# Regularization of Gaussian Free Fields Based on the Fourier-Bessel Expansion

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## ABSTRACT

In this work, we regularize Gaussian free fields based on the Fourier-Bessel expansion to extend part of the results on the Liouville quantum gravity measure and the KPZ relation in even dimensions [1, 2] to odd dimensions. We adopt the definition of Gaussian free field on $\mathbb{R}^n$ viewed as an abstract Wiener space with the underlying Hilbert space $H^{\frac{n}{2}}(\mathbb{R}^n)$. In particular, we can show that the Liouville quantum gravity measure on $\mathbb{R}^3$ is the weak limit of the measures associated with a weighted series of spherical averages of the Gaussian free field. We also prove the KPZ formula on $\mathbb{R}^3$, which gives the quadratic relation between the geometric properties of models in the quantum gravity setting and its counterpart in the Euclidean setting.

# ABRÉGÉ

Dans ce travail, nous régularisons des champs libres Gaussiens basés sur l'expansion de Fourier-Besse pour étendre une partie des résultats sur la mesure de gravité quantique de Liouville et la relation KPZ en dimensions [1, 2] et même en dimensions impaires. Nous adoptons la définition du champ libre Gaussien sur $\mathbb{R}^n$ considéré comme un espace de Wiener abstrait avec l'espace de Hilbert sous-jacent $H^{\frac{n}{2}}(\mathbb{R}^n)$. En particulier, nous pouvons montrer que la mesure de la gravité quantique de Liouville sur $\mathbb{R}^3$ est la limite faible des mesures associées à une série pondérée de moyennes sphériques du champ libre gaussien. Nous montrons aussi la formule de KPZ sur $\mathbb{R}^3$, qui donne la relation quadratique entre les propriétés géométriques des modèles dans le cadre de la gravitation quantique et sa contrepartie dans le contexte Euclidien.

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# Chapter 1

# Introduction

Many recent developments in quantum physics and probability theory have seen the notion of Gaussian free field (GFF) as an indispensable tool. Mathematically speaking, GFFs can be viewed as analogues of the Brownian motion with multi-dimensional time parameters. Just like the Brownian motion being a natural model of random curves, GFFs are promising candidates in modelling random surfaces or random manifolds. Being in the center of the intersection of probability theory and random geometry, the study of GFFs' geometrical properties and applications is one of the fastest developing fields in probability. For example, properties of extrema and near extrema of GFFs in discrete settings, i.e., on discrete lattices, have been extensively studied [3, 4, 5]. In the continuum settings, GFF models that are most relevant to applications in physics and geometry consist of generalized functions as generic elements. In other words, an instance of such a GFF is only a tempered distribution instead of a function.

Due to this singularity, it is challenging to establish analytic results regarding geometry of GFFs in continuum settings. To overcome the singularity, one would need to apply a regularization procedure. There are various ways to regularize a singular GFF. One commonly adopted procedure is based on the multiplicative chaos theory (MCT). The MCT was originally proposed by Kahane [6] and later revived by Rhodes and Vargas in a series of work on the geometry of log-correlated GFFs [7, 8, 9]. Another procedure that is also natural to treat singular GFFs is to average GFFs over certain Borel sets such as circles in two dimensions or spheres in higher dimensions. Both the MCT and the averaging procedure lead to numerous important results on the geometry of GFFs in continuum settings. For example, via either procedure, researchers have determined the Hausdorff dimension of thick point sets of continuum GFFs, where thick points are the counterparts of extrema in discrete settings [10, 11, 12].

One important application of GFFs in quantum field theory is to provide a mathematical approach towards the study of Liouville quantum gravity. In his orginal work on the MCT, Kahane had already constructed a random measure which could be interpreted as the Liouville quantum gravity measure. More recently, Duplantier and Sheffield [1] gave another construction of the Liouville quantum gravity measure based on the circular averages of the log-correlated GFF in two dimensions, and moreover, they gave the first mathematically rigorous proof of the celebrated formula conjectured by Knizhnik, Polyakov and Zamolodchikov [13], known as the KPZ formula. Heuristically speaking, the KPZ formula provides the exact correspondence between certain geometric parameters in the Euclidean setting and their counterparts in the quantum setting. In [1], the KPZ formula is the extract relation between volume scaling exponent between the Lebesgue measure and the Liouville quantum gravity measure. Later, Rhodes and Vargas proved the KPZ formula for log-correlated GFFs in any dimension based on the MCT [9], and independently, Chen and Jakobson proved the same results for log-correlated GFFs in even dimensions based on spherical averages [2].

The goal of this thesis is to investigate further the sphere averaging procedure of GFFs in arbitrary dimensions, and, via spherical averages of GFFs, to extend the results on the Liouville quantum gravity measure and the KPZ formula further to odd dimensions. The motivation of proposing such a project is two-folded: first, we hope to obtain the desired results without relying on the constraints of the MCT, e.g., the sigma-positiveness of the kernel of the GFF, which means that we can potentially treat more general types of Gaussian random fields; second, in the study of the geometry of a generic instance of the GFF, averaging the instance locally is a natural geometric action, and reflects of the local behavior of the instance in some sense.

In this thesis, we treat log-correlated GFFs in $\mathbb{R}^n$ for arbitrary $n \geq 2$ based on an imporant tool from the special function theory, known as the Fourier-Bessel expansion. To be specific, for every $x \in \mathbb{R}^n$, $\varepsilon > 0$, we consider the weighted average over a family of spheres centered at $x$ with radius growing to infinity in a certain way, i.e., $\mu_\varepsilon^x := \sum_{m=1}^{\infty} c_m \sigma_{j_m\varepsilon}^x$, where $j_1, j_2...$ are the positive zeros of $J_{\frac{n-2}{2}}(x)$ arranged in ascending order. By choosing

$c_m$ properly according the theory of Fourier-Bessel expansion, we can make $\hat{\mu_\epsilon^x}(\xi) = 1_{[0,1)}(\varepsilon|\xi|)e^{i(x,\xi)_{\mathbb{R}^n}}$. We can show that $\{\mathcal{I}(h_{\mu_\varepsilon^x}) : x \in \mathbb{R}^n, \varepsilon > 0\}$ is a reasonable candidate to construct a Liouville measure on $\mathbb{R}^n$. In particualr, we give an example of the construction of random measures on $\mathbb{R}^3$. We also prove that the scaling exponent $\rho$ of a bounded Borel set in $\mathbb{R}^3$ under the Lebesgue measure and the scaling exponent $Q$ of the same set under the random measure defined above satisfy the quadratic relation

$$\rho = \frac{\gamma^2}{12\pi^2}Q^2 + \left(1 - \frac{\gamma^2}{12\pi^2}\right)Q.$$

This thesis is organized as follows. Chapter 2 reviews some fundations of the theory of abstract Wiener spaces (AWS), which provides mathematical foundation of GFF models. Then, in chapter 3, we introduce the GFF and its basic properties. In addition, we review the work of constructing Liouville quantum gravity measures and proving the KPZ formula in even dimensional Euclidean spaces, which are introduced in [1, 2], and propose a possible alternative regularization of the GFF in four dimensions based on weighted averages of the GFF over two spheres. The core of the thesis, which will be presented in the fifth chapter, will be dedicated to establishing a theoretical analysis of the regularity of GFFs based on the Fourier-Bessel expansion. Further on, chapter 6 presents the construction of Liouville quantum gravity measures with GFFs via the tool of Fourier-Bessel expansion and proves the KPZ formula on $\mathbb{R}^3$. Appendix contains all the formulas associated with Bessel functions that that are invoked in this work.

# Chapter 2

# Background: Abstract Wiener spaces

In this chapter, we present a brief review of the theory of AWS, originally introduced by Gross in [14], as a mathematical construction of Gaussian measures in infinite dimensions. More recently, the theory of AWS was revisited by Stroock [15, 16]. The framework of AWS adopted in this thesis follows the one from Chapter 8 of [15].

## 2.1 Preliminaries

Let's first consider Gaussian measures in finite dimensions. Let $\mathcal{W}$ be a centered Gaussian measure on $\mathbb{R}^n$ with non-degenerate covariance matrix $C$, i.e.,

$$\mathcal{W}(dh) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \frac{1}{\sqrt{\det C}} \exp\left(\frac{(h, C^{-1}h)_{\mathbb{R}^n}}{2}\right) dh. \tag{2.1}$$

Then, re-write $\mathbb{R}^n$ as $H$, and for every $h, g \in H$, define $(h, g)_H = (h, C^{-1}g)_{\mathbb{R}^n}$. Let $\lambda_H(dh)$ be the Borel measure under which the unit ball under $(\cdot, \cdot)_H$ has unit volume, i.e., $\lambda_H(dh) = \frac{dh}{\sqrt{\det C}}$. Then, $\mathcal{W}_H$ takes a the form of a standard Gaussian measure, i.e.,

$$\mathcal{W}_H(dh) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{\|x\|_H^2}{2}\right) \lambda_H(dh).$$

This is to say that, in finite dimensions, a natural hosting space of a centered Gaussian measure is a Hilbert space. Moreover, its Fourier transform is given by

$$\widehat{\mathcal{W}_H}(h) \equiv \mathbb{E}^{\mathcal{W}_H}[e^{\sqrt{-1}(\cdot,h)_H}] = \mathbb{E}^{\mathcal{W}_H}[e^{\sqrt{-1}(\cdot,C^{-1}h)_{\mathbb{R}^n}}] = e^{-\frac{(C^{-1}h,h)_{\mathbb{R}^n}}{2}} = e^{-\frac{\|h\|_H^2}{2}},$$

It is clear that $\mathcal{W}_H$ is strictly positive and locally finite. In finite dimensions, any translation of $\mathcal{W}_H$ is equivalent to $\mathcal{W}_H$ in the sense that the translated $\mathcal{W}_H$ is absolutely continuous with respect $\mathcal{W}_H$ and vice versa.

However, if $n = \infty$, then the previous construction fails, i.e., a measure $\mathcal{W}_H$ as defined in (2.1) fails to exist when $H$ is infinite dimensional. The reason is that for any orthonormal basis $\{h_m : m \geq 1\}$ of $H$, the mappings $h \in H \longmapsto (h, h_m)_H \in \mathbb{R}$ are independent, centered Gaussian random variables, and thus $\|h\|_H^2 = \sum_{m=0}^{\infty} \big|(h, h_m)_H\big|^2$ is infinite $\mathcal{W}_H$-almost surely by the strong law of large numbers.

## 2.2 Abstract Wiener measures

To construct a strictly positive and locally finite Gaussian measure $\mathcal{W}$ on a infinite dimensional space, Gross [14] introduced the idea by embedding $H$ in a larger Banach space.

**Lemma 2.1.** Let $\Theta$ be a separable, real Banach space, and $H$ be a real Hilbert space which is continuously embedded as a dense subspace of $\Theta$. If $\lambda^* \in \Theta^*$, then there is a unique $h_{\lambda^*} \in H$ such that for all $h \in H$, $(h, h_{\lambda^*})_H = \langle h, \lambda^* \rangle$ and $i : \lambda^* \in \Theta^* \longmapsto h_{\lambda^*} \in H$ is an injective bounded linear map which has dense image in $H$.

*Proof.* Since $H$ is continuously embedded in $\Theta$, there exists a constant $C$ such that $\|h\|_\Theta \leq C\|h\|_H$ for all $h \in H$. Thus, if $\lambda^* \in \Theta^*$, then $\lambda^* \in H^*$ and $\big|\langle h, \lambda^* \rangle\big| \leq \|h\|_\Theta \|\lambda^*\|_{\Theta^*} \leq C\|h\|_H \|\lambda^*\|_{\Theta^*}$, where the formula $f(h) = \langle h, \lambda^* \rangle$ defines a bounded linear functional $f$ on $H$. By the Riesz Representation Theorem, there exists a unique $h_{\lambda^*} \in H$ such that $f(h) = (h, h_{\lambda^*})_H$ and therefore $\|h_{\lambda^*}\|_H \leq C\|\lambda^*\|_{\Theta^*}$. Now, if $h_{\lambda^*} = 0$, then $(h, h_{\lambda^*})_H = \langle h, \lambda^* \rangle = 0$ for every $h \in H$. Since $H$ is dense in $\Theta$, we have $\lambda^* = 0$ and thus $i$ is one-to-one. Moreover, for all $a, b \in \mathbb{R}$ and $\lambda_1^*, \lambda_2^* \in H$, the uniqueness of $h_{\lambda^*}$ yields

$$(h, h_{a\lambda_1^*+b\lambda_2^*})_H = \langle h, a\lambda_1^* + b\lambda_2^* \rangle = (h, ah_{\lambda_1^*} + bh_{\lambda_2^*})_H.$$

To see that $\{h_{\lambda^*} : \lambda^* \in H\}$ is dense in $H$, it suffices to show that for any weak* dense subset $E^*$ of $\Theta^*$, $\{h_{\lambda^*} : \lambda^* \in E^*\}$ is dense in $H$. If it is not, then there exists $h \in H \setminus \{0\}$ with $\langle h, \lambda^* \rangle = (h, h_{\lambda^*})_H = 0$ for all $\lambda^* \in E$. However, since $E^*$ is weak* dense in $\Theta^*$, we get $h = 0$, which contradicts the assumption $h \in H \setminus \{0\}$. □

Given a separable Banach space $\Theta$ and a separable Hilbert space $H$ which is continuously embedded in $\Theta$ as a dense subspace, $\mathcal{W}$ denotes a Borel probability measure on $\Theta$. The triple $(H, \Theta, \mathcal{W})$ is said to be an *abstract Wiener space*(AWS) if $\mathcal{W}$ has Fourier transform

$$\widehat{\mathcal{W}}(\lambda^*) = \exp\left(-\frac{\|h_{\lambda^*}\|_H^2}{2}\right) \text{ for all } \lambda^* \in \Theta^*.$$

This Borel measure $\mathcal{W}$, named as the *Wiener measure*, is strictly positive and locally finite with $\mathcal{W}(H) = 0$ if $\Theta$ is infinite dimensional.

**Theorem 2.1.** Given any separable infinite dimensional Hilbert space $H$, there exists a separable Banach space $\Theta$ and a centered non-degenerate Wiener measure $\mathcal{W}$ on $\Theta$ such that $(H, \Theta, \mathcal{W})$ forms an AWS, and the choice of $\Theta$ is not unique.

On the other hand, given any separable Banach space $\Theta$ and a centered non-degenerate Gaussian measure on $\Theta$, there exists a unique separable Hilbert space $H$ such that $H$ is the Cameron-Martin space for $\Theta$ and $\mathcal{W}$.

## 2.3 Cameron–Martin spaces

Given an abstract Wiener space $(H, \Theta, \mathcal{W})$ , it can be shown that there exists a unique linear isometry $\mathcal{I} : H \longrightarrow L^2(\mathcal{W})$ such that $\mathcal{I}(h_{\lambda^*}) = \langle \cdot, \lambda^* \rangle$ for all $\lambda^* \in \Theta^*$ and $\{\mathcal{I}(h) : h \in H\}$ is a centered Gaussian family in $L^2(\mathcal{W})$ with covariance structure

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_1)\mathcal{I}(h_2)] = (h_1, h_2)_H \text{ for all } h_1, h_2 \in H.$$

We call $\mathcal{I}$ the *Payley-Wiener map* and its image $\mathcal{I}(h)$ the *Paley Wiener integral*. One important application of Paley-Wiener maps is to describe the behavior of Gaussian measures under translation.

**Theorem 2.2.** (**Cameron–Martin Theorem**) If $\mathcal{W}_H$ is a centered, non-degenerate Gaussian measure and $(H, \Theta, \mathcal{W}_H)$ forms an abstract Wiener space , then for each $h \in H$, the pushforward measure $(\tau_h) * \mathcal{W}_H$ is equivalent to the Gaussian measure $\mathcal{W}_H$ with respect to the Radon-Nikodym derivative

$$\frac{d(\tau_h) * \mathcal{W}_H}{d\mathcal{W}_H} = \exp[\mathcal{I}(h) - \frac{1}{2}\|h\|_H^2], \tag{2.2}$$

where $\tau_h : \Theta \to \Theta$ refers to the translation map $\tau_h(x) = x + h$. Note that the *Cameron–Martin formula* (2.2) is only valid for translations in $H$ and the support of $\mathcal{W}_H$ is the whole of $\Theta$.

## 2.4 Example: Classical Wiener spaces

A typical example of an abstract Wiener space is the classical Wiener space, which is the collection of continuous paths on a given domain.

Recall that an *n-dimensional Brownian motion* is an $\mathbb{R}^n$-valued, continuous-time stochastic process $\{B(t) : t \geq 0\}$ with the properties that $B(0) = x$ with $x \in \mathbb{R}^n$, $t \mapsto B(t)$ is almost surely continous, and for all $0 \leq s \leq t$, the increment $B(t) - B(s)$ is independent of $\mathcal{F}_s = \sigma(\{B(\tau) : \tau \in [0, s]\})$ and has a centered Gaussian distribution with variance $t - s$. When $x = 0$, $\{B(t) : t \geq 0\}$ is *standard* and $\mathbb{E}[B(s)B(t)] = (s \wedge t)I_n$ for all $s, t \geq 0$, where $I_n$ is the identity matrix. The distribution of a Brownian motion is known as a *Wiener measure*.

Since a Brownian motion is continuous in $t$, it is natural to consider the space of continuous paths as the underlying space. Define $\Theta \equiv \Theta(\mathbb{R}^n)$ to be the space of continous paths $\theta : [0, \infty) \to \mathbb{R}^n$ such that $\theta(0) = 0$ and $\lim_{t\to\infty} t^{-1}\big|\theta(t)\big| = 0$, equipped with the norm

$$\|\theta\|_\Theta := \sup_{t\geq 0} (1 + t)^{-1}\big|\theta(t)\big|.$$

Then $\Theta$ is a separable Banach space which is continuously embedded in $C(\mathbb{R}^n)$, and the process $\{B(t) : t \geq 0\}$ is in $\Theta$ almost surely. In addition, the distribution of $\{B(t) : t \geq 0\}$ induces a Borel measure $\mathcal{W}$ on $\Theta$, which is called the *classical Wiener measure*. Finally, we set $H \equiv H^1(\mathbb{R}^n)$ to be the space of continuous path $h : [0, \infty) \to \mathbb{R}^n$ such that $h(0) = 0$ and $\|h\|_H^2 = \int_0^\infty \left|\dot{h(t)}\right|^2 dt < \infty$. Then, $(H, \Theta, \mathcal{W})$ forms an abstract Wiener space, and $H$ is the Cameron-Martin space for the classical Wiener space.

## 2.5 Wiener series

Given a Hilbert space $H$ with an orthonormal basis $\{h_m : m \geq 1\}$, based on the above discussion, $\{\mathcal{I}(h_m) : m \geq 1\}$ is a family of independent standard Gaussian random variables. As we have pointed out earlier, $\sum_{m=1}^{\infty} \mathcal{I}(h_m)h_m$ cannot be an element in $H$ since if it had been in $H$, then $\left\| \sum_{m=1}^{\infty} \mathcal{I}(h_m)h_m \right\|_H^2 = \infty$ almost surely. However, one may construct a bigger space $\Theta$ such that the series converges in $\Theta$ almost surely.

**Theorem 2.3.** Let $H$ be an infinite-dimensional separable real Hilbert space which is continuously embedded in a Banach space $\Theta$ as a dense subspace. If there is an orthonormal basis $\{h_m : m \geq 0\}$ in $H$ such that $\sum_{m=0}^{\infty} x_m h_m$ is $\gamma_{0,1}^{\mathbb{N}}$-almost surely convergent in $\Theta$ , where $\mathbf{x} = (x_0, x_1, \dots, x_m, \dots) \in \mathbb{R}^{\mathbb{N}}$ and $\Lambda : \mathbb{R}^{\mathbb{N}} \longrightarrow \Theta$ is defined by

$$\Lambda(\mathbf{x}) = \begin{cases} \sum_{m=0}^{\infty} x_m h_m & \text{if the series converges in } \Theta \\ 0 & \text{otherwise,} \end{cases}$$

then $(H, \Theta, \mathcal{W})$ with $\mathcal{W} = \Lambda_* \gamma_{0,1}^{\mathbb{N}}$ is an abstract Wiener space. Conversely, if $(H, \Theta, \mathcal{W})$ is an abstract Wiener space and $\{h_m : m \geq 0\}$ is an orthonormal basis in $H$, then $\sum_{m=0}^{\infty} \mathcal{I}(h_m)h_m$ converges $\mathcal{W}$-almost surely in $\Theta$ as well as in $L^p(\mathcal{W})$ for every $p \in [1, \infty)$.

*Proof.* Let $\Lambda_n(\mathbf{x}) = \sum_{m=0}^{n} x_m h_m$ and $\mathcal{W} = \Lambda_* \gamma_{0,1}^{\mathbb{N}}$. Since $\Lambda_n(\mathbf{x})$ converges to $\Lambda(\mathbf{x})$ in $\Theta$ for $\gamma_{0,1}^{\mathbb{N}}$-almost surely $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$, for $\lambda^* \in \Theta^*$,

$$\begin{aligned} \widehat{\mathcal{W}}(\lambda^*) &= \lim_{n \to \infty} \mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}[\exp(\sqrt{-1}\,\langle \Lambda_n, \lambda^* \rangle)] \\ &= \lim_{n \to \infty} \exp(-\frac{1}{2} \sum_{m=0}^{n} (h_m, h_{\lambda^*})^2) = e^{-\frac{\|h_{\lambda^*}\|_H^2}{2}}. \end{aligned}$$

Suppose that $(H, \Theta, \mathcal{W})$ is an abstract Wiener space. First, for each $n \in \mathbb{N}$, define $\mathcal{F}_n = \sigma(\{\mathcal{I}(h_m) : m \in [0, n]\})$. Then $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$, and $\mathcal{F} \equiv \cup_{n=0}^{\infty} \mathcal{F}_n$ is the $\sigma$-algebra generated by $\{\mathcal{I}(h_m) : m \geq 0\}$. Second, we show that $\mathcal{B}_\Theta$ is contained in the $\mathcal{W}$=completion $\overline{\mathcal{F}}$ of $\mathcal{F}$. Since $\sum_{m=0}^{n} (h, h_m)_H h_m$ converges to $h$ in $H$,

$$\sum_{m=0}^{n} (h, h_m)_H \mathcal{I}(h_m) = \mathcal{I}\left( \sum_{m=0}^{n} (h, h_m)_H h_m \right) \longrightarrow \mathcal{I}(h) \ \text{ in } L^2(\mathcal{W}).$$

Thus, $\mathcal{I}(h)$ is $\overline{\mathcal{F}}$-measurable for every $h \in H$. Third, if $S_n = \sum_{m=0}^{n} \mathcal{I}(h_m)h_m$, then for $\theta \in \Theta$, $\langle \theta - S_n(\theta), \lambda^* \rangle$ is perpendicular to $\mathcal{I}(h_m)$ in $L^2(\mathcal{W})$ for all $\lambda^* \in \Theta^*$ and $0 \leq m \leq n$ since $\{\mathcal{I}(h_m) : m \geq 0\}$ is a Gaussian family. Hence, $\theta - S_n(\theta)$ is independent of $\mathcal{F}_n$ and $S_n = \mathbb{E}^{\mathcal{W}}[\theta|\mathcal{F}_n]$. Finally, use Doob's martingale convergence theorem to conclude that $S_n(\theta) \to \theta$ for $\mathcal{W}$-almost surely $\theta$ as $n \to \infty$. □

AWS plays an important role in the construction of quantum fields. One of the quantum fields that will arise afterwards in the exposition of this thesis is called *Gaussian free field*(GFF), which is a natural generalization of Brownian motion with multidimensional time parameters. Detailed information on GFFs will be explored in the next chapter.

# Chapter 3

# Gaussian free fields essentials

In this chapter, we introduce some basic concepts of GFFs in accordance with [1, 15, 17, 18].

## 3.1 Introduction to GFF

### 3.1.1 GFF on a bounded domain $D$

We start by constructing the GFF on a bounded domain $D \subset \mathbb{R}^n$ with the Laplace operator $L = -\Delta$. Consider the Hilbert space $H^1 \equiv H_0^1(D)$, which is the completion of the space $\mathcal{D}(D)$ of all smooth real-valued functions with compact support in $D$, endowed with the Dirichlet inner product

$$(f, g)_{H^1} \equiv \frac{1}{2\pi} \int_D \nabla f \cdot \nabla g \; dx \quad \text{for all } f, g \in \mathcal{D}(D).$$

As seen earlier, there exist a separable Banach space $\Theta^1 \equiv \Theta^1(D)$ and a Borel measure $\mathcal{W}^1 \equiv \mathcal{W}^1(D)$ on $\Theta^1$ such that the triple $(H^1, \Theta^1, \mathcal{W}^1)$ forms an AWS.

The GFF on $D$ is a random distribution $h := \sum_{m=0}^{\infty} X_m h_m$, where $X_m$'s are independent identically distributed (i.i.d.) standard Gaussian random variables, and the sequence $\{h_m : m \geq 0\}$ is an orthonormal basis for $H^1$. One can use Theorem 2.3 to check it is well-defined in $(H^1, \Theta^1, \mathcal{W}^1)$.

Note that for any $f = \sum_{m=0}^{\infty} Y_m h_m \in H^1$, $(h, f)_{H^1} = \sum_{m=0}^{\infty} X_m Y_m$ is a centered Gaussian variable. Consequently, if $h$ is a GFF on $D$, then $\{(h, f)_{H^1}\}_{f \in H^1}$ forms a centered Gaussian process with covariance

$$\mathrm{Cov}[(h, f)_{H^1}, (h, g)_{H^1}] = \mathbb{E}^{\mathcal{W}^1}[(h, f)_{H^1} (h, g)_{H^1}] = (f, g)_{H^1} \quad \text{for all } f, g \in H^1.$$

This fact implies that a GFF can be interpreted as a Gaussian process $\{(h,f)_{H^1}\}_{f\in H^1}$ indexed by $H^1$ such that $(h,f)_{H^1}$ is a centered Gaussian with variance $(f,f)_{H^1}$ for each $f \in H^1$.

On the other hand, integration by parts yields

$$\int_D \nabla f \cdot \nabla g \, dx = \int_D f(-\Delta)g \, dx = \int_D (-\Delta)^{1/2} f \cdot (-\Delta)^{1/2} g \, dx,$$

and the map $(-\Delta)^{-1/2}$ is an isomorphism from $L^2(D)$ with the $L^2$ inner product to $H^1$ with the Dirichlet inner product. We may write $H^1 = \mathcal{L}_{-1/2}(D)$.

Similarly, for any $s \in \mathbb{R}$, $\mathcal{L}_s(D)$ is a Hilbert space such that the inner product $(\cdot,\cdot)_s$ is the pullback of $L^2$ inner product, where $(\cdot,\cdot)_s$ is defined by

$$(f,g)_s = \left((-\Delta)^{-s} f, (-\Delta)^{-s} g\right)_{L^2}.$$

It is easy to check that although the formal sum $h = \sum_{m=0}^{\infty} X_m h_m$ may not be convergent in $H^1$, it is well-defined as a random distribution in $\mathcal{L}_s(D)$ with $s > \frac{n-2}{4}$. In particular, if $n = 2$, then one may be allowed to define $h$ as a random distribution in $\mathcal{L}_s(D)$ with $s > 0$.

### 3.1.2 GFF on $\mathbb{R}^n$

Given $s > 1$, the Bessel-type operator $L = (I - \Delta)^s$ allows GFFs to be constructed on the entire Euclidean space $\mathbb{R}^n$.

Consider the Sobolev space $H^s \equiv H^s(\mathbb{R}^n)$ with $n \in \mathbb{N}$, which is the completion of the space $\mathcal{D}(\mathbb{R}^n)$ of all smooth compactly supported real-valued functions under the inner product

$$(f,g)_{H^s} \equiv \left((I-\Delta)^s f, g\right)_{L^2} = \frac{1}{(2\pi)^\nu} \int_{\mathbb{R}} (1+|\xi|^2)^s \hat{f}(\xi)\hat{g}(\xi) d\xi$$

for all $f, g \in \mathcal{D}(\mathbb{R}^n)$. Then, the separable Hilbert space $(H^s, (\cdot,\cdot)_{H^s})$ is a Cameron-Martin space for some abstract Wiener spaces, and thus there exist a separable Banach space $\Theta^s \equiv \Theta^s(\mathbb{R}^n)$ and a Borel probability measure $\mathcal{W}^s \equiv \mathcal{W}^s(\mathbb{R}^n)$ on $\Theta^s$ such that $(H^s, \Theta^s, \mathcal{W}^s)$ forms an AWS. In particular, when $s = \frac{n+1}{2}$, $\Theta^{\frac{n+1}{2}}$ can be taken as the space of continuous paths $\theta : \Theta^n \to \mathbb{R}$ such that $\lim_{|x|\to\infty} \log(e+|x|)^{-1} \left|\theta(x)\right| = 0$ with the norm

$$\|\theta\|_{\Theta^{\frac{n+1}{2}}} = \sup_{x\in\mathbb{R}^n} (\log(e+|x|)^{-1} \left|\theta(x)\right|.$$

In this case, $\theta$ is $\alpha$-Hölder continuous $\mathcal{W}^{\frac{n+1}{2}}$-almost surely for $\alpha \in (0, \frac{1}{2})$.

More generally, given $s \in \mathbb{R}$ and set

$$\begin{aligned}\Theta^s &= \{(I-\Delta)^{-\frac{n+1-2s}{4}}\theta : \theta \in \Theta^{\frac{n+1}{2}}\},\\ \|\theta\|_{\Theta^s} &= \left\|(I-\Delta)^{\frac{n+1-2s}{4}}\theta\right\|_{\Theta^{\frac{n+1}{2}}},\\ \mathcal{W}^s &= ((I-\Delta)^{-\frac{n+1-2s}{4}})_*\mathcal{W}^{\frac{n+1}{2}}.\end{aligned}$$

Then, $\Theta^s$ is a separable Banach space in which $H^s$ is continuously embedded as a dense subspace, and the triple $(H^s, \Theta^s, \mathcal{W}^s)$ forms an abstract Wiener space, to which we refer as the GFF. In addition, $(\Theta^s)^*$ is a subspace of $H^{-s}$ (which is the dual space of $H^s$), and for each $\lambda^* \in (\Theta^s)^*$, $h_{\lambda^*} := (I-\Delta)^{-s}\lambda^*$ is the unique element in $H^s$ such that $(h, h_{\lambda^*})_{H^s} = \langle h, \lambda^* \rangle$ for all $h \in H^s$. It is not hard to see that the Paley Wiener integrals $\{\mathcal{I}(h_\lambda) : \lambda \in H^{-s}\}$ form a centered Gaussian family with the covariance

$$\mathbb{E}^{\mathcal{W}^s}[\mathcal{I}(h_\lambda)\mathcal{I}(h_\nu)] = (h_\lambda, h_\nu)_{H^s} = (\lambda, \nu)_{H^{-s}} \text{ for all } \lambda, \nu \in H^{-s}.$$

## 3.2 Green's function

The Green's function is an integral kernel representing the inverse operator $L^{-1}$ on a given domain, and it plays an important role in quantum field theory. It is a basic fact that GFFs can always be characterized by Green's functions.

### 3.2.1 Green's function of $-\Delta$ in a bounded domain $D$

The *Green's function* $G_D : D \times D \to \mathbb{R}$ of the Laplace operator $L = -\Delta$ in a bounded domain $D \subset \mathbb{R}^n$ is defined by

$$G_D(x,y) = \Phi(y-x) - \widetilde{G}^x(y) \quad (x, y \in D, x \neq y),$$

where $\Phi$ is given by

$$\Phi(x) = \begin{cases} -\log|x| & (n=2)\\ \frac{2\Gamma(\frac{n}{2}+1)}{n(n-2)\pi^{\frac{n}{2}-1}}\frac{1}{|x|^{n-2}} & (n \geq 3)\end{cases}$$

for $x \in D$ with $x \neq 0$, and $\widetilde{G}^x$ is the harmonic extension to $D$ of $\Phi(\cdot - x)$ on the boundary $\partial D$. We write $G(x,y) = G_D(x,y)$. Fix $x \in D$, $G(x,x) = \infty$ and $G(x,\cdot)$ is harmonic in $D \setminus \{x\}$ with $G = 0$ on $\partial D$.

The Green's function $G$ in a domain $D$ has the following properties:

- Fix $x \in D$, $-\frac{1}{2\pi}\Delta G(x, \cdot) = \delta_x$ in the sense of distributions, where $\delta_x$ is the Dirac point mass at $x$.
- For any $x \in D$, $G(x, \cdot) \in H^1$.
- (Symmetry) For all $x, y \in D$, $G(x, y) = G(y, x)$.
- (Conformal invariance) If $D \subset \mathbb{R}^2$ and $f : D \to D'$ is a conformal map, then for any $x, y \in D$, $G_{f(D)}(f(x), f(y)) = G(x, y)$. Moreover, the harmonic extension $\widetilde{G}^x(y)$ satisfies $\widetilde{G}^x(y) = \log C(x, D)$, where $C(x, D)$ is the *conformal radius* of $D$.

If $u \in C^2(\bar{D})$ solves $-\Delta u = \rho$ for $\rho \in \mathcal{D}(D)$ with $u = 0$ on $\partial D$, then

$$u(x) = -\Delta^{-1}\rho(x) = \frac{1}{2\pi}\int_D G(x, y)\rho(y)dy \quad (x \in D). \tag{3.1}$$

For any $\rho_1, \rho_2 \in \mathcal{D}(D)$, use integration by parts and (3.1) to write

$$\begin{aligned} \mathrm{Cov}[(h, \rho_1), (h, \rho_2)] &= \mathbb{E}[\left(h, -2\pi\Delta^{-1}\rho_1\right)_{H^1}\left(h, -2\pi\Delta^{-1}\rho_2\right)_{H^1}] \\ &= \left(-2\pi\Delta^{-1}\rho_1, -2\pi\Delta^{-1}\rho_2\right)_{H^1} \\ &= \left(-2\pi\Delta^{-1}\rho_1, \rho_2\right)_{L^2} \\ &= \int\int_{D\times D} \rho_1(x)G(x, y)\rho_2(y)dxdy. \end{aligned}$$

Therefore, the collection $\{(h, \rho)\}_{\rho\in\mathcal{D}(D)}$ is a centered Gaussian process with the following convariance structure

$$\mathrm{Cov}[(h, \rho_1), (h, \rho_2)] = \int\int_{D\times D} \rho_1(x)G(x, y)\rho_2(y)dxdy.$$

### 3.2.2 Green's function of $(I - \Delta)^s$ on $\mathbb{R}^n$

Now consider the equation

$$(I - \Delta)^s u = \rho \text{ for all } x \in \mathbb{R}^n. \tag{3.2}$$

We solve this equation by computing the Fourier transform of $u$ in the spatial variables $x$ and derive

$$\hat{u} = \frac{\hat{\rho}}{(1 + |\xi|^2)^s}.$$

It follows that

$$u(x) = \mathcal{F}^{-1}\left(\hat{u}\right) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{e^{i(x,\xi)_{\mathbb{R}^n}}}{(1+|\xi|^2)^s} \hat{\rho}(\xi) d\xi.$$

The fundamental solution of Equation (3.2) is given by

$$\begin{aligned}
\Phi(x) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^s} e^{i(x,\xi)_{\mathbb{R}^n}} d\xi \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_0^\infty \frac{r^{n-1}}{(1+r^2)^s} \int_{S^{n-1}} e^{i(x,rx')_{\mathbb{R}^n}} d\sigma(x') dr \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_0^\infty \frac{r^{n-1}}{(1+r^2)^s} \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_{-1}^1 e^{i(|x|,rs)_{\mathbb{R}^n}} (1-s^2)^{\frac{n-3}{2}} ds dr \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_0^\infty \frac{r^{n-1}}{(1+r^2)^s} \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}{(\frac{|x|r}{2})^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(|x|\, r) dr \\
&= \int_0^\infty \frac{r^{n-1}}{(1+r^2)^s} \frac{J_{\frac{n-2}{2}}(|x|\, r)}{(|x|\, r)^{\frac{n-2}{2}}} dr,
\end{aligned}$$

and, therefore, the Green's function of (3.2) is given by

$$G(x,y) = \Phi(x-y) = \int_0^\infty \frac{r^{n-1}}{(1+r^2)^s} \frac{J_{\frac{n-2}{2}}(|x-y|\, r)}{(|x-y|\, r)^{\frac{n-2}{2}}} dr.$$

Note that for $n \geq 2$, if $s = \frac{n}{2}$, then the Green's function of $(I-\Delta)^s$ on $\mathbb{R}^n$ has logarithmic singularity and hence the corresponding GFF is logarithmic correlated; if $s \in \frac{1}{2}\mathbb{N}$ with $s < \frac{n}{2}$, then the Green's function of $(I-\Delta)^s$ on $\mathbb{R}^n$ has polynomial singularity with degree $n-2s$ and thus the corresponding GFF is polynomially correlated.

## 3.3 Field averages on a bounded domain $D$

A useful tool in the study of the GFF is the field average, which gives the mean value of the GFF over a circle or a sphere centered around a point in the domain.

For any $\varepsilon > 0$ and $x \in D$, where $D \subseteq \mathbb{R}^n$ is a bounded domain, define

$$G_\varepsilon^x(y) = -\log(|x-y| \vee \varepsilon) + \tilde{G}^x(y).$$

Then, $G_\varepsilon^x \in H^1$ and

$$-\frac{1}{2\pi}\Delta[G_\varepsilon^x(\cdot)] = \sigma_\varepsilon^x,$$

where $\sigma_\varepsilon^x \in H^{-1}$ denotes the spherical average measure over the sphere $\partial B(x,\varepsilon)$ centered at $x \in \mathbb{R}^n$ with radius $\varepsilon > 0$. If $h$ is a sample of a GFF in $D$, then for every $x \in D$, let

$$h_{\sigma_x^\varepsilon} \equiv (h, G_\varepsilon^x)_{H^1} = \frac{1}{2\pi}\left\langle h, (-\Delta)G_\varepsilon^x(\cdot)\right\rangle = \langle h, \sigma_\varepsilon^x\rangle .$$

In particular, when $n = 2$, $h_{\sigma_x^\varepsilon}$ is the circle average of GFF on $\mathbb{R}^2$ with variance

$$\begin{aligned}\mathrm{Var}(h_{\sigma_x^\varepsilon}) &= \mathrm{Var}[(h, G_\varepsilon^x)_{H^1}] = (G_\varepsilon^x, G_\varepsilon^x)_{H^1} = \langle G_\varepsilon^x, \sigma_\varepsilon^x\rangle \\ &= -\log\varepsilon + \int_D \tilde{G}^x(y)\sigma_\varepsilon^x(dy) \\ &= -\log\varepsilon + \log C(x, D).\end{aligned}$$

**Proposition 3.1.** (Circle average is a Brownian motion) Let $h$ be a GFF with zero-boundary conditions in $D$. For any $x \in D$, let

$$t_0^x = \inf\{t \geq 0; B_{e^{-t}}(x) \subset D\}.$$

Then, the stochastic process

$$\mathcal{B}_t(x) = h_{e^{-(t_0^x+t)}} - h_{e^{-t_0^x}},$$

is a standard Brownian motion.

*Proof.* It is clear that the collection of random variables $\{B_t(x)\}_{t\geq 0}$ is a Gaussian process. For any $0 \leq s \leq t$, we first derive

$$\begin{aligned}\mathrm{Cov}[h_{e^{-(t_0^x+s)}}, h_{e^{-(t_0^x+t)}}] &= \mathbb{E}\left[\left(h, G^x_{e^{-(t_0^x+s)}}\right)_{H^1}, \left(h, G^x_{e^{-(t_0^x+t)}}\right)_{H^1}\right] \\ &= \left(G^x_{e^{-(t_0^x+s)}}, G^x_{e^{-(t_0^x+t)}}\right)_{H^1} = \left\langle G^x_{e^{-(t_0^x+s)}}, \sigma^x_{e^{-(t_0^x+t)}}\right\rangle \\ &= -\log e^{-(t_0^x+s)} + \log C(x, D) \\ &= t_0^x + s + \log C(x, D).\end{aligned}$$

Similarly, one can compute $\mathrm{Cov}[h_{e^{-(t_0^x)}}, h_{e^{-(t_0^x+s)}}]$ and $\mathrm{Cov}[h_{e^{-(t_0^x)}}, h_{e^{-(t_0^x+t)}}]$. There, we have

$$\begin{aligned}\mathrm{Cov}[\mathcal{B}_s(x), \mathcal{B}_t(x)] &= \mathrm{Cov}[h_{e^{-(t_0^x+s)}} - h_{e^{-t_0^x}}, h_{e^{-(t_0^x+t)}} - h_{e^{-t_0^x}}] \\ &= t_0^x + s - t_0^x - t_0^x + t_0^x = s.\end{aligned}$$

□

In fact, the process $h_{\sigma_\varepsilon^x}$ determines a random continuous function of $x$ and $\varepsilon$.

**Proposition 3.2.** (Cirlce average is jointly Hölder) There exists a modification of $h$ such that $h_{\sigma_\varepsilon^x}$ is locally Hölder jointly continuous of order $\alpha > \frac{1}{2}$.

This proposition is a consequence of the Kolmogorov-Centsov continuity theorem, and the proof can be found in [1].

# Chapter 4

# GFF in even dimensions

## 4.1 Circle averages of GFF on $\mathbb{R}^2$

We begin the study of Liouville quantum gravity introduced by Duplantier and Sheffield [1]. Informally, a random measure can be expressed by "$e^{\gamma h(x)}dx$". When $h$ is an instance of the GFF, such a measure is referred to as a Liouville measure.

### 4.1.1 Liouville measures

For any bounded domain $D$ in $\mathbb{R}^2$, $\gamma > 0$ and $\varepsilon > 0$, let $m_\varepsilon$ be the measure absolutely continuous with respect to the Lebesgue measure and

$$m_\varepsilon(dx) = \varepsilon^{\gamma^2/2} e^{\gamma h_{\sigma_\varepsilon^x}} dx.$$

If $\gamma \in [0, 2)$, then for almost surely $\varepsilon \downarrow 0$, Duplantier and Sheffield [1] prove that the measure $m_\varepsilon$ weakly converges inside $D$ toward the Liouville quantum gravity measure "$m_\gamma(dx) = e^{\gamma h(x)}dx$"as $\varepsilon \downarrow 0$, where $h_{\sigma_\varepsilon^x}$ is the circular average of GFF over the circle centered at $x$ with radius $\varepsilon$ and $dx$ denotes the Lebesgue measure on $D$.

### 4.1.2 KPZ relation in $\mathbb{R}^2$

For the measure $m_\gamma$ on $D$, the *isothermal quantum ball* $B^\delta(x)$ of area $\delta$ centered at $x \in \mathbb{R}^2$ is defined by $m_\gamma(B^\delta(x)) = \delta$. Given a subset $\Omega \subset D$, denote the $\varepsilon$ neighborhood of $\Omega$ by

$$B_\varepsilon(\Omega) = \{x \in \mathbb{R}^2 : B_\varepsilon(x) \cap \Omega \neq \emptyset\}.$$

We also define the *isothermal quantum* $\delta$ neighborhood of $\Omega$ by

$$\Omega^\delta = \{x \in \mathbb{R}^2 : B^\delta(x) \cap \Omega \neq \emptyset\}.$$

Then, we say that $\Omega$ has the *Euclidean scaling exponent* $\rho$ if

$$\lim_{\varepsilon \downarrow 0} \frac{\log \mathrm{Vol}(B_\varepsilon(\Omega))}{\log \varepsilon^2} = \rho,$$

and the *quantum scaling exponent* $Q$ if

$$\lim_{\delta \downarrow 0} \frac{\log \mathbb{E}[m_\gamma(\Omega^\delta)]}{\log \delta} = Q.$$

**Theorem 4.1.** Fix $\gamma \in [0, 2)$ and a compact subset $E$ of $D$. If $X \cap E$ has the Euclidean scaling exponent $\rho \geq 0$ then it has the quantum scaling exponent $Q$ where

$$\rho = \frac{\gamma^2}{4} Q^2 + \left(1 - \frac{\gamma^2}{4}\right) Q.$$

## 4.2 Spherical averages of GFF on $\mathbb{R}^4$

Chen and Jakobson [2] first introduced the generalization of the 2D results by Duplantier and Sheffield [1] to four dimensions by viewing GFF as an AWS.

### 4.2.1 Construction of random measures

Consider the Hilbert space $H \equiv H^2(\mathbb{R}^4)$, which is the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^4)$ equipped with the inner product

$$(f, g)_H = \int_{\mathbb{R}^4} (I - \Delta)^2 f(x) g(x) dx \quad \text{for all } f, g \in \mathcal{S}(\mathbb{R}^4).$$

Then the Gaussian free field on $\mathbb{R}^4$ refers to the probability space $(\Theta, \mathcal{B}_(\Theta), \mathcal{W})$ such that $(H, \Theta, \mathcal{W})$ is an AWS, and $H^{-2} = H^{-2}(\mathbb{R}^4)$ is the Hilbert space consisting of tempered distributions $\mu$ such that

$$\|\mu\|_{H^{-2}}^2 \equiv \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \frac{1}{(1 + |\xi|^2)^2} \left|\hat{\mu}(\xi)\right|^2 d\xi < \infty,$$

where $\hat{\mu}$ is the Fourier transform of $\mu$. As $(I-\Delta)^{-2} : H^{-2} \to H$ is linear isometRic, one may identify $H$ with $H^{-2}$ and therefore $h_\nu \equiv (I-\Delta)^{-2}\nu$ is the unique element in $H$ such that $\langle h, \nu \rangle = (h, h_\nu)_H$ for all $h \in H$. Moreover, $\{\mathcal{I}(h_\nu) : \nu \in H^{-2}\}$ forms a Gaussian family with covariance

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\nu_1})\mathcal{I}(h_{\nu_2})] = (h_{\nu_1}, h_{\nu_2})_H = (\nu_1, \nu_2)_{H^{-2}}.$$

Given $x \in \mathbb{R}^4$ and $\varepsilon > 0$, define the tempered distribution $\sigma_\varepsilon^x \in H^{-2}$ by

$$\langle f, \sigma_\varepsilon^x \rangle = \frac{1}{2\pi^2\varepsilon^3} \int_{\partial B_\varepsilon(x)} f(y) d\sigma(y) \text{ for all } f \in \mathcal{S}(\mathbb{R}^4).$$

In other words, $\sigma_\varepsilon^x$ is the spherical average measure of $B_\varepsilon(x)$ in the sense of tempered distribution.

Introduce the matrix

$$\mathbf{A}(\varepsilon) \equiv \begin{pmatrix} K_1'(\varepsilon) & K_1(\varepsilon)/\varepsilon \end{pmatrix} \text{ and } \mathbf{B}(\varepsilon) \equiv \begin{pmatrix} I_1(\varepsilon)/\varepsilon & I_1'(\varepsilon) \end{pmatrix}.$$

If $x \in \mathbb{R}^4$ and $\varepsilon_1 \geq \varepsilon_2 > 0$, then

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\sigma_{\varepsilon_1}^x})\mathcal{I}(h_{\sigma_{\varepsilon_2}^x})] = (-\frac{1}{4\pi^2})\mathbf{A}(\varepsilon_1)\mathbf{B}^T(\varepsilon_2). \quad (4.1)$$

Note that the family of spherical averages in $\mathbb{R}^4$ does not possess the "good" properties as in the 2D case. In particular, the concentric family of spherical average is not even a Markov process. The solution to fix this issue is to "collect" more information about the GFF. Particularly, not only do we collect the average of GFF over a sphere, but we also take into account the "rate of change" of the spherical average at the same sphere. Therefore, Chen and Jakobson introduced

$$\{\mathcal{I}(h_{\sigma_\varepsilon^x}), \mathcal{I}(h_{d\sigma_\varepsilon^x}) : x \in \mathbb{R}^4, \varepsilon > 0\},$$

where $d\sigma_\varepsilon^x$ denotes the tempered distribution given by $\langle f, d\sigma_\varepsilon^x \rangle = \frac{d}{d\varepsilon} \langle f, \sigma_\varepsilon^x \rangle$ for all $f \in \mathcal{S}(\mathbb{R}^4)$.

Define

$$\mathbf{C}(\varepsilon) \equiv \begin{pmatrix} I_1(\varepsilon)/\varepsilon & I_1'(\varepsilon) \\ I_2(\varepsilon)/\varepsilon & I_1''(\varepsilon) \end{pmatrix} \text{ and } \mathbf{V}_\varepsilon^x \equiv \begin{pmatrix} \mathcal{I}(h_{\sigma_\varepsilon^x}) \\ \mathcal{I}(h_{d\sigma_\varepsilon^x}) \end{pmatrix}.$$

It is shown by Chen and Jakobson [2] that if we define the "normalized" vector $\mathbf{U}_\varepsilon^x \equiv \mathbf{C}^{-1}(\varepsilon)\mathbf{V}_\varepsilon^x$, then the Gaussian family $\{\mathbf{U}_\varepsilon^x : \varepsilon > 0\}$ is a backward Markovian. Let $\zeta = (1,1)^T$ and denote

$$\mu_\epsilon^x \equiv \zeta^T \mathbf{C}^{-1}(\varepsilon) \begin{pmatrix} \sigma_\epsilon^x \\ d\sigma_\epsilon^x \end{pmatrix} = f_1(\epsilon)\sigma_\epsilon^x + f_2(\epsilon) d\sigma_\epsilon^x$$

with

$$f_1(\varepsilon) = \frac{\varepsilon I_1(\varepsilon) - 2I_2(\varepsilon)}{I_1^2(\varepsilon) - I_0(\varepsilon)I_2(\varepsilon)} \text{ and } f_2(\varepsilon) = \frac{-\varepsilon I_2(\varepsilon)}{I_1^2(\varepsilon) - I_0(\varepsilon)I_2(\varepsilon)},$$

then $\mu_\varepsilon^x$ converges to $2\delta_x$ as $\varepsilon \downarrow 0$. Moreover, it is proven that the mapping $\mathcal{I}(h_{\mu_\varepsilon^x})(\theta)$ is almost surely $\alpha$-Hölder continuous with respect to $(x,\theta) \in \mathbb{R}^4 \times \Theta$ for every $\alpha < \frac{1}{2}$.

Define $G : (0,\infty) \to (0,\infty)$ by

$$G(\varepsilon) = \frac{2I_1(\varepsilon)K_1(\varepsilon) + 2I_2(\varepsilon)K_0(\varepsilon) - 1}{I_1^2(\varepsilon) - I_0(\varepsilon)I_2(\varepsilon)}.$$

It can be checked that $G$ is strictly decreasing and smooth with $\lim_{\varepsilon\downarrow 0} G(\varepsilon) = +\infty$ and $\lim_{\varepsilon\uparrow\infty} G(\varepsilon) = 0$ . In addition, we have the following:

(1) Given $x \in \mathbb{R}^4$ and $\varepsilon_1 \geq \varepsilon_2 > 0$,

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu_{\varepsilon_1}^x})\mathcal{I}(h_{\mu_{\varepsilon_2}^x})] = \mathbb{E}^{\mathcal{W}}[\mathcal{I}^2(h_{\mu_{\varepsilon_1}^x})] = G(\varepsilon_1),$$

which is asymptotic to $-\frac{1}{2\pi^2}\log\varepsilon_1$ for arbitrary small $\varepsilon_1$. Fix $\varepsilon_0 > 0$, the Gaussian family $\{\mathcal{I}(h_{\mu_\varepsilon^x}) : 0 < \varepsilon \leq \varepsilon_0\}$ is a Brownian motion up to a non-random time change.

(2) Given $x, y \in \mathbb{R}^4$ with $x \neq y$ and $\varepsilon_1, \varepsilon_2 > 0$ with $\varepsilon_1 > |x-y| + \varepsilon_2$,

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu_{\varepsilon_1}^x})\mathcal{I}(h_{\mu_{\varepsilon_2}^y})] = I_0(|x-y|)G(\varepsilon_1) - \frac{1}{4\pi^2}\frac{I_2(|x-y|)}{I_1^2(\varepsilon_1) - I_0(\varepsilon_1)I_2(\varepsilon_1)},$$

which is asymptotic to $-\frac{1}{2\pi^2}\log\varepsilon_1$ for an arbitrary small $\varepsilon_1$.

(3) Given $x, y \in \mathbb{R}^4$ with $x \neq y$ and $\varepsilon_1, \varepsilon_2 > 0$ with $|x-y| > \epsilon_1 + \epsilon_2$,

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu_{\varepsilon_1}^x})\mathcal{I}(h_{\mu_{\varepsilon_2}^y})] = \frac{1}{2\pi^2}K_0(|x-y|)$$

which is asymptotic to $-\frac{1}{2\pi^2}\log|x-y|$ for arbitrary small $\varepsilon_1$.

Now it is convincing that $\{\mathcal{I}(h_{\mu_\varepsilon^x}) : x \in \mathbb{R}^4, \varepsilon > 0\}$ is a suitable candidate to replace the circular average in $\mathbb{R}^4$. Chen and Jakobson proved that if a sequence of approximating measures is defined by

$$m^\theta_{\varepsilon_n}(dx) \equiv \exp(\gamma \mathcal{I}(h_{\mu^x_{\varepsilon_n}}) - \frac{\gamma^2}{2} G(\varepsilon_n))dx \tag{4.2}$$

with $0 < \gamma^2 < 2\pi^2$ and $\varepsilon_n \equiv \varepsilon_0^n$ for fixed $\varepsilon_0 \in (0,1)$, then the sequence $\{m^\theta_{\varepsilon_n} : n \geq 1\}$ is weakly convergent almost surely as $\varepsilon_n \downarrow 0$.

### 4.2.2 KPZ relation in $\mathbb{R}^4$

Given a bounded domain $\Omega \subseteq \mathbb{R}^4$, we say that $B_\varepsilon(\Omega)$ is the *$\varepsilon$-neighborhood* of $\Omega$ if

$$B_\varepsilon(\Omega) \equiv \{x \in \mathbb{R}^4 : B_\varepsilon(x) \cap \Omega \neq \emptyset\}.$$

Note that if $0 < \gamma^2 < \pi^2$ and $x \in \mathbb{R}^4$ is fixed, then it can be shown that

$$\mathbb{E}^{\mathcal{W}}[\limsup_{n\to\infty} e^{8\gamma^2 G(\varepsilon_n)} m^\theta(\overline{B_{\varepsilon_n}(x)})] = 0.$$

Set

$$\Theta_x \equiv \{\theta \in \Theta : \limsup_{n\to\infty} e^{8\gamma^2 G(\varepsilon_n)} m^\theta(\overline{B_{\varepsilon_n}(x)}) = 0\}, \tag{4.3}$$

$\Theta_x$ is a measurable set and $\mathcal{W}(\Theta_x) = 1$. For $\omega > 0$ and $\theta \in \Theta$, denote

$$R(x,\theta;\omega) \equiv \begin{cases} \sup\{r > 0 : m^\theta(B_r(x)) \leq \omega\} & \text{if } \theta \in \Theta_x, \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

and introduce the *isothermal $\omega$- neighborhood* of $\Omega$

$$\Omega^\theta(\omega) \equiv \left\{x \in \Omega \text{ and } R(x,\theta;\omega) = 0 \text{ or } 0 < \operatorname{dist}(x,\Omega) < R(x,\theta;\omega)\right\}. \tag{4.5}$$

We call $\rho$ the *Euclidean scaling exponent* of $\Omega$ if

$$\lim_{\epsilon\downarrow 0} \frac{\log \operatorname{Vol}(\Omega_\epsilon)}{\log \epsilon^4} = \rho, \tag{4.6}$$

where $\epsilon > 0$ and $\Omega_\epsilon \equiv \cup_{z\in\Omega} B_\epsilon(z)$ is the canonical $\epsilon$-neighborhood of $\Omega$, and we call $Q$ the *quantum scaling exponent* of $\Omega$ if

$$\lim_{\omega\downarrow 0} \frac{\log \mathbb{E}^{\mathcal{W}}[m^\theta(\Omega^\theta(\omega))]}{\log \omega} = Q. \tag{4.7}$$

Assume is a bounded Borel set with the Euclidean scaling exponent $\rho \in [0,1]$. Then, $\Omega$ has the quantum scaling exponent $Q \in [0,1]$ as defined in. where $Q$ is determined by the following quadratic relation with $\rho$ :

$$\rho = \frac{\gamma^2}{16\pi^2} Q^2 + \left(1 - \frac{\gamma^2}{16\pi^2}\right) Q.$$

## 4.3 Possible Replacement of $\mathbb{R}^4$

The approach outlined above is to collect information about the spherical average as well as its "rate of change". We now propose another way to collect more information on the GFF, i.e., to consider the averages of the GFF over two different spheres rather than a single "sphere". In other words, our goal is find proper radiuses $r_1(\varepsilon)$ and $r_2(\varepsilon)$ and the weighted coefficients $f_1(\varepsilon)$ and $f_2(\varepsilon)$ such that

$$\mu_\varepsilon^x = f_1(\varepsilon)\sigma^x_{r_1(\varepsilon)} + f_2(\varepsilon)\sigma^x_{r_2(\varepsilon)},$$

where $\mu_\varepsilon^x$ converges to a constant multiple of $\delta_x$ and the Gaussian family $\{\mathcal{I}(h_{\mu^x_{\varepsilon_n}}) : \varepsilon > 0\}$ satisfies the Markov property.

Our first attempt is to take $r_1(\varepsilon) = \varepsilon$ and $r_2(\varepsilon) = c\varepsilon$ with undefined constant $c \in (0,1)$.

Given $x \in \mathbb{R}^4$ and $\varepsilon_1 \geq \varepsilon_2 > 0$, Formula (5.6) implies

$$\begin{aligned}
& \mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu^x_{\varepsilon_1}})\mathcal{I}(h_{\mu^x_{\varepsilon_2}})] \\
= \ & f_1(\varepsilon_1)f_1(\varepsilon_2)\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\sigma^x_{\varepsilon_1}})\mathcal{I}(h_{\sigma^x_{\varepsilon_2}})] + f_1(\varepsilon_1)f_2(\varepsilon_2)\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\sigma^x_{\varepsilon_1}})\mathcal{I}(h_{\sigma^x_{c\varepsilon_2}})] \\
& f_2(\varepsilon_1)f_1(\varepsilon_2)\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\sigma^x_{c\varepsilon_1}})\mathcal{I}(h_{\sigma^x_{\varepsilon_2}})] + f_2(\varepsilon_1)f_2(\varepsilon_2)\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\sigma^x_{c\varepsilon_1}})\mathcal{I}(h_{\sigma^x_{c\varepsilon_2}})] \\
= \ & (-\frac{1}{4\pi^2})[f_1(\varepsilon_1)\mathbf{A}(\varepsilon_1) + f_2(\varepsilon_1)\mathbf{A}(c\varepsilon_1)][f_1(\varepsilon_2)\mathbf{B}^T(\varepsilon_2) + f_2(\varepsilon_2)\mathbf{B}^T(c\varepsilon_2)],
\end{aligned}$$

where $c\varepsilon_1 > \varepsilon_2$ is required for the third covariance function.

To preserve the Markov property, we can simply require

$$\begin{aligned}\begin{pmatrix} a \\ b \end{pmatrix} &= f_1(\varepsilon_2)\mathbf{B}^T(\varepsilon_2) + f_2(\varepsilon_2)\mathbf{B}^T(c\varepsilon_2) \\ &= \begin{pmatrix} f_1(\varepsilon_2)I_1(\varepsilon_2)/\varepsilon_2 + f_2(\varepsilon_2)I_1(c\varepsilon_2)/c\varepsilon_2 \\ f_1(\varepsilon_2)I_1'(\varepsilon_2) + f_2(\varepsilon_2)I_1'(c\varepsilon_2) \end{pmatrix} \\ &= \begin{pmatrix} I_1(\varepsilon_2)/\varepsilon_2 & I_1(c\varepsilon_2)/c\varepsilon_2 \\ I_1'(\varepsilon_2) & I_1'(c\varepsilon_2) \end{pmatrix} \begin{pmatrix} f_1(\varepsilon_2) \\ f_2(\varepsilon_2) \end{pmatrix}\end{aligned}$$

with $a, b \in \mathbb{R}$. Here we let $a = 1, b = -1$ and denote

$$\mathbf{C}(\varepsilon) = \begin{pmatrix} I_1(\varepsilon)/\varepsilon & I_1(c\varepsilon)/c\varepsilon \\ I_1'(\varepsilon) & I_1'(c\varepsilon) \end{pmatrix}.$$

For every $\varepsilon > 0$, the formulas $(A.11)$ and $(A.12)$ yield

$$\det \mathbf{C}(\varepsilon) = \frac{cI_1(\varepsilon)I_0(c\varepsilon) - I_1(c\varepsilon)I_0(\varepsilon)}{c\varepsilon}.$$

Hence,

$$\mathbf{C}^{-1}(\varepsilon) = \frac{1}{cI_1(\varepsilon)I_0(c\varepsilon) - I_1(c\varepsilon)I_0(\varepsilon)} \begin{pmatrix} c\varepsilon I_1'(c\varepsilon) & -I_1(c\varepsilon) \\ -c\varepsilon I_1'(\varepsilon) & cI_1(\varepsilon) \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} f_1(\varepsilon_2) \\ f_2(\varepsilon_2) \end{pmatrix} = \frac{1}{cI_1(\varepsilon_2)I_0(c\varepsilon_2) - I_1(c\varepsilon_2)I_0(\varepsilon_2)} \begin{pmatrix} c\varepsilon_2 I_0(c\varepsilon_2) \\ -cI_0(\varepsilon_2) \end{pmatrix}.$$

Then for each $x \in \mathbb{R}^4$, if $\mu_\varepsilon^x \in H^{-2}$ is defined by

$$\mu_\varepsilon^x = f_1(\varepsilon)\sigma_\varepsilon^x + f_2(\varepsilon)\sigma_{c\varepsilon}^x$$

with

$$f_1(\varepsilon) = \frac{c\varepsilon I_0(c\varepsilon)}{cI_1(\varepsilon)I_0(c\varepsilon) - I_1(c\varepsilon)I_0(\varepsilon)} \text{ and } f_2(\varepsilon) = \frac{-cI_0(\varepsilon)}{cI_1(\varepsilon)I_0(c\varepsilon) - I_1(c\varepsilon)I_0(\varepsilon)},$$

then $\mu_\varepsilon^x$ converges to $2\delta_x$ as $\varepsilon \downarrow 0$.

To construct the approximating measure $m^\theta_{\varepsilon_n}(dx)$ defined by (4.2) with $\varepsilon_n \equiv \varepsilon_0^n$ for $n \geq 1$, one can choose $c \in (\varepsilon_0, 1)$ so that $\{\mathcal{I}(h_{\mu^x_{\varepsilon_n}}) : x \in \mathbb{R}^4\}$ possesses the Markov property.

As we have seen in the 4D case, it is possible to obtain an "ideal" regularization of the GFF by taking a linear combination of the GFF's average over two spheres. Similarly, in $\mathbb{R}^{2n}$, one will need to consider the averages of GFF over n families of spheres simultaneously. However, this approach will fail in odd dimensions, as we will demonstrate in the next chapter.

# Chapter 5

# GFF with Fourier-Bessel series

The main goal of this chapter is to generalize part of the results from even dimensional Euclidean spaces [1, 2] to all finite dimensions $\mathbb{R}^n$ with $n \geq 2$ by the Fourier-Bessel expansion approach.

## 5.1 Introduction to Fourier-Bessel Series

We start with a brief introduction of Fourier-Bessel series. More information can be found in [19, Chapter 18].

Consider a function $f : (0, \alpha) \longrightarrow \mathbb{R}$ where $\alpha > 0$, which can be written in terms of series of Bessel functions of the first kind of non-negative order $\nu$

$$f(x) = \sum_{k=1}^{\infty} a_k J_\nu(j_k \frac{x}{\alpha}), \tag{5.1}$$

where $j_1 < j_2 < \ldots$ denote the positive zeros of $J_\nu$ and $\alpha > 0$.

To obtain the coefficients $a_k$, use the orthogonality of Bessel function zeros to write

$$\begin{aligned}
\int_0^\alpha x J_\nu(j_m \frac{x}{\alpha}) f(x) dx &= \sum_{k=1}^{\infty} a_k \int_0^\alpha x J_\nu(j_m \frac{x}{\alpha}) J_\nu(j_k \frac{x}{\alpha}) dx \\
&= a_m \int_0^\alpha x J_\nu^2(j_m \frac{x}{\alpha}) dx \\
&= a_m \frac{\alpha^2 J_{\nu+1}^2(j_m)}{2}.
\end{aligned}$$

It follows that

$$a_m = \frac{2}{\alpha^2 J_{\nu+1}^2(j_m)} \int_0^\alpha x J_\nu(j_m \frac{x}{\alpha}) f(x) dx \quad \text{for} \quad m \geq 1. \tag{5.2}$$

The series (5.1) with coefficients defined by (5.2) is called the *Fourier-Bessel expansion* of $f$ on $(0, \alpha)$. To simplify, we only consider the case of $\alpha = 1$.

**Theorem 5.1.** Let $f$ be a function defined in the interval $(0, 1)$ and let $\int_0^1 x^{\frac{1}{2}} \left| f(x) \right| dx$ exist and (if it is an improper integral) let it be absolutely convergent, then the series $\sum_{m=1}^\infty a_m J_\nu(j_m x)$ is convergent and its sum is equal to $[f(x+0) + f(x-0)]/2$.

The following theorem indicates the order of magnitude of the coefficients in the Fourier-Bessel series.

**Proposition 5.1.** If $x^{\frac{1}{2}} f(x)$ has limited total fluctuation in (0,1), then the coefficient $a_m$ is asymptotic to $j_m^{-\frac{1}{2}}$ when $m$ is large enough.

*Proof.* First, based on [19, pp595], we have

$$\int_0^1 x^{\frac{1}{2}} f(x) J(j_m x) dx = O(j_m^{-\frac{3}{2}}) \text{ as } m \to \infty. \tag{5.3}$$

Then from [20], we see

$$j_m = m\pi + \frac{\pi}{2}(\nu - \frac{1}{2}) - \frac{4\nu^2 - 1}{8(m\pi + \pi/2(\nu - 1/2))} + O(\frac{1}{m^3}) \text{ as } m \to \infty.$$

Recall that $J_\nu(x)$ is asymptotic to $\sqrt{\frac{2}{\pi x}} \cos(x - \frac{\pi}{4} - \frac{\nu\pi}{2})$ as long as $x$ is large enough. Thus, for sufficiently large $m$, it is easy to show that $2/J_{\nu+1}^2(j_m)$ is asymptotic to $\pi j_m$. Combining this result with the formula (5.3), we can conclude that the coefficient $a_m$ is asymptotic to $j_m^{-\frac{1}{2}}$ when $m$ is large enough. □

## 5.2 Fourier-Bessel series on $\mathbb{R}^n$

Consider the underlying Hilbert space $H^{\frac{n}{2}} \equiv H^{\frac{n}{2}}(\mathbb{R}^n)$ with $n \geq 2$, which is the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ with the inner product

$$(f, g)_{H^{\frac{n}{2}}} \equiv \int_{\mathbb{R}^n} (I - \Delta)^{\frac{n}{2}} f(x) g(x) dx \quad \text{for all } f, g \in S(\mathbb{R}^n).$$

As mentioned before, there exists a separable Banach space $\Theta^{\frac{n}{2}} \equiv \Theta^{\frac{n}{2}}(\mathbb{R}^n)$ and a Gaussian measure $\mathcal{W}^{\frac{n}{2}} \equiv \mathcal{W}^{\frac{n}{2}}(\mathbb{R}^n)$ such that the triple $(H^{\frac{n}{2}}, \Theta^{\frac{n}{2}}, \mathcal{W}^{\frac{n}{2}})$ forms an AWS. The GFF $h$ on $\mathbb{R}^n$ refers to the probability space $(\Theta^{\frac{n}{2}}, \mathcal{B}_{\Theta^{\frac{n}{2}}}, \mathcal{W}^{\frac{n}{2}})$ where $\mathcal{B}_{\Theta^{\frac{n}{2}}}$ denotes the Borel $\sigma$-algebra over $\Theta^{\frac{n}{2}}$.

Moreover, $H^{-\frac{n}{2}} \equiv H^{-\frac{n}{2}}(\mathbb{R}^n)$ is the Hilbert space which consists of all tempered distributions $\mu_1, \mu_2$ such that

$$(\mu_1, \mu_2)_{H^{-\frac{n}{2}}} \equiv \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-\frac{n}{2}} \hat{\mu_1}(\xi) \overline{\hat{\mu_2}(\xi)} d\xi < \infty.$$

If $h_\nu \equiv (I - \Delta)^{-\frac{n}{2}} \nu$ for some $\nu \in H^{-\frac{n}{2}}$, then $h_\nu$ is the unique element in $H^{\frac{n}{2}}$ such that $\langle h, \nu \rangle = (h, h_\nu)_{H^{-\frac{n}{2}}}$ for all $h \in H^{\frac{n}{2}}$ and the Paley-Wiener integrals $\{\mathcal{I}(h_\nu) : \nu \in H^{-\frac{n}{2}}\}$ form a Gaussian family with the covariance structure

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\nu_1})\mathcal{I}(h_{\nu_2})] = (h_{\nu_1}, h_{\nu_2})_{H^{\frac{n}{2}}} = (\nu_1, \nu_2)_{H^{-\frac{n}{2}}}.$$

Furthermore, given $x \in \mathbb{R}^n$ and $\varepsilon > 0$, $\sigma_\varepsilon^x \in H^{-\frac{n}{2}}$ denotes the spherical average measure over $\partial B(x, \varepsilon)$ and its Fourier transform referred to appendix A.4 is given by

$$\hat{\sigma_\varepsilon^x}(\xi) = \frac{2^{\frac{n-2}{2}} \Gamma(\frac{n}{2})}{(\varepsilon|\xi|)^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(\varepsilon|\xi|) e^{i(x,\xi)_{\mathbb{R}^n}}. \tag{5.4}$$

In particular, for every $x \in \mathbb{R}^n$ and $\varepsilon > 0$, the spherical average $\mathcal{I}(h_{\sigma_\varepsilon^x})$ of the GFF is well defined and $\mathcal{I}(h_{\sigma_\varepsilon^x})$ approximates $\delta_x$ as $\varepsilon \downarrow 0$.

### 5.2.1 Motivation

**Lemma 5.1.** Given $x \in \mathbb{R}^2$ and $\varepsilon_1 \geq \varepsilon_2 > 0$,

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\sigma_{\varepsilon_1}^x})\mathcal{I}(h_{\sigma_{\varepsilon_2}^x})] = \frac{1}{2\pi} K_0(\varepsilon_1) I_0(\varepsilon_2). \tag{5.5}$$

Therefore, the family $\{\mathcal{I}(h_{\sigma_\varepsilon^x}) : 0 < \varepsilon < 1\}$ is a backward Markov process.

*Proof.* A basic computation referred to formulas (5.4) and (A.10) implies

$$\begin{aligned}
\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\sigma_{\varepsilon_1}^x})\mathcal{I}(h_{\sigma_{\varepsilon_2}^x})] &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{1}{1 + |\xi|^2} \hat{\sigma_{\varepsilon_1}^x}(\xi) \hat{\sigma_{\varepsilon_2}^x}(\xi) d\xi \\
&= \frac{1}{2\pi} \int_0^\infty \frac{r}{1 + r^2} J_0(\varepsilon_1 r) J_0(\varepsilon_2 r) dr \\
&= \frac{1}{2\pi} K_0(\varepsilon_1) I_0(\varepsilon_2).
\end{aligned}$$

□

However, such facts about spherical averages fail in higher dimensions. For example, in [2], Chen and Jakobson pointed out that the covariance function of the Gaussian family consisting of all the spherical averages of GFF at a fixed point $x$ in $\mathbb{R}^{2n}$ with $n \geq 2$ fails the backward Markov property. But, as we will see in next section, family of certain functionals of spherical averages of GFF in $\mathbb{R}^{2n}$ can be chosen to possess such backward Markov property.

**Lemma 5.2.** Given $x \in \mathbb{R}^3$ and $\varepsilon_1 \geq \varepsilon_2 > 0$,

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\sigma^x_{\varepsilon_1}})\mathcal{I}(h_{\sigma^x_{\varepsilon_2}})] = \frac{1}{4\pi^2}[\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1\varepsilon_2}K_1(\varepsilon_1 - \varepsilon_2) - \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1\varepsilon_2}K_1(\varepsilon_1 + \varepsilon_2)]. \quad (5.6)$$

So, the family $\{\mathcal{I}(h_{\sigma^x_\varepsilon}) : 0 < \varepsilon < 1\}$ does not possess backward Markov property.

*Proof.* Under this assumption, the formula (A.9) implies

$$\begin{aligned}
&\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\sigma^x_{\varepsilon_1}})\mathcal{I}(h_{\sigma^x_{\varepsilon_2}})] \\
= \; & \frac{1}{16\pi^2}\int_{\mathbb{R}^3} \frac{1}{(1+|\xi|^2)^{\frac{3}{2}}} \frac{J_{\frac{1}{2}}(\varepsilon_1|\xi|)}{\sqrt{\varepsilon_1|\xi|}} \frac{J_{\frac{1}{2}}(\varepsilon_2|\xi|)}{\sqrt{\varepsilon_2|\xi|}} d\xi \\
= \; & \frac{1}{4\pi}\int_0^\infty \frac{r}{(1+r^2)^{\frac{3}{2}}} \frac{J_{\frac{1}{2}}(\varepsilon_1 r)}{\sqrt{\varepsilon_1}} \frac{J_{\frac{1}{2}}(\varepsilon_2 r)}{\sqrt{\varepsilon_2}} dr \\
= \; & \frac{1}{2\pi^2}\int_0^\infty \frac{1}{(1+r^2)^{\frac{3}{2}}} \frac{\sin(\varepsilon_1 r)}{\varepsilon_1} \frac{\sin(\varepsilon_2 r)}{\varepsilon_2} dr \\
= \; & \frac{1}{4\pi^2\epsilon_1\epsilon_2}\int_0^\infty \frac{1}{(1+r^2)^{\frac{3}{2}}} [\cos((\varepsilon_1 - \epsilon_2)r) - \cos((\varepsilon_1 + \varepsilon_2)r)] dr \\
= \; & \frac{1}{4\pi^2}[\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1\varepsilon_2}K_1(\varepsilon_1 - \varepsilon_2) - \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1\varepsilon_2}K_1(\varepsilon_1 + \varepsilon_2)] \\
= \; & \frac{1}{4\pi^2}\left(\frac{1}{\varepsilon_2}[K_1(\varepsilon_1 - \varepsilon_2) - K_1(\varepsilon_1 + \varepsilon_2)] - \frac{1}{\varepsilon_1}[K_1(\varepsilon_1 - \varepsilon_2) + K_1(\varepsilon_1 + \varepsilon_2)]\right).
\end{aligned}$$

□

More generally, the next computation shows that the same phenomenon is expected to occur in $\mathbb{R}^{2n+1}$ with $n \geq 1$.

**Theorem 5.2.** Given $x \in \mathbb{R}^{2n+1}$ with $n \geq 1$, $\{\mathcal{I}(h_{\sigma_\varepsilon^x}) : \varepsilon > 0\}$ fails to be a backward Markov Gaussian process.

*Proof.* Given $x \in \mathbb{R}^{2n+1}$ with $n \geq 1$ and $\varepsilon_1 \geq \varepsilon_2 > 0$, formulas (A.2) and (A.7) yield

$$\begin{aligned}
&\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\sigma_{\varepsilon_1}^x})\mathcal{I}(h_{\sigma_{\varepsilon_2}^x})] \\
= \;& \frac{2^{2n-1}\Gamma^2(\frac{2n+1}{2})}{(2\pi)^{2n+1}} \int_{\mathbb{R}^{2n+1}} \frac{1}{(1+|\xi|^2)^{n+\frac{1}{2}}} \frac{J_{\frac{2n-1}{2}}(\varepsilon_1|\xi|)}{(\varepsilon_1|\xi|)^{\frac{2n-1}{2}}} \frac{J_{\frac{2n-1}{2}}(\varepsilon_2|\xi|)}{(\varepsilon_2|\xi|)^{\frac{2n-1}{2}}} d\xi \\
= \;& \frac{2^{2n-1}\Gamma^2(\frac{2n+1}{2})}{(2\pi)^{2n+1}} \frac{2\pi^{\frac{2n+1}{2}}}{\Gamma(\frac{2n+1}{2})} \int_0^\infty \frac{r^{2n}}{(1+r^2)^{n+\frac{1}{2}}} \frac{J_{\frac{2n-1}{2}}(\varepsilon_1 r)}{(\varepsilon_1 r)^{\frac{2n-1}{2}}} \frac{J_{\frac{2n-1}{2}}(\varepsilon_2 r)}{(\varepsilon_2 r)^{\frac{2n-1}{2}}} dr \\
= \;& \frac{\Gamma(\frac{2n+1}{2})}{2\pi^{\frac{2n+1}{2}}} (\frac{d}{\varepsilon_1 d\varepsilon_1})^{n-1} (\frac{d}{\varepsilon_2 d\varepsilon_2})^{n-1} [\int_0^\infty \frac{r^{2n}}{(1+r^2)^{n+\frac{1}{2}}} \frac{J_{\frac{1}{2}}(\varepsilon_1 r)}{(\varepsilon_1 r)^{\frac{1}{2}}} \frac{J_{\frac{1}{2}}(\varepsilon_2 r)}{(\varepsilon_2 r)^{\frac{1}{2}}} dr] \\
= \;& \frac{\Gamma(\frac{2n+1}{2})}{2\pi^{\frac{2n+1}{2}}} (\frac{d}{\varepsilon_1 d\varepsilon_1})^{n-1} (\frac{d}{\varepsilon_2 d\varepsilon_2})^{n-1} [\frac{2}{\pi\varepsilon_1\varepsilon_2} \int_0^\infty \frac{r^{2n-2}}{(1+r^2)^{n+\frac{1}{2}}} \sin(\varepsilon_1 r)\sin(\varepsilon_2 r) dr].
\end{aligned}$$

Note that for $\varepsilon > 0$, the formula (A.9) implies

$$\begin{aligned}
\int_0^\infty \frac{r^{2n-2}}{(1+r^2)^{n+\frac{1}{2}}} \cos(\varepsilon r) dr &= \frac{d}{d\varepsilon^{2n-2}} [\int_0^\infty \frac{1}{(1+r^2)^{n+\frac{1}{2}}} \cos(\varepsilon r) dr] \\
&= \frac{d}{d\varepsilon^{2n-2}} [\frac{\varepsilon^n \pi^{1/2}}{2^n \Gamma(n+1/2)} K_n(\varepsilon)].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\sigma_{\varepsilon_1}^x})\mathcal{I}(h_{\sigma_{\varepsilon_2}^x})] \\
= \;& \frac{\Gamma(\frac{2n+1}{2})}{2\pi^{\frac{2n+1}{2}}} (\frac{d}{\varepsilon_1 d\varepsilon_1})^{n-1} (\frac{d}{\varepsilon_2 d\varepsilon_2})^{n-1} \\
& \times [\frac{1}{\pi\varepsilon_1\varepsilon_2} \int_0^\infty \frac{r^{2n-2}}{(1+r^2)^{n+\frac{1}{2}}} \left\{\cos((\varepsilon_1-\varepsilon_2)r) - \cos((\varepsilon_1+\varepsilon_2)r)\right\} dr] \\
= \;& \frac{\Gamma(\frac{2n+1}{2})}{2\pi^{\frac{2n+1}{2}}} (\frac{d}{\varepsilon_1 d\varepsilon_1})^{n-1} (\frac{d}{\varepsilon_2 d\varepsilon_2})^{n-1} \times \frac{(-1)^{n-1}}{\pi\varepsilon_1\varepsilon_2} \\
& \frac{d}{d\varepsilon_1^{2n-2}} \left[ \frac{\pi^{1/2}}{2^n\Gamma(n+1/2)} [(\varepsilon_1-\varepsilon_2)^n K_n(\varepsilon_1-\varepsilon_2) - (\varepsilon_1+\varepsilon_2)^n K_n(\varepsilon_1+\varepsilon_2)] \right].
\end{aligned}$$

Since the modified bessel functions $K_\nu(\varepsilon_1 \pm \varepsilon_2)$ are clearly not "separable" in the sense that it cannot be written as the product of a function of $\varepsilon_1$ and a function of $\varepsilon_2$ for $\nu \in \mathbb{R}$, we complete the proof. $\square$

Given $x \in \mathbb{R}^{2n+1}$ with $n \geq 1$, the Gaussian process $\{\mathcal{I}(h_{\sigma_\varepsilon^x}) : \varepsilon > 0\}$ not only fails the reversed Markov property, but also is not "separable". Thus, if we set up the approximating measure $\mu_\varepsilon^x$ as the linear combination of finite spherical measures $\sigma_{r_m(\varepsilon)}^x \in H^{-\frac{2n+1}{2}}$ with different radiuses $r_m(\varepsilon) > 0$, then the process also fails to be a backward Markovian. In other words, the finite spherical averages do not provide enough information to indicate how the Gaussian process will behave. More specically, we will collect information about the GFF from infinitely many spheres.

Now consider infinitely many spherical averages. We will show that the Fourier-Bessel expansion of certain function $f$ may be viewed as the collection of information about GFF on infinitely many spheres at a fixed point $x$ as the radius of the spheres increasing to infinity, which is a potential candidate for our project. Therefore, we assume $\mu_\varepsilon^x = \sum_{m=1}^\infty c_m \sigma_{r_m(\varepsilon)}^x$ is in the form of the Fourier-Bessel series by taking $r_m(\varepsilon) = j_m \varepsilon$ for all $m \geq 1$, where $j_m$'s are zeros of Bessel functions of order $\nu$ in ascending order of magnitude.

### 5.2.2 Spherical Averages of GFFs on $\mathbb{R}^n$

Assume that

$$\mu_\varepsilon^x = \sum_{m=1}^\infty c_m \sigma_{j_m\varepsilon}^x$$

with

$$\hat{\mu_\varepsilon^x}(\xi) = 1_{[0,\frac{1}{\varepsilon})}(|\xi|)e^{i(x,\xi)_{\mathbb{R}^n}} = 1_{[0,1)}(\varepsilon|\xi|)e^{i(x,\xi)_{\mathbb{R}^n}}, \tag{5.7}$$

where $j_1, j_2, , \ldots$ are the positive zeros of $J_{\frac{n-2}{2}}$ arranged in ascending order. Since the coefficients in the Fourier-Bessel expansion of $f(x)$ do not depend on the location $x$, we have

$$\hat{\mu_\varepsilon^x}(\xi) = \frac{1}{(\varepsilon|\xi|)^{\frac{n-2}{2}}} \sum_{m=1}^\infty c_m \Gamma(\frac{n}{2}) \left(\frac{2}{j_m}\right)^{\frac{n-2}{2}} J_{\frac{n-2}{2}}(j_m\varepsilon|\xi|)e^{i(x,\xi)_{\mathbb{R}^n}} \tag{5.8}$$

with

$$c_m = \frac{1}{\Gamma(\frac{n}{2})}\left(\frac{j_m}{2}\right)^{\frac{n-2}{2}} \frac{2}{J^2_{\frac{n}{2}}(j_m)} \int_0^1 y^{\frac{n}{2}} J_{\frac{n-2}{2}}(j_m y)dy \tag{5.9}$$

where $f(y) = y^{\frac{n-2}{2}} 1_{[0,1)}(y)$. Note that the coefficient $c_m$ is asymtotic to $j_m^{\frac{n-3}{2}}$ when $m$ is large.

By Theorem 5.1, since $f(y)$ is bounded and continuous in $(0,1)$ and $\int_0^1 t^{\frac{1}{2}} f(t)dt < \infty$, the series $\sum_{m=1}^{\infty} c_m \widehat{\sigma^x_{j_m\varepsilon}}(\xi)$ is convergent and its sum is equal to $\hat{\mu^x_\varepsilon}(\xi)$ for each $\xi \in \mathbb{R}^n$. In this case, $\mu^x_\varepsilon \in H^{-\frac{n}{2}}$ and $\mu^x_\varepsilon$ tends to the point mass $\delta_x$ as $\varepsilon \downarrow 0$ in the sense of distributions.

**Lemma 5.3.** Given $x \in \mathbb{R}^n$, let $S^x_N = \sum_{m=1}^{N} c_m \sigma^x_{j_m\varepsilon}$. Then $S^x_N$ converges to $\mu^x_\varepsilon$ in $H^{-\frac{n}{2}}$ as $N \to \infty$.

*Proof.* It suffices to show

$$\int_{B(0,\frac{1}{\varepsilon})} \frac{1}{(1+|\xi|^2)^{\frac{n}{2}}} \left| \hat{\mu^x_\varepsilon}(\xi) - \sum_{m=1}^{N} c_m \widehat{\sigma^x_{j_m\varepsilon}}(\xi) \right|^2 d\xi \longrightarrow 0 \quad \text{as} \quad N \to \infty.$$

Note that

$$\begin{aligned}
&\int_{B(0,\frac{1}{\varepsilon})} \frac{1}{(1+|\xi|^2)^{\frac{n}{2}}} \left| \hat{\mu^x_\varepsilon}(\xi) - \sum_{m=1}^{N} c_m \widehat{\sigma^x_{j_m\varepsilon}}(\xi) \right|^2 d\xi \\
= \;& \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^{\frac{1}{\varepsilon}} \frac{r^{n-1}}{(1+r^2)^{\frac{n}{2}}} \left| \sum_{m=N+1}^{\infty} c_m \Gamma(\frac{n}{2}) \left(\frac{2}{j_m\varepsilon r}\right)^{\frac{n-2}{2}} J_{\frac{n-2}{2}}(j_m\varepsilon r) \right|^2 dr \\
= \;& \frac{2^{n-1}\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}{\varepsilon^{n-2}} \int_0^{\frac{1}{\varepsilon}} \frac{r}{(1+r^2)^{\frac{n}{2}}} \left| \sum_{m=N+1}^{\infty} \frac{c_m}{j_m^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(j_m\varepsilon r) \right|^2 dr \\
= \;& \frac{2^{n-1}\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}{\varepsilon^{n-2}} \sum_{m=N+1}^{\infty} \frac{c_m^2}{j_m^{n-2}} \int_0^{\frac{1}{\varepsilon}} \frac{r}{(1+r^2)^{\frac{n}{2}}} J^2_{\frac{n-2}{2}}(j_m\varepsilon r)dr \\
&+ \frac{2^n\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}{\varepsilon^{n-2}} \sum_{m=N+1}^{\infty} \sum_{k=m+1}^{\infty} \frac{c_m c_k}{j_m^{\frac{n-2}{2}} j_k^{\frac{n-2}{2}}} \int_0^{\frac{1}{\varepsilon}} \frac{r}{(1+r^2)^{\frac{n}{2}}} J_{\frac{n-2}{2}}(j_m\varepsilon r) J_{\frac{n-2}{2}}(j_k\varepsilon r)dr \\
= \;& I + J.
\end{aligned}$$

Using the fact that $c_m$ is asymtotic to $j_m^{\frac{n-3}{2}}$ when $m$ is large and $J_{\frac{n-2}{2}}(r)$ is asymtotic to $r^{-\frac{1}{2}}$ when $m$ is large, we have

$$\begin{aligned} |I| &= \frac{2^{n-1}\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}{\varepsilon^{n-2}} \sum_{m=N+1}^{\infty} \frac{c_m^2}{j_m^{n-2}} \int_0^{\frac{1}{\varepsilon}} \frac{r}{(1+r^2)^{\frac{n}{2}}} J^2_{\frac{n-2}{2}}(j_m\varepsilon r)dr \\ &\le \frac{2^{n-1}\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}{\varepsilon^{n-2}} C_0 \sum_{m=N+1}^{\infty} \frac{j_m^{n-3}}{j_m^{n-2}} \int_0^{\frac{1}{\varepsilon}} \frac{r}{(1+r^2)^{\frac{n}{2}}} \frac{1}{j_m\varepsilon r} dr \\ &= \frac{2^{n-1}\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}{\varepsilon^{n-1}} C_0 \sum_{m=N+1}^{\infty} \frac{1}{j_m^2} \int_0^{\frac{1}{\varepsilon}} \frac{1}{(1+r^2)^{\frac{n}{2}}} dr \\ &\le C \sum_{m=N+1}^{\infty} \frac{1}{j_m^2} < \infty \end{aligned}$$

for large enough $m$. Since $\sum_{m=N+1}^{\infty} j_m^{-2}$ is convergent for all $N$, the first part $I$ tends to zero as $N \to \infty$.

On the other hand,

$$J = \frac{2^n\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}{\varepsilon^n} \sum_{m=N+1}^{\infty} \frac{c_m}{j_m^{\frac{n-2}{2}}} \sum_{k=m+1}^{\infty} \frac{c_k}{j_k^{\frac{n-2}{2}}} \int_0^1 \frac{u}{(1+(\frac{u}{\varepsilon})^2)^{\frac{n}{2}}} J_{\frac{n-2}{2}}(j_m u) J_{\frac{n-2}{2}}(j_k u) du.$$

Let $g(u) = \frac{1}{(1+(\frac{u}{\varepsilon})^2)^{\frac{n}{2}}} J_{\frac{n-2}{2}}(j_m u)$ and define

$$F_{\frac{n-2}{2},m}(j_k) = \int_0^1 g(u) J_{\frac{n-2}{2}}(j_k u) u du.$$

It is easy to see that $\int_0^1 u^{\frac{1}{2}} \left|f(u)\right| du < \infty$. By (5.3), we have $F_{\frac{n-2}{2},m}(j_k) = O(j_k^{-\frac{3}{2}})$ as $k \to \infty$. Therefore,

$$\begin{aligned} |J| &\le \frac{2^n\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}{\varepsilon^n} C \sum_{m=N+1}^{\infty} \frac{j_m^{\frac{n-3}{2}}}{j_m^{\frac{n-2}{2}}} \sum_{k=m+1}^{\infty} \frac{j_k^{\frac{n-3}{2}}}{j_k^{\frac{n-2}{2}}} j_k^{-\frac{3}{2}} \\ &= \frac{2^n\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}{\varepsilon^n} C \sum_{m=N+1}^{\infty} \frac{j_m^{\frac{n-3}{2}}}{j_m^{\frac{n-2}{2}}} \sum_{k=m+1}^{\infty} j_k^{-2} \\ &\le \frac{2^n\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}{\varepsilon^n} C \sum_{m=N+1}^{\infty} j_m^{-\frac{3}{2}} < \infty \end{aligned}$$

for large enough $m$. Since $\sum_{m=N+1}^{\infty} j_m^{-\frac{3}{2}} < \infty$ for all $N$, the second part $J$ tends to zero as $N \to \infty$. $\square$

At this point, one may allow to take $\hat{\mu_\varepsilon^x}(\xi) = 1_{[0,\frac{1}{\varepsilon})}(|\xi|)e^{i(x,\xi)_{\mathbb{R}^n}}$ into the covariance of the Gaussian family $\{\mathcal{I}(h_{\mu_\varepsilon^x}) : x \in \mathbb{R}^n, \varepsilon > 0\}$.

**Theorem 5.3.** (1) Given $x \in \mathbb{R}^n$ and $\varepsilon_1 \geq \varepsilon_2 > 0$,

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu_{\varepsilon_1}^x})\mathcal{I}(h_{\mu_{\varepsilon_2}^x})] = \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}\int_0^{\frac{1}{\epsilon_1}} \frac{r^{n-1}}{(1+r^2)^{\frac{n}{2}}}dr = G(\varepsilon_1).$$

The covariance function $\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu_{\varepsilon_1}^x})\mathcal{I}(h_{\mu_{\varepsilon_2}^x})]$ is asymptotic to $-\frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}\log(\varepsilon_1)$ when $\varepsilon_1$ is small . Given $\varepsilon_0 > 0$, $\{\mathcal{I}(h_{\mu_\varepsilon^x}) : 0 < \varepsilon \leq \varepsilon_0\}$ has the same distribution of a Brownian motion up to a change of variable.
(2) Given $x, y \in \mathbb{R}^n$ with $x \neq y$,

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu_{\varepsilon_1}^x})\mathcal{I}(h_{\mu_{\varepsilon_2}^y})] = \frac{1}{(2\pi)^{\frac{n}{2}}}\frac{1}{|x-y|^{\frac{n-2}{2}}}\int_0^{\frac{1}{\varepsilon_1}} \frac{r^{\frac{n}{2}}}{(1+r^2)^{\frac{n}{2}}}J_{\frac{n-2}{2}}(|x-y|\, r)dr \tag{5.10}$$

If $|x-y| \leq \varepsilon_1$, then $\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu_{\varepsilon_1}^x})\mathcal{I}(h_{\mu_{\varepsilon_2}^y})]$ is asymptotic to $-\frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}\log(\varepsilon_1)$ when $\varepsilon_1$ is small. If $|x-y| > \varepsilon_1$, then $\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu_{\varepsilon_1}^x})\mathcal{I}(h_{\mu_{\varepsilon_2}^y})]$ is asymptotic to $-\frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}\log|x-y|$ when $\varepsilon_1$ is small.

*Proof.* (1) Given $x \in \mathbb{R}^n$ and $\varepsilon_1 \geq \varepsilon_2 > 0$,

$$\begin{aligned}
\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu_{\varepsilon_1}^x})\mathcal{I}(h_{\mu_{\varepsilon_2}^x})] &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(1+|\xi|^2)^{-\frac{n}{2}}\hat{\mu_{\varepsilon_1}^x}(|\xi|)\hat{\mu_{\varepsilon_2}^x}(|\xi|)d\xi \\
&= \frac{1}{(2\pi)^n}\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_0^{\frac{1}{\epsilon_1}}\frac{r^{n-1}}{(1+r^2)^{\frac{n}{2}}}dr \\
&= \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}\int_0^{\frac{1}{\epsilon_1}}\frac{r^{n-1}}{(1+r^2)^{\frac{n}{2}}}dr = G(\varepsilon_1) < \infty.
\end{aligned}$$

Straightforward computations show that $G$ is strictly decreasing on $(0,\infty)$ with $\lim_{\varepsilon\downarrow 0} G(\varepsilon) = +\infty$ and $\lim_{\varepsilon\uparrow\infty} G(\varepsilon) = 0$. Then $G^{-1}$ is also a decreasing function and $G(\varepsilon)$ is asymptotic to $-\frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}\log(\varepsilon)$ as $\varepsilon$ decreases to 0. For $x \in \mathbb{R}^n$, let $t_0 > 0$ and define

$$B(t) = \mathcal{I}(h_{\mu_{G^{-1}(t)}^x}), t \geq t_0.$$

For every $t \geq s \geq t_0$,

$$\mathrm{Var}(B(t) - B(s)) = G(G^{-1}(t)) - 2G(G^{-1}(s)) + G(G^{-1}(s)) = t - s,$$

which yields $B(t) - B(t_0)$ has the Gaussian distribution with mean 0 and variance $t - t_0$. Moreover, if $t_2 > s_2 \geq t_1 > s_1 \geq t_0$, then

$$\mathrm{Cov}(B(t_2) - B(s_2), B(t_1) - B(s_1)) = t_1 - s_1 - t_1 + s_1 = 0.$$

Hence, $B(t_2) - B(s_2)$ is independent of $B(t_1) - B(s_1)$. Let $\varepsilon_0 = G^{-1}(t_0)$, then $\mathcal{I}(h_{\mu_\varepsilon^x}) - \mathcal{I}(h_{\mu_{\varepsilon_0}^x})$ has the same distribution as a standard Brownian motion for $0 < \varepsilon \leq \varepsilon_0$.
(2) Given $x, y \in \mathbb{R}^n$ with $x \neq y$ and $\epsilon_1 \geq \epsilon_2 > 0$,

$$\begin{aligned} & \mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu_{\varepsilon_1}^x})\mathcal{I}(h_{\mu_{\varepsilon_2}^y})] \\ = & \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-\frac{n}{2}} 1_{[0, \frac{1}{\varepsilon_1})}(|\xi|) 1_{[0, \frac{1}{\varepsilon_2})}(|\xi|) e^{i(x-y,\xi)_{\mathbb{R}^n}} d\xi \\ = & \frac{1}{(2\pi)^n} \int_0^\infty \frac{r^{n-1}}{(1+r^2)^{\frac{n}{2}}} 1_{[0, \frac{1}{\varepsilon_1})} \int_{S^{n-1}} e^{i(x-y, rx')_{\mathbb{R}^n}} d\sigma(x') dr \\ = & \frac{1}{(2\pi)^n} \int_0^{\frac{1}{\varepsilon_1}} \frac{r^{n-1}}{(1+r^2)^{\frac{n}{2}}} \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_{-1}^1 e^{i(x-y, rs)_{\mathbb{R}^n}} (1 - s^2)^{\frac{n-2}{2}} \frac{ds}{\sqrt{1-s^2}} dr \\ = & \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{|x-y|^{\frac{n-2}{2}}} \int_0^{\frac{1}{\varepsilon_1}} \frac{r^{\frac{n}{2}}}{(1+r^2)^{\frac{n}{2}}} J_{\frac{n-2}{2}}(|x-y|\, r) dr. \end{aligned}$$

Comparing the covariance function $G$ with the formula (A.8), let $\varepsilon_1 \downarrow 0$, if $|x - y| \leq \varepsilon_1$, then $|x - y|$ is sufficiently close to $\varepsilon_1$ and

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu_{\varepsilon_1}^x})\mathcal{I}(h_{\mu_{\varepsilon_2}^y})] \longrightarrow \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})} K_0(\varepsilon_1), \tag{5.11}$$

where the right-hand side is asymptotic to $-\frac{1}{2^{n-1}\pi^{\frac{n}{2}}} \log(\varepsilon_1)$; if $|x - y| > \varepsilon_1$, then

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu_{\varepsilon_1}^x})\mathcal{I}(h_{\mu_{\varepsilon_2}^y})] \longrightarrow \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})} K_0(|x-y|), \tag{5.12}$$

where the right-hand side is asymptotic to $-\frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})} \log|x - y|$ when $|x - y|$ is sufficiently small. □

By now, one should believe that $\{\mathcal{I}(h_{\mu_\varepsilon^x}) : x \in \mathbb{R}^n, \varepsilon > 0\}$ is a suitable choice on $\mathbb{R}^n$.

# Chapter 6

# Fourier-Bessel series in $\mathbb{R}^3$

This chapter presents an explicit example of the Fourier-Bessel expansion approach to construct random measures and prove a KPZ relation on $\mathbb{R}^3$.

## 6.1 Spherical averages of GFF on $\mathbb{R}^3$

For $x \in \mathbb{R}^3$ and $\varepsilon > 0$, the Fourier transform of the spherical average measure $\sigma_\varepsilon^x \in H^{-\frac{3}{2}}$ referred to (5.4) is given by

$$\hat{\sigma_\varepsilon^x}(\xi) = \sqrt{\frac{\pi}{2\varepsilon|\xi|}} J_{\frac{1}{2}}(\varepsilon|\xi|) e^{i(x,\xi)_{\mathbb{R}^3}}.$$

Assume that

$$\mu_\varepsilon^x = \sum_{m=1}^{\infty} c_m \sigma_{j_m \varepsilon}^x$$

with

$$\hat{\mu_\varepsilon^x}(\xi) = 1_{[0,\frac{1}{\varepsilon})}(|\xi|) e^{i(x,\xi)_{\mathbb{R}^3}} = 1_{[0,1)}(\varepsilon|\xi|) e^{i(x,\xi)_{\mathbb{R}^3}},$$

where $j_1, j_2, , \ldots$ are the positive zeros of $J_{\frac{1}{2}}$ arranged in ascending order. Note that the coefficient $a_m$ in $f(x) = \sum_{m=1}^{\infty} a_m J_{\frac{1}{2}}(j_m x)$ does not depend on the point $x$. Hence, by (5.1) and (5.2), we have

$$\hat{\mu_\varepsilon^x}(\xi) = \frac{1}{\sqrt{\varepsilon|\xi|}} \sum_{m=1}^{\infty} c_m \sqrt{\frac{\pi}{2j_m}} J_{\frac{1}{2}}(j_m \varepsilon|\xi|) e^{i(x,\xi)_{\mathbb{R}^3}}.$$

with

$$c_m = \sqrt{\frac{2j_m}{\pi}} \frac{2}{J^2_{\frac{3}{2}}(j_m)} \int_0^1 y^{\frac{3}{2}} J_{\frac{1}{2}}(j_m y) dy.$$

Since $j_m = m\pi$ for all $m \geq 1$, use the formula (A.2) to write

$$J_{\frac{3}{2}}(j_m) = \sqrt{\frac{2}{m\pi^2}} [\frac{\sin(m\pi)}{m\pi} - \cos(m\pi)] = (-1)^{m+1} \sqrt{\frac{2}{m\pi^2}}.$$

It follows that

$$c_m = \frac{2}{m\pi} \int_0^{m\pi} u \sin(u) du = 2(-1)^{m+1}.$$

Therefore,

$$\hat{\mu^x_\varepsilon}(\xi) = 2 \sum_{m=1}^{\infty} (-1)^{m+1} \hat{\sigma}^x_{j_m \varepsilon}(\xi).$$

By Theorem 5.1, the series $\sum_{m=1}^{\infty} (-1)^{m+1} \hat{\sigma}^x_{j_m \varepsilon}(\xi)$ is convergent and its sum is equal to $\hat{\mu^x_\varepsilon}(\xi)$ for each $\xi \in \mathbb{R}^3$. In particular, $\mu^x_\varepsilon \in H^{-\frac{3}{2}}$ and $\mu^x_\varepsilon \to \delta^x_\varepsilon$ as $\varepsilon \downarrow 0$ for every $x \in \mathbb{R}^3$. In addition, by Lemma 5.3, $S^x_N = \sum_{m=1}^{N} c_m \sigma^x_{j_m \varepsilon}$ converges to $\mu^x_\varepsilon$ in $H^{-\frac{3}{2}}$ as $N \to \infty$ for given $x \in \mathbb{R}^3$. Finally, by Theorem 5.3, we have the following result in $\mathbb{R}^3$:

**Theorem 6.1.** (1) Given $x \in \mathbb{R}^3$ and $\varepsilon_1 \geq \varepsilon_2 > 0$,

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu^x_{\varepsilon_1}}) \mathcal{I}(h_{\mu^x_{\varepsilon_2}})] = \frac{1}{2\pi^2} [-\frac{1}{\sqrt{1+{\varepsilon_1}^2}} + \log(1 + \sqrt{1+\varepsilon_1^2}) - \log \varepsilon_1] = G(\varepsilon_1),$$

where the function $\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu^x_{\varepsilon_1}}) \mathcal{I}(h_{\mu^x_{\varepsilon_2}})]$ is asymptotic to $-\frac{1}{2\pi^2} \log \varepsilon_1$ when $\varepsilon_1$ is small. Moreover, $\{\mathcal{I}(h_{\mu^x_\varepsilon}) : 0 < \varepsilon \leq \varepsilon_0\}$ has the same distribution as a Brownian motion up to a non-random time change.
(2) Given $x, y \in \mathbb{R}^3$ with $x \neq y$ and $\varepsilon_1 \geq \varepsilon_2 > 0$,

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu^x_{\varepsilon_1}}) \mathcal{I}(h_{\mu^y_{\varepsilon_2}})] = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{|x-y|}} \int_0^{\frac{1}{\varepsilon_1}} \frac{r^{\frac{3}{2}}}{(1+r^2)^{\frac{3}{2}}} J_{\frac{1}{2}}(|x-y|\, r) dr. \quad (6.1)$$

If $|x-y| \leq \varepsilon_1$, then $\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu^x_{\varepsilon_1}}) \mathcal{I}(h_{\mu^y_{\varepsilon_2}})]$ is asymptotic to $-\frac{1}{2\pi^2} \log \varepsilon_1$ when $\varepsilon_1$ is small. If $|x-y| > \varepsilon_1$, then $\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu^x_{\varepsilon_1}}) \mathcal{I}(h_{\mu^y_{\varepsilon_2}})]$ is asymptotic to $-\frac{1}{2\pi^2} \log|x-y|$ when $\varepsilon_1$ is small.

This theorem indicates that the collection of Paley-Wiener integrals $\{\mathcal{I}(h_{\mu_\varepsilon^x}) : x \in \mathbb{R}^3, \varepsilon > 0\}$ forms a Gaussian family with the corresponding backward Markovian properties and is a suitable candidate for the spherical average process.

*Remark.* For $x, y \in \mathbb{R}^3$ with $x \neq y$ and $\varepsilon_1 \geq \varepsilon_2 > 0$, by the formula (A.8),

$$\begin{aligned}
\left|\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu_{\varepsilon_1}^x})\mathcal{I}(h_{\mu_{\varepsilon_2}^y})] - \frac{1}{2\pi^2}K_0(|x-y|)\right| &\leq \frac{1}{2\pi^2|x-y|}\int_{\frac{1}{\varepsilon_1}}^{\infty}\frac{r}{(1+r^2)^{\frac{3}{2}}}dr \\
&= \frac{1}{2\pi^2|x-y|}\frac{1}{\sqrt{1+\frac{1}{\varepsilon_1^2}}} \\
&\leq \frac{\varepsilon_1}{2\pi^2|x-y|}.
\end{aligned}$$

If $0 < |x-y| \leq \varepsilon_1$, then

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu_{\varepsilon_1}^x})\mathcal{I}(h_{\mu_{\varepsilon_2}^y})] = \frac{1}{2\pi^2}K_0(\varepsilon_1) + O(\varepsilon_1) \quad \text{as} \quad \varepsilon_1 \downarrow 0. \tag{6.2}$$

If $|x-y| > \varepsilon_1$, then

$$\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu_{\varepsilon_1}^x})\mathcal{I}(h_{\mu_{\varepsilon_2}^y})] = \frac{1}{2\pi^2}K_0(|x-y|) + O(\varepsilon_1) \quad \text{as} \quad |x-y| \downarrow 0. \tag{6.3}$$

Moreover, the Payley-Wiener integral $\mathcal{I}(h_{\mu_\varepsilon^x})$ determines a continuous function almost surely.

**Corollary 6.1.** Given $\varepsilon > 0$, the mapping $x \in \mathbb{R}^3 \longmapsto \mathcal{I}(h_{\mu_\varepsilon^x}) \in L^2(\mathcal{W})$ is continuous. Moreover, for almost every $\theta \in \Theta$, $x \in \mathbb{R}^3 \mapsto \mathcal{I}(h_{\mu_\varepsilon^x})(\theta) \in \mathbb{R}$ is $\alpha$-Hölder continuous for every $\alpha \in (0, \frac{1}{2})$.

*Proof.* By Kolmogorov's Continuity Criterion, it suffices to show there exist constant $\beta \in (0,1)$ such that for every $x, y \in \mathbb{R}^3$,

$$\left\|\mathcal{I}(h_{\mu_\varepsilon^x}) - \mathcal{I}(h_{\mu_\varepsilon^y})\right\|_{L^2(\mathcal{W})}^2 \leq C_{\beta,\varepsilon}|x-y|^\beta.$$

Note that

$$\begin{aligned}
\mathbb{E}^{\mathcal{W}}[|\mathcal{I}(h_{\mu_\varepsilon^x}) - \mathcal{I}(h_{\mu_\varepsilon^y})|^2] &\leq \frac{1}{\pi^2}\int_0^{\frac{1}{\varepsilon}}\frac{r^2}{(1+r^2)^{\frac{3}{2}}}\left|1 - J_{\frac{1}{2}}(|x-y|\,r)\sqrt{\frac{\pi}{2|x-y|\,r}}\right|dr \\
&= \frac{1}{\pi^2}\int_0^{\frac{1}{\varepsilon}}\frac{r^2}{(1+r^2)^{\frac{3}{2}}}\left|1 - \frac{\sin(|x-y|\,r)}{|x-y|\,r}\right|dr.
\end{aligned}$$

For $\beta \in (0,2)$, if $t \in [1,\infty)$, then $\left|1-\frac{\sin t}{t}\right| \leq 2 \leq 2t^\beta$; if $t \in (0,1)$, then $\left|1-\frac{\sin t}{t}\right| \leq \frac{1}{6}t^2 \leq \frac{1}{6}t^\beta$. So, we have

$$\left|1-\frac{\sin t}{t}\right| \leq 2t^\beta \text{ for } \beta \in (0,2).$$

and thus

$$\begin{aligned}
\mathbb{E}^{\mathcal{W}}[\left|\mathcal{I}(h_{\mu_\varepsilon^x}) - \mathcal{I}(h_{\mu_\varepsilon^y})\right|^2] &\leq \frac{2}{\pi^2}\int_0^{\frac{1}{\varepsilon}} \frac{r^2}{(1+r^2)^{\frac{3}{2}}}[|x-y|\, r]^\beta dr \\
&\leq |x-y|^\beta \frac{2}{\pi^2}\int_0^{\frac{1}{\varepsilon}} r^{\beta-1}dr \\
&= \frac{2}{\beta\pi^2\varepsilon^\beta}|x-y|^\beta = C_{\beta,\varepsilon}|x-y|^\beta .
\end{aligned}$$

□

## 6.2 Construction of random measure

In this section, we use the Gaussian family $\{\mathcal{I}(h_{\mu_\varepsilon^x}) : x \in \mathbb{R}^3, \varepsilon > 0\}$ to construct random measures on $\mathbb{R}^3$.

Recall that for each $x \in \mathbb{R}^3$, $\mu_\varepsilon^x$ converges to $\delta_x$ weakly as $\varepsilon \downarrow 0$, so $\theta(x)$ is the limit of $\mathcal{I}(h_{\mu_\varepsilon^x})(\theta) = (\theta, \mu_\varepsilon^x)_{H^{\frac{3}{2}}}$ as $\varepsilon \downarrow 0$. Define a random measure on $\mathbb{R}^3$ by

$$m_\varepsilon^\theta(dx) = E_\varepsilon^\theta(x)dx,$$

where

$$E_\varepsilon^\theta(x) = \exp(\gamma\mathcal{I}(h_{\mu_\varepsilon^x})(\theta) - \frac{\gamma^2}{2}G(\varepsilon)).$$

Corollary 6.1 guarantees that the mapping $(x,\theta) \mapsto \mathcal{I}(h_{\mu_\varepsilon^x})(\theta)$ is measurable with respect to $\mathcal{B}_{\mathbb{R}^3} \times \mathcal{B}_\Theta$ if $\varepsilon > 0$ is given. Additionally, for every $x \in \mathbb{R}^3$ and $\varepsilon > 0$, the Paley-Wiener integral $\mathcal{I}(h_{\mu_\varepsilon^x})$ has standard Gaussian distribution under $\mathcal{W}$ with variance $G(\varepsilon)$, thus $\mathbb{E}^{\mathcal{W}}[E_\varepsilon^\theta(x)] = 1$. Moreover, given any Borel set $X$ in $\mathbb{R}^3$, denote

$$m_\varepsilon^\theta(X) = \int_X E_\varepsilon^\theta(x)dx \in \mathbb{R}.$$

Then the mapping $\theta \mapsto m_\varepsilon^\theta(X)$ is measurable, and by Tonelli's Theorem,

$$\mathbb{E}^{\mathcal{W}}[m_\varepsilon^\theta(X)] = \int_X \mathbb{E}^{\mathcal{W}}[E_\varepsilon^\theta(x)]dx = \mathrm{vol}(X).$$

**Theorem 6.2.** If $0 < \gamma^2 < \pi^2$ and $\varepsilon_n \equiv \varepsilon^{n^2}$ for $n \geq 1$ with $0 < \varepsilon < 1$, then for almost every $\theta \in \Theta$, there exists a non-negative Borel measure $m^\theta(dx)$ on $\mathbb{R}^3$ such that for every $f \in C_c(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} f(x) m_{\varepsilon_n}^\theta(dx) \longrightarrow \int_{\mathbb{R}^3} f(x) m^\theta(dx) \text{ as } n \to \infty$$

almost surely and also in $L^2$.

*Proof.* Since $f \in C_c(\mathbb{R}^3)$, it suffices to show the weak convergence of $m_{\epsilon_n}^\theta(dx)$ on a compact set $X \subseteq \mathbb{R}^3$ with $\mathrm{supp}(f) \subseteq X$.

First, we show there exists a non-negative random variable $\theta \mapsto m^\theta(X)$ such that

$$m^\theta(X) = \lim_{N\to\infty} \sum_{n=0}^{N-1} \left| m_{\varepsilon_{n+1}}^\theta(X) - m_{\varepsilon_n}^\theta(X) \right|$$

almost surely and in $L^2$. For $n \geq 1$, by Tonelli's Theorem, we have

$$\begin{aligned} \mathbb{E}^{\mathcal{W}}\left[\left| m_{\varepsilon_{n+1}}^\theta(X) - m_{\varepsilon_n}^\theta(X) \right|^2\right] &= \int\int_{X^2} e^{\gamma^2 \mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu^x_{\varepsilon_{n+1}}})\mathcal{I}(h_{\mu^y_{\varepsilon_{n+1}}})]}\, dxdy \\ &+ \int\int_{X^2} e^{\gamma^2 \mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu^x_{\varepsilon_n}})\mathcal{I}(h_{\mu^y_{\varepsilon_n}})]}\, dxdy \\ &- 2\int\int_{X^2} e^{\gamma^2 \mathbb{E}^{\mathcal{W}}[\mathcal{I}(h_{\mu^x_{\varepsilon_{n+1}}})\mathcal{I}(h_{\mu^y_{\varepsilon_n}})]}\, dxdy. \end{aligned}$$

The covariance formula (6.1) yields

$$\begin{aligned} &\mathbb{E}^{\mathcal{W}}\left[\left| m_{\varepsilon_{n+1}}^\theta(X) - m_{\varepsilon_n}^\theta(X) \right|^2\right] \\ = &\int\int_{X^2} \exp\left( \frac{\gamma^2}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{|x-y|}} \int_0^{\frac{1}{\varepsilon_n}} \frac{r^{\frac{3}{2}}}{(1+r^2)^{\frac{3}{2}}} J_{\frac{1}{2}}(|x-y|\, r) dr \right) \\ &\cdot \left[ \exp\left( \frac{\gamma^2}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{|x-y|}} \int_{\frac{1}{\varepsilon_n}}^{\frac{1}{\varepsilon_{n+1}}} \frac{r^{\frac{3}{2}}}{(1+r^2)^{\frac{3}{2}}} J_{\frac{1}{2}}(|x-y|\, r) dr \right) - 1 \right] \, dxdy. \end{aligned}$$

Dividing the domain into two parts:

$$I = \int\int_{|x-y|>\varepsilon_{n-1}} \quad \text{and} \quad J = \int\int_{|x-y|\leq\varepsilon_{n-1}}.$$

First, if $|x-y| > \varepsilon_{n-1}$, then $|x-y|\,r > 1$ for $\frac{1}{|x-y|} \leq r < \frac{1}{\varepsilon_{n+1}}$. It follows that

$$\begin{aligned}
&\left|\frac{\gamma^2}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{|x-y|}}\int_0^{\frac{1}{\varepsilon_n}}\frac{r^{\frac{3}{2}}}{(1+r^2)^{\frac{3}{2}}}J_{\frac{1}{2}}(|x-y|\,r)dr\right| \\
\leq\ &\left|\frac{\gamma^2}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{|x-y|}}\int_0^{\frac{1}{|x-y|}}\frac{r^{\frac{3}{2}}}{(1+r^2)^{\frac{3}{2}}}J_{\frac{1}{2}}(|x-y|\,r)dr\right| \\
&+\left|\frac{\gamma^2}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{|x-y|}}\int_{\frac{1}{|x-y|}}^{\frac{1}{\varepsilon_n}}\frac{r^{\frac{3}{2}}}{(1+r^2)^{\frac{3}{2}}}J_{\frac{1}{2}}(|x-y|\,r)dr\right| \\
\leq\ & C_1\int_0^{\frac{1}{|x-y|}}\frac{r}{(1+r^2)^{\frac{3}{2}}}dr + \frac{\gamma^2}{(2\pi)^{\frac{3}{2}}}\int_{\frac{1}{|x-y|}}^{\frac{1}{\varepsilon_n}}\frac{r^2}{(1+r^2)^{\frac{3}{2}}}dr \\
=\ & C_1\left[-\frac{1}{\sqrt{x^2+1}}\right]_0^{\frac{1}{|x-y|}} + \frac{\gamma^2}{(2\pi)^{\frac{3}{2}}}\left[\ln\left(\left|\sqrt{x^2+1}+x\right|\right) - \frac{x}{\sqrt{x^2+1}}\right]_{\frac{1}{|x-y|}}^{\frac{1}{\varepsilon_n}},
\end{aligned}$$

where the square root of the right-hand side is summable in $n \geq 1$. Second, if $|x-y| \leq \varepsilon_n$, then we have

$$J \leq C\varepsilon_n^3 \cdot 2e^{\gamma^2 G(\varepsilon_{n+1})} \leq Ce^{-(6\pi^2-\gamma^2)G(\varepsilon_n)}$$

and thus

$$\mathbb{E}^{\mathcal{W}}\left[\left|m^\theta_{\varepsilon_{n+1}}(X) - m^\theta_{\varepsilon_n}(X)\right|^2\right] \leq Ce^{-(6\pi^2-\gamma^2)G(\varepsilon_n)}. \tag{6.4}$$

The square root of the right-hand side of (6.4) is summable in $n \geq 1$. Since it is clear that $m^\theta_{\varepsilon_1}(X)$ is square integrable, one conclude that

$$m^\theta(X) \equiv \sum_{n=0}^{\infty}\left|m^\theta_{\varepsilon_{n+1}}(X) - m^\theta_{\varepsilon_n}(X)\right|$$

is square integrable and converges in $L^2$. Furthermore, there exists a subsequence $\{\varepsilon_{n_k}\}_{k=1}^\infty$ such that $\sum_{n=0}^M\left|m^\theta_{\varepsilon_{n_{k+1}}}(X) - m^\theta_{\varepsilon_{n_k}}(X)\right|$ converges to $m^\theta(X)$

almost surely. However, since the series is bounded monotonic increasing, it has to converges almost surely along $\{\varepsilon_n\}_{n=1}^{\infty}$.

Next, let $f$ be a continuous function with $\text{supp}(f) \subseteq X$ and define

$$M_{\varepsilon_n}^{\theta}(f) \equiv \int_{\mathbb{R}^3} f(x) m_{\varepsilon_n}^{\theta}(dx), n \geq 1.$$

Then one obtain

$$\sum_{n=0}^{\infty} \left| M_{\varepsilon_{n+1}}^{\theta}(f) - M_{\varepsilon_n}^{\theta}(f) \right| \leq \|f\|_u \sum_{n=0}^{\infty} \left| m_{\varepsilon_{n+1}}^{\theta}(X) - m_{\varepsilon_n}^{\theta}(X) \right| < \infty$$

which is also square integrable and converges almost surely. Thus,

$$M^{\theta}(f) \equiv \lim_{n\to\infty} M_{\varepsilon_n}^{\theta}(f) \text{ exists almost surely and } \left| M_{\varepsilon}^{\theta}(f) \right| \leq \|f\|_u \, m_{\varepsilon}^{\theta}(X). \tag{6.5}$$

Finally, let $\{f_k \in C_c(\mathbb{R}^3) : k \geq 1\}$ with $\text{supp}(f_k) \subseteq X$ for all $k \geq 1$ and choose a subsquence $\{f_{k_j} : j \geq 1\}$ such that $f_{k_j} \to f$ uniformly. Then for every $m, n \geq 1$,

$$\begin{aligned} \left| M_{\varepsilon_m}^{\theta}(f) - M_{\varepsilon_n}^{\theta}(f) \right| &\leq \left| M_{\varepsilon_m}^{\theta}(f) - M_{\varepsilon_m}^{\theta}(f_{k_j}) \right| + \left| M_{\varepsilon_m}^{\theta}(f_{k_j}) - M_{\varepsilon_n}^{\theta}(f_{k_j}) \right| \\ &\quad + \left| M_{\varepsilon_n}^{\theta}(f_{k_j}) - M_{\varepsilon_n}^{\theta}(f) \right| \\ &\leq 2m^{\theta}(X) \left\| f_{k_j} - f \right\|_u + \left| M_{\varepsilon_m}^{\theta}(f_{k_j}) - M_{\varepsilon_n}^{\theta}(f_{k_j}) \right|. \end{aligned}$$

Therefore, $\{M_{\varepsilon_n}^{\theta}(f) : n \geq 1\}$ is a Cauchy sequence in $\mathbb{R}$. It follows that $f \in C_c(X)$ and $M^{\theta}(f) \equiv \lim_{j\to\infty} M^{\theta}(f_{k_j})$. By the Riesz representation theorem, there exists a unique Borel measure $m^{\theta}(dx)$ such that

$$M^{\theta}(f) = \int_{\mathbb{R}^3} f(x) m^{\theta}(dx)$$

and the total variation of $m^{\theta}(dx)$ is bounded by $m^{\theta}(X)$. □

## 6.3 KPZ relation

Given a bounded domain $\Omega \subseteq \mathbb{R}^3$, we say that $B_{\varepsilon}(\Omega)$ is the $\varepsilon$- *neighborhood* of $\Omega$ if

$$B_{\varepsilon}(\Omega) \equiv \{x \in \mathbb{R}^3 : B_{\varepsilon}(x) \cap \Omega \neq \emptyset\}.$$

Note that if $0 < \gamma^2 < \pi^2$ and $x \in \mathbb{R}^3$ is fixed, then

$$\mathbb{E}^{\mathcal{W}}[\limsup_{n\to\infty} e^{6\gamma^2 G(\varepsilon_n)} m^\theta(\overline{B_{\varepsilon_n}(x)})] = 0.$$

Set

$$\Theta_x \equiv \{\theta \in \Theta : \limsup_{n\to\infty} e^{6\gamma^2 G(\varepsilon_n)} m^\theta(\overline{B_{\varepsilon_n}(x)}) = 0\}, \tag{6.6}$$

$\Theta_x$ is a measurable set and $\mathcal{W}(\Theta_x) = 1$. For $\omega > 0$ and $\theta \in \Theta$, denote

$$R(x,\theta;\omega) \equiv \begin{cases} \sup\{r > 0 : m^\theta(B_r(x)) \le \omega\} & \text{if } \theta \in \Theta_x, \\ 0 & \text{otherwise} \end{cases} \tag{6.7}$$

and introduce the *isothermal* $\omega$*- neighborhood* of $\Omega$

$$\Omega^\theta(\omega) \equiv \left\{x \in \Omega \text{ and } R(x,\theta;\omega) = 0 \text{ or } 0 < \operatorname{dist}(x,\Omega) < R(x,\theta;\omega)\right\}. \tag{6.8}$$

We call $\rho$ the *Euclidean scaling exponent* of $\Omega$ if

$$\lim_{\epsilon\downarrow 0} \frac{\log \operatorname{Vol}(\Omega_\epsilon)}{\log \epsilon^3} = \rho, \tag{6.9}$$

where $\epsilon > 0$ and $\Omega_\epsilon \equiv \cup_{z\in\Omega} B_\epsilon(z)$ is the canonical $\epsilon$-neighborhood of $\Omega$, and we call $Q$ the *quantum scaling exponent* of $\Omega$ if

$$\lim_{\omega\downarrow 0} \frac{\log \mathbb{E}^{\mathcal{W}}[m^\theta(\Omega^\theta(\omega))]}{\log \omega} = Q. \tag{6.10}$$

**Lemma 6.1.** For every $x \in \mathbb{R}^3$, define

$$\theta \mapsto \hat{m}^x_\theta(dy) \equiv \begin{cases} \exp\left(\dfrac{\gamma^2}{2\pi^2} K_0(|x-y|)\right) m^\theta(dy) & \text{if } \theta \in \Theta_x, \\ m^\theta(dy) & \text{otherwise.} \end{cases}$$

Then $m^{\theta,x}(dy)$ is a non-negative regular and $\sigma$-finite Borel measure on $\mathbb{R}^3$. Moreover, for every $r \in [0,\infty)$, compact set $X \subseteq \mathbb{R}^3$ and $F \in C_0(\mathbb{R}^3 \times [0,\infty))$,

$$\int_\Theta \int_X F(x, m^\theta(B_r(x))) m^\theta(dx) \mathcal{W}(d\theta) = \int_X \int_\Theta F(x, m^{\theta,x}(B_r(x))) \mathcal{W}(d\theta) m^\theta(dx).$$

*Proof.* Since the function $\exp\left(\frac{\gamma^2}{2\pi^2}K_0(|x-\cdot|)\right)$ is locally integrable with respect to $m^\theta(dx)$ for $\theta \in \Theta_x$ and

$$\begin{aligned}\int_{B_{\varepsilon_0}(x)} e^{\left(\frac{\gamma^2}{2\pi^2}K_0(|x-y|)\right)} m^\theta(dy) &= \sum_{k=0}^{\infty}\int_{\varepsilon_k\le|z|<\varepsilon_{k-1}} e^{\left(\frac{\gamma^2}{2\pi^2}K_0(|z|)\right)} m^\theta(dz) \\ &\le \sum_{k=0}^{\infty} e^{\left(\frac{\gamma^2}{2\pi^2}K_0(|z|)\right)} m^\theta(B_{\varepsilon_{k-1}}(0)) < \infty,\end{aligned}$$

it is clear that $\hat{m}_\theta^x(dy)$ is non-negative regular and $\sigma$-finite almost surely.

Let $f^x \in C_0(\mathbb{R}^3)$ be a continuous function in $x \in X$ with $0 \le f^x < \chi_{B_r(x)}$. Since $F(x, M^\theta(f^x))$ is continuous in $x \in X$, the weak convergence result, the dominated converegence theorem and Fubini's theorem that

$$\begin{aligned}&\int_\Theta\int_X F(x, M^\theta(f^x))m^\theta(dx)\mathcal{W}(d\theta) \\ =\ &\int_\Theta \lim_{n\to\infty}\left[\int_X F(x, M^\theta(f^x))m^\theta_{\varepsilon_n}(dx)\right]\mathcal{W}(d\theta) \\ =\ &\lim_{n\to\infty}\int_\Theta\int_X F(x, M^\theta(f^x))E^\theta_{\varepsilon_n}(x)dx\mathcal{W}(d\theta) \\ =\ &\lim_{n\to\infty}\int_X\int_\Theta F\left(x, M^{\theta+\gamma h_{\mu^x_{\varepsilon_n}}}(f^x)\right)\mathcal{W}(d\theta)dx.\end{aligned}$$

Moreover, given $x \in \mathbb{R}^3$, the Cameron-Martin theorem guarantees that $m_{\varepsilon_k}^{\theta+\gamma h_{\mu^x_{\varepsilon_n}}}(dy)$ converges to $m^{\theta+\gamma h_{\mu^x_{\varepsilon_n}}}(dy)$ weakly as $k \to \infty$ for each $k \ge 1$. Thus, we have

$$\begin{aligned}M^{\theta+\gamma h_{\mu^x_{\epsilon_n}}}(f^x) &= \lim_{k\to\infty} M_{\varepsilon_k}^{\theta+\gamma h_{\mu^x_{\epsilon_n}}}(f^x) \\ &= \lim_{k\to\infty}\int_{\mathbb{R}^3} f^x(y)\exp\left(\gamma^2\mathbb{E}\left[\mathcal{I}(h_{\mu^x_{\varepsilon_n}})\mathcal{I}(h_{\mu^y_{\varepsilon_k}})\right]\right)E^\theta_{\varepsilon_k}(y)dy.\end{aligned}$$

Assume $n \ge 1$ is large enough and $n \le k$, divide the integral in the right-hand side into two parts:

$$I = \int_{|y-x|\le\varepsilon_n} \quad \text{and} \quad J = \int_{|y-x|>\varepsilon_n}.$$

Note that $\mathbb{E}\left[\mathcal{I}(h_{\mu^x_{\varepsilon_n}})\mathcal{I}(h_{\mu^y_{\varepsilon_k}})\right]$ is bounded by $\beta G(\varepsilon_n)$ for some constant $\beta \in (1, 2)$ for large enough $n$. Therefore, the first integral $I$ is bounded by

$e^{\beta\gamma^2 G(\varepsilon_n)} m^\theta(\overline{B_{\varepsilon_n}(x)})$ as $k \to \infty$ and then converges to 0 as $n \to \infty$. Moreover,

$$\begin{aligned} &\int_{|y-x|>\varepsilon_n} f^x(y) \exp\left(\gamma^2 \mathbb{E}\left[\mathcal{I}(h_{\mu^x_{\varepsilon_n}})\mathcal{I}(h_{\mu^y_{\varepsilon_k}})\right]\right) E^\theta_{\varepsilon_k}(y)dy \\ = \;& M^{\theta+\gamma h_{\mu^x_{\varepsilon_n}}}_{\varepsilon_k}\left(f^x \exp\left(\frac{\gamma^2}{2\pi^2} K_0(|x-y|) + O(\varepsilon_n)\right)\right) \\ & -e^{\frac{\gamma^2}{2\pi^2}K_0(|x-y|)+O(\varepsilon_n)} \int_{|y-x|\le\varepsilon_n} f^x(y) E^\theta_{\varepsilon_k}(y)dy, \end{aligned}$$

which is convergent to $\int_{\mathbb{R}^3} f^x(y) \exp\left(\frac{\gamma^2}{2\pi^2} K_0(|x-y|)\right) m^\theta(dy)$. So for $\theta \in \Theta_x$,

$$\begin{aligned} &\lim_{n\to\infty}\lim_{k\to\infty} \int_{|y-x|>\varepsilon_n} f^x(y) \exp\left(\gamma^2 \mathbb{E}\left[\mathcal{I}(h_{\mu^x_{\varepsilon_n}})\mathcal{I}(h_{\mu^y_{\varepsilon_k}})\right]\right) E^\theta_{\varepsilon_k}(y)dy \\ = \;& \int_{\mathbb{R}^3} f^x(y) \exp\left(\frac{\gamma^2}{2\pi^2} K_0(|x-y|)\right) m^\theta(dy) < \infty. \end{aligned}$$

It follows that

$$\begin{aligned} &\int_\Theta \int_X F(x, M^\theta(f^x)) m^\theta(dx) \mathcal{W}(d\theta) \\ = \;& \int_X \int_\Theta F(x, M^\theta(f^x e^{\frac{\gamma^2}{2\pi^2}K_0(|x-\cdot|)}) \mathcal{W}(d\theta) dx. \end{aligned}$$

Next, choose a sequence $\{f^x_j : j \ge 1\} \subseteq C^\infty_c(\mathbb{R}^3)$ such that $0 \le f^x_j \nearrow \chi_{B_r(x)}$ as $j \to \infty$ and $f^x_j$ is continuous in $x \in \mathbb{R}^3$ for every $j \ge 1$. Then (6.11) holds for each $j \ge 1$. Let $j \to \infty$, the limit can be passed all the way inside. We complete the proof. $\square$

Now we prove the KPZ results. Note that for each $r > 0$ and $\omega > 0$, since the distribution of $\hat{m}^\theta_x(B_r(x))$ and $R(x, \theta; \omega)$ under $\mathcal{W}$ does not depend on $x \in \mathbb{R}^3$, without loss of generality, we will assume $x = 0$ and denote $\hat{m}^\theta \equiv \hat{m}^\theta_0(B_r(0))$.

**Lemma 6.2.** Let $B$ be the closed ball in $\mathbb{R}^3$ centered at the origin with unit volume under $e^{\frac{\gamma^2}{2\pi^2}K_0(|y|)}dy$. If $\eta$ and $\kappa$ are constants with

$$0 < \eta < 3\pi^2 - 3\gamma^2 \quad \text{and} \quad \frac{3\pi^2+\gamma^2}{6\pi^2-\gamma^2-\eta} < \kappa < 1,$$

then there exists $C > 0$ such that for sufficiently large $K > 0$,

$$\mathcal{W}(m^\theta(B) \le e^{-K\gamma}) \le C \exp\left[-\frac{2\kappa}{\gamma}\left(6\pi^2 - \gamma^2 - \frac{\gamma^2}{\kappa} - \eta\right)K\right].$$

*Proof.* Since $B$ is closed, it suffices to estimate $\mathcal{W}(\limsup_{n\to\infty} \hat{m}^\theta_{\varepsilon_n}(B) \le e^{-K\gamma})$ where $\hat{m}^\theta_{\varepsilon_n}(B) = e^{\frac{\gamma^2}{2\pi^2}K_0(|y|)} m^\theta_{\varepsilon_n}(dy)$. By the same argument in (6.4), we can prove that

$$\mathbb{E}^{\mathcal{W}}\left[\left|\hat{m}^\theta_{\varepsilon_{n+1}}(B) - \hat{m}^\theta_{\varepsilon_n}(B)\right|^2\right] \le Ce^{-(6\pi^2-\gamma^2)G(\varepsilon_n)}.$$

For any $0 < \eta < 3\pi^2 - 3\gamma^2$ and $n \ge 1$, let

$$\mathcal{A}'_n = \left\{\left|\hat{m}^\theta_{\varepsilon_{i+1}}(B) - \hat{m}^\theta_{\varepsilon_i}(B)\right| \le Ce^{-\frac{\eta}{2}G(\varepsilon_i)-K\gamma}, \forall i \ge n\right\}.$$

Then the Borel-Cantelli Lemma implies that $\mathcal{W}(\cup_{n=1}^\infty \mathcal{A}'_n) = 1$. In addition, if $\mathcal{A}_1 = \mathcal{A}'_1$ and $\mathcal{A}_n = \mathcal{A}'_n \setminus \mathcal{A}'_{n+1}$ for $n \ge 2$, then there exists constant $C > 0$ such that for all $n \ge 2$,

$$\mathcal{W}(\mathcal{A}_n) \le Ce^{2K\gamma}e^{-(6\pi^2-\gamma^2-\eta)G(\varepsilon_n)}.$$

Set $\mathcal{B} = \{\limsup_{n\to\infty} \hat{m}^\theta_{\varepsilon_n}(B) \le e^{-K\gamma}\}$, then $\mathcal{W}(\mathcal{B}) = \sum_{n=1}^\infty \mathcal{W}(\mathcal{B} \cap \mathcal{A}_n)$ and $\hat{m}^\theta_{\varepsilon_n}(B) \le C_\eta e^{-K\gamma}$ with $C_\eta = 1 + \sum_{n=1}^\infty e^{-\frac{\eta}{2}G(\varepsilon_n)}$. Given $\kappa$ with $\frac{3\pi^2+\gamma^2}{6\pi^2-\gamma^2-\eta} < \kappa < 1$, there exists a unique $N \in \mathbb{N}$ such that

$$G(\varepsilon_N) < \frac{2\kappa K}{\gamma} \quad \text{and} \quad G(\varepsilon_{N+1}) \ge \frac{2\kappa K}{\gamma}.$$

Then we have

$$\begin{aligned}\sum_{n=N+1}^\infty \mathcal{W}(\mathcal{B} \cap \mathcal{A}_n) &\le Ce^{2K\gamma}e^{-(6\pi^2-\gamma^2-\eta)G(\varepsilon_{N+1})} \\ &\le C \exp\left[-\frac{2\kappa}{\gamma}\left(6\pi^2 - \gamma^2 - \frac{\gamma^2}{\kappa} - \eta\right)K\right].\end{aligned}$$

Moreover, for $n = 1, 2, \ldots, N$, Jensen's inequality yields

$$\begin{aligned}
&\mathcal{W}(\hat{m}^{\theta}_{\varepsilon_n}(B) \leq C_\eta e^{-K\gamma}) \\
\leq\ & \mathcal{W}\left(\exp\left[\int_B \left(\gamma \mathcal{I}(h_{\mu^y_{\varepsilon_n}})(\theta) - \frac{\gamma^2}{2} G(\varepsilon_n)\right) e^{\frac{\gamma^2}{2\pi^2} K_0(|y|)} dy\right] \leq C_\eta e^{-K\gamma}\right) \\
\leq\ & \mathcal{W}\left(\int_B \mathcal{I}(h_{\mu^y_{\varepsilon_n}})(\theta) e^{\frac{\gamma^2}{2\pi^2} K_0(|y|)} dy \leq -K + \frac{\gamma}{2} G(\varepsilon_n) + \frac{\log C_\eta}{\gamma}\right).
\end{aligned}$$

It is easy to show that for each $n \geq 1$,

$$\theta \in \Theta \longmapsto \int_B \mathcal{I}(h_{\mu^y_{\varepsilon_n}})(\theta) e^{\frac{\gamma^2}{2\pi^2} K_0(|y|)} dy \in \mathbb{R}$$

is a centered Gaussian random variable with bounded variance. Furthermore, when $K$ is sufficiently large,

$$\begin{aligned}
\mathcal{W}(\hat{m}^{\theta}_{\varepsilon_n}(B) \leq C_\eta e^{-K\gamma}) &\leq \exp\left[-\frac{1}{2M}\left(K - \frac{\gamma}{2} G(\varepsilon_n) - \frac{\log C_\eta}{\gamma}\right)^2\right] \\
&\leq \exp\left[-\frac{1}{2M}\left((1-\kappa)K - \frac{\log C_\eta}{\gamma}\right)^2\right] \\
&\leq \exp\left[-\frac{1}{4M}(1-\kappa)^2 K^2\right].
\end{aligned}$$

Therefore, when $K$ is large enough,

$$\sum_{n=1}^{N} \mathcal{W}(\mathcal{B} \cap \mathcal{A}_n) \leq \sum_{n=1}^{N} \mathcal{W}(\hat{m}^{\theta}_{\varepsilon_n}(B) \leq C_\eta e^{-K\gamma}) \leq CKe^{-\frac{1}{4M}(1-\kappa)^2 K^2}.$$

So $\sum_{n=1}^{N} \mathcal{W}(\mathcal{B} \cap \mathcal{A}_n)$ tends to 0 as $K \to \infty$ and the result follows immediately. $\square$

Recall that if $r(t) \equiv G^{-1}(t + G(R))$, then the process $\{X_t : t \geq 0\}$ with $X_t = \mathcal{I}(h_{\mu^0_{r(t)}}) - \mathcal{I}(h_{\mu^0_R})$ has the same distribution as the standard Brownian motion. Let $\{\chi_l : l \geq 1\} \subseteq C_c(\overline{B_R})$ be a sequence of indicator functions such that $0 \leq \chi_l \nearrow \chi_{B_{r(t)}}$ and

$$f_l(\cdot) \equiv \chi_l(\cdot) e^{\frac{\gamma^2}{2\pi^2}(K_0(|\cdot|) \wedge l)} \nearrow \chi_{B_{r(t)}}(\cdot) e^{\frac{\gamma^2}{2\pi^2}(K_0(|\cdot|))} \text{ as } l \to \infty.$$

Then by the monotone convergence theorem, one can see that for any $t \geq 0$,

$$\mathbb{E}^{\mathcal{W}}[\hat{m}^{\theta}(B_{r(t)})|X_t] = \lim_{l\to\infty}\lim_{n\to\infty}\mathbb{E}^{\mathcal{W}}[M^{\theta}_{\varepsilon_n}(B_{r(t)})|X_t]. \tag{6.11}$$

It is not hard to see that when $t$ is large,

$$\mathbb{E}^{\mathcal{W}}[\hat{m}^{\theta_*}(B_{r(t)})|X_t] \equiv \exp\left(\gamma X_t - \left(6\pi^2 - \frac{\gamma^2}{2}\right)t\right). \tag{6.12}$$

**Lemma 6.3.** Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Borel set with Euclidean scaling exponent $\rho \in [0,1]$. For each $\omega > 0$, define the stopping time

$$T^*_\omega = \inf\left\{t \geq 0 : \hat{m}^{\theta_*}(B_{R(t)}) = \exp\left(\gamma X_t - (6\pi^2 - \frac{\gamma^2}{2})t\right) \leq \omega\right\}. \tag{6.13}$$

Also define the random radius

$$\theta \mapsto r^*_\Omega(\theta) \equiv G^{-1}(T^*_\omega(\theta) + G(R))$$

and the random neighbourhood

$$\theta \mapsto \Omega^{\omega,\theta} \equiv \cup_{z\in\Omega} B_{r^*_\Omega(\theta)}(z).$$

Then we have

$$\lim_{\omega\downarrow 0}\frac{\log \mathbb{E}^{\mathcal{W}}[m^\theta(\Omega^\theta(\omega))]}{\log\omega} = \lim_{\omega\downarrow 0}\frac{\log \mathbb{E}^{\mathcal{W}}[(r^*_\omega)^{3k}]}{\log \omega} = Q$$

where $Q \in [0,1]$ is determined by the following quadratic relation with $\varrho$:

$$\varrho = \frac{\gamma^2}{12\pi^2}Q^2 + (1 - \frac{\gamma^2}{12\pi^2})Q. \tag{6.14}$$

*Proof.* For every $\omega > 0$, define the stopping time $T^*_\omega$ by (6.13). Then for each $s \leq 0$, by Doob's stopping time theorem, $\left\{\exp\left[sY_{t\wedge T^*_\omega} - \frac{S^2}{2}(t\wedge T^*_\omega)\right] : t \geq 0\right\}$ is a uniformly bounded martingale. In addition, the continuity of Brownian motion implies that

$$Y_{T^*_\omega} = \frac{\log\omega}{\gamma} + \left(6\pi^2 - \frac{\gamma^2}{2}\right)\frac{T^*_\omega}{\gamma}. \tag{6.15}$$

Therefore, we have

$$\mathbb{E}^{\mathcal{W}}\left[\exp\left(-\frac{\gamma s^2 - 2s(6\pi^2 - \frac{\gamma^2}{2})}{2\gamma}\right) T^*_\omega\right] = \omega^{-s/\gamma}. \tag{6.16}$$

That is

$$\mathbb{E}^{\mathcal{W}}[m^\theta(\Omega^\theta(\omega))] \approx \mathbb{E}^{\mathcal{W}}[(r^*_\omega)^{3\varrho}] \approx \mathbb{E}^{\mathcal{W}}[\exp(-6\pi^2 \varrho T^*_\omega)]..$$

Given (6.16), for some $s \in [-\gamma, 0]$, set $6\pi^2\varrho = \frac{s^2}{2} - \frac{s}{\gamma}(6\pi^2 - \frac{\gamma^2}{2})$. Then we have

$$\begin{aligned} Q = \lim_{\omega\downarrow 0} \frac{\log \mathbb{E}^{\mathcal{W}}[m^\theta(\Omega^\theta(\omega))]}{\log \omega} &= \lim_{\omega\downarrow 0} \frac{\log \mathbb{E}^{\mathcal{W}}[\exp(-6\pi^2\varrho T^*_\omega)]}{\log \omega} \\ &= \lim_{\omega\downarrow 0} \frac{\log \omega^{-s/\gamma}}{\log \omega} = -\frac{s}{\gamma}, \end{aligned}$$

and the result in (6.14) follows immediately. □

**Lemma 6.4.** Assume the pair $(\rho, Q) \in [0,1]^2$ satisfies the quadratic relation in (6.14). Then

$$\lim_{\omega\downarrow 0} \frac{\log \mathbb{E}^{\mathcal{W}}[(r^*_\omega)^{3\rho}]}{\log \omega} = Q.$$

*Proof.* It suffices to show that there exists some constant $C > 0$ such that

$$C^{-1} \le \omega^{-Q} \mathbb{E}^{\mathcal{W}}[(r^*_\omega)^{3\rho}] \le C.$$

First, we show the existence of upper bound. Let $T_\omega \equiv G(\hat{r}_\omega) - G(R)$. by Lemma 6.2 and the fact that $\mathcal{W}(\Theta_0) = 1$, we have $T_\omega > -G(R)$ almost surely. Define

$$\Gamma = \frac{-\log \omega}{6\pi^2 - \gamma^2} \frac{\Delta - Q\gamma^2}{\Delta}$$

where $\Delta = 2\kappa(6\pi^2 - \gamma^2 - \frac{\gamma^2}{\kappa} - \eta)$. Then

$$\begin{aligned} &\mathbb{E}^{\mathcal{W}}[\exp(-6\pi^2\rho T_\omega)] \\ =\ &\mathbb{E}^{\mathcal{W}}[\exp(-6\pi^2\rho T_\omega)\chi_{\{-G(R)<T_\omega<\Gamma\}}] + \mathbb{E}^{\mathcal{W}}[\exp(-6\pi^2\rho T_\omega)\chi_{\{\Gamma\le T_\omega<\infty\}}]. \end{aligned}$$

For the first part, the volume $\mathrm{Vol}(B_{r(\Gamma)})$ with $r(\Gamma) = G^{-1}(\Gamma + G(R)) \approx \exp(-4\pi\Gamma)$ is bounded by $\Gamma$ under $\hat{m}^\theta(dy)$. Then

$$\mathrm{Vol}(B_{r(\Gamma)}) \approx (r(\Gamma))^{3-\frac{\gamma^2}{2\pi^2}} \approx \exp(-(6\pi^2 - \gamma^2)\Gamma)$$

under the measure $e^{\frac{\gamma^2}{2\pi^2}K_0(|y|)}dy$. Thus by Lemma 6.2, the probability of this event is bounded by

$$C\exp\left(-\frac{\Delta}{\gamma}\left(\frac{-\log\Gamma}{\gamma} - \frac{6\pi^2-\gamma^2}{\gamma}\Gamma\right)\right),$$

and the result follows immediately. For the second part, since the integral is bounded by $e^{-6\pi^2\rho\Gamma}$, we only need to show that for all $\omega > 0$, $Q\log\omega + 6\pi^2\rho\Gamma \geq 0$, that is

$$Q \leq \frac{6\pi^2\rho}{8\pi^2-\gamma^2}\frac{\Delta - Q\gamma^2}{\Delta}, \tag{6.17}$$

where $\Delta = 2\kappa(6\pi^2 - \gamma^2 - \frac{\gamma^2}{\kappa} - \eta)$. By Lemma 6.2, the statemnt in (6.17) is equivalent to

$$P(Q) \equiv \gamma^2 Q^2 + (12\pi^2 - \gamma^2 - \Delta)Q - \Delta \leq 0.$$

for all $Q \in [0,1]$. However, it is clearly true since both $P(0)$ and $P(1)$ are negative.

Second, we prove the lower bound. It is not hard to see that

$$\begin{aligned}\mathbb{E}^{\mathcal{W}}[\exp(-6\pi^2\rho T_\omega)] &\geq \mathbb{E}^{\mathcal{W}}[\exp(-6\pi^2\rho T_\omega^*)\chi_{\{}T_\omega \leq T_{\omega^*}\}] \\ &\geq \mathbb{E}^{\mathcal{W}}[\exp(-6\pi^2\rho T_\omega^*)] - \mathbb{E}^{\mathcal{W}}[\exp(-6\pi^2\rho T_\omega)\chi_{\{}T_\omega > T_{\omega^*}\}]\end{aligned}$$

Since $\mathbb{E}^{\mathcal{W}}[\exp(-6\pi^2\rho T_\omega^*)] = \omega^Q$, we only need to show that there exists some constant $0 < c < 1$ such that

$$\omega^{-Q}\mathbb{E}^{\mathcal{W}}[\exp(-6\pi^2\rho T_\omega)\chi_{\{}T_\omega > T_{\omega^*}\}] \leq C$$

uniformly in small $\omega$. Conditional on $T_\omega^* = T$, the event $\{T_\omega > T_{\omega^*}\}$ yields

$$\hat{m}^\theta(B_{r(T)}) > \omega \quad \text{and} \quad \hat{m}^\theta(B_{r(T)}) > \hat{m}^{\theta_*}(B_{r(T)}).$$

By Chebyshev's inequality, we have

$$\mathbb{P}(T_\omega > T | T^*_\omega) \leq C \cdot \mathbb{E}^{\mathcal{W}} \left[ \frac{\hat{m}^\theta(B_{r(T)})}{\hat{m}^{\theta_*}(B_{r(T)})} \right],$$

and the result follows. □

Now we are ready to present the KPZ relation for $m^\theta(dx)$.

**Theorem 6.3.** Assume $\Omega \subseteq \mathbb{R}^3$ be a bounded Borel set with Euclidean scaling exponent $\rho \in [0,1]$. Then $\Omega$ has quantum scaling exponent $Q \in [0,1]$ as defined in (6.10), where $Q$ is related to $\rho$ by (6.14).

*Proof.* Assume $\Omega \subseteq \overline{B_N(0)}$ for some large enough $N \geq 1$. Let $R(x,\theta;\omega)$ and $\Omega^\theta(\omega)$ be as defined in (6.7) and (6.8). Denote $\mathcal{N}(d\theta dx) \equiv \mathcal{W}(d\theta)dx$. Based on Lemma 6.1,

$$\begin{aligned} & \mathcal{M}(\{(x,\theta): \text{ either } x \in D \text{ or } \operatorname{dist}(x,\Omega) < R(x,\theta;\omega)\}) && (6.18) \\ = \; & \mathcal{N}(\{(x,\theta): |x| \leq 2N, \operatorname{dist}(x,\Omega) < \hat{R}(x,\theta;\omega)\}) && (6.19) \\ & + \lim_{2N \leq M \to \infty} \mathcal{N}(\{(x,\theta): 2N \leq |x| \leq M, \operatorname{dist}(x,\Omega) < \hat{R}(x,\theta;\omega)\}) && (6.20) \end{aligned}$$

For the first term in (6.18), we have

$$\begin{aligned} & \mathbb{E}^{\mathcal{W}} \left[ m^\theta(\Omega^\theta(\omega)) \right] \\ = \; & \mathcal{N}(\{(x,\theta): |x| \leq 2N, \operatorname{dist}(x,\Omega) < \hat{R}(x,\theta;\omega)\}) \\ = \; & \mathbb{E}^{\mathcal{W}} \left[ \operatorname{vol}\left(\Omega_{\hat{R}(x,\theta;\omega)}\right) \chi_{\hat{R}(x,\theta;\omega) \leq N} \right] + \mathbb{E}^{\mathcal{W}} \left[ \operatorname{vol}\left(\Omega_{\hat{R}(x,\theta;\omega)} \cap \overline{B_{2N}(0)}\right) \chi_{\hat{R}(x,\theta;\omega) \leq N} \right] \\ = \; & \mathbb{E}^{\mathcal{W}} \left[ \operatorname{vol}\left(\Omega_{\hat{R}(x,\theta;\omega)}\right) \right] - \mathbb{E}^{\mathcal{W}} \left[ \operatorname{vol}\left(\Omega_{\hat{R}(x,\theta;\omega)}\right) \chi_{\hat{R}(x,\theta;\omega) > N} \right] \\ & + \mathbb{E}^{\mathcal{W}} \left[ \operatorname{vol}\left(\Omega_{\hat{R}(x,\theta;\omega)} \cap \overline{B_{2N}(0)}\right) \chi_{\hat{R}(x,\theta;\omega) \leq N} \right]. \end{aligned}$$

When $\omega$ is sufficiently small,

$$\mathbb{E}^{\mathcal{W}} \left[ \operatorname{vol}\left(\Omega_{\hat{R}(x,\theta;\omega)}\right) \right] \approx \omega^Q.$$

On the other hand, the last terms in the right-hand side of the equation above are both bounded by

$$\mathbb{E}^{\mathcal{W}}\left[\left(\hat{R}(x,\theta;\omega)\right)^3 \chi_{\{\hat{R}(x,\theta;\omega)>N\}}\right] \leq 4\int_{[1,\infty)} z^2 \mathcal{W}(\hat{R}(x,\theta;\omega)>z)dz. \quad (6.21)$$

If $\Delta = 2\kappa(6\pi^2 - \gamma^2 - \frac{\gamma^2}{\kappa} - \eta)$, then

$$\mathcal{W}\left(\hat{R}(x,\theta;\omega)>s\right) \leq \mathcal{W}(\hat{m}_x^\theta(B_s(x)) \leq \omega) \leq C\omega^{\frac{\Delta}{\gamma^2}} s^{\frac{-\Delta}{\gamma^2}(3-\frac{\gamma^2}{2\pi^2})}.$$

Given $\eta, \kappa$ and $\Delta$, the right-hand side integral in (6.21) converges to zero faster than $\omega^Q$ as $\omega \downarrow 0$ for any $Q \in [0,1]$.

For the second term in (6.18), since $\Omega \subseteq \overline{B_N(0)}$, we have

$$\mathbb{P}\left(\hat{R}(x,\theta;\omega) > \frac{1}{2}|x|\right) \leq C\omega^{\frac{\Delta}{\gamma^2}} s^{-\frac{\Delta}{\gamma^2}(3-\frac{\gamma^2}{2\pi^2})}$$

with $\int_{|x|\geq 2N} \omega^{\frac{\Delta}{\gamma^2}} s^{-\frac{\Delta}{\gamma^2}(3-\frac{\gamma^2}{2\pi^2})} d\omega < \infty$. Therefore, the second term also converges to zero faster than $\omega^Q$ as $\omega \downarrow 0$. Combining the results of two terms, we finish the proof. □

# Chapter 7

# Conclusion

In this work, we invoke the tool of Fourier-Bessel expansion from the special function theory to extend and adapt sphere averaging method to treat log-correlated GFFs from even dimensions to arbitrary dimensions. In particular, via a regularization based on a weighted series of spherical averages of the GFF, we re-construct the Liouville quantum gravity measure and prove the KPZ formula in any dimension. This way of combining special function theory and probabilistic techniques proves to be robust and can be applied to treat more general types of GFFs. The broad connnections between Bessel functions and the Euclidean GFFs are also revisited and refreshed.

Although the sphere averaging procedure typically results in heavier technicality compared with the MCT approach in treating log-correlated GFFs, it is promising that it can be extended to treat more general types of GFFs, including polynomial-correlated GFFs, and more potential connections between Bessel-type functions and Gaussian random fields may be revealed, a possibility we hope to explore in the near future.

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# Appendices

# Appendix A

# The Bessel Differential Equation

In this appendix, we present some basics of Bessel functions, which are involved in this thesis. Most of the content can be found in [19].

## A.1 Bessel functions

The Bessel's equation is a linear second-order ordinary differential equation of type

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \nu \in \mathbb{R}.$$

If $\nu$ is not a negative integer, one solution of this differential equation using power series approach is

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu + k + 1)} (\frac{x}{2})^{\nu+2k},$$

which is known as the *Bessel function of the first kind.* If $\nu = -n$ with $n \in \mathbb{N}$, then $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$ solves the Bessel's equation. When $x \geq 0$ and $\nu > -1/2$, the *Poisson representation formula* of $J_\nu$ is given by

$$J_\nu(x) = \frac{(\frac{x}{2})^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} e^{ixs}(1 - s^2)^\nu \frac{ds}{\sqrt{1 - s^2}}. \tag{A.1}$$

As a special case, Bessel functions of half integer order $\nu = n + \frac{1}{2}$ with $n \in \mathbb{Z}$ can be expressed explicitly by trigonometric functions. In particular,

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x, J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left( \frac{\sin x}{x} - \cos x \right). \tag{A.2}$$

For $x \in \mathbb{R}$, if $x \to \infty$, then

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - \frac{\pi}{4} - \frac{\nu\pi}{2}). \tag{A.3}$$

**Recurrence formulas**

Some recurrence formulae of Bessel function $J_\nu$ are summarized as follows:

$$\nu J_\nu(x) + x J_\nu'(x) = x J_{\nu-1}(x), \tag{A.4}$$

$$\nu J_\nu(x) - x J_\nu'(x) = x J_{\nu+1}(x), \tag{A.5}$$

$$(\frac{d}{xdx})^n [x^\nu J_\nu(x)] = x^{\nu-n} J_{\nu-n}(x), \tag{A.6}$$

$$(\frac{d}{xdx})^n [x^{-\nu} J_\nu(x)] = (-1)^n x^{-\nu-n} J_{\nu+n}(x). \tag{A.7}$$

**Infinite integrals**

For $-a < a + 2 < 2b + \frac{7}{2}$,

$$\int_0^\infty \frac{x^{a+1} J_\nu(xy)}{(x^2 + \alpha^2)^{b+1}} dx = \frac{\alpha^{a-b} y^b}{2^b \Gamma(b+1)} K_{a-b}(\alpha y). \tag{A.8}$$

For $\alpha > 0$ and $\nu > -\frac{1}{2}$,

$$\int_0^\infty \frac{1}{(x^2 + \alpha^2)^{\nu + \frac{1}{2}}} \cos(xy) dx = \frac{y^\nu \pi^{1/2}}{(2\alpha)^\nu \Gamma(\nu + 1/2)} K_\nu(\alpha y). \tag{A.9}$$

For $a \geq b > 0$ and $y > 0$,

$$\int_0^\infty \frac{x}{x^2 + y^2} J_\nu(ax) J_\nu(bx) dx = K_\nu(ay) I_\nu(by). \tag{A.10}$$

## A.2 Modified Bessel functions

The modified Bessel's equation is given by

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0.$$

The method of power series gives a solution

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(\nu+k+1)} (\frac{x}{2})^{\nu+2k}.$$

Since this is a seond order differential equation, there exists two linearly independent solutions. The second solution is related to $I_\nu(x)$ by

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu x} \text{ for } \nu \notin \mathbb{Z}$$

and

$$K_n(x) = \lim_{\nu \to n} K_\nu(x) \text{ for } n \in \mathbb{Z}.$$

We call $I_\nu$ and $K_\nu$ the *modified Bessel functions of the first and second kind.*

### Recurrence formulas

Some recurrence relations in a similar form of $(A.4) - (A.7)$ hold for modified Bessel function of the first kind

$$\nu I_\nu(x) + xI'_\nu(x) = xI_{\nu-1}(x), \quad (A.11)$$

$$\nu I_\nu(x) - xI'_\nu(x) = -xI_{\nu+1}(x), \quad (A.12)$$

$$(\frac{d}{xdx})^n[x^\nu I_\nu(x)] = x^{\nu-n}I_{\nu-n}(x), \quad (A.13)$$

$$(\frac{d}{xdx})^n[x^{-\nu} I_\nu(x)] = x^{-\nu-n}I_{\nu+n}(x), \quad (A.14)$$

and modified Bessel function of the second kind

$$\nu K_\nu(x) + xK'_\nu(x) = -xK_{\nu-1}(x), \quad (A.15)$$

$$\nu K_\nu(x) - xK'_\nu(x) = xK_{\nu+1}(x), \quad (A.16)$$

$$(\frac{d}{xdx})^n[x^\nu K_\nu(x)] = (-1)^n x^{\nu-n}K_{\nu-n}(x), \quad (A.17)$$

$$(\frac{d}{xdx})^n[x^{-\nu} K_\nu(x)] = (-1)^n x^{-\nu-n}K_{\nu+n}(x). \quad (A.18)$$

## A.3 Zeros of Bessel functions

Suppose $j_{\nu,1}, j_{\nu,2}, \ldots$ are the positive zeros of $J_\nu$, arranged in ascending order of magnitude. The zeros of a Bessel function $J_\nu$ have the following properties:

- $J_\nu$ has infinitely many positive zeroes $j_{\nu,1}, j_{\nu,2}, \ldots$.
- If $\nu > -1$, then $\nu < j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < \ldots$.
- $j_{\frac{1}{2},k} = k\pi$ for $k \geq 1$.

The orthogonal property of Bessel function zeros is known as

$$\int_0^\alpha x J_\nu(j_{\nu,m}\frac{x}{\alpha}) J_\nu(j_{\nu,n}\frac{x}{\alpha}) dx = \begin{cases} \frac{\alpha}{2} J_{\nu+1}^2(j_{\nu,m}), & \text{if } m = n, \\ 0 & , \text{ if } m \neq n, \end{cases}$$

where $j_{\nu,1}, j_{\nu,2}, j_{\nu,3} \ldots$ denote the positive zeros of $J_\nu$ and $\alpha > 0$.

## A.4 The Fourier transform of spherical measures on $S_\varepsilon^{n-1}$

Let $\sigma$ denote the surface area measure on $S^{n-1}$ with $n \geq 3$ and $\sigma_\varepsilon^x$ denote the spherical measure whose action is to give the average value of test functions over the sphere $S_\varepsilon^{n-1}(x)$ centerd at $x \in \mathbb{R}^n$ with radius $\varepsilon > 0$. It can be found in [21, pp442] that

$$\int_{S^{n-1}} F(x \cdot y)\, d\sigma(y) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_{-1}^{+1} F(|x|\, s)(1-s^2)^{\frac{n-3}{2}} ds.$$

Then use this formula and (A.1) to write

$$\begin{aligned}
\hat{\sigma_\varepsilon^x}(\xi) &= \frac{1}{\sigma(S_\varepsilon^{n-1}(x))} \int_{S_\varepsilon^{n-1}(x)} e^{i(y,\xi)_{\mathbb{R}^n}} d\sigma(y) \\
&= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} \varepsilon^{n-1}} \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \varepsilon^{n-2} e^{i(x,\xi)_{\mathbb{R}^n}} \int_{-1}^{1} e^{i(s,\varepsilon|\xi|)_{\mathbb{R}^n}} (1-s^2)^{\frac{n-2}{2}} \frac{\varepsilon ds}{\sqrt{1-s^2}} \\
&= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \frac{\Gamma(\frac{n-2}{2}+\frac{1}{2})\Gamma(\frac{1}{2})}{(\frac{\varepsilon|\xi|}{2})^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(\varepsilon|\xi|) e^{i(x,\xi)_{\mathbb{R}^n}} \\
&= \frac{2^{\frac{n-2}{2}}\Gamma(\frac{n}{2})}{(\varepsilon|\xi|)^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(\varepsilon|\xi|) e^{i(x,\xi)_{\mathbb{R}^n}}.
\end{aligned}$$