $\cos \frac{x_b}{2}$. There is however one small but important difference. In the region $t < 0$, for t fixed, there are two solutions to the equation $\phi(s,t) = 0$, which are called $s_>$ and $s_<$ and satisfy $s_> > (m_a + m_b)^2 > (m_a - m_b)^2 > s_<$. At $\phi(s_>, t)$, $\sin x_a = 0$, $\cos x_a = +1$, $\sin x_b = 0$ and $+ \cos x_b = -1$ while the cosines change sign at $\phi(s_<, t) = 0$. An examination of $\sin x_a$ reveals that it is pure imaginary for $t < 0$, $s_> \geq s \geq (m_a + m_b)^2$. Hence $|\cos x_a| > 1$ in this region. It follows that near $s = (m_a + m_b)^2$, $\cos x_a$ can be represented as $R\, e^{i\phi}$ where $R_A > 0$ because $\cos x_a = +1$ at $s_>$. Similarly $\cos x_b = -R_B\, e^{i\phi'}$ where $R_B > 0$. In order to decide whether a particular function has a branch point at $s = (m_a + m_b)^2$ we compute the function at the point A in Figure 1 and again after a rotation of 2π in a clockwise sense, that is, at B. A discontinuity indicates a branch point. Now

at point A , $\cos x_a = R_A \exp(-i\pi/2)$; $\cos x_b = -R_B \exp(-i\pi/2)$

at point B , $\cos x_b = -R_B \exp(i\pi/2)$; $\cos x_a = R_A \exp(i\pi/2)$

$R_A > 0 \quad R_B > 0$

It is then straightforward to examine the half angles and arrive at the following table:

	$\sin\frac{1}{2}x_a$	$\cos\frac{1}{2}x_a$	$\sin\frac{1}{2}x_b$	$\cos\frac{1}{2}x_b$
At point A	$e^{i\alpha/2}K_A$	$e^{-i\alpha/2}K_A$	$-e^{-i\beta/2}K_B$	$e^{i\beta/2}K_B$
At point B	$-e^{-i\alpha/2}K_A$	$e^{i\alpha/2}K_A$	$e^{i\beta/2}K_B$	$e^{-i\beta/2}K_B$

	$\cos\frac{1}{2}x_a \ \cos\frac{1}{2}x_b \pm \sin\frac{1}{2}x_a \ \sin\frac{1}{2}x_b$
At point A	$K_A K_B\left[e^{-i\alpha/2}\,e^{i\beta/2} \mp e^{-i\beta/2}\,e^{i\alpha/2}\right]$
At point B	$K_A K_B\left[e^{i\alpha/2}\,e^{-i\beta/2} \mp e^{-i\alpha/2}\,e^{i\beta/2}\right]$

where

$$K_A = \sqrt{1 + R_A^{\,2}} \qquad \alpha = \tan R_A \qquad K_B = \sqrt{1 + R_B^{\,2}} \qquad \beta = \tan R_B$$

This table shows $(\cos\frac{1}{2}x_a \ \cos\frac{1}{2}x_b - \sin\frac{1}{2}x_a \ \sin\frac{1}{2}x_b)$ to have no branch point at $s = (m_a + m_b)^2$, while $(\cos\frac{1}{2}x_a \ \cos\frac{1}{2}x_b + \sin\frac{1}{2}x_b \ \sin\frac{1}{2}x_b)$ has a branch point at this point.

Using similar methods the following functions can be shown to be analytic at both $s = (m_a + m_b)^2$ and $s = (m_a - m_b)^2$; $S_{ab}\sin x_a$, $S_{ab}\cos x_a$, $S_{ab}\sin x_b$, $S_{ab}\cos x_b$, $\left[\cos\frac{1}{2}x_a \ \cos\frac{1}{2}x_b + \sin\frac{1}{2}x_a \ \sin\frac{1}{2}x_b\right] \times \left[s - (m_a + m_b)^2\right]^{1/2}$, $\left[\cos\frac{1}{2}x_a \ \cos\frac{1}{2}x_b - \sin\frac{1}{2}x_a \ \sin\frac{1}{2}x_b\right] \times \left[s - (m_a - m_b)^2\right]^{1/2}$, etc. A full list of such quantities has been given by Wang.

Using the analytic properties of such quantities along with equations (3.14), (3.15) and (3.16), it is then straightforward, if very tedious, to show that the following prescription gives the amplitudes in the case $m_a = m_b \quad m_c \neq m_d$. The expressions for the crossing matrices from which these prescriptions are deduced are listed in Appendix D.

Then the kinematic singularity free amplitudes are

$$\left[\bar{f}^{s}_{cd;ab} \pm \bar{f}^{s}_{-c-d;ab}\right]\left(s-4m_a^2\right)^{\frac{1}{2}\alpha_1}\left(s-(m_c+m_d)^2\right)^{\frac{1}{2}\beta_1} s^{\frac{1}{2}\alpha_2'}\left[s-(m_c-m_d)^2\right]^{\frac{1}{2}\beta_2} \tag{3.17}$$

where $\alpha_2' = \max\left(\mp\eta_{ab}\right) \text{ of } \left[J_a + J_b - \frac{1}{2}(\nu_a+\nu_b)\right] + \frac{1}{2}(\nu_a+\nu_b) \quad \text{if } \nu_a = \nu_b = 1$

and if $\nu_a = \nu_b = 0, \quad \alpha_2' = \max\left(\pm\eta_{ab}\right) \text{ of } \left[J_a + J_b\right]$

where

$v = 1$ if J = half integer, $\quad v = 0$ if J = integer,

$$\eta_{ab} = \frac{\eta_A \eta_C}{\eta_B \eta_D}(-1)^{J_c+J_d+c+d} \qquad \eta_{cd} = (-1)^{J_c+J_d+c+d}$$

and the expressions for

	BB $\rightarrow$ F(c)$\overline{F}$(d)	F(a)$\overline{F}$(b) $\rightarrow$ BB	FF $\rightarrow$ FF	BB $\rightarrow$ BB
α_1	$\alpha_g\ (\pm)$	$\alpha_g\ (\pm)$	$\alpha_g\ (\pm)$	$\alpha_g\ (\pm)$
β_1	$\beta_g\ (\pm)$	$\beta_g\ (\pm)$	$\beta_g\ (\pm)$	$\beta_g\ (\pm)$
β_2	$\beta_g\ (\mp)$	$\beta_g\ (\pm)$	$\beta_g\ (\mp)$	$\beta_g\ (\pm)$

(B = Boson) (F = Fermion)

where

$$\alpha_g(\pm) = \frac{1}{2}(v_a+v_b) - |\lambda - \mu| + \max\ (\pm\eta_{ab})\ \text{of}\ \left[J_a+J_b-\frac{1}{2}(v_a+v_b)+\frac{1}{2}(|\lambda-\mu|-|\lambda+\mu|)\right]$$

and

$$\beta_g(\pm) = \frac{1}{2}(v_c+v_d) - |\lambda - \mu| + \max\ (\pm\eta_{cd})\ \text{of}\ \left[J_c+J_d-\frac{1}{2}(v_c+v_d)+\frac{1}{2}(|\lambda-\mu| - |\lambda+\mu|)\right]$$

where max η of N means greatest even integer $\leqslant N$ if $\eta = +1$ and means greatest odd integer $\leqslant N$ if $\eta = -1$.

The physical consequences of this expression will be discussed in the next section and some modifications suggested. However, although this formulation appears to have found all kinematic singularities in the helicity amplitudes it certainly does not contain all the zeros. These zeros will appear in the form of constraints between the amplitudes.(13)

CHAPTER 4

In Chapters 2 and 3 a method of Reggeization of processes involving spin has been developed and the kinematic singularities of the helicity amplitudes under consideration have been explicitly calculated. In this section we will discuss some ansatz for the residues and give the form of the helicity amplitudes for specific processes.

In order to discuss the residues, it is necessary to examine the constraints which can be applied to these residues. The first of these constraints is the factorization theorem for the Regge residues, first put forward by Gellman[14] and independently by Gribov and Pomeranchuk.[15] While Gellman demonstrated this hypothesis only for the case of a coupled channel Schrodinger equation, Gribov and Pomeranchuk explored the relativistic problem, using unitarity directly, for the coupled $\pi\pi$ and $K\bar{K}$ channels in the region $4m_\pi^2 < s < 4m_K^2$. Oehme[16] shows how to continue this relation back to the t channel physical region. This simple situation can be extended to the case of a large number of two body channels quite easily but requires more careful consideration if many particle channels are included. The proof for N two body channels is given by Squires[6] and is reproduced in Appendix B for the case of helicity amplitudes. It should be recognized that since this theorem is a consequence of unitarity, the factorization property is carried by the residues of the poles in $F^J_{cd;ab}$, the amplitudes for which unitarity assumes a simple form.

This theorem then states

$$\beta_{cd;ab} \quad \beta_{ef;gh} \quad = \quad \beta_{cd;gh} \quad \beta_{ef;ab} \tag{4.1}$$

Fox and Leader[17] have recently shown that, because the leading term for large Z of the $d^J_{\lambda\mu}(Z)$ also can be factored, this factorization property also applies to the contribution to the t channel helicity amplitudes from a single Regge pole.

That is,

$$f^t_{cd;ab} \quad f^t_{ef;gh} \quad = \quad f^t_{cd;gh} \quad f^t_{ef;ab} \tag{4.2}$$

The "sense-nonsense" restrictions, which are a consequence of this factorization, will now be discussed. We remark that the definition of $F^{J\pm}_{cd;ab}$ in equation (2.19) contains the functions $C^{J\pm}_{\lambda\mu}$, which have factors such as $\left[J(J+1)\right]^{1/2}$ if exactly one of $|\lambda|$, $|\mu|$ exceeds 0. It is necessary that the $F^{J\pm}$ contain these factors in order that they cancel with similar factors in the functions $\mathcal{E}^{J\pm}_{\lambda\mu}(Z)$. Then the partial wave sum does not contain fixed branch points and no difficulties are encountered in the Sommerfeld-Watson transformation.

In general an amplitude is called a sense-nonsense amplitude at $J = M$ if one of $|\lambda|$, $|\mu| > M$. Such a sense-nonsense amplitude contains a factor $\left[(\alpha - M)(\alpha + M + 1)\right]^{1/2}$ in the residue of the Regge pole occurring in the amplitude. An amplitude where both $|\lambda|$, $|\mu| > M$ is defined to be a nonsense-nonsense amplitude at $J = M$ and an amplitude where both $|\lambda|$, $|\mu| \leqslant M$, a sense-sense amplitude at $J = M$. The factorization theorem for the Regge residues states that the square of the residue of a nonsense-sense amplitude is equal to the product of the sense-sense and nonsense-nonsense residues. Thus, either the sense-sense residue or the nonsense-nonsense residue must contain a factor $[(J - M)(J + M + 1)]$. Such a factor would cancel the pole in the amplitude at J = M. If the sense-sense (nonsense-nonsense) residue does not approach zero as $J \to M$ the trajectory is said to have chosen sense (nonsense) at J = M. If the trajectory chooses sense the pole in the nonsense-nonsense amplitude that occurs as a Regge trajectory passes through M is cancelled by the zero in the residue. If the trajectory chooses nonsense, it appears that $f^{\pm}_{cdab}$ has a singularity at a nonsensical value of $\alpha^{\pm}$. However, it follows from (2.32) that this singularity is cancelled by a compensating trajectory of opposite J parity that passes through $J = -1 - \alpha$ as the original trajectory goes through $J = \alpha$.

In the previous chapter the exact form of the kinematic singularities of the amplitudes was calculated. These singularities were explicitly included in the full residue of the Regge pole. However, there are modifications of these kinematic forms which will be discussed here.

Now from equation (2.11)

$$F'^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} = z^{1/2}\left(\frac{k_i}{k_f}\right)^{1/2}\int_{-1}^{1} dz\left[C^{J+}_{\lambda\mu}\, f'^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} + C^{J-}_{\lambda\mu}\, f'^{\mp}_{\lambda_c\lambda_d;\lambda_a\lambda_b}\right] \quad (4.3)$$

By definition $f'^{\pm}$ is free of kinematic singularities. From Appendix A

$$s = z/2t \cdot (T_{ac}\, T_{bd}) + L(t) \quad (4.4)$$

where $$L(t) = -\frac{1}{2t}\left(t^2 - t\sum m_i^2 + (m_d^2 - m_b^2)(m_c^2 - m_a^2)\right)$$

The integrations in the above expressions are carried out at fixed t. If there is a power series expansion of $f'^{\pm}_{cd;ab}(t,s)$ in s near $L(t)$ which converges for some range of s, that is

$$f'^{\pm}_{\{\lambda\}}(t,s) = f'^{\pm}_{\{\lambda\}}\left(t, L(t)\right) + \frac{(\tau_{ac}\,\tau_{bd})}{2t}\frac{\partial}{\partial s}\, f'^{\pm}_{\{\lambda\}}\left(t, L(t)\right) + \ldots\ldots \tag{4.5}$$

then it follows, since $c^{J+}_{\lambda\mu}$ is a linear combination of P_N, $N = J + \lambda_M \ldots$ $\ldots J - \lambda_M$ and $c^{J-}_{\lambda\mu}$ is a linear combination of P_N, $N = J + \lambda_M - 1 \ldots J - \lambda_M + 1$, that, near τ_{ac} or $\tau_{bd} = 0$,

$$F'^{J\pm}_{\{\lambda\}} \sim \left[\tau_{ac}\,\tau_{bd}/2ts_0\right]^{J-\lambda_m} \tag{4.6}$$

Some specific examples will now be considered in the light of the previous points. Consider the process $\pi^+ p \longrightarrow \pi^0 N^{*++}$, making the identification

$$a \leftrightarrow p,\ b \leftrightarrow \pi^+,\ c \leftrightarrow N^{*++},\ d \leftrightarrow \pi^0$$

In the crossed channel this process can be written:

$$D + B \longrightarrow C + A \quad \text{or} \quad \pi^0 + \pi^+ \longrightarrow N^{*++} + \bar{p}$$

Now relabel the process $\pi\pi \longrightarrow N^* \bar{p}$ as $a' + b' \longrightarrow c' + d'$

Then

$$m_{a'} = m_{b'} \quad \text{and} \quad m_{c'} \neq m_{d'} \quad \text{and} \quad t = (p_{a'} + p_{b'})^2 .$$

Using the results of Chapter 3 it is found that the following combinations of helicity amplitudes are free of kinematic singularities:

$$\left[\bar{f}_{c'd';00} \pm \bar{f}_{-c'-d';00} \right] (t-4\mu^2)^{\frac{1}{2}\alpha_1} \left[t-(m+M)^2\right]^{\frac{1}{2}\beta_1} t^{\frac{1}{2}\alpha_2'} \left[t-(M-m)^2\right]^{\frac{1}{2}\beta_2} \tag{4.7}$$

where μ = pion mass m = nucleon mass M = isobar mass

The quantities $\alpha_1, \alpha_2, \beta_1, \beta_2$ were defined in Chapter 3. Since any trajectory that couples to the $\pi^+\pi^0$ system must have positive J parity, G = +1, and I = 1 or 2, there is a severe limitation on the trajectories which can contribute. In fact the only known, zero strangeness, boson trajectory with these quantum numbers is the ρ trajectory. Now

$$f^{\pm} = \bar{f}_{c'd';00} \pm (-1)^{\lambda+\lambda_m} \eta_{c'} \eta_{d'} (-1)^{S_c+S_d-\nu} \bar{f}_{-c'-d';00} \tag{4.8}$$

Further since the ρ trajectory has positive J parity, it couples only to f^+ in the large Z limit. Therefore, $f^- \rightarrow 0$ in large Z limit.

Then $\bar{f}_{c'd';00} = \frac{1}{2} f^+_{c'd';00}$

and $\bar{f}_{c'd';00} = -(-1)^{c'-d'} \bar{f}_{-c'-d';00}$ (4.9)

The above expression for f^+ can be simplified to the form

$$f^+_{cd;00} = \bar{f}_{cd;00} - (-1)^{c-d} \bar{f}_{-c-d;00} \tag{4.10}$$

Using the results of the previous pages, we expect F^{J+} and hence the Regge residues to contain the following terms.

$$\beta_{\frac{1}{2}-\frac{1}{2};00} \sim \left[\alpha(\alpha+1)\right]^{\frac{1}{2}} (t-4\mu^2)^{\frac{1}{2}} \left[t-(M-m)^2\right]^{-\frac{1}{2}} \left(\frac{p_{\pi\pi}\, p_{NN^*}}{S_0}\right)^{\alpha-1} \tag{4.11}$$

$\beta_{\frac{3}{2}\frac{1}{2};00}$ contains identical factors.

The factor $(p_{\pi\pi}\, p_{\bar{N}\Delta})^{\alpha-1}$ describes the threshold behavior of $F^{J+}_{\{\lambda\}}$ as described on the previous pages. The $[\alpha(\alpha+1)]^{1/2}$ arises because at $\alpha = 0$ these amplitudes are sense-nonsense amplitudes. The factors of $(t - 4\mu^2)^{1/2}$ and $[t - (M - m)^2]^{-1/2}$ are the kinematic singularities as derived in Chapter 3. Analogously the factors contained in the remaining residues are

$$\beta_{3/2\,-1/2;00}(t) \sim \left[\alpha(\alpha+1)(\alpha-1)(\alpha+2)\right]^{1/2}\left[t-(m+M)^2\right]^{1/2}(t-4\mu^2)\left(\frac{p_{\pi\pi}\, p_{\bar{N}N^*}}{s_0}\right)^{\alpha-2} \tag{4.12}$$

and

$$\beta_{1/2\,1/2;00}(t) \sim \left[t-(m+M)^2\right]^{-1/2}\left[t-(m-M)^2\right]^{-1} \left(\frac{p_{\pi\pi}\, p_{\bar{N}N^*}}{s_0}\right)^{\alpha-1} \tag{4.13}$$

The asymptotic limit for the E functions can be derived by substituting the expression for $\mathcal{D}_J$ in the expressions given in reference 1 for $e^{J\pm}_{\lambda\mu}$ and expanding the hypergeometric functions.

Then

$$E^{\alpha,+}_{00}(z) \underset{\text{LARGE } z}{\simeq} \frac{\Gamma(\alpha+1/2)}{\pi^{1/2}\,\Gamma(\alpha+1)}\left(\frac{s'}{p_{\pi\pi}\, p_{\bar{N}N^*}}\right)^{\alpha} \tag{4.14}$$

$$E^{\alpha,+}_{01}(z) \underset{\text{LARGE } z}{\simeq} \frac{2\alpha}{\left[\alpha(\alpha+1)\right]^{1/2}}\,\frac{\Gamma(\alpha+1/2)}{\pi^{1/2}\,\Gamma(\alpha+1)}\left(\frac{s'}{p_{\pi\pi}\, p_{\bar{N}N^*}}\right)^{\alpha-1} \tag{4.15}$$

$$E^{\alpha,+}_{02}(z) \underset{\text{LARGE } z}{\simeq} \frac{4\alpha(\alpha-1)}{\left[\alpha(\alpha+1)(\alpha-1)(\alpha+2)\right]^{1/2}}\,\frac{\Gamma(\alpha+1/2)}{\pi^{1/2}\,\Gamma(\alpha+1)}\left(\frac{s'}{p_{\pi\pi}\, p_{\bar{N}N^*}}\right)^{\alpha-2} \tag{4.16}$$

where $s' = \frac{1}{2}(s - u)$.

The final form for the helicity amplitudes is then

$$f_{\frac{3}{2}\frac{1}{2};00} = \frac{\left[1-\exp(-i\pi\alpha)\right] \sin\theta_t \;\; \alpha \;\; (4\mu^2-t)^{1/2} \;\; \gamma_{\frac{3}{2}\frac{1}{2};00}(t) \;\; \left(\frac{s'}{s_0}\right)^{\alpha-1}}{\Gamma(\alpha+1) \;\; \sin\pi\alpha \;\; \left[(M-m)^2-t\right]^{1/2}} \tag{4.17}$$

$$f_{\frac{1}{2}-\frac{1}{2};00} = \frac{\left[1-\exp(-i\pi\alpha)\right] \sin\theta_t \;\; \alpha \;\; (4\mu^2-t)^{1/2} \;\; \gamma_{\frac{1}{2}-\frac{1}{2};00}(t)}{\Gamma(\alpha+1) \;\; \sin\pi\alpha \;\; \left[(M-m)^2-t\right]^{1/2}} \left(\frac{s'}{s_0}\right)^{\alpha-1} \tag{4.18}$$

$$f_{\frac{1}{2}\frac{1}{2};00} = \frac{\left[1-\exp(-i\pi\alpha)\right] \;\; \gamma_{\frac{1}{2}\frac{1}{2};00}(t) \;\; \left(\frac{s'}{s_0}\right)^{\alpha}}{\Gamma(\alpha+1) \;\; \sin\pi\alpha \;\; \left[t-(M-m)^2\right] \left[t-(M+m)^2\right]^{1/2}} \tag{4.19}$$

$$f_{\frac{3}{2}-\frac{1}{2};00} = \frac{\left[1-\exp(-i\pi\alpha)\right]}{\Gamma(\alpha+1) \sin\pi\alpha} \sin^2\theta_t \, \alpha(\alpha-1)(t-4\mu^2)\left[t-(M+m)^2\right]^{+1/2} \gamma_{\frac{3}{2}-\frac{1}{2};00}(t) \left(\frac{s'}{s_0}\right)^{\alpha-2} \tag{4.20}$$

The factors $(p_{\pi\pi}\, p_{\overline{N}\Delta})^{\alpha-\lambda_{max}}$ and the factors $\left[\alpha(\alpha+1)\right]^{1/2}$ have cancelled out of the final expression. The terms $\frac{1}{\Gamma(\alpha+1)}$ from the $\mathcal{P}_\alpha(Z)$ have been combined with the $\frac{1}{\sin\pi\alpha}$ from the Sommerfeld-Watson transformation in order to cancel the poles which occur in $(\sin\pi\alpha)^{-1}$ when α passes through a negative integer. The reduced residue $\gamma_{\{\lambda\}}(t)$ is formed by combining the remainder of the residue, after explictly removing the kinematic terms described earlier, with the factor $\frac{(2\alpha+1)\,\Gamma(\alpha+1/2)}{\pi^{1/2}}$.

The poles of $\Gamma(\alpha + 1/2)$ at $\alpha = -3/2, -5/2$ are cancelled by the zeros of F^{J+} at these points.

Further understanding of these expressions may be obtained by considering the threshold properties* of the amplitudes $f_{cd;oo}$ at the thresholds in the t channel which occur whenever $p_{\pi\pi}$ or $p_{\overline{N}\Delta} \rightarrow 0$. From Appendix A, we see that $p_{\pi\pi}$ goes to zero as $t \rightarrow 4m^2$ and $p_{\overline{N}\Delta}$ goes to zero as $t \rightarrow (M + m)^2$ or $t \rightarrow (M - m)^2$. This second point is called a pseudo-threshold.

In order to facilitate the treatment of the threshold properties, it is useful to define amplitudes governing transitions to the eigenstates of spin labelled by S, S_λ for the $\overline{N}\Delta$ system where S is the total spin of $\overline{N}\Delta$ system and S_λ is its component along the direction of motion. These amplitudes are

$$A_{22} = f_{3/2 -1/2;00} \tag{4.21}$$

$$A_{21} = (3/4)^{1/2} f_{1/2 -1/2\, 00} + (1/4)^{1/2} f_{3/2\; 1/2;\, 00} \tag{4.22}$$

$$A_{11} = -(1/4)^{1/2} f_{1/2 -1/2;00} + (3/4)^{1/2} f_{3/2\; 1/2;00} \tag{4.23}$$

$$A_{10} = \left(\frac{1}{2}\right)^{1/2}\left(f_{1/2\; 1/2;00} - f_{-1/2\, -1/2;00} \right) \tag{4.24}$$

But from Jacob and Wick[4] :

$$A^J_{L'S';00}(W) = \left[\frac{(2L'+1)}{(2J+1)}\right]^{1/2} \sum_{\lambda'} \left(L' S' 0 \lambda' | J \lambda' \right) A^J_{S'\lambda';00} \tag{4.25}$$

Franklin,[18] in an earlier paper, has given expressions for the behavior of $A^J_{L'S';00}$ at both thresholds and pseudo-thresholds.

The equation (4.25) above can be inverted by introducing a sum over L'. Further, $A_{S'\lambda';00}$ has the following expansion

$$A_{s'\lambda';00} = \sum_J (2J+1)\; d^J_{\lambda'\lambda}\; A^J_{s'\lambda';00} \tag{4.26}$$

*I am greatly indebted to Dr. Jerrold Franklin for showing me this result prior to publication.

Moreover $d^J_{\mu\mu'}(\theta)$ can be expanded as

$$(1-z)^{\frac{\mu'-\mu}{2}} (1+z)^{\frac{\mu'+\mu}{2}} d^J_{\mu\mu'}(\theta) = \sum_{\ell=J-n}^{J+n} C_\ell(\mu,\mu' J)\, P_\ell(z) \tag{4.27}$$

where

$$C_\ell(\mu,\mu',J) = (2\ell+1)\, 2^{\mu'} \left(\frac{(\mu'-\mu)!\,(\mu'+\mu)!}{(2\mu')!}\right)^{1/2} \begin{pmatrix} J & \ell & \mu' \\ \mu' & 0 & -\mu' \end{pmatrix} \begin{pmatrix} J & \ell & \mu' \\ \mu & 0 & -\mu \end{pmatrix} \tag{4.28}$$

where $\mu' \geq 0$ and $\mu' \geq \mu$ and $n = \max(|\mu|, |\mu'|)$.

The other $d^J_{\mu\mu'}(Z)$ can be defined using symmetry properties of these functions.

This procedure leads to the following expression for $A_{S'\lambda', S\lambda}$, generalized to the case where both initial and final states carry intrinsic spin.

$$A_{S'\lambda'; S\lambda} = \sum_{J\ell L L'} (2J+1)(2\ell+1)\, 2^{\lambda} \left[\frac{(\lambda-\lambda')!\,(\lambda+\lambda')!}{(2\lambda)!}\right]^{1/2} \begin{pmatrix} J & \ell & \lambda \\ \lambda & 0 & -\lambda \end{pmatrix} \begin{pmatrix} J & \ell & \lambda \\ \lambda' & 0 & -\lambda' \end{pmatrix} \times$$

$$(-1)^{L+L'-S-S'+\lambda+\lambda'} \sqrt{(2L+1)(2L'+1)} \begin{pmatrix} L' & S' & J \\ 0 & \lambda' & -\lambda' \end{pmatrix} \begin{pmatrix} L & S & J \\ 0 & \lambda & -\lambda \end{pmatrix} \frac{P_\ell(z)}{(1-z)^{\frac{\lambda-\lambda'}{2}} (1+z)^{\frac{\lambda+\lambda'}{2}}} A^J_{L'S'; LS} \tag{4.29}$$

Now consider $t \to (M+m)^2$ in the example $\pi N \to \pi N^*$. For J fixed only a single L will contribute, $L = J-S$, and $A^J_{L'S';00} \sim q^{J-S}$. Further as $q \to 0$, $Z \to \infty$ and $P_\ell(Z) \to Z^\ell$, which causes only $\ell = J + N$ to contribute to the sum over ℓ.

These simplifications imply

$$A_{S'\lambda';00} = (-1)^{-\lambda/2} \frac{1}{q^S} \left[\frac{1}{(S-\lambda)!\,(S+\lambda)!}\right]^{1/2} \times \sum_J \left(\begin{array}{l}\text{Quantities} \\ \text{Independent} \\ \text{of } \lambda \text{ and } q \\ \text{as } q \to \infty\end{array}\right) \tag{4.30}$$

If, for reasons of parity, the lowest angular momentum which can contribute is $L = J-S+1$

$$A_{S'\lambda';00} = (-1)^{-\lambda/2} \frac{\lambda}{q^{S-1}} \left[\frac{1}{(S-\lambda)!(S+\lambda)!}\right]^{1/2} \times \text{ factors independent of } \lambda \text{ and } q \tag{4.31}$$

and for $L = J-S+2$

$$A_{S'\lambda';00} = (-1)^{-\lambda/2} \frac{1}{q^{S-2}} \left[\frac{1}{(S-\lambda)!(S+\lambda)!}\right]^{1/2} \times (A+B\lambda^2) \tag{4.32}$$

where A and B are independent of λ and q.

This direct threshold behavior was specified earlier in our expressions for the ρ Regge pole contribution. For example, at the direct threshold $t = 4\mu^2$ Wang's factors are just those necessary to cancel a similar factor in the factor $\sin \theta_t$. This behavior is verified by this calculation since it predicts a $1/q^S$, i.e. a q^0, behavior at this threshold. Similar results hold at the threshold $t = (M + m)^2$. It is important to note that there are just two independent amplitudes at this threshold instead of the usual four, because of the relations just derived.

Specifically, for ρ trajectory for the S=2 amplitudes at the threshold $t = (M + m)^2$, we have

$$A_{22} = -K_1 \times \frac{2}{q} \left[\frac{1}{24}\right]^{1/2} \tag{4.33}$$

$$A_{21} = -i K_1 \frac{1}{q} \left[\frac{1}{6}\right]^{1/2} \tag{4.34}$$

If, for reasons of parity, the lowest angular momentum which can contribute is $L = J-S+1$

$$A_{S'\lambda';00} = (-1)^{-\lambda/2} \frac{\lambda}{q^{S-1}} \left[\frac{1}{(S-\lambda)!(S+\lambda)!}\right]^{1/2} \times \text{ factors independent of } \lambda \text{ and } q \tag{4.31}$$

and for $L = J-S+2$

$$A_{S'\lambda';00} = (-1)^{-\lambda/2} \frac{1}{q^{S-2}} \left[\frac{1}{(S-\lambda)!(S+\lambda)!}\right]^{1/2} \times (A+\beta\lambda^2) \tag{4.32}$$

where A and B are independent of λ and q.

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$$A_{21} = -i K_1 \frac{1}{q} \left[\frac{1}{6}\right]^{1/2} \tag{4.34}$$

and for S = 1

$$A_{11} = -i\,\frac{1}{q}\left[\frac{1}{2}\right]^{1/2} K_2 \tag{4.35}$$

$$A_{10} = \frac{1}{q}\,K_2 \tag{4.36}$$

where K_1 and K_2 are two independent amplitudes.

The behavior of these amplitudes at the pseudo-thresholds can be considered in a similar fashion. Franklin has given a prescription for the behavior of the amplitudes $A^J_{LS,L'S'}$ at these points. At the pseudo-threshold only a single L = J-S is expected to contribute, as before. However, the dependence on q is modified at $t = (M - m)^2$. Then

$$A_{22} = -K_1 \times \frac{3}{q^2}\left[\frac{1}{24}\right]^{1/2} \tag{4.37}$$

$$A_{21} = -i K_1 \times \frac{1}{q^2}\left[\frac{1}{6}\right]^{1/2} \tag{4.38}$$

$$A_{10} = \frac{1}{q^2} K_2 \tag{4.39}$$

$$A_{11} = -i\,\frac{1}{q^2}\left[\frac{1}{2}\right]^{1/2} K_2 \tag{4.40}$$

This pseudo-threshold occurs at t = 0.09 Bev^2, which is near the physical region in the present example, and so we will investigate the consequences of letting such conditions apply, at least approximately, at the edge of physical region.

If we define

$$\gamma_{22}(t) = \gamma_{3/2\,-1/2;00}(t) \qquad \gamma_{10}(t) = \sqrt{2}\,\gamma_{1/2\,1/2;00}(t) \tag{4.41}$$

$$\gamma_{21}(t) = (3/4)^{1/2}\gamma_{1/2\,-1/2;00}(t) + (1/4)^{1/2}\gamma_{3/2\,1/2\,00}(t) \tag{4.42}$$

$$\gamma_{11}(t) = (3/4)^{1/2}\gamma_{3/2\,1/2;00}(t) - (1/4)^{1/2}\gamma_{1/2\,-1/2;00}(t) \tag{4.43}$$

Then it follows from equations (4.41), (4.42), (4.37), (4.38), (4.21), (4.22), (4.18), (4.19) and (4.20) that

$$\frac{\gamma_{21}(t)}{\gamma_{22}(t)} = \left.\frac{\sin\theta_t\,(\alpha-1)\,(t-4\mu^2)^{1/2}\left(t-(M+m)^2\right)^{1/2}\left((M-m)^2-t\right)^{1/2}}{\left(s'/s_0\right)}\right]_{t=(M-m)^2} \tag{4.44}$$

and for large s

$$\gamma_{21} = 2s_0(M-m)(\alpha-1)\gamma_{22} \tag{4.45}$$

Similarly,

$$\gamma_{10} = 2\sqrt{2}\,s_0(M-m)\alpha\,\gamma_{11} \tag{4.46}$$

These last two relations are the result of applying the constraint only to the leading power of S' of the Regge amplitudes. All t dependent quantities in these equations are to be evaluated at $t = (M - m)^2$.

Rho Trajectory Contribution to $PP \longrightarrow PN^*$

The treatment of this process is similar to that of the previous process excepting two complications, the occurrence of nonsense-nonsense amplitudes and of restrictions due to factorization, as well as some difficulties associated with the kinematic singularities at $t = 0$. We list the singular factors that are extracted from the Regge residue of the ρ trajectory for the process $pp \longrightarrow pN^*$. Then

$$\beta^-_{\frac12\frac12;\frac12-\frac12},\ \beta^+_{\frac12-\frac12;\frac12\frac12},\ \beta^+_{\frac32\frac12;\frac12\frac12} \sim \left[t-(m-M)^2\right]^{-1/2}\left(\frac{p_{\bar N N}\,p_{\bar N\Delta}}{s_0}\right)^{\alpha-1}\left[\alpha(\alpha+1)\right]^{1/2} \tag{4.47}$$

$$\beta^-_{\frac32-\frac12;\frac12\frac12} \sim (t-4m^2)^{1/2}\left[t-(M+m)^2\right]^{1/2}\left(\frac{p_{N\bar N}\,p_{\bar N\Delta}}{s_0}\right)^{\alpha-2}\left[\alpha(\alpha+1)(\alpha-1)(\alpha+2)\right]^{1/2} \tag{4.48}$$

$$\beta^-_{\frac12\frac12;\frac12\frac12} \sim (t-4m^2)^{-1/2}\left[t-(M+m)^2\right]^{-1/2}\left[t-(M-m)^2\right]^{-1}\left(\frac{p_{N\bar N}\,p_{\bar N\Delta}}{s_0}\right)^{\alpha} \tag{4.49}$$

$$\beta^-_{\frac32\frac12;\frac12-\frac12},\ \beta^-_{\frac12-\frac12;\frac12-\frac12} \sim \left[t-(M-m)^2\right]^{-1/2} t^{-1/2}\left(\frac{p_{N\bar N}\,p_{\bar N\Delta}}{s_0}\right)^{\alpha-1}\alpha(\alpha+1) \tag{4.50}$$

$$\beta^+_{\frac32-\frac12;\frac12-\frac12} \sim t^{-1/2}(t-4m^2)^{1/2}\left[t-(m+M)^2\right]^{1/2}\left(\frac{p_{N\bar N}\,p_{\bar N\Delta}}{s_0}\right)^{\alpha-2}\alpha(\alpha+1)\left[(\alpha-1)(\alpha+2)\right]^{1/2} \tag{4.51}$$

The superscripts on the β indicate whether the positive J parity combination occurs in $f_{cd;ab} + f_{-c-d;ab}$ or in $f_{cd;ab} - f_{-c-d;ab}$ as $Z \longrightarrow \infty$. Moreover, the factorization theorem states

$$\frac{\beta_{\frac12\frac12;\frac12\frac12}}{\beta_{\frac12\frac12;\frac12-\frac12}} = \frac{\beta_{\frac12-\frac12;\frac12\frac12}}{\beta_{\frac12-\frac12;\frac12-\frac12}} = \frac{\beta_{\frac32-\frac12;\frac12\frac12}}{\beta_{\frac32\frac12;\frac12-\frac12}} = \frac{\beta_{\frac32-\frac12;\frac12+\frac12}}{\beta_{\frac32-\frac12;\frac12-\frac12}} \tag{4.52}$$

Evaluating these ratios we have

$$\beta_{\frac{3}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}/\beta_{\frac{3}{2}\frac{1}{2};\frac{1}{2}-\frac{1}{2}} \sim t^{1/2} \qquad \beta_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}}/\beta_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}} \sim t^{1/2}$$

$$\beta_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}/\beta_{\frac{1}{2}\frac{1}{2};\frac{1}{2}-\frac{1}{2}} \sim t^{-1/2} \qquad \beta_{\frac{3}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}}/\beta_{\frac{3}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}} \sim t^{1/2} \tag{4.53}$$

In addition, factorization enables one to relate these helicity amplitudes to processes like $\pi N \to \pi N^*$ with relations such as

$$\beta_{\frac{3}{2}\frac{1}{2};00}/\beta_{\frac{1}{2}\frac{1}{2};00} = \beta^{\rho}_{\frac{3}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}/\beta^{\rho}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}} \sim t^{1/2} \tag{4.54}$$

The left-hand side of this relation refers to the residues of $\pi N \to \pi N^*$, the right-hand side to those of $NN \to NN^*$. The simplest solution to these equations is to place a further factor of t in the residues $\beta_{\frac{3}{2}\frac{1}{2};\frac{1}{2}-\frac{1}{2}}$ $\beta_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}}$ and $\beta_{\frac{3}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}}$.

A different approach to this matter is to return to Wang's derivation of the kinematic singularities. The $t^{-1/2}$ singularities come entirely from the crossing matrix, arising out of the singularities of $\cos x_a$ and $\cos x_c$ at $t = 0$. (The channel process $A + c \to b + D$ has been identified with $\bar{p} + p \to \bar{p} + N^*$.) Then

$$\cos x_a = \frac{+1}{S_{ab} \quad t^{1/2}(t-4m^2)^{1/2}} \left[t(s + m^2 - M^2) + 2m^2(M^2 - m^2) \right] \tag{4.55}$$

The crucial point is that although $\cos x_a$ has a $1/t^{1/2}$ behavior as $t \to 0$ the leading term as $s \to \infty$ behaves as $t^{1/2}$ at small t. The first behavior leads to a structure

$$\bar{f}^t \propto t^{-1/2(S_A+S_C)} \propto t^{-1/2} \quad \text{independent of helicities.}$$

This behavior can be verified if one examines both linear combinations $f^{+}_{cd;ab}$ and $f^{-}_{cd;ab}$. For all helicity amplitudes, one of these combinations contains the $t^{-1/2}$ factor. We argue that, since we are interested in the t dependence of the leading terms as $s \to \infty$, this behavior should be modified to $t^{1/2}$. In any case, $\cos x_a$ is bounded by 1 when we are in the S channel physical region and the $t^{-1/2}$ singularity cannot be supposed to be effective when we are in the physical region, even when $t_{min} \longrightarrow 0$ as S becomes very large.

Among the factors which have been extracted from the Regge residues are some factors of α. The occurrence of such factors in the sense-nonsense amplitudes was explained previously, as being due to the analytic properties of these amplitudes in the J plane. The factorization theorem for the Regge residues can be written in a manner which relates the product of two sense-nonsense residues at $\alpha = J_o$ to the product of a sense-sense residue and a nonsense-nonsense residue.

For example

$$\beta_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}}\,\beta_{\frac{1}{2}\frac{1}{2};\frac{1}{2}-\frac{1}{2}} = \beta_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}\,\beta_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}}$$

This relation immediately implies that one of the residues on the right-hand side of this equation should contain a factor $\alpha(\alpha + 1)$. If, as in the present work, this factor is placed in $\beta_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}}$, the pole in the nonsense-nonsense amplitude is cancelled and the trajectory is said to have chosen sense at this point.

It should be pointed out that Trueman and Muellar[7] have recently shown that the existence of fixed poles in the scattering amplitudes alleviates the necessity that a trajectory choose sense or nonsense at the point J_o, where $F^{J^{\pm}}$ is of the wrong signature at J_o. Mandelstam and Wang[19] have demonstrated that such fixed poles are possible in any theory that possesses a third double spectral function. In particular, Trueman and Muellar showed that it is no longer necessary to have the helicity amplitude for a process decouple from the trajectory when α passes through an integer $< |\lambda|, |\lambda|$. However, in the present calculations it is always assumed that the effects of the third double spectral function are small and that the trajectories will choose sense at such points.

If the edge of the s channel physical region is approached at large fixed s by letting $t \longrightarrow t_{min}$, the expression given in Appendix A reveals that $\lim_{t \longrightarrow t_{min}} \cos \theta_t = 1$. This limiting value of $\cos \theta_t$ would seem to imply that the Regge approximation to the scattering amplitude should break down as t becomes very small. However, in a recent paper Freedman and Wang[21] have shown that, for the similar problem of backward πp scattering, the assumption of analyticity in the momentum transfer is a strong enough condition to enable one to prove that the S'^{α} behavior can be extended back to zero momentum transfer. This behavior was realized through the intervention of daughter trajectories with singular residues. In the present work it is assumed, in the same spirit, that the amplitudes which are free of kinematic singularities, the $f'^{\pm}_{cd;ab}$, behave as $S'^{\alpha-\lambda_m}$ everywhere in the physical region.

It follows that the helicity amplitudes for this process can be expressed as

$$f_{\frac{3}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}} = \frac{\left[1-\exp(-i\pi\alpha)\right]}{\Gamma(\alpha+1)\sin\pi\alpha} \left\{ \frac{\sin\left(\frac{\theta_t}{2}\right)\cos\left(\frac{\theta_t}{2}\right)\ \alpha\ \gamma_{\frac{3}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}}{\left[t-(M-m)^2\right]^{1/2}} \right\} \left(\frac{s_t}{s_0}\right)^{\alpha-1} \tag{4.56}$$

Similar expressions hold for $f_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}}$ and $f_{\frac{1}{2}\frac{1}{2};\frac{1}{2}-\frac{1}{2}}$ with the appropriate residues .

$$f_{\frac{3}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}} = \frac{\left[1-\exp(-i\pi\alpha)\right]}{\Gamma(\alpha+1)\sin\pi\alpha}\ \frac{1}{4}\sin^2\theta_t\ \alpha(\alpha-1)\,\gamma_{\frac{3}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}}\left(\frac{s_t}{s_0}\right)^{\alpha-2}\left(t-4m^2\right)^{1/2}\left(t-(M+m)^2\right)^{1/2} \tag{4.57}$$

$$f_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}} = \frac{\left[1-\exp(-i\pi\alpha)\right]}{\Gamma(\alpha+1)\sin\pi\alpha}\ \frac{\gamma_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}(t)\ \left(s_t/s_0\right)^{\alpha}}{\left[t-(M-m)^2\right]\left[t-(M+m)^2\right]^{1/2}\left(t-4m^2\right)^{1/2}} \tag{4.58}$$

$$f_{\frac{3}{2}\frac{1}{2};\frac{1}{2}-\frac{1}{2}} = \frac{\left[1-\exp(-i\pi\alpha)\right]}{\Gamma(\alpha+1)\sin\pi\alpha}\ \frac{t^{1/2}}{\left[t-(M-m)^2\right]^{1/2}}\cos^2\frac{\theta_t}{2}\ \alpha^2\ \gamma_{\frac{3}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2}}\left(\frac{s_t}{s_0}\right)^{\alpha-1} \tag{4.59}$$

A similar expression holds for $f_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}}$

$$f_{\frac{3}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}} = \left[\frac{1-\exp(-i\pi\alpha)}{\Gamma(\alpha+1)\sin\pi\alpha}\right] t^{1/2}\left(t-4m^2\right)^{1/2}\left(t-(M+m)^2\right)^{1/2}\cos^3\left(\frac{\theta_t}{2}\right)\sin\frac{\theta_t}{2}\ \alpha^2(\alpha-1)\,\gamma_{\frac{3}{2}-\frac{1}{2}\frac{1}{2}-\frac{1}{2}}\left(\frac{s_t}{s_0}\right)^{\alpha-2} \tag{4.60}$$

where

$$s_t = (s-u)/2 .$$

The other amplitudes are implied by parity conservation and the fact that only positive J parity contributes.

Pion Exchange Contribution to $PP \rightarrow PN^*$

The calculations of the amplitudes for this process are very similar to that of the previous process except that there is the added restriction that the pion can couple to the $N\overline{N}$ system only when they have the same helicity. This restriction follows immediately from the fact that $G = (-1)^{L+S+T}$ for an $N\bar{N}$ system. In order to couple to the pion, L must be even and T = 1 for the $N\bar{N}$ system. Now, if the nucleons have opposite helicities, S = 1, which implies G = +. Hence the $N\bar{N}$ pair with opposite helicities can't couple to the pion.

Then

$$f_{\frac{3}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}} = \frac{[1+\exp(-i\pi\alpha)]}{\Gamma(\alpha+1)\sin\pi\alpha} \sin\theta_t \, \alpha \left(\frac{s_t}{s_0}\right)^{\alpha-1} \frac{(t-4m^2)^{1/2} t^{1/2}}{[t-(M+m)^2]^{1/2}} \gamma_{\frac{3}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}(t) \tag{4.61}$$

$$f_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}} = \frac{[1+\exp(-i\pi\alpha)]}{\Gamma(\alpha+1)\sin\pi\alpha} \sin\theta_t \, \alpha \left(\frac{s_t}{s_0}\right)^{\alpha-1} \frac{(t-4m^2)^{1/2} t^{1/2}}{[t-(M+m)^2]^{1/2}} \gamma_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}}(t) \tag{4.62}$$

$$f_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}} = \frac{[1+\exp(-i\pi\alpha)]}{\Gamma(\alpha+1)\sin\pi\alpha} \left(\frac{s_t}{s_0}\right)^{\alpha} \frac{t^{1/2}\,\gamma_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}(t)}{[t-(M-m)^2]^{1/2}[t-(M+m)^2]^{1/2}} \tag{4.63}$$

$$f_{\frac{3}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}} = \frac{[1+\exp(-i\pi\alpha)]}{\Gamma(\alpha+1)\sin\pi\alpha} \sin^2\theta_t \,[t-(M-m)^2]^{1/2} \left(\frac{s_t}{s_0}\right)^{\alpha-2} (t-4m^2)\, t^{1/2}\, \alpha(\alpha-1)\, \gamma_{\frac{3}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}}(t) \tag{4.64}$$

Again parity and J parity can be used to deduce the other amplitudes from those given. There are threshold constraints for this process. In order to apply such constraints, we must consider amplitudes with total spin a good quantum number. Such amplitudes are

$$A_{22} = f_{\frac{3}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}} \tag{4.65}$$

$$A_{21} = (3/4)^{1/2} f_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}} + (1/4)^{1/2} f_{\frac{3}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}} \tag{4.66}$$

$$A_{20} = \frac{1}{\sqrt{2}} \left(f_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}} + f_{-\frac{1}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}} \right) \tag{4.67}$$

$$A_{11} = -\left(\tfrac{1}{4}\right)^{1/2} f_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}} + \left(\tfrac{3}{4}\right)^{1/2} f_{\frac{3}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}} \tag{4.68}$$

Now for pion exchange, the $\bar{N}\Delta$ system has L even and J even, implying that for S = 2 and J fixed, the minimal value which L can attain is J - S. For S = 1 the minimal value of L = J - S + 1. If these amplitudes are examined at the direct thresholds, the behavior is exactly that contained in the expressions for $f_{\frac{3}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}}$ etc. There are some constraints on the amplitudes at these points but we neglect such conditions since the thresholds occur a long distance from the S channel physical region. When the analogous procedure is attempted at the pseudo threshold only one constraint arises. The reason for this is that for L = J - 2 and L = J we have exactly the same pseudo-threshold behavior q^{J-1}. These two partial waves give an overall helicity dependence of the form A + B λ^2 at this threshold to describe the three S = 2 amplitudes. This occurrence leads to a single constraint. This constraint can be written

$$f_{22;\frac{1}{2}\frac{1}{2}} - 4 f_{21;\frac{1}{2}\frac{1}{2}} + 3 f_{20;\frac{1}{2}\frac{1}{2}} = 0 \tag{4.69}$$

This constraint then reads, dropping all terms except the leading power of S'

$$-(M-m)^2 \alpha(\alpha-1)\,\gamma_{\frac{3}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}} - 4\left(\frac{M-m}{S_0}\right)\alpha\left[\left(\tfrac{3}{4}\right)^{1/2}\gamma_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}} + \left(\tfrac{1}{4}\right)^{1/2}\gamma_{\frac{3}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}\right] + 3\sqrt{2}\;\gamma_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}\big/S_0^2 = 0 \tag{4.70}$$

CHAPTER 5

THE PROCESS $\pi^+ p \rightarrow \pi^0 N^{*++}$

A parameterization of the ρ trajectory contribution to the helicity amplitudes for this process was given in Chapter 4, including some possible ansatz for the t dependence of the Regge residues and possible relations between these residues. In this section the relation of these ansatz to the experimental data will be explored.

In terms of the amplitudes given in Chapter 4 we have

$$\frac{d\sigma}{dt} = \sum_{c,d} \left| f^t_{00;cd} \right|^2 \Big/ \left[\left(s-(m+\mu)^2\right)\left(s-(m-\mu)^2\right) \right] \tag{5.1}$$

This relation follows from the unitary nature of the crossing matrix.

In addition

$$2 \frac{d\sigma}{dt} \rho_{mm'} = \sum_{d} f^t_{00;md} \, f^{*t}_{00;m'd} \tag{5.2}$$

In deriving equations (5.1) and (5.2) we have made use of the relation

$$f_{00;-c-d} = \frac{\eta_c \eta_d}{\eta_a \eta_b} (-1)^{s_c + s_d} (-1)^{-c+d} f_{00;cd} \tag{5.3}$$

which comes from parity conservation.

This relation simplifies to

$$f_{00;-c-d} = -(-1)^{d-c} f_{00;cd}. \tag{5.4}$$

By making use of this relation the sums in equation (5.1) can be restricted to those amplitudes which are listed in section 4. This procedure introduces a factor of two, but this factor is immediately absorbed into the residues by defining $\gamma'_{oo;cd} = \sqrt{2}\,\gamma_{oo;cd}$. This factor introduces the two on the left-hand side of equation (5.2). Further the relation (5.4) introduces a minus sign between the two terms of equation (5.2) for $\rho_{3,-1}$ if this equation is expressed in terms of the four amplitudes listed in Chapter 4. Moreover, since the phase of all helicity amplitudes is given by the signature factor, $Re\,(\rho_{m\,m'}) = \rho_{m,m'}$

for this single trajectory model.

The t dependence of the amplitudes $f_{oo,cd}$ in the physical region is examined next. For this process the edge of the physical region is given by $\phi(s,t) = 0$ where $\phi(s,t)$ is given in Appendix A. For s large it follows that

$$|t|_{min.} \approx \frac{\mu^2 (M^2 - m^2)^2}{S'^2} \tag{5.5}$$

where S' was defined in Chapter 4.

Hence for practical purposes we can set $t_{min} = 0$ for this process. Furthermore, in the region of interest, that is, $t \lesssim 0.5\ (\text{Bev})^2$

$$|\sin\theta_t| = \frac{2S'}{(M+m)} \left[\frac{|t - t_{min}|}{(t-4\mu^2)\,((M-m)^2 - t)} \right]^{1/2} \tag{5.6}$$

Figure 1 shows a plot of sin θ_t as a function of t in the s channel physical region. Note that it vanishes at the edge of the physical region and peaks at about $t = -.10\ (\text{Bev})^2$. Since all amplitudes include a factor $|\sin\theta_t|^{\lambda}$ where λ = no. of units of helicity flip, all amplitudes involving helicity flip vanish in the forward direction and should peak at small t if the other t dependences in the problem are not unreasonably strong. If the contribution of the non-helicity flip amplitude is not equal to zero, but is of the same order of magnitude as the helicity flip amplitudes the density matrix elements ρ_{33}, $\rho_{3,1}$ and $\rho_{3,-1}$ should approach zero, for $t < 0.10$ (Bev)^2 as $|t| \rightarrow 0$. This situation can be modified however if the non-flip amplitude is approximately equal to zero. In this case only $\rho_{3,-1}$ goes to zero, since only $\rho_{3,1}$ contains a factor $\sin \theta_t$. These conclusions are, of course, dependent on the validity of the Freedman and Wang extension of the Regge expansion to the forward direction.

There exists data for this process at incident pion laboratory momenta of 2.75 Gev/c, 3.54 GeV/c, 4.0 Gev/c and 8.0 Gev/c. The differential cross sections at these energies have been given by D. R. O. Morrison[22] while the density matrix elements at 4 and 8 Gev/c are given in the paper of Crijns et al.[23]

The differential cross section data given by Morrison contains a forward peak at about $|t| = 0.05$. It appears to fall off for $|t| < 0.05$ at all energies. The existence of this forward dip in our fits will be used as one criterion for differentiating between the "good" and "bad" fits. The second interesting feature of the data is the energy variation of the density matrix elements between 4 and 8 Gev/c. One of the features of a single Regge pole model is that it gives all the helicity amplitudes the same energy behavior, provided, of course, that they can couple to the pole under consideration. This situation leads immediately to the prediction that the density matrix elements, as predicted by a single Regge pole model are energy independent. This prediction is clearly incompatible with the experimental data for ρ_{11} by at least one standard deviation. It is tempting to propose the mechanism of direct channel resonances interfering with the Regge pole terms to explain this energy variation of the data. However, recent work involving finite energy sum rules[24] seem to indicate that the Regge pole analysis and the partial wave analysis in the direct channel involving resonating partial waves are each complete representations of the scattering amplitude and that these representations should not be mixed. In this case it is necessary to attempt to explain this energy variation of the density matrix elements with terms arising out of the Regge expansion; that is, with lower Regge poles or Regge cuts or even a term from the background integral interfering with the ρ contribution. There is evidence for a ρ' trajectory contributing to πN elastic charge exchange scattering.[24,25,26] This trajectory is supposed to have an intercept about 0.4 below the ρ trajectory and so might lead to such interference terms. These considerations naturally suggest fitting the single Regge pole model primarily to the 8 GeV/c data. The fit that is achieved at this energy for the various ansatz for the residues will be compared to experiment at lower energies however.

A numerical least-squares fit to the experimental data has been attempted for several sets of assumptions.

In the first fit it was assumed that α (t) = 0.56 + 0.81 t and s_o = 2.0 Bev^2, in agreement with the Arbab and Chiu[27] fit to pion-nucleon charge exchange scattering. Furthermore, it was assumed that all the reduced residues are independent of the momentum transfer and of one another. These reduced residues were varied to achieve the fit which is shown in Table 5.1 and in Figures 3, 4, 5, and 6. It is interesting to note that this fit has the parameter $\gamma_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}} = 0.0$, which implies that Re $\rho_{3,1}$ vanishes in the forward direction as $\sin\theta_t$, while all other density matrix elements remain finite at the edge of the physical region. At 8 GeV/c incident momentum the x^2 per point is about 0.75 for the differential cross section data where x^2 is defined in the usual manner to be

$$\chi^2 = \sum_{\substack{\text{experimental}\\ \text{points}}} \left(\frac{f_{\text{THEORY}} - f_{\text{EXPERIMENT}}}{\Delta} \right)^2 \tag{5.7}$$

Δ = experimental error assigned.

Furthermore, the density matrix elements are in good agreement with experiment, all lying near or within 1 standard deviation of experiment. At 4 GeV/c incident momentum the x^2 for the differential cross section is about 1.5 per point, which represents a barely adequate fit. At lower incident momenta the experimental values are noticeably exceeded by the calculated curves. This circumstance necessitates a secondary contribution interfering destructively with the ρ trajectory contributions. It is noted, as a matter of interest, that the fit to the 8 GeV/c incident momentum data is relatively insensitive to about a 10% change in the parameters.

It is perhaps amusing to note that a second, relatively simple fit to the data can be achieved in the following manner. It is assumed that α (t) = 0.56 + 0.81 t and the reduced residues are independent of momentum transfer. These residues are chosen such that

$$\gamma^2_{00;\frac{1}{2}-\frac{1}{2}} \Big/ \gamma^2_{00;\frac{3}{2}\frac{1}{2}} = 0.29/0.21$$

while $\gamma_{00;1/2\,1/2} = \gamma_{00;3/2\,-1/2} = 0$. The scale parameter S_o and $\gamma_{00;3/2\,1/2}$ are then varied to achieve a best fit to the data. Therefore, by construction, this fit gives $\rho_{11} = 0.29$, Re $\rho_{3,1} = 0.0$, and Re $\rho_{3,-1} = 0.21$. Since only spin flip amplitudes are used, the forward dip, which we have accepted as one of the criteria for a fit, appears automatically. The dependence on momentum transfer is fixed except for a term $\exp(-(\ln s_o)\alpha' t)$ which grows exponentially in the scattering region if $s_o > 1$. The best fit, for $s_o = 5.64$, has a $x^2 = 0.9$ per point for the differential cross section. Again the results at lower energies become progressively worse, although both the 4 Gev/c and 8 Gev/c data can be said to be adequately fit.

The third fit to this process explores the threshold constraints at the pseudo-threshold $t = (M - m)^2$, as discussed in Chapter 4. Since these constraints occur much closer to the physical region than the ρ trajectory pole at $\alpha_\rho = 1$, it is possible that they might be at least approximately satisfied for $t \lesssim 0$.

The constraints, as listed in (4.11.8) and (4.11.9), reduce to two the number of parameters in the fit. If it is demanded that the helicity non-flip amplitude be much smaller than the helicity flip amplitudes in order to insure the appearance of the forward dip, only a single parameter remains to fit the data. This parameter is the strength of the S = 2 amplitudes. In this case, the density matrix elements are determined to be $\rho_{11} = 0.13$, $\rho_{3,1} = 0.18$ and $\rho_{3,-1} = 0.07$. These numbers bear no resemblance to the experimental values. Further, the differential cross section disagrees with experiment. If we drop our requirement of a forward dip, then much better numerical fits can be achieved and x^2 per point reduced to about 1 per point for both the differential cross sections and density matrix elements. However these fits are rejected since they do not contain the dip, which was accepted as a criterion for an adequate fit.

If the residues are chosen, not as constants, but as linear polynomials in t the number of parameters rises to 6. There are a large number of adequate fits in this case, since there is not enough experimental information to determine 6 parameters.

Pion Nucleon Charge Exchange Scattering

This reaction has been studied by many authors and satisfactory fits to the differential cross section have been obtained by several authors. The most useful of these fits is that of Arbab and Chiu[27] since their parameterization is closest to the helicity amplitude formalism. Although the Regge pole model appears at first sight to give a vanishing polarization for the recoil neutron, Durand[28] has shown in a recent review article that a closer examination reveals a variety of plausible reasons which might account for the non zero value of the polarization.

Chiu and Arbab parameterize their fit as follows

$$\frac{d\sigma}{dt}(s,t) = \frac{1}{\pi s}\left(\frac{M}{4k}\right)^2\left[\left(1-\frac{t}{4M^2}\right)|A|^2 + \frac{t}{4M^2}\left(s - \frac{s+p^2}{(1-t/4M^2)}\right)|B|^2\right] \tag{5.8}$$

where

$$A = C(t)\left[\frac{1-\exp(-i\pi\alpha)}{\sin\pi\alpha}\right]\left(\frac{E}{E_0}\right)^{\alpha}$$

$$B = D(t)\left[\frac{1-\exp(-i\pi\alpha)}{\sin\pi\alpha}\right]\left(\frac{E}{E_0}\right)^{\alpha-1}$$

In the above expression B represents the helicity flip amplitude and A the helicity non-flip amplitude. The symbols S and t are the invariant squares of energy and momentum transfer, p and E are the incident pion momentum and total energy in the laboratory system, k is the centre of mass momentum, M is the nucleon mass and E_o is a scale factor which Chiu and Arbab take to be 1 GeV. Chiu and Arbab find that the data is well represented by the following set of parameters

$$\alpha(t) = 0.56 + 0.81\, t \qquad C(t) = (\alpha + 1)\, C_o \exp(C_1 t)$$

$$D(t) = \alpha(\alpha + 1)\, D_o \exp(D_1 t)$$

where

$$C_o = 2.3 \text{ mb GeV} \qquad D_o = 38.9 \text{ mb}$$

$$C_1 = 0.01 \text{ GeV}^{-2} \qquad D_1 = 0.01 \text{ GeV}^{-2}$$

Our parameterization will be slightly different from that of the previous reference in order to facilitate use of the factorization theorem.

In the t channel we consider the process

$$\pi\,\pi \rightarrow N\bar{N}$$

Then the following t channel helicity amplitudes are free of kinematical singularities

$$\left(t - 4m_N^2\right)^{1/2} f^t_{oo;++} \tag{5.9}$$

$$(\sin\theta_t)^{-1} \left[t\left(t - 4m_\pi^2\right)\right]^{-1/2} f^t_{oo;+-} \tag{5.10}$$

Using the method developed in the previous chapter the following form for the differential cross section can be derived.

$$\frac{d\sigma}{dt} = \frac{1}{\mathscr{S}_{\pi N}^2} \left|1 - \exp(-i\pi\alpha)\right|^2 \times \left[\frac{1}{\Gamma(\alpha+1)\sin\pi\alpha}\right]^2$$

$$\times \left\{ \frac{\left|\gamma_{00;\frac{1}{2}\frac{1}{2}}\right|^2 \left(\frac{s'}{s_0}\right)^{2\alpha}}{(t-4m_N^2)} + \left|\gamma_{00;\frac{1}{2}-\frac{1}{2}}\right|^2 \left(\frac{s'}{s_0}\right)^{2\alpha-2} \left|\sin\theta_t\right|^2 \alpha^2 t\left(t-4m_\pi^2\right) \right\} \tag{5.11}$$

where

$$\mathscr{S}_{\pi N}^2 = \left[s-(m+\mu)^2\right]\left[s-(m-\mu)^2\right]$$

For $t \ll 4m_N^2$ the following identification can be made

$$\left|\gamma_{00;\frac{1}{2}\frac{1}{2}}\right|^2 = 4(4m_N^2)\left[\Gamma(\alpha+2)\right]^2 \tfrac{1}{4\pi}\, C_0^2 \tag{5.12}$$

$$\left|\gamma_{00;\frac{1}{2}-\frac{1}{2}}\right|^2 = \frac{1}{4\pi}\left[\Gamma(\alpha+2)\right]^2 D_0^2 \tag{5.13}$$

Therefore

$$\gamma_{00;\frac{1}{2}-\frac{1}{2}} \Big/ \gamma_{00;\frac{1}{2}\frac{1}{2}} = 2.40\ \mathrm{GeV}^{-3} \tag{5.14}$$

We note that the helicity flip amplitude dominates the helicity non-flip away from t = 0. Because of the factorization theorem this feature will be found in all processes where the nucleon-nucleon Regge ρ vertex occurs. This helicity flip dominance will result in a forward minima in all processes where ρ coupling to $N\bar{N}$ is the dominant coupling.

<u>The Process $PP \rightarrow PN^*$</u>

Only the ρ and π trajectory contributions to this process will be discussed in a quantitative fashion. The ρ contribution will be discussed first, making use of the factorization theorem. The possible contribution of the pion trajectory will then be explored.

(a) ρ Trajectory Contribution

In Chapter 4 expressions for the helicity amplitudes for this process were derived under the assumption of domination by the ρ trajectory. The fits for this process are determined up to a single parameter by constraints from the factorization theorem for Regge residues. An examination of the definition of the reduced residues, the $\gamma_{\{\lambda\}}(t)$, reveals that they carry the factorization property attributed to $\beta_{\{\lambda\}}(t)$. These reduced residues are then determined up to a single parameter by the fits given previously for $\pi^+ p \rightarrow \pi^0 N$ and $\pi^+ p \longrightarrow \pi^0 N^{*++}$. This parameter would be determined by a knowledge of the π-π charge exchange scattering in the Regge pole region. The expression for the differential cross section and density matrix elements will be compared to experiment at the highest energy possible since the ρ trajectory is most likely to be dominant at high energy. The single parameter remaining determines overall normalization of the cross section and is chosen to give the best fit to experiment, which is shown in Figure 7. The experimental data is that of Anderson et al[29] at 15 Bev/c incident proton momentum. The resulting curves are obviously in very poor agreement with the experimental data. At all incident momenta where ρ dominance is hypothesized, this procedure leads to the prediction of the forward dip and leads to expressions for the cross section which have a flatter t dependence than is found experimentally. There is no experimental data which would indicate a forward dip at any energy as can be seen in Figures 8 to 14. Furthermore, the density matrix elements of the N* produced by nucleon excitation will be equal to those of the N* from pion excitation under the assumptions of constant residues and the factorization theorem. This prediction is tested by the data on nucleon excitation at 4 GeV/c[30] and 5.5 GeV/c[31] and the data at 4 GeV/c and 8 GeV/c on pion excitation. Moreover, since a single Regge pole model predicts that the high energy values of the density matrix elements will become independent of energy, these results can be expected to be independent of the small energy differences. An examination of the experimental values, as listed in Tables 2 and 3, shows the density matrix elements for the two processes to be widely at variance. Finally, it is noted that the dependence of the cross section on the incident momentum will be approximately $(P_{incident})^{-1}$ if the ρ trajectory is dominant. This dependence is weaker than that found experimentally.

Some of the previous results have been inferred using the factorization theorem for the residues and the assumption that the reduced residues are independent of momentum transfer. However, in Chapter 4 an extension of the factorization theorem to the leading power of s of the full t channel helicity amplitude was given. This theorem points out that the conclusions concerning the density matrix elements and the forward dip are independent of the assumption of constant residues. This conclusion follows from the relation

$$f_{00;00}^{\pi\pi\to\pi\pi} \; f_{cd;ab}^{\bar{N}N\to\bar{N}N^*} = f_{cd;00}^{\pi\pi\to\bar{N}N^*} \; f_{00;ab}^{\pi\pi\to\bar{N}N} \qquad (5.15)$$

If this expression is multiplied by its complex conjugate, with d replaced by d', and summed over a b c, the equality of the density matrix elements of the N* follows immediately. A similar procedure yields equality for the products of cross sections. This procedure makes obvious the assertion that the dip is predicted not in the differential cross section for NN $\rightarrow$ NN* but in the product of differential cross sections for NN $\rightarrow$ NN* and $\pi\pi \rightarrow \pi\pi$ charge exchange. It is only after the assumption of a specific ansatz for the residues, which causes the $\pi\pi$ charge exchange cross section to have no forward dip, that the earlier prediction for the ρ contributions to NN $\rightarrow$ NN* is valid. However, there is no reason to expect such a dip in $\pi\pi$ charge exchange scattering.

We are of the opinion that the arguments presented in this section constitute a strong case against ρ dominance.

(b) Pion Trajectory

The pion exchange amplitudes for NN $\rightarrow$ NN* have been listed in Chapter 4. In this section we wish to demonstrate that it is plausible that the pion trajectory is giving the major contribution to this process.

The first indication of pion trajectory domination comes from an examination of the density matrix elements of N* produced in the bubble chamber experiments of Alexander et al.[31] and Coletti et al.[30] at 5.5 GeV/c and 4.0 GeV/c respectively. These values are not substantially different than those which would be found from elementary pion exchange with pure derivative coupling to a Rarita-Schwinger wave function for the nucleon isobar.[32]

Again the factorization theorem for the t channel helicity amplitudes can be used to write down the relation

$$f^{\pi\rho\to\bar{N}N}_{cd;ab}\; f^{\bar{N}N\to\bar{N}N^*}_{ef;gh} = f^{\pi\rho\to\bar{N}N^*}_{cd;gh}\; f^{\bar{N}N\to N\bar{N}}_{ef;ab} \tag{5.16}$$

The above expression is multiplied by the complex conjugate of the identical expression with $h \to h'$ and summed over c, d, g, e, f, a, b. This procedure leads immediately to the prediction of equality of the N* matrix elements produced in the processes $NN \to NN^*$ and $\pi N \to \rho N^*$ provided that the pion trajectory dominates in both these processes. It is not necessary that the pion dominate in $\pi N \to \rho N$ and $NN \to NN$; however, it must make a finite contribution. The process $\pi N \to \rho N^*$ can have a contribution only from the A_2 or π trajectories among the physically verified trajectories. However, the dominance of the A_2 trajectory would give $\rho_{11} = 0.5$, $\rho_{oo} = 0.0$, and Re $\rho_{10} = 0.0$ for the ρ meson density matrix elements. Experimental values(23) are $\rho_{oo} = .77$ and Re $\rho_{10} = 0.12 \pm 0.025$. These numbers seem to indicate that the A_2 trajectory does not play a major role in $\pi N \to \rho N^*$. If the π trajectory is assumed to dominate the process $\pi N \to \rho N^*$, an assumption which is at least plausible in the light of the previous arguments, a test of π dominance in $NN \to NN^*$ is found in the comparison of the density matrix elements of the N* produced in these two processes. An examination of these numbers, which are listed in Tables 3 and 4, reveals that there is much better agreement between these sets of numbers than those examined in the test of ρ pole dominance. The agreement between the two processes is at least as good as that in each process at the two energies measured. Since this single π trajectory model predicts the density matrices to be energy independent, these variations must be ascribed to experimental error or background effects of the type described earlier.

Using similar arguments to those used earlier in discussing $\pi N \rightarrow \pi N^*$, it can be shown that the density matrix elements $\rho_{3,3}$, $\rho_{3,1}$ and $\rho_{3,-1}$ of the N* produced in the nucleon-nucleon collisions vanish in the forward direction if the π trajectory dominates. This situation arises because G parity allows the pion to couple to the $N\bar{N}$ pair only when they have the same helicity. This condition then causes the factor $(\sin \theta_t)^{\lambda}$, λ being the number of units of helicity flip at the $\bar{N}N^*$ vertex, to occur in the amplitudes given in Chapter 4 for this process. There is no evidence for this behavior in the Coletti et al. and Alexander et al. experiments. However, since in both cases, the data is averaged over the region from $-t = 0.0$ to 0.1 Bev^2, it is not obvious that this behavior would have been detected.

The data on the differential cross sections comes from two sources. The two aforementioned bubble chamber experiments and the missing mass experiments of Anderson et al[29] and Blair et al.[33] This data ranges in incident momentum from 2.85 to 15 Gev/c and in momentum transfer from $-t \approx 0.0$ Bev^2 to $-t = 0.30$ Bev^2. A least squares fit has been made to all the data including the data on the density matrix elements. In terms of the amplitudes listed in Chapter 4, the differential cross section and density matrices are

$$\frac{d\sigma}{dt} = \frac{1}{s(s-4m^2)}\left\{\left|f_{\frac{3}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}\right|^2 + \left|f_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}}\right|^2 + \left|f_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}}\right|^2 + \left|f_{\frac{3}{2}-\frac{1}{2}\frac{1}{2}\frac{1}{2}}\right|^2\right\} \tag{5.17}$$

$$\rho_{mm'} = \sum_b f_{mb;\frac{1}{2}\frac{1}{2}}\, f^*_{m'b;\frac{1}{2}\frac{1}{2}} \Big/ \sum_{a,b} \left|f_{ab;\frac{1}{2}\frac{1}{2}}\right|^2 \tag{5.18}$$

where

$$f_{-a-b;\frac{1}{2}\frac{1}{2}} = (-1)^{a-b}\, f_{ab;\frac{1}{2}\frac{1}{2}} \tag{5.19}$$

There are no particularly distinctive features in the experimental data and so the only criterion used to discriminate whether a fit is "good" is the value of x^2. In some cases, notably in the fits to the data from a single laboratory, low values of x^2 are rather easy to achieve and the fit is not unique. However, when the data from all laboratories is included, the task of fitting the data becomes more difficult, primarily because the data from the two missing mass experiments cover different ranges of kinematic variables, but also because the data from the two laboratories are somewhat inconsistent. This partial inconsistency can be most easily recognized in the data at 6 Bev/c incident momentum.

The least squares fits to these processes are shown in Figures 8 to 14. This best fit depends weakly on the relative weights given to the x^2 from the density matrix elements and the x^2 from the differential cross section. In most cases, weighting factors are given to these two terms such that their contribution to the total x^2 are about equal.

The first set of fits presented were calculated using the formula given in Chapter 4, adding the assumption that the reduced residues are four linearly independent quantities and are independent of momentum transfer. We then describe three different fits; in the first we have fixed the pion trajectory to have the value α (t) = -0.02 + t and S_o = 2.7 Bev^2 and varied the four residues to achieve our fit. This procedure allows us to arrive at the fit in Table 4 and Figures 8 to 14. The density matrix elements are shown in Table 4 along with the values of the reduced residues. There is not good agreement with experiment. The difficulty arises in attempting to attain a large enough value for ρ_{33} without causing Re $\rho_{3,1}$ to be much larger than experiment. Further, it is difficult to fit all the data points simultaneously for the differential cross section although there does not appear to be any systematic trend in the deviations. In particular, the data at 6.0 Bev/c disagrees with the calculated curves although the agreement both above and below this energy is satisfactory. Moreover, the data points at 7.88 GeV/c which have $-t > 0.30$ Bev^2 are markedly below the calculated values but since these are the only points with $-t > 0.30$ Bev^2; and because these points have such a large scatter, there has been no attempt to fit these points. For the remaining data points x^2 is of the order of one per point. The assignment of an error to these fitted parameters would be meaningless because of the arbitrary manner in which some of the data points have been excluded. However, there is a region in the parameter space which has a size of about 10% of the value of each parameter where x^2 for the differential cross section does not vary substantially.

It is interesting to note the cause of the difficulty in obtaining good fits simultaneously to the differential cross section and the density matrix elements. This difficulty arises because all contributions to ρ_{33}, $\rho_{3,1}$ and $\rho_{3,-1}$ contain at least 1 spin flip amplitude and all spin flip amplitudes contain the sin θ_t factor. Thus, in order to keep the ρ_{33}, $\rho_{3,1}$ and $\rho_{3,-1}$ nonzero, it is necessary to include finite amounts of these amplitudes with strong forward dips and weaker t dependence at large t . Since we have fixed s_o and α (t), the t dependence of these amplitudes is fixed and tends to exceed experiment. Thus, a fit to the density matrix elements implies that the slope of the differential cross section is less than measured experimentally.

In order to explore fully the content of the previous remarks, two other non-constrained fits will be presented. In the first of these fits, the slope of the pion trajectory has been changed to 1.5/Bev^2 and the reduced residues varied to obtain fit. In the second, no attempt to fit the density matrix elements has been made, and only the amplitude $f_{1/2\ 1/2;1/2\ 1/2}$ has been retained and the fit to the differential cross section attained by varying the slope of the pion trajectory. This second fit is approximately equivalent to that of Margolis and Rotsstein[33] except that we have included the threshold factors which differ from those of the Born approximation. These fits are shown in Figures 8 to 14. The parameters, density matrix elements and x^2 for these fits are shown in Table 4. By choosing a larger slope for the pion trajectory, larger slopes have been induced for all the helicity amplitudes. This change demands greater amounts of the helicity flip contributions to fit the differential cross sections. This procedure allows somewhat better fits to the density matrix elements with $\alpha(t) = -0.02 + t\,(1.5)$. However, this data would have to be regarded as scanty evidence, at best, for assigning the pion trajectory a larger slope. Moreover, in a recent paper Frautschi and Jones[34] have found a slope of 1./B.V. for the pion trajectory quite adequate to attain a fit for the processes $\pi p \to \rho N^*$, $\pi^- p \to \rho^0 N$, $Kp \to K^*\Delta$, $\pi p \to f^0 N$, and $\pi p \to f^0 N^*$.

The second of these fits is presented in order to accommodate the possibilities that there is a substantial contribution from states of the opposite G parity or that the density matrix elements can be accounted for by kinematically reflected isobar events. The second possibility is discussed in Appendix C. The possibility of contributions with opposite G parity allows for helicity flip amplitudes with $\lambda = \mu$, and hence allows for contributions to the density matrix elements $\rho_{3,3}$, Re $\rho_{3,1}$ and Re $\rho_{3,-1}$ which are non-vanishing as t approaches t_{min}. The parameters for this fit are listed in Table 4.

In Chapter 4 a single constraint between the four Regge residues for this process was developed. It is badly violated for each of the fits which were presented on the previous pages. The assumption of independence of t for the Regge residues makes this constraint inconsistent with experiment. This inconsistency is most easily seen by noting the t dependence of the ratios of the helicity amplitudes is given by simple products of $\phi(s,t)$, α, and $\alpha - 1$. In the region of small t, $\alpha - 1$ is approximately constant. Moreover α is antisymmetric about $t = +0.02$ and $\phi(s,t)$ is approximately antisymmetric about $t = 0$. Therefore, these ratios of helicity amplitudes are approximately the same at $t = -(M - m)^2$ and $t = +(M - m)^2$. But at $t = -.1$ the helicity non-flip amplitude dominates the other amplitudes by at least one order of magnitude if the shape of the differential cross sections is to resemble experiment. Thus, it appears impossible to fit the shape of the differential cross section and the constraints simultaneously.

The Process $NN \to N^*N^*$

This process will be considered only briefly. The factorization theorem can be used to relate this process to others thought to be dominated by pion exchange, through the relation

$$f^{t(\pi)\,(\pi\rho \to \bar{N}N^*)}_{cd;ab} \; f^{t(\pi)\,(\bar{N}N^* \to \bar{N}N)}_{ef;gh} = f^{t(\pi)\,(\pi\rho \to \bar{N}N)}_{cd;ab} \; f^{t(\pi)\,(\bar{N}N^* \to \bar{N}N^*)}_{ef;ab} \tag{5.20}$$

If this relation is multiplied by the complex conjugate relation and summed over all helicities, it is found

$$\left(\frac{d\sigma}{dt}\right)^{\pi p \to \rho^0 N^*} \left(\frac{d\sigma}{dt}\right)^{np \to N^{*+} n} = \left(\frac{d\sigma}{dt}\right)^{\pi^- p \to \rho^0 N} \left(\frac{d\sigma}{dt}\right)^{np \to N^{+*} N^{*0}} \tag{5.21}$$

Of course, this relation can only be expected to be valid when the π trajectory dominates all four processes. The assumption of isospin invariance allows the above relation to be transformed to

$$\frac{1}{3}\left(\frac{d\sigma}{dt}\right)^{\pi^+ p \to \rho^0 N^{*++}} \left(\frac{d\sigma}{dt}\right)^{pp \to N^{*++} n} = \left(\frac{d\sigma}{dt}\right)^{\pi^- p \to \rho^0 n} \left(\frac{d\sigma}{dt}\right)^{pp \to N^{*++} N^{*0}} \tag{5.22}$$

Unfortunately data is available on the process $\pi^+ p \rightarrow \rho^0 N^{*++}$ only at 4 and 8 GeV/c[23], on $pp \rightarrow N^{*++} N^{*0}$ at 5.5 Gev/c[35], and on $\pi^- p \rightarrow \rho^0 \eta$ at about 3 GeV/c. This scarcity of data makes the comparison with experiment of relation (5.22) extremely difficult and any conclusions drawn will be only tentative. However, the assumption of π dominance provides us with an energy extrapolation for the cross sections. Then, it is only required that this relation be evaluated at energies such that the product of the s values on each side of equation (5.22) are equal. This condition is realized if $\pi^+ p \rightarrow \pi^0 \Delta^{++}$ is evaluated at 4 GeV/c incident momentum, $pp \rightarrow pN^*$ at 4 GeV/c incident momentum, $\pi^- p \rightarrow \rho^0 \eta$ at 3 GeV/c incident momentum, and $pp \rightarrow N^* N^*$ at 5.5 GeV/c incident momentum. Then, if relation (5.22) is evaluated for $0.1\ \mathrm{BeV}^2 \leq -t \leq 0.2\ \mathrm{BeV}^2$, it is found that the left-hand side of the relation is about one order of magnitude greater than the right-hand side. Therefore, the assumption of π dominance for the left-hand side implies that there must be a large cancellation of the pion contribution in at least one term on the right-hand side. The other possibility is that the pion does not dominate both the terms on the left-hand side of equation (5.22). However, this relation should be tested with better data at an energy near 5 Bev/c before any definite conclusions are drawn.

Relation (5.20) can be manipulated in the usual manner to predict the equality of the density matrix elements of all the isobars produced in the various processes entering this relation. The density matrix elements are given in Tables 2, 3, 5. These relations appear to be satisfied.

Furthermore, an examination of the two body density matrix elements[36] for double isobar production reveals that these can be factored into a product of one body density matrix elements if the t channel helicity amplitudes factor. This result has experimental implications. However, there is no data available to test the result.

Using the techniques developed in earlier chapters, expressions have been derived for the differential cross section for $NN \longrightarrow N^*N^*$ assuming dominance of either the π or ρ trajectories. If the reduced residues are assumed to be independent of momentum transfer and if the factorization theorem is used to relate these residues to other processes, the number of undetermined parameters in each case may be reduced to a single normalization parameter. This simplification makes use of $NN \longrightarrow NN^*$ for the π trajectory and $\pi N \rightarrow \pi N^*$ for the ρ trajectory. The results are shown in the last figure. In both cases the fit is poor, although the pion trajectory definitely gives a superior result.

CHAPTER 6

CONCLUSIONS

Before any conclusions are drawn, the major assumptions necessary for the derivation of the phenomenological forms used in this work will be reiterated. These assumptions can be lumped into three classes. The first class, upon which the formalism depends critically, contains the assumption of the Jacob-Wick crossing relations for the helicity amplitude or sufficient analyticity in the Mandlestam s - t variables to guarantee these relations, the assumption that the helicity amplitudes for such quasi two body inelastic processes with kinematic singularities removed satisfy dispersion relations in both energy and momentum transfer and the assumption that the definition of $F^J_{\{\lambda\}}(t)$ can be continued to smaller values of J than those for which the original Froissart-Gribov definition converged. The second set of assumptions concerned the nature of the singularities in F^J found in this continuation. It was assumed that only simple poles, dependent on the variable t, were encountered and that there were no poles of higher order or Regge cuts or fixed poles that could contribute to the asymptotic behavior of the helicity amplitudes. Thirdly, there is the set of assumptions used in fixing upon some ansatz to determine the overall t dependence of our helicity amplitudes. This group includes the assumption that the t channel helicity amplitudes still have the usual t channel threshold behavior when continued to large Z, the assumption that the continuation for Z large from $t = (M - m)^2$ or $t = m_\pi^2$ or m_ρ^2 to t negative can be approximated by a reduced residue independent of t and the assumption that effects due to third Mandlestam double spectral function are small, forcing a trajectory to choose "sense" or "nonsense." Furthermore, it was assumed the π and ρ trajectories always choose sense and that their trajectories can be approximated by straight lines in the region of interest.

With the aid of these assumptions, phenomenological forms have been derived and fits to the experimental data have been made. The fit to the process $\pi N \rightarrow \pi N^*$ using only ρ exchange is successful. However, more detailed information about the energy dependence of the cross section and the t dependence of the density matrix elements is necessary to test the model in more than a perfunctory fashion.

The situation for the production of isobars in nucleon-nucleon collisions presents a much more complicated picture. However, the factorization property has allowed us to conclude that it is not the ρ trajectory which dominates this process below 15 Bev/c. The possibility that it is the pion trajectory which is dominant has been explored and satisfactory fits have been obtained for both the differential cross section and density matrices. It is only when this process is related to double isobar production that difficulties of a non-trivial nature are encountered. These difficulties are encountered as a consequence of factorization and are dependent on experimental results which have large errors.

These procedures can be extended to a larger number of processes and trajectories, all of which should be fit simultaneously. Detailed knowledge of the dependence on momentum transfer of the density matrices along with constraints from the factorization theorem should provide stringent tests for the Regge phenomenology.

The threshold constraints derived by Franklin were found inconsistent with experiment if ρ dominance is assumed for $\pi N \rightarrow \pi N^*$. A similar inconsistency was found if π dominance was assumed for $NN \longrightarrow NN^*$. Therefore, if these constraints are accepted, either the simple π or ρ dominance model or the assumption of constant residues must be abandoned.

APPENDIX A

The boundary of the physical region is the curve $\phi(s,t) = 0$ for all three processes.

$$\phi(s,t) = st\left(\sum_i m_i^2 - s - t\right) - s\left(m_b^2 - m_d^2\right)\left(m_a^2 - m_c^2\right)$$
$$- t\left(m_a^2 - m_b^2\right)\left(m_c^2 - m_d^2\right) - \left(m_a^2 m_d^2 - m_c^2 m_b^2\right)\left(m_a^2 + m_d^2 - m_c^2 - m_b^2\right) \quad \text{(A.1)}$$

Then in the s channel $(a + b \rightarrow c + d)$ where θ_s is defined as the angle between $\bar{p}_a$ and $\bar{p}_c$.

$$\cos\theta_s = \left[2st + s^2 - s\sum_i m_i^2 + \left(m_a^2 - m_b^2\right)\left(m_c^2 - m_d^2\right)\right] \Big/ \mathcal{S}_{ab}\,\mathcal{S}_{cd} \quad \text{(A.2)}$$

where

$$\mathcal{S}_{ab}^2 = \left[s - \left(m_a - m_b\right)^2\right]\left[s - \left(m_a + m_b\right)^2\right] = 4s\,p_{ab}^2 \quad \text{(A.3)}$$

and

$$\sin\theta_s = 2\left[s\,\phi(s,t)\right]^{1/2} \Big/ \mathcal{S}_{ab}\,\mathcal{S}_{cd} \quad \text{(A.4)}$$

Further, if in the t channel $(D + b \rightarrow c + A)$ barycentric system θ_t is defined as the scattering angle between D and C.

$$\cos\theta_t = \left[2st + t^2 - t\sum_i m_i^2 + \left(m_d^2 - m_b^2\right)\left(m_c^2 - m_a^2\right)\right] \Big/ \mathcal{T}_{ac}\,\mathcal{T}_{bd} \quad \text{(A.5)}$$

$$\sin\theta_t = 2\left[t\,\phi(s,t)\right]^{1/2} \Big/ \mathcal{T}_{ac}\,\mathcal{T}_{bd} \quad \text{(A.6)}$$

where

$$\Upsilon_{ac}^{2} = \left[t-(m_a+m_c)^2\right]\left[t-(m_a-m_c)^2\right] \tag{A.7}$$

$$= 4t\, p_{ac}^{2}$$

APPENDIX B

The factorization theorem for the Regge residues is proved herein, the proof being that of Squires,[6] generalized to helicity amplitudes.

The unitary relation in the physical region for partial wave helicity amplitudes can be written

$$\sum_{N,\lambda_{N_1},\lambda_{N_2}} \langle f\,\lambda_c\lambda_d|S^J|N\,\lambda_{N_1}\lambda_{N_2}\rangle^* \langle i\,\lambda_a\lambda_b|S^J|N\,\lambda_{N_1}\lambda_{N_2}\rangle = \delta_{fi}\,\delta_{\lambda_a\lambda_c}\,\delta_{\lambda_b\lambda_d} \tag{B.1}$$

If S_I is the I particle threshold, then for $s_I \leqslant s \leqslant s_{I+1}$, the sum over intermediate states contains all states for which $\left(\sum_{i=1}^{I+1} m_i\right)^2 \leqslant s_{I+1}$.
The above equation can be written in matrix notation as

$$\left(S^J(s)\right)^\dagger S^J(s) = \mathbb{1} \tag{B.2}$$

Then Carlson's theorem allows the definition of a <u>unique</u> interpolation in the complex J plane of this relation, provided, of course, that an interpolation $S(\lambda,s)$, λ = complex angular momentum, satisfying the necessary condition for Carlson's theorem can be found. Then the continuation of (B.2) reads

$$S(\lambda^*,s)^\dagger S(\lambda,s) = \mathbb{1} \quad \text{for} \quad s_N < s < s_{N+1} \tag{B.3}$$

Therefore

$$S(\lambda,s) = \operatorname{cof}\left(S^\dagger(\lambda^*,s)\right) \Big/ \operatorname{Det.}\left(S^\dagger(\lambda^*,s)\right) \tag{B.4}$$

It is recalled that the rank of a matrix equals the number of linearly independent rows or columns. Further the reader is reminded of two well known theorems: (1) the rank of an N by N matrix whose determinant is nonzero is N (2) the rank of the product of two square matrices is at least as large as the sum of the ranks minus the number of rows.

Therefore

$$\gamma\left[S_N^{\dagger}\right] + \gamma\left[\mathrm{cof}\, S_N^{\dagger}\right] - N \leq \gamma\left[\mathrm{Det}\left(S_N^{\dagger}\right)\cdot \mathbb{1}\right] \tag{B.5}$$

where

$$\gamma\left[A\right] = \text{rank of matrix A.}$$

Now for general s, it is assumed that S_N^{+} contains only simple zeros.

Hence

$$\gamma\left[S_N^{\dagger}\right] = N-1 \qquad \text{Therefore} \quad \gamma\left[\mathrm{cof}\, S_N^{\dagger}\right] \leq 1$$

That is, the matrix of residues has rank one.

Therefore

$$\frac{\beta_{ij}(s)}{\beta_{ij'}(s)} = \frac{\beta_{i'j}(s)}{\beta_{i'j'}(s)} \qquad \text{where} \quad S_{ij}(J,s) = \beta_{ij}(s)\Big/\alpha(s)-J$$

This relation (B.6) can be continued in s since the matrix $S_N(J,s)$ is assumed to have reasonable continuations in both J and s.

APPENDIX C

In the previous chapters extensive use has been made of the values determined from experiment for the density matrix elements of the N^* (1238) produced in various reactions. These numbers have been computed using the well-known result of Gottfried and Jackson[20] which relates the t channel barycentric helicity amplitudes, continued into the s channel physical region, to the decay distribution of the resonance in its s channel rest frame. Therefore, if it is assumed that the contribution due to the resonance as a final state interaction dominates the scattering amplitude completely in the particular region of the kinematic variables being studied, these decay distributions may be used to determine the t channel density matrix elements.

In order to examine possible complications, a more careful study of the reaction $pp \rightarrow pn\pi^+$ at an incident proton momentum of about 5 Bev/c will be made in this appendix. At this energy the final state contains a very large bump in the $p\pi^+$ mass plot, the N^{*++}(1238) resonance, which contains roughly one-half of the events in the $pn\pi^+$ final state. Since this channel, that is, the $N^{*++}n$ channel, appears to have a background contribution of less than 10% of the value of the cross section at the resonance peak as estimated from the shape of the resonance, all the events in the mass region $M(p\pi^+) \approx 1238$ MeV are usually analyzed as being controlled by a final state interaction in the $p\pi^+$ channel. Usually this channel is specialized to the P_{33} $p\pi^+$ channel. Since the values of the density matrix elements depend critically on this assumption, it is of interest to estimate, in a phenomenological manner, the effects of other contributions.

The first contributions that are considered are the other possible π^+p channels such as P_{31} and S_{31}. However, in the region of the 33 resonance the phase shifts in these channels are very small, being of the order of 5 or 10 degrees. Since the phase shift in the 33 channel is about 90^o, the interference terms will be small because the two contributions are about 90^o out of phase. Actually both the interference contribution and the squared contribution due to the non-resonant phase shifts will be of the order of δ_{31}^2 when compared to the resonance contribution. This contribution is negligible.

The second set of possible contributions are those which involve a $(n\pi^+)$ final state interaction and which are usually described in the $(n\pi^+)p$ channel. These events will be examined on the "kinematically reflected" mass plot, that is, on $M(p\pi^+)$ in order to ascertain their effect on the decay distribution of the $p\pi^+$ system in its rest frame. It should be noted that the descriptions of the final state as a $p(n\pi^+)$ or $n(p\pi^+)$ system are both complete and that the states of the two channels are not orthogonal. This duplication would cause difficulties if the final state interactions in the $n\pi^+$ and $p\pi^+$ states were computed coherently.

The kinematical variables for the process $pp \rightarrow pn\pi^+$ are described as follows: let the initial proton four momenta be p_1 and p_2, the final proton, neutron and pion four momenta be p_f, p_n and k respectively.

Then the following invariants are defined

$$S = (p_1 + p_2)^2 \qquad M_{++}^2 = (p_f + k)^2 \qquad M_+^2 = (p_n + k)^2$$

$$t = (p_2 - p_n)^2 \qquad \bar{t} = (p_1 - p_f)^2 \qquad \text{(C.1)}$$

At energies of a few BeV the production process is usually described by an exchange mechanism, with bosonic quantum numbers in the crossed channel and is extremely peripheral, that is, for most of the events either t or $\bar{t}$ is small. In this appendix, however, a purely phenomenological model is assumed which is thought to describe, albeit very roughly, the contribution to the cross section of the final state interactions $n\pi^+$ and $p\pi^+$. Moreover, it is assumed, without justification, that the effects of these two final state interactions of the pion with each of the nucleons add incoherently. However this assumption simplifies the calculation enormously.

This simplification is due to the relation

$$\frac{d\sigma}{d\Omega_{M_{++}}\, dt\, dM_{++}} = \frac{p_{M_{++}}}{p_{M_+}} \frac{d\sigma}{d\Omega_{M_+}\, d\bar{t}\, dM_+} \qquad \text{(C.2)}$$

where $d\Omega_{M_{++}}$

is evaluated in the frame $\bar{p}_f + \bar{k} = 0$ and $d\Omega_{M+}$ is evaluated in the frame $\bar{p}_n + \bar{k} = 0$, p_{M_+} ($p_{M_{++}}$) is the three momentum of the pion or nucleon from the decaying $n\pi^+$ ($p\pi^+$) system in the $n\pi^+$ ($p\pi^+$) rest frame. In each case the z direction is defined to be that of the incident proton, for example, $\bar{p}_2$ in the M_+ rest frame. The identity (C.2) can be derived by working out the phase space integrals for these differential cross sections and then making use of identities generated by evaluating the invariant

$$\epsilon_{\mu\nu\rho\sigma}\, p_1^{\mu}\, p_2^{\nu}\, p_f^{\rho}\, p_n^{\sigma}$$

in the two isobar rest frames.

The contribution of $n\pi^+$ final state interactions is then approximated as

$$\frac{d\sigma}{d\bar{t}\, dM_+\, d\Omega_{M_+}} = \sum_i \frac{(\Gamma_i/2\pi)}{(M_+ - M_i)^2 + \Gamma_i^2/4}\, b\, A_i\, e^{-b\bar{t}}\, F_i(\theta_{M_+}, \phi_{M_+}) \quad (C.3)$$

where

F_i is usually a low order polynomial in cos θ

Since $\int dM_+ \, (\Gamma_i/2\pi) \Big/ \left[(M_+ - M_i)^2 + \Gamma_i^2/4\right] = 1$

it follows that $\sigma = \sum_i A_i$, A_i being identified with the total cross section for the production of the $i^{\underline{th}}$ isobar.

In order to fix the parameters of this phenomenological ansatz, the experiment of Alexander et al [31] at 5.5 Bev/c is considered. This experiment recorded some 1500 events in the $pn\pi^+$ channel of which about 500 have $1180 \lesssim M(p\pi^+) \lesssim 1300$. The remaining events, those outside of $1180 \lesssim M(p\pi^+) \lesssim 1300$, have considerable low mass enhancement and structure on the $M(n\pi^+)$ mass plot. We feel this indicates a final state interaction in this channel. However, the magnitude of such an effect is difficult to estimate.

The angular distributions are plotted in Figure 10 of the paper of Alexander et al, along with the curves generated in the usual Jackson-Gottfried analysis. However, this analysis does not appear to generate a good fit to the experiment. In particular the Jackson-Gottfried analysis predicts

$$W(\cos\theta) = \frac{3}{2}\left[\rho_{33}\sin^2\theta + \rho_{11}\left(\frac{1}{3}+\cos^2\theta\right)\right] \tag{C.4}$$

$$W(\phi) = \frac{1}{2\pi}\left[1 - 4/\sqrt{3}\,\cos 2\phi\ \mathrm{Re}\,\rho_{3,-1}\right] \tag{C.5}$$

Now relation (C.4) predicts that there should be equal numbers of events with $\theta < \pi/2$ and $\theta > \pi/2$. Experimentally, there are 232 events with $\theta > \pi/2$ and 174 events with $\theta < \pi/2$ in the region $1180 \leq M(p\pi^+) \leq 1300$. These numbers are 2.5 standard deviations from equality and there is less than a 10% chance that the Jackson-Gottfried analysis will fit the data. Further $\omega(\phi)$ is peaked at $\phi = \pi$, an effect which cannot be duplicated by the Jackson-Gottfried analysis. Moreover, it is of interest to note that the events for $1300 \leq M(p\pi^+) \leq 1800$ accentuate these features.

Using the prescription described earlier, the angular distributions of the $p\pi^+$ pair have been computed. The $n\pi^+$ spectrum has been represented by two peaks which are normalized to give roughly the number of events in $n\pi^+$ spectrum with mass < 1600 MeV. These peaks were placed at 1236 and 1512 MeV and assigned widths of 200 MeV. This procedure, which is rather arbitrary, is justified by the fact that the result is not very sensitive to the mass of the $n\pi^+$ pair chosen. In addition, the contribution of these $n\pi^+$ events to the angular distribution is not sensitive to the form chosen for F_i; in this calculation $F_1 = 1 + 3\cos^2\theta/8\pi$ and $F_2 = 1/4\pi$. The t dependence of the differential cross section controls the amount of backward peaking. In the present calculations, b = 7.0 Bev^{-2}, a value lying between the experimental values at 4 and 5.5 GeV/c.

It is reasonable to question how the approximately 400 events of the $n\pi^+$ spectrum which have been included can affect the angular distributions since only about 10-20% of these are expected to lie in the region $1180 \leq M(p\pi^+) \leq 1300$. This situation arises because these events, decaying approximately isotropically in the $(n\pi^+)$ rest frame, have rather sharp angular characteristics in the $(p\pi^+)$ rest frame. Most of the events occur near $\emptyset = \pi$ and all occur with $\theta > \pi/2$. Thus, some 800 events in the $n\pi^+$ spectrum would have the effect shown in Figures A1 to A4. In Figure A1 and A2 the fits to the events for $1300 \leq M(p\pi^+) \leq 1800$ are shown. The results are excellent. However, the results for $M(p\pi^+)$ between 1180 and 1300 MeV are too small although the shape resembles experiment. However, this calculation has been performed incoherently and it is expected that interference effects will increase the magnitude of the effect markedly in the region $1180 \leq M(p\pi^+) \leq 1300$. Moreover, such interference effects would not be visible as background since they would have the shape of the resonance. We hope to include interference effects in future calculations.

In conclusion, the simple isobar model is probably adequate to account for gross features of the process $pp \rightarrow pn\pi^+$ but more extensive calculations should be made to check the validity of the isobar model for features of the cross section, such as angular distributions, which might be sensitive to background effects.

Leaves 81, 82 and 83 omitted in page numbering.

APPENDIX D

The singularity structure at $S_{ab} = 0$ can be deduced from the following form.

$$M^{\pm} \propto \left[\sin\theta_{t/2}\right]^{|\lambda'-\mu'|}\left[\cos\theta_{t/2}\right]^{|\lambda'+\mu'|}\left[\sin\theta_s\right]^{-|\lambda-\mu|}\Bigg\{\Bigg[\Bigg[\left(\frac{1-\cos\chi_a}{\sin\chi_a}\right)^{|A'-a|}\left(\frac{1-\cos\chi_b}{\sin\chi_b}\right)^{|b'-b|}\mathcal{P}^{(J_a-\nu_a/2)}(\cos\chi_a)\,\mathcal{P}^{(J_b-\nu_b/2)}(\cos\chi_b)\,(1+\cos\theta_s)^{m}\left(\cos\tfrac{1}{2}\chi_a\right)^{\nu_a}\left(\cos\tfrac{1}{2}\chi_b\right)^{\nu_b}$$

$$\eta_{ab}\left(\frac{1+\cos\chi_a}{-\sin\chi_a}\right)^{|A'-a|}\left(\frac{1+\cos\chi_b}{-\sin\chi_b}\right)^{|b'-b|}\mathcal{P}^{(J_a-\nu_a/2)}(-\cos\chi_a)\,\mathcal{P}^{(J_b-\nu_b/2)}(-\cos\chi_b)\,(1-\cos\theta_s)^{m}\left(\sin\tfrac{1}{2}\theta_s\right)^{\nu}\left(\sin\tfrac{1}{2}\chi_a\right)^{\nu_a}\left(\sin\tfrac{1}{2}\chi_b\right)^{\nu_b}\Bigg]\,d^{J_c}_{c'c}(\chi_c)\,d^{J_d}_{D'd}(\chi_d)$$

$$+\Bigg[\Bigg[\left(\frac{1-\cos\chi_b}{\sin\chi_b}\right)^{|b'-b|}\left(\frac{1-\cos\chi_a}{\sin\chi_a}\right)^{|A'-a|}\mathcal{P}^{J_a-\nu_a/2}(\cos\chi_a)\,\mathcal{P}^{(J_b-\nu_b/2)}(\cos\chi_b)\,(1-\cos\theta_s)^{m}\left(\sin\tfrac{1}{2}\theta_s\right)^{\nu}\left(\cos\tfrac{1}{2}\chi_a\right)^{\nu_a}\left(\cos\tfrac{1}{2}\chi_b\right)^{\nu_b}\pm\eta_{ab}(-1)^{2(c+d)}\times$$

$$\times\left(\frac{1+\cos\chi_a}{-\sin\chi_a}\right)^{|A'-a|}\left(\frac{1+\cos\chi_b}{-\sin\chi_b}\right)^{|b'-b|}\mathcal{P}^{(J_a-\nu_a/2)}(-\cos\chi_a)\,\mathcal{P}^{(J_b-\nu_b/2)}(-\cos\chi_b)\,(1+\cos\theta_s)^{m}\left(\cos\tfrac{1}{2}\theta_s\right)^{\nu}\left(\sin\tfrac{1}{2}\chi_a\right)^{\nu_a}\left(\sin\tfrac{1}{2}\chi_b\right)^{\nu_b}\Bigg]\,d^{J_c}_{c'c}(\pi-\chi_c)\,d^{J_d}_{D'd}(\pi-\chi_d$$

where

$$2m+\nu = |\lambda-\mu| - |\lambda+\mu| \qquad \eta_{ab} = \frac{\eta_A\,\eta_c}{\eta_b\,\eta_D}(-1)^{J_c+J_D+c+d}.$$

$\nu_i = 1$ if i^{th} particle is a fermion

$\nu_i = 0$ if i^{th} particle is a boson

$\nu = 0$ if the exchanged particles are bosonic

$\nu = 1$ if the exchanged particles are fermionic

A similar form gives the singularity structure at $S_{cd} = 0$.

TABLE 1

Density Matrix Elements	8 Bev/c Theory	8 Bev/c Experiment	4 Bev/c Theory	4 Bev/c Experiment
$\rho_{1,1}$	0.284	0.286 ± 0.06	0.273	0.10 ± 0.06
$\rho_{3,3}$	0.216	0.214 ± 0.06	0.227	0.40 ± 0.06
$\mathrm{Re}\,\rho_{3,1}$	0.143	0.066 ± 0.07	0.150	-0.03 ± 0.07
$\mathrm{Re}\,\rho_{3,-1}$	0.190	0.13 ± 0.067	0.181	0.21 ± 0.08

$\gamma^2_{oo,3/2\ 1/2} = 0.599$

$\gamma^2_{oo,3/2\ -1/2} = 0.679$

$\gamma^2_{oo,1/2\ -1/2} = 1.36$

$\gamma^2_{oo,1/2\ 1/2} = 0.0$

TABLE 2

Density Matrix Elements of N* Produced in NN $\longrightarrow$ NN*

Incident Momentum 4.0 Bev/c

$\cos\theta_{N^*}$ (in barycentric system)	1.0 to .98	.98 to .95	.95 to .90	.90 to 0
ρ_{33}	.17 $\pm$ .04	.22 $\pm$ .06	.14 $\pm$ .06	.25 $\pm$ .04
Re $\rho_{3,1}$	0.10 $\pm$ .04	0.10 $\pm$ 0.05	0.16 $\pm$ 0.04	0.05 $\pm$ 0.01
Re $\rho_{3,-1}$	0.00 $\pm$ 0.08	0.01 $\pm$ 0.09	0.09 $\pm$ 0.08	0.04 $\pm$ 0.08

Incident Momentum 5.5 Bev/c

	Averaged over $\cos\theta_{N^*}$
ρ_{33}	0.13 $\pm$ 0.04
Re $\rho_{3,1}$	0.02 $\pm$ 0.04
Re $\rho_{3,-1}$	-0.03 $\pm$ 0.04

TABLE 3

Density Matrix Elements of N* Produced in $\pi N \longrightarrow \rho N^*$

Incident Momentum 4 Bev/c

ρ_{33}	0.08 ± 0.03
Re $\rho_{3,1}$	0.01 ± 0.03
Re $\rho_{3,-1}$	-0.01 ± 0.03

Incident Momentum 8 Bev/c

	average over t	t $\rightarrow$	$0.0 \rightarrow -0.05$	$-.05 \rightarrow -.10$	$-.10 \rightarrow -.20$	$-.20 \rightarrow \infty$
ρ_{33}	0.05 ± 0.03	ρ_{33}	0.05 ± 0.08	0.0 ± 0.06	0.13 ± 0.06	0.20 ± 0.10
Re $\rho_{3,1}$	0.015 ± 0.028	Re $\rho_{3,1}$	0.10 ± 0.07	-0.18 ± 0.08	$-.10 \pm .10$	$-.07 \pm .10$
Re $\rho_{3,-1}$	-0.076 ± 0.033	Re $\rho_{3,-1}$	0.04 ± 0.05	-0.05 ± 0.05	-0.02 ± 0.06	0.04 ± 0.10

TABLE 4

Parameters and Density Matrix Elements for Three Fits to NN $\longrightarrow$ NN*

Fit 1

$\gamma^2_{1/2\ 1/2\ 1/2\ 1/2} = 157.3$ $\rho_{11} = 0.5$ $\rho_{33} = \rho_{3,-1} = \rho_{3,1} = 0.0$

$\gamma^2_{1/2\ -1/2;1/2\ 1/2} = 0.0$ $\alpha\ (t) = -0.02 + .61\ t$

$\gamma^2_{3/2\ 1/2;1/2\ 1/2} = 0.0$ $x^2 = 0.67$ per point on $\frac{d\sigma}{dt}$

$\gamma^2_{3/2\ -1/2;1/2\ 1/2} = 0.0$

Represented by . — . — . — . in Figures 8 to 14

Fit 2

$\gamma^2_{1/2\ 1/2\ 1/2\ 1/2} = 472.4$ $\rho_{11} = 0.38$ Re $\rho_{3,-1} = -0.01$

Re $\rho_{3,1} = 0.08$

$\gamma^2_{1/2\ -1/2;1/2\ 1/2} = 129.0$ $\alpha\ (t) = -0.02 + 1.5\ t$

$\gamma^2_{3/2\ 1/2;1/2\ 1/2} = 2490.0$ $x^2 = 1.2$ per point on $\frac{d\sigma}{dt}$

$\gamma^2_{3/2\ -1/2;1/2\ 1/2} = 535.0$

Represented by — — — — — — in Figures 8 to 14

Fit 3

$\gamma^2_{1/2\ 1/2\ 1/2\ 1/2} = 376.0$ $\rho_{11} = .43$ Re $\rho_{3,-1} = 0.0$

Re $\rho_{3,1} = 0.09$

$\gamma^2_{1/2\ 1/2;1/2\ -1/2} = 0.0$ $\alpha\ (t) = -0.02 + 1.0\ t$

$\gamma^2_{3/2\ 1/2;\ 1/2\ 1/2} = 1668.0$ $x^2 = 1.4$ per point on $\frac{d\sigma}{dt}$

$\gamma^2_{3/2\ -1/2;1/2\ 1/2} = 0.0$

Represented by .... in Figures 8 to 14

TABLE 5

Density Matrix Elements of N* Produced in $NN \longrightarrow N^{*++}N^{*o}$

	N^{*++}	N^{*o}
ρ_{33}	0.04 ± 0.04	0.05 ± 0.04
$\mathrm{Re}(\rho_{3,-1})$	$0.04 - 0.02$	0.03 ± 0.02

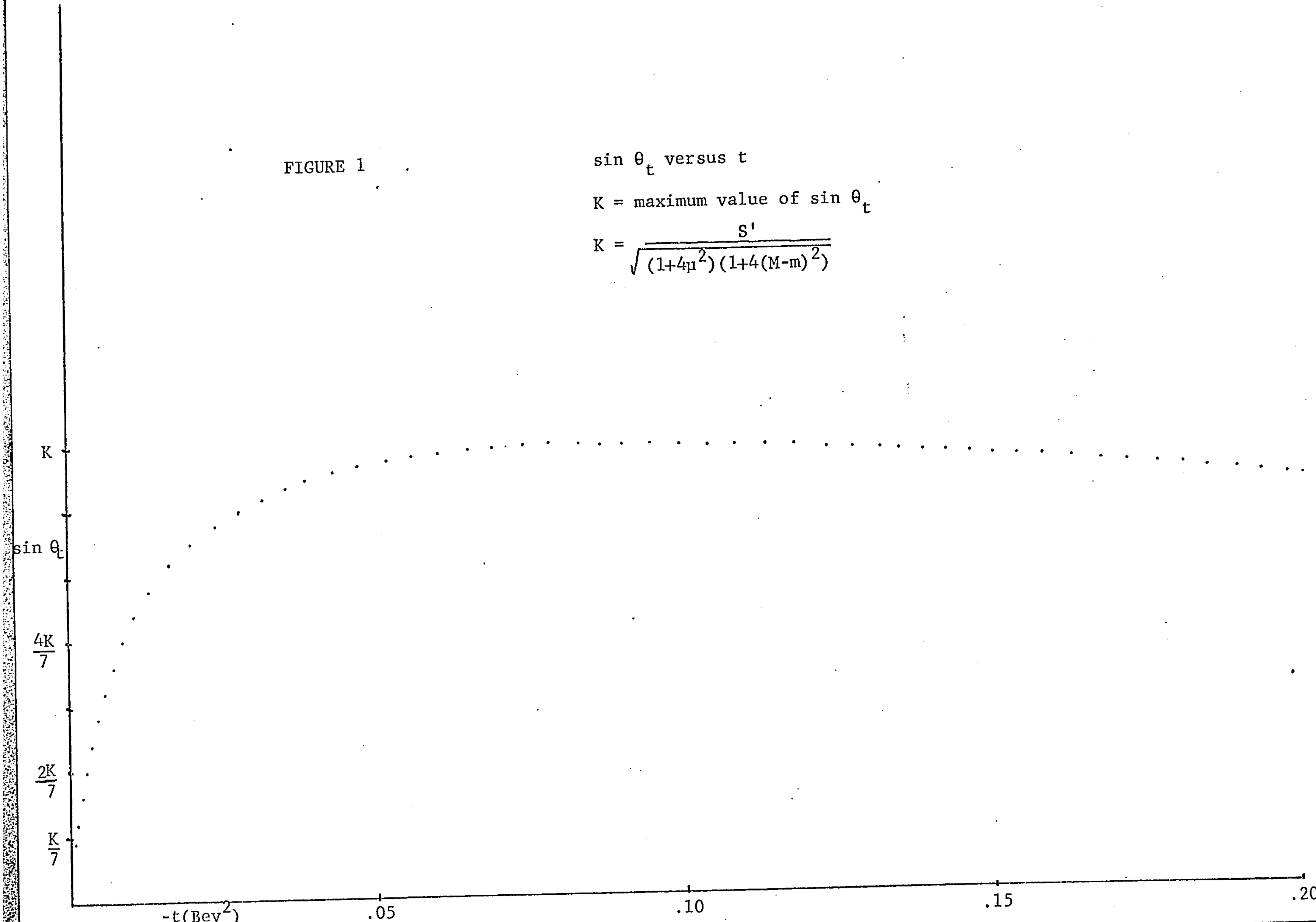


FIGURE 1

$\sin\theta_t$ versus t

K = maximum value of $\sin\theta_t$

$$K = \frac{S'}{\sqrt{(1+4\mu^2)(1+4(M-m)^2)}}$$

FIGURE 2

Complex s plane.

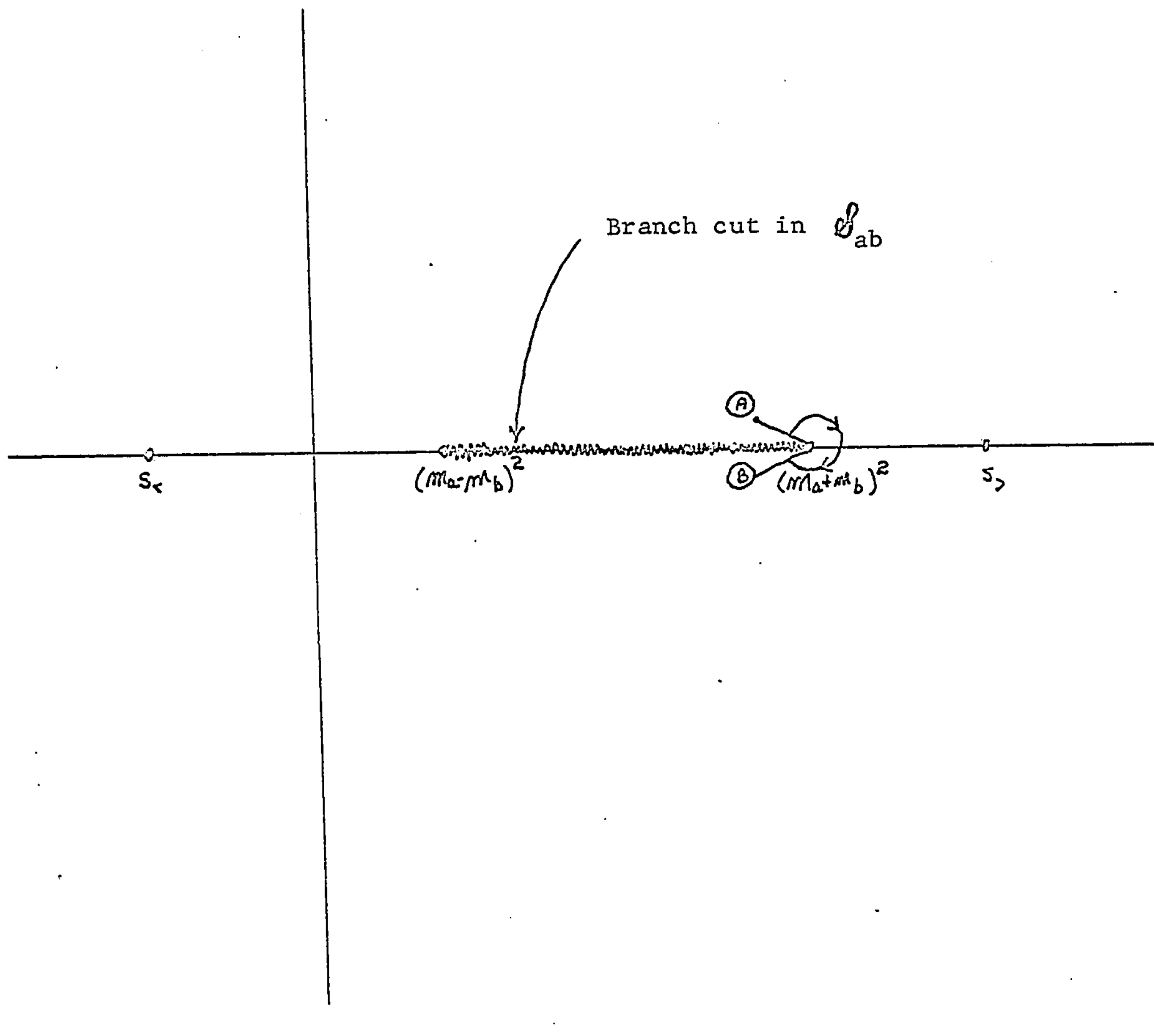
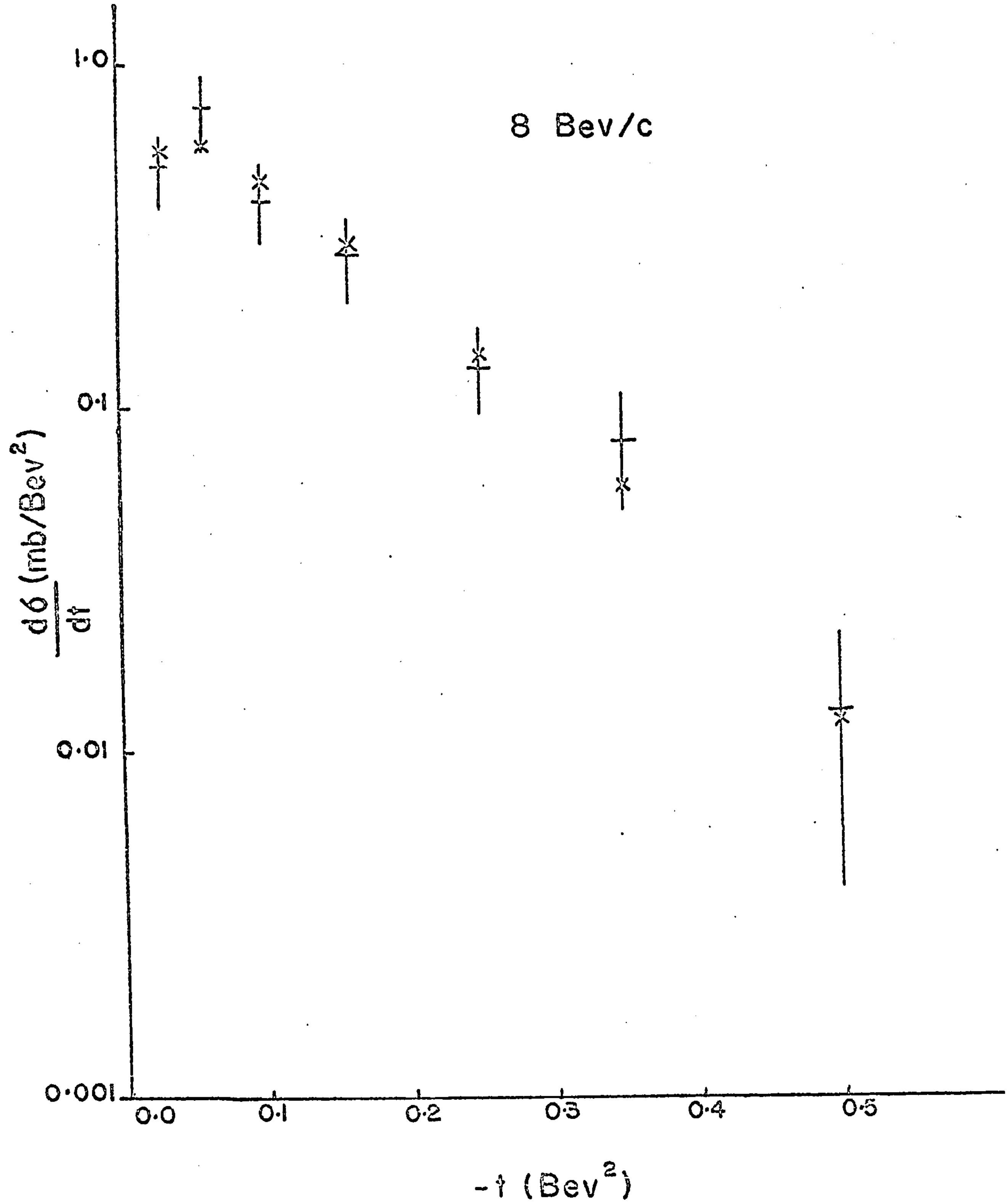


Figure 3



8 Bev/c
1·0
0·1
0·01
0·001
$\frac{d\sigma}{dt}$ (mb/Bev^2)
0·0
0·1
0·2
0·3
0·4
0·5
-t (Bev^2)

Figure 4

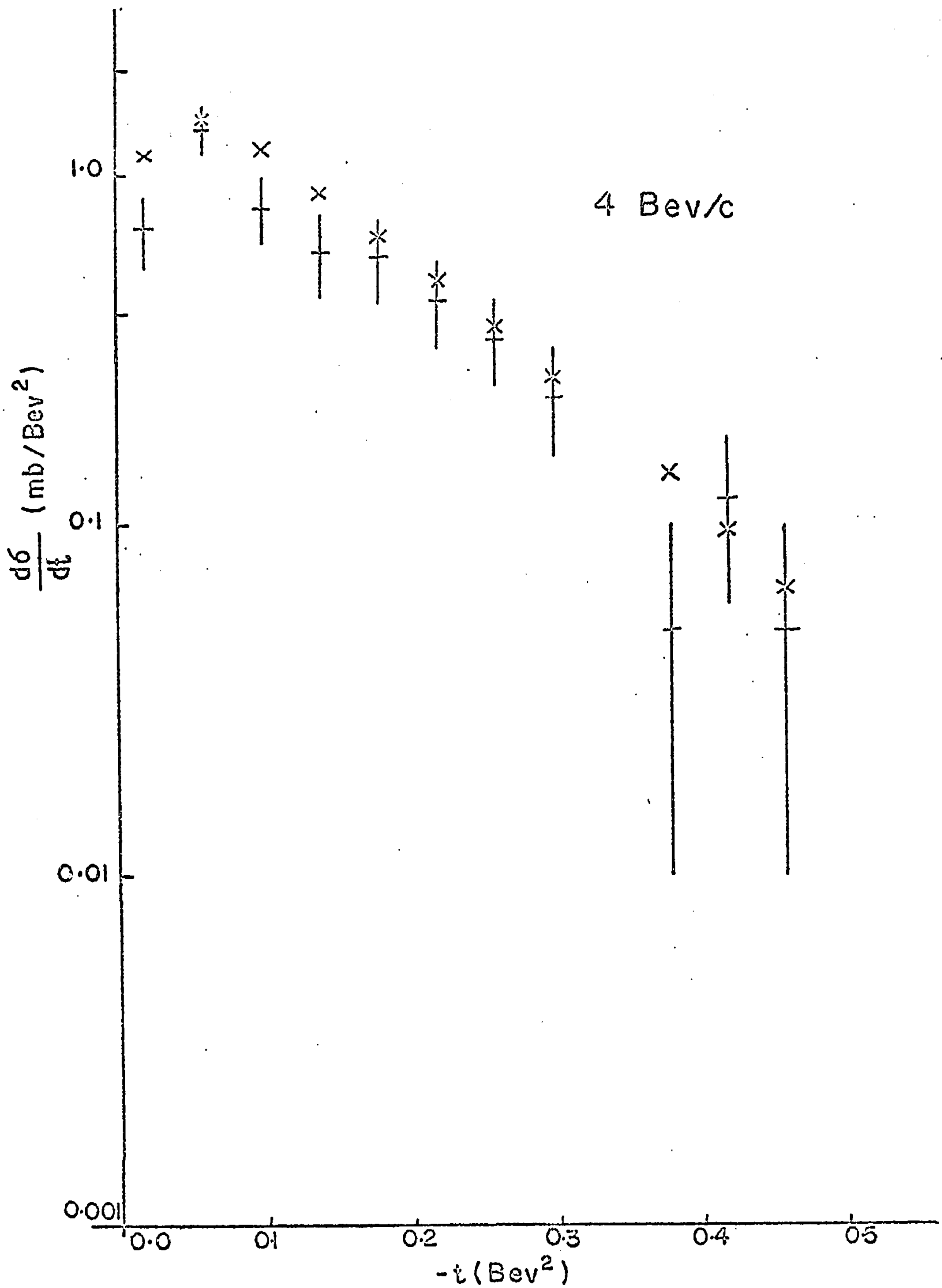

4 Bev/c
1·0
0·1
0·01
0·001
$\frac{d\sigma}{dt}$ (mb/Bev^2)
0·0
0·1
0·2
0·3
0·4
0·5
-t (Bev^2)

FIGURE 5

Incident Momentum 2.75 Bev/c.

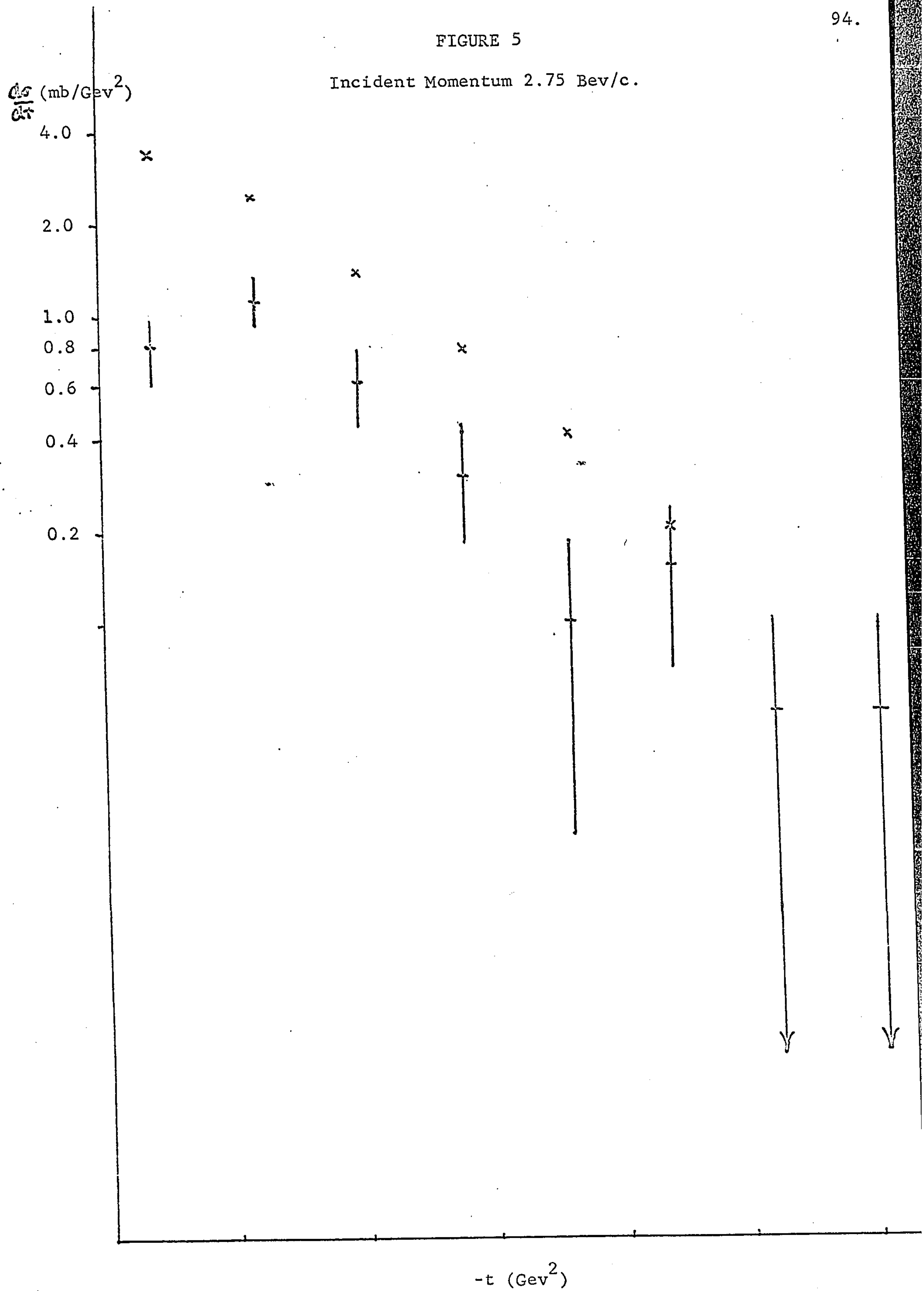


FIGURE 6

Incident Momentum 3.54 Bev/c.

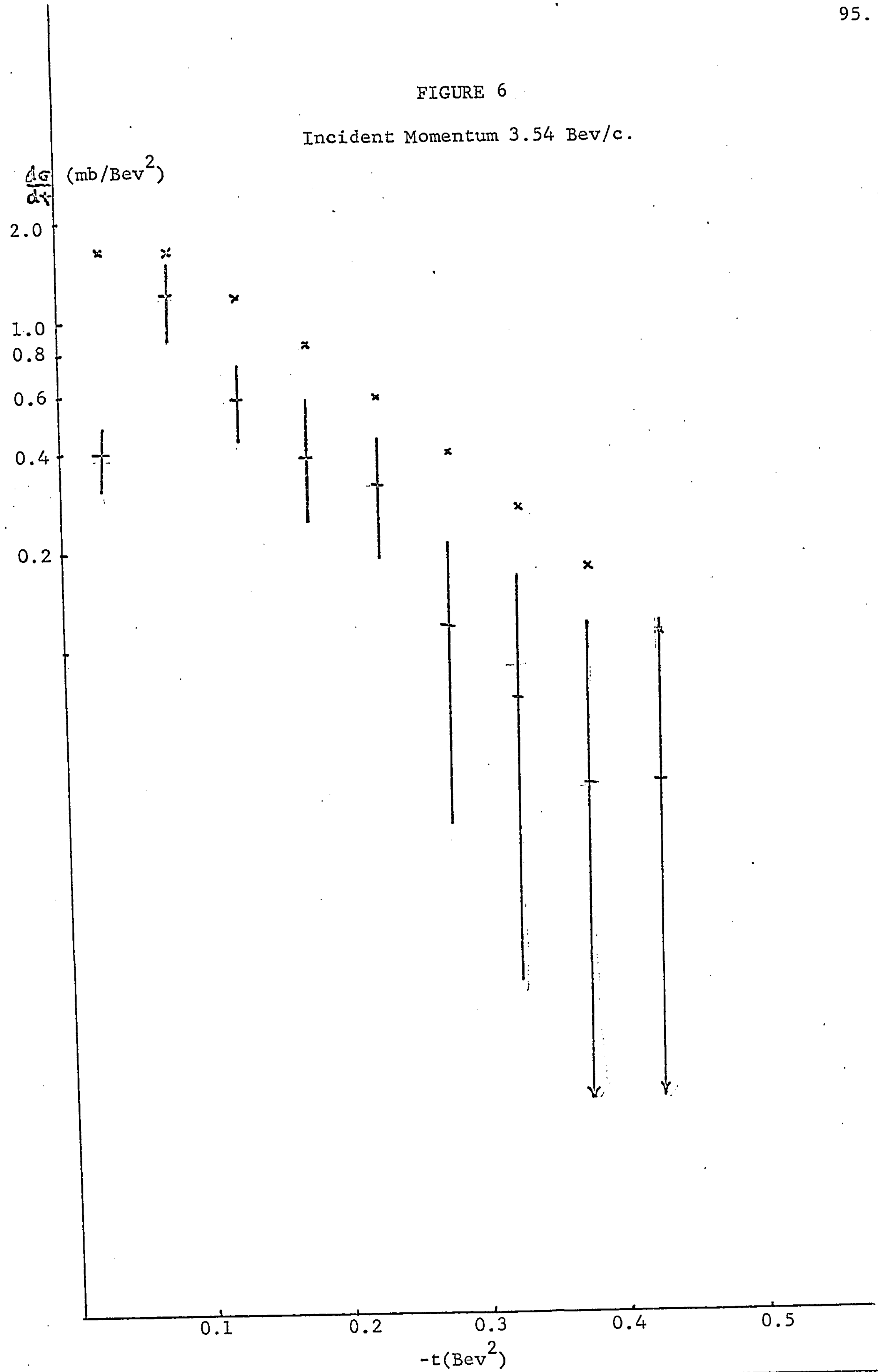


Figure 7

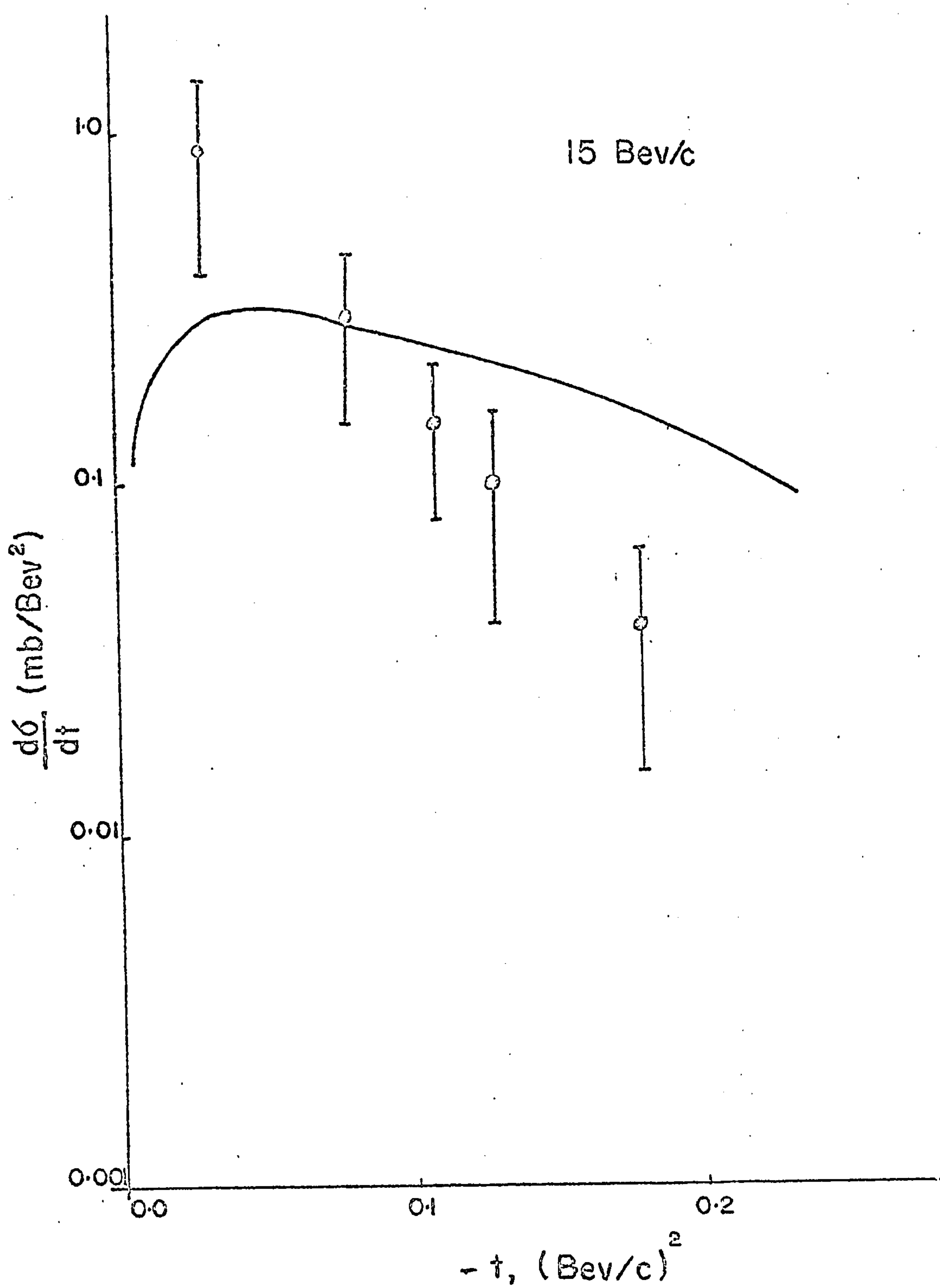


FIGURE 8

Incident Momentum 2.85 Bev/c.

The legend for Figures 8-14 is given in Table 4.

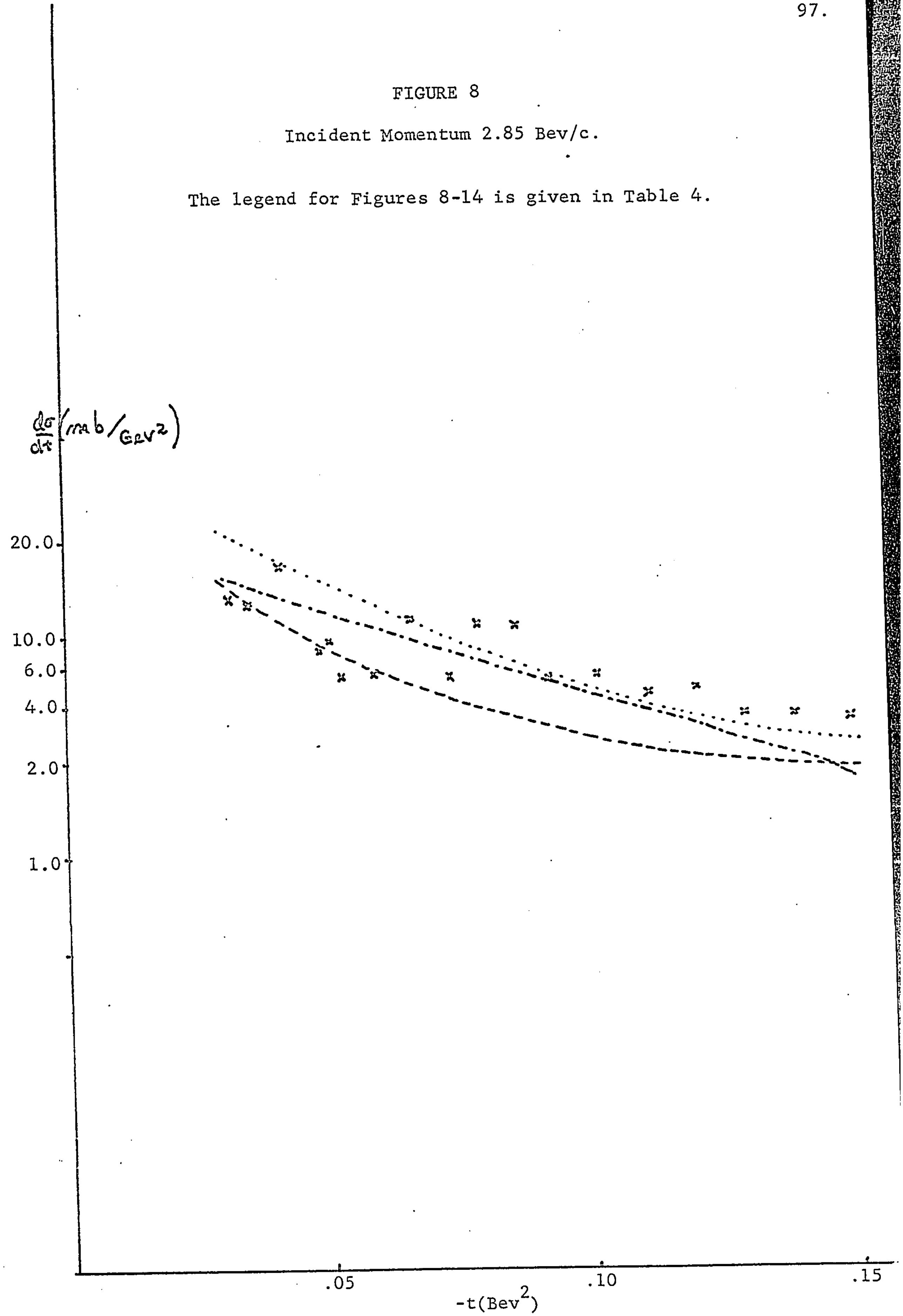


FIGURE 9

Incident Momentum 4.55 Bev/c.

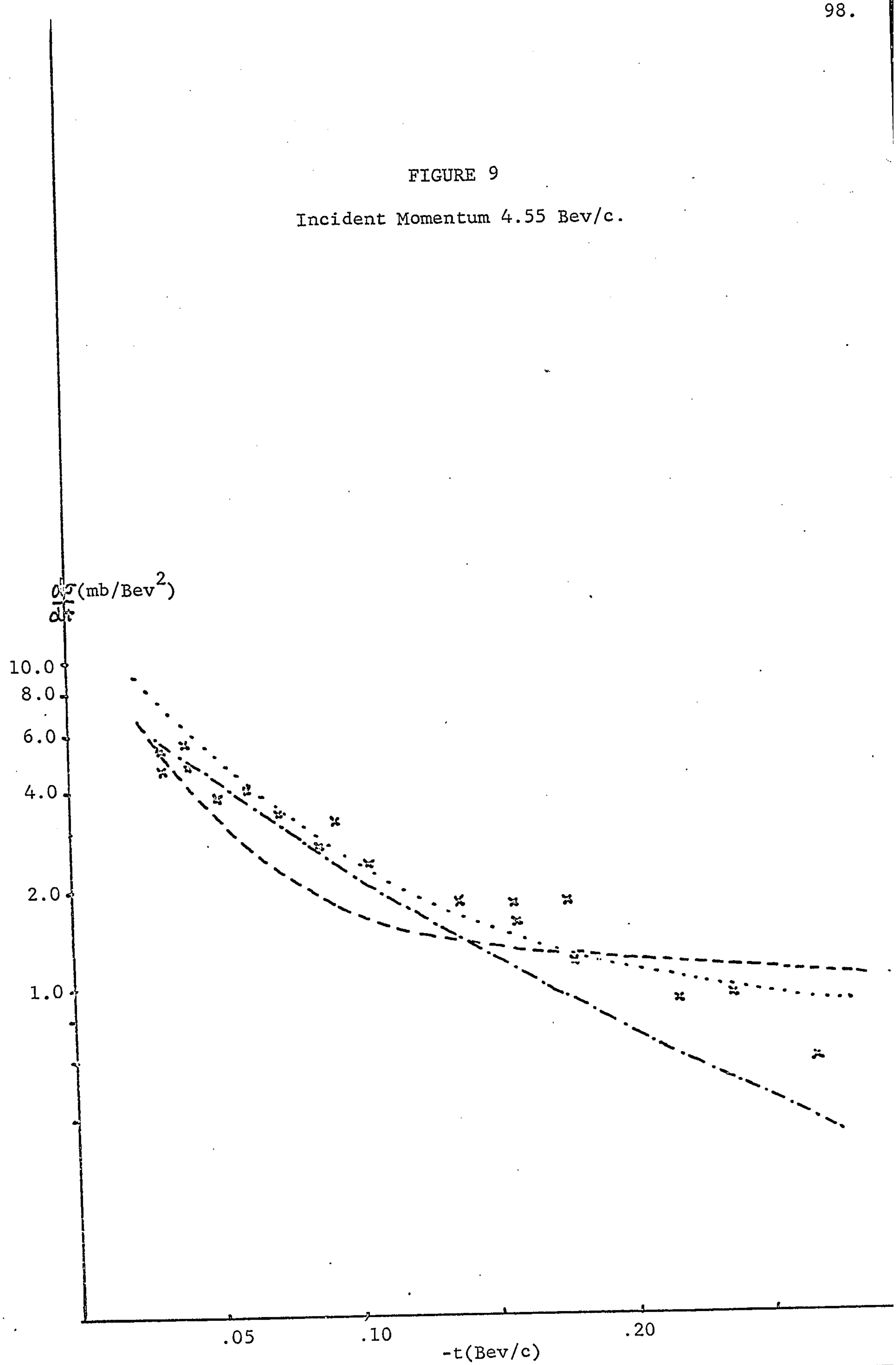


FIGURE 10

Incident Momentum 5.5 Bev/c.

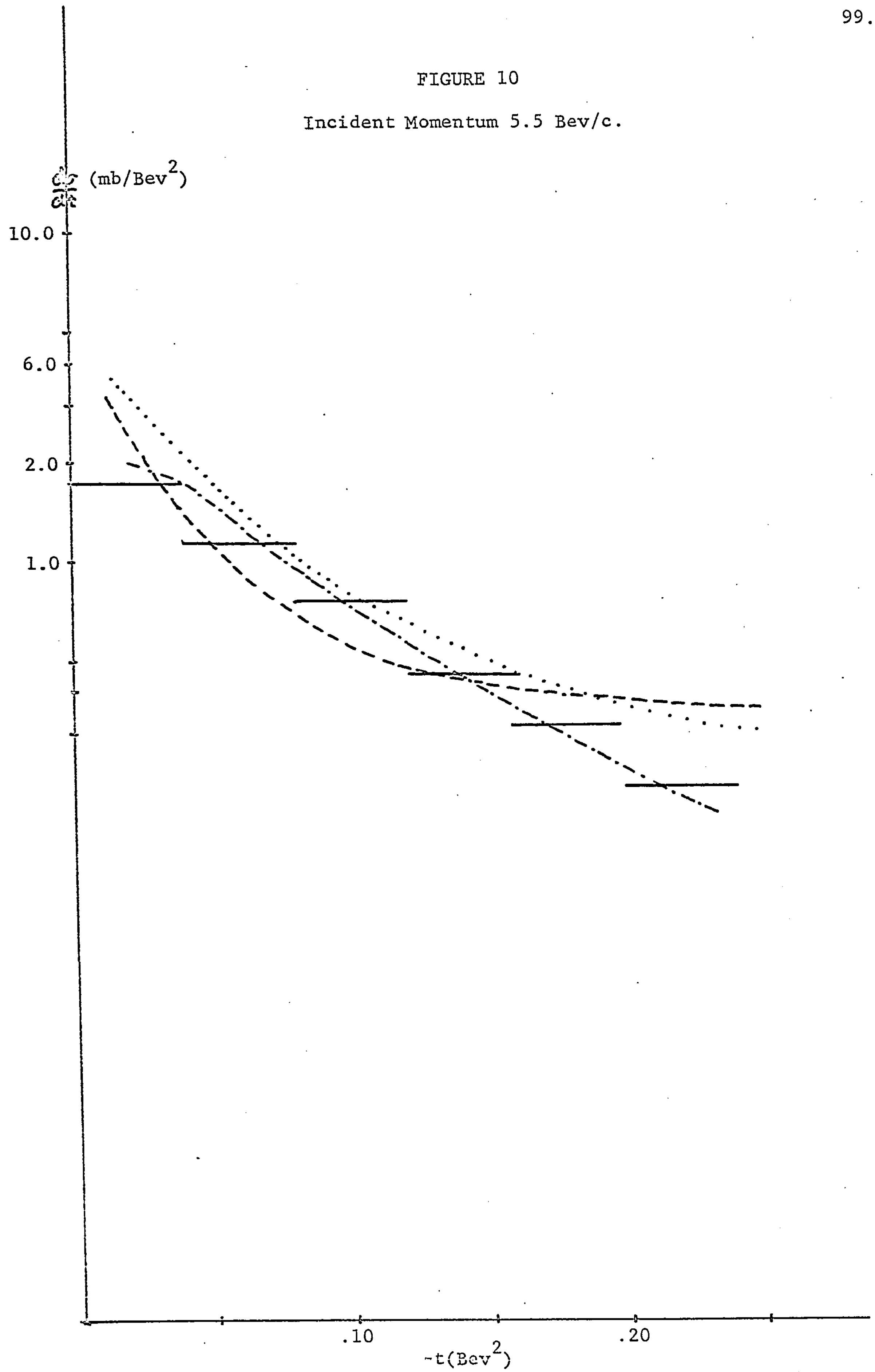


FIGURE 11

Incident Momentum 6.0 Bev/c.

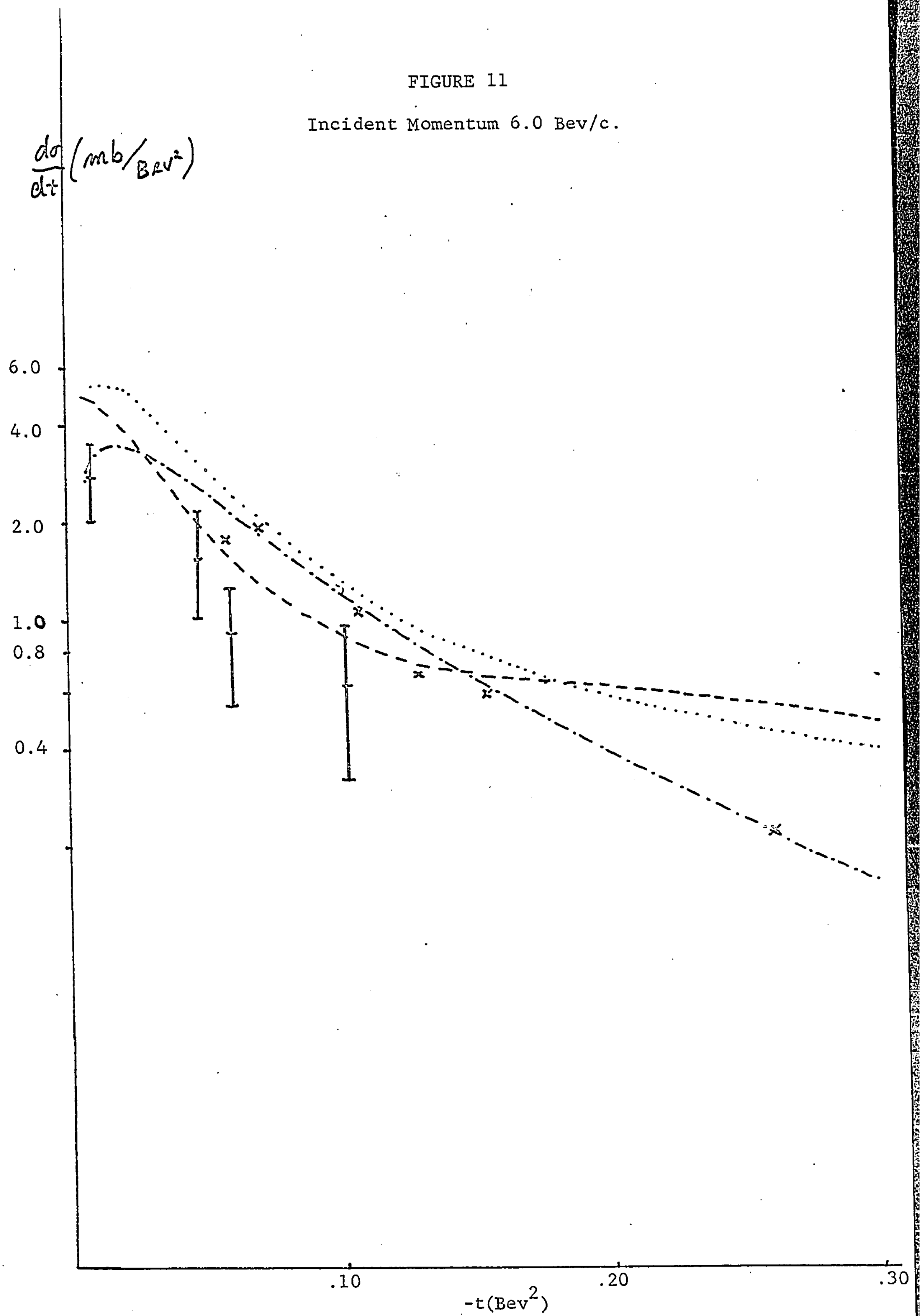


FIGURE 12

Incident Momentum 7.88 Bev/c.

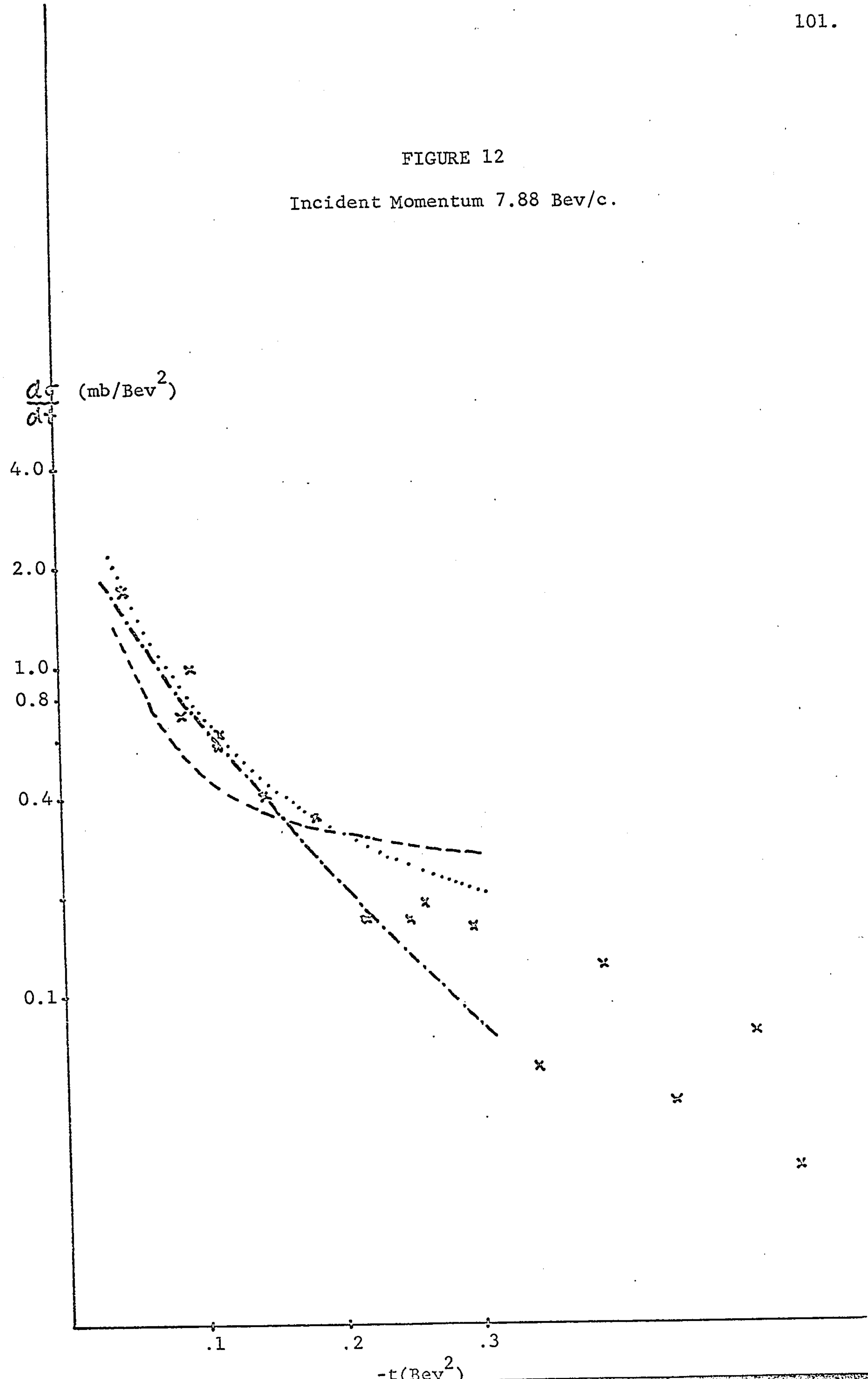


FIGURE 13

Incident Momentum 10 Gev/c.

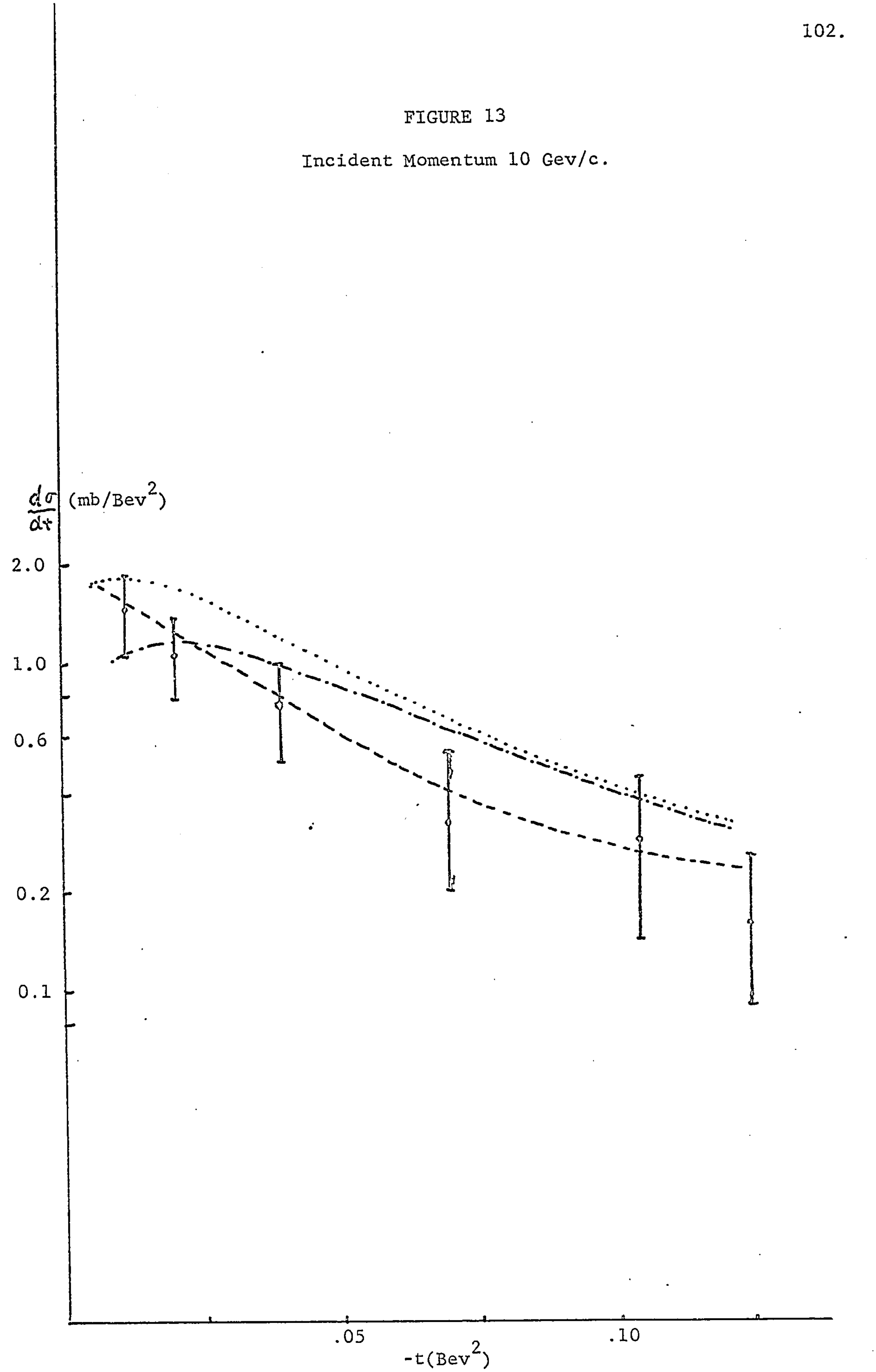


FIGURE 14

Incident Momentum 15 Bev/c.

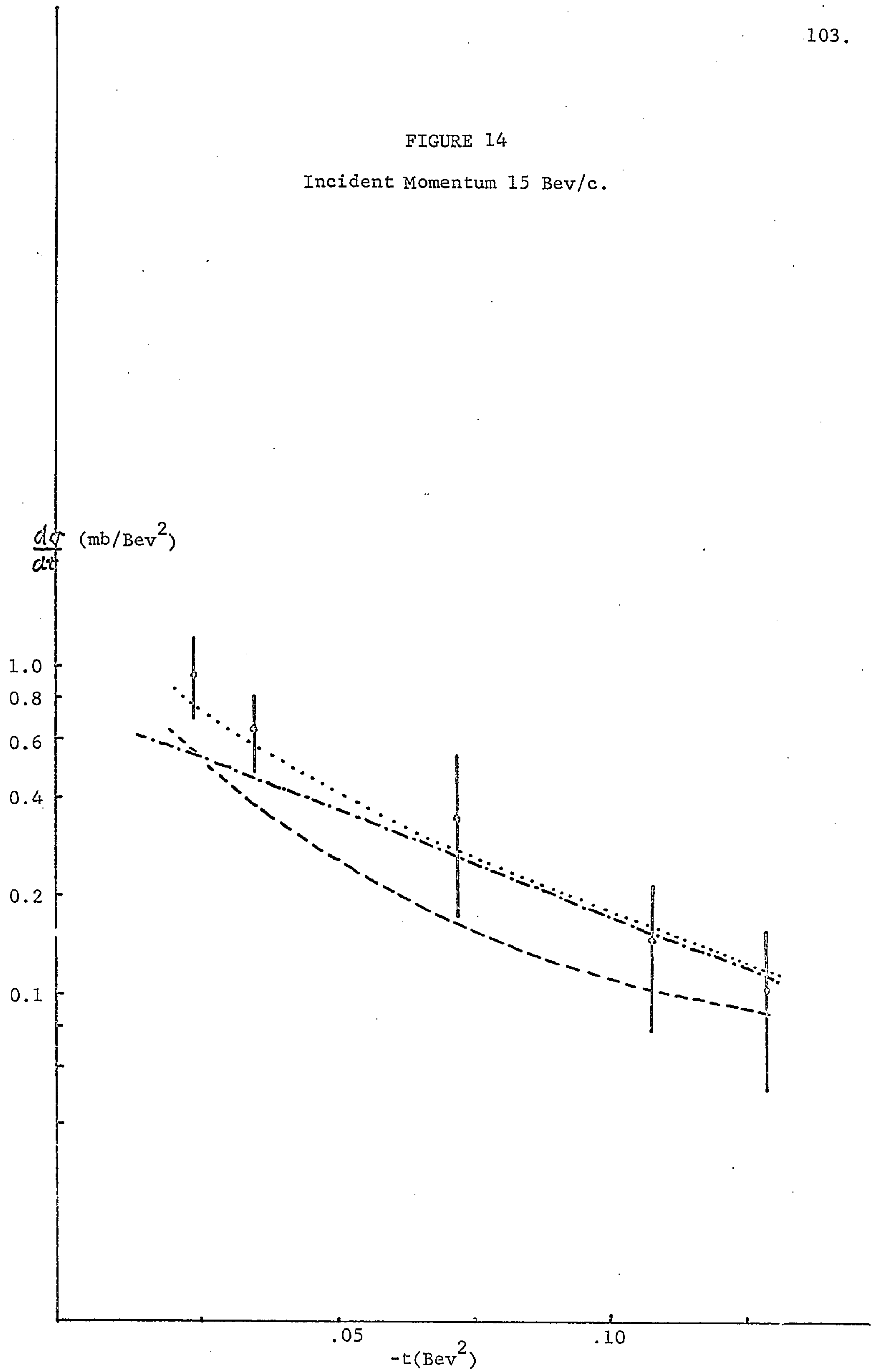
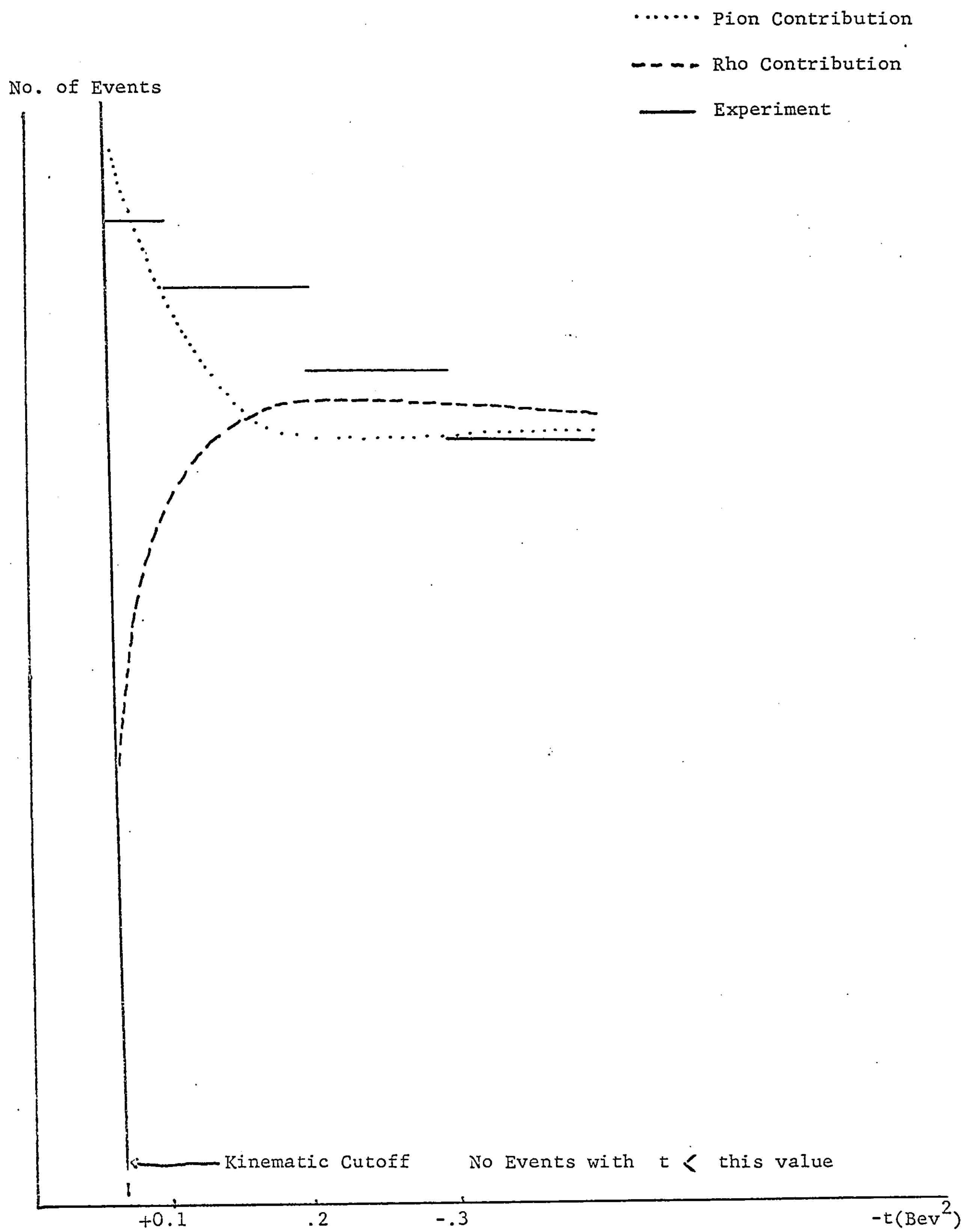


FIGURE 16



Pion Contribution
Rho Contribution
Experiment
No. of Events
Kinematic Cutoff
No Events with t < this value
+0.1
.2
-.3
-t(Bev²)

FIGURE A1 Number of Events with $1300 \leq M(p\pi^+) \leq 1800$

As a Function of θ.

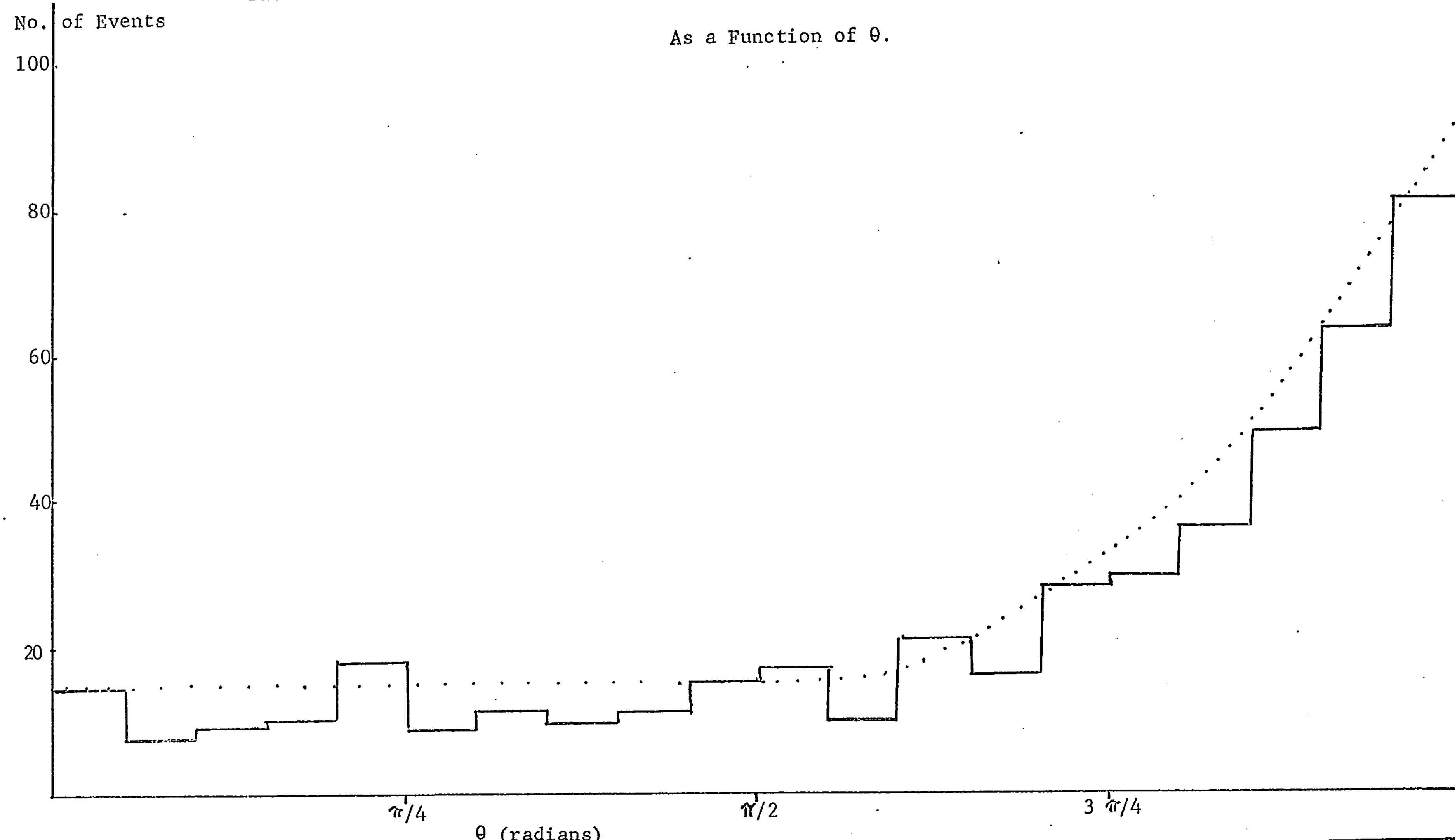


FIGURE A2

ϕ Dependence of the Events with $1300 < M(p\pi^+) < 1800$

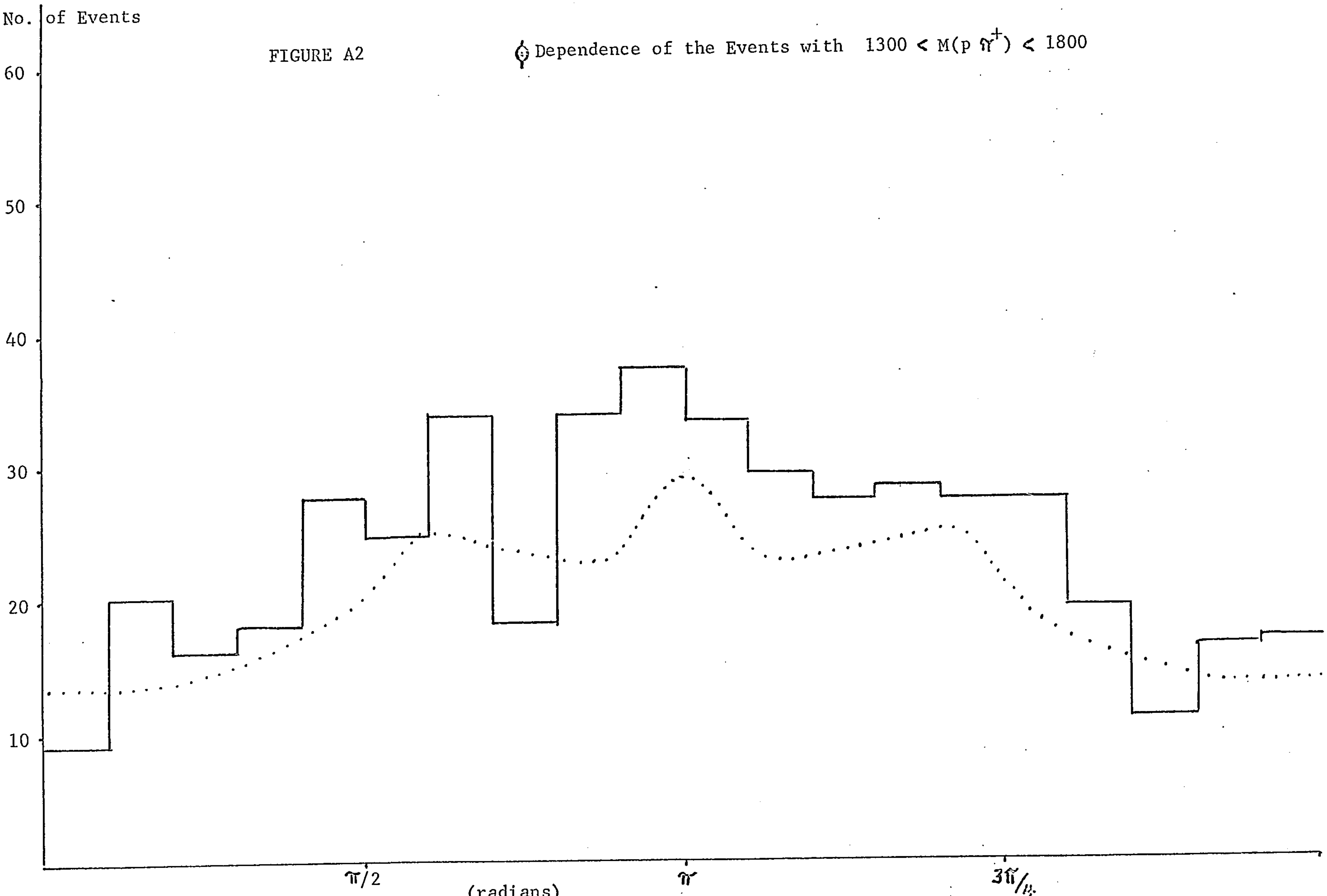


FIGURE A3 θ Dependence of Events with 1180 < M(p π^+) < 1300.

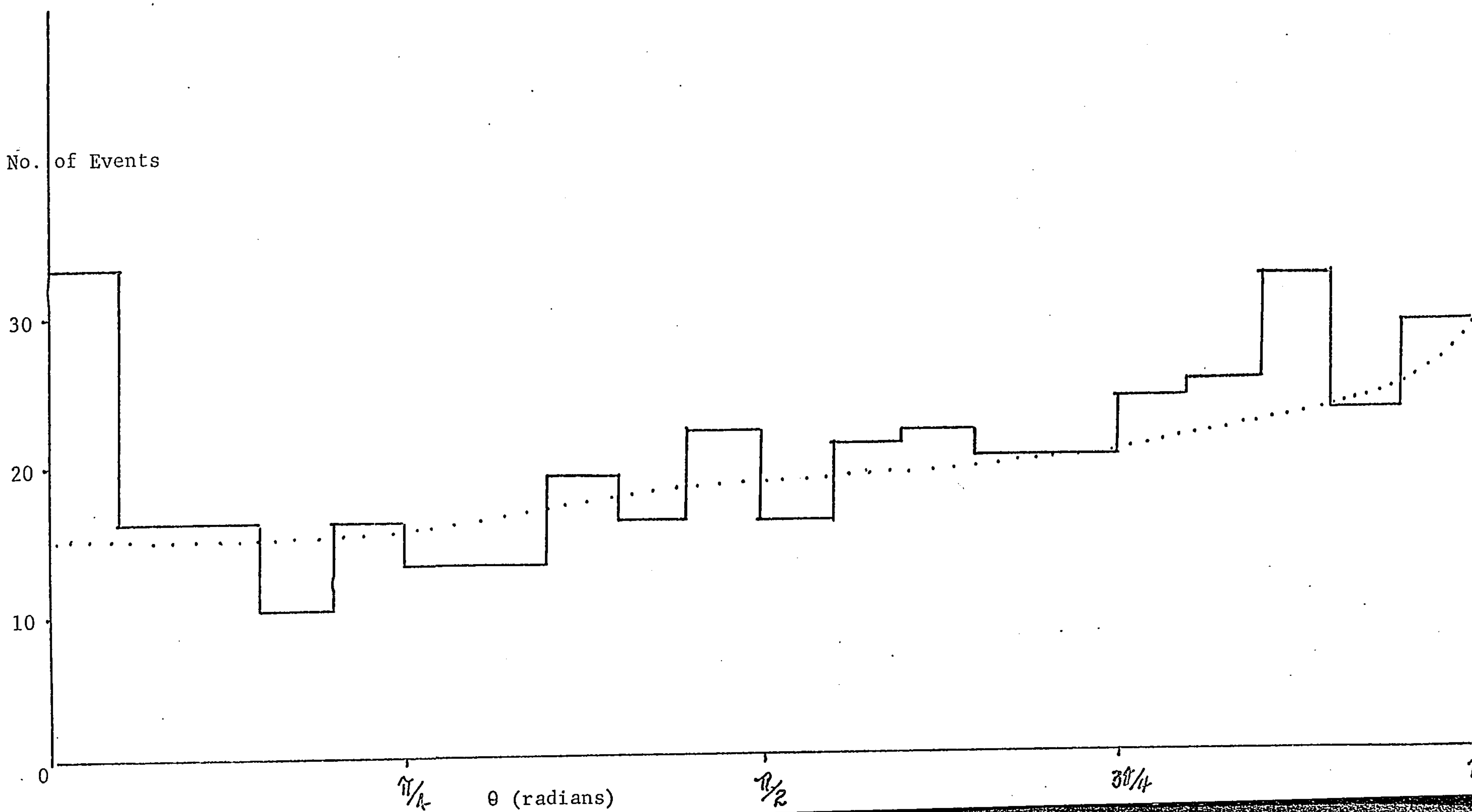
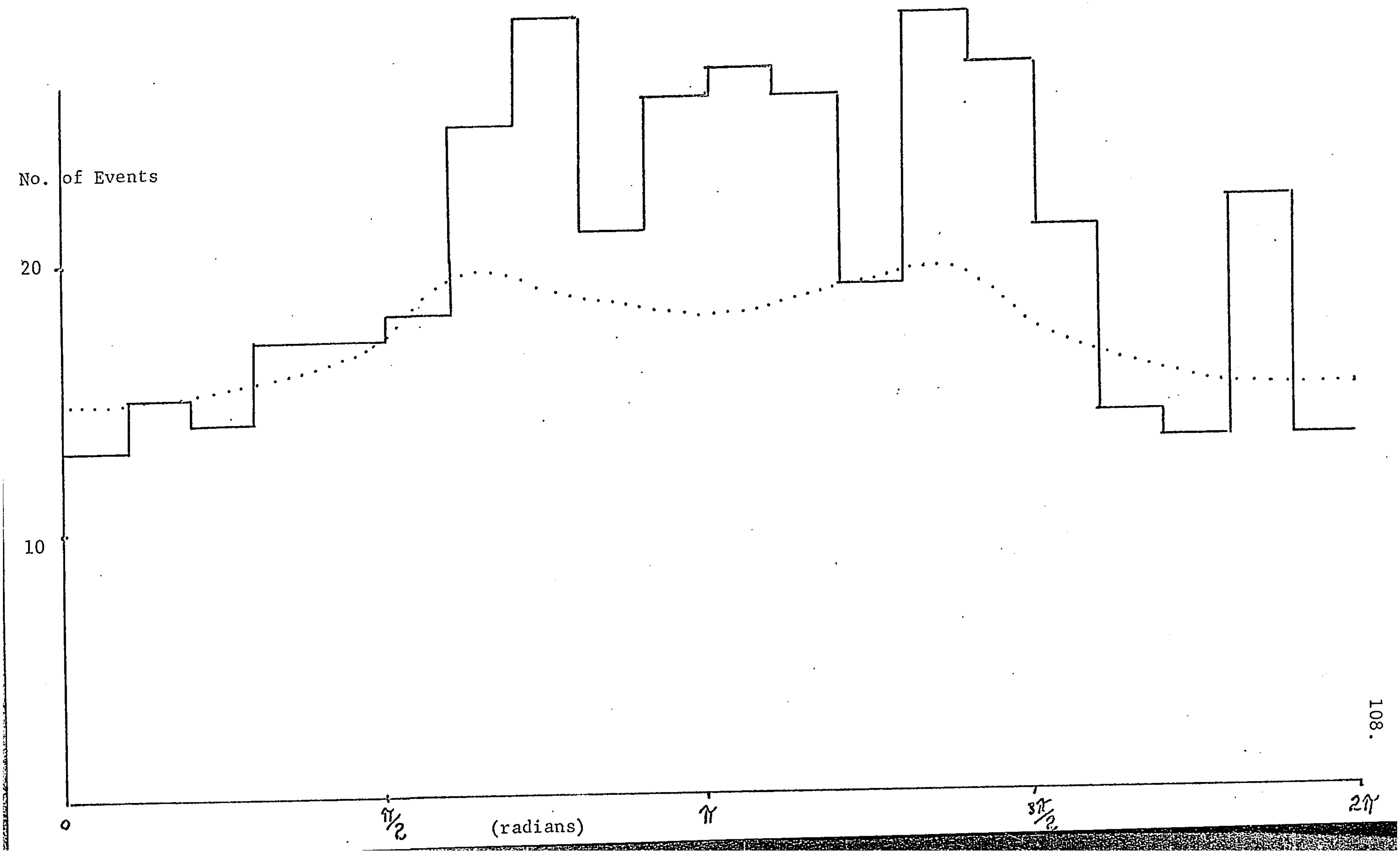


FIGURE A4 Φ Dependence of Events with $1180 < M(p\pi^+) < 1300$



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