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Four–Dimensional String Theories via Nonabelian Twists

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To my parents, and my wife

Abstract

The construction of four-dimensional string models via nonabelian twist is discussed in an operator formalism. Features of Hilbert space related to nonabelian twists are studied from the group theoretical point of view. This enables global anomalies to be removed if one insists the vacuum states to be a representation of the nonabelian group. We present a systematic procedure for the identification of the final gauge group, whose rank is generically reduced in a nonabelian twist. This general method of model-building is applied to obtain all minimal-rank strings resulting from twists by finite nonabelian subgroups of $SU(2)$. Their partition functions, vacuum representations, gauge groups, and the elimination of global anomalies are considered individually for each case.

Résumé

Nous présentons dans cette thèse une construction de la théorie des cordes dans un espace-temps quadri-dimensionel. La méthode utilisée, dans cette construction, est basée sur des transformations non-abéliennes appliquées sur le champs d'opérateurs de cordes. Les caractéristiques de l'espace Hilbertien, reliées à ces transformations, sont étudiées du point de vue de la théorie des groupes. Nous constatons que les anomalies du type global disparaissent en demandant que les états du vide forment une représentation du groupe non-abélien considéré. D'une façon systématique, nous énumérons les étapes à suivre pour identifier le groupe de symetrie de gauge de la théorie des cordes que nous obtenons après avoir performer les transformations non-abéliennes. Comme example, nous examinons tous les sous-groupes non-abéliens du groupe $SU(2)$. Chaque sous-groupe donne une théorie des cordes avec un groupe de symetrie de gauge d'ordre minimal. Finalement, les fonctions de partitions, les représentations du vide, le groupe de gauge, et l'absence des anomalies globales sont aussi revus pour chacun de ces sous-groupes.

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Original Contributions

Contributions to original knowledge are listed as follows:

in Chapter 2

[i] constraints from space-time supersymmetry are derived in the operator formalism;

in Chapter 3

[ii] the formulation for a consistent operator structure in a nonabelian-twist Hilbert space is presented; GSO projection is worked out;

[iii] a general formula of the partition function for nonabelian twist is derived. Special cases of this formula appeared in the literature but not the general formula;

[iv] conditions for the elimination of global anomalies within one-loop in the operator formalism are presented. Global anomalies are known to exist in the path integral formalism when multi-loop string amplitudes are taken into account. Using operator formalism, they can be handled in one-loop;

[v] a procedure for the identification of the final gauge group from the nonabelian twist is presented;

in Chapter 4

[vi] all minimal-rank models resulting from twists by finite nonabelian subgroups of $SU(2)$ are classified. Their partition functions, vacuum representations, gauge groups and the elimination of global anomalies are discussed for each case.

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Chapter 1

Introduction

§1.1 The Road to Superstring Models

It is generally believed that the Standard Model is an effective theory of the weak, electromagnetic and strong forces valid below certain energy scale. The Standard Model is a quantum field theory with a nonabelian gauge symmetry $SU(3) \times SU(2) \times U(1)$, in which the gauge group $SU(3)$ describes the strong interaction while the $SU(2) \times U(1)$ unifies electromagnetic and weak interactions. It is also a renormalizable theory which enables us to do calculations without encountering too many divergences. Despite its remarkable success in fitting the present experimental data, the Standard Model still does not satisfy us since the theory itself has many drawbacks. For example, it does not provide answers to the following questions: why are there three generations? Why do masses of quarks differ so greatly? Also there are about twenty free parameters in the theory. Therefore, many efforts have been made to go beyond the Standard Model. There are various scenarios being proposed, such as grand unified theories (GUT), composite models, supersymmetry and so on [1]. GUT claims that the three forces in the Standard Model are unified into one force at the energy scale 10^{15}GeV. Since proton decay has not been conclusively observed, the minimal $SU(5)$ GUT is ruled out. Other

GUT models are possible, but they have very few measurable consequences and they suffer from the gauge hierarchy problem. This problem is essentially that, the quantum correction of the Higgs mass is affected by the GUT energy scale which as a consequence either gives an unacceptable large Higgs mass or requires an unnatural fine tuning. There are two ways to avoid this unnaturalness. The first is to assume the Higgs to be a composite rather than an elementary particle; this leads to a theory without the hierarchy problem from the very beginning. The second is supersymmetry. If this symmetry is exact, bosons and fermions of equal masses must exist in pairs. Since no supersymmetric particles have been observed, supersymmetry must be broken. The broken scale has been suggested to be around the order of 1TeV [2]. Even when supersymmetry is broken, its bosonic and fermionic contributions nearly cancel each other, hence the quantum correction of the Higgs mass is still small. In this way the hierarchy problem would again be avoided.

The two scenarios of solving the hierarchy problem lead to thousands of self-consistent theories. Whether they are correct or not can be judged only experimentally. As supercolliders and ultrahigh energy machines are built and run, new phenomena may be discovered which may provide us with a means to select certain class of models. More accurate and more sensitive measurements may also indicate a deviation from the Standard Model as well. Recently the remarkable results [3] has been obtained by LEP, where the electroweak couplings were measured very accurately and the error-bars of the strong coupling have been significantly reduced. A renormalization group analysis shows that the strong coupling misses the crossing point of the other two couplings by almost four orders of magnitude or, equivalently, by more than five standard deviations. This is an independent way of ruling out minimal grand unification. However, in the presence of supersymmetry the three couplings meet spectacularly at a single point around 10^{16}GeV. The particular mechanism of supersymmetry breaking has no significant effect if one assumes a breaking scale in the range between the mass of W boson and a few TeV. Although no explicit supersymmetric particle has yet been seen, the LEP data implicitly favors the existence of supersymmetry.

Theoretically, once supersymmetry is acknowledged as a major building block of modern unified theories, the gauge hierarchy problem is automatically solved as aforementioned. However, the dynamical question of why the scale of the supersymmetry breaking is $\sim$1TeV arises. This question is addressed in no-scale supergravity theories [4] where a flat classical potential is tuned to give the vanishing of the cosmological constant (even after the supersymmetry breaking). But these supergravity theories are not renormalizable. To solve the problem of quantum gravity, there is only one known solution, namely the superstring theory [5].

A superstring theory is a string theory with supersymmetry. A string theory is a theory in which the elementary constituents are not point particles, but are rather one-dimensional string-like objects. String theory was first proposed for describing the strong interaction of the hadronic physics [5,6] before the $SU(3)$ QCD came into the world. However, it was soon realized that the theory suffers from two major drawbacks. First, it involves a massless spin-two particle which is not present in the hadronic spectrum. Second, the theory is not consistent in four space-time dimensions but rather in critical dimensions 26 and 10 for the bosonic string and the superstring respectively [7,8,9]. Later on it was suggested that this spin-two particle can be identified with the graviton and the string theory can be a potential candidate to describe all interactions including gravity if the energy is pushed further to the Planck scale (10^{19}GeV) [10]. A revolution in string theory took place in 1984 when Green and Schwarz found that certain superstring theories are anomaly-free [11]. The most promising string theory is the heterotic string [12], whose left-moving component is taken to be a bosonic string and the right-moving component to be a superstring. The theory can be formulated as a ten-dimensional theory with a rank 16 symmetry group on top. The internal consistency requirements of the theory dictates the dimension ten, as well as the possible symmetry group to be either $E_8 \times E_8'$ or $SO(32)$. The massless spectrum of the theory as well as its interaction give exactly ten-dimensional supergravity and super-Yang-Mills fields with gauge group $E_8 \times E_8'$ or $SO(32)$. It contains and is capable of unifying all four interactions, and more importantly it is also a finite quantum field theory.

The problem with the original heterotic string is that it requires ten-dimensional space-time, which obviously fails to satisfy the fact that our universe is in four-dimensional space-time. This problem can be bypassed by requiring, in the context of the old theory of Kaluza-Klein [13], that the extra dimensions be compactified, that is, by letting them live on a tiny compact object with a size of the order of the Planck length (10^{-33}cm). This would account for the experimental failure to see these hidden dimensions.

Although efforts have been made on the nonperturbative effect of string theory, it is still too primitive to rigorously answer the question of whether the theory undergoes spontaneous dimensional breaking [14]. Rather than waiting for the full development of the true quantum vacuum of the theory, we can instead look for various classical vacua of the theory by compactifying the extra six dimensions and see whether a reasonable phenomenology can be performed. Actually, much progress has been achieved in this direction [15].

There are many schemes in compactifying the extra six dimensions. The simplest is toroidal compactification of six dimensions. This means that six dimensions are described as forming a direct product of six circles with some radius. This ends up with an N=4 space-time supersymmetry, and the absence of chiral fermions. In recent years a vast number of other exactly soluble models has been obtained. This includes Calabi-Yau space compactification [16], Gepner models [17,18], fermion formulation [19,20,21], lattice approach [22,23], orbifold [24,25,26] and so on. However only relatively few may be called realistic models. It has been proved that the models from Type II superstring, in which both left- and right-moving are taken to be superstring, can never produce the Standard Model containing triplet quarks and doublet leptons [27]. The models from both $E_8 \times E_8'$ and $SO(32)$ heterotic strings can contain the Standard Model, but it seems that the $E_8 \times E_8'$ string is more viable phenomenologically, since among other things we can interpret the matter fields broken from E_8' as a hidden sector (cryptons). Technically regarding all the compactification schemes, it is relatively easy to construct realistic models by using the real fermion formulation or orbifold.

The compactification by nonabelian orbifold was proposed by Dixon, Harvey, Vafa and Witten [24], where they considered the simple case of "standard embedding", in which twists are the same for the left- and right-moving fermions. Later, geometrical analysis showed that global anomalies will generally be present even when the modular invariance conditions are satisfied [28]. The lack of a simple way to identify the global anomalies is the main obstacle for model-building from a nonabelian orbifold. That is one reason why there are very few studies on the nonabelian orbifold [29,30,31]. In order to avoid global anomalies, the usual procedure is to confine ourselves to "standard embedding", possibly adding to that some abelian twists or shifts. This is because global anomalies from the left and right are cancelled among each other in the "standard embedding". Some of the work based on the "standard embedding" plus some abelian shift has been done by Chang and Li [30]. They constructed models of the nonabelian orbifold corresponding to the dihedral-like group $\Delta(3 \cdot 3^2)$. It is interesting that the models provide us with three or four families of quarks and leptons with gauge group $SU(3) \times SU(2) \times U(1)$ multiplying some other group.

Recently the construction of realistic string models has been rapidly developed. The class of flipped $SU(5)$ models has been constructed from the real fermion formulation [32,33] and the class of $SU(3) \times SU(2) \times U(1)^n$ string models has been obtained from the abelian orbifold [26,34]. These models give many interesting results. Certainly they are still at their early stage, and need to be fully explored. One important generalization of the real fermion formulation and abelian orbifold is the nonabelian twist (or nonabelian orbifold). We expect many new features will emerge in string models from the nonabelian twist. Since we are able to solve the problem of global anomalies in a simple way, construction of nonabelian twist models can be carried out successfully, in particular the interesting class of the minimal-rank models [35].

Technically there are at least two advantages in the framework of the non-abelian twist. First, it can dramatically break the huge gauge group down to a low-rank gauge group. Second, the allowed twist representations are very restricted, since not only should they satisfy all the modular invariance conditions of abelian twists of the corresponding abelian subgroups, but also they should be free from any global anomalies. Therefore the number of allowed models in the nonabelian twist is much less than that in the real fermion construction or abelian orbifolds. This fact in turn makes it easier to classify the models by nonabelian twists.

It is also interesting to note that the nonabelian nature actually exists even in the real fermion formulation and abelian orbifold. The real fermion formalism allows the twist group to be the nonabelian group $O(2)$ since the twist can change the sign of the real part but not the imaginary part of a complex fermion which is made from two real fermions. In the abelian orbifold, the combination of abelian twist and abelian shift is generically a nonabelian space group. It is because of this nonabelian nature that the rank of the gauge group could sometimes be reduced in the real fermion formulation [19] or abelian orbifold [26] . Therefore the general consideration of the real fermion formulation or abelian orbifold needs the knowledge of nonabelian twist, such as the case of allowing the interchange of the real fermions in the real fermion formulation or adding different Wilson lines to the left and to the right of the abelian orbifold.

§1.2 Present Work

The aim of this thesis is to give a detailed description of how to construct string models by means of nonabelian twists (orbifolds). The main obstacle of global anomaly is removed by a representation requirement in the consistent Hilbert space. This means not only that the twist of string fields should form a representation of the underlying finite twist group, but also the vacua of all sectors should form a representation of the twist group as well. Unlike other studies, we do not assume here "standard embedding".

In order to have a realistic string model, one would like to construct a four-dimensional string theory to be as close to the Standard Model as possible. Among other things, the rank of the gauge group is 4 in the Standard Model and 16 in the heterotic string in ten dimensions, and it has an even larger rank when compactified to four space-time dimensions. In order to dramatically reduce their rank difference, I choose the nonabelian twist representation to give a minimal rank to the resulting gauge group. Furthermore, since the supersymmetry is expected to be broken at very low energy ($\sim$ 1TeV), one would like to keep the space-time supersymmetry at the compactification scale. This in turn restricts the twist group to be a finite subgroup of $SU(3)$ [5,36]. A relatively simple class of those twist nonabelian groups is the finite nonabelian subgroups of $SU(2)$. We will restrict the twist group to be in this simple class throughout all this thesis. Technically, in order to eliminate unphysical degrees of freedom in the Hilbert space, one may choose light-cone gauge, which reduces the ten space-time coordinates into eight transverse coordinates. Contrary to the path integral formalism, we shall build our theory starting from the Hilbert space instead of starting from the partition function, and this is called the operator formalism in the literature. Also we shall begin with the phenomenologically viable $E_8 \times E_8'$ heterotic string, and confine ourselves to symmetric orbifolds. The symmetric orbifold means that the twists are the same for the left- and right-compactified space-time coordinates.

The plan of the thesis is as follows. In Chapter 2, the basic features of abelian twists are reviewed and expanded in three subsections. General prescription of string field boundary condition and GSO projection [37] is presented in Section 2.1, in which the modular invariance conditions are derived and vacuum phases in all sectors are determined. In Section 2.2, Lorentz symmetry and space-time supersymmetry are considered. In particular the constraints to the vacuum phases by these symmetries are obtained. Abelian orbifolds are reviewed in Section 2.3, where the general constraints as well as a table of acceptable representations are presented.

Chapter 3, which spreads out in three sections, is devoted to the detailed description of the construction of nonabelian twists. Some examples are given to illustrate how to carry out the model-building in practice. Specifically, in Section 3.1, the boundary condition and GSO projection are described in terms of the consistent Hilbert space, which is very different from the case of the abelian twist. From the Hilbert space, the partition function of the nonabelian twist is obtained, which turns out to be a linear combination of the partition functions of the abelian twists corresponding to the abelian subgroups of the underlying nonabelian group. It then follows that the condition of modular invariance for the nonabelian twist is nothing but the modular invariance conditions of all corresponding abelian twists. Within each of the abelian subgroups, the twist is obtained from the diagonalization of the nonabelian twist representation, and the vacuum phases are calculated based on the formulae derived in Chapter 2. The consistency of Hilbert space demands the vacua of all sectors to form a representation of the nonabelian group, a condition which is not a priori satisfied. The failure for the vacua to be in a representation indicates the existence of global anomalies, which will be discussed in Section 3.2. Also some toy models are presented in this subsection in order to demonstrate how the global anomalies arise in the operator formalism. In Section 3.3, the general method of identifying the final gauge group is described. The rank of the gauge group that can be obtained from the model is found by looking at the twist representation. Emphasis is made on the effects of the discrete torsions and the rank enhancement.

In Chapter 4, I systematically study and classify all minimal-rank nonabelian twists arising from all the finite nonabelian subgroups of $SU(2)$. These finite nonabelian groups are the dihedral groups D_l ($l = 3, 4, 6$), the tetrahedral group T, the octahedral group O, the icosahedral group I, and their double groups $D_l^{(d)}, T^{(d)}, O^{(d)}, I^{(d)}$ [38]. The nonabelian twist of each of these groups is considered in a separate section. The partition function and the vacuum representations are calculated for each nonabelian group, and global anomalies are eliminated for each case. The twist representations and the final gauge groups of all minimal-rank models are presented in various Tables.

Finally some discussions and conclusions are made in Chapter 5. The detail of modular transformations is derived in Appendix A, while the modular invariance conditions and the formulae of vacuum phases are obtained in Appendix B.

Chapter 2

Constraints on Abelian Twist

String compactification via orbifolds is simple to construct and interesting phenomenologically. In order to construct string theories from nonabelian twists (or orbifolds), we have to establish some basic features of abelian twists, which are used in later chapters. So in this chapter we plan to discuss the constraints given by an abelian twist. In Section 2.1 we review the conditions of modular invariance and the vacuum parameters. The Lorentz symmetry and space-time supersymmetry are considered in Section 2.2. We will see that those symmetries further restrict the twist and vacuum parameters. In Section 2.3 the crystallographic constraints are investigated. It shows that only very limited number of twist groups are allowed.

§2.1 Modular Invariance of Abelian Twists

As we mentioned in Chapter 1, the left-moving component is taken to be a bosonic string and the right-moving to be a superstring in the ten-dimensional heterotic string. In the light-cone gauge, there are eight real left-moving bosons and eight real right-moving bosons for the transverse space-time coordinates, eight real right-moving fermions for the world-sheet supersymmetric partners, plus sixteen real left-moving bosons for the gauge coordinates. As it is well known a real boson field in two dimensions may be replaced by a complex fermion field in a process which is called fermionization [39]. In this way one may represent the gauge coordinates by sixteen complex fermions. Also two real bosons or two real fermions can be made into one complex boson or one complex fermion. By doing that, the fields of the heterotic string become

$$\psi_L^a(t+\sigma),\ X_L^b(t+\sigma),\ X_R^b(t-\sigma),\ \psi_R^b(t-\sigma)\quad (1\leq a\leq 16, 0\leq b\leq 3), \tag{2.1.1}$$

where ψ_L^a, ψ_R^b are complex fermion fields, and X_L^b, X_R^b are complex boson fields. The subscript L stands for left-movers and R for right-movers.

In order to have a four-dimensional heterotic string, let us take the fields $X_R^{(0)}(\sigma,t), X_L^{(0)}(\sigma,t)$ to describe the transverse components of our four-dimensional space-time coordinates, which are periodic, and define the following field vectors,

$$\begin{gathered}\chi\equiv\psi_R^{(0)},\quad \eta\equiv\begin{pmatrix}\psi_R^{(1)}\\ \psi_R^{(2)}\\ \psi_R^{(3)}\end{pmatrix},\quad \lambda\equiv\begin{pmatrix}\psi_L^{(1)}\\ \vdots\\ \psi_L^{(16)}\end{pmatrix},\\ Y\equiv\begin{pmatrix}X_R^{(1)}\\ X_R^{(2)}\\ X_R^{(3)}\end{pmatrix},\quad Z\equiv\begin{pmatrix}X_L^{(1)}\\ X_L^{(2)}\\ X_L^{(3)}\end{pmatrix},\end{gathered} \tag{2.1.2}$$

where Y and Z correspond to the six compactified dimensions. In the construction of an abelian twist, the compactification is achieved by the general abelian boundary conditions of the string fields,

$$\begin{aligned}\chi(\sigma+2\pi,t)&=R_\chi^*(g)\chi(\sigma,t),\\ \eta(\sigma+2\pi,t)&=R_\eta^*(g)\eta(\sigma,t),\\ \lambda(\sigma+2\pi,t)&=R_\lambda(g)\lambda(\sigma,t),\\ Y(\sigma+2\pi,t)&=R_Y^*(g)Y(\sigma,t)+V_Y(g),\\ Z(\sigma+2\pi,t)&=R_Z(g)Z(\sigma,t)+V_Z(g),\end{aligned} \tag{2.1.3}$$

where g is a group element of an abelian group G_a, $R_i(g)$ for $i = \chi, \eta, \lambda, Y, Z$ are unitary representations of g, and $V_j(g)$ $(j = Y, Z)$ are shift vectors. Since only symmetric orbifold is considered in this thesis, one has $R_Y(g) = R_Z(g) \equiv R_X(g)$ and $V_Y(g) = V_Z(g) \equiv V(g)$. For simplicity, we will take $V(g) = 0$. We shall use the word *sector* to mean a particular set of boundary conditions, and we denote the one specified in (2.1.3) as sector g.

The world-sheet supercurrent is given by [5,21]

$$T_-^F(\sigma, t) = (\chi(\sigma, t))^\dagger \cdot \partial_- X_R^0(\sigma, t) + (\eta(\sigma, t))^\dagger \cdot \partial_- Y(\sigma, t), \tag{2.1.4}$$

where $\partial_- \equiv \partial/\partial\sigma^-$ with $\sigma^- \equiv t - \sigma$. We know that $T_-^F(\sigma, t)$ is a fermionic current, so it must be periodic or antiperiodic,

$$T_-^F(\sigma + 2\pi, t) = \pm T_-^F(\sigma, t). \tag{2.1.5}$$

In order for the string boundary conditions to maintain the world-sheet supersymmetry, we have to demand

$$\begin{aligned} R_\chi(g) &= \pm 1, \\ R_X(g) &= R_\chi(g) R_\eta(g). \end{aligned} \tag{2.1.6}$$

In the abelian boundary conditions, the representations $R_i(g)$ of the abelian group G_a can be expressed in a diagonal form. Therefore we can discuss the boundary condition of each field independently. For convenience, let us denote $(\lambda; \eta, \chi)$ as a 20-component vector ψ, whose first 16 components are λ and last 4 components are (η, χ). The boundary condition for a fermion $\psi^a(\sigma, t)$ can be written as

$$\psi^a(\sigma + 2\pi, t) = e^{-i2\pi\epsilon^a v^a} \psi^a(\sigma, t), \tag{2.1.7}$$

where $\epsilon^a = +1$ $(1 \le a \le 16)$ and $\epsilon^a = -1$ $(17 \le a \le 20)$. The parameter v^a may be confined to the range $0 \le v^a < 1$ and we will do so from now on.

It is straightforward to obtain the mode expansion of the free fermionic field with these boundary conditions. They are

$$\begin{aligned}\psi^a(\sigma,t) &= \sum_{m=-\infty}^{\infty} \psi^a_{m+v^a-1} \exp[-i(t+\epsilon^a\sigma)(m+v^a-1)],\\ \bar{\psi}^a(\sigma,t) &= \sum_{m=-\infty}^{\infty} \bar{\psi}^a_{m-v^a} \exp[-i(t+\epsilon^a\sigma)(m-v^a)],\end{aligned} \tag{2.1.8}$$

where $\bar{\psi}^a = (\psi^a)^\dagger$ and $\bar{\psi}^a_{-m-v^a} = (\psi^a_{m+v^a})^\dagger$. A similar expansion for the complex boson field $X^b(\sigma,t)$ can be made in terms of the oscillator modes $X^b_{m+v^b}$ and $\bar{X}^b_{m-v^b}$, where X^b is one of the fields X^b_R or X^b_L.

A momentum eigenstate $|p\rangle$ can be annihilated by any mode with positive indices,

$$\psi^{a_1}_{m_1+v_1}|p\rangle = \bar{\psi}^{a_2}_{\bar{m}_2-v_2}|p\rangle = X^{b_3}_{m_3+v_3}|p\rangle = \bar{X}^{b_4}_{\bar{m}_4-v_4}|p\rangle = 0 \quad (m_i+v_i>0, \bar{m}_i-v_i>0). \tag{2.1.9}$$

States in a sector are linear combinations of states of the form

$$\psi^{a_1}_{m_1+v_1}\psi^{a_2}_{m_2+v_2}\cdots\psi^{\bar{a}_3}_{\bar{m}_3-v_3}\cdots X^b_{m_4+v_4}\cdots|p\rangle \tag{2.1.10}$$

with all $m_i + v_i \le 0$ and $\bar{m}_i - v_i \le 0$.

The ground state energy for a complex fermion is $E^a_F = -\sum_{n>0}(n-v^a)$ by the normal ordering procedure [5]. Using zeta function regularization, this is known to be

$$E^a_F = -\frac{1}{2}v^a(1-v^a) + \frac{1}{12}. \tag{2.1.11}$$

For a complex boson the ground state energy is the same formula as above but with the sign reversed, i.e. $E_B^b = \frac{1}{2}v^b(1-v^b) - \frac{1}{12}$. The vacuum energy for a given string theory is therefore

$$E_0 = \sum_a E_F^a + \sum_b E_B^b. \tag{2.1.12}$$

Closed string fields may obey aperiodic boundary conditions because these are not directly observable. Physical states however must be one-valued. To satisfy this requirement, we may first define an operation by a group element h on the fields of any sector,

$$\mathbf{h}\lambda(\sigma,t)\mathbf{h}^{-1} = R_\lambda(h)^{-1}\lambda(\sigma,t), \tag{2.1.13}$$

which is consistent with the string boundary conditions in the abelian twist. Then we can define a singlet projection operator $\mathbf{P} = \sum_{g\in G_a} \mathbf{g}/|G_a|$ where $|G_a|$ is the order of the group G_a, and demand all physical states in each sector to be invariant under this $\mathbf{P}$. This turns out to be equivalent to the one-valued requirement for the physical states. This scheme is called GSO projection in the literature [37].

Having established the string boundary conditions and the GSO projection, we actually have given the Hilbert space of our string theory. The next important task is to calculate the one-loop string amplitude (or partition function). The one-loop diagram in string theories is a torus. In order to globally describe the coordinate transformations on the torus, we have to define the modular parameter τ to be $\tau = t/\sigma$, where t and σ are world-sheet coordinates. The torus admits coordinate transformations that are not continuously connected to the identity. They are transformation [5]

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \tag{2.1.14}$$

where a, b, c, d are integers with $ad - bc = 1$. In other words the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.1.15}$$

forms a group $SL(2, \mathbf{Z})/\pm 1$ called the modular group. The transformation (2.1.14) is called a modular transformation. The whole modular group is generated by the two modular transformations, $\tau \rightarrow \tau + 1, \quad \tau \rightarrow -1/\tau$. The $SL(2, \mathbf{Z})/\pm 1$ invariance on a toroidal world-sheet is formally a consequence of the underlying reparametrization invariance in the classical action; we want to maintain this invariance in the quantum theory. The modular invariance is crucial for the absence of ultraviolet divergences [5] (see also Appendix A). Therefore we must check the partition function for a string theory to see whether it is invariant under the modular transformations. We shall carry out this to one-loop order.

Let $q = e^{i2\pi\tau}$ and $\bar{q} = e^{-i2\pi\bar{\tau}}$, where $\bar{\tau}$ stands for the complex conjugate of τ. The partition function is given by

$$\begin{aligned} \mathcal{Z}(G_a) &= \mathrm{Tr}[q^{H_L}\bar{q}^{H_R}(-1)^{\mathbf{F}}] \\ &= \sum_{h \in G_a} \mathrm{Tr}_h[q^{H_L(h)}\bar{q}^{H_R(h)}(-1)^{\mathbf{F}}\mathbf{P}] \\ &= \frac{1}{|G_a|}\sum_{g,h \in G_a} \mathrm{Tr}_h[q^{H_L(h)}\bar{q}^{H_R(h)}(-1)^{\mathbf{F}}\mathbf{g}]. \end{aligned} \tag{2.1.16}$$

Here Tr and Tr_h are respectively the traces taken over all *physical* states, and the states in the sector h. H_L and H_R are the respective Hamiltonians for the left-movers and right-movers, and $\mathbf{F}$ is the fermionic number operator. It is interesting to note that this partition function for left-movers can be simply written down by using the elementary quantum statistical mechanics with the identification $e^{-\beta H} \sim e^{i2\pi\tau H}$.

Consider the general abelian twist group G_a, which can always be written as $\mathbf{Z}_{n_1} \times \mathbf{Z}_{n_2} \times \cdots \times \mathbf{Z}_{n_A}$ for some A. Also the group element can be represented by a vector $s = (s_1, s_2, \cdots, s_A)$, with s_i being integers defined modulo n_i, where the multiplication of two group elements becomes the addition of two s vectors. The generator g_i of G_a is represented by the vector s with all entries zero except the ith, which is equal to 1. In sector s the boundary condition and the operation by the group element for the 16 left-moving and 4 right-moving complex fermions are

$$\begin{aligned} \psi^a(\sigma + 2\pi, t) &= \exp[-i2\pi\epsilon^a\omega^a(s)]\psi^a(\sigma, t), \\ \mathrm{s}\psi^a(\sigma, t)\mathrm{s}^{-1} &= \exp[i2\pi\epsilon^a\omega^a(s)]\psi^a(\sigma, t), \end{aligned} \tag{2.1.17}$$

and similar formula is applied to the complex bosons $X^b(\sigma, t)$ with the twist parameters $\hat{\omega}^b(s)$ instead of $\omega^a(s)$. Let $\omega(s) = (\omega^a(s))$ be a Lorentzian vector with $(16; 4)$ components specifying the twist of the fermions ψ. The group structure demands $\omega(s) = \sum_{i=1}^{A} s_i\omega_i - \omega'(s)$ with $0 \le \omega_i^a, \omega(s) < 1$, and $\omega'(s) \in \mathbf{Z}$ is chosen to maintain this bound on $\omega(s)$. Notice that $\omega_i = \omega(g_i)$. Let the eigenvalues of the vacuum of s sector $|\Omega_s\rangle$ for the operator $\mathbf{F}$ and $\mathrm{g_i}$ $(g_i \in G_a)$ to be

$$\begin{aligned} \mathrm{g_i}|\Omega_s\rangle &= \exp[-i2\pi\zeta_i(s)]|\Omega_s\rangle, \\ (-1)^{\mathbf{F}}|\Omega_s\rangle &= \exp[-i\pi F(s)]|\Omega_s\rangle. \end{aligned} \tag{2.1.18}$$

The partition function (2.1.16) is calculated to be the following [21],

$$\begin{aligned}\mathcal{Z}(G_a) = & \frac{1}{|G_a|}\sum_{r,s} C(r,s)\prod_{a=1}^{16} f(\omega^a(r),\omega^a(s)|\tau)\prod_{a=17}^{20}[f(\omega^a(r),\omega^a(s)|\tau)]^* \\ & \cdot\prod_{b=0}^{3} b(\hat{\omega}^b(r),\hat{\omega}^b(s)|\tau)[b(\hat{\omega}^b(r),\hat{\omega}^b(s)|\tau)]^*,\end{aligned} \tag{2.1.19}$$

where the sum is taken over all $0 \le r_i, s_i \le n_i - 1$,

$$\begin{aligned} f(u,v|\tau) &= q^{-\frac{1}{2}v(1-v)+\frac{1}{12}}\prod_{m=1}^{\infty}(1-q^{m-v}e^{i2\pi u})(1-q^{m-1+v}e^{-i2\pi u}), \\ b(0,0|\tau) &= \eta^{-2}(\tau)\sum_p \exp[i\pi\tau p^2], \\ b(u,v|\tau) &= f^{-1}(u,v|\tau) \qquad (u \text{ or } v \neq 0), \\ C(r,s) &= \exp\{-i\pi[F(s)+2r\cdot\zeta(s)]\}, \end{aligned} \tag{2.1.20}$$

$r\cdot\zeta(s) = \sum_{i=1}^{A} r_i\zeta_i(s)$, $q = e^{i2\pi\tau}$, and $\eta(\tau)$ is the Dedekind function defined in Appendix A. By using the results in Appendix A, the function $f(u,v|\tau)$ can be expressed by the Jacobi function $\vartheta_1(\nu|\tau)$ and the Dedekind function,

$$f(u,v|\tau) = \exp[i\pi(-u+v^2\tau-\frac{1}{2})]\vartheta_1(-u+v\tau|\tau)/\eta(\tau). \tag{2.1.21}$$

The modular transformations, $\tau\to\tau+1$ and $\tau\to-\frac{1}{\tau}$ for the function $f(u,v|\tau)$ are

$$\begin{aligned} f(u,v|\tau+1) &= \exp[i\pi(v^2-v+\frac{1}{6})]f(u-v,v|\tau), \\ f(u,v|-\frac{1}{\tau}) &= \exp[i\pi(2uv-u-v-\frac{1}{2})]f(1-v,u|\tau). \end{aligned} \tag{2.1.22}$$

This shows that if one has a sector twisting by v, then one must have all sectors twisting by $u, u-v, 1-v, \cdots$, to enable the partition function to be modular invariant. Since the left-moving and right-moving bosons have the same boundary conditions, their phases generated by the modular transformations to the partition function are actually cancelled each other. For the case u or $v \neq 0$, the cancellation is obvious since there is no phase generated in $|b(u,v|\tau)|^2 = |f(u,v|\tau)|^{-2}$ by using (2.1.22). The cancellation is also true for the case $u = v = 0$, which is shown in Appendix A. Therefore, as far as modular invariance is concerned, we can neglect the bosonic part of the partition function in our discussion. The requirement for modular invariance of the partition function is

$$\begin{aligned}
&\sum_{a=1}^{20} \epsilon^a[\omega^a(s)\omega^a(s) - \omega^a(s)] - 2s \cdot \zeta(s) \in 2\mathbf{Z}, \\
&\sum_{a=1}^{20} \epsilon^a[\omega^a(r)\omega^a(s) - \frac{1}{2}(\omega^a(r) + \omega^a(s))] - s \cdot \zeta(r) - r \cdot \zeta(s) + \frac{1}{2}[F(r) - F(s)] \in \mathbf{Z}.
\end{aligned} \tag{2.1.23}$$

After some calculations given in the Appendix B, the above two equations turn out to be the following two non-trivial conditions, which are equivalent to level matching conditions in the literature [24]:

$$n_i(\omega_i^2 + F_i) \in 2\mathbf{Z}, \tag{2.1.24}$$

$$n_i n_j \omega_i \cdot \omega_j \in D_{ij}\mathbf{Z} \qquad (i \neq j). \tag{2.1.25}$$

In these equations, $F_i \in \mathbf{Z}, \omega_i \cdot \omega_j \equiv \sum_{a=1}^{a=20} \epsilon^a \omega_i^a \omega_j^a$, $\omega_i^2 \equiv \omega_i \cdot \omega_i$ and D_{ij} is the common divisor of n_i and n_j. The vacuum parameters are found to be:

$$\begin{aligned}
F(s) &= \sum_{i=1}^{A} s_i F_i - 2\omega'(s)\cdot T + 2\mathbf{Z}, \\
\zeta_i(s) &= \zeta_i(0) + \sum_{i=1}^{A} s_j \zeta_{ij} - \omega_i \cdot \omega'(s) + \mathbf{Z}, \\
\zeta_i(0) &= -\omega_i \cdot T - \frac{F_i}{2} + \mathbf{Z}, \\
\zeta_{ii} &= \frac{1}{2}(\omega_i^2 + F_i) + \mathbf{Z}, \\
\zeta_{ij} &= (n_j Y_{ij} \omega_i \cdot \omega_j + Q_{ij})/D_{ij} + \mathbf{Z} \qquad (i \neq j),
\end{aligned} \tag{2.1.26}$$

where $T = ((\frac{1}{2})^{16}; (\frac{1}{2})^4)$, $Y_{ij} \in \mathbf{Z}$ is defined so that $Y_{ij}n_j + Y_{ji}n_i = D_{ij}$, and the discrete torsion parameters $Q_{ij} = -Q_{ji}$ $(1 \leq i < j \leq A)$ can be taken to be arbitrary integer from 0 to $D_{ij} - 1$. The above vacuum parameters are the most general solutions for a four-dimensional string to have modular invariance. The importance of the discrete torsions was first noticed by Vafa [40]. We will see later that these torsions may change the final gauge group in the nonabelian twist.

In order to produce the original $E_8 \times E_8'$ heterotic string, one may simply take the twist group $G_a = \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ with $\omega_1 = T, \omega_2 = ((\frac{1}{2})^8, 0^8; 0^4), \omega_3 = (0^8, (\frac{1}{2})^8; 0^4)$. That implies the three groups of the first eight, the second eight and the last four fermions can be chosen independently to be either periodic or antiperiodic. We will use the symbol $(N, N; N)$ to denote all three groups being antiperiodic (Neveu-Schwarz fermions). For periodic fermions (Ramond fermions), we will write R instead of N. On the other hand, if taking $G_a = \mathbf{Z}_2 \times \mathbf{Z}_2$ with $\omega_1 = T, \omega_2 = ((\frac{1}{2})^{16}; 0^4)$ instead, one gets $SO(32)$ heterotic string.

Since we would like to compactify the original ten-dimensional $E_8 \times E_8'$ heterotic string, the twist group should be taken to be $Z_2 \times Z_2 \times Z_2 \times G'_a$, where G'_a is some point group. Therefore, in describing the four-dimensional string, we have the eight sectors from the $E_8 \times E'_8$ string and also different sectors generated from the abelian group G'_a. We will label the vacuum of the sector $g \in G'_a$ with initial (N, N; N) boundary conditions as $|\Omega_g^{NNN}\rangle$. Other vacua are labelled in a similar way.

Before leaving this section, we would like to point out that although the conditions (2.1.24) and (2.1.25) are derived by the requirement of modular invariance at one-loop, these are actually the sufficient conditions for modular invariance to all-loop in the abelian twist. The proof [41,42] can be done by writing down n-loop string amplitudes and n-loop modular transformations, and by showing that no more conditions can be obtained beyond the level matching conditions.

§2.2 Lorentz Symmetry and Space-Time Supersymmetry

So far we have only considered the constraints on the world-sheet. In this section we will discuss two important space-time symmetries, i.e. Lorentz symmetry and space-time supersymmetry. They will give more constraints on the twist and vacuum parameters.

It is known that Lorentz symmetry is essential for a theory to make sense physically. The critical dimensions 26 for bosonic string and 10 for superstring can be determined by requiring Lorentz invariance in the first quantization of the string [5]. In the ten-dimensional heterotic string, the Lorentz algebra is $SO(9,1)$. It becomes $SO(8)$ in the light-cone gauge where only eight transverse space-time coordinates remain. The Lorentz algebra is generated by the zero-modes of the right-handed Ramond fermions [5], whose anti-commutation relations are

$$
\begin{aligned}
&\{\psi_0^a, \bar{\psi}_0^b\} = \delta^{ab}, \\
&\{\psi_0^a, \psi_0^b\} = \{\bar{\psi}_0^a, \bar{\psi}_0^b\} = 0,
\end{aligned}
\tag{2.2.1}
$$

and the appropriate space-time indices are $17 \leq a, b \leq 20$. The anti-commutation relations (2.2.1) generates the so-called Clifford algebra. The gauge particles reside in the sectors $(\cdot\cdot, \cdot\cdot;\ \mathrm{N})$ since the modes $\psi^a_{-\frac{1}{2}}$ and $\bar{\psi}^a_{-\frac{1}{2}}$ $(17 \leq a \leq 20)$ form a vector representation of the Lorentz algebra $SO(8)$. On the other hand, the gaugino particles reside in the sectors $(\cdot\cdot, \cdot\cdot;\ \mathrm{R})$ since ψ_0^a and $\psi_0^a\psi_0^b\psi_0^c$ $(17 \leq a, b, c \leq 20)$ form a spinor representation of the Lorentz algebra.

Lorentz invariance must also be maintained in string compactifications. The Lorentz algebra after compactifying the extra six dimensions becomes $SO(2)$, which is generated by the zero modes of $20th$ fermion $\psi_0^{(20)}$ and its complex conjugate. Similar to the case in the ten-dimensional string, the states in $(\cdot\cdot, \cdot\cdot; R)$ sectors with or without $\psi_0^{(20)}$ are Lorentz spinors, since they form spinor representations of the Lorentz algebra. Therefore they are physical fermions. On the other hand the states in $(\cdot\cdot, \cdot\cdot; N)$ sectors with or without the modes $\psi^{(20)}_{-\frac{1}{2}}, \bar{\psi}^{(20)}_{-\frac{1}{2}}$ correspond to physical bosons. Hence all fermion are in the sectors with twist parameter $\omega^{(20)} = 0$, and all bosons are in the sectors with $\omega^{(20)} = \frac{1}{2}$. That implies, in order to have correct spin-statistics relation, the total fermionic number of a state in sector s has to be

$$
F_{tot}(s) = 1 + 2\omega^{(20)}(s) + 2\mathbb{Z}. \tag{2.2.2}
$$

We can also determine the total fermionic number of a state in sector s from its explicit construction, and this gives

$$F_{tot}(s) = 2N(s) \cdot T + F(s) + 2\mathbf{Z}, \tag{2.2.3}$$

where $N(s)$ is a (16; 4) Lorentzian vector, which labels the number of different fermion modes in that state. Since we start from $E_8 \times E_8'$ heterotic string, the twist group is $G_a = \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_{n_4} \times \cdots \times \mathbf{Z}_{n_A}$ with first three twist vectors fixed, i.e. $\omega_1 = T, \omega_2 = ((\frac{1}{2})^8, 0^8; 0^4), \omega_3 = (0^8, (\frac{1}{2})^8; 0^4)$. Without loss of generality we can always choose $\omega_i^{(20)} = 0$ $(i > 3)$ since we can add the twist vector ω_1 to ω_i any way. Therefore we have $R_\eta(g_i) = R_X(g_i)$ $(i > 3)$. The GSO invariance under the group of the first $\mathbf{Z}_2$ implies

$$N(s) \cdot T + \zeta_1(s) = 0 + \mathbf{Z}. \tag{2.2.4}$$

From eqs. (2.2.2)—(2.2.4) one can obtain

$$F(s) = 1 + 2\omega^{(20)}(s) + 2\zeta_1(s) + 2\mathbf{Z}. \tag{2.2.5}$$

Expanding both sides of the above equation by use of (2.1.26), one may deduce the following,

$$\begin{aligned} F_1 &= 1 + 2\mathbf{Z}, \\ F_i &= 2\zeta_{1i} + 2\mathbf{Z}. \end{aligned} \tag{2.2.6}$$

That is the constraints to the vacuum parameters by the Lorentz symmetry.

Let us turn to the discussion of space-time supersymmetry. As we mentioned in Chapter 1, we want to keep the space-time supersymmetry at the compactification scale, since it is expected to be broken at very low energy ($\sim$ 1TeV). It also has been shown in the lattice construction that it is very difficult to have string theories having vanishing cosmological constant without space-time supersymmetry [43]. This is another indication that a physical string theory should have at least $N = 1$ space-time supersymmetry, in which case there must be the same mass spectra of space-time bosons as their fermionic partners.

The experimentally measurable spectrum is the string massless states because the energy scale of the massive states is 10^{19}GeV. The negative energy states (tachyons) are unacceptable for a physical theory. Fortunately the existence of space-time supersymmetry can guarantee a string model to be free of tachyons [5]. From Section 2.1 one can calculate that the vacuum energy for right-moving fields is greater than or equal to $-\frac{1}{2}$, while that for left-moving fields is greater than or equal to -1. The gauge particles are created by the modes $\psi^{(20)}_{-\frac{1}{2}}, \bar{\psi}^{(20)}_{-\frac{1}{2}}$. To have a massless gauge bosons, we must take the minimal vacuum energy of right-moving fields, which occurs in the identity (e) sector of the abelian group G'_a. All other sectors have no contribution to the vector bosons. More precisely, the gauge particles are given by the vacua $|\Omega_e^{NNN}\rangle$, $|\Omega_e^{RNN}\rangle$ and $|\Omega_e^{NRN}\rangle$ excited by some string modes. On the other hand, the gaugino particles come from the vacua $|\Omega_e^{NNR}\rangle$, $|\Omega_e^{RNR}\rangle$ and $|\Omega_e^{NRR}\rangle$ excited by some string modes. The excitation modes of the right-fermions can be either $\psi_0^{(20)}$ or without it. Notice that there is no twist to the fermion $\psi^{(20)}$ except by the first $\mathbf{Z}_2$ group of the whole twist group $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times G'_a$. In order for the massless gauge bosons and their partner gauginos to be matched, it is necessary that the phases generated by $\mathbf{g}_i$ ($i > 1$) on the vacua $|\Omega_e^{NNN}\rangle$ and $|\Omega_e^{NNR}\rangle$

are the same. It also should be true for the vacua $|\Omega_e^{RNN}\rangle$ and $|\Omega_e^{RNR}\rangle$, or $|\Omega_e^{NRN}\rangle$ and $|\Omega_e^{NRR}\rangle$. In terms of the vacuum parameters, it has to satisfy the following for $i > 1$,

$$\zeta_i(\delta_1) = \zeta_i(\delta_2 + \delta_3), \tag{2.2.7}$$

$$\zeta_i(\delta_2) = \zeta_i(\delta_1 + \delta_3), \tag{2.2.8}$$

$$\zeta_i(\delta_3) = \zeta_i(\delta_1 + \delta_2), \tag{2.2.9}$$

where δ_i is a vector of $s = (s_1, s_2, \cdots, s_A)$ with all entries zero except the ith, which is equal to 1. Using (2.1.26), the above turns out to be

$$\zeta_{i1} = \zeta_{i2} + \zeta_{i3} + \mathbf{Z}, \tag{2.2.10}$$

$$\zeta_{i2} = \zeta_{i1} + \zeta_{i3} - 2\omega_i \cdot \omega_3 + \mathbf{Z}, \tag{2.2.11}$$

$$\zeta_{i3} = \zeta_{i1} + \zeta_{i2} - 2\omega_i \cdot \omega_2 + \mathbf{Z}. \tag{2.2.12}$$

By use of eq. (B.11) in Appendix B, which is $\zeta_{ij} + \zeta_{ji} = \omega_i \cdot \omega_j + \mathbf{Z}$, the three equations above can be written as:

$$\zeta_{1i} = \zeta_{2i} + \zeta_{3i} + \omega_i \cdot w + \mathbf{Z}, \tag{2.2.13}$$

$$\zeta_{2i} = \zeta_{1i} + \zeta_{3i} - \omega_i \cdot w + \mathbf{Z}, \tag{2.2.14}$$

$$\zeta_{3i} = \zeta_{1i} + \zeta_{2i} - \omega_i \cdot w + \mathbf{Z}, \tag{2.2.15}$$

where $w \equiv T - \omega_2 - \omega_3 = (0^{16}; (\frac{1}{2})^4)$. Obviously (2.2.13)—(2.2.15) are equivalent. Therefore there is only one non-trivial equation,

$$\zeta_{1i} = \omega_i \cdot w + \zeta_{2i} + \zeta_{3i} + \mathbf{Z} \qquad (i > 1). \tag{2.2.16}$$

Setting $i = 2, 3$ and using (2.1.26) and (2.2.6), one can obtain

$$\zeta_{23} = 0, \quad \zeta_{32} = 0. \tag{2.2.17}$$

The parameters ζ_{2i}, ζ_{3i} $(i > 3)$ can be obtained from (2.1.26). In summary, the vacuum parameters for $(i > 3)$ are

$$\begin{aligned}
&F_1 = 1 + 2\mathbf{Z}, \\
&\zeta_{23} = 0, \quad \zeta_{32} = 0, \\
&\zeta_{2i} = n_i \omega_i \cdot \omega_2, \quad \zeta_{3i} = n_i \omega_i \cdot \omega_3 \qquad\qquad (n_i \ \text{odd}), \\
&\zeta_{2i} = \frac{1}{2} n_i \omega_i \cdot \omega_2 + \frac{1}{2} Q_{2i}, \quad \zeta_{3i} = \frac{1}{2} n_i \omega_i \cdot \omega_3 + \frac{1}{2} Q_{3i} \quad (n_i \ \text{even}), \\
&\zeta_{1i} = \omega_i \cdot w + \zeta_{2i} + \zeta_{3i}, \\
&F_i = 2\zeta_{1i} + 2\mathbf{Z},
\end{aligned} \tag{2.2.18}$$

where $F_2 = 2\zeta_{12}, F_3 = 2\zeta_{13}$, Q_{2i} and Q_{3i} can be chosen to be either 0 or 1. Those are corresponding to $\mathbf{Z}_2$ torsions among $\mathbf{Z}_2$'s and other even cyclic groups. Eqs. (2.2.18) and (2.1.26) are our formulae for calculating vacuum parameters.

Imposing boundary condition $2\zeta_{1i} \in \mathbf{Z}$ in (2.2.16), one has

$$2\omega_i \cdot w \in \mathbf{Z} \qquad (i > 3), \tag{2.2.19}$$

which is a new constraint on the twist parameter. In general (2.2.19) can be represented by $\det(R_X(g_i)) = 1$ $(i > 3)$. It is this constraint that restricts the twist point group to be a subgroup of $SU(3)$ rather than $U(3)$. In other words, we have just demonstrated from the operator formalism the requirement of $SU(3)$ holonomy which has been shown previously in the path-integral formalism [5]. It can also be shown that the condition (2.2.19) is sufficient to guarantee at least $N = 1$ space-time supersymmetry [36]. In our approach this conclusion can be reached by comparing the spectra of the sector with twist $\omega(s) + w$ and the sector with twist $\omega(s)$ for a given vector s. It turns out that these are mutually supersymmetric partners. Therefore the requirements (2.2.7)—(2.2.9) are necessary and also sufficient to have space-time supersymmetry.

Comparing this fermionic formalism with the lattice approach [22,23], we find that F_2 and F_3 are always set to zero if one starts from an $E_8 \times E_8'$ self-dual lattice. In abelian twist the different values of F_2 and F_3 make no difference because they are just a matter of choosing spinors versus conjugate spinors, which are actually equivalent. However, we will see that this equivalence no longer exists in the nonabelian case. In that sense the fermionic formalism is more general than the lattice approach.

§2.3 Crystallographic Constraints on Abelian Orbifolds

The propagation of strings on the ten-dimensional space-time can be regarded as on $M^4 \times T^6$, with M^4 being four-dimensional Minkowski space and T^6 being a six-dimensional torus. An orbifold is essentially the torus modding some discrete symmetry. Rather than being abstract, let us consider $\mathbf{Z}_3$ orbifold as a concrete example. To begin with, we consider a special torus T_0 ($SU(3)$ root lattice) made by the following identifications of points in the complex z plane,

$$z \simeq z + 1 \simeq z + e^{i2\pi/3}. \tag{2.3.1}$$

This lattice admits a $\mathbf{Z}_3$ symmetry generated by

$$\alpha_0 : \quad z \to e^{i2\pi/3} z. \tag{2.3.2}$$

On the torus (the fundamental region) there are three points left invariant by this transformation (see Figure 1). They are the points

$$z = \frac{k}{\sqrt{3}} e^{i\pi/6} \qquad (k = 0, 1, 2), \tag{2.3.3}$$

with the understanding that points shifted by lattice vectors are considered to be equivalent.

Now consider three complex variables z_i ($i = 1, 2, 3$). With the identification $z_i \simeq z_i + 1 \simeq z_i + e^{i2\pi/3}$, we obtain three tori T_i ($i = 1, 2, 3$). The product $T = T_1 \times T_2 \times T_3$ is a torus of real dimension six (complex dimension three). T admits the $\mathbf{Z}_3$ symmetry:

$$\alpha : \quad z_i \to e^{i2\pi/3} z_i \qquad (i = 1, 2, 3). \tag{2.3.4}$$

This symmetry has $3^3 = 27$ *fixed points*, which are left invariant by the discrete (such as $\mathbf{Z}_3$) transformations. We denote the point group (such as $\mathbf{Z}_3$ group generated by α) as P. We would like to supplement (2.3.1) with an additional equivalence relation, this being the statement that two points on T are considered equivalent if they are related by the P action. Thus, we impose the condition

$$z_i \simeq e^{i2\pi/3} z_i \qquad (i = 1, 2, 3). \tag{2.3.5}$$

We denote as O the space T/P of equivalent points on T subject to this equivalence relation. Since (2.3.5) introduces conical singularities at the 27 fixed points, O is not a manifold. It is called an "orbifold". If the point group P is Z_3, then the space T/Z_3 is called Z_3 orbifold. Even though O is not a manifold, string propagation on $M^4 \times O$ seems to make good sense since one can resolve the singularities by blowing up the fixed points [24].

The discrete symmetry can be seen from the boundary conditions of abelian twists for complex bosons, which was discussed in Section 2.1. Let $X^a(\sigma, t)$ be one of the components of the field vector Y or Z defined in (2.1.2). Its boundary condition is

$$X^a(\sigma + 2\pi, t) = e^{-i2\pi\epsilon^a v^a} X^a(\sigma, t), \tag{2.3.6}$$

where $V_Y(g)$ or $V_Z(g)$ in (2.1.3) has been taken to zero, and ϵ^a is equal to $+1$ for left-movers and -1 for right-movers. The mode expansion of the bosonic field is

$$X^a(\sigma, t) = q^a + \frac{1}{2\sqrt{2\pi}} p^a(t + \epsilon^a \sigma) + i \sum_{n \neq 0} \frac{1}{n} X^a_{n+v^a} \exp[-i(t + \epsilon^a \sigma)(n + v^a)]. \tag{2.3.7}$$

Imposing (2.3.6) on the mode expansion for $v^a \neq 0$, we may obtain

$$p^a = 0, \tag{2.3.8}$$

$$q^a = e^{-i2\pi\epsilon^a v^a} q^a, \tag{2.3.9}$$

where p^a will form a lattice for the case $v^a = 0$. We know that q^a $(a = 1, 2, 3)$ are complex variables. The equation (2.3.9) for the case $v^a = \frac{1}{3}$ and $\epsilon^a = -1$ is exactly the same as (2.3.5). Therefore the point group P which is used to define the orbifold is nothing but the abelian group G_a without the first three Z_2's.

In general, we may take the (6; 6) real bosons as a lattice $T^6_L \otimes T^6_R$, which can be thought as free bosons modding some shift vectors $\vec{l}$. The elements of space group therefore can be represented by $(\theta, \vec{l})$, where θ is an element of point group P. Since we only deal with symmetric orbifold, if Λ is referred to either T_L^6 or T_R^6 for short, then the orbifolds we will consider can be represented by $O = \Lambda/P$. Let $\vec{x}$ be a vector describing 6 real bosons. The modding by the space group implies that the points $\vec{x}$ and $\theta\vec{x} + \vec{l}$ are identified. Under this identification there are points, such as the origin, left invariant up to a lattice vector. These invariant points are called fixed points. Since the twist θ is an automorphism of the lattice, i.e. $\theta\vec{l} \in \Lambda$ for $\vec{l} \in \Lambda$, this implies that there is a basis in which all the θ entries are integers. Thus in any basis $\det\theta$ and $\mathrm{Tr}\theta$ must be integers. Since the number of fixed points, which are invariant under θ up to a lattice vector, is given by $\det(1-\theta)$ according to Lefschetz theorem, $\det(1-\theta)$ must also be an integer. In summary we have to have the following constraints,

$$\mathrm{Tr}\theta \in \mathbf{Z}, \tag{2.3.10}$$

$$\det\theta \in \mathbf{Z}, \tag{2.3.11}$$

$$\det(1-\theta) \in \mathbf{Z}. \tag{2.3.12}$$

The six real bosons can be viewed as three complex bosons. Let $z_i = x_i + ix_{i+3}$ $(i = 1,2,3)$. Therefore (z_1, z_2, z_3) are nothing but the q^a $(a = 1,2,3)$ in (2.3.9). Let z_i, z_i^* $(i = 1,2,3)$ be a basis of the lattice Λ, the equations (2.3.10)—(2.3.12) can be written to be

$$\mathrm{Tr}(R_X(g) + R_X^*(g)) \in \mathbf{Z}, \tag{2.3.13}$$

$$\det(R_X(g)R_X^*(g)) \in \mathbf{Z}, \tag{2.3.14}$$

$$\det(1 - R_X(g))(1 - R_X^*(g)) \in \mathbb{Z}, \tag{2.3.15}$$

where $R_X(g)$ is a 3×3 complex matrix, which is either R_Y or R_Z defined in (2.1.3). Notice that (2.3.14) is trivial for any abelian representation $R_X(g)$.

Recall that $R_X(g_i) = R_\eta(g_i)$ for $i > 3$ and $g_i \in G'_a$. Let $a = \omega_i^{(17)}$, $b = \omega_i^{(18)}$, $c = \omega_i^{(19)}$ ($i > 3$). The group element $g_i \in G'_a$ acting on the complex bosons of left-movers is

$$\mathrm{g}_i \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mathrm{g}_i^{-1} = R_X^{-1}(g_i) \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} e^{i2\pi a} & & \\ & e^{i2\pi b} & \\ & & e^{i2\pi c} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}. \tag{2.3.16}$$

From (2.2.19) we know $a+b+c \in \mathbf{Z}$. If we consider $\mathbf{Z}_N$ abelian group, i.e. $g_i \in \mathbf{Z}_N$, there are only six values of N, $N = 3, 4, 6, 7, 8, 12$ satisfied (2.3.13) and (2.3.15). The total number of different representations is thirteen, which is the number of different (a, b, c) values modulo the equivalences of the interchange of different sectors. The acceptable representations are classified in Table 1.

Chapter 3

Construction of Nonabelian Twist

Nonabelian twists have the nice feature that they can reduce the rank of a gauge group. However they suffer from global anomalies [28]. Instead of the rather impractical cohomological conditions given by the geometrical analysis, we propose a method to ensure a vanishing global anomaly from a group theoretical point of view. This forms the basis of model constructions in the nonabelian twists. The organization of this chapter is the following. In Section 3.1 we discuss the Hilbert space of the nonabelian twist, which is very different from the abelian counterpart, then consider the structure of the partition function which in turn depends on the Hilbert space. In Section 3.2 the representation requirement for vanishing global anomalies will be presented, which is actually a requirement for a consistent Hilbert space. The identification of the final gauge group after the GSO projection will be carried out in Section 3.3.

§3.1 Hilbert Space and Partition Function

In the construction of the abelian twist we have different sectors, each of which has a one-to-one correspondence with a group element. We also require a GSO projection which ensures all physical states to be group invariant states. Given a state of one sector of an abelian twist, the operation of GSO projection is always within that sector. That means each sector itself is a sub-Hilbert space. Therefore the whole Hilbert space is just the direct sum of the sub-Hilbert spaces of every sector. However the Hilbert space structure of nonabelian twist differs greatly from the abelian case.

Let us start with the general boundary conditions (twist representation) of a nonabelian twist by a nonabelian group G. Similar to the abelian case, one may define a g-sector ($g \in G$) by the boundary condition as the following,

$$\begin{aligned} \chi_g(\sigma+2\pi,t) &= D^*_\chi(g)\chi_g(\sigma,t), \\ \eta_g(\sigma+2\pi,t) &= D^*_\eta(g)\eta_g(\sigma,t), \\ \lambda_g(\sigma+2\pi,t) &= D_\lambda(g)\lambda_g(\sigma,t), \\ Y_g(\sigma+2\pi,t) &= D^*_X(g)Y_g(\sigma,t), \\ Z_g(\sigma+2\pi,t) &= D_X(g)Z_g(\sigma,t), \end{aligned} \tag{3.1.1}$$

where $D_i(g)$ for $i = \chi, \eta, \lambda, X$ are unitary representations of g while assuming there is no shift vector for simplicity. Also only symmetric orbifold is considered. In order to maintain world-sheet supersymmetry, one has

$$\begin{aligned} D_\chi(g) &= \pm 1, \\ D_X(g) &= D_\chi(g)D_\eta(g). \end{aligned} \tag{3.1.2}$$

Due to the nonabelian nature of G, the representations $D_i(g)$ are generally non-diagonal. In order to have the mode expansion of the string fields in a particular sector, one has to diagonalize the representations within that sector. The construction of the string states and the formulation of the vacuum energy are the same as the abelian case within each sector.

One might think that the field operators under a nonabelian twists would transform in the same way as in the abelian case,

$$\mathbf{h}\lambda_g(\sigma,t)\mathbf{h}^{-1} = D_\lambda(h)^{-1}\lambda_g(\sigma,t), \tag{3.1.3}$$

where $\mathbf{h}$ is an operator corresponding to a group element h. However this is not correct. To understand why, let us assume it to be correct, and see what kind of contradiction we may derive. To be specific, let us take the string fields $\lambda(\sigma,t)$ whose boundary condition is defined in (3.1.1),

$$\lambda_g(\sigma+2\pi,t) = D_\lambda(g)\lambda_g(\sigma,t). \tag{3.1.4}$$

Applying the operator $\mathbf{h}$ on its both sides, one has

$$\mathbf{h}\lambda_g(\sigma+2\pi,t)\mathbf{h}^{-1} = D_\lambda(g)\mathbf{h}\lambda_g(\sigma,t)\mathbf{h}^{-1}. \tag{3.1.5}$$

Applying (3.1.3) to the above, one obtains

$$D_\lambda(h)^{-1}\lambda_g(\sigma+2\pi,t) = D_\lambda(g)D_\lambda(h)^{-1}\lambda_g(\sigma,t). \tag{3.1.6}$$

Substituting the boundary condition, it leads to

$$D_\lambda(h)^{-1}D_\lambda(g)\lambda_g(\sigma,t) = D_\lambda(g)D_\lambda(h)^{-1}\lambda_g(\sigma,t). \tag{3.1.7}$$

This implies

$$[D_\lambda(g) - D_\lambda(hgh^{-1})]\lambda_g(\sigma,t) = 0, \tag{3.1.8}$$

where $D_\lambda(h)D_\lambda(g)D_\lambda(h)^{-1} = D_\lambda(hgh^{-1})$ has been used. However, generally $D_\lambda(g) \neq D_\lambda(hgh^{-1})$ for two arbitrary group elements h, g will lead to a non-vanishing left side of (3.1.8).

We have seen that the simple-minded definition for the GSO projection does not work in the nonabelian twist when $[g,h] \neq 0$. There is however no contradiction if $[g,h]=0$. Let us define therefore

$$\mathbf{c}\lambda_g(\sigma,t)\mathbf{c}^{-1} = D_\lambda(c)^{-1}\lambda_g(\sigma,t) \qquad ([c,g]=0) \tag{3.1.9}$$

for any group element c which commutes with g. This is consistent with the abelian case where all group elements mutually commute and therefore c actually is any group element of the abelian group. Setting c equal to g, one has

$$\mathbf{g}\lambda_g(\sigma,t)\mathbf{g}^{-1} = D_\lambda(g)^{-1}\lambda_g(\sigma,t). \tag{3.1.10}$$

Take a group element a, and apply the operator $\mathbf{a}$ on both sides of (3.1.10),

$$(\mathbf{aga}^{-1})\mathbf{a}\lambda_g(\sigma,t)\mathbf{a}^{-1}(\mathbf{aga}^{-1})^{-1} = D_\lambda(g)^{-1}\mathbf{a}\lambda_g(\sigma,t)\mathbf{a}^{-1}. \tag{3.1.11}$$

Multiplying $D_\lambda(a)$ to the left of its both sides, one obtains

$$(\mathbf{aga}^{-1})D_\lambda(a)\mathbf{a}\lambda_g(\sigma,t)\mathbf{a}^{-1}(\mathbf{aga}^{-1})^{-1} = D_\lambda(aga^{-1})^{-1}D_\lambda(a)\mathbf{a}\lambda_g(\sigma,t)\mathbf{a}^{-1}, \tag{3.1.12}$$

where $D_\lambda(aga^{-1})^{-1} = D_\lambda(a)D_\lambda(g)^{-1}D_\lambda(a)^{-1}$ has been used. Substituting g with aga^{-1} in (3.1.10), one has

$$(\mathbf{aga}^{-1})\lambda_{aga^{-1}}(\sigma,t)(\mathbf{aga}^{-1})^{-1} = D_\lambda(aga^{-1})^{-1}\lambda_{aga^{-1}}(\sigma,t). \tag{3.1.13}$$

Compared (3.1.13) with (3.1.12), one obtains

$$D_\lambda(a)\mathbf{a}\lambda_g(\sigma,t)\mathbf{a}^{-1} = C_\lambda(aga^{-1})\lambda_{aga^{-1}}(\sigma,t), \tag{3.1.14}$$

where $C_\lambda(h)$ is a unitary representation that commutes with $D_\lambda(h)$. Since $C_\lambda(h)$ can be always absorbed by the redefinition of the string fields $\lambda_h(\sigma, t)$ without affecting the string boundary condition, one may set $C_\lambda(h) = 1$. Then the equation (3.1.14) can be written as

$$\mathbf{a}\lambda_g(\sigma, t)\mathbf{a}^{-1} = D_\lambda(a)^{-1}\lambda_{aga^{-1}}(\sigma, t). \tag{3.1.15}$$

Note that (3.1.9) is a special case of (3.1.15). It is straightforward to show that the formula (3.1.15) is the general string field transformation that is consistent with the general string boundary condition in the nonabelian twist. It is important to notice that all sectors will be mixed with each other within one class of the nonabelian group. Therefore all sectors within one class form a sub-Hilbert space. The whole Hilbert space can be represented by the direct sum of all sub-Hilbert spaces of corresponding group classes, i.e.

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{K-1}, \tag{3.1.16}$$

where K is the number of classes of the twist group, and $\mathcal{H}_0$ denotes the sub-Hilbert space of the identity (e) which forms a group class by itself. It is interesting to notice that this Hilbert space structure also applies to the abelian case where each group element itself forms a class in an abelian group. Therefore one may think that abelian twist is nothing but a certain limit of the nonabelian twist.

Let us give an example to illustrate the structure of the Hilbert space. Take a simplest nonabelian group D_3, which has six elements a, b, c, d, e, f, and three classes $(e), (a, b, c), (d, f)$. The two-dimensional irreducible representation **2** of the elements a and f generates the group [38], i.e.

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \tag{3.1.17}$$

with $\omega = e^{-i2\pi/3}$. The two one-dimensional irreducible representations are described by $\mathbf{1}(a = 1, f = 1)$ and $\mathbf{1}'(a = -1, f = 1)$. For elements multiplication to be specific, one may take $d = f^2, b = dad^{-1}$ and $c = faf^{-1}$. We list the multiplications in Table 2.

In order to simplify our discussion, consider two complex fermion fields $\psi^{(1)}, \psi^{(2)}$. Suppose the boundary condition is given by the two-dimensional irreducible representation $D(g)$ of D_3 that is given in (3.1.17), i.e.

$$\begin{pmatrix} \psi_g^{(1)}(\sigma + 2\pi, t) \\ \psi_g^{(2)}(\sigma + 2\pi, t) \end{pmatrix} = D(g) \begin{pmatrix} \psi_g^{(1)}(\sigma, t) \\ \psi_g^{(2)}(\sigma, t) \end{pmatrix}. \tag{3.1.18}$$

Consider the identity (e) sector. Since the identity element commutes with all elements of the group, one obtains

$$\begin{aligned} \mathbf{f} \begin{pmatrix} \psi_e^{(1)} \\ \psi_e^{(2)} \end{pmatrix} \mathbf{f}^{-1} &= D(f)^{-1} \begin{pmatrix} \psi_e^{(1)} \\ \psi_e^{(2)} \end{pmatrix} = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} \begin{pmatrix} \psi_e^{(1)} \\ \psi_e^{(2)} \end{pmatrix}, \\ \mathbf{a} \begin{pmatrix} \psi_e^{(1)} \\ \psi_e^{(2)} \end{pmatrix} \mathbf{a}^{-1} &= D(a)^{-1} \begin{pmatrix} \psi_e^{(1)} \\ \psi_e^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_e^{(1)} \\ \psi_e^{(2)} \end{pmatrix}, \end{aligned} \tag{3.1.19}$$

where the operations by other group elements can be obtained from the above two operations. In the f sector, since the operation of the group element a will bring the fields of the f sector to the fields of the sector $afa^{-1} = d$, one has

$$\begin{aligned} \mathbf{f} \begin{pmatrix} \psi_f^{(1)} \\ \psi_f^{(2)} \end{pmatrix} \mathbf{f}^{-1} &= D(f)^{-1} \begin{pmatrix} \psi_f^{(1)} \\ \psi_f^{(2)} \end{pmatrix} = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} \begin{pmatrix} \psi_f^{(1)} \\ \psi_f^{(2)} \end{pmatrix}, \\ \mathbf{a} \begin{pmatrix} \psi_f^{(1)} \\ \psi_f^{(2)} \end{pmatrix} \mathbf{a}^{-1} &= D(a)^{-1} \begin{pmatrix} \psi_d^{(1)} \\ \psi_d^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_d^{(1)} \\ \psi_d^{(2)} \end{pmatrix}. \end{aligned} \tag{3.1.20}$$

In the sub-Hilbert space of the class (f,d), the string field transformation is

$$
\begin{aligned}
\mathbf{f}\begin{pmatrix}\psi_f^{(1)}\\ \psi_f^{(2)}\\ \psi_d^{(1)}\\ \psi_d^{(2)}\end{pmatrix}\mathbf{f}^{-1} &= \begin{pmatrix}\omega^2 & & & \\ & \omega & & \\ & & \omega^2 & \\ & & & \omega\end{pmatrix}\begin{pmatrix}\psi_f^{(1)}\\ \psi_f^{(2)}\\ \psi_d^{(1)}\\ \psi_d^{(2)}\end{pmatrix},\\
\mathbf{a}\begin{pmatrix}\psi_f^{(1)}\\ \psi_f^{(2)}\\ \psi_d^{(1)}\\ \psi_d^{(2)}\end{pmatrix}\mathbf{a}^{-1} &= \begin{pmatrix} & & & 1\\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}\begin{pmatrix}\psi_f^{(1)}\\ \psi_f^{(2)}\\ \psi_d^{(1)}\\ \psi_d^{(2)}\end{pmatrix},
\end{aligned}
\tag{3.1.21}
$$

where all blank entries are zero. That is the group representation for the string fields in the sub-Hilbert space of the class (f,d). With the same procedure, the group representation for the fields in the sub-Hilbert space of the class (a,b,c) is found to be

$$
\begin{aligned}
\mathbf{f}\begin{pmatrix}\psi_a^{(1)}\\ \psi_a^{(2)}\\ \psi_b^{(1)}\\ \psi_b^{(2)}\\ \psi_c^{(1)}\\ \psi_c^{(2)}\end{pmatrix}\mathbf{f}^{-1} &= \begin{pmatrix} & & & & \omega^2 & \\ & & & & & \omega\\ \omega^2 & & & & & \\ & \omega & & & & \\ & & \omega^2 & & & \\ & & & \omega & & \end{pmatrix}\begin{pmatrix}\psi_a^{(1)}\\ \psi_a^{(2)}\\ \psi_b^{(1)}\\ \psi_b^{(2)}\\ \psi_c^{(1)}\\ \psi_c^{(2)}\end{pmatrix},\\
\mathbf{a}\begin{pmatrix}\psi_a^{(1)}\\ \psi_a^{(2)}\\ \psi_b^{(1)}\\ \psi_b^{(2)}\\ \psi_c^{(1)}\\ \psi_c^{(2)}\end{pmatrix}\mathbf{a}^{-1} &= \begin{pmatrix} & 1 & & & & \\ 1 & & & & & \\ & & & & & 1\\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \end{pmatrix}\begin{pmatrix}\psi_a^{(1)}\\ \psi_a^{(2)}\\ \psi_b^{(1)}\\ \psi_b^{(2)}\\ \psi_c^{(1)}\\ \psi_c^{(2)}\end{pmatrix}.
\end{aligned}
\tag{3.1.22}
$$

Through this simple example, we have demonstrated that the whole Hilbert space can be constructed by using (3.1.15) for a given boundary condition (twist representation). Also we have seen that the string fields in each sub-Hilbert space should form a representation of the nonabelian group.

In order to complete the construction of the Hilbert space, we have to consider the vacuum states. Take a vacuum state of the g sector $|\Omega_g\rangle$. Following the definition (2.1.9) for the vacuum state and the structure of the string fields, if $[h,g]=0$, then the operation of the operator $\mathbf{h}$ on $|\Omega_g\rangle$ will remain in the g sector. On the other hand, if $[u,g]\neq 0$, then the operation on $|\Omega_g\rangle$ will result in the ugu^{-1} sector. This implies

$$\begin{aligned} \mathbf{h}|\Omega_g\rangle &= \varepsilon(h,g)|\Omega_g\rangle \qquad ([h,g]=0), \\ \mathbf{u}|\Omega_g\rangle &= \rho(u,g)|\Omega_{ugu^{-1}}\rangle \qquad ([u,g]\neq 0), \end{aligned} \tag{3.1.23}$$

where $\varepsilon(h,g),\rho(u,g)$ are some phases. These phases are not arbitrary because of two reasons. Firstly, the vacuum states should form a representation of the nonabelian group in order for a theory to be consistent. This will be referred to the *representation requirement*. Secondly, as we have seen in Chapter 2, the vacuum phases are determined by the twist parameter via the modular invariance requirement of the partition function. Similar constraints will be expected from the modular invariance of the nonabelian twist.

In order to discuss the modular invariance, one has to consider the partition function of the nonabelian twist. From the structure of the Hilbert space, the operation of the group element c on the fields of the g sector will result in the fields of the cgc^{-1} sector. Also it is obvious that the fields of different sectors are mutual orthogonal. Therefore the net contributions to the partition function is given by the commuting pair of group elements,

$$
\begin{aligned}
\mathcal{Z}(G) &\equiv \frac{1}{|G|} \sum_{g,h\in G} \mathrm{Tr}\{\exp[i2\pi\tau H_L(g) - i2\pi\bar{\tau} H_R(g)](-1)^{\mathrm{F}}\mathrm{h}\} \\
&\equiv \sum_{g,h\in G} \overset{g}{\square}\, h = \sum_{[g,h]=0} \overset{g}{\square}\, h,
\end{aligned}
\tag{3.1.24}
$$

where $|G|$ is the order of the nonabelian point group G, and the box symbol attached with g and h denotes the contribution to the partition function by the projector of a group element h in the g sector. This formula was first obtained in the path integral formalism [24].

As an example, consider the partition function of the nonabelian group D_3,

$$
\begin{aligned}
\mathcal{Z}(D_3) = &\overset{e}{\square}\,(e,a,b,c,d,f) + \overset{a}{\square}\,(e,a) + \overset{b}{\square}\,(e,b) + \overset{c}{\square}\,(e,c) \\
&+ \overset{d}{\square}\,(e,d,f) + \overset{f}{\square}\,(e,d,f),
\end{aligned}
\tag{3.1.25}
$$

where the $(e,\cdots)$ on the right of the box is referred to the sum of all projectors in the parenthesis. Rewrite the above as

$$
\begin{aligned}
\mathcal{Z}(D_3) = &\left\{\overset{e}{\square}\,(e,a) + \overset{a}{\square}\,(e,a)\right\} + \left\{\overset{e}{\square}\,(e,b) + \overset{b}{\square}\,(e,b)\right\} \\
&+ \left\{\overset{e}{\square}\,(e,c) + \overset{c}{\square}\,(e,c)\right\} \\
&+ \left\{\overset{e}{\square}\,(e,d,f) + \overset{d}{\square}\,(e,d,f) + \overset{f}{\square}\,(e,d,f)\right\} - 3\,\overset{e}{\square}\,e \\
= &\frac{1}{3}\mathcal{Z}(\mathrm{Z}_2') + \frac{1}{3}\mathcal{Z}(\mathrm{Z}_2'') + \frac{1}{3}\mathcal{Z}(\mathrm{Z}_2''') + \frac{1}{2}\mathcal{Z}(\mathrm{Z}_3) - \frac{1}{2}\mathcal{Z}(\mathrm{Z}_1),
\end{aligned}
\tag{3.1.26}
$$

where the fraction factor is due to the definition of partition function involving a factor $1/|G|$, which is the inverse of the order of point group. Notice that each $\{\cdots\}$ in (3.1.26) in itself is a partition function of an abelian twist. The three Z_2 abelian groups $(e, a), (e, b), (e, c)$ have equal contributions to the partition function because of the sub-Hilbert space structure. This can be seen by the fact that the group element b twisting string fields of the b-sector and c twisting those of the c-sector are exactly the same as the group element a twisting string fields of the a-sector. Therefore it is no longer necessary to distinguish these three partition functions in (3.1.26). Hence the partition function of D_3 is

$$\mathcal{Z}(D_3) = \mathcal{Z}(\mathrm{Z}_2) + \frac{1}{2}\mathcal{Z}(\mathrm{Z}_3) - \frac{1}{2}\mathcal{Z}(\mathrm{Z}_1). \tag{3.1.27}$$

The partition functions of nonabelian twists in $SU(2)$ WZW model were first obtained by P. Ginsparg in the path integral formalism [29], where he considered the $c = 1$ conformal field theories based on modding out string propagation on the $SU(2)$ group manifold by its finite subgroups. Among other things he obtained the same formula as (3.1.27) for D_3 orbifold. However, since he considered the orbifolds of the $c = 1$ case, our partition functions are more general. Furthermore our partition functions constructed from the operator formalism can give more information. For example, we know that the three partition functions of $\mathrm{Z}_2', \mathrm{Z}_2''$ and Z_2''' in (3.1.26) are exactly equal even without any phase differences. From the path integral formalism one can figure out that the partition function of a nonabelian orbifold is a linear combination of the partition functions of some abelian orbifolds [28], but the relative phases among those partition functions can be only fixed by the knowledge of the accurate Hilbert space. We will see later that many things become more transparent in our operator formalism.

Given the formula (3.1.24) for the partition function, in principle one can calculate it for any nonabelian group. However in practice, it is not easy to figure out what kind of linear combination of the partition functions of the abelian subgroups forms the partition function of the nonabelian group, especially for a large group. There is however a straightforward approach to obtain the nonabelian partition function which requires no guess work. This we shall discuss presently. Consider the partition function of a general nonabelian twist, and let G_i be the maximal abelian subgroups of G. We will denote M as the total number of G_i's. We might guess that the partition function of G would be the linear combination of the partition functions of all the G_i's. However this turns out not to be true because we have over-counted the sectors generated by the group elements of $G_{ij} \equiv G_i \cap G_j$. If we subtract the contributions from the group G_{ij}, then we would have over-subtracted the sectors given by the group $G_{ijk} \equiv G_i \cap G_j \cap G_k$, etc. This process will go on until the group $G_{ijk\dots}$ being the intersection of all M subgroups. Let $\mathcal{Z}(G_{ijk\dots}) \equiv \mathcal{Z}'(G_{ijk\dots})/|G_{ijk\dots}|$, then the partition function of the nonabelian twist can be expressed in terms of the partition functions of the abelian twists,

$$\mathcal{Z}'(G) = \sum_i \mathcal{Z}'(G_i) - \sum_{\{i<j\}} \mathcal{Z}'(G_{ij}) + \sum_{\{i<j<k\}} \mathcal{Z}'(G_{ijk}) - \cdots, \tag{3.1.28}$$

where the summation is truncated with the last term being the partition function of the intersection of all M subgroups. This is the general formula for the partition function of a nonabelian twist. For a relatively large group, it is far more efficient to use this formula instead of (3.1.24).

As a check, consider D_3 group where there are four maximal abelian subgroups, $G_1 = (e, f, d), G_2 = (e, a), G_3 = (e, b), G_4 = (e, c)$. It is easy to obtain $G_{ij} =$

$G_{ijk} = G_{ijkl} = (e)$ $(1 \leq i < j < k < l \leq 4)$. Therefore we have $\mathcal{Z}'(D_3) = \mathcal{Z}'(\mathrm{Z}_3)+\mathcal{Z}'(\mathrm{Z}_2')+\mathcal{Z}'(\mathrm{Z}_2'')+\mathcal{Z}'(\mathrm{Z}_2''')-6\mathcal{Z}'(\mathrm{Z}_1)+4\mathcal{Z}'(\mathrm{Z}_1)-\mathcal{Z}'(\mathrm{Z}_1)$, which is the same as (3.1.26) if the relation $\mathcal{Z}(G_{ijk\ldots}) = \mathcal{Z}'(G_{ijk\ldots})/|G_{ijk\ldots}|$ is taken into account.

From the general formula for the partition function it is rather obvious that the partition function $\mathcal{Z}(G)$ will be guaranteed modular invariance if the partition functions $\mathcal{Z}(G_{ijk\ldots})$ for all the related abelian twists are modular invariant. From the modular invariance of these abelian twists, we can obtain the vacuum parameters within the corresponding abelian groups. From the group structure of the Hilbert space we may construct the operations of group elements on the whole vacua. However these operations on the vacua do not necessarily form a representation of the group. The conflict between the modular invariance and the representation requirement is referred to global anomaly, which will be discussed in the next section.

Let us briefly summarize the discussion we have presented. For a given nonabelian twist representation of the string boundary condition, one may construct the Hilbert space and one may also compute its partition function from the partition functions of the appropriate abelian twists. Vacuum phases in a nonabelian twist are constrained by modular invariance requirements of the twists by its abelian subgroups. On the other hand, the representation requirement demands that the vacua form a group representation. If a conflict arises between these two requirements, then global anomaly is present. If not, the representation requirement would save to further constrain the discrete torsion parameters.

For an illustration, let us give a concrete example to show how to carry out all these in practice. Similar to the case of the abelian twist, the whole nonabelian twist group here is $G = \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times G'$, where G' is some finite nonabelian subgroup of $SU(3)$. The twist parameters of the first three $\mathbf{Z}_2$'s are the same as those in the case of the abelian twist. Take a nonabelian group G' to be D_3 and assign the D_3 representation

$$D \equiv (D_\lambda; D_\eta, D_\chi) = (3 \times \mathbf{2} + \mathbf{1}' + \mathbf{1}, \mathbf{2} + 5 \times \mathbf{1}' + \mathbf{1}; \mathbf{2} + \mathbf{1}' + \mathbf{1}), \qquad (3.1.29)$$

where $D_\lambda, D_\eta, D_\chi$ are defined in (3.1.1), and the irreducible representations of D_3 are given in (3.1.17) and the paragraph it following.

Consider the $\mathbf{Z}_3$ abelian subgroup of D_3 generated by f, then the twist parameter is

$$\omega_f = (\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, 0, 0, \frac{1}{3}, \frac{2}{3}, 0, 0, 0, 0, 0, 0; \frac{1}{3}, \frac{2}{3}, 0, 0). \qquad (3.1.30)$$

It is easy to show that it satisfies the modular invariance condition (2.1.24) with $F_f = 1$. It also satisfies the condition (2.2.19) of space-time supersymmetry and Table 1 of crystallographic constraints. We now consider the phase that $\mathbf{f}$ generates on the vacuum $|\Omega_f^{NNN}\rangle$ of sector f. From (2.1.26) we have

$$\zeta_f^{NNN}(f) = \zeta_f(0) + \zeta_{ff} + \zeta_{f1} - \omega_f \cdot \lfloor \omega_f + T \rfloor, \qquad (3.1.31)$$

$$\zeta_f(0) = -\omega_f \cdot T - \frac{F_f}{2} = 0, \qquad (3.1.32)$$

$$\zeta_{ff} = \frac{1}{2}(\omega_f^2 + F_f) = \frac{1}{3}. \qquad (3.1.33)$$

From (B.11) in Appendix B, and (2.2.18) one has

$$\zeta_{f1} = \omega_f \cdot T - \zeta_{1f} = \omega_f \cdot T - \frac{F_f}{2} = 0. \tag{3.1.34}$$

Therefore (3.1.31) becomes

$$\zeta_f^{NNN}(f) = \frac{1}{3} + \mathbf{Z}. \tag{3.1.35}$$

For the phase generated by **f** on vacuum $|\Omega_d^{NNN}\rangle$ in sector d, one obtains the following from (2.1.26)

$$\zeta_f^{NNN}(d = f^2) = \zeta_f(0) + 2\zeta_{ff} + \zeta_{f1} - \omega_f \cdot \lfloor 2\omega_f + T \rfloor = \frac{2}{3} + \mathbf{Z}. \tag{3.1.36}$$

Eqs. (3.1.35) and (3.1.36) imply

$$\mathbf{f}\begin{pmatrix} |\Omega_f^{NNN}\rangle \\ |\Omega_d^{NNN}\rangle \end{pmatrix} = \begin{pmatrix} e^{-i2\pi\frac{1}{3}} & 0 \\ 0 & e^{-i2\pi\frac{2}{3}} \end{pmatrix} \begin{pmatrix} |\Omega_f^{NNN}\rangle \\ |\Omega_d^{NNN}\rangle \end{pmatrix}. \tag{3.1.37}$$

Consider now the operation on these two vacuum states by **a**. Since $afa^{-1} = d$, **a** takes the vacuum of the f sector into the vacuum of the d sector. The phase is arbitrary which can be absorbed in the vacuum, so we have

$$\mathbf{a}|\Omega_f^{NNN}\rangle = |\Omega_d^{NNN}\rangle. \tag{3.1.38}$$

Therefore we have

$$\mathbf{a}\begin{pmatrix} |\Omega_f^{NNN}\rangle \\ |\Omega_d^{NNN}\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_f^{NNN}\rangle \\ |\Omega_d^{NNN}\rangle \end{pmatrix}. \tag{3.1.39}$$

Eqs. (3.1.37) and (3.1.39) describe vacua representation of f, d sectors.

Consider the $\mathbf{Z}_2$ abelian subgroup of D_3 generated by a. From the diagonal form of the twist representation with respect to the group element a, the twist parameter is found to be

$$\omega_a = (\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0;\frac{1}{2},0,\frac{1}{2},0). \tag{3.1.40}$$

It satisfies modular invariance conditions (2.1.24) and (2.1.25), as well as space-time supersymmetry and crystallographic constraints. The phase of vacuum $|\Omega_a^{NNN}\rangle$ generated by a is

$$\zeta_a^{NNN}(a) = \zeta_a(0) + \zeta_{aa} + \zeta_{a1} - \omega_a \cdot \lfloor \omega_a + T \rfloor = -\frac{1}{2}(Q_{2a} + Q_{3a}), \tag{3.1.41}$$

where Q_{2i}, Q_{3i} are the discrete torsions. Similarly one may obtain

$$\begin{aligned} \zeta_b^{NNN}(b) &= -\frac{1}{2}(Q_{2b} + Q_{3b}), \\ \zeta_c^{NNN}(c) &= -\frac{1}{2}(Q_{2c} + Q_{3c}). \end{aligned} \tag{3.1.42}$$

From the sub-Hilbert space structure of group class (a,b,c), one has

$$\begin{aligned} \mathrm{a}\begin{pmatrix} |\Omega_a^{NNN}\rangle \\ |\Omega_b^{NNN}\rangle \\ |\Omega_c^{NNN}\rangle \end{pmatrix} &= V^{-1}(a)\begin{pmatrix} |\Omega_a^{NNN}\rangle \\ |\Omega_b^{NNN}\rangle \\ |\Omega_c^{NNN}\rangle \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} |\Omega_a^{NNN}\rangle \\ |\Omega_b^{NNN}\rangle \\ |\Omega_c^{NNN}\rangle \end{pmatrix}, \\ \mathrm{f}\begin{pmatrix} |\Omega_a^{NNN}\rangle \\ |\Omega_b^{NNN}\rangle, \\ |\Omega_c^{NNN}\rangle \end{pmatrix} &= V^{-1}(f)\begin{pmatrix} |\Omega_a^{NNN}\rangle \\ |\Omega_b^{NNN}\rangle \\ |\Omega_c^{NNN}\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} |\Omega_a^{NNN}\rangle \\ |\Omega_b^{NNN}\rangle \\ |\Omega_c^{NNN}\rangle \end{pmatrix}, \end{aligned} \tag{3.1.43}$$

where V stands for the vacuum representation. This turns out to require

$$Q_{2a} + Q_{3a} = Q_{2b} + Q_{3b} + 2\mathbf{Z} = Q_{2c} + Q_{3c} + 2\mathbf{Z}. \tag{3.1.44}$$

Similarly if we consider the vacua $|\Omega_i^{NRN}\rangle$ and $|\Omega_i^{RNN}\rangle$ instead of $|\Omega_i^{NNN}\rangle$ ($i = a, b, c$), then the sub-Hilbert space structure requires

$$
\begin{aligned}
Q_{2a} &= Q_{2b} = Q_{2c}, \\
Q_{3a} &= Q_{3b} = Q_{3c}.
\end{aligned}
\tag{3.1.45}
$$

The different values of Q_{2a}, Q_{3a} are the $\mathbf{Z}_2$'s discrete torsions. Generally the discrete torsions of the sectors within one class should be the same in order to satisfy the requirement of sub-Hilbert space structure. This example shows that the consistency of the Hilbert space can help us to fix many free parameters. The representations of other vacua, such as $|\Omega_i^{RRR}\rangle, |\Omega_i^{NRR}\rangle, |\Omega_i^{NRR}\rangle$ and so on are found with the same procedure. There is no inconsistency raised in this nonabelian twist.

§3.2 Global Anomalies

Global anomalies in nonabelian twists were discovered in the path integral formalism [28]. Cohomological considerations restrict the possible phases gained going around a loop. Contradictions, or global anomalies, will result if these phases are not consistently matched up. This is known to occur if higher loops are taken into consideration. In our approach the global anomalies turn out to be the inconsistency of the vacuum representation, and this additional information enables us to stay within one-loop order to detect these anomalies. In this sense our approach is much simpler than the path-integral cohomology analysis.

In principle, one may calculate the vacuum phases from the modular invariance requirement of the partition function, and see whether these phases fit the *representation requirement.* However it is a huge labor to work out all the vacuum representations since the number of the sectors is 8 multiplying the order of the nonabelian group G'. In order to simplify the calculation, let us reconsider the formula defined in (3.1.23), i.e.

$$\mathbf{h}|\Omega_g\rangle = \varepsilon(h,g)|\Omega_g\rangle \qquad ([h,g]=0), \tag{3.2.1}$$

$$\mathbf{u}|\Omega_g\rangle = \rho(u,g)|\Omega_{ugu^{-1}}\rangle \qquad ([u,g]\neq 0). \tag{3.2.2}$$

Notice that the phases $\varepsilon(h,g)$ can be calculated from the requirement of the modular invariance of all partition functions of the abelian subgroups of the underlying nonabelian group. Also they should satisfy the representation requirement. However the phases $\rho(u,g)$ cannot be calculated and only need to satisfy the group representation.

Consider a group element c. Operating $\mathbf{c}$ on both sides of (3.2.1), one has

$$(\mathbf{chc}^{-1})\mathbf{c}|\Omega_g\rangle = \varepsilon(h,g)\mathbf{c}|\Omega_g\rangle \qquad ([h,g]=0). \tag{3.2.3}$$

Using either (3.2.1) for the case $[c,g]=0$ or (3.2.2) for the case $[c,g]\neq 0$, it leads to

$$(\mathbf{chc}^{-1})|\Omega_{cgc^{-1}}\rangle = \varepsilon(h,g)|\Omega_{cgc^{-1}}\rangle. \tag{3.2.4}$$

Since $[h,g]=0$, one can show $[chc^{-1},cgc^{-1}]=0$. Following the definition (3.2.1), one has

$$(\mathbf{chc}^{-1})|\Omega_{cgc^{-1}}\rangle = \varepsilon(chc^{-1},cgc^{-1})|\Omega_{cgc^{-1}}\rangle. \tag{3.2.5}$$

Comparing (3.2.5) with (3.2.4), one concludes

$$\varepsilon(chc^{-1},cgc^{-1}) = \varepsilon(h,g). \tag{3.2.6}$$

Consider two group elements h_1, h_2 such that $[h_1, g] = [h_2, g] = 0$. Operating $\mathbf{h_1}, \mathbf{h_2}$ successively on the vacuum $|\Omega_g\rangle$, one obtains

$$\mathbf{h_1}\mathbf{h_2}|\Omega_g\rangle = \mathbf{h_1}\varepsilon(h_2, g)|\Omega_g\rangle = \varepsilon(h_1, g)\varepsilon(h_2, g)|\Omega_g\rangle. \tag{3.2.7}$$

It is obvious $[h_1 h_2, g] = 0$, so if thinking $h_1 h_2$ as a single element, then one has

$$\mathbf{h_1}\mathbf{h_2}|\Omega_g\rangle = \varepsilon(h_1 h_2, g)|\Omega_g\rangle. \tag{3.2.8}$$

From (3.2.7) and (3.2.8), one obtains

$$\varepsilon(h_1 h_2, g) = \varepsilon(h_1, g)\varepsilon(h_2, g). \tag{3.2.9}$$

The two conditions (3.2.6) and (3.2.9) are necessary for a theory to be consistent. Since the phases $\varepsilon(h, g)$ can be calculated from the twist representation, these two conditions are not a priori satisfied. The failure for the phases to satisfy the two conditions indicates the presence of global anomalies. Notice that no more conditions can be obtained from the operation of group elements on the both sides of (3.2.1).

One may also consider (3.2.2) to try to derive other conditions. However, it turns out that these conditions are either included in (3.2.6) and (3.2.9), or can be satisfied by choosing appropriate values of $\rho(u, g)$. Actually the representation requirement cannot fix all the phases $\rho(u, g)$, such as the vacuum representations in the D_3 twist considered in the last section. Therefore the representation requirement can be satisfied as long as the two conditions (3.2.6) and (3.2.9) are satisfied. This implies that (3.2.6) and (3.2.9) are equivalent to the representation requirement. In practice, checking these two conditions are much simpler than to construct all the vacuum representations.

The two conditions that we have presented are necessary and sufficient to eliminate global anomalies at least within one-loop in the operator formalism. To show the presence of global anomalies, we need only to find one condition that is violated. However to show the absence of global anomalies, we have to check all the conditions given by (3.2.6) and (3.2.9) being satisfied. If $[h_1, h_2] = 0$ in the (3.2.9), then h_1, h_2, g belong to the same maximal abelian subgroup. Since there is no global anomaly within an abelian group, the condition (3.2.9) for the case $[h_1, h_2] = 0$ is always trivial. Therefore one only needs to check the non-trivial case $[h_1, h_2] \neq 0$ in the (3.2.9).

In the following, we will illustrate this method with two toy examples that was discussed by Freed and Vafa in the path integral formalism [28] to show how these inconsistencies can already be detected in one-loop. Many other cases of global anomalies will be discussed in Chapter 4.

1. Extra-special p-group

This group G' is generated by three elements g, h, c defined by the following relations

$$
\begin{aligned}
&g^p = h^p = c^p = 1, \qquad gh = hg, \\
&cg = gc, \qquad ch = ghc.
\end{aligned}
\tag{3.2.10}
$$

We consider the case $p = 5$ to be specific (but what we shall discuss applies to any prime number $p \geq 5$). This group contains 125 elements and 29 classes. It has six maximal abelian subgroups $G_i \cong \mathbf{Z}_5 \times \mathbf{Z}_5$ $(i = 1, \cdots, 6)$, generated respectively by $\{g, h\}, \{g, c\}, \{g, ch\}, \{g, c^2h\}, \{g, c^3h\}, \{g, c^4h\}$. The intersections of the different G_i's as well as their further intersections are given by the group generated by $\{g\}$ which is isomorphic to $\mathbf{Z}_5$. If we consider this nonabelian twist based on the $E_8 \times E_8'$ heterotic string, then the whole twist group is $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times G'$. Let us define the partition function $\mathcal{Z}_h(G) \equiv \mathcal{Z}(\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times G)$ from now on for any finite group G, where the subscript h stands for choosing the heterotic string initially. From the general formula (3.1.28) for the partition function, one obtains,

$$\mathcal{Z}_h(G') = \frac{1}{5} \sum_{i=1}^{6} \mathcal{Z}_h(\mathbf{Z}_5 \times \mathbf{Z}_5^{(i)}) - \frac{1}{5} \mathcal{Z}_h(\mathbf{Z}_5). \tag{3.2.11}$$

Let us consider a five-dimensional irreducible representation of G' described by

$$g = \begin{pmatrix} \alpha & & & & \\ & \alpha & & & \\ & & \alpha & & \\ & & & \alpha & \\ & & & & \alpha \end{pmatrix}, \quad h = \begin{pmatrix} 1 & & & & \\ & \alpha & & & \\ & & \alpha^2 & & \\ & & & \alpha^3 & \\ & & & & \alpha^4 \end{pmatrix},$$
$$c = \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ 1 & & & & \end{pmatrix}, \tag{3.2.12}$$

where $\alpha = e^{-i2\pi/5}$. Take the twist representation as one copy of this to apply on the five complex fermions $\psi^{(9)}, \cdots, \psi^{(13)}$, while leaving the other fermions untwisted. It is easy to check that all the modular invariant conditions for all the $\mathbf{Z}_5 \times \mathbf{Z}_5$ and $\mathbf{Z}_5$ abelian subgroups of G' are satisfied. In the case when the abelian subgroup involves c, which is non-diagonal, this matrix should first be diagonalized before these conditions are applied. We now consider the abelian subgroup $\mathbf{Z}_5$ generated by g. In this case, $\omega_g = (0^8, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0^3; 0^4)$. The modular invariance condition (2.1.24) demands $F_g = 1$. From (2.1.26) it's easy to obtain the phase that g generates on the vacuum $|\Omega_g^{RRR}\rangle$ in sector g,

$$\zeta_g^{RRR}(g) = \zeta_g(0) + \zeta_{gg} = -\omega_g \cdot T + \frac{1}{2}\omega_g^2 = -\frac{2}{5} + \mathbf{Z}. \tag{3.2.13}$$

Hence

$$\mathbf{g}|\Omega_g^{RRR}\rangle = e^{i2\pi\frac{2}{5}}|\Omega_g^{RRR}\rangle. \tag{3.2.14}$$

We know that g commutes with every element of the group G', and $cgc^{-1} = hg$. From the condition (3.2.6), one has $\varepsilon(h,g) = \varepsilon(chc^{-1}, cgc^{-1}) = \varepsilon(hg, g)$. That means $\mathbf{h}|\Omega_g^{RRR}\rangle = \mathbf{hg}|\Omega_g^{RRR}\rangle$. On the other hand, we get from (3.2.14) that

$$\mathbf{hg}|\Omega_g^{RRR}\rangle = e^{i2\pi\frac{2}{5}}\mathbf{h}|\Omega_g^{RRR}\rangle \neq \mathbf{h}|\Omega_g^{RRR}\rangle. \tag{3.2.15}$$

This contradiction shows up as a global anomaly in the path-integral formalism.

2. Quaternion group (Double dihedral group $D_2^{(d)}$)

This group has eight elements and five classes (e), (a^2) , (a, a^3), $(b, a^2 b)$, $(ab, a^3 b)$, generated by [38]

$$a = \begin{pmatrix} e^{-i2\pi\frac{1}{4}} & 0 \\ 0 & e^{-i2\pi\frac{3}{4}} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & e^{-i2\pi\frac{1}{4}} \\ e^{-i2\pi\frac{1}{4}} & 0 \end{pmatrix}. \tag{3.2.16}$$

It has one two-dimensional irreducible representation $\mathbf{2}$ described by the above matrix, and four one-dimensional irreducible representations described by $\mathbf{1}(a = 1, b = 1), \mathbf{1}'(a = 1, b = -1), \mathbf{1}''(a = -1, b = -1)$, and $\mathbf{1}'''(a = -1, b = 1)$. The partition function of this nonabelian group is computed to be

$$\mathcal{Z}_h(D_2^{(d)}) = \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_4) + \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_4') + \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_4'') - \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_2), \tag{3.2.17}$$

where the abelian group $\mathbf{Z}_4$ is generated by a, $\mathbf{Z}_4'$ by b, $\mathbf{Z}_4''$ by ab, and $\mathbf{Z}_2$ by a^2. Let us take the twist representation as four copies of the two-dimensional representation to apply on the fermions $\psi^{(9)}, \cdots, \psi^{(16)}$. We check that all conditions of abelian twists are satisfied. Consider the $\mathbf{Z}_4$ group $\{a, a^2, a^3, e\}$. The twist vector is $\omega_a = (0^8, \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}; 0^4)$ in (R,R;R) sector. From (2.1.26) one has

$$\begin{aligned} \zeta_a^{RRR}(0) &= -\frac{F_a}{2} + \mathbf{Z}, \\ \zeta_a^{RRR}(a^2) &= \frac{1}{2} + \frac{F_a}{2} + \mathbf{Z}, \end{aligned} \tag{3.2.18}$$

and this means

$$\begin{aligned} \mathbf{a}|\Omega_e^{RRR}\rangle &= e^{i\pi F_a}|\Omega_e^{RRR}\rangle, \\ \mathbf{a}|\Omega_{a^2}^{RRR}\rangle &= e^{-i\pi(1+F_a)}|\Omega_{a^2}^{RRR}\rangle. \end{aligned} \tag{3.2.19}$$

With the same procedure to the $\mathbf{Z}_4'$ and $\mathbf{Z}_4''$ groups, one has

$$\begin{aligned} \mathbf{b}|\Omega_e^{RRR}\rangle &= e^{i\pi F_b}|\Omega_e^{RRR}\rangle, \\ \mathbf{b}|\Omega_{b^2=a^2}^{RRR}\rangle &= e^{-i\pi(1+F_b)}|\Omega_{b^2=a^2}^{RRR}\rangle, \\ \mathbf{ab}|\Omega_e^{RRR}\rangle &= e^{i\pi F_{ab}}|\Omega_e^{RRR}\rangle, \\ \mathbf{ab}|\Omega_{(ab)^2=a^2}^{RRR}\rangle &= e^{-i\pi(1+F_{ab})}|\Omega_{(ab)^2=a^2}^{RRR}\rangle. \end{aligned} \tag{3.2.20}$$

From the condition (3.2.9), one has $\varepsilon(ab, g) = \varepsilon(a, g)\varepsilon(b, g)$ with g being e or a^2. This implies

$$F_a + F_b = F_{ab} + 2\mathbf{Z}, \tag{3.2.21}$$

$$F_a + 1 + F_b + 1 = F_{ab} + 1 + 2\mathbf{Z}. \tag{3.2.22}$$

Obviously eqs. (3.2.21) and (3.2.22) are mutually contradictory. In the path integral formalism, this contradiction again shows up as a global anomaly. On the other hand, if we take the twist representation as eight copies of two-dimensional representation and apply on $\psi^{(1)}, \cdots, \psi^{(16)}$, the above contradiction is removed. It straightforward to check that there is also no inconsistency in other vacua. Therefore global anomaly is absent.

Before leaving this section, let us make some comments on the relation between the present representation requirement and the cohomological conditions given by Freed and Vafa [28] in the path integral formalism. Our conditions of the absence of the global anomalies was derived at the one-loop level in the operator formalism, which could detect the high-loop global anomalies of the path integral formalism. Explicit examples have been shown that the representation requirement and the cohomological conditions are mutually equivalent to detect and eliminate the global anomalies. Obviously our conditions are much simpler and easier to calculate. That is because the operator formalism gives more information than the path integral formalism. Following Seiberg and Witten [44], the whole partition function of spin structures cannot be determined at one-loop level in the path integral formalism, but the factorization and unitarity at two-loop can provide a definite answer to the partition function. As discussed in Chapter 2, the partition function is fully determined in the operator formalism without any two-loop information. Therefore to some extent, the operator formalism at one-loop level is more or less equivalent to the path integral formalism at two-loop. Furthermore, the non-trivial Dehn twists [41] of high-loops are essentially caused by the non-trivial Dehn twist of two-loop, i.e. the generator linking the two handles [45]. That implies the nontrivial features, such as the global anomalies, may fully show up at two-loop level in the path integral formalism. From this argument it is very suggestive that the representation requirement at one-loop in the operator formalism is the sufficient condition for the absence of global anomalies at all-loops.

§3.3 The Massless Spectrum

As we know the relevant spectrum at low energy is given by the string massless states. Suppose a given twist representation satisfies all the constraints that include the absence of global anomalies discussed in the last section. From the structure of the Hilbert space, it is straightforward to find out all the massless string states. What is new here in the nonabelian twist compared to the abelian case is that the different string fields will be mixed with each other after the GSO projection. This complexity often makes it difficult to identify the final gauge group. In this section we will first present the general features of the massless spectrum, then concentrate on the identification of the final gauge group.

The massless states are created by the string creation operators acting on vacua. Similar to the abelian case, massless vector bosons are created by the modes $\psi^{(20)}_{-\frac{1}{2}}, \bar{\psi}^{(20)}_{-\frac{1}{2}}$. Furthermore they are in the identity (e) sector of the nonabelian group G'. All other sectors have no contribution to the vector bosons. Thus, to get massless vector bosons of the nonabelian twist, we simply perform the GSO projection on the states of the $E_8 \times E_8'$ adjoint representation. To do that, we have to know the vacuum phases of the states $|\Omega_e^{NNN}\rangle, |\Omega_e^{RNN}\rangle$ and $|\Omega_e^{NRN}\rangle$ when operated on by any $\mathbf{g} \in \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times G'$. Following the formula (2.1.26) and (2.2.18) derived in Chapter 2, one can evaluate the vacuum phases of the e-sector of G' for each associated abelian twist with space-time supersymmetry ($i > 3$):

$$\zeta_i^{NNN}(e) = \zeta_i(0) + \zeta_{i1} = 0, \tag{3.3.1}$$

$$\zeta_i^{RNN}(e) = \zeta_i(0) + \zeta_{i1} + \zeta_{i2} - 2\omega_i \cdot \omega_2 = -\omega_i \cdot \omega_2 - \zeta_{2i}, \tag{3.3.2}$$

$$\zeta_i^{NRN}(e) = \zeta_i(0) + \zeta_{i1} + \zeta_{i3} - 2\omega_i \cdot \omega_3 = -\omega_i \cdot \omega_3 - \zeta_{3i}, \tag{3.3.3}$$

where ζ_{2i} and ζ_{3i} are determined by (2.2.18). One must get consistent result no matter which abelian twist these vacuum states belong to. We see that the vacuum $|\Omega_e^{NNN}\rangle$ is always invariant under all operations $\mathbf{g} \in G'$. Since the GSO projection involves not only the group element of G', but also the group element of the first three $\mathbf{Z}_2$'s, it is necessary to calculate the phases generated by the $\mathbf{Z}_2$'s elements as well. These phases can be used to determine whether an even number or an odd number of string excitation modes is selected in GSO-invariant states. From (2.1.26) and (2.2.18) in Chapter 2 one can evaluate the phases generated by the non-identity element of the first $\mathbf{Z}_2$ group:

$$\begin{aligned}
\zeta_1^{NNN}(e) &= \zeta_1(0) + \zeta_{11} = \frac{1}{2}, \\
\zeta_1^{RNN}(e) &= \zeta_1(0) + \zeta_{11} + \zeta_{12} - 2\omega_1 \cdot \omega_2 = \frac{1}{2} + \frac{F_2}{2}, \\
\zeta_1^{NRN}(e) &= \zeta_1(0) + \zeta_{11} + \zeta_{13} - 2\omega_1 \cdot \omega_3 = \frac{1}{2} + \frac{F_3}{2}.
\end{aligned} \tag{3.3.4}$$

Recall $\omega_1 = T = ((\frac{1}{2})^{16}; (\frac{1}{2})^{8})$, so there should be an odd number of string modes from fields $\psi^{(1)}, \cdots, \psi^{(20)}$ acting on the vacuum $|\Omega_e^{NNN}\rangle$ in order for a state to be GSO-invariant under the first $\mathbf{Z}_2$ group. The number of string modes acting on the vacuum $|\Omega_e^{RNN}\rangle$, or $|\Omega_e^{NRN}\rangle$ is odd or even depends on the fermionic number F_2 or F_3 chosen to be 0 or 1. Similarly one can obtain the phases generated by the non-identity element of the second $\mathbf{Z}_2$ group:

$$\begin{aligned}
\zeta_2^{NNN}(e) &= \zeta_2(0) + \zeta_{21} = 0, \\
\zeta_2^{RNN}(e) &= \zeta_2(0) + \zeta_{22} + \zeta_{12} - 2\omega_2 \cdot \omega_2 = \frac{F_2}{2}, \\
\zeta_2^{NRN}(e) &= \zeta_2(0) + \zeta_{21} + \zeta_{23} - 2\omega_2 \cdot \omega_3 = 0.
\end{aligned} \tag{3.3.5}$$

We know the twist parameter $\omega_2 = ((\frac{1}{2})^8, 0^8; 0^4)$, so there should be an even number of string modes from the fields $\psi^{(1)}, \cdots, \psi^{(8)}$ acting on the vacua $|\Omega_e^{NNN}\rangle$ and $|\Omega_e^{NRN}\rangle$ in order to get GSO-invariant states under the second $\mathbf{Z}_2$ group. The number of string modes from $\psi^{(1)}, \cdots, \psi^{(8)}$ acting on $|\Omega_e^{RNN}\rangle$ depends on the value of F_2. For the third $\mathbf{Z}_2$ group, one has

$$\begin{aligned}
\zeta_3^{NNN}(e) &= \zeta_3(0) + \zeta_{31} = 0, \\
\zeta_3^{RNN}(e) &= \zeta_3(0) + \zeta_{31} + \zeta_{32} - 2\omega_3 \cdot \omega_2 = 0, \\
\zeta_3^{NRN}(e) &= \zeta_3(0) + \zeta_{31} + \zeta_{33} - 2\omega_3 \cdot \omega_3 = \frac{F_3}{2}.
\end{aligned} \tag{3.3.6}$$

The number of string modes from the fields $\psi^{(9)}, \cdots, \psi^{(16)}$ acting on vacua $|\Omega_e^{NNN}\rangle$ and $|\Omega_e^{RNN}\rangle$ should be even, while the number of those acting on $|\Omega_e^{NRN}\rangle$ depends on the value of F_3. These criteria demand that the physical state cannot be such a state like $\psi_{-\frac{1}{2}}^{(1)}\psi_{-\frac{1}{2}}^{(9)}\psi_{-\frac{1}{2}}^{(20)}|\Omega_e^{NNN}\rangle$ since it is not GSO-invariant under the second and the third $\mathbf{Z}_2$ groups.

Knowing all the vacuum phases generated by the group elements of $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times G'$, in principle we can obtain all the GSO-invariant massless states. The identification of a final gauge group is often the most important and also most difficult issue in the nonabelian twist, since the rank of the gauge group is generically reduced, which is different from the abelian case.

In order to figure out what a final gauge group is, let us first define a complex variable $z = e^{i(t+\sigma)}$, then the mode expansion for the left-moving fermionic fields in (2.1.8) becomes

$$\begin{aligned} \psi^a(z) &= \sum_{m=-\infty}^{+\infty} \psi^a_{m+v^a} z^{-(m+v^a)}, \\ \bar{\psi}^a(z) &= \sum_{m=-\infty}^{+\infty} \bar{\psi}^a_{m-v^a} z^{-(m-v^a)}. \end{aligned} \tag{3.3.7}$$

The modes satisfy anti-commutation relations,

$$\begin{aligned} \{\psi^a_p, \bar{\psi}^b_q\} &= \delta_{p+q,0}\delta_{a,b}, \\ \{\psi^a_p, \psi^b_q\} &= \{\bar{\psi}^a_p, \bar{\psi}^b_q\} = 0. \end{aligned} \tag{3.3.8}$$

One may define generators of $SO(16)$ Lie algebra in terms of Neveu-Schwarz fermions for $a \neq b$ [46],

$$\begin{aligned} H_a &= \oint \frac{dz}{2\pi i z} : \psi^a(z)\bar{\psi}^a(z) : = \sum_{i=0}^{\infty} (\psi^a_{-i-\frac{1}{2}} \bar{\psi}^a_{i+\frac{1}{2}} + \bar{\psi}^a_{-i-\frac{1}{2}} \psi^a_{i+\frac{1}{2}}), \\ E_{e_a+e_b} &= \pi(a,b) \oint \frac{dz}{2\pi i z} : \bar{\psi}^a(z)\bar{\psi}^b(z) :, \\ E_{e_a-e_b} &= \pi(a,b) \oint \frac{dz}{2\pi i z} : \bar{\psi}^a(z)\psi^b(z) :, \\ E_{-e_a-e_b} &= \pi(a,b) \oint \frac{dz}{2\pi i z} : \psi^a(z)\psi^b(z) :, \end{aligned} \tag{3.3.9}$$

where $1 \leq a, b \leq 8$ (or $9 \leq a, b \leq 16$), $\pi(a,b) = +1$ ($a < b$) and $\pi(a,b) = -1$ ($a > b$). It can be shown that these generators satisfy the standard commutation relations of Lie algebra,

$$[H_i, E_\alpha] = \alpha_i E_\alpha,$$

$$[E_\alpha, E_\beta] = \epsilon(\alpha,\beta) E_{\alpha+\beta} \qquad \text{(if } \alpha+\beta \text{ is a root)}, \tag{3.3.10}$$

$$[E_\alpha, E_{-\alpha}] = \alpha \cdot H,$$

where all other commutations are equal to zero. Notice that the cocycle factor $\epsilon(\alpha,\beta)$ can be determined by $\pi(a,b)$.

In the ten-dimensional $E_8 \times E_8'$ heterotic string, the 120 GSO-invariant massless states of Neveu-Schwarz fermions form an adjoint representation of the $SO(16)$ defined in (3.3.9), and the 128 massless states of Ramond fermions form a spinor representation of the $SO(16)$. They together form an adjoint representation of E_8. This also applies to E_8'. In the construction of four-dimensional strings via nonabelian twists, we also have to identify the representation of the states from Neveu-Schwarz fermions first, which generally is an adjoint representation of a subalgebra of $SO(16)$, then consider the states from Ramond fermions. An appropriate linear combination of the $SO(16)$ generators can fulfil the first part of our task. As for the second part, i.e. constructing the remaining generators of the final Lie algebra, one has to consider generators corresponding to the massless states from the $(R,N;N)$ and $(N,R;N)$ fermions. Since there involve the states of the different sectors, generally one needs a picture change operator to relate these states. However, there is a simple way to do this by using the bosonization procedure [39]. In the $E_8 \times E_8'$ heterotic string, there is a one-to-one correspondence between a state in the fermionic formalism and that in the $E_8 \times E_8'$ lattice approach [5]. For example, the state $\psi^{(20)}_{-\frac{1}{2}}|\Omega^{RNN}_e\rangle$ is equivalent to a momentum state

$p = \frac{1}{2}(-e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8)$, and $\psi_0^{(1)}\psi_0^{(6)}\psi_{-\frac{1}{2}}^{(20)}|\Omega_e^{RNN}\rangle$ is equivalent to $p = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 + e_6 - e_7 - e_8)$. The presence of the above two states in turn implies there exist two generators $E_{\frac{1}{2}(-e_1-e_2-e_3-e_4-e_5-e_6-e_7-e_8)}$ and $E_{\frac{1}{2}(e_1-e_2-e_3-e_4-e_5+e_6-e_7-e_8)}$ respectively in the E_8 Lie algebra. Since the massless states of four-dimensional strings via a nonabelian twist in the fermionic formalism are some linear combination of the states of the $E_8 \times E_8'$ heterotic string, the corresponding generators are expected to be a similar kind of linear combination of the E_8 generators. The representation of the massless states from $(R, N; N)$ and $(N, R; N)$ fermions can be identified through the commutation relations of these generators and the generators of the subalgebra of the $SO(16)$. Then the gauge group via the nonabelian twist can be found.

In the following we will present the procedure of the identification of the final gauge group through two concrete examples.

Let us take as the first example the D_3 orbifold, whose twist representation was defined in (3.1.29) of Section 3.1. There exists no global anomaly in this case. Consider the massless state from $(N, N; N)$ fermions of E_8:

$$\psi_{-\frac{1}{2}}^{(1)}\psi_{-\frac{1}{2}}^{(4)}\psi_{-\frac{1}{2}}^{(20)}|\Omega_e^{NNN}\rangle, \tag{3.3.11}$$

where the subscript e for e-sector fermion fields is omitted since one can know it from the vacuum symbol. When the GSO projection is applied, this state becomes

$$(\psi_{-\frac{1}{2}}^{(1)}\psi_{-\frac{1}{2}}^{(4)} + \psi_{-\frac{1}{2}}^{(2)}\psi_{-\frac{1}{2}}^{(3)})\psi_{-\frac{1}{2}}^{(20)}|\Omega_e^{NNN}\rangle. \tag{3.3.12}$$

In order to simplify counting the number of massless states, we only allow $\psi_{-\frac{1}{2}}^{(20)}$

to represent one of the two string modes $\psi^{(20)}_{-\frac{1}{2}}, \bar{\psi}^{(20)}_{-\frac{1}{2}}$ for the degree of freedom of $SO(2)$ Lorentz algebra. We will do so from now on. There are 17 GSO-invariant massless states from $(N,N;N)$ fermions of E_8:

$$\begin{aligned}
&(\psi^I_{-\frac{1}{2}}\psi^{J+1}_{-\frac{1}{2}} + \psi^{I+1}_{-\frac{1}{2}}\psi^J_{-\frac{1}{2}})\psi^{(20)}_{-\frac{1}{2}}|\Omega^{NNN}_e\rangle \qquad (I < J \in \{1,3,5\}),\\
&(\bar{\psi}^I_{-\frac{1}{2}}\bar{\psi}^{J+1}_{-\frac{1}{2}} + \bar{\psi}^{I+1}_{-\frac{1}{2}}\bar{\psi}^J_{-\frac{1}{2}})\psi^{(20)}_{-\frac{1}{2}}|\Omega^{NNN}_e\rangle \qquad (I < J \in \{1,3,5\}),\\
&(\psi^I_{-\frac{1}{2}}\bar{\psi}^J_{-\frac{1}{2}} + \psi^{I+1}_{-\frac{1}{2}}\bar{\psi}^{J+1}_{-\frac{1}{2}})\psi^{(20)}_{-\frac{1}{2}}|\Omega^{NNN}_e\rangle \qquad (I, J \in \{1,3,5\}),\\
&\psi^{(7)}_{-\frac{1}{2}}\bar{\psi}^{(7)}_{-\frac{1}{2}}\psi^{(20)}_{-\frac{1}{2}}|\Omega^{NNN}_e\rangle, \quad \psi^{(8)}_{-\frac{1}{2}}\bar{\psi}^{(8)}_{-\frac{1}{2}}\psi^{(20)}_{-\frac{1}{2}}|\Omega^{NNN}_e\rangle.
\end{aligned} \tag{3.3.13}$$

For the above massless states, one may define

$$\begin{aligned}
&\mathcal{H}_i = H_{2i-1} + H_{2i} \ \ (i = 1,2,3), \ \mathcal{H}_4 = 2(H_7 + H_8), \ \mathcal{H}_5 = 2(H_7 - H_8),\\
&\mathcal{E}_{b_1-b_2} = E_{e_1-e_3} + E_{e_2-e_4}, \ \mathcal{E}_{b_2-b_3} = E_{e_3-e_5} + E_{e_4-e_6}, \ \mathcal{E}_{b_2+b_3} = E_{e_3+e_6} + E_{e_4+e_5}.
\end{aligned} \tag{3.3.14}$$

Then these generators generate a Lie algebra $SO(6) \times U_{\mathcal{H}_4}(1) \times U_{\mathcal{H}_5}(1)$. The first 15 GSO-invariant states in (3.3.13) belong to an adjoint representation of this $SO(6)$, and the remaining 2 GSO-invariant states belong to an adjoint representation of this $U_{\mathcal{H}_4}(1) \times U_{\mathcal{H}_5}(1)$.

Consider massless states from the $(R,N;N)$ fermions of E_8. We know that there involve discrete torsions in calculating vacuum phases. The choice $F_2 = 0$ corresponds to an even number of E_8 string modes in the GSO-invariant states, while $F_2 = 1$ corresponds to an odd number of E_8 string modes in the GSO-invariant states. We know in abelian twist that the difference between $F_2 = 0$ and $F_2 = 1$ is the matter of choosing the spinor representation or the conjugate spinor representation [5] as the GSO-invariant states, so the physical consequence is the same. In the current example of the nonabelian twist, the different choice of discrete torsions makes no difference to the final gauge group. We will see in next example that this statement is no longer correct. In order for our discussion to be specific, one may set $F_2 = Q_{2a} = 0$ and compute the phases generated by $\mathbf{f}$ and $\mathbf{a}$ to be

$$\begin{aligned} \zeta_f^{RNN}(e) &= -\,\omega_f \cdot \omega_2 - \zeta_{2f} = 0, \\ \zeta_a^{RNN}(e) &= -\,\omega_a \cdot \omega_2 - \zeta_{2a} = 0. \end{aligned} \tag{3.3.15}$$

This implies

$$\begin{aligned} \mathbf{f}|\Omega_e^{RNN}\rangle &= |\Omega_e^{RNN}\rangle, \\ \mathbf{a}|\Omega_e^{RNN}\rangle &= |\Omega_e^{RNN}\rangle. \end{aligned} \tag{3.3.16}$$

There are 22 GSO-invariant massless states from $(R,N;N)$ fermions of E_8,

$$\begin{aligned}
&\psi_{-\frac{1}{2}}^{(20)}|\Omega_e^{RNN}\rangle, \quad \psi_0^{(1)}\psi_0^{(2)}\psi_0^{(3)}\psi_0^{(4)}\psi_0^{(5)}\psi_0^{(6)}\psi_0^{(7)}\psi_0^{(8)}\psi_{-\frac{1}{2}}^{(20)}|\Omega_e^{RNN}\rangle, \\
&(\psi_0^I\psi_0^{J+1} + \psi_0^{I+1}\psi_0^J)\psi_{-\frac{1}{2}}^{(20)}|\Omega_e^{RNN}\rangle \quad (I<J \in \{1,3,5\}), \\
&(\psi_0^I\psi_0^{J+1} - \psi_0^{I+1}\psi_0^J)\psi_0^{(7)}\psi_0^{(8)}\psi_{-\frac{1}{2}}^{(20)}|\Omega_e^{RNN}\rangle \quad (I<J \in \{1,3,5\}), \\
&\psi_0^I\psi_0^{I+1}\psi_0^J\psi_0^{J+1}\psi_{-\frac{1}{2}}^{(20)}|\Omega_e^{RNN}\rangle \quad (I<J \in \{1,3,5\}), \\
&\psi_0^I\psi_0^{I+1}\psi_0^{(7)}\psi_0^{(8)}\psi_{-\frac{1}{2}}^{(20)}|\Omega_e^{RNN}\rangle \quad (I=1,3,5), \\
&(\psi_0^I\psi_0^{J+1} - \psi_0^{I+1}\psi_0^J)\psi_0^K\psi_0^{K+1}\psi_{-\frac{1}{2}}^{(20)}|\Omega_e^{RNN}\rangle \quad (I<J \neq K \neq I \in \{1,3,5\}), \\
&(\psi_0^I\psi_0^{J+1} + \psi_0^{I+1}\psi_0^J)\psi_0^K\psi_0^{K+1}\psi_0^{(7)}\psi_0^{(8)}\psi_{-\frac{1}{2}}^{(20)}|\Omega_e^{RNN}\rangle \ (I<J \neq K \neq I \in \{1,3,5\}), \\
&(\psi_0^{(1)}\psi_0^{(3)}\psi_0^{(5)} + \psi_0^{(2)}\psi_0^{(4)}\psi_0^{(6)})\psi_0^{(8)}\psi_{-\frac{1}{2}}^{(20)}|\Omega_e^{RNN}\rangle, \\
&(\psi_0^{(1)}\psi_0^{(3)}\psi_0^{(5)} - \psi_0^{(2)}\psi_0^{(4)}\psi_0^{(6)})\psi_0^{(7)}\psi_{-\frac{1}{2}}^{(20)}|\Omega_e^{RNN}\rangle.
\end{aligned} \tag{3.3.17}$$

By using the bosonization procedure, one may find the corresponding momentum states to the above states then one may write down the generators of the Lie algebra in terms of the E_8 generators. From the commutation relations, it is straightforward to identify the representation of these states. It turns out that the first 20 of these states form a $\overline{10}(-2)+10(2)$ representation of $SO(6)\times U_{\mathcal{H}_4}(1)$, and the other 2 together with $U_{\mathcal{H}_8}$ form $SO(3)$. They assemble to form an adjoint representation [47] of $SP(8)\times SO(3)$.

The same procedure can be applied to E_8'. Considering all GSO-invariant massless states from E_8' fermions acting on vacua $|\Omega_e^{NNN}\rangle$ and $|\Omega_e^{NRN}\rangle$, one can see that there are 47 states with vacuum $|\Omega_e^{NNN}\rangle$ forming an adjoint representation of $SO(10)'\times U^2(1)'$, and there are 32 states with vacuum $|\Omega_e^{NRN}\rangle$ forming $\mathbf{16}(3)+\overline{\mathbf{16}}(-3)$ of $SO(10)'\times U(1)'$. Therefore they assemble to form an adjoint representation of $E_6'\times U(1)'$. The discrete torsions also have no physical consequence. In summary the original gauge group $E_8\times E_8'$ in the heterotic string is now broken down to the gauge group $SP(8)\times SO(3)\times E_6'\times U(1)'$. It is interesting to notice that the rank of the gauge group has been reduced by 4. The reason for this is because there are 4 two-dimensional irreducible representations in our twist representation D_λ. It is usually true that each n-dimensional irreducible representation reduces the rank of the gauge group by $n-1$, except in some special cases with rank enhancement, which will be discussed in the next example.

Knowing the gauge group, other massless spectra can be easily found. Gaugino particles will be the states in (3.3.13) under the substitutions $\psi_{-\frac{1}{2}}^{(20)}\to\psi_0^{(20)}$ or $\psi_0^{(17)}\psi_0^{(18)}\psi_0^{(19)}$, $|\Omega_e^{NNN}\rangle\to|\Omega_e^{NNR}\rangle$, and states in (3.3.17) under $\psi_{-\frac{1}{2}}^{(20)}\to\psi_0^{(20)}$ or $\psi_0^{(17)}\psi_0^{(18)}\psi_0^{(19)}$, $|\Omega_e^{RNN}\rangle\to|\Omega_e^{RNR}\rangle$. The two choices $\psi_0^{(20)}$ and $\psi_0^{(17)}\psi_0^{(18)}\psi_0^{(19)}$

reflect the spinor representation of the Lorentz algebra $SO(2)$. The graviton is represented by the states $\psi^{(20)}_{-\frac{1}{2}}(X^{(0)}_L)_{-1}|\Omega^{NNN}_e\rangle$, $\bar{\psi}^{(20)}_{-\frac{1}{2}}(X^{(0)}_L)_{-1}|\Omega^{NNN}_e\rangle$, $\psi^{(20)}_{-\frac{1}{2}}(\bar{X}^{(0)}_L)_{-1}|\Omega^{NNN}_e\rangle$, $\bar{\psi}^{(20)}_{-\frac{1}{2}}(\bar{X}^{(0)}_L)_{-1}|\Omega^{NNN}_e\rangle$, where $(X^{(0)}_L)_{-1}$, $(\bar{X}^{(0)}_L)_{-1}$ are the excitation modes of the left-moving complex boson $X^{(0)}_L$ for the uncompactified space-time coordinates. The partner gravitino is given by the graviton states under the same substitution as that for the gaugino particles. Chiral fermions and their partners could be found in the similar way. Unlike those of gauge bosons, the states of chiral fermions may exist in the identity (e) sector, and in the twist sectors as well. The number of generations of chiral fermions depends on the number of fixed points of the nonabelian orbifold, which in turn depends on an initial lattice being chosen. Notice that since our twist representation is not the "standard embedding", we cannot simply use Lefschetz theorem to calculate the number of generations. From the above discussion we know that the $N = 1$ space-time supersymmetry is manifest.

The next example is still D_3 orbifold but with a slight change of the twist representation for the left-moving fermion fields. We assign the following representation,

$$D = (4 \times \mathbf{2}, 6 \times \mathbf{1'} + 2 \times \mathbf{1}; \mathbf{2} + \mathbf{1'} + \mathbf{1}). \tag{3.3.18}$$

Compared it with (3.1.29), we have just interchanged $\mathbf{1'} + \mathbf{1}$ in E_8 with $\mathbf{2}$ in E_8'. Again it satisfies all constraints including the absence of global anomalies. The twist representation for E_8' is just the abelian representation. It's very easy to write down all the GSO-invariant massless states from E_8' fermions. They form an adjoint representation of $E_7' \times SU(2)'$. Let us focus on the identification of the broken gauge group from E_8. There are 28 GSO-invariant massless states from $(N,N;N)$ fermions of E_8,

$$
\begin{aligned}
&(\psi^I_{-\frac{1}{2}}\psi^{J+1}_{-\frac{1}{2}} + \psi^{I+1}_{-\frac{1}{2}}\psi^{J}_{-\frac{1}{2}})\psi^{(20)}_{-\frac{1}{2}}|\Omega^{NNN}_e\rangle \qquad (I < J \in \{1,3,5,7\}),\\
&(\bar\psi^I_{-\frac{1}{2}}\bar\psi^{J+1}_{-\frac{1}{2}} + \bar\psi^{I+1}_{-\frac{1}{2}}\bar\psi^{J}_{-\frac{1}{2}})\psi^{(20)}_{-\frac{1}{2}}|\Omega^{NNN}_e\rangle \qquad (I < J \in \{1,3,5,7\}),\\
&(\psi^I_{-\frac{1}{2}}\bar\psi^{J}_{-\frac{1}{2}} + \psi^{I+1}_{-\frac{1}{2}}\bar\psi^{J+1}_{-\frac{1}{2}})\psi^{(20)}_{-\frac{1}{2}}|\Omega^{NNN}_e\rangle \qquad (I, J \in \{1,3,5,7\}).
\end{aligned}
\tag{3.3.19}
$$

Similar to the first example one may define

$$
\begin{aligned}
&\mathcal{H}_i = H_{2i-1} + H_{2i} \quad (i = 1,2,3,4),\\
&\mathcal{E}_{b_1-b_2} = E_{e_1-e_3} + E_{e_2-e_4}, \quad \mathcal{E}_{b_2-b_3} = E_{e_3-e_5} + E_{e_4-e_6},\\
&\mathcal{E}_{b_3-b_4} = E_{e_5-e_7} + E_{e_6-e_8}, \quad \mathcal{E}_{b_3+b_4} = E_{e_5+e_8} + E_{e_6+e_7}.
\end{aligned}
\tag{3.3.20}
$$

These generators generates $SO(8)$ Lie algebra. The 28 states in (3.3.19) belong to an adjoint representation of this $SO(8)$.

As we mentioned before, the GSO-invariant states from $(R,N;N)$ fermions of E_8 depend on the discrete torsions being chosen. Let us set $F_2 = 1, Q_{2a} = 0$. The phases generated by **f** and **a** are

$$
\begin{aligned}
\zeta^{RNN}_f(e) &= -\,\omega_f \cdot \omega_2 - \zeta_{2f} = 0,\\
\zeta^{RNN}_a(e) &= -\,\omega_a \cdot \omega_2 - \zeta_{2a} = 0.
\end{aligned}
\tag{3.3.21}
$$

The choice $F_2 = 1$ implies that there must be an odd number of string modes from the fields $\psi^{(1)}, \cdots, \psi^{(8)}$ in the GSO-invariant states. There are 8 those states,

$$
\begin{aligned}
&(\psi^I_0\psi^J_0\psi^K_0 + \psi^{I+1}_0\psi^{J+1}_0\psi^{K+1}_0)\psi^{(20)}_{-\frac{1}{2}}|\Omega^{RNN}_e\rangle \; (I < J < K \in \{1,3,5,7\}),\\
&(\psi^I_0\psi^J_0\psi^K_0 - \psi^{I+1}_0\psi^{J+1}_0\psi^{K+1}_0)\psi^L_0\psi^{L+1}_0\psi^{(20)}_{-\frac{1}{2}}|\Omega^{RNN}_e\rangle \; (I,J,K,L \in \{1,3,5,7\}),
\end{aligned}
\tag{3.3.22}
$$

where no pair of the labels I, J, K, L is the same in $(I, J, K, L \in \{1,3,5,7\})$ and $I < J < K$. These states form a representation 8 of $SO(8)$. Therefore the broken gauge group [47] from E_8 is $SO(9)$. Including the part from E_8', the final gauge group is $SO(9) \times E_7' \times SU(2)'$.

If we choose Q_{2a} equal to 1 instead of equal to 0, the gauge group broken from E_8 is still $SO(9)$, so the different choice of Q_{2a} does not affect the gauge particles. It is interesting to know that the rank of the gauge group is 4. The reason for this is because there are 4 two-dimensional irreducible representations in the twist representation and each one reduces the rank of the gauge group by 1. However this is no longer held if we set $F_2 = 0$. Again there is no physical difference for the different choice of Q_{2a}. One may set $Q_{2a} = 0$ for the vacuum phases to be specific. There are 35 GSO-invariant massless states from $(R, N; N)$ fermions of E_8,

$$\begin{aligned}
&\psi^{(20)}_{-\frac{1}{2}}|\Omega^{RNN}_e\rangle, \quad \psi^{(1)}_0\psi^{(2)}_0\psi^{(3)}_0\psi^{(4)}_0\psi^{(5)}_0\psi^{(6)}_0\psi^{(7)}_0\psi^{(8)}_0\psi^{(20)}_{-\frac{1}{2}}|\Omega^{RNN}_e\rangle,\\
&\psi^I_0\psi^{I+1}_0\psi^J_0\psi^{J+1}_0\psi^{(20)}_{-\frac{1}{2}}|\Omega^{RNN}_e\rangle \quad (I < J \in \{1,3,5,7\}),\\
&(\psi^I_0\psi^{J+1}_0 + \psi^{I+1}_0\psi^J_0)\psi^{(20)}_{-\frac{1}{2}}|\Omega^{RNN}_e\rangle \quad (I < J \in \{1,3,5,7\}),\\
&(\psi^I_0\psi^{J+1}_0 - \psi^{I+1}_0\psi^J_0)\psi^K_0\psi^{K+1}_0\psi^{(20)}_{-\frac{1}{2}}|\Omega^{RNN}_e\rangle \quad (I < J \neq K \neq I \in \{1,3,5,7\}),\\
&(\psi^I_0\psi^{J+1}_0 + \psi^{I+1}_0\psi^J_0)\psi^K_0\psi^{K+1}_0\psi^L_0\psi^{L+1}_0\psi^{(20)}_{-\frac{1}{2}}|\Omega^{RNN}_e\rangle \ (I, J, K, L \in \{1,3,5,7\}),\\
&(\psi^{(1)}_0\psi^{(4)}_0\psi^{(5)}_0\psi^{(8)}_0 + \psi^{(2)}_0\psi^{(3)}_0\psi^{(6)}_0\psi^{(7)}_0)\psi^{(20)}_{-\frac{1}{2}}|\Omega^{RNN}_e\rangle,\\
&(\psi^{(1)}_0\psi^{(4)}_0\psi^{(6)}_0\psi^{(7)}_0 + \psi^{(2)}_0\psi^{(3)}_0\psi^{(5)}_0\psi^{(8)}_0)\psi^{(20)}_{-\frac{1}{2}}|\Omega^{RNN}_e\rangle,\\
&(\psi^{(1)}_0\psi^{(3)}_0\psi^{(6)}_0\psi^{(8)}_0 + \psi^{(2)}_0\psi^{(4)}_0\psi^{(5)}_0\psi^{(7)}_0)\psi^{(20)}_{-\frac{1}{2}}|\Omega^{RNN}_e\rangle,
\end{aligned} \tag{3.3.23}$$

where no pair of the labels I, J, K, L is the same in $(I, J, K, L \in \{1,3,5,7\})$ and

$I < J, K < L$. These 35 states form a $\mathbf{35}_v$ representation of $SO(8)$. Together with the adjoint representation $\mathbf{28}$ of $SO(8)$, they form an adjoint representation of $SU(8)$.

Notice that the states from $(R, N; N)$ fermions of E_8 enhance the rank of the gauge group by 3. That can be verified by the following observation. From the bosonization procedure, the state $\psi_0^{(1)}\psi_0^{(4)}\psi_0^{(5)}\psi_0^{(8)}\psi_{-\frac{1}{2}}^{(20)}|\Omega_e^{RNN}\rangle$ is equivalent to the momentum state $p = \frac{1}{2}(e_1 - e_2 - e_3 + e_4 + e_5 - e_6 - e_7 + e_8)$ in the E_8 lattice [5], which in turn implies there exists a generator $E_{\frac{1}{2}(e_1-e_2-e_3+e_4+e_5-e_6-e_7+e_8)}$ in the Lie algebra. The existence of the last three states in (3.3.23) implies that there are three generators in the final Lie algebra,

$$\mathcal{H}_5 = E_\alpha + E_{-\alpha},\ \mathcal{H}_6 = E_\beta + E_{-\beta},\ \mathcal{H}_7 = E_\gamma + E_{-\gamma},$$

with

$$\begin{aligned}\alpha &= \frac{1}{2}(e_1 - e_2 - e_3 + e_4 + e_5 - e_6 - e_7 + e_8),\\ \beta &= \frac{1}{2}(e_1 - e_2 - e_3 + e_4 - e_5 + e_6 + e_7 - e_8),\\ \gamma &= \frac{1}{2}(e_1 - e_2 + e_3 - e_4 - e_5 + e_6 - e_7 + e_8).\end{aligned} \tag{3.3.24}$$

It's easy to show that $\mathcal{H}_5, \mathcal{H}_6, \mathcal{H}_7$ mutually commute and also commute with $\mathcal{H}_i$ ($i = 1, \cdots, 4$) defined in (3.3.20). Obviously $\mathcal{H}_i$ ($i = 1, \cdots, 7$) are linear independent. Also the other 32 generators corresponding to the first 32 states in (3.3.23) do not commute with $\mathcal{H}_i$ ($i = 1, \cdots, 4$). Therefore the rank of the Lie algebra is 7, which is exactly the rank of $SU(8)$. In practice we use the above argument to figure out the rank of a Lie algebra for a complicated case, which often helps us to identify a specific Lie algebra. In this case the final gauge group broken from $E_8 \times E_8'$ is

$SU(8) \times E_7' \times SU(2)'$. It is clear that we have to specify the twist representation and the discrete torsions for a particular nonabelian twist model. The torsion parameters play a non-trivial role in our construction of minimal-rank models.

Chapter 4

Minimal-Rank Models from Twists of Finite Groups

The construction of four-dimensional strings from an abelian twist suffers from having too large a gauge group. This in principle can be cured by nonabelian twists because generically they can reduce the rank of the gauge group, which is desirable phenomenologically. As we know, there is a large difference between the rank of the gauge groups in the Standard Model and the original ten-dimensional heterotic string, and even more so when compactified to four space-time dimensions. In order for a theory to be as realistic as possible, one would like to choose a twist representation such that the resulting gauge group has as small a rank as possible. This kind of string theories will be referred to as *minimal-rank models*.

With the constraints for the nonabelian twist established in the previous chapters, we have all the tools necessary to construct minimal-rank models from twists of finite nonabelian groups. Given a finite group, we desire to choose a twist representation such that it satisfies all the constraints and it also gives a minimal-rank gauge group. Specifically we need to perform the following. Since we start with the $E_8 \times E_8'$ heterotic string in the fermionic formalism, the whole twist group is $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times G'$, where G' is a finite nonabelian group. From the discussion in Section 3.1, we know that the twist parameters of the first three $\mathbf{Z}_2$'s are the same as in the abelian case and the whole twist representation is given by (3.1.1). Let us first consider the twist representation for right movers. One can always choose $D_\chi(g) = +1$ which results in $D_\eta(g) = D_X(g)$ for $g \in G'$. In order to have a qualified nonabelian orbifold with space-time supersymmetry as well, one has to ensure all abelian orbifolds corresponding to the abelian subgroups of the underlying nonabelian group to be in Table 1. This turns out to give a very tight constraint to limit possible twist representations of $D_X(g)$. Having chosen a representation for the right movers, one may then find a twist representation $D_\lambda(g)$ ($g \in G'$) for the left fermions by requiring modular invariance of the partition function for the nonabelian twist. The rank of the final gauge group can be estimated by the argument given in Section 3.3, that each n-dimensional irreducible representation may reduce the rank by $n-1$ assuming that there is no rank enhancement. By using this estimate, all twist representations that result in a minimal-rank gauge group may be selected. Remember now that global anomalies can be present even when modular invariance is satisfied. One should then eliminate the global anomalies case by case for the twist representations under consideration. Finally, for those consistent minimal-rank models, we can find all the massless GSO-invariant states that correspond to gauge particles, and thereby identify the final gauge group. Notice that since there are many more constraints in a nonabelian twist than an abelian one, the number of nonabelian twist models is much less than that of abelian models. This in turn makes the classification of nonabelian twist models much more feasible.

As we discussed in Chapter 2, the nonabelian twist group G' should be a finite subgroup of $SU(3)$ in order to maintain space-time supersymmetry. A simple class of twist groups of this type is the finite nonabelian subgroups of $SU(2)$. We shall confine ourselves to this simple class. In this chapter we shall classify *all* minimal-rank models of nonabelian twists by ***finite nonabelian subgroups of*** $SU(2)$. These finite nonabelian groups include the dihedral groups D_l ($l = 3, 4, 6$), the tetrahedral group T, the octahedral group O, the icosahedral group I, and their double groups $D_l^{(d)}, T^{(d)}, O^{(d)}, I^{(d)}$ [38], each of which will be discussed in a separate section. The twist representations and the final gauge groups will be listed in the various Tables. In each representation, the $\mathbf{Z}_2$ torsions are sometimes left free (F_2, F_3, Q_{2i} and Q_{3i} can be equal to 0 or 1), in that case we shall list all models of the same representation with different choices of torsions in one block. The first line of each block always refers to a regular model in which $F_2 = F_3 = Q_{2i} = Q_{3i} = 0$, unless otherwise specified. Since we are interested in obtaining different gauge groups, we will not list those torsion models in which different $\mathbf{Z}_2$ torsions do not lead to a change in the final gauge group in our tables. We shall also ignore the possible contributions to the final gauge group from the excitations of X_L, which depends on the detail of an initial lattice being chosen for these compactified spatial coordinates.

§4.1 Nonabelian Twists of the Dihedral Group D_3

The group D_3 is the simplest finite nonabelian group. Its irreducible representations are given in (3.1.17) and the paragraph it following. The partition function of this nonabelian twist is similar to (3.1.27) except that we start now from the heterotic string. It is

$$\mathcal{Z}_h(D_3) = \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_3) + \mathcal{Z}_h(\mathbf{Z}_2) - \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_1). \tag{4.1.1}$$

This contains $\mathbf{Z}_3$ and $\mathbf{Z}_2$ abelian twists constructed from the $\mathbf{Z}_3$ and $\mathbf{Z}_2$ abelian subgroups of D_3 respectively. The representation $D_\eta = D_X$ should be $\mathbf{2}+\mathbf{1}'$, which is unique in the D_3 orbifold. In this case the $\mathbf{Z}_3$ orbifold belongs to Number 2 of Table 1 with twist parameter $(\frac{1}{3}, \frac{2}{3}, 0)$, while the $\mathbf{Z}_2$ orbifold belongs to Number 1 of Table 1 with twist parameter $(\frac{1}{2}, 0, \frac{1}{2})$. When we consider the $\mathbf{Z}_2$ abelian orbifold, we should first diagonalize the matrix of the two-dimensional irreducible representation, then we would proceed to obtain the twist parameters.

Let us assign the following twist representation for the sixteen left-fermions,

$$D_\lambda = (m \times \mathbf{2} + n_0 \times \mathbf{1} + n_1 \times \mathbf{1}'), \tag{4.1.2}$$

where the non-negative numbers m, n_i satisfy $2m+n_0+n_1 = 16$. Modular invariance of the corresponding $\mathbf{Z}_3$ abelian twist demands

$$\frac{5}{3}(m-1) \in \mathbf{Z}. \tag{4.1.3}$$

Modular invariance of the $\mathbf{Z}_2$ abelian twist requires

$$\frac{1}{2}(m+n_1-2) \in 2\mathbf{Z}. \tag{4.1.4}$$

Since each two-dimensional irreducible representation may reduce the rank of the gauge group by 1, for maximal rank reduction we choose the solutions to (4.1.3) and (4.1.4) with the largest value of m. It is easy to obtain the minimal-rank twist representations in this nonabelian twist. They are

$$\begin{aligned} &(4\times\mathbf{2}+6\times\mathbf{1}'+2\times\mathbf{1}),\\ &(4\times\mathbf{2}+6\times\mathbf{1}+2\times\mathbf{1}'). \end{aligned} \tag{4.1.5}$$

The above two representations are equivalent because $\mathbf{1}\leftrightarrow\mathbf{1}'$ corresponds to the interchange of NS (Neveu-Schwarz) fermions with R (Ramond) fermions in the a,b,c sectors. Since the twist is on $E_8\times E_8'$, we have to divide the representation into two pieces while maintaining modular invariance. There are three non-equivalent divisions,

$$\begin{aligned} &(3\times\mathbf{2}+\mathbf{1}'+\mathbf{1},\mathbf{2}+5\times\mathbf{1}'+\mathbf{1}),\\ &(2\times\mathbf{2}+2\times\mathbf{1}'+2\times\mathbf{1},2\times\mathbf{2}+4\times\mathbf{1}'),\\ &(4\times\mathbf{2},6\times\mathbf{1}'+2\times\mathbf{1}), \end{aligned} \tag{4.1.6}$$

which may result in minimal-rank models. The vacuum phases $|\Omega_g^{NNN}\rangle$ ($g\in D_3$) of the first representation of (4.1.6) were calculated in Section 3.1. With similar computation one can obtain the vacuum phases for the second and the third representations as well, and no global anomalies are present in all three cases. The final gauge groups for the first and the third representations were obtained in Section 3.3. Notice that the third representation with all discrete torsions vanishing (the regular model) enhances the rank of the final gauge group. Therefore it does not correspond to a minimal-rank model. In this nonabelian twists there are three minimal-rank models, which are listed in Table 3.

§4.2 Nonabelian Twists of the Dihedral Group D_4

The nonabelian group D_4 has eight elements, five classes (e), (θ^2), (θ, θ^3), $(r, r\theta^2)$, $(r\theta, \theta r)$, generated by

$$\theta = \begin{pmatrix} e^{-i2\pi(1/4)} & 0 \\ 0 & e^{-i2\pi(3/4)} \end{pmatrix}, \quad r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.2.1}$$

It has one two-dimensional irreducible representation $\mathbf{2}$ described by the above defining matrix, and four one-dimensional irreducible representations described by $\mathbf{1}(\theta = 1, r = 1), \mathbf{1}'(\theta = 1, r = -1), \mathbf{1}''(\theta = -1, r = -1), \mathbf{1}'''(\theta = -1, r = 1)$. The partition function of this nonabelian twist is found to be

$$\mathcal{Z}_h(D_4) = \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_4) + \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_2 \times \mathbf{Z}_2) + \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_2' \times \mathbf{Z}_2') - \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_2) \tag{4.2.2}$$

where the abelian group $\mathbf{Z}_4$ is generated by θ, $\mathbf{Z}_2 \times \mathbf{Z}_2$ by $r, r\theta^2$, $\mathbf{Z}_2' \times \mathbf{Z}_2'$ by $r\theta, \theta r$, and $\mathbf{Z}_2$ by θ^2.

The representation $D_\eta = D_X$ is unique. It is $\mathbf{2} + \mathbf{1}'$. In this nonabelian twist there are the following two equivalences. The representation under $\mathbf{1} \leftrightarrow \mathbf{1}''', \mathbf{1}' \leftrightarrow \mathbf{1}''$ corresponds to the interchange of NS fermions with R fermions in $\theta, \theta^3, r\theta, \theta r$ sectors. The representation under $\mathbf{1} \leftrightarrow \mathbf{1}', \mathbf{1}'' \leftrightarrow \mathbf{1}'''$ corresponds to the interchange of NS fermions with R fermions in $r, r\theta^2, r\theta, \theta r$ sectors.

We assign the twist representation of the sixteen left-fermions as follows,

$$D_\lambda = (m \times \mathbf{2} + n_0 \times \mathbf{1} + n_1 \times \mathbf{1}' + n_2 \times \mathbf{1}'' + n_3 \times \mathbf{1}'''), \tag{4.2.3}$$

where the non-negative integers satisfy $2m + n_0 + n_1 + n_2 + n_3 = 16$. Modular invariance of a $\mathbf{Z}_4$ abelian twist leads to

$$\frac{5}{2}(m-1)+n_2+n_3 \in 2\mathbf{Z}. \tag{4.2.4}$$

Modular invariance of a $\mathbf{Z}_2 \times \mathbf{Z}_2$ abelian twist demands

$$\begin{aligned} &\frac{1}{2}(m+n_1+n_2-2) \in 2\mathbf{Z}, \\ &n_1+n_2-1 \in 2\mathbf{Z}. \end{aligned} \tag{4.2.5}$$

Finally, modular invariance of a $\mathbf{Z}_2' \times \mathbf{Z}_2'$ abelian twist requires

$$\begin{aligned} &\frac{1}{2}(m+n_1+n_3-2) \in 2\mathbf{Z}, \\ &n_1+n_3-1 \in 2\mathbf{Z}. \end{aligned} \tag{4.2.6}$$

The minimal-rank models correspond to the maximal value of m. We may easily obtain m to be 5. Knowing that, equations (4.2.4)—(4.2.6) are equivalent to the following,

$$\begin{aligned} &n_1+n_2-1 \in 4\mathbf{Z}, \\ &n_1+n_3-1 \in 4\mathbf{Z}. \end{aligned} \tag{4.2.7}$$

Solving these equations, we get two twist representations,

$$\begin{aligned} &(5 \times \mathbf{2} + \mathbf{1}''' + \mathbf{1}'' + 4 \times \mathbf{1}), \\ &(5 \times \mathbf{2} + 5 \times \mathbf{1}' + \mathbf{1}), \end{aligned} \tag{4.2.8}$$

where the two equivalences of the twist representation mentioned above have been taken into account. From that, the possible representations acted on (E_8, E_8') are

$$\begin{aligned} &(3 \times \mathbf{2} + \mathbf{1}''' + \mathbf{1}'', 2 \times \mathbf{2} + 4 \times \mathbf{1}), \\ &(4 \times \mathbf{2}, \mathbf{2} + \mathbf{1}''' + \mathbf{1}'' + 4 \times \mathbf{1}), \\ &(4 \times \mathbf{2}, \mathbf{2} + 5 \times \mathbf{1}' + \mathbf{1}). \end{aligned} \tag{4.2.9}$$

In order to see the structure of vacuum representations, one would like to take the first twist representation in (4.2.9) as an example for the D_4 twist. Consider the $\mathbf{Z}_4$ abelian subgroup of D_4 generated by θ. The twist parameter is

$$\omega_\theta = (\frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, 0, 0, 0, 0; \frac{1}{4}, \frac{3}{4}, 0, 0). \tag{4.2.10}$$

From (2.1.26) and (2.2.18) we have

$$\begin{aligned}
\zeta_\theta^{NNN}(e) &= \zeta_\theta(0) + \zeta_{\theta 1} = 0, \\
\zeta_\theta^{NNN}(\theta) &= \zeta_\theta(0) + \zeta_{\theta 1} + \zeta_{\theta\theta} - \omega_\theta \cdot \lfloor \omega_\theta + T \rfloor = \frac{1}{2}(Q_{2\theta} + Q_{3\theta}), \\
\zeta_\theta^{NNN}(\theta^2) &= \zeta_\theta(0) + \zeta_{\theta 1} + 2\zeta_{\theta\theta} - \omega_\theta \cdot \lfloor 2\omega_\theta + T \rfloor = 0, \\
\zeta_\theta^{NNN}(\theta^3) &= \zeta_\theta(0) + \zeta_{\theta 1} + 3\zeta_{\theta\theta} - \omega_\theta \cdot \lfloor 3\omega_\theta + T \rfloor = \frac{1}{2}(Q_{2\theta} + Q_{3\theta}),
\end{aligned} \tag{4.2.11}$$

where $Q_{2\theta}, Q_{3\theta}$ are discrete torsions. From the Hilbert space structure we can obtain the representation of the vacuum $|\Omega_\theta^{NNN}\rangle \oplus |\Omega_{\theta^3}^{NNN}\rangle$,

$$\begin{aligned}
\theta \begin{pmatrix} |\Omega_\theta^{NNN}\rangle \\ |\Omega_{\theta^3}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} e^{-i\pi(Q_{2\theta}+Q_{3\theta})} & 0 \\ 0 & e^{-i\pi(Q_{2\theta}+Q_{3\theta})} \end{pmatrix} \begin{pmatrix} |\Omega_\theta^{NNN}\rangle \\ |\Omega_{\theta^3}^{NNN}\rangle \end{pmatrix}, \\
\mathbf{r} \begin{pmatrix} |\Omega_\theta^{NNN}\rangle \\ |\Omega_{\theta^3}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_\theta^{NNN}\rangle \\ |\Omega_{\theta^3}^{NNN}\rangle \end{pmatrix}.
\end{aligned} \tag{4.2.12}$$

With diagonalizing the twist representation within the $\mathbf{Z}_2 \times \mathbf{Z}_2$ abelian subgroup generated by $r, r\theta^2$, we have

$$\begin{aligned}
\omega_r &= (0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0, 0; 0, \frac{1}{2}, \frac{1}{2}, 0), \\
\omega_{r\theta^2} &= (\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0, 0, 0, 0; \frac{1}{2}, 0, \frac{1}{2}, 0).
\end{aligned} \tag{4.2.13}$$

The vacuum phases are

$$
\begin{aligned}
&\zeta_r^{NNN}(e) = \zeta_r(0) + \zeta_{r1} = 0,\\
&\zeta_r^{NNN}(r) = \zeta_r(0) + \zeta_{r1} + \zeta_{rr} - \omega_r \cdot \lfloor \omega_r + T \rfloor = \frac{1}{2}(1 + Q_{2r} + Q_{3r}),\\
&\zeta_{r\theta^2}^{NNN}(e) = \zeta_{r\theta^2}(0) + \zeta_{(r\theta^2)1} = 0,\\
&\zeta_{r\theta^2}^{NNN}(r\theta^2) = \frac{1}{2}(1 + Q_{2(r\theta^2)} + Q_{3(r\theta^2)}),\\
&\zeta_r^{NNN}(r\theta^2) = \zeta_r(0) + \zeta_{r1} + \zeta_{r(r\theta^2)} - \omega_r \cdot \lfloor \omega_{r\theta^2} + T \rfloor = \frac{1}{2} Q_{r(r\theta^2)},\\
&\zeta_{r\theta^2}^{NNN}(r) = \zeta_{r\theta^2}(0) + \zeta_{(r\theta^2)1} + \zeta_{(r\theta^2)r} - \omega_{r\theta^2} \cdot \lfloor \omega_r + T \rfloor = \frac{1}{2} Q_{r(r\theta^2)},
\end{aligned}
\tag{4.2.14}
$$

where all Q's are discrete torsions. From Hilbert space structure we have

$$
\begin{aligned}
&\theta \begin{pmatrix} |\Omega_r^{NNN}\rangle \\ |\Omega_{r\theta^2}^{NNN}\rangle \end{pmatrix} = \begin{pmatrix} 0 & e^{i\pi Q_{r(r\theta^2)}} \\ e^{i\pi(1+Q_{2(r\theta^2)}+Q_{3(r\theta^2)})} & 0 \end{pmatrix} \begin{pmatrix} |\Omega_r^{NNN}\rangle \\ |\Omega_{r\theta^2}^{NNN}\rangle \end{pmatrix},\\
&\mathbf{r} \begin{pmatrix} |\Omega_r^{NNN}\rangle \\ |\Omega_{r\theta^2}^{NNN}\rangle \end{pmatrix} = \begin{pmatrix} e^{-i\pi(1+Q_{2r}+Q_{3r})} & 0 \\ 0 & e^{-i\pi Q_{r(r\theta^2)}} \end{pmatrix} \begin{pmatrix} |\Omega_r^{NNN}\rangle \\ |\Omega_{r\theta^2}^{NNN}\rangle \end{pmatrix},
\end{aligned}
\tag{4.2.15}
$$

where the discrete torsions satisfy $Q_{2r} + Q_{3r} = Q_{2(r\theta^2)} + Q_{3(r\theta^2)}$. With the same procedure for the abelian group $\mathbf{Z}_2' \times \mathbf{Z}_2'$ generated by $r\theta, \theta r$, we obtain

$$
\begin{aligned}
&\theta \begin{pmatrix} |\Omega_{r\theta}^{NNN}\rangle \\ |\Omega_{\theta r}^{NNN}\rangle \end{pmatrix} = \begin{pmatrix} 0 & e^{-i\pi(Q_{2(r\theta)}+Q_{3(r\theta)}+Q_{(r\theta)(\theta r)})} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{r\theta}^{NNN}\rangle \\ |\Omega_{\theta r}^{NNN}\rangle \end{pmatrix},\\
&\mathbf{r} \begin{pmatrix} |\Omega_{r\theta}^{NNN}\rangle \\ |\Omega_{\theta r}^{NNN}\rangle \end{pmatrix} = \begin{pmatrix} 0 & -e^{-i\pi Q_{(r\theta)(\theta r)}} \\ -e^{i\pi Q_{(r\theta)(\theta r)}} & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{r\theta}^{NNN}\rangle \\ |\Omega_{\theta r}^{NNN}\rangle \end{pmatrix},
\end{aligned}
\tag{4.2.16}
$$

where the discrete torsions satisfy $Q_{2(r\theta)} + Q_{3(r\theta)} = Q_{2(\theta r)} + Q_{3(\theta r)}$.

For the vacuum phase $\mathbf{r}$ generated on $|\Omega_{\theta^2}^{NNN}\rangle$, we can think of θ^2 as an element of the $\mathbf{Z}_2 \times \mathbf{Z}_2$ abelian group generated by $r, r\theta^2$, or as an element of the $\mathbf{Z}_2' \times \mathbf{Z}_2'$ abelian group generated by $r\theta, \theta r$. We evaluate the phase in both ways and results must be the same,

$$
\begin{aligned}
&\zeta_r^{NNN}(\theta^2) = \zeta_r^{NNN}(r \cdot r\theta^2) = \frac{1}{2}(1 + Q_{2r} + Q_{3r} + Q_{r(r\theta^2)}),\\
&\zeta_{r\theta}^{NNN}(\theta^2) = \zeta_{r\theta}^{NNN}(r\theta \cdot \theta r) = \frac{1}{2}(1 + Q_{2(r\theta)} + Q_{3(r\theta)} + Q_{(r\theta)(\theta r)}).
\end{aligned}
\tag{4.2.17}
$$

That implies

$$\begin{aligned}
\mathbf{r}|\Omega_{\theta^2}^{NNN}\rangle &= e^{-i\pi(1+Q_{2r}+Q_{3r}+Q_{r(r\theta^2)})}|\Omega_{\theta^2}^{NNN}\rangle, \\
\mathbf{r}\theta|\Omega_{\theta^2}^{NNN}\rangle &= e^{-i\pi(1+Q_{2(r\theta)}+Q_{3(r\theta)}+Q_{(r\theta)(\theta r)})}|\Omega_{\theta^2}^{NNN}\rangle.
\end{aligned} \tag{4.2.18}$$

From (4.2.11) we have

$$\theta|\Omega_{\theta^2}^{NNN}\rangle = |\Omega_{\theta^2}^{NNN}\rangle, \tag{4.2.19}$$

so the consistency of (4.2.18) and (4.2.19) leads to

$$Q_{2r} + Q_{3r} + Q_{r(r\theta^2)} = Q_{2(r\theta)} + Q_{3(r\theta)} + Q_{(r\theta)(\theta r)} + 2\mathbf{Z}. \tag{4.2.20}$$

If we consider the vacua $|\Omega_r^{RNN}\rangle \oplus |\Omega_{r\theta^2}^{RNN}\rangle$ and $|\Omega_r^{NRN}\rangle \oplus |\Omega_{r\theta^2}^{NRN}\rangle$, we may obtain the relations of the discrete torsions,

$$Q_{2r} = Q_{2(r\theta^2)}, \quad Q_{3r} = Q_{3(r\theta^2)}. \tag{4.2.21}$$

Similar consideration to the vacua $|\Omega_{r\theta}^{RNN}\rangle \oplus |\Omega_{\theta r}^{RNN}\rangle$ and $|\Omega_{r\theta}^{NRN}\rangle \oplus |\Omega_{\theta r}^{NRN}\rangle$ leads to

$$Q_{2(r\theta)} = Q_{2(\theta r)}, \quad Q_{3(r\theta)} = Q_{3(\theta r)}. \tag{4.2.22}$$

The results of (4.2.21) and (4.2.22) are expected since the discrete torsions between the $\mathbf{Z}_2$ abelian group element and the nonabelian group elements within one class should be equal. Also the elements $r, r\theta^2$ belong to the same class, and so do the elements $r\theta, \theta r$.

Consider the vacua $|\Omega_{\theta^2}^{RNN}\rangle$ and $|\Omega_{\theta^2}^{NRN}\rangle$, the representation requirements demand

$$\begin{aligned}
Q_{2\theta} + Q_{3r} + Q_{r(r\theta^2)} &= Q_{3(r\theta)} + Q_{(r\theta)(\theta r)} + 2\mathbf{Z}, \\
Q_{3\theta} + Q_{2r} + Q_{r(r\theta^2)} &= Q_{2(r\theta)} + Q_{(r\theta)(\theta r)} + 2\mathbf{Z},
\end{aligned} \tag{4.2.23}$$

From (4.2.20) and (4.2.23) we have

$$\begin{aligned} Q_{2(r\theta)} &= Q_{2r} + Q_{2\theta} + 2\mathrm{Z}, \\ Q_{3(r\theta)} &= Q_{3r} + Q_{3\theta} + 2\mathrm{Z}, \\ Q_{(r\theta)(\theta r)} &= Q_{r(r\theta^2)} + Q_{2\theta} + Q_{3\theta} + 2\mathrm{Z}. \end{aligned} \tag{4.2.24}$$

We see in this example that the representation requirement not only naturally leads to the class structure (requiring the equal discrete torsion for all sectors within one class), but also gives more constraints to the discrete torsions of different classes. In some cases if these constraints are mutually contradictory, then global anomalies arise. There are no global anomalies in all three twist representations in (4.2.9). We list final gauge groups of all minimal-rank models in Table 4.

§4.3 Nonabelian Twists of the Dihedral Group D_6

The nonabelian group D_6 has twelve elements, six classes $(e), (\delta^3)$, (δ^2, δ^4), (δ, δ^5), $(p, p\delta^2, p\delta^4)$, $(p\delta, p\delta^3, p\delta^5)$, generated by

$$\delta = \begin{pmatrix} e^{-i2\pi(1/6)} & 0 \\ 0 & e^{-i2\pi(5/6)} \end{pmatrix}, \quad p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.3.1}$$

It has two inequivalent two-dimensional irreducible representations, **2** described by the above defining matrix, **2′** by

$$\delta = \begin{pmatrix} e^{-i2\pi(1/3)} & 0 \\ 0 & e^{-i2\pi(2/3)} \end{pmatrix}, \quad p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.3.2}$$

It also has four one-dimensional irreducible representations described by $\mathbf{1}(\delta = 1, p = 1), \mathbf{1}'(\delta = 1, p = -1), \mathbf{1}''(\delta = -1, p = -1), \mathbf{1}'''(\delta = -1, p = 1)$. The partition function of this nonabelian twist is

$$\mathcal{Z}_h(D_6) = \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_6) + \mathcal{Z}_h(\mathbf{Z}_2 \times \mathbf{Z}_2) - \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_2), \tag{4.3.3}$$

where the abelian group $\mathbf{Z}_6$ is generated by δ, $\mathbf{Z}_2 \times \mathbf{Z}_2$ by $p, p\delta^3$ or $p\delta, p\delta^4$ or $p\delta^2, p\delta^5$, and $\mathbf{Z}_2$ by δ^3. Since the elements $p, p\delta^2, p\delta^4$ belong to one class and $p\delta, p\delta^3, p\delta^5$ belong to another class, the partition function of the abelian group $\mathbf{Z}_2 \times \mathbf{Z}_2$ generated by $p, p\delta^3$ is the same as that generated by $p\delta, p\delta^4$ or $p\delta^2, p\delta^5$. Therefore we can just write a single partition function $\mathcal{Z}_h(\mathbf{Z}_2 \times \mathbf{Z}_2)$.

For the representation D_X, we could choose $\mathbf{2}'+\mathbf{1}'$, but it corresponds to the D_3 nonabelian orbifold, since $\mathbf{2}'+\mathbf{1}'$ is a representation of D_3. In order to qualify a D_6 orbifold, the representation D_X has to be $\mathbf{2}+\mathbf{1}'$. In this nonabelian twist there are following equivalences: $\mathbf{2} \leftrightarrow \mathbf{2}', \mathbf{1} \leftrightarrow \mathbf{1}''', \mathbf{1}' \leftrightarrow \mathbf{1}''$ corresponding to the interchange of NS fermions with R fermions in $\delta, \delta^3, \delta^5$ sectors; $\mathbf{1} \leftrightarrow \mathbf{1}', \mathbf{1}'' \leftrightarrow \mathbf{1}'''$ corresponding to the interchange of NS fermions with R fermions in $p, p\delta, p\delta^2, p\delta^3, p\delta^4, p\delta^5$ sectors; and $\mathbf{1}'' \leftrightarrow \mathbf{1}'''$ corresponding to the interchange of $p, p\delta^2, p\delta^4$ sectors with $p\delta, p\delta^3, p\delta^5$ sectors. As for the twist representation of the sixteen left-fermions, we assign the following,

$$D_\lambda = (m_0 \times \mathbf{2} + m_1 \times \mathbf{2}' + n_0 \times \mathbf{1} + n_1 \times \mathbf{1}' + n_2 \times \mathbf{1}'' + n_3 \times \mathbf{1}'''), \tag{4.3.4}$$

where the non-negative integer numbers m_i, n_j satisfy $2(m_0 + m_1) + \sum_{j=0}^{3} n_j = 16$. Modular invariance of a $\mathbf{Z}_6$ abelian twist demands

$$\frac{13}{6}(m_0 - 1) + \frac{5}{3}m_1 + \frac{3}{4}(n_2 + n_3) \in \mathbf{Z}. \tag{4.3.5}$$

Modular invariance of a $\mathbf{Z}_2 \times \mathbf{Z}_2$ abelian twist leads to

$$\begin{aligned}
&m_0 + m_1 + n_1 + n_2 - 2 \in 4\mathbf{Z}, \\
&m_0 + m_1 + n_1 + n_3 - 2 \in 4\mathbf{Z}, \\
&m_1 + n_1 - 1 \in 2\mathbf{Z},
\end{aligned} \tag{4.3.6}$$

and modular invariance of a $\mathbf{Z}_2$ abelian twist demands

$$2m_0 + n_2 + n_3 - 2 \in 4\mathbf{Z}. \tag{4.3.7}$$

Since there are three equivalences, we can restrict the solutions to be $m_0 \geq m_1, n_3 \geq n_2, n_0 \geq n_1$. Also we want solutions with a maximal value of $m_0 + m_1$ because the minimal-rank models are desired. Solving (4.3.5) and (4.3.6), we can obtain the maximal value of $m_0 + m_1$ to be 4. Knowing that we can find all the solutions by solving the following,

$$\begin{aligned}
&n_1 + n_2 \in 4\mathbf{Z} + 2, \\
&n_1 + n_3 \in 4\mathbf{Z} + 2, \\
&2m_0 + n_2 + n_3 \in 4\mathbf{Z} + 2, \\
&n_0 + n_1 + n_2 + n_3 = 8, \\
&m_0 + m_1 = 4, \\
&m_0 \geq m_1, \quad n_3 \geq n_2, \quad n_0 \geq n_1.
\end{aligned} \tag{4.3.8}$$

We list all non-negative solutions of m_i, n_i modulo the three equivalences in Table 5. We should maintain the modular invariance when we split the sixteen-dimensional twist representation into two eight-dimensional representations. Also the three equivalences should be considered in order to avoid duplications of our models.

To see what the vacuum representations look like, one may take the following twist representation as a typical example,

$$(2 \times \mathbf{2} + \mathbf{2}' + \mathbf{1}''' + \mathbf{1}'', \mathbf{2}' + \mathbf{1}' + 5 \times \mathbf{1}; \mathbf{2} + \mathbf{1}' + \mathbf{1}). \tag{4.3.9}$$

Considering the length of this thesis, we only present results of vacuum representations for the twist of (4.3.9). For vacuum $|\Omega_{\delta}^{NNN}\rangle \oplus |\Omega_{\delta^5}^{NNN}\rangle$ we have

$$\begin{aligned}
\delta \begin{pmatrix} |\Omega_{\delta}^{NNN}\rangle \\ |\Omega_{\delta^5}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} e^{-i\pi(Q_{2\delta}+Q_{3\delta})} & 0 \\ 0 & e^{-i\pi(Q_{2\delta}+Q_{3\delta})} \end{pmatrix} \begin{pmatrix} |\Omega_{\delta}^{NNN}\rangle \\ |\Omega_{\delta^5}^{NNN}\rangle \end{pmatrix}, \\
\mathbf{p} \begin{pmatrix} |\Omega_{\delta}^{NNN}\rangle \\ |\Omega_{\delta^5}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{\delta}^{NNN}\rangle \\ |\Omega_{\delta^5}^{NNN}\rangle \end{pmatrix}.
\end{aligned} \tag{4.3.10}$$

Similarly, for the vacuum $|\Omega_{\delta^2}^{NNN}\rangle \oplus |\Omega_{\delta^4}^{NNN}\rangle$, we have

$$\begin{aligned}
\delta \begin{pmatrix} |\Omega_{\delta^2}^{NNN}\rangle \\ |\Omega_{\delta^4}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} e^{-i2\pi(2/3)} & 0 \\ 0 & e^{-i2\pi(1/3)} \end{pmatrix} \begin{pmatrix} |\Omega_{\delta^2}^{NNN}\rangle \\ |\Omega_{\delta^4}^{NNN}\rangle \end{pmatrix}, \\
\mathbf{p} \begin{pmatrix} |\Omega_{\delta^2}^{NNN}\rangle \\ |\Omega_{\delta^4}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{\delta^2}^{NNN}\rangle \\ |\Omega_{\delta^4}^{NNN}\rangle \end{pmatrix}.
\end{aligned} \tag{4.3.11}$$

The representation of the vacuum $|\Omega_{\delta^3}^{NNN}\rangle$ is

$$\begin{aligned}
\delta|\Omega_{\delta^3}^{NNN}\rangle &= e^{-i\pi(1+Q_{2\delta}+Q_{3\delta})}|\Omega_{\delta^3}^{NNN}\rangle, \\
\mathbf{p}|\Omega_{\delta^3}^{NNN}\rangle &= e^{-i\pi(1+Q_{2p}+Q_{3p}+Q_{p(p\delta^3)})}|\Omega_{\delta^3}^{NNN}\rangle.
\end{aligned} \tag{4.3.12}$$

For the vacuum $|\Omega_{p}^{NNN}\rangle \oplus |\Omega_{p\delta^2}^{NNN}\rangle \oplus |\Omega_{p\delta^4}^{NNN}\rangle$, we have

$$\begin{aligned}
\delta \begin{pmatrix} |\Omega_{p}^{NNN}\rangle \\ |\Omega_{p\delta^2}^{NNN}\rangle \\ |\Omega_{p\delta^4}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 0 & e^{-i\pi(1+Q_{2p}+Q_{3p})} \\ 1 & 0 & 0 \\ 0 & e^{-i\pi(1+Q_{p(p\delta^3)})} & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{p}^{NNN}\rangle \\ |\Omega_{p\delta^2}^{NNN}\rangle \\ |\Omega_{p\delta^4}^{NNN}\rangle \end{pmatrix}, \\
\mathbf{p} \begin{pmatrix} |\Omega_{p}^{NNN}\rangle \\ |\Omega_{p\delta^2}^{NNN}\rangle \\ |\Omega_{p\delta^4}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} e^{-i\pi(1+Q_{2p}+Q_{3p})} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{p}^{NNN}\rangle \\ |\Omega_{p\delta^2}^{NNN}\rangle \\ |\Omega_{p\delta^4}^{NNN}\rangle \end{pmatrix}.
\end{aligned} \tag{4.3.13}$$

Finally, the representation of vacuum $|\Omega_{p\delta}^{NNN}\rangle \oplus |\Omega_{p\delta^3}^{NNN}\rangle \oplus |\Omega_{p\delta^5}^{NNN}\rangle$ is

$$
\begin{aligned}
\delta \begin{pmatrix} |\Omega_{p\delta}^{NNN}\rangle \\ |\Omega_{p\delta^3}^{NNN}\rangle \\ |\Omega_{p\delta^5}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 0 & e^{-i\pi(1+Q_{2(p\delta)}+Q_{3(p\delta)})} \\ 1 & 0 & 0 \\ 0 & e^{-i\pi(1+Q_{(p\delta)(p\delta^4)})} & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{p\delta}^{NNN}\rangle \\ |\Omega_{p\delta^3}^{NNN}\rangle \\ |\Omega_{p\delta^5}^{NNN}\rangle \end{pmatrix}, \\
\mathbf{p} \begin{pmatrix} |\Omega_{p\delta}^{NNN}\rangle \\ |\Omega_{p\delta^3}^{NNN}\rangle \\ |\Omega_{p\delta^5}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & e^{-i\pi(1+Q_{(p\delta)(p\delta^4)})} & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{p\delta}^{NNN}\rangle \\ |\Omega_{p\delta^3}^{NNN}\rangle \\ |\Omega_{p\delta^5}^{NNN}\rangle \end{pmatrix}.
\end{aligned}
\tag{4.3.14}
$$

All Q's are discrete torsions. We have checked that there are no global anomalies in all minimal-rank models. Final gauge groups together with the twist representations of the minimal-rank models are listed in Table 6.

§4.4 Nonabelian Twists of the Tetrahedral Group T

The nonabelian group T has twelve elements, four classes (e), (r, r', rr'), $(u, ru, r'u, rr'u)$, $(u^2, (ru)^2, (r'u)^2, (rr'u)^2)$, generated by

$$
u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4.4.1}
$$

where $r' = uru^{-1}$. It has one three-dimensional irreducible representations $\mathbf{3}$ described by the above defining matrix. It also has three one-dimensional irreducible representations described by $\mathbf{1}(u = 1, r = 1), \mathbf{1}'(u = e^{-i2\pi\frac{1}{3}}, r = 1), \mathbf{1}''(u = e^{-i2\pi\frac{2}{3}}, r = 1)$. The partition function of this nonabelian twist is

$$
\mathcal{Z}_h(T) = \mathcal{Z}_h(\mathbf{Z}_3) + \frac{1}{3}\mathcal{Z}_h(\mathbf{Z}_2 \times \mathbf{Z}_2) - \frac{1}{3}\mathcal{Z}_h(\mathbf{Z}_1), \tag{4.4.2}
$$

where the abelian group $\mathbf{Z}_3$ is generated by u or ru or $r'u$ or $rr'u$, $\mathbf{Z}_2 \times \mathbf{Z}_2$ by r, r', and $\mathbf{Z}_1$ by the identity.

The representation D_X has to be $\mathbf{3}$, which is unique in this nonabelian group. In this nonabelian twist there is one equivalence $\mathbf{1}' \leftrightarrow \mathbf{1}''$, which is the interchange of a fermion with its complex conjugate. Let us assign the twist representation of sixteen left-fermions as

$$D_\lambda = (m \times \mathbf{3} + n_0 \times \mathbf{1} + n_1 \times \mathbf{1}' + n_2 \times \mathbf{1}''), \tag{4.4.3}$$

where the non-negative integers m, n_i satisfy $3m + \sum_{i=0}^{2} n_i = 16$. Modular invariance of corresponding $\mathbf{Z}_3$ and $\mathbf{Z}_2 \times \mathbf{Z}_2$ abelian twists demands

$$\begin{aligned} &\frac{5}{3}(m-1) + \frac{1}{3}n_1 + \frac{4}{3}n_2 \in \mathbf{Z}, \\ &m - 1 \in 2\mathbf{Z}. \end{aligned} \tag{4.4.4}$$

Solving (4.4.4) with the maximal value of m, we obtain the twist representations,

$$\begin{aligned} &(3 \times \mathbf{3} + \mathbf{1}'' + \mathbf{1}' + 5 \times \mathbf{1}), \\ &(3 \times \mathbf{3} + 5 \times \mathbf{1}'' + 2 \times \mathbf{1}), \end{aligned} \tag{4.4.5}$$

where the equivalence has been considered. To split the sixteen-dimensional representation into two eight-dimensional representations, we should keep in mind the modular invariance conditions.

We can take the following representation,

$$(2 \times \mathbf{3} + 2 \times \mathbf{1}, \mathbf{3} + \mathbf{1}'' + \mathbf{1}' + 3 \times \mathbf{1}; \mathbf{3} + \mathbf{1}), \tag{4.4.6}$$

to work out vacuum representations. After some calculation we will obtain the representation for $|\Omega_{u}^{NNN}\rangle \oplus |\Omega_{ru}^{NNN}\rangle \oplus |\Omega_{r'u}^{NNN}\rangle \oplus |\Omega_{rr'u}^{NNN}\rangle$. It is

$$\mathbf{u}\begin{pmatrix} |\Omega_{u}^{NNN}\rangle \\ |\Omega_{ru}^{NNN}\rangle \\ |\Omega_{r'u}^{NNN}\rangle \\ |\Omega_{rr'u}^{NNN}\rangle \end{pmatrix} = \begin{pmatrix} e^{-i2\pi(1/3)} & 0 & 0 & 0 \\ 0 & 0 & e^{-i2\pi(1/3)} & 0 \\ 0 & 0 & 0 & e^{-i2\pi(2/3)} \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{u}^{NNN}\rangle \\ |\Omega_{ru}^{NNN}\rangle \\ |\Omega_{r'u}^{NNN}\rangle \\ |\Omega_{rr'u}^{NNN}\rangle \end{pmatrix},$$
$$\mathbf{r}\begin{pmatrix} |\Omega_{u}^{NNN}\rangle \\ |\Omega_{ru}^{NNN}\rangle \\ |\Omega_{r'u}^{NNN}\rangle \\ |\Omega_{rr'u}^{NNN}\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{u}^{NNN}\rangle \\ |\Omega_{ru}^{NNN}\rangle \\ |\Omega_{r'u}^{NNN}\rangle \\ |\Omega_{rr'u}^{NNN}\rangle \end{pmatrix}. \tag{4.4.7}$$

For vacuum $|\Omega_{u^2}^{NNN}\rangle \oplus |\Omega_{(ru)^2}^{NNN}\rangle \oplus |\Omega_{(r'u)^2}^{NNN}\rangle \oplus |\Omega_{(rr'u)^2}^{NNN}\rangle$, we have

$$\mathbf{u}\begin{pmatrix} |\Omega_{u^2}^{NNN}\rangle \\ |\Omega_{(ru)^2}^{NNN}\rangle \\ |\Omega_{(r'u)^2}^{NNN}\rangle \\ |\Omega_{(rr'u)^2}^{NNN}\rangle \end{pmatrix} = \begin{pmatrix} e^{-i2\pi(2/3)} & 0 & 0 & 0 \\ 0 & 0 & e^{-i2\pi(2/3)} & 0 \\ 0 & 0 & 0 & e^{-i2\pi(1/3)} \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{u^2}^{NNN}\rangle \\ |\Omega_{(ru)^2}^{NNN}\rangle \\ |\Omega_{(r'u)^2}^{NNN}\rangle \\ |\Omega_{(rr'u)^2}^{NNN}\rangle \end{pmatrix},$$
$$\mathbf{r}\begin{pmatrix} |\Omega_{u^2}^{NNN}\rangle \\ |\Omega_{(ru)^2}^{NNN}\rangle \\ |\Omega_{(r'u)^2}^{NNN}\rangle \\ |\Omega_{(rr'u)^2}^{NNN}\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{u^2}^{NNN}\rangle \\ |\Omega_{(ru)^2}^{NNN}\rangle \\ |\Omega_{(r'u)^2}^{NNN}\rangle \\ |\Omega_{(rr'u)^2}^{NNN}\rangle \end{pmatrix}. \tag{4.4.8}$$

The representation of $|\Omega_{r}^{NNN}\rangle \oplus |\Omega_{r'}^{NNN}\rangle \oplus |\Omega_{rr'}^{NNN}\rangle$ is

$$\mathbf{u}\begin{pmatrix} |\Omega_{r}^{NNN}\rangle \\ |\Omega_{r'}^{NNN}\rangle \\ |\Omega_{rr'}^{NNN}\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{r}^{NNN}\rangle \\ |\Omega_{r'}^{NNN}\rangle \\ |\Omega_{rr'}^{NNN}\rangle \end{pmatrix},$$
$$\mathbf{r}\begin{pmatrix} |\Omega_{r}^{NNN}\rangle \\ |\Omega_{r'}^{NNN}\rangle \\ |\Omega_{rr'}^{NNN}\rangle \end{pmatrix} = \begin{pmatrix} e^{i\pi(Q_{2r}+Q_{3r})} & 0 & 0 \\ 0 & e^{i\pi Q_{rr'}} & 0 \\ 0 & 0 & e^{i\pi(Q_{2r}+Q_{3r}+Q_{rr'})} \end{pmatrix} \begin{pmatrix} |\Omega_{r}^{NNN}\rangle \\ |\Omega_{r'}^{NNN}\rangle \\ |\Omega_{rr'}^{NNN}\rangle \end{pmatrix}. \tag{4.4.9}$$

All Q's are discrete torsions. There are no global anomalies in the minimal-rank twist representations. We present all minimal-rank models in Table 7.

§4.5 Nonabelian Twists of the Octahedral Group O

The nonabelian group O has twenty-four elements, five classes (e), (r, r', rr'), $(u, ru, r'u, rr'u, u^2, (ru)^2, (r'u)^2, (rr'u)^2)$, $(r\theta, r\theta^3, r\theta', r\theta'^3, r'\theta'', r'\theta''^3)$, $(\theta, \theta^3, \theta', \theta'^3, \theta'', \theta''^3)$ generated by

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.5.1}$$

where $\theta' = u\theta u^{-1}$, $\theta'' = u^{-1}\theta u$, $r = \theta''^2 = diag(1, -1, -1)$, $r' = \theta^2 = diag(-1, -1, 1)$. It has irreducible representations $\mathbf{3}$ described by the defining matrix, $\mathbf{3'}$ by

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4.5.2}$$

$\mathbf{2}$ by

$$u = \begin{pmatrix} e^{-i2\pi(1/3)} & 0 \\ 0 & e^{-i2\pi(2/3)} \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.5.3}$$

and $\mathbf{1}(u = 1, \theta = 1), \mathbf{1'}(u = 1, \theta = -1)$. The partition function of this nonabelian twist is

$$\begin{aligned} \mathcal{Z}_h(O) = &\frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_4) + \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_3) + \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_2 \times \mathbf{Z}_2) \\ &+ \frac{1}{6}\mathcal{Z}_h(\mathbf{Z}'_2 \times \mathbf{Z}'_2) - \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_2) - \frac{1}{6}\mathcal{Z}_h(\mathbf{Z}_1), \end{aligned} \tag{4.5.4}$$

where the abelian group $\mathbf{Z}_4$ is generated by θ or θ' or θ'', $\mathbf{Z}_3$ by u or ru or $r'u$ or $rr'u$, $\mathbf{Z}_2 \times \mathbf{Z}_2$ by $r\theta, r\theta^3$ or $r\theta', r\theta'^3$ or $r'\theta'', r'\theta''^3$, $\mathbf{Z}'_2 \times \mathbf{Z}'_2$ by r, r' , $\mathbf{Z}_2$ by r or r' or rr', and $\mathbf{Z}_1$ by the identity.

Let us first consider the twist representation of the right-moving fields. The representation $D_X(g)$ could be chosen to be $\mathbf{2} + \mathbf{1'}$, but it actually corresponds to the nonabelian orbifold of D_3, since $\mathbf{2}+\mathbf{1}'$ is a representation of D_3. To be a faithful representation of O, either $\mathbf{3}$ or $\mathbf{3}'$can be chosen. But $\mathbf{3}'$ is ruled out by the space-time supersymmetry condition (2.2.19) since the diagonalization of group element θ gives $diag(e^{i2\pi\frac{1}{4}}, e^{i2\pi\frac{3}{4}}, e^{i2\pi\frac{1}{2}})$ and $\frac{1}{4}+\frac{3}{4}+\frac{1}{2} \notin \mathbf{Z}$. Therefore only $\mathbf{3}$ is qualified to be a nonabelian orbifold of O. In this nonabelian twist there is one equivalence. The representation under $\mathbf{3} \leftrightarrow \mathbf{3}', \mathbf{1} \leftrightarrow \mathbf{1}'$ corresponds to the interchange of NS fermions with R fermions in $\theta, \theta^3, \theta', \theta'^3, r\theta, r\theta^3, r\theta', r\theta'^3, r'\theta'', r'\theta''^3$ sectors.

For the representation of sixteen left-fermions, assign the twist representation of O,

$$D_\lambda = (m_0 \times \mathbf{3} + m_1 \times \mathbf{3}' + m_2 \times \mathbf{2} + n_0 \times \mathbf{1} + n_1 \times \mathbf{1}'), \tag{4.5.5}$$

where all non-negative integers m_i, n_j satisfy $3(m_0+m_1)+2m_2+n_0+n_1 = 16$. For $\mathbf{Z}_4$ and $\mathbf{Z}_3$ abelian twists to be modular invariance, we have to have

$$\begin{aligned} &\frac{5}{2}(m_0-1)+\frac{7}{2}m_1+m_2+n_1 \in 2\mathbf{Z}, \\ &\frac{5}{3}(m_0-1+m_1+m_2) \in \mathbf{Z}. \end{aligned} \tag{4.5.6}$$

Modular invariance of a $\mathbf{Z}_2 \times \mathbf{Z}_2$ abelian twist leads to

$$\begin{aligned} &m_0-1+\frac{1}{2}(m_1+m_2+n_1) \in 2\mathbf{Z}, \\ &m_0-1+m_2+n_1 \in 2\mathbf{Z}. \end{aligned} \tag{4.5.7}$$

Finally, for a $\mathbf{Z}_2' \times \mathbf{Z}_2'$ abelian twist to be modular invariance, we obtain

$$m_0-1+m_1 \in 2\mathbf{Z}, \tag{4.5.8}$$

where modular invariance condition of a $\mathbf{Z}_2$ abelian twist is already included in the above. In order to have minimal-rank models, we have to find solutions to the equations (4.5.6)—(4.5.8) with maximal value of $2(m_0 + m_1) + m_2$, which is the amount of the rank of the gauge group might be reduced. We rewrite (4.5.6)–(4.5.8) to be

$$\begin{aligned} &\frac{1}{2}(m_0 - 1 - m_1) + m_2 + n_1 \in 2\mathbf{Z}, \\ &m_0 - 1 + m_1 + m_2 \in 3\mathbf{Z}, \\ &m_0 - 1 + \frac{1}{2}(m_1 + m_2 + n_1) \in 2\mathbf{Z}. \end{aligned} \tag{4.5.9}$$

After some calculation we obtain possible minimal-rank twist representations,

$$\begin{aligned} &(\mathbf{3} + 3 \times \mathbf{2} + \mathbf{1}' + 6 \times \mathbf{1}), \\ &(\mathbf{3} + 3 \times \mathbf{2} + 5 \times \mathbf{1}' + 2 \times \mathbf{1}), \end{aligned} \tag{4.5.10}$$

where the equivalence of the twist representation has been considered. Apply the representations of (4.5.10) on (E_8, E_8'), and we have

$$\begin{aligned} &(\mathbf{3} + 2 \times \mathbf{2} + \mathbf{1}, \mathbf{2} + \mathbf{1}' + 5 \times \mathbf{1}), \\ &(\mathbf{3} + \mathbf{2} + \mathbf{1}' + 2 \times \mathbf{1}, 2 \times \mathbf{2} + 4 \times \mathbf{1}), \\ &(\mathbf{3} + 5 \times \mathbf{1}, 3 \times \mathbf{2} + \mathbf{1}' + \mathbf{1}), \\ &(\mathbf{3} + \mathbf{2} + 3 \times \mathbf{1}', 2 \times \mathbf{2} + 2 \times \mathbf{1}' + 2 \times \mathbf{1}), \\ &(\mathbf{3} + 4 \times \mathbf{1}' + \mathbf{1}, 3 \times \mathbf{2} + \mathbf{1}' + \mathbf{1}), \end{aligned} \tag{4.5.11}$$

where the equivalence and modular invariance have been taken into account.

To see the vacuum structure, one may take the following twist representation,

$$(\mathbf{3} + 2 \times \mathbf{2} + \mathbf{1}, \mathbf{2} + \mathbf{1}' + 5 \times \mathbf{1}; \mathbf{3} + \mathbf{1}). \tag{4.5.12}$$

For the vacuum $|\Omega_u^{NNN}\rangle \oplus |\Omega_{ru}^{NNN}\rangle \oplus |\Omega_{r'u}^{NNN}\rangle \oplus |\Omega_{rr'u}^{NNN}\rangle \oplus |\Omega_{u^2}^{NNN}\rangle \oplus |\Omega_{(ru)^2}^{NNN}\rangle \oplus$

$|\Omega^{NNN}_{(r'u)^2}\rangle \oplus |\Omega^{NNN}_{(rr'u)^2}\rangle$, the representation is calculated to be

$$\mathbf{u}\begin{pmatrix} |\Omega^{NNN}_{u}\rangle \\ |\Omega^{NNN}_{ru}\rangle \\ |\Omega^{NNN}_{r'u}\rangle \\ |\Omega^{NNN}_{rr'u}\rangle \\ |\Omega^{NNN}_{u^2}\rangle \\ |\Omega^{NNN}_{(ru)^2}\rangle \\ |\Omega^{NNN}_{(r'u)^2}\rangle \\ |\Omega^{NNN}_{(rr'u)^2}\rangle \end{pmatrix} = \begin{pmatrix} \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} |\Omega^{NNN}_{u}\rangle \\ |\Omega^{NNN}_{ru}\rangle \\ |\Omega^{NNN}_{r'u}\rangle \\ |\Omega^{NNN}_{rr'u}\rangle \\ |\Omega^{NNN}_{u^2}\rangle \\ |\Omega^{NNN}_{(ru)^2}\rangle \\ |\Omega^{NNN}_{(r'u)^2}\rangle \\ |\Omega^{NNN}_{(rr'u)^2}\rangle \end{pmatrix},$$

$$\theta\begin{pmatrix} |\Omega^{NNN}_{u}\rangle \\ |\Omega^{NNN}_{ru}\rangle \\ |\Omega^{NNN}_{r'u}\rangle \\ |\Omega^{NNN}_{rr'u}\rangle \\ |\Omega^{NNN}_{u^2}\rangle \\ |\Omega^{NNN}_{(ru)^2}\rangle \\ |\Omega^{NNN}_{(r'u)^2}\rangle \\ |\Omega^{NNN}_{(rr'u)^2}\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} |\Omega^{NNN}_{u}\rangle \\ |\Omega^{NNN}_{ru}\rangle \\ |\Omega^{NNN}_{r'u}\rangle \\ |\Omega^{NNN}_{rr'u}\rangle \\ |\Omega^{NNN}_{u^2}\rangle \\ |\Omega^{NNN}_{(ru)^2}\rangle \\ |\Omega^{NNN}_{(r'u)^2}\rangle \\ |\Omega^{NNN}_{(rr'u)^2}\rangle \end{pmatrix}, \tag{4.5.13}$$

where $\omega = e^{-i2\pi\frac{1}{3}}$. The representation of the vacuum $|\Omega^{NNN}_{\theta}\rangle \oplus |\Omega^{NNN}_{\theta'}\rangle \oplus |\Omega^{NNN}_{\theta''}\rangle \oplus |\Omega^{NNN}_{\theta 3}\rangle \oplus |\Omega^{NNN}_{\theta' 3}\rangle \oplus |\Omega^{NNN}_{\theta'' 3}\rangle$ is

$$\mathbf{u}\begin{pmatrix} |\Omega^{NNN}_{\theta}\rangle \\ |\Omega^{NNN}_{\theta'}\rangle \\ |\Omega^{NNN}_{\theta''}\rangle \\ |\Omega^{NNN}_{\theta 3}\rangle \\ |\Omega^{NNN}_{\theta' 3}\rangle \\ |\Omega^{NNN}_{\theta'' 3}\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} |\Omega^{NNN}_{\theta}\rangle \\ |\Omega^{NNN}_{\theta'}\rangle \\ |\Omega^{NNN}_{\theta''}\rangle \\ |\Omega^{NNN}_{\theta 3}\rangle \\ |\Omega^{NNN}_{\theta' 3}\rangle \\ |\Omega^{NNN}_{\theta'' 3}\rangle \end{pmatrix},$$

$$\theta\begin{pmatrix} |\Omega^{NNN}_{\theta}\rangle \\ |\Omega^{NNN}_{\theta'}\rangle \\ |\Omega^{NNN}_{\theta''}\rangle \\ |\Omega^{NNN}_{\theta 3}\rangle \\ |\Omega^{NNN}_{\theta' 3}\rangle \\ |\Omega^{NNN}_{\theta'' 3}\rangle \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} |\Omega^{NNN}_{\theta}\rangle \\ |\Omega^{NNN}_{\theta'}\rangle \\ |\Omega^{NNN}_{\theta''}\rangle \\ |\Omega^{NNN}_{\theta 3}\rangle \\ |\Omega^{NNN}_{\theta' 3}\rangle \\ |\Omega^{NNN}_{\theta'' 3}\rangle \end{pmatrix}, \tag{4.5.14}$$

where $\alpha = e^{-i\pi(Q_{2\theta}+Q_{3\theta})}$, and Q's are discrete torsions. The representation of $|\Omega^{NNN}_{r\theta}\rangle \oplus |\Omega^{NNN}_{r\theta'^3}\rangle \oplus |\Omega^{NNN}_{r'\theta''^3}\rangle \oplus |\Omega^{NNN}_{r\theta'}\rangle \oplus |\Omega^{NNN}_{r'\theta''}\rangle \oplus |\Omega^{NNN}_{r\theta^3}\rangle$ is

$$
\begin{aligned}
\mathbf{u}\begin{pmatrix} |\Omega^{NNN}_{r\theta}\rangle \\ |\Omega^{NNN}_{r\theta'^3}\rangle \\ |\Omega^{NNN}_{r'\theta''^3}\rangle \\ |\Omega^{NNN}_{r\theta'}\rangle \\ |\Omega^{NNN}_{r'\theta''}\rangle \\ |\Omega^{NNN}_{r\theta^3}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\Omega^{NNN}_{r\theta}\rangle \\ |\Omega^{NNN}_{r\theta'^3}\rangle \\ |\Omega^{NNN}_{r'\theta''^3}\rangle \\ |\Omega^{NNN}_{r\theta'}\rangle \\ |\Omega^{NNN}_{r'\theta''}\rangle \\ |\Omega^{NNN}_{r\theta^3}\rangle \end{pmatrix}, \\
\theta\begin{pmatrix} |\Omega^{NNN}_{r\theta}\rangle \\ |\Omega^{NNN}_{r\theta'^3}\rangle \\ |\Omega^{NNN}_{r'\theta''^3}\rangle \\ |\Omega^{NNN}_{r\theta'}\rangle \\ |\Omega^{NNN}_{r'\theta''}\rangle \\ |\Omega^{NNN}_{r\theta^3}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\Omega^{NNN}_{r\theta}\rangle \\ |\Omega^{NNN}_{r\theta'^3}\rangle \\ |\Omega^{NNN}_{r'\theta''^3}\rangle \\ |\Omega^{NNN}_{r\theta'}\rangle \\ |\Omega^{NNN}_{r'\theta''}\rangle \\ |\Omega^{NNN}_{r\theta^3}\rangle \end{pmatrix},
\end{aligned}
\tag{4.5.15}
$$

where $\beta = e^{-i\pi Q_{(r\theta)(r\theta^3)}}$ and $Q_{2(r\theta)}+Q_{3(r\theta)} = 1$ in order to satisfy the representation requirement. For the vacuum $|\Omega^{NNN}_{r}\rangle \oplus |\Omega^{NNN}_{r'}\rangle \oplus |\Omega^{NNN}_{rr'}\rangle$ we have

$$
\begin{aligned}
\mathbf{u}\begin{pmatrix} |\Omega^{NNN}_{r}\rangle \\ |\Omega^{NNN}_{r'}\rangle \\ |\Omega^{NNN}_{rr'}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\Omega^{NNN}_{r}\rangle \\ |\Omega^{NNN}_{r'}\rangle \\ |\Omega^{NNN}_{rr'}\rangle \end{pmatrix}, \\
\theta\begin{pmatrix} |\Omega^{NNN}_{r}\rangle \\ |\Omega^{NNN}_{r'}\rangle \\ |\Omega^{NNN}_{rr'}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 0 & e^{-i\pi Q_{rr'}} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\Omega^{NNN}_{r}\rangle \\ |\Omega^{NNN}_{r'}\rangle \\ |\Omega^{NNN}_{rr'}\rangle \end{pmatrix},
\end{aligned}
\tag{4.5.16}
$$

where $Q_{2r} + Q_{3r} = 0$ and $Q_{rr'} = Q_{(r\theta)(r\theta^3)} = Q_{(r\theta')(r\theta'^3)} = Q_{(r'\theta'')(r'\theta''^3)}$ in order to satisfy the representation requirement. Notice that the discrete torsions between the abelian $\mathbf{Z}_2$ group element and the nonabelian group elements within one class are always the same, such as $Q_{2r} = Q_{2r'}, Q_{3r} = Q_{3r'}$.

In a similar way we can calculate the group O representations of vacua $|\Omega^{NRN}_{s}\rangle, |\Omega^{RNN}_{s}\rangle, |\Omega^{RRR}_{s}\rangle$ and so on, and the representation requirement will give more constraints to the discrete torsions. However there are no contradictions on these constraints, so no global anomalies arise. We have checked that there exist no global anomalies in all minimal-rank models in this nonabelian group. The results of the minimal-rank models are listed in Table 8.

§4.6 Nonabelian Twists of the Quaternion Group $D_2^{(d)}$

This nonabelian group was considered in Section 3.2 when we dealt with global anomalies. The irreducible representations are listed in (3.2.16) and the paragraph it following. The partition function of this nonabelian twist is given by (3.2.17). The representation D_X is $\mathbf{2}+\mathbf{1}$. Notice that this nonabelian orbifold keeps $N=2$ space-time supersymmetry [36]. We will see that the nonabelian orbifold of $D_3^{(d)}$ shares the same feature. There are four equivalences in the twist representation. The representation under $\mathbf{1} \leftrightarrow \mathbf{1}', \mathbf{1}'' \leftrightarrow \mathbf{1}'''$ corresponds to the interchange of NS fermions with R fermions in $b, a^2 b, ab, a^3 b$ sectors, while it under $\mathbf{1} \leftrightarrow \mathbf{1}''', \mathbf{1}' \leftrightarrow \mathbf{1}''$ corresponds to the interchange of NS fermions with R fermions in $a, a^3, ab, a^3 b$ sectors. The equivalence $\mathbf{1}'' \leftrightarrow \mathbf{1}'''$ corresponds to the interchange of $b, a^2 b$ sectors with $ab, a^3 b$ sectors, while $\mathbf{1}' \leftrightarrow \mathbf{1}''$ corresponds to the interchange of a, a^3 sectors with $ab, a^3 b$ sectors.

Consider the twist representation of the sixteen left-fermions,

$$D_\lambda = (m \times \mathbf{2} + n_0 \times \mathbf{1} + n_1 \times \mathbf{1}' + n_2 \times \mathbf{1}'' + n_3 \times \mathbf{1}'''), \qquad (4.6.1)$$

where the non-negative numbers m, n_i satisfy $2m + \sum_{i=0}^{3} n_i = 16$. In order for

Z_4, Z_4', Z_4'' and Z_2 abelian twists to be modular invariant, we have to demand

$$\begin{aligned}&\frac{5}{2}(m-1)+n_2+n_3 \in 2Z,\\&\frac{5}{2}(m-1)+n_1+n_2 \in 2Z,\\&\frac{5}{2}(m-1)+n_1+n_3 \in 2Z.\end{aligned} \tag{4.6.2}$$

The minimal-rank models correspond to solutions with m being a maximal value. It is easy to get $m=5$. The possible minimal-rank twist representations are

$$\begin{aligned}&(5\times\mathbf{2}+6\times\mathbf{1}),\\&(5\times\mathbf{2}+4\times\mathbf{1}+2\times\mathbf{1}'),\\&(5\times\mathbf{2}+2\times\mathbf{1}+2\times\mathbf{1}'+2\times\mathbf{1}''),\\&(5\times\mathbf{2}+3\times\mathbf{1}+\mathbf{1}'+\mathbf{1}''+\mathbf{1}'''),\end{aligned} \tag{4.6.3}$$

where the equivalences of the twist representation have been considered. Compared the first representation of (4.6.3) with that discussed in Section 3.2, we have one more copy of the two-dimensional representation acting on the left-fermions, but we also have the same extra copy acting on the right-fermions. Their extra contributions cancel each other when the vacuum phases are evaluated. Therefore we expect to have the same vacuum phases as (3.2.19) and (3.2.20), and global anomalies arise. Let us consider the second twist representation of (4.6.3), i.e.

$$(5\times\mathbf{2}+4\times\mathbf{1}+2\times\mathbf{1}';\mathbf{2}+\mathbf{1}+\mathbf{1}). \tag{4.6.4}$$

Although we leave the sixteen-dimensional representation undivided, we still can calculate some vacuum phases. Consider the Z_4 abelian twist, and we obtain

$$\begin{aligned}&\mathbf{a}|\Omega_e^{RRR}\rangle = e^{i\pi F_a}|\Omega_e^{RRR}\rangle,\\&\mathbf{a}|\Omega_{a^2}^{RRR}\rangle = e^{-i\pi(1+F_a)}|\Omega_{a^2}^{RRR}\rangle.\end{aligned} \tag{4.6.5}$$

For the $\mathbf{Z}_4'$ abelian twist, we have

$$\begin{aligned} \mathbf{b}|\Omega_e^{RRR}\rangle &= e^{i\pi(1+F_b)}|\Omega_e^{RRR}\rangle, \\ \mathbf{b}|\Omega_{b^2=a^2}^{RRR}\rangle &= e^{-i\pi(1+F_b)}|\Omega_{b^2=a^2}^{RRR}\rangle. \end{aligned} \tag{4.6.6}$$

Similar computation for the $\mathbf{Z}_4''$ abelian twist leads to

$$\begin{aligned} \mathbf{ab}|\Omega_e^{RRR}\rangle &= e^{i\pi(1+F_{ab})}|\Omega_e^{RRR}\rangle, \\ \mathbf{ab}|\Omega_{(ab)^2=a^2}^{RRR}\rangle &= e^{-i\pi(1+F_{ab})}|\Omega_{(ab)^2=a^2}^{RRR}\rangle. \end{aligned} \tag{4.6.7}$$

From (3.2.9), we have $\varepsilon(ab,g)=\varepsilon(a,g)\varepsilon(b,g)$ with g being e or a^2. This implies

$$\begin{aligned} F_a+1+F_b &= 1+F_{ab}+2\mathrm{Z}, \\ 1+F_a+1+F_b &= 1+F_{ab}+2\mathrm{Z}. \end{aligned} \tag{4.6.8}$$

Obviously global anomalies arise. With the same procedure we can show that the third representation of (4.6.3) also suffers from global anomalies. Therefore only the last of (4.6.3) is left free from global anomalies at this moment.

There are several ways to split it into two eight-dimensional representations,

$$\begin{aligned} &(3\times\mathbf{2}+\mathbf{1}'+\mathbf{1}, 2\times\mathbf{2}+\mathbf{1}'''+\mathbf{1}''+2\times\mathbf{1}), \\ &(3\times\mathbf{2}+\mathbf{1}'''+\mathbf{1}'', 2\times\mathbf{2}+\mathbf{1}'+3\times\mathbf{1}), \\ &(3\times\mathbf{2}+2\times\mathbf{1}, 2\times\mathbf{2}+\mathbf{1}'''+\mathbf{1}''+\mathbf{1}'+\mathbf{1}), \\ &(4\times\mathbf{2}, \mathbf{2}+\mathbf{1}'''+\mathbf{1}''+\mathbf{1}'+3\times\mathbf{1}), \end{aligned} \tag{4.6.9}$$

where the equivalences and modular invariance have been considered. We will show that the first two representations in (4.6.9) have global anomalies. Take the twist representation,

$$(3\times\mathbf{2}+\mathbf{1}'+\mathbf{1}, 2\times\mathbf{2}+\mathbf{1}'''+\mathbf{1}''+2\times\mathbf{1}; \mathbf{2}+\mathbf{1}+\mathbf{1}). \tag{4.6.10}$$

The phase that $\mathbf{b}$ generates on $|\Omega_e^{NRR}\rangle$ is

$$\zeta_b^{NRR}(e) = \zeta_{b2} = \omega_b \cdot \omega_2 - \zeta_{2b} = \frac{7}{4} - (\frac{7}{2} + \frac{1}{2}Q_{2b}) = -\frac{3}{4} - \frac{1}{2}Q_{2b} + \mathrm{Z}, \quad (4.6.11)$$

where the discrete torsion parameter Q_{2b} can be equal to 0 or 1. That implies

$$\mathbf{b}|\Omega_e^{NRR}\rangle = e^{i\pi(\frac{3}{2}+Q_{2b})}|\Omega_e^{NRR}\rangle. \quad (4.6.12)$$

From (3.2.6), one has $\varepsilon(aba^{-1}, aea^{-1}) = \varepsilon(b,e)$, which is $\varepsilon(a^2b = b^3, e) = \varepsilon(b,e)$. On the other hand, one has $\varepsilon(a^2b, e) = (\varepsilon(b,e))^3$ from (3.2.9). Therefore the representation requirement leads to $(\varepsilon(b,e))^2 = 1$. However, one obtains $(\varepsilon(b,e))^2 = \exp[i2\pi(\frac{3}{2} + Q_{2b})] = -1$ from (4.6.12). Obviously, the inconsistency occurs. As for the second twist representation in (4.6.9), one obtains the same formula as (4.6.12), so there also exist global anomalies.

Let us take the twist representation,

$$(\mathbf{3} \times \mathbf{2} + \mathbf{2} \times \mathbf{1}, \mathbf{2} \times \mathbf{2} + \mathbf{1}''' + \mathbf{1}'' + \mathbf{1}' + \mathbf{1}; \mathbf{2} + \mathbf{1} + \mathbf{1}), \quad (4.6.13)$$

to see the vacuum structure. The representation of the vacuum $|\Omega_a^{NNN}\rangle \oplus |\Omega_{a^3}^{NNN}\rangle$ can be calculated to be

$$\begin{aligned}
\mathbf{a}\begin{pmatrix} |\Omega_a^{NNN}\rangle \\ |\Omega_{a^3}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} e^{i\pi(Q_{2a}+Q_{3a})} & 0 \\ 0 & e^{i\pi(Q_{2a}+Q_{3a})} \end{pmatrix} \begin{pmatrix} |\Omega_a^{NNN}\rangle \\ |\Omega_{a^3}^{NNN}\rangle \end{pmatrix}, \\
\mathbf{b}\begin{pmatrix} |\Omega_a^{NNN}\rangle \\ |\Omega_{a^3}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_a^{NNN}\rangle \\ |\Omega_{a^3}^{NNN}\rangle \end{pmatrix}.
\end{aligned} \quad (4.6.14)$$

Similarly for the vacuum $|\Omega_b^{NNN}\rangle \oplus |\Omega_{a^2b}^{NNN}\rangle$, we have

$$\begin{aligned}
\mathbf{a}\begin{pmatrix} |\Omega_b^{NNN}\rangle \\ |\Omega_{a^2b}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_b^{NNN}\rangle \\ |\Omega_{a^2b}^{NNN}\rangle \end{pmatrix}, \\
\mathbf{b}\begin{pmatrix} |\Omega_b^{NNN}\rangle \\ |\Omega_{a^2b}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} e^{i\pi(Q_{2b}+Q_{3b})} & 0 \\ 0 & e^{i\pi(Q_{2b}+Q_{3b})} \end{pmatrix} \begin{pmatrix} |\Omega_b^{NNN}\rangle \\ |\Omega_{a^2b}^{NNN}\rangle \end{pmatrix},
\end{aligned} \quad (4.6.15)$$

and the representation of $|\Omega_{ab}^{NNN}\rangle \oplus |\Omega_{a^3b}^{NNN}\rangle$ is

$$\mathbf{a}\begin{pmatrix}|\Omega_{ab}^{NNN}\rangle\\|\Omega_{a^3b}^{NNN}\rangle\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}|\Omega_{ab}^{NNN}\rangle\\|\Omega_{a^3b}^{NNN}\rangle\end{pmatrix},$$
$$\mathbf{b}\begin{pmatrix}|\Omega_{ab}^{NNN}\rangle\\|\Omega_{a^3b}^{NNN}\rangle\end{pmatrix} = \begin{pmatrix}0 & e^{i\pi(Q_{2(ab)}+Q_{3(ab)})}\\e^{i\pi(Q_{2(ab)}+Q_{3(ab)})} & 0\end{pmatrix}\begin{pmatrix}|\Omega_{ab}^{NNN}\rangle\\|\Omega_{a^3b}^{NNN}\rangle\end{pmatrix}. \tag{4.6.16}$$

For the vacuum $|\Omega_{a^2}^{NNN}\rangle$ we obtain

$$\mathbf{a}|\Omega_{a^2}^{NNN}\rangle = |\Omega_{a^2}^{NNN}\rangle, \quad \mathbf{b}|\Omega_{a^2}^{NNN}\rangle = |\Omega_{a^2}^{NNN}\rangle. \tag{4.6.17}$$

We have checked that there are no global anomalies in the last two twist representations in (4.6.9). The minimal-rank models are presented in Table 9.

§4.7 Nonabelian Twists of the Double Dihedral Group $D_3^{(d)}$

The nonabelian group $D_3^{(d)}$ has twelve elements spread out in six classes: (e), (a^3), (a,a^5), (a^2,a^4), (b,a^2b,a^4b), (ab,a^3b,a^5b). It has two inequivalent two-dimensional irreducible representations. The $\mathbf{2}'$ can be taken as the definition of the group and is given by

$$a = \begin{pmatrix}e^{-i2\pi(1/6)} & 0\\0 & e^{-i2\pi(5/6)}\end{pmatrix}, \quad b = \begin{pmatrix}0 & e^{-i2\pi(1/4)}\\e^{-i2\pi(1/4)} & 0\end{pmatrix}, \tag{4.7.1}$$

while the $\mathbf{2}$ given by

$$a = \begin{pmatrix}e^{-i2\pi(1/3)} & 0\\0 & e^{-i2\pi(2/3)}\end{pmatrix}, \quad b = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}. \tag{4.7.2}$$

It also has four one-dimensional irreducible representations described by $\mathbf{1}(a=1,b=1), \mathbf{1}'(a=1,b=-1), \mathbf{1}''(a=-1,b=e^{-i2\pi\frac{1}{4}}), \mathbf{1}'''(a=-1,b=e^{-i2\pi\frac{3}{4}})$. The partition function of this nonabelian twist is

$$\mathcal{Z}_h(D_3^{(d)}) = \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_6) + \mathcal{Z}_h(\mathbf{Z}_4) - \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_2), \tag{4.7.3}$$

where the abelian group $\mathbf{Z}_6$ is generated by a, $\mathbf{Z}_4$ by b or a^2b or a^4b , and $\mathbf{Z}_2$ by a^3. For the representation D_X, we could choose $\mathbf{2}+\mathbf{1}'$, but it corresponds to the D_3 orbifold. The twist representation D_X has to be $\mathbf{2}'+\mathbf{1}$ in order to qualify a $D_3^{(d)}$ orbifold. There are two equivalences in twist representations. The representation under $\mathbf{1}\leftrightarrow\mathbf{1}', \mathbf{1}''\leftrightarrow\mathbf{1}'''$ corresponds to the interchange of NS fermions with R fermions in $b, a^2b, a^4b, ab, a^3b, a^5b$ sectors. The representation under $\mathbf{1}''\leftrightarrow\mathbf{1}'''$ corresponds to the interchange of one fermion with its complex conjugate.

Let us assign the twist representation of $D_3^{(d)}$ for the sixteen left-fermions,

$$D_\lambda = (m_0\times\mathbf{2}' + m_1\times\mathbf{2} + n_0\times\mathbf{1} + n_1\times\mathbf{1}' + n_2\times\mathbf{1}'' + n_3\times\mathbf{1}'''), \tag{4.7.4}$$

where the non-negative integers m_i, n_j satisfy $2(m_0+m_1)+\sum_{j=0}^{3} n_j = 16$. Modular invariance of $\mathbf{Z}_6$ and $\mathbf{Z}_4$ abelian twists demands

$$\begin{aligned} &\frac{13}{3}(m_0-1) + \frac{10}{3}m_1 + \frac{3}{2}(n_2+n_3) \in 2\mathbf{Z}, \\ &\frac{5}{2}(m_0-1) + m_1 + n_1 + \frac{1}{4}n_2 + \frac{9}{4}n_3 \in 2\mathbf{Z}, \end{aligned} \tag{4.7.5}$$

where modular invariance of a $\mathbf{Z}_2$ abelian twist is already included in the above. We rewrite (4.7.5) as

$$\begin{aligned} &\frac{1}{3}(m_0-1-2m_1) - \frac{1}{2}(n_2+n_3) \in 2\mathbf{Z}, \\ &\frac{1}{2}(m_0-1+\frac{1}{2}(n_2+n_3)) + m_1 + n_1 \in 2\mathbf{Z}. \end{aligned} \tag{4.7.6}$$

We want to find solutions with a maximal value of m_0+m_1 because they correspond to minimal-rank models. The necessary but *not* sufficient condition for holding (4.7.6) is

$$\begin{aligned} &n_2+n_3 \in 2\mathbf{Z}, \\ &m_0-1+m_1 \in 3\mathbf{Z}. \end{aligned} \tag{4.7.7}$$

One may guess that the maximal value of m_0+m_1 is 7. In fact it is correct. We list all the solutions of m_i, n_j of the minimal-rank models in Table 10, while the two equivalences have been taken into account. The split of the sixteen-dimensional twist representation into two eight-dimensional representations can also be worked out as long as modular invariance is considered and model duplications are eliminated.

To see the vacuum structure, one may take the following twist representation,

$$(4\times\mathbf{2}', 2\times\mathbf{2}'+\mathbf{2}+2\times\mathbf{1}''; \mathbf{2}'+\mathbf{1}+\mathbf{1}). \tag{4.7.8}$$

We present the result of vacuum representations for the twist of (4.7.8). The $D_3^{(d)}$ representation of the vacuum $|\Omega_a^{NNN}\rangle \oplus |\Omega_{a^5}^{NNN}\rangle$ is

$$\begin{aligned} \mathbf{a}\begin{pmatrix} |\Omega_a^{NNN}\rangle \\ |\Omega_{a^5}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |\Omega_a^{NNN}\rangle \\ |\Omega_{a^5}^{NNN}\rangle \end{pmatrix}, \\ \mathbf{b}\begin{pmatrix} |\Omega_a^{NNN}\rangle \\ |\Omega_{a^5}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_a^{NNN}\rangle \\ |\Omega_{a^5}^{NNN}\rangle \end{pmatrix}, \end{aligned} \tag{4.7.9}$$

where discrete torsions satisfy $Q_{2a} = Q_{3a} = 0$ because of the representation requirement. The representation of the vacuum $|\Omega_{a^2}^{NNN}\rangle \oplus |\Omega_{a^4}^{NNN}\rangle$ is

$$\begin{aligned} \mathbf{a}\begin{pmatrix} |\Omega_{a^2}^{NNN}\rangle \\ |\Omega_{a^4}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} e^{-i2\pi(1/3)} & 0 \\ 0 & e^{-i2\pi(2/3)} \end{pmatrix} \begin{pmatrix} |\Omega_{a^2}^{NNN}\rangle \\ |\Omega_{a^4}^{NNN}\rangle \end{pmatrix}, \\ \mathbf{b}\begin{pmatrix} |\Omega_{a^2}^{NNN}\rangle \\ |\Omega_{a^4}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{a^2}^{NNN}\rangle \\ |\Omega_{a^4}^{NNN}\rangle \end{pmatrix}. \end{aligned} \tag{4.7.10}$$

For the vacuum $|\Omega^{NNN}_{a^3}\rangle$ we have

$$\mathbf{a}|\Omega^{NNN}_{a^3}\rangle = -|\Omega^{NNN}_{a^3}\rangle, \quad \mathbf{b}|\Omega^{NNN}_{a^3}\rangle = e^{-i2\pi\frac{3}{4}}|\Omega^{NNN}_{a^3}\rangle. \tag{4.7.11}$$

The $D_3^{(d)}$ representation of $|\Omega^{NNN}_{b}\rangle \oplus |\Omega^{NNN}_{a^2b}\rangle \oplus |\Omega^{NNN}_{a^4b}\rangle$ is

$$\begin{aligned}
\mathbf{a}\begin{pmatrix}|\Omega^{NNN}_{b}\rangle\\ |\Omega^{NNN}_{a^2b}\rangle\\ |\Omega^{NNN}_{a^4b}\rangle\end{pmatrix} &= \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}|\Omega^{NNN}_{b}\rangle\\ |\Omega^{NNN}_{a^2b}\rangle\\ |\Omega^{NNN}_{a^4b}\rangle\end{pmatrix},\\
\mathbf{b}\begin{pmatrix}|\Omega^{NNN}_{b}\rangle\\ |\Omega^{NNN}_{a^2b}\rangle\\ |\Omega^{NNN}_{a^4b}\rangle\end{pmatrix} &= \begin{pmatrix}\alpha&0&0\\0&0&\alpha\\0&\alpha&0\end{pmatrix}\begin{pmatrix}|\Omega^{NNN}_{b}\rangle\\ |\Omega^{NNN}_{a^2b}\rangle\\ |\Omega^{NNN}_{a^4b}\rangle\end{pmatrix},
\end{aligned} \tag{4.7.12}$$

where $\alpha = e^{i\pi(1+Q_{2b}+Q_{3b})}$, and discrete torsions satisfy $Q_{2b} = Q_{2(a^2b)} = Q_{2(a^4b)}, Q_{3b} = Q_{3(a^2b)} = Q_{3(a^4b)}$. For the vacuum $|\Omega^{NNN}_{ab}\rangle \oplus |\Omega^{NNN}_{a^3b}\rangle \oplus |\Omega^{NNN}_{a^5b}\rangle$, we obtain

$$\begin{aligned}
\mathbf{a}\begin{pmatrix}|\Omega^{NNN}_{ab}\rangle\\ |\Omega^{NNN}_{a^3b}\rangle\\ |\Omega^{NNN}_{a^5b}\rangle\end{pmatrix} &= \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}|\Omega^{NNN}_{ab}\rangle\\ |\Omega^{NNN}_{a^3b}\rangle\\ |\Omega^{NNN}_{a^5b}\rangle\end{pmatrix},\\
\mathbf{b}\begin{pmatrix}|\Omega^{NNN}_{ab}\rangle\\ |\Omega^{NNN}_{a^3b}\rangle\\ |\Omega^{NNN}_{a^5b}\rangle\end{pmatrix} &= \begin{pmatrix}0&0&\alpha\\0&\alpha&0\\\alpha&0&0\end{pmatrix}\begin{pmatrix}|\Omega^{NNN}_{ab}\rangle\\ |\Omega^{NNN}_{a^3b}\rangle\\ |\Omega^{NNN}_{a^5b}\rangle\end{pmatrix}.
\end{aligned} \tag{4.7.13}$$

There are no global anomalies in this twist representation. Furthermore we have checked that there exist no global anomalies in all minimal-rank models. The twist representations and final gauge groups are listed in Table 11.

§4.8 Nonabelian Twists of the Double Tetrahedral Group $T^{(d)}$

The nonabelian group $T^{(d)}$ has twenty-four elements, seven classes (e), (a^3), (a, ab, ab', ab''), $(a^2, a^5b, a^5b', a^5b'')$, $(a^4, a^4b, a^4b', a^4b'')$, $(a^5, a^2b, a^2b', a^2b'')$,

$(b, b', b'', a^3b, a^3b', a^3b'')$, generated by

$$a = \frac{(1+i)}{2}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \tag{4.8.1}$$

where $b' = aba^{-1}, b'' = ab'a^{-1}$. It has three inequivalent two-dimensional irreducible representations, one of which $\mathbf{2}$ is described by the above defining matrix, and other two are $\mathbf{2}'$ by

$$a = \frac{(1+i)e^{-i2\pi\frac{1}{3}}}{2}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \tag{4.8.2}$$

and $\mathbf{2}''$ by

$$a = \frac{(1+i)e^{-i2\pi\frac{2}{3}}}{2}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \tag{4.8.3}$$

It has one three-dimensional irreducible representation $\mathbf{3}$ described by

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.8.4}$$

It also has three one-dimensional irreducible representations described by $\mathbf{1}(a = 1, b = 1), \mathbf{1}'(a = e^{-i2\pi\frac{1}{3}}, b = 1), \mathbf{1}''(a = e^{-i2\pi\frac{2}{3}}, b = 1)$. The partition function of this nonabelian twist is

$$\mathcal{Z}_h(T^{(d)}) = \mathcal{Z}_h(\mathbf{Z}_6) + \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_4) - \frac{1}{2}\mathcal{Z}_h(\mathbf{Z}_2), \tag{4.8.5}$$

where the abelian group $\mathbf{Z}_6$ is generated by a or ab or ab' or ab'', $\mathbf{Z}_4$ by b or b' or b'' , and $\mathbf{Z}_2$ by a^3.

Let us first consider the twist representation of the right-moving fields. We could choose D_X to be $\mathbf{3}$, but it actually corresponds to the nonabelian orbifold of

T, since $\mathbf{3}$ is a representation of T. To be a faithful representation of $T^{(d)}$, either $\mathbf{2}$ or $\mathbf{2}'$ or $\mathbf{2}''$ plus suitable one-dimensional representation can be chosen. Since the corresponding abelian twists should be in Table 1, there are three options for the twist representation D_X, which are $\mathbf{2}+\mathbf{1}, \mathbf{2}'+\mathbf{1}'$ and $\mathbf{2}''+\mathbf{1}''$. In these three cases, the twist parameters of a $\mathbf{Z}_4$ abelian orbifold are the same and equal to $(\frac{1}{4}, \frac{3}{4}, 0)$, while the twist parameters of a $\mathbf{Z}_6$ orbifold are $(\frac{1}{6}, \frac{5}{6}, 0), (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$ and $(\frac{5}{6}, \frac{1}{2}, \frac{2}{3})$ respectively. There are two equivalences in the twist representation. These two equivalences are the representation under $\mathbf{1}' \leftrightarrow \mathbf{1}''$ or $\mathbf{2}' \leftrightarrow \mathbf{2}''$ corresponding to the interchange of fermions with their complex conjugate. Therefore the two representations $\mathbf{2}'+\mathbf{1}'$ and $\mathbf{2}''+\mathbf{1}''$ are equivalent. Notice that the two twist representations $\mathbf{2}+\mathbf{1}$ and $\mathbf{2}'+\mathbf{1}'$ have $N=2$ and $N=1$ space-time supersymmetry respectively [36].

Let us first consider the case of the twist representation D_X being $\mathbf{2}+\mathbf{1}$, and assign the following representation for (16; 4) fermions,

$$(m_0 \times \mathbf{2} + m_1 \times \mathbf{2}' + m_2 \times \mathbf{2}'' + m_3 \times \mathbf{3} + n_0 \times \mathbf{1} + n_1 \times \mathbf{1}' + n_2 \times \mathbf{1}''; \mathbf{2}+\mathbf{1}+\mathbf{1}), \quad (4.8.6)$$

where the non-negative integers m_i, n_j satisfy $\sum_{i=0}^{2}(2m_i + n_i) + 3m_3 = 16$. Since the partition function of $T^{(d)}$ twist is given by (4.8.5), the modular invariance of the nonabelian twist requires $\mathbf{Z}_6, \mathbf{Z}_4$ and $\mathbf{Z}_2$ abelian twists to be modular invariant. The abelian twist parameters can be found by diagonalizing (4.8.6) with respect to a specific abelian group. Modular invariance conditions of the $\mathbf{Z}_6$ and $\mathbf{Z}_4$ abelian twists are

$$\begin{aligned} &\frac{13}{3}(m_0-1) + \frac{5}{3}m_1 + \frac{17}{3}m_2 + \frac{10}{3}m_3 + \frac{2}{3}n_1 + \frac{8}{3}n_2 \in 2\mathbf{Z}, \\ &\frac{5}{2}(m_0 - 1 + m_1 + m_2) \in 2\mathbf{Z}, \end{aligned} \quad (4.8.7)$$

where the condition from the $\mathbf{Z}_2$ twist is already included in the above equations. We can rewrite (4.8.7) as

$$\begin{aligned} &m_0 - 1 + m_1 + m_2 \in 4\mathbf{Z}, \\ &m_0 - 1 - m_1 - m_2 - 2m_3 + 2(n_1 + n_2) \in 6\mathbf{Z}. \end{aligned} \tag{4.8.8}$$

The minimal-rank models correspond to the twist representation with a maximal value of $m_0 + m_1 + m_2 + 2m_3$. After some calculation one can find solutions of (4.8.8) corresponding to the minimal-rank models. The twist representations are

$$\begin{aligned} &(5 \times \mathbf{2} + 2 \times \mathbf{3}; \mathbf{2} + \mathbf{1} + \mathbf{1}), \\ &(2 \times \mathbf{2} + 3 \times \mathbf{2'} + 2 \times \mathbf{3}; \mathbf{2} + \mathbf{1} + \mathbf{1}), \\ &(\mathbf{2} + 4 \times \mathbf{3} + \mathbf{1'} + \mathbf{1}; \mathbf{2} + \mathbf{1} + \mathbf{1}), \\ &(\mathbf{2'} + 4 \times \mathbf{3} + \mathbf{1'} + \mathbf{1''}; \mathbf{2} + \mathbf{1} + \mathbf{1}), \end{aligned} \tag{4.8.9}$$

where we have considered the two equivalences corresponding to the interchange of fermions with their complex conjugate. Notice that the twist representations in (4.8.9) can possibly reduce the rank of the gauge group by 9. Now we should check that whether global anomalies arise in these twist representations. Let us take the first one of (4.8.9), i.e.

$$(5 \times \mathbf{2} + 2 \times \mathbf{3}; \mathbf{2} + \mathbf{1} + \mathbf{1}). \tag{4.8.10}$$

The twist parameter of the $\mathbf{Z}_4$ abelian twist is

$$\omega_b = (\frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}; \frac{1}{4}, \frac{3}{4}, 0, 0). \tag{4.8.11}$$

The vacuum phases that $\mathbf{b}$ generates on $|\Omega_e^{RRR}\rangle$ and $|\Omega_{a^3}^{RRR}\rangle$ can be calculated,

$$\begin{aligned} &\zeta_b^{RRR}(e) = -\omega_b \cdot T - \frac{F_b}{2} = -\frac{F_b}{2} + \mathbf{Z}, \\ &\zeta_b^{RRR}(b^2 = a^3) = \zeta(0) + 2\zeta_{bb} - \omega_b \cdot \lfloor 2\omega_b \rfloor = \frac{F_b}{2} + \frac{1}{2} + \mathbf{Z}. \end{aligned} \tag{2.8.12}$$

This implies

$$\mathbf{b}|\Omega_e^{RRR}\rangle = e^{i\pi F_b}|\Omega_e^{RRR}\rangle,$$
$$\mathbf{b}|\Omega_{a^3}^{RRR}\rangle = e^{i\pi(F_b+1)}|\Omega_{a^3}^{RRR}\rangle. \tag{4.8.13}$$

Since $|\Omega_e^{RRR}\rangle$ is in a one-dimensional representation and $b = 1$ in all three one-dimensional representations, we have to demand $F_b = 0 + 2\mathbf{Z}$. On the other hand, $|\Omega_{a^3}^{RRR}\rangle$ should also be in a one-dimensional representation since the group element a^3 itself forms a class, so we have to set $F_b = 1+2\mathbf{Z}$. This contradiction indicates the existence of global anomalies. With the same procedure, i.e. calculating the phases that $\mathbf{b}$ generates on the vacua $|\Omega_e^{RRR}\rangle$ and $|\Omega_{a^3}^{RRR}\rangle$, we know that the second twist representation in (4.8.9) also suffers from global anomalies. The third and the last ones in (4.8.9) are free from global anomalies at this moment, but we should check them further when the sixteen-dimensional twist representation is divided into two eight-dimensional ones. The divided representations are

$$(\mathbf{2} + 2\times\mathbf{3}, 2\times\mathbf{3} + \mathbf{1}' + \mathbf{1}; \mathbf{2} + \mathbf{1} + \mathbf{1}),$$
$$(\mathbf{2}' + 2\times\mathbf{3}, 2\times\mathbf{3} + \mathbf{1}'' + \mathbf{1}'; \mathbf{2} + \mathbf{1} + \mathbf{1}). \tag{4.8.14}$$

Consider the first one of (4.8.14), i.e.

$$(\mathbf{2} + 2\times\mathbf{3}, 2\times\mathbf{3} + \mathbf{1}' + \mathbf{1}; \mathbf{2} + \mathbf{1} + \mathbf{1}). \tag{4.8.15}$$

We can calculate the vacuum phases. In order for the vacua to form representations of $T^{(d)}$, we have to demand $Q_{2a} = Q_{3a} = Q_{3b} = 0$ and $Q_{2b} = 1$. The same procedure for the second twist representation of (4.8.14) also leads to $Q_{2a} = Q_{3a} = Q_{3b} = 0$ and $Q_{2b} = 1$. There are no global anomalies in these two twist representations.

So far we only considered the case with D_X being $\mathbf{2} + \mathbf{1}$. Let us look at the other case with D_X being $\mathbf{2}' + \mathbf{1}'$, and assign the twist representation as

$$(m'_0 \times \mathbf{2} + m'_1 \times \mathbf{2}' + m'_2 \times \mathbf{2}'' + m'_3 \times \mathbf{3} + n'_0 \times \mathbf{1} + n'_1 \times \mathbf{1}' + n'_2 \times \mathbf{1}''; \mathbf{2}' + \mathbf{1}' + \mathbf{1}), \quad (4.8.16)$$

where the non-negative integers m'_i, n'_j satisfy $\sum_{i=0}^{2}(2m'_i + n'_i) + 3m'_3 = 16$. Since a phase for the partition function generated from modular transformations by the right-fermions is always the opposite sign of that by the left-fermions, the moduiar invariance conditions (4.8.7) have a term with a factor $(m_0 - 1)$ corresponding to the contribution by m_0 copies of $\mathbf{2}$ in the twist of the left-fermions and one copy of $\mathbf{2}$ in the twist of the right-fermions. This observation allows us to obtain modular invariance conditions for the twist (4.8.16) from (4.8.8) with the following substitutions: $m_0 - 1 \sim m'_0, m_1 \sim m'_1 - 1, n_1 \sim n'_1 - 1, m_i \sim m'_i$ $(i = 2, 3), n_2 \sim n'_2$. It turns out

$$\begin{aligned} &m'_0 + m'_1 - 1 + m'_2 \in 4\mathbf{Z}, \\ &m'_0 - (m'_1 - 1) - m'_2 - 2m'_3 + 2(n'_1 - 1 + n'_2) \in 6\mathbf{Z}. \end{aligned} \quad (4.8.17)$$

We rewrite the above as

$$\begin{aligned} &m'_0 - 1 + m'_1 + m'_2 \in 4\mathbf{Z}, \\ &m'_0 - 1 - m'_1 - m'_2 - 2m'_3 + 2(n'_1 + n'_2) \in 6\mathbf{Z}. \end{aligned} \quad (4.8.18)$$

Obviously (4.8.18) is the same as (4.8.8). By using (4.8.9) we may write down the twist representations of the minimal-rank models with D_X being $\mathbf{2}' + \mathbf{1}'$,

$$\begin{aligned} &(5 \times \mathbf{2} + 2 \times \mathbf{3}; \mathbf{2}' + \mathbf{1}' + \mathbf{1}), \\ &(2 \times \mathbf{2} + 3 \times \mathbf{2}' + 2 \times \mathbf{3}; \mathbf{2}' + \mathbf{1}' + \mathbf{1}), \\ &(\mathbf{2} + 4 \times \mathbf{3} + \mathbf{1}' + \mathbf{1}; \mathbf{2}' + \mathbf{1}' + \mathbf{1}), \\ &(\mathbf{2}' + 4 \times \mathbf{3} + \mathbf{1}' + \mathbf{1}''; \mathbf{2}' + \mathbf{1}' + \mathbf{1}). \end{aligned} \quad (4.8.19)$$

Similarly, we can show that the first two representations in (4.8.19) suffer from global anomalies. Splitting the sixteen-dimensional twist representations, we have

$$\begin{aligned}&(\mathbf{2}+2\times\mathbf{3},2\times\mathbf{3}+\mathbf{1}'+\mathbf{1};\mathbf{2}'+\mathbf{1}'+\mathbf{1}),\\&(\mathbf{2}'+2\times\mathbf{3},2\times\mathbf{3}+\mathbf{1}''+\mathbf{1}';\mathbf{2}'+\mathbf{1}'+\mathbf{1}).\end{aligned}\tag{4.8.20}$$

In the both above twist representations, the representation requirement leads to $Q_{2a}=Q_{3a}=Q_{3b}=0, Q_{2b}=1$. There are also no global anomalies.

To see the vacuum structure, we present the result of vacua $|\Omega_s^{NNN}\rangle$ for the model with the twist (4.8.15). For vacuum $|\Omega_{a^3}^{NNN}\rangle$ we have

$$\mathbf{a}|\Omega_{a^3}^{NNN}\rangle=|\Omega_{a^3}^{NNN}\rangle,\quad \mathbf{b}|\Omega_{a^3}^{NNN}\rangle=|\Omega_{a^3}^{NNN}\rangle.\tag{4.8.21}$$

The representation of $|\Omega_{a}^{NNN}\rangle\oplus|\Omega_{ab}^{NNN}\rangle\oplus|\Omega_{ab'}^{NNN}\rangle\oplus|\Omega_{ab''}^{NNN}\rangle$ is calculated to be

$$\begin{aligned}\mathbf{a}\begin{pmatrix}|\Omega_{a}^{NNN}\rangle\\|\Omega_{ab}^{NNN}\rangle\\|\Omega_{ab'}^{NNN}\rangle\\|\Omega_{ab''}^{NNN}\rangle\end{pmatrix}&=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{pmatrix}\begin{pmatrix}|\Omega_{a}^{NNN}\rangle\\|\Omega_{ab}^{NNN}\rangle\\|\Omega_{ab'}^{NNN}\rangle\\|\Omega_{ab''}^{NNN}\rangle\end{pmatrix},\\
\mathbf{b}\begin{pmatrix}|\Omega_{a}^{NNN}\rangle\\|\Omega_{ab}^{NNN}\rangle\\|\Omega_{ab'}^{NNN}\rangle\\|\Omega_{ab''}^{NNN}\rangle\end{pmatrix}&=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}|\Omega_{a}^{NNN}\rangle\\|\Omega_{ab}^{NNN}\rangle\\|\Omega_{ab'}^{NNN}\rangle\\|\Omega_{ab''}^{NNN}\rangle\end{pmatrix},\end{aligned}\tag{4.8.22}$$

while the representation of $|\Omega_{a^5}^{NNN}\rangle\oplus|\Omega_{a^2b}^{NNN}\rangle\oplus|\Omega_{a^2b'}^{NNN}\rangle\oplus|\Omega_{a^2b''}^{NNN}\rangle$ is

$$\begin{aligned}\mathbf{a}\begin{pmatrix}|\Omega_{a^5}^{NNN}\rangle\\|\Omega_{a^2b}^{NNN}\rangle\\|\Omega_{a^2b'}^{NNN}\rangle\\|\Omega_{a^2b''}^{NNN}\rangle\end{pmatrix}&=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{pmatrix}\begin{pmatrix}|\Omega_{a^5}^{NNN}\rangle\\|\Omega_{a^2b}^{NNN}\rangle\\|\Omega_{a^2b'}^{NNN}\rangle\\|\Omega_{a^2b''}^{NNN}\rangle\end{pmatrix},\\
\mathbf{b}\begin{pmatrix}|\Omega_{a^5}^{NNN}\rangle\\|\Omega_{a^2b}^{NNN}\rangle\\|\Omega_{a^2b'}^{NNN}\rangle\\|\Omega_{a^2b''}^{NNN}\rangle\end{pmatrix}&=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}|\Omega_{a^5}^{NNN}\rangle\\|\Omega_{a^2b}^{NNN}\rangle\\|\Omega_{a^2b'}^{NNN}\rangle\\|\Omega_{a^2b''}^{NNN}\rangle\end{pmatrix}.\end{aligned}\tag{4.8.23}$$

For the vacuum $|\Omega_{a^2}^{NNN}\rangle \oplus |\Omega_{a^5 b}^{NNN}\rangle \oplus |\Omega_{a^5 b'}^{NNN}\rangle \oplus |\Omega_{a^5 b''}^{NNN}\rangle$, we obtain

$$
\begin{aligned}
\mathbf{a}\begin{pmatrix} |\Omega_{a^2}^{NNN}\rangle \\ |\Omega_{a^5 b}^{NNN}\rangle \\ |\Omega_{a^5 b'}^{NNN}\rangle \\ |\Omega_{a^5 b''}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{a^2}^{NNN}\rangle \\ |\Omega_{a^5 b}^{NNN}\rangle \\ |\Omega_{a^5 b'}^{NNN}\rangle \\ |\Omega_{a^5 b''}^{NNN}\rangle \end{pmatrix}, \\
\mathbf{b}\begin{pmatrix} |\Omega_{a^2}^{NNN}\rangle \\ |\Omega_{a^5 b}^{NNN}\rangle \\ |\Omega_{a^5 b'}^{NNN}\rangle \\ |\Omega_{a^5 b''}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{a^2}^{NNN}\rangle \\ |\Omega_{a^5 b}^{NNN}\rangle \\ |\Omega_{a^5 b'}^{NNN}\rangle \\ |\Omega_{a^5 b''}^{NNN}\rangle \end{pmatrix},
\end{aligned}
\tag{4.8.24}
$$

while for the vacuum $|\Omega_{a^4}^{NNN}\rangle \oplus |\Omega_{a^4 b}^{NNN}\rangle \oplus |\Omega_{a^4 b'}^{NNN}\rangle \oplus |\Omega_{a^4 b''}^{NNN}\rangle$, we have

$$
\begin{aligned}
\mathbf{a}\begin{pmatrix} |\Omega_{a^4}^{NNN}\rangle \\ |\Omega_{a^4 b}^{NNN}\rangle \\ |\Omega_{a^4 b'}^{NNN}\rangle \\ |\Omega_{a^4 b''}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{a^4}^{NNN}\rangle \\ |\Omega_{a^4 b}^{NNN}\rangle \\ |\Omega_{a^4 b'}^{NNN}\rangle \\ |\Omega_{a^4 b''}^{NNN}\rangle \end{pmatrix}, \\
\mathbf{b}\begin{pmatrix} |\Omega_{a^4}^{NNN}\rangle \\ |\Omega_{a^4 b}^{NNN}\rangle \\ |\Omega_{a^4 b'}^{NNN}\rangle \\ |\Omega_{a^4 b''}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{a^4}^{NNN}\rangle \\ |\Omega_{a^4 b}^{NNN}\rangle \\ |\Omega_{a^4 b'}^{NNN}\rangle \\ |\Omega_{a^4 b''}^{NNN}\rangle \end{pmatrix}.
\end{aligned}
\tag{4.8.25}
$$

Finally, the representation of the vacuum $|\Omega_{b}^{NNN}\rangle \oplus |\Omega_{b'}^{NNN}\rangle \oplus |\Omega_{b''}^{NNN}\rangle \oplus |\Omega_{a^3 b}^{NNN}\rangle \oplus$ $|\Omega_{a^3 b'}^{NNN}\rangle \oplus |\Omega_{a^3 b''}^{NNN}\rangle$ is calculated to be

$$
\begin{aligned}
\mathbf{a}\begin{pmatrix} |\Omega_{b}^{NNN}\rangle \\ |\Omega_{b'}^{NNN}\rangle \\ |\Omega_{b''}^{NNN}\rangle \\ |\Omega_{a^3 b}^{NNN}\rangle \\ |\Omega_{a^3 b'}^{NNN}\rangle \\ |\Omega_{a^3 b''}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{b}^{NNN}\rangle \\ |\Omega_{b'}^{NNN}\rangle \\ |\Omega_{b''}^{NNN}\rangle \\ |\Omega_{a^3 b}^{NNN}\rangle \\ |\Omega_{a^3 b'}^{NNN}\rangle \\ |\Omega_{a^3 b''}^{NNN}\rangle \end{pmatrix}, \\
\mathbf{b}\begin{pmatrix} |\Omega_{b}^{NNN}\rangle \\ |\Omega_{b'}^{NNN}\rangle \\ |\Omega_{b''}^{NNN}\rangle \\ |\Omega_{a^3 b}^{NNN}\rangle \\ |\Omega_{a^3 b'}^{NNN}\rangle \\ |\Omega_{a^3 b''}^{NNN}\rangle \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\Omega_{b}^{NNN}\rangle \\ |\Omega_{b'}^{NNN}\rangle \\ |\Omega_{b''}^{NNN}\rangle \\ |\Omega_{a^3 b}^{NNN}\rangle \\ |\Omega_{a^3 b'}^{NNN}\rangle \\ |\Omega_{a^3 b''}^{NNN}\rangle \end{pmatrix}.
\end{aligned}
\tag{4.8.26}
$$

We present all minimal-rank models of this nonabelian group in Table 12.

§4.9 Discussions on Twists by Other Finite Subgroups of $SU(3)$

As we argued in Section 2.2, the twist group should be finite subgroups of $SU(3)$ in order to have space-time supersymmetry. The finite subgroups of $SU(3)$ can be divided by two categories [48]: the finite subgroups of $SU(2)$ and the finite subgroups of $SU(3)$ but not of $SU(2)$. Concerning the first category, we have not considered the groups $D_4^{(d)}, D_6^{(d)}, O^{(d)}, I$ and $I^{(d)}$. It turns out that there exist no nonabelian orbifolds for these groups because there are no abelian orbifolds corresponding to their abelian subgroups.

Let us take the $D_4^{(d)}$ nonabelian group as an example to show that the $D_4^{(d)}$ nonabelian orbifold does not exist. This group has sixteen elements, seven classes (e), (a^4), (a^2, a^6), (a, a^7), (a^3, a^5), (b, a^2b, a^4b, a^6b), (ab, a^3b, a^5b, a^7b), generated by

$$a = \begin{pmatrix} e^{-i2\pi(1/8)} & 0 \\ 0 & e^{-i2\pi(7/8)} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & e^{-i2\pi(1/4)} \\ e^{-i2\pi(1/4)} & 0 \end{pmatrix}. \tag{4.9.1}$$

It has irreducible representations $\mathbf{2}$ described by the above defining matrix, $\mathbf{2}'$ by

$$a = \begin{pmatrix} e^{-i2\pi(3/8)} & 0 \\ 0 & e^{-i2\pi(5/8)} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & e^{-i2\pi(1/4)} \\ e^{-i2\pi(1/4)} & 0 \end{pmatrix}, \tag{4.9.2}$$

$\mathbf{2}''$ by

$$a = \begin{pmatrix} e^{-i2\pi(1/4)} & 0 \\ 0 & e^{-i2\pi(3/4)} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \tag{4.9.3}$$

and $\mathbf{1}(a=1, b=1), \mathbf{1}'(a=1, b=-1), \mathbf{1}''(a=-1, b=1), \mathbf{1}'''(a=-1, b=-1)$. We could take the twist representation D_X to be $\mathbf{2}'' + \mathbf{1}'$, but it is the D_4 nonabelian orbifold, since $\mathbf{2}'' + \mathbf{1}'$ is a representation of D_4. If we choose D_X to be $\mathbf{2}$ or $\mathbf{2}'$ plus some one-dimensional representation, the twist parameters corresponding to the abelian subgroup $\mathbf{Z}_8$ are not in Table 1. Therefore there is no $D_4^{(d)}$ nonabelian orbifold. With the same procedure for the groups $D_6^{(d)}, O^{(d)}, I$ and $I^{(d)}$, we found that there exist no orbifolds corresponding to these groups.

Results from twists by any other finite nonabelian subgroups of $SU(3)$ can be obtained similarly. We expect that more promising models can be obtained if the nonabelian subgroup is chosen appropriately.

Chapter 5

Summary and Conclusions

In this chapter we recapitulate the major findings of this thesis and make some comments about the possible applications and extensions of the present work.

We discuss in this thesis four-dimensional string theories defined by free fields on the world-sheet satisfying a set of twisted boundary conditions. In Chapter 2, conditions leading to a consistent and desirable string are discussed for abelian twists. The partition function of the string is required to be modular invariant, which in turn limits the allowed twist parameters, vacuum phases, and fermionic numbers. Four-dimensional Lorentz invariance is satisfied only when fermionic numbers and discrete torsions are suitably related, and space-time supersymmetry places further restrictions on the twist parameters and the vacuum phases. Furthermore, crystallographic requirements reduce the consistent twist parameters for the right-moving fields to those listed in Table 1.

Nonabelian boundary conditions have been discussed in Chapter 3. Instead of restricting ourselves to "standard embedding", we allow the most general twist for right-moving and left-moving fermions. The Hilbert space of a nonabelian twist is constructed, which has many new features compared to the abelian case. A general formula of the partition function for a nonabelian twist is derived, and it turns out to be a linear combination of partition functions of abelian twists by abelian subgroups of the nonabelian group. Therefore, one can discuss the partition functions, vacuum phases, and mass spectra from its abelian subgroups by using the technique developed in Chapter 2. It is necessary for a consistent string to require all the constraints given by the twists from the abelian subgroups. However, this is not sufficient. The consistent structure of the Hilbert space requires not only the string fields to form a representation of the nonabelian group, but the vacua as well. The vacuum phases are calculated from the twist parameters and they are not necessarily in a representation. The failure for the vacua to be a representation leads to global anomalies, which have been known to exist in the path integral formalism. This problem does not arise in the abelian case because the Hilbert space structure there is much simpler. The advantage of the present operator formalism is that one can identify and correct for the global anomalies purely within one loop, unlike the path integral formalism where multi-loop information is needed. Once the well-behaved Hilbert space has been constructed, one can extract from the model the massless spectrum. The rank of the gauge group is generally reduced, with the generators of the final Lie algebra being some nontrivial combination of the generators of the Lie algebra $E_8 \times E_8'$. We have given a general procedure to identify the Lie algebra from the massless states. The resulting gauge group is not always embedded in $E_8 \times E_8'$ in a regular way.

Four-dimensional string models obtained from the heterotic string by an abelian twist suffer from having too high a rank for the gauge group. Non-abelian twist models can potentially correct this problem since such twists can reduce the rank of the gauge group, whereas abelian twists cannot. For this reason one would like to choose the nonabelian twist representation to give a minimal rank to the resulting gauge group. These minimal-rank models are systematically analysed in Chapter 4 for twists from all finite nonabelian subgroups of $SU(2)$. Twist parameters, vacuum phases, and gauge groups are obtained; systematic elimination of global anomalies are discussed in each case. Although the Standard Model gauge group has not emerged from this analysis, we obtain gauge groups such as $SU(5) \times U(1) \times SO(7)'^2$ (from the octahedral group) and $SU(3) \times SU(2) \times G_2' \times SU(3)'$ (from the double tetrahedral group) which have small enough ranks in the non-hidden sector and are close enough to the Standard Model to make them interesting.

In summary, we have presented systematically the methods and the results for model building from the nonabelian twist in the operator formalism. The Hilbert space of the nonabelian twist is very different from that of the abelian case, and as a result, group theory is required to keep a consistent structure in the Hilbert space. Global anomalies are eliminated by requiring the vacua to be in a representation of the nonabelian group. The generators of the final Lie algebra describing the gauge symmetry of the theory are obtained by making appropriate linear combination of the generators of the algebra $E_8 \times E_8'$. Since it is desirable to obtain models with a low-rank gauge group, we have classified all the minimal-rank string from the twists of finite nonabelian subgroups of $SU(2)$. This provides an important step forward to the construction of realistic models from a nonabelian twist.

Much further work remains to be done with the nonabelian twist models. As aforementioned, the twist group is to be a finite subgroup of $SU(3)$ in order to have space-time supersymmetry. It is straightforward to extend the present work to twist models by finite nonabelian subgroups of $SU(3)$ not in $SU(2)$.

In our discussions above we have presented the gauge group in each case, which is actually the most difficult part to compute among the whole spectra. Other consequences, such as the number of generations and representations of the chiral fermions, can also be worked out and is usually straightforward. Furthermore, paralleling the discussion in the realistic models obtained from abelian twist or the real fermion formulation, one can also study the phenomenologies of nonabelian twist models, including the Yukawa couplings, proton stability, neutrino masses, quark-lepton masses and so on.

A slight variation of our formalism could also lead to another interesting class of models. Recall that we fermionized the sixteen left-moving real bosons into sixteen complex fermions at the beginning in describing the boundary conditions. It is known that the bosonic shift is equivalent to the fermionic twist in the abelian case. However this is no longer true in the nonabelian twist. Therefore we may keep left-moving fields to be all bosonic and impose instead the nonabelian twist and also shift on these fields at the beginning. We expect that the Hilbert space structure will be similar, but the spectrum will be different since there will be bosonic modes and momenta instead of fermionic modes in the massless states. Therefore a new class of models will be expected. In principle, there will be no technical difficulties in building this kind of models.

As we have seen, in constructing low-rank realistic models, four-dimensional strings obtained via nonabelian twists have their advantages. Much more work still need to be done in order to have a complete phenomenological analysis of this kind of strings. The present thesis shows how far we can go using only the twist of a single nonabelian subgroup of $SU(2)$, so more promising models are expected if one considers other nonabelian subgroups of $SU(3)$. Inspired by the recent LEP data and the realistic models obtained by abelian twist and the real fermion formulation, it is reasonable to hope that the nonabelian twist technique will lead to its own realistic models, which may eventually be tested by the experiment.

Appendix A

Modular Transformations

In this appendix we will derive modular transformations of the function $f(u,v|\tau)$ and discuss the function $b(u,v|\tau)$, which are used in Chapter 2.

The Jacobi function $\vartheta_1(\nu|\tau)$ is defined by [5,49]

$$\vartheta_1(\nu|\tau) = 2Gq^{\frac{1}{8}} \sin \pi v \prod_{n=1}^{\infty} (1 - q^n e^{i2\pi\nu})(1 - q^n e^{-i2\pi\nu}), \tag{A.1}$$

where $G = \prod_{n=1}^{\infty}(1-q^n)$, and $q = e^{i2\pi\tau}$. The modular transformations for this Jacobi function are given by

$$\begin{aligned} \vartheta_1(\nu|\tau+1) &= e^{i\frac{\pi}{4}}\vartheta_1(\nu|\tau), \\ \vartheta_1(-\frac{\nu}{\tau}|-\frac{1}{\tau}) &= e^{i\frac{\pi}{4}}\tau^{\frac{1}{2}}e^{i\frac{\pi\nu^2}{\tau}}\vartheta_1(\nu|\tau). \end{aligned} \tag{A.2}$$

The Dedekind function $\eta(\tau)$ is defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n). \tag{A.3}$$

It has the following modular transformations

$$\begin{aligned} \eta(\tau+1) &= e^{i\frac{\pi}{12}}\eta(\tau), \\ \eta(-\frac{1}{\tau}) &= e^{-i\frac{\pi}{4}}\tau^{\frac{1}{2}}\eta(\tau). \end{aligned} \tag{A.4}$$

In order to represent the function $f(u,v|\tau)$ in terms of the Jacobi function and the Dedekind function, one may rewrite the Jacobi function as

$$\begin{aligned} \vartheta_1(\nu|\tau) &= i\eta(\tau)q^{\frac{1}{12}}(e^{-i\pi\nu} - e^{i\pi\nu}) \prod_{n=1}^{\infty}(1 - q^n e^{i2\pi\nu})(1 - q^n e^{-i2\pi\nu}) \\ &= i\eta(\tau)q^{\frac{1}{12}}e^{-i\pi\nu} \prod_{n=1}^{\infty}(1 - q^{n-1}e^{i2\pi\nu})(1 - q^n e^{-i2\pi\nu}). \end{aligned} \tag{A.5}$$

Therefore the function $f(u,v|\tau)$ can be expressed by $\vartheta_1(\nu|\tau)$ and $\eta(\tau)$,

$$\begin{aligned} f(u,v|\tau) &= q^{-\frac{1}{2}v(1-v)+\frac{1}{12}}\prod_{n=1}^{\infty}(1-q^{n-v}e^{i2\pi u})(1-q^{n-1+v}e^{-i2\pi u}) \\ &= e^{i\pi\tau(v^2-v)}q^{\frac{1}{12}}\prod_{n=1}^{\infty}(1-q^{n}e^{-i2\pi(v\tau-u)})(1-q^{n-1}e^{i2\pi(v\tau-u)}) \\ &= e^{i\pi(\tau v^2-u-\frac{1}{2})}\vartheta_1(-u+v\tau|\tau)/\eta(\tau). \end{aligned} \tag{A.6}$$

Since we know the modular transformations of the Jacobi function and the Dedekind function, it is straightforward to derive the modular transformations of the function $f(u,v|\tau)$. The modular transformation $\tau\to\tau+1$ for the function $f(u,v|\tau)$ is

$$\begin{aligned} f(u,v|\tau+1) &= e^{i\pi(\tau v^2-u-\frac{1}{2}+v^2)}\vartheta_1(-u+v+v\tau|\tau+1)/\eta(\tau+1) \\ &= e^{i\pi(\tau v^2-(u-v)-\frac{1}{2}+v^2-v+\frac{1}{6})}\vartheta_1(-u+v+v\tau|\tau)/\eta(\tau) \\ &= e^{i\pi(v^2-v+\frac{1}{6})}f(u-v,v|\tau), \end{aligned} \tag{A.7}$$

while the modular transformation $\tau\to-\frac{1}{\tau}$ is

$$\begin{aligned} f(u,v|-\frac{1}{\tau}) &= e^{i\pi(-\frac{v^2}{\tau}-u-\frac{1}{2})}\vartheta_1(-u-\frac{v}{\tau}|-\frac{1}{\tau})/\eta(-\frac{1}{\tau}) \\ &= e^{i\pi(\tau u^2+2uv-u)}\vartheta_1(v+u\tau|\tau)/\eta(\tau) \\ &= e^{i\pi(\tau u^2+2uv-u-1)}\vartheta_1(-1+v+u\tau|\tau)/\eta(\tau) \\ &= e^{i\pi(2uv-u-v+\frac{1}{2})}f(1-v,u|\tau). \end{aligned} \tag{A.8}$$

Let us discuss the modular transformations of $b(u,v|\tau)$. For the case u or $v\neq 0$, since $b(u,v|\tau)=f^{-1}(u,v|\tau)$, the modular transformations for $b(u,v|\tau)$ are just followed those of $f(u,v|\tau)$. This is obvious. However, there are two cases for the modular transformations of

$$b(0,0|\tau)=\eta^{-2}(\tau)\sum_p \exp(i\pi p^2). \tag{A.9}$$

First the summation will be interpreted as an integration for the four-dimensional space-time coordinates, since the physical momentun can be any value. Second the summation will be over some momentum lattice for the compactified coordinates. In the first case, notice that the physical momenta for the left- and right-moving should be the same, so the contribution to the partition function by both a left-moving complex boson and a right-moving complex boson is calculated to be

$$\begin{aligned} b(0,0|\tau)[b(0,0|\tau)]^* =& \eta^{-2}(\tau)\eta^{-2}(\bar{\tau}) \int e^{i\pi\tau p^2} e^{-i\pi\bar{\tau}p^2} d^2p \\ =& \eta^{-2}(\tau)\eta^{-2}(\bar{\tau}) \int e^{-2\pi Im(\tau)p^2} d^2p \\ =& [2Im(\tau)]^{-1}\eta^{-2}(\tau)\eta^{-2}(\bar{\tau}) \qquad (Im(\tau) > 0), \end{aligned} \tag{A.10}$$

where $\bar{\tau}$ is the complex conjugate of τ, and $Im(\tau)$ is the imaginary part of τ. Notice that $Im(-1/\tau) = Im(\tau)/|\tau|^2$. Using (A.4), it is easy to show that $|b(0,0|\tau)|^2$ is modular invariance. In the second case, it is known that $|b(0,0|\tau)|^2$ is also modular invariance as long as the momentum lattice is a self-dual integral lattice [23]. From (A.10) one also can know that there is a ultraviolet divergence when $Im(\tau) \leq 0$. If a theory has modular invariance, one can restrict the integral over τ within the fundamental domain $(Im(\tau) > 0, \{-\frac{1}{2} \leq Re(\tau) \leq 0, |\tau| \geq 1\} \cup \{0 < Re(\tau) < \frac{1}{2}, |\tau| > 1\})$ [50], then the divergence can be avoided.

Appendix B

Derivation of Vacuum Parameters

In this appendix we would like to derive the equations (2.1.24)-(2.1.26) by solving (2.1.23). Let $T = ((\frac{1}{2})^{16}; (\frac{1}{2})^{4})$, $\sum_{i=1}^{20} \epsilon^a \omega^a(r)\omega^a(s) \equiv \omega(r)\cdot\omega(s)$, and we rewrite (2.1.23) as the following [21],

$$\omega(s)\cdot\omega(s) - 2\omega^a(s)\cdot T - 2\sum_{i=1}^{A} s_i\zeta_i(s) \in 2\mathbf{Z}, \tag{B.1}$$

$$\omega(r)\cdot\omega(s) - [\omega(r)+\omega(s)]\cdot T - \sum_{i=1}^{A}[s_i\zeta_i(r) + r_i\zeta_i(s)] + \frac{1}{2}[F(r) - F(s)] \in \mathbf{Z}. \tag{B.2}$$

Let δ_i be a vector of $s = (s_1, s_2, \cdots, s_A)$ with all entries zero except the ith, which is equal to 1. We can decompose $\omega(s), F(s)$ and $\zeta_i(s)$ as

$$\begin{aligned}
\omega(s) &= \sum_{i=1}^{A} s_i\omega_i - \omega'(s), \\
F(s) &= \sum_{i=1}^{A} s_i F_i + F'(s), \qquad\qquad (B.3)\\
\zeta_i(s) &= \zeta_i(0) + \sum_{i=1}^{A} s_j\zeta_{ij} + \zeta_i'(s),
\end{aligned}$$

where $0 \le \omega_i, \omega(s) < 1, \omega'(s) \in \mathbf{Z}, \omega'(0) = \omega'(\delta_i) = 0, F'(0) = F'(\delta_i) = 0$ and $\zeta_i'(0) = \zeta_i'(\delta_j) = 0$.

Let us substitute respectively $r = 0, s = \delta_i$, and $r = \delta_i, s = 0$ as well in eq. (B.2), and we obtain

$$-\omega_i\cdot T - \zeta_i(0) \mp \frac{F_i}{2} \in \mathbf{Z}. \tag{B.4}$$

From the above one obtains

$$F_i \in \mathbf{Z},$$
$$\zeta_i(0) = -\omega_i \cdot T - \frac{F_i}{2} + \mathbf{Z}. \tag{B.5}$$

Setting $s = \delta_i$ in (B.1), one has

$$\omega_i^2 - 2\omega_i \cdot T - 2\zeta_i(0) - 2\zeta_{ii} \in 2\mathbf{Z}. \tag{B.6}$$

Substituting (B.5) into (B.6), one obtains

$$\zeta_{ii} = \frac{1}{2}(\omega_i^2 + F_i) + \mathbf{Z}. \tag{B.7}$$

Let $r = 0$ in (B.2), and it leads to

$$\sum_{i=1}^{A} s_i(-\omega_i \cdot T - \frac{F_i}{2} - \zeta_i(0)) - \omega'(s) \cdot T - \frac{F'(s)}{2} \in \mathbf{Z}. \tag{B.8}$$

Applying (B.4) to the above equation, one obtains

$$F'(s) = -2\omega'(s) \cdot T + 2\mathbf{Z}. \tag{B.9}$$

Let $r = \delta_i, s = \delta_j$, and eq. (B.2) becomes

$$\omega_i \cdot \omega_j - \omega_i \cdot T - \omega_j \cdot T - \zeta_j(0) - \zeta_{ji} - \zeta_i(0) - \zeta_{ij} + \frac{F_i}{2} - \frac{F_j}{2} \in \mathbf{Z}. \tag{B.10}$$

Using (B.4), the above equation can be simplified as

$$\zeta_{ij} + \zeta_{ji} = \omega_i \cdot \omega_j + \mathbf{Z}. \tag{B.11}$$

Imposing boundary conditions $n_i\zeta_{ij} \in \mathbf{Z}$ in (B.11), which means that an operation on vacua by the identity ($\mathbf{e}$) operator does not pick up a phase, one may obtain

$$n_i n_j \omega_i \cdot \omega_j \in D_{ij}\mathbf{Z}, \tag{B.12}$$

$$\zeta_{ij} = (n_j Y_{ij} \omega_i \cdot \omega_j + Q_{ij})/D_{ij} + \mathbf{Z} \qquad (i \neq j), \tag{B.13}$$

where D_{ij} is a common divisor of n_i and n_j, $Y_{ij} \in \mathbf{Z}$ is defined such that $Y_{ij}n_j + Y_{ji}n_i = D_{ij}$, and the discrete torsion parameters $Q_{ij} = -Q_{ji}$ $(1 \leq i < j \leq A)$ can be taken to be an arbitrary integer from 0 to $D_{ij} - 1$. Setting $r = \delta_i$ in (B.2), then imposing (B.4) and (B.11), one has

$$\zeta_i'(s) = -\omega_i \cdot \omega'(s) + \mathbf{Z}. \tag{B.14}$$

Using boundary conditions $n_i\zeta_{ii} \in \mathbf{Z}$ in (B.7), one obtains

$$n_i(\omega_i^2 + F_i) \in 2\mathbf{Z}. \tag{B.15}$$

Eqs. (B.3) and (B.9) lead to

$$F(s) = \sum_{i=1}^{A} s_i F_i - 2\omega'(s) \cdot T + 2\mathbf{Z}, \tag{B.16}$$

and eqs. (B.3) and (B.14) lead to

$$\zeta_i(s) = \zeta(0) + \sum_{i=1}^{A} s_j \zeta_{ij} - \omega_i \cdot \omega'(s) + \mathbf{Z}. \tag{B.17}$$

Obviously eqs. (B.15), (B.12) give modular invariance conditions (2.1.14) and (2.1.15) respectively, and eqs. (B.16), (B.17), (B.5), (B.7), (B.13) give vacuum parameters (2.1.16). One might ask whether these solutions are general solutions of eqs. (B.1) and (B.2) for any vectors s and r, since we only use the special vectors s and r to obtain the solutions. It is straightforward to check them by substituting them back to (B.1) and (B.2) in arbitrary s and r. It turns out that these are the most general solutions.

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Figure and Table Captions

Figure 1. An $SU(3)$ root lattice in the complex z plane is used in constructing a torus with $\mathbf{Z}_3$ symmetry, i.e. $\mathbf{Z}_3$ orbifold. The "fundamental region" in this construction contains three fixed points.

Table 1. Thirteen acceptable representations $R_X(g_i) = diag(e^{i2\pi a}, e^{i2\pi b}, e^{i2\pi c})$ for the right-moving bosons satisfy all the constraints in the twist by a single abelian group.

Table 2. This is the multiplication table of the nonabelian group D_3 with e being an identity element.

Table 3. This table describes minimal-rank models of the D_3 nonabelian twist. We list the representations D_λ of the underlying nonabelian group acted on the (E_8, E_8'), and present the final gauge groups $\mathcal{G}$, which are broken from $E_8 \times E_8'$. For each model, if only a representation is listed, then we mean the regular model with all torsions setting to zero ($F_2 = F_3 = Q_{2i} = Q_{3i} = 0$ for $i > 3$), otherwise we specify the torsion values. Furthermore only those torsion models that give different final gauge groups are listed. The above also applies to other tables that list minimal-rank models for other nonabelian groups.

Table 4. This table lists all minimal-rank models of the D_4 nonabelian twist. The models in one block are the same representation, which is characterized

by the first line of the representation, and possibly followed by other lines with different Z_2 torsions, i.e. different $F_2, F_3, Q_{2\theta}, Q_{3\theta}$ values, where θ generates even cyclic group $Z(\theta)$. The first line in one block is always the regular model, unless the torsion values are indicated. The above also applies to other tables of minimal-rank models.

Table 5. The twist representations (4.3.4) of the nonabelian group D_6 for the sixteen left-moving fermions are given in this table, which are the solutions by requiring modular invariance of the partition function.

Table 6. This table lists all minimal-rank models of the D_6 nonabelian twist.

Table 7. All minimal-rank models of the T nonabelian twist are listed in this table.

Table 8. This table presents all minimal-rank models of the O nonabelian twist.

Table 9. All minimal-rank models of the $D_2^{(d)}$ nonabelian twist are given in this table.

Table 10. The twist representations (4.7.4) of the nonabelian group $D_3^{(d)}$ for the sixteen left-moving fermions are listed in this table, which are the solutions by requiring modular invariance of the partition function.

Table 11. All minimal-rank models of the $D_3^{(d)}$ nonabelian twist are listed in this table. Notice that the torsion parameter $F_2 = 1$ in all models of this table.

Table 12. All minimal-rank models of the $T^{(d)}$ nonabelian twist are presented in this table. Notice that we also list the twist representation D_X for the right-moving fields since there are two inequivalent representations of D_X ($\mathbf{2}+\mathbf{1}$ and $\mathbf{2}'+\mathbf{1}'$) in this nonabelian twist.

Figure 1

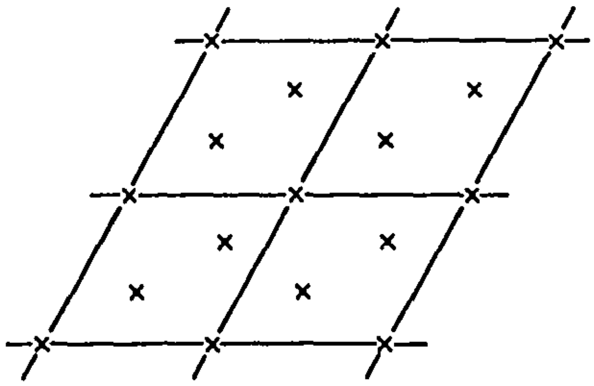

Table 1

No	(a,b,c)	Group Order N	# of Fixed Points
1	$(\frac{1}{2},\frac{1}{2},0)$	2	0
2	$(\frac{1}{3},\frac{2}{3},0)$	3	0
3	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	3	27
4	$(\frac{1}{4},\frac{3}{4},0)$	4	0
5	$(\frac{1}{4},\frac{1}{4},\frac{1}{2})$	4	16
6	$(\frac{1}{6},\frac{5}{6},0)$	6	0
7	$(\frac{1}{6},\frac{1}{6},\frac{2}{3})$	6	3
8	$(\frac{1}{6},\frac{1}{3},\frac{1}{2})$	6	12
9	$(\frac{1}{7},\frac{2}{7},\frac{4}{7})$	7	7
10	$(\frac{1}{8},\frac{1}{4},\frac{5}{8})$	8	4
11	$(\frac{1}{8},\frac{3}{8},\frac{1}{2})$	8	8
12	$(\frac{1}{12},\frac{1}{3},\frac{7}{12})$	12	3
13	$(\frac{1}{12},\frac{5}{12},\frac{1}{2})$	12	4

Table 2

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	f	b	c
b	b	f	e	d	c	a
c	c	d	f	e	a	b
d	d	c	a	b	f	e
f	f	b	c	a	e	d

Table 3

No	D_3 Representation D_λ and Z_2 Torsion	Gauge Group $\mathcal{G}$
1	$3\cdot\mathbf{2}+\mathbf{1}'+\mathbf{1},\mathbf{2}+5\cdot\mathbf{1}'+\mathbf{1}$	$Sp_8\cdot SO_3\cdot E_6'\cdot U_1'$
2	$2\cdot\mathbf{2}+2\cdot\mathbf{1}'+2\cdot\mathbf{1},2\cdot\mathbf{2}+4\cdot\mathbf{1}'$	$SO_7^2\cdot SO_{11}'\cdot SU_2'$
3	$4\cdot\mathbf{2},6\cdot\mathbf{1}'+2\cdot\mathbf{1};F_2=1$	$SO_9\cdot E_7'\cdot SU_2'$

Table 4

No	D_4 Representation D_λ and Z_2 Torsion	Gauge Group $\mathcal{G}$
1	$3\cdot\mathbf{2}+\mathbf{1}'''+\mathbf{1}'',2\cdot\mathbf{2}+4\cdot\mathbf{1}'$	$Sp_8\cdot U_1\cdot SO_{11}'\cdot SU_2'$
2	$Q_{2\theta}=1$	$SO_8\cdot U_1\cdot SO_{11}'\cdot SU_2'$
3	$Q_{3\theta}=1$	$Sp_8\cdot U_1\cdot SO_9'\cdot SU_2'^2$
4	$Q_{2\theta}=Q_{3\theta}=1$	$SO_8\cdot U_1\cdot SO_9'\cdot SU_2'^2$
5	$4\cdot\mathbf{2},\mathbf{2}+5\cdot\mathbf{1}'+\mathbf{1};F_2=1$	$SO_8\cdot SO_{10}'\cdot U_1'^2$
6	$F_2=Q_{3\theta}=1$	$SO_8\cdot E_6'\cdot U_1'$
7	$4\cdot\mathbf{2},\mathbf{2}+\mathbf{1}'''+\mathbf{1}''+4\cdot\mathbf{1};F_2=1$	$SO_8\cdot SO_{10}'\cdot U_1'^2$

Table 5

No	m_0	m_1	n_0	n_1	n_2	n_3
1	2	2	5	1	1	1
2	4	0	5	1	1	1
3	4	0	1	1	1	5
4	3	1	6	2	0	0
5	3	1	4	0	2	2
6	3	1	2	2	0	4
7	3	1	0	0	2	6

Table 6

No	D_6 Representation D_λ and Z_2 Torsion	Gauge Group $\mathcal{G}$
1	$2\cdot 2+2'+1'''+1'',2'+1'+5\cdot 1$	$SO_5^2\cdot U_1\cdot E_6'\cdot U_1'$
2	$Q_{2\delta}=1$	$SU_4\cdot SO_3\cdot U_1\cdot E_6'\cdot U_1'$
3	$Q_{3\delta}=1$	$SO_5^2\cdot U_1\cdot SO_{10}'\cdot U_1'^2$
4	$Q_{2\delta}=Q_{3\delta}=1$	$SU_4\cdot SO_3\cdot U_1\cdot SO_{10}'\cdot U_1'^2$
5	$2\cdot 2+2'+1'+1,2'+1'''+1''+4\cdot 1$	$SO_5^2\cdot SO_3\cdot SO_{10}'\cdot U_1'^2$
6	$Q_{2\delta}=1$	$SU_4\cdot U_1^2\cdot SO_{10}'\cdot U_1'^2$
7	$2\cdot 2+1'''+1''+1'+1,2\cdot 2'+4\cdot 1$	$SO_5^2\cdot U_1^2\cdot SO_{11}'\cdot SU_2'$
8	$Q_{3\delta}=1$	$SO_5^2\cdot U_1^2\cdot SO_8'\cdot SU_2'^2$
9	$2\cdot 2+1'''+1''+1'+1,2\cdot 2+4\cdot 1$	$SO_5^2\cdot U_1^2\cdot SO_{11}'\cdot SU_2'$
10	$Q_{3\delta}=1$	$SO_5^2\cdot U_1^2\cdot SO_8'\cdot SU_2'^2$
11	$2\cdot 2+2'+1'+1,2+1'+5\cdot 1$	$SO_5^2\cdot SO_3\cdot E_6'\cdot U_1'$
12	$Q_{2\delta}=1$	$SU_4\cdot U_1^2\cdot E_6'\cdot U_1'$
13	$Q_{3\delta}=1$	$SO_5^2\cdot SO_3\cdot SO_{10}'\cdot U_1'^2$
14	$Q_{2\delta}=Q_{3\delta}=1$	$SU_4\cdot U_1^2\cdot SO_{10}'\cdot U_1'^2$
15	$2\cdot 2+2'+1'''+1'',2+1'''+1''+4\cdot 1$	$SO_5^2\cdot U_1\cdot SO_{10}'\cdot U_1'^2$
16	$Q_{2\delta}=1$	$SU_4\cdot SO_3\cdot U_1\cdot SO_{10}'\cdot U_1'^2$
17	$2+2'+1'''+1''+1'+1,2+2'+4\cdot 1$	$SU_2^6\cdot SO_{10}'\cdot U_1'$
18	$Q_{3\delta}=1$	$SU_2^6\cdot SO_9'\cdot U_1'^2$
19	$3\cdot 2+1'''+1'',2+1'+5\cdot 1$	$Sp_8\cdot SO_3\cdot E_6'\cdot U_1'$
20	$Q_{2\delta}=1$	$SU_4\cdot U_1^2\cdot E_6'\cdot U_1'$
21	$Q_{3\delta}=1$	$Sp_8\cdot SO_3\cdot SO_{10}'\cdot U_1'^2$
22	$Q_{2\delta}=Q_{3\delta}=1$	$SU_4\cdot U_1^2\cdot SO_{10}'\cdot U_1'^2$
23	$3\cdot 2+1'+1,2+1'''+1''+4\cdot 1$	$Sp_8\cdot U_1\cdot SO_{10}'\cdot U_1'^2$
24	$Q_{2\delta}=1$	$SU_4\cdot SO_3\cdot U_1\cdot SO_{10}'\cdot U_1'^2$
25	$3\cdot 2+1'''+1'',2'+1'''+1''+4\cdot 1$	$Sp_8\cdot SO_3\cdot SO_{10}'\cdot U_1'^2$
26	$Q_{2\delta}=1$	$SU_4\cdot U_1^2\cdot SO_{10}'\cdot U_1'^2$

Table 6 (cont'd)

27	$4 \cdot \mathbf{2}, \mathbf{1}''' + \mathbf{1}'' + \mathbf{1}' + 5 \cdot \mathbf{1}; F_2 = 1$	$SO_8 \cdot E_6' \cdot U_1'^2$
28	$F_2 = Q_{2\delta} = 1$	$SO_9 \cdot E_6' \cdot U_1'^2$
29	$3 \cdot \mathbf{2} + \mathbf{1}' + \mathbf{1}, \mathbf{2}' + \mathbf{1}' + 5 \cdot \mathbf{1}$	$Sp_8 \cdot U_1 \cdot E_6' \cdot U_1'$
30	$Q_{2\delta} = 1$	$SU_4 \cdot SO_3 \cdot U_1 \cdot E_6' \cdot U_1'$
31	$Q_{3\delta} = 1$	$Sp_8 \cdot U_1 \cdot SO_{10}' \cdot U_1'^2$
32	$Q_{2\delta} = Q_{3\delta} = 1$	$SU_4 \cdot SO_3 \cdot U_1 \cdot SO_{10}' \cdot U_1'^2$
33	$2 \cdot \mathbf{2} + 2 \cdot \mathbf{1}' + 2 \cdot \mathbf{1}, \mathbf{2} + \mathbf{2}' + 4 \cdot \mathbf{1}$	$SO_7^2 \cdot SO_{10}' \cdot U_1'$
34	$Q_{2\delta} = 1$	$SU_2^6 \cdot SO_{10}' \cdot U_1'$
35	$Q_{3\delta} = 1$	$SO_7^2 \cdot SO_9' \cdot U_1'^2$
36	$Q_{2\delta} = Q_{3\delta} = 1$	$SU_2^6 \cdot SO_9' \cdot U_1'^2$
37	$2 \cdot \mathbf{2} + 4 \cdot \mathbf{1}, \mathbf{2} + \mathbf{2}' + 2 \cdot \mathbf{1}' + 2 \cdot \mathbf{1}$	$SO_{11} \cdot SU_2 \cdot SU_4'^2$
38	$Q_{2\delta} = 1$	$SO_8 \cdot SU_2^2 \cdot SU_4'^2$
39	$Q_{3\delta} = 1$	$SO_{11} \cdot SU_2 \cdot SO_5'^2 \cdot U_1'^2$
40	$Q_{2\delta} = Q_{3\delta} = 1$	$SO_8 \cdot SU_2^2 \cdot SO_5'^2 \cdot U_1'^2$
41	$2 \cdot \mathbf{2} + 2 \cdot \mathbf{1}''' + 2 \cdot \mathbf{1}', \mathbf{2} + \mathbf{2}' + 2 \cdot \mathbf{1}'' + 2 \cdot \mathbf{1}'$	$SO_7 \cdot SU_2^3 \cdot SU_4' \cdot SO_5' \cdot U_1'$
42	$2 \cdot \mathbf{2} + 2 \cdot \mathbf{1}''' + 2 \cdot \mathbf{1}'', \mathbf{2} + \mathbf{2}' + 4 \cdot \mathbf{1}'$	$SO_7^2 \cdot SO_{10}' \cdot U_1'$
43	$Q_{2\delta} = 1$	$SU_2^6 \cdot SO_{10}' \cdot U_1'$
44	$Q_{3\delta} = 1$	$SO_7^2 \cdot SO_9' \cdot U_1'^2$
45	$Q_{2\delta} = Q_{3\delta} = 1$	$SU_2^6 \cdot SO_9' \cdot U_1'^2$
46	$2 \cdot \mathbf{2} + 4 \cdot \mathbf{1}''', \mathbf{2} + \mathbf{2}' + 2 \cdot \mathbf{1}' + 2 \cdot \mathbf{1}$	$SO_{11} \cdot SU_2 \cdot SU_4'^2$
47	$Q_{2\delta} = 1$	$SO_8 \cdot SU_2^2 \cdot SU_4'^2$
48	$Q_{3\delta} = 1$	$SO_{11} \cdot SU_2 \cdot SO_5'^2 \cdot U_1'^2$
49	$Q_{2\delta} = Q_{3\delta} = 1$	$SO_8 \cdot SU_2^2 \cdot SO_5'^2 \cdot U_1'^2$
50	$3 \cdot \mathbf{2} + \mathbf{2}', 2 \cdot \mathbf{1}''' + 2 \cdot \mathbf{1}'' + 4 \cdot \mathbf{1}'$	$Sp_8 \cdot SO_{12}' \cdot SU_2'^2$
51	$F_2 = 1$	$SO_7 \cdot U_1 \cdot SO_{12}' \cdot SU_2'^2$
52	$F_2 = Q_{2\delta} = 1$	$SO_6 \cdot SU_2 \cdot SO_{12}' \cdot SU_2'^2$

Table 6 (cont'd)

53	$3 \cdot 2 + 2', 2 \cdot 1' + 6 \cdot 1$	$Sp_8 \cdot E_7' \cdot SU_2'$
54	$Q_{3\delta} = 1$	$Sp_8 \cdot SO_{12}' \cdot SU_2'^2$
55	$F_2 = 1$	$SO_7 \cdot U_1 \cdot E_7' \cdot SU_2'$
56	$F_2 = Q_{3\delta} = 1$	$SO_7 \cdot U_1 \cdot SO_{12}' \cdot SU_2'^2$
57	$F_2 = Q_{2\delta} = 1$	$SO_6 \cdot SU_2 \cdot E_7' \cdot SU_2'$
58	$F_2 = Q_{2\delta} = Q_{3\delta} = 1$	$SO_6 \cdot SU_2 \cdot SO_{12}' \cdot SU_2'^2$
59	$2 \cdot 2 + 2 \cdot 2', 1''' + 1'' + 1' + 5 \cdot 1; F_2 = 1$	$SO_4 \cdot SO_5 \cdot E_6' \cdot U_1'^2$
60	$Q_{2\delta} = 1$	$SO_8 \cdot E_6' \cdot U_1'^2$

Table 7

No	T Representation D_λ	Gauge Group $\mathcal{G}$
1	$2 \cdot 3 + 2 \cdot 1, 3 + 1'' + 1' + 3 \cdot 1$	$G_2 \cdot G_2 \cdot SU_6' \cdot U_1'$
2	$2 \cdot 3 + 1'' + 1', 3 + 5 \cdot 1$	$G_2 \cdot SU_3 \cdot E_6'$
3	$2 \cdot 3 + 1' + 1, 3 + 1'' + 4 \cdot 1$	$SU_3^2 \cdot SO_8' \cdot U_1'^2$
4	$2 \cdot 3 + 2 \cdot 1, 3 + 5 \cdot 1''$	$G_2 \cdot G_2 \cdot SU_6' \cdot U_1'$
5	$2 \cdot 3 + 1'' + 1', 3 + 3 \cdot 1'' + 2 \cdot 1$	$G_2 \cdot SU_3 \cdot SU_3'^3$
6	$2 \cdot 3 + 1' + 1, 3 + 4 \cdot 1'' + 1$	$SU_3^2 \cdot SO_8' \cdot U_1'^2$

Table 8

No	O Representation D_λ and Z_2 Torsion	Gauge Group $\mathcal{G}$
1	$\mathbf{3}+2\cdot\mathbf{2}+\mathbf{1},\mathbf{2}+\mathbf{1}'+5\cdot\mathbf{1}$	$SO_5\cdot SU_2\cdot U_1\cdot E_6'\cdot U_1'$
2	$\mathbf{3}+\mathbf{2}+\mathbf{1}'+2\cdot\mathbf{1},2\cdot\mathbf{2}+4\cdot\mathbf{1}$	$SU_3^2\cdot U_1\cdot SO_{11}'\cdot SU_2'$
3 4	$\mathbf{3}+5\cdot\mathbf{1},3\cdot\mathbf{2}+\mathbf{1}'+\mathbf{1}$ $Q_{2\theta}=1$	$SO_{10}\cdot U_1\cdot SP_8'\cdot SO_3'$ $E_6\cdot SP_8'\cdot SO_3'$
5	$\mathbf{3}+\mathbf{2}+3\cdot\mathbf{1}',2\cdot\mathbf{2}+2\cdot\mathbf{1}'+2\cdot\mathbf{1}$	$SU_5\cdot U_1\cdot SO_7'^2$
6	$\mathbf{3}+4\cdot\mathbf{1}'+\mathbf{1},3\cdot\mathbf{2}+\mathbf{1}'+\mathbf{1}$	$SO_{10}\cdot U_1\cdot SP_8'\cdot SO_3'$

Table 9

No	$D_2^{(d)}$ Representation D_λ and Z_2 Torsion	Gauge Group $\mathcal{G}$
1 2	$2\cdot\mathbf{2}+\mathbf{1}'''+\mathbf{1}''+\mathbf{1}'+\mathbf{1},3\cdot\mathbf{2}+2\cdot\mathbf{1}$ $Q_{3q}=1$	$SO_7\cdot SU_2^3\cdot Sp_8'\cdot SU_2'$ $SO_7\cdot SU_2^3\cdot F_4'\cdot SU_2'$
3	$4\cdot\mathbf{2},\mathbf{2}+\mathbf{1}'''+\mathbf{1}''+\mathbf{1}'+3\cdot\mathbf{1};F_2=1$	$Sp_8\cdot SU_6'\cdot U_1'^2$

Table 10

No	m_0	m_1	n_0	n_1	n_2	n_3
1	7	0	1	1	0	0
2	6	1	0	0	2	0
3	5	2	2	0	0	0
4	3	4	1	1	0	0
5	2	5	0	0	2	0
6	1	6	2	0	0	0

Table 11 (The torsion parameter $F_2 = 1$ in all models of this table.)

No	$D_3^{(d)}$ Representation D_λ and $\mathbf{Z}_2$ Torsion	Gauge Group $\mathcal{G}$
1	$\mathbf{4 \cdot 2', 3 \cdot 2' + 1' + 1}; Q_{3a} = 1$	$Sp_8 \cdot Sp'_6 \cdot SO'_3 \cdot U'_1$
2	$\mathbf{4 \cdot 2', 2 \cdot 2' + 2 + 2 \cdot 1''}$	$Sp_8 \cdot SO'_7 \cdot SU'_2 \cdot U'_1$
3	$F_3 = 1$	$Sp_8 \cdot Sp'_6 \cdot SO'_3 \cdot U'_1$
4	$\mathbf{4 \cdot 2', 2' + 2 \cdot 2 + 2 \cdot 1}$	$Sp_8 \cdot SO'_7 \cdot SU'^2_2$
5	$\mathbf{4 \cdot 2, 3 \cdot 2' + 1' + 1}; Q_{3a} = 1$	$SO_9 \cdot Sp'_6 \cdot SO'_3 \cdot U'_1$
6	$\mathbf{4 \cdot 2, 2 \cdot 2' + 2 + 2 \cdot 1''}$	$SO_9 \cdot SO'_7 \cdot SU'_2 \cdot U'_1$
7	$F_3 = 1$	$SO_9 \cdot Sp'_6 \cdot SO'_3 \cdot U'_1$
8	$\mathbf{4 \cdot 2, 2' + 2 \cdot 2 + 2 \cdot 1}$	$SO_9 \cdot SO'_7 \cdot SU'^2_2$
9	$\mathbf{3 \cdot 2' + 2, 3 \cdot 2' + 1''' + 1''}; Q_{2a} = 1$	$Sp_6 \cdot SO_3 \cdot F'_4 \cdot SO'_3$
10	$Q_{2a} = Q_{3b} = 1$	$Sp_6 \cdot SO_3 \cdot Sp'_8 \cdot SO'_3$
11	$F_3 = Q_{2a} = 1$	$Sp_6 \cdot SO_3 \cdot Sp'_6 \cdot SU'_2 \cdot U'_1$
12	$\mathbf{3 \cdot 2' + 2, 2 \cdot 2' + 2 + 2 \cdot 1}; Q_{2a} = Q_{3a} = 1$	$Sp_6 \cdot SO_3 \cdot Sp'_6 \cdot SU'_2 \cdot U'_1$
13	$\mathbf{3 \cdot 2' + 2, 3 \cdot 2 + 1' + 1}; Q_{2a} = 1$	$Sp_6 \cdot SO_3 \cdot Sp'_8 \cdot SO'_3$
14	$\mathbf{2 \cdot 2' + 2 \cdot 2, 3 \cdot 2' + 2 \cdot 1}$	$Sp^2_4 \cdot Sp'_8 \cdot SU'_2$
15	$Q_{3b} = 1$	$Sp^2_4 \cdot F'_4 \cdot SU'_2$
16	$\mathbf{2 \cdot 2' + 2 \cdot 2, 2' + 2 \cdot 2 + 1' + 1}; Q_{3a} = 1$	$Sp^2_4 \cdot SU'_4 \cdot SU'_2 \cdot SO'_3$
17	$\mathbf{2 \cdot 2' + 2 \cdot 2, 3 \cdot 2 + 2 \cdot 1''}$	$Sp^2_4 \cdot Sp'_8 \cdot SU'_2$
18	$F_3 = 1$	$Sp^2_4 \cdot SO'_6 \cdot SU'_2 \cdot SO'_3$
19	$\mathbf{2' + 3 \cdot 2, 2 \cdot 2' + 2 + 1' + 1}; Q_{2a} = 1$	$SO_7 \cdot SU_2 \cdot SO'_7 \cdot SU'_2 \cdot SO'_3$
20	$\mathbf{2' + 3 \cdot 2, 2' + 2 \cdot 2 + 1''' + 1''}; Q_{2a} = 1$	$SO_7 \cdot SU_2 \cdot SO'_7 \cdot SU'_2 \cdot SO'_3$
21	$F_3 = Q_{2a} = 1$	$SO_7 \cdot SU_2 \cdot SU'_4 \cdot SU'^2_2$
22	$\mathbf{2' + 3 \cdot 2, 3 \cdot 2 + 2 \cdot 1}; Q_{2a} = Q_{3a} = 1$	$SO_7 \cdot SU_2 \cdot SO'_6 \cdot SO'_4$

Table 12

No	$T^{(d)}$ Representation $(D_\lambda; D_X)$ and $\mathbf{Z}_2$ Torsion	Gauge Group $\mathcal{G}$
1	$\mathbf{2}+2\cdot\mathbf{3},2\cdot\mathbf{3}+\mathbf{1}'+\mathbf{1};\mathbf{2}+\mathbf{1};Q_{2b}=1$	$G_2\cdot SU_2\cdot SU_3'^2$
2	$\mathbf{2}'+2\cdot\mathbf{3},2\cdot\mathbf{3}+\mathbf{1}''+\mathbf{1}';\mathbf{2}+\mathbf{1};Q_{2b}=1$	$SU_3\cdot SU_2\cdot G_2'\cdot SU_3'$
3	$\mathbf{2}+2\cdot\mathbf{3},2\cdot\mathbf{3}+\mathbf{1}'+\mathbf{1};\mathbf{2}'+\mathbf{1}';Q_{2b}=1$	$G_2\cdot SU_2\cdot SU_3'^2$
4	$\mathbf{2}'+2\cdot\mathbf{3},2\cdot\mathbf{3}+\mathbf{1}''+\mathbf{1}';\mathbf{2}'+\mathbf{1}';Q_{2b}=1$	$SU_3\cdot SU_2\cdot G_2'\cdot SU_3'$