From sailing ships to subtraction symbols: Multiple representations to support abstraction

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Teachers are tasked with supporting students' learning of abstract mathematical concepts. Students can represent their mathematical understanding in a variety of modes, for example: manipulatives, pictures, diagrams, spoken languages, and written symbols. Although most students easily pick up rudimentary knowledge through the use of concrete objects, we ask our students to use symbols and other mathematical notation to represent their understanding. Thus, teaching strategies that support abstraction are important for teachers' arsenals. In this paper, I use a case study of a Grade one teacher to illustrate how she uses multiple representations as a learning progression for the purposes of abstraction. I present a detailed description of one specific lesson that incorporated multiple representations and discuss her pedagogy with her four chosen representation forms. Administrators and mathematics teachers can use this case as a model for how multiple representations can be used to move students to abstraction.

Construction of knowledge has been studied by psychologists and educational theorists for many years (e.g., Alagic, 2003; Perkins, 1993; Piaget, 1995; Reys, Suydam, Lindquist, & Smith, 1998; Vygotsky, 1978). Researchers have examined how students learn and highlight strategies and environments that students should experience in order to make their construction of knowledge most effective and meaningful (e.g., De Bock, Deprez, Van Dooren, Roelens, & Verschaffel, 2011; Kim & Baylor, 2006; Mevarech & Kramarski, 1997; Van de Walle & Folk, 2005; Vygotsky, 1978).

The National Council of Teachers of Mathematics (NCTM, 2000) encourages teachers to create an environment where students learn mathematics with understanding. True understanding occurs when students are able to use what they know and apply it to new situations (Perkins, 1993). Students demonstrate understanding by "being able to carry out a variety of actions or performances with the topic by the ways of critical thinking: explaining, applying, generalizing, representing in new ways, making analogies and metaphors" (Alagic, 2003, p. 384). For this reason, teachers must create situations where students are given the opportunity to show their understanding in a variety of contexts. This is to ensure that students are successfully constructing knowledge. Students with a deep understanding of concepts are able to grasp subsequent concepts more efficiently (Alagic, 2003). These types of students will be more successful in their academic career and beyond. Instilling in students the goal of deeper understanding is to prepare them for the future.

Teachers are tasked with supporting students' learning of abstract mathematical concepts. Although most students easily pick up rudimentary knowledge through the use of concrete objects, we ask our students to use symbols and other mathematical notation to represent their understanding. These symbols are foreign and for some students, are difficult to grasp, especially if the students have not fully understood the mathematics concept that they represent (Uttal, Scudder, & DeLoache, 1997).

The concept of "abstraction" involves moving students from a concrete level of mathematical understanding of a concept to a generalized, more abstract understanding of the concept (Sfard, 1991). Research has found that the use of multiple representations can be used to support abstraction (e.g., Cooper & Warren, 2011; Ross & Willson, 2012). Specifically, representation forms that scaffold the students' understanding by moving the student from using real-world and concrete representation forms to those more abstract can be fruitful.

My research question is as follows: How do teachers use multiple representations for the purposes of abstraction? I use a case study of a Grade One teacher as an example of one teacher's beliefs about multiple representations and how she integrated them into her teaching practice. I will also recount one specific lesson that incorporated multiple representations. Although this paper does not focus on students' outcomes, this narrative of a teacher's use of multiple representations allows the reader to understand this teacher's perspective and how multiple representations can be used in the mathematics classroom. Educators can transfer ideas captured from this case study to practices within their own classroom.

Literature Review

Many researchers have discussed the strength of using multiple representation forms as a vehicle to construct students' mathematical knowledge and to support a deeper, more abstract understanding of mathematics. In this section, I first describe the literature regarding representations, then present the benefits of representations on student learning. I then present different perspectives about student learning and their connection to multiple representations, and conclude by presenting multiple representations as a means to scaffold learning and describe an instructional strategy that incorporates multiple representations as a learning progression.

Representations

Researchers have discussed the notion that mathematical ideas can be represented externally and internally (Putnam, Lampert, & Peterson, 1990). External representations include manipulatives, pictures, diagrams, spoken languages, and written symbols (Lesh, Post, & Behr, 1987) and internal representations include mental models and cognitive representations of the mathematical concept (Putnam et al., 1990). Goldin and Janvier (1998) classified representations under four interpretations: external, linguistic, formal, and internal. Their external, linguistic, and formal representations align with Putnam et al.'s (1990) external representations and Goldin and Janvier's internal representations parallel the namesake representation of Putnam et al. (1990).

Kaput (1985, 1987) wrote of the variety of forms of representation and their roles in learning, knowing and doing mathematics. Among the types of representations described, Kaput presented mathematical representations (using one mathematical structure to represent another), and external symbolic representation (using concrete objects to represent abstract ideas). Thus Kaput suggested that different representation forms may ease communication of mathematical ideas (the words "one-half", the symbolic notation of "1/2", and a picture of half of an object all represent the same concept) and more specifically, that concrete objects are a legitimate format for communicating abstract concepts. The variety of representation forms that exist opens the possibilities of how students can communicate their mathematical understanding (Putnam et al., 1990; Sternberg & Grigorenko, 2004).

It is important to clarify the distinction between models and representations. Cooper and Warren (2011) describe the difference as the following: "models are ways of thinking about abstract concepts (e.g., balance for equivalence) and representations are various forms of the models (e.g., physical balances, balance diagrams, balance language, equations as balance)" (p. 191). This paper focuses on the representation forms that teachers can use in their teaching practice.

Students are most comfortable with concrete ideas over those that are abstract (Reys et al., 1998). In mathematics, when first exposing students to new ideas, getting the students to interact with concrete tools (such as manipulatives) can ease them into the concept and develop the basic connections needed for them to progress to abstract ideas and the use of more complex mathematical language (Reys et al., 1998). It is a step-by-step progression as learning does not happen instantaneously and takes time to develop. Since concrete tools make learning accessible and the use of abstract mathematical language demonstrates deeper knowledge, it is important to incorporate each into a mathematics lesson. Teachers can use multiple representations to lead their students to make connections between concrete and abstract representation forms. By scaffolding the forms and guiding students to see the progressive lean towards abstract representation forms, students may be able to make the connections more easily (Alagic & Palenz, 2006).

Representation forms can vary in levels of abstractness. Using multiple representation forms provide a chance for a group of students with diverse ability levels to become engaged. Students can choose to use the representation form that is most meaningful to them and can move to higher levels of understanding using increasingly complex forms (Alagic, 2003). Teachers can also use this fact when they are assessing their students. If a student chooses a simpler representation, chances are the student has a more basic understanding of the concept.

Multiple Perspectives of Multiple Representations

The work of Lesh and various colleagues (e.g., Lesh, Landau, & Hamilton, 1983; Lesh, Post, & Behr, 1987) highlighted five representation forms that teachers should use in their teaching: real life experience, manipulative models, pictures or diagrams, spoken symbols and written symbols. The researchers did not describe the representation forms in a way that one form is more complex or requires more mathematical understanding than the others. They asserted that if students have the ability to translate between the different representation forms, students can be said to understand the mathematical idea (Lesh et al., 1987). The term “translation” was also used by Janvier (1987) to describe the process of moving among different representations of the same concept in order to support students’ mathematical development, specifically for problem solving. Lesh and his colleagues further elaborated that it is not the type of representation form used by the students that is of importance, but the intricacies of the “translation/transformation networks” that indicate the strength of the students’ mathematical understanding (Lesh et al., 1987, p. 36).

Piaget’s (1970) Four Stages of Cognitive Development describes the stages in which students advance their learning. In the early school years, Piaget believed that students are in a Pre-operational Stage (ages 2 to 7) during which students are developing their motor skills. The subsequent Concrete Operational Stage (ages 7 to 12) lasts throughout the majority of a students’ elementary school and during this stage, students’ mathematical understanding is best demonstrated through the use of manipulatives and symbols related to concrete objects. Piaget asserted that not until the Formal Operational Stage (age 12 onwards) that students develop abstract thought and that mathematical understanding is represented through symbols, yet even during elementary school, students are asked to use symbols to communicate their mathematical understanding. Thus, elementary school teachers may consider maximizing the natural tendencies of their students (using concrete objects) as a stepping stone to beginning to develop their mathematical fluency and usage of symbolic notation, and abstract thought (McNeil & Jarvin, 2007).

Dreyfus (1991) stated that learning happens as a result of students transitioning between multiple representation forms. He explained that students should use multiple representations in parallel, explore the connections between the representation forms and be able to fluently move between them. Gentner and Ratterman (1991) also suggested that students need to understand how the different representation forms are related to one another in order for effective mathematics learning to take place. Cooper and Warren (2011) described that "following the general sequence concrete to dynamic diagram to static diagram to symbols (e.g., physical balance to drawing of balance, blocks for number to symbols)" (p. 210) promotes flexibility in student thinking. They also highlighted the importance of making clear the connection that the representation form has to the real-world. For example, describing a real-world context in which a balance scale can be used.

Research has indicated that students benefit when their learning is connected to their practical and real-world knowledge (Baranes, Perry, & Stigler, 1989; Rittle-Johnson & Koedinger, 2005; Tepper, 1999). Uttal et al. (1997) stated that teachers should take advantage of their students' lived experienced and real-world examples of mathematics to develop mathematical knowledge. Uttal and his colleagues used a real world example of dividing pie or candy between friends as an informal experience with the mathematical concept of fractions, and suggested that teachers build on this context and ask their students to describe their experiences, re-create the scenario with physical objects, and model them through other representation forms.

Bruner (1966) theorized that representations fall under three categories: enactive, iconic, and symbolic. Bruner declared that students need three levels of engagement to build a complete understanding of a mathematics concept. Within the enactive category of representation forms, students use manipulatives and other hands-on objects (e.g., technological tools) to represent the mathematics concept. Bruner believed that the more forms of concrete representations with which the students can engage, the greater the opportunity for students to diminish their fixation on the physical object itself and instead focus on the mathematics the forms represent. Due to their tactile and physical nature, ionic representation forms have been found to increase student memory and mathematical understanding (Martin & Schwartz, 2005). Enactive representation forms are the most accessible and rudimentary representation form.

Iconic representations are in the form of pictures. Research has shown that images are effective in increasing students' mathematical understanding (e.g., Ferrucci, Kaur, Carter, & Yeap, 2008; Ng & Lee, 2009). For example, a study by Ainsworth, Bibby, and Wood (2002) determined that the use of pictorial representations increased the students' ages 8 to 10 abilities of estimation. Ainsworth et al. used a computer program, Computational Estimation Notation-Based Teaching System (CENTS), to let students pictorially investigate the concept of estimation.

Symbolic representations are the most complex representation form and stretch students to consider their mathematics understanding in a different way. Research has found that symbolic representations strengthen students' conceptual knowledge (e.g., Ainsworth et al., 2002; Nason & Woodruff, 2003; Sharp & Adams, 2002).

Similarly, Dienes (1960) created levels that students move through to create a thorough mathematical understanding. He created five levels in total: free play, generalization, representation, symbolization and formalization. During free play, students work with physical materials and manipulatives to discover basics about the concept. In generalization, students notice patterns and commonalities and then take these ideas to be represented by images in the representation level. Next, students describe their representation using mathematical language and symbols. Finally, they create a set of rules and algorithms to match their understanding of the concept.

With the structural levels proposed by Bruner and Dienes, one can imagine them to be rungs on a ladder, where the first rung (level) is the most basic and the upper rungs are most abstract. In contrast to the model of Lesh et al. (1983), Bruner and Dienes' models are hierarchical. Bruner and Dienes described their first levels to be accessible to students at all levels of understanding. The levels can be seen as scaffolding on top of one another and earlier levels can be used to support progress towards a deeper level of understanding that can be represented through a higher level. In both structures, the first level involves the use of concrete material, often in the form of manipulatives.

An Example of the use of Multiple Representations as a Learning Progression

The concrete-representational-abstract (CRA) instructional sequence is one teaching strategy used by mathematics educators to use multiple representations to increase students' mathematical competency (Flores, 2009). This strategy is especially documented to be effective with struggling mathematics students and students with learning disabilities (e.g., Avant & Heller, 2011; Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Cole & Washburn-Moses, 2010; Hudson & Miller, 2006; Maccini & Ruhl, 2000; Miller & Mercer, 1993; Witzel, 2005). As the name suggests, the CRA sequence is comprised of three phases: concrete (manipulatives), representational (pictures/drawings), abstract (numbers) and this is intended to be a multistep graduated instructional approach. CRA can also be referred to as CSA, concrete-semiconcrete-abstract sequence (Strickland & Maccini, 2010).

During the concrete phase of the instructional sequence, the teacher models the mathematical skill using manipulatives. Next, students work with the manipulatives and practice the mathematical skill as the teacher provides prompts and cue to support their progress. Examples of manipulatives that are often used in this phase include counters, blocks, balance scales, fraction tiles, algebra tiles, and geoboards. Once students are able to use the manipulatives independently, the teacher moves on to the next phase. In the representational or semiconcrete phase, a similar process to the first phase is repeated (teacher modeling followed by student practice with teacher support then independent student performance) however manipulatives are replaced with pictures, drawings, or virtual manipulatives. This phase is intended to act as a transition phase between the concrete and abstract phases. In this phase, students are encouraged to come up with their own pictures or drawings, often the pictures in this phase closely resemble the concrete objects that were used in the first phase. At the completion of this phase of the instructional sequence, some teachers present their students with memory aid in the form of a mnemonic. This mnemonic allows students to more easily remember the steps of the mathematical skill. The final phase of the instructional sequence, the abstract phase, promotes students' fluency with the mathematical task. In this phase students may only use numbers, symbols or variables to represent the mathematics.

CRA encourages students' conceptual understanding and emphasizes mastery of mathematics skills. In conjunction with each phase, the teacher is actively involved in CRA thus this strategy does not directly align with constructivist approaches. The teacher play a vital role in CRA as (s)he needs to monitor the students' progress and mastery of each phase. It is important to note that not all students will move at the same pace, thus repeated and informal assessment should occur regularly so that the teacher can determine when a student is ready to move on to the next phase (Witzel, Riccomini, & Schneider, 2008). Additionally, the teacher needs to value the transitions between phases as much as the phases themselves. "Without explicit awareness of how each stage connects with the next interconnected stage, the students may feel as though they are memorizing separate and arbitrary procedures to solve the same mathematical skill" (Witzel et al., 2008, p. 271). It is only until the final abstract phase where the teacher completely removes themselves from the learning process.

Representations can take many forms. Teachers who allow their students to discover mathematics concepts through multiple representation forms provide their students an increased opportunity to explore the concept in different ways. Additionally, a purposefully scaffolded sequence of representation forms can help develop students' mathematical understanding. Students have a strong grasp of a mathematics concept if they are able to represent their understanding through an abstract form. Teachers should consider using multiple representations to guide students to abstraction.

Methodology

Case study research was conducted (Stake, 1995), focusing on Sabrina, a Grade One teacher, and how she used multiple representations to scaffold students' mathematical learning. As a participant of the School Improvement in Mathematics project (McDougall, 2009; McDougall, Jao, Kwan, & Yan, 2011), Sabrina was actively seeking to improve her mathematics teaching. This two-year project focused on peer coaching as a model for professional development and investigated how participants improved and reflected upon their teaching practice by partnering up with a colleague and engaging classroom observations (as both observer and teacher) and a sequence of organized interviews (pre- and post-observation). During the project, a team of researchers (including the author of this paper) collected data in the form of field notes from classroom observations and transcripts from individual teacher and peer coaching interviews.

Sabrina started her teaching career in Central Canada, teaching at the elementary level. She then moved to Western Canada and continued to teach in this division. At the completion of the School Improvement in Mathematics Project, Sabrina had been teaching for 25 years. She has experience teaching Kindergarten, Grade One and Grade Two, and as a part-time resource teacher. During the study, she was the only teacher teaching Grade One at St. Brendan elementary school.

Sabrina was selected as a case study for this paper as it was observed that she regularly used multiple representations in her teaching practice. In peer coaching and individual interviews with Sabrina, she spoke of her rationale for using multiple representations. This prompted the author to believe that this was a carefully crafted strategy rather than coincidental occurrences. The data collected from the School Improvement in Mathematics project formed the data for this paper. Findings from data analysis are reported below. These findings reflect the themes described in the theoretical framework that are most relevant to beliefs expressed by Sabrina and observed teaching practices.

Themes

One of Sabrina's goals was for all of her students to have a "basic understanding" of mathematics. This basic understanding was just the beginning of what she hoped her students develop. Sabrina articulated that she had chosen to use multiple representations as a way for students to develop versatility with which to express their understanding of a mathematics concept. She believed that students should be able to represent concepts in different ways. She wanted her students to "see that real things can be represented [by] drawings too. You are trying to move them in their ability to think [in different] ways." Thus, Sabrina highlighted that the variety of representation forms was important to extend student mathematical understanding.

Sabrina elaborated on this statement by saying that students may start by expressing their mathematical understanding with one representation, but she hoped that through the use of multiple representations, students would be able to use a different representation that required a deeper mathematical understanding of the concept. Specifically, she believed that manipulatives are an accessible representation form and that students should move from “the concrete to the abstract”.

In this section, I present one example in which Sabrina used multiple representations in a lesson about subtraction. I use this example to illustrate the variety of representation forms that a teacher may use and the progression from basic representation forms to those more abstract.

Representation Form 1: Drama and Storytelling

Sabrina started her lesson about subtraction by pulling out a pile of cards in preparation for her first representation form: drama and storytelling. Sabrina selected one card at random from the pile and this card had a picture of a boat. Sabrina explained the logic behind the cards:

> It’s sometimes hard to find kids who’ll want to tell an addition or subtraction story, so all I did was make little blank cards and I said, “You can draw whatever you want on the card. Just one object on the card, whatever you want.” And then we shuffle them and then we say, “Okay, it’s story time”. Pick out a card, “Who wants to do the addition story?” They love it.

Sabrina explained that whenever she needs to create a story in her classroom, she turns to the student-drawn cards for inspiration. Sabrina found that her students showed more enthusiasm for a main character created by a student and that she does not have the burden of needing to come up with a topic herself each time.

After picking out the card with a picture of boat on it, Sabrina asked her students who created the card. A little boy, Sam, excitedly raised his hand and explained that the boat was in fact a pirate ship. Thus, the context for the story was set. Sabrina invited students to volunteer to play the part of pirates sailing on their ship. Almost all of the students put their hand up to volunteer, including Sam, and Sabrina selected Sam and five additional students to act as pirates. Sabrina told a story of six pirates out at sea, and the pirates enacted various duties including hoisting sails and scrubbing the deck as their peers watched with glee. Sabrina stated that she always creates a story that involved students moving about. She described this: “Moving bodies around, that’s a good thing to do.” Sabrina believed that physical participation in the story is important.

Sabrina continued her story by telling the class that a large wave struck the side of the pirate ship, causing four pirates to be swept out to sea. The six students enacted the scenario that Sabrina created and four students dramatically ‘fell off’ the ship when the ‘large wave’ swept them into the water. Some students collapsed on the ground while others hurled themselves across the room. Many of the students (‘actors’ and ‘audience members’ alike) were giggling. At this point in the story, Sabrina paused to ask the ‘audience’ how many ‘pirates’ were left on the ship. The students surveyed the scene and stated that there were still two ‘pirates’ on the ship. Sabrina concluded this representation form by summarizing the story as a subtraction sentence: “Six pirates take away four pirates equals two pirates”.

Representation Form 2: Manipulatives

Next, Sabrina asked the students to work in pairs and collect bags of manipulatives, including coloured stones, toy dinosaurs and stars. Sabrina asked the students to use the manipulatives to create subtraction sentences just as she had done through her storytelling about pirates. As the pairs recreated the subtraction stories, they personalized them based on the manipulative with which they were working. Instead of the students acting out the stories themselves, they used their manipulatives to model the stories.

I observed a pair of students with toy dinosaurs gleefully recount a story of a '*Tyrannosaurus Rex*' coming to eat some '*Stegosaurus* dinosaurs'. After an animated discussion about how dinosaurs attack their prey, the students enacted such an attack with their toys complete with sound effects including growling, shrieking and chomping noises. Once they were satisfied with their enactment, the students grouped their '*Stegosaurus* dinosaurs' such that those who were still alive remained in the centre of their 'scene' and those that had been 'eaten' were set aside. Students finished their story by summarizing the events. This summary was similar to that of the subtraction sentence that Sabrina had modeled at the end of her story, but used less mathematical terminology and focused on the storyline. "There were seven *Stegosauruses* [sic] in the forest. A T-rex ate four of them and three were left."

Sabrina explained that concrete objects are accessible for students of this age to illustrate mathematics concepts. She said, "The young students need a lot of concrete materials so that they can connect to the abstract concept. And it helps them to understand what it means and what they have to do with these materials." Sabrina continued to explain the strength of manipulatives and introduced the next representation form that she had the students use. She described that the specific progression of representation forms allows students to strengthen their mathematical understanding. She said: "It (manipulatives) helps them to solidify their understanding, they go from a model and then they'll go, you know, to representational drawings. You know, just help them to gain skills and increase their skills." When the students had finished using the manipulatives to create their subtraction sentence, Sabrina asked her students to chart their work by drawing their scenarios on paper.

Representation Form 3: Drawings

Using their manipulatives as a guide, Sabrina's students drew their subtraction sentences in different ways. Some students drew pictures of their manipulatives while others drew dots representing the initial quantity. There was also variety in the approaches used for the subtraction itself. Some students crossed out the number of items that corresponded with the subtrahend, while others erased the subtrahend. The resulting difference was either the non-crossed out items or those that remained on the page. As I observed different pairs of students, I noticed that some students had difficulty with this representation form. Sabrina mentioned that this particular representation form often caused students problems. She said,

> But I know that subtraction is – to show it in picture form is really difficult. Really hard, but we'll just practice. Maybe the line through the groups or crossing them out or whatever is easier for them to understand, but I know subtraction is really much harder– in terms of illustration.

Although the students used the grouped manipulatives as a visual guide, instead of subtracting (crossing out/erasing) the subtrahend, I observed many students subtract the difference. In other words, the students would mix up the two groups (subtrahend and difference) of manipulatives in their drawings. Upon completion of their drawings, Sabrina brought the pairs of students together for a full-class discussion to introduce the final representation form, mathematical symbols.

Representation Form 4: Introduction to Symbols

The final segment of the lesson brought the class together in a group discussion to share their pictures and subtraction sentences. The students were able to see each other's approaches to the visual representations and Sabrina concluded the lesson by describing that, when the numbers got larger, it would be tedious to draw each item for the minuend and thus, mathematical symbols (numbers, '–', and '=') could be used in place of pictures.

For further consolidation of the use of mathematical symbols, Sabrina recited a few more examples of subtraction sentences and invited students to volunteer and write out the sentences using symbols. If a student was unsure of the symbolic representation of the sentence, Sabrina would ask the student to first create a concrete or pictoral representation of the sentence. Sabrina explained that by using a more basic representation form first, the students could "use what they know already and go from what they already know (concrete or pictoral forms) to where they are going (symbolic form)".

In the post-lesson discussion, Sabrina said that she would build from this idea in the next lesson and have the students practice representing more subtraction sentences in pictures as well as with mathematical symbols. Sabrina says that, when using representations, a subsequent lesson always begins with a review, not only of the mathematics concepts that were studied previously, but also of the representation form. She says that her students are developing the skills and knowledge of how to use the various representation forms while using the forms to construct their understanding of mathematics concepts. Sabrina believes that the more her students can practice a concept, the better they will understand it.

Discussion

The purpose of this paper is to provide an example of how one teacher used multiple representations to scaffold students' mathematical learning. The case of Sabrina shows that she purposefully used a sequence of representation forms to lead students to abstraction.

Her students were exposed to a variety of strategies. Some of these strategies came naturally to the students and different students had an affinity to different strategies (for example, the students who were able to correctly use manipulatives to represent their subtraction sentence but got confused when asked to draw their subtraction sentences). Although students may struggle with certain tasks, it is important for them to be exposed to a variety of tasks. The students learned more about themselves as learners, finding out their strengths and weaknesses and hopefully focusing on their weaknesses so that they would become more well-rounded learners (Pape, Bell, & Yetkin, 2003).

This exposure to a variety of strategies enhanced the students' aptitude for representational thinking. She asked her students to work with and between the different forms to stimulate them to understand mathematics concepts at a deeper level (Lesgold, 1998; Pape & Tchoshanov, 2001). Her students learned to interpret the different forms and to develop a fluency to use different forms in different contexts.

Sabrina's scaffolding of the multiple representation forms that she uses in her class parallels Bruner's (1966) levels of engagement: enactive, iconic and symbolic. Sabrina used manipulatives as a starting point with her students (enactive). This level was accessible to all students no matter their ability level. Sabrina then moved her students to more abstract representations in the form of drawings and pictures (iconic). The final stage in Sabrina's progression occurred once her students were ready to represent their mathematical understanding using symbols and more advanced mathematical terminology (symbolic). By going through Bruner's three levels, Sabrina's students had a chance to build their mathematical understanding in a progressive manner so that they were not expected to quickly make the drastic jump to understanding abstract concepts. While some students may not have needed the first level to be competent at the second level, exposing her students to all levels gave students of varied abilities a safe zone where they could join the group and move up to the most abstract level together (Alagic, 2003).

Alagic and Palenz (2006) asserted that multiple representations allow all learners a chance to become engaged in learning. Multiple representations give students of all ability levels somewhere to start. The type of representation form used will draw out certain types of learners. No matter the ability level, students have an access point at which to start their learning. Sabrina put into place a model of progression where all of her students could use these multiple representation forms to build their mathematical understanding.

By introducing her students to a variety of forms from which they could select to express their mathematical understanding, Sabrina's students were able to explore different ways to communicate their knowledge. Students will not feel limited by the constraints of how they are able to communicate their learning and can experiment with different formats that will help them to grow as a learner. Sabrina's practice echoes work by Pape et al. (2003) that stated that students who have had to learn by using a variety of representation forms have developed skills to support their emerging mathematical understanding. These students had the creativity to take risks and try other routes to solving their problems.

Sabrina had chosen to use multiple representations to help her students move from concrete understanding to a deeper more abstract understanding of a concept. She empowered and engaged her students by exposing them to different representation forms that made the learning of the concepts more accessible. Concrete representations in the form of manipulatives are a fundamental representation form.

Although there has been much evidence that promotes the use of manipulatives to aid student learning, there has been an equal amount of research to show its flaws (e.g., Ball, 1992; Hiebert & Carpenter, 1992). Some examples of reasons against using manipulatives include: a lack of usefulness after Grade 1 (Friedman, 1978), a limited impact on students' mathematical understanding (Thompson, 1992), using the manipulative as a toy instead of as a learning tool (Moyer, 2001), and an inefficacy when students solve word problems and are not explicitly reminded to consider how manipulatives would represent the concept (Fuson & Briars, 1990).

Uttal et al. (1997) suggested that teachers be thoughtful in their choice of manipulative. They caution the use of items that are familiar to students (e.g., toys, food) as students may place too much focus on the objects rather on the mathematics that they represent. Uttal et al. believed that objects that are only used as mathematical representations should be used so that when presented to the students in a lesson, the students are cued to the knowledge that these objects will represent a new mathematics concept. The manipulatives that Sabrina used range from strictly mathematical tools (coloured stones) to toys (dinosaurs). Although the students with the dinosaurs were initially preoccupied with the physical objects that they had as opposed to the mathematics that they represented, after deciding on a particular context for their dinosaurs were able to focus on the mathematics. Perhaps this side-tracked discussion about context was in fact worthwhile to the students' learning process as the application of concept of subtraction in the real world (dinosaurs being eaten) provided a relevant connection between the mathematical task and the real world (Tepper, 1999). Witzel et al. (2008) stated that giving students choice and ownership over a representation form allows for a more meaningful experience and provides a context that students can more easily recall and with which they could connect.

Witzel et al. (2008) also indicated that teachers should pay attention to transitions between and relationships across representation forms. Sabrina kept the representation forms relatively independent other than between the "manipulatives" and "drawings" phases. Although her students started fresh in the manipulatives phase and could create their own subtraction story using their manipulatives, Sabrina asked the students to represent this same story through drawings in the next phase. This intentional overlap between phases mimics CRA. In the representational phase of CRA, students draw pictures of the concrete objects that they had just used in the concrete phase (Flores, 2009).

In general, multiple representations allow students to experience a variety of modes to communicate mathematics. The multimodality of multiple representations (visual, auditory, kinesthetic, and tactile) allows for multisensory experiences. Additionally, when using multiple representations, students of all learning styles will have a better chance of finding a representation of interest to them (Oberer, 2003). In terms of supporting students to understand abstract concepts, purposefully selected representation forms allow students understand the concept using more accessible forms (Noice & Noice, 2001). A focused sequence of representation forms that scaffolds the students' experience and learning from more basic form to a final most abstract representation can aid students' developing mathematics understanding.

This case study is an example of how a Grade One teacher used multiple representations to teach mathematics. Administrators and mathematics teachers can use Sabrina as a model for how multiple representations can be introduced to students in a mathematics context. The case of Sabrina illustrates how one educator used multiple representations in the classroom to move students to abstraction. Sabrina chose to incorporate the various representation forms in a specific order so that students first experienced the mathematics concept in an approachable and accessible form (storytelling and drama) and concluding to the most abstract form (mathematics symbols). As per Bruner (1966) and Dienes (1960), when using multiple representations in their own classroom, educators should consider following the lead of Sabrina and scaffold the use of multiple representations as a means to take students' mathematical understanding to a higher level. Further research could investigate students' performance to determine if this purposeful sequence of events in fact yields increased student understanding.

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